Malaya Journal of Matematik, Vol. 9, No. 1, 124-129, 2021

https://doi.org/10.26637/MJM0901/0021

Oscillation of first order delay differential equations

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Abstract

In this article, we establish some new criteria for the oscillation of the delay differential equation

$$y'(x)+q(x)y(\tau(x))=0, x\geq x_0, \tau(x)<x.$$

For the case where

$$\int_{\tau(x)}^{x} q(t)dt \geq \frac{1}{e} \text{ and } \lim_{x\to\infty}\int_{\tau(x)}^{x} q(t)dt=\frac{1}{e}.$$

An open problem by A. Elbert and I. P. Stavroulakis (1995, *Proc. Amer. Math. Soc.,* 123, 1503–1510) is solved.

Keywords

Oscillation, Non oscillation, Delay Differential equation.

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Article History: Received **24** September **2020**; Accepted **19** December **2020**

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1. Introduction

Consider the delay differential equation

$$y'(x)+q(x)y(\tau(x))=0, x\geq x_0, \tag{1.1}$$

where

$$q(x)\geq 0, \tau(x)<x \text{ and } \lim_{x\to\infty}\tau(x)=\infty$$

The study of oscillatory properties of solutions of (1.1) has been the subject of interest for many investigators. The first systematic study of the problem was made by myskis [12]. In 1950, myskis [11] obtained the first criterion for the oscillation of equation (1.1). since then the oscillatory properties of equation (1.1) have been studied extensively. Now it is well known that every solution of (1.1) oscillates provided that either

$$\alpha=\liminf_{x\to\infty}\int_{\tau(x)}^{x} q(t)dt>\frac{1}{e} \tag{1.2}$$

$$\beta=\limsup_{x\to\infty}\int_{\tau(x)}^{x} q(t)dt>1 \tag{1.3}$$

Condition (1.2) has been extensively used in the study of oscillation of various functional differential equations. For example, see [1,2,3,9]. There have been many papers improving the conditions (1.2) and (1.3). See, for example, [5-9]. In particular, Yu and Wang [9] established that every solution (1.1) is oscillatory if

$$\alpha\leq\frac{1}{e} \text{ and } \beta=1-\frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha-\alpha^2}\right) \tag{1.4}$$

which improves condition (1.3). Elbert and Stavroulakis [5] and Li [6] have proved that every solution of equation (1.1) is oscillatory if

$$\int_{\tau(x)}^{x} q(t)dt\geq\frac{1}{e} \tag{1.5}$$

which improves condition (1.2).

Li [14] and Tang and Shen [10] established another new criteria for the oscillation of (1.1) without condition (1.5). These results further improve many known results in literature.

We assume that the delay function $\tau(x)$ is strictly increasing on $[x_0,\infty)$ and $\tau^{-1}(x)>x$ is its inverse. Define $\tau^{-k}(x)$ on $[x_0,\infty)$ by

$$\tau^{-(k+1)}(x)=\tau^{-1}\left(\tau^{-k}(x)\right), \quad k=1,2\ldots \tag{1.6}$$

and let

$$x_k=\tau^{-k}(x_0), \quad k=1,2\ldots \tag{1.7}$$

Clearly $x_k \to \infty$ as $k \to \infty$.
Assume that the coefficient $q(x)$ is a piecewise continuous function and satisfies condition (1.5). Elbert and Stavroulakis [5] obtained best results on the oscillation of (1.1), when

$$\int_{\tau(x)}^{x} q(t)dt \geq \frac{1}{e} \text{ and } \lim_{x\to\infty}\int_{\tau(x)}^{x} q(t)dt = \frac{1}{e}. \tag{1.8}$$

They are the following Theorems 1.1 and 1.2.

Theorem 1.1. *Assume that $q(x) \in A_\lambda$ for some $\lambda \in (0,1]$ the definition of A_λ is found in* [5] *and either*

$$\lambda \limsup_{k\to\infty} k \sum_{i=k}^{\infty}\left(\int_{x_{i-1}}^{x_i} q(t)dt - \frac{1}{e}\right) > \frac{2}{e} \tag{1.9}$$

or

$$\lambda \liminf_{k\to\infty} k \sum_{i=k}^{\infty}\left(\int_{x_{i-1}}^{x_i} q(t)dt - \frac{1}{e}\right) > \frac{1}{2e}. \tag{1.10}$$

Then every solution of (1.1) *oscillates.*

Theorem 1.2. *Let $\tau(x) = x-1$, $q(x) = \frac{1}{e} + a(x)$ and $x_0 = 1$ in* (1.1). *Assume that*

$$a(x) \leq \frac{1}{8ex^2}. \tag{1.11}$$

Then (1.1) *has a solution $y(x) \geq \sqrt{x}e^{-x}$.*

Remark 1.3. *For the following equation*

$$y'(x) + \left(\frac{1}{e} + Kx^{-\alpha}\right)y(x-1) = 0, x \geq 1. \tag{1.12}$$

By Theorems 1.1 *and* 1.2*, we see that every solution of* (1.12) *oscillates for any $k > 0$ if*

$$0 \leq \alpha < 2 \text{ or } K > \frac{1}{2e} \text{ if } \alpha = 2$$

On the other hand, (1.12) *has a nonoscillatory solution for any*

$$0 < K < \frac{1}{8e} \text{ if } \alpha = 2 \text{ or } K > 0 \text{ if } \alpha > 2 \text{ or } K \leq 0.$$

Clearly there is a gap between $\frac{1}{2e}$ and $\frac{1}{8e}$; therefore, Elbert and Stavroulakis posted the following open problem in [5].

Example 1.4. *Can the bounds in conditions* (1.9) *and* (1.10) *of Theorem* 1.1. *can be replaced by smaller ones?*
Our aim in this paper is to solve the above open problem. The main results are the following.

Theorem 1.5. *Assume that* (1.5) *holds, and*

$$\limsup_{k\to\infty} k \int_{x_k}^{\infty} q(x)\left(\int_{\tau(x)}^{x} q(t)dt - \frac{1}{e}\right)dx > \frac{1}{e^2}. \tag{1.13}$$

Then every solution of (1.1) *oscillates.*

Theorem 1.6. *Let $\tau(x) = x - \tau, q(x) = \frac{1}{e\tau} + a(x), \tau > 0$ and $a(x) \geq 0$ in* (1.1). *Assume that*

$$\liminf_{x\to\infty} x \int_{x}^{\infty} a(t)dt > \frac{\tau}{8e}. \tag{1.14}$$

Then every solution of (1.1) *oscillates.*

Remark 1.7. *Theorems* 1.5 *and* 1.6 *remove the condition $q(x) \in \mathrm{A}_\lambda$ in Theorem* 1.1. *If $q(x) \in \mathrm{A}_\lambda$ for some $\lambda \in (0,1]$ then*

$$\int_{\tau(x)}^{x} q(t)dt - \frac{1}{e} \geq \lambda_k\left(\int_{x_k}^{x_{k+1}} q(t)dt - \frac{1}{e}\right) \geq 0,$$

for $x_k < x < x_{k+1}$, $k = 1, 2, \ldots$. Where $\lim_{k\to\infty}\lambda_k = \lambda$. It follows that

$$\begin{aligned}
&(k-1)\int_{x_{k-1}}^{\infty} q(x)\left(\int_{\tau(x)}^{x} q(t)dt - \frac{1}{e}\right)dx \\
&\quad = (k-1)\sum_{i=k-1}^{\infty}\int_{x_i}^{x_i+1} q(x)\left(\int_{\tau(x)}^{x} q(t)dt - \frac{1}{e}\right)dx \\
&\quad \geq (k-1)\sum_{i=k-1}^{\infty}\lambda_i\left(\int_{x_i}^{x_{i+1}} q(t)dt - \frac{1}{e}\right)\int_{x_i}^{x_i+1} q(x)dx \\
&\quad \geq \frac{k-1}{e}\sum_{i=k-1}^{\infty}\lambda_i\left(\int_{x_i}^{x_{i+1}} q(t)dt - \frac{1}{e}\right) \\
&\quad = \frac{k-1}{ek}k\sum_{i=k}^{\infty}\lambda_{i-1}\left(\int_{x_{i-1}}^{x_i} q(t)dt - \frac{1}{e}\right).
\end{aligned}$$

Therefore

$$\lambda \limsup_{k\to\infty} k \sum_{i=k}^{\infty}\left(\int_{x_{i-1}}^{x_i} q(t)dt - \frac{1}{e}\right) > \frac{1}{e} \tag{1.15}$$

implies (1.13). *We remark that the bound $\frac{1}{e}$ in* (1.15) *is a half of $\frac{2}{e}$ in* (1.9).

Remark 1.8. *If $\tau(x) = x - \tau$ and $q(x) = \frac{1}{e\tau} + a(x)$, then $x_k = x_0 + k\tau$ for $k = 1, 2, \ldots$. For,*

$$x_0 + (k-2)\tau = x_{k-2} < x \leq x_{k-1} = x_0 + (k-1)\tau, k = 2, 3\ldots$$

$$\begin{aligned}
\frac{x}{\tau}\int_{x}^{\infty} a(t)dt &\geq \frac{x_0 + (k-2)\tau}{\tau}\int_{x_0+(k-1)\tau}^{\infty} a(t)dt \\
&= \left(1 - \frac{2}{k} + \frac{x_0}{k\tau}\right)k\sum_{i=k}^{\infty}\left(\int_{x_{i-1}}^{x_i} q(t)dt - \frac{1}{e}\right).
\end{aligned}$$

Therefore

$$\liminf_{k\to\infty} k \sum_{i=k}^{\infty}\left(\int_{x_{i-1}}^{x_i} q(t)dt - \frac{1}{e}\right) > \frac{1}{8e} \tag{1.16}$$

implies (1.14). *We remark that the bound $\frac{1}{8e}$ in* (1.16) *is also a quarter of $\frac{1}{2e}$ in* (1.10).

Remark 1.9. *For equation* (1.12)*, by Theorem* 1.6 *and Theorem* 1.2 *we have that every solution of* (1.12) *oscillates if and only if $k > 0$ and $\alpha < 2$ or $k > \frac{1}{8e}$ and $\alpha = 2$. This shows that condition* (1.14) *is unimprovable in that sense.*

Remark 1.10. *If*

$$\int_{x_0}^{\infty} q(x)\left(\int_{\tau(x)}^{x} q(t)dt - \frac{1}{e}\right)dx = \infty$$

condition (1.13) *holds naturally. Therefore, Theorem* 1.5 *also improves* [5, *Theorem 1*], *and the main result in* [6].

2. Some Lemmas

To prove Theorems 1.5 and 1.6, we need the following lemmas.

Lemma 2.1 ([5]). *Assume that* (1.5) *holds and* $y(x)$ *is a positive solution of* (1.1) *on* $[x_{k-3}, x_{k+1}]$ *for some* $k \geq 3$. *Let* M, N *be defined by*

$$N = \min_{x_{k-1}\leq x\leq x_k} \frac{y(\tau(x))}{y(x)}, \quad M = \min_{x_k\leq x\leq x_{k+1}} \frac{y(\tau(x))}{y(x)}.$$

Then, $1 < N \leq M < (2e)^2$. *Let the sequence* $\{r_i\}_{i=0}^{\infty}$ *be defined by the recurrence relation*

$$r_0 = 1, r_{i+1} = e^{r_i/e} \text{ for } i = 0, 1\ldots \tag{2.1}$$

Lemma 2.2 ([5]). *For the sequence* $\{r_i\}_{i=0}^{\infty}$ *in* (2.1) *the following relations hold*

(a) $r_i < r_{i+1}$

(b) $r_i < e$

(c) $\lim_{i\to\infty} r_i = e$

(d) $r_i > e - 2e/(i+2)$.

Lemma 2.3. *Assume that* $\tau > 0$ *and* $a(x) : [x_0, \infty) \to [0, \infty)$ *is a piecewise Continuous function and* (1.14) *holds. Then every solution of the equation*

$$z'(x) + \left(\tau^{-1} + ea(x)\right)z(x-\tau) - \tau^{-1}z(x) = 0, \quad x \geq x_0 \tag{2.2}$$

oscillates.

Proof. Suppose the contrary. Then we may assume, without loss of generality, that there exists a solution $z(x)$ such that $z(x) > 0$ for $x \geq x_k = x_0 + k\tau$ for some $k > 0$. Rewrite (2.2) as

$$\left(z(x) - \tau^{-1}\int_{x-\tau}^{x} z(t)dt\right)' + ea(x)z(x-\tau) = 0, \quad x \geq x_k. \tag{2.3}$$

By [15, Lemma 1] we have eventually

$$s(x) \triangleq z(x) - \tau^{-1}\int_{x-\tau}^{x} z(t)dt > 0 \text{ and } s'(x) \leq 0. \tag{2.4}$$

Let $k_1 \geq k+1$ such that the inequalities above hold for $x \geq x_{k_1} = x_0 + k_1\tau$. Set $N = 2^{-1}\min\left\{z(x) : x_{k_1} - \tau \leq x \leq x_{k_1}\right\}$. We claim that

$$z(x) > N \text{ for } x \geq x_{k_1} - \tau. \tag{2.5}$$

In fact, if (2.5) does not hold, we may let $x^* = \inf\{x > x_{k_1} : z(x) \leq N\}$ such that $z(x) > N$ for $x_{k_1} - \tau \leq x \leq x^*$ and $z(x^*) = N$. By (2.4) we have

$$N = s(x^*) + \tau^{-1}\int_{x^*-\tau}^{x^*} z(t)dt > N$$

This is a contradiction and so (2.5) holds.
Let $\lim_{x\to\infty} s(x) = m$. Then there exist two possible cases:
Case 1. $m = 0$. Let $k_2 > k_1$ such that $s(x) < \frac{N}{4}$ for $x \geq x_{k_2} = x_0 + k_2\tau$. Then for any $\bar{x} \geq x_{k_2}$, we have

$$z(x) > 2\tau^{-1}\int_{\bar{x}}^{x+\tau} s(t)dt_{\text{for }} x \in [\bar{x}, \bar{x}+\tau].$$

Case 2. $m > 0$. Noting that $s'(x) \leq 0$, for $x \geq x_{k_1}\cdot$ We have $s(x) \geq m$ for $x \geq x_{k_1}$. From (2.4) and (2.5) we get

$$z(x) \geq m + \tau^{-1}\int_{x-\tau}^{x} z(t)dt \geq m + N, x \geq x_{k_1}$$

By induction, one can easily show that

$$z(x) \geq nm + N, x \geq x_{k_1} + (n-1)\tau, n = 1, 2\ldots$$

and so $\lim_{x\to\infty} z(x) = \infty$, which implies that there is a $k_3 > k_2$ such that

$$z(x) > 2\tau^{-1}\int_{x_{k_3}}^{x+\tau} s(t)dt_x \in \left[x_{k_3}, x_{k_3}+\tau\right]$$

Combining cases 1 and 2, we see that there exists a $X > x_{k_2}$ such that

$$z(x) > 2\tau^{-1}\int_{X}^{x+\tau} s(t)dt \quad x \in [X, X+\tau].$$

Now we prove that

$$z(x) > 2\tau^{-1}\int_{X}^{x+\tau} s(t)dt \quad x \geq X. \tag{2.6}$$

If (2.6) does not hold, then we may define X^* by

$$X^* = \inf\left\{x \geq X+\tau : z(x) \leq 2\tau^{-1}\int_{X}^{x+\tau} s(t)dt\right\}$$

and so

$$z(x) > 2\tau^{-1}\int_{X}^{x+\tau} s(t)dt \text{ for } x \in [X, X^*)$$

and

$$z(X^*) = 2\tau^{-1}\int_{X}^{X^*+\tau} s(t)dt.$$

By (2.4) we have

$$\begin{aligned}
&2\tau^{-1}\int_X^{X^*+\tau} s(t)dt\\
&= s(X^*)+\tau^{-1}\int_{X^*-\tau}^{X^*} z(t)dt\\
&> s(X^*)+2\tau^{-2}\int_{X^*-\tau}^{X^*} dt\int_X^{t+\tau} s(\xi)d\xi\\
&= s(X^*)+2\tau^{-1}\int_X^{X^*+\tau} s(t)dt-2\tau^{-2}\int_{X^*}^{X^*+\tau}(t-X^*)s(t)dt\\
&\geq s(X^*)+2\tau^{-1}\int_X^{X^*+\tau} s(t)dt-2\tau^{-2}s(X^*)\int_{X^*}^{X^*+\tau}(t-X^*)\,dt\\
&= 2\tau^{-1}\int_X^{X^*+\tau} s(t)dt.
\end{aligned}$$

This is a contradiction and so (2.6) holds. From (2.6) we obtain

$$z(x-\tau) > 2\tau^{-1}\int_X^x s(t)dt \quad for \quad x\geq X+\tau \tag{2.7}$$

Let $v(x)=\int_X^x s(t)dt$; Then, $v'(x)=s(x), v''(x)=s'(x)$. From (2.3), (2.4) and (2.7), we have

$$v''(x)+2\tau^{-1}ea(x)v(x)\leq 0, x\geq X+\tau. \tag{2.8}$$

This shows that inequality (2.8) has an eventually positive solution. On the other hand, by a known result in [16], condition (1.14) implies that (2.8) has no eventually positive solution. This contradiction completes the proof. □

3. Proofs of Theorems

Proof of Theorem 1.5.
Suppose the contrary. Then we may assume, without loss of generality, that there exists a solution $y(x)$ such that

$$y(x)>0 \text{ for } x\geq x_{m-3} \text{ for some } m\geq 3.$$

Set,

$$u(x)=\frac{y(\tau(x))}{y(x)} \text{ for } x\geq x_{m-2}. \tag{3.1}$$

Then $u(x)\geq 1$ for $x\geq x_{m-2}$. From (1.1) we have

$$u(x)=\exp\left(\int_{\tau(x)}^x q(t)u(t)dt\right), \quad x\geq x_{m-1}. \tag{3.2}$$

Let the sequence $\{M_i\}_{i=0}^{\infty}$ be defined by

$$M_i=\min\{u(x): x_{m+i-1}\leq x\leq x_{m+i}\}\, i=0,1\ldots. \tag{3.3}$$

By Lemma 2.1, we have

$$1<M_i\leq M_{i+1}<4e^2 \text{ for } i=0,1,2\ldots. \tag{3.4}$$

This shows that the sequence $\{M_i\}_{i=0}^{\infty}$ converges. Let

$$\lim_{i\to\infty} M_i = M. \tag{3.5}$$

By (1.5), (3.2) and (3.3), we have

$$M_{i+1}\geq \exp\left(\frac{M_i}{e}\right) \text{ for } i=0,1,2,\ldots$$

which, together with (3.5), yields $M\geq \exp\left(\frac{M}{e}\right)$. It is easy to check that

$$e^{y/e}>y \quad \text{for} \quad y\neq e$$

This inequality, (3.6), and (3.5) imply that $M=e$ and

$$1<M_1\leq M_2<\ldots<e$$

From (3.3) and (3.5) we have

$$u(x)\geq M_i \text{ for } x\geq x_{m+i-1}, \quad i=0,1,2,3\ldots. \tag{3.6}$$

It follows from (3.2) that for $x\geq x_{m+i}$

$$\begin{aligned}
&u(x)=\exp\left(\int_{\tau(x)}^x q(t)u(t)dt\right)\\
&=\exp\left(\int_{\tau(x)}^x q(t)\left(u(t)-M_i\right)dt+\frac{M_i}{e}\right)\\
&\qquad\times\exp\left[M_i\left(\int_{\tau(x)}^x q(t)dt-\frac{1}{e}\right)\right]\\
&\geq\left[e\int_{\tau(x)}^x q(t)\left(u(t)-M_i\right)dt+M_i\right]\left[1+M_i\left(\int_{\tau(x)}^x q(t)dt-\frac{1}{e}\right)\right]\\
&\geq e\int_{\tau(x)}^x q(t)\left(u(t)-M_i\right)dt+M_i+M_i^2\left(\int_{\tau(x)}^x q(t)dt-\frac{1}{e}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&q(x)\left(u(x)-M_i\right)-eq(x)\int_{\tau(x)}^x q(t)\left(u(t)-M_i\right)dt\\
&\qquad\geq M_i^2 q(x)\left(\int_{\tau(x)}^x q(t)dt-\frac{1}{e}\right), x\geq x_{m+i}, i=0,1\ldots.
\end{aligned} \tag{3.7}$$

Integrating both sides of (3.7) from x_{m+i} to $X>x_{m+n}$ with $n\geq i+1$

$$\begin{aligned}
&\int_{x_{m+i}}^X q(x)\left(u(x)-M_i\right)dx-e\int_{x_{m+i}}^X q(x)\left(\int_{\tau(x)}^x q(t)\left(u(t)-M_i\right)dt\right)dx\\
&\qquad\geq M_i^2\int_{x_{m+i}}^X q(x)\left(\int_{\tau(x)}^x q(t)dt-\frac{1}{e}\right)dx, \text{ for } i=0,1\ldots
\end{aligned} \tag{3.8}$$

By interchanging the order of integration and using (1.5) and (3.6), we have

$$\begin{aligned}
& e\int_{x_{m+i}}^{X} q(x)\int_{\tau(x)}^{x} q(t)\,(u(t)-M_i)\,dt\,dx \\
&= e\int_{x_{m+i}}^{\tau(X)} q(x)\,(u(x)-M_i)\int_{x}^{\tau^{-1}(x)} q(t)dt\,dx \\
&\quad + e\int_{x_{m+i-1}}^{x_{m+i}} q(x)\,(u(x)-M_i)\int_{x_{m+i}}^{\tau^{-1}(x)} q(t)dt\,dx \\
&\quad + e\int_{\tau(X)}^{X} q(x)\,(u(x)-M_i)\int_{x}^{X} q(t)dt\,dx \\
&\geq \int_{x_{m+i}}^{\tau(X)} q(x)\,(u(x)-M_i)\,dx \\
&\quad + e\,(M_n-M_i)\int_{\tau(X)}^{X} q(x)\int_{x}^{X} q(t)dt\,dx \\
&\geq \int_{x_{m+i}}^{\tau(X)} q(x)\,(u(x)-M_i)\,dx + (M_n-M_i)\,/2e
\end{aligned}$$

It follows from (3.8) that

$$\begin{aligned}
&\int_{\tau(X)}^{X} q(x)\,(u(x)-M_i)\,dx + \frac{(M_n-M_i)}{2e} \\
&\geq M_i^2\int_{x_{m+i}}^{X} q(x)\left(\int_{\tau(x)}^{x} q(t)dt-\frac{1}{e}\right)dx,\ \text{for } i=0,1\ldots
\end{aligned}$$

or

$$\begin{aligned}
&\ln u(x)-\frac{(M_n+M_i)}{2e} \\
&\quad\geq M_i^2\int_{x_{m+i}}^{X} q(x)\left(\int_{\tau(x)}^{x} q(t)dt-\frac{1}{e}\right)dx,\ \text{for } i=0,1\ldots. \qquad (3.9)
\end{aligned}$$

Let $X_n \in [x_{m+n}, x_{m+n+1}]$ such that $u(X_n)=M_{n+1}$ for $n=1,2\ldots.$ Set $X=X_n$ in (3.9) and taking the superior limit as $n\to\infty$ we get from (3.9)

$$\frac{\mathrm{e}-\mathrm{M_i}}{2\mathrm{e}} \geq M_i^2\int_{x_{m+i}}^{\infty} q(x)\left(\int_{\tau(x)}^{x} q(t)dt-\frac{1}{e}\right)dx,\ \text{for } i=0,1\ldots \qquad (3.10)$$

Comparing (2.1) with (3.6) we can obtain by induction

$$M_0>r_0, M_i>r_i \text{ for } i=1,2,3,\ldots$$

Then by Lemma 2.2 (d) we have

$$e-M_i<e-r_i<\frac{2e}{i+2}.$$

Multiplying (3.10) by $m+i$ we obtain

$$\frac{m+i}{i+2} \geq M_i^2(m+i)\int_{x_{m+i}}^{\infty} q(x)\left(\int_{\tau(x)}^{x} q(t)dt-\frac{1}{e}\right)dx,\ \text{for } i=0,1\ldots$$

Taking the superior limit as $i\to\infty$ we get

$$1\geq e^2\limsup_{k\to\infty} k\int_{x_k}^{\infty} q(x)\left(\int_{\tau(x)}^{x} q(t)dt-\frac{1}{e}\right)dx.$$

which contradicts (1.13). The proof is complete. □

Proof of Theorem 1.6.
Suppose the contrary. Then we may assume that there exists a solution $y(x)$ such that $y(x)>0$ for $x\geq x_k$ for some $k>0$. Set

$$w(x)=y(x)e^{\frac{x}{\tau}} \text{ for } x\geq x_k. \qquad (3.11)$$

Then from (1.1) we have,

$$w'(x)+\left(\tau^{-1}+ea(x)\right)w(x-\tau)-\tau^{-1}w(x)=0, x\geq x_k. \qquad (3.12)$$

This shows that (2.2) has an eventually positive solution $w(x)$. On the other hand, by Lemma 2.3, (1.14) implies that (2.2) has no eventually positivesolution. This contradiction completes the proof. □

4. Example

Consider the equation

$$y'(x)+\left(\frac{1}{e}+kx^{-\alpha}\sin^2(\beta x)\right)y(x-1)=0, x\geq 1, \qquad (4.1)$$

where $q(x)=\frac{1}{e}+kx^{-\alpha}\sin^2(\beta x), k,\alpha,\beta>0$, and $\tau=1$. By a simple calculation, one can obtain

$$\begin{aligned}
& x\int_{x}^{\infty} a(t)dt = kx\int_{x}^{\infty} t^{-\alpha}\sin^2(\beta t)dt \\
&= kx\sum_{k=0}^{\infty}\int_{x+k\pi/\beta}^{x+(k+1)\pi/\beta} t^{-\alpha}\sin^2(\beta t)dt \\
&\geq (2\beta)^{-1}k\pi x\sum_{k=0}^{\infty}[x+(k+1)\pi/\beta]^{-\alpha} \\
&\geq \left\{\left[2(\alpha-1)^{-1}kx^{2-\alpha}(1-\pi/\beta x)^{1-\alpha}, \alpha>1\right]\right.
\end{aligned}$$

Therefore

$$\liminf_{x\to\infty} x\int_{x}^{\infty} a(t)dt = \begin{cases} \infty, \alpha<2 \\ \frac{k}{2}, \alpha=0 \end{cases}$$

Applying Theorem 1.6, we see that every solution of (4.1) oscillates for any

$$k>0 \quad \text{if} \quad 0\leq\alpha<2 \text{ or } k>\frac{1}{4e} \text{ if } \alpha=2.$$

The same conclusion, however, cannot be inferred from Theorem 1.1 when $\beta\neq n\pi$ or $\beta=n\pi$ and $\frac{1}{4e}<k\leq\frac{1}{e}$ and $\alpha=2$.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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