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Second order Volterra-Fredholm functional integrodifferential equations

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Abstract

This paper deals with the study of global existence of solutions to initial value problem for second order nonlinear mixed Volterra-Fredholm functional integrodifferential equations in Banach spaces. The technique used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative and rely on a priori bounds of solution.

Keywords: Global solution; Volterra-Fredholm functional integrodifferential equation; Leray-Schauder alternative; Fixed point; priori bounds.

2010 MSC: 5G10, 47H10, 35R10.

1 Introduction

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = C([-r,0],X)$, $0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r,0] \to X$ endowed with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(\theta)\| : -r \le \theta \le 0\}.$$

Let $B = C([-r,T],X)$, $T > 0$, be the Banach space of all continuous functions $x : [-r,T] \to X$ with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : -r \le t \le T\}$. For any $x \in B$ and $t \in [0,T]$ we denote x_t the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$.

In this paper we prove the global existence for second order abstract nonlinear mixed Volterra-Fredholm functional integrodifferential equation of the form

$$(\varrho(t)x'(t))' = f\left(t, x_t, \int_0^t a(t,s)g(s,x_s)ds, \int_0^T b(t,s)h(s,x_s)ds\right),\ t \in [0,T], \tag{1.1}$$

$$x(t) = \phi(t),\ -r \le t \le 0,\ \ x'(0) = \delta, \tag{1.2}$$

where $f : [0,T] \times C \times X \times X \to X$, $a,\ b : [0,T] \times [0,T] \to \mathbb{R}$, $g,h : [0,T] \times C \to X$ are continuous functions, $\varrho(t)$ is real valued positive sufficiently smooth function on $[0,T]$, $\phi \in C$ and $\delta \in X$ are given.

Equation of the form (1.1)-(1.2) and their special forms serve as an abstract formulation of many partial differential equations or partial integrodifferential equations which arising in heat flow in materials with memory, viscoelasticity and many other physical phenomena see [6, 8, 12] and the references given therein.

The problem of existence, uniqueness and other properties of solutions of the special forms of (1.1)-(1.2) have been studied by many authors by using different techniques, see [1–4, 7, 9–11, 14–17] and some of the references given therein. In an interesting paper [13], Ntouyas have investigated the global existence for Volterra functional integro-differential equations in $\mathbb{R}^n$ by using classical application of Leray-Schauder alternative. The

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present paper generalizes the result of [13]. The aim of this paper is to study the existence of global solutions of (1.1)-(1.2). The main tool used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence.

2 Preliminaries and Main Results

Firstly, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Definition 2.1. *A function* $x : [-r, T] \to X$ *is called solution of initial value problem (1.1)-(1.2) if* $x \in C([-r, T], X) \bigcap C^2([0, T], X)$ *and satisfies (1.1)-(1.2) on* $[-r, T]$.

Our results are based on the following lemma, which is a version of the topological transversality theorem given by Granas [5, p. 61].

Lemma 2.1. *Let* S *be a convex subset of a normed linear space* E *and assume* $0 \in S$. *Let* $F : S \to S$ *be a completely continuous operator, and let*

$$\varepsilon(F) = \{x \in S : x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\varepsilon(F)$ *is unbounded or* F *has a fixed point.*

We list the following hypotheses for our convenience.

($\mathbf{H_1}$) There exists a continuous function $p : [0, T] \to \mathbb{R}_+ = [0, \infty)$ such that

$$\|f(t, \psi, x, y)\| \leq p(t)(\|\psi\|_C + \|x\| + \|y\|),$$

for every $t \in [0, T], \psi \in C$ and $x, y \in X$.

($\mathbf{H_2}$) There exists a continuous function $m : [0, T] \to \mathbb{R}_+$ such that

$$\|g(t, \psi)\| \leq m(t)G(\|\psi\|_C),$$

for every $t \in [0, T]$ and $\psi \in C$, where $G : \mathbb{R}_+ \to (0, \infty)$ is continuous nondecreasing function.

($\mathbf{H_3}$) There exists a continuous function $n : [0, T] \to \mathbb{R}_+$ such that

$$\|h(t, \psi)\| \leq n(t)H(\|\psi\|_C),$$

for every $t \in [0, T]$ and $\psi \in C$, where $H : \mathbb{R}_+ \to (0, \infty)$ is continuous nondecreasing function.

($\mathbf{H_4}$) There exists a constants K and L such that

$$|a(t, s)| \leq K, \text{ for } t \geq s \geq 0, \quad \text{and} \quad |b(t, s)| \leq L, \text{ for } s, t \in [0, T].$$

($\mathbf{H_5}$) For each $t \in [0, T]$ the function $f(t, ., ., .) : C \times X \times X \to X$ is continuous and for each $(\psi,\ x,\ y) \in C \times X \times X$ the function $f(., \psi,\ x,\ y) : [0, T] \to X$ is strongly measurable.

($\mathbf{H_6}$) For each $t \in [0, T]$ the functions $g(t, .),\ \ h(t, .) : C \to X$ are continuous and for each $\psi \in C$ the functions $g(., \psi),\ \ h(., \psi) : [0, T] \to X$ are strongly measurable.

With these preparations we state and prove our main results.

Theorem 2.1. *Suppose that the hypothesis* ($\mathbf{H_1}$)-($\mathbf{H_6}$) *holds. Then the initial-value problem (1.1)-(1.2) has a solution* x *on* $[-r, T]$ *if* T *satisfies*

$$\int_0^T M(s)ds < \int_\alpha^\infty \frac{ds}{s + G(s)}, \tag{2.1}$$

where

$$M(t) = max\left\{\frac{1}{R}\int_0^t p(s)ds, Km(t), Ln(t)\right\},\ t \in [0, T],$$

$$\alpha = \beta + \|\phi\|_C + \|\delta\|\varrho(0)\int_0^T \frac{ds}{\varrho(s)},$$

$R = min\{\varrho(t) : t \in [0,T]\}$ and β is constant such that $\int_0^T M(s)H(J(s))ds \le \beta$ for any continuous function $J : [0,T] \to \mathbb{R}_+$.

Proof. First we establish the priori bounds on the solutions of the initial value problem

$$(\varrho(t)x'(t))' = \lambda f\left(t, x_t, \int_0^t a(t,s)g(s,x_s)ds, \int_0^T b(t,s)h(s,x_s)ds\right), \quad t \in [0,T], \tag{2.2}$$

with the initial condition (1.2) for $\lambda \in (0,1)$. Let $x(t)$ be a solution of the problem (2.2)-(1.2) then it satisfies the equivalent integral equation

$$\begin{aligned} x(t) &= \phi(0) + \delta\varrho(0)\int_0^t \frac{ds}{\varrho(s)} \\ &\quad + \lambda\int_0^t \frac{1}{\varrho(s)}\int_0^s f\left(\tau, x_\tau, \int_0^\tau a(\tau,\eta)g(\eta,x_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta,x_\eta)d\eta\right)d\tau ds, t \in [0,T] \qquad (2.3)\\ x(t) &= \phi(t), \ -r \le t \le 0, \ x'(0) = \delta. \qquad (2.4) \end{aligned}$$

Using (2.3), hypotheses $(\mathbf{H_1}) - (\mathbf{H_4})$ and the fact that $\lambda \in (0,1)$, for $t \in [0,T]$ we have

$$\begin{aligned} \|x(t)\| &\le \|\phi\|_C + \|\delta\|\varrho(0)\int_0^t \frac{ds}{\varrho(s)} \\ &\quad + |\lambda|\int_0^t \frac{1}{\varrho(s)}\int_0^s \left\|f\left(\tau, x_\tau, \int_0^\tau a(\tau,\eta)g(\eta,x_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta,x_\eta)d\eta\right)\right\| d\tau ds \\ &\le \|\phi\|_C + \|\delta\|\varrho(0)\int_0^t \frac{ds}{\varrho(s)} \\ &\quad + \int_0^t \frac{1}{\varrho(s)}\int_0^s p(\tau)\left[\|x_\tau\|_C + \int_0^\tau Km(\eta)G(\|x_\eta\|_C)d\eta + \int_0^T Ln(\eta)H(\|x_\eta\|_C)d\eta\right] d\tau ds. \qquad (2.5) \end{aligned}$$

Consider the function Z given by $Z(t) = \sup\{\|x(s)\| : -r \le s \le t\}, \ t \in [0,T]$. Let $t^* \in [-r,t]$ be such that $Z(t) = \|x(t^*)\|$. If $t^* \in [0,t]$ then from (2.5), we have

$$\begin{aligned} Z(t) &\le \|\phi\|_C + \|\delta\|\varrho(0)\int_0^{t^*} \frac{ds}{\varrho(s)} \\ &\quad + \int_0^{t^*} \frac{1}{\varrho(s)}\int_0^s p(\tau)\left[\|x_\tau\|_C + \int_0^\tau Km(\eta)G(\|x_\eta\|_C)d\eta + \int_0^T Ln(\eta)H(\|x_\eta\|_C)d\eta\right] d\tau ds \\ &\le \|\phi\|_C + \|\delta\|\varrho(0)\int_0^T \frac{ds}{\varrho(s)} \\ &\quad + \int_0^t \frac{1}{\varrho(s)}\int_0^s p(\tau)\left[Z(\tau) + \int_0^\tau M(\eta)G(Z(\eta))d\eta + \int_0^T M(\eta)H(Z(\eta))d\eta\right] d\tau ds. \qquad (2.6) \end{aligned}$$

If $t^* \in [-r,0]$ then $Z(t) = \|\phi\|_C$ and the previous inequality (2.6) obviously holds. Denoting by $u(t)$ the right-hand side of the inequality (2.6), we have

$$u(0) = \|\phi\|_C + \|\delta\|\varrho(0)\int_0^T \frac{ds}{\varrho(s)}, \quad Z(t) \le u(t), \quad t \in [0,T]$$

and

$$\begin{aligned} u'(t) &= \frac{1}{\varrho(t)}\int_0^t p(s)\left[Z(s) + \int_0^s M(\tau)G(Z(\tau))d\tau + \int_0^T M(\tau)H(Z(\tau))d\tau\right] ds \\ &\le \frac{1}{R}\int_0^t p(s)\left[u(s) + \int_0^s M(\tau)G(u(\tau))d\tau + \int_0^T M(\tau)H(u(\tau))d\tau\right] ds \\ &\le \frac{1}{R}\int_0^t p(s)\left[u(s) + \int_0^s M(\tau)G(u(\tau))d\tau + \beta\right] ds. \end{aligned}$$

Let $w(t) = u(t) + \int_0^t M(\tau)G(u(\tau))d\tau + \beta$. Then we have $u(t) \leq w(t),\ t \in [0,T]$. Since $u(t)$ is increasing, $w(t)$ is also increasing on [0,T] and $w(0) = \beta + \|\phi\|_C + \|\delta\|\varrho(0)\int_0^T \frac{ds}{\varrho(s)} = \alpha$. Now,

$$\begin{aligned} w'(t) = &\ u'(t) + M(t)G(u(t)) \\ &\leq \frac{1}{R}\int_0^t p(s)w(s)ds + M(t)G(u(t)) \\ &\leq w(t)\frac{1}{R}\int_0^t p(s)ds + M(t)G(u(t)) \\ &\leq M(t)[w(t) + G(w(t))]. \end{aligned}$$

Therefore

$$\frac{w'(t)}{w(t) + G(w(t))} \leq M(t),\ t \in [0,T].$$

Integrating from 0 to t and using change of variables $t \to s = w(t)$ and the condition (2.1), we obtain

$$\int_\alpha^{w(t)} \frac{ds}{s + G(s)} \leq \int_0^t M(s)ds \leq \int_0^T M(s)ds < \int_\alpha^\infty \frac{ds}{s + G(s)},\ t \in [0,T]. \tag{2.7}$$

From the inequality (2.7) there exists a constant γ, independent of $\lambda \in (0,1)$ such that $w(t) \leq \gamma$ for $t \in [0,T]$. Hence $Z(t) \leq u(t) \leq w(t) \leq \gamma,\ t \in [0,T]$. Since for every $t \in [0,T]$, $\|x_t\|_C \leq Z(t)$, we have

$$\|x\|_B = \sup\{\|x(t)\| : t \in [-r,T]\} \leq \gamma.$$

Now, we rewrite initial value problem (1.1)-(1.2) as follows: For $\phi \in C$, define $\widehat{\phi} \in B, B = C([-r,T],X)$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t) & \text{if } -r \leq t \leq 0 \\ \phi(0) + \delta\varrho(0)\int_0^t \frac{ds}{\varrho(s)} & \text{if } 0 \leq t \leq T. \end{cases}$$

If $y \in B$ and $x(t) = y(t) + \widehat{\phi}(t)$, $t \in [-r,T]$ then it is easy to see that $y(t)$ satisfies

$$\begin{aligned} &y(t) = y_0 = 0; \quad -r \leq t \leq 0 \quad \text{and} \\ &y(t) = \int_0^t \frac{1}{\varrho(s)}\int_0^s f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, y_\eta + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, y_\eta + \widehat{\phi}_\eta)d\eta\right) d\tau ds,\ t \in [0,T], \end{aligned}$$

if and only if $x(t)$ satisfies

$$\begin{aligned} x(t) = &\ \phi(0) + \delta\varrho(0)\int_0^t \frac{ds}{\varrho(s)} \\ &+ \int_0^t \frac{1}{\varrho(s)}\int_0^s f\left(\tau, x_\tau, \int_0^\tau a(\tau,\eta)g(\eta, x_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, x_\eta)d\eta\right) d\tau ds, t \in [0,T], \end{aligned} \tag{2.8}$$

$$x(t) = \phi(t),\ -r \leq t \leq 0,\ x'(0) = \delta. \tag{2.9}$$

We define the operator $F : B_0 \to B_0$, $B_0 = \{y \in B : y_0 = 0\}$ by

$$(Fy)(t) = \begin{cases} 0 \quad \text{if } -r \leq t \leq 0 \\ \int_0^t \frac{1}{\varrho(s)}\int_0^s f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, y_\eta + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, y_\eta + \widehat{\phi}_\eta)d\eta\right) d\tau ds,\ \text{if } t \in [0,T]. \end{cases} \tag{2.10}$$

From the definition of operator F equations (2.8)-(2.9) can be written as $y = Fy$, and the equations (2.3)-(2.4) can be written as $y = \lambda Fy$.

Now, we prove that F is completely continuous. First, we prove that $F : B_0 \to B_0$ is continuous. Let $\{u_m\}$ be a sequence of elements of B_0 converging to u in B_0. Then by using hypothesis $(\mathbf{H_5})$ and $(\mathbf{H_6})$ we have

$$f\left(t, u_{m_t} + \widehat{\phi}_t, \int_0^t a(t,s)g(s, u_{m_s} + \widehat{\phi}_s)ds, \int_0^T b(t,s)h(s, u_{m_s} + \widehat{\phi}_s)ds\right)$$

$$\to f\left(t, u_t + \widehat{\phi}_t, \int_0^t a(t,s)g(s, u_s + \widehat{\phi}_s)ds, \int_0^T b(t,s)h(s, u_s + \widehat{\phi}_s)ds\right)$$

for each $t \in [0,T]$. Then by dominated convergence theorem, we have

$$\begin{aligned}
&\|(Fu_m)(t) - (Fu)(t)\| \\
&\le \int_0^t \frac{1}{\varrho(s)} \int_0^s \left\| f\left(\tau, u_{m_\tau} + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, u_{m_\eta} + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, u_{m_\eta} + \widehat{\phi}_\eta)d\eta\right)\right. \\
&\qquad \left. - f\left(\tau, u_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, u_\eta + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, u_\eta + \widehat{\phi}_\eta)d\eta\right)\right\| d\tau ds \\
&\to 0 \quad \text{as} \quad n \to \infty, \forall\, t \in [0,T].
\end{aligned}$$

Since, $\|Fu_m - Fu\|_B = \sup_{t\in[-r,T]} \|(Fu_m)(t) - (Fu)(t)\|$, it follows that $\|Fu_m - Fu\|_B \to 0$ as $n \to \infty$ which implies $Fu_m \to Fu$ in B_0 as $u_m \to u$ in B_0. Therefore, F is continuous.

We prove that F maps a bounded set of B_0 into a precompact set of B_0. Let $B_k = \{y \in B_0 : \|y\|_B \le k\}$ for $k \ge 1$. We show that FB_k is uniformly bounded. Let $M^* = \sup\{M(t) : t \in [0,T]\}$ and $\|\phi\|_C = c$. Then from the definition of F in (2.10) and using hypotheses $(\mathbf{H_1}) - (\mathbf{H_4})$ and the fact that $\|y\|_B \le k, y \in B_k$ implies $\|y_t\|_C \le k, t \in [0,T]$ we obtain

$$\begin{aligned}
&\|(Fy)(t)\| \\
&\le \int_0^t \frac{1}{\varrho(s)} \int_0^s p(\tau)\left[\|y_\tau + \widehat{\phi}_\tau\|_C + \int_0^\tau Km(\eta)G(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta + \int_0^T Ln(\eta)H(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta\right] d\tau ds \\
&\le \int_0^t \frac{1}{\varrho(s)} \int_0^s p(\tau)\left[\|k + c + \int_0^\tau M(\eta)G(k+c)d\eta + \int_0^T M(\eta)H(k+c)d\eta\right] d\tau ds \\
&\le [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_0^t \frac{1}{R} \int_0^s p(\tau)d\tau ds \\
&\le [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_0^T M(s)ds.
\end{aligned}$$

This implies that the set $\{(Fy)(t) : \|y\|_B \le k,\ -r \le t \le T\}$ is uniformly bounded in X and hence FB_k is uniformly bounded.

Next we show that F maps B_k into an equicontinuous family of functions with values in X. Let $y \in B_k$ and $t_1, t_2 \in [-r,T]$. Then from the equation (2.10) and using the hypotheses $(\mathbf{H_1}) - (\mathbf{H_4})$ we have three cases:

Case 1 : Suppose $0 \le t_1 \le t_2 \le T$

$$\begin{aligned}
&\|(Fy)(t_2) - (Fy)(t_1)\| \\
&\le \int_{t_1}^{t_2} \frac{1}{\varrho(s)} \int_0^s \left\| f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, y_\eta + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, y_\eta + \widehat{\phi}_\eta\right) d\eta\right\| d\tau ds \\
&\le \int_{t_1}^{t_2} \frac{1}{\varrho(s)} \int_0^s p(\tau)\left[\|y_\tau + \widehat{\phi}_\tau\|_C + \int_0^\tau Km(\eta)G(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta + \int_0^T Ln(\eta)H(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta\right] d\tau ds \\
&\le \int_{t_1}^{t_2} \frac{1}{\varrho(s)} \int_0^s p(\tau)\left[k + c + \int_0^\tau M(\eta)G(k+c)d\eta + \int_0^T M(\eta)H(k+c)d\eta\right] d\tau ds \\
&\le [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_{t_1}^{t_2} \frac{1}{R} \int_0^s p(\tau)d\tau ds \\
&\le [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_{t_1}^{t_2} M(s)ds.
\end{aligned}$$

Case 2 : Suppose $-r \leq t_1 \leq 0 \leq t_2 \leq T$. Proceeding as in Case 1, we get

$$\begin{aligned}
&\|(Fy)(t_2) - (Fy)(t_1)\| \\
&\leq \int_0^{t_2} \frac{1}{\varrho(s)} \int_0^s \left\| f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, y_\eta + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, y_\eta + \widehat{\phi}_\eta\right) d\eta \right\| d\tau ds \\
&\leq [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_0^{t_2} M(s)ds.
\end{aligned}$$

Case 3 : Suppose$-r \leq t_1 \leq t_2 \leq 0$. Then$\|(Fy)(t_2) - (Fy)(t_1)\| = 0$.

From Cases 1-3, we see that $\|(Fy)(t_2) - (Fy)(t_1)\| \to 0$ as $(t_2 - t_1) \to 0$ and we conclude that FB_k is an equicontinuous family of functions with values in X.

We have already shown that FB_k is an equicontinuous and uniformly bounded collection. To prove the set FB_k is precompact in B, it is sufficient, by Arzela-Ascoli's argument, to show that the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X for each $t \in [-r, T]$. Since $(Fy)(t) = 0$ for $t \in [-r, 0]$ and $y \in B_k$, it is sufficient to show this for $0 < t \leq T$. Let $0 < t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_k$, we define

$$(F_\epsilon y)(t) = \int_0^{t-\epsilon} \frac{1}{\varrho(s)} \int_0^s f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, y_\eta + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, y_\eta + \widehat{\phi}_\eta)d\eta\right) d\tau ds.$$

Since the set FB_k is bounded in B, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $y \in B_k$, we have

$$(Fy)(t) - (F_\epsilon y)(t) = \int_{t-\epsilon}^t \frac{1}{\varrho(s)} \int_0^s f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, y_\eta + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, y_\eta + \widehat{\phi}_\eta)d\eta\right) d\tau ds.$$

By making use of hypotheses $(\mathbf{H_1}) - (\mathbf{H_4})$ and the fact that $\|y\|_B \leq k$, $y \in B_k$ implies $\|y_t\|_C \leq k$, $t \in [0, T]$, we have

$$\begin{aligned}
&\|(Fy)(t) - (F_\epsilon y)(t))\| \\
&\leq \int_{t-\epsilon}^t \frac{1}{\varrho(s)} \int_0^s p(\tau) \left[\|y_\tau + \widehat{\phi}_\tau\|_C + \int_0^\tau Km(\eta)G(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta + \int_0^T Ln(\eta)H(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta \right] d\tau ds \\
&\leq \int_{t-\epsilon}^t \frac{1}{\varrho(s)} \int_0^s p(\tau) \left[k + c + \int_0^\tau M(\eta)G(k+c)d\eta + \int_0^T M(\eta)H(k+c)d\eta \right] d\tau ds \\
&\leq [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_{t-\epsilon}^t \frac{1}{\varrho(s)} \int_0^s p(\tau) d\tau ds \\
&\leq [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_{t-\epsilon}^t M(s)ds.
\end{aligned}$$

This shows that there exists precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_k\}$ hence the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X. Thus we have shown that F is completely continuous operator. Moreover, the set

$$\varepsilon(F) = \{y \in B_0 : y = \lambda Fy, \ 0 < \lambda < 1\},$$

is bounded in B, since for every y in $\varepsilon(F)$, the function $x(t) = y(t) + \widehat{\phi}(t)$ is a solution of initial value problem (2.2)-(1.2) for which we have proved that $\|x\|_B \leq \gamma$ and hence $\|y\|_B \leq \gamma + c$. Now, by virtue of Lemma 2.1, the operator F has a fixed point $\tilde{y}$ in B_0. Then $\tilde{x} = \tilde{y} + \widehat{\phi}$ is a solution of the initial value problem (1.1)-(1.2) . This completes the proof of the Theorem 2.1. □

In concluding this paper, we remark that one can easily extend the ideas of this paper to study the global existence of solutions to second order nonlinear mixed Volterra-Fredholm functional integrodifferential equation of the form

$$(\varrho(t)x'(t))' = f\left(t, x_t, x'(t), \int_0^t a(t,s)g(s, x_s, x'(s))ds, \int_0^T b(t,s)h(s, x_s, x'(s))ds\right), \ t \in [0, T],$$

$x(t) = \phi(t),\ -r \leq t \leq 0,\ x'(0) = \delta.$

with conditions given in $(\mathbf{H_1}) - (\mathbf{H_6})$ and suitable condition similar to that given in (2.1). The precise formulation of this result is very close to that of the result given in our Theorem 2.1 with suitable modification and hence we omit details.

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Existence and uniqueness of solutions for random impulsive differential equation

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Abstract

In this paper, we study the existence and uniqueness of the mild solutions for random impulsive differential equations through fixed point technique. An example is provided to illustrate the theory.

Keywords: Existence, Uniqueness, Random impulses, Fixed point theorem.

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1 Introduction

Many evolution processes from fields as diverse as physics, population dynamics, aeronautics, economics, telecommunications and engineering are characterized by the fact that they undergo abrupt change of state at certain moments of time between intervals of continuous evolution. The duration of these changes are often negligible compared to the total duration of process act instantaneously in the form of impulses. The impulses may be deterministic or random. There are lot of papers which investigate the qualitative properties of deterministic impulses see [1, 5, 7, 8] and the references therein.

When the impulses are exist at random points, the solutions of the differential systems are stochastic processes. It is very different from deterministic impulsive differential systems and also it is different from stochastic differential equations. Thus the random impulsive systems give more realistic than deterministic impulsive systems. The study of random impulsive differential equations is a new research area. There are few publications in this field, Iwankievicz and Nielsen [6], investigated dynamic response of non-linear systems to poisson distributed random impulses. Sanz-Serna and Stuart [9] first brought dissipative differential equations with random impulses and used Markov chains to simulate such systems. Tatsuyuki et al [10] presented a mathematical model of random impulse to depict drift motion of granules in chara cells due to myosin-actin interaction. Shujin Wu and Meng [2004] first brought forward random impulsive ordinary differential equations and investigated boundedness of solutions to these models by Liapunov's direct function in [13]. Shujin Wu et al. [14, 15, 16, 17], studied some qualitative properties of random impulses. In [3], the author studied the existence and uniqueness of random impulsive differential system by relaxing the linear growth conditions, sufficient conditions for stability through continuous dependence on initial conditions and exponential stability via fixed point theory. In [2, 4, 11, 12] the author has studied some properties of random type impulsive differential systems.

Motivated by the above mentioned works, the main purpose of this paper is to study the random impulsive differential equations. We utilize the technique developed by [7, 8, 15].

This paper is organized as follows: Some preliminaries are presented in Section 2. In Section 3, we investigate the existence and uniqueness of solution of random impulsive differential equation by reducing the linear growth condition. Moreover, Lipschitz condition has to be relaxed on the impulsive terms in the deriving results. Finally in Section 4, we give an example to motivate our results.

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2 Preliminaries

Let X be a real separable Hilbert space and Ω a nonempty set. Assume that τ_k is a random variable defined from Ω to $D_k \overset{def.}{=} (0, d_k)$ for all $k = 1, 2, \cdots$, where $0 < d_k < +\infty$. Furthermore, assume that τ_i and τ_j are independent with each other as $i \neq j$ for $i, j = 1, 2, \cdots$. Let $\tau \in \Re$ be a constant. For the sake of simplicity, we denote $\Re_\tau = [\tau, T]$. We consider the differential equations with random impulses of the form

$$
\begin{aligned}
x'(t) &= Ax(t) + f(t, x(t)), \quad t \neq \xi_k, \quad t \geq \tau, && (2.1)\\
x(\xi_k) &= b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, \cdots, && (2.2)\\
x_{t_0} &= x_0, && (2.3)
\end{aligned}
$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X; $f : \Re_\tau \times X \to X$, $b_k : D_k \to \Re$ for each $k = 1, 2, \cdots$; $\xi_0 = t_0 \in [\tau, T]$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \cdots$, here $t_0 \in \Re_\tau$ is arbitrary real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots$; $x(\xi_k^-) = \lim_{t \uparrow \xi_k} x(t)$ according to their paths with the norm $\|x\| = \sup_{\tau \leq s \leq t} |x(s)|$ for each t satisfying $\tau \leq t \leq T$.

Let us denote $\{B_t, t \geq 0\}$ be the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote $\mathcal{F}_t$ the σ-algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $L_2 = L_2(\Omega, \{\mathcal{F}_t\}, X)$ denote the Hilbert space of all $\{\mathcal{F}_t\}$- measurable square integrable random variables with values in X.

Let $\mathcal{B}$ denote Banach space $\mathcal{B}([\tau, T], L_2)$, the family of all $\{\mathcal{F}_t\}$- measurable random variables ψ with the norm

$$\|\psi\|^2 = \sup_{t \in [\tau, T]} E\|\psi(t)\|^2.$$

Definition 2.1. *Consider the inhomogeneous initial value problem where* $f : [0, T] \to X$*.*

$$
\begin{aligned}
x'(t) &= Ax(t) + f(t)\\
x(0) &= x_0.
\end{aligned}
$$

Let A be the infinitesimal generator of a C_0 semigroup $T(t)$. Let $x_0 \in X$ and $f \in L^1(0, T; X)$. Then the function $x \in C([0, T]; X)$ is given by

$$x(t) = T(t)x_0 + \int_0^t T(t - s)f(s)\mathrm{d}s, \quad 0 \leq t \leq T$$

is the mild solution of the above initial value problem for $t \in [0, T]$.

Definition 2.2. *A semigroup $\{T(t),\ t \geq 0\}$ is said to be uniformly bounded if there exists a constant $M \geq 1$ such that*

$$\|T(t)\| \leq M, \quad \textit{for } t \geq 0.$$

Definition 2.3. *For a given $T \in (\tau, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B},\ \tau \leq t \leq T\}$ is called a mild solution to equation* (2.1)-(2.3) *in $(\Omega, P, \{\mathcal{F}_t\})$, if*

(i) $x(t) \in X$ *is* $\mathcal{F}_t$*- adapted;*

(ii)

$$
\begin{aligned}
x(t) = \sum_{k=0}^{+\infty} \Bigg[& \prod_{i=1}^{k} b_i(\tau_i)T(t - t_0)x_0 + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t - s)f(s, x(s))ds \\
& + \int_{\xi_k}^{t} T(t - s)f(s, x(s))ds \Bigg] I_{[\xi_k, \xi_{k+1})}(t), \qquad t \in [\tau, T],
\end{aligned}
\tag{2.4}
$$

where $\prod_{j=m}^{n} (\cdot) = 1$ *as* $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1}) \cdots b_i(\tau_i)$*, and* $I_A(\cdot)$ *is the index function, i.e.,*

$$I_A(t) = \begin{cases} 1, & \textit{if} \quad t \in A, \\ 0, & \textit{if} \quad t \notin A. \end{cases}$$

3 Existence and Uniqueness

In this section, we discuss the existence and uniqueness of the mild solution for the system (2.1)-(2.3). Before proving the main results, we introduce the following hypotheses which are used in our results.

($\mathbf{H_1}$) The function f satisfies the Lipschitz condition. That is., for $\zeta,\ \varsigma \in X$ and $\tau \le t \le T$ there exits a constant $L > 0$ such that

$$\begin{aligned} E\left\|f(t,\zeta)-f(t,\varsigma)\right\|^2 &\le L\, E\left\|\zeta-\varsigma\right\|^2, \\ E\left\|f(t,0)\right\|^2 &\le \kappa, \quad \text{where } \kappa \ge 0 \text{ is a constant.} \end{aligned}$$

($\mathbf{H_2}$) The condition $\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}$ is uniformly bounded if, there is a constant $C > 0$ such that

$$\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\} \le C \qquad \text{for all } \tau_j \in D_j, \quad j = 1,2,\cdots.$$

Theorem 3.1. *Let the hypotheses* ($\mathbf{H_1}$) $-$ ($\mathbf{H_2}$) *be hold. If the following inequality*

$$\Lambda = M^2\max\{1,C^2\}(T-\tau)^2L < 1, \tag{3.1}$$

is satisfied, then the system (2.1)-(2.3) has a unique mild solution in $\mathcal{B}$.

Proof. Let T be an arbitrary number $\tau < T < +\infty$. First we define the nonlinear operator $\mathcal{S} : \mathcal{B} \to \mathcal{B}$ as follows

$$\begin{aligned} (\mathcal{S}x)(t) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i)T(t-t_0)x_0 + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} T(t-s)f(s,x(s))ds \\ &\quad + \int_{\xi_k}^{t} T(t-s)f(s,x(s))ds\Bigg] I_{[\xi_k,\xi_{k+1})}(t), \qquad t \in [\tau,T]. \end{aligned}$$

It is easy to prove the continuity of $\mathcal{S}$. Now, we have to show that $\mathcal{S}$ maps $\mathcal{B}$ into itself.

$$\begin{aligned} \|(\mathcal{S}x)(t)\|^2 &\le \Big[\sum_{k=0}^{+\infty}\Big[\|\prod_{i=1}^{k} b_i(\tau_i)\|\|T(t-t_0)\|\|x_0\| \\ &\quad + \sum_{i=1}^{k}\|\prod_{j=i}^{k} b_j(\tau_j)\|\Big\{\int_{\xi_{i-1}}^{\xi_i}\|T(t-s)f(s,x(s))\|ds\Big\} \\ &\quad + \int_{\xi_k}^{t}\|T(t-s)f(s,x(s))\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2 \\ &\le 2\Big[\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\|T(t-t_0)\|^2\|x_0\|^2 I_{[\xi_k,\xi_{k+1})}(t)\Big] \\ &\quad + \Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\|\prod_{j=i}^{k} b_j(\tau_j)\|\Big\{\int_{\xi_{i-1}}^{\xi_i}\|T(t-s)\|\|f(s,x(s))\|ds\Big\} \\ &\quad + \int_{\xi_k}^{t}\|T(t-s)\|\|f(s,x(s))\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\Big] \\ &\le 2M^2\max_{k}\Big\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\Big\}\|x_0\|^2 \\ &\quad + 2M^2\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\Big]^2 \\ &\quad \times\Big(\int_{t_0}^{t}\|f(s,x(s))\|\,ds I_{[\xi_k,\xi_{k+1})}(t)\Big)^2 \end{aligned}$$

$$\begin{aligned}
&\leq 2M^2C^2\|x_0\|^2 + 2M^2\max\{1,C^2\}\Big(\int_{t_0}^{t}\|f(s,x(s))\|\,ds\Big)^2\\
&\leq 2M^2C^2\|x_0\|^2 + 2M^2\max\{1,C^2\}(t-t_0)\int_{t_0}^{t}\|f(s,x(s))\|^2\,ds\\
E\|(\mathcal{S}x)(t)\|^2 &\leq 2M^2C^2\|x_0\|^2 + 2M^2\max\{1,C^2\}(T-\tau)\int_{t_0}^{t}E\,\|f(s,x(s))\|^2\,ds\\
&\leq 2M^2C^2\|x_0\|^2 + 4M^2\max\{1,C^2\}(T-\tau)^2\kappa\\
&\quad +4M^2\max\{1,C^2\}(T-\tau)L\int_{t_0}^{t}E\,\|x(s)\|^2\,ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{t\in[\tau,T]}E\|(\mathcal{S}x)(t)\|^2 &\leq 2M^2C^2\|x_0\|^2 + 4M^2\max\{1,C^2\}(T-\tau)^2\kappa\\
&\quad +4M^2\max\{1,C^2\}(T-\tau)L\int_{t_0}^{t}\sup_{s\in[\tau,t]}E\,\|x(s)\|^2\,ds\\
&\leq 2M^2C^2\|x_0\|^2 + 4M^2\max\{1,C^2\}(T-\tau)^2\kappa\\
&\quad +4M^2\max\{1,C^2\}(T-\tau)^2L\sup_{t\in[\tau,T]}E\,\|x(t)\|^2
\end{aligned}$$

for all $t\in[\tau,T]$, therefore $\mathcal{S}$ maps $\mathcal{B}$ into itself.

Now, we have to show $\mathcal{S}$ is a contraction mapping

$$\begin{aligned}
\|(\mathcal{S}x)(t)-(\mathcal{S}y)(t)\|^2 &\leq \Big[\sum_{k=0}^{+\infty}\big[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\\
&\qquad\times\int_{\xi_{i-1}}^{\xi_i}\|T(t-s)\|\|f(s,x(s))-f(s,y(s))\|ds\\
&\quad+\int_{\xi_k}^{t}\|T(t-s)\|\|f(s,x(s))-f(s,y(s))\|ds\big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\\
&\leq M^2\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\Big]^2\\
&\quad\times\Big(\int_{t_0}^{t}\|f(s,x(s))-f(s,y(s))\|dsI_{[\xi_k,\xi_{k+1})}(t)\Big)^2\\
&\leq M^2\max\{1,C^2\}(t-t_0)\int_{t_0}^{t}\|f(s,x(s))-f(s,y(s))\|^2ds\\
E\|(\mathcal{S}x)(t)-(\mathcal{S}y)(t)\|^2 &\leq M^2\max\{1,C^2\}(t-t_0)\int_{t_0}^{t}E\|f(s,x(s))-f(s,y(s))\|^2ds\\
&\leq M^2\max\{1,C^2\}(T-\tau)L\int_{t_0}^{t}E\|x(s)-y(s)\|^2ds.
\end{aligned}$$

Taking supremum over t, we get,

$$\|(\mathcal{S}x)-(\mathcal{S}y)\|^2 \leq M^2\max\{1,C^2\}(T-\tau)^2L\ \|x-y\|^2.$$

Thus,

$$\|(\mathcal{S}x)-(\mathcal{S}y)\|^2 \leq \Lambda\ \|x-y\|^2,$$

since $0<\Lambda<1$. This shows that the operator $\mathcal{S}$ satisfies the contraction mapping principle and therefore, $\mathcal{S}$ has a unique fixed point which is the mild solution of the system (2.1)-(2.3). This completes the proof. □

4 Example

As an application for the problem (2.1)-(2.3), consider a one dimensional rod of length π whose ends are maintained at 0^0 and whose sides are insulated. Suppose there is an exothermic reaction taking place inside the rod with heat being produced proportionally to the temperature at a previous time $t-r$ (for the sake of simplicity, we assume the delay $r \geq 0$ is constant). Consequently, the temperature in the rod may be modeled to satisfy

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + \rho u(x, t-r), \quad 0 < x < \pi, \ t > 0, \\ u(0,t) &= u(\pi, t) = 0, \\ u(x,t) &= \varphi(x,t), \quad -r \leq t \leq 0, \quad 0 \leq x \leq \pi. \end{cases} \tag{4.1}$$

where ρ depends on the rate of reaction and $\varphi : [-r, 0] \times [0, \pi] \to \Re$ is a given function. We observe that, when there is no heat production (i.e., $\rho = 0$), the problem (4.1) has solution given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx,$$
$$\text{where } r = 0 \text{ and } \varphi(x,0) = \sum_{n=1}^{\infty} a_n \sin nx.$$

However, it often occurs that the exothermic reaction can be related with random impulses. In some cases, the equation (4.1) may be written in the generalized form with $r = 0$,

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + \rho u(x,t), \quad 0 < x < \pi, \ t > 0, \ t \neq \xi_k, \\ u(x, \xi_k) &= q(k) \ \tau_k \ u(x, \xi_k^-), \ t = \xi_k, \\ u(0,t) &= u(\pi,t) = 0, \\ u(x,t) &= \varphi(x,t), \quad 0 \leq x \leq \pi, \end{cases} \tag{4.2}$$

and setting $X = L^2[0,\pi]$ and the operator $A = \frac{\partial^2}{\partial x^2}$ with the domain

$$D(A) = \Big\{ u \in X \, \Big| \, u \text{ and } \frac{\partial u}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 u}{\partial x^2} \in X, \ u(0) = u(\pi) = 0 \Big\}.$$

It is well known that A generates a strongly continuous semigroup $T(t)$ which is compact, analytic and self adjoint and

$$\|T(t)\| \leq M, \quad \text{for } t \geq 0, \text{ where } M > 0.$$

Thus $T(t)$ is bounded.

Furthermore, we may assume that the impulsive nature satisfy the following condition

$$E\Big[\max_{i,k} \Big\{ \prod_{j=i}^{k} \|q(j)(\tau_j)\|^2 \Big\} \Big] < \infty.$$

Under these condition, we can define the functions f and b_k as

$$f(t, x(t)) = \rho u(x,t) \quad \text{and} \quad b_k(\tau_k) = q(k)\tau_k.$$

Then the problem (4.2) can be modeled as the abstract random impulsive differential equations of the form (2.1)-(2.3) .

The next result is consequence of Theorem 3.1

Proposition 4.1. *Let the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_2})$ *be hold. Then there exist a unique mild solution* u *of the system (4.2) provided,*

$$M^2 \max\{1, C^2\}(T-\tau)^2 L < 1, \tag{4.3}$$

is satisfied.

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Chaos and bifurcation of discontinuous dynamical systems with piecewise constant arguments

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Abstract

In this paper we are concerned with the definition and some properties of the discontinuous dynamical systems generated by piecewise constant arguments. Then we study a discontinuous dynamical system of the Riccati type equation as an example. The local stability at the fixed points is studied. The bifurcation analysis and chaos are discussed. In addition, we compare our results with the discrete dynamical system of the Riccati type equation.

Keywords: Discontinuous dynamical systems, piecewise constant arguments, Riccati type equation, fixed points, bifurcation, chaos.

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1 Introduction

The discontinuous dynamical systems generated by the retarded functional equations has been defined in [1]-[4]. The dynamical systems with piecewise constant arguments has been studied in [5]-[7] and references therein. In this work we define the discontinuous dynamical systems generated by functional equations with piecewise constant arguments. The dynamical properties of the discontinuous dynamical system of the Riccati type equation will be discussed. Comparison with the corresponding discrete dynamical system of the Riccati type equation

$$x_n = 1 - \rho x_{n-1}^2, \quad n = 1, 2, 3, ...,$$

will be given.

2 Piecewise constant arguments

Consider the problem of functional equation with piecewise constant arguments

$$x(t) = f(x(r[\frac{t}{r}])), \quad t > 0, r > 0. \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

where [.] denotes the greatest integer function.
Let $n = 1, 2, 3, ...$ and $t \in [nr, (n+1)r)$, then

$$x(t) = f(x_n(nr)), \quad t \in [nr, (n+1)r).$$

Let $r = 1$ and take the limit as $t \to n+1$, we get

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, ...$$

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This shows that the discrete dynamical system

$$x_n = f(n, x_{n-1}), \quad n = 1, 2, 3, ..., T.$$

$$x(0) = x_o,$$

is a special case of the problem of functional equation with piecewise constant arguments (2.1)-(2.2). Now let $t \in [0, r)$, then $\frac{t}{r} \in [0, 1)$, $x(r[\frac{t}{r}]) = x(0)$ and the solution of (2.1)-(2.2) is given by

$$x(t) = x_1(r) = f(x(0)), \quad t \in [0, r),$$

with

$$x_1(r) = \lim_{t \to r^-} x(t) = f(x(0)).$$

For $t \in [r, 2r)$, then $\frac{t}{r} \in [1, 2)$, $x(r[\frac{t}{r}]) = x(r)$ and the solution of (2.1)-(2.2) is given by

$$x(t) = x_2(t) = f(x_1(r)), \quad t \in [r, 2r).$$

Repeating the process we can easily deduce that the solution of (2.1)-(2.2) is given by

$$x(t) = x_{(n+1)}(t) = f(x_n(nr)), \quad t \in [nr, (n+1)r),$$

which is continuous on each subinterval $(k, (k+1))$, $k = 1, 2, 3, ..., n$, but

$$\lim_{t \to kr^+} x_{(k+1)}(t) = f(x_k(kr)) \neq x_k(kr).$$

Hence the problem (2.1)-(2.2) is a discontinuous and we have proved the following theorem.

Theorem 2.1. *The solution of the problem of functional equation with piecewise constant arguments (2.1)-(2.2) is discontinuous (sectionally continuous) even if the function f is continuous.*

Now let $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $r \in \mathbb{R}^+$. Then, the following definition can be given.

Definition 2.1. *The discontinuous dynamical system generated by piecewise constant arguments is the problem*

$$x(t) = f(t, x(r[\frac{t}{r}]), x(r[\frac{t-1}{r}]), ..., x(r[\frac{t-n}{r}])), \quad t \in [0, T], \tag{2.3}$$

$$x(t) = x_0, \quad t \leq 0. \tag{2.4}$$

Definition 2.2. *The fixed points of the discontinuous dynamical system (2.3) and (2.4) are the solution of the equation*

$$x(t) = f(t, x, x, ..., x).$$

3 Main Problem

Consider the discontinuous dynamical system generated by piecewise constant arguments of Riccati type equation

$$x(t) = 1 - \rho x^2(r[\frac{t}{r}]), \quad t, r > 0, \text{ and } x(0) = x_0. \tag{3.1}$$

Here we study the stability at the fixed points. In order to study bifurcation and chaos we take firstly $r = 1$ and we compare the results with the results of the discrete dynamical system of Riccati type difference equation

$$x_{n+1} = 1 - \rho x_n^2, \quad n = 1, 2, 3, ..., \text{ and } x_0 = x_o. \tag{3.2}$$

Secondly, we take some other values of r and T and study some examples.

3.1 Fixed points and stability

As in the case of discrete dynamical systems, the fixed points of the dynamical system (3.1) are the solution of the equation $f(x) = x$. Thus there are two fixed points which are

$$(x_{fixed})_1 = \frac{-1 + \sqrt{1+4\rho}}{2\rho},$$

$$(x_{fixed})_2 = \frac{-1 - \sqrt{1+4\rho}}{2\rho}.$$

To study the stability of these fixed points, we take into account the following theorem.

Theorem 3.1. *[8] Let f be a smooth map on $\mathbb{R}$, and assume that x_0 is a fixed point of f.*
1. If $|f^{'}(x_0)| < 1$, then x_0 is stable.
2. If $|f^{'}(x_0)| > 1$, then x_0 is unstable.

Now since in our case $f(x) = 1 - \rho x^2$, the first fixed point $(x_{fixed})_1 = \frac{-1+\sqrt{1+4\rho}}{2\rho}$ is stable if

$$|1 - \sqrt{1+4\rho}| < 1,$$

that is, $\frac{-1}{4} < \rho < \frac{3}{4}$.
The second fixed point $(x_{fixed})_2 = \frac{-1-\sqrt{1+4\rho}}{2\rho}$ is stable if

$$|1 + \sqrt{1+4\rho}| < 1,$$

which can never happen since $1 + \sqrt{1+4\rho}$ is always > 1. So, the second fixed point is unstable.
Figure (1) shows the trajectories of (3.1) when $r = 1$, while Figure (2) shows the trajectories of (3.2).

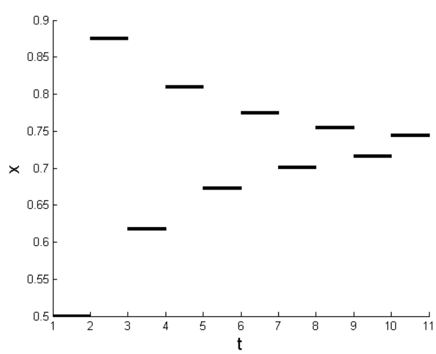

Figure 1: Trajectories of (3.1), r=1.

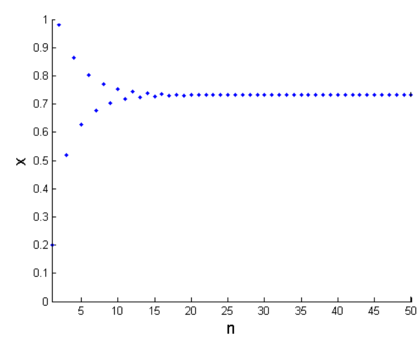

Figure 2: Trajectories of (3.2).

4 Bifurcation and Chaos

In this section, the numerical experiments show that the dynamical behaviors of the discontinuous dynamical system (3.1) depends completely on both r and T as follows:

1. Take $r = 1$ and $t \in [0, 30]$, in this case the dynamical behaviors of the two dynamical systems (3.1) and (3.2) are identical (Figure 4).

2. Take $r = 0.25$ and $t \in [0, 2]$ in the dynamical system (3.1) (Figure 5).

3. Take $r = 0.5$ and $t \in [0, 2]$ in the dynamical system (3.1) (Figure 6).

4. Take $r = 0.25$ and $T = N = 13$ in the dynamical system (3.1) (Figure 7).

5. Take $r = 0.5$ and $T = N = 35$ in the dynamical system (3.1) (Figure 3).

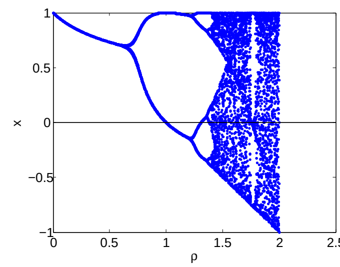

Figure 3: Bifurcation diagram of the dynamical systems (3.1) with $r = 1$ and (3.2) where $N = T = 70$.

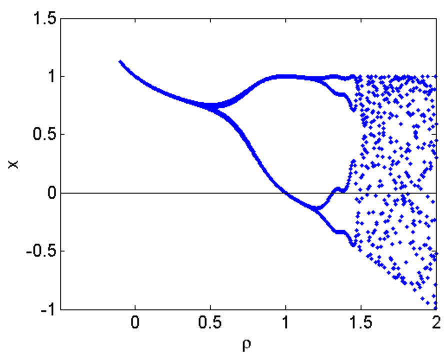

Figure 4: Bifurcation diagram for (3.1), $r = 0.5$, $t = [0, 3]$.

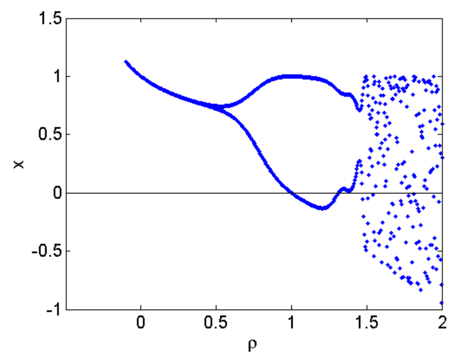

Figure 5: Bifurcation diagram for (3.1), $r = 0.25$, $t = [0, 3]$.

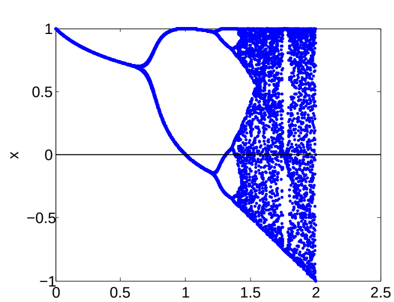

Figure 6: Bifurcation diagram for (3.1), $r = 0.5$, $T = N = 13$.

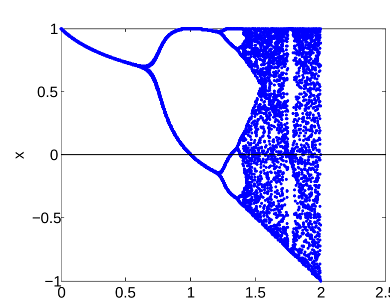

Figure 7: Bifurcation diagram for (3.1), $r = 0.25$, $T = N = 13$.

5 Conclusion

The discontinuous dynamical system models generated by piecewise constant arguments have the same behavior as its discrete version when $r = 1$.

On the other hand, changing the parameter r together with the time $t \in [0, T]$ affects the chaos behavior of the dynamical system generated by the piecewise constant arguments model as it is shown clearly in the above figures.

□

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Optimal intervals for uniqueness of solutions for lipschitz nonlocal boundary value problems

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Abstract

For the nth order differential equation, $y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$, where $f(t, r_1, r_2, \dots, r_n)$ satisfies a Lipschitz condition in terms of $r_i, 1 \le i \le n$, we obtain optimal bounds on the length of intervals on which solutions are unique for certain nonlocal three point boundary value problems. These bounds are obtained through an application of the Pontryagin Maximum Principle.

Keywords: Nonlocal boundary value problem, optimal length intervals, Pontryagin maximum principle.

2010 MSC: 34B15, 49J15.

1 Introduction

In this paper, we shall be concerned with the nth order differential equation,

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}), \quad a < t < b, \tag{1.1}$$

where we assume throughout that

(A) $f(t, r_1, \dots, r_3n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and

(B) f satisfies the Lipschitz condition

$$|f(t, r_1, \dots, r_n) - f(t, s_1, \dots, s_n)| \le \sum_{i=1}^{n} k_i |r_i - s_i|$$

for each $(t, r_1, \dots, r_n), (t, s_1, \dots, s_3) \in (a, b) \times \mathbb{R}^3$.

Let $0 \le p \le n - 2$ be fixed throughout the paper.

We characterize optimal length for subintervals of (a, b), in terms of the Lipschitz coefficients $k_i, 1 \le i \le n$, on which solutions are unique for problems involving (1.1) and satisfying the *nonlocal* three point boundary conditions,

$$y^{(i)}(t_1) = y_{i+1}, \quad i \in \{0, \dots, n-1\} \setminus \{p+1\}, \quad y^{(p)}(t_2) - y^{(p)}(t_3) = y_{p+2}, \tag{1.2}$$

where $a < t_1 < t_2 < t_3 < b$, and $y_1, \dots, y_n \in \mathbb{R}$.

Namely, we characterize optimal length for subintervals of (a, b) on which solutions of (1.1), (1.2) are unique. Such uniqueness results are of interest, because in many cases, uniqueness of solutions implies existence of

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solutions for boundary value problems; see, for example, the papers [5, 7, 9, 20, 21, 24, 26, 27, 35] and the references therein.

There is a close connection between the boundary value problem (1.1), (1.2) and certain focal boundary value problems for (1.1). From this relationship, we will eventually establish that it suffices for us to characterize optimal length subintervals of (a, b) on which solutions are unique for (1.1) satisfying the focal boundary conditions,

$$y^{(i)}(t_1) = y_{i+1}, \quad i \in \{0, \dots, n-1\} \setminus \{p+1\}, \quad y^{(p+1)}(t_2) = y_{p+2}, \tag{1.3}$$

where $a < t_1 < t_2 < b$, and $y_1, \dots, y_n \in \mathbb{R}$. The connection between this characterization and the characterization for our three point nonlocal problems is through a simple application of the Mean Value Theorem.

Theorem 1.1. *If solutions for* (1.1), (1.3) *are unique, when they exist on* (a, b), *then solutions for* (1.1), (1.2) *are unique, when they exist on* (a, b).

In view of Theorem 1.1, conditions sufficient to provide uniqueness of solutions, when they exist on (a, b), for two point focal boundary value problems (1.1), (1.3), are sufficient to provide uniqueness of solutions, when they exist on (a, b) for three point nonlocal boundary value problems (1.1), (1.2).

Our process will involve development of a situation in which the Pontryagin Maximum Principle can be applied. We follow a pattern that has an extensive history, with first motivation found in the papers by Melentsova [39] and Melentsova and Mil'shtein [40, 41]. Those papers were subsequently adapted to the context of several types of boundary value problems, with classical papers including Jackson [31, 32], Eloe and Henderson [8], Hankerson and Henderson [19] and Henderson *et al.* [22, 23, 28], and more recent results have appeared in [6, 10, 11, 25]

Interest in nonlocal boundary value problems also has a long history, both in application and theory, as can be seen in this list of papers and the references therein: [1] -[4], [12, 13], [15] - [18], [25], [29, 30], [33, 34], [37, 38], [42] - [50].

2 Optimal Intervals for Uniqueness of Solutions

In this section, we characterize in terms of the Lipschitz constants $k_i, 1 \le i \le n$, optimal length for subintervals of (a, b) on which solutions are unique, when they exist for the focal boundary value problem (1.1), (1.3). This length, it will be argued later, is optimal for uniqueness of solutions for the three point nonlocal boundary value problem (1.1), (1.2). Our characterization involves an application of the Pontryagin Maximum Principle.

We begin by defining a set $\mathcal{U}$ of vector-valued *control functions*

$$\begin{aligned} \mathcal{U} \;\; := \;\; & \{\mathbf{v}(t) = (v_1(t), \dots, v_n(t))^T \in \mathbb{R}^n \mid v_i(t) \text{ are Lebesgue} \\ & \text{measurable and } |v_i(t)| \le k_i \text{ on } (a, b), i = 1, \dots, n\}. \end{aligned}$$

We will be concerned with boundary value problems associated with linear differential equations of the form

$$x^{(n)} = \sum_{i=1}^{n} u_i(t) x^{(i-1)}, \tag{2.1}$$

where $\mathbf{u}(t) = (u_1(t), \dots, u_n(t))^T \in \mathcal{U}$. We immediately make a connection of these linear differential equations in the context of solutions of (1.1), (1.3). Much of our analysis will be based upon our choosing, if they exist, distinct solutions $y(t)$ and $z(t)$ of (1.1), (1.3).

If $y(t)$ and $z(t)$ are distinct solutions of (1.1), (1.3), then their difference $x(t) := y(t) - z(t)$ satisfies

$$x^{(i)}(t_1) = x^{(p+1)}(t_2) = 0, \quad i \in \{0, \dots, n-1\} \setminus \{p+1\}, \tag{2.2}$$

for some $a < t_1 < t_2 < b$, and $x(t)$ is a nontrivial solution of (2.1), for $\mathbf{u}(t) = (u_1(t), \dots, u_n(t))^T \in \mathcal{U}$, where for $1 \le i \le n$,

$$u_i(t) := \begin{cases} \frac{f(t, z(t), \dots, z^{(i-2)}(t), y^{(i-1)}(t), \dots, y^{(n-1)}(t)) - f(t, z(t), \dots, z^{(i-1)}(t), y^{(i)}(t), \dots, y^{(n-1)}(t))}{y^{(i-1)}(t) - z^{(i-1)}(t)}, & \\ & y^{(i-1)}(t) \neq z^{(i-1)}(t), \\ 0, & y^{(i-1)}(t) = z^{(i-1)}(t). \end{cases}$$

From optimal control theory (cf. Gamkrelidze [14, p. 147] and Lee and Markus [36, p. 259]), there is a boundary value problem in the class (2.1), (2.2), which has a nontrivial time optimal solution; that is, there exists at least one nontrivial $\mathbf{u}^* \in \mathcal{U}$ and points $t_1 \le c < d \le t_2$ such that

$$x^{(n)} = \sum_{i=1}^{n} u_i^*(t)x^{(i-1)}, \tag{2.3}$$

$$x^{(i)}(c) = x^{(p+1)}(d) = 0, \quad i \in \{0, \dots, n-1\} \setminus \{p+1\}, \tag{2.4}$$

has a nontrivial solution, $x_0(t)$, and $d-c$ is a minimum over all such solutions. For this time optimal solution, $x_0(t)$, set $\mathbf{x_0}(t) = (x_0(t), \dots, x_0^{(n-1)}(t))^T$. Then $\mathbf{x_0}(t)$ is a solution of a first order system,

$$\mathbf{r}' = A[\mathbf{u}^*(t)]\mathbf{r}, \quad a < t < b.$$

By the Pontryagin Maximum Principle, the adjoint system, whose form is given by

$$\mathbf{x}' = -A^T[\mathbf{u}^*(t)]\mathbf{x}, \quad a < t < b, \tag{2.5}$$

has a nontrivial optimal solution, $\mathbf{x}^*(t) = (x_1^*(t), \dots, x_n^*(t))^T$ such that, for *a. e.* $t \in [c, d]$,

(i) $\sum_{i=1}^n x_0^{(i)}(t)x_i^*(t) = \langle \mathbf{x_0'}(t), \mathbf{x}^*(t)\rangle = \max_{\mathbf{u}\in\mathcal{U}}\{\langle A[\mathbf{u}(t)]\mathbf{x_0}(t), \mathbf{x}^*(t)\rangle\}$,

(ii) $\langle \mathbf{x_0'}(t), \mathbf{x}^*(t)\rangle$ is a nonnegative constant,

(iii) $x_{p+2}^*(c) = x_1^*(d) = \cdots = x_{p+1}^*(d) = x_{p+3}^*(d) = \cdots = x_n^*(d) = 0$.

The maximum condition in (i) can be rewritten as

$$x_n^*(t)\sum_{i=1}^{n} u_i^*(t)x_0^{(i-1)}(t) = \max_{\mathbf{u}\in\mathcal{U}}\{x_n^*(t)\sum_{i=1}^{n} u_i(t)x_0^{(i-1)}(t)\}, \tag{2.6}$$

for *a. e.* $t \in [c, d]$.

By its time optimality and repeated applications of Rolle's Theorem, $x_0(t) \ne 0$, $t \in (c, d]$. In fact, for each $0 \le i \le p+1$, $x_0^{(i)}(t) \ne 0$ on (c, d). We may assume without loss of generality that $x_0(t) > 0$ on $(c, d]$. Moreover, by its own time optimality, $x_n^*(t)$ has no zeros on (c, d). In view of that, we can use (2.6) to determine an optimal control $\mathbf{u}^*(t)$, for *a. e.* $t \in [c, d]$.

Now, $x_0(t) > 0$ on $(c, d]$, and so we have from (2.6) that, if $x_n^*(t) < 0$ on (c, d), then the time optimal solution $x_0(t)$ is a solution of

$$x^{(n)} = -k_1 x - \sum_{i=2}^{n} k_i|x^{(i-1)}| \tag{2.7}$$

on $[c, d]$, while if $x_n^*(t) > 0$ on (c, d), then the time optimal solution $x_0(t)$ is a solution of

$$x^{(n)} = k_1 x + \sum_{i=2}^{n} k_i|x^{(i-1)}| \tag{2.8}$$

on $[c, d]$. In particular, from either (2.7) or (2.8), $x_0^{(n)}(t)$ is of one sign. It follows from the assumed positivity of $x_0(t)$ and the nature of the boundary conditions (2.4) that $x_0^{(n-1)}(t)$ is decreasing so that $x_n^*(t) < 0$ and $x_0(t)$ is a solution of (2.7). In addition, from the boundary conditions (2.4), $x_0^{(i)}(t) > 0$ on (c, d), $0 \le i \le p+1$, and $x_0^{(i)}(t) < 0$ on (c, d), $p+2 \le i \le n-1$. As a consequence, not only is $x_0(t)$ is a solution of (2.7), but also where (2.7) takes the form

$$x^{(n)} = -\sum_{i=1}^{p+2} k_i x^{(i-1)} + \sum_{i=p+3}^{n} k_i x^{(i-1)}. \tag{2.9}$$

Our discussion to this point has been based on (1.1) having distinct solutions whose difference satisfies (2.2). This led to optimal intervals being determined on which only trivial solutions exist for boundary value problems (2.7), (2.2) or (2.8), (2.2). A more detailed sign analysis led to determination of optimal intervals on which only trivial solutions exist for only the boundary value problem (2.9), (2.2). As a consequence, solutions of the boundary value problem (1.1), (1.3) will be unique on such subintervals.

Theorem 2.1. *If there is a vector-valued* $\mathbf{u}(t) \in \mathcal{U}$ *for all* $a < t < b$*, for which the boundary value problem* (2.1), (2.2) *has a nontrivial solution for some* $a < t_1 < t_2 < b$*, and if* $x_0(t)$ *is a time optimal solution satisfing* (2.4)*, where* $d - c$ *is a minimum, then* $x_0(t)$ *is a solution of* (2.9) *on* $[c, d]$.

Theorem 2.2. *Let* $\ell = \ell(k_1, \ldots, k_n) > 0$ *be the smallest positive number such that there exists a solution* $x(t)$ *of the boundary value problem for* (2.9) *satisfying*

$$x^{(i)}(0) = 0,\ i \in \{0, \ldots, n-1\} \setminus \{p+1\},\ x^{(p+1)}(\ell) = 0, \tag{2.10}$$

with $x(t) > 0$ *on* $(0, \ell]$*, or* $\ell = \infty$ *if no such solution exists. If* $y(t)$ *and* $z(t)$ *are solutions of the boundary value problem* (1.1), (1.3)*, for some* $a < t_1 < t_2 < b$*, and if* $t_2 - t_1 < \ell$*, it follows that* $y(t) \equiv z(t)$ *on* $[t_1, t_2]$*, and this is best possible for the class of all differential equations satisfying the Lipschitz condition* (B).

Proof. Since equation (2.9) is autonomous, translations of solutions are again solutions of (2.9). Hence, it suffices to apply Theorem 2.1 with respect to the boundary conditions at 0 and ℓ.

Now, if $y(t)$ and $z(t)$ are distinct solutions of (1.1) whose difference $w(t) := y(t) - z(t)$ satisfies (2.2), where $t_2 - t_1 < \ell$, then $w(t)$ is a nontrivial solution of the boundary value problem (2.1), (2.2), for appropriately defined $\mathbf{u} \in \mathcal{U}$. Then, from the discussion and Theorem 2.1, equation (2.9) has a nontrivial solution on a subinterval of length less than ℓ. But, by the minimality of ℓ, such a boundary value problem can have only the trivial solution; this is a contradiction. Therefore, solutions of the boundary value problem (1.1), (1.3) are unique, whenever $t_2 - t_1 < \ell$.

That this is best possible from the fact that (2.9) satisfies the Lipschitz condition (B), and if $\ell \neq \infty$, then $x(t)$ is a nontrivial solution of (2.9) and (2.2) on $[0, \ell]$. The boundary value problem also has the trivial solution. □

Remark 2.1. *Since* (2.9) *is a linear equation, we observe that, if* $x(t)$ *is the solution, of the initial value problem for* (2.9)*, satisfying,*

$$x^{(i)}(0) = 0,\ i \in \{0, \ldots, n-1\} \setminus \{p+1\}, \quad x^{(p+1)}(0) = 1,$$

and if $\eta > 0$ *is the first positive number such that* $x^{(p+1)}(\eta) = 0$*, then* $\eta = \ell(k_1, \ldots, k_n)$ *of Theorem 2.2.*

Because of the uniqueness relationships stated in Theorem 1.1, we can apply Theorem 2.2 to obtain optimal intervals for uniqueness of solutions of the boundary value problem (1.1), (1.2).

Theorem 2.3. *Let* ℓ *be as in Theorem 2.2. If* $y(t)$ *and* $z(t)$ *are solutions of the boundary value problem* (1.1), (1.2)*, for some* $a < t_1 < t_2 < t_3 < b$*, and if* $t_3 - t_1 \leq \ell$*, it follows that* $y(t) \equiv z(t)$ *on* $[t_1, t_3]$*, and this is best possible for the class of all differential equations satisfying the Lipschitz condition* (B).

Proof. In view of Theorem 1.1 and Theorem 2.2, solutions of the boundary value problem (1.1), (1.2) are unique, when $t_3 - t_1 \leq \ell$. To see again that this is best possible, consider the nontrivial solution $x(t)$ of (2.9) and (2.10) in Theorem 2.2.

Let $\epsilon > 0$ be sufficiently small that $x(t)$ is a solution of (2.9) on $[0, \ell + \epsilon]$. Now, $x^{(p+2)}(t) < 0$ on $[0, \ell + \epsilon]$. From (2.10), $x^{(p+1)}(\ell) = 0$, and since $x^{(p+2)}(\ell) < 0$, we have that $x^{(p)}(t)$ has a positive maximum at ℓ. So, there exist $0 < \tau_1 < \ell < \tau_2 < \ell + \epsilon$ such that $x(t)$ is a nontrivial solution of (2.9) satisfying $x^{(i)}(0) = 0, i \in \{0, \ldots, n-1\} \setminus \{p+1\}$, and $x^{(p)}(\tau_1) - x^{(p)}(\tau_2) = 0$. This boundary value problem also has the trivial solution. Since $\epsilon > 0$ was arbitrary, the "best possible" statement follows for uniqueness of solutions of the boundary value problem (1.1), (1.2). □

3 Optimal Intervals of Existence for Linear Equations

In the case of boundary value problem (1.1), (1.2), we do not have a "uniqueness implies existence" theorem to appeal to, since this is an open question for this type of boundary value problem. However, uniqueness does imply existence for linear differential equations, and so the following corollary can be stated.

Corollary 3.1. *Let* ℓ *be as in Theorem 2.2. Assume* $r_i(t), 1 \leq i \leq n$*, and* $q(t)$ *are continuous on* (a, b) *and that* $|r_i(t)| \leq k_i$ *on* $(a, b), 1 \leq i \leq n$*. If* $a < t_1 < t_2 < t_3 < b$ *and* $t_3 - t_1 < \ell$*, then the boundary value problem,*

$$y^{(n)} = \sum_{i=1}^{n} r_i(t) y^{(i-1)} + q(t),$$

$$y^{(i)}(t_1) = y_{i+1},\ i \in \{0, \ldots, n-1\} \setminus \{p+1\}, \quad y^{(p)}(t_2) - y^{(p)}(t_3) = y_{p+2},$$

has a solution for any assignment of values of $y_i \in \mathbb{R}, 1 \leq i \leq n$.

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Existence results for impulsive neutral stochastic functional integrodifferential systems with infinite delay

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Abstract

This paper is devoted to build the existence of mild solutions of impulsive neutral stochastic functional integrodifferential equations (INSFIDEs) with infinite delay at abstract phase space in Hilbert spaces. Under the uniform Lipschitz condition, we obtain the solution for INSFIDEs. Sufficient conditions for the existence results are derived with the help of Krasnoselski-Schaefer type fixed point theorem. An example is provided to illustrate the theory.

Keywords: Impulsive neutral stochastic integrodifferential equations, infinite delay, Krasnoselski-Schaefer type fixed point theorem, semigroup theory.

2010 MSC: 34A37, 37H10, 60H20, 34K50, 34K05.

1 Introduction

Stochastic differential equations are well known to model problems from many areas of science and engineering, wherein, quite often the future state of such systems depends not only on the present state but also on its past history (delay) leading to stochastic functional differential equations and it has played an important role in many ways such as option pricing, forecast of the growth of population, etc., [32, 36, 37]. Random differential and integral equations play an important role in characterizing numerous social, physical, biological and engineering problems and for more details reader may refer [16, 25] and reference therein.

From time in memory, the theory of nonlinear functional differential or integrodifferential equations has become an active area of investigation due to their application in many physical phenomena. Several authors [3, 7, 8, 22] have investigated the integrodifferential equations with or without impulsive conditions in Banach spaces. Recently impulsive neutral differential and integrodifferential equations have generated considerable interest among the researchers [20].

Impulsive dynamical systems exhibit the various evolutionary process, including those in engineering, biology and population dynamics, undergo abrupt changes in their state at certain moments between intervals of continuous evolution. Since many evolution process, optimal control models in economics, stimulated neutral networks, frequency- modulated systems and some motions of missiles or aircrafts are characterized by the impulsive dynamical behavior. Nowadays, there has been increasing interest in the analysis and synthesis of impulsive systems due to their significance both in theory and applications. Thus the theory of impulsive differential equations has seen considerable development. For instance, see the monograph of Lakshmikantham et al. [35], Bainov and Simeonov [6] and Somoilenko and Perestuk [44] for the ordinary impulsive differential system and [26, 27, 28, 29, 41, 42] for the partial differential and partial functional differential equations with impulses and for more details reader may refer [2, 3, 4, 10, 11, 18, 19, 39, 45] and reference therein. The stochastic differential equations combined with impulsive conditions with unbounded delay have been studied

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by few authors, [1, 5, 12, 15, 24, 40] and the papers of [8, 13, 14, 31, 33, 43], where the numerous properties of their solutions are studied.

In [9] Balachandran et al. studied the existence for impulsive neutral evolution integrodifferential equations with infinite delay and Krasnoselski-Schaefer type fixed point theorem, whereas A. Lin et al. [34] proved on neutral impulsive stochastic integrodifferential equations with infinite delay via fractional operators and Sadovskii fixed point theorem, and Yong Ren et al. [40] established the controllability of impulsive neutral stochastic functional differential inclusions with infinite delay and Dhage's fixed point theorem. Recently, Jing Cui et al. [23] derived nonlocal Cauchy problem for some stochastic integrodifferential equations in Hilbert spaces and Leray-Schauder nonlinear alternative fixed point theorem.

Inspired by the above mentioned works [9, 23, 34, 40], in this paper, we are interested in studying the existence of solutions of the following impulsive neutral stochastic differential equations with infinite delay;

$$d[x(t)-g(t,x_t)] = A\Big[x(t)+e\Big(t,x_t,\int_0^t h_1(t,s,x_s)ds\Big)\Big]dt + f(t,x_t)dt + \sigma\Big(t,x_t,\int_0^t h_2(t,s,x_s)ds\Big)dw(t),$$

$$t\in J:=[0,b],\quad t\neq t_k,\quad k=1,2,\dots,m, \tag{1.1}$$

$$\Delta x(t_k) = x(t_k^+)-x(t_k^-) = I_k(x(t_k^-)),\quad k=1,2,\dots,m, \tag{1.2}$$

$$x_0=\phi\in\mathcal{B}_h,\quad t\in J_0=(-\infty,0], \tag{1.3}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operator $\{T(t)\}_{t\geq 0}$ in the Hilbert space H. The history $x_t:(-\infty,0]\to H, x_t(s)=x(t+s), s\leq 0$, belong to an abstract phase space $\mathcal{B}_h$, which will be described axiomatically in Section 2. Let K be the another separable Hilbert space with inner product $(\cdot,\cdot)_K$ and the norm $\|\|_K$. Suppose $\{w(t):t\geq 0\}$ is a given K- valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q\geq 0$ defined on a complete probability space $(\Omega,\mathcal{F},P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which generated by the Wiener process w. We now employing the same notation $\|\cdot\|$ for the norm $L(K;H)$, where $L(K;H)$ denotes the space of all bounded linear operator from K into H. Here $g,f:J\times\mathcal{B}_h\to H$, $e:J\times\mathcal{B}_h\times H\to H$, $h_1,h_2:J\times J\times\mathcal{B}_h\to H$ and $\sigma:J\times\mathcal{B}_h\times H\to L_Q(K,H)$ are given functions, where $L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operator from K into H which will be defined in Section 2. The initial data $\phi=\{\phi(t):-\infty<t\leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}_h$- valued random variable independent of the Wiener process w with finite second moment. Furthermore, the fixed times t_k satisfies $0=t_0<t_1<t_2<\cdots<t_m<b$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t=t_k$. And $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$ represents the jump in the state x at time t_k, where I_k determines the size of the jump.

The outline of the paper is as follows. We review some basic facts about semigroups, the theory of SDEs, as preliminaries in Section 2. Then, Section 3 is devoted to the development of our main existence results and our basic tool include Krasnoselski-Schaefer fixed point theorem. Finally, the paper is conclude with an example to illustrate the obtained results.

2 Preliminaries

Let $(K,\|\cdot\|_K)$ and $(H,\|\cdot\|_H)$ be the two separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and $\langle\cdot,\cdot\rangle_H$, respectively. We denote by $\mathcal{L}(K,H)$ be the set of all linear bounded operator from K into H, equipped with the usual operator norm $\|\cdot\|$. In this article, we use the symbol $\|\cdot\|$ to denote norms of operator regardless of the space involved when no confusion possibly arises.

Let $(\Omega,\mathcal{F},P,H)$ be the complete probability space furnished with a complete family of right continuous increasing σ- algebra $\{\mathcal{F}_t, t\in J\}$ satisfying $\mathcal{F}_t\subset\mathcal{F}$. An H- valued random variable is an $\mathcal{F}$- measurable function $x(t):\Omega\to H$ and a collection of random variables $S=\{x(t,\omega):\Omega\to H\setminus t\in J\}$ is called stochastic process. Usually we write $x(t)$ instead of $x(t,\omega)$ and $x(t):J\to H$ in the space of S. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t):t\geq 0\}$ is a cylindrical K-valued wiener process with a finite trace nuclear covariance operator $Q\geq 0$, denote $\mathrm{Tr}(Q)=\sum_{i=1}^{\infty}\lambda_i=\lambda<\infty$, which satisfies that $Qe_i=\lambda_i e_i$. So, actually $\omega(t)=\sum_{i=1}^{\infty}\sqrt{\lambda_i}\omega_i(t)e_i$, where $\{\omega_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t=\sigma\{\omega(s):0\leq s\leq t\}$ is the σ-algebra generated by ω and $\mathcal{F}_t=\mathcal{F}$. Let $\Psi\in\mathcal{L}(K,H)$ and define

$$\|\Psi\|_Q^2 = \mathrm{Tr}(\Psi Q\Psi^*) = \sum_{n=1}^{\infty}\|\sqrt{\lambda_n}\Psi e_n\|^2.$$

If $\|\Psi\|_Q < \infty$, then Ψ is called a Q-Hilbert-Schmidt operator. Let $\mathcal{L}_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\Psi : K \to H$. The completion $\mathcal{L}_Q(K,H)$ of $\mathcal{L}(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\Psi\|_Q^2 = \langle \Psi, \Psi\rangle$ is a Hilbert space with the above norm topology.

The collections of all strongly measurable, square integrable, H-valued random variable, denoted by $L_2(\Omega, \mathcal{F}, P, H) \equiv L_2(\Omega, H)$, is a Banach space equppied with norm $\|x(\cdot)\|_{L_2} = (E\|x(\cdot,\omega)\|^2)^{\frac{1}{2}}$, where the expectation, E is defined by $Ex = \int_\Omega x(\omega)dP$. Let $C(J, L_2(\Omega, H))$ be the Banach space of all continuous map from J into $L_2(\Omega, H)$ satisfying the condition $\sup_{t\in J} E\|x(t)\|^2 < \infty$. An important subspace is given by $L_2^0(\Omega, H) = \{f \in L_2(\Omega, H) : f \quad \text{is} \quad \mathcal{F}_0 - \text{measurable}\}$.

Let A be the infinitesimal generator of an analytic semigroup $T(t)$ in H. Suppose that $0 \in \rho(A)$ where $\rho(A)$ denotes the resolvent set of A and that semigroup $T(\cdot)$ is uniformly bounded that is to say, $\|T(t)\| \le M_1$ for some constant $M_1 \ge 1$ and for every $t \ge 0$. Then for $\alpha \in (0,1]$, it is possible to define the fractional power operator $((-A)^\alpha)$ as a closed linear invertible operator on its domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \; x \in D((-A)^\alpha),$$

defines the norm on $H_\alpha = D((-A)^\alpha)$.

It should be pointed out that, to study of abstract impulsive functional differential systems with infinite delay, the abstract phase space $\mathcal{B}_h$(which is similar to that used in [46]) is very appropriate. Now we present we present the abstract phase space $\mathcal{B}_h$ as given in [21].

Assume that $h : (-\infty, 0] \to (0, +\infty)$ is a continuous function with $l = \int_{-\infty}^0 h(s)ds < +\infty$. For any $a > 0$, we define,

$$\mathcal{B} = \{\psi : [-a, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable}\},$$

and equip the space $\mathcal{B}$ with the norm,

$$\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|, \quad \forall \psi \in \mathcal{B}.$$

Let us define,

$$\mathcal{B}_h = \Big\{\psi : (-\infty, 0] \to H : \; (E\|\psi(\theta)\|^2)^{\frac{1}{2}} \text{is a bounded and measurable function on } [-a, 0]$$
$$\text{and} \int_{-\infty}^0 h(s) \sup_{s\le\theta\le 0} (E\|\psi(\theta)\|^2)^{\frac{1}{2}} ds < +\infty\Big\}.$$

If $\mathcal{B}_h$ is endowed with the norm,

$$\|\psi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s\le\theta\le 0} (E\|\psi(\theta)\|^2)^{\frac{1}{2}} ds, \quad \text{for all} \quad \psi \in \mathcal{B}_h,$$

then, it is easy to see that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space [30].

Now, we consider the space,

$$\mathcal{B}_h' = \Big\{x : (-\infty, b] \to H \text{ such that } x_k \in C(J_k, H) \quad \text{and there exist} \quad x(t_k^+)$$
$$\text{and} \quad x(t_k^-)\text{with} \quad x(t_k^+) = x(t_k^-), \quad x_0 = \phi \in \mathcal{B}_h, k = 1, 2, \cdots, m\Big\},$$

where, x_k is the restrictions of x to $J_k = (t_k, t_{k+1}],\; k = 1, 2, \cdots, m$. Set $\|\cdot\|_b$ be a seminorm in $\mathcal{B}_h'$ defined by,

$$\|x\|_b = \|\phi\|_{\mathcal{B}_h} + \sup\{(E\|x(s)\|^2)^{\frac{1}{2}} : s \in [0, b]\}, \quad x \in \mathcal{B}_h'.$$

Next, we recall some basic definitions and lemmas which are used throughout this paper.

Lemma 2.1. *([21]) Assume that $x \in \mathcal{B}_h'$, then for $t \in J$, $x_t \in \mathcal{B}_h$. Moreover,*

$$l\Big(E\|x(t)\|^2\Big)^{\frac{1}{2}} \le \|x_t\|_{\mathcal{B}_h} \le \|x_0\|_{\mathcal{B}_h} + l \sup_{s\in[0,t]} \Big(E\|x(s)\|^2\Big)^{\frac{1}{2}},$$

where $l = \int_{-\infty}^0 h(t)dt < +\infty$.

Lemma 2.2. *([17]) Let H be a Hilbert space and Φ_1, Φ_2 be the two operator on H such that*

(a) Φ_1 *is a contraction and*

(b) Φ_2 *is completely continuous.*

Then either

(i) the operator equation $\Phi_1 x + \Phi_2 x = x$ *has a solution or*

(ii) the set $G = \{x \in H : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ *is unbounded for* $\lambda \in (0,1)$.

Lemma 2.3. *([27]) Let* $v(\cdot)$, $w(\cdot) : [0,b] \to [0,\infty)$ *be continuous function. If* $w(\cdot)$ *is nondecreasing and there exist two constants* $\theta \geq 0$ *and* $0 < \alpha < 1$ *such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \quad t \in J,$$

then

$$v(t) \leq e^{\theta^n (\Gamma(\alpha))^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1} \Big(\frac{\theta b^\alpha}{\alpha}\Big)^j w(t),$$

for every $t \in [0,b]$ *and every* $n \in N$ *such that* $n\alpha > 1$ *and* $\Gamma(\cdot)$ *is the Gamma function.*

Lemma 2.4. *([38]) Suppose the following properties are satisfied.*

(i) Let $0 \leq \alpha \leq 1$. *Then* H_α *is a banach space.*

(ii) If $0 < \beta < \alpha \leq 1$, *then* $H_\alpha \subset H_\beta$ *and the imbedding is compact whenever the resolvent operator of* A *is compact.*

(iii) For every $0 < \alpha \leq 1$, *there exists a positive constant* $M_\alpha > 0$ *such that;*

$$\|(-A)^\alpha T(t)\| \leq \frac{M_\alpha}{t^\alpha}, \ for \ all \ 0 < t \leq b. \tag{2.4}$$

Definition 2.1. *A map* $F : J \times \mathcal{B}_h \to H$ *is said to be* L^2*- Caratheodory if*

(i) $t \to F(t,v)$ *is a measurable for each* $v \in \mathcal{B}_h$;

(ii) $v \to F(t,v)$ *is continuous for almost all* $t \in J$;

(iii) for each $q > 0$, *there exist* $h_q \in L^1(J, R_+)$ *such that*

$$\|F(t,v)\|^2 = \sup_{f \in F(t,v)} E\|f\|^2 \leq h_q(t), \quad forall \quad \|v\|^2_{\mathcal{B}_h} \leq q \quad and \ for \ a.e. \ t \in J.$$

Definition 2.2. *An* $\mathcal{F}_t$*-adapted stochastic process* $x : (-\infty, b] \to H$ *is called mild solution of the system (1.1)-(1.3) if* $x_0 = \phi \in \mathcal{B}_h$ *satisfying* $x_0 \in L^0_2(\Omega, H)$, *for each* $s \in [0,b)$ *the function* $AT(t-s)e\Big(s, x_s, \int_0^s h_1(s,\tau,x_\tau)d\tau\Big)$ *is integrable and the following conditions hold:*

(i) $\{x_t : t \in J\}$ *is* $\mathcal{B}_h$ *valued and the restrictions of* $x(\cdot)$ *to the interval* $(t_k, t_{k+1}]$, $k = 1,2,\ldots,m$ *is continuous;*

(ii) $\Delta x(t_k) = I_k(x_{t_k}), \quad k = 1,2,\ldots,m;$

(iii) for each $t \in J$, $x(t)$ *satisfies the following integral equation*

$$\begin{aligned} x(t) &= T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t T(t-s)f(s,x_s)ds + \int_0^t AT(t-s)g(s,x_s)ds \\ &+ \int_0^t AT(t-s)e\Big(s,x_s,\int_0^s h_1(s,\tau,x_\tau)d\tau\Big)ds + \int_0^t T(t-s)\sigma\Big(s,x_s,\int_0^s h_2(s,\tau,x_\tau)d\tau\Big)dw(s) \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)), \quad t \in J. \end{aligned} \tag{2.5}$$

3 Existence Results

In this section, we present and prove the existence results for the problem (1.1) – (1.3). In order to prove the main theorem of this section, we list the following hypotheses:

(H₁) The function $f : J \times \mathcal{B}_h \to X$ satisfies the following coditions:

(i) For $x : (-\infty, b] \to H$ such that $x_0 \in \mathcal{B}_h$ and $x|_J \in \mathcal{B}'_h$, the function $t \to f(t, x_t)$ is strongly measurable. i.e., $f(., x_t) : J \to H$ is a strongly measurable.

(ii) For each $t \in J$, the function $f(t, .) : \mathcal{B}_h \to H$ is continuous.

(iii) There exists integrable function $m(t) : J \to [0, \infty)$ and a continuous nondecreasing function Ω : $[0, \infty) \to (0, \infty)$ such that,

$$E\|f(t, \psi)\|^2 \leq m(t)\Omega_1(E\|\psi\|^2_{\mathcal{B}_h}); \quad (t, \psi) \in J \times \mathcal{B}_h.$$

(H₂) A is the infinitesimal generator of a compact analytic semigroup and $0 \in \rho(A)$ such that

$$\|T(t)\|^2 \leq M_1, \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad \|(-A)^{1-\beta}T(t-s)\|^2 \leq \frac{M^2_{1-\beta}}{(t-s)^{2(1-\beta)}} \quad 0 \leq t \leq b.$$

(H₃) There exists a constant $M_{h_1} \geq 0$, such that

$$\| \int_0^t [h_1(t, s, x) - h_1(t, s, y)]\|^2 \leq M_{h_1}\|x - y\|^2_{\mathcal{B}_h}.$$

(H₄)) There exists constants $0 < \beta < 1$, such that e is H_β-valued, $(-A)^\beta e : J \times \mathcal{B}_h \to H$ is completely continuous,

(i) The function $e : J \times \mathcal{B}_h \times H \to H$ for $t \in J$, $x_1, x_2 \in \mathcal{B}_h$ and $y_1, y_2 \in H$ such that the function M_e satisfies the Lipschitz condition:

$$E\|(-A)^\beta e(t, x_1, y_1) - (-A)^\beta e(t, x_2, y_2)\|^2 \leq M_e[\|x_1 - x_2\|^2_{\mathcal{B}_h} + \|y_1 - y_2\|^2].$$

Let $\tilde{c}_1 = b \ \sup_{t\in J} \|h_1(t, s, 0)\|^2$, $\tilde{c}_2 = \|(-A)^\beta\|^2 \sup_{t\in J} \|e(t, 0, 0)\|^2$, $\|(-A)^{-\beta}\|^2 = M_0$.

(ii) There exist constants $0 < \beta < 1$, C_0, c_1, c_2, M_g such that g is H_β-valued, $(-A)^\beta g$ is continuous, and

$$E\|(-A)^\beta g(t, x)\|^2 \leq c_1\|x\|^2_{\mathcal{B}_h} + c_2, \quad t \in J, \quad x \in \mathcal{B}_h,$$
$$E\|(-A)^\beta g(t, x_1) - (-A)^\beta g(t, x_2)\|^2 \leq M_g\|x - y\|^2_{\mathcal{B}_h}, \quad t \in J, \quad x_1, x_2 \in \mathcal{B}_h, \text{with}$$
$$C_0 \equiv l^2 \left\{M_g M_0 + \left[M_g + M_e(1 + M_{h_1})\right]\frac{(M_{1-\beta}b^\beta)^2}{2\beta - 1}\right\} < 1.$$

(H₅) There exist constants d_k such that $\|I_k(x)\|^2 \leq d_k$, k= 1, 2, ..., m, for each $x \in H$.

(H₆) Foe each $(t, s) \in J \times J$, the function $h_2(t, s, \cdot) : \mathcal{B}_h \to H$ is continuous for each $x \in \mathcal{B}_h$, the function $h_2(\cdot, \cdot, x) : J \times J \to H$ is strongly measurable. There exists an integrable function $m : J \to [0, \infty)$ and a constant $\gamma \geq 0$, such that

$$\|h_2(t, s, x)\|^2 \leq \gamma m(s)\Omega_3(\|x\|^2_{\mathcal{B}_h}),$$

where $\Omega_3 : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing functions. Let us assume that the finite bound of $\int_0^t \gamma m(s)ds$ is L_0.

(H₇) The function $\sigma : J \times \mathcal{B}_h \times H \to H$ satisfies the following Caratheodory conditions:

(i) $t \to \sigma(t, x, y)$ is measurable for each $(x, y) \in \mathcal{B}_h \times H$,

(ii) $(x, y) \to \sigma(t, x, y)$ is continuous for almost all $t \in J$.

($\mathbf{H_8}$) $E\|\sigma(t,x,y)\|^2 \le p(t)\Omega_2(\|x\|^2_{\mathcal{B}_h} + \|y\|^2)$ for almost all $t \in J$ and all $x \in \mathcal{B}_h$, $y \in H$, where $p \in L^2(J, R_+)$ and $\Omega_2 : R_+ \to (0,\infty)$ is continuous and increasing with

$$\widehat{m}(s) \le \int_{B_0K_1}^{\infty} \frac{ds}{\Omega_1(s)+\Omega_2(s)+\Omega_3(s)}, \quad \text{where}$$

$$N_0 = 2l^2\Big\{64\|(-A)^{-\beta}\|^2 c_1\Big\}, \tag{3.6}$$

$$N_1 = 2\|\phi\|^2_{\mathcal{B}_h} + 2l^2\overline{F}, \tag{3.7}$$

$$N_2 = 128 l^2 b M^2_{1-\beta}(c_1 + M_e(1+M_{h_1})),$$

$$\widehat{m}(t) = max[B_0K_3m(t), B_0K_4p(t), \gamma m(t)],$$

$$B_0 = e^{K_2^n(\Gamma(2\beta-1))^n b^{n2\beta-1}/\Gamma(n(2\beta-1))} \sum_{j=0}^{n-1}\Big(\frac{K_2 b^{2\beta-1}}{2\beta-1}\Big)^j,$$

$$N_3 = 128 l^2 M_1, \qquad N_4 = 128 l^2 M_1 Tr(Q), \tag{3.8}$$

$$K_1 = \frac{N_1}{(1-N_0)}, \quad K_2 = \frac{N_2}{(1-N_0)}, \quad K_3 = \frac{N_3}{(1-N_0)}, \quad K_4 = \frac{N_4}{(1-N_0)}, \tag{3.9}$$

$$\overline{F} = 64M_1\|\phi\|^2_{\mathcal{B}_h} + 64\|(-A)^{-\beta}\|^2 M_1(c_1\|\phi\|^2_{\mathcal{B}_h} + c_2) + 64\|(-A)^{-\beta}\|^2 M_g c_2$$
$$+ 64\frac{M^2_{1-\beta}c_2 b^{2\beta}}{2\beta-1} + 64(M_e\tilde{c}_1 + \tilde{c}_2)\frac{M^2_{1-\beta}b^{2\beta}}{2\beta-1} + 64M_1\sum_{k=1}^{m} d_k. \tag{3.10}$$

We consider the operator $\Phi : \mathcal{B}'_h \to \mathcal{B}'_h$ defined by

$$\Phi x(t) = \begin{cases} \phi(t), \ t \in (-\infty, 0], \\ T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t T(t-s)f(s,x_s)ds \\ \quad + \int_0^t AT(t-s)g(s,x_s)ds \\ \quad + \int_0^t AT(t-s)e\Big(s, x_s, \int_0^s h_1(s,\tau,x_\tau)d\tau\Big)ds \\ \quad + \int_0^t T(t-s)\sigma\Big(s, x_s, \int_0^s h_2(s,\tau,x_\tau)d\tau\Big)dw(s) \\ \quad + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)), \quad t \in J. \end{cases} \tag{3.11}$$

From, hypothesis ($\mathbf{H_3}$) – (**H4**) and Lemma 2.4, the following inequality holds:

$$\|AT(t-s)e(s,x_s,\int_0^t h_1(s,\tau,x_\tau)d\tau)\|^2 \le \|(-A)^{1-\beta}T(t-s)(-A)^{\beta}e(s,x_s,\int_0^t h_1(s,\tau,x_\tau)d\tau)\|^2$$
$$\le \frac{M^2_{1-\beta}}{(t-s)^{2(1-\beta)}}\big[M_e(1+M_{h_1})\|x_s\|^2_{\mathcal{B}_h} + M_e\tilde{c}_1 + \tilde{c}_2].$$

Then, from the Bochner theorem, it follows that $AT(t-s)e(s,x_s,\int_0^t h_1(s,\tau,x_\tau)d\tau)$ is integrable on $[0,t)$. For $\phi \in \mathcal{B}_h$, we defined $\tilde{\phi}$ by

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ T(t)\phi(0), & t \in J, \end{cases}$$

and then, $\tilde{\phi} \in \mathcal{B}'_h$. Let $x(t) = y(t) + \tilde{\phi}(t)$, $-\infty < t \le b$. It is easy to see that x satisfies (2.5) if and only if y satisfies $y_0 = 0$ and

$$y(t) = -T(t)g(0,\phi) + g(t, y_t + \tilde{\phi}_t) + \int_0^t T(t-s)f(s, y_s + \tilde{\phi}_s)ds$$
$$+ \int_0^t AT(t-s)g(s, y_s + \tilde{\phi}_s)ds$$
$$+ \int_0^t AT(t-s)e\Big(s, y_s + \tilde{\phi}_t, \int_0^s h_1(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\Big)ds$$

$$+\int_0^t T(t-s)\sigma\Big(s, y_s+\tilde{\phi}_s, \int_0^s h_2(s,\tau,y_\tau+\tilde{\phi})d\tau\Big)dw(s)$$
$$+\sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)+\tilde{\phi}(t_k^-)).$$

Let $\mathcal{B}_h''=\{y\in\mathcal{B}_h' : y_0=0\in\mathcal{B}_h\}$. For any $y\in\mathcal{B}_h''$, we have

$$\|y\|_b=\|y_0\|_{\mathcal{B}_h}+\sup_{0\le s\le b}(E\|y(s)\|^2)^{\frac{1}{2}}=\sup_{0\le s\le b}(E\|y(s)\|^2)^{\frac{1}{2}}.$$

Thus, $(\mathcal{B}_h'', \|\cdot\|_b)$ is a Banach space. Set

$$\mathcal{B}_q=\{y\in\mathcal{B}_h'' : \|y\|_b\le q\} \quad \text{for some } q\ge 0,$$

then $\mathcal{B}_q\subseteq\mathcal{B}_h''$ is uniformly bounded. Moreover, for $y\in\mathcal{B}_q$, from Lemma 2.1, we have

$$\begin{aligned}E(\|y_t+\tilde{\phi}_t\|^2_{\mathcal{B}_h}) &\le 2(\|y_t\|^2_{\mathcal{B}_h}+\tilde{\phi}_t\|^2_{\mathcal{B}_h})\\ &\le 2l^2\sup_{0\le s\le t}E\|y(s)\|^2+2\|y_0\|^2_{\mathcal{B}_h}+2l^2\sup_{0\le s\le t}E\|\tilde{\phi}(s)\|^2+2\|\tilde{\phi}_0\|^2_{\mathcal{B}_h}\\ &\le 2l^2(q^2+M_1E\|\phi(0)\|^2)+2\|\phi\|^2_{\mathcal{B}_h}\\ &=q'. \end{aligned} \tag{3.12}$$

Define the operator $\tilde{\Phi}:\mathcal{B}_h''\to\mathcal{B}_h''$ by

$$\tilde{\Phi}y(t)=\begin{cases}0, \quad t\in(-\infty,0],\\ -T(t)g(0,\phi)+g(t,y_t+\tilde{\phi}_t)+\int_0^t T(t-s)f(s,y_s+\tilde{\phi}_s)ds\\ +\int_0^t AT(t-s)g(s,y_s+\tilde{\phi}_s)ds\\ +\int_0^t AT(t-s)e\Big(s,y_s+\tilde{\phi}_t,\int_0^s h_1(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)ds\\ +\int_0^t T(t-s)\sigma\Big(s,y_s+\tilde{\phi}_s,\int_0^s h_2(s,\tau,y_\tau+\tilde{\phi})d\tau\Big)dw(s)\\ +\sum_{0<t_k<t}T(t-t_k)I_k(y(t_k^-)+\tilde{\phi}(t_k^-)), \quad t\in J.\end{cases}$$

Now, we decompose $\tilde{\Phi}$ as $\tilde{\Phi}_1+\tilde{\Phi}_2$ where

$$\begin{aligned}\tilde{\Phi}_1y(t)&=-T(t)g(0,\phi)+g(t,y_t+\tilde{\phi}_t)+\int_0^t AT(t-s)g(s,y_s+\tilde{\phi}_s)ds\\ &\quad+\int_0^t AT(t-s)e\Big(s,y_s+\tilde{\phi}_s,\int_0^s h_1(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)ds, \quad t\in J,\\ \tilde{\Phi}_2y(t)&=\int_0^t T(t-s)\sigma\Big(s,y_s+\tilde{\phi}_s,\int_0^s h_2(s,\tau,x_\tau+\tilde{\phi})d\tau\Big)dw(s)+\int_0^t T(t-s)f(s,y_s+\tilde{\phi}_s)ds\\ &\quad+\sum_{0<t_k<t}T(t-t_k)I_k(y(t_k^-)+\tilde{\phi}(t_k^-)), \quad t\in J.\end{aligned}$$

Obviously, the operator Φ having a fixed point is equivalent to $\tilde{\Phi}$ having one. Now, we shall show that the operator $\tilde{\Phi}_1,\tilde{\Phi}_2$ satisfy all the conditions of Lemma 2.2.

Theorem 3.1. *If assumption* $(\mathbf{H_1})-(\mathbf{H_8})$ *hold, then* $\tilde{\Phi}_1$ *is a contraction and* $\tilde{\Phi}_2$ *is completely continuous.*

Proof. Let $u,v\in\mathcal{B}_h''$. Then, we have to show that $\tilde{\Phi}_1$ is a contraction on $\mathcal{B}_h''$, we have

$$\begin{aligned}&E\|\tilde{\Phi}_1u(t)-\tilde{\Phi}_1v(t)\|^2\\ &\le E\|g(t,u_t+\tilde{\phi}_t)-g(t,v_t+\tilde{\phi}_t)\|^2+E\|\int_0^t AT(t-s)\Big[g(s,u_s+\tilde{\phi}_s)-g(s,v_s+\tilde{\phi}_s)\Big]ds\|^2\\ &\quad+E\|\int_0^t AT(t-s)\Big[e\Big(s,u_s+\tilde{\phi}_s,\int_0^s h_1(s,\tau,u_\tau+\tilde{\phi}_\tau)d\tau\Big)-e\Big(s,v_s+\tilde{\phi}_s,\int_0^s h_1(s,\tau,v_\tau+\tilde{\phi}_\tau)d\tau\Big)\Big]ds\|^2\end{aligned}$$

$$\leq 16\Big\{E\|g(t,u_t+\tilde{\phi}_t)-g(t,v_t+\tilde{\phi}_t)\|^2+E\|\int_0^t AT(t-s)[g(s,u_s+\tilde{\phi}_s)-g(s,v_s+\tilde{\phi}_s)]ds\|^2$$
$$+E\|\int_0^t AT(t-s)\Big[e\Big(s,u_s+\tilde{\phi}_s,\int_0^s h_1(s,\tau,u_\tau+\tilde{\phi}_\tau)d\tau\Big)-e\Big(s,v_s+\tilde{\phi}_s,\int_0^s h_1(s,\tau,v_\tau+\tilde{\phi}_\tau)d\tau\Big)\Big]ds\|^2\Big\}$$
$$\leq 16\Big\{M_g\|(-A)^{-\beta}\|^2E\|u_t-v_t\|^2_{\mathcal{B}_h}+M_gE\|u_t-v_t\|^2_{\mathcal{B}_h}\frac{(M_{1-\beta}b^\beta)^2}{2\beta-1}$$
$$+\frac{(M_{1-\beta}b^\beta)^2}{2\beta-1}M_e[E\|u_t-v_t\|^2_{\mathcal{B}_h}+M_{h_1}E\|u_t-v_t\|^2_{\mathcal{B}_h}]\Big\}$$
$$\leq 16\Big\{M_gM_0+\Big[M_g+M_e(1+M_{h_1})\Big]\frac{(M_{1-\beta}b^\beta)^2}{2\beta-1}\Big\}E\|u_t-v_t\|^2_{\mathcal{B}_h}$$
$$\leq 16\Big\{M_gM_0+\Big[M_g+M_e(1+M_{h_1})\Big]\frac{(M_{1-\beta}b^\beta)^2}{2\beta-1}\Big\}\Big[2l^2\sup_{s\in[0,t]}E\|u(s)-v(s)\|^2+2\|u_0\|^2_{\mathcal{B}_h}+2\|v_0\|^2_{\mathcal{B}_h}\Big]$$
$$\leq 32l^2\Big\{M_gM_0+\Big[M_g+M_e(1+M_{h_1})\Big]\frac{(M_{1-\beta}b^\beta)^2}{2\beta-1}\Big\}E\|u(s)-v(s)\|^2$$
$$\leq \sup_{s\in[0,b]}C_0E\|u(s)-v(s)\|^2.$$

Since, $\|u_0\|^2_{\mathcal{B}_h}=0$, $\|v_0\|^2_{\mathcal{B}_h}=0$. Taking the supremum over t,

$$\|\tilde{\Phi}_1u-\tilde{\Phi}_1v\|^2\leq C_0\|u-v\|^2,$$

and so, by assumption $0\leq C_0\leq 1$, we see that $\tilde{\Phi}_1$ is a contraction on $\mathcal{B}''_h$.

Now, we show that the operator $\tilde{\Phi}_2$ is completely continuous. First, we show that $\tilde{\Phi}_2$ maps bounded sets into bounded sets in $\mathcal{B}''_h$. It is enough to show that there exists a positive constants r such that for each $y\in B_q=\{y\in\mathcal{B}''_h:\|y\|^2_b\leq q\}$ one has $E\|\tilde{\Phi}_2y\|^2_b\leq r$. Now for $t\in J$,

$$\tilde{\Phi}_2y(t)=\int_0^t T(t-s)f(s,y_s+\tilde{\phi}_s)ds+\int_0^t T(t-s)\sigma\Big(s,y_s+\tilde{\phi}_s,\int_0^s h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)dw(s)$$
$$+\sum_{0<t_k<t}T(t-t_k)I_k(y(t_k^-)+\tilde{\phi}(t_k^-)),\quad t\in J.$$

Therefore, by the assumption, for each $t\in J$, we have

$$E\|\tilde{\Phi}_2y(t)\|^2\leq 9M_1\int_0^b m(s)\Omega_1(E\|y_s+\tilde{\phi}_s\|^2_{\mathcal{B}_h})ds+9M_1Tr(Q)\int_0^t p(s)\Omega_2\Big(E\|y_s+\tilde{\phi}_s\|^2_{\mathcal{B}_h}$$
$$+\int_0^s\gamma m(\tau)\Omega_3(E\|y_\tau+\tilde{\phi}_\tau\|^2_{\mathcal{B}_h})d\tau\Big)ds+9M_1\sum_{k=1}^m d_k$$
$$\leq 9M_1\Omega_1(q')\int_0^b m(s)ds+9M_1Tr(Q)\Omega_2(q'+L_0\Omega_3(q'))\int_0^b p(s)ds+9M_1\sum_{k=1}^m d_k$$
$$=r.$$

Then, for each $y\in\tilde{\Phi}_2y(B_q)$, we have $\|\tilde{\Phi}_2y\|^2_b\leq r$.

Next, we show that $\tilde{\Phi}_2$ maps bounded set into equicontinuous sets of $\mathcal{B}''_h$.
Let $0<\tau_1<\tau_2\leq b$. Then for each $y\in\mathcal{B}_q=\{y\in\mathcal{B}''_h:\|y\|_b\leq q\}$ and $y\in\tilde{\Phi}_2y$. Then for each $t\in J$, we have

$$\tilde{\Phi}_2y(t)=\int_0^t T(t-s)f(s,y_s+\tilde{\phi}_s)ds+\int_0^t T(t-s)\sigma\Big(s,y_s+\tilde{\phi}_s,\int_0^s h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)dw(s)$$
$$+\sum_{0<t_k<t}T(t-t_k)I_k(y(t_k^-)+\tilde{\phi}(t_k^-)),\quad t\in J.$$

Let $\tau_1,\tau_1\in J-\{t_1,t_2,\cdots,t_m\}$. Then, we have

$$E\|\tilde{\Phi}_2y(\tau_2)-\tilde{\Phi}_2y(\tau_1)\|^2$$
$$\leq 9E\|\int_0^{\tau_1-\epsilon}[T(\tau_2-s)-T(\tau_1-s)]f(s,y_s+\tilde{\phi}_s)ds\|^2$$

$$
\begin{aligned}
&+9E\Big\|\int_{\tau_1-\epsilon}^{\tau_1}[T(\tau_2-s)-T(\tau_1-s)]f(s,y_s+\tilde{\phi}_s)ds\Big\|^2\\
&+9E\Big\|\int_{\tau_1}^{\tau_2}[T(\tau_2-s)]f(s,y_s+\tilde{\phi}_s)ds\Big\|^2\\
&+9E\Big\|\int_0^{\tau_1-\epsilon}[T(\tau_2-s)-T(\tau_1-s)]\sigma\Big(s,y_s+\tilde{\phi}_s,\int_0^s h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)dw(s)\Big\|^2\\
&+9E\Big\|\int_{\tau_1-\epsilon}^{\tau_1}[T(\tau_2-s)-T(\tau_1-s)]\sigma\Big(s,y_s+\tilde{\phi}_s,\int_0^s h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)dw(s)\Big\|^2\\
&+9E\Big\|\int_{\tau_1}^{\tau_2}[T(\tau_2-s)]\sigma\Big(s,y_s+\tilde{\phi}_s,\int_0^s h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)dw(s)\Big\|^2\\
&+9E\Big\|\sum_{0<t_k<\tau_1}\|[T(\tau_2-t_k)-T(\tau_1-t_k)]I_k(y(t_k^-)+\tilde{\phi}(t_k^-))\Big\|^2\\
&+9E\Big\|\sum_{\tau_1\le t_k<\tau_2}\|T(\tau_2-t_k)I_k(y(t_k^-)+\tilde{\phi}(t_k^-))\Big\|^2\\
\le\ &9\int_0^{\tau_1-\epsilon}E\|T(\tau_2-s)-T(\tau_1-s)\|^2h_{q'}(s)+9\int_{\tau_1-\epsilon}^{\tau_1}E\|T(\tau_2-s)-T(\tau_1-s)\|^2h_{q'}(s)ds\\
&+9\int_{\tau_1}^{\tau_2}E\|T(\tau_2-s)\|^2h_{q'}(s)ds+9bTr(Q)\int_0^{\tau_1-\epsilon}E\|T(\tau_2-s)-T(\tau_1-s)\|^2p(s)\Omega_2(q')ds\\
&+9bTr(Q)\int_{\tau_1-\epsilon}^{\tau_1}E\|T(\tau_2-s)-T(\tau_1-s)\|^2p(s)\Omega_2(q')ds\\
&+9bTr(Q)\int_{\tau_1}^{\tau_2}E\|T(\tau_2-s)\|^2p(s)\Omega_2(q')ds+9\sum_{0<t_k<\tau_1}E\|T(\tau_2-t_k)-T(\tau_1-t_k)\|^2d_k\\
&+9M_1\sum_{\tau_1\le t_k<\tau_2}d_k.
\end{aligned}
$$

The right-hand side of the above inequality is independent of $y\in B_q$ tends to zero as $\tau_2-\tau_1\to 0$, and for ϵ sufficiently small, since the compactness of $\{T(t)\}_{t\ge 0}$ implies the continuity in the uniform operator topology. Thus the set $\{\tilde{\Phi}_2 y : y\in B_q\}$ is equicontinuous. Here we consider only the case $0<\tau_1\le\tau_2\le b$, since the other cases $\tau_1\le\tau_2\le 0$ 0r $\tau_1\le 0\le\tau_2\le b$ are very simple.

Next, we show that $\tilde{\Phi}_2:\mathcal{B}_h''\to\mathcal{B}_h''$ is continuous.

Let $\{y^{(n)}(t)\}_{n=0}^{\infty}\subseteq\mathcal{B}_h''$, with $y^{(n)}\to y$ in $\mathcal{B}_h''$. Then, there is a number $q\ge 0$ such that $|y^{(n)}(t)|\le q$ for all n and a.e. $t\in J$, so $y^{(n)}\in B_q$ and $y\in B_q$. Using (3.12), we have $\|y_t^{(n)}+\tilde{\phi}_t\|_{\mathcal{B}_h}^2\le q', t\in J$. By Definition 2.1, $(\mathbf{H_8})$, I_k, $k=1,2,\cdots,m$, is continuous

$$
\begin{gathered}
f(t,y_t^{(n)}+\tilde{\phi}_t)\to f(t,y_t+\tilde{\phi}_t),\\
\sigma\Big(t,y_t^{(n)}+\tilde{\phi}_t,\int_0^t h_2(s,\tau,y_\tau^{(n)}+\tilde{\phi}_\tau)d\tau\Big)\to\sigma\Big(t,y_t+\tilde{\phi}_t,\int_0^t h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big),
\end{gathered}
$$

for each $t\in J$, and since

$$
\begin{aligned}
&E\|f(t,y_t^{(n)}+\tilde{\phi}_t)-f(t,y_t+\tilde{\phi}_t)\|^2\le 2\alpha_{q'}(t),\\
&E\Big\|\sigma\Big(t,y_t^{(n)}+\tilde{\phi}_t,\int_0^t h_2(s,\tau,y_\tau^{(n)}+\tilde{\phi}_\tau)d\tau\Big)-\sigma\Big(t,y_t+\tilde{\phi}_t,\int_0^t h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)\Big\|^2\le 2p(t)\Omega_2(q').
\end{aligned}
$$

By the dominated convergence theorem that,

$$
\begin{aligned}
E\|\tilde{\Phi}_2y^{(n)}-\tilde{\Phi}_2y\|^2=\sup_{t\in J}E\Big\|&\int_0^t T(t-s)\Big[\sigma\Big(t,y_s^{(n)}+\tilde{\phi}_s,\int_0^t h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)\\
&-\sigma\Big(t,y_s+\tilde{\phi}_s,\int_0^t h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)\Big]dw(s)\\
&+\int_0^t T(t-s)\Big[f(t,y_s^{(n)}+\tilde{\phi}_s)-f(t,y_s+\tilde{\phi}_s)\Big]ds\\
&+\sum_{0\le t_k<t}T(t-t_k)\Big[I_k(y^n(t_k^-)+\tilde{\phi}(t_k^-))-I_k(y(t_k^-)+\tilde{\phi}(t_k^-))\Big]\Big\|^2
\end{aligned}
$$

$$\begin{aligned}
&\leq M_1 Tr(Q)\int_0^t E\|\sigma\Big(t, y_s^{(n)}+\tilde{\phi}_s, \int_0^t h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)\\
&\quad -\sigma\Big(t, y_s+\tilde{\phi}_s, \int_0^t h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)\|^2 ds\\
&\quad + M_1\int_0^t E\|f(t, y_s^{(n)}+\tilde{\phi}_s)-f(t,y_s+\tilde{\phi}_s)\|^2 ds\\
&\quad + \sum_{0\leq t_k<t}\|T(t-t_k)\|^2 E\|I_k(y^n(t_k^-)+\tilde{\phi}(t_k^-))-I_k(y(t_k^-)+\tilde{\phi}(t_k^-))\|^2\\
&\quad \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Thus, $\tilde{\Phi}_2$ is continuous.

Next, we show that $\tilde{\Phi}_2$ maps B_q into a precompact set in H. Let $0<t\leq b$ be fixed and ϵ be a real number satisfying $0<\epsilon\leq t$. For $y\in B_q$, we define

$$\begin{aligned}
(\tilde{\Phi}_2^{\epsilon} y)(t) &= \int_0^{t-\epsilon} T(t-s)\sigma\Big(t, y_s+\tilde{\phi}_s, \int_0^t h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)dw(s)\\
&\quad + \int_0^{t-\epsilon} T(t-s)f(t,y_s+\tilde{\phi}_s)ds + \sum_{0\leq t_k<t-\epsilon} T(t-t_k)I_k(y(t_k^-)+\tilde{\phi}(t_k^-))\\
&= T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)\sigma\Big(t, y_s+\tilde{\phi}_s, \int_0^t h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)dw(s)\\
&\quad + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)f(t,y_s+\tilde{\phi}_s)ds\\
&\quad + T(\epsilon)\sum_{0\leq t_k<t-\epsilon} T(t-t_k-\epsilon)I_k(y(t_k^-)+\tilde{\phi}(t_k^-)).
\end{aligned}$$

Since $T(t)$ is a compact operator, the set $V_\epsilon(t)=\{(\tilde{\Phi}_2^{\epsilon} y)(t): y\in B_q\}$ is relatively compact in H for every ϵ, for every $0<\epsilon<t$. Moreover, for each $y\in B_q$, we have

$$\begin{aligned}
&E\|(\tilde{\Phi}_2 y)(t)-(\tilde{\Phi}_2^{\epsilon} y)(t)\|^2\\
&\leq \int_{t-\epsilon}^t \|T(t-s)\|^2 E\|\sigma\Big(t, y_s+\tilde{\phi}_s, \int_0^t h_2(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)\|^2 dw(s)\\
&\quad + \int_{t-\epsilon}^t \|T(t-s)\|^2 E\|f(t,y_s+\tilde{\phi}_s)\|^2 ds + \sum_{t-\epsilon\leq t_k<t}\|T(t-t_k)\|^2 E\|I_k(y(t_k^-)+\tilde{\phi}(t_k^-))\|^2\\
&\leq M_1 Tr(Q)\int_{t-\epsilon}^t p(s)\Omega_2(q')d(s) + M_1\int_{t-\epsilon}^t \alpha_{q'}(s)ds + M_1\sum_{t-\epsilon\leq t_k<t} d_k.
\end{aligned}$$

Therefore,

$$E\|(\tilde{\Phi}_2 y)(t)-(\tilde{\Phi}_2^{\epsilon} y)(t)\|^2\to 0, \quad \text{as} \quad \epsilon\to 0.$$

and there are precompact sets arbitrarily close to the set $\{(\tilde{\Phi}_2 y)(t): y\in B_q\}$. Thus, the set $\{(\tilde{\Phi}_2^{\epsilon} y)(t): y\in B_q\}$ is precompact in H. Therefore, from Arzela- Ascoli theorem, the operator $\tilde{\Phi}_2$ is completely continuous.

In order to study the existence results for the problem (1.1)-(1.3), we consider the following nonlinear operator equation,

$$\begin{aligned}
x(t) &= \lambda T(t)[\phi(0)-g(0,\phi)] + \lambda g(t,x_t) + \lambda\int_0^t AT(t-s)g(s,x_s)ds\\
&\quad + \lambda\int_0^t T(t-s)f(s,x_s)ds + \lambda\int_0^t AT(t-s)e(s,x_s,\int_0^s h_1(s,\tau,x_\tau)d\tau)ds\\
&\quad + \lambda\int_0^t T(t-s)\sigma(s,x_s,\int_0^s h_2(s,\tau,x_\tau)d\tau)dw(s)\\
&\quad + \lambda\sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)), \quad t\in J, \qquad (3.13)
\end{aligned}$$

for some $0<\lambda<1$. The following lemma proves that an a priori bound exists for the solution of the above equation. □

Theorem 3.2. *If hypothesis* $(\mathbf{H_1})-(\mathbf{H_8})$ *are satisfied, then there exist an a priori bound* $K\geq 0$ *such that* $\|x_t\|^2_{\mathcal{B}_h}\leq K, t\in J$*, where* K *depends only on* b *and on the function* $\Omega_1,\Omega_2,\hat{m}$ *and* Ω_3.

Proof. From (3.13), we have

$$\begin{aligned}E\|x(t)\|^2 &\leq 64M_1\|\phi\|^2_{\mathcal{B}_h}+64\|(-A)^{-\beta}\|^2M_1(c_1\|\phi\|^2_{\mathcal{B}_h}+c_2)\\ &+64\|(-A)\|^{-\beta}\|^2M_g(c_1\|x_t\|^2_{\mathcal{B}_h}+c_2)+64\left[\int_0^t\frac{M^2_{1-\beta}bc_1}{(t-s)^{2(1-\beta)}}\|x_s\|^2_{\mathcal{B}_h}ds+\frac{M^2_{1-\beta}c_2b^{2\beta}}{2\beta-1}\right]\\ &+64M_1\int_0^t m(s)\Omega_1(\|x_s\|^2_{\mathcal{B}_h})ds+64(M_e\tilde{c}_1+\tilde{c}_2)\frac{M^2_{1-\beta}b^{2\beta}}{2\beta-1}\\ &+64bM^2_{1-\beta}M_e(1+M_{h_1})\int_0^t\frac{\|x_s\|^2_{\mathcal{B}_h}}{(t-s)^{2(1-\beta)}}ds\\ &+64M_1Tr(Q)\int_0^t p(s)\Omega_2\Big(\|x_s\|^2_{\mathcal{B}_h}+\int_0^s\gamma m(\tau)\Omega_3(\|x_\tau\|^2)d\tau\Big)ds+64\ M_1\sum_{k=1}^m d_k.\end{aligned}$$

Now, we consider the function μ defined by

$$\mu(t)=\sup_{0\leq s\leq t}E\|x(s)\|^2,\quad 0\leq t\leq b.$$

From, Lemma 2.1 and the above inequality, we have

$$E\|x(t)\|^2=2\|\phi\|^2_{\mathcal{B}_h}+2l^2\sup_{0\leq s\leq t}(E\|x(s)\|^2).$$

Therefore, we get

$$\begin{aligned}\mu(t)&\leq 2\|\phi\|^2_{\mathcal{B}_h}+2l^2\Big\{\overline{F}+64\|(-A)^{-\beta}\|^2c_1\mu(t)+64bM^2_{1-\beta}c_1\int_0^t\frac{\mu(s)}{(t-s)^{2(1-\beta)}}ds\\ &+64\ M_1\int_0^t m(s)\Omega_1\mu(s)ds+64\|(-A)^{-\beta}\|^2M_e(1+M_{h_1})\mu(t)\\ &+64bM^2_{1-\beta}M_e(1+M_{h_1})\int_0^t\frac{\mu(s)}{(t-s)^{2(1-\beta)}}ds\\ &+64\ M_1Tr(Q)\int_0^t p(s)\Omega_2\Big(\mu(s)+\int_0^s\gamma m(\tau)\Omega_3(\mu(\tau))d\tau\Big)ds\Big\},\end{aligned}$$

where $\overline{F}$ is given in (3.10). Thus, we have

$$\begin{aligned}\mu(t)\leq K_1&+K_2\int_0^t\frac{\mu(s)}{(t-s)^{2(1-\beta)}}ds+K_3\int_0^t m(s)\Omega_1\mu(s)ds\\ &+K_4\int_0^t p(s)\Omega_2\Big(\mu(s)+\int_0^s\gamma m(\tau)\Omega_3(\mu(\tau))d\tau\Big)ds,\end{aligned}$$

where K_1,K_2,K_3 are given in (3.9). By Lemma 2.3, we have

$$\begin{aligned}\mu(t)\leq B_0\Bigg(&K_1+K_3\int_0^t m(s)\Omega_1\mu(s)ds\\ &+K_4\int_0^t p(s)\Omega_2\Big(\mu(s)+\int_0^s\gamma m(\tau)\Omega_3\mu(\tau)d\tau\Big)ds\Bigg),\end{aligned}$$

where

$$B_0=e^{K_2^n(\Gamma(2\beta-1))^nb^{n2\beta-1}/\Gamma(n(2\beta-1)}\sum_{j=0}^{n-1}\Big(\frac{K_2b^{2\beta-1}}{2\beta-1}\Big)^j.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then $v(0)=B_0K_1,\ \mu(t)\leq v(t), 0\leq t\leq b$, and

$$v'(t)\leq B_0\Big[K_3m(t)\Omega_1\mu(t)+K_4p(t)\Omega_2\Big(\mu(t)+\int_0^t\gamma m(s)\Omega_3(\mu(s))ds\Big)\Big].$$

Since, ψ is nondecreasing,

$$v'(t) \leq B_0\Big[K_3 m(t)\Omega_1 v(t) + K_4 p(t)\Omega_2\Big(v(t) + \int_0^t \gamma m(s)\Omega_3(v(s))ds\Big)\Big].$$

Let $w(t) = v(t) + \int_0^t \gamma m(s)\Omega_3(v(s))ds$. Then $w(0) = v(0)$ and $v(t) \leq w(t)$.

$$\begin{aligned} w'(t) &= v'(t) + \gamma m(t)\Omega_3(v(t)) \\ &\leq B_0K_3m(t)\Omega_1(w(t)) + B_0K_4p(t)\Omega_2(w(t)) + \gamma m(t)\Omega_3(w(t)) \\ &\leq \widehat{m}(t)[\Omega_1(w(t)) + \Omega_2(w(t)) + \Omega_3(w(t))]. \end{aligned}$$

This implies that,

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega_1(s) + \Omega_2(s) + \Omega_3(s)} \leq \int_0^b \widehat{m}(s) \leq \int_{B_0K_1}^{\infty} \frac{ds}{\Omega_1(s) + \Omega_2(s) + \Omega_3(s)}.$$

This implies that $v(t) < \infty$. So the inequality shows that there is a constant K such that $v(t) \leq K,\ t \in J$. So, $\|x_t\|^2_{\mathcal{B}_h} \leq \mu(t) \leq v(t) \leq K,\ t \in J$, where K depends only on b and on the functions $\Omega_1, \Omega_2, \Omega_3$ and $\widehat{m}$. □

Theorem 3.3. *Assume that the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_8})$ *hold. Then problem (1.1)-(1.3) has at least one mild solution on* J.

Proof. Let us take the set,

$$G(\tilde{\Phi}) = \{y \in \mathcal{B}''_h : y = \lambda\tilde{\Phi}_1(\frac{y}{x}) + \lambda\tilde{\Phi}_2 y, \quad \text{for some} \quad \lambda \in (0,1)\}. \tag{3.14}$$

Then, for any $y \in G(\tilde{\Phi})$, we have by Theorem 3.2 that $\|x_t\|^2_{\mathcal{B}_h} \leq K, t \in J$, and hence

$$\begin{aligned} \|y\|_b^2 &= \|y_0\|^2_{\mathcal{B}_h} + \sup\{E\|y(t)\|^2 : 0 \leq t \leq b\} \\ &= \sup\{E\|y(t)\|^2 : 0 \leq t \leq b\} \\ &\leq \sup\{E\|x(t)\|^2 : 0 \leq t \leq b\} + \sup\{\|\tilde{\phi}(t)\|^2 : 0 \leq t \leq b\} \\ &\leq \sup\{l^-\|x_t\|^2_{\mathcal{B}_h} : 0 \leq t \leq b\} + \sup\{\|T(t)\phi(0)\|^2 : 0 \leq t \leq b\} \\ &\leq l^-K + M_1\|\phi(0)\|^2. \end{aligned}$$

This implies that G is bounded on J. Consequently, by the Krasnoselski-Schaefer type fixed point theorem the operator $\tilde{\Phi}$ has a fixed point $y^* \in \mathcal{B}''_h$. Since $x(t) = y^*(t) + \tilde{\phi}(t)$, $t \in (-\infty, b]$, x is a fixed point of the operator Φ which is a mild solution of problem (1.1)-(1.3). □

4 Example

In this, we present the application for the problem (1.1)-(1.3), we consider the following impulsive neutral stochastic partial integrodifferential equation of the form

$$\begin{aligned} \frac{\partial}{\partial t}\Big[v(t,y) - \int_{-\infty}^t \int_0^\pi a(s-t,\eta,y)d\eta ds\Big] &= \frac{\partial^2}{\partial y^2}\Big[v(t,y) + \int_0^t a_1(t,y,s-t)P_1(v(s,y))ds \\ &+ \int_0^t \int_{-\infty}^s k(s-\tau)P_2(v(\tau,y))d\tau\Big]ds + k_0(y)v(t,y) + \int_0^t a_2(t,y,s-t)Q_1(v(s,y))ds \\ &+ \int_0^t \int_{-\infty}^s k(s-\tau)Q_2(v(\tau,y))d\tau d\beta(s), \quad y \in [0,\pi], \qquad t \in [0,b], \quad t \neq t_k. \end{aligned} \tag{4.1}$$

$$v(t,0) = v(t,\pi) = 0, \qquad t \geq 0, \tag{4.2}$$

$$v(t,y) = \phi(t,y), \quad t \in (-\infty, 0], \quad y \in [0,\pi], \tag{4.3}$$

$$\Delta v(t_i)(y) = \int_{-\infty}^{t_i} q_i(t_i - s)v(s,y)ds, \quad y \in [0,\pi], \tag{4.4}$$

where $0 < t_1 < \cdots < t_n < b$ are prefixed numbers and $\psi \in \mathcal{B}_h$ and $\beta(t)$ is a one-dimensional standard Wiener process. Let us take $H = L^2[0,\pi]$ with the norm $\|\cdot\|$. Define $A : H \to H$ by $A(t)z = -a(t,y)z''$ with domain,

$$D(A) = \{z(\cdot) \in H : z, z', \text{ are absolutely continuous, } z'' \in H, z(0) = z(\pi) = 0\},$$

Then

$$Az = \sum_{n=1}^{\infty} n^2 \langle z, z_n \rangle z_n, \quad z \in D(A),$$

where $z_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns)$, $n = 1, 2, \cdots$ is the orthonormal set of eigenvector of A. It is well known that A is the infinitesimal generator of an analytic semigroup $T(t),\ t \geq 0$ in H and is given by

$$T(t)z = \sum_{n=1}^{\infty} exp^{-n^2 t} \langle z, z_n \rangle z_n, \quad z \in H.$$

For every $z \in H$, $(-A)^{\frac{1}{2}} z = \sum_{n=1}^{\infty} \frac{1}{n} \langle z, z_n \rangle z_n$, and $\|(-A)^{\frac{1}{2}}\|^2 = 1$. The operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}} z = \sum_{n=1}^{\infty} n \langle z, z_n \rangle z_n,$$

on the space $D((-A)^{\frac{1}{2}}) = \{z \in H : \sum_{n=1}^{\infty} n \langle z, z_n \rangle z_n \in H\}$. Since, the analytic semigroup $T(t)$ is compact [38], there exists a constant $M_1 \geq 0$ such that $\|T(t\|^2 \leq M_1$ and satisfies (H_2).

Now, we give a special $\mathcal{B}_h$- space. Let $h(s) = e^{2s}, s \leq 0$, then $l = \displaystyle\int_{-\infty}^{0} h(s)ds = \frac{1}{2}$ and let

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} E\Big(\|\phi(\theta)\|^2\Big)^{\frac{1}{2}} ds.$$

It follows from [30], that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.
Hence, for $(t,\phi) \in [0,b] \times \mathcal{B}_h$, where $\phi(\theta)(y) = \phi(\theta, y), (\theta, y) \in (-\infty, 0] \times [0,\pi]$. Set

$$\begin{aligned} v(t)(y) = v(t,y), \quad g(t,\phi)y &= \int_{-\infty}^{0} \int_{0}^{\phi} a(s-t,\eta,y)d\eta ds, \\ f(t,\phi)(y) &= k_0(y)\phi(t,y), \\ b(t,\phi,B_1\phi)(y) &= \int_{-\infty}^{0} a_1(t,y,\theta)P_1(\phi(\theta)(y))d\theta + B_1\phi(y), \end{aligned}$$

and

$$\sigma(t,\phi,B_2\phi)(y)) = \int_{-\infty}^{0} a_2(t,y,\theta)Q_1(\phi(\theta)(y))d\theta + B_2\phi(y),$$

where

$$\begin{aligned} B_1\phi(y) &= \int_0^t \int_{-\infty}^0 k(s-\theta)P_2(\phi(\theta)(y))d\theta ds, \\ B_2\phi(y) &= \int_0^t \int_{-\infty}^0 k(s-\theta)Q_2(\phi(\theta)(y))d\theta d\beta(s). \end{aligned}$$

Then, the above equation can be written in the abstract form as system (1.1)-(1.3). The function a_1, k and P_1, P_2 are assumed to satisfy the conditions of [27] and $q_i : R \to R$ are continuous and $d_i = \displaystyle\int_{-\infty}^{0} h(s)q_i^2(s)ds < \infty$ for $i = 1, 2, \cdots, n$. Moreover, $e([0,b] \times \mathcal{B}_h \times L^2) \subseteq D((-A)^{\frac{1}{2}})$ and $\|(-A)^{\frac{1}{2}} e(t,\phi_1,u_1)(y) - (-A)^{\frac{1}{2}} e(t,\phi_2,u_2)(y)\|^2 \leq M_e[\|\phi_1 - \phi_2\|^2_{\mathcal{B}_h} + |u_1 - u_2|^2]$ for some constants $M_e > 0$ depending on a_1, k, P_1, P_2 and $|u_1 - u_2|^2 = \|B_1\phi_1 - B_1\phi_2\|^2 \leq M_{h_1}\|\phi_1 - \phi_2\|^2_{\mathcal{B}_h}$ for $M_{h_1} > 0$ such that $\frac{1}{2}M_e(1 + M_{h_1})(1 + 2C_{\frac{1}{2}}\sqrt{e}) < 1$.
Suppose further that :

(i) The function $a_2(t, y, \theta)$ is continuous in $[0, b] \times [0, \pi] \times (-\infty, 0]$ and $a_2(t, y, \theta) \geq 0$, $\int_{-\infty}^{0} a_2(t, y, \theta)d\theta = p_1(t, y) < \infty$.

(ii) The function $k(t-s)$ is continuous in $[0, b]$ and $k(t-s) \geq 0$, $\int_0^t \int_{-\infty}^0 k(s-\theta)d\theta ds = p_2(t) < \infty$.

(iii) The function $Q_i(\cdot), i = 1, 2$ are continuous and for each $(\theta, y) \in (-\infty, 0] \times [0, \pi], 0 \leq Q_i(v(\theta)(y)) \leq \Phi(\int_{-\infty}^0 e^{2s}\|v(s, \cdot)\|_{L_2} ds)$, where $\Phi : [0, +\infty) \to (0, +\infty)$ is a continuous and nondecreasing function.

Now, we can see that,

$$\begin{aligned}
& E|\sigma(t, \phi, B_2\phi)|_{L_2} \\
&= \Big[\int_0^\pi \Big(\int_{-\infty}^0 a_2(t, y, \theta)Q_1(\phi(\theta)(y))d\theta + B_2\phi(\theta)(y)\Big)^2 dy\Big]^{\frac{1}{2}} \\
&\quad \leq \sqrt{2}\Big[\int_0^\pi \Big(\int_{-\infty}^0 a_2(t, y, \theta)\Phi\Big(\int_{-\infty}^0 e^{2s}\|\phi(s), (\cdot)\|_{L_2} ds\Big)d\theta\Big)^2 dy\Big]^{\frac{1}{2}} \\
&\quad + \sqrt{2}\Big[\int_0^\pi \Big(\int_0^t \int_{-\infty}^0 k(\tau-\theta)\Phi\Big(\int_{-\infty}^0 e^{2s}\|\phi(s), (\cdot)\|_{L_2} ds\Big)d\theta d\beta(\tau)\Big)^2 dy\Big]^{\frac{1}{2}} \\
&\leq \sqrt{2}\Big[\int_0^\pi \Big(\int_{-\infty}^0 a_2(t, y, \theta)\Phi\Big(\int_{-\infty}^0 e^{2s} \sup_{s\in[\theta,0]} \|\phi(s)\|_{L_2} ds\Big)d\theta\Big)^2 dy\Big]^{\frac{1}{2}} \\
&\quad + \sqrt{2}\ Tr(Q)\Big[\int_0^\pi \Big(\int_0^t \int_{-\infty}^0 k(\tau-\theta)\Phi\Big(\int_{-\infty}^0 e^{2s} \sup_{s\in[\theta,0]} \|\phi(s)\|_{L_2} ds\Big)d\theta d\tau\Big)^2 dy\Big]^{\frac{1}{2}} \\
&= \sqrt{2}\Big[\int_0^\pi \Big(\int_{-\infty}^0 a_2(t, y, \theta)d\theta\Big)^2 dy\Big]^{\frac{1}{2}} \Phi(\|\phi\|_h^2) \\
&\quad + \sqrt{2}\ Tr(Q)\Big[\int_0^\pi \Big(\int_0^t \int_{-\infty}^0 k(s-\theta)d\theta ds\Big)^2 dy\Big]^{\frac{1}{2}} \Phi(\|\phi\|_h^2) \\
&= \sqrt{2}\bigg(\Big[\int_0^\pi (p_1(t, y))^2 dy\Big]^{\frac{1}{2}} + Tr(Q)\Big[\int_0^\pi (p_2(t, y))^2 dy\Big]^{\frac{1}{2}}\bigg)\Phi(\|\phi\|_h^2) \\
&= \sqrt{2}[\bar{p}_1(t) + \sqrt{\pi}Tr(Q)\bar{p}_2(t)]\Phi(\|\phi\|_h^2).
\end{aligned}$$

Since, $\Phi : [0, +\infty) \to [0, +\infty)$ is continuous and nondecreasing functions, we can take $p(t) = \sqrt{2}[\bar{p}_1(t) + \sqrt{\pi}Tr(Q)\bar{p}_2(t)]$ and $\Omega_2(r) = \Omega_3(r) = \Phi(r)$ in $(\mathbf{H_8})$. If $(\mathbf{H_5})$, $(\mathbf{H_7})$ and the bounds in $(\mathbf{H_8})$ are satisfied then equations (4.1)-(4.4) have a mild solution on $[0, b]$.

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Existence and stability for fractional order integral equations with multiple time delay in Fréchet spaces

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Abstract

In this paper, we present some results concerning the existence of solutions for a system of integral equations of Riemann-Liouville fractional order with multiple time delay in Fréchet spaces, we use an extension of the Burton-Kirk fixed point theorem. Also we investigate the stability of solutions of this system.

Keywords: Functional integral equation, left-sided mixed Riemann-Liouville integral of fractional order, solution, stability, multiple time delay, Fréchet space, fixed point.

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1 Introduction

Integral equations occur in mechanics and many related fields of engineering and mathematical physics and others. They also form one of useful mathematical tools in many branches of pure analysis such as functional analysis [21, 27, 29]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [7], Baleanu et al. [12], Kilbas et al. [22], Lakshmikantham et al. [23], Podlubny [26]. Recently some interesting results on the attractivity of the solutions of some classes of integral equations have been obtained by Abbas et al. [1, 2, 3, 5, 6, 8], Banaś et al. [13, 14, 15], Darwish et al. [16], Dhage [17, 18, 19], Pachpatte [24, 25] and the references therein.

In [10], Avramescu and Vladimirescu presented an existence result of asymptotically stable solutions for the integral equation

$$x(t) = q(t) + \int_0^t K(t,s,x(s))ds + \int_0^\infty G(t,s,x(s))ds; \text{ if } t \in R_+. \tag{1.1}$$

They used two fixed point theorems in Fréchet spaces, the Banach's contraction principle and the fixed point theorem of Burton-Kirk. In [11], the same authors studied the existence and the stability of solutions of the integral equation

$$x(t) = f(t,x(t)) + \int_0^{\nu(t)} u(t,s,x(\mu(s)))ds; \text{ if } t \in R_+, \tag{1.2}$$

by using the Schauder-Tychonoff fixed point theorem (see, e.g., [29]) in some Fréchet spaces. Recently, in [4], Abbas and Benchohra investigated the existence and uniqueness of solutions for the following fractional order integral equations for the system

$$u(x,y) = \sum_{i=1}^m g_i(x,y)u(x-\xi_i, y-\mu_i) + I_\theta^r f(x,y,u(x,y)); \text{ if } (x,y) \in J_1, \tag{1.3}$$

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$$u(x,y) = \Phi(x,y);\ \text{if } (x,y) \in \tilde{J}_1 := [-\xi, a] \times [-\mu, b] \backslash (0,a] \times (0,b], \tag{1.4}$$

where $J_1 = [0,a] \times [0,b],\ a,b > 0,\ \theta = (0,0),\ \xi_i, \mu_i \geq 0;\ i = 1\ldots,m,\ \xi = \max_{i=1\ldots,m}\{\xi_i\},\ \mu = \max_{i=1\ldots,m}\{\mu_i\},\ I_\theta^r$ is the left-sided mixed Riemann-Liouville integral of order $r = (r_1, r_2) \in (0,\infty) \times (0,\infty),\ f : J_1 \times R^n \to R^n,\ g_i : J_1 \to R;\ i = 1\ldots m$ are given continuous functions, and $\Phi : \tilde{J}_1 \to R^n$ is a given continuous function such that $\Phi(x,0) = \sum_{i=1}^m g_i(x,0)\Phi(x-\xi_i, -\mu_i)$ and $\Phi(0,y) = \sum_{i=1}^m g_i(0,y)\Phi(-\xi_i, y-\mu_i)$.

Motivated by those papers, this work deals with the existence and the stability of solutions of a class of functional integral equations of Riemann-Liouville fractional order with multiple time delay. We establish some sufficient conditions for the existence and the stability of solutions of the following fractional order integral equations for the system

$$u(t,x) = \sum_{i=1}^m g_i(t,x)u(t-\tau_i, x-\xi_i) + f(t,x,I_\theta^r u(t,x), u(t,x));\ (t,x) \in J, \tag{1.5}$$

$$u(t,x) = \Phi(t,x);\ \text{if } (t,x) \in \tilde{J} := [-\tau, \infty) \times [-\xi, b] \backslash (0,\infty) \times (0,b], \tag{1.6}$$

where $J := R_+ \times [0,b],\ b > 0,\ R_+ = [0,\infty),\ \theta = (0,0),\ r = (r_1,r_2),\ r_1,r_2 \in (0,\infty),\ \tau_i, \xi_i \geq 0;\ i = 1\ldots,m,\ \tau = \max_{i=1\ldots,m}\{\tau_i\},\ \xi = \max_{i=1\ldots,m}\{\xi_i\},\ f : J \times R \times R \to R,\ g_i : J \to R_+;\ i = 1\ldots m,\ \Phi : \tilde{J} \to R$ are given continuous functions such that

$$\Phi(t,0) = \sum_{i=1}^m g_i(t,0)\Phi(t-\tau_i, -\xi_i) + f(t,0,0,\Phi(t,0));\ t \in [0,\infty),$$

and

$$\Phi(0,x) = \sum_{i=1}^m g_i(0,x)\Phi(-\tau_i, x-\xi_i) + f(0,x,0,\Phi(0,x));\ x \in [0,b].$$

Our investigations are conducted in Fréchet spaces with an application of the fixed point theorem of Burton-Kirk for the existence of solutions of our problem, and we prove that all solutions are globally asymptotically stable. Also, we present an example illustrating the applicability of the imposed conditions.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([0,a] \times [0,b])$, for $a,b > 0$, we denote the space of Lebesgue-integrable functions $u : [0,a] \times [0,b] \to R$ with the norm

$$\|u\|_1 = \int_0^a \int_0^b |u(t,x)| dx dt.$$

As usual, $\mathcal{C} := C([-\tau,\infty) \times [-\xi, b])$ is the space of all continuous functions from $[-\tau,\infty) \times [-\xi,b]$ into R.

Definition 2.1. *([28]) Let $r = (r_1,r_2) \in (0,\infty) \times (0,\infty),\ \theta = (0,0)$ and $u \in L^1([0,a] \times [0,b]);\ a,b > 0$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} u(s,y) dy ds,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\zeta) = \int_0^\infty t^{\zeta-1}e^{-t}dt;\ \zeta > 0$.

In particular, for almost all $(t,x) \in [0,a] \times [0,b]$,

$$(I_\theta^\theta u)(t,x) = u(t,x),\ \text{and } (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x u(s,y) dy ds,$$

where $\sigma = (1,1)$.
For instance, $I_\theta^r u$ exists almost everywhere for all $r_1, r_2 > 0$, when $u \in L^1([0,a] \times [0,b])$. Moreover

$$(I_\theta^r u)(t,0) = 0;\ \text{for a.a. } t \in [0,a],$$

and

$$(I_\theta^r u)(0,x)=0, \text{ for a.a. } x\in[0,b].$$

Example 2.1. *Let* $\lambda,\omega\in(-1,\infty)$ *and* $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$. *Then*

$$I_\theta^r t^\lambda x^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}t^{\lambda+r_1}x^{\omega+r_2}, \text{ for a.a. } (t,x)\in[0,a]\times[0,b].$$

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}^*:=\{1,2,\ldots\}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1\le\|x\|_2\le\ \|x\|_3\le\ldots \quad \text{for every } x\in X.$$

Let $Y\subset X$, we say that Y is bounded if for every $n\in\mathbb{N}^*$, there exists $\overline{M}_n>0$ such that

$$\|y\|_n\le\overline{M}_n \quad \text{for all } y\in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n,\|\cdot\|_n)\}$ as follows: For every $n\in\mathbb{N}^*$, we consider the equivalence relation $\sim_n$ defined by: $x\sim_n y$ if and only if $\|x-y\|_n=0$ for $x,y\in X$. We denote $X^n=(X|_{\sim_n},\|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y\subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n\subset X^n$ as follows: For every $x\in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n=\{[x]_n : x\in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [20].

For each $p\in\mathbb{N}^*$ we consider following set, $C_p=C([-\tau,p]\times[-\xi,b])$, and we define in $\mathcal{C}$ the semi-norms by

$$\|u\|_p=\sup_{(t,x)\in[-\tau,p]\times[-\xi,b]}\|u(t,x)\|.$$

Then $\mathcal{C}$ is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$.

Definition 2.2. *Let X be a Fréchet space. A function $N : X\longrightarrow X$ is said to be a contraction if for each $n\in\mathbb{N}^*$ there exists $k_n\in[0,1)$ such that*

$$\|N(u)-N(v)\|_n\le k_n\|u-v\|_n \quad \text{for all } u,v\in X.$$

We need the following extension of the Burton-Kirk fixed point theorem in the case of a Fréchet space.

Theorem 2.1. *[9] Let $(X,\|.\|_n)$ be a Fréchet space and let $A,B : X\to X$ be two operators such that*

(a) *A is a compact operator;*

(b) *B is a contraction operator with respect to a family of seminorms $\{\|.\|_n\}$;*

(c) *the set $\left\{x\in X : x=\lambda A(x)+\lambda B\left(\frac{x}{\lambda}\right),\ \lambda\in(0,1)\right\}$ is bounded.*

Then the operator equation $A(u)+B(u)=u$ has a solution in X.

Let $\emptyset\ne\Omega\subset\mathcal{C}$, and let $G:\Omega\to\Omega$, and consider the solutions of equation

$$(Gu)(t,x)=u(t,x). \tag{2.1}$$

Now we introduce the concept of attractivity of solutions for our equations.

Definition 2.3. *([6, 7]) Solutions of equation (2.1) are locally attractive if there exists a ball $B(u_0,\eta)$ in the space $\mathcal{C}$ such that, for arbitrary solutions $v=v(t,x)$ and $w=w(t,x)$ of equation (2.1) belonging to $B(u_0,\eta)\cap\Omega$, we have that, for each $x\in[0,b]$,*

$$\lim_{t\to\infty}\Big(v(t,x)-w(t,x)\Big)=0. \tag{2.2}$$

When the limit (2.2) is uniform with respect to $B(u_0,\eta)\cap\Omega$, solutions of equation (2.1) are said to be uniformly locally attractive (or equivalently that solutions of (2.1) are locally asymptotically stable).

Definition 2.4. *([6, 7]) The solution $v=v(t,x)$ of equation (2.1) is said to be globally attractive if (2.2) holds for each solution $w=w(t,x)$ of (2.1). If condition (2.2) is satisfied uniformly with respect to the set Ω, solutions of equation (2.1) are said to be globally asymptotically stable (or uniformly globally attractive).*

3 Existence and Stability Results

Let us start by defining what we mean by a solution of the problem (1.5)-(1.6).

Definition 3.1. *A function $u \in \mathcal{C}$ is said to be a solution of (1.5)-(1.6) if u satisfies equation (1.5) on J and condition (1.6) on $\tilde{J}$.*

Now, we are concerned with the existence and the stability of solutions for the problem (1.5)-(1.6). Set

$$B_p = \max_{i=1\dots m}\left\{\sup_{(t,x)\in[0,p]\times[0,b]} g_i(t,x)\right\};\ p\in\mathbb{N}^*,$$

and

$$B^* = \max_{i=1\dots m}\left\{\sup_{(t,x)\in J} g_i(t,x)\right\}.$$

Theorem 3.1. *Assume that the following hypothesis holds:*

(H) The function f is continuous and there exist functions $P,Q : J \to R_+$ such that

$$|f(t,x,u,v)| \le \frac{P(t,x)|u| + Q(t,x)|v|}{1+|u|+|v|},\ \textit{for } (t,x)\in J \textit{ and } u,v\in R.$$

Moreover, assume that

$$\lim_{t\to\infty} P(t,x) = \lim_{t\to\infty} Q(t,x) = 0;\ for\ x\in[0,b].$$

If $mB_p < 1;\ p\in\mathbb{N}^*$, then the problem (1.5)-(1.6) has at least one solution in the space $\mathcal{C}$. Moreover, if the functions $g_i;\ i=1\dots m$ are bounded on J, and $mB^* < 1$, then solutions of (1.5)-(1.6) are globally asymptotically stable.

Proof. Let us define the operators $A, B : \mathcal{C} \to \mathcal{C}$ by

$$(Au)(t,x) = \begin{cases} 0; & (t,x)\in\tilde{J},\\ f(t,x,I_\theta^r u(t,x),u(t,x)); & (t,x)\in J,\end{cases} \tag{3.1}$$

$$(Bu)(t,x) = \begin{cases} \Phi(t,x); & (t,x)\in\tilde{J},\\ \sum_{i=1}^m g_i(t,x)u(t-\tau_i,x-\xi_i); & (t,x)\in J.\end{cases} \tag{3.2}$$

The problem of finding the solutions of (1.5)-(1.6) is reduced to finding the solutions of the operator equation $A(u)+B(u)=u$. We shall show that the operators A and B satisfied all the conditions of Theorem 2.1. The proof will be given in several steps.

Step 1: *A is compact.*

To this aim, we must prove that A is continuous and it transforms every bounded set into a relatively compact set. Recall that $M\subset\mathcal{C}$ is bounded if and only if

$$\forall p\in\mathbb{N}^*,\ \exists \ell_p>0 : \forall u\in M,\ \|u\|_p \le \ell_p,$$

and $M = \{u(t,x);\ (t,x))\in[-\tau,\infty)\times[-\xi,b]\}\subset\mathcal{C}$ is relatively compact if and only if for any $p\in\mathbb{N}^*$, the family $\{u(t,x)|_{(t,x)]\in[-\tau,p]\times[-\xi,b]}\}$ is equicontinuous and uniformly bounded on $[-\tau,p]\times[-\xi,b]$. The proof will be given in several claims.

Claim 1: *A is continuous.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n\to u$ in $\mathcal{C}$. Then, for each $(t,x)\in[-\tau,\infty)\times[-\xi,b]$, we have

$$|(Au_n)(t,x)-(Au)(t,x)| \le |f(t,x,I_\theta^r u_n(t,x),u_n(t,x)) - f(t,x,I_\theta^r u(t,x),u(t,x))|. \tag{3.3}$$

If $(t,x)\in[-\tau,p]\times[-\xi,b];\ p\in\mathbb{N}^*$, then, since $u_n\to u$ as $n\to\infty$, then (3.3) gives

$$\|A(u_n)-A(u)\|_p \to 0 \quad \text{as } n\to\infty.$$

Claim 2: *A maps bounded sets into bonded sets in C.*

Let M be a bounded set in $\mathcal{C}$, then, for each $p \in \mathbb{N}^*$, there exists $\ell_p > 0$, such that for all $u \in \mathcal{C}$ we have $\|u\|_p \leq \ell_p$. Then, for arbitrarily fixed $(t,x) \in [-\tau, p] \times [-\xi, b]$ we have

$$\begin{aligned} |(Au)(t,x)| &\leq |f(t,x,I_\theta^r u(t,x), u(t,x))| \\ &\leq (P(t,x)|I_\theta^r u(t,x)| + Q(t,x)|u(t,x))|) \\ &\times (1 + |I_\theta^r u(t,x)| + |u(t,x))|)^{-1} \\ &\leq P(t,x) + Q(t,x) \\ &\leq P_p + Q_p, \end{aligned}$$

where

$$P_p = \sup_{(t,x)\in[0,p]\times[0,b]} P(t,x) \;\; and \;\; Q_p = \sup_{(t,x)\in[0,p]\times[0,b]} Q(t,x).$$

Thus

$$\|A(u)\|_p \leq P_p^* + Q_p^* := \ell'_p. \tag{3.4}$$

Claim 3: *A maps bounded sets into equicontinuous sets in C.*

Let $(t_1,x_1),(t_2,x_2) \in [0,p] \times [0,b]$, $t_1 < t_2,\; x_1 < x_2$ and let $u \in M$, thus we have

$$|(Au)(t_2,x_2) - (Au)(t_1,x_1)| \leq$$

$$|f(t_2,x_2,I_\theta^r u(t_2,x_2), u(t_2,x_2)) - f(t_1,x_1,I_\theta^r u(t_1,x_1), u(t_1,x_1))|.$$

From continuity of f, I_θ^r, u and as $t_1 \to t_2,\; x_1 \to x_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 < 0$, $x_1 < x_2 < 0$ and $t_1 \leq 0 \leq t_2$, $x_1 \leq 0 \leq x_2$ is obvious. As a consequence of claims 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that A is continuous and compact.

Step 2: *B is a contraction.*

Consider $v, w \in \mathcal{C}$. Then, for any $p \in \mathbb{N}$ and each $(t,x) \in [-\tau, p] \times [-\xi, b]$, we have

$$\begin{aligned} |(Bv)(t,x) - (Bw)(t,x)| &\leq \sum_{i=1}^m g_i(t,x)|v(t-\tau_i, x-\xi_i) - w(t-\tau_i, x-\xi_i)| \\ &\leq mB_p\|v-w\|_p, \end{aligned}$$

then

$$\|(B(v) - B(w)\|_p \leq mB_p\|v-w\|_p.$$

Since $mB_p < 1;\; p \in \mathbb{N}^*$, then; the operator B is a contraction.

Step 3: *the set* $\mathcal{E} := \left\{u \in \mathcal{C} : u = \lambda A(u) + \lambda B\left(\frac{u}{\lambda}\right),\; \lambda \in (0,1)\right\}$ *is bounded.*

Let $u \in \mathcal{C}$, such that $u = \lambda A(u) + \lambda B\left(\frac{u}{\lambda}\right)$ for some $\lambda \in (0,1)$. Then, for any $p \in \mathbb{N}^*$ and each $(t,x) \in [0,p] \times [0,b]$, we have

$$\begin{aligned} |u(t,x)| &\leq \lambda|(Au)(t,x)| + \lambda\left|B\left(\frac{u(t,x)}{\lambda}\right)\right| \\ &\leq mB_p|u(t,x)| + Q(t,x) + P(t,x) \\ &\leq mB_p\|u\|_p + P_p + Q_p, \end{aligned}$$

then,

$$\|u\|_p \leq \frac{P_p + Q_p}{1 - mB_p}.$$

On the other hand, for each $(t,x) \in [-\tau, p] \times [-\xi, b] \backslash (0,p] \times (0,b]$, we get

$$|u(t,x)| \leq |\Phi(t,x)| \leq \sup_{(t,x)\in[-\tau,p]\times[-\xi,b]\backslash(0,p]\times(0,b]} |\Phi(t,x)| := \Phi_p.$$

Thus

$$\|u\|_p \leq \max\left\{\frac{P_p + Q_p}{1 - mB_p}, \Phi_p\right\} =: \ell_p^*.$$

Hence, the set $\mathcal{E}$ is bounded. As a consequence of steps 1 and 3 together with Theorem 2.1, we deduce that $A+B$ has a fixed point u in $\mathcal{C}$ which is a solution to problem (1.5)-(1.6).

Now, we show the stability of solutions of the problem (1.5)-(1.6). Let u and v be any two solutions of (1.5)-(1.6), then for each $(t,x)\in[-\tau,\infty)\times[-\xi,b]$, we have

$$\begin{aligned}|u(t,x)-v(t,x)| &= |(Au)(t,x)-(Av)(t,x)+(Bu)(t,x)-(Bv)(t,x)|\\ &\le \sum_{i=1}^{m} g_i(t,x)|u(t-\tau_i,x-\xi_i)-v(t-\tau_i,x-\xi_i)|\\ &+ |f(t,x,I_\theta^r u(t,x),u(t,x))-f(t,x,I_\theta^r v(t,x),v(t,x)))|\\ &\le mB^*|u(t,x)-v(t,x)|+2P(t,x)+2Q(t,x).\end{aligned}$$

Thus

$$|u(t,x)-v(t,x)|\le\frac{2(P(t,x)+Q(t,x))}{1-mB^*}. \tag{3.5}$$

By using (3.5), we deduce that

$$\lim_{t\to\infty}(u(t,x)-v(t,x))=0.$$

Consequently, the problem (1.5)-(1.6) has a least one solution and all solutions are globally asymptotically stable. □

4 Example

Consider the following system of fractional order integral equation of the form

$$u(t,x)=\frac{t^3x}{1+8t^3}u\left(t-\frac{3}{4},x-3\right)+\frac{t^4x^2}{1+12t^4}u\left(t-2,x-\frac{1}{2}\right)+\frac{1}{4}u\left(t-1,x-\frac{3}{2}\right)$$

$$+\frac{\frac{1}{1+t+x}|I_\theta^r u(t,x)|+e^{2-t+x}|u(t,x)|}{1+\frac{1}{1+t+x}|I_\theta^r u(t,x)|+e^{2-t+x}|u(t,x)|};\ (t,x)\in R_+\times[0,1], \tag{4.6}$$

$$u(t,x)=0;\ \text{if } (t,x)\in\tilde{J}:=[-2,\infty)\times[-3,1]\backslash(0,\infty)\times(0,1], \tag{4.7}$$

where $r=\left(\frac{1}{2},\frac{33}{5}\right)$. Set

$$(\tau_1,\xi_1)=\left(\frac{3}{4},3\right),\ (\tau_2,\xi_2)=\left(2,\frac{1}{2}\right),\ (\tau_3,\xi_3)=\left(1,\frac{3}{2}\right),$$

$$g_1(t,x)=\frac{t^3x}{1+8t^3},\ g_2(t,x)=\frac{t^4x^2}{1+12t^4},\ g_3(t,x)=\frac{1}{4},$$

and

$$f(t,x,u,v)=\frac{\frac{|u|}{1+t+x}+|v|e^{2-t+x}}{1+\frac{|u|}{1+t+x}+|v|e^{2-t+x}};\ (t,x)\in R_+\times[0,1].$$

We have $m=3,\ (\tau,\xi)=(2,3)$ and $B_p\le B^*=\frac{1}{4};\ p\in\mathbb{N}$.
The function f is continuous and satisfies assumption (H), with

$$P(t,x)=\frac{1}{1+t+x}\text{ and }Q(t,x)=e^{2-t+x}.$$

Hence by Theorem 3.1, the problem (4.6)-(4.7) has a solution defined on $[-2,\infty)\times[-3,1]$ and all solutions are globally asymptotically stable.

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Existence results for neutral functional fractional differential equations with state dependent-delay

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Abstract

In this paper, we provide sufficient conditions for the existence of mild solutions for a class of fractional differential equations with state-dependent delay. The results are obtained by using the nonlinear alternative of Leray-Schauder type [14] fixed point theorem. An example is provided to illustrate the main results.

Keywords: Functional differential equation, fractional derivative, fractional integral, state-dependent delay, fixed point.

2010 MSC: 26A32, 26A42, 34K30.

1 Introduction

In the last two decades, the theory of fractional calculus has gained importance and popularity, due to its wide range of applications in varied fields of sciences and engineering. In [1, 3, 6, 7, 13, 19, 25, 27, 31, 32, 33] applications are mentioned to fluid flow, rheology, dynamical processes in self-similar and porous structures, electrical networks, control theory of dynamical systems and so on.

In this work, we establish the existence of mild solutions for a class of fractional abstract differential equations with state-dependent delay described by

$$^{c}D^{q}x(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in J = [0, a],\ 0 < q < 1, \tag{1.1}$$

$$x(t) = \varphi(t) \in \mathcal{B},\ t \in (-\infty, 0], \tag{1.2}$$

where the unknown $x(\cdot)$ takes values in Banach space X with norm $\|\cdot\|$, $^{c}D^{q}$ is the Caputo fractional derivative of order $0 < q < 1$, A is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t), t \geq 0\}$ in X, $f : J \times \mathcal{B} \to X$ and $\rho : J \times \mathcal{B} \to (-\infty, a]$ are appropriate given functions, $\varphi \in \mathcal{B}$, $\varphi(0) = 0$ and $\mathcal{B}$ is called a phase space that will be defined in preliminaries.

An important point to note here it that when the delay is infinite the right notion is phase space. This concept was introduced by Hale and Kato [15] (see also Kappel and Schappacher [26] and Schumacher [34]) which enables to deduce important information about qualitative properties of differential equations with unbounded delay. For a detailed discussion on this topic, we refer the reader to the book by Hino et al. [24].

On the other hand, functional differential equations with state-dependent delay appears frequently in applications as models of equations. Investigations of these classes of delay equations essentially differ from once of equations with constant or time-dependent delay. For these reasons the theory of differential equations with state-dependent delay has drawn the attention of researchers in the recent years, see for instance [4, 5, 16, 17, 18, 20, 21, 22, 23, 28, 29] and the references therein. The investigation of the exitnece of mild

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solutions of fractional functional differential equations with state-dependent delay is very recent and limited, see for instance [2, 8, 9, 10].

The results in the present work are, on one side, an extension of results in [10] and [35] and, one the otherside, an interesting contribution to the study of qualitative properties for fractional differential equations with state-dependent delay. The topological method that we have choosen to study existence of mild solutions of the fractiona differential equations (1.1)-(1.2) is the theory of fixed points, which has been a very powerful and important tool to study the nonlilnear phenomena.

Our approach and techniques here are based on the nonlinear alternative of Leray-Schauder type [14] and probability density function given by EI-Borai [11] and was then developed by Zhou et al. [35, 36].

2 Preliminaries

In this section, we introduce notation, definitions and preliminary facts which are used throughtout this paper.

By $C(J,X)$ we denote the Banach space of continuous functions from J into X with the norm

$$\|x\|_\infty := \sup\{|x(t)| : t \in J\}.$$

Definition 2.1. *The fractional integral of order α with the lower limit 0 for the function $f : (0,a] \to X$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \ \alpha > 0,$$

provided the right hand side exists pointwise on $(0,a]$, where Γ is the gamma function.

For instance, $I^\alpha f$ exists for all $\alpha > 0$, where $f \in C((0,a],X) \cup L^1((0,a],X)$; note also that when $f \in C((0,a],X)$ then $I^\alpha f \in C((0,a],X)$ and moreover $I^\alpha f(0) = 0$.

Definition 2.2. *The Caputo derivative of order α with the lower limit zero for a function $f : (0,a] \to X$ can be written as*

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \ n-1 < \alpha < n.$$

If f is an abstract function with values in X, then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner's sence.

In this paper, we will employ an axiomatic definition, for the phase space $\mathcal{B}$ which is similar to those introduced in [24]. More precisely, $\mathcal{B}$ will be a linear space of all functions from $(-\infty, 0]$ to X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ satisfying the following axioms:

(A1) If $x : (-\infty, a] \to X, \ a > 0$ is continuous on J and $x_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:

  (i) x_t is in $\mathcal{B}$,

  (ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$,

  (iii) $\| x_t \|_{\mathcal{B}} \leq K(t) \sup\{\| x(s) \| : 0 \leq s \leq t\} + M(t) \| x_0 \|_{\mathcal{B}}$, where $H > 0$ is a constant, $K : [0,\infty) \to [1,\infty)$ is continuous, $M : [0,\infty) \to [1,\infty)$ is locally bounded and $H, \ K, \ M$ are independent of $x(\cdot)$.

(A2) For the function $x(\cdot)$ in (A1), x_t is a $\mathcal{B}$-valued continuous function on J.

(A3) The space $\mathcal{B}$ is complete.

The next lemma is a consequence of the phase space axioms and is proved in [20].

Lemma 2.1. *Let $\varphi \in \mathcal{B}$ and $I = (\gamma, 0]$ be such that $\varphi_t \in \mathcal{B}$ for every $t \in I$. Assume that there exists a locally bounded function $J^\varphi : I \to [0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in I$. If $x : (\infty, a] \to \mathbb{R}$ is continuous on J and $x_0 = \varphi$, then*

$$\|x_t\|_{\mathcal{B}} \leq (M_a + J^\varphi(\max\{\gamma, -|s|\})\|\varphi\|_{\mathcal{B}} + K_a \sup\{|x(\theta)| : \theta \in [0, \max\{0,s\}]\},$$

for $s \in (\gamma, a]$, where we denoted $K_a = \sup_{t\in J} K(t)$ and $M_a = \sup_{t \in J} M(t)$.

3 Existence results for functional fractional differential equations with state-dependent delay

In this section, we discuss the existence of mild solutions for the fractional differential equations with state-dependent delay of the form (1.1)-(1.2). Following [11, 12, 35], we will introduce now the definition of mild solution to (1.1)-(1.2).

Definition 3.1. *A function $x : (-\infty, a] \to X$ is said to be a mild solution of (1.1)-(1.2) if $x_0 = \varphi,\ x_{\rho(s,x_s)} \in \mathcal{B}$ for each $s \in J$ and*

$$x(t) = S_q(t)\varphi(0) + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_{\rho(s,x_s)})ds, \quad t \in J,$$

where

$$\begin{aligned} S_q(t) &= \int_0^\infty \xi_q(\theta) T(t^q\theta)d\theta, \\ T_q(t) &= q\int_0^\infty \theta\xi_q(\theta)T(t^q\theta)d\theta, \\ \xi_q(\theta) &= \frac{1}{q}\theta^{-1-\frac{1}{q}}\overline{w}_q(\theta^{-\frac{1}{q}}) \geq 0, \\ \overline{w}_q(\theta) &= \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \theta \in (0,\infty), \end{aligned}$$

ξ_q is a probability density function on $(0, \infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0,\infty) \text{ and } \int_0^\infty \xi_q(\theta)d\theta = 1.$$

Remark 3.1. *It is not difficult to verify that for $v \in [0, 1]$*

$$\int_0^\infty \theta^v \xi_q(\theta)d\theta = \int_0^\infty \theta^{-qv}\overline{w}_q(\theta)d\theta = \frac{\Gamma(1+v)}{\Gamma(1+qv)}.$$

Lemma 3.1. *[35] For any $t \geq 0$, The operators $S_q(t)$ and $T_q(t)$ have the following properties:*

(a) For any fixed $t \geq 0$, S_q and T_q are linear and bounded operators, ie., for any $x \in X$,

$$\|S_q(t)x\| \leq M\|x\|, \quad \|T_q(t)x\| \leq \frac{qM}{\Gamma(1+q)}\|x\|.$$

(b) $\{S_q(t), t \geq 0\}$ and $\{T_q(t), t \geq 0\}$ are strongly continuous.

(c) For every $t > 0$, $S_q(t)$ and $T_q(t)$ are also compact operators.

To prove our results, we always assume that $\rho : J \times \mathcal{B} \to (-\infty, a]$ is continuous. In addition, we introduce the following conditions.

(H₁) The semigroup $T(t)$ is compact for $t > 0$.

(H₂) For each $t \in J$, the function $f(t, \cdot) : \mathcal{B} \to X$ is continuous and for each $\psi \in \mathcal{B}$, the function $f(\cdot, \psi) : J \to X$ is strongly measurable.

(H₃) There exist $p : J \to [0, \infty]$ and a continuous non-decreasing function $\Omega : [0, \infty) \to (0, \infty)$ such that

$$\|f(t,\psi)\| \leq p(t)\Omega(\|\psi\|_{\mathcal{B}}) \text{ for } t \in J, \text{ and each } \psi \in \mathcal{B}.$$

(H₄) The function $t \to \varphi_t$ is well defined and continuous from the set

$$\mathcal{R}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in J \times \mathcal{B},\ \rho(s,\psi) \leq 0\}$$

into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

Remark 3.2. *We point out here that the condition* $(\mathbf{H_4})$ *is usually satisfied by functions that are continuous and bounded. For complementary details related this matter the reader can see [20].*

Theorem 3.1. *Let conditions* $(\mathbf{H_1}) - (\mathbf{H_4})$ *hold with* $\rho(t,x) \leq t$ *for every* $(t,x) \in J \times \mathcal{B}$ *and*

$$\frac{\|\xi\|_\infty}{(M_a + \overline{J}^\varphi)\|\varphi\|_\mathcal{B} + MK_a\Omega(\|\xi\|_\infty)\|I^q p\|_\infty} > 1$$

then there exists a mild solution of (1.1)-(1.2) on $(-\infty, a]$.

Proof. Let $Y = \{u \in C(J,X) : u(0) = \varphi(0) = 0\}$ endowed with the uniform operator topology and $\Phi : Y \to Y$ be the operator defined by

$$\Phi(x)(t) = \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_{\rho(s,x(s))}) ds, \quad t \in J,$$

where $\overline{x} : (-\infty, a] \to X$ is such that $\overline{x}_0 = \varphi$ and $\overline{x} = x$ on J. From axiom (A_1) and our assumption on φ, we infer that $\Phi(x)(\cdot)$ is well defined and continuous.

Let $\overline{\varphi} : (-\infty, a] \to X$ be the extension of φ to $(-\infty, a]$ such that $\overline{\varphi}(\theta) = \phi(0) = 0$ on J and $\overline{J}^\varphi = \sup\{J^\varphi : s \in \mathcal{R}(\rho^-)\}$.

We will prove that $\Phi(\cdot)$ is completely continuous from $B_r(\overline{\varphi}|_J, Y)$ to $B_r(\overline{\varphi}|_J, Y)$.

We break the proof into several steps.

Step 1: Φ is continuous on $B_r(\overline{\varphi}|_J, Y)$.

Let $\{x^n\} \subset B_r(\overline{\varphi}|_J, Y)$ and $x \in B_r(\overline{\varphi}|_J, Y)$ with $x^n \to x$ $(n \to \infty)$. From axiom $A1$, it is easy to see that $(\overline{x^n})_s \to \overline{x}_s$ uniformly for $s \in (-\infty, a]$ as $n \to \infty$. By $(H3)$, we have

$$\begin{aligned}
&\|f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) - f(s, \overline{x}_{\rho(s,(\overline{x})_s)})\| \\
&\leq \|f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) - f(s, \overline{x}_{\rho(s,(\overline{x^n})_s)})\| + \|f(s, \overline{x}_{\rho(s,(\overline{x^n})_s)}) - f(s, \overline{x}_{\rho(s,(\overline{x})_s)})\|
\end{aligned}$$

which implies that $f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) \to f(s, \overline{x}_{\rho(s,(\overline{x})_s)})$ as $n \to \infty$ for each $s \in J$. By axiom A1(ii), Lemma(2.1) and the dominated convergence theorem, we obtain

$$\begin{aligned}
\|\Phi(x^n) - \Phi(x)\| &\leq \left\| \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_{\rho(s,x_s)}) ds \right\| \\
&\to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

Therefore, Φ is continuous.

Step 2: Φ maps bounded sets into bounded sets. If $x \in B_r(\overline{\phi}|_J, Y)$, from Lemma(2.1), it follows that

$$\|\overline{x}_{\rho(t,\overline{x}_t)}\|_\mathcal{B} \leq r^* := (M_a + \overline{J}^\phi)\|\phi\|_\mathcal{B} + K_a r$$

and so

$$\begin{aligned}
|\Phi(x)(t)| &\leq \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds \\
&\leq \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} p(s) \Omega(\|\overline{x}_{\rho(s,\overline{x}_s)}\|_\mathcal{B}) ds \\
&\leq \frac{qM}{\Gamma(1+q)} \|p\|_\infty \Omega(r^*) \int_0^t (t-s)^{q-1} ds \\
&\leq \frac{Ma^q}{\Gamma(1+q)} \|p\|_\infty \Omega(r^*).
\end{aligned}$$

Thus,

$$\|\Phi(x)\|_\infty \leq \frac{Ma^q}{\Gamma(1+q)} \|p\|_\infty \Omega(r^*) := l.$$

Step 3: Φ maps bounded sets into equicontinuous sets.

Let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$ and B_r be a bounded set as in step 2. Let $\epsilon > 0$ be given. For each $t \in J$, we have

$$\begin{aligned}
&\|\Phi(x)(\tau_2) - \Phi(x)(\tau_1)\| \\
&\leq \int_0^{\tau_1-\epsilon} \Big\| \Big[(\tau_2 - s)^{q-1} T_q(\tau_2 - s) - (\tau_1 - s)^{q-1} T_q(\tau_1 - s)\Big] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\
&+ \int_{\tau_1-\epsilon}^{\tau_1} \Big\| \Big[(\tau_2 - s)^{q-1} T_q(\tau_2 - s) - (\tau_1 - s)^{q-1} T_q(\tau_1 - s)\Big] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\
&+ \int_{\tau_1}^{\tau_2} \Big\| (\tau_2 - s)^{q-1} T_q(\tau_2 - s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\
&\leq \|p\|_\infty \Omega(r^*) \Big[\int_0^{\tau_1-\epsilon} \Big[|(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}| \|T_q(\tau_2 - s)\| \\
&+ |(\tau_1 - s)^{q-1}| \|T_q(\tau_2 - s) - T_q(\tau_1 - s)\| \Big] ds \\
&+ \int_{\tau_1-\epsilon}^{\tau_1} \Big[|(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}| \|T_q(\tau_2 - s)\| \\
&+ |(\tau_1 - s)^{q-1}| \|T_q(\tau_2 - s) - T_q(\tau_1 - s)\| \Big] ds + \frac{M}{\Gamma(1+q)} (\tau_2 - \tau_1)^q \Big].
\end{aligned}$$

The right hand side tends to zero as $\tau_2 - \tau_1 \to 0$, since $T_q(t),\ t \geq 0$ is a strongly continuous semigroup and $T_q(t)$ is compact for $t > 0$ (so $T_q(t)$ is continuous in the uniform operator topology for $t > 0$). The equicontinuous for the other cases $\tau_1 < \tau_2 \leq 0$ or $\tau_1 \leq 0 \leq \tau_2 \leq a$ are very simple.

Step 4: Φ is precompact.

Let $0 < t \leq s \leq a$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$, and $\delta > 0$. For $x \in B_r$, we define,

$$\begin{aligned}
\Phi_{\epsilon,\delta}(x)(t) &= q \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \\
&= T(\epsilon^q \delta) q \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta - \epsilon^q \delta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds,
\end{aligned}$$

Since $T(\epsilon^q \delta)$ is a compact operator for $\epsilon^q \delta > 0$, the set $Y_{\epsilon,\delta}(t) = \{\Phi_{\epsilon,\delta}(x)(t) : x \in B_r\}$ is precompact in X for every $\epsilon,\ 0 < \epsilon < t$. Moreover

$$\begin{aligned}
&\|\Phi(x)(t) - \Phi_{\epsilon,\delta}(x)(t)\| \\
&= q \Big[\Big\| \int_0^{t-\epsilon} \int_0^\delta \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \Big\| \\
&\quad + \Big\| \int_{t-\epsilon}^t \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \Big\| \Big] \\
&\leq \|p\|_\infty \Omega(r^*) \frac{qM}{\Gamma(1+q)} \Big[\int_0^{t-\epsilon} (t-s)^{q-1} ds \int_0^\delta \theta \xi_q(\theta) d\theta + \int_{t-\epsilon}^t (t-s)^{q-1} ds \int_\delta^\infty \theta \xi_q(\theta) d\theta \Big].
\end{aligned}$$

Therefore, there are precompact sets arbitrarly close to the set $Y(t) = \{\Phi_{\epsilon,\delta}(x)(t) : x \in B_r\}$ is precompact. Hence the set $Y(t) = \{\Phi_{\epsilon,\delta}(x)(t) : x \in B_r\}$ is precompact in X.

As a consequence of the Step 1 to Step 4 and the Arzela-Ascoli theorem, we can conclude that the operator Φ is completely continuous.

Step 5: We now show there exists an open set $U \subset Y$ with $x \neq \lambda \Phi(x)$ for $\lambda \in (0,1)$ and $x \in \partial U$. Let $x \in Y$ and $x = \lambda \Phi(x)$ for some $0 < \lambda < 1$. Then for each $t \in J$ we have,

$$x(t) = \lambda \int_0^t (t-s)^{q-1} T_q(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds.$$

This implies by (H3) and Lemma(2.1) that

$$\begin{aligned}
|x(t)| &\leq \int_0^t (t-s)^{q-1} \|T_q(t-s)\| \|f(s, \overline{x}_{\rho(s,\overline{x}(s))})\| ds \\
&\leq \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} p(s) \Omega\Big((M_a + \overline{J}^\phi) \|\phi\|_{\mathcal{B}} + K_a \sup\{|\overline{x}(s)| : s \in [0,t]\} \Big) ds,
\end{aligned}$$

since $\rho(s,\overline{x}_s) \leq s$ for every $s \in J$. Here $\overline{J}^{\phi} = \sup\{J^{\phi}(s) : s \in \mathcal{R}(\rho^{-})\}$.

Set $\mu(t) = \sup\{|x(s)| : 0 \leq s \leq t\},\ t \in J$. Then we have

$$\mu(t) \leq \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} p(s)\Omega\Big((M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + K_a\mu(s)\Big)ds.$$

If $\xi(t) = (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + K_a\mu(t)$ then we have,

$$\begin{aligned}\xi(t) &\leq (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + \frac{qMK_a}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} p(s)\Omega(\xi(s))ds \\ &\leq (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + MK_a\Omega(\|\xi\|_{\infty})\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} p(s)ds \\ &\leq (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + MK_a\Omega(\|\xi\|_{\infty})\|I^q p\|_{\infty}.\end{aligned}$$

Then

$$\frac{\|\xi\|_{\infty}}{(M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + MK_a\Omega(\|\xi\|_{\infty})\|I^q p\|_{\infty}} \leq 1.$$

Then there exists M^* such that $\|x\|_{\infty} \neq M^*$. Set $U = \{x \in Y : \|x\|_{\infty} < M^* + 1\}$.

Then $\Phi : \overline{U} \to Y$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda\Phi(x)$ for $\lambda \in (0,1)$. As a consequence of the nonlinear aiternative of Leray- Schauder type [14], we deduce that Φ has a fixed point x in U, which is a solution of (1.1)-(1.2). □

4 Existence results for netural functional fractional differential equations with state-dependent delay

In this section, we study existence results for netural fractional differential equations with state-dependent delay of the form

$$^cD^q[x(t) - g(t,x_t)] = A[x(t) - g(t,x_t)] + f(t, x_{\rho(t,x_t)}), \quad t \in J = [0,a],\ 0 < q < 1, \tag{4.1}$$

$$x(t) = \phi(t) \in \mathcal{B},\ t \in (-\infty, 0], \tag{4.2}$$

where $A, f, \rho,$ and ϕ are same as defined in (1.1)-(1.2) and $g : J \times \mathcal{B} \to X$ is appropriate given function.

Definition 4.1. *A function $x : (-\infty, a] \to X$ is said to be a mild solution of (4.1)-(4.2) if $x_0 = \phi,\ x_{\rho(s,x_s)} \in \mathcal{B}$ for each $s \in J$ and*

$$x(t) = S_q(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_{\rho(s,x_s)})ds, \quad t \in J,$$

where

$$\begin{aligned} S_q(t) &= \int_0^{\infty} \xi_q(\theta) T(t^q\theta)d\theta, \\ T_q(t) &= q\int_0^{\infty} \theta\xi_q(\theta) T(t^q\theta)d\theta, \\ \xi_q(\theta) &= \frac{1}{q}\theta^{-1-\frac{1}{q}}\overline{w}_q(\theta^{-\frac{1}{q}}) \geq 0, \\ \overline{w}_q(\theta) &= \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \theta \in (0,\infty), \end{aligned}$$

ξ_q is a probability density function on $(0,\infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0,\infty) \text{ and } \int_0^{\infty} \xi_q(\theta)d\theta = 1.$$

To prove the next theorems, in addition, we need the following hypotheses:

($\mathbf{H_5}$) The function $g : J \times \mathcal{B} \to X$ is completely continuous and there exist positive constants c_1 and c_2 such that

$$\|g(t,\psi)\| \le c_1\|\psi\|_{\mathcal{B}} + c_2, \quad t \in J,\ \psi \in \mathcal{B}.$$

($\mathbf{H_5}$)* The function $g : J \times \mathcal{B} \to X$ is continuous and there exists $L_f > 0$ such that

$$\|g(t,\psi_1) - g(t,\psi_2)\| \le L_f\|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad t \in J,\ \psi_i \in \mathcal{B},\ i = 1,2.$$

Theorem 4.1. *Assume that the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_5})$ *are fulfilled. If*

$$K_a\Big[L_f + \frac{qM}{\Gamma(1+q)} \liminf_{\xi\to\infty} \frac{\Omega(\xi)}{\xi}\int_0^a (t-s)^{q-1}p(s)ds\Big] < 1$$

then there exist a mild solution of (4.1)-(4.2) on J.

Proof. Let $\overline{\phi} : (-\infty, a] \to X$ be the extension of ϕ to $(-\infty, a]$ such that $\overline{\phi}(\theta) = \phi(0)$ on $J = [0, a]$. Consider the space $S(a) = \{u \in C(J;X) : u(0) = \phi(0)\}$ endowed with the uniform operator topology and define the operator $\Upsilon : S(a) \to S(a)$ by

$$\Upsilon x(t) = S_q(t)[\phi(0) - g(0,\phi(0)) + g(t,\overline{x}_t) + \int_0^t (t-s)^{q-1}T_q(t-s)f(s,\overline{x}_{\rho(s,x_s)})ds, \quad t \in J,$$

where $\overline{x} : (-\infty, a] \to X$ is such that $\overline{x}_0 = \phi$ and $\overline{x} = x$ on J. From our assumptions, it is easy to see that $\Upsilon S(a) \subset S(a)$.

We shall prove that there exists a $r > 0$ such that $\Upsilon(B_r(\overline{\phi}|_J, S(a))) \subset B_r(\overline{\phi}|_J, S(a))$. If this property is false, then for every $r > 0$ there exist $x^r \in B_r(\overline{\phi}|_J, S(a))$ and $t^r \in J$ such that $r < \|\Upsilon x^r(t^r) - \phi(0)\|$. Then from Lemma (2.1), we find,

$$\begin{aligned}
r &\le \|\Upsilon x^r(t^r) - \phi(0)\| \\
&\le \|S_q(t^r)\phi(0) - \phi(0)\| + \|S_q(t^r)g(0,\phi) - g(0,\phi)\| + \|g(t,(\overline{x^r})_{t^r} - g(0,\phi)\| \\
&\quad + \int_0^{t^r} \big\|(t-s)^{q-1}T_q(t^r-s)f(s,\overline{x^r}_{\rho(s,(\overline{x^r})_s)})\big\|ds \\
&\le (M+1)H\|\phi\|_{\mathcal{B}} + \|S_q(t^r)g(0,\phi) - g(0,\phi)\| + L_f\Big(K_a r + (M_a + HK_a + 1)\|\phi\|_{\mathcal{B}}\Big) \\
&\quad + \frac{qM}{\Gamma(1+q)}\Omega\Big(\big(M_a + \overline{J}^{\phi}\big)\|\phi\|_{\mathcal{B}} + K_a\big(r + \|\phi(0)\|\big)\Big)\int_0^a (t-s)^{q-1}p(s)ds
\end{aligned}$$

and hence

$$1 \le K_a\Big[L_f + \frac{qM}{\Gamma(1+q)} \liminf_{\xi\to\infty} \frac{\Omega(\xi)}{\xi}\int_0^a (t-s)^{q-1}p(s)ds\Big]$$

which contradicts our assumption.

Let $r > 0$ be such that $\Upsilon(B_r(\overline{\phi}|_J, S(a))) \subset B_r(\overline{\phi}|_J, S(a))$, in what follows, r^* is the number defined by $r^* := (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + K_a\big(r + \|\phi(0)\|\big)$. To prove that Υ is condensing operator, we introduce the decomposition $\Upsilon = \Upsilon_1 + \Upsilon_2$, where

$$\begin{aligned}
\Upsilon_1 x(t) &= S_q(t)[\phi(0) - g(0,\phi) + g(t,\overline{x}_t), \\
\Upsilon_2 x(t) &= \int_0^t (t-s)^{q-1}T_q(t-s)f(s,\overline{x}_{\rho(s,x_s)})ds, \quad t \in J.
\end{aligned}$$

Step 1: $\Upsilon_1(\cdot)$ is contraction on $B_r(\overline{\phi}|_J, S(a))$.

If $x, y \in B_r(\overline{\phi}|_J, S(a))$ and $t \in J$, then we have

$$\begin{aligned}
\|\Upsilon_1 x(t) - \Upsilon_1 y(t)\| &\le \|g(t,\overline{x}_t) - g(t,\overline{y}_t)\| \\
&\le L_f K_a\|x - y\|_a,
\end{aligned}$$

which proves that $\Upsilon_1(\cdot)$ is a contraction on $B_r(\overline{\phi}|_J, S(a))$.

Next we prove that $\Upsilon_2(\cdot)$ is completely continuous from $B_r(\overline{\phi}|_J, S(a))$ into $B_r(\overline{\phi}|_J, S(a))$.

Step 2: Υ_2 is continuous on $B_r(\overline{\phi}|_J, S(a))$.

Let $\{x^n\} \subset B_r(\overline{\phi}|_J, S(a))$ and $x \in B_r(\overline{\phi}|_J, S(a))$ with $x^n \to x$ $(n \to \infty)$. From axiom A_1, it is easy to see that $(\overline{x^n})_s \to \overline{x}_s$ uniformly for $s \in (-\infty, a]$ as $n \to \infty$. By $(H3)$, we have

$$\begin{aligned}&\|f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) - f(s, \overline{x}_{\rho(s,(\overline{x})_s)})\| \\ &\leq \|f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) - f(s, \overline{x}_{\rho(s,(\overline{x^n})_s})\| + \|f(s, \overline{x}_{\rho(s,(\overline{x^n})_s)}) - f(s, \overline{x}_{\rho(s,(\overline{x})_s)})\|\end{aligned}$$

which implies that $f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) \to f(s, \overline{x}_{\rho(s,(\overline{x})_s})$ as $n \to \infty$ for each $s \in J$. By axiom A1(ii), Lemma(2.1) and the dominated convergence theorem we obtain

$$\begin{aligned}\|\Upsilon_2 x^n - \Upsilon_2 x\| &\leq \Big\| \int_0^t (t-s)^{q-1} T_q(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds \Big\| \\ &\to 0 \quad \text{as} \quad n \to \infty\end{aligned}$$

Therefore, Υ_2 is continuous.

Step 3: $\Upsilon_2(\cdot)$ is equicontinuous on J.

Let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$ and B_r be a bounded set as in step 2. Let $\epsilon > 0$ be given. For each $t \in J$, we have

$$\begin{aligned}&\|\Upsilon_2(x)(\tau_2) - \Upsilon_2(x)(\tau_1)\| \\ &\leq \int_0^{\tau_1-\epsilon} \Big\| \Big[(\tau_2-s)^{q-1}T_q(\tau_2-s) - (\tau_1-s)^{q-1}T_q(\tau_1-s)\Big] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\ &+ \int_{\tau_1-\epsilon}^{\tau_1} \Big\| \Big[(\tau_2-s)^{q-1}T_q(\tau_2-s) - (\tau_1-s)^{q-1}T_q(\tau_1-s)\Big] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\ &+ \int_{\tau_1}^{\tau_2} \Big\| (\tau_2-s)^{q-1}T_q(\tau_2-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\ &\leq \Omega(r^*) \Big[\int_0^{\tau_1-\epsilon} \Big[|(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}| \|T_q(\tau_2-s)\| \\ &+ |(\tau_1-s)^{q-1}| \|T_q(\tau_2-s) - T_q(\tau_1-s)\| \Big] p(s) ds \\ &+ \int_{\tau_1-\epsilon}^{\tau_1} \Big[|(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}| \|T_q(\tau_2-s)\| \\ &+ |(\tau_1-s)^{q-1}| \|T_q(\tau_2-s) - T_q(\tau_1-s)\| \Big] p(s) ds + \frac{qM}{\Gamma(1+q)} \int_{\tau_1}^{\tau_2} |(\tau_2-s)^{q-1}| p(s) ds \Big].\end{aligned}$$

The right hand side tends to zero as $\tau_2 - \tau_1 \to 0$, since $T_q(t),\ t \geq 0$ is a strongly continuous semigroup and $T_q(t)$ is compact for $t > 0$ (so $T_q(t)$ is continuous in the uniform operator topology for $t > 0$). The equicontinuous for the other cases $\tau_1 < \tau_2 \leq 0$ or $\tau_1 \leq 0 \leq \tau_2 \leq a$ are very simple.

Step 4: Υ_2 is precompact.

Let $0 < t \leq s \leq a$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$, and $\delta > 0$. For $x \in B_r$, we define,

$$\begin{aligned}\Upsilon_{2_{\epsilon,\delta}}(x)(t) &= q \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \\ &= T(\epsilon^q \delta) q \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta - \epsilon^q \delta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds,\end{aligned}$$

Since $T(\epsilon^q \delta)$ is a compact operator for $\epsilon^q \delta > 0$, the set $V_{\epsilon,\delta}(t) = \{\Upsilon_{2_{\epsilon,\delta}}(x)(t) : x \in B_r\}$ is precompact in X for every $\epsilon,\ 0 < \epsilon < t$. Moreover

$$\begin{aligned}&\|\Upsilon_2(x)(t) - \Upsilon_{2_{\epsilon,\delta}}(x)(t)\| \\ &= q \Big[\Big\| \int_0^{t-\epsilon} \int_0^\delta \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \Big\| \\ &\quad + \Big\| \int_{t-\epsilon}^t \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \Big\| \Big]\end{aligned}$$

$$\leq \Omega(r^*)\frac{qM}{\Gamma(1+q)}\Big[\int_0^{t-\epsilon}(t-s)^{q-1}p(s)ds\int_0^{\delta}\theta\xi_q(\theta)d\theta+\int_{t-\epsilon}^{t}(t-s)^{q-1}p(s)ds\int_{\delta}^{\infty}\theta\xi_q(\theta)d\theta\Big].$$

Therefore, there are precompact sets arbitrarly close to the set $V(t)=\{\Upsilon_{2_{\epsilon,\delta}}(x)(t): x\in B_r\}$ is precompact. Hence the set $V(t)=\{\Upsilon_{2_{\epsilon,\delta}}(x)(t): x\in B_r\}$ is precompact in X.

As a consequence of the Step 2 to Step 4 and the Arzela-Ascoli theorem, we can conclude that the operator Υ_2 is completely continuous.

These arguments enable us to conclude that $\Upsilon=\Upsilon_1+\Upsilon_2$ is a condensing mapping on $B_r(\overline{\phi}|_J, S(a))$ and the existence of a mild solution for (4.1)-(4.2) is now a consequence of [[30], Theorem 4.3.2]. This completes the proof. □

Theorem 4.2. *Assume that the hypotheses* $(\mathbf{H_1})-(\mathbf{H_5})$ *and* $(\mathbf{H_5})^*$ *are fulfilled with* $\rho(t,\psi)\leq t$ *for every* $t\in J,\ \psi\in\mathcal{B}$. *If*

$$\frac{\|\xi\|_\infty}{(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+\frac{K_a}{1-c_1}\big[M(H+c_1)\|\phi\|_{\mathcal{B}}+c_2(1+M)+M\Omega(\|\xi\|_\infty)\|I^qp\|_\infty\big]}>1$$

then there exists a mild solution of (4.1)-(4.2) on J.

Proof. Let Υ be a function given in the proof of Theorem 4.1.

We show that there exists an open set $U_1\subset S(a)$ with $x\neq\lambda\Upsilon(x)$ for $\lambda\in(0,1)$ and $x\in\partial U_1$. Let $x\in S(a)$ and $x=\lambda\Upsilon(x)$ for some $0<\lambda<1$. Then

$$x(t)=\lambda\Big[S_q(t)[\phi(0)-g(0,\phi)+g(t,\overline{x}_t)+\int_0^t(t-s)^{q-1}T_q(t-s)f(s,\overline{x}_{\rho(s,x_s)})ds\Big],\quad t\in J,$$

and

$$\begin{aligned}|x(t)|&\leq MH\|\phi\|_{\mathcal{B}}+M[c_1\|\phi\|_{\mathcal{B}}+c_2]+c_1\|\overline{x}_t\|_{\mathcal{B}}+c_2\\&\quad+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega(\|\overline{x}_{\rho(s,(\overline{x}))}\|_{\mathcal{B}})ds\\&\leq M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]+c_1\|\overline{x}_t\|_{\mathcal{B}}\\&\quad+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega\Big((M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+K_a\|\overline{x}\|\Big)ds.\end{aligned}$$

If $\mu(t)=\sup\{|x(s)|: s\in[0,t]\}$ then

$$\begin{aligned}\mu(t)&\leq M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]+c_1\mu(t)\\&\quad+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega\Big((M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+K_a\mu(s)\Big)ds.\end{aligned}$$

Since $0<c_1<1$, we have

$$\begin{aligned}\mu(t)\leq\frac{1}{1-c_1}\Big[&M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]\\&+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega\Big((M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+K_a\mu(s)\Big)ds\Big],\quad t\in J.\end{aligned}$$

If $\xi(t)=(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+K_a\mu(s)$ then we have

$$\begin{aligned}\xi(t)&=(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+\frac{K_a}{1-c_1}\Big[M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]\\&\quad+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega(\xi(s))ds\Big]\\&\leq(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+\frac{K_a}{1-c_1}\Big[M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]+M\Omega(\|\xi\|_\infty)\|I^qp\|_\infty\Big].\end{aligned}$$

Consequently,

$$\frac{\|\xi\|_\infty}{(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+\frac{K_a}{1-c_1}\big[M(H+c_1)\|\phi\|_{\mathcal{B}}+c_2(1+M)+M\Omega(\|\xi\|_\infty)\|I^qp\|_\infty\big]}\leq 1.$$

Now, there exist L^* such that $\|x\|_\infty \neq L^*$. set

$$U_1 = \{x \in Y : \|x\|_\infty < L^* + 1\}.$$

From the choice of U_1 there is no $x \in \partial U_1$ such that $x = \lambda \Upsilon(x)$ for $\lambda \in (0,1)$.

To prove that Υ is completely continuous on $S(a)$, we introduce the decomposition $\Upsilon = \Upsilon_1 + \Upsilon_2$ introduced in the proof of the Theorem 4.1. From the proof of Theorem 4.1, we obtain that Υ_2 is completely continuous on $S(a)$ and from the condition $(\mathbf{H_5})$ it follows that Υ_1 is completely continuous on $S(a)$. As a consequence of the nonlinear alternative of Leray-Schauder type [14], we deduce that Υ has a fixed point x in U_1. Then Υ has a fixed poinf, whixh is a solution of (4.1)-(4.2). □

5 Example

In this section, we consider an applications of our abstract results. At first we introduce the required technical framework. In the rest of this secion, $X = L^2([0,\pi])$ and $A : D(A) \subset X \to X$ be the operator $Aw = w''$ with domain $D(A) := \{w \in X : w'' \in X,\ w(0) = w(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup on X.

Then

$$Aw = -\sum_{n=1}^{\infty} n^2 < w, e_n > e_n, \quad w \in D(A),$$

where $e_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi),\ 0 \le \xi \le \pi,\ n = 1, 2, \ldots$. Clearly A generates a compact semigroup $T(t), t > 0$ in X and it is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} < w, e_n > e_n, \text{ for every } w \in X.$$

Clearly the assumption $(\mathbf{H_1})$ is satisfied. Consider the fractional differential system

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t,\xi) = \frac{\partial^2}{\partial \xi^2} u(t,\xi) + \int_{-\infty}^{t} a_2(s-t) u(s - \rho_1(t)\rho_2(\|u(t)\|), \xi) ds,\ t \in J,\ \xi \in [0,\pi], \tag{5.1}$$

submitted to the conditions

$$u(t,0) = u(t,\pi) = 0, \quad t \ge 0, \tag{5.2}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \quad \theta \le 0,\ 0 \le \xi \le \pi, \tag{5.3}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is a Caputo fractional partial derivative of order $0 < \alpha < 1$. In the sequel, $\mathcal{B} = C_0 \times L^2(g, X)$ is the space introduced in [20]; $\varphi \in \mathcal{B}$ with the identification $\varphi(s)(\theta) = \varphi(s,\theta)$.

To treat this system, we assume that $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1, 2$, are continuous functions and the following condition.

(a) The functions $a_1 : \mathbb{R} \to \mathbb{R}$, are continuous and $L_1 = \left(\int_{-\infty}^{0} \frac{(a_1(s))^2}{g(s)} ds\right)^{1/2} < \infty$.

Under these conditions, we can define the operators $f : J \times \mathcal{B} \to X$, and $\rho : J \times \mathcal{B} \to \mathbb{R}$ by

$$f(t,\varphi)(\xi) = \int_{-\infty}^{0} a_1(s)\varphi(s,\xi) ds, \tag{5.4}$$

$$\rho(s,\varphi) = s - \rho_1(s)\rho_2(\| \varphi(0) \|), \tag{5.5}$$

which permit to transform system (5.1)-(5.3) into the abstract Cauchy problem (1.1)-(1.2). Moreover, the maps f is bounded linear operators with $\| f \|_{\mathcal{L}(\mathcal{B},X)} \le L_1$. The following result is a direct consequence of Theorem 3.1.

Proposition 5.1. *Let $\varphi \in \mathcal{B}$ be such that condition (H4) holds. Then there exists a mild solution of (5.1)-(5.3).*

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On some integral inequalities using Hadamard fractional integral

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Abstract

In this paper, using Hadamard fractional integral, we establish two main new result on fractional integral inequalities by considering the extended Chebyshev functional in case of synchronous function. The first result concerns with some inequalities using one fractional parameter and other with two parameter.

Keywords: Chebyshev functional, Hadamard fractional integral, Hadamard fractional derivative and fractional integral inequality.

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1 Introduction

In recent years, many authors have worked on fractional integral inequalities and its application which plays important role in classical differential and integral equations, see [3, 5, 6, 7, 8, 9, 10]. Dahmani gave the following fractional integral inequalities, using the Riemann-Liouville fractional integral for extended Chebyshev functional, see for instance [6].

Theorem 1.1. *Let f and g be two synchronous function on $[0,\infty[$ and let $r,p,q : [0,\infty[\to [0,\infty[$ for all $t>0, \alpha>0$ and then*

$$\begin{aligned}&2J^{\alpha}r(t)\left[J^{\alpha}p(t)J^{\alpha}(qfg)(t)+J^{\alpha}q(t)J^{\alpha}(pfg)(t)\right]+2J^{\alpha}p(t)J^{\alpha}q(t)J^{\alpha}(rfg)(t)\geq\\&J^{\alpha}r(t)\left[J^{\alpha}(pf)(t)J^{\alpha}(qg)(t)+J^{\alpha}(qf)(t)J^{\alpha}(pg)(t)\right]J^{\alpha}p(t)[J^{\alpha}(rf)(t)J^{\alpha}(qg)(t)+J^{\alpha}(qf)(t)J^{\alpha}(rg)(t)]\\&+J^{\alpha}q(t)\left[J^{\alpha}(rf)(t)J^{\alpha}(pg)(t)+J^{\alpha}(pf)(t)J^{\alpha}(rg)(t)\right].\end{aligned} \tag{1.1}$$

Theorem 1.2. *Let f and g be two synchronous function on $[0,\infty[$ and let $r,p,q : [0,\infty[\to [0,\infty[$ for all $t>0, \alpha>0$ $\beta>0$ then we have,*

$$\begin{aligned}&J^{\alpha}r(t)\left[J^{\alpha}q(t)J^{\beta}(pfg)(t)+2J^{\alpha}p(t)J^{\beta}(qfg)(t)+J^{\beta}q(t)J^{\alpha}(pfg)(t)\right]\\&+\left[J^{\alpha}p(t)J^{\beta}q(t)+J^{\beta}p(t)J^{\alpha}q(t)\right]J^{\alpha}(rfg)(t)\geq\\&J^{\alpha}r(t)\left[J^{\alpha}(pf)(t)J^{\beta}(qg)(t)+J^{\beta}(qf)(t)J^{\alpha}(pg)(t)\right]J^{\alpha}p(t)[J^{\alpha}(rf)(t)J^{\beta}(qg)(t)+J^{\beta}(qf)(t)J^{\alpha}(rg)(t)]\\&+J^{\alpha}q(t)\left[J^{\alpha}(rf)(t)J^{\beta}(pg)(t)+J^{\beta}(pf)(t)J^{\alpha}(rg)(t)\right].\end{aligned} \tag{1.2}$$

The main objective of this paper is to establish some inequalities for the extended Chebyshev functional given in [6], using Hadamard fractional integrals. The paper has been organized as follows. In Section 2, we define basic definitions and proposition related to Hadamard fractional derivatives and integrals. In Section 3, we give the main results.

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2 Preliminaries

Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville, Caputo derivative, see [3, 5, 6, 7, 8, 9, 10]. The necessary background details are given in the book A.A. Kilbas [1], and in book of S.G. Samko et al. [4], here we present some definitions of Hadamard derivative and integral as given in [2, p.159-171].

Definition 2.1. *The Hadamard fractional integral of order* $\alpha \in R^+$ *of function* $f(x)$*, for all* $x > 1$ *is defined as,*

$$ {}_HD_{1,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x \ln(\frac{x}{t})^{\alpha-1} f(t) \frac{dt}{t}, \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$*.*

Definition 2.2. *The Hadmard fractional derivative of order* $\alpha \in [n-1, n)$*,* $n \in Z^+$*, of function* $f(x)$ *is given as follows*

$$ {}_HD_{1,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} (x\frac{d}{dx})^n \int_1^x \ln(\frac{x}{t})^{n-\alpha-1} f(t) \frac{dt}{t}. \tag{2.2}$$

From the above definitions, we can see obviously the difference between Hadamard fractional and Riemann-Liouville fractional derivative and integrals, which include two aspects. The kernel in the Hadamard integral has the form of $\ln(\frac{x}{t})$ instead of the form of $(x-t)$, which is involves both in the Riemann-Liouville and Caputo integral. The Hadamard derivative has the operator$(x\frac{d}{dx})^n$, whose construction is well suited to the case of the half-axis and is invariant relation to dilation [4, p.330], while the Riemann-Liouville derivative has the operator $(\frac{d}{dx})^n$.

We give some image formulas under the operator (2.1) and (2.2), which would be used in the derivation of our main result.

Proposition 2.1. *[2] If* $0 < \alpha < 1$*, the following relation hold:*

$$ {}_HD_{1,x}^{-\alpha} (\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\ln x)^{\beta+\alpha-1}, \tag{2.3}$$

$$ {}_HD_{1,x}^{\alpha} (\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\ln x)^{\beta-\alpha-1}, \tag{2.4}$$

respectively.

For the convenience of establishing the result, we give the semigroup property,

$$ ({}_HD_{1,x}^{-\alpha})({}_HD_{1,x}^{-\beta}) f(x) = {}_HD_{1,x}^{-(\alpha+\beta)} f(x). \tag{2.5}$$

3 Main Results

In this section, we present and prove the main results.

Lemma 3.1. *Let* f *and* g *be two synchronous function on* $[0,\infty[$*. and* $x, y : [0,\infty) \to [0,\infty)$*. Then for all* $t > 0$*,* $\alpha > 0$*, we have,*

$$ \begin{aligned} &{}_HD_{1,t}^{-\alpha} x(t) {}_HD_{1,t}^{-\alpha} (yfg)(t) + {}_HD_{1,t}^{-\alpha} y(t) {}_HD_{1,t}^{-\alpha} (xfg)(t) \geq \\ &{}_HD_{1,t}^{-\alpha} (xf)(t) {}_HD_{1,t}^{-\alpha} (yg)(t) + {}_HD_{1,t}^{-\alpha} (yf)(t) {}_HD_{1,t}^{-\alpha} (xg)(t). \end{aligned} \tag{3.1}$$

Proof. Since f and g are synchronous on $[0,\infty[$ for all $\tau \geq 0$, $\rho \geq 0$, we have

$$ (f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0. \tag{3.2}$$

From (3.2),

$$ f(\tau).g(\tau) + f(\rho).g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{3.3}$$

Now, multiplying both side of (3.3) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}x(\tau)}{\tau\Gamma(\alpha)}$, $\tau\in(0,t)$, $t>0$. Then the integrating resulting identity with respect to τ from 1 to t we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\tau).g(\tau)\frac{d\tau}{\tau}+\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\rho).g(\rho)\frac{d\tau}{\tau}\geq\\&\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\tau).g(\rho)\frac{d\tau}{\tau}+\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\rho).g(\tau)\frac{d\tau}{\tau}.\end{aligned}\tag{3.4}$$

Consequently,

$${}_HD_{1,t}^{-\alpha}(xfg)(t)+f(\rho).g(\rho)_HD_{1,t}^{-\alpha}(x)(t)\geq g(\rho)_HD_{1,t}^{-\alpha}(xf)(t)+f(\rho)_HD_{1,t}^{-\alpha}(xg)(t).\tag{3.5}$$

Multiplying both side of (3.5) by $\frac{(\ln(\frac{t}{\rho}))^{\alpha-1}y(\rho)}{\rho\Gamma(\alpha)}$, $\rho\in(0,t)$, $t>0$. Then integrating resulting identity with respect to ρ from 1 to t we obtain

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}(xfg)(t)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}y(\rho)\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}(x)(t)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}y(\rho)f(\rho)g(\rho)\frac{d\rho}{\rho}\\&\geq_H D_{1,t}^{-\alpha}(xf)(t)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}y(\rho)g(\rho)\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}(xg)(t)\frac{1}{\Gamma(\alpha)},\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}y(\rho)f(\rho)\frac{d\rho}{\rho},\end{aligned}\tag{3.6}$$

and this ends the proof of inequality 3.1. □

Now, we gave our main result here.

Theorem 3.2. *Let f and g be two synchronous function on $[0,\infty[$, and $r,p,q:[0,\infty)\to[0,\infty)$. Then for all $t>0$, $\alpha>0$, we have*

$$\begin{aligned}&2_HD_{1,t}^{-\alpha}r(t)\left[_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_H D_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(pfg)(t)\right]+\\&2_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(rfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}r(t)\left[_HD_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(pg)(t)\right]+\\&{}_HD_{1,t}^{-\alpha}p(t)\left[_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right]+\\&{}_HD_{1,t}^{-\alpha}q(t)\left[_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(pg)(t)+_H D_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right]\end{aligned}\tag{3.7}$$

Proof. To prove above theorem, putting $x=p,\ y=q$, and using lemma 3.1, we get

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_H D_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(pfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(pg)(t).\end{aligned}\tag{3.8}$$

Now, multiplying both side of (3.8) by ${}_HD_{1,t}^{-\alpha}r(t)$, we have

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}r(t)\left[_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_H D_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(pfg)(t)\right]\geq\\&{}_HD_{1,t}^{-\alpha}r(t)\left[_HD_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(pg)(t)\right],\end{aligned}\tag{3.9}$$

putting $x=r,y=q$, and using lemma 3.1, we get

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}r(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_{1,t}^{-\alpha} q(t)_HD_{1,t}^{-\alpha}(rfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(rg)(t),\end{aligned}\tag{3.10}$$

multiplying both side of (3.10) by ${}_HD_{1,t}^{-\alpha}p(t)$, we have

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}p(t)\left[_HD_{1,t}^{-\alpha}r(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_{1,t}^{-\alpha} q(t)_HD_{1,t}^{-\alpha}(rfg)(t)\right]\geq\\&{}_HD_{1,t}^{-\alpha}p(t)\left[_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right].\end{aligned}\tag{3.11}$$

With the same arguments as before, we can write

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}q(t)\left[_HD_{1,t}^{-\alpha}r(t)_{1,t}^{-\alpha}(pfg)(t)+_H D_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}(rfg)(t)\right]\geq\\&{}_HD_{1,t}^{-\alpha}q(t)\left[_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(pg)(t)+_H D_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right].\end{aligned}\tag{3.12}$$

Adding the inequalities (3.9), (3.11) and (3.12), we get required inequality (3.7). □

Lemma 3.3. *Let f and g be two synchronous function on $[0,\infty[$. and $x, y : [0,\infty[\to[0,\infty[$. Then for all $t>0$, $\alpha>0$, we have*

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}x(t)_HD_{1,t}^{-\beta}(yfg)(t)+_H D_{1,t}^{-\beta}y(t)_HD_{1,t}^{-\alpha}(xfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}(xf)(t)_HD_{1,t}^{-\beta}(yg)(t)+_H D_{1,t}^{-\beta}(yf)(t)_HD_{1,t}^{-\alpha}(xg)(t).\end{aligned}\tag{3.13}$$

Proof. Now multiplying both side of (3.5) by $\frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}$, $\rho\in(0,t)$, $t>0$ we obtain:

$$\begin{aligned}&\frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}.{}_HD_{1,t}^{-\alpha}(xfg)(t)+\frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}.f(\rho)g(\rho)_HD_{1,t}^{-\alpha}x(t)\geq\\&\frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}.g(\rho)_HD_{1,t}^{-\alpha}(xf)(t)+\frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}.f(\rho)_HD_{1,t}^{-\alpha}(xg)(t),\end{aligned}\tag{3.14}$$

then integrating (3.14) over (1,t), we obtain

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}(xfg)(t)\frac{1}{\Gamma(\beta)}\int_1^t\ln(\frac{t}{\rho})^{\beta-1}y(\rho)\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}(x)(t)\frac{1}{\Gamma(\beta)}\int_1^t\ln(\frac{t}{\rho})^{\beta-1}y(\rho)f(\rho)g(\rho)\frac{d\rho}{\rho}\\&\geq_H D_{1,t}^{-\alpha}(xf)(t)\frac{1}{\Gamma(\beta)}\int_1^t\ln(\frac{t}{\rho})^{\beta-1}y(\rho)g(\rho)\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}(xg)(t)\frac{1}{\Gamma(\beta)},\int_1^t\ln(\frac{t}{\rho})^{\beta-1}y(\rho)f(\rho)\frac{d\rho}{\rho},\end{aligned}\tag{3.15}$$

this ends the proof of inequality (3.13). □

Theorem 3.4. *Let f and g be two synchronous function on $[0,\infty[$, and $r,p,q:[0,\infty)\to[0,\infty)$. Then for all $t>0$, $\alpha>0$, we have*

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\beta}(pfg)(t)+2_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\beta}(qfg)(t)+_H D_{1,t}^{-\beta}q(t)_HD_{1,t}^{-\alpha}(pfg)(t)\right]\\&+\left[{}_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\beta}q(t)+_H D_{1,t}^{-\beta}p(t)_HD_{1,t}^{-\alpha}q(t)\right]_H D_{1,t}^{-\alpha}(rfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\beta}(qg)(t)+_H D_{1,t}^{-\beta}(qf)(t)_HD_{1,t}^{-\alpha}(pg)(t)\right]+\\&{}_HD_{1,t}^{-\alpha}p(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\beta}(qg)(t)+_H D_{1,t}^{-\beta}(qf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right]+\\&{}_HD_{1,t}^{-\alpha}q(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\beta}(pg)(t)+_H D_{1,t}^{-\beta}(pf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right].\end{aligned}\tag{3.16}$$

Proof. To prove above theorem, putting $x=p$, $y=q$, and using lemma 3.3 we get

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\beta}(qfg)(t)+_H D_{1,t}^{-\beta}q(t)_HD_{1,t}^{-\alpha}(pfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\beta}(qg)(t)+_H D_{1,t}^{-\beta}(qf)(t)_HD_{1,t}^{-\alpha}(pg)(t).\end{aligned}\tag{3.17}$$

Now, multiplying both side of (3.16) by ${}_HD_{1,t}^{-\alpha}r(t)$, we have

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\beta}(qfg)(t)+_H D_{1,t}^{-\beta}q(t)_HD_{1,t}^{-\alpha}(pfg)(t)\right]\geq\\&{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\beta}(qg)(t)+_H D_{1,t}^{-\beta}(qf)(t)_HD_{1,t}^{-\alpha}(pg)(t)\right],\end{aligned}\tag{3.18}$$

putting $x=r$, $y=q$, and using lemma 3.3, we get

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}r(t)_HD_{1,t}^{-\beta}(qfg)(t)+_{1,t}^{-\beta}q(t)_HD_{1,t}^{-\alpha}(rfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\beta}(qg)(t)+_H D_{1,t}^{-\beta}(qf)(t)_HD_{1,t}^{-\alpha}(rg)(t),\end{aligned}\tag{3.19}$$

multiplying both side of (3.19) by ${}_HD_{1,t}^{-\alpha}p(t)$, we have

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}p(t)\left[{}_HD_{1,t}^{-\alpha}r(t)_HD_{1,t}^{-\beta}(qfg)(t)+_H D_{1,t}^{-\beta}q(t)_HD_{1,t}^{-\alpha}(rfg)(t)\geq\right]\\&{}_HD_{1,t}^{-\alpha}p(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\beta}(qg)(t)+_H D_{1,t}^{-\beta}(qf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right].\end{aligned}\tag{3.20}$$

With the same argument as before, we obtain

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}q(t)\left[{}_HD_{1,t}^{-\alpha}r(t)_{1,t}^{-\beta}(pfg)(t)+_H D_{1,t}^{-\beta}p(t)_HD_{1,t}^{-\alpha}(rfg)(t)\right]\geq\\&{}_HD_{1,t}^{-\alpha}q(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\beta}(pg)(t)+_H D_{1,t}^{-\beta}(pf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right].\end{aligned}\tag{3.21}$$

Adding the inequalities (3.18), (3.20) and (3.21), we follows the inequality (3.16). □

Remark 3.1. *Applying theorem 3.4 for $\alpha = \beta$, we obtain Theorem 3.2.*

Remark 3.2. *If f, g, r, p and q satisfies the following condition,*

1. *The function f and g is asynchronous on $[0, \infty)$.*
2. *The function r,p,q are negative on $[0, \infty)$.*
3. *Two of the function r,p,q are positive and the third is negative on $[0, \infty)$.*

then the inequality 3.7 and 3.16 are reversed.

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New oscillation criteria for forced superlinear neutral type differential equations

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Abstract

Some new oscillation criteria are established for the neutral type differential equation

$$(a(t)((x(t)+p(t)x(\tau(t)))')^{\alpha})' + q(t)x^{\beta}(t) = e(t),\ t \geq t_0,$$

which are applicable to equations with nonnegative forcing term. Examples are provided to illustrate the results.

Keywords: Neutral differential equation, second order, oscillation, superlinear.

2010 MSC: 34C15.

1 Introduction

Consider the forced second order neutral type differential equation of the form

$$(a(t)((x(t)+p(t)x(\tau(t)))')^{\alpha})' + q(t)x^{\beta}(t) = e(t), \quad t \geq t_0, \tag{1.1}$$

where $\alpha > 0,\ \beta > 0$ are the quotient of odd positive integers, $a(t), p(t), q(t), \tau(t)$, $e(t) \in C([t_0,\infty))$ and $a(t) > 0,\ \int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\alpha}}(t)}dt = \infty,\quad 0 \leq p(t) \leq p < 1,\quad q(t) > 0, e(t) \geq 0,\ \tau(t) \leq t,\ \tau'(t) \geq 0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

Set $z(t) = x(t) + px(\tau(t))$. By a solution of equation (1.1) we mean a function $x(t) \in C([T_x,\infty)),\ T_x \geq t_0$, which has the properties $z(t) \in C^1([T_x,\infty)),\ a(t)(z'(t))^{\alpha} \in C^1([T_x,\infty))$, and satisfies equation (1.1) on $[T_x,\infty)$.

We consider only those solutions $x(t)$ of equation (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that equation (1.1) possess such a solution. A solution of equation (1.1) is called oscillatory if it has infinitely many zeros on $[t_x,\infty)$ and otherwise it is said to be nonoscillatory. Also a solution $x(t)$ is said to be almost oscillatory if either $x(t)$ is oscillatory or $x'(t)$ is oscillatory or $x(t) \to 0$ as $t \to \infty$.

When $p(t) = 0$ and $\alpha = 1$ then equation (1.1) reduces to the following equation

$$(a(t)x'(t))' + q(t)x^{\beta}(t) = e(t), \quad t \geq t_0. \tag{1.2}$$

The oscillatory behavior of solutions of equation (1.2) has been discussed in many papers, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and the references cited therein. In [2, 14], the authors studied oscillatory behavior of equation (1.1) or (1.2) with the assumption that $e(t)$ changes sign and therefore in this paper we establish conditions for the oscillatory behavior of equation (1.1) when $e(t)$ does not changes sign.

In Section 2, we present some oscillation criteria for equation (1.1) and in Section 3, we provide several examples to illustrate our main results.

In the sequel, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

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2 Oscillation Results

We begin with a lemma which can be easily proved using differential calculus.

Lemma 2.1. *Set* $F(x) = ax^{\beta-\alpha} + \frac{b}{x^{\alpha}}$ *for* $x > 0$. *If* $a \geq 0$, $b \geq 0$ *and* $\beta > \alpha \geq 1$ *then* $F(x)$ *attains its minimum with*

$$F_{min} = \frac{\beta a^{\frac{\alpha}{\beta}} b^{1-\frac{\alpha}{\beta}}}{\alpha^{\frac{\alpha}{\beta}}(\beta-\alpha)^{1-\frac{\alpha}{\beta}}}.$$

Theorem 2.1. *Assume that there exists a real valued positive function* $\rho(t)$ *such that*

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left(\rho(s)Q^*(s) - \frac{a(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right) ds = \infty, \tag{2.1}$$

and

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left(\int_{t_0}^{s} (Mq(u) \pm e(u))du\right) ds = \infty \tag{2.2}$$

where

$$Q(t) = \frac{\beta q^{\frac{\alpha}{\beta}}(t) e^{1-\frac{\alpha}{\beta}}(t)(1-p)^{\alpha}}{\alpha^{\frac{\alpha}{\beta}}(\beta-\alpha)^{1-\frac{\alpha}{\beta}}},$$

$$Q^*(t) = \min\{Q(t), d^{(\beta-\alpha)}q(t)(1-p)^{\beta} - d^{-\alpha}e(t)\},$$

$M > 0$ *and* $d > 0$. *Then every solution of equation* (1.1) *is almost oscillatory.*

Proof. Suppose that $x(t)$ is not almost oscillatory.Then there is a positive solution of equation (1.1) such that $x(\tau(t)) > 0$ and $x(t) > 0$ for all $t \geq t_1 \geq t_0$. Then by the definition of not almost oscillatory there are two possibilities to consider: (I) $x'(t) > 0$ for all $t \geq t_1$ and (II) $x'(t) < 0$ for all $t \geq t_1$.

Case (I). Assume that $x'(t) > 0$ for all $t \geq t_1$. Set

$$z(t) = x(t) + p(t)x(\tau(t)) \tag{2.3}$$

then $z'(t) > 0$ for all$t \geq t_1$, and $x(t) \geq (1-p)z(t)$. Then from equation (1.1), we have

$$(a(t)(z'(t))^{\alpha})' + q(t)(1-p)^{\beta}z^{\beta}(t) \leq e(t). \tag{2.4}$$

Define

$$w(t) = \frac{\rho(t)a(t)(z'(t))^{\alpha}}{z^{\alpha}(t)}, \quad t \geq t_1. \tag{2.5}$$

Then inview of (2.4), we obtain

$$w'(t) \leq -\rho(t)\left(q(t)(1-p)^{\beta}z^{\beta-\alpha}(t) - \frac{e(t)}{z^{\alpha}(t)}\right) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha}{(a(t)\rho(t))^{\frac{1}{\alpha}}}w^{1+\frac{1}{\alpha}}(t). \tag{2.6}$$

Set $F(u) = q(t)(1-p)^{\beta}u^{(\beta-\alpha)} - \frac{e(t)}{u^{\alpha}}$. Then, since u is increasing , there is a constant $d > 0$ such that $u \geq d > 0$ and

$$F(u) \geq d^{\beta-\alpha}(1-p)^{\beta}q(t) - d^{-\alpha}e(t). \tag{2.7}$$

Using the inequality

$$Bu - Au^{1+\frac{1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \ A > 0, \tag{2.8}$$

we have

$$\frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha}{(a(t)\rho(t))^{\frac{1}{\alpha}}}w^{1+\frac{1}{\alpha}}(t) \leq \frac{a(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}. \tag{2.9}$$

From (2.6),(2.7) and (2.9), we have

$$w'(t) \leq -\left[\rho(t)Q^*(t) - \frac{a(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}\right]. \tag{2.10}$$

Integrating (2.10) from t_1 *to* t , we obtain

$$\int_{t_1}^{t}\left(\rho(s)Q^*(s)-\frac{a(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right)ds\le w(t_1)-w(t)\le w(t_1)$$

for all large t, and this contradicts (2.1). Next, assume $x(t)<0$ for all $t\ge t_1$, and we use the transformation $y(t)=-x(t)$, then we have $y(t)$ is an eventually positive solution of the equation

$$(a(t)((y(t)+p(t)y(\tau(t)))')^{\alpha})'+q(t)y^{\beta}(t)=-e(t).$$

Define

$$w(t)=\rho(t)\frac{a(t)(z'(t))^{\alpha}}{z^{\alpha}(t)},t\ge t_1, \tag{2.11}$$

where $z(t)=y(t)+p(t)y(\tau(t))$. Then $w(t)>0$ and satisfies

$$w'(t)\le-\rho(t)\left(q(t)(1-p)^{\beta}z^{\beta-\alpha}(t)+\frac{e(t)}{z^{\alpha}(t)}\right)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\alpha w^{1+\frac{1}{\alpha}}(t)}{(a(t)\rho(t))^{\frac{1}{\alpha}}}. \tag{2.12}$$

Set $F(u)=q(t)(1-p)^{\beta}u^{\beta-\alpha}+\frac{e(t)}{u^{\alpha}}$. Using Lemma 2.1, we see that

$$F(u)\ge\frac{\beta q^{\frac{\alpha}{\beta}}(t)e^{1-\frac{\alpha}{\beta}}(t)}{\alpha^{\frac{\alpha}{\beta}}(\beta-\alpha)^{1-\frac{\alpha}{\beta}}}(1-p)^{\alpha}$$

and also (2.8) holds. Then the rest of the proof is similar to that of the above and hence is omitted.

Case (II). Assume that $x'(t)$ is negative for all $t\ge t_1$. From the definition of $z(t)$ we obtain $z'(t)=x'(t)+px'(\tau(t))\tau'(t)$. Since $p\ge0$ and $\tau'(t)>0$ we have $z'(t)<0$ for all $t\ge t_1$. From $x'(t)<0$ we obtain $\lim_{t\to\infty}x(t)=b$. We assert that $b=0$. If not then $x^{\beta}(t)\to b^{\beta}>0$ as $t\to\infty$, and hence there exists a $t_2\ge t_1$ such that $x^{\beta}(t)\ge b^{\beta}$ for $t\ge t_2$. Therefore, we have

$$(a(t)(z'(t))^{\alpha})'\le-q(t)b^{\beta}+e(t).$$

Integrating the last inequality from t_2 *to* t, we obtain

$$a(t)(z'(t))^{\alpha}<a(t)(z'(t))^{\alpha}-a(t_2)(z'(t_2))^{\alpha}\le-\int_{t_2}^{t}(b^{\beta}q(s)-e(s))ds$$

and then

$$z'(t)\le-\left(\frac{1}{a(t)}\int_{t_2}^{t}(b^{\beta}q(s)-e(s))ds\right)^{\frac{1}{\alpha}},\quad t\ge t_2.$$

Again integrating the above inequality from t_2 to t, we obtain

$$z(t)\le z(t_2)-\int_{t_2}^{t}\left(\frac{1}{a(s)}\int_{t_2}^{s}(b^{\beta}q(u)-e(u))du\right)^{\frac{1}{\alpha}}ds.$$

Condition (2.2) implies that $z(t)$ is negative for all $t\ge t_2$, a contradiction. Finally, for $x(t)<0$ for all $t\ge t_1$, we use the transformation $y(t)=-x(t)$ then we have $y(t)$ is an eventually positive solution of the equation

$$(a(t)(z'(t))^{\alpha})'+q(t)y^{\beta}(t)=-e(t)$$

where $z(t)=y(t)+p(t)y(\tau(t))>0$. The rest of the proof is similar to the above and hence omitted. The proof is now complete. □

Corollary 2.1. *Assume that all the conditions of Theorem 2.2 hold, except the condition (2.1) is replaced by*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\rho(s)Q^*(s)ds=\infty,$$

and

$$\lim_{t\to\infty}\sup\int\limits_{t_0}^{t}\frac{a(s)(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)}ds<\infty.$$

Then every solution of equation (1.1) *is almost oscillatory.*

In the following theorem, we provide another sufficient condition for almost oscillation of equation (1.1).

Definition 2.1. *Consider the sets* $D_0=\{(t,s):t>s\geq t_0\}$ *and* $D=\{(t,s):t\geq s\geq t_0\}$. *Assume that* $H\in C(D,R)$ *satisfies the following assumptions:*
(A_1) $H(t,t)=0,\quad t\geq t_0;\quad H(t,s)>0,\quad (t,s)\in D_0;$
(A_2) H *has a nonpositive continuous partial derivative with respect to the second variable in* D_0.
Then the function H *has the property* P.

Theorem 2.2. *Assume that condition* (2.2) *holds. Further assume that* $H\in C(D,R)$ *has the property* P *and there exists a function* $\rho\in C'([t_0,\infty),(0,\infty))$ *such that for all sufficiently large* $t_1\geq t_0$

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_1)}\int\limits_{t_1}^{t}\left[H(t,s)\rho(s)Q^*(s)-\frac{a(s)\rho(s)}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'(s)}{\rho(s)}H^{\frac{1}{\alpha+1}}(t,s)-h(t,s)\right)^{\alpha+1}\right]ds=\infty, \tag{2.13}$$

where $h(t,s)=\frac{1}{H^{\frac{\alpha}{\alpha+1}}(t,s)}\frac{\partial}{\partial s}H(t,s),\quad (t,s)\in D_0$. *Then every solution of equation* (1.1) *is almost oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.1 we have two cases to consider. First assume that $x'(t)>0$ for all $t\geq t_1$. Define $w(t)$ by (2.5) , then $w(t)>0$ and satisfies

$$w'(t)\leq-\rho(t)Q^*(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\alpha}{(a(t)\rho(t))^{\frac{1}{\alpha}}}w^{1+\frac{1}{\alpha}}(t). \tag{2.14}$$

In (2.14), replace t by s and then multiply both sides by $H(t,s)$, and integrate with respect to s from t_1 to t, we have

$$\int\limits_{t_1}^{t}H(t,s)\rho(s)Q^*(s)ds\leq-\int\limits_{t_1}^{t}H(t,s)w'(s)ds+\int\limits_{t_1}^{t}H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds-\alpha\int\limits_{t_1}^{t}\frac{H(t,s)}{(a(s)\rho(s))^{\frac{1}{\alpha}}}w^{1+\frac{1}{\alpha}}(s)ds.$$

Thus we obtain

$$\begin{aligned}\int\limits_{t_1}^{t}H(t,s)\rho(s)Q^*(s)ds\leq H(t,t_1)w(t_1)&-\int\limits_{t_1}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-\frac{\rho'(s)}{\rho(s)}H(t,s)\right]w(s)ds\\&-\alpha\int\limits_{t_1}^{t}\frac{H(t,s)}{(a(s)\rho(s))^{\frac{1}{\alpha}}}w^{1+\frac{1}{\alpha}}(s)ds.\end{aligned} \tag{2.15}$$

From the last inequality and (2.8), we obtain

$$\frac{1}{H(t,t_1)}\int\limits_{t_1}^{t}\left[H(t,s)\rho(s)Q^*(s)-\frac{a(s)\rho(s)}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'(s)}{\rho(s)}H^{\frac{1}{\alpha+1}}(t,s)-h(t,s)\right)^{\alpha+1}\right]ds$$
$$\leq w(t_1)$$

which contradicts (2.13). Next we consider the case when $x(t)<0$ for all $t\geq t_1$ and we use the transformation $y(t)=-x(t)$ then $y(t)$ is a positive solution of the equation

$$(a(t)(z'(t))^{\alpha})'+q(t)y^{\beta}(t)=-e(t)$$

where $z(t)=y(t)+p(t)y(\tau(t))$. Define $w(t)$ by (2.11), then (2.12) holds. The remainder of the proof is similar to that of first case and hence omitted. The proof for the case (II) is similar to that of Theorem 2.2. The proof is now complete. □

Corollary 2.2. *Assume that all the conditions of Theorem 2.2 hold except the condition (2.13) is replaced by*

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)\rho(s)Q^*(s)ds = \infty, \tag{2.16}$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_1)} \int_{t_1}^{t} a(s)\rho(s) \left(\frac{\rho'(s)}{\rho(s)} H^{\frac{1}{\alpha+1}}(t,s) - h(t,s) \right)^{\alpha+1} ds < \infty. \tag{2.17}$$

Then the conclusion of Theorem 2.2 holds.

Remark 2.1. *By choosing the function $H(t,s)$ in appropriate manners, we can derive several oscillation criteria for equation (1.1). For example, set*

$$H(t,s) = (t-s)^m, \quad m \geq 1, \quad (t,s) \in D_0$$

we have the following result.

Corollary 2.3. *Assume that all the conditions of Corollary 2.2 are satisfied except the conditions (2.16) and (2.17) replaced by*

$$\lim_{t\to\infty} \sup \frac{1}{(t-t_1)^m} \int_{t_1}^{t} (t-s)^m \rho(s)Q^*(s)ds = \infty$$

and

$$\lim_{t\to\infty} \sup \frac{1}{(t-t_1)^m} \int_{t_1}^{t} a(s)\rho(s) \left(\frac{\rho'(s)}{\rho(s)} (t-s)^{\frac{m}{\alpha+1}} + m(t-s)^{\frac{m}{\alpha+1}-11} \right)^{\alpha+1} ds < \infty.$$

Then the conclusion of Theorem 2.1 holds.

3 Examples

In this section we present some examples to illustrate the main results.

Example 3.1 Consider the differential equation

$$(((x(t) + 2x(t-2))')^3)' + tx^5(t) = \frac{1}{t^2}, \quad t \geq 1. \tag{3.1}$$

Here $p = 2,\ \alpha = 3,\ \beta = 5,\ \tau(t) = t-2,\ q(t) = t$ and $e(t) = \frac{1}{t^2}$. By taking $\rho(t) = 1$, we see that all conditions of Theorem 2.1 are satisfied. Hence every solution of equation (3.1) is almost oscillatory.

Example 3.2 Consider the differential equation

$$(t(x(t) + \frac{1}{2}x(\frac{t}{2}))')' + t^3(t+1)x^3(t) = t + 1 + \frac{2}{t^2}, \quad t \geq 1. \tag{3.2}$$

Here $p = \frac{1}{2},\ \alpha = 1,\ \beta = 3,\ \tau(t) = \frac{t}{2},\ q(t) = t^3(t+1)$ and $e(t) = t+1+\frac{2}{t^2}$. By taking $\rho(t) = 1$, we see that all conditions of Theorem 2.1 are satisfied and hence every solution of equation (3.2) is almost oscillatory. Infact $x(t) = \frac{1}{t}$ is one such solution of equation (3.2) since it satisfies the equation.

Example 3.3 Consider the differential equation

$$(x(t) + 2x(\frac{t}{2}))'' + t^2x^3(t) = t, \quad t \geq 1. \tag{3.3}$$

Here $p = 2,\ \alpha = 1,\ \beta = 3,\ \tau(t) = \frac{t}{2},\ q(t) = t^2$ and $e(t) = t$. By taking $\rho(t) = 1$ and $H(t,s) = (t-s)^2$ we see that all conditions of Corollary 2.3 are satisfied, and hence every solution of equation (3.3) is almost oscillatory.

Remark 3.1. *Since the forcing terms $e(t)$ in the above examples are positive, the results obtained in [2-14] cannot be applied to these examples. So our results are new and applicable to neutral differential equations with positive forcing terms.*

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Semilinear functional differential equations with fractional order and finite delay

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Abstract

In this paper, we establish sufficient conditions for existence and uniqueness of solutions for semilinear functional differential equations with finite delay involving the Riemann-Liouville fractional derivative. Our approach is based on resolvent operators, the Banach contraction principle, and the nonlinear alternative of Leray-Schauder type.

Keywords: Semilinear functional differential equation, fractional derivative, fractional integral, fixed point, mild solutions, resolvent operator.

2010 MSC: 34A08, 34K05.

1 Introduction

This paper is concerned with existence of solutions defined on a compact real interval for fractional order semilinear functional differential equations of the form

$$D^{\alpha}y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b], \ 0 < \alpha < 1, \tag{1.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $f : J \times C([-r, 0], E) \to E$ is a continuous function, $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E, $\phi : [-r, 0] \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Differential equations with fractional order have recently proved to be valuable tools for the description of hereditary properties of various materials and systems. For more details, see [9].

Fractional calculus appears in rheology, viscoelasticity, electrochemistry, electromagnetism, etc. For details, see the monographs of Kilbas et al. [8], Miller and Ross [10], Podlubny [13], Oldham et al. [12]. For some recent developments on the subject, see for instance [1, 2, 3, 4, 7, 11] and references cited therein.

The purpose of this paper is to study the existence and uniqueness of mild solutions for (1.1)-(1.2) by virtue of resolvent operator. In Section 2 we recall some definitions and preliminary facts which will be used in the sequel. In Section 3, we give our main existence and uniqueness results. An example will be presented in the last section illustrating the abstract theory.

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2 Preliminaries

In this section, we recall some definitions and propositions of fractional calculus and resolvent operators. Let E be a Banach space. By $C(J,E)$ we denote the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

For $\phi \in C([-r,b],E)$ the norm of ϕ is defined by

$$\|\phi\|_{\mathfrak{D}} = \sup\{|\phi(\theta)| : \theta \in [-r,b]\}.$$

$C([-r,0],E)$ is endowed with norm defined by

$$\|\psi\|_C = \sup\{|\psi(\theta)| : \theta \in [-r,0]\}.$$

$\mathcal{L}(E)$ denotes the space of bounded linear operators from E into E, with norm

$$\|N\|_{\mathcal{L}(E)} = \sup\{|N(y)| : |y| = 1\}.$$

Definition 2.1. *[8, 13] The Riemann-Liouville fractional primitive of order $\alpha \in \mathbb{R}^+$ of a function $h : (0,b] \to E$ is defined by*

$$I_0^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

provided the right hand side exists pointwise on $(0,b]$, where Γ is the gamma function.

Definition 2.2. *[8, 13] The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a continuous function $h : (0,b] \to E$ is defined by*

$$\begin{aligned} \frac{d^\alpha h(t)}{dt^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds \\ &= \frac{d}{dt} I_0^{1-\alpha} h(t). \end{aligned}$$

Consider the fractional differential equation

$$D^\alpha y(t) = Ay(t) + f(t), \quad t \in J,\ 0 < \alpha < 1,\ y(0) = 0, \tag{2.1}$$

where A is a closed linear unbounded operator in E and $f \in C(J,E)$. Equation (2.1) is equivalent to the following integral equation [8]

$$y(t) = \frac{1}{\Gamma(\alpha)} A \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,\ \ t \in J. \tag{2.2}$$

This equation can be written in the following form of integral equation

$$y(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ay(s)ds,\ \ t \ge 0, \tag{2.3}$$

where

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds. \tag{2.4}$$

Examples where the exact solution of (2.1) and the integral equation (2.2) are the same, are given in [4]. Let us assume that the integral equation (2.3) has an associated resolvent operator $(S(t))_{t\ge0}$ on E.

Next we define the resolvent operator of the integral equation (2.3).

Definition 2.3. *[14, Definition 1.1.3] A one parameter family of bounded linear operators $(S(t))_{t\ge0}$ on E is called a resolvent operator for (2.2) if the following conditions hold:*

(a) $S(\cdot)x \in C([0,\infty),E)$ and $S(0)x = x$ for all $x \in E$;

(b) $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and every $t \ge 0$;

(c) for every $x \in D(A)$ and $t \geq 0$,

$$S(t)x = x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} AS(s)x ds. \tag{2.5}$$

Here and hereafter we assume that the resolvent operator $(S(t))_{t\geq 0}$ is analytic [14, Chapter 2], and there exist a function $\phi_A \in L^1_{loc}([0,\infty), \mathbb{R}^+)$ such that $\|S'(t)x\| \leq \phi_A(t)\|x\|_{[D(A)]}$ for all $t > 0$ and each $x \in D(A)$.

We have the following concept of solution using Definition 1.1.1 in [14].

Definition 2.4. *A function $u \in C(J, E)$ is called a mild solution of the integral equation (2.3) on J if $\int_0^t (t-s)^{\alpha-1} u(s)ds \in D(A)$ for all $t \in J$, $h(t) \in C(J, E)$ and*

$$u(t) = \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds + h(t), \quad \forall t \in J.$$

The next result follows from [14, Proposition I.1.2, Theorem II.2.4, Corollary II.2.6].

Lemma 2.1. *Under the above conditions the following properties are valid.*

(i) If $u(\cdot)$ is a mild solution of (2.3) on J, then the function $t \to \int_0^t S(t-s)h(s)ds$ is continuously differentiable on J, and

$$u(t) = \frac{d}{dt} \int_0^t S(t-s)h(s)ds, \quad \forall t \in J.$$

(ii) If $h \in C^\beta(J, E)$ for some $\beta \in (0,1)$, then the function defined by

$$u(t) = S(t)(h(t) - h(0)) + \int_0^t S'(t-s)[h(s) - h(t)]ds + S(t)h(0), \quad t \in J,$$

is a mild solution of (2.3) on J.

(iii) If $h \in C(J, [D(A)])$ then the function $u : J \to E$ defined by

$$u(t) = \int_0^t S'(t-s)h(s)ds + h(t), \quad t \in J,$$

is a mild solution of (2.3) on J.

3 Main Results

In this section we give our main existence results for problem (1.1)-(1.2). This problem is equivalent to the following integral equation

$$y(t) = \begin{cases} \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} y(s)ds + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y_s)ds, & t \in J, \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

Motivated by Lemma 2.1 and the above representation, we introduce the concept of mild solution.

Definition 3.1. *We say that a continuous function $y : [-r, b] \to E$ is a mild solution of problem (1.1)-(1.2) if:*

1. $\int_0^t (t-s)^{\alpha-1} y(s)ds \in D(A)$ *for $t \in J$,*

2. $y(t) = \phi(t)$, $t \in [-r, 0]$, *and*

3. $y(t) = \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} y(s)ds + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y_s)ds$, $t \in J$.

Suppose that there exists a resolvent $(S(t))_{t\geq 0}$ which is differentiable and the function f is continuous. Then by Lemma 2.1 (iii), if $y : [-r, b] \to E$ is a mild solution of (1.1)-(1.2), then

$$y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_s)ds + \int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau)d\tau\right)ds, & t \in J, \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

Our first existence result for problem (1.1)-(1.2) is based on the Banach's contraction principle.

Theorem 3.1. *Let $f : J \times C([-r, 0], E) \to E$ be continuous and there exists a constant $L > 0$ such that*

$$|f(t, u) - f(t, v)| \leq L\|u - v\|_C, \quad for \quad t \in J \quad and \ u, v \in C([-r, 0], E).$$

If

$$\frac{Lb^\alpha}{\Gamma(\alpha+1)}\left(1 + \|\phi_A\|_{L^1}\right) < 1, \tag{3.1}$$

then the problem (1.1)-(1.2) has a unique mild solution on $[-r, b]$.

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $F : C([-r, b], E) \to C([-r, b], E)$ defined by:

$$F(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_s)ds + \int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau)d\tau\right)ds, & t \in [0, b]. \end{cases}$$

We need to prove that F has a fixed point, which is a unique mild solution of (1.1)-(1.2) on $[-r, b]$. We shall show that F is a contraction. Let $y, z \in C([-r, b], E)$. For $t \in [0, b]$, we have

$$\begin{aligned}
& |F(y)(t) - F(z)(t)| \\
= & \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f(s, y_s) - f(s, z_s)]ds \right. \\
& \left. + \int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[f(\tau, y_\tau) - f(\tau, z_\tau]d\tau\right)ds\right| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, y_s) - f(s, z_s)|ds \\
& + \int_0^t \phi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^\tau (s-\tau)^{\alpha-1}|f(\tau, y_\tau) - f(\tau, z_\tau)|d\tau ds \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L\|y_\tau - z_\tau\|_C ds + \frac{1}{\Gamma(\alpha)}\int_0^t \phi_A(t-s)\int_0^s (s-\tau)^{\alpha-1} L\|y_\tau - z_\tau\|_C d\tau ds \\
\leq & \frac{L}{\Gamma(\alpha)}\|y - z\|_{\mathfrak{D}}\int_0^t (t-s)^{\alpha-1}ds + \frac{L}{\Gamma(\alpha)}\|y - z\|_{\mathfrak{D}}\int_0^t \phi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}d\tau ds \\
\leq & \frac{Lb^\alpha}{\Gamma(\alpha+1)}\|y - z\|_{\mathfrak{D}} + \frac{\|\phi_A\|_{L^1} Lb^\alpha}{\Gamma(\alpha+1)}\|y - z\|_{\mathfrak{D}}.
\end{aligned}$$

Taking the supremum over $t \in [-r, b]$, we get

$$\|F(y) - F(z)\|_{\mathfrak{D}} \leq \frac{Lb^\alpha}{\Gamma(\alpha+1)}\left(1 + \|\phi_A\|_{L^1}\right)\|y - z\|_{\mathfrak{D}}.$$

By (3.1) F is a contraction and thus, by the contraction mapping theorem, we deduce that F has a unique fixed point. This fixed point is the mild solution of (1.1)-(1.2). □

Next, we give an existence result based upon the following nonlinear alternative of Leray-Schauder applied to completely continuous operators [5].

Theorem 3.2. *Let E a Banach space, and $U \subset E$ convex with $0 \in U$. Let $F : U \to U$ be a completely continuous operator. Then either* *(a) F has a fixed point, or*

(b) The set $\mathcal{E} = \{x \in U : x = \lambda F(x),\ 0 < \lambda < 1\}$ *is unbounded.*

Our main result here reads:

Theorem 3.3. *Let* $f : J \times C([-r, 0], E) \to E$ *be continuous. Assume that:*

(i). $S(t)$ *is compact for all* $t > 0$;

(ii). there exist functions $p, q \in C(J, \mathbb{R}_+)$ *such that*

$$|f(t, u)| \leq p(t) + q(t)\|u\|_C, \quad t \in J \text{ and } u \in C([-r, 0], E).$$

Then, the problem (1.1)-(1.2) has at least one mild solution on $[-r, b]$*, provident that*

$$\frac{b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)} (1 + \|\phi_A\|_{L^1}) < 1.$$

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $F : C([-r, b], E) \to C([-r, b], E)$ defined in Theorem 3.1, namely,

$$F(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y_s) ds + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau) d\tau \right) ds, & t \in [0, b]. \end{cases}$$

In order to prove that F is completely continuous, we divide the operator F into two operators:

$$F_1(y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds,$$

and

$$F_2(y)(t) = \int_0^t S'(t-s) F_1(y)(s) ds.$$

We prove that F_1 and F_2 are completely continuous. We note that the condition (i) implies that $S'(t)$ is compact for all $t > 0$ (see [6, Lemma 2.2]).

Step 1: F_1 is completely continuous.
At first, we prove that F_1 is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ as $n \to \infty$ in $C([-r, b], E)$. Then for $t \in [0, b]$ we have

$$\begin{aligned} |F_1(y_n)(t) - F_1(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f(s, y_{ns}) - f(s, y_s) \Big| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \|f(\cdot, y_{n\cdot}) - f(\cdot, y_\cdot)\|_\infty \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{b^\alpha}{\Gamma(\alpha+1)} \|f(\cdot, y_{n\cdot}) - f(\cdot, y_\cdot)\|_\infty. \end{aligned}$$

Since f is a continuous function, we have

$$\|F_1(y_n) - F_1(y)\|_{\mathfrak{D}} \to 0 \text{ as } n \to \infty.$$

Thus F_1 is continuous.
Next, we prove that F_1 maps bounded sets into bounded sets in $C([-r, b], E)$. Indeed, it is enough to show that for any $\rho > 0$, there exists a positive constant δ such that for each $y \in B_\rho = \{y \in C([-r, b], E) : \|y\|_{\mathfrak{D}} \leq \rho\}$ one has $F_1(y) \in B_\delta$. Let $y \in B_\rho$. Since f is a continuous function, we have for each $t \in [0, b]$

$$|F_1(y)(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds \right|$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|f(s,y_s)\big|ds \\ &\leq \frac{b^\alpha}{\Gamma(\alpha+1)}\big(\|p\|_\infty+\rho\|q\|_\infty\big)=\delta^*<\infty. \end{aligned}$$

Then, $\|F_1(y)\|_{\mathfrak{D}} = \max\{\|\phi\|_C, \delta^*\} = \delta$, and hence $F_1(y) \in B_\delta$.
Now, we prove that F_1 maps bounded sets into equicontinuous sets of $C([-r, b], E)$. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set. Let $y \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned} & |F_1(y)(\tau_2)-F_1(y)(\tau_1)| \\ = & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}(\tau_2-s)^{\alpha-1}f(s,y_s)ds-\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}f(s,y_s)ds\right| \\ \leq & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1-\epsilon}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]f(s,y_s)ds\right| \\ & +\left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]f(s,y_s)ds\right| \\ & +\left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}f(s,y_s)ds\right| \\ \leq & \frac{\|p\|_\infty+\rho\|q\|_\infty}{\Gamma(\alpha)}\bigg(\int_0^{\tau_1-\epsilon}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]ds \\ & +\int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]ds+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\bigg). \end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_1 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$ we define

$$F_{1\epsilon}(y)(t)=\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}f(s,y_s)ds.$$

Note that the set

$$\left\{\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}f(s,y_s)ds : y\in B_\rho\right\}$$

is bounded since

$$\begin{aligned} \left|\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}f(s,y_s)ds\right| &\leq (\|p\|_\infty+\rho\|q\|_\infty)\left|\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}ds\right| \\ &\leq \frac{\|p\|_\infty+\rho\|q\|_\infty}{\Gamma(\alpha+1)}(t-\epsilon)^\alpha. \end{aligned}$$

Then for $t > 0$, the set

$$Y_\epsilon(t)=\{F_{1\epsilon}(y)(t) : y\in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned} \Big|F_1(y)(t)-F_{1\epsilon}(y)(t)\Big| &\leq \frac{\|p\|_\infty+\rho\|q\|_\infty}{\Gamma(\alpha)}\bigg(\int_0^{t-\epsilon}[(t-s)^{\alpha-1}-(t-s-\epsilon)^{\alpha-1}]ds+\int_{t-\epsilon}^t(t-s)^{\alpha-1}ds\bigg) \\ &\leq \frac{\|p\|_\infty+\rho\|q\|_\infty}{\Gamma(\alpha+1)}(t^\alpha-(t-\epsilon)^\alpha). \end{aligned}$$

Therefore, the set $Y(t) = \{F_1(y)(t) : y \in B_\rho\}$ is precompact in E. Hence the operator F_1 is completely continuous.
Step 2: F_2 is completely continuous.
The operator F_2 is continuous, since $S'(\cdot) \in C([0, b], \mathcal{L}(E))$ and F_1 is continuous as proved in Step 1.
Now, let B_ρ be a bounded set as in Step 1. For $y \in B_\rho$ we have

$$|F_2(y)(t)| \leq \int_0^t |S'(t-s)||F_1(y)(s)|ds$$

$$\leq \int_0^t \phi_A(t-s)\|F_1(y)(s)\|_{[D(A)]}ds$$
$$\leq \frac{\|\phi\|_{L^1} b^\alpha(\|p\|_\infty + \rho\|q\|_\infty)}{\Gamma(\alpha+1)} = \delta'.$$

Thus, there exists a positive number δ' such that $\|F_2(y)\|_{\mathfrak{D}} \leq \delta'$. This means that $F_2(y) \in B_{\delta'}$.
Next, we shall show that F_2 maps bounded sets into equicontinuous sets in $C([-r, b], E)$. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set as in Step 1. Let $y \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned}
&|F_2(y)(\tau_2) - F_2(y)(\tau_1)| \\
= &\left|\int_0^{\tau_2} S'(\tau_2 - s)F_1(y)(\tau_2)ds - \int_0^{\tau_1} S'(\tau_1 - s)F_1(y)(\tau_1)ds\right| \\
\leq &\frac{b^\alpha\left(\|p\|_\infty + \rho\|q\|_\infty\right)}{\Gamma(\alpha+1)}\Bigg(\int_0^{\tau_1-\epsilon} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds \\
&+ \int_{\tau_1-\epsilon}^{\tau_1} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds + \int_{\tau_1}^{\tau_2} |S'(\tau_2 - s)|ds\Bigg).
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_2 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$ we define

$$F_{2\epsilon}(y)(t) = S'(\epsilon)\int_0^{t-\epsilon} S'(t-s-\epsilon)F_1(y)(s)ds.$$

Since $S'(t)$ is a compact operator for $t > 0$, the set

$$Y_\epsilon(t) = \{F_{2\epsilon}(y)(t) : y \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\left|F_2(y)(t) - F_{2\epsilon}(y)(t)\right| \leq \frac{\|\phi_A\|_{L^1}\left(\|p\|_\infty + \rho\|q\|_\infty\right)}{\Gamma(\alpha+1)}\left(t^\alpha - (t-\epsilon)^\alpha\right).$$

Then $Y(t) = \{F_2(y)(t) : y \in B_\rho\}$ is precompact in E. Hence the operator F_2 is completely continuous.
Step 3: A priori bound on solutions.
Now, it remains to show that the set

$$\mathcal{E} = \{y \in \mathcal{C}([-r, b], E) : y = \lambda F(y), \quad 0 < \lambda < 1\}$$

is bounded.
Let $y \in \mathcal{E}$ be any element. Then, for each $t \in [0, b]$,

$$y(t) = \lambda F(y)(t) \quad = \quad \lambda\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, y_s)ds + \lambda\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}f(\tau, y_\tau)d\tau\right)ds.$$

Then

$$\begin{aligned}
|y(t)| \;\leq\; &\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, y_s)ds + \int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}f(\tau, y_\tau)d\tau\right)ds\right| \\
\leq\; &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, y_s)|ds + \int_0^t \phi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}|f(\tau, y_\tau)|d\tau ds \\
\leq\; &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\|p\|_\infty + \|q\|_\infty\|y_s\|_C]ds \\
&+ \int_0^t \phi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\|p\|_\infty + \|q\|_\infty\|y_s\|_C]d\tau ds \\
\leq\; &\frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\|y_s\|_C + \frac{\|\phi_A\|_{L^1}b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{\|\phi_A\|_{L^1}b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\|y_s\|_C \\
\leq\; &\frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}\left(1 + \|\phi_A\|_{L^1}\right) + \frac{b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\left(1 + \|\phi_A\|_{L^1}\right)\|y\|_{\mathfrak{D}},
\end{aligned}$$

and consequently

$$\|y\|_{\mathfrak{D}} \leq \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} (1 + \|\phi_A\|_{L^1}) \left\{1 - \frac{b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)} (1 + \|\phi_A\|_{L^1})\right\}^{-1}.$$

Hence the set $\mathcal{E}$ is bounded. As a consequence of Theorem 3.2 we deduce that F has at least a fixed point which gives rise to a mild solution of problem (1.1)-(1.2) on $[-r, b]$. □

4 Example

As an application of our results we consider the following fractional time partial functional differential equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + Q(t, u(t-r, x)), \; x \in [0,\pi], t \in [0,b], \alpha \in (0,1), \tag{4.1}$$

$$u(t,0) = u(t,\pi) = 0, \; t \in [0,b], \tag{4.2}$$

$$u(t,x) = \phi(t,x), \quad x \in [0,\pi], \quad t \in [-r, 0], \tag{4.3}$$

where $r > 0$, $\phi : [-r,0] \times [0,\pi] \to \mathbb{R}$ is continuous and $Q : [0,b] \times \mathbb{R} \to \mathbb{R}$ is a given function.

To study this system, we take $E = L^2[0,\pi]$ and let A be the operator given by $Aw = w''$ with domain $D(A) = \{w \in E, w, w' \text{ are absolutely continuous}, w'' \in E, w(0) = w(\pi) = 0\}$.

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \qquad w \in D(A),$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $w_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(nx)$, $n = 1, 2, \ldots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on E and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} (w, w_n) w_n, \qquad w \in E.$$

From these expressions it follows that $(T(t))_{t\geq 0}$ is uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is compact operator for all $\lambda \in \rho(A)$.

From [14, Example 2.2.1] we know that the integral equation

$$u(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A u(s) ds, \; s \geq 0,$$

has an associated analytic resolvent operator $(S(t))_{t\geq 0}$ on E given by

$$S(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda, & t > 0, \\ I, & t = 0, \end{cases}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in (\pi, \frac{\pi}{2})$. $S(t)$ is differentiable (Proposition 2.15 in [3]) and there exists a constant $M > 0$ such that $\|S'(t)x\| \leq M\|x\|$, for $x \in D(A)$, $t > 0$.

To represent the differential system (4.1) $-$ (4.3) in the abstract form (1.1)-(1.2), let

$$\begin{aligned} y(t)(x) &= u(t,x), \quad t \in [0,b], \; x \in [0,\pi] \\ \phi(\theta)(x) &= \phi(\theta, x), \quad \theta \in [-r, 0], \; x \in [0,\pi] \\ f(t,\phi)(x) &= Q(t, \phi(\theta, x)), \quad \theta \in [-r,0], \; x \in [0,\pi] \end{aligned}$$

Choose b such that

$$\frac{Lb^\alpha}{\Gamma(\alpha+1)} (1 + M) < 1.$$

Since the conditions of Theorem 3.1 are satisfied, there is a function $u \in C([-r,b], \; L^2[0,\pi])$ which is a mild solution of (4.1)-(4.3).

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