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Cauchy Riemann-lightlike submanifolds in the aspect of an indefinite Kaehler statistical manifold

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Abstract

The present work aims to analyse the lightlike geometry of an indefinite Kaehler statistical manifold and develop properties based on the structure of Cauchy Riemann (CR)-lightlike submanifolds of the same.The results related to induced geometric objects corresponding to the dual connections in these submanifolds have been established. We also characterize the geodesicity and integrability of distributions of the tangent bundle in the CR lightlike submanifolds of the indefinite Kaehler statistical manifold. Further, some conditions on totally umbilical Cauchy Riemann-lightlike submanifolds have been derived.

Keywords

Statistical manifold, CR-lightlike submanifolds, mixed geodesic, foliation, indefinite Kaehler statistical manifold

AMS Subject Classification

53B05, 53C40, 53B30.

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Contents

1. Introduction

In complex geometry, the theory of lightlike submanifolds has proved very useful in defining various induced geometric objects upon extensive investigation by many researchers.The non applicability of the geometry available earlier in some disciplines of mathematics where the metric is not meant to be definite led to the development of the lightlike geometry. [5] introduced the theory of Lorentz Cauchy Riemann (CR)-submanifolds with lightlike distributions. In order to bridge the gap between geometry and mathematical physics, he also developed a new class of globally framed manifolds and established a relation between the spacetime geometry and framed structures in [6]. Further [7] brought out striking differences between the Riemannian and Lorentzian geometries which resulted in a predilection for exploring this sphere of knowledge in the framework of indefinite metrics. With the motive of dealing with the lightlike (degenerate) submanifolds, they introduced the notion of CR lightlike submanifolds in an indefinite Kaehler manifold. Thereafter, the concept of indefinite metric gathered great interest in the field of geometry due to its requisition in the realm of general relativity. Now, we apply this lightlike geometry to the theory of statistical manifolds which are abstract generalizations of statistical models and are geometrically composed as Riemannian manifolds equpped with a specific connection. It is a contemporary field of research having applications in neural networks and control systems. This theory was introduced by [19] and thereafter developed extensively by [1] and [2]. An appreciable amount of work has been done in the field of CR-statistical submanifolds and hypersurfaces of a statistical manifold by [10],[16],[11],[12],[17], [21]. Recently, [3]and [4] have studied the lightlike geometry of an indefinite statistical manifold.

This paper aims to extend the geometry of the combined notion of the Kaehler statistical manifold and lightlike geometry introduced by [14] . We have studied the structural properties of Cauchy Riemann-lightlike submanifolds of the indefinite Kaehler statistical manifold. Some characterizations of totally geodesic foliation and mixed geodesic CR-lightlike submanifolds of the same have been established. Also, some results for the totally umbilical CR-lightlike submanifolds of the indefinite Kaehler statistical manifold have been developed.

2. Cauchy Riemann-lightlike submanifolds

The concept of lightlike submanifolds as [8] is structured as follows:
Let $(\bar{M},\bar{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold with semi-Riemannian metric $\bar{g}$ and a constant index q where $m,n \geq 1$, $1 \leq q \leq m+n-1$.
Let (M,g) be a lightlike submanifold of $\bar{M}$ of dimension m. There exists a smooth distribution $RadTM$ on M of rank $r>0$, known as Radical distribution on M such that $RadTM_p = TM_p \cap TM_p^{\perp}, \forall\, p \in M$ where TM_p and $TM_p^{\perp}$ are degenerate orthogonal spaces but not complementary. Then M is called an r-lightlike submanifold of $\bar{M}$.
Consider $S(TM)$, known as Screen distribution, as a complementary distribution of radical distribution in TM ,i.e.,

$$TM = RadTM \perp S(TM)$$

and $S(TM^{\perp})$, called screen transversal vector bundle, as a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$ i.e.,

$$TM^{\perp} = RadTM \perp S(TM^{\perp})$$

As $S(TM)$ is non degenerate vector subbundle of $T\bar{M}|_M$, we have

$$T\bar{M}|_M = S(TM) \perp S(TM)^{\perp}$$

where $S(TM)^{\perp}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\bar{M}|_M$.

If $tr(TM)$ and $ltr(TM)$ denote the complementary vector bundles to TM in $T\bar{M}|_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$, then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\bar{M}|_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.1}$$

Theorem 2.1. *[8] Let $(M,g,S(TM),S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ called a lightlike transversal bundle of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and basis of $\Gamma(ltr(TM)|_U)$ consisting of smooth sections $\{N_1,\cdots,N_r\}$ $S(TM^{\perp})^{\perp}|_U$ such that*

$$\bar{g}(N_i,\xi_j) = \delta_{ij}, \quad \bar{g}(N_i,N_j) = 0, \quad i,j = 0,1,\cdots,r$$

where $\{\xi_1,\cdots,\xi_r\}$ is a lightlike basis of $\Gamma(RadTM)|_U$.

If $\hat{\nabla}$ denotes the Levi-Civitia connection on $\bar{M}$, then from the above mentioned theory, the Guass and Weingarten formulae are as below:

$$\hat{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall\, X,Y \in \Gamma(TM) \tag{2.2}$$

and

$$\hat{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \quad \forall\, X \in \Gamma(TM), V \in \Gamma(tr(TM))$$

From the projections $L : tr(TM) \longrightarrow ltr(TM)$ and $S : tr(TM) \longrightarrow S(TM^{\perp})$, we have the following equations specified by [8]:

$$\hat{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y)$$

$$\hat{\nabla}_X V = -A_V X + D^l_X V + D^s_X V$$

In particular,

$$\hat{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N)$$

$$\hat{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W)$$

for any $X,Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$ here $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $D^l_X V = L(\nabla^{\perp}_X V)$, $D^s_X V = S(\nabla^{\perp}_X V)$, $\nabla^l_X N$, $D^l(X,W) \in \Gamma(ltr(TM))$, $\nabla^s_X W$, $D^s(X,N) \in \Gamma(S(TM^{\perp}))$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$.

Denoting by P, the projection morphism of tangent bundle TM to the screen distribution, we consider the following decomposition:

$$\nabla_X PY = \nabla'_X PY + h'(X,PY)$$

$$\nabla_X \xi = -A'_{\xi} X + \nabla'^t_X \xi$$

for any $\xi \in \Gamma(Rad(TM))$, where $\{\nabla'_X PY, A'_{\xi} X\}$ and $\{h'(X,PY), \nabla'^t_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively; ∇' and ∇'^t are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$ respectively. The screen distribution $S(TM)$ is totally geodesic if $h'(X,Y) = 0$ for any $X,Y \in \Gamma(TM)$. Also, we have the following equations:

$$\bar{g}(h^l(X,PY),\xi) = g(A'_{\xi} X,PY), \quad \bar{g}(h'(X,PY),N) = g(A_N X,PY)$$

$$g(A'_{\xi} PX,PY) = g(PX,A'_{\xi} PY), \quad A'_{\xi}\xi = 0$$

for any $X,Y \in \Gamma((TM))$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

As per the structure of submanifolds decribed above, the CR-lightlike submanifold is defined as follows:

Definition 2.2. *[8] A submanifold M of an indefinite Kaehler manifold $\bar{M}$ is said to be a CR-lightlike submanifold if the following two conditions are fullfilled:*
(i) $\bar{J}(Rad(TM))$ is a distribution on M such that

$$Rad(TM)\cap\bar{J}Rad(TM)=\{0\}$$

(ii) There exist vector bundles $S(TM)$, $S(TM^{\perp})$, $ltr(TM)$, $D_{\circ}$ and D' over M such that

$$S(TM)=\{\bar{J}(RadTM)\oplus D'\}\perp D_{\circ},\ \bar{J}D_{\circ}=D_{\circ},\ \bar{J}D'=L_1\perp L_2$$

where $D_{\circ}$ is a nondegenerate distribution on M and L_1 , L_2 are vector bundles of $ltr(TM)$ and $S(TM^{\perp})$, respectively.

Using the above definition , the tangent bundle TM of M is decomposed as:

$$TM=D\oplus D'$$

where

$$D=RadTM\perp\bar{J}RadTM\perp D_{\circ}$$

The projections on D and D' are respectively denoted by S and Q so that

$$\bar{J}X=fX+wX \tag{2.3}$$

for any $X,Y\in\Gamma(TM)$, where f is a tensor field of type $(1,1)$ such that $fX = \bar{J}SX$ and w is $\Gamma(L_1\perp L_2)$ - valued 1-form on M such that $wX = \bar{J}QX$. Also $X\in\Gamma(D)$ iff $wX=0$.
On the other hand, we set

$$\bar{J}V=BV+CV \tag{2.4}$$

for any $V\in\Gamma(tr(TM))$. where $BV\in\Gamma(TM)$ and $CV\in\Gamma(tr(TM))$.

Unless otherwise stated, M_1 and M_2 are supposed to be as $\bar{J}L_1$ and $\bar{J}L_2$ where $\bar{J}(L_1)=M_1\subset D'$ and $\bar{J}(L_2)=M_2\subset D'$ respectively.

3. Lightlike structure in an indefinite statistical manifold

Some basic results related to the theory of lightlike submanifolds of an indefinite statistical manifold developed so far are as follows:

3.1 Indefinite statistical manifold

Consider a semi-Riemannian manifold $(\bar{M},\bar{g})$ where $\bar{g}$ is a semi-Riemannian metric of constant index q on $\bar{M}$, If $\bar{M}$ admits an affine connection $\bar{\nabla}$ such that for all $X,Y,Z\in\Gamma(T\bar{M})$
(i) $\bar{\nabla}_XY-\bar{\nabla}_YX=[X,Y]$;
(ii) $(\bar{\nabla}_X\bar{g})(Y,Z)=(\bar{\nabla}_Y\bar{g})(X,Z)$ hold,
then $(\bar{M},\bar{g})$ is said to be an indefinite statistical manifold.
Also, if

$$X\bar{g}(Y,Z)=\bar{g}(\bar{\nabla}_XY,Z)+\bar{g}(Y,\bar{\nabla}^*_XZ)\quad X,Y,Z\in\Gamma(T\bar{M})$$

then $\bar{\nabla}^*$ is referred to as a dual connection of $\bar{\nabla}$.

If $(\bar{M},\bar{g},\bar{\nabla})$ is an indefinite statistical manifold, then so is $(\bar{M},\bar{g},\bar{\nabla}^*)$. Therefore, the indefinite statistical manifold is denoted by $(\bar{M},\bar{g},\bar{\nabla},\bar{\nabla}^*)$.

Let M be a submanifold of a statistical manifold $(\bar{M}$,$\bar{\nabla}$,$\bar{g})$ and g be the induced metric on M. An affine connection ∇ on M is defined by ([16], [11]) as:

$$\nabla_XY=(\bar{\nabla}_XY)^T$$

where $(\bar{\nabla}_XY)^T$ denotes the orthogonal projection of ∇_XY on the tangent space with respect to $\bar{g}$, that is $\langle\nabla_XY,Z\rangle=\langle\bar{\nabla}_XY,Z\rangle$ for $X,Y,Z\in\Gamma(TM)$.Then $(M,\nabla$,g) becomes a statistical manifold and (∇,g) is called the induced statistical structure on M.
(M,∇,g) is said to be a statistical submanifold in ($\bar{M}$,$\bar{\nabla}$,$\bar{g}$) if (∇,g) is induced statistical structure on M.
Now $T_x^{\perp}\ M$ denote the normal space of M i.e.
$T_x^{\perp}\ M :=\{v\in T_x\bar{M}\mid\bar{g}(v,w)=0,w\in T_xM\}$ and g, the induced metric on M. It follows that

$$\begin{aligned}
&\nabla,\nabla^*:\Gamma(TM)\times\Gamma(TM)\longrightarrow\Gamma(TM)\\
&h,h^*:\Gamma(TM)\times\Gamma(TM)\longrightarrow\Gamma(T^{\perp}M)\\
&A,A^*:\Gamma(T^{\perp}M)\times\Gamma(TM)\longrightarrow\Gamma(TM)\\
&\nabla^{\perp},\nabla^{\perp*}:\Gamma(TM)\times\Gamma(T^{\perp}M)\longrightarrow\Gamma(T^{\perp}M)\\
&\bar{\nabla}_XY=\nabla_XY+h(X,Y),\quad\bar{\nabla}_XV=-A_VX+\nabla^{\perp}_XV,
\end{aligned}$$

$$\bar{\nabla}^*_XY=\nabla^*_XY+h^*(X,Y),\quad\bar{\nabla}^*_XV=-A^*_VX+\nabla^{\perp*}_XV, \tag{3.1}$$

for X, $Y\in\Gamma(TM)$, $V\in\Gamma(T^{\perp}M)$.

Then the following hold for X, $Y\in\Gamma(TM)$, $V\in\Gamma(T^{\perp}M)$:

$$\bar{g}(h(X,Y),V)=g(A^*_VX,Y),\quad\bar{g}(h^*(X,Y),V)=g(A_VX,Y) \tag{3.2}$$

The structure of lightlike submanifolds developed hitherto implies that the Gauss and Weingarten formulae for a lightlike submanifold of an indefinite statistical manifold are as under:

$$\begin{aligned}
\bar{\nabla}_XY&=\nabla_XY+h^l(X,Y)+h^s(X,Y),\\
\bar{\nabla}^*_XY&=\nabla^*_XY+h^{*l}(X,Y)+h^{*s}(X,Y)
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\bar{\nabla}_XV&=-A_VX+D^l_XV+D^s_XV,\\
\bar{\nabla}^*_XV&=-A^*_VX+D^{*l}_XV+D^{*s}_XV,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\bar{\nabla}_XN&=-A_NX+\nabla^l_XN+D^s(X,N),\\
\bar{\nabla}^*_XN&=-A^*_NX+\nabla^{*l}_XN+D^{*s}(X,N)
\end{aligned} \tag{3.5}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W),$$
$$\bar{\nabla}^*_X W = -A^*_W X + \nabla^{*s}_X W + D^{*l}(X,W) \tag{3.6}$$

for any $X,Y \in \Gamma(TM)$, $V \in \Gamma(tr(TM))$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(STM^{\perp})$.

The concept of indefinte statistical manifold and the equations (2.1), (3.3), (3.4), (3.5), (3.6), result in the following:

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y,D^{*l}(X,W)) = \bar{g}(Y,A^*_W X),$$
$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y,\nabla^*_X \xi) + \bar{g}(Y,h^{*l}(X,\xi)) = 0,$$
$$\bar{g}(D^s(X,N),W) = \bar{g}(N,A^*_W X),$$
$$\bar{g}(A_N X,PY) = \bar{g}(N,\bar{\nabla}^*_X PY),$$

and

$$\bar{g}(A_N X,N') + \bar{g}(A^*_{N'} X,N) = 0.$$

From the theory of non-degenerate submanifolds of a statistical manifold, it is known that submanifold of statistical manifold is a statistical manifold but this is not true for lightlike submanifolds as from the definition of statistical manifold and the equations (2.1) and (3.3), we have

$$(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = \bar{g}(Y,h^l(X,Z)) - \bar{g}(X,h^l(Y,Z)).$$

and

$$Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla^*_X Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(Y,h^{*l}(X,Z))$$

If P is the projection morphism of the tangent bundle TM to the screen distribution, then the following decomposition w.r.t ∇ and ∇^* holds:

$$\nabla_X PY = \nabla'_X PY + h'(X,PY), \quad \nabla^*_X PY = \nabla^{*\prime}_X PY + h^{*\prime}(X,PY) \tag{3.7}$$

$$\nabla_X \xi = -A'_\xi X + \nabla'^t_X \xi, \quad \nabla^*_X \xi = -A^{*\prime}_\xi X + \nabla^{*\prime t}_X \xi \tag{3.8}$$

for any $X,Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$.
Then the equations (3.3),(3.4),(3.7) and (3.8) imply that

$$\bar{g}(h^l(X,PY),\xi) = g(A^{*\prime}_\xi X,PY), \bar{g}(h^{*l}(X,PY),\xi) = g(A'_\xi X,PY) \tag{3.9}$$

$$\bar{g}(h'(X,PY),N) = g(A^*_N X,PY), \bar{g}(h^{*\prime}(X,PY),N) = g(A_N X,PY) \tag{3.10}$$

for any $X,Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$. As h^l and h^{*l} are symmetric, (3.9) leads to the following:

$$g(A'_\xi PX,PY) = g(PX,A'_\xi PY), g(A^{*\prime}_\xi PX,PY) = g(PX,A^{*\prime}_\xi PY).$$

3.2 Indefinite Kaehler statistical manifold

Let $\bar{\nabla}^\circ$ be the Levi-Civita connection w.r.t $\bar{g}$. Then, we have $\bar{\nabla}^\circ = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$.
For a statistical manifold $(\bar{M},\bar{g},\bar{\nabla},\bar{\nabla}^*)$, the difference $(1,2)$ tensor K of a torsion free affine connection $\bar{\nabla}$ and Levi-civita connection $\bar{\nabla}^\circ$ is defined as

$$K(X,Y) = K_X Y = \bar{\nabla}_X Y - \bar{\nabla}^\circ_X Y \tag{3.11}$$

Since $\bar{\nabla}$ and $\bar{\nabla}^\circ$ are torsion free, we have

$$K(X,Y) = K(Y,X) \quad , \quad \bar{g}(K_X Y,Z) = \bar{g}(Y,K_X Z)$$

for any $X,Y,Z \in \Gamma(TM)$. Also we have

$$K(X,Y) = \bar{\nabla}^\circ_X Y - \bar{\nabla}^*_X Y.$$

From the above equations, we have

$$K(X,Y) = \frac{1}{2}(\bar{\nabla}_X Y - \bar{\nabla}^*_X Y).$$

Also, from (3.11), we have

$$\bar{g}(\bar{\nabla}_X Y,Z) = \bar{g}(K(X,Y),Z) + \bar{g}(\bar{\nabla}^\circ_X Y,Z)$$

We have the following result from [18]:

$$\bar{g}((\bar{\nabla}_X \bar{J})Y,Z) = -\bar{g}(Y,(\bar{\nabla}^*_X \bar{J})Z) \tag{3.12}$$

holds for any $X,Y,Z \in \Gamma(TM)$ for an almost Hermitian manifold $(\bar{M},\bar{g},\bar{J},\bar{\nabla},\bar{\nabla}^*)$. Now , from [21] , we have the following equations for the almost Hermitian manifold:

$$(\bar{\nabla}_X \bar{J})Y = (\bar{\nabla}^\circ_X \bar{J})Y + (K_X \bar{J})Y$$

$$(\bar{\nabla}^*_X \bar{J})Y = (\bar{\nabla}^\circ_X \bar{J})Y - (K_X \bar{J})Y$$

for any $X,Y,Z \in \Gamma(TM)$. This implies

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}^*_X \bar{J})Y = 2(\bar{\nabla}^\circ_X \bar{J})Y$$

Let $(\bar{M},\bar{J},\bar{g})$ be an indefinite almost Hermitian manifold with an almost complex structure $\bar{J}$ and Hermitian metric $\bar{g}$ such that for all $X,Y \in \Gamma(T\bar{M})$,

$$\bar{J}^2 = -I, \quad \bar{g}(\bar{J}X,\bar{J}Y) = \bar{g}(X,Y). \tag{3.13}$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ with respect to metric $\bar{g}$, then the covariant derivative of $\bar{J}$ is defined by

$$(\bar{\nabla}_X \bar{J})Y = \bar{\nabla}_X \bar{J}Y - \bar{J}\bar{\nabla}_X Y$$

An indefinite almost Hermitian manifold $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, i.e., $(\bar{\nabla}_X \bar{J})Y = 0$

Definition 3.1. *Let $(\bar{g},\bar{J})$ be an indefinite Hermitian structure on $\bar{M}$. A triplet $(\bar{\nabla}=\bar{\nabla}^{\circ}+K,\bar{g},\bar{J})$ is called an indefinite Hermitian statistical structure on $\bar{M}$ if $(\bar{\nabla},\bar{g})$ is a statistical structure on $\bar{M}$.Then $(\bar{M},\bar{\nabla},\bar{\nabla}^*,\bar{g},\bar{J})$ is called an indefinite Hermitian statistical manifold.*

In this context, we have the following definition:

Definition 3.2. *[14] An indefinite Hermitian statistical manifold is called indefinite Kaehler statistical manifold if its almost complex structure is parallel with respect to Levi-Civita connection i.e. if,*

$$(\bar{\nabla}^{\circ}_X\bar{J})Y=0$$

Equivalently

$$(\bar{\nabla}_X\bar{J})Y+(\bar{\nabla}^*_X\bar{J})Y=0$$

for all $X,Y\in\Gamma(T\bar{M})$.

4. CR-lightlike submanifolds in an indefinite Kaehler statistical manifold

The lightlike geometry of CR-lightlike submanifolds in the indefinite Kaehler statistical manifold have been worked upon and thus some results related to its structure and geodesicity have been derived. Also, some conditions for screen transversal curvature vector fields in the totally umbilical CR-lightlike submanifolds with respect to the dual connections have been developed.

4.1 Results on structure of Cauchy Riemann-lightlike submanifolds

Lemma 4.1. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then*

$$(\nabla_X f)Y+(\nabla^*_X f)Y=A_{wY}X+A^*_{wY}X+Bh(X,Y)+Bh^*(X,Y), \tag{4.1}$$

$$(\nabla^{\perp}_X w)Y+(\nabla^{*\perp}_X w)Y=Ch(X,Y)+Ch^*(X,Y)-h(X,fY)-h^*(X,fY), \tag{4.2}$$

holds for any $X,Y\in\Gamma(TM)$, where

$$(\nabla_X f)Y=\nabla_X fY-f(\nabla_X Y),\quad (\nabla^*_X f)Y=\nabla^*_X fY-f(\nabla^*_X Y)$$

$$(\nabla^{\perp}_X w)Y=\nabla^{\perp}_X wY-w(\nabla_X Y),\quad (\nabla^{*\perp}_X w)Y=\nabla^{*\perp}_X wY-w(\nabla^*_X Y)$$

Proof: Since $\bar{M}$ is a Kaehler statistical manifold. For any $X,Y\in\Gamma(TM)$,

$$(\bar{\nabla}_X\bar{J})Y+(\bar{\nabla}^*_X\bar{J})Y=0$$

$$\bar{\nabla}_X\bar{J}Y+\bar{\nabla}^*_X\bar{J}Y=\bar{J}\bar{\nabla}_X Y+\bar{J}\bar{\nabla}^*_X Y$$

Using the equations (2.3) and (3.1) , we get

$$\bar{\nabla}_X(fY+wY)+\bar{\nabla}^*_X(fY+wY)=\bar{J}(\nabla_X Y+h(X,Y)) +\bar{J}(\nabla^*_X Y+h^*(X,Y))$$

Now from (2.3),(2.4),(3.1) and (3.2) , we get

$$\nabla_X fY+h(X,fY)+\nabla^*_X fY+h^*(X,fY)-A_{wY}X+\nabla^{\perp}_X wY-A^*_{wY}X +\nabla^{*\perp}_X wY=f\nabla_X Y+w\nabla_X Y+f\nabla^*_X Y+w\nabla^*_X Y+Bh(X,Y)+ Ch(X,Y)+Bh^*(X,Y)+Ch^*(X,Y)$$

Comparing the tangential and normal components, we get the desired result.

Lemma 4.2. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then we have*

$$(\nabla_X B)V+(\nabla^*_X B)V=-fA_V X-fA^*_V X+A_{CV}X+A^*_{CV}X, \tag{4.3}$$

$$(\nabla_X C)V+(\nabla^*_X C)V=-wA_V X-wA^*_V X-h(X,BV)-h^*(X,BV), \tag{4.4}$$

for any $X\in\Gamma(TM)$ and $V\in\Gamma(tr(TM))$

$$(\nabla_X B)V=\nabla_X BV-B\nabla^{\perp}_X V,\quad (\nabla^*_X B)V=\nabla^*_X BV-B\nabla^{*\perp}_X V$$

$$(\nabla_X C)V=\nabla^{\perp}_X CV-C\nabla^{\perp}_X V,\quad (\nabla^*_X C)V=\nabla^{*\perp}_X CV-C\nabla^{*\perp}_X V$$

Proof: The proof follows using the same hypothesis as in Lemma 4.

Theorem 4.3. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then we have the following conditions:*
(i) the distribution D is integrable, if and only if the second fundamental form satisfies

$$h(X,\bar{J}Y)+h^*(X,\bar{J}Y)=h(Y,\bar{J}X)+h^*(Y,\bar{J}X)\qquad\forall X,Y\in\Gamma(D).$$

(ii) The totally real distribution D' is integrable, if and only if, the shape operator of M satisfies

$$A_{\bar{J}Z}U+A^*_{\bar{J}Z}U=A_{\bar{J}U}Z+A^*_{\bar{J}U}Z\quad\forall U,Z\in\Gamma(D')$$

Proof: From equation (4.2), we obtain

$$h(X,\bar{J}Y)+h^*(X,\bar{J}Y)=Ch(X,Y)+Ch^*(X,Y)+w(\nabla_X Y)+w(\nabla^*_X Y)$$

Now using the fact that h and h^* are symmetric and connections ∇ and ∇^* are torsion free, it follows that

$$h(X,\bar{J}Y)+h^*(X,\bar{J}Y)-h(Y,\bar{J}X)-h^*(Y,\bar{J}X)=2w[X,Y]$$

which proves condition (i).
Now from (4.1), we obtain

$$A_{\bar{J}Z}U+A^*_{\bar{J}Z}U=-Bh(U,Z)-Bh^*(U,Z)-f(\nabla_U Z)-f(\nabla^*_U Z)$$

Hence

$$A_{\bar{J}Z}U + A^*_{\bar{J}Z}U - A_{\bar{J}U}Z - A^*_{\bar{J}U}Z = -2f([Z,U])$$

Using the given hypothesis, we obtain the condition (ii).

Corollary 4.4. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then the distribution D is integrable, if and only if the second fundamental form satisfies*

$$\bar{g}\big(h(X,\bar{J}Y) + h^*(X,\bar{J}Y) - h(Y,\bar{J}X) - h^*(Y,\bar{J}X), \xi\big) = 0\,; \quad \xi \in \Gamma(RadTM)$$

and

$$\bar{g}\big(h(X,\bar{J}Y) + h^*(X,\bar{J}Y) - h(Y,\bar{J}X) - h^*(Y,\bar{J}X), W\big) = 0\,; \quad W \in \Gamma(L_2)$$

Lemma 4.5. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold. Then $\nabla_X \bar{J}X + \nabla^*_X \bar{J}X = \bar{J}\nabla_X X + \bar{J}\nabla^*_X X$ for any $X \in \Gamma(D_\circ)$.*

Proof: Let $X, Y \in \Gamma(D_\circ)$. Then from the theory of Kaehler statistical manifold, we obtain

$$\bar{g}(\nabla_X \bar{J}X + \nabla^*_X \bar{J}X, Y) = \bar{g}(\bar{\nabla}_X \bar{J}X - h(X,\bar{J}X) + \bar{\nabla}^*_X \bar{J}X - h^*(X,\bar{J}X), Y)$$

$$= \bar{g}(\bar{J}\bar{\nabla}_X X + \bar{J}\bar{\nabla}^*_X X, Y) = -\bar{g}(\bar{\nabla}_X X, \bar{J}Y) - \bar{g}(\bar{\nabla}^*_X X, \bar{J}Y)$$

$$= -\bar{g}(\nabla_X X, \bar{J}Y) - \bar{g}(\nabla^*_X X, \bar{J}Y) = \bar{g}(\bar{J}\nabla_X X + \bar{J}\nabla^*_X X, Y)$$

Therefore $\bar{g}(\nabla_X \bar{J}X + \nabla^*_X \bar{J}X - \bar{J}\nabla_X X - \bar{J}\nabla^*_X X, Y) = 0$.
Hence the result followsfrom the non-degeneracy of $D_\circ$.

4.2 Characterizations of Geodesic CR-lightlike submanifolds

Definition 4.6. *A CR-lightlike submanifold of an indefinite Kaehler statistical manifold is called D-totally geodesic with respect to $\bar{\nabla}$ (respectively $\bar{\nabla}^*$) if $h(X,Y) = 0$ (respectively $h^*(X,Y) = 0$) for all $X, Y \in D$.*

Definition 4.7. *A CR-lightlike submanifold of an indefinite Kaehler statistical manifold is called mixed totally geodesic with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$) if $h(X,Y) = 0$ (resp. $h^*(X,Y) = 0$) for $X \in D$ and $Y \in D'$.*

Theorem 4.8. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then the distribution D defines a totally geodesic foliation with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ if M is D-geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$.*

Proof: The distribution D defines a totally geodesic foliation respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ if and only if,

$$\nabla_X Y + \nabla^*_X Y \in \Gamma(D), \quad \forall\, X, Y \in \Gamma(D) \tag{4.5}$$

Since $D' = \bar{J}(L_1 \perp L_2)$, therefore the above equation holds, if and only if, we have

$$\bar{g}(\nabla_X Y + \nabla^*_X Y, \bar{J}\xi) = 0,$$

$$\bar{g}(\nabla_X Y + \nabla^*_X Y, \bar{J}W) = 0.$$

Let $X, Y \in \Gamma(D)$, Using the fact that $\bar{M}$ is a Kaehler statistical manifold, we derive

$$\bar{g}(\nabla_X Y + \nabla^*_X Y, \bar{J}\xi) = \bar{g}(\bar{\nabla}_X Y - h(X,Y) + \bar{\nabla}^*_X Y - h^*(X,Y), \bar{J}\xi)$$

$$= \bar{g}(\bar{\nabla}_X Y + \bar{\nabla}^*_X Y, \bar{J}\xi) - \bar{g}(h(X,Y), \bar{J}\xi) - \bar{g}(h^*(X,Y), \bar{J}\xi)$$

$$= -\bar{g}(\bar{J}\bar{\nabla}_X Y + \bar{J}\bar{\nabla}^*_X Y, \xi) = -\bar{g}(\bar{\nabla}_X \bar{J}Y, \xi) - \bar{g}(\bar{\nabla}^*_X \bar{J}Y, \xi)$$

$$= -\bar{g}(h(X,\bar{J}Y), \xi) - \bar{g}(h^*(X,\bar{J}Y), \xi)$$

Similarly, we obtain

$$\bar{g}(\nabla_X Y + \nabla^*_X Y, \bar{J}W) = -\bar{g}(h(X,\bar{J}Y), W) - \bar{g}(h^*(X,\bar{J}Y), W)$$

Then the result follows from the given hypothesis.

Theorem 4.9. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then the distribution D defines a totally geodesic foliation with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ if and only if D is integrable.*

Proof: Since $\bar{M}$ is a Kaehler statistical manifold, therefore using equation (4.2), we have

$$h(X,\bar{J}Y) + h^*(X,\bar{J}Y) = Ch(X,Y) + Ch^*(X,Y) + w(\nabla_X Y) + w(\nabla^*_X Y)$$

Now the fact that h and h^* are symmetric and connections ∇ and ∇^* are torsion free proves the assertion.

Theorem 4.10. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then, M is mixed geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$, if and only if, $wA_{wY}X + wA^*_{wY}X = 0$, and $C\nabla^{\perp}_X wY - C\nabla^{*\perp}_X wY = 0$ for any $X \in \Gamma(D), Y \in \Gamma(D')$.*

Proof: Since $\bar{M}$ is a Kaehler statistical manifold, therefore we derive

$$h(X,Y) + h^*(X,Y) = -\bar{J}^2\bar{\nabla}_X Y - \nabla_X Y - \bar{J}^2\bar{\nabla}^*_X Y - \nabla^*_X Y$$

$$= -\bar{J}(\bar{\nabla}_X \bar{J}Y) - \bar{J}(\bar{\nabla}^*_X \bar{J}Y) - \nabla_X Y - \nabla^*_X Y$$

$$= -\bar{J}(-A_{\bar{J}Y}X + \nabla^{\perp}_X \bar{J}Y) - \bar{J}(-A^*_{\bar{J}Y}X + \nabla^{*\perp}_X \bar{J}Y) - \nabla_X Y - \nabla^*_X Y$$

$$= \bar{J}(A_{\bar{J}Y}X) - \bar{J}(\nabla^{\perp}_X \bar{J}Y) + \bar{J}(A^*_{\bar{J}Y}X) - \bar{J}(\nabla^{*\perp}_X \bar{J}Y) - \nabla_X Y - \nabla^*_X Y$$

$$= fA_{wY}X + wA_{wY}X - B\nabla^{\perp}_X wY - C\nabla^{\perp}_X wY + fA^*_{wY}X + wA^*_{wY}X - B\nabla^{*\perp}_X wY - C\nabla^{*\perp}_X wY - \nabla_X Y - \nabla^*_X Y$$

Equating transversal parts on both sides, we have

$$h(X,Y) + h^*(X,Y) = wA_{wY}X + wA^*_{wY}X - C\nabla^{\perp}_X wY - C\nabla^{*\perp}_X wY$$

Thus, M is mixed geodesic w.r.t to $\bar{\nabla}$ and $\bar{\nabla}^*$, if and only if

$$wA_{wY}X + wA^*_{wY}X = 0, \quad C\nabla^{\perp}_X wY - C\nabla^{*\perp}_X wY = 0.$$

Theorem 4.11. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then,*

$$\nabla_X Z + \nabla^*_X Z = -fA_W X + B\nabla^s_X W + BD^l(X,W) - fA^*_W X + B\nabla^{*s}_X W + BD^{*l}(X,W)$$

for any $X \in \Gamma(TM)$, $Z \in \Gamma(\bar{J}L_2)$ and $W \in \Gamma(L_2)$.

Proof: Let $W \in \Gamma(L_2)$ such that $Z = \bar{J}W$. Since $\bar{M}$ is a Kaehler statistical manifold, therefore

$$\bar{\nabla}_X \bar{J}W + \bar{\nabla}^*_X \bar{J}W = \bar{J}\bar{\nabla}_X W + \bar{J}\bar{\nabla}^*_X W$$

Then,

$$\begin{aligned}\nabla_X Z + h(X,Z) + \nabla^*_X Z + h^*(X,Z) &= \bar{J}(-A_W X + \nabla^s_X W + D^l(X,W)) \\ &\quad + \bar{J}(-A^*_W X + \nabla^{*s}_X W + D^{*l}(X,W)) \\ &= -fA_W X - wA_W X + B\nabla^s_X W + C\nabla^s_X W + BD^l(X,W) \\ &\quad + CD^l(X,W) - fA^*_W X - wA^*_W X + B\nabla^{*s}_X W + C\nabla^{*s}_X W \\ &\quad + BD^{*l}(X,W) + CD^{*l}(X,W)\end{aligned}$$

Equating tangential parts, we get

$$\nabla_X Z + \nabla^*_X Z = -fA_W X + B\nabla^s_X W + BD^l(X,W) - fA^*_W X + B\nabla^{*s}_X W + BD^{*l}(X,W)$$

Lemma 4.12. *Let M be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold and screen distribution be totally geodesic w.r.t to ∇^* (resp. ∇). Then $\nabla_X Y \in \Gamma(S(TM))$ (resp. $\nabla^*_X Y \in \Gamma(S(TM))$) for any $X, Y \in \Gamma(S(TM))$.*

Proof: For any $X, Y \in \Gamma(S(TM))$, using the concept of statistical manifold, we derive

$$\bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, N) = -\bar{g}(Y, \bar{\nabla}^*_X N) = \bar{g}(Y, A^*_N X) = \bar{g}(h'(X,Y), N)$$

Hence, the lemma follows using the given hypothesis alongwith theorem (1). Similarly, the corresponding result for dual connection ∇^* holds.

4.3 Conditions on totally umbilical CR-lightlike submanifolds

Definition 4.13. *Let M be a lightlike submanifold of a indefinite Kaehler statistical manifold $\bar{M}$. Then M is said to be a totally umbilical with respect to $\bar{\nabla}$ (respectively $\bar{\nabla}^*$) if $h(X,Y) = H\bar{g}(X,Y)$ (respectively $h^*(X,Y) = H^*\bar{g}(X,Y)$) for all $X, Y \in \Gamma(TM)$, where $H \in \Gamma(tr(TM))$ (resp. $H^* \in \Gamma(tr(TM))$) stands for transversal curvature vector fields of M in $\bar{M}$ with respect to $\bar{\nabla}$(respectively $\bar{\nabla}^*$).*

Also, M is totally umbilical with respect to $\bar{\nabla}$ (respectively $\bar{\nabla}^*$) if and only if on each co-ordinate neighbourhood, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ ($H^{*l} \in \Gamma(ltr(TM))$ and $H^{*s} \in \Gamma(S(TM^\perp))$ respectively) such that $h^l(X,Y) = H^l\bar{g}(X,Y)$, $h^s(X,Y) = H^s\bar{g}(X,Y)$ and $h^{*l}(X,Y) = H^{*l}\bar{g}(X,Y)$, $h^{*s}(X,Y) = H^{*s}\bar{g}(X,Y)$ respectively with respect to $\bar{\nabla}$(respectively $\bar{\nabla}^*$).

Theorem 4.14. *Let M be a totally umbilical CR-lightlike submanifold of an indefinite Kaehler statistical manifold and screen distribution be totally geodesic w.r.t to ∇ and ∇^*. Then, $\bar{g}(H^s, W) + \bar{g}(H^{*s}, W) = 0$ for the screen transversal curvature vector fields $H^s(resp. H^{*s})$ with respect to the connections $\bar{\nabla}_X$ and $\bar{\nabla}^*_X$ respectively.*

Proof: Let $W \in \Gamma(L_2)$ and $X \in \Gamma(D_\circ)$
For a totally umbilical CR-lightlike submanifold of indefinite Kaehler statistical manifold, we have

$$\begin{aligned}\bar{g}(\bar{J}\bar{\nabla}_X X + \bar{J}\bar{\nabla}^*_X X, \bar{J}W) &= \bar{g}(\bar{\nabla}_X \bar{J}X + \bar{\nabla}^*_X \bar{J}X, \bar{J}W) \\ &= \bar{g}(\nabla_X \bar{J}X, \bar{J}W) + \bar{g}(\nabla^*_X \bar{J}X, \bar{J}W) \\ &= \bar{g}(\bar{J}\nabla_X X + \bar{J}\nabla^*_X X, \bar{J}W) \\ &= \bar{g}(\nabla_X X + \nabla^*_X X, W) = 0\end{aligned}$$

and

$$\begin{aligned}\bar{g}(\bar{J}\bar{\nabla}_X X + \bar{J}\bar{\nabla}^*_X X, \bar{J}W) &= \bar{g}(\bar{\nabla}_X X + \bar{\nabla}^*_X X, W) \\ &= \bar{g}(h^s(X,X), W) + \bar{g}(h^{*s}(X,X), W) \\ &= \bar{g}(X,X)\bar{g}(H^s, W) + \bar{g}(X,X)\bar{g}(H^{*s}, W) = 0\end{aligned}$$

Now the non-degeneracy of $D_\circ$ implies that $\bar{g}(H^s, W) + \bar{g}(H^{*s}, W) = 0$.

Theorem 4.15. *Let M be a totally umbilical CR-lightlike submanifold of an indefinite Kaehler statistical manifold and screen distribution be totally geodesic. Then, $A_{\bar{J}Z}W - A_{\bar{J}W}Z = A^*_{\bar{J}Z}W - A^*_{\bar{J}W}Z, \quad \forall\, W, Z \in \Gamma(D')$*

Proof: $\bar{M}$, being an indefinite Kaehler statistical manifold, implies

$$\bar{J}\bar{\nabla}_Z W + \bar{J}\bar{\nabla}^*_Z W = \bar{\nabla}_Z \bar{J}W + \bar{\nabla}^*_Z \bar{J}W,$$

$$\bar{J}\nabla_Z W + \bar{J}h(Z,W) + \bar{J}\nabla^*_Z W + \bar{J}h^*(Z,W) = -A_{\bar{J}W}Z + \nabla^\perp_Z \bar{J}W - A^*_{\bar{J}W}Z + \nabla^{*\perp}_Z \bar{J}W$$

Interchanging the role of Z and W in the above equation and then subtracting the resulting equation from it, we obtain

$$A_{\bar{J}Z}W - A_{\bar{J}W}Z + A^*_{\bar{J}Z}W - A^*_{\bar{J}W}Z = \bar{J}\nabla_Z W - \bar{J}\nabla_W Z + \bar{J}\nabla^*_Z W - \bar{J}\nabla^*_W Z$$

Taking inner product with $X \in \Gamma(D_\circ)$, we get

$$\bar{g}(A_{\bar{J}Z}W - A_{\bar{J}W}Z + A^*_{\bar{J}Z}W - A^*_{\bar{J}W}Z, X) = \bar{g}(\bar{J}\nabla_Z W - \bar{J}\nabla_W Z + \bar{J}\nabla^*_Z W - \bar{J}\nabla^*_W Z, X)$$

$$= -\bar{g}(\nabla_Z W, \bar{J}X) + \bar{g}(\nabla_W Z, \bar{J}X) - \bar{g}(\nabla^*_Z W, \bar{J}X) + \bar{g}(\nabla^*_W Z, \bar{J}X)$$

Now

$$\begin{aligned}\bar{g}(\nabla_W Z + \nabla^*_W Z, \bar{J}X) &= \bar{g}(\bar{\nabla}_W Z + \bar{\nabla}^*_W Z, \bar{J}X) \\ &= \bar{g}(Z, \bar{\nabla}^*_W \bar{J}X + \bar{\nabla}_W \bar{J}X)\end{aligned}$$

$$= \bar{g}(Z, \bar{J}(\bar{\nabla}^*_W X + \bar{\nabla}_W X)) = \bar{g}(\bar{J}Z, \bar{\nabla}_W X) + \bar{g}(\bar{J}Z, \bar{\nabla}^*_W X)$$

$$= \bar{g}(\bar{J}Z, \nabla_W X) + \bar{g}(\bar{J}Z, h^s(W,X)) + \bar{g}(\bar{J}Z, h^l(W,X))$$
$$+\bar{g}(\bar{J}Z, \nabla^*_W X) + \bar{g}(\bar{J}Z, h^{*s}(W,X)) + \bar{g}(\bar{J}Z, h^{*l}(W,X))$$

Since M is totally umbilical CR-lightlike submanifold, hence for any $W \in \Gamma(D')$ and $X \in \Gamma(D_\circ)$, we have

$$h^s(W,X) = H^s \bar{g}(W,X) = 0, \quad h^{*s}(W,X) = H^{*s} \bar{g}(W,X) = 0$$

$$h^l(W,X) = H^l \bar{g}(W,X) = 0, \quad h^{*l}(W,X) = H^{*l} \bar{g}(W,X) = 0$$

$$\bar{g}(\nabla_W Z + \nabla^*_W Z, \bar{J}X) = \bar{g}(\bar{J}Z, \nabla_W X + \nabla^*_W X) = 0$$

$$\bar{g}(\nabla_W Z + \nabla^*_W Z, \bar{J}X) = 0$$

Similarly, $\bar{g}(\nabla_Z W + \nabla^*_Z W, \bar{J}X) = 0$.
So, $\bar{g}(A_{\bar{J}Z}W - A_{\bar{J}W}Z + A^*_{\bar{J}Z}W - A^*_{\bar{J}W}Z, X) = 0$
Hence the result.

5. Conclusion and Scope

This research work explores the properties of Cauchy Riemann-lightlike submanifolds in an indefinite Kaehler statistical manifold and thus characterize the geodesicity and integrability of the distributions therein. Results for screen transversal curvature vector fields in the totally umbilical CR-lightlike submanifolds have also been worked upon . Since the paper inspects the structure of lightlike submanifolds in the Kaehler statistical manifold, it can motivate the geometers to explore further properties and characterizations in the same as well as in its odd dimensional counterpart. Also, due to wide applications of lightlike geometry and statistical manifolds in mathematical physics and neural networks, the present study can be considered as a tool to work further on the structure of the indefinite Kaehler statistical manifold.

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