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On lacunary $\mathcal{I}$-invariant arithmetic convergence

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Abstract. In this study, we investigate the notion of lacunary $\mathcal{I}_\sigma$ arithmetic convergence for real sequences and examine relations between this new type convergence notion and the notions of lacunary invariant arithmetic summability, lacunary strongly q-invariant arithmetic summability and lacunary σ-statistical arithmetic convergence which are defined in this study. Finally, giving the notions of lacunary $\mathcal{I}_\sigma$ arithmetic statistically convergence, lacunary strongly $\mathcal{I}_\sigma$ arithmetic summability, we prove the inclusion relation between them.

AMS Subject Classifications: 40A05, 40A99, 46A70, 46A99.

Keywords: Lacunary sequence, statistical convergence, invariant, arithmetic convergence.

Contents

1. Introduction and Background

The idea of arithmetic convergence was firstly originated by Ruckle [22]. Then, it was further investigated by many authors (for examples, see [9, 10, 34–38]).

A sequence $x = (x_m)$ is called arithmetically convergent if for each $\varepsilon > 0$, there is an integer n such that for every integer m we have $|x_m - x_{\langle m,n\rangle}| < \varepsilon$, where the symbol $\langle m, n\rangle$ denotes the greatest common divisior of two integers m and n. We denote the sequence space of all arithmetic convergent sequence by AC.

Statistical convergence of a real number sequence was firstly originated by Fast [2]. It became a notable topic in summability theory after the work of Fridy [3] and Šalát [23].

By a lacunary sequence, we mean an increasing integer sequence $\theta = \{k_r\}$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty.$$

The intervals determined by θ is denoted by $I_r = (k_{r-1}, k_r]$. The idea of lacunary statistical convergence was investigated by Fridy and Orhan [4] and then studied by several authors (for examples, see [5, 6, 13, 17, 27]).

In the wake of the study of ideal convergence defined by Kostyrko et al. [11], there has been comprehensive research to discover applications and summability studies of the classical theories. A lot of development have been seen in area about $\mathcal{I}$-convergence of sequences after the work of [1, 7, 8, 12, 16, 24, 28–30, 32].

An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$.

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A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if (i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$$

is a filter of $\mathbb{N}$, it is called the filter associated with the ideal.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. The sequence (x_k) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}.$$

If (x_k) is $\mathcal{I}$-convergent to L, then we write $\mathcal{I} - \lim x = L$.

An admissible ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, ...\}$ of mutually disjoint sets of I, there is sequence $\{B_1, B_2, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$ $(i = 1, 2, ...)$ is finite and $\bigcup\limits_{i=1}^{\infty} B_i \in \mathcal{I}$.

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of all positive integers). A continuous linear functional Φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if it satisfies the following conditions:

(1) $\Phi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all $n \in \mathbb{N}$;

(2) $\Phi(e) = 1$, where $e = (1, 1, 1, ...)$;

(3) $\Phi\left(x_{\sigma(n)}\right) = \Phi(x_n)$ for all $(x_n) \in l_\infty$.

The mappings Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x_n) = \lim x_n$, for all $(x_n) \in c$.

In case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit.

The space V_σ, the set of bounded sequences whose invariant means are equal, can be shown that

$$V_\sigma = \left\{(x_k) \in l_\infty : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L\right\}$$

uniformly in n.

Several authors studied invariant mean and invariant convergent sequence (for examples, see [14, 15, 18–21, 25, 26, 31, 33]).

Savaş and Nuray [18] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Nuray et al. [20] defined the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence for real sequences and investigated relationships between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Ulusu and Nuray [33] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers. Recently, the concept of strong σ-convergence was generalized by Savaş [25]. The concept of strongly σ-convergence was defined by Mursaleen [14].

Let θ be a lacunary sequence, $E \subseteq \mathbb{N}$ and

$$s_r := \min_n \{|E \cap \{\sigma^m(n) : m \in I_r\}|\}$$
$$S_r := \max_n \{|E \cap \{\sigma^m(n) : m \in I_r\}|\}.$$

If the following limits exist

$$\underline{V}_\theta(E) = \lim_{r \to \infty} \frac{s_r}{h_r},\ \overline{V}_\theta(E) = \lim_{r \to \infty} \frac{S_r}{h_r},$$

then they are called a lower lacunary invariant uniform density and an upper lacunary invariant uniform density of the set E, respectively. If $\underline{V}_\theta(E) = \overline{V}_\theta(E)$, then $V_\theta(E) = \underline{V}_\theta(E) = \overline{V}_\theta(E)$ is called the lacunary invariant uniform density of E.

The class of all $E \subseteq \mathbb{N}$ with $\underline{V}_\theta(E) = 0$ will be denoted by $\mathcal{I}_{\sigma\theta}$. Note that $\mathcal{I}_{\sigma\theta}$ is an admissible ideal.

A sequence (x_m) is lacunary $\mathcal{I}_\sigma$-convergent to L, if for each $\varepsilon > 0$,

$$E(\varepsilon) := \{m \in \mathbb{N} : |x_m - L| \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta},$$

i.e., $V_\theta(E(\varepsilon)) = 0$. In this case, we write $\mathcal{I}_{\sigma\theta} - \lim x_m = L$.

The arithmetic statistically convergence and lacunary arithmetic statistically convergence was examined by Yaying and Hazarika [38].

A sequence $x = (x_m)$ is said to be arithmetic statistically convergent if for $\varepsilon > 0$, there is an integer n such that

$$\lim_{t\to\infty} \frac{1}{t}|\{m \leq t : |x_m - x_{\langle m,n\rangle}| \geq \varepsilon\}| = 0.$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences. We shall write $ASC - \lim x_m = x_{\langle m,n\rangle}$ to denote the sequence (x_m) is arithmetic statistically convergent to $x_{\langle m,n\rangle}$.

A sequence $x = (x_m)$ is said to be lacunary arithmetic statistically convergent if for $\varepsilon > 0$ there is an integer n such that

$$\lim_{r\to\infty} \frac{1}{h_r}| \left\{m \in I_r : |x_m - x_{\langle m,n\rangle}| \geq \varepsilon\right\} = 0.$$

We will use $ASC_\theta - \lim x_m = x_{\langle m,n\rangle}$ to denote the sequence (x_m) is lacunary arithmetic statistically convergent to $x_{\langle m,n\rangle}$.

Kişi [9] investigated the concepts of invariant arithmetic convergence, strongly invariant arithmetic convergence, invariant arithmetic statistically convergence, lacunary invariant arithmetic statistical convergence and obtained interesting results.

In [10], arithmetic $\mathcal{I}$-statistically convergent sequence space and $\mathcal{I}$-lacunary arithmetic statistically convergent sequence space were given and established interesting results.

Kişi [10] examined $\mathcal{I}$-invariant arithmetic convergence, $\mathcal{I}^*$-invariant arithmetic convergence, q-strongly invariant arithmetic convergence of sequences.

A sequence $x = (x_p)$ is said to be invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty} \frac{1}{m}\sum_{p=1}^{m} x_{\sigma^p(s)} = x_{\langle p,n\rangle}$$

uniformly in s. In this case we write $x_p \to x_{\langle p,n\rangle}\,(AV_\sigma)$ and the set of all invariant arithmetic convergent sequences will be demostrated by AV_σ.

A sequence $x = (x_p)$ is said to be strongly invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty} \frac{1}{m}\sum_{p=1}^{m} |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| = 0$$

uniformly in s. In this case we write $x_p \to x_{\langle p,n\rangle}\,[AV_\sigma]$ to denote the sequence (x_p) is strongly invariant arithmetic convergent to $x_{\langle p,n\rangle}$ and the set of all invariant arithmetic convergent sequences will be demostrated by $[AV_\sigma]$.

A sequence $x = (x_p)$ is said to be invariant arithmetic statistically convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{m\to\infty} \frac{1}{m}\left|\{p \leq m : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right| = 0$$

uniformly in s. We shall use $AS_{\sigma}C$ to denote the set of all invariant arithmetic statistical convergent sequences. In this case we write $AS_{\sigma}C - \lim x_p = x_{\langle p,n\rangle}$ or $x_p \to x_{\langle p,n\rangle}\left(AS_{\sigma}C\right)$.

A sequence $x=(x_p)$ is said to be lacunary invariant arithmetic statistical convergent if for every $\varepsilon>0$, there is an integer n such that

$$\lim_{r\to\infty}\frac{1}{h_r}|\{p\in I_r : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}| = 0$$

uniformly in s. We shall use $AS_{\sigma\theta}C$ to denote the set of all lacunary invariant arithmetic statistical convergent sequences. In this case we write $AS_{\sigma\theta}C - \lim x_p = x_{\langle p,n\rangle}$.

The $\mathcal{I}$-invariant arithmetic convergence was defined by [10] as below:

A sequence $x=(x_p)$ is said to be $\mathcal{I}$-invariant arithmetic convergent if for every $\varepsilon>0$, there is an integer η such that

$$\left\{p\in\mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\} \in \mathcal{I}_{\sigma}.$$

In this case we write $A\mathcal{I}_{\sigma}C - \lim x_p = x_{\langle p,\eta\rangle}$. We shall use $A\mathcal{I}_{\sigma}C$ to denote the set of all $\mathcal{I}$-invariant arithmetic convergent sequences.

2. Main Results

Definition 2.1. *A sequence $x=(x_p)$ is said to be lacunary invariant arithmetic summable to $x_{\langle p,\eta\rangle}$ if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{p\in I_r} x_{\sigma^p(s)} = x_{\langle p,\eta\rangle},$$

uniformly in s, for an integer η.

Also, the set of lacunary strongly invariant arithmetic convergence sequences is defined as below:

$$[AV_{\sigma\theta}] = \left\{x=(x_p) : \lim_{r\to\infty}\frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| = 0\right\}$$

uniformly in s. In this case, we write $x_p \to x_{\langle p,\eta\rangle}\ ([AV_{\sigma\theta}])$ to demonstrate the sequence (x_p) is lacunary strongly invariant arithmetic summable to $x_{\langle p,\eta\rangle}$.

Definition 2.2. *A sequence $x=(x_p)$ is said to be lacunary strongly q-invariant arithmetic summable $(0<q<\infty)$ to $x_{\langle p,\eta\rangle}$ if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|^q = 0,$$

uniformly in s and it is indicated by $x_p \to x_{\langle p,\eta\rangle}\left([AV_{\sigma\theta}]_q\right)$.

Definition 2.3. *A sequence $x=(x_p)$ is said to be lacunary σ-statistical arithmetic convergent to $x_{\langle p,\eta\rangle}$ if for every $\varepsilon>0$, there is an integer η such that*

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{p\in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\}\right| = 0,$$

uniformly in s.

Definition 2.4. *A sequence $x=(x_p)$ is lacunary $\mathcal{I}_{\sigma}$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, if for each $\varepsilon>0$, there is an integer η such that*

$$K(\varepsilon) := \left\{p\in\mathbb{N} : \left|x_p - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\} \in \mathcal{I}_{\sigma\theta},$$

i.e., $V_{\theta}(K(\varepsilon)) = 0$. In this case, we write $x_p \to x_{\langle p,\eta\rangle}\ (A\mathcal{I}_{\sigma\theta})$ or $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p,\eta\rangle}$.

Theorem 2.5. *Let (x_p) is bounded sequence. If (x_p) is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, then (x_p) is lacunary invariant arithmetic summable to $x_{\langle p,\eta\rangle}$.*

Proof. Let $s \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. Also, we suppose that (x_p) is bounded sequence and (x_p) is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$. Now, we estimate

$$t_\theta(s) := \left| \frac{1}{h_r} \sum_{p \in I_r} x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|.$$

For every $s = 1, 2, ...$, we have

$$t_\theta(s) \le t_\theta^1(s) + t_\theta^2(s),$$

where

$$t_\theta^1(s) := \frac{1}{h_r} \sum_{p \in I_r,\ \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \ge \varepsilon} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|$$

and

$$t_\theta^2(s) := \frac{1}{h_r} \sum_{p \in I_r,\ \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| < \varepsilon} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|.$$

For every $s = 1, 2, ...$, it is obvious that $t_\theta^2(s) < \varepsilon$. Since (x_p) is bounded sequence, there is a $M > 0$ such that

$$\left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \le M,\ (p \in I_r,\ s = 1, 2, ...)$$

and so we have

$$t_\theta^1(s) = \tfrac{1}{h_r} \sum_{p \in I_r,\ \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \ge \varepsilon} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|$$

$$\le \tfrac{M}{h_r} \left| \left\{ p \in I_r : \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \ge \varepsilon \right\} \right|$$

$$\le M \tfrac{\max_s \left| \left\{ p \in I_r : \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \ge \varepsilon \right\} \right|}{h_r} = M \tfrac{S_r}{h_r}.$$

Hence, due to our assumption, (x_p) is lacunary invariant arithmetic summable to $x_{\langle p,\eta\rangle}$. ■

In general, the converse of the Theorem 2.5 does not hold. For example, let $x = (x_p)$ be the sequence defined as follows:

$$x_p := \begin{cases} 1, & \text{if } p_{r-1} < p < p_{r-1} + \left[\sqrt{h_r}\right], \\ & \text{and } p \text{ is an even integer,} \\ 0, & \text{if } p_{r-1} < p < p_{r-1} + \left[\sqrt{h_r}\right], \\ & \text{and } p \text{ is an odd integer.} \end{cases}$$

When $\sigma(s) = s + 1$, this sequence is lacunary invariant arithmetic summable to $\frac{1}{2}$ but it is not lacunary $\mathcal{I}_\sigma$ arithmetic convergent.

Now, we will give the following theorems which state relations between the notions of lacunary $\mathcal{I}_\sigma$ arithmetic convergence and lacunary strongly q-invariant arithmetic summability, and we will denote that these notions are equivalent for bounded sequences.

Theorem 2.6. *If a sequence $x = (x_p)$ is lacunary strongly q-invariant arithmetic summable to $x_{\langle p,\eta\rangle}$, then it is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. Let $0 < q < \infty$. Suppose that $x_p \to x_{\langle p,\eta\rangle}\left([AV_{\sigma\theta}]_q\right)$ for an integer η. Then, for every $s = 1, 2, ...$ and $\varepsilon > 0$ we have

$$\begin{aligned}&\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|^q \\ &\geq \sum_{p\in I_r,\left|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}\right|\geq\varepsilon}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|^q \\ &\geq \varepsilon^q\left|\left\{p\in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\}\right| \\ &\geq \varepsilon^q \max_s\left|\left\{p\in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\}\right|\end{aligned}$$

and so

$$\frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|^q \geq \varepsilon^q\frac{\max_s\left|\left\{p\in I_r:\left|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}\right|\geq\varepsilon\right\}\right|}{h_r} = \varepsilon^q\frac{S_r}{h_r}.$$

Hence, due to our assumption, $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p,\eta\rangle}$. ■

Theorem 2.7. *Let (x_p) is bounded sequence. If $x = (x_p)$ is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, then it is lacunary strongly q-invariant arithmetic summable to $x_{\langle p,\eta\rangle}$.*

Proof. Assume that $(x_p) \in l_\infty$ and $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p,\eta\rangle}$. Let $0 < q < \infty$ and $\varepsilon > 0$. The boundedness of (x_p) implies that there exists a $M > 0$ such that $\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \leq M$, $(p \in I_r,\ s = 1, 2, ...)$. Therefore, we obtain

$$\begin{aligned}&\frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|^q = \frac{1}{h_r}\sum_{\substack{p\in I_r\\ \left|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}\right|\geq\varepsilon}}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|^q + \frac{1}{h_r}\sum_{\substack{p\in I_r\\ \left|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}\right|<\varepsilon}}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|^q \\ &\leq M\frac{\max_s\left|\left\{p\in I_r:\left|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}\right|\geq\varepsilon\right\}\right|}{h_r} + \varepsilon^q \\ &= M\frac{S_r}{h_r} + \varepsilon^q.\end{aligned}$$

Therefore, we obtain

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|^q = 0,$$

uniformly in s. Hence, we get $x_p \to x_{\langle p,\eta\rangle}\left([AV_{\sigma\theta}]_q\right)$. ■

Theorem 2.8. *A sequence $(x_p) \in l_\infty$. Then, $x = (x_p)$ to lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$ iff it is lacunary strongly q-invariant arithmetic summable to $x_{\langle p,\eta\rangle}$.*

Proof. This is an immediate consequence of Theorem 2.6 and Theorem 2.7. ■

Now, without proof, we will state a theorem that gives a relation between the notions of lacunary $\mathcal{I}_\sigma$ arithmetic convergence and lacunary σ-statistical arithmetic convergence.

Theorem 2.9. *A sequence $x = (x_p)$ is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$ iff this sequence is lacunary σ-statistical arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Finally, introducing the notion of lacunary $\mathcal{I}_\sigma^*$ arithmetic convergence, we will give the relation between this notion and the notion of lacunary $\mathcal{I}_\sigma$ arithmetic convergence.

Definition 2.10. *A sequence $x=(x_p)$ is said to be lacunary $\mathcal{I}_\sigma^*$ arithmetic convergent or $A\mathcal{I}_{\sigma\theta}^*$-convergent to $x_{\langle p,\eta\rangle}$, if there exists a set $M=\{m_1<m_2<...<m_p<...\}\in\mathcal{F}(\mathcal{I}_{\sigma\theta})$ $(\mathbb{N}\setminus M=H\in\mathcal{I}_{\sigma\theta})$ and there is an integer η such that*

$$\lim_{p\to\infty} x_{m_p}=x_{\langle p,\eta\rangle}.$$

In this case, we write $A\mathcal{I}_{\sigma\theta}^*-\lim x_p=x_{\langle p,\eta\rangle}$ or $x_p\to x_{\langle p,\eta\rangle}\left(A\mathcal{I}_{\sigma\theta}^*\right)$.

Theorem 2.11. *If a sequence $x=(x_p)$ is lacunary $\mathcal{I}_\sigma^*$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, then this sequence is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. Let $\varepsilon>0$. Since $A\mathcal{I}_{\sigma\theta}^*-\lim x_p=x_{\langle p,\eta\rangle}$, there exists a set $H\in\mathcal{I}_{\sigma\theta}$ such that for

$$M=\mathbb{N}\backslash H=\{m_1<m_2<...<m_p<...\}$$

and so there exists a $p_0\in\mathbb{N}$ such that $|x_{m_p}-x_{\langle p,\eta\rangle}|<\varepsilon$ for every $p>p_0$. Then, for every $\varepsilon>0$, we have

$$K(\varepsilon)=\{p\in\mathbb{N}:\left|x_p-x_{\langle p,\eta\rangle}\right|\geq\varepsilon\}$$

$$\subset H\cup\{m_1<m_2<...<m_p<...\}.$$

Since $\mathcal{I}_{\sigma\theta}$ is admissible ideal,

$$H\cup\{m_1<m_2<...<m_p<...\}\in\mathcal{I}_{\sigma\theta}$$

and so we have $K(\varepsilon)\in\mathcal{I}_{\sigma\theta}$. Hence, we get $A\mathcal{I}_{\sigma\theta}-\lim x_p=x_{\langle p,\eta\rangle}$. ■

The converse of the Theorem 2.11 holds if the ideal $\mathcal{I}_{\sigma\theta}$ has the property (AP).

Theorem 2.12. *Let the ideal $\mathcal{I}_{\sigma\theta}$ be with property (AP). If a sequence $x=(x_p)$ is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, then this sequence is lacunary $\mathcal{I}_\sigma^*$ arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. Let the ideal $\mathcal{I}_{\sigma\theta}$ be with the property (AP) and $\varepsilon>0$. Also, we suppose that $A\mathcal{I}_{\sigma\theta}-\lim x_p=x_{\langle p,\eta\rangle}$. Then, for every $\varepsilon>0$ we have

$$K(\varepsilon)=\left\{p\in\mathbb{N}:\left|x_p-x_{\langle p,\eta\rangle}\right|\geq\varepsilon\right\}\in\mathcal{I}_{\sigma\theta}.$$

Denote $K_1,K_2,...,K_n$ as following

$$K_1:=\left\{p\in\mathbb{N}:\left|x_p-x_{\langle p,\eta\rangle}\right|\geq 1\right\}$$

and

$$K_n:=\left\{p\in\mathbb{N}:\frac{1}{n}\leq\left|x_p-x_{\langle p,\eta\rangle}\right|<\frac{1}{n-1}\right\},$$

where $n\geq 2$ $(n\in\mathbb{N})$. Note that $K_i\cap K_j=\emptyset$ $(i\neq j)$ and $K_i\in\mathcal{I}_{\sigma\theta}$ (for each $i\in\mathbb{N}$). Since $\mathcal{I}_{\sigma\theta}$ has the property (AP), there exists a set sequence $\{F_n\}_{n\in\mathbb{N}}$ such that the symmetric differences $K_i\Delta F_i$ are finite (for each $i\in\mathbb{N}$) and $F=\bigcup_{j=1}^{\infty}F_j\in\mathcal{I}_{\sigma\theta}$. Now, to complete the proof, it is enough to prove that

$$\lim_{p\to\infty}x_p=x_{\langle p,\eta\rangle},\ p\in M, \tag{2.1}$$

where $M=\mathbb{N}\setminus F$. Let $\gamma>0$. Select $n\in\mathbb{N}$ such that $\frac{1}{n+1}<\gamma$. Then, we get

$$\left\{p\in\mathbb{N}:\left|x_p-x_{\langle p,\eta\rangle}\right|\geq\gamma\right\}\subset\bigcup_{i=1}^{n+1}K_i.$$

Since the symmetric differences $K_i \Delta F_i$ $(i = 1, 2, ..., n+1)$ are finite, there exists a $p_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\left(\bigcup_{i=1}^{n+1} K_i \right) \cap \{p \in \mathbb{N} : p > p_0\} \\ &= \left(\bigcup_{i=1}^{n+1} F_i \right) \cap \{p \in \mathbb{N} : p > p_0\}. \end{aligned} \tag{2.2}$$

If $p > p_0$ and $p \notin F$, then

$$p \notin \bigcup_{i=1}^{n+1} F_i \text{ and by } (2.2)\ \ p \notin \bigcup_{i=1}^{n+1} K_i.$$

This give that

$$\left| x_p - x_{\langle p,\eta \rangle} \right| < \frac{1}{n+1} < \gamma$$

and so (2.1) holds. As a result, $A\mathcal{I}^*_{\sigma\theta} - \lim x_p = x_{\langle p,\eta \rangle}$. ■

Definition 2.13. *A sequence $x = (x_p)$ is said to be lacunary $\mathcal{I}$ invariant arithmetic statistically convergent to $x_{\langle p,\eta \rangle}$, for each $\varepsilon > 0$ and $\delta > 0$, there is an integer η such that*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ p \in I_r : \left| x_p - x_{\langle p,\eta \rangle} \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}_{\sigma\theta}.$$

In this case, we write $x_p \to x_{\langle p,\eta \rangle}\ (A\mathcal{I}_{\sigma\theta}(S))$.

Definition 2.14. *A sequence $x = (x_p)$ is said to be lacunary strongly $\mathcal{I}_\sigma$ arithmetic summable to $x_{\langle p,\eta \rangle}$ if for each $\varepsilon > 0$, there is an integer η such that*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{p \in I_r} \left| x_p - x_{\langle p,\eta \rangle} \right| \geq \varepsilon \right\} \in \mathcal{I}_{\sigma\theta}.$$

We will use $[A(\mathcal{I}_{\sigma\theta})] - \lim x_p = x_{\langle p,\eta \rangle}$ or $x_p \to x_{\langle p,\eta \rangle}\ ([A(\mathcal{I}_{\sigma\theta})])$ to indicate the sequence (x_m) is lacunary strongly $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle m,n \rangle}$.

Theorem 2.15. *Let $\theta = \{k_r\}$ be a lacunary sequence.*

(i) If $x_p \to x_{\langle p,\eta \rangle}\ ([A(\mathcal{I}_{\sigma\theta})])$, then $x_p \to x_{\langle p,\eta \rangle}\ (A\mathcal{I}_{\sigma\theta}(S))$.

(ii) If $x \in l_\infty$ and $x_p \to x_{\langle p,\eta \rangle}\ (A\mathcal{I}_{\sigma\theta}(S))$, then $x_p \to x_{\langle p,\eta \rangle}\ ([A(\mathcal{I}_{\sigma\theta})])$.

(iii) $(A\mathcal{I}_{\sigma\theta}(S)) \cap l_\infty = [A(\mathcal{I}_{\sigma\theta})] \cap l_\infty$.

Proof. (i) Let $\varepsilon > 0$ and $x_p \to x_{\langle p,\eta \rangle}\ ([A(\mathcal{I}_{\sigma\theta})])$. Then, we can write

$$\begin{aligned} \tfrac{1}{h_r} \sum_{p \in I_r} \left| x_{\sigma^p(s)} - x_{\langle p,\eta \rangle} \right| &\geq \tfrac{1}{h_r} \sum_{\substack{p \in I_r \\ \left| x_{\sigma^p(s)} - x_{\langle p,\eta \rangle} \right| \geq \varepsilon}} \left| x_{\sigma^p(s)} - x_{\langle p,\eta \rangle} \right| \\ &\geq \varepsilon . \tfrac{1}{h_r} \left| \left\{ p \in I_r : \left| x_{\sigma^p(s)} - x_{\langle p,\eta \rangle} \right| \geq \varepsilon \right\} \right| \end{aligned}$$

for $s = 1, 2, ...$ So, for any $\delta > 0$,

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \tfrac{1}{h_r} \left| \left\{ p \in I_r : \left| x_{\sigma^p(s)} - x_{\langle p,\eta \rangle} \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \\ &\subseteq \left\{ r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{p \in I_r} \left| x_{\sigma^p(s)} - x_{\langle p,\eta \rangle} \right| \geq \varepsilon . \delta \right\} \end{aligned}$$

uniformly in s. Since $x_p \to x_{\langle p,\eta\rangle}$ $([A(\mathcal{I}_{\sigma\theta})])$, the set on the right-hand side belongs to $\mathcal{I}_{\sigma\theta}$ and so we obtain $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_{\sigma\theta}(S))$.

(ii) Suppose that $x \in l_\infty$ and $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_{\sigma\theta}(S))$. Then, there exists a $M > 0$ such that

$$\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \leq M$$

for $s = 1, 2, ...$

Given $\varepsilon > 0$, we obtain

$$\frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| = \frac{1}{h_r}\sum_{\substack{p\in I_r\\ \left|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}\right|\geq\varepsilon}}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| + \frac{1}{h_r}\sum_{\substack{p\in I_r\\ \left|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}\right|<\varepsilon}}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|$$

$$\leq \frac{M}{h_r}\left|\left\{p \in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\}\right| + \varepsilon$$

uniformly in s. Note that

$$A(\varepsilon) = \left\{r \in \mathbb{N} : \tfrac{1}{h_r}\left|\left\{p \in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\}\right| \geq \frac{\varepsilon}{M}\right\}.$$

It is obvious that $A(\varepsilon) \in \mathcal{I}_{\sigma\theta}$. If $r \in (A(\varepsilon))^c$ then

$$\frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| < 2\varepsilon.$$

Hence

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq 2\varepsilon\right\} \subset A(\varepsilon)$$

and so belongs to $\mathcal{I}_{\sigma\theta}$. This shows that $x_p \to x_{\langle p,\eta\rangle}$ $([A(\mathcal{I}_{\sigma\theta})])$. This completes the proof.

(iii) This is an immediate consequence of (i) ve (ii). ■

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Approximating positive solutions of nonlinear IVPs of ordinary second order hybrid differential equations

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Abstract. In this paper we prove the existence and approximation of solution for a nonlinear initial value problem of ordinary second order hybrid differential equation. The right hand side of the differential equation is assumed to be Carathoèodory and the proof is based on a Dhage iteration method.

AMS Subject Classifications: 34A45, 34B15, 47H07, 47H10.

Keywords: Nonlinear initial value problems, Hybrid differential equation, Dhage iteration method, Existence and approximation theorem.

Contents

1. Introduction and Background

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}_+$ the set of all nonnegative reals. Given a closed and bounded interval $J = [0, T] \subset \mathbb{R}$, consider the nonlinear hybrid initial value problem (in short HIVP) of ordinary second order hybrid differential equation (in short HDE),

$$\left.\begin{aligned} \frac{d^2}{dt^2}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad \text{a.e. } t \in J, \\ x(0) = 0, \quad x'(0) = 0, \end{aligned}\right\} \tag{1.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous and $f : J \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

When $f \equiv 1$ on $J \times \mathbb{R}$, the HIVP (1.1) reduces to the well-known nonlinear ordinary second order differential equation

$$\left.\begin{aligned} x''(t) = g(t,x(t)) \quad \text{a.e. } t \in J, \\ x(0) = 0, \ x'(0) = 0, \end{aligned}\right\} \tag{1.2}$$

which is studied earlier extensively in the literature (see Dhage and Dhage [5]).

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Definition 1.1. *A function $x \in AC^1(J,\mathbb{R})$ is said to be a* lower solution *of the IVP (1.1) if*

$$\left.\begin{aligned} \frac{d^2}{dt^2}\left(\frac{x(t)}{f(t,x(t))}\right) &\leq g(t,x(t)) \quad a.e. \;\; t \in J, \\ x(0) = 0, \;\; x'(0) &= 0, \end{aligned}\right\} \tag{1.3}$$

where, $AC^1(J,\mathbb{R})$ is the space of functions $x \in C(J,\mathbb{R})$ whose first derivative exists and is absolutely continuous on I. Similarly, $x \in AC^1(J,\mathbb{R})$ is called an upper solution *of (1.1) on J if the reversed inequalities hold in (1.3). If equalities hold in (1.3), we say that x is a* solution *of (1.1) on J.*

The existence of the solution to the problem (1.1) may be proved by using hybrid fixed point theorems of Dhage in a Banach algebra as did in Dhage [2] and Dhage and Imdad [7]. The existence of positive solution to a nonlinear equation is generally proved using the properties of cones in a partially ordered Banach space (see Deimling [1] and Granas [8]). However, the existence and approximation result for the second order IVPs and PBVPs are already proved in Dhage and Dhage [5, 6] without using the properties of the cones via a new Dhage iteration method developed in [3]. In the present paper, we shall extend above Dhage iteration method to the HIVP (1.1) and study the existence and approximation of positive solutions of under certain hybrid conditions on the nonlinearities f and g from algebra, analysis and topology.

2. Auxiliary Results

We need the following definition in what follows.

Definition 2.1. *A function $\beta \,:\, J \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) the map $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \mapsto \beta(t,x)$ is continuous for each $t \in J$.

The following lemma is often used in the study of nonlinear differential equations (see Dhage [2] and references therein).

Lemma 2.2 (Carathéodory)**.** *Let $\beta : J \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the map $(t,x) \mapsto \beta(t,x)$ is jointly measurable. In particular the map $t \mapsto \beta(t,x(t))$ is measurable on J for each $x \in C(J,\mathbb{R})$.*

We need the following hypotheses in the sequel.

(H_1) f defines a continuous bounded function $f \,:\, J \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$ with bound M_f.

(H_2) There exists a $\mathcal{D}$-function $\varphi_f \in \mathfrak{D}$ such that

$$0 \leq f(t,x) - f(t,y) \leq \varphi_f\big(x-y\big)$$

for all $\in J$ and $x, y \in \mathbb{R}$ with $x \geq y$. Moreover, $T^2 M_g \varphi_f(r) < r, \;\; r > 0$.

(H_3) The function g is Carathéodory on $J \times \mathbb{R}$ into $\mathbb{R}_+$.

(H_4) g is bounded on $J \times \mathbb{R}$ with bound M_g.

(H_5) $g(t,x)$ is nondecreasing in x for each $t \in J$.

(LS) The HIVP (1.1) and (1.3) has a lower solution $u \in AC^1(J,\mathbb{R})$.

(US) The HIVP (1.1) and (1.3) has an upper solution $v \in AC^1(J,\mathbb{R})$.

Lemma 2.3. *Given any function $h \in L^1(J, \mathbb{R})$, the HIVP*

$$\left.\begin{aligned} \frac{d^2}{dt^2}\left(\frac{x(t)}{f(t,x(t))}\right) &= h(t) \quad a.e.\ \ t \in J, \\ x(0) = 0, \quad x'(0) &= 0, \end{aligned}\right\} \tag{2.1}$$

is equivalent to the quadratic hybrid integral equation (in short HIE)

$$x(t) = \big[f(t,x(t))\big]\left(\int_0^t (t-s)h(s)\,ds\right), \ \ t \in J. \tag{2.2}$$

The proof of our main result will be based on the **Dhage monotone iteration principle** or **Dhage monotone iteration method** contained in a applicable hybrid fixed point theorem in the partially ordered Banach algebras.

A non-empty closed convex subset K of the Banach algebra E is called a cone if it satisfies i) $K + K \subset K$, ii) $\lambda K \subseteq K$ for $\lambda > 0$ and iii) $\{-K\} \cap K = \{0\}$. We define a partial order $\preceq$ in E by the relation $x \preceq y \iff y - x \in K$. The cone K is called positive if iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplicative composition in E. In what follows we assume that the cone K in a partially ordered Banach algebra (E, K) is always positive. Then the following results are known in the literature.

Lemma 2.4 (Dhage [4])**.** *Every ordered Banach space (E, K) is regular.*

Lemma 2.5 (Dhage [4])**.** *Every partially compact subset S of an ordered Banach space (E, K) is a Janhavi set in E.*

Theorem 2.6 (Dhage [3])**.** *Let $\big(E, K, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra and let every chain C in E be a Janhavi set. Suppose that $\mathcal{A}, \mathcal{B} : E \to K$ are two monotone nondecreasing operators such that*

(a) $\mathcal{A}$ is partially bounded and partial $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\varphi_{\mathcal{A}}$,

(b) $\mathcal{B}$ is partially continuous and uniformly partially compact,

(c) $M_{\mathcal{B}}\, \varphi_{\mathcal{A}}(r) < r$, $r > 0$, where $M_{\mathcal{B}} = \sup\{\|\mathcal{B}(C)\| : C$ is a chain in $E\}$, and

(d) there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0\, \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\, \mathcal{B}x_0$.

Then the hybrid operator equation $\mathcal{A}x\, \mathcal{B}x = x$ has a solution x^* in K and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\, \mathcal{B}x_n$ converges monotonically to x^*.

The details of Dhage monotone iteration principle or method and related definitions of Janhavi set and uniformly partially compact operator along with some applications may be found in Dhage [3, 4] and the references therein.

3. Existence and Approximation Result

Let $C_+(J, \mathbb{R})$ denote the space of all nonnegative-valued functions of $C(J, \mathbb{R})$. We assume that the space $C(J, \mathbb{R})$ is endowed with the norm $\|\cdot\|$ and the multiplication "$\cdot$" defined by

$$\|x\| = \max_{t \in J} |x(t)| \quad \text{and } (x \cdot y)(t) = x(t)y(t) \ \ t \in J. \tag{3.1}$$

We define a partial order $\preceq$ in E with the help of the cone K in E defined by

$$K = \big\{x \in E \mid x(t) \geq 0 \ \text{ for all } \ t \in J\big\} = C_+(J, \mathbb{R}), \tag{3.2}$$

which is obviously a positive cone in $C(J,\mathbb{R})$. Thus, we have $x \preceq y \iff y - x \in K$.

Clearly, $C(J,\mathbb{R})$ is a partially ordered Banach algebra with respect to above supremum norm, multiplication and the partially order relation in $C(J,\mathbb{R})$. A solution ξ^* of the HIVP (1.1) is *positive* if it belongs to the class of function space $C_+(J,\mathbb{R})$.

Theorem 3.1. *Suppose that hypotheses (H$_1$)-(H$_5$) and (LS) hold. Then the BVP (1.1) has a positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= u(t), \ \ t \in J, \\ x_{n+1}(t) &= \big[f(t, x_n(t))\big] \left(\int_0^t (t-s) g(s, x_n(t))\, ds\right), \ \ t \in J, \end{aligned}\right\} \tag{3.3}$$

converges monotone nondecreasingly to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemmas 2.4 and 2.5, E is regular and every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\preceq$ so that every compact chain C is a Janhavi set in E.

Now by Lemma 2.2, the BVP (1.1) is equivalent to the HIE

$$x(t) = \big[f(t, x(t))\big] \left(\int_0^t (t-s) g(s, x(t))\, ds\right), \ t \in J. \tag{3.4}$$

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = f(t, x(t)), \ t \in J, \tag{3.5}$$

and

$$\mathcal{B}x(t) = \int_0^t (t-s) g(s, x(t))\, ds, \ \ t \in J. \tag{3.6}$$

From hypotheses (H$_1$) and (H$_3$), it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \to K$. Now the HIE (3.4) is equivalent to the quadratic hybrid operator equation

$$\mathcal{A}x(t)\, \mathcal{B}x(t) = x(t), \ t \in J. \tag{3.7}$$

Now, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.6 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on E.*

Let $x, y \in E$ be such that $x \succeq y$. Then, from the hypothesis (H$_2$) it follows that

$$\mathcal{A}x(t) = f(t, x(t)) \geq f(t, y(t)) = \mathcal{A}x(t)$$

for all $t \in J$. Hence $\mathcal{A}x \succeq \mathcal{A}(y)$ and that $\mathcal{A}$ is nondecreasing on E. Similarly, we have by hypothesis (H$_5$),

$$\mathcal{B}x(t) = \int_0^t (t-s) g(s, x(s)) d \geq \int_0^t (t-s) g(s, y(s))\, ds = \mathcal{B}y(t)$$

for all $t \in J$. This implies that $\mathcal{B}x \succeq \mathcal{B}y$ whenever $x \succeq y$. Thus, $\mathcal{B}$ is also nondecreasing operator on E.

Step II: *Next we show that $\mathcal{A}$ is partially bounded and partial $\mathcal{D}$- Lipschitz on E.*

Now, for any $x \in E$, one has

$$\|\mathcal{A}x\| = \sup_{t \in J} |f(t, x(t))| \leq M_f$$

and so $\mathcal{A}$ is bounded and consequently partially bounded on E. Nxt let $x, y \in E$ be such that $x \succeq y$. Then, by hypotesis (H$_2$),

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq \varphi_f(|x(t) - y(t)|) \leq \varphi_f(\|x - y\|)$$

for all $t \in J$. Taking the supremum over t, we get

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \varphi_f(\|x - y\|)$$

which shows that $\mathcal{A}$ is a $\mathcal{D}$-Lipschitz on E with $\mathcal{D}$-function φ_f.

Step III: *$\mathcal{B}$ is a partially contiuous and partially compact on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Since the f is continuous, by dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \int_0^t (t - s) g(s, x_n(s))\, ds \\ &= \int_0^t (t - s) \left[\lim_{n\to\infty} g(s, x_n(s))\right] \, ds = \mathcal{B}x(t),\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J. Next, we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now for any $t_1, t_2 \in J$, one obtains

$$|\mathcal{B}x_n(t_1) - \mathcal{B}x_n(t_2)| \leq M_g T\, |t_1 - t_2| + |p(t_1) - p(t_2)|\, ds \tag{3.8}$$

uniformly for all $n \in \mathbb{N}$, where $p(t) = \displaystyle\int_0^t M_g(T - s)\, ds$.

Since the functions $t \to |t|$ and $t \to p(t)$ is continuous on compact J, they are uniformly continuous there. Therefore, we have

$$|p(t_1) - p(t_2)| \to 0 \quad \text{as } t_1 \to t_2$$

uniformly on J. As a result, we have that

$$|\mathcal{B}x_n(t_1) - \mathcal{B}x_n(t_2)| \to 0 \quad \text{as } t_1 \to t_2,$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and that $\mathcal{B}$ is a partially continuous operator on E into itself.

Next, we show that $\mathcal{B}$ is a uniformly partially compact operator on E. Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (H$_2$)

$$|y(t)| = |\mathcal{B}x(t)| \leq \int_0^t (t - s)|g(s, x(s))|\, ds \leq T^2\, M_g,$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| = \|\mathcal{B}x\| \leq M_g\, T^2$ for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next, proceeding with the arguments that given in Step II it can be shown that

$$\left|y(t_2) - y(t_1)\right| = |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathcal{B}(C)$. This shows that $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{B}$ is a uniformly partially compact operator on E into itself.

Step IV: *$\mathcal{A}$ and $\mathcal{B}$ satisfy the growth inequality $M_B\, \varphi_A(r) < r$, $r > 0$.*

Now, it can be shown $\|\mathcal{B}(C)\| \leq T^2 M_g = M_{\mathcal{B}}$ for all chain C in E. Therefore, we obtain

$$M_{\mathcal{B}}\varphi_{\mathcal{A}}(r) = T^2 M_g \varphi_f(r) < r$$

for all $r > 0$ and so the hypothesis (c) of Theorem 2.6 is satisfied.

Step VI: *The function* u *satisfies the operator inequality* $u \preceq \mathcal{A}u\,\mathcal{B}u$.

By hypothesis (LS), the HIVP (1.1) has a lower solution u defined on J. Then, we have

$$\left.\begin{aligned} &\frac{d^2}{dt^2}\left(\frac{u(t)}{f(t,u(t))}\right) \leq g(t,u(t)) \quad \text{a.e. } t \in J, \\ &\frac{u(0)}{f(0,u(0))} = 0, \qquad \left(\frac{u(t)}{f(t,u(t))}\right)'\Bigg|_{t=0} = 0, \end{aligned}\right\} \tag{3.9}$$

By using this, the fundamental theorem of calculus and the definitions of the operators $\mathcal{A}$ and $\mathcal{B}$, it can be shown that the function $u \in C(J,\mathbb{R})$ satisfies the relation $u \preceq \mathcal{A}u\,\mathcal{B}u$ on J.

Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.6 and so the quadratic hybrid operator equation $\mathcal{A}x\,\mathcal{B}x = x$ has a positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n$ with initial term $x_0 = u$ converges monotone nondecreasingly to x^*. Therefore, the HIE (3.4) and consequently the HIVP (1.1) has a positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) with $x_0 = u$, converges monotone nondecreasingly to x^*. This completes the proof. ■

Remark 3.2. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (LS) with (US). The proof of Theorem 3.1 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications. In this case the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (3.3) with $x_0(t) = v(t),\ t \in [0,T]$, converges montone nonincrasingly to the solution x^* of he HIVP (1.1) on J. Again, the existence and approximation result, Theorem 3.1 includes similar result for the positive solution of the HIVP (1.2) as a special case.

Remark 3.3. We note that if the HIVP (1.1) has a lower solution $u \in AC^1(J,\mathbb{R})$ as well as an upper solution $v \in AC^1(J,\mathbb{R})$ such that $u \preceq v$, then under the given conditions of Theorem 3.1 it has corresponding solutions x_* and y^* and these solutions satisfy the inequality

$$u = x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_* \preceq y^* \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0 = v.$$

Hence x_* and y^* are respectively the minimal and maximal impulsive solutions of the HIVP (1.1) in the vector segment $[u,v]$ of the Banach space $E = C(J,\mathbb{R})$, where the vector segment $[u,v]$ is a set of elements in $C(J,\mathbb{R})$ defined by

$$[u,v] = \{x \in C(J,\mathbb{R}) \mid u \preceq x \preceq v\}.$$

This is because of the order cone K defined by (3.2) is a closed convex subset of $C(J,\mathbb{R})$. However, we have not used any property of the cone K in the main existence results of this paper. A few details concerning the order relation by the order cones and the Janhavi sets in an ordered Banach space are given in Dhage [4].

4. An Example

Example 4.1.

Given a closed interval $J = [0,1]$ in $\mathbb{R}$, consider the nonlinear HIVP of hybrid differential equations

$$\left.\begin{aligned} &\frac{d^2}{dt^2}\left(\frac{x(t)}{f(t,x(t))}\right) = \tanh x(t) + 1 \quad \text{a.e. } t \in J, \\ &x(0) = 0, \quad x'(0) = 0, \end{aligned}\right\} \tag{4.1}$$

where the function $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is defined by

$$f(t,x) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 + \dfrac{x}{1+x}, & \text{if } x > 0. \end{cases}$$

Then the function f satisfies the hypotheses (H$_1$)-(H$_2$) with $M_f = 2$ and $\varphi_f(r) = \dfrac{r}{1+\xi^2}$, $0 \leq \xi \leq r$. Here $g(t,x) = \tanh x + 1$ and satisfies the hypotheses (H$_3$)-(H$_5$) with $M_g = 2$. Now the HIVP (4.1) is equivalent to the HIE

$$x(t) = \big[f(t,x(t))\big]\left(\int_0^t (t-s)\big[\tanh x(s) + 1\big]\,ds\right), \;\; t \in [-1,1],$$

It can be verified that the function $u \in C(J,\mathbb{R})$ defined by $u(t) = -t^2$ and $v(t) = 4t^2$ are respectively the lower and upper solutions of the HIVP (4.1) on $[0,1]$. Hence, by an application of Theorem 3.1, the HIVP (4.1) has a positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} x_0(t) &= -t^2, \;\; t \in [0,1], \\ x_{n+1}(t) &= \big[f(t,x_n(t))\big]\left(\int_0^t (t-s)\big[\tanh x_n(s) + 1\big]\,ds\right), \;\; t \in [0,1], \end{aligned}$$

converges monotone nondecreasingly to x^*. Similarly, by Remark 3.2, the sequence $\{y_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} y_0(t) &= 4t^2, \;\; t \in [0,1], \\ y_{n+1}(t) &= \big[f(t,y_n(t))\big]\left(\int_0^t (t-s)\big[\tanh y_n(s) + 1\big]\,ds\right), \;\; t \in [0,1], \end{aligned}$$

converges monotone non-increasingly to the positive solution y^* of the HIVP (4.1) on $[0,1]$.

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On the spherical magnetic trajectories

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Abstract. We consider spherical indicatrix magnetic trajectories of a magnetic field in Euclidean 3−space. From classical formulation of Killing magnetic flow equations, we derive the differential equation systems for tangent spherical indicatrix magnetic trajectories in Euclidean 3−space. Then we solve these equations by using Jacobi elliptic functions. Finally, we make similar calculations for curves whose principal normal and binormal spherical indicatrix are magnetic curves.

AMS Subject Classifications: 53A04, 53A05.

Keywords: Magnetic curve, spherical indicatrix, Killing magnetic flow.

Contents

1. Introduction

Any magnetic vector field is known divergence zero vector field in three dimensional spaces. A magnetic trajectory of a magnetic flow created by magnetic vector field are curves called as magnetic curves. Although the problem of investigating magnetic trajectories appears to be physical problem, recent studies show that the characterization of magnetic flow in a magnetic field have brought variational perspective in more geometrical manner [2, 8]. Let S be a surface in Euclidean 3−space $\mathbb{R}^3$ and F denote a complete differential 2−form in a open subset U of S. Then we can write $F = d\omega$ for some potential 1−form ω. If we define Γ as smooth curves that connect two fixed point of U, the Lorentz force equation is known a minimizer of the functional $\mathcal{L} : \Gamma \to \mathbb{R}$ defined by

$$\mathcal{L}(\gamma) : \frac{1}{2}\int_{\gamma} < \gamma', \gamma' > dt + \omega(\gamma')\, dt. \tag{1.1}$$

The Euler-Lagrange equation of the functional $\mathcal{L}$ is derived as

$$\phi(\gamma') = \nabla_{\gamma'}\gamma', \tag{1.2}$$

where ϕ is the skew-symmetric operator. The critical point of the functional $\mathcal{L}$ corresponds to the Lorentz force equation [2, 4].

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Any function defined from a space curve to a suitable sphere in Euclidean $3-$space is called the spherical indicatrix (spherical image) of the curve. The spherical indicatrix of a curve in Euclidean $3-$space emerges in three types: the tangent indicatrix (tangential indicatrix or tangent spherical indicatrix), the principle normal indicatrix and the binormal indicatrix of the curve. The spherical indicatrix is a nice way to envision the motion of the curve on a sphere by using components of the Frenet Frame. Furthermore, the movement of a spherical indicatrix describes the changes in the original direction of the curve [6, 7].

In this paper we consider the magnetic trajectories which are the tangent, principal and binormal indicatrices, separately. We first investigate the tangent indicatrix magnetic trajectories and we derive the Killing magnetic flow equations for tangent indicatrix magnetic vector field. Then we solve these equations by using elliptic functions. Then we apply this method the other imagine types of curves by using same calculations. But we do not dwell on variational and differential calculations of the problem of finding curves whose principle normal and binormal indicatrix are magnetic since the same procedure would repeat.

2. Preliminaries

We consider a regular curve γ in Euclidean $3-$space $\mathbb{R}^3$, parametrized by arc length s, $0 \leq s \leq \ell$. Let $T = \gamma'(s)$ denote the unit tangent vector field, $N(s)$ the unit principle normal vector field and $B = T \times N$ binormal vector field at point $\gamma(s)$. Then we have the Frenet frame $\{T, N, B\}$ along the curve γ and Frenet equations given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.1}$$

where $\kappa > 0$ and τ are respectively curvature and torsion of γ [6].

If the Frenet frame of the tangent indicatrix $\gamma_t = T$ of a space curve γ is $\{T_t, N_t, B_t\}$, then we have the following Frenet equations

$$\begin{pmatrix} T_t'(s_t) \\ N_t'(s_t) \\ B_t'(s_t) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_t & 0 \\ -\kappa_t & 0 & \tau_t \\ 0 & -\tau_t & 0 \end{pmatrix} \begin{pmatrix} T_t \\ N_t \\ B_t \end{pmatrix}, \tag{2.2}$$

where

$$T_t = N, \; N_t = \frac{-T + fB}{\sqrt{1+f^2}}, \; B_t = \frac{fT + B}{\sqrt{1+f^2}} \tag{2.3}$$

and

$$s_t = \int \kappa(s)\, ds, \; \kappa_t = \sqrt{1+f^2}, \; \tau_t = \sigma\sqrt{1+f^2}, \tag{2.4}$$

where

$$f(s) = \frac{\tau(s)}{\kappa(s)} \text{ and } \sigma = \frac{f'(s)}{\kappa(s)\left(1+f^2\right)^{3/2}} = \frac{\tau_t}{\kappa_t}. \tag{2.5}$$

σ is the geodesic curvature of the principal image of the principal normal indicatrix of the curve γ, s_t is natural representation of the tangent indicatrix of the curve γ and equal the total curvature of the curve γ and κ_t and τ_t are the curvature and torsion of γ_t.

If the Frenet frame of the normal indicatrix $\gamma_n = N$ of a space curve γ is $\{T_n, N_n, B_n\}$, then we have the following Frenet equations

$$\begin{pmatrix} T_n'(s_n) \\ N_n'(s_n) \\ B_n'(s_n) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_n & 0 \\ -\kappa_n & 0 & \tau_n \\ 0 & -\tau_n & 0 \end{pmatrix} \begin{pmatrix} T_n \\ N_n \\ B_n \end{pmatrix}, \tag{2.6}$$

where

$$T_n = \frac{-T + fB}{\sqrt{1+f^2}},\ N_t = \frac{\sigma}{\sqrt{1+\sigma^2}}\left[\frac{-T+fB}{\sqrt{1+f^2}} - \frac{N}{\sigma}\right],$$
$$B_t = \frac{1}{\sqrt{1+\sigma^2}}\left[\frac{fT+B}{\sqrt{1+f^2}} + \sigma N\right].$$

and

$$s_n = \int \kappa(s)\left(\sqrt{1+f^2(s)}\right) ds,\ \kappa_n = \sqrt{1+\sigma^2},\ \tau_t = \Gamma\sqrt{1+\sigma^2},$$

where

$$\Gamma = \frac{\sigma'(s)}{\kappa(s)\sqrt{(1+f^2)}\,(1+\sigma^2)^{3/2}} = \frac{\tau_n}{\kappa_n},$$

where s_n is natural representation of the principal normal indicatrix of the curve γ and κ_n and τ_n are the curvature and torsion of γ_n.

If the Frenet frame of the binormal indicatrix $\gamma_b = N$ of a space curve γ is $\{T_b, N_b, B_b\}$, then we have Frenet formula:

$$\begin{pmatrix} T_b'(s_b) \\ N_b'(s_b) \\ B_b'(s_b) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_b & 0 \\ -\kappa_b & 0 & \tau_b \\ 0 & -\tau_b & 0 \end{pmatrix} \begin{pmatrix} T_b \\ N_b \\ B_b \end{pmatrix}, \tag{2.7}$$

where

$$T_b = -N,\ N_t = \frac{T - fB}{\sqrt{1+f^2}},\ B_t = \frac{fT+B}{\sqrt{1+f^2}}.$$

and

$$s_b = \int \tau(s)\, ds,\ \kappa_b = \frac{\sqrt{1+f^2}}{f},\ \tau_b = -\sigma\frac{\sqrt{1+f^2}}{f},$$

where

$$\sigma = \frac{\tau_b}{\kappa_b},$$

where s_b is the natural representation of the binormal indicatrix of the curve γ and κ_b and τ_b are the curvature and torsion of γ_b [1].

3. Spherical indicatrix magnetic fields

Let V be a divergence-free vector field in Euclidean 3-space $\mathbb{R}^3$. Then it defines a magnetic vector field. Given a differential $2-$form F is a magnetic field on $\mathbb{R}^3$. The Lorentz force of F is defined to be the skew-symmetric operator ϕ given by

$$< \phi(X), Y >= F(X, Y) \tag{3.1}$$

for all vector field $X, Y \in \chi\left(\mathbb{R}^3\right)$. The associated magnetic trajectories are curves γ on $\mathbb{R}^3$ that satisfies the Lorentz force equation (1.2). On the other hand the Lorentz force ϕ can be write as follows

$$\phi(X) = V \times X, \tag{3.2}$$

that is, the Lorentz force ϕ of V is defined via cross product on $\mathbb{R}^3$. Combining (1.2) and (3.2), the Lorentz equation can be written by

$$\phi(\gamma') = \nabla_{\gamma'}\gamma' = V \times \gamma'$$

for a curve γ on $\mathbb{R}^3$.

By means of these structures defined on $\mathbb{R}^3$, the Killing magnetic flow equations corresponding to spherical indicatrix for a unit-speed curve γ on $\mathbb{R}^3$ will be found.

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a reparametrized curve in Euclidean $3-$space and $\{T_t, N_t, B_t\}$ is the Frenet frame along γ_t. Then the Lorentz force in the frame $\{T_t, N_t, B_t\}$ is written as

$$\phi(T_t) = \kappa_t N_t, \tag{3.3}$$
$$\phi(N_t) = -\kappa_t T_t + \omega_t B_t \tag{3.4}$$

and

$$\phi(B_t) = -\omega_t B_t, \tag{3.5}$$

where the function $\omega_t(s_t)$ associated with each tangent indicatrix magnetic curve is quasislope measured with respect to the magnetic field V_t.

Then we can give the following propositions.

Proposition 3.1. The tangential indicatrix γ_t is a magnetic trajectory of a magnetic field V_t if and only if V_t can be written along γ_t as

$$V_t = \omega_t T_t + \kappa_t B_t. \tag{3.6}$$

Proof. Assume that γ_t is a magnetic curve along a magnetic field V_t and the orthogonal frame along γ_t is given by $\{T_t, N_t, B_t\}$. Then, V_t can be written as

$$V_t =< V_t, T_t > T_t + < V_t, N_t > N_t + < V_t, B_t > B_t.$$

To find coefficient of V_t, we use the Lorentz force in orthogonal frame equations (3.3), (3.4) and (3.5):

$$\omega_t =< \phi(N_t), B_t >=< V_t \times N_t, B_t >=< V_t, T_t >,$$
$$0 =< \phi(T_t), B_t >=< V_t \times T_t, B_t >= - < V_t, B_t >$$

and

$$\kappa_t =< \phi(T_t), N_t >=< V_t \times T_t, N_t >=< V_t, B_t > .$$

Proposition 3.2. The principle indicatrix γ_n is a magnetic trajectory of a magnetic field V_n if and only if V_n can be written along γ_n as

$$V_n = \omega_n T_n + \kappa_n B_n.$$

Proposition 3.3. The binormal indicatrix γ_b is a magnetic trajectory of a magnetic field V_b if and only if V_b can be written along γ_b as

$$V_b = \omega_b T_b + \kappa_b B_b.$$

4. Killing magnetic flow equations for spherical indicatrix magnetic curves

Let γ_t be a tangential indicatrix of γ in $\mathbb{R}^3$ and V_t be a vector field along that curve. One can take a variation of γ_t in the direction of V_t, say a map

$$\begin{aligned} \Gamma : [0,1] \times (-\varepsilon, \varepsilon) &\to \mathbb{S}^2 \\ (s, w) &\to \Gamma(s, w) \end{aligned}$$

which satisfies

$$\Gamma(s, 0) = \gamma_t(s), \ \left(\frac{\partial \Gamma(s,w)}{\partial w}\right)_{w=0} = V_t(s),$$

and

$$\left(\frac{\partial \Gamma(s,w)}{\partial s}\right)_{w=0} = \gamma_t'(s).$$

One can write the speed function $\upsilon_t(s, w) = \left\| \frac{\partial \Gamma(s,w)}{\partial s} \right\|$, the curvature function $\kappa_t(s, w)$ and the torsion function $\tau_t(s, w)$ [2, 5].

Lemma 4.1 (see [2, 3]). Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$, γ_t denote the tangent indicatrices and V_t be a vector field along the curve γ_t. Then we have the following equalities

$$V_t(\upsilon_t) = \left(\frac{\partial \upsilon_t(s,w)}{\partial w}\right)_{w=0} = < \nabla_{T_t} V_t, T_t > \upsilon_t, \tag{4.1}$$

$$V_t(\kappa_t) = \left(\frac{\partial \kappa_t(s,w)}{\partial w}\right)_{w=0} = \frac{1}{\kappa_t} < \nabla^2_{T_t} V_t, \nabla_{T_t} T_t > -2\kappa_t < \nabla_{T_t} V_t, T_t > \tag{4.2}$$

and

$$V_t(\tau_t) = \left(\frac{\partial \tau_t(s,w)}{\partial w}\right)_{w=0} = \left(\frac{1}{\kappa_t^2} < \nabla^2_{T_t} V_t, T_t \times \nabla_{T_t} T_t >\right)' + \tau_t < \nabla_{T_t} V_t, T_t > + < \nabla_{T_t} V_t, T_t \times \nabla_{T_t} T_t > . \tag{4.3}$$

Proposition 4.1. Let V_t be the restriction to the tangent indicatrix γ_t of a Killing vector field, say V_t of $\mathbb{R}^3$; then

$$V_t(\upsilon_t) = V_t(\kappa_t) = V_t(\tau_t) = 0. \tag{4.4}$$

Then we can give the Killing magnetic flow equations of the tangential indicatrix.

Theorem 4.1. Let γ_t be the tangential indicatrix of a regular curve γ. Suppose that $V_t = \omega_t T_t + \kappa_t B_t$ is a Killing vector field along γ_t. Then the tangential indicatrix magnetic trajectories are curves on $\mathbb{S}^2$ satisfying following differential equations

$$\kappa_t^2 \left(\frac{1}{2}\omega_t - \tau_t\right) = A_1. \tag{4.5}$$

and

$$\kappa_t'' + \kappa_t \tau_t (\omega_t - \tau_t) + C\kappa_t + \frac{1}{2}\kappa_t^3 - A_2 \kappa_t = 0, \tag{4.6}$$

where A_1, A_2 and C are undetermined constants.

Proof. Assume that V_t is a Killing vector field along γ_t on S^2. Along any spherical magnetic trajectory γ_t, we have $V_t = \omega_t T_t + \kappa_t B_t$. If V_t is Killing vector field, we calculate

$$\omega_t' = 0,$$

that is ω_t is a constant, and

$$\nabla_{T_t} V_t = \kappa_t (\omega_t - \tau_t) N_t + \kappa_t' B_t. \tag{4.7}$$

By using the first derivative of (4.7), (4.2) and (4.4), we get

$$\left(\kappa_t^2 \left(\frac{1}{2}\omega_t - \tau_t\right)\right)' = 0.$$

Similarly, from (4.2) and (4.4), we find to $V(\tau)$ as follows

$$V_t(\tau_t) = \left(\frac{\partial \tau_t(s,w)}{\partial w}\right)_{w=0} = \left(\frac{1}{\kappa_t^2} < \nabla^2_{T_t} V_t, T_t \times \nabla_{T_t} T_t >\right)' + \tau_t < \nabla_{T_t} V_t, T_t > + < \nabla_{T_t} V_t, T_t \times \nabla_{T_t} T_t > .$$

Definition 4.1. Any tangent indicatrix of a Euclidean curve is called the tangent indicatrix magnetic trajectory of a magnetic field V_t if it satisfies the differential equation system (4.5) and (4.6).

We can combine Eqs. (4.5) and (4.6) as follows

$$\kappa_t'' + \frac{1}{2}\kappa_t^3 + \left(C - A_2 + \frac{1}{4}\omega_t^2\right)\kappa_t - \frac{A_1^2}{\kappa_t^3} = 0.$$

This equation admits an obvious first integral. In fact, just multiply by $2\kappa_t'$ and integrate to get

$$(\kappa_t')^2+\frac{1}{4}\kappa_t^4+\left(C-A_2+\frac{1}{4}\omega_t^2\right)\kappa_t^2-\frac{A_1^2}{\kappa_t^2}=A_3.$$

Since this equation is of the type $(u')^2=P\,(h)$, where P is a polynomial of degree 3 in u, it can be solved using elliptic functions as follows

$$\kappa_t\,(s)=\sqrt{a_3\,(1-q^2 sn\,(rs,p))},$$
$$\tau_t\,(s)=\frac{1}{2}\omega_t-\frac{A_1}{\kappa_t^2},$$

when $\kappa_t\neq const.$, where

$$(u_t')^2+(u-a_1)\,(u-a_2)\,(u-a_3)=0,\quad u=\kappa_t^2,$$
$$p=\frac{a_3-a_2}{a_3-a_1},\ q^2=\frac{a_3-a_2}{a_3}\text{ and }r=\frac{1}{2}\sqrt{a_3-a_1}.$$

So, the curvature and the curvature of γ must satisfy the equations

$$\frac{\tau}{\kappa}=\sqrt{\sqrt{a_3\,(1-q^2 sn\,(rs,p))}-1},$$

and

$$\left(\frac{\tau}{\kappa}\right)'=\kappa_t\left(1+\frac{\tau^2}{\kappa^2}\right)\left(1+\omega_t-\frac{1}{\sqrt{a_3\,(1-q^2 sn\,(rs,p))}}\right).$$

Making similar calculations we can give the Killing magnetic flow equations of the principle normal and binormal indicatrix.

Theorem 4.2. Let γ_n be the normal indicatrix of a regular curve γ. Suppose that $V_n=\omega_n T_n+\kappa_n B_n$ is a Killing vector field along γ_n. Then the normal indicatrix magnetic trajectories are curves on $\mathbb{S}^2$ satisfying following differential equations

$$\kappa_n^2\left(\frac{1}{2}\omega_n-\tau_n\right)=A_4. \tag{4.8}$$

and

$$\kappa_n''+\kappa_n\tau_n\,(\omega_n-\tau_n)+C_1\kappa_n+\frac{1}{2}\kappa_n^3-A_5\kappa_n=0, \tag{4.9}$$

where A_4, A_5 and C_1 are undetermined constants.

Theorem 4.3. Let γ_b be the binormal indicatrix of a regular curve γ. Suppose that $V_b=\omega_b T_b+\kappa_b B_b$ is a Killing vector field along γ_b. Then the binormal indicatrix magnetic trajectories are curves on $\mathbb{S}^2$ satisfying following differential equations

$$\kappa_b^2\left(\frac{1}{2}\omega_b-\tau_b\right)=A_6. \tag{4.10}$$

and

$$\kappa_b''+\kappa_b\tau_b\,(\omega_b-\tau_b)+C_2\kappa_b+\frac{1}{2}\kappa_b^3-A_7\kappa_b=0, \tag{4.11}$$

where A_6, A_7 and C_2 are undetermined constants.

Definition 4.1. Any principle normal (binormal) indicatrix of a Euclidean curve is called the principle (binormal) indicatrix magnetic trajectory of a magnetic field V_n (V_b) if it satisfies the differential equation system (4.8) and (4.9) (resp., (4.10) and (4.11)).

Example 4.1. We consider a unit-speed circular helix $\beta\,(s)=\left(\cos\frac{s}{\sqrt{2}},\sin\frac{s}{\sqrt{2}},-\frac{s}{\sqrt{2}}\right)$ [2]. The curve $\beta\,(s)$ can be seen on Fig. 1.

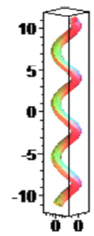

Figure 1: The helix $\beta(s)$

Curvature and torsion of $\beta(s)$ are found as $\kappa = -\tau = \frac{1}{2}$. Then, tangent indicatrix of the circular helix is

$$\beta_t \approx \beta'(s) = \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

β_t is a circle cut from unit sphere by the plane $z = -\frac{1}{\sqrt{2}}$. The curvature and the torsion of the tangent indicatrix of the circular helix are found as $\kappa_t = \sqrt{2}$, $\tau_t = 0$. We can see from (4.5) and (4.6), the tangent indicatrix β_t of β is a tangent indicatrix magnetic trajectory with $A_2 = C + 1$ of the Killing magnetic field $V_t = A_1 T_t + \sqrt{2} B_t$. The principal normal indicatrix of the circular helix is

$$\beta_n \approx N(s) = \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right),$$

that is, β_n lies on the great circle lines on the sphere with $\kappa_n = 1$ and $\tau_n = 0$. From (4.8) and (4.9), we show that β_n is a principle normal indicatrix magnetic trajectory with $A_5 = C_1 + \frac{1}{2}$ of the Killing magnetic field $V_t = 2A_4 T_n + B_n$. Finally, the binormal indicatrix of the circular helix is

$$\beta_b \approx B(s) = \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}\right),$$

that is, β_b is a circle cut from unit sphere by the plane $y = \frac{1}{\sqrt{2}}$. From (4.10) and (4.11), β_b is a binormal indicatrix magnetic trajectory with $A_7 = C_2 + 1$ of the Killing magnetic field $V_b = A_6 T_b + \sqrt{2} B_b$. The graphs of β_t, β_n and β_b are given as follows.

Fig. 2. The magnetic trajectory β_t

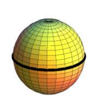

Fig. 3. The magnetic trajectory β_n

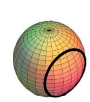

Fig. 4. The magnetic trajectory β_b

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The performance of the secant method in the field of p-adic numbers

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Abstract. In this paper, we compute the square roots of p-adic numbers in $\mathbb{Q}_p$, using the secant method. We also study the performance of this method: the speed of its convergence and the number of iterations necessary to obtain the desired precision M which represents the number of p-adic digits in the development of $\sqrt{a}$.

AMS Subject Classifications: 26E30, 11E95, 34K28.

Keywords: p-adic numbers, square roots, secant method, hensel's lemma, speed of convergence.

Contents

1. Introduction and Background

For a few hundred years theoretical physics has been developed on the basis of real and, later, complex numbers. This mathematical model of physical reality survived even in the process of the transition from classical to quantum physics, complex numbers became more important than real, but not essentially more so than in the Fourier analysis which was already being used, e.g., in classical electrodynamics and acoustics. However, in the last 20 years the field of p-adic numbers $\mathbb{Q}_p$ (as well as its algebraic extensions, including the field of complex p-adic numbers $\mathbb{C}_p$) has been intensively used in theoretical and mathematical physics. Thus, notwithstanding the fact that p-adic numbers were only discovered by K. Hensel around the end of the nineteenth century, the theory of p-adic numbers has already penetrated intensively into several areas of mathematics and its applications.

For each prime p, we will get a new field called the field of p-adic numbers denoted by $\mathbb{Q}_p$. These fields will be constructed in a manner analogous to the way the real number system $\mathbb{R}$ is constructed from $\mathbb{Q}$ (see [1, 4, 6, 7]). The p-adic numbers can be used to consider and study congruences modulo p and modulo p^n and have many applications in classical number theory.

The root-finding problem is one of the most important computational problems. It arises in a wide variety of practical applications in physics, chemistry, biosciences, engineering, etc. As a matter of fact, determination of any unknown appearing implicitly in scientific or engineering formulas gives rise to a root-finding problem. The

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Root-Finding Problem is the problem of finding a root of the equation $f(x) = 0$, where f is a function of a single variable x. Specifically, the problem is stated as follows: Given a function f. Find a number $x = \alpha$ such that $f(\alpha) = 0$.

Except for some very special functions, it is not possible to find an analytical expression for the root, from where the solution can be exactly determined. Thus, most computational methods for the root-finding problem have to be iterative in nature. Two important aspects of an iterative method are convergence and stopping criterion.

The idea behind an iterative method is the following: Starting with an initial approximation x_0, construct a sequence of iterates $(x_n)_n$ using an iteration formula with a hope that this sequence converges to a root of $f(x) = 0$.

In this present paper we will see how we can use classical root-finding method (secant method) and explore a very interesting application of tools from numerical analysis to number theory. We use this method to calculate the zero noted α of a p-adic continuous function f defined on $\mathbb{Q}_p$. The number α represents the square root of a p-adic number $a \in \mathbb{Q}_p^*$.

To calculate the square root of a p-adic number $a \in \mathbb{Q}_p^*$, one studies the following problem

$$f(x) = x^2 - a = 0, a \in \mathbb{Q}_p^*. \tag{1.1}$$

Our goal is to calculate the first numbers of the p-adic development of the solution of the previous equation, and this solution is approached by a sequence of the p-adic numbers $(x_n)_n \subset \mathbb{Q}_p$ constructed by the secant method.

In fact, several studies have been made with regards to finding square roots and cubic roots of p-adic numbers. In 2010, for instance, Knapp and Xenophontos [12] showed how classical root-finding methods from numerical analysis can be used to calculate inverses of units modulo prime powers. In the same year, Zerzaihi, Kecies and Knapp [15] applied some classical root-finding methods, such as the fixed-point method, in finding square roots of p-adic numbers through Hensel's lemma. In 2011, Zerzaihi and Kecies [13] used secant method to find the cubic roots of p-adic numbers. These authors [14] then applied the Newton method to find the cubic roots of p-adic numbers in $\mathbb{Q}_p$. A similar problem also appeared in [8] wherein Ignacio et al. computed the square roots of p-adic numbers via Newton-Raphson method.

The paper is organized as follows. The next section recalls several concepts about $\mathbb{Q}_p$ which will be used through the paper. Our main contribution is formally stated and proved in section 3. The paper ends with conclusions and final remarks.

2. Preliminaries

Definition 2.1. *Let p be a prime number. We define the p-adic valuation $v_p(\cdot)$ of a rational number $x \in \mathbb{Q}$ by the following definition:*
(i) *If $x \in \mathbb{Z}^*$, then $v_p(x)$ is equal to the highest power of p which divides x.*
(ii) *If $x = \frac{a}{b} \in \mathbb{Q}^*$, then $v_p(x) = v_p(a) - v_p(b)$. The $p-$adic valuation of $x \in \mathbb{Q}$ is also called the p-adic order and denoted as $ord(x)$.*
(iii) *We set $v_p(0) = +\infty$. The reason to set $v_p(0) = +\infty$ is that we can divide 0 by p^n for each $n \in \mathbb{N}$.*

Definition 2.2. *Let the function $|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{R}$ be defined as*

$$|x|_p = \begin{cases} p^{-v_p(x)}, \text{ if } x \neq 0, \\ \\ 0, \text{ if } x = 0. \end{cases} \tag{2.1}$$

$|\cdot|_p$ is called the p-adic norm on $\mathbb{Q}$.

Remark 2.3.
1) *The p-adic norm satisfies the non-archimedean property*

$$|x + y|_p \leq \max\left\{|x|_p , |y|_p\right\} \text{ for all } x, y \in \mathbb{Q}, \tag{2.2}$$

and we say that the p-adic norm is ultra-metric or non-archimedean.
2) *An important property of the p-adic norm is the discreteness of its image. It is clear that the function $|\cdot|_p$ takes its values in a discrete subset of $\mathbb{R}^+$(namely $\{0\} \cup \{p^n, n \in \mathbb{Z}\}$).*

Since for any prime p the p-adic norm is a norm hence it defines a p-adic distance function on $\mathbb{Q}$ given by

Definition 2.4. *The p-adic norm induces a metric $d_p : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{R}^+$ given by*

$$d_p(x, y) = |x - y|_p \text{ for all } x, y \in \mathbb{Q}, \tag{2.3}$$

this metric is called the p-adic metric.

Further since the p-adic norm is non-archimedean it follows that the p-adic distance function is an ultrametric and satisfies

$$d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\} \text{ for all } x, y, z \in \mathbb{Q}. \tag{2.4}$$

Definition 2.5. *For each prime p, the normed field $\mathbb{Q}_p$ of p-adic numbers is the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ which contains the rational numbers $\mathbb{Q}$ as a dense subset. The norm on $\mathbb{Q}_p$ induced by the p-adic norm on $\mathbb{Q}$, will be considered an extension of the p-adic norm, and will therefore also be denoted by $|\cdot|_p$. Further each of these fields is distinct from the real numbers $\mathbb{R}$ and for different primes p_1, p_2 the fields are distinct.*

Remark 2.6. *The elements of $\mathbb{Q}_p$ are equivalent classes of Cauchy sequences in $\mathbb{Q}$ with respect to the extension of the p-adic norm. For some $x \in \mathbb{Q}_p$, let $(x_n)_n$ be a Cauchy sequence of rational numbers representing a. Then by definition*

$$|x|_p = \lim_{n \longrightarrow +\infty} |x_n|_p \,. \tag{2.5}$$

Proposition 2.7. *[2] Let p be a fixed prime and $\mathbb{Q}_p$ the field of p-adic numbers. Given $x \in \mathbb{Q}_p$, there exists a unique sequence of integers $(\beta_N)_{n \geq N}$, with $N = v_p(x)$, such that $0 \leq \beta_n \leq p - 1$ for all n and*

$$x = \beta_N p^N + \beta_{N+1} p^{N+1} + ... + \beta_n p^n + ... = \sum_{k=N}^{\infty} \beta_k p^k. \tag{2.6}$$

Remark 2.8.
1) *The representation* (2.6) *is called the canonical p-adic expansion of p-adic number x.*
2) *There is a one-to-one correspondence between the power series expansion*

$$\beta_N p^N + \beta_{N+1} p^{N+1} + ... + \beta_n p^n + ... \tag{2.7}$$

and the short representation $\beta_N \beta_{N+1} \beta_{N+2} ...$, where only the coefficients of the powers of p are shown. We can use the p-adic point as a device for displaying the sign of N.

$$\begin{array}{ll} \beta_N \beta_{N+1} \beta_{N+2} ... \beta_{-2} \beta_{-1} \cdot \beta_0 \beta_1 \beta_2 ... & \text{for } N < 0, \\ \cdot \beta_0 \beta_1 \beta_2 ... & \text{for } N = 0, \\ \cdot 000...0 \beta_0 \beta_1 \beta_2 ... & \text{for } N > 0. \end{array} \tag{2.8}$$

The most important fact has already been noted: $\mathbb{Q}_p$ is a complete metric space, hence every Cauchy sequence converges. Cauchy sequences are characterized as follows

Theorem 2.9. *[10] A sequence (a_n) in $\mathbb{Q}_p$ is a Cauchy sequence, and therefore convergent, if and only if it satisfies*

$$\lim_{n \longrightarrow +\infty} |a_{n+1} - a_n|_p = 0. \tag{2.9}$$

The following result is an important tool for determining whether a series of p-adic numbers converge in $\mathbb{Q}_p$ or not.

Proposition 2.10. *[10] A series* $\sum_{n=0}^{\infty} a_n$ *with* $a_n \in \mathbb{Q}_p$ *converges in* $\mathbb{Q}_p$ *if and only if* $\lim_{n \longrightarrow +\infty} a_n = 0$*, in which case*

$$\left| \sum_{n=0}^{\infty} a_n \right|_p \leq \max_n |a_n|_p . \tag{2.10}$$

Definition 2.11. *A p-adic number* $x \in \mathbb{Q}_p$ *is a p-adic integer if its p-adic norm is less than or equal to* 1*,* $|x|_p \leq 1$*. We denote the set of p-adic integers by* $\mathbb{Z}_p$ *and hence*

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\} . \tag{2.11}$$

Lemma 2.12. *[6] A p-adic number* $x \in \mathbb{Q}_p$ *is a p-adic integer if and only if its canonical expansion has only positive powers of p. That is*

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : x = \sum_{n=0}^{\infty} \beta_n p^n \right\} . \tag{2.12}$$

The p-adic integers form a subring of $\mathbb{Q}_p$ *which contains* $\mathbb{Z}$*.*

Recall that a unit in a ring R with identity is an element which has amultiplicative inverse. In the rational integers $\mathbb{Z}$ the only units are $\{-1, 1\}$. The situation is quite different in $\mathbb{Z}_p$ where there are many units and in fact every rational integer m relatively prime to p is invertible.

Definition 2.13. *A p-adic integer* $x \in \mathbb{Z}_p$ *is said to be a p-adic unit (or invertible) if the first digit* β_0 *in the p-adic p-adic expansion is different from zero. The set of p-adic units is denoted by* $\mathbb{Z}_p^{\times}$ *or* $U(\mathbb{Z}_p)$*. Hence we have*

$$\mathbb{Z}_p^{\times} = \left\{ x = \sum_{n=0}^{\infty} \beta_n p^n : \beta_0 \neq 0 \right\} . \tag{2.13}$$

It is also easy to see that

$$\mathbb{Z}_p^{\times} = \left\{ x \in \mathbb{Z}_p : |x|_p = 1 \right\} . \tag{2.14}$$

$\mathbb{Z}_p^{\times}$ *is also called the group of p-adic units.*

The next result shows that any element of $\mathbb{Q}_p$ is a product of an invertible p-adic integer and a power of p.

Proposition 2.14. *[10] Let* x *be a p-adic number of norm* p^{-n}*. Then* x *can be written as the product* $x = p^n u$*, where* $u \in \mathbb{Z}_p^{\times}$*.*

The following result is very useful for our work.

Proposition 2.15. *[10] We say that* a *and* $b \in \mathbb{Q}_p$ *are congruent mod* p^n *and write* $a \equiv b \mod p^n$ *if and only if* $|a - b|_p \leq \frac{1}{p^n}$*.*

Proposition 2.16. *[1] Let* $(x_n)_n$ *be a p-adic number sequence. If*

$$\lim_{n \longrightarrow +\infty} x_n = x, x \in \mathbb{Q}_p, |x|_p \neq 0,$$

then the sequence of norms $\left\{ |x_n|_p : n \in \mathbb{N} \right\}$ *must stabilize for sufficiently large* n*, i.e., there exists* N *such that*

$$|x_n|_p = |x|_p , \forall n \geq N. \tag{2.15}$$

The following proposition is modestly known as Hensel's lemma.

Theorem 2.17. *[3] (Hensel's Lemma, first form). Let $F(x) \in \mathbb{Z}_p[x]$ be a p-adic polynomial and assume there exists $\alpha_0 \in \mathbb{Z}_p$ such that $F(\alpha_0) \equiv 0 \mod p$ but $F'(\alpha_0) \not\equiv 0 \mod p$. Then there exists a unique $\alpha \in \mathbb{Z}_p$ such that $F(\alpha) = 0$ and $\alpha \equiv \alpha_0 \mod p$.*

Sometimes the stated Hensel's lemma is not enough and one should use its generalization:

Theorem 2.18. *[3] (Hensel's Lemma, strong form). Let $F(x) \in \mathbb{Z}_p[x]$ be a p-adic polynomial and assume there exists $\alpha_0 \in \mathbb{Z}_p$ such that $F(\alpha_0) \equiv 0 \mod p^{2k+1}$ but $F'(\alpha_0) \not\equiv 0 \mod p^{k+1}$. Then there exists a unique $\alpha \in \mathbb{Z}_p$ such that $F(\alpha) = 0$ and $\alpha \equiv \alpha_0 \mod p^{k+1}$.*

Actually Hensel's lemma is valid for any complete nonarchimedian field.

As an application of the Hensel's lemma, we investigate the squares in $\mathbb{Q}_p$.

Proposition 2.19. *Let p be a prime number, then*
1) If $p \neq 2$, then a p-adic number $a \in \mathbb{Q}_p^*$ is a square if and only if $a = p^{2n}v^2$ for some $n \in \mathbb{Z}$ and $v \in \mathbb{Z}_p^\times$.
2) If $p = 2$, then a 2-adic number $a \in \mathbb{Q}_2^*$ is a square if and only if $a = 2^{2n}v^2 = 2^{2n}u$ for some $n \in \mathbb{Z}$ and $u \equiv 1 \mod 8$.

Now, we are ready to give our main results.

3. Main Results

Solving non linear equations is one of the most important and challenging problems in science and engineering applications. The root finding problem is one of the most relevant computational problems. It arises in a wide variety of practical applications in Physics, Chemistry, Biosciences, Engineering, etc.

The Newton-Raphson method, or Newton Method, is a powerful technique for solving a nonlinear equations $f(x) = 0$ numerically. We start with an initial approximation x_0 and generate a sequence of approximations $(x_n)_n$ through the iterative formula

$$\forall n \in \mathbb{N} : x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3.1}$$

A major disadvantage of the Newton Method is the requirement of finding the value of the derivative of $f'(x)$ at each approximation, which may not be practical for some choices of f. When the derivative of $f(x)$ is either hard or impossible to write down (and hence, to program), or when the computational effort required to evaluate $f'(x)$ is very large compared to that for $f(x)$, Newton method is impossible or costly to carry out. An alternative is to approximate the derivative by a finite difference, that is, to write

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}. \tag{3.2}$$

The approximate Newton iteration can then be expressed in the following algorithm

$$\forall n \in \mathbb{N}^* : x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}. \tag{3.3}$$

This iteration is called the secant method because $f(x)$ is approximated by a secant line through two points on the graph of f, rather than a tangent line through one point on the graph. In the secant method, we always use x_n and x_{n-1} to generate x_{n+1}.

We also study the performance of the secant method. The performance of the method is estimated by:

$a)$ The speed of convergence which is an important factor of the quality of the algorithms, if the speed of convergence is high, the algorithm converges quickly and the computation time is less. To measure the speed of convergence, we study the evolution of the sequence $(e_n)_n$ defined by

$$e_n = x_{n+n_0+1} - x_{n+n_0}. \tag{3.4}$$

with $n_0 \in \mathbb{N}$. Roughly speaking, if the rate of convergence of a method is s, then after each iteration the number of correct significant digits in the approximation increases by a factor of approximately s.

$b)$ The number of iterations necessary to obtain the desired precision M which represents the number of p-adic digits in the development of $\sqrt{a}$. So, it's all about finding n such that

$$|x_{n+n_0+1} - x_{n+n_0}|_p \leq p^{-M}, \tag{3.5}$$

this is equivalent to

$$v_p(e_n) \geq p^M. \tag{3.6}$$

The general principle of calculation is as follows,
Let $a \in \mathbb{Q}_p^*$ a p-adic number such that

$$|a|_p = p^{-v_p(a)} = p^{-2m}, m \in \mathbb{Z}, \tag{3.7}$$

If $(x_n)_n$ is a sequence of p-adic numbers that converges to a p-adic number $\alpha \neq 0$, then from a certain rank one has

$$|x_n|_p = |\alpha|_p\,,$$

We also know that if there exists a p-adic number α such that $\alpha^2 = a$, then $v_p(a)$ is even and

$$|x_n|_p = |\alpha|_p = p^{-m}. \tag{3.8}$$

We consider the following equation

$$f(x) = x^2 - a. \tag{3.9}$$

Then, the iteration of the secant method associated with the function f is written in the form

$$\forall n \in \mathbb{N}^* : x_{n+1} = \frac{x_n x_{n-1} + a}{x_n + x_{n-1}}. \tag{3.10}$$

The performance of the Secant method is given by the following theorem.

Theorem 3.1. *If x_{n_0-1} is the square root of a of order α and x_{n_0} is the square root of a of order β, then*
$1)$ *If $p \neq 2$, then x_{n+n_0-1} is the square root of a of order π_n, where the sequence $(\pi_n)_n$ is defined by, for all $n \in \mathbb{N}$*

$$\begin{aligned}\pi_n = &\left(\tfrac{1}{\sqrt{5}}\left(\beta - \alpha(1-\Phi)\right)\Phi^n + \tfrac{1}{\sqrt{5}}\left(-\beta + \alpha\Phi\right)(1-\Phi)^n\right) \\ &-2\left(\left(\tfrac{1}{\sqrt{5}}\left(\Phi^{n+1} - (1-\Phi)^{n+1}\right)\right) - 1\right)m.\end{aligned} \tag{3.11}$$

Furthermore

$$\forall n \in \mathbb{N} : x_{n+n_0} - x_{n+n_0-1} \equiv 0 \mod p^{\eta_n}, \tag{3.12}$$

such as

$$\forall n \in \mathbb{N} : \eta_n = \pi_n - m. \tag{3.13}$$

Where $\Phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.
$2)$ *If $p = 2$, then x_{n+n_0-1} is the square root of a of order π'_n, where the sequence $(\pi'_n)_n$ is defined by, for all $n \in \mathbb{N}$*

$$\pi'_n = \pi_n - 2\left(\left(\frac{1}{\sqrt{5}}\left(\Phi^{n+1} - (1-\Phi)^{n+1}\right)\right) - 1\right). \tag{3.14}$$

Furthermore

$$\forall n \in \mathbb{N} : x_{n+n_0} - x_{n+n_0-1} \equiv 0 \mod 2^{\eta'_n}, \tag{3.15}$$

such as

$$\forall n \in \mathbb{N} : \eta'_n = \pi'_n - (m+1). \tag{3.16}$$

Proof. Let $(x_n)_n$ be the sequence defined by (3.10). We have

$$\forall n \in \mathbb{N}^* : x_{n+1}^2 - a = \frac{(x_n^2 - a)(x_{n-1}^2 - a)}{(x_n + x_{n-1})^2}. \tag{3.17}$$

We assume that x_{n_0-1} (resp: x_{n_0}) is the square root of a of order α (resp: β), i.e,

$$\begin{cases} x_{n_0-1}^2 \equiv a \mod p^{\alpha}, \alpha \in \mathbb{N}, \\ x_{n_0}^2 \equiv a \mod p^{\beta}, \beta \in \mathbb{N}. \end{cases}$$

Then

$$\begin{cases} v_p\left(x_{n_0-1}^2 - a\right) \geq \alpha, \\ v_p\left(x_{n_0}^2 - a\right) \geq \beta. \end{cases}$$

Hence we obtain

$$\begin{cases} \left|x_{n_0-1}^2 - a\right|_p \leq p^{-\alpha}, \\ \left|x_{n_0}^2 - a\right|_p \leq p^{-\beta}. \end{cases}$$

Therefore, using the proposition (2.16) , we get

$$\begin{aligned} \left|x_{n_0+1}^2 - a\right|_p &= \frac{\left|(x_{n_0}^2 - a)(x_{n_0-1}^2 - a)\right|_p}{|x_{n_0} + x_{n_0-1}|_p^2} \\ &= \frac{1}{|4|_p} \frac{\left|x_{n_0}^2 - a\right|_p \left|x_{n_0-1}^2 - a\right|_p}{p^{-2m}}. \end{aligned}$$

Since

$$|4|_p = \begin{cases} 1, \text{ if } p \neq 2, \\ \frac{1}{4}, \text{ if } p = 2. \end{cases} \tag{3.18}$$

We have

$$\begin{cases} \left|x_{n_0+1}^2 - a\right|_p \leq p^{2m} p^{-\alpha} p^{-\beta}, \text{ if } p \neq 2, \\ \left|x_{n_0+1}^2 - a\right|_2 \leq 2^2 2^{2m} 2^{-\alpha} 2^{-\beta}, \text{ if } p \neq 2. \end{cases}$$

Consequently

$$\begin{cases} \left|x_{n_0+1}^2 - a\right|_p \leq p^{-(\alpha+\beta-2m)}, \text{ if } p \neq 2, \\ \left|x_{n_0+1}^2 - a\right|_2 \leq 2^{-(\alpha+\beta-2m-2)}, \text{ if } p \neq 2. \end{cases}$$

This gives

$$\begin{cases} x_{n_0+1}^2 - a \equiv 0 \mod p^{(\alpha+\beta)-2m} \text{ if } p \neq 2, \\ x_{n_0+1}^2 - a \equiv 0 \mod 2^{(\alpha+\beta)-2(m+1)} \text{ if } p = 2. \end{cases}$$

In this manner, we find that if $p \neq 2$, then

$$\forall n \in \mathbb{N} : x_{n+n_0-1}^2 - a \equiv 0 \mod p^{\pi_n}. \tag{3.19}$$

The sequence $(\pi_n)_n$ is defined by

$$\forall n \in \mathbb{N} : \pi_n = J_n - mA_n, \tag{3.20}$$

Such that $(J_n)_n$ and $(A_n)_n$ are two linear recurrence sequences defined by

$$\begin{cases} J_0 = \alpha, J_1 = \beta, \\ \\ \forall n \in \mathbb{N}^* : J_{n+1} = J_{n-1} + J_n, \end{cases}, \tag{3.21}$$

and

$$\begin{cases} A_0 = A_1 = 0, \\ \\ \forall n \in \mathbb{N}^* : A_{n+1} = A_{n-1} + A_n + 2. \end{cases} \tag{3.22}$$

The general terms of the sequences $(J_n)_n$ and $(A_n)_n$ are given respectively by

$$\forall n \in \mathbb{N} : J_n = \frac{1}{\sqrt{5}} \left(\beta - \alpha(1-\Phi)\right) \Phi^n + \frac{1}{\sqrt{5}} \left(-\beta + \alpha\Phi\right) (1-\Phi)^n. \tag{3.23}$$

and

$$\forall n \in \mathbb{N} : A_n = 2 \left(\left(\frac{1}{\sqrt{5}} \left(\Phi^{n+1} - (1-\Phi)^{n+1} \right) \right) - 1 \right). \tag{3.24}$$

We obtain, for all $n \in \mathbb{N}$

$$\begin{aligned} \pi_n = &\left(\tfrac{1}{\sqrt{5}} \left(\beta - \alpha(1-\Phi)\right) \Phi^n + \tfrac{1}{\sqrt{5}} \left(-\beta + \alpha\Phi\right)(1-\Phi)^n \right) \\ &-2 \left(\left(\tfrac{1}{\sqrt{5}} \left(\Phi^{n+1} - (1-\Phi)^{n+1} \right) \right) - 1 \right) m. \end{aligned} \tag{3.25}$$

Furthermore

$$v_p(x_{n+n_0-1}^2 - a) \geq \pi_n. \tag{3.26}$$

On the other hand, if $p = 2$, then

$$\forall n \in \mathbb{N} : x_{n+n_0-1}^2 - a \equiv 0 \mod 2^{\pi'_n}. \tag{3.27}$$

The sequence $(\pi'_n)_n$ is defined by

$$\forall n \in \mathbb{N} : \pi'_n = J_n - (m+1)A_n, \tag{3.28}$$

Then, for all $n \in \mathbb{N}$

$$\begin{aligned} \pi'_n = &\left(\tfrac{1}{\sqrt{5}} \left(\beta - \alpha(1-\Phi)\right) \Phi^n + \tfrac{1}{\sqrt{5}} \left(-\beta + \alpha\Phi\right)(1-\Phi)^n \right) \\ &-2 \left(\left(\tfrac{1}{\sqrt{5}} \left(\Phi^{n+1} - (1-\Phi)^{n+1} \right) \right) - 1 \right)(m+1). \end{aligned} \tag{3.29}$$

Therefore

$$\forall n \in \mathbb{N} : \pi'_n = \pi_n - 2 \left(\left(\frac{1}{\sqrt{5}} \left(\Phi^{n+1} - (1-\Phi)^{n+1} \right) \right) - 1 \right). \tag{3.30}$$

Furthermore

$$v_2(x_{n+n_0-1}^2 - a) \geq \pi'_n. \tag{3.31}$$

On the other hand, we have

$$\forall n \in \mathbb{N}^* : x_{n+1} - x_n = \frac{a - x_n^2}{x_n + x_{n-1}}. \tag{3.32}$$

Since

$$|2|_p = \begin{cases} 1, \text{ if } p \neq 2, \\ \\ \frac{1}{2}, \text{ if } p = 2. \end{cases} \tag{3.33}$$

We have

$$|x_{n+n_0} - x_{n+n_0-1}|_p = \frac{|a - x^2_{n+n_0-1}|_p}{|x_{n+n_0-1} + x_{n+n_0-2}|_p}, \tag{3.34}$$

Hence we obtain

$$\begin{cases} |x_{n+n_0} - x_{n+n_0-1}|_p \leq p^m p^{-\pi_n}, \text{ if } p \neq 2, \\ \\ |x_{n+n_0} - x_{n+n_0-1}|_2 \leq 22^m 2^{-\pi'_n}, \text{ if } p = 2. \end{cases} \tag{3.35}$$

and so

$$\begin{cases} x_{n+n_0} - x_{n+n_0-1} \equiv 0 \mod p^{\pi_n - m}, \text{ if } p \neq 2, \\ \\ x_{n+n_0} - x_{n+n_0-1} \equiv 0 \mod 2^{\pi'_n - (m+1)}, \text{ if } p = 2. \end{cases} \tag{3.36}$$

Therefore, if $p \neq 2$, then

$$\forall n \in \mathbb{N} : x_{n+n_0} - x_{n+n_0-1} \equiv 0 \mod p^{\eta_n}. \tag{3.37}$$

Where

$$\forall n \in \mathbb{N} : \eta_n = \pi_n - m. \tag{3.38}$$

Which give

$$v_p(x_{n+n_0} - x_{n+n_0-1}) \geq \eta_n. \tag{3.39}$$

If $p = 2$, then

$$\forall n \in \mathbb{N} : x_{n+n_0} - x_{n+n_0-1} \equiv 0 \mod 2^{\eta'_n}, \tag{3.40}$$

Where

$$\forall n \in \mathbb{N} : \eta'_n = \pi'_n - (m+1). \tag{3.41}$$

It's clear that

$$\forall n \in \mathbb{N} : \eta'_n = \eta_n - \left(2\left(\frac{1}{\sqrt{5}}\left(\Phi^{n+1} - (1-\Phi)^{n+1}\right)\right) - 1\right), \tag{3.42}$$

Which give

$$v_2(x_{n+n_0} - x_{n+n_0-1}) \geq \eta'_n. \tag{3.43}$$

This completes the proof. ■

The results obtained are presented here.

1. If $p \neq 2$, then the following are true.

   (a) The speed of convergence of the sequence $(x_n)_n$ is the order η_n.

   (b) Since $|1 - \Phi| < 1$, then

$$\eta_n \simeq \frac{1}{\sqrt{5}}\left(\beta - \alpha(1-\Phi)\right)\Phi^n - \frac{2}{\sqrt{5}}(\Phi^{n+1} - 1)m, \tag{3.44}$$

   and if $(\beta - \alpha(1-\Phi) - 2\Phi m) > 0$, then the number of iterations n to obtain M correct digits is

$$n = \left[\frac{\ln\left(\frac{\sqrt{5}(M-m)}{\beta - \alpha(1-\Phi) - 2\Phi m}\right)}{\ln \Phi}\right]. \tag{3.45}$$

2. If $p \neq 2$, then the following are true.
   (a) The speed of convergence of the sequence $(x_n)_n$ is the order η'_n.
   (b) If $\beta - \alpha(1-\Phi) - 2\Phi(m+1) > 0$, then the number of iterations n to obtain M correct digits is

$$n = \left[\frac{\ln\left(\frac{\sqrt{5}(M-(m+1))}{\beta-\alpha(1-\Phi)-2\Phi(m+1)}\right)}{\ln \Phi} \right]. \tag{3.46}$$

According to the results obtained in this section, we conclude the following corollary.

Corollary 3.2. *The order of convergence of the secant method is given by the positive number* $\Phi = \frac{1+\sqrt{5}}{2}$ *(superlinear order of convergence), this means the number of correct digits increases by a factor of approximately* Φ.

4. Conclusions

Let's consider for $p \neq 2$ the sets defined by

$$\begin{aligned} S_1 &= \left\{a \in \mathbb{Q}_p : |a|_p = 1\right\} \text{ if } m = 0, \\ S_2 &= \left\{a \in \mathbb{Q}_p : |a|_p < 1\right\} \text{ if } m > 0, \\ S_3 &= \left\{a \in \mathbb{Q}_p : |a|_p > 1\right\} \text{ if } m < 0. \end{aligned} \tag{4.1}$$

For $p = 2$, we consider the sets defined by

$$\begin{aligned} B_1 &= \{a \in \mathbb{Q}_2 : |a|_2 = 4\} \text{ if } m = -1, \\ B_2 &= \{a \in \mathbb{Q}_2 : |a|_2 < 4\} \text{ if } m > -1, \\ B_3 &= \{a \in \mathbb{Q}_2 : |a|_2 > 4\} \text{ if } m < -1. \end{aligned} \tag{4.2}$$

Then we have the following conclusion.

1. If $m < 0$, then the convergence for any p-adic number (Resp: 2-adic) belongs to the set S_3 (Resp: B_3) is faster than that of S_1 (Resp: B_1).

2. If $m > 0$, then the speed of convergence for any p-adic number (Resp: 2-adic) belongs to the set S_2 (Resp: B_2) is slower than that of S_1 (Resp: B_1).

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On dual π-endo Baer modules

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Abstract. We introduce the concept of dual π-endo Baer modules. We evolve several structural properties such as direct summands and direct sums. Moreover, we prove that the endomorphism ring of a dual π-endo Baer module is a π-Baer ring. The examples are presented to exhibit the results.

AMS Subject Classifications: 16D10, 16D80.

Keywords: Baer ring; π-Baer ring; dual-Baer module; projection invariant submodule; endomorphism ring.

Contents

1. Introduction

All rings are associate with unity and modules are unital right R-modules. R and M stand by such a ring and such a module, respectively. Throughout the paper, $\mathcal{H}$ denotes the endomorphism ring of M. A ring R is called *Baer* (*quasi-Baer*) [10], [8], if the right annihilator of each nonempty subset (ideal) of R is generated by an idempotent element of R. A kind of generalization of this condition is introduced in [4]. R is π*-Baer* [4], if the right annihilator of each projection invariant left ideal is generated by an idempotent of R. Observe that R is Baer implies that R is π-Baer implies that R is quasi-Baer.

A module M is *e.Baer* (*quasi-e.Baer*) [14], if for each $A_R \leq M_R$ ($A_R \trianglelefteq M_R$), $l_{\mathcal{H}}(A) = \mathcal{H}h$ for some $h = h^2 \in \mathcal{H}$. Recently, the authors in [5] have defined a module M is π*-endo Baer*, if for each for each $A_R \trianglelefteq_p M_R$, $l_{\mathcal{H}}(A) = \mathcal{H}p$ for some $p = p^2 \in \mathcal{H}$. In [12] and [2], the authors dualized the concept of e.Baer and quasi-e.Baer modules. M is called *dual Baer* (*quasi-dual Baer*), if for each ($A_R \trianglelefteq M_R$) $A_R \leq M_R$, $D_{\mathcal{H}}(A) = p\mathcal{H}$ for some $p = p^2 \in \mathcal{H}$, where $D_{\mathcal{H}}(A) = \{\psi \in \mathcal{H} \mid \psi(M) \subseteq A\}$. Following the ideas in [12] and [2], we explore the dual concept of π-e.Baer modules. We call a module M is *dual* π*-e.Baer*, if for each $A_R \trianglelefteq_p M_R$, $D_{\mathcal{H}}(A) = h\mathcal{H}$ for some $h = h^2 \in \mathcal{H}$. We indicate the fundamental results and connections between related notions in Section 2. Moreover, we study on the direct summands and direct sums properties for the former class of modules. In general, this class is neither closed under direct summands nor direct sums (see, Example 2.13 and Example 3.8). However, Proposition 2.11 and Corollary 2.12 explain some conditions when the dual π-e.Baer module property is inherited by direct summands. Further, we give a complete characterization for the direct sums of dual π-e.Baer modules in Theorem 2.14.

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In Section 3, we obtain the results related to the endomorphism rings. We prove that $\mathcal{H}$ is π-Baer if M is dual π-e.Baer in Proposition 3.1. Theorem 3.3 and Corollaries 3.4-3.5 provide some conditions which ensure the reverse of Proposition 3.1 fulfills. Finally, we represent the relations between e.Baer and dual-Baer modules when the module has a countable regular endomorphism ring in Proposition 3.7.

The notations $L_R \leq M_R$ ($L_R \leq R_R$), $L_R \trianglelefteq_p M_R$ ($L_R \trianglelefteq_p R_R$), $L_R \trianglelefteq M_R$ ($L \trianglelefteq R$) and $L_R \leq^{\oplus} M_R$ mean that L is a right R-submodule of M (L is a right ideal of R), L is a projection invariant right R-submodule of M (L is a projection invariant right ideal of R), L is a fully invariant submodule of M (L is an ideal of R), and L is a direct summand of M, respectively. $r_M(-)$ ($l_{\mathcal{H}}(-)$), $\mathbf{I}$ and $Mat_n(R)$ show the right (left) annihilator in M ($\mathcal{H}$), the subring of $\mathcal{H}$ generated by the idempotent elements of $\mathcal{H}$ and the n-by-n matrix ring over R, respectively. Recall that a right submodule A of M is called *projection (fully) invariant* in M, if $p(A) \subseteq A$ for all $p = p^2 \in \mathcal{H}$ ($p \in \mathcal{H}$). A ring R is *Abelian* if every idempotent of R is central. An idempotent $e \in R$ is called *left* (*right*) *semicentral* if $re = ere$ ($er = ere$) for each $r \in R$. $S_l(R)$ ($S_r(R)$) denotes the set of left (right) semicentral idempotents of R. A module M has *FI-SSSP*, if the sum of any number of fully invariant direct summands is a direct summand. For undefined notation or terminology, see [3, 6, 13].

2. Structural Properties

We evolve principal results and relations between the dual π-e.Baer modules and connected notions. Recall that $D_{\mathcal{H}}(A) = \{\psi \in \mathcal{H} \mid \psi(M) \subseteq A\}$ for some $A_R \leq M_R$ and $E_M(\mathcal{Y}) = \sum_{\psi \in \mathcal{Y}} \psi(M)$ for some $\mathcal{Y}_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$.

Lemma 2.1. *Assume $\mathcal{I}$ is a right ideal of $\mathcal{H}$ and A is a right submodule of M. Then*
(i) $E_M(D_{\mathcal{H}}(E_M(\mathcal{I}))) = E_M(\mathcal{I})$.
(ii) $D_{\mathcal{H}}(E_M(D_{\mathcal{H}}(A))) = D_{\mathcal{H}}(A)$.
(iii) $D_{\mathcal{H}}(hM) = h\mathcal{H}$ *for some* $h = h^2 \in \mathcal{H}$.
(iv) $E_M(h\mathcal{H}) = hM$ *for some* $h = h^2 \in \mathcal{H}$.

Proof. (i) and (ii) These parts follow from [2, Lemma 1.3].

(iii) Let $g \in h\mathcal{H}$. Then $g = hg$ and $g(M) = hg(M) \subseteq h(M)$. Thus $g \in D_{\mathcal{H}}(hM)$, so $h\mathcal{H} \subseteq D_{\mathcal{H}}(hM)$. Conversely, assume $f \in D_{\mathcal{H}}(hM)$. Then $f(M) \subseteq hM$, so $(1-h)f = 0$ and hence $f = hf + (1-h)f = hf \in h\mathcal{H}$. Therefore $D_{\mathcal{H}}(hM) \subseteq h\mathcal{H}$. It follows that $D_{\mathcal{H}}(hM) = h\mathcal{H}$.

(iv) Observe that $hM \subseteq E_M(h\mathcal{H})$. Let $m \in E_M(h\mathcal{H})$. Then $m = \alpha_1(m_1) + \alpha_2(m_2) + \cdots + \alpha_n(m_n)$, where $\alpha_i \in h\mathcal{H}$ and $m_i \in M$. Note that $h\alpha_i = \alpha_i$, so $m \in hM$. Thus $E_M(h\mathcal{H}) \subseteq hM$. ■

Lemma 2.2. (i) *$D_{\mathcal{H}}(A)$ is a projection invariant right ideal of $\mathcal{H}$, for each $A_R \trianglelefteq_p M_R$.*
(ii) *$E_M(\mathcal{Y})_R$ is a projection invariant submodule of M_R, for each $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$.*

Proof. (i) Let $A_R \trianglelefteq_p M_R$. Then $D_{\mathcal{H}}(A)$ is a right ideal of $\mathcal{H}$. Consider $e = e^2 \in \mathcal{H}$ and $\alpha \in D_{\mathcal{H}}(A)$. Then $e\alpha(M) = e(\alpha(M)) \subseteq e(A) \subseteq A$, as $A_R \trianglelefteq_p M_R$. It follows that $D_{\mathcal{H}}(A)_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$.

(ii) Note that $E_M(\mathcal{Y})$ is a submodule of M. Assume $f = f^2 \in \mathcal{H}$. Since $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$, $f(E_M(\mathcal{Y})) = f(\sum_{\psi \in Y} \psi(M)) = \sum_{\psi \in Y} (f\psi)M \subseteq \sum_{\beta \in \mathcal{Y}} \beta(M)$ for some $\beta \in \mathcal{Y}$. Thence $f(E_M(\mathcal{Y})) \subseteq E_M(\mathcal{Y})$, so $E_M(\mathcal{Y})_R \trianglelefteq_p M_R$. ■

Definition 2.3. *We call a module M dual endo π-Baer (denoted, dual π-e.Baer), provided that for all $A_R \trianglelefteq_p M_R$, there exists $h = h^2 \in \mathcal{H}$ such that $D_{\mathcal{H}}(A) = h\mathcal{H}$.*

Proposition 2.4. *M is dual π-e.Baer if and only if there exists $h = h^2 \in \mathcal{H}$ such that $E_M(\mathcal{Y}) = hM$ for each $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$.*

Proof. Suppose M is dual π-e.Baer and $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Then $E_M(\mathcal{Y}) \trianglelefteq_p M_R$ by Lemma 2.2. Thus $D_{\mathcal{H}}(E_M(\mathcal{Y})) = h\mathcal{H}$ for some $h = h^2 \in \mathcal{H}$. It follows from Lemma 2.1 that $E_M(\mathcal{Y}) = E_M(D_{\mathcal{H}}(E_M(\mathcal{Y}))) = E_M(f\mathcal{H}) = hM$.

Conversely, let $A_R \trianglelefteq_p M_R$. Observe that $D_{\mathcal{H}}(A) \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ by Lemma 2.2. Then there exists $p = p^2 \in \mathcal{H}$ such that $E_M(D_{\mathcal{H}}(A)) = pM$. Therefore $D_{\mathcal{H}}(A) = D_{\mathcal{H}}(pM) = p\mathcal{H}$ by Lemma 2.1. Hence M is dual π-e.Baer. ■

Since $D_{\mathcal{H}}(A) \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $E_M(\mathcal{Y}) \trianglelefteq_p M_R$, $h, p \in S_l(\mathcal{H})$ in Proposition 2.4 by [7, Proposition 4.12].

Lemma 2.5. *Suppose M is a dual π-e.Baer module.*
(i) *If $\psi(M)_R \trianglelefteq_p M_R$ for some $\psi \in \mathcal{H}$, then $\psi(M)_R \leq^{\oplus} M_R$.*
(ii) *If $N_R \cong D_R \leq^{\oplus} M_R$ for each $N_R \trianglelefteq_p M_R$, then $N_R \leq^{\oplus} M_R$.*

Proof. (i) Assume $\psi(M)_R \trianglelefteq_p M_R$ for some $\psi \in \mathcal{H}$. Then $\mathbf{I}(\psi(M)) = \psi(M)$, and $\mathbf{I}\psi\mathcal{H} = \psi\mathcal{H}$. It follows that $(\mathbf{I}\psi\mathcal{H})_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $\psi(M) = E_M(\mathbf{I}\psi\mathcal{H})$. Thus there exists $h = h^2 \in \mathcal{H}$ such that $\psi(M) = hM$ by Proposition 2.4.

(ii) Let $N_R \trianglelefteq_p M_R$ and $N \cong hM$ for some $h = h^2 \in \mathcal{H}$. Then there exists an isomorphism $\alpha : hM \to N$. Consider the map $\psi = \iota\alpha\pi$, where $\pi : M \to hM$ is projection, and $\iota : N \to M$ is inclusion. Observe that $\psi \in \mathcal{H}$ and $\psi(M) = \iota\alpha\pi(M) = \alpha(hM) = N$. Since $N_R \trianglelefteq_p M_R$, part (i) yields that $N_R \leq^{\oplus} M_R$. ■

Theorem 2.6. *M is dual Baer implies that M is dual π-e.Baer implies that M is quasi-dual Baer.*

Proof. Suppose M is dual Baer and $A_R \trianglelefteq_p M_R$. Then there exists $h = h^2 \in \mathcal{H}$ such that $D_{\mathcal{H}}(A) = h\mathcal{H}$. Hence M is dual π-e.Baer. Observe that fully invariant submodules are projection invariant. Therefore the second part follows the similar arguments in the above. ■

At the end of the paper, we provide examples which shows that the implications in Theorem 2.6 are irreversible (see, Example 3.8).

Proposition 2.7. *Assume that $\psi(M)_R \trianglelefteq M_R$ for each $\psi \in \mathcal{H}$. Then M is dual π-e.Baer if and only if M has FI-SSSP and $\psi(M)_R \leq^{\oplus} M_R$ for all $\psi \in \mathcal{H}$.*

Proof. Suppose that $\psi(M)_R \trianglelefteq M_R$ for all $\psi \in \mathcal{H}$, and M is dual π-e.Baer. Then Lemma 2.5, Theorem 2.6, and [2, Lemma 2.2] complete the result. Conversely, assume M has the stated property. Let $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $E_M(\mathcal{Y}) = \sum_{\psi \in \mathcal{Y}} \psi(M)$. By hypothesis, $\psi(M) \trianglelefteq M$ and $\psi(M) \leq^{\oplus} M$ for all $\psi \in \mathcal{H}$. Then $E_M(\mathcal{Y}) \leq^{\oplus} M$ by the FI-SSSP condition. Therefore the proof is done. ■

Proposition 2.8. (i) *If $\psi(M)_R \trianglelefteq M_R$ for all $\psi \in \mathcal{H}$, then M is dual Baer $\Leftrightarrow$ M is dual π-e.Baer $\Leftrightarrow$ M is quasi-dual Baer.*
(ii) *If M is indecomposable, then M is dual Baer $\Leftrightarrow$ M is dual π-e.Baer.*
(iii) *Assume $\mathcal{H}$ is an Abelian ring. Then M is dual Baer $\Leftrightarrow$ M is dual π-e.Baer.*
(iv) *Assume $\mathcal{H} = \mathbf{I}$. Then M is dual π-e.Baer $\Leftrightarrow$ M is quasi-dual Baer.*

Proof. (i) [2, Theorem 2.3] and Theorem 2.6 complete the proof.

(ii) Observe that every submodule of an indecomposable module is projection invariant. Therefore Theorem 2.6 yields the result.

(iii) Suppose M is dual π-e.Baer and $\mathcal{Y}_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$. Then $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ by [4, Lemma 2.3]. Thus $E_M(\mathcal{Y}) = hM$ for some $h = h^2 \in \mathcal{H}$ by Proposition 2.4. It follows from [12, Theorem 2.1] that M is dual Baer. Theorem 2.6 yields the converse.

(iv) Suppose $\mathcal{H} = \mathbf{I}$ and M is quasi-dual Baer. Let $A_R \trianglelefteq_p M_R$. Since $\mathcal{H} = \mathbf{I}$, $A_R \trianglelefteq M_R$. Thus $D_{\mathcal{H}}(A) = h\mathcal{H}$ for some $h = h^2 \in \mathcal{H}$, so M is dual π-e.Baer. Converse is clear from Theorem 2.6. ■

Corollary 2.9. *The free R-module F with a finite rank is dual π-e.Baer if and only if it is quasi-dual Baer.*

Proof. Suppose $F_R = \bigoplus_{t=1}^{n} R_t$ where $n > 1$ and $R_t \cong R$. Then $\mathcal{H} \cong Mat_n(R)$ and $\mathbf{I}(Mat_n(R)) = Mat_n(R)$. Therefore Proposition 2.8(iv) ensures the result. ■

Now, we study on the direct summands and directs sums properties for the former class of modules. A module M is *retractable*, if $Hom_R(M,A) \neq 0$ for all $0 \neq A \leq M$.

Lemma 2.10. *Assume M is a dual π-e.Baer and retractable module. Then every $0 \neq A_R \trianglelefteq_p M_R$ includes a nonzero direct summand of M.*

Proof. Suppose M satisfies the stated property. Let $0 \neq A_R \trianglelefteq_p M_R$. Then $D_{\mathcal{H}}(A) = h\mathcal{H}$ for some $h = h^2 \in \mathcal{H}$ by Proposition 2.4. Note that $h \in S_l(\mathcal{H})$. Since M is retractable, $\psi M \subseteq A$ for some $0 \neq \psi \in \mathcal{H}$. Thus $\psi \in D_{\mathcal{H}}(A)$, so $\psi = h\psi$. Observe that $(\psi h)^2 = \psi(h\psi h) = \psi h \in \mathcal{H}$, as $h \in S_l(\mathcal{H})$. Moreover $0 \neq \psi hM \subseteq \psi M \subseteq A$, so $\psi hM \leq^{\oplus} M$. ■

We mention in Example 3.8 that a direct summand of a dual π-e.Baer module need not to be dual π-e.Baer. To this end, we investigate when the direct summands fulfill the property.

Proposition 2.11. *Assume M is dual π-e.Baer and $(hM)_R \trianglelefteq M_R$ for all $h = h^2 \in \mathcal{H}$. Then $(hM)_R$ and $((1-h)M)_R$ are dual π-e.Baer.*

Proof. Let M be dual π-e.Baer, $(hM)_R \trianglelefteq M_R$ and $A_R \trianglelefteq_p (hM)_R$. Then $A_R \trianglelefteq_p M_R$ by [5, Lemma 3.1]. Hence $D_{\mathcal{H}}(A) = p\mathcal{H}$ for some $p \in S_l(\mathcal{H})$. Notice that $\mathcal{H}_{hM} \cong h\mathcal{H}h$ and $h \in S_l(\mathcal{H})$. Moreover, $(hph)^2 = hph \in h\mathcal{H}h$ and $(hph)M \subseteq hp(M) \subseteq h(A) \subseteq A$. Hence $hph \in D_{h\mathcal{H}h}(A)$. Thus $(hph)(h\mathcal{H}h) \subseteq D_{h\mathcal{H}h}(A)$. Let $\psi \in D_{h\mathcal{H}h}(A)$. Then $\psi(M) \subseteq A$ and $\psi \in h\mathcal{H}h$. It follows that $\psi \in D_{\mathcal{H}}(A) = f\mathcal{H}$, so $\psi = f\psi$. Since $\psi \in h\mathcal{H}h$ and $h \in S_l(\mathcal{H})$, we obtain that $\psi = fh\psi = (hfh)\psi \in (hfh)(h\mathcal{H}h)$. Therefore $D_{h\mathcal{H}h}(A) \subseteq (hfh)(h\mathcal{H}h)$. It follows that $D_{h\mathcal{H}h}(A) = (hfh)(h\mathcal{H}h)$, where $(hfh)^2 = hfh \in h\mathcal{H}h$. Consequently, $(hM)_R$ is dual π-e.Baer.

Now, let $B \trianglelefteq_p ((1-h)M)_R$. Then $(hM \oplus B)_R \trianglelefteq_p M_R$ by [7, Lemma 4.13]. Then $J = D_{\mathcal{H}}(hM \oplus B) = g\mathcal{H}$ for some $g \in S_l(\mathcal{H})$. Note that $\mathcal{H}_{(1-h)M} \cong (1-h)\mathcal{H}(1-h)$ and $(1-h)J(1-h) = J \cap (1-h)\mathcal{H}(1-h)$. Since $1-h \in S_r(\mathcal{H})$, $(1-h)J(1-h) = (1-h)g\mathcal{H}(1-h) = (1-h)g(1-h)\mathcal{H}(1-h) = (1-h)g(1-h)((1-h)\mathcal{H}(1-h))$. Further, $(1-h)g(1-h) = ((1-h)g(1-h))^2 \in (1-h)\mathcal{H}(1-h)$. Our claim is $(1-h)J(1-h) = D_{(1-h)\mathcal{H}(1-h)}(B)$. Let $\alpha \in J$. Then $(1-h)\alpha(1-h)(M) \subseteq (1-h)\alpha(M) \subseteq (1-h)(hM \oplus B) = (1-h)B \subseteq B$, as $B_R \trianglelefteq_p (1-h)M_R$. It follows that $(1-h)J(1-h) \subseteq D_{(1-h)\mathcal{H}(1-h)}(B)$. Assume that $(1-h)\beta(1-h) \in (1-h)\mathcal{H}(1-h)$ such that $(1-h)\beta(1-h)(M) \subseteq B$. Hence $(1-h)\beta(1-h) \in J$. But $(1-h)\beta(1-h) \in (1-h)\mathcal{H}(1-h)$, so $(1-h)\beta(1-h) \in J \cap (1-h)\mathcal{H}(1-h) = (1-h)J(1-h)$. It follows that $D_{(1-h)\mathcal{H}(1-h)}(B) \subseteq (1-h)J(1-h)$, so $((1-h)M)_R$ is dual π-e.Baer. ■

Corollary 2.12. *Suppose M is dual π-e.Baer and $\mathcal{H}$ is Abelian. Then $(hM)_R$ and $((1-h)M)_R$ are dual π-e.Baer for all $h = h^2 \in \mathcal{H}$.*

Proof. Since $\mathcal{H}$ is Abelian, $(hM)_R \trianglelefteq M_R$ for all $h = h^2 \in \mathcal{H}$. Hence Proposition 2.11 completes the proof. ■

The following example illustrates the direct sums of dual π-e.Baer modules.

Example 2.13. *For any prime p, consider $M_{\mathbb{Z}} = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p$. Then $\mathbb{Z}(p^\infty)$ and $\mathbb{Z}_p$ are dual π-e.Baer modules. On the other hand, $M_{\mathbb{Z}}$ is not dual π-e.Baer by [2, Example 2.3] and Theorem 2.6.*

Theorem 2.14. *Suppose $M = \bigoplus_{\kappa \in \mathcal{K}} M_\kappa$ such that $(M_\kappa)_R \trianglelefteq M_R$ for all $\kappa \in \mathcal{K}$. Then M is dual π-e.Baer if and only if M_κ is dual π-e.Baer for all $\kappa \in \mathcal{K}$.*

Proof. Assume that for each $\kappa \in \mathcal{K}$, M_κ is dual π-e.Baer. Since $(M_\kappa)_R \trianglelefteq M_R$, $Hom_R(M_\kappa, M_\chi) = 0$ for all $\kappa \neq \chi \in \mathcal{K}$. Observe that $\mathcal{H} = \prod_{\kappa \in \mathcal{K}} \mathcal{H}_\kappa$, where $\mathcal{H}_\kappa = \mathcal{H}_{M_\kappa}$. Let $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Then $\mathcal{Y} = \prod_{\kappa \in \mathcal{K}} (\mathcal{Y} \cap \mathcal{H}_\kappa) = \prod_{\kappa \in \mathcal{K}} \mathcal{Y}_\kappa$, where $\mathcal{Y}_\kappa = \mathcal{Y} \cap \mathcal{H}_\kappa$ for $\kappa \in \mathcal{K}$. Notice that $(\mathcal{Y}_\kappa)_{\mathcal{H}_\kappa} \trianglelefteq_p (\mathcal{H}_\kappa)_{\mathcal{H}_\kappa}$. Since M_κ is dual π-e.Baer, $E_{M_\kappa}(\mathcal{Y}_\kappa) = h_\kappa M_\kappa$ for some $h_\kappa = h_\kappa^2 \in \mathcal{H}_\kappa$. Note that $E_M(\mathcal{Y}) = \sum_{\psi \in \mathcal{Y}} \psi(M) = \sum_{\kappa \in \mathcal{K}} E_{M_\kappa}(\mathcal{Y}_\kappa) = \bigoplus_{\kappa=1} h_\kappa M_\kappa$, as $h_\kappa M_\kappa \cap h_\chi M_\chi = 0$ for all $\kappa \neq \chi \in \mathcal{K}$. It gives that $E_M(\mathcal{Y}) \leq^{\oplus} M$, so M is dual π-e.Baer. Converse is a consequence of Proposition 2.11. ■

3. Endomorphism Rings of Dual π-e.Baer Modules

Our goal is to analyze the properties of the endomorphism ring of a dual π-endo Baer module.

Proposition 3.1. *The endomorphism ring of a dual π-e.Baer module is a π-Baer ring.*

Proof. Suppose M is dual π-e.Baer and $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Then $E_M(\mathcal{Y}) = \sum_{\psi \in \mathcal{Y}} \psi(M) = hM$ for some $h = h^2 \in \mathcal{H}$ by Proposition 2.4. Observe that $\psi(M) \subseteq hM$, so $(1-h)\psi(M) = 0$. Thus $(1-h)\psi = 0$ which gives that $1 - h \in l_{\mathcal{H}}(\mathcal{Y})$. Hence $\mathcal{H}(1-h) \subseteq l_{\mathcal{H}}(\mathcal{Y})$. Let $\alpha \in l_{\mathcal{H}}(\mathcal{Y})$. Then $\alpha\mathcal{Y} = 0$, so $\alpha\psi(M) = 0$ for all $\psi \in \mathcal{Y}$. Thence $\alpha(E_M(\mathcal{Y})) = 0$, hence $(\alpha h)M = 0$, so $\alpha h = 0$. Therefore $\alpha = \alpha h + \alpha(1-h) = \alpha(1-h) \in \mathcal{H}(1-h)$, so $l_{\mathcal{H}}(\mathcal{Y}) \subseteq \mathcal{H}(1-h)$. Thus $\mathcal{H}$ is π-Baer. ■

The next example validates the reverse of Proposition 3.1 may not be true, in general.

Example 3.2. (i) *Assume $M_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}}$. Then $\mathcal{H} \cong \mathbb{Z}$ is a π-Baer ring, but $M_{\mathbb{Z}}$ is not dual π-e.Baer.*

(ii) *Let $R = \prod_{\iota=1}^{\infty} \mathcal{F}_\iota$, where $\mathcal{F}$ is a field and $\mathcal{F}_\iota = \mathcal{F}$ for $\iota = 1, 2, \cdots$. Then $M_R = R_R$ is not dual Baer by [12, Corollary 2.9]. Since R is a commutative ring, M_R is not dual π-e.Baer. However, $\mathcal{H} \cong R$ and R is a π-Baer ring by [4, Proposition 2.10].*

A module M_R is called *coretractable* (*quasi-coretractable*) [1], [11], provided that $Hom_R(M/A, M) \neq 0$ ($Hom_R(M/\sum_{\psi \in I} \psi(M), M) \neq 0$) for all proper $A \leq M$ ($I_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$ with $\sum_{\psi \in I} \psi(M) \neq M$). Notice that every coretractable module is quasi-coretractable. In the following result, we characterize a dual π-e.Baer (resp., quasi-dual Baer) module and its endomorphism ring being π-Baer (resp., quasi-Baer).

Theorem 3.3. *Assume M is quasi-coretractable. Then M is dual π-e.Baer (resp., quasi-dual Baer) if and only if $\mathcal{H}$ is π-Baer (resp., quasi-Baer).*

Proof. Assume M is dual π-e.Baer. By Proposition 3.1, $\mathcal{H}$ is π-Baer. Let $\mathcal{H}$ is π-Baer and $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. We claim that $E_M(\mathcal{Y}) = \sum_{\psi \in \mathcal{Y}} \psi(M) \leq^{\oplus} M_R$. Since $\mathcal{H}$ is π-Baer, there is $h = h^2 \in \mathcal{H}$ such that $l_{\mathcal{H}}(\mathcal{Y}) = \mathcal{H}h$. Observe $\mathcal{Y} \subseteq r_{\mathcal{H}}(l_{\mathcal{H}}(\mathcal{Y})) = (1-h)\mathcal{H}$. Consider the right ideal $\mathcal{A} = \mathcal{Y} + h\mathcal{H}$. Notice that $l_{\mathcal{H}}(\mathcal{A}) = l_{\mathcal{H}}(\mathcal{Y}) \cap l_{\mathcal{H}}(h\mathcal{H}) = \mathcal{H}h \cap \mathcal{H}(1-h) = 0$. Thus, $l_{\mathcal{H}}(\mathcal{A}) = 0$. By [11, Lemma 3.3], $\sum_{\psi \in \mathcal{A}} \psi(M) = M$. Furthermore, $M = \sum_{\psi \in \mathcal{A}} \psi(M) = \sum_{\psi \in I} \psi(M) \oplus \sum_{\psi \in h\mathcal{H}} \psi(M)$ which gives that $M = E_M(\mathcal{Y}) \oplus \sum_{\psi \in h\mathcal{H}} \psi(M)$. Hence M is dual π-e.Baer. The quasi-dual Baer case follows from the similar arguments and [2, Proposition 3.1]. ■

Corollary 3.4. *M is dual π-e.Baer if and only if $E_M(\mathcal{Y}) = r_M(l_{\mathcal{H}}(\mathcal{Y}))$ is a direct summand of M_R for all $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $\mathcal{H}$ is π-Baer.*

Proof. Suppose M is dual π-e.Baer. By Proposition 3.1, $\mathcal{H}$ is π-Baer. Let $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Then $E_M(\mathcal{Y}) = pM$ for some $p \in S_l(\mathcal{H})$. Thus $(1-p)\psi(M) = 0$ for all $\psi \in \mathcal{Y}$ by Proposition 2.4. Then $1 - p \in l_{\mathcal{H}}(\mathcal{Y})$, so $\mathcal{H}(1-p) \subseteq l_{\mathcal{H}}(\mathcal{Y})$. It follows that $r_M(l_{\mathcal{H}}(\mathcal{Y})) \subseteq r_M(\mathcal{H}(1-p)) = pM = E_M(\mathcal{Y})$. We claim that $l_{\mathcal{H}}(\mathcal{Y})pM = 0$. Observe that $g\mathcal{Y} = 0$ for all $g \in l_{\mathcal{H}}(\mathcal{Y})$. Then $0 = g(\sum_{\psi \in \mathcal{Y}} \psi(M)) = g(E_M(\mathcal{Y})) = g(pM)$. Therefore $l_{\mathcal{H}}(\mathcal{Y})pM = 0$, so $pM \subseteq r_M(l_{\mathcal{H}}(\mathcal{Y}))$. It follows that $E_M(\mathcal{Y}) = r_M(l_{\mathcal{H}}(\mathcal{Y}) = pM$. Conversely, let $E_M(\mathcal{Y}) = r_M(l_{\mathcal{H}}(\mathcal{Y})) \leq^{\oplus} M_R$ for all $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $\mathcal{H}$ be π-Baer. Thus $l_{\mathcal{H}}(\mathcal{Y}) = \mathcal{H}q$ for some $q \in S_r(\mathcal{H})$ by [4, Proposition 2.1] Hence $q\nu = 0$ for all $\nu \in \mathcal{Y}$. Thus $\nu = q\nu + (1-q)\nu = (1-q)\nu$ and $\nu(M) \subseteq (1-q)M$. Thence $E_M(\mathcal{Y}) \subseteq (1-q)M$. However, $(1-q)M = r_M(\mathcal{H}q) = r_M(l_{\mathcal{H}}(\mathcal{Y}))$. By hypothesis, $(1-q)M = E_M(\mathcal{Y})$, so M is dual π-e.Baer. ■

A ring R is called *right Kasch* [13], if every simple right R-module can be embedded in R_R.

Corollary 3.5. (i) *Suppose $\mathcal{H}$ is right Kasch. Then M is dual π-e.Baer if and only if $\mathcal{H}$ is π-Baer.*
(ii) *If M is an indecomposable dual π-e.Baer module with finite uniform dimension, then $\mathcal{H}$ is semilocal.*

Proof. (i) Since $\mathcal{H}$ is a right Kasch ring, $\mathcal{H}_{\mathcal{H}}$ is coretractable by [1, Theorem 2.14]. Then $\mathcal{H}_{\mathcal{H}}$ is quasi-coretractable. Therefore Theorem 3.3 yields the result.
(ii) Proposition 2.8(ii) and [12, Proposition 2.17] complete the proof. ∎

Proposition 3.6. *The followings are equivalent.*
(i) *M is an indecomposable dual π-e.Baer module.*
(ii) *M is a quasi-coretractable module and $\mathcal{H}$ is a domain.*
(iii) *Every $0 \neq \tau \in \mathcal{H}$ is an epimorphism.*
(iv) *$E_M(\mathcal{Y}) = M$ for all $0 \neq \mathcal{Y}_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$.*
(v) *$D_{\mathcal{H}}(A) = \mathcal{H}$ for all $0 \neq A_R \leq M_R$.*

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ Proposition 2.8(ii), [11, Corollary 2.7] and [12, Corollary 2.2] yield the implications.
$(i) \Rightarrow (iv)$ Let $0 \neq \mathcal{Y}_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$. Since M is indecomposable, $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Hence $E_M(\mathcal{Y}) = pM$ for some $p = p^2 \in \mathcal{H}$. Thence $E_M(\mathcal{Y}) = 0$ or $E_M(\mathcal{Y}) = M$. If $E_M(\mathcal{Y}) = 0$, then $\mathcal{Y} \subseteq D_{\mathcal{H}}(E_M(\mathcal{Y})) = 0$, a contradiction. Therefore $E_M(\mathcal{Y}) = M$.
$(iv) \Rightarrow (i)$ Suppose $\mathcal{X}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. If $\mathcal{X} = 0$, then we are done. Let $0 \neq \mathcal{X}$. By part (iv), $E_M(\mathcal{X}) = M$ so M_R is dual π-e.Baer. Moreover, $E_M(h\mathcal{H}) = M$ for some $0 \neq h = h^2 \in \mathcal{H}$ by part (iv). Hence $M = E_M(h\mathcal{H}) = hM$, so $h = 1$. Therefore M is indecomposable.
$(i) \Leftrightarrow (v)$ This part follows from the similar steps in part $(i) \Rightarrow (iv)$ and part $(iv) \Rightarrow (i)$. ∎

Assume T is the $\mathbb{Z}_2$-subalgebra of $\prod_{\varpi=1}^{\infty} F_{\varpi}$ generated by $\bigoplus_{\varpi=1}^{\infty} F_{\varpi}$ and 1, where $F_{\varpi} = \mathbb{Z}_2$. Then T is a countable von Neumann regular ring [6]. In the following result, we make connections between the related notions when the module has a countable regular endomorphism ring.

Proposition 3.7. *Assume $\mathcal{H}$ is countable regular. Then the following statements are equivalent.*
(i) *$\mathcal{H}$ is a Baer ring.*
(ii) *M_R is a dual Baer module.*
(iii) *$\mathcal{H}_{\mathcal{H}}$ is a dual Baer module.*
(iv) *M_R is an e.Baer module.*

Proof. $(i) \Rightarrow (2)$ $\mathcal{H}$ is a semisimple Artinian ring by [6, Corollary 3.1.13]. Then $D_{\mathcal{H}}(X) \leq^{\oplus} \mathcal{H}_{\mathcal{H}}$ for any $\emptyset \neq X \subseteq M$, so M is dual Baer.
$(ii) \Rightarrow (iii)$ By [11, Theorem 3.6], $\mathcal{H}$ is Baer. Thence $\mathcal{H}_{\mathcal{H}}$ is a dual Baer module by [6, Corollary 3.1.13] and [12, Corollary 2.9].
$(iii) \Rightarrow (iv)$ Observe that ${}_{\mathcal{H}}\mathcal{H}$ is semisimple by [12, Corollary 2.9]. Hence ${}_{\mathcal{H}}(l_{\mathcal{H}}(B)) \leq^{\oplus} {}_{\mathcal{H}}\mathcal{H}$ for all $\emptyset \neq B \subseteq M$. Thus M is e.Baer.
$(iv) \Rightarrow (i)$ This part follows from [14, Theorem 4.1]. ∎

The following example explains dual Baer, dual π-e.Baer and quasi-dual Baer modules are strictly different from each other. Furthermore, it gives an answer to the question: is the dual π-e.Baer module property inherited by direct summands?

Example 3.8. *Assume that R be a simple Noetherian ring with $\{0, 1\}$ as its only idempotents and not Morita equivalent to a domain [9]. Observe from [4, Theorem 2.1], R is quasi-Baer but not π-Baer. Then consider the following examples:*
(1) *Let $M_R = R_R$. Observe that R is a quasi-Baer ring, and R_R is coretractable. Hence R_R is quasi-dual Baer by Theorem 3.3. Since R is not a π-Baer ring by [4, Theorem 2.1], R_R is not dual π-e.Baer.*

(2) *Let* $T_R = \bigoplus_{\kappa=1}^{n} R_\kappa$ *where* $R_\kappa \cong R$. *Hence* T_R *is dual* π*-e.Baer, but not dual Baer. To see this, observe that* T_R *is a coretractable module by [1, Proposition 2.6]. Notice that* T_R *is quasi-e.Baer by [14, Proposition 3.19], and hence* $\mathcal{H} \cong Mat_n(R)$ *is also a quasi-Baer ring by [14, Theorem 4.1]. It follows from Theorem 3.3 that* T_R *is quasi-dual Baer. Moreover,* T_R *is dual* π*-e.Baer by Corollary 2.9*(i)*. However,* T_R *is not dual Baer. Because* $\mathcal{H} \cong Mat_n(R)$ *is not a Baer ring by [10, Exercise 3].*

(3) *Note that* $T_R = \bigoplus_{\kappa=1}^{n} R_\kappa$ *in part* (2) *includes a direct summand,* R_R*, which is not dual* π*-e.Baer.*

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Existence results for ψ-Caputo hybrid fractional integro-differential equations

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Abstract. In this paper, we study the existence of solutions for hybrid fractional integro-differential equations involving ψ-Caputo derivative. We use an hybrid fixed point theorem for a sum of three operators due to Dhage for proving the main results. An example is provided to illustrate main results.

AMS Subject Classifications: 34A08, 26A33, 34A34.

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Contents

1. Introduction

Fractional differential equations are a generalization of the classical ordinary differential equations, they play a very important role in modeling of various fields of science and engineering, chemistry, physics, economics, biology, control, etc..., see [2, 3, 11, 12, 17] and the references cited therein. The existence and uniqueness solution of fractional differential equations are studies by many authors, with different approaches, such as Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard, with various boundary conditions as nonlocal, integral, multipoint and hybrid, see for example [1, 6]. In recent years, many researchers focused on developing the theoretical aspects and methods of solution of the hybrid fractional differential equations by using different kinds of fixed point, we refer the reader to the works [5, 9, 16].
In [8] Almeida presented a new fractional differentiation operator named ψ-Caputo fractional operator. This type of differentiation depends on a kernel, and for particular choices of ψ, we obtain some well known fractional derivatives like Caputo, Caputo-Hadamard or Caputo, Erdélyi-Kober fractional derivatives. One can find some recent works on ψ-Caputo derivative in the following published papers and the references cited therein [5, 8, 10, 18].

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However, this domain reported on the existence solution for hybrid differential equations with ψ-Caputo fractional derivative still new.
Motivated by this fact, in this paper, we study the existence of solutions for ψ-Caputo hybrid fractional integro-differential equations of the form

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\nu;\psi}\left[\frac{z(\tau)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau,z(\tau))}{\mathbb{G}(\tau,z(\tau))}\right] = \mathbb{H}(\tau, z(\tau)), \tau \in \mathrm{J} := [a, b], \\ z(a) = 0, \end{cases} \tag{1.1}$$

where ${}^{c}\mathbb{D}_{a^+}^{\nu;\psi}$ is the ψ-Caputo fractional derivative of order $\nu \in (0, 1]$, $\mathbb{I}_{a^+}^{\theta;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $\theta > 0, \theta \in \{\sigma_1, \sigma_2, \ldots, \sigma_m\}, \sigma_k > 0, k = 1, 2, \ldots, m$. $\mathbb{G} \in C(\mathrm{J} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $\mathbb{F}_k, \mathbb{H} \in C(\mathrm{J} \times \mathbb{R}, \mathbb{R})$,$(k = 1, 2, \ldots, m)$.

The paper is organized as follows. In section 2, we present some definitions of fractional calculus and lemmas. In section 3, we prove the existence of solutions for problem (1.1) via hybrid fixed point theorems in Banach algebra due to Dhage. In section 4, an example is provided to check the applicability of the theoretical findings.

2. Background material

First, we introduce the essential functional spaces that we will adopt in this paper. We denote by $\mathfrak{C}([a, b], \mathbb{R})$ the Banach space of all continuous functions z from $[a, b]$ into $\mathbb{R}$ with the supremum norm

$$\|z\|_{\mathfrak{C}} = \sup_{\tau\in[a,b]} |z(\tau)|,$$

and the multiplication in $\mathfrak{C}$ by

$$(zy)(\tau) = z(\tau)y(\tau).$$

Clearly, $\mathfrak{C}$ is a Banach algebra with respect to the supremum norm and multiplication in it.
Now, we present some facts from the theory of fractional calculus.

Definition 2.1 ([7, 12])**.** *For $\nu > 0$, the left-sided ψ–Riemann-Liouville fractional integral of order ν for an integrable function $z\colon [a, b] \longrightarrow \mathbb{R}$ with respect to another function $\psi\colon [a, b] \longrightarrow \mathbb{R}$ that is an increasing differentiable function such that $\psi'(\tau) \neq 0$, for all $\tau \in \mathrm{J}$ is defined as follows*

$$\mathbb{I}_{a^+}^{\nu;\psi} z(\tau) = \frac{1}{\Gamma(\nu)} \int_a^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\nu-1} z(s)\mathrm{d}s, \tag{2.1}$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\nu) = \int_0^{+\infty} \tau^{\nu-1}e^{-\tau}\mathrm{d}\tau, \quad \nu > 0.$$

Definition 2.2 ([7])**.** *Let $n \in \mathbb{N}$ and let $\psi, z \in \mathfrak{C}^n([a, b], \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(\tau) \neq 0$, for all $\tau \in \mathrm{J}$. The left-sided ψ–Riemann–Liouville fractional derivative of a function z of order ν is defined by*

$$\begin{aligned} \mathbb{D}_{a^+}^{\nu;\psi} z(\tau) &= \left(\frac{1}{\psi'(\tau)}\frac{d}{dt}\right)^n \mathbb{I}_{a^+}^{n-\nu;\psi} z(\tau) \\ &= \frac{1}{\Gamma(n-\nu)} \left(\frac{1}{\psi'(\tau)}\frac{d}{dt}\right)^n \int_a^\tau \Xi(s)\mathrm{d}s, \end{aligned}$$

where $n = [\nu] + 1$ and $\Xi(s) = \psi'(s)(\psi(\tau) - \psi(s))^{n-\nu-1}z(s)$.

Definition 2.3 ([7])**.** *Let $n \in \mathbb{N}$ and let $\psi, z \in \mathfrak{C}^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(\tau) \neq 0$, for all $\tau \in \mathrm{J}$. The left-sided ψ-Caputo fractional derivative of z of order ν is defined by*

$$^{c}\mathbb{D}_{a^+}^{\nu;\psi} z(\tau) = \mathbb{I}_{a^+}^{n-\nu;\psi}\left(\frac{1}{\psi'(\tau)}\frac{d}{dt}\right)^n z(\tau),$$

where $n = [\nu]+1$ for $\nu \notin \mathbb{N}$, $n = \nu$ for $\nu \in \mathbb{N}$.

Lemma 2.4 ([7])**.** *Let $\nu, \beta > 0$, and $z \in \mathfrak{C}([a,b],\mathbb{R})$. Then for each $\tau \in \mathrm{J}$ we have*
(1) $^{c}\mathbb{D}_{a^+}^{\nu;\psi}\mathbb{I}_{a^+}^{\nu;\psi} z(\tau) = z(\tau)$,
(2) $\mathbb{I}_{a^+}^{\nu;\psi}\,{}^{c}\mathbb{D}_{a^+}^{\nu;\psi} z(\tau) = z(\tau) - z(a), \quad 0 < \nu \leq 1$,
(3) $\mathbb{I}_{a^+}^{\nu;\psi}(\psi(\tau)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\nu)}(\psi(\tau)-\psi(a))^{\beta+\nu-1}$,
(4) $^{c}\mathbb{D}_{a^+}^{\nu;\psi}(\psi(\tau)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\nu)}(\psi(\tau)-\psi(a))^{\beta-\nu-1}$,
(5) $^{c}\mathbb{D}_{a^+}^{\nu;\psi}(\psi(\tau)-\psi(a))^{k} = 0$, *for all $k \in \{0,\dots,n-1\}, n \in \mathbb{N}$.*

Theorem 2.5. *Let S be a closed convex , bounded and nonempty subset of a Banach algebra $\mathcal{X}$, and let $\mathcal{A},\mathcal{C} : \mathcal{X} \longrightarrow \mathcal{X}$ and $\mathcal{B} : S \longrightarrow \mathcal{X}$ be three operators such that*
(a) *$\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian with Lipschitz constants δ and ξ, respectively,*
(b) *$\mathcal{B}$ is compact and continuous,*
(c) *$x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x \Rightarrow x \in S$ for all $y \in S$,*
(d) *$\delta M + \xi < 1$ where $M = \left\|\mathcal{B}(S)\right\|$.*
Then the operator equation $\mathcal{A}x\mathcal{B}x + \mathcal{C}x = x$ has a solution in S.

3. Main Results

Before proceeding to the main results, we start by the following lemma.

Lemma 3.1. *Let $\nu \in (0,1]$ be fixed and functions $\mathbb{F}_i, (i = 1,\cdots,n), \mathbb{G}, \mathbb{H}$ satisfy problem* (1.1)*. Then the function $z \in \mathfrak{C}([a,b],\mathbb{R})$ is a solution of the hybrid fractional integro-differential problem* (1.1) *if and only if it satisfies the integral equation*

$$\begin{aligned} z(\tau) = \mathbb{G}(\tau, z(\tau))&\left[\mathbb{M}_\psi + \mathbb{I}_{a^+}^{\nu;\psi}\mathbb{H}(\tau, z(\tau))\right] \\ &+ \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau, z(\tau)), \quad \tau \in [a,b], \end{aligned} \tag{3.1}$$

where

$$\mathbb{M}_\psi = \frac{-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(a,0)}{\mathbb{G}(a,0)} \tag{3.2}$$

For the proof of Lemma 3.1, it is useful to refer to [13, 14].

Theorem 3.2. *Assume that:*

(H_1) *Let the functions $\mathbb{G} : \mathrm{J}\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ and, $\mathbb{F}_k, \mathbb{H} : \mathrm{J}\times\mathbb{R} \to \mathbb{R}$, $k = 0,1,2,...,m$ are continuous*

(H_2) *There exists two positive functions $\mathbb{L}_{\mathbb{F}_k}, \mathbb{L}_{\mathbb{G}}$, $k = 0,1,...,m$ with bounds $\|\mathbb{L}_{\mathbb{F}_k}\|$ and $\|\mathbb{L}_{\mathbb{G}}\|$, $k = 0,1,2,...,m$, respectively, such that*

$$|\mathbb{F}_k(\tau, z(\tau)) - \mathbb{F}_k(\tau, \overline{z}(\tau))| \leq \mathbb{L}_{\mathbb{F}_k}(\tau)|z - \overline{z}| \qquad (3.3)$$
$$k = 0, 1..., m$$

and

$$|\mathbb{G}(\tau, z(\tau)) - \mathbb{G}(\tau, \overline{z}(\tau))| \leq \mathbb{L}_{\mathbb{G}}(\tau)|z - \overline{z}|, \qquad (3.4)$$

for all $(\tau, z, \overline{z}) \in \mathrm{J} \times \mathbb{R} \times \mathbb{R}$.

(H_3) *There exist a function* $p \in \mathfrak{C}(\mathrm{J}, \mathbb{R}_+)$ *and a continuous nondecreasing function* $\Omega : [0, \infty) \to (0, \infty)$ *such that*

$$|\mathbb{H}(\tau, z(\tau))| \leq p(\tau)\Omega(|z|), \qquad (3.5)$$

for all $\tau \in \mathrm{J}$ *and* $z \in \mathbb{R}$.

(H_4) *There exists a number* $r > 0$ *such that*

$$r \geq \frac{\mathbb{G}^*\Lambda + \ell_\psi^{\sigma_k}\mathbb{F}_k^*}{1 - \|\mathbb{L}_{\mathbb{G}}\|\Lambda - \ell_\psi^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\|}, \qquad (3.6)$$

and

$$\|\mathbb{L}_{\mathbb{G}}\|\Lambda + l_\psi^{\sigma_k}\mathbb{F}_k^* < 1, \qquad (3.7)$$

where $\mathbb{F}_k^* = sup_{\tau\in \mathrm{J}}|\mathbb{F}_k(\tau, 0)|$, *and* $\mathbb{G}^* = sup_{\tau\in \mathrm{J}}|\mathbb{G}(\tau, 0)|$, $k = 0, 1, 2, ..., m$, *and*

$$\Lambda = |\mathbb{M}_\psi| + \Omega(r)\|p\|l_\psi^\nu, \qquad (3.8)$$

Then hybrid fractional integro-differential problem (1.1) has a least one solution defined on J.

Proof. In order to use Dhage's fixed-point theorem to prove our main result, we define a subset $\mathbb{S}_r$ of $\mathfrak{C}$ by

$$\mathbb{S}_r = \{z \in \mathfrak{C} : \|z\|_{\mathfrak{C}} \leq r\},$$

with r is a constant defined by hypothesis H_4.
Notice that $\mathbb{S}_r$ is closed, convex and bounded subset of $\mathfrak{C}$. Define three operators $\mathbb{A}, \mathbb{C} : \mathfrak{C} \longrightarrow \mathfrak{C}$ and $\mathbb{B} : \mathbb{S}_r \longrightarrow \mathfrak{C}$ by

$$\begin{cases} \mathbb{A}z(\tau) & = \mathbb{G}(\tau, z(\tau)), \\ \mathbb{B}z(\tau) & = \mathbb{M}_\psi + \mathbb{I}_{a^+}^{\nu;\psi}\mathbb{H}(\tau, z(\tau)), \end{cases} \quad \tau \in \mathrm{J},$$

and

$$\mathbb{C}z(\tau) = \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau, z(\tau)), \quad \tau \in \mathrm{J}.$$

Then (3.1) in operator form becomes

$$z(\tau) = \mathbb{A}z(\tau)\mathbb{B}z(\tau) + \mathbb{C}z(\tau), \quad \tau \in \mathrm{J}.$$

We shall prove that the operators $\mathbb{A},\mathbb{B}$ and $\mathbb{C}$ satisfy the conditions of Theorem 2.5. For the sake of clarity, we split the proof into a sequence of steps.

Step 1: First, we show that $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian on $\mathfrak{C}$. Let $z,\bar{z}\in\mathfrak{C}$. then by (H2), for $\tau\in[a,b]$, we have

$$\begin{aligned}|\mathbb{A}z(\tau)-\mathbb{A}\bar{z}(\tau)| &= |\mathbb{G}(\tau,z_(\tau))-\mathbb{G}(\tau,\bar{z}(\tau))|\\ &\leq \mathbb{L}_{\mathbb{G}}(\tau)\|z(\tau)-\bar{z}(\tau)\|_{\mathfrak{C}}.\end{aligned}$$

Taking supremum over $\tau\in[a,b]$, we obtain

$$\|\mathbb{A}z-\mathbb{A}\bar{z}\|_{\mathfrak{C}}\leq\|\mathbb{L}_{\mathbb{G}}\|\|z(\tau)-\bar{z}(\tau)\|_{\mathfrak{C}},$$

for all $z,\bar{z}\in\mathfrak{C}$. Therefore, $\mathbb{A}$ is a Lipschitzian on $\mathfrak{C}$ with Lipschitz constant $\mathbb{L}_{\mathbb{G}}$. Also, for any $z,\bar{z}\in\mathfrak{C}$., we have

$$\begin{aligned}|\mathbb{C}z(\tau)-\mathbb{C}\bar{z}(\tau)| &\leq \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}|\mathbb{F}_k(\tau,z(\tau))-\mathbb{F}_k(\tau,\bar{z}(\tau))|\\ &\leq \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{L}_{\mathbb{F}_k}(\tau)\|z(\tau)-\bar{z}(\tau)\|_{\mathfrak{C}}\\ &\leq \sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\|\mathbb{L}_{\mathbb{F}_k}\|\|z(\tau)-\bar{z}(\tau)\|_{\mathfrak{C}}.\end{aligned}$$

Hence, we have

$$\|\mathbb{C}z-\mathbb{C}\bar{z}\|_{\mathfrak{C}}\leq\ell_{\psi}^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\|\|z(\tau)-\bar{z}(\tau)\|_{\mathfrak{C}}.$$

Which means that $\mathbb{C}$ is a Lipschitzian on $\mathfrak{C}$ with Lipschitz constant $\ell_{\psi}^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\|$.

Step 2: We show that $\mathbb{B}$ is completely continuous on $\mathbb{S}_r$. The continuity of $\mathbb{B}$ follows by the continuity of $\mathbb{H}$. Now, it is sufficient to show that $\mathbb{B}$ is uniformly bounded and equicontinuous on $\mathbb{S}_r$. On the other hand, Keeping in mind the definition of the operator $\mathbb{B}$ on $[a,b]$ together with assumption (H3). For any $z\in\mathbb{S}_r$ we can get

$$\begin{aligned}|\mathbb{B}z(\tau)| \leq&|\mathbb{M}_{\psi}|+\int_a^{\tau}\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}|\mathbb{H}(s,z(s)|\mathrm{ds}\\ \leq&|\mathbb{M}_{\psi}|\int_a^{\tau}\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\Omega(r)p(s)\mathrm{ds}\\ \leq&|\mathbb{M}_{\psi}|+\Omega(r)\|p\|\int_a^{\tau}\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathrm{ds}\\ \leq&|\mathbb{M}_{\psi}|+\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\Omega(r)\|p\|\\ =&|\mathbb{M}_{\psi}|+\Omega(r)\|p\|\ell_{\psi}^{\nu}.\end{aligned}$$

Hence

$$\|\mathbb{B}z\|_{\mathfrak{C}}\leq|\mathbb{M}_{\psi}|+\Omega(r)\|p\|\ell_{\psi}^{\nu}.$$

Thus $\|\mathbb{B}z\|\leq\Lambda$ with Λ given in (3.8), for all $z\in\mathbb{S}_r$. This shows that $\mathbb{B}$ is uniformly bounded on $\mathbb{S}_r$.

Now, we will show that $\mathbb{B}(\mathbb{S}_r)$ is an equicontinuous set in $\mathfrak{C}$.

Let $\tau_1,\tau_2\in$ J with $\tau_1<\tau_2$. Then for any $z\in\mathbb{S}_r$, by (3.5) we get

$$\begin{aligned}|\mathbb{B}z(\tau_2)-\mathbb{B}z(\tau_1)| &\leq \left|\int_a^{\tau_2}\Theta_{\tau_2}(s)\mathrm{ds}-\int_a^{\tau_1}\Theta_{\tau_1}(s)\mathrm{ds}\right|\\ &\leq \frac{\Omega(r)\|p\|}{\Gamma(\nu)}\int_a^{\tau_1}\Delta_1(s)\mathrm{ds}+\frac{\Omega(r)\|p\|}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\Delta_2(s)\mathrm{ds},\end{aligned}\tag{3.9}$$

where $\Theta_{\tau_2}(s) = \frac{\psi(s)(\psi(\tau_2)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathbb{H}(\tau, z(\tau))$, $\Theta_{\tau_1}(s) = \frac{\psi(s)(\psi(\tau_1)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathbb{H}(\tau, z(\tau))$,
$\Delta_1(s) = \psi'(s)\left[(\psi(\tau_1) - \psi(s))^{\nu-1} - (\psi(\tau_2) - \psi(s))^{\nu-1}\right]$ and $\Delta_2(s) = \psi'(s)\left(\psi(\tau_2) - \psi(s)\right)^{\nu-1}$.
It is clear that the right-hand side of (3.9) is independent of z. Therefore, as $\tau_2 \to \tau_1$, inéquality (3.9) tends zeros. As consequence of the Arzela-Ascoli theorem, $\mathbb{B}$ is a completely continuous operator on $\mathbb{S}_r$.
Step 3: The hypothesis (c) of Theorem 2.5 is satisfied.
Let $z \in \mathfrak{C}$ and $y \in \mathbb{S}_r$ be arbitrary elements such that $z = \mathbb{A}z\mathbb{B}y + \mathbb{C}z$. Then we have

$$\begin{aligned}
|z(\tau)| &\leq |\mathbb{A}z(\tau)|\ |\mathbb{B}y(\tau)| + |\mathbb{C}z(\tau)| \\
&\leq |\mathbb{G}(\tau, z(\tau))| \left\{\mathbb{M}_\psi + \mathbb{I}_{a^+}^{\nu;\psi}|\mathbb{H}(\tau, y(\tau))|\right\} + \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\psi}|\mathbb{F}_k(\tau, z(\tau))| \\
&\leq \big(|\mathbb{G}(\tau, z(\tau)) - \mathbb{G}(\tau, 0)| \\
&\quad + |\mathbb{G}(\tau, 0)|\big) \left\{\mathbb{M}_\psi + \mathbb{I}_{a^+}^{\nu;\psi}|\mathbb{H}(\tau, y(\tau))|\right\} + \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\psi}\left|\big(|\mathbb{F}_k(\tau, z(\tau)) - \mathbb{F}_k(\tau, 0)| + |\mathbb{F}_k(\tau, 0)|\big)\right. \\
&\leq \big(\|\mathbb{L}_{\mathbb{G}}\|\|z\|_{\mathfrak{C}} + \mathbb{G}^*\big)\left[|\mathbb{M}_\psi| + \Omega(r)\|p\|\ell_\psi^{\nu}\right] + \ell_\psi^{\sigma_k}\big(\|\mathbb{L}_{\mathbb{F}_k}\|\|z\|_{\mathfrak{C}} + \mathbb{F}_k^*\big).
\end{aligned}$$

Thus,

$$|z(\tau)| \leq \big(\|\mathbb{L}_{\mathbb{G}}\|\|z\|_{\mathfrak{C}} + \mathbb{G}^*\big)\Lambda + \ell_\psi^{\sigma_k}\big(\|\mathbb{L}_{\mathbb{F}_k}\|\|z\|_{\mathfrak{C}} + \mathbb{F}_k^*\big).$$

Taking the supremum over τ,

$$\|z\| \leq \frac{\mathbb{G}^*\Lambda + \ell_\psi^{\sigma_k}\mathbb{F}_k^*}{1 - \|\mathbb{L}_{\mathbb{G}}\|\Lambda - \ell_\psi^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\|} \leq r.$$

Step 4: Finally we show that $\delta M + \xi < 1$, that is, (d) of Theorem 2.5 holds.
Since

$$M = \|B(S)\| = \sup_{z\in S}\left\{\sup_{\tau\in J}|Bz(t)|\right\} \leq \Lambda,$$

and so

$$\|\mathbb{L}_{\mathbb{G}}\|M + \ell_\psi^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\| \leq \|\mathbb{L}_{\mathbb{G}}\|\Lambda + \ell_\psi^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\| < 1,$$

with $\delta = \|\mathbb{L}_{\mathbb{G}}\|, \xi = \ell_\psi^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\|$. Thus all the conditions of Theorem 2.5 are satisfied and hence the operator equation $z = \mathcal{A}z\mathcal{B}z + \mathcal{C}z$ has a solution in $\mathbb{S}_r$. As a result, problem (1.1) has a solution on J. ■

4. Application

In this section, we present an example to show the applicability of the main result.

Example 4.1. *Consider the following hybrid fractional integro-differential equation:*

$$\begin{cases}
{}^{c}\mathbb{D}_{a^+}^{\frac{1}{2};\psi}\left[\frac{z(\tau)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau,z(\tau))}{\mathbb{G}(\tau,z(\tau))}\right] = \frac{1}{\sqrt{25+t^2}}\left(\frac{|z|}{(4|z|+1)} + \frac{z^2}{|z|+1} + \frac{1}{4}\right), \\
\tau \in \mathrm{J} := [0, 1], \\
z(a) = 0.
\end{cases} \tag{4.1}$$

We take

$$\nu = \frac{1}{2}, \quad m = 3, \quad \sigma_1 = \frac{1}{2}, \quad \sigma_2 = \frac{3}{2}, \quad \sigma_3 = \frac{5}{2},$$

$$\sum_{k=1}^{3} \mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau, z(\tau)) = \mathbb{I}_{a^+}^{\frac{1}{2};\psi}\frac{\tau}{10}\left(z(\tau)+e^{-\tau}\right) + \mathbb{I}_{a^+}^{\frac{3}{2};\psi}\frac{\tau\cos\tau}{12(1+e^{\tau})}\left(\frac{|z(\tau)|}{1+|z(\tau)|}+\frac{\tau}{\tau+1}\right) + \mathbb{I}_{a^+}^{\frac{5}{2};\psi}\frac{3\sin\pi\tau}{4+\tau}\left(\frac{|z(\tau)|}{5+|z(\tau)|}+\cos\tau\right),$$

$$\psi(\tau, z(\tau)) = \frac{\tau}{2}(\tau+1), \tau \in [0,1],$$
$$\mathbb{G}(\tau, z(\tau)) = \frac{6\sqrt{\pi}\sin^2(\pi\tau)}{(\tau+5)}\frac{z(\tau)}{1+z(\tau)}+\frac{1}{2},$$
$$\mathbb{H}(\tau, z(\tau))) = \frac{1}{\sqrt{36+t^2}}\left(\frac{|z|}{(4|z|+1)}+\frac{z^2}{|z|+1}+\frac{1}{4}\right).$$

We can show that

$$|\mathbb{F}_1(\tau, z(\tau)) - \mathbb{F}_1(\tau, \overline{z}(\tau))| \leq \frac{\tau}{10}|z-\overline{z}|,$$
$$|\mathbb{F}_2(\tau, z(\tau)) - \mathbb{F}_2(\tau, \overline{z}(\tau))| \leq \frac{\tau}{12(1+e^{\tau})}|z-\overline{z}|,$$
$$|\mathbb{F}_3(\tau, z(\tau)) - \mathbb{F}_3(\tau, \overline{z}(\tau))| \leq \frac{3}{20+5\tau}|z-\overline{z}|,$$
$$|\mathbb{G}(\tau, z(\tau))\mathbb{G}(\tau, \overline{z}(\tau))-| \leq \frac{6\sqrt{\pi}}{(\tau+5)}|z-\overline{z}|,$$
$$|\mathbb{H}(\tau, z(\tau))) - \mathbb{H}(\tau, \overline{z}(\tau)))| = \frac{1}{\sqrt{36+t^2}}(|z|+\frac{1}{2}),$$

where

$$\Omega(|z|) = |z|+1, \quad p(\tau) = \frac{1}{\sqrt{36+t^2}}.$$

Hence we have

$$\mathbb{L}_{\mathbb{G}}(\tau) = \frac{6\sqrt{\pi}}{(\tau+5)}, \quad \mathbb{F}_1 = \frac{\tau}{10}, \quad \mathbb{F}_2 = \frac{\tau}{12(1+e^{\tau})}, \quad \mathbb{F}_3 = \frac{3}{20+\tau}.$$

Then

$$\|\mathbb{L}_{\mathbb{G}}\| = \frac{6\sqrt{\pi}}{5}, \quad \|\mathbb{L}_{\mathbb{F}_1}\| = \frac{1}{10}, \quad \|\mathbb{L}_{\mathbb{F}_2}\| = \frac{1}{12(1+e)},$$
$$\|\mathbb{L}_{\mathbb{F}_3}\| = \frac{3}{20}, \quad \|p\| = \frac{1}{6}, \quad l_{\psi}^{\nu} = \frac{2}{\sqrt{\pi}},$$

$$l_{\psi}^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\| = \frac{81(1+e)+25}{450\sqrt{\pi}(1+e)}, \quad l_{\psi}^{\sigma_k}\mathbb{F}_k^* = \frac{58}{75\sqrt{\pi}e^2}$$
$$\mathbb{M}_{\psi} = \frac{234(1+e)+100}{225\sqrt{\pi}(1+e)},$$

and

$$\mathbb{F}_k^* = \sup_{z\in J}|\mathbb{F}_k(\tau, 0)| = \frac{1}{5e^2}, \quad \mathbb{G}^* = \sup_{z\in J}|\mathbb{G}(\tau, 0)| = \frac{1}{2}, \quad k = 1,2,3.$$

By using Matlab program, it follows by (3.6) *and* (3.7) *that the constant* r *satisfies the inequality* $0.7411 < r < 0.9970$. *As all the assumptions of Theorem* (3.2) *are satisfied then the problem* (4.1) *has at least one solution on* J.

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On nearly Ricci recurrent manifolds

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Abstract. The object of the present paper is to introduce a new type of Ricci recurrent manifold called nearly Ricci recurrent manifold . Some geometric properties of nearly Ricci recurrent manifold have been studied. Finally we give an example of nearly Ricci recurrent manifold.

AMS Subject Classifications: 53C15 and 53C25.

Keywords: Nearly Ricci recurrent manifold, Constant scalar curvature tensor, Conformally flat manifold.

Contents

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold with the matric g. A tensor field T of type $(0, q)$ is said to recurrent [1] if the relation

$$(D_X T)(Y_1, Y_2, ..., Y_q)T(Z_1, Z_2, ..., Z_q) - T(Y_1, Y_2, ..., Y_q)(D_X T)(Z_1, Z_2, ..., Z_q) = 0$$

holds on (M^n, g). From definition it follows that if at a point $x \in M$; $T(X) \neq 0$, then on some neighbourhood of x, there exits a unique 1-form A satisfying

$$(D_X T)(Y_1, Y_2, ..., Y_q) = A(X)T(Y_1, Y_2, ..., Y_q)$$

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In 1952, Patterson [2] introduced a Ricci recurrent manifolds. According to him, a manifold (M^n, g) of dimension n, was called Ricci recurrent if

$$(D_X S)(Y, Z) = A(X)S(Y, Z)$$

for some 1-form A. He denoted such a manifold by R_n. Ricci recurrent manifolds have been studied by several authors ([3], [4], [1], [5]) and many others. In a recent paper De, Guha and Kamilya [6] introduced the notion of generalized Ricci recurrent manifold as follows:

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called generalized Ricci recurrent if the Ricci tensor S is non-zero and satisfies the condition:

$$(D_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z),$$

where A and B non-zero 1-forms. Such a manifold where denoted by them as GR_n. If the associated 1-form B becomes zero, then the manifold GR_n reduces to a Ricci recurrent manifold R_n. This justifies the name generalized Ricci recurrent manifold and the symbols GR_n for it. Also in a paper De, and Guha [7] introduced a non flat Riemannian $(M^n, g)(n > 2)$ called a generalized recurrent manifold if its curvature tensor $R(X, Y)Z$ of type (1,3) satisfies the condition:

$$(D_U R)(X, Y)Z = A(U)R(X, Y)Z + B(U)[g(Y, Z)X - g(X, Z)Y],$$

where A and B are two non-zero 1-forms and D denotes the operator of covariant differentiation with respect to metric tensor g. Such a manifold has been denoted by GK_n. If the associated 1-form B becomes zero, then the manifold GK_n reduces to recurrent manifold introduced by Ruse [8] and Waker [9] which was denoted by K_n. In recent papers Arslan etal [10], Shaikh and Patra [11], Mallick, De and De [12], Khairnar [Kh], Shaikh, Prakasha and Ahmad [14], Kumar, Singh and Chowdhary [15], Hui [16], Singh and Mayanglambam [17], Singh and Kishor [18] etc. explored various geometrical propertis by using generlaized recurrent and generlaized Ricci recurrent manifold on Riemannian manifolds , Lorentzian Trans-Sasakian manifolds, LP-Sasakian manifolds, $(k - \mu)$ contact metric manifolds.

Further the authors Prasad and Yadav [19] considered a non-flat Riemannian manifold $(M^n, g)(n > 3)$ whose curvature tensor R satisfies the following condition:

$$(D_U R)(X, Y)Z = [A(U) + B(U)]R(X, Y)Z + B(U)[g(Y, Z)X - g(X, Z)Y],$$

where A and B are two non-zero 1-forms and D has the meaning already mentioned. Such a manifold where called by them as nearly recurrent Riemannian manifold and denoted by $(NR)_n$.

The motivation of the above studies, we define a new type of non flat Riemannian manifold is called nearly Ricci recurrent manifolds if the Ricci tensor S is non zero and satisfies the condition:

$$(D_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + B(X)g(Y, Z) \tag{1.1}$$

where A and B non-zero 1-forms, P and Q be two vector fields such that

$$A(X) = g(P, X), \qquad B(X) = g(Q, X) \tag{1.2}$$

Such a manifold shall be called as a nearly Ricci recurrent manifold and 1-forms A and B shall be called its associated 1-form and n dimensional nearly Ricci recurrent manifold of this kind shall be denoted by $N\{R(R_n)\}$. The name nearly Ricci recurrent Riemannian manifold was chosen because if $B = 0$ in (1.1) then the manifold reduces to a Ricci recurrent manifold which is very close to Ricci recurrent space. This justifies the name $Nearly\ Ricci\ recurrent\ manifold$ for the manifold defined by (1.1) and the use of the symbol $N\{R(R_n)\}$ for it.

In this paper, after preliminaries, the existence of a $N\{R(R_n)\}$ is first established and then it proved that the scalar curvature of $N\{R(R_n)\}$ cannot be zero. In section 4, the necessary and sufficient condition for constant scalar curvature of $N\{R(R_n)\}$ is obtained. Here it is established if A is closed then B is also closed and conversely in section 5. In section 6, it is shown that if the scalar curvature is constant in $N\{R(R_n)\}$ then the eigen value of the Ricci tensor S corresponding to the given eigen vector not exist. In section 7, it is proved that in Conformally flat $N\{R(R_n)\}$ with constant scalar curvature if the 1-form A is closed then $R(X,Y).S = 0$ if and only if $\{A(X) + B(X)\}\,A(LZ) = \{A(Z) + B(Z)\}\,A(LX)$. In section 8, a necessary and sufficient condition for $N\{R(R_n)\}$ to be a $(NR)_n$ is obtained. Finally the existence of nearly Ricci recurrent manifold $N\{R(R_n)\}$ is ensured by a non trivial example.

2. Preliminaries

Let L denotes the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S that is $g(LX,Y) = S(X,Y)$ for every vector field X, Y. Therefore,

$$g((D_X L)Y, Z) = (D_X S)(Y, Z). \tag{2.1}$$

From (1.1), we have

$$dr(X) = [A(X) + B(X)]r + nB(X). \tag{2.2}$$

3. Existence of a $N\{R(R_n)\}$ $(n \geq 2)$

In this section, it show that there exist a Riemannian manifold $(M^n, g)(n \geq 2)$ whose Ricci tensor S of type (0,2) satisfies the condition

$$(D_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + B(X)g(Y, Z)$$

and for which $(D_X S)(Y, Z) \neq A(X)S(Y, Z)$. For this we consider a Riemannian manifold (M^n, g) which admits a linear connection $\overline{D}$ defined by

$$\overline{D}_X Y = D_X Y + \frac{1}{2}B(X)LY + \frac{1}{2}B(X)Y \tag{3.1}$$

where B is non zero 1-form L is a symmetric endomorphism of the tangent space at each point (M^n, g) corresponding to the Ricci tensor S defined by $g(LX, Y) = S(X, Y)$ and $L^2 X = X$ and which satisfies the condition

$$(\overline{D}_X S)(Y, Z) = A(X)S(Y, Z) \tag{3.2}$$

If (3.2) holds, then

$$\begin{aligned} &XS(Y,Z) - S(\overline{D}_X Y, Z) - S(Y, \overline{D}_X Z) = A(X)S(Y,Z) \\ \Rightarrow &XS(Y,Z) - S\left(D_X Y + \frac{1}{2}B(X)LY + \frac{1}{2}B(X)Y, Z\right) - \\ &S\left(Y, D_X Z + \frac{1}{2}B(X)LZ + \frac{1}{2}B(X)Z\right) = A(X)S(Y,Z) \end{aligned}$$

From this, we get

$$(D_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + B(X)g(Y, Z)$$

The connection D is not identical with $\overline{D}$. Hence $(D_X S)(Y, Z) \neq A(X)S(Y, Z)$. Thus a Riemannian manifold $(M^n, g)(n \geq 2)$ admits a linear connection $\overline{D}$ which satisfies (3.1) and (3.2) then the manifold is a $N\{R(R_n)\}$.

4. Nature of scalar curvature of a $N\{R(R_n)\}$

From (2.2), we get if $r=0$, then $B=0$. Since the 1-form B cannot be zero.
Hence we can state the following theorem:

Theorem 4.1. *The scalar curvature of a $N\{R(R_n)\}$ $(n \geq 2)$ cannot be zero.*

Now suppose that $N\{R(R_n)\}$ is of constant scalar curvature. Then from (2.2) it follows that

$$[A(X)+B(X)]r+nB(X)=0$$

Hence we have

$$A(X)=-\left(1+\frac{n}{r}\right)B(X) \tag{4.1}$$

Again if (4.1) holds, then from (2.2) we get $r=$ constant.
Hence we have the following theorem:

Theorem 4.2. *A $N\{R(R_n)\}$ $(n \geq 2)$ is of constant scalar curvature if and only if the condition (4.1) holds.*

5. Nature of the 1-forms A and B

We have

$$d^2r(X,Y)=\frac{1}{2}\left[Xdr(Y)-Ydr(X)-dr([X,Y])\right] \tag{5.1}$$

Now in virtue of (2.2), we get from (5.1)

$$\begin{aligned}&\frac{1}{2}\left[X\{(A(Y)+B(Y))r+nB(Y)\}-Y\{(A(X)+B(X))r+nB(X)\}-nB([X,Y])\right]\\ &\text{or } rdA(X,Y)+(n+r)dB(X,Y)=0\end{aligned}$$

Since B is closed then $rdA(X,Y)=0$. But $r\neq 0$, A is closed.
Conversely if A is closed then B is closed.
Hence we have the following theorem:

Theorem 5.1. *In a $N\{R(R_n)\}$ if B is closed then A is closed. Conversely if A is closed then B is closed, provided $r\neq 0$.*

6. $N\{R(R_n)\}$ with constant scalar curvature

Let us suppose that the scalar curvature r of a $N\{R(R_n)\}$ be constant. Now from (1.1), we have

$$\begin{aligned}(D_XS)(Y,Z)-(D_ZS)(Y,X)=&[A(X)+B(X)]S(Y,Z)-[A(Z)+B(Z)]S(Y,X)\\ &+B(X)g(Y,Z)-B(Z)g(Y,X)\end{aligned} \tag{6.1}$$

In view of (2.1), we have from (6.1)

$$\begin{aligned}&g((D_XL)Z,Y)-g((D_ZL)X,Y)=[A(X)+B(X)]g(LZ,Y)-[A(Z)+B(Z)]g(LX,Y)\\ &\qquad+B(X)g(Z,Y)-B(Z)g(X,Y)\\ &\text{or } (D_XL)Z-(D_ZL)X=[A(X)+B(X)]LZ-[A(Z)+B(Z)]LX+B(X)Z-B(Z)X\end{aligned}$$

which on contraction gives

$$dr(X)=2[A(X)+B(X)]r-2[A(LX)+B(LX)]+2(n-1)B(X). \tag{6.2}$$

From (2.2) and (6.2), we have

$$2[A(LX)+B(LX)] = [A(X)+B(X)]r + (n-2)B(X)$$
$$or\ \ B(X) = \frac{2}{r+n-2}[A(LX)+B(LX)] - \frac{r}{r+n-2}A(X) \tag{6.3}$$

In view of (2.2) and (6.3), we get

$$dr(X) = -\frac{2r}{r+n-2}A(X) + \frac{2(r+n)}{r+n-2}[A(LX)+B(LX)].$$

Now if r is constant then

$$S(X,P)+S(X,Q) = \frac{1}{1+\frac{n}{r}}g(X,P) \tag{6.4}$$

Hence we can state the following theorem:

Theorem 6.1. *In a $N\{R(R_n)\}$, none of P and Q can be an eigen vector corresponding to any eigen values.*

7. Conformally flat $N\{R(R_n)\}$ with constant scalar curvature

In Conformally flat (M^n, g) it known [20]

$$(D_X S)(Y,Z) - (D_Z S)(Y,X) = \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Z)g(X,Y)], \tag{7.1}$$

From (2.2) and (7.1), we get

$$\begin{aligned}(D_X S)(Y,Z) - (D_Z S)(Y,X) =& \frac{1}{2(n-1)}[\{A(X)+B(X)\}rg(Y,Z) + nB(X)g(Y,Z) \\ &- \{A(Z)+B(Z)\}rg(Y,X) - nB(Z)g(Y,X)].\end{aligned} \tag{7.2}$$

Putting $Y = P$ in (7.2), we get

$$[(A(X)+B(X)]A(LZ) - [(A(Z)+B(Z)]A(LX) + [B(X)A(Z) - B(Z)A(X)] =$$
$$\frac{1}{2(n-1)}[\{A(X)+B(X)\}rA(Z) + nB(X)A(Z) - \{A(Z)+B(Z)\}rA(X) - nB(Z)A(X)],$$

$$\begin{aligned}or\ \ A(X)B(Z) - A(Z)B(X) = & \frac{2(n-1)}{r-n+2}[\{A(X)+B(X)\}A(LZ) \\ & - \{(A(Z)+B(Z)\}A(LX)]\end{aligned} \tag{7.3}$$

Now from (1.1),we get

$$\begin{aligned}(D_U D_V S)(Y,Z) =& [(D_U A)(V) + A(D_U V) + (D_U B)(V) + B(D_U V)]S(Y,Z) \\ &+ [A(U)+B(U)][A(V)+B(V)]S(Y,Z) + \\ &[A(V)B(U) + B(U)B(V) + (D_U B)(V) + B(D_U V)]g(Y,Z)\end{aligned}$$

From above, we have

$$(D_U D_V S)(Y,Z) - (D_V D_U S)(Y,Z) - (D_{[U,V]}S)(Y,Z) =$$
$$[(D_U A)(V) - (D_V A)(U) + (D_U B)(V) - (D_V B)(U)]S(Y,Z)$$
$$[(D_U B)(V) - (D_V B)(U) + A(V)B(U - A(U)B(V))]g(Y,Z)$$

which gives

$$(R(U,V).S)(Y,Z) = [(dA(U,V) + dB(U,V)]S(Y,Z) + dB(U,V)g(Y,Z) \\ [A(V)B(U) - A(U)B(V)]g(Y,Z). \tag{7.4}$$

Suppose the 1-form A is closed. Then in virtue of theorem (5.1) and (7.3) we get from (7.4)

$$(R(U,V).S)(Y,Z) = \frac{2(n-1)}{r-n+2}[\{A(X)+B(X)\}A(LZ) - \{A(Z)+B(Z)\}A(LX)]$$

Hence we have the following theorem:

Theorem 7.1. *In a Conformally flat $N\{R(R_n)\}$with constant scalar curvature, R(X,Y).S = 0 if and only if* $\{A(X)+B(X)\}A(LZ) = \{A(Z)+B(Z)\}A(LX)$.

8. Necessary and sufficient condition for a $N\{R(R_n)\}$ to be a $(NR)_n$

It is known that the Conformal curvature tensor $'C$ of type (0, 4) of a Riemannian manifold $(M^n, g)(n > 3)$ is given by

$$'C(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] - \\ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{8.1}$$

where $'C(X,Y,Z,W) = g(C(X,Y)Z,W)$, $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and C is the Conformal curvature tensor of type (1,3). Now let M^n be a nearly Ricci recurrent manifold $N\{R(R_n)\}$ specified by a non-zero1-form B.

Then in view of (1.1), (2.2) and (8.1), we get

$$(D_U\,'C)(X,Y,Z,W) - [A(U)+B(U)]\,'C(X,Y,Z,W) = \\ (D_U\,'R)(X,Y,Z,W) - [A(U)+B(U)]\,'R(X,Y,Z,W) - \\ \frac{B(U)}{(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{8.2}$$

Conversely if (8.2) holds, then putting $Y = Z = e_j$ in (8.2) where $\{e_j\}, j = 1,2,3,...,n$ is orthonormal basis of the tangent space at each point of the manifold and l is summed for $l \leq j \leq n$, we get

$$(D_U C)(X,W) - [A(U)+B(U)]C(X,W) = \\ (D_U S)(X,W) - [A(U)+B(U)]S(X,W) - B(U)g(X,W) \tag{8.3}$$

But in view of $C(X,W) = 0$, we get from (8.3) that

$$(D_U S)(X,W) - [A(U)+B(U)]S(X,W) - B(U)g(X,W).$$

From (8.2) and (8.3), we can state the following theorem:

Theorem 8.1. *A necessary and sufficient condition that Riemannian manifold M^n be a $N\{R(R_n)\}$ is that (8.2) holds.*

In particular, if the M^n Conformal to a flat space or if $n = 3$ then $C = 0$. In the first case it follows (8.2) that the $N\{R(R_n)\}$ is a $(NR)_n$. In the second case it follows that $N\{R(R_3)\}$ is a $(NR)_3$.
Thus we can state the following theorem:

Theorem 8.2. *Every $N\{R(R_n)\}$ $(n > 3)$ is a $(NR)_n$ if it is Conformal to a flat space and every $N\{R(R_3)\}$ is a $(NR)_3$.*

9. Example

Let us consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3, z \neq 0\}$, where (x,y,z) are standard coordinate of R^3.

We choose the vector fields

$$e_1 = e^{iy}\frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y},\ e_3 = e^{-iy}\frac{\partial}{\partial z} \tag{9.1}$$

which is linearly independently at each point of M.

Let g be the Riemannian metric denoted by

$$g(e_i, e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{9.2}$$

Let D be the Levi-Civita connection with respect to metric g. Then from equation (9.1), we have

$$[e_1, e_2] = -ie_1, [e_1, e_3] = 0, [e_2, e_3] = -ie_3. \tag{9.3}$$

The Riemannian connection D of the metric g is given by

$$\begin{aligned} 2g\left(D_X Y, Z\right) =& Xg\left(Y, Z\right) + Yg\left(X, Z\right) - Zg\left(X, Y\right) - g\left(X, [Y, Z]\right) \\ & - g\left(Y, [X, Z]\right) + g\left(Z, [X, Y]\right), \end{aligned} \tag{9.4}$$

which is known as Koszul's formula. Using (9.2) and (9.3) in (9.4), we get

$$\begin{aligned} &D_{e_1}e_1 = ie_2, \quad D_{e_1}e_2 = -ie_1, \quad D_{e_1}e_3 = 0, \\ &D_{e_2}e_1 = 0, \quad D_{e_2}e_2 = 0, \quad D_{e_2}e_1 = 0, \\ &D_{e_3}e_1 = 0, \quad D_{e_3}e_2 = ie_3, \quad D_{e_3}e_3 = -ie_2. \end{aligned} \tag{9.5}$$

The curvature tensor is given by

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \tag{9.6}$$

Using (9.3) and (9.5) in (9.6) , we get

$$\begin{aligned} &R\left(e_1, e_2\right)e_1 = -e_2, \quad R\left(e_1, e_2\right)e_2 = e_1, \quad R\left(e_1, e_2\right)e_3 = 0 \\ &R\left(e_2, e_3\right)e_1 = 0, \quad R\left(e_2, e_3\right)e_2 = -e_3, \quad R\left(e_2, e_3\right)e_3 = e_2 \\ &R\left(e_1, e_3\right)e_1 = e_3, \quad R\left(e_1, e_3\right)e_2 = 0 \quad R\left(e_1, e_3\right)e_3 = -e_1 \\ &R\left(e_1, e_1\right)e_1 = R\left(e_1, e_1\right)e_2 = R\left(e_1, e_1\right)e_3 = 0 \\ &R\left(e_2, e_2\right)e_1 = R\left(e_2, e_2\right)e_2 = R\left(e_2, e_2\right)e_3 = 0 \\ &R\left(e_3, e_3\right)e_1 = R\left(e_3, e_3\right)e_2 = R\left(e_3, e_3\right)e_3 = 0. \end{aligned} \tag{9.7}$$

The Ricci tensor is given by

$$S(e_i, e_i) = \sum_{i=1}^{3} g(R(e_i, X)Y, e_i) \tag{9.8}$$

From (9.7) and (9.8), we get

$$S(e_1, e_1) = 0, \quad S(e_2, e_2) = 2, \quad S(e_3, e_3) = 0 \tag{9.9}$$

and the scalar curvature is $r = 2$.

Since $\{e_1, e_2, e_3\}$ forms a basis of Riemannian manifold any vector field $X, Y, Z \in \chi(M)$ can be written as

$$X = a_1e_1 + b_1e_2 + c_1e_3, \;\; Y = a_2e_1 + b_2e_2 + c_2e_3,$$

where $a_j, b_j, c_j \in \gtrless^+$ (the set of all positive real numbers), $j = 1, 2, 3$.
Hence

$$S(X, Y) = b_1b_2 \tag{9.10}$$

$$g(X, Y) = a_1a_2 + b_1b_2 + c_1c_2 \tag{9.11}$$

By view of (9.10), we get

$$(D_{e_j}S)(X, Y) = D_{e_j}S(X, Y) - S(D_{e_j}X, Y) - S(X, D_{e_j}Y)$$

$$(D_{e_1}S)(X, Y) = -i(a_1b_2 + a_2b_1)$$

$$(D_{e_2}S)(X, Y) = 0$$

$$(D_{e_3}S)(X, Y) = -i(b_1c_2 + b_2c_1)$$

Consequently, the manifold under consideration is neither Ricci symmetric nor Ricci recurrent. Let us now consider 1-form non vanishes

$$\begin{aligned}
&A(e_1) = \frac{5i(a_1b_2 + a_2b_1)}{3a_1a_2 - b_1b_2 + 3c_1c_2}, \quad B(e_1) = \frac{-3i(a_1b_2 + a_2b_1)}{3a_1a_2 - b_1b_2 + 3c_1c_2} \\
&A(e_2) = 0, \qquad\qquad\qquad\qquad\;\; B(e_2) = 0 \\
&A(e_3) = \frac{5i(b_1c_2 + b_2c_1)}{3a_1a_2 - b_1b_2 + 3c_1c_2}, \quad B(e_3) = \frac{-3i(b_1c_2 + b_2c_1)}{3a_1a_2 - b_1b_2 + 3c_1c_2}
\end{aligned} \tag{9.12}$$

at any point $x \in M$. From (1.1), we have

$$\left(D_{e_j}S\right)(X, Y) = [A(e_j) + B(e_j)]S(X, Y) + B(e_j)g(X, Y), \;\; j = 1, 2, 3. \tag{9.13}$$

It can be easily seen that the Riemannian manifold with 1-forms satisfies relation (9.13). Hence the manifold under consideration is a nearly Ricci recurrent manifold (M^3, g), which is neither Ricci recurrent nor Ricci symmetric. Thus we have the following theorem:

Theorem 9.1. *There exist a nearly Ricci recurrent manifold (M^3, g), which is neither Ricci recurrent nor Ricci symmetric.*

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