ISSN 2319-3786
VOLUME 1, ISSUE 1, JANUARY 2013

M
J
M

# Malaya Journal of Matematik

an international journal of mathematical sciences

UNIVERSITY PRESS
5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),
Coimbatore- 641114, Tamil Nadu, India.
www.malayajournal.org

The Malaya Journal of Matematik is published quarterly in single volume annually and four issues constitute one volume appearing in the months of January, April, July and October.

**Subscription**

The subscription fee is as follows:

USD 350.00 For USA and Canada

Euro 190.00 For rest of the world

Rs. 4000.00 In India. (For Indian Institutions in India only)

Prices are inclusive of handling and postage; and issues will be delivered by Registered Air-Mail for subscribers outside India.

**Subscription Order**

Subscription orders should be sent along with payment by Cheque/ D.D. favoring "Malaya Journal of Matematik" payable at COIMBATORE at the following address:

**MKD Publishing House**

5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),

Coimbatore- 641114, Tamil Nadu, India.

Contact No. : +91-9585408402

E-mail : info@mkdpress.com; editorinchief@malayajournal.org; publishingeditor@malayajournal.org

Website : https://mkdpress.com/index.php/index/index

**Vol. 1 No. 01 (2013): Malaya Journal of Matematik (MJM)**

Malaya Journal of Matematik 1(1)(2013) 1–9

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

# On graph differential equations and its associated matrix differential equations

J. Vasundhara Devi,[a,*] R.V.G. Ravi Kumar[b] and N. Giribabu[c]

[a,b,c] *GVP-Prof.V.Lakshmikantham Institute for Advanced Studies, Department of Mathematics, GVP College of Engineering, Visakhapatnam-530048, Andhra Pradesh, India.*

**Abstract**

Networks are one of the basic structures in many physical phenomena pertaining to engineering applications. As a network can be represented by a graph which is isomorphic to its adjacency matrix, the study of analysis of networks involving rate of change with respect to time reduces to the study of graph differential equations or equivalently matrix differential equations. In this paper, we develop the basic infrastructure to study the IVP of a graph differential equation and the corresponding matrix differential equation. Criteria are obtained to guarantee the existence of a solution and an iterative technique for convergence to the solution of a matrix differential equation is developed.

*Keywords:* Dynamic graph, adjacency matrix, graph linear space, graph differential equations, matrix differential equations, existence of a solution, monotone iterative technique.

*2010 MSC:* 34G20.



## 1 Introduction

A graph [1] represents a network of a natural or a man-made system, wherein interconnections between its constituents play an important role. Graphs have been utilized to model organizational structures in social sciences. It has been observed that the graphs which are static in nature limit the study in social phenomena where changes with time are natural. Hence, it was thought that a dynamic graph will be more appropriate in modeling such social behavior [2, 4]. The concept of a dynamic graph was introduced in [2] and a graph differential equation was utilized to describe the famous prey predator model and its stability properties were studied [2].

The importance of networks in engineering fields and the representation of a network by a graph led us to consider a graph differential equation as an important topic of study. Thus we plan to study the existence of solutions through monotone iterative technique [3] for the graph differential equation through its associated matrix differential equations.

## 2 Preliminaries

In this section we introduce the notions and concepts that are necessary to develop graph differential equations and and the corresponding matrix differential equations. All the basic definitions and results are taken from [2] and suitable changes are made to suit our set up. Consider a weighted directed simple graph (called digraph) $D = (V, E)$ an ordered pair, where $V$ is a non-empty finite set of $N$ vertices and $E$ is the set of all directed edges. To each directed edge $(v_i, v_j)$ we assign a nonzero weight $e_{ij} \in \mathbb{R}$ if $(v_i, v_j) \in E$ while $e_{ij} = 0$ if $(v_i, v_j) \notin E$. Corresponding to a digraph $D$ we associate an adjacency matrix $E = (e_{ij})$. This association is

*Corresponding author.
*E-mail address*: jvdevi@gmail.com (J. Vasundhara Devi).

an isomorphism.

**Graph linear space.** Let $v_1, v_2, \dots v_N$ be $N$ vertices, $N$ fixed and $\mathcal{D}_N$ be the set of all weighted directed simple graphs ( called digraphs), $D = (V, E)$. Then $(\mathcal{D}_N, +, .)$ is a linear space over the field of real numbers with the following definition of the addition and scalar multiplication.
Let $D_1, D_2$ be two digraphs $D_1 = (V, E_1)$ and $D_2 = (V, E_2)$.
Then the sum $D_1 + D_2$ is defined as

$$D_1 + D_2 = (V, E_1 + E_2)$$

where $E_1 + E_2$ is the set of all edges $(v_i, v_j) \in E_1 \cup E_2$ where the weight of $(v_i, v_j)$ is defined as the sum of the weights of the edges $(v_i, v_j)$ in the respective digraphs $D_1$ and $D_2$.
Let $D = (V, E)$ be a graph then by $\alpha D = (V, \alpha E)$ where $\alpha E$ is the set of all edges $(v_i, v_j)$ whose weight is $\alpha$ times the weight of $(v_i, v_j)$. Observe that if $\alpha = 0$ then $\alpha D = 0 \in \mathcal{D}_N$ is the graph consisting of $N$ isolated vertices. Hence the set of edges is empty. With the fore mentioned operations, $(\mathcal{D}_N, +, .)$ is a linear space. This space is isomorphic to the linear space $\mathcal{M}_N$ of all $N \times N$ adjacency matrices with entries of the principal diagonal being zero, defined over the field of real numbers, with the usual definition of matrix addition and scalar multiplication.
Let $\gamma$ be a matrix norm defined as

$$\gamma : \mathcal{M}_N \to \mathbb{R}_+ \quad \text{satisfying}$$

(i) $\gamma(m) > 0 \;\; \forall \;\; m \in \mathcal{M}_N, m \neq 0$
(ii) $\gamma(\alpha m) = |\alpha| \, \gamma(m), \;\; \forall \;\; m \in \mathcal{M}_N, \alpha \in \mathbb{R}$
(iii) $\gamma(m_1 + m_2) \leq \gamma(m_1) + \gamma(m_2), \;\; \forall \;\; m_1, m_2 \in \mathcal{M}_N$.

Once a matrix norm is chosen we can define an associated matrix norm on $\mathcal{D}_N$ and induced metric $\eta$ is given by

$$\eta(m_1, m_2) = \gamma\,(m_1 - m_2), \;\; \forall \, m_1, m_2 \in \mathcal{M}_N.$$

In order to study graph functions that vary over time, we use an axiomatic definition of the abstract linear space $\mathcal{D}_N$ into itself.
Consider the space $\mathcal{D}_N$ and a family of mappings $\Phi : \mathbb{R}_+ \times \mathcal{D}_N \to \mathcal{D}_N$, where to any graph $D \in \mathcal{D}_N$ and any parameter (time) $t \in \mathbb{R}_+$ assigns a graph $\Phi\,(t, D) \in \mathcal{D}_N$.

**Dynamic graph.** A dynamic graph $\widehat{D} = \Phi_D(t)$ is a one parameter mapping $\Phi_D : \mathbb{R}_+ \to \mathcal{D}_N$ with $\Phi_D(t) = \Phi\,(t, D) \in \mathcal{D}_N$ satisfying the following axioms.
(i). $\Phi\,(t_0, D_0) = D_0$
(ii). $\Phi$ is continuous
(iii). $\Phi(t_2, \Phi\,(t_1, D)) = \Phi(t_1 + t_2, D), \;\; \forall \;\; t_1, t_2 \in \mathbb{R}_+, \forall \, D \in \mathcal{D}_N$.

The first axiom establishes $D(t_0) = D_0$ as the initial graph. The second axiom requires continuity of mapping $\Phi\,(t, D)$ with respect to $t$ and $D$ which includes continuity with respect to $t_0$ and $D_0$. The third axiom establishes that dynamic graph $D$ as a one parameter graph $\Phi\,(t, D)$ of transformations of the space $\mathcal{D}_N$ into itself. Corresponding to a dynamic graph the dynamic adjacency matrix is defined as follows.

**Definition 2.1.** *A dynamic adjacency matrix $\widehat{E}$ is a one-parameter mapping $\psi : \mathbb{R}_+ \times \mathcal{E}_N \to \mathcal{E}_N$ of the space $\mathcal{E}_N$ into itself satisfying the following axioms.*

*(i)* $\psi(t_0, E_0) = E_0$.

*(ii)* $\psi(t, E)$ *is continuous.*

*(iii)* $\psi(t_2, \psi(t_1, E)) = \psi(t_1 + t_2, E), \; \forall \; t_1, t_2 \in \mathbb{R}_+$ *and* $\forall \; E \in \mathcal{E}_N$.

**Examples.**

A dynamic graph can be defined by the corresponding adjacency matrix and a few examples are given below.
(1) Let $\psi(t, E) = E, \;\; \forall \;\; t \in \mathbb{R}_+, \; \forall \; E \; \in \mathcal{D}_N$

Then $\psi(t_0, E_0) = E_0$ and $\psi(t, E) = E$ is continuous $\forall \; t$ and $\forall \; E$ and

$$\psi(t_2, \psi(t_1, E)) \;\; = \;\; \psi(t_2, E) = E$$

$$= \psi(t_1 + t_2, E)$$

Therefore, the dynamic graph $\widehat{D} = D$ for all $t \in \mathbb{R}_+$.

(2) Let $\psi(t, E) = E_0 \;\; \forall \;\; t \in \mathbb{R}_+, \; \forall \; E \;\in \mathcal{E}_N$. Then the dynamic graph $\widehat{D} = E_0$ for all $t \in \mathbb{R}_+$.

(3) Let $\psi(t, E) = t\,E_0 + E, \; \forall \; t \in \mathbb{R}_+, \; \forall \; E \;\in\; \mathcal{E}_N$ and $E_0$ be any initial adjacency matrix.
Then $\psi(0, E_0) = E_0$
and $\psi(t_n, E) \to \psi(t_0, E)$ whenever $t_n \to t_0$
and $\psi(t, E_n) \to \psi(t, E_0)$ whenever $E_n \to E_0$

$$\begin{aligned} \text{Further} \quad \psi(t_2, \psi(t_1, E)) &= \psi(t_2, t_1 E_0 + E) \\ &= t_2\,E_0 + (t_1 E_0 + E) \\ &= (t_1 + t_2)\,E_0 + E \\ &= \psi(t_1 + t_2, E). \end{aligned}$$

Therefore, the dynamic graph $\widehat{D} = tE_0 + D$ for all $t \in \mathbb{R}_+$.
**Motion of the graph.** The mapping $\Phi(t, D) = \widehat{D}$ is called the motion of the graph. The mapping $\psi(t, E)$ is called as the motion of the adjacency matrix $\widehat{E}$. A graph $D^e$ satisfying $\Phi(t, D^e) = D^e$ is called as the equilibrium graph.

In order to define the time evolution of a graph one needs the concept of a derivative in the abstract space, we can use the theory of abstract differential equations. Introducing the concept of Frechet derivative, if it exists on the notion of a generalized derivative we consider the time-evolution of a dynamic graph abstractly by the equation $\Delta D = \mathcal{G}(t, D)$ where $\Delta D$ represents the tendency of the graph to change in time $t$.

In order to introduce the corresponding concept in the adjacency matrices we need the following notions.

(1) The adjacency matrix $\widehat{E} = E(t)$ is said to be continuous if the entry $e_{ij}(t)$ is continuous for all $i, j = 1, 2, \ldots N$.

(2) The continuous adjacency matrix $\widehat{E} = E(t)$ is said to be differentiable if each continuous entry $e_{ij}(t)$ is differentiable for all $i, j = 1, 2, \ldots N$, and is denoted by $E' = (e'_{ij})_{N \times N}$. With the above definitions in place we can express the corresponding changes in an adjacency matrix that evolved in time '$t$' for a dynamic graph by the equation
$$\frac{dE}{dt} = F(t, E).$$
With the concept of rate of change of a graph with respect to time $t$, one can consider the differentiable equation in the abstract space $\mathcal{D}_N$. Using the theory of differential equations in abstract spaces one can study the graph differential equations.

An alternative approach that is more useful for practical purposes would be to consider the corresponding adjacency matrix differential equation or simply the matrix differential equation.

## 3 Linear Matrix differential equations

In this section, we study a graph differential equation that can be expressed as a linear matrix differential equation. Now consider a matrix differential equation ($MDE$) given by
$$E' = F(t, E).$$
where $F(t, E)$ is a $N \times N$ matrix in which each entry $f_{ij}(t)$ is a function of $t, e_{ij}$ where $i, j = 1, 2, \ldots, N$ and satisfies certain smoothness conditions.

In order to analyze the graph differential equation through the Matrix differential equation ($MDE$) we first consider those equations that can be transformed to a linear system.

Consider the IVP of a MDE, corresponding to some graph differential equation, given by

$$\left.\begin{aligned} E' &= F(t, E) \\ E(t_0) &= E_0 = (k_{ij})_{N \times N} \end{aligned}\right\} \tag{3.1}$$

where $F : I \times \mathcal{E}_N \to \mathcal{E}_N$ is continuous, $I = [t_0, T]$. This means that $F(t,E) = (f_{ij}(t, e_{11}, e_{12}, \ldots, e_{1N}, e_{21}, e_{22}, \ldots, e_{2N}, \ldots, e_{N1}, e_{N2}, \ldots, e_{NN}))_{N\times N}$ and $f_{ij}$ is a continuous, real valued function. Suppose that $f_{ij}(t)$ are linear combinations of the functions $e_{ij}(t)$. Then the system (3.1) can be written as a linear system

$$\left.\begin{array}{rcl} X' &=& AX \\ X(t_0) &=& X_0 \end{array}\right\} \tag{3.2}$$

where $X$ is the vector given by

$$X^T = [e_{11}, e_{12}, \ldots, e_{1N}, e_{21}, e_{22}, \ldots, e_{2N}, \ldots, e_{N1}, e_{N2}, \ldots, e_{NN}],$$

$A$ is $N^2 \times N^2$ coefficient matrix and

$$X_0^T = [k_{11}, k_{12}, \ldots, k_{1N}, k_{21}, k_{22}, \ldots, k_{2N}, \ldots, k_{N1}, k_{N2}, \ldots, k_{NN}].$$

As the qualitative theory of the system (3.2) is well established, using it one can easily analyze the linear system (3.2) and the corresponding graph differential equation.
Next suppose that MDE(3.2) along with its initial condition is of the form

$$\left.\begin{array}{rcl} E' &=& AE \\ E(t_0) &=& E_0 = (k_{ij})_{N\times N} \end{array}\right\} \tag{3.3}$$

where $A$ is the coefficient matrix of order $N \times N$ . The system (3.2) can be considered as $N$ subsystems given by

$$X_j' = AX_j, \quad X_j(t_0) = k_j, \; j = 1, 2, \ldots, N \tag{3.4}$$

where $X_j = \begin{pmatrix} e_{1j} \\ e_{2j} \\ \vdots \\ e_{Nj} \end{pmatrix}$ and $K_j = \begin{pmatrix} k_{1j} \\ k_{2j} \\ \vdots \\ k_{Nj} \end{pmatrix}$.
The $N$ subsystems given by (3.4) can be completely understood through the theory of ordinary differential systems and the corresponding graph differential equation can be analyzed.

## 4 Nonlinear matrix differential equation

We proceed to introduce an initial value problem of the nonlinear matrix differential equation in this section. Further we prove some basic inequality theorems.
Consider the Matrix differential equation ($MDE$) given by,

$$\left.\begin{array}{rcl} E' &=& F(t,E) \\ E(t_0) &=& E_0, \end{array}\right\} \tag{4.1}$$

where $E' = (e'_{ij})_{N\times N}$ and $F(t,E)$ is the matrix given by $F(t,E) = (f_{ij}(t, e_{rs}))$ $i,j = 1,2,\ldots,N$ and $r,s = 1,2,\ldots,N$ and $f_{ij}$ are real valued functions which are nonlinear in terms of the entries $e_{rs}$. In order to study the MDE (4.1) we need to devlop new notions that would help us to develop basic Matrix differential inequality results. We begin with a Partial Order $\leq$.

**Definition 4.1.** *Consider two matrices $A$ and $B$ of order $N$. We say that $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i,j = 1,2,\ldots,N$.*

**Definition 4.2.** *A matrix function $E : I \to \mathcal{E}_N$ defined by $E(t) = (e_{ij}(t))$ is said to be continuous if and only if $e_{ij} : I \to \mathbb{R}$ is continuous for all $i,j = 1,2,\ldots,N$.*

**Definition 4.3.** *A matrix function $E : I \to \mathcal{E}_N$ is said to be continuous and differentiable if and only if $e_{ij} : I \to \mathbb{R}$ is continuous and differentiable for all $i,j = 1,2,\ldots,N$.*

**Definition 4.4.** *By a solution of the IVP(4.1) we mean a matrix function $E : I \to \mathcal{E}_N$ which is continuous, differentiable and satisfies the equation(4.1) along with the initial condition.*

In order to state the basic differential inequality theorem we introduce the following notions.

**Definition 4.5.** *By a lower solution of the MDE(4.1) we mean a continuous differentiable matrix function $V(t)$ satisfying the inequalities*

$$V' \leq F(t, V), \;\; V(t_0) \leq E_0 \tag{4.2}$$

**Definition 4.6.** *By a upper solution of the MDE(4.1) we mean a continuous differentiable matrix function $W(t)$ satisfying the inequalities*

$$W' \geq F(t, W), \;\; W(t_0) \geq E_0 \tag{4.3}$$

**Definition 4.7.** *A function $F(t, U) \in C[I \times \mathcal{E}_N, \mathcal{E}_N]$ is said to be quasi monotone nondecreasing in $U$ for each $t$, if and only if $V \leq W$ and $v_{mn} = w_{mn}$ for some $m, n$ implies $f_{ij}(t, V(t)) \leq f_{ij}(t, W(t))$ for all $i, j = 1, 2, \ldots, N$.*

The basic matrix differential inequlity results is given below.

**Theorem 4.1.** *Assume that*

$$V' \leq F(t, V) \tag{4.4}$$

$$W' \geq F(t, W) \tag{4.5}$$

*where $V, W \in C^1[I, \mathcal{E}_N]$ and $F \in C[I \times \mathcal{E}_N, \mathcal{E}_N]$ and $F(t, U)$ be quasi monotone nondecreasing in $U$ for each $t$. Further assume that $V_0 < W_0$ where $V(t_0) = V_0 \in \mathcal{E}_N$ and $W(t_0) = W_0 \in \mathcal{E}_N$. Then $V(t) < W(t),\ t \in I$,where $I = [t_0, T]$ provided one of the inequalities in (4.4) and (4.5) is strict.*

*Proof.* Assume that $V' \leq F(t, V),\ W' > F(t, W)$. Suppose that the conclusion does not hold. Then there exists an element $t_1 \in I$ such that $V(t) < W(t)$ for $t_0 < t < t_1$ and there exists a pair of indices $k$ and $l$ such that $v_{kl}(t_1) = w_{kl}(t_1)$. Now since $F(t, U)$ is quasi monotone nondecreasing in $U$, this implies that

$$f_{ij}(t, V(t)) \leq f_{ij}(t, W(t)),\ t \in I \tag{4.6}$$

for $i, j = 1, 2, \ldots, N$. Further $v_{kl}(t) < w_{kl}(t), t_0 < t < t_1$ and $v_{kl}(t_1) = w_{kl}(t_1)$ implies for small $h < 0,\ \ v_{kl}(t_1 + h) - v_{kl}(t_1) < w_{kl}(t_1 + h) - w_{kl}(t_1)$, which further implies that

$$\frac{v_{kl}(t_1 + h) - v_{kl}(t_1)}{h} > \frac{w_{kl}(t_1 + h) - w_{kl}(t_1)}{h}$$

taking limit as $h \to 0$, we get

$$v'_{kl}(t_1) \geq w'_{kl}(t_1) \tag{4.7}$$

Using the inequalities (4.4), (4.5) and (4.7), yield
$f_{kl}(t_1, V(t_1)) \geq v'_{kl}(t_1) \geq w'_{kl}(t_1) > f_{kl}(t_1, W(t_1)) = f_{kl}(t_1, V(t_1))$, which is a contradiction. Hence the conclusion holds and the proof is complete. □

Next we state and prove a theorem involving non strict inequalities in this set up.

**Theorem 4.2.** *Suppose (4.4) and (4.5) holds and that $F(t, U)$ is quasi monotone nondecreasing in $U$ for each $t$. Further, suppose that $F$ satisfies,*

$$F(t, W) - F(t, V) \leq L\,(W - V) \text{ for } W \geq V, \text{ where } L > 0 \text{ is a } N \times N \text{ matrix.}$$

*Then $V_0 \leq W_0$ implies that $V(t) \leq W(t), t \in I$.*

*Proof.* Let us define

$$\begin{aligned} W_\epsilon(t) &= W(t) + \epsilon\, e^{2Lt}, \text{ where } \epsilon > 0 \text{ is sufficiently small.} \\ \text{Then } W'_\epsilon(t) &= W'(t) + 2L\,\epsilon\, e^{2\,Lt} \end{aligned}$$

$$\begin{aligned} &\geq F(t, W(t)) + 2L\,\epsilon\, e^{2Lt} \\ &\geq F(t, W(t)) - F(t, W_\epsilon(t)) + F(t, W_\epsilon(t)) + 2L\,\epsilon\, e^{2Lt} \end{aligned}$$

$$\begin{aligned} &\geq -L(W_\epsilon(t) - W(t)) + F(t, W_\epsilon(t)) + 2L\,\epsilon\, e^{2Lt} \\ &= F(t, W_\epsilon(t)) + L\,\epsilon\, e^{2Lt} \\ &> F(t, W_\epsilon(t)). \end{aligned}$$

$$\begin{aligned} \text{Further, } W_\epsilon(t_0) &= W(t_0) + \epsilon\, e^{2Lt_0} \\ &> W_0 \\ &\geq V_0 \end{aligned}$$

Hence we are in a position to apply the result for strict differential inequalities which yields $V(t) < W_\epsilon(t)$, $t \in I$ which implies as $\epsilon \to 0$,
$V(t) \leq W(t)$ and the proof is complete. □

The study of existence of a solution in a sector is essential to develop the monotone iterative technique. The following theorem deals with the existence of a solution in a sector.

**Theorem 4.3.** *Let $V, W \in C^1[I, \mathcal{E}_N]$ be lower and upper solutions of the Matrix differential equation*

$$\left.\begin{aligned} E' &= F(t, E) \\ E(t_0) &= E_0 \end{aligned}\right\} \tag{4.8}$$

*such that $V(t) \leq W(t)$ on $I$ and $F \in C[\Omega, \mathcal{E}_N]$, where*
$\Omega = \{(t, E) : V(t) \leq E \leq W(t), t \in I\}$. *Then there exists a solution $E(t)$ of (4.8) such that*
$V(t) \leq E(t) \leq W(t)$ *on $I$.*

*Proof.* Let $P : I \times \mathcal{E}_N \to \mathcal{E}_N$ be defined by $P(t, E) = (p_{ij}(t))_{N \times N}$ where
$p_{ij}(t) = Max\{v_{ij}(t), Min\{e_{ij}, w_{ij}(t)\}\}$

Then $F(t, P) = (f_{ij}(t, P(t, E))$ defines a continuous extension of $F$ to $I \times \mathcal{E}_N$ and is also bounded since $F$ is bounded on $\Omega$, which implies that $E'$ is bounded on $\Omega$. Hence the system
$E' = F(t, P(t, E))$, $E(t_0) = E_0$ has a solution $E(t)$ on $I$.

For $\epsilon > 0$, consider
$w_{\epsilon_{ij}}(t) = w_{ij}(t) + \epsilon(1 + t)$ and $v_{\epsilon_{ij}}(t) = v_{ij}(t) - \epsilon(1 + t)$ for $i, j = 1, 2, \ldots, N$.
We claim that $V_\epsilon(t) < E(t) < W_\epsilon(t)$. Since $v_{\epsilon_{ij}}(0) < e_{ij}(0) < w_{\epsilon_{ij}}(0)$ for any $i$ and $j$ we have $V_\epsilon(0) < E(0) < W_\epsilon(0)$. Suppose that there exists an element $t_1 \in (t_0, T]$ and a pair of indices $k$ and $l$ such that $v_{\epsilon_{kl}}(t) < e_{kl}(t) < w_{\epsilon_{kl}}(t)$ on $[t_0, t_1)$ and $e_{kl}(t_1) = w_{\epsilon_{kl}}(t_1)$.
Then $e_{kl}(t_1) > w_{kl}(t_1)$ and hence $p_{kl}(t_1) = w_{kl}(t_1)$.
Also we have $V(t_1) \leq P(t_1, E(t_1)) \leq W(t_1)$.
Since $F$ is quasi monotone nondecreasing, we have

$$F(t_1, P(t_1, E(t_1))) \leq F(t_1, W(t_1))$$

$$\begin{aligned} \text{Then } w'_{kl}(t_1) &\geq f_{kl}(t_1, W(t_1)) \\ &\geq f_{kl}(t_1, P(t_1, E(t_1)) \\ &= e'_{kl}(t_1) \end{aligned}$$

Since $w'_{\epsilon_{kl}}(t_1) > w'_{kl}(t_1)$, we have $w'_{\epsilon_{kl}}(t_1) > e'_{kl}(t_1)$, which is a contradiction to the fact that $e_{kl}(t) < w_{\epsilon_{kl}}(t)$ for $t \in [t_0, t_1)$ and $e_{kl}(t_1) = w_{\epsilon_{kl}}(t_1)$.
Therefore $V_\epsilon(t) < E(t) < W_\epsilon(t)$ on $I$.
Now as $\epsilon \to 0$, we obtain that $V(t) \leq E(t) \leq W(t)$ and the proof is complete. □

## 5 Monotone iterative technique

In this section we shall construct monotone sequences that converges to the solutions of

$$\left.\begin{aligned} E' &= F(t, E) \\ E(t_0) &= E_0 \end{aligned}\right\} \tag{5.1}$$

**Theorem 5.1.** *Assume that $V_0, W_0 \in C^1[I, \mathcal{E}_N]$, $I = [t_0, T]$ are lower and upper solutions of the IVP (5.1) such that $V_0 \leq W_0$ on $I$. Let $F \in C[I \times \mathcal{E}_N, \mathcal{E}_N]$. Suppose further that $F(t, X) - F(t, Y) \geq -M(X - Y)$, for $V_0 \leq Y \leq X \leq W_0$, $M \in \mathbb{R}^{N \times N}, M \geq 0$. Then there exists monotone sequences $\{V_n\}, \{W_n\}$ such that $\{V_n\}$ converges to $\rho$ and $\{W_n\}$ converges to $R$ as $n \to \infty$ uniformly and monotonically on $I$ and that $\rho$ and $R$ are the minimal and maximal solutions of IVP (5.1) respectively.*

*Proof.* For any $Y \in C^1[I, \mathcal{E}_N]$ such that $V_0 \leq Y \leq W_0$, we consider the linear Matrix differential equation

$$X' = F(t, Y) - M(X - Y),\ X(t_0) = X_0. \tag{5.2}$$

Then there exists a unique solution of (5.2) given by

$$X(t) = e^{M(t-t_0)} X_0 + \int_{t_0}^{t} e^{M(t-s)}[F(s, Y(s)) + MY(s)]ds$$

Define a sequence $\{V_n\}$ by

$$V_n' = F(t, V_{n-1}) - M(V_n - V_{n-1}), V_n(t_0) = X_0, \quad n = 1, 2, \ldots, \tag{5.3}$$

Let $V_1$ be the solution of (5.3) for $n = 1$.

$$\begin{aligned} \text{Consider} \quad P &= V_0 - V_1 \\ \text{Then} \quad P' &= V_0' - V_1' \\ &\leq F(t, V_0) - F(t, V_0) + M(V_1 - V_0), \\ &\leq -MP. \end{aligned}$$

and $P(t_0) \leq 0$ which implies that $P \leq 0$ on $I$, and thus $V_0 \leq V_1$ on $I$.
Similarly, we consider a sequence $\{W_n\}$ by

$$W_n' = F(t, W_{n-1}) - M(W_n - W_{n-1}), W_n(t_0) = X_0 \tag{5.4}$$

Let $W_1$ be the solution of (5.4) for $n = 1$.

$$\begin{aligned} \text{Consider} \quad Q &= W_1 - W_0 \\ \text{Then} \quad Q' &= W_1' - W_0' \\ &\leq F(t, W_0) - M(W_1 - W_0) - F(t, W_0) \\ &= -MQ \end{aligned}$$

and $Q(t_0) \leq 0$ which implies that $Q(t) \leq 0$. Hence $W_1 \leq W_0$ on $I$.
Now we proceed to show that $V_1 \leq W_1$ on $I$.

$$\begin{aligned} \text{Set} \quad R &= V_1 - W_1 \\ \text{Then} \quad R' &= V_1' - W_1' \\ &= F(t, V_0) - M(V_1 - V_0) - F(t, W_0) + M(W_1 - W_0) \\ &\leq M(W_0 - V_0) - M(V_1 - V_0 - W_1 + W_0) \\ &= -MR \end{aligned}$$

and $R(t_0) = 0$, which implies that $R \leq 0$ on $I$ and thus $V_1 \leq W_1$ on $I$.
Hence we have shown that $V_0 \leq V_1 \leq W_1 \leq W_0$ on $I$.
Now suppose that for some $n = k$, the result $V_{k-1} \leq V_k \leq W_k \leq W_{k-1}$ holds on $I$. We claim that $V_k \leq V_{k+1} \leq W_{k+1} \leq W_k$ on $I$. To prove this we first set $n = k$ in (5.3) and (5.4). Then clearly there exists unique solutions $V_{k+1}(t)$ and $W_{k+1}(t)$ satisfying (5.3) and (5.4) respectively on $I$.

$$\begin{aligned} \text{Consider} \quad S &= V_k - V_{k+1} \\ \text{Then} \quad S' &= V_k' - V_{k+1}' \\ &= F(t, V_{k-1}) - M(V_k - V_{k-1}) - F(t, V_k) + M(V_{k+1} - V_k), \end{aligned}$$

$$\begin{aligned} &\leq M(V_k - V_{k-1}) + M\left(V_{k+1} - V_k - V_k + V_{k-1}\right), \\ &\leq -MS. \end{aligned}$$

and $S(t_0) = 0$ which implies that $S \leq 0$ on $I$ and thus $V_k \leq V_{k+1}$ on $I$.
Similarly we can show that $W_{k+1} \leq W_k$ on $I$.

$$\begin{aligned} \text{Set} \quad T &= V_{k+1} - W_{k+1} \\ \text{Then} \quad T' &= V'_{k+1} - W'_{k+1} \\ &= F(t, V_k) - M(V_{k+1} - V_k) - F(t, W_k) + M\left(W_{k+1} - W_k\right), \\ &\leq M(W_k - V_k) + M\left(W_{k+1} - W_k - V_{k+1} + V_k\right), \\ &\leq -MT. \end{aligned}$$

and $T\,(t_0) = 0$, which implies that $T \leq 0$ on $I$ and thus $V_{k+1} \leq W_{k+1}$ on $I$.
We have shown that $V_k \leq V_{k+1} \leq W_{k+1} \leq W_k$ on $I$.
Therefore we have

$$V_0 \leq V_1 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_1 \leq W_0 \text{ on } [t_0, T]. \tag{5.5}$$

The sequences $\{V_n\}, \{W_n\}$ are uniformly bounded on $[t_0, T]$ and by (5.3) and (5.4) it follows that $\{|V'_n|\}, \{|W'_n|\}$ are also uniformly bounded. As a result, the sequences $\{V_n\}$ and $\{W_n\}$ are equicontinuous on $[t_0, T]$ and consequently by Ascoli-Arzela's Theorem there exists subsequences $\{V_{n_k}\}$, $\{W_{n_k}\}$ that converge uniformly on $[t_0, T]$. In view of (5.5) it also follows that the entire sequences $\{V_n\}, \{W_n\}$ converge uniformly and monotonically to $\rho$ and $R$ respectively as $n \to \infty$. By considering the integral equations corresponding to the IVP of MDE (5.3) and (5.4) respectively, we can show that $\rho$ and $R$ are solutions of IVP(5.1). The proof uses the concepts of uniform convergence and uniform continuity and is well established.
To prove that $\rho, R$ are respectively the minimal and maximal solutions of (5.1) we have to show that if $X$ is any solution of (5.1) such that $V_0 \leq X \leq W_0$ on $I$, then $V_0 \leq \rho \leq X \leq R \leq W_0$ on $I$. To do this, suppose that for some $n, V_n \leq X \leq W_n$ on $I$ and set $\phi = X - V_{n+1}$ so that

$$\begin{aligned} &\phi' = F(t, X) - F(t, V_n) + M(V_{n+1} - V_n) \\ &\geq -M(X - V_n) + M(V_{n+1} - V_n) = -M\phi; \end{aligned}$$

and $\phi(t_0) = 0$.
Hence, it follows that $V_{n+1} \leq X$ on $I$. Similarly $X \leq W_{n+1}$ on $I$.
Hence $\qquad V_{n+1} \leq X \leq W_{n+1}$ on $I$.
Since $V_0 \leq X \leq W_0$ on $I$, this proves by induction that $V_n \leq X \leq W_n$ on $I$ for all $n$. Taking the limit as $n \to \infty$, we conclude that $\rho \leq X \leq R$ on $I$ and the proof is complete. □

**Corollary 5.1.** *If in addition to the assumption Theorem 5.1, if $F$ satisfies the following condition*

$$F(t, X) - F(t, Y) \leq M(X - Y), X \geq Y$$

*then the solution is unique.*

*Proof.* We have $\rho \leq R$ on $I$.

$$\begin{aligned} \text{Consider} \quad \phi(t) &= R(t) - \rho(t) \\ \text{Then} \quad \phi' &= R'(t) - \rho'(t) \\ &= F(t, R) - F(t, \rho) \\ &\leq M(R - \rho) \\ &\leq M\phi. \end{aligned}$$

and $\phi(t_0) = 0$ which implies that $\phi(t) \leq 0$ on $I$ and thus $R(t) \leq \rho(t)$ on $I$. Hence $\rho(t) = X(t) = R(t)$ on $I$, and the proof is complete. □

## 6 Acknowledgements

This work was done under the project no. 2/48(8)/2011/-R&D II/1600 sanctioned by National Board of Higher Mathematics, Department of Atomic Energy, Government of India. The authors acknowledge their support.

## References

[1] Narsingh Deo, *Graph Theory with Application to Engineering and Computer Science*

[2] D.D.Siljak, *Dynamic graphs*, Nnlinear Analysis: Hybrid Systems 2(2008) 544-567.

[3] G.S.Ladde, V.Lakshmikantham and A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman Publishing INC., 1985.*

[4] A.R. Radcliffe-Brown, On social structure, Journal of the Royal Anthropological Institute of Great Britain and Ireland 70(1940)1-12.

*Received*: September 12, 2012; *Accepted*: October 30, 2012

UNIVERSITY PRESS

Malaya Journal of Matematik 1(1)(2013) 10–17

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

# Additive functional equation and inequality are stable in Banach space and its applications

M. Arunkumar,[a,*] and P. Agilan[b]

[a] *Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, Tamil Nadu, India.*

[b] *Department of Mathematics, S.K.P. Engineering College, Tiruvannamalai - 606 611, Tamil Nadu, India.*

## Abstract

In this paper, the authors established the solution of the additive functional equation and inequality

$$f(x) + f(y + z) - f(x + y) = f(z)$$

and

$$||f(x) + f(y + z) - f(x + y)|| \leq ||f(z)||.$$

We also prove that the above functional equation and inequality are stable in Banach space in the sense of Ulam, Hyers, Rassias. An application of this functional equation is also studied.

*Keywords:* Additive functional equations, generalized Hyers - Ulam - Rassias stability.

*2010 MSC:* 39B52, 32B72, 32B82.



## 1 Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [21] concerning the stability of group homomorphisms. D.H. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [20] for linear mappings by considering an unbounded Cauchy difference.

The paper of Th.M. Rassias [20] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Th.M. Rassias theorem was obtained by P. Gavruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach.

In 1982, J.M. Rassias [14] followed the innovative approach of the Th.M. Rassias theorem [20] in which he replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in R$ with $p + q = 1$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [19] by considering the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 4, 5, 6, 8, 9, 11, 12, 16, 17]).

The solution and stability of the following additive functional equations

$$f(x + y) = f(x) + f(y), \tag{1.1}$$

$$f(2x - y) + f(x - 2y) = 3f(x) - 3f(y), \tag{1.2}$$

$$f(x + y - 2z) + f(2x + 2y - z) = 3f(x) + 3f(y) - 3f(z), \tag{1.3}$$

*Corresponding author.
*E-mail addresses*: annarun2002@yahoo.co.in (M. Arunkumar) and agilram@gmail.com (P. Agilan).

$$f(2x \pm y \pm z) = f(x \pm y) + f(x \pm z), \tag{1.4}$$

were discussed in [1, 3, 13, 18].

One of the most famous functional equations is the additive functional equation (1.1). In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of A.L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function.

It is well known that if an additive function $f : R \to R$ satisfies one of the following conditions:

$(a)$ $f$ is continuous at a point;

$(b)$ $f$ is monotonic on an interval of positive length;

$(c)$ $f$ is bounded on an interval of positive length;

$(d)$ $f$ is integrable;

$(e)$ $f$ is measurable,

then $f$ is of the form $f(x) = cx$ with a real constant $c$. That is to say $f$ has the linearity. That is, if a solution of the additive equation (1.1) satisfies one of the very weak conditions $(a)$ to $(e)$, then it does have the linearity. But every additive functional which is not linear displays a very strange behavior. More precisely, the graph of every additive functional $f : R \to R$ which is not of the form $f(x) = cx$ is dense in $R^2$ .

In this paper, the authors established the solution and generalized Ulam-Hyers stability of the additive functional equation and inequality

$$f(x) + f(y + z) - f(x + y) = f(z) \tag{1.5}$$

$$||f(x) + f(y + z) - f(x + y)|| \leq ||f(z)||. \tag{1.6}$$

In Section 2, we proved the general solution of (1.5) and (1.6) is provided.
In Section 3, the generalized Ulam-Hyers stability of the functional equation (1.5) is investigated.
The generalized Ulam-Hyers stability of the functional inequality (1.6) is discussed in section 4.
In Section 5, the application of functional equation (1.5) is studied.

## 2 General Solution of (1.5) and (1.6)

In this section, the general solution of (1.5) and (1.6) are given. Through out this section let $X$ and $Y$ be real vector spaces.

**Theorem 2.1.** *The mapping $f : X \to Y$ satisfies the functional equation*

$$f(x + y) = f(x) + f(y) \tag{2.1}$$

*if and only if $f : X \to Y$ satisfies the functional equation*

$$f(x) + f(y + z) - f(x + y) = f(z) \tag{2.2}$$

*for all $x, y, z \in X$ with $f(0) = 0$.*

*Proof.* Let $f : X \to Y$ satisfies the functional equation (2.1). Setting $x = y = 0$ in (2.1), we get $f(0) = 0$. Set $x = -y$ in (2.1), we get $f(-y) = -f(y)$ for all $y \in X$. Therefore $f$ is an odd function. Replacing $y$ by $x$ and $y$ by $2x$ in (2.1), we obtain

$$f(2x) = 2f(x) \quad \text{and} \quad f(3x) = 3f(x) \tag{2.3}$$

for all $x \in X$. In general for any positive integer $a$, we have $f(ax) = af(x)$.

Replacing $(x, y)$ by $(x, y + z)$ in (2.1), we get

$$f(x) + f(y + z) = f(x + y + z) \tag{2.4}$$

for all $x, y, z \in X$. Again replacing $(x, y)$ by $(x + y, z)$ in (2.1), we obtain

$$f(x + y) + f(z) = f(x + y + z) \tag{2.5}$$

for all $x, y, z \in X$. From (2.4) and (2.5), we derive (1.5) for all $x, y, z \in X$.

Conversely, assume $f : X \to Y$ satisfies the functional equation (2.2) with $f(0) = 0$. Set $(x, z)$ by $(-y, 0)$ in (2.2), we get $f(-y) = -f(y)$ for all $y \in X$. Therefore $f$ is an odd function. Replacing $(y, z)$ by $(x, 0)$ and $(2x, 0)$ respectively, in (2.2), we obtain

$$f(2x) = 2f(x) \quad \text{and} \quad f(3x) = 3f(x) \tag{2.6}$$

for all $x \in X$. In general for any positive integer $a$, we have $f(ax) = af(x)$.

Replacing $z$ by 0 in (2.2), we derive (2.1) for all $x, y \in X$. □

**Theorem 2.2.** *The mapping $f : X \to Y$ satisfies the functional equation (2.1) if and only if $f : X \to Y$ satisfies the functional inequality*

$$||f(x) + f(y + z) - f(x + y)|| \le ||f(z)|| \tag{2.7}$$

*for all $x, y, z \in X$ with $||f(0)|| = 0$.*

*Proof.* Let $f : X \to Y$ satisfies the functional equation (2.7). Setting $z = 0$ in (2.7), we get

$$||f(x) + f(y) - f(x + y)|| \le ||f(0)|| \tag{2.8}$$

for all $x, y, z \in X$. It follows from (2.8) our result is desired.

Conversely, assume $f : X \to Y$ satisfies the functional equation (2.1). Adding $f(z)$ on both sides of (2.1) and using (2.1) and rewrite the equation, we have

$$f(x) + f(y + z) - f(x + y) = f(z) \tag{2.9}$$

for all $x, y, z \in X$. It follows from (2.9) our result is desired. □

**Corollary 2.3.** *For a mapping $f : X \to Y$ the following conditions are equivalent.*

*(i)* *$f$ is additive*

*(ii)* $f(x) + f(y + z) - f(x + y) = f(z)$

*(iii)* $||f(x) + f(y + z) - f(x + y)|| \le ||f(z)||$.

Hereafter through out this paper, let us consider $X$ and $Y$ to be a normed linear space and a Banach space, respectively.

# 3 Stability Results for Functional Equation (1.5)

In this section, the generalized Ulam-Hyers stability of the functional equation (1.5) is investigated.

**Theorem 3.1.** *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\alpha\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{nj}} \quad \text{converges in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{nj}} = 0 \tag{3.1}$$

*for all $x, y, z \in X$. Let $f : X \to Y$ be a function satisfying the inequality*

$$\|f(x) + f(y + z) - f(x + y) - f(z)\| \le \alpha(x, y, z) \tag{3.2}$$

*for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and satisfying the functional equation (1.5) such that*

$$\|f(x) - A(x)\| \le \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x, 2^{kj}x, 0)}{2^{kj}} \tag{3.3}$$

*for all $x \in X$. The mapping $A(x)$ is defined by*

$$A(x) = \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{nj}} \tag{3.4}$$

*for all $x \in X$.*

*Proof.* Assume $j = 1$. Replacing $(x, y, z)$ by $(x, x, 0)$ in (3.2), we get

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \leq \frac{\alpha\,(x, x, 0)}{2} \tag{3.5}$$

for all $x \in X$. Now replacing $x$ by $2x$ and dividing by 2 in (3.5), we get

$$\left\| \frac{f(2x)}{2} - \frac{f(2^2 x)}{2^2} \right\| \leq \frac{\alpha\,(2x, 2x, 0)}{2^2} \tag{3.6}$$

for all $x \in X$. From (3.5) and (3.6), we obtain

$$\begin{aligned} \left\| f(x) - \frac{f(2^2 x)}{2^2} \right\| &\leq \left\| f(x) - \frac{f(2x)}{2} \right\| + \left\| \frac{f(2x)}{2} - \frac{f(2^2 x)}{2^2} \right\| \\ &\leq \frac{1}{2}\left[ \alpha(x, x, 0) + \frac{\alpha(2x, 2x, 0)}{2} \right] \end{aligned} \tag{3.7}$$

for all $x \in X$. In general for any positive integer $n$ , we get

$$\begin{aligned} \left\| f(x) - \frac{f(2^n x)}{2^n} \right\| &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\alpha(2^k x, 2^k x, 0)}{2^k} \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\alpha(2^k x, 2^k x, 0)}{2^k} \end{aligned} \tag{3.8}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{ \frac{f(2^n x)}{2^n} \right\},$$

replace $x$ by $2^m x$ and dividing by $2^m$ in (3.8), for any $m, n > 0$ , we deduce

$$\begin{aligned} \left\| \frac{f(2^m x)}{2^m} - \frac{f(2^{n+m} x)}{2^{(n+m)}} \right\| &= \frac{1}{2^m} \left\| f(2^m x) - \frac{f(2^n \cdot 2^m x)}{2^n} \right\| \\ &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\alpha(2^{k+m} x, 2^{k+m} x, 0)}{2^{k+m}} \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\alpha(2^{k+m} x, 2^{k+m} x, 0)}{2^{k+m}} \\ &\to 0 \quad as \ \ m \to \infty \end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{f(2^n x)}{2^n} \right\}$ is Cauchy sequence. Since $Y$ is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \quad \forall \ x \in X.$$

Letting $n \to \infty$ in (3.8) we see that (3.3) holds for all $x \in X$. To prove that $A$ satisfies (1.5), replacing $(x, y, z)$ by $(2^n x, 2^n y, 2^n z)$ and dividing by $2^n$ in (3.2), we obtain

$$\frac{1}{2^n} \left\| f(2^n x) + f(2^n (y + z)) - f(2^n (x + y)) - f(2^n z) \right\| \leq \frac{1}{2^n} \alpha(2^n x, 2^n y, 2^n z)$$

for all $x, y, z \in X$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$A(x) + A(y + z) = A(x + y) + A(z).$$

Hence $A$ satisfies (1.5) for all $x, y, z \in X$. To prove $A$ is unique, we let $B(x)$ be another mapping satisfying (1.5) and (3.3), then

$$\|A(x) - B(x)\| = \frac{1}{2^n} \|A(2^n x) - B(2^n x)\|$$

$$
\begin{aligned}
&\leq \frac{1}{2^n}\left\{\|A(2^n x)-f(2^n x)\|+\|f(2^n x)-B(2^n x)\|\right\}\\
&\leq \sum_{k=0}^{\infty}\frac{2\ \alpha(2^{k+n}x)}{2^{(k+n)}}\\
&\to 0 \quad as \ \ n\to\infty
\end{aligned}
$$

for all $x \in X$. Hence $A$ is unique.

For $j=-1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers [10], Ulam-Hyers-Rassias [20] and Ulam-JRassias [19] stabilities of (1.5).

**Corollary 3.2.** *Let $\lambda$ and $s$ be nonnegative real numbers. Let a function $f : X \to Y$ satisfies the inequality*

$$
\begin{aligned}
&\|f(x)+f(y+z)-f(x+y)-f(z)\|\\
&\leq \begin{cases}\lambda, & \\ \lambda\left(||x||^s+||y||^s+||z||^s\right), & s<1 \quad or \quad s>1;\\ \lambda\left\{||x||^s||y||^s||z||^s+\left(||x||^{3s}+||y||^{3s}+||z||^{3s}\right)\right\}, & s<\frac{1}{3} \quad or \quad s>\frac{1}{3};\end{cases}
\end{aligned}
\tag{3.9}
$$

*for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ such that*

$$
\|f(x)-A(x)\| \leq \begin{cases}\lambda,\\ \dfrac{2\ \lambda||x||^s}{|2-2^s|},\\ \dfrac{2\ \lambda||x||^{3s}}{|2-2^{3s}|},\end{cases}
\tag{3.10}
$$

*for all $x \in X$.*

## 4 Stability Results for Functional Inequality (1.6)

In this section, we discussed the generalized Ulam-Hyers stability of the functional inequality (1.6).

**Theorem 4.1.** *Let $j \in \{-1,1\}$ and $\beta : X^3 \to [0,\infty)$ be a function such that*

$$
\sum_{n=0}^{\infty}\frac{\beta\left(2^{nj}x,2^{nj}y,2^{nj}z\right)}{2^{nj}} \ \ converges \ \ in \ \ \mathbb{R} \quad and \quad \lim_{n\to\infty}\frac{\beta\left(2^{nj}x,2^{nj}y,2^{nj}z\right)}{2^{nj}}=0
\tag{4.1}
$$

*for all $x, y, z \in X$. Let $f : X \to Y$ be a function satisfying the functional inequality*

$$
\|f(x)+f(y+z)-f(x+y)\| \leq ||f(z)||+\beta\left(x,y,z\right)
\tag{4.2}
$$

*for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and satisfying the functional equation (1.6) such that*

$$
\|f(x)-A(x)\| \leq \frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x,2^{kj}x,0)}{2^{kj}}
\tag{4.3}
$$

*for all $x \in X$. The mapping $A(x)$ is defined by*

$$
A(x)=\lim_{n\to\infty}\frac{f(2^{nj}x)}{2^{nj}}
\tag{4.4}
$$

*for all $x \in X$.*

*Proof.* Assume $j=1$. Replacing $(x, y, z)$ by $(x, x, 0)$ in (4.2), we get

$$
\left\|f(x)-\frac{f(2x)}{2}\right\| \leq \frac{\beta\left(x,x,0\right)}{2}
\tag{4.5}
$$

for all $x \in X$. Now replacing $x$ by $2x$ and dividing by 2 in (4.5), we get

$$\left\| \frac{f(2x)}{2} - \frac{f(2^2x)}{2^2} \right\| \leq \frac{\beta(2x, 2x, 0)}{2^2} \tag{4.6}$$

for all $x \in X$. From (4.5) and (4.6), we obtain

$$\begin{aligned} \left\| f(x) - \frac{f(2^2x)}{2^2} \right\| &\leq \left\| f(x) - \frac{f(2x)}{2} \right\| + \left\| \frac{f(2x)}{2} - \frac{f(2^2x)}{2^2} \right\| \\ &\leq \frac{1}{2}\left[\beta(x, x, 0) + \frac{\beta(2x, 2x, 0)}{2}\right] \end{aligned} \tag{4.7}$$

for all $x \in X$. In general for any positive integer $n$ , we get

$$\begin{aligned} \left\| f(x) - \frac{f(2^nx)}{2^n} \right\| &\leq \frac{1}{2}\sum_{k=0}^{n-1} \frac{\beta(2^kx, 2^kx, 0)}{2^k} \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty} \frac{\beta(2^kx, 2^kx, 0)}{2^k} \end{aligned} \tag{4.8}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{ \frac{f(2^nx)}{2^n} \right\},$$

replace $x$ by $2^mx$ and dividing by $2^m$ in (4.8), for any $m, n > 0$ , we deduce

$$\begin{aligned} \left\| \frac{f(2^mx)}{2^m} - \frac{f(2^{n+m}x)}{2^{(n+m)}} \right\| &= \frac{1}{2^m} \left\| f(2^mx) - \frac{f(2^n \cdot 2^mx)}{2^n} \right\| \\ &\leq \frac{1}{2}\sum_{k=0}^{n-1} \frac{\beta(2^{k+m}x, 2^{k+m}x, 0)}{2^{k+m}} \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty} \frac{\beta(2^{k+m}x, 2^{k+m}x, 0)}{2^{k+m}} \\ &\to 0 \quad as \;\; m \to \infty \end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{f(2^nx)}{2^n} \right\}$ is Cauchy sequence. Since $Y$ is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{n\to\infty} \frac{f(2^nx)}{2^n} \quad \forall \; x \in X.$$

Letting $n \to \infty$ in (4.8) we see that (4.3) holds for all $x \in X$. In order to prove that $A$ satisfies (1.6) and it is unique, the proof is similar to that of Theorem 3.1.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 4.1 concerning the Ulam-Hyers [10], Ulam-Hyers-Rassias [20] and Ulam-JRassias [19] stabilities of (1.6).

**Corollary 4.2.** *Let $\lambda$ and $r, s, t$ be nonnegative real numbers. Let a function $f : X \to Y$ satisfies the functional inequality*

$$\begin{aligned} &\|f(x) + f(y+z) - f(x+y)\| \\ &\leq ||f(z)|| + \begin{cases} \lambda, & \\ \lambda\left(||x||^r + ||y||^s + ||z||^t\right), & r, s < 1 \quad or \quad r, s > 1; \\ \lambda\left\{||x||^r||y||^s||z||^t + \left(||x||^{r+s+t} + ||y||^{r+s+t} + ||z||^{r+s+t}\right)\right\}, & \\ \qquad r+s+t < \frac{1}{3} \quad or \quad r+s+t > \frac{1}{3}; & \end{cases} \end{aligned} \tag{4.9}$$

*for all* $x, y, z \in X$. *Then there exists a unique additive function* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq \begin{cases} \lambda, \\ \dfrac{\lambda||x||^r}{|2-2^r|} + \dfrac{\lambda||x||^s}{|2-2^s|}, \\ \dfrac{\lambda||x||^{r+s+t}}{|2-2^{r+s+t}|}, \end{cases} \tag{4.10}$$

*for all* $x \in X$.

# 5 Application of Functional Equation (1.5)

Consider the additive functional equation (1.5), that is

$$f(x) + f(y+z) - f(x+y) = f(z).$$

The above functional equation can be rewritten as

$$f(x) + f(y+z) = f(x+y) + f(z).$$

This functional equation is originating from an excellent definition of Group Theory which states the associative law for the binary operation "+".

Since $f(x) = x$ is the solution of the functional equation, the above equation is written as follows

$$x + (y+z) = (x+y) + z$$

# References


[1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables,* Cambridge University Press, 1989.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, 2(1950), 64-66.

[3] M. Arunkumar, Solution and stability of Arun-Additive functional equations, *International Journal Mathematical Sciences and Engineering Applications*, 4(3)(2010), 33-46.

[4] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[5] G.Z. Eskandani, P. Găvruţă, J.M. Rassias and R. Zarghami, Generalized Hyers-Ulam stability for a general mixed functional equation in quasi-$\beta$-normed Spaces, *Mediterr. J. Math.*, 8(2011), 331-348.

[6] G.Z. Eskandani, P. Găvruţă, On the stability problem in quasi-Banach spaces, *Nonlinear Funct. Anal. Appl.,* (to appear).

[7] P. Găvruţă, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, 184(1994), 431-436.

[8] P. Găvruţă, An answer to a question of J.M. Rassias concerning the stability of Cauchy functional equation, *Advances in Equations and Inequalities, Hadronic Math. Ser.*, (1999), 67-71.

[9] P. Găvruţă, On a problem of G. Isac and Th. M. Rassias concerning the stability of mappings, *J. Math. Anal. Appl.*, 261(2001), 543-553.

[10] D.H. Hyers, On the stability of the linear functional equation, *Proc.Nat. Acad.Sci.*, 27(1941), 222-224.

[11] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, Basel, 1998.

[12] S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[13] D.O. Lee, Hyers-Ulam stability of an addtiive type functional equation, *J. Appl. Math. and Computing*, 13(1)(2)(2003), 471-477.

[14] J.M. Rassias, On approximately of approximately linear mappings by linear mappings, *J. Funct. Anal.,* 46(1982), 126-130.

[15] J.M. Rassias, On approximately of approximately linear mappings by linear mappings, *Bull. Sc. Math.*, 108(1984), 445-446.

[16] J.M. Rassias and H.M. Kim, Generalized HyersUlam stability for general additive functional equations in quasi-$\beta$-normed spaces, *J. Math. Anal. Appl.*, 356(2009), 302-309.

[17] J.M. Rassias, K.W. Jun and H.M. Kim, Approximate $(m, n)-$Cauchy - Jensen additive mappings in C*-algebras, *Acta Mathematica Sinica*, 27(10)(2011), 1907-1922.

[18] K. Ravi and M. Arunkumar, On a n- dimensional additive Functional Equation with fixed point Alternative, *Proceedings of International Conference on Mathematical Sciences*, 2007, Malaysia.

[19] K. Ravi, M. Arunkumar and J.M. Rassias, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *International Journal of Mathematical Sciences*, 3(8)(2008), 36-47.

[20] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, 72(1978), 297-300.

[21] S.M. Ulam, *Problems in Modern Mathematics,* Science Editions, Wiley, New York, 1964. (Chapter VI, Some Questions in Analysis: 1, Stability).


*Received*: September 29, 2012; *Accepted*: November 3, 2012

UNIVERSITY PRESS

Malaya Journal of Matematik 1(1)(2013) 18–26

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

# Certain subclasses of uniformly convex functions and corresponding class of starlike functions

N. Magesh,[a,*] and V. Prameela[b]

[a] P.G. and Research Department of Mathematics, Govt Arts College for Men , Krishnagiri - 635001, India.

[b] Department of Mathematics, Adhiyamaan College of Engineering (Autonomous), Hosur - 635109, Tamilnadu, India.

**Abstract**

In this paper, we defined a new subclass of uniformly convex functions and corresponding subclass of starlike functions with negative coefficients and obtain coefficient estimates. Further we investigate extreme points, growth and distortion bounds, radii of starlikeness and convexity and modified Hadamard products.

*Keywords:* Univalent functions, convex functions, starlike functions, uniformly convex functions, uniformly starlike functions.

*2010 MSC:* 30C45. 

## 1 Introduction

Denoted by $S$ the class of functions of the form

$$f(z) = z + \sum_{n=j+1}^{\infty} a_n z^n \tag{1.1}$$

that are analytic and univalent in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ and by $ST$ and $CV$ the subclasses of $S$ that are respectively, starlike and convex. Goodman [5, 6] introduced and defined the following subclasses of $CV$ and $ST$.

A function $f(z)$ is uniformly convex (uniformly starlike) in $\mathcal{U}$ if $f(z)$ is in $CV$ $(ST)$ and has the property that for every circular arc $\gamma$ contained in $\mathcal{U}$, with center $\xi$ also in $\mathcal{U}$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions denoted by $UCV$ and the class of uniformly starlike functions by $UST$ (for details see [5]). It is well known from [8, 11] that

$$f \in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| \leq \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}.$$

In [11], Running introduced a new class of starlike functions related to $UCV$ and defined as

$$f \in S_p \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}.$$

Note that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$ . Further Running generalized the class $S_p$ by introducing a parameter $\alpha, \ -1 \leq \alpha < 1$,

$$f \in S_p(\alpha) \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\}.$$

*Corresponding author.
*E-mail addresses*: nmagi_2000@yahoo.co.in (N. Magesh) and vasuprameelak@gmail.com (V. Prameela).

Motivated by the works of Bharati et al [2], Frasin [3, 4], Murugusundaramoorthy and Magesh [10] and others [5, 6, 8, 11, 12, 17, 18], we define the following class:

For $\beta \geq 0$, $-1 \leq \alpha < 1$ and $0 \leq \lambda < 1$, we let $S(\lambda,\alpha,\beta,j)$ denote the subclass of $S$ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$\text{Re}\left\{\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-\alpha\right\} \tag{1.2}$$

$$>\beta\left|\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right|, \quad z\in\mathcal{U}. \tag{1.3}$$

We also let $TS(\lambda,\alpha,\beta,j)=S(\lambda,\alpha,\beta,j)\bigcap T$ where $T$, the subclass of $S$ consisting of functions of the form

$$f(z)=z-\sum_{n=j+1}^{\infty}a_nz^n, a_n\geq 0, \forall\ n\geq j+1 \tag{1.4}$$

introduced and studied by Silverman [14].

We note that, by specializing the parameters $j$, $\lambda$, $\alpha$, and $\beta$ we obtain the following subclasses studied by various authors.

1. $TS(0,\alpha,0,1)=T^*(\alpha)$ and $TS(1,\alpha,0,1)=\mathcal{K}(\alpha)$ (Silverman [14])
2. $TS(0,\alpha,0,j)=T^*(\alpha,j)$ and $TS(1,\alpha,0,j)=\mathcal{K}(\alpha,j)$ (Srivastava et al. [15])
3. $TS(1/2,\alpha,0,1)=\mathcal{P}(\alpha)$ (Al-Amiri [1], Gupta and Jain [7] and Sarangi and Uralegaddi [13])
4. $TS(\lambda,\alpha,0,j)=\mathcal{B}_{\mathcal{T}}(\lambda,\alpha,j)$ (Frasin [3, 4] and Magesh [9])
5. $TS(1/2,\alpha,\beta,1)=TR(\alpha,\beta)$ (Rosy [12] and Stephen and Subramanian [16])
6. $TS(0,\alpha,\beta,1)=TS(\alpha,\beta)$ and $TS(1,\alpha,\beta,1)=UCV(\alpha,\beta)$ (Bharati et al. [2])
7. $TS(0,0,\beta,1)=TS_p(\beta)$ ( Subramanian et al. [17])
8. $TS(1,0,\beta,1)=UCV(\beta)$ (Subramanian et al. [18])

The main object of this paper is to obtain a necessary and sufficient conditions for the functions $f(z)$ in the generalized class $TS(\lambda,\alpha,\beta,j)$. Further we investigate extreme points, growth and distortion bounds, radii of starlikeness and convexity and modified Hadamard products for class $TS(\lambda,\alpha,\beta,j)$.

## 2 Coefficient Estimates

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $TS(\lambda,\alpha,\beta,j)$.

**Theorem 2.1.** *A function $f(z)$ of the form (1.1) is in $S(\lambda,\alpha,\beta,j)$ if*

$$\sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]|a_n|\leq 1-\alpha, \tag{2.1}$$

*where*

$$M_n=(2\lambda^2-\lambda)n^2+(1+\lambda-2\lambda^2)n,\quad F_n=(2\lambda^2-\lambda)n+(1+2\lambda^2-3\lambda) \tag{2.2}$$

*and $-1\leq\alpha<1$, $\frac{1}{2}\leq\lambda<1$, $\beta\geq 0$.*

*Proof.* It suffices to show that

$$\beta\left|\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right|$$
$$-\text{Re}\left\{\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right\}\leq 1-\alpha.$$

We have

$$\begin{aligned}
&\beta\left|\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right| \\
&- \quad \mathrm{Re}\left\{\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right\} \\
&\leq \quad (1+\beta)\left|\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right| \\
&\leq \quad \frac{(1+\beta)\sum\limits_{n=j+1}^{\infty}(M_n-F_n)|a_n|}{1-\sum\limits_{n=j+1}^{\infty}|a_n|}.
\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]|a_n|\leq 1-\alpha,$$

and hence the proof is complete. □

**Theorem 2.2.** *A necessary and sufficient condition for $f(z)$ of the form (1.4) to be in the class $TS(\lambda,\alpha,\beta,j)$, is that*

$$\sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]\ a_n\leq 1-\alpha. \tag{2.3}$$

*Proof.* In view of Theorem 2.1, we need only to prove the necessity. If $f(z)\in TS(\lambda,\alpha,\beta,j)$ and $z$ is real then

$$\frac{1-\sum\limits_{n=j+1}^{\infty}M_n\ a_nz^{n-1}}{1-\sum\limits_{n=j+1}^{\infty}F_n\ a_nz^{n-1}}-\alpha\geq\beta\left|\frac{\sum\limits_{n=j+1}^{\infty}(M_n-F_n)\ a_nz^{n-1}}{1-\sum\limits_{n=j+1}^{\infty}F_n\ a_nz^{n-1}}\right|.$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]\ a_n\leq 1-\alpha,\ -1\leq\alpha<1,\ \beta\geq 0.$$

Finally, the function $f(z)$ given by

$$f(z)=z-\frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}z^{j+1}, \tag{2.4}$$

where $M_{j+1}$ and $F_{j+1}$ as written in (2.2), is extremal for the function. □

**Corollary 2.3.** *Let the function $f(z)$ defined by (1.4) be in the class $TS(\lambda,\alpha,\beta,j)$. Then*

$$a_n\leq\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]},\ \ n\geq j+1. \tag{2.5}$$

*This equality in (2.5) is attained for the function $f(z)$ given by (2.4).*

## 3 Growth and Distortion Theorem

**Theorem 3.1.** *Let the function $f(z)$ defined by (1.4) be in the class $TS(\lambda,\alpha,\beta,j)$. Then for $|z|<r=1$*

$$r-\frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^{j+1}\leq|f(z)|\leq r+\frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^{j+1}. \tag{3.1}$$

*The result (3.1) is attained for the function $f(z)$ given by (2.4) for $z=\pm r$.*

*Proof.* Note that

$$[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]\sum_{n=j+1}^{\infty}a_n \le \sum_{n=j+1}^{\infty}\left[M_n(1+\beta)-(\alpha+\beta)F_n\right]a_n \le 1-\alpha,$$

this last inequality follows from Theorem 2.2. Thus

$$|f(z)| \ge |z| - \sum_{n=j+1}^{\infty}a_n|z|^n \ge r - r^{j+1}\sum_{n=j+1}^{\infty}a_n \ge r - \frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^{j+1}.$$

Similarly,

$$|f(z)| \le |z| + \sum_{n=j+1}^{\infty}a_n|z|^n \le r + r^{j+1}\sum_{n=j+1}^{\infty}a_n \le r + \frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^{j+1}.$$

This completes the proof. □

**Theorem 3.2.** *Let the function $f(z)$ defined by (1.4) be in the class $TS(\lambda,\alpha,\beta,j)$. Then for $|z|<r=1$*

$$r - \frac{(j+1)(1-\alpha)}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^j \le |f'(z)| \le r + \frac{(j+1)(1-\alpha)}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^j. \tag{3.2}$$

*Proof.* We have

$$|f'(z)| \ge 1 - \sum_{n=j+1}^{\infty}na_n|z|^{n-1} \ge 1 - r^j\sum_{n=j+1}^{\infty}na_n \tag{3.3}$$

and

$$|f'(z)| \le 1 + \sum_{n=j+1}^{\infty}na_n|z|^{n-1} \le 1 + r^j\sum_{n=j+1}^{\infty}na_n. \tag{3.4}$$

In view of Theorem 2.2,

$$\frac{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}{j+1}\sum_{n=j+1}^{\infty}na_n \le \sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]a_n \le 1-\alpha, \tag{3.5}$$

or, equivalently

$$\sum_{n=j+1}^{\infty}na_n \le \frac{(j+1)(1-\alpha)}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}. \tag{3.6}$$

A substitution of (3.6) into (3.3) and (3.4) yields the inequality (3.2). This completes the proof. □

**Theorem 3.3.** *Let $f_j(z)=z$, and*

$$f_n(z) = z - \frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}z^n, \quad n \ge j+1 \tag{3.7}$$

*for $0\le\lambda\le 1, \beta\ge 0, -1\le\alpha<1$. Then $f(z)$ is in the class $TS(\lambda,\alpha,\beta,j)$ if and only if it can be expressed in the form*

$$f(z) = \sum_{n=j}^{\infty}\mu_n f_n(z), \tag{3.8}$$

*where $\mu_n \ge 0 (n\ge j)$ and $\sum_{n=j}^{\infty}\mu_n = 1$.*

*Proof.* Assume that

$$\begin{aligned} f(z) &= \mu_j f_j(z) + \sum_{n=j+1}^{\infty}\mu_n\left[z - \frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}z^n\right] \\ &= \sum_{n=j}^{\infty}\mu_n z - \sum_{n=j+1}^{\infty}\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}\mu_n z^n. \end{aligned}$$

Then it follows that

$$\sum_{n=j+1}^{\infty}\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}\mu_n\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha}=\sum_{n=j+1}^{\infty}\mu_n\leq 1,$$

so by Theorem 2.2, $f(z)\in TS(\lambda,\alpha,\beta,j)$.

Conversely, assume that the function $f(z)$ defined by (1.4) belongs to the class $TS(\lambda,\alpha,\beta,j)$, then

$$a_n\leq\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]},\quad n\geq j+1.$$

Setting $\mu_n=\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha}\,a_n,(n\geq j+1)$ and $\mu_j=1-\sum_{n=j+1}^{\infty}\mu_n$, we have,

$$\begin{aligned} f(z) &= z-\sum_{n=j+1}^{\infty}a_nz^n \\ f(z) &= z-\sum_{n=j+1}^{\infty}\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}\mu_nz^n. \end{aligned}\tag{3.9}$$

Then (3.8) gives

$$\begin{aligned} f(z) &= z+\sum_{n=j+1}^{\infty}(f_n(z)-z)\mu_n \\ &= f_j(z)\mu_j+\sum_{n=j+1}^{\infty}f_n(z)\mu_n \\ &= \sum_{n=j}^{\infty}\mu_nf_n(z) \end{aligned}$$

and hence the proof is complete. □

## 4 Radii of close-to-convexity, Starlikeness and Convexity

In this subsection, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS(\lambda,\alpha,\beta,j)$.

**Theorem 4.1.** *Let $f\in TS(\lambda,\alpha,\beta,j)$. Then $f(z)$ is close-to-convex of order $\sigma$ $(0\leq\sigma<1)$ in the disc $|z|<r_1$, where*

$$r_1:=\inf\left[\frac{(1-\sigma)[M_n(1+\beta)-(\alpha+\beta)F_n]}{n(1-\alpha)}\right]^{\frac{1}{n-1}},\quad n\geq j+1.\tag{4.1}$$

*The result is sharp, with extremal function $f(z)$ given by (2.4).*

*Proof.* Given $f\in T$, and $f$ is close-to-convex of order $\sigma$, we have

$$|f'(z)-1|<1-\sigma.\tag{4.2}$$

For the left hand side of (4.2) we have

$$|f'(z)-1|\leq\sum_{n=j+1}^{\infty}na_n|z|^{n-1}.$$

The last expression is less than $1-\sigma$ if

$$\sum_{n=j+1}^{\infty}\frac{n}{1-\sigma}a_n|z|^{n-1}<1.$$

Using the fact, that $f\in TS(\lambda,\alpha,\beta,j)$, if and only if

$$\sum_{n=j+1}^{\infty}\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}a_n\leq 1.$$

We can say (4.2) is true if

$$\frac{n}{1-\sigma}|z|^{n-1} \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\frac{(1-\sigma)[M_n(1+\beta)-(\alpha+\beta)F_n]}{n(1-\alpha)}\right],$$

which completes the proof. □

**Theorem 4.2.** *Let $f \in TS(\lambda, \alpha, \beta, j)$. Then*

*(i) $f$ is starlike of order $\sigma (0 \leq \sigma < 1)$ in the disc $|z| < r_2$; where*

$$r_2 = \inf\left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}\right]^{\frac{1}{n-1}}, \ n \geq j+1, \tag{4.3}$$

*(ii) $f$ is convex of order $\sigma$ $(0 \leq \sigma < 1)$ in the unit disc $|z| < r_3$, where*

$$r_3 = \inf\left[\left(\frac{1-\sigma}{n(n-\sigma)}\right)\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}\right]^{\frac{1}{n-1}}, \ n \geq j+1. \tag{4.4}$$

*Each of these results are sharp for the extremal function $f(z)$ given by (2.4).*

*Proof.* (i) Given $f \in T$, and $f$ is starlike of order $\sigma$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\sigma. \tag{4.5}$$

For the left hand side of (4.5) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=j+1}^{\infty}(n-1)a_n\ |z|^{n-1}}{1-\sum\limits_{n=j+1}^{\infty}a_n\ |z|^{n-1}}.$$

The last expression is less than $1-\sigma$ if

$$\sum_{n=j+1}^{\infty}\frac{n-\sigma}{1-\sigma}a_n\ |z|^{n-1} < 1.$$

Using the fact, that $f \in TS(\lambda, \alpha, \beta, j)$ if and only if

$$\sum_{n=j+1}^{\infty}\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}a_n \leq 1.$$

We can say (4.5) is true if

$$\frac{n-\sigma}{1-\sigma}|z|^{n-1} < \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}\right]$$

which yields the starlikeness of the family.
(ii) Using the fact that $f$ is convex if and only if $zf'$ is starlike, we can prove (ii), on lines similar to the proof of (i). □

## 5 Modified Hadamard Product

Let the functions $f_i(z)(i=1,2)$ be defined by

$$f_i(z) = z - \sum_{n=j+1}^{\infty} a_{n,i} z^n, \quad a_{n,i} \geq 0; j \in \mathbb{N}, \tag{5.1}$$

then we define the modified Hadamard product of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z - \sum_{n=j+1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{5.2}$$

Now, we prove the following.

**Theorem 5.1.** *Let each of the functions $f_i(z)(i=1,2)$ defined by (5.1) be in the class $TS(\lambda,\alpha,\beta,j)$. Then $(f_1 * f_2) \in TS(\lambda,\delta_1,\beta,j)$, for*

$$\delta_1 = \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]^2 - [M_n(1+\beta)-\beta F_n](1-\alpha)^2}{[M_n(1+\beta)-(\alpha+\beta)F_n]^2 - F_n(1-\alpha)^2}. \tag{5.3}$$

*The result is sharp.*

*Proof.* We need to prove the largest $\delta_1$ such that

$$\sum_{n=j+1}^{\infty} \frac{[M_n(1+\beta)-(\delta_1+\beta)F_n]}{1-\delta_1} a_{n,1} a_{n,2} \leq 1. \tag{5.4}$$

From Theorem 2.2, we have

$$\sum_{n=j+1}^{\infty} \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha} a_{n,1} \leq 1$$

and

$$\sum_{n=j+1}^{\infty} \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha} a_{n,2} \leq 1,$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{n=j+1}^{\infty} \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha} \sqrt{a_{n,1} a_{n,2}} \leq 1. \tag{5.5}$$

Thus it is sufficient to show that

$$\frac{[M_n(1+\beta)-(\delta_1+\beta)F_n]}{1-\delta_1} a_{n,1} a_{n,2} \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha} \sqrt{a_{n,1} a_{n,2}}, \ n \geq j+1 \tag{5.6}$$

that is

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n](1-\delta_1)}{[M_n(1+\beta)-(\delta_1+\beta)F_n](1-\alpha)}, \ n \geq j+1. \tag{5.7}$$

Note that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(1-\alpha)}{[M_n(1+\beta)-(\alpha+\beta)F_n]}, \ n \geq j+1. \tag{5.8}$$

Consequently, we need only to prove that

$$\frac{(1-\alpha)}{[M_n(1+\beta)-(\alpha+\beta)F_n]} \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n](1-\delta_1)}{[M_n(1+\beta)-(\delta_1+\beta)F_n](1-\alpha)}, \tag{5.9}$$

or equivalently

$$\delta_1 \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]^2 - [M_n(1+\beta)-\beta F_n](1-\alpha)^2}{[M_n(1+\beta)-(\alpha+\beta)F_n]^2 - F_n(1-\alpha)^2} = \Delta(n). \tag{5.10}$$

Since $\Delta(n)$ is an increasing function of $n(n \geq j+1)$, letting $n=j+1$ in (5.10) we obtain

$$\delta_1 \leq \Delta(j+1) = \frac{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]^2 - [M_{j+1}(1+\beta)-\beta F_{j+1}](1-\alpha)^2}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]^2 - F_{j+1}(1-\alpha)^2} \tag{5.11}$$

which proves the main assertion of Theorem 5.1. The result is sharp for the functions defined by (2.4). □

**Theorem 5.2.** *Let the function $f_i(z)(i = 1, 2)$ defined by (5.1) be in the class $TS(\lambda, \alpha, \beta, j)$. If the sequence $\{[M_n(1+\beta) - (\alpha+\beta)F_n]\}$ is non-decreasing. Then the function*

$$h(z) = z - \sum_{n=j+1}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n \tag{5.12}$$

*belongs to the class $TS(\lambda, \delta_2, \beta, j)$ where*

$$\delta_2 = \frac{[M_n(1+\beta) - (\alpha+\beta)F_n]^2 - 2[M_n(1+\beta) - \beta F_n](1-\alpha)^2}{[M_n(1+\beta) - (\alpha+\beta)F_n]^2 - 2F_n(1-\alpha)^2}.$$

*Proof.* By virtue of Theorem 2.2, we have for $f_j(z)(j = 1, 2) \in TS(\lambda, \alpha, \beta, j)$ we have

$$\sum_{n=j+1}^{\infty} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha}\right]^2 a_{n,1}^2 \le \sum_{n=j+1}^{\infty} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha} a_{n,1}\right]^2 \le 1 \tag{5.13}$$

and

$$\sum_{n=j+1}^{\infty} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha}\right]^2 a_{n,2}^2 \le \sum_{n=j+1}^{\infty} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha} a_{n,2}\right]^2 \le 1. \tag{5.14}$$

It follows from (5.13) and (5.14) that

$$\sum_{n=j+1}^{\infty} \frac{1}{2} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha}\right]^2 (a_{n,1}^2 + a_{n,2}^2) \le 1. \tag{5.15}$$

Therefore we need to find the largest $\delta_2$, such that

$$\frac{[M_n(1+\beta) - (\delta_2+\beta)F_n]}{1-\delta_2} \le \frac{1}{2} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha}\right]^2, \quad n \ge j+1$$

that is

$$\delta_2 \le \frac{[M_n(1+\beta) - (\alpha+\beta)F_n]^2 - 2[M_n(1+\beta) - \beta F_n](1-\alpha)^2}{[M_n(1+\beta) - (\alpha+\beta)F_n]^2 - 2F_n(1-\alpha)^2} = \Psi(n).$$

Since $\Psi(n)$ is an increasing function of $n, (n \ge j+1)$, we readily have

$$\delta_2 \le \Psi(j+1) = \frac{[M_{j+1}(1+\beta) - (\alpha+\beta)F_{j+1}]^2 - 2[M_{j+1}(1+\beta) - \beta F_{j+1}](1-\alpha)^2}{[M_{j+1}(1+\beta) - (\alpha+\beta)F_{j+1}]^2 - 2F_{j+1}(1-\alpha)^2}$$

which completes the proof. □

# 6 Acknowledgement

The authors are very much thankful to the referee for his/her useful suggestions and comments.

# References

[1] H.S. Al-Amiri, On a subckass of close-to-convex functions with negative coefficients, Mathematics, *Tome*, 31(1)(54)(1989), 1–7.

[2] R. Bharati, R. Parvatham, and A.Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, *Tamkang J. Math.*, 28(1)(1997), 17–32.

[3] B.A. Frasin, On the analytic functions with negative coefficients, *Soochow J. Math.*, 31(3)(2005), 349–359.

[4] B.A. Frasin, Some applications of fractional calculus operators to certain subclass of analytic functions with negative coefficients, *Acta Universitatis Apulensis*, 23(2010), 123-132.

[5] A.W. Goodman, On uniformly convex functions, *Ann. polon. Math.*, 56(1991), 87-92.

[6] A.W. Goodman, On uniformly starlike functions, *J. Math. Anal. Appl.*, 155(1991), 364-370.

[7] V.P. Gupta and P.K. Jain, Certain classes univalent analytic functions with negative coefficients II, *Bull. Austral. Math. Soc.*, 15(1976), 467-473.

[8] W. Ma and D. Minda, Uniformly convex functions, *Ann. Polon. Math.*, 57(1992), 165-175.

[9] N. Magesh, On certain subclasses of analytic and bi-univalent functions, *Preprint*.

[10] G. Murugusundaramoorthy and N. Magesh, On certain subclasses of analytic functions associated with hypergeometric functions, *Applied Mathematics Letters*, 24(2011), 494-500.

[11] F. Running, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, 118(1993), 189-196.

[12] T. Rosy, *Study on subclasses of univalent functions*, Thesis, University of Madras, 2002.

[13] S. M. Sarangi and B.A. Uralegaddi, The radius of convexity and starlikeness of certain classes of analytic functions with negative coefficients I, *Atti. Accad. Naz. Lincei rend. Cl. Sci. Fis. Mat. Natur.*, 65(8)(1978), 34-42.

[14] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, 51(1975), 109–116.

[15] H. M. Srivastava, S. Owa and S. K. Chatterjea, A note on certain classes of starlike func- tions, *Rend. Sem. Mat. Univ. Padova*, 77(1987), 115–124.

[16] B.A. Stephand and K.G. Subramanian, On a subclass of Noshiro-Type analytic functions with negative coefficients, *Conference Proc. Ramanujan Mathematical Society*, 1998.

[17] K.G. Subramanian, T.V. Sudharsan, P. Balasubrahmanyam and H. Silverman, Class of uniformly starlike functions, *Publ. Math. Debercen.*, 53(4)(1998), 309 - 315.

[18] K.G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahmanyam and H. Silverman, Subclasses of uniformly convex and uniformly starlike functions, *Math. Japonica*, 42(3)(1995), 517–522.

*Received*: October 17, 2012; *Accepted*: November 30, 2012

UNIVERSITY PRESS

Malaya Journal of Matematik 1(1)(2013) 27–37

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

# On a nonlinear functional second order integrodifferential equation in Banach Spaces

P. M. Dhakane,[a,*] and D. B. Pachpatte[b]

[a] Department of Mathematics, S. B. Science College, Aurangabad - 431 001, India.

[b] Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431 001, India.

**Abstract**

In this paper, we study the existence, uniqueness and other properties of solutions of a nonlinear functional second order Volterra integrodifferential equation in a general Banach space. The techniques used in our analysis are the theory of the strongly continuous cosine family, Schauder fixed point theorem and Pachpatte's integral inequality.

*Keywords:* Integrodifferential equations, Cosine family, Schauder's fixed point theorem, Pachpatte's integral inequality.

*2010 MSC:* 45D05, 34G20, 45N05,45P05, 37C25. 

## 1 Introduction

Let $X$ denotes a Banach space with norm $\|\cdot\|$. Let $C = C([-r,0],X),\ 0 < r < \infty$, be the Banach space of all continuous functions from $\psi : [-r,0] \longrightarrow X$ with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(\theta)\| : -r \leq \theta \leq 0\}.$$

If $x$ is a continuous function from $[-r,\, T]$, $T > 0$, to $X$ and $t \in [0,\, T]$ then $x_t$ stands for the element of $C$ given by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$. Let $B = C\,([-r,T],\, X)$ denotes the Banach space of all continuous functions $x : [-r,\, T] \to X$ endowed with supremum norm $||\,x\,||_B \;=\; sup\ \{||\,x(t)\,|| : \;\; -r \leq t \leq T\}$. We investigate the abstract nonlinear functional second order Volterra integrodifferential equation of the form

$$x''(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t,s), g(s,x_s)ds\right), \quad t \in [0,T] \tag{1.1}$$

$$x_0(t) = \phi(t), \quad -r \leq t \leq 0, \tag{1.2}$$

$$x'(0) = \delta \tag{1.3}$$

where $A$ is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space $X$, $f : [0,T] \times C \times X \to X,\ k : [0,T] \times [0,T] \to \mathbb{R},\ g : [0,T] \times C \to X$ are continuous functions, $\phi$ and $\delta$ are given elements of $C = C([-r,0],X)$ and $X$ respectively.

Equations of these types (1.1)-(1.3) are their special forms commonly come across in almost all phases of physics and applied mathematics, see, for example [1-6] and the references cited therein. Many authors have been investigated the problems such as existence, uniqueness and other properties of solutions of equations (1.1)-(1.3) or their special forms by using various methods, see, for example [7, 8, 13, 17-22] and the references given therein. Our attempt is to generalize some results obtained by A. Pazy [15], and C. C. Travis and G. F. Webb [20]. It is advantageous to treat second order abstract differential equations directly rather than to convert into first order systems, see, for example Fitzgibbon [10]. In [10], Fitzgibbon used the second order

*Corresponding author.

*E-mail addresses*: pmdhakane@gmail.com (P. M. Dhakane) and pachpatte@gmail.com (D. B. Pachpatte).

abstract system for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. Our work in the present chapter is motivated by the interesting results obtained by Fattorini H. O. in [9] and is influenced by the work of Patcheu S. K. [14] and Travis C. C. and Webb G. F. [21].

The paper is organized as follows: In section 2, we present the preliminaries and statements of our results. Section 3 proves the Theorems 2.4 and 2.5 In section 4, we discuss the proofs of Theorems 2.6 - 2.8. Finally, section 5 presents an example to illustrate the application of our theorem.

## 2 Preliminaries

Before proceeding to the statements of our main results, we setforth some preliminaries from [11, 18, 20] and hypotheses used in our further discussion.

**Definition 2.1.** *A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space $X$ is called a strongly continuous cosine family if and only if*

(a) $C(0) = I$ *($I$ is the identity operator);*

(b) $C(t)x$ *is strongly continuous in $t$ on $\mathbb{R}$ for each fixed $x \in X$;*

(c) $C(t + s) + C(t - s) = 2C(t)C(s)$ *for all $t,\ s \in \mathbb{R}$.*

The associated strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}. \tag{2.1}$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is the operator $A : X \longrightarrow X$ defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(.)x \in C^2(\mathbb{R}, X)\}$.

**Definition 2.2.** *Let $f \in L^1(0, T; X)$. The function $x \in B$ defined by*

$$x(t) = C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s) f\Big(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\Big) ds, \quad t \in [0, T] \tag{2.2}$$

$$x_0(t) = \phi(t), \quad -r \le t \le 0 \tag{2.3}$$

*is called mild solution of the initial value problem (1.1)-(1.3).*

**Definition 2.3.** *A set $S$ in a Banach space $X$ is said to be relatively compact set if its closure is compact.*

**Definition 2.4.** *An operator $T : X \to X$ is called compact if it maps bounded sets into relatively compact sets.*

Consider the following initial value problems

$$x''(t) = Ax(t) + h\Big(t, x_t, \int_{t_0}^t k(t,s), g(s, x_s) ds, \mu_1\Big), \quad t \in [0, T] \tag{2.4}$$

$$x_0(t) = \phi(t), \quad -r \le t \le 0, \tag{2.5}$$

$$x'(0) = \delta \tag{2.6}$$

and

$$x''(t) = Ax(t) + h\Big(t, x_t, \int_{t_0}^t k(t,s), g(s, x_s) ds, \mu_2\Big), \quad t \in [0, T] \tag{2.7}$$

$$x_0(t) = \phi(t), \quad -r \le t \le 0, \tag{2.8}$$

$$x'(0) = \delta \tag{2.9}$$

where $A$ is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space $X$, $h : [0, T] \times C \times X \times \mathbb{R} \to X$, $k : [0, T] \times [0, T] \to \mathbb{R}$, $g : [0, T] \times C \to X$ are continuous functions, $\mu_1$, $\mu_2$ are real parameters, $\phi \in C$ and $\delta \in X$ are given elements.

For our convenience, we list the following hypotheses.

$(H_1)$ There are constants $K \geq 1$ and $K_1 > 0$ such that

$$\|C(t)\| \leq K \quad \text{and} \quad \|S(t)\| \leq K_1,$$

for all $t \in [0, T]$.

$(H_2)$ For every $t \in [0, T],\ \psi \in C$ and $x \in X$, there exist a continuous function $p : [0, T] \to \mathbb{R}_+$ such that

$$\|f(t, \psi, x)\| \leq p(t)\big[\|\psi\|_C + \|x\|\big].$$

$(H_3)$ There exist a continuous function $q : [0, T] \to \mathbb{R}_+$ such that

$$\|g(t, \psi)\| \leq q(t)\|\psi\|_C$$

for every $t \in [0, T]$ and $\psi \in C$.

$(H_4)$ For every $t \in [0, T], \psi_1, \psi_2 \in C$ and $x_1, x_2 \in X$, there exists a constant $M$ such that

$$\|f(t, \psi_1, x_1) - f(t, \psi_2, x_2)\| \leq M[\|\psi_1 - \psi_2\|_C + \|x_1 - x_2\|]$$

$(H_5)$ There exists a constant $N$ such that

$$\|g(t, \psi_1) - g(t, \psi_2)\| \leq N\|\psi_1 - \psi_2\|_C,$$

for all $t \in [0, T]$ and $\psi_1, \psi_2 \in C$.

$(H_6)$ For each $t \in [0, T]$ the function $f(t, ., .) : [0, T] \times C \times X \to X$ is continuous and for each $\psi \in C$ and for each $x \in X$, the function $f(., \psi, x) : [0, T] \times C \times X \to X$ is strongly measurable.

$(H_7)$ For each $t \in [0, T]$ the function $g(t, .) : [0, T] \times C \to X$ is continuous and for each $\psi \in C$, the function $g(., \psi) : [0, T] \times C \to X$ is strongly measurable.

$(H_8)$ For every positive integer $q$ there exists $\alpha_q \in L^1([0, T], [0, \infty))$ such that for a.e. $t \in [0, T]$ and $x \in B$

$$\sup_{\|x\|_B \leq q} \|f\left(t, x_t, \int_0^t k(t, s) g(s, x_s) ds\right)\| \leq \alpha_q(t)$$

and

$$\liminf_{q \to +\infty} \frac{1}{q} \int_0^T \alpha_q(s) ds = \zeta < \infty.$$

$(H_9)$ There exist constants $M_1$ and $M_2$ such that

$$\|h(t, \psi_1, y_1, \rho) - h(t, \psi_2, y_2, \rho)\| \leq M_1[\|\psi_1 - \psi_2\|_C + \|y_1 - y_2\|]$$

and

$$\|h(t, \psi, y, \rho_1) - h(t, \psi, y, \rho_2)\| \leq M_2|\rho_1 - \rho_2|.$$

We use Schauder fixed point theorem to prove our results.

**Lemma 2.1.** *(Schauder fixed point theorem [16], p-37) Let $S$ be a bounded, closed and convex subset of a Banach space $X$. If $f \in \mathcal{C}(S, S)$, where $\mathcal{C}(S, S)$ is the set of all compact maps from $S$ into $S$, then $f$ has at least one fixed point.*

The following Pachpatte's inequality is the key instrument in our subsequent discussion.

**Lemma 2.2 ([12], p. 758).** *Let $u(t), p(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on $\mathbb{R}_+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t p(s)\Big[u(s) + \int_0^s q(\tau)u(\tau)d\tau\Big]ds,$$

*holds for all $t \in \mathbb{R}_+$, where $u_0$ is a nonnegative constant, then*

$$u(t) \leq u_0\Big[1 + \int_0^t p(s) \exp\Big(\int_0^s \big(p(\tau) + q(\tau)\big)d\tau\Big)ds\Big],$$

*holds for all $t \in \mathbb{R}_+$.*

We need the following result in the sequel.

**Lemma 2.3.** *([16], p.76) Let $C(t)$, (resp. $S(t)$), $t \in \mathbb{R}$ be a strongly continuous cosine (resp. sine) family on $X$. Then there exists constants $N \geq 1$ and $\omega \geq 0$ such that*

$$\|C(t)\| \leq Ne^{\omega|t|}, \text{ for } t \in \mathbb{R},$$

$$\|S(t_1) - S(t_2)\| \leq N \left| \int_{t_1}^{t_2} e^{\omega|s|} ds \right|, \text{ for } t_1, t_2 \in \mathbb{R}$$

For more details on strongly continuous cosine and sine families, we refer the reader to [19] and [21]. With these preperations, now, we are in position to state our main results.

**Theorem 2.4.** *Suppose that the hypotheses $(H_1)$, $(H_6) - (H_8)$ hold. Then initial value problem (1.1)-(1.3) has at least one mild solution on $[-r, T]$ if $K_1\zeta < 1$.*

**Theorem 2.5.** *Suppose that the hypotheses $(H_1), (H_4)$ and $(H_5)$ hold. Then initial value problem (1.1)-(1.3) has at most one mild solution on $[-r, T]$.*

**Theorem 2.6.** *Suppose that the hypotheses $(H_1) - (H_3)$ hold. Then, every solution of the initial value problem (1.1)-(1.3) is bounded on $[-r, T]$.*

**Theorem 2.7.** *Suppose that the hypotheses $(H_1), (H_4)$ and $(H_5)$ hold. Let $x_1(t)$ and $x_2(t)$ be two solutions of the initial value problem (1.1) with initial conditions*

$$x_{1_0}(t) = \phi(t), \quad -r \leq t \leq 0, \ x_1'(0) = \delta$$

*and*

$$x_{2_0}(t) = \chi(t), \quad -r \leq t \leq 0, \ x_2'(0) = \sigma$$

*respectively. Then*

$$\|x_1 - x_2\|_B \leq \left[ K\|\phi - \chi\|_C + K_1\|\delta - \sigma\| \right] \left[ 1 + K_1MT \exp\{(K_1M + LN)T\} \right].$$

The following theorem investigates the continuous dependency of solutions of initial value problems (2.4) - (2.6) and (2.7) - (2.9) on parameters.

**Theorem 2.8.** *Suppose that the hypotheses $(H_1)$, $(H_5)$ and $(H_9)$ hold. Let $x_1(t)$ and $x_2(t)$ be the solutions of initial value problem (2.4) - (2.6) and (2.7) - (2.9) respectively on $[-r, T]$. Then*

$$\|x_1 - x_2\|_B \leq K_1M_2T|\mu_1 - \mu_2| \left[ 1 + K_1M_1T \exp\{(K_1M_1 + LN)T\} \right].$$

## 3 Proofs of the Theorems 2.4 and 2.5

*Proof of Theorem 2.4.* Define the operator $F : B \to B$ by

$$(Fx)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s)f\left(s, x_s, \int_0^s k(s,\tau)g(\tau, x_\tau)d\tau\right) ds, & t \in [0, T]. \end{cases}$$

Then the equivalent integral equation for the system (1.1) - (1.3) can be written as the fixed point problem $x = Fx$. We prove that $F$ has a fixed point $x(\cdot)$ by applying the Schauder fixed point theorem. For each positive integer $q$, let

$$B_q = \{x \in B : x(t) = \phi(t), t \in [-r, 0] \text{ and } \|x\|_B \leq q\}.$$

Then for each $q$, $B_q$ is clearly closed, convex and bounded subset in $B$. Obviously, $F$ is well defined on $B_q$. We claim that there exists a positive integer $q$ such that $FB_q \subseteq B_q$. If this were not true for some $q$, then for each positive integer $q$, there is a function $x_q \in B_q$ with $Fx_q \notin B_q$, that is $\|Fx_q\| > q$. Then $1 < \frac{1}{q}\|Fx_q\|$, and hence

$$1 \leq \liminf_{q\to+\infty} \frac{1}{q}\|Fx_q(t)\|,\ t \in [0,T] \tag{3.1}$$

However, on the other hand by using the hypotheses $(H_1)$, $(H_8)$ and condition in Theorem, we have

$$\begin{aligned}
&\liminf_{q\to+\infty} \frac{1}{q}\,\|Fx_q(t)\| \\
=&\liminf_{q\to+\infty} \frac{1}{q}\,\|C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s)f\left(s, x_{q_s}, \int_0^s k(s,\tau)g(\tau, x_{q_\tau})d\tau\right)ds\| \\
\leq&\liminf_{q\to+\infty} \frac{1}{q}\left[\|C(t)\|\|\phi(0)\| + \|S(t)\|\|\delta\| + \int_0^t \|S(t-s)\|\|f\left(s, x_{q_s}, \int_0^s k(s,\tau)g(\tau, x_{q_\tau})d\tau\right)\|ds\right] \\
\leq&\liminf_{q\to+\infty} \frac{1}{q}\left[K\|\phi\|_C + K_1\|\delta\| + \int_0^t K_1\|f\Big(s, x_{q_s}, \int_0^s k(s,\tau)g(\tau, x_{q_\tau})d\tau\Big)\|ds\right] \\
\leq&\liminf_{q\to+\infty}\left[\frac{K\|\phi\|_C + K_1\|\delta\|}{q} + K_1\frac{1}{q}\int_0^t \alpha_q(s)ds\right] \\
=&K_1\zeta < 1,
\end{aligned}$$

which contradicts the condition (3.1). Therefore, for some positive integer $q$, we must have $FB_q \subseteq B_q$.

Next we prove that $F$ is a compact operator on $B_q$ . For this purpose, first we prove that $F$ is continuous on $B_q$. Let $\{x_n\} \subseteq B_q$ with $x_n \to x$ in $B_q$. By using hypotheses $(H_6)$ and $(H_7)$, we have

$$f\Big(t, x_{n_t}, \int_0^t k(t,s)g(s, x_{n_s})ds\Big) \to f\Big(t, x_t, \int_0^t k(t,s)g(s, x_s)ds\Big) \text{ as } n \to \infty,$$

for each $t \in [0,T]$. Therefore by dominated convergence theorem,

$$\begin{aligned}
&\|(Fx_n)(t) - (Fx)(t)\| \\
&= \|\int_0^t S(t-s)\Big[f\Big(s, x_{n_s}, \int_0^s k(s,\tau)g(\tau, x_{n_\tau})d\tau\Big) \\
&\qquad - f\Big(s, x_s, \int_0^s k(s,\tau)g(\tau, x_\tau)d\tau\Big)\Big]ds\| \\
&= \int_0^t \|S(t-s)\|\ \|f\Big(s, x_{n_s}, \int_0^s k(s,\tau)g(\tau, x_{n_\tau})d\tau\Big) \\
&\qquad - f\Big(s, x_s, \int_0^s k(s,\tau)g(\tau, x_\tau)d\tau\Big)\|ds \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This implies that $\|Fx_n - Fx\|_B \to 0$ as $n \to \infty$. Therefore, $F$ is continuous.

Next we prove that the family $\{Fx : x \in B_q\}$ is an equicontinuous family of functions. To do this, let $0 < t_1 < t_2 \leq T$; then

$$\begin{aligned}
&\|(Fx)(t_1) - (Fx)(t_2)\| \\
&\leq \|[C(t_1) - C(t_2)]\phi(0)\| + \|[S(t_1) - S(t_2)]\delta\| \\
&\quad + \|\int_0^{t_1} [S(t_1-s) - S(t_2-s)]f\Big(s, x_s, \int_0^s k(s,\tau)g(\tau, x_\tau)d\tau\Big)ds\| \\
&\quad + \|\int_{t_1}^{t_2} S(t_2-s)f\Big(s, x_s, \int_0^s k(s,\tau)g(\tau, x_\tau)d\tau\Big)ds\| \\
&\leq \|[C(t_1) - C(t_2)]\|\|\phi\|_C + \|S(t_1) - S(t_2)\|\|\delta\| \\
&\quad + \int_0^{t_1} \|S(t_1-s) - S(t_2-s)\|\alpha_q(s)ds + \int_{t_1}^{t_2} \|S(t_2-s)\|\alpha_q(s)ds
\end{aligned}$$

The right hand side of above inequality is independent of $x \in B_q$ and tends to zero as $(t_2 - t_1) \to 0$, since $C(t), S(t)$ are uniformly continuous for $t \in [0,T]$. The compactness of $C(t), S(t)$ for $t > 0$ imply the continuity in the uniform operator topology (see lemma 2.3). The compactness of $S(t)$ follows from that of $C(t)$. Thus $F$ maps $B_q$ into an equicontinuous family of functions. The equicontinuity for the cases $t_1 \leq t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ follows from the uniform continuity of $\phi$ on $[-r, 0]$ and from the relation

$$\|(Fy)(t_1) - (Fy)(t_2)\| \leq \|\phi(t_1) - \phi(0)\| + \|(Fy)(0) - (Fy)(t_2)\|$$

respectively.

It remains to prove that $V(t) = \{(Fx)(t) : x \in B_q\}$ is relatively compact in $X$ for each $t \in [-r, T]$. This is trivial for $t \in [-r, 0]$, since $V(t) = \{\phi(t)\}$ which is singleton set. So let $0 < t \leq T$ be fixed and $\epsilon$ a real number satisfying $0 < \epsilon < t$; for $x \in B_q$, we define

$$(F_\epsilon x)(t) = C(t)\phi(0) + S(t)\delta + \int_0^{t-\epsilon} S(t-s) f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right) ds$$

Since $C(t)$ and $S(t)$ are compact operators the set $V\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_q\}$ is relative compact in $X$ for every $\epsilon$, $0 < \epsilon < t$. Moreover by making use of hypotheses $(H_8)$, for every $x \in B_q$, we have

$$\begin{aligned}\|(Fx)(t) - (F_\epsilon x)(t)\| &= \int_{t-\epsilon}^t \|S(t-s) f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right)\| ds \\ &\leq \int_{t-\epsilon}^t \|S(t-s)\| \alpha_q(s) ds\end{aligned}$$

Therefore there are relative compact sets arbitrarily close to the set $V(t) = \{(Fx)(t) : x \in B_q\}$; hence the set $V(t)$ is also relative compact in $X$. Thus, by the Arzela-Ascoli theorem $F$ is a compact operator and by Schauder's fixed point theorem there exists a fixed point $x(\cdot)$ for $F$, which is a solution of (1.1) - (1.3) satisfying $x(t) = \phi(t), -r \leq t \leq 0$. This completes proof of the Theorem 2.4. □

*Proof of Theorem 2.5.* Assume that $x$ and $y$ are two solutions of the initial value problem (1.1) - (1.3) on $[-r, T]$. The function $k : [0, T] \times [0, T] \longrightarrow \mathbb{R}$ being continuous on compact set, there exists a constant $L > 0$ such thawt

$$\|k(t,s)\| \leq L \quad \text{for} \quad 0 \leq s \leq t \leq T \tag{3.2}$$

From definition of mild solution given in (2.2) - (2.3) and using hypotheses $(H_1), (H_4), (H_5)$ and condition 3.2, we have

$$\begin{aligned}\|x(t) - y(t)\| &\leq \int_0^t \|S(t-s)\| \, \|f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right) - f\left(s, y_s, \int_0^s k(s,\tau) g(\tau, y_\tau) d\tau\right)\| ds \\ &\leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \end{aligned} \tag{3.3}$$

**Case 1:** Suppose $t \geq r$. Then, for every $\theta \in [-r, 0]$, we have $t + \theta \geq 0$. For such $\theta' s$, from (3.3) we have

$$\begin{aligned}\|x(t+\theta) - y(t+\theta)\| &\leq K_1 M \int_0^{t+\theta} \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \\ &\leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds,\end{aligned}$$

which implies

$$\|x_t - y_t\|_C \leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \tag{3.4}$$

**Case 2:** Suppose $0 \leq t < r$. Then for all $\theta \in [-r, -t)$, we have $t + \theta < 0$. For such $\theta' s$, we observe, from (2.2)-(2.3), that

$$\begin{aligned}\|x(t+\theta) - y(t+\theta)\| &= \|x_t(\theta) - y_t(\theta)\| \\ &= 0,\end{aligned}$$

which yields

$$\|x_t - y_t\|_C = 0. \tag{3.5}$$

For $\theta \in [-t, 0],\ t + \theta \geq 0$. Then, for such $\theta' s$ we obtain as in the case 1,

$$\|x_t - y_t\|_C \leq K_1 M \int_0^t \Big[ \|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \Big] ds \tag{3.6}$$

Thus, for every $\theta \in [-r, 0]$, $(0 \leq t < r)$, from (3.5) and (3.6), we get

$$\|x_t - y_t\|_C \leq K_1 M \int_0^t \Big[ \|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \Big] ds \tag{3.7}$$

For every $t \in [0, T]$, from inequalities (3.4) and (3.7), we have

$$\begin{aligned} \|x_t - y_t\|_C &\leq K_1 M \int_0^t \Big[ \|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \Big] ds \\ &< \epsilon + K_1 M \int_0^t \Big[ \|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \Big] ds \end{aligned} \tag{3.8}$$

for an arbitrary $\epsilon > 0$. Thanks to Pachpatte's integral inequality given in Lemma 2.2 and applying it to (3.8) with $u(t) = \|x_t - y_t\|_C$ we get

$$\begin{aligned} \|x_t - y_t\|_C &\leq \epsilon \Big[ 1 + \int_0^t K_1 M \exp\Big( \int_0^s (K_1 M + LN) d\tau \Big) ds \Big] \\ &< \epsilon \Big[ 1 + K_1 M T \exp\Big( \{K_1 M + LN\} T \Big) \Big] \end{aligned}$$

Since $\|x(t) - y(t)\| = 0\ \forall\ t \in [-r, 0]$, it follows, for $t \in [-r, T]$, that

$$\|x(t) - y(t)\| \leq \epsilon \Big[ 1 + K_1 M T \exp\big\{ (K_1 M + LN) T \big\} \Big]$$

which yields

$$\|x - y\|_B \leq \epsilon \Big[ 1 + K_1 M T \exp\big\{ (K_1 M + LN) T \big\} \Big]$$

Since $\epsilon > 0$ is an arbitrary, it follows that

$$\|x - y\|_B = 0$$

which implies $x(t) = y(t),\ \forall\ t \in [-r, T]$. This proves that the initial value problem (1.1)-(1.3) has at most one solution. □

## 4 Proofs of Theorems 2.6 and 2.8

*Proof of Theorem 2.6.* The solution of the initial value problem (1.1)-(1.3) is given by

$$x(t) = C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s) f\Big( s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau \Big) ds \qquad t \in [o, T] \tag{4.1}$$

$$x_0(t) = \phi(t), \quad -r \leq t \leq 0 \tag{4.2}$$

If $t \in [0, T]$ then from (4.1) and using the hypotheses $(H_1) - (H_3)$ and condition (3.2), we have

$$\begin{aligned} \|x(t)\| &\leq \|C(t)\|\ \|\phi(0)\| + \|S(t)\|\ \|\delta\| + \int_0^t \|S(t-s)\|\ \|f\Big( s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau \Big)\| ds \\ &\leq K\|\phi(0)\| + K_1\|\delta\| + \int_0^t K_1 p(s) \Big[ \|x_s\|_C + L \int_0^s q(\tau) \|x_\tau\|_C d\tau \Big] ds \end{aligned}$$

Since $K \geq 1$, for $-r \leq t \leq T$, we get

$$\|x(t)\| \leq K\|\phi\|_C + K_1\|\delta\| + \int_0^t K_1 p(s) \Big[ \|x_s\|_C + L \int_0^s q(\tau) \|x_\tau\|_C d\tau \Big] ds \tag{4.3}$$

From (4.3) and considering cases 1 and 2 as in the proof of the Theorem 2.5, we obtain

$$\|x_t\|_C \leq K\|\phi\|_C + K_1\|\delta\| + \int_0^t K_1 p(s)\|x_s\|_C ds + \int_0^t K_1 p(s)\int_0^s Lq(\tau)\|x_\tau\|_C d\tau ds \tag{4.4}$$

Thanks to Pachpatte's integral inequality given in Lemma 2.2 and applying it to (4.4) with $u(t) = \|x_t\|_C$, we get

$$\begin{aligned}\|x_t\|_C &\leq \Big[K\|\phi\|_C + K_1\|\delta\|\Big]\Big[1 + \int_0^t K_1 p(s)\exp\Big(\int_0^s (K_1 p(\tau) + Lq(\tau))d\tau\Big)ds\Big]\\ &\leq \Big[K\|\phi\|_C + K_1\|\delta\|\Big]\Big[1 + \big\{K_1 P\exp(K_1 P + LQ)T\big\}T\Big] \end{aligned}\tag{4.5}$$

where

$$P = \max_{t\in[0,T]} p(t),\ \ Q = \max_{t\in[0,T]} q(t).$$

It follows that solutions $x(t)$ of initial value problem (1.1) - (1.3) are bounded on closed interval $[-r, T]$ and proof of the Theorem 2.6 is complete. □

**Remark 4.1.** *We remark that our result in Theorem 2.6 also proves the stability of a solution $x(t)$ if $\|\phi\|_C$, $\|\delta\|$ are small enough.*

**Remark 4.2.** *We note that cosine family $C(t)$ and sine family $S(t)$ are not bounded in $\mathbb{R}$. $C(t)$ and $S(t)$ are bounded only in finite interval and may have exponential growth in $\mathbb{R}$. Consequently, all solutions of initial value problem (1.1)-(1.3) are not bounded on $\mathbb{R}_+$.*

*Proof of Theorem 2.7.* By making use of the definition of mild solution given in (2.2) - (2.3), the condition (3.2) and hypothesis $(H_1), (H_4)$ and $(H_5)$, we get

$$\begin{aligned}\|x_1(t) - x_2(t)\| &\leq \|C(t)\|\ \|\phi(0) - \chi(0)\| + \|S(t)\|\ \|\delta - \sigma\|\\ &\quad + \int_0^t \|S(t-s)\|\ \|f\Big(s, x_{1_s}, \int_0^s k(s,\tau), g(\tau, x_{1_\tau})d\tau\Big)\\ &\quad - f\Big(s, x_{2_s}, \int_0^s k(s,\tau), g(\tau, x_{2_\tau})d\tau\Big)\|ds\\ &\leq K\|\phi(0) - \chi(0)\| + K_1\|\delta - \sigma\| + \int_0^t K_1 M\Big[\|x_{1_s} - x_{2_s}\|_C + LN\int_0^s \|x_{1_\tau} - x_{2_\tau}\|_C d\tau\Big]ds \end{aligned}\tag{4.6}$$

From (4.6) and considering cases 1 and 2 as in the proof of Theorem 2.5, for every $t \in [0, T]$, we get

$$\|x_{1_t} - x_{2_t}\|_C \leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big] + \int_0^t K_1 M\Big[\|x_{1_s} - x_{2_s}\|_C + LN\int_0^s \|x_{1_\tau} - x_{2_\tau}\|_C d\tau\Big]ds \tag{4.7}$$

Applying Pachpatte's inequality given in Lemma 2.2, to the inequality (4.7) with $u(t) = \|x_{1_t} - x_{2_t}\|_C$, we obtain

$$\begin{aligned}\|x_{1_t} - x_{2_t}\|_C &\leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big]\Big[1 + \int_0^t K_1 M\exp\Big(\int_0^s (K_1 M + LN)d\tau\Big)ds\Big]\\ &\leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big]\Big[1 + K_1 MT\exp\Big\{(K_1 M + LN)T\Big\}\Big]\end{aligned}$$

which yields, for every $t \in [-r, T]$,

$$\|x_1(t) - x_2(t)\| \leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big]\Big[1 + K_1 MT\exp\Big\{(K_1 M + LN)T\Big\}\Big]$$

and therefore, we have

$$\|x_1 - x_2\|_B \leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big]\Big[1 + K_1 MT\exp\Big\{(K_1 M + LN)T\Big\}\Big]$$

This completes the proof of the Theorem 2.7. □

*Proof of Theorem 2.8.* Using the hypotheses $(H_1), (H_5), (H_9)$ and condition (3.2) we have

$$
\begin{aligned}
&\|x_1(t)-x_2(t)\| \qquad (4.8)\\
&= \int_0^t \|S(t-s)\|\ \|h\Big(s, x_{1_s}, \int_0^s k(s,\tau)g(\tau, x_{1_\tau})d\tau, \mu_1\Big) - h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_1\Big)\\
&\quad + h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_1\Big) - h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_2\Big)\|ds\\
&\le \int_0^t \|S(t-s)\|\ \|h\Big(s, x_{1_s}, \int_0^s k(s,\tau)g(\tau, x_{1_\tau})d\tau, \mu_1\Big) - h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_1\Big)\|ds\\
&\quad + \int_0^t \|S(t-s)\|\ \|h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_1\Big) - h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_2\Big)\|ds\\
&\le \int_0^t K_1M_1\Big[\|x_{1_s}-x_{2_s}\|_C + \int_0^s LN\|x_{1_\tau}-x_{1_\tau}\|_C d\tau\Big]ds + \int_0^t K_1M_2|\mu_1-\mu_2|ds\\
&\le K_1M_2T|\mu_1-\mu_2| + \int_0^t K_1M_1\Big[\|x_{1_s}-x_{2_s}\|_C + \int_0^s LN\|x_{1_\tau}-x_{1_\tau}\|_C d\tau\Big]ds \qquad (4.9)
\end{aligned}
$$

From (4.9) and considering cases 1 and 2 as in the proof of the Theorem 2.5, we get

$$
\|x_{1_t}-x_{2_t}\|_C \le K_1M_2T|\mu_1-\mu_2| + \int_0^t K_1M_1\Big[\|x_{1_s}-x_{2_s}\|_C + \int_0^s LN\|x_{1_\tau}-x_{1_\tau}\|_C d\tau\Big]ds \qquad (4.10)
$$

Once again, thanks to Pachpatte's integral inequality given in Lemma 2.2 and applying it to (4.10) with $u(t) = \|x_{1_t}-x_{2_t}\|_C$, we obtain

$$
\begin{aligned}
\|x_{1_t}-x_{2_t}\|_C &\le K_1M_2T|\mu_1-\mu_2|\Big[1 + \int_0^t K_1M_1 \exp\Big(\int_0^s (K_1M_1+LN)d\tau\Big)ds\Big]\\
&\le |\mu_1-\mu_2|K_1M_2T\Big[1 + K_1M_1T\exp(\{K_1M_1+LN\}T)\Big]
\end{aligned} \qquad (4.11)
$$

Thus, for $t \in [-r, T]$, we have

$$
\|x_1(t)-x_2(t)\| \le K_1M_2T|\mu_1-\mu_2|\Big[1 + K_1M_1T\exp(\{K_1M_1+LN\}T)\Big]
$$

and hence

$$
\|x_1-x_2\|_B \le K_1M_2T|\mu_1-\mu_2|\Big[1 + K_1M_1T\exp(\{K_1M_1+LN\}T)\Big]
$$

This follows that the solutions of initial value problem (2.4) - (2.6) and (2.7) - (2.9) depend continuously on the parameters. This completes the proof of the Theorem 2.8 □

# 5 Application

To illustrate the application of our main result, consider the following nonlinear partial integrodifferential equation of the form

$$
z_{tt}(w,t) = z_{ww}(w,t) + Q\Big(t, z(w,t-r), \int_0^t k_1(t,s)g_1(s, z(w,s-r))ds\Big)ds,
$$
$$
t \in [0,T], \quad 0 \le w \le \pi \qquad (5.1)
$$
$$
z(0,t) = z(\pi,t) = 0,\ t \in [0,T], \qquad (5.2)
$$
$$
z(w,t) = \phi(w,t),\ 0 \le w \le \pi, -r \le t \le 0, \qquad (5.3)
$$
$$
z_t(w,0) = z_0(w),\ 0 \le w \le \pi, \qquad (5.4)
$$

where $\phi$ is continuous, $Q : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g_1 : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous and strongly measurable and $k_1 : [0,T] \times [0,T] \to \mathbb{R}$ is continuous. We assume that the following condition is satisfied.

(1) For every positive integer $q_1$ there exists $\alpha'_{q_1} \in L^1([0,T],[0,\infty))$ such that for a.e. $t \in [0,T]$ and $z \in \mathbb{R}$

$$\sup_{|z|\leq k_1}\left|Q\left(t, z(w,t-r), \int_0^t k_1(t,s)g_1(s,z(w,s-r))ds\right)\right| \leq \alpha'_{q_1}(t),$$

and

$$\liminf_{q_1\to+\infty}\frac{1}{q_1}\int_0^b \alpha'_{q_1}(s)ds = \zeta' < \infty.$$

Let $X = L^2[0,\pi]$ be endowed with usual norm $\|\cdot\|_{L^2}$. Define the operator $A : X \to X$ by $Ay = y''$ with domain $D(A) = \{y \in X : y, y'$ are absolutely continuous, $y'' \in X$ and $y(0) = y(\pi) = 0\}$. Then

$$Ay = \sum_{n=1}^{\infty} -n^2(y,y_n)y_n, \quad y \in D(A),$$

where $y_n(s) = (\sqrt{2/\pi})\sin ns,\ n = 1,2,3,...$ is the orthogonal set of eigenvectors of $A$ and it can be easily shown that $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in \mathbb{R}$, in $X$ and is given by (see[18])

$$C(t)y = \sum_{n=1}^{\infty} \cos nt(y,y_n)y_n, \quad y \in X.$$

The associated sine family is given by

$$S(t)y = \sum_{n=1}^{\infty} \frac{1}{n}\sin nt(y,y_n)y_n, \quad y \in X.$$

Further assume that $K_1\zeta' < 1$, where $K_1 = \sup\{\|S(t)\| : t \in [0,T]\}$.

Define the functions $f : [0,T]\times C\times X \to X$, $g : [0,T]\times C \to X$, $k : [0,T]\times[0,T] \to \mathbb{R}$, as follows

$$\begin{aligned} f(t,\psi,x)(v) &= Q(t,\psi(-r)(v),x(v)),\\ g(t,\psi)(v) &= g_1(t\psi(-r)(v)),\\ k(t,s) &= k_1(t,s), \end{aligned}$$

for $t \in [0,T], x \in X, \psi \in C$ and $v \in \mathbb{R}$. Then the above partial differential system (5.1)-(5.4) can be formulated abstractly as

$$x''(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t,s)g(s,x_s)ds\right), \quad t \in [0,T] \tag{5.5}$$

$$x_0(t) = \phi(t), \quad -r \leq t \leq 0 \tag{5.6}$$

$$x'(0) = \delta \tag{5.7}$$

Since all the hypotheses of the Theorem 2.4 are satisfied, and hence, by an application of the Theorem 2.4, the partial differential equations (5.1) - (5.4) have at least one solution on $[-r,T]$.

# References

[1] P. Aviles and J. Sandefur, Nolinear second order equations wtih applications to partial differential equations, *J. Differential Equations*, 58(1985), 404-427.

[2] A. Belleni Morante and G. F. Roach, A mathematical model for Gamma ray transport in the cardiac region, *J. Math. Anal. Appl.*, 244(2000), 498-514.

[3] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, 1991.

[4] A. Constantin, Topological Transversality: Application to an integrodifferential equation, *J. Math. Anal. Appl.*, 197(1996), 855-863.

[5] Dalintang and Samuel M. Rankin III, Peristaltic transport of a heat conducting viscous fluid as an application of abstract differential equations and semigroup of operators, *J. Math. Anal. Appl.,* 169(1992), 391-407.

[6] S. G. Deo V. Lakshmikantham and V. Raghavendra, *Text Book of Ordinary Differential Equations*, Tata McGraw-Hill Publishing Company Ltd., New Delhi, 1997.

[7] P. M. Dhakane, On global existence of solutions of an abstract nonlinear functional second order integrodifferential equation, Proceeding of International Conference on Mathematical Science in Honour of Prof. A. M. Mathai [3-5 January 2011]. *St. Thomas College, Palai Mahatma Gandhi University, Kottayam, Kerala*, 67-74.

[8] M. B. Dhakne and G. B. Lamb, On an abstract nonlinear second order integrodifferential equation, *J. Function Spaces and Applications*, 5(2)(2007), 167-174.

[9] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces: Mathematical Studies*, North-Holland, Amsterdam, Vol. 108, 1985.

[10] W. E. Fitzgibbon, Global existence and boundedness of solutions to the extensible beam equation, *SIAM J. Math. Anal.*, 13(1982), 739-745.

[11] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, Oxford University Press, New York, 1985.

[12] B. G. Pachpatte, A note on Gronwall- Bellman inequality, *J. Math. Anal. Appl.*, 44(1973), 758-762.

[13] B. G. Pachpatte, On abstarct second order differential equations, *Demonstratio Mathematica*, XXIII(2)(1990), 357-366.

[14] S. K. Patcheu, On the global solution and asymptotic behaviour for the generalized damped extensible beam equation, *J. Differential Equations*, 135(1996), 679-687.

[15] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, New York, 1983.

[16] Teschl Gerald, *Nonlinear Functional Analysis*, Vienna, Austria, 2001.

[17] H. L. Tidke and M. B. Dhakne, Existence and uniqueness of solutions of certain second order nonlinear equations, *Note di Mathematica*, 30(2)(2010), 73-81.

[18] C. C. Travis and G. F. Webb, Compactness, regularity, and uniform continuity properties of strongly continuous cosine families, *Houston J. Math.,* 3(4)(1977), 555–567.

[19] C. C. Travis and G. F. Webb, Second order differential equations in Banach spaces, *Proc. Int. Symp. on Nonlinear Equations in Abstract spaces, Academic Press, New York*, (1978), 331-361.

[20] C. C. Travis and G. F. Webb, Cosine families and abstract nonlinear second order differential equations, *Acta. Math. Hung.*, (32)(1978), 75-96.

[21] C. C. Travis and G. F. Webb, An abstract second order semilinear Volterra integrodifferential equation, *SIAM J. Math. Anal.* , 10(1979), 412-424.

[22] R. Ye and G. Zhang, Neutral functional differential equations of second order with infinite delays, *Electronic Jurnal of Differential Equations*, 2010(36)(2010), 1-12.


*Received*: October 16, 2012; *Accepted*: December 17, 2012

Malaya Journal of Matematik 1(1)(2013) 38–49

Malaya Journal of Matematik
MJM
an international journal of mathematical sciences with computer applications...
www.malayajournal.org
ISSN : 2319-3786

# Oscillation results for third order nonlinear neutral differential equations of mixed type

Ethiraju Thandapani,[a,*] and Renu Rama[b]

[a,b] *Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai -600 005, India.*

**Abstract**

Some oscillation results are obtained for the third order nonlinear mixed type neutral differential equations of the form

$$((x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2))^{\alpha})''' = q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2),\ t\geq t_0$$

where $\alpha,\ \beta$ and $\gamma$ are ratios of odd positive integers $\tau_1,\ \tau_2,\ \sigma_1$ and $\sigma_2$ are positive constants.

*Keywords:* Oscillation, third order, neutral differential equations.

*2010 MSC:* 34C15. 

## 1 Introduction

In this paper, we study the oscillatory nature of the third order nonlinear mixed type neutral differential equations of the form

$$((x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2))^{\alpha})''' = q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2),\ t\geq t_0 \tag{1.1}$$

subject to the following conditions:

$(c_1)$ $\tau_1,\ \tau_2,\ \sigma_1$ and $\sigma_2$ are positive constants;

$(c_2)$ $q(t)$ and $p(t)$ are real valued positive continuous functions on $[t_0,\infty)$;

$(c_3)$ $\alpha,\ \beta$ and $\gamma$ are ratios of odd positive integers;

$(c_4)$ $b(t)$ and $c(t)$ are real valued and thrice continuously differentiable functions with $0\leq b(t)<b<\infty$ and $0\leq c(t)<c<\infty$.

Let $\theta=\max\{\tau_1,\sigma_1\}$. By a solution of equation (1.1), we mean a real valued continuous function $x(t)$ defined for all $t\geq t_0-\theta$ and satisfying the equation (1.1) for all $t\geq t_0$. A nontrivial solution of equation (1.1) is called oscillatory if it has infinitely many zeros on $[t_0,\infty)$, otherwise it is called nonoscillatory.

Recently there has been a great interest in studying the oscillatory and asymptotic behavior of third order differential equations, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], and the references cited therein. In [1, 4, 7, 8, 9, 15, 20, 23], the authors studied the oscillatory behavior of solutions of equation (1.1) when $b(t)\equiv 0,\ c(t)\equiv 0$ and $p(t)\equiv 0$. In [5, 6, 10, 11, 17, 18, 19, 21], the authors studied the oscillatory behavior of solutions of equation (1.1) when $c(t)\equiv 0$ and $p(t)\equiv 0$. In [2, 13, 14, 22], the authors discussed the oscillatory behavior of all solutions of equation (1.1) when $\alpha=\beta=\gamma=1$.

Motivated by this observation, in this paper we study the oscillatory and asymptotic behavior of all solutions of equation (1.1) for different values of $\alpha,\beta$ and $\gamma$. So the purpose of this paper is to present some new oscillatory

*Corresponding author.
*E-mail addresses*: ethandapani@yahoo.co.in (Ethiraju Thandapani) and renurama68@gmail.com (Renu Rama).

and asymptotic criteria for equation (1.1). In Section 2, we present criteria for equation (1.1) to be either oscillatory or all its nonoscillatory solutions tend to zero as $t \to \infty$. Examples are provided in Section 3 to illustrate the results presented in Section 2.

## 2 Oscillation results

In this section, we present some new oscillation criteria for the equation (1.1). For convenience we use the following notations:
$Q(t)=\min(q(t), q(t-\tau_1), q(t+\tau_2))$, $P(t)=\min(p(t), p(t-\tau_1), p(t+\tau_2))$,
and $z(t)=[x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2)]^{\alpha}$.

**Lemma 2.1.** *If $x(t)$ is a positive solution of equation (1.1), then the corresponding function $z(t)$ satisfies only the following two cases*

$$\textit{Case (I)} \quad z(t)>0,\ z'(t)>0,\ z''(t)>0, z'''(t)>0; \tag{2.1}$$
$$\textit{Case (II)} \quad z(t)>0,\ z'(t)>0,\ z''(t)<0, z'''(t)>0. \tag{2.2}$$

*Proof.* Assume that $x(t)$ is a positive solution of equation (1.1). Then there exists a $t_1 \geq t_0$ such that $x(t-\theta)>0$ for all $t \geq t_1$. From the definition of $z(t)$, it is clear that $z(t)>0$ for all $t \geq t_1$. From equation (1.1), we have $z'''(t)>0$ for all $t \geq t_1$. Therefore $z''(t)$ is strictly increasing for all $t \geq t_1$ and $z''(t)$ and $z'(t)$ are of one sign for all $t \geq t_1$. We prove that $z'(t)>0$ for all $t \geq t_1$. If not, there exists a $t_2 \geq t_1$ and $M<0$ such that $z'(t)<M$ for all $t \geq t_2$. Integrating the last inequality from $t_2$ to $t$, we get

$$z(t)-z(t_2)<M(t-t_2).$$

Letting $t \to \infty$, we see that $z(t) \to -\infty$, which is a contradiction. Hence $z'(t)>0$ for all $t \geq t_1$.This completes the proof of the lemma. □

**Lemma 2.2.** *If $A \geq 0,\ B \geq 0$ and $0<\delta \leq 1$, then*

$$A^{\delta}+B^{\delta} \geq (A+B)^{\delta} \tag{2.3}$$

*If $\delta \geq 1$ then*

$$(A^{\delta}+B^{\delta}) \geq \frac{1}{2^{\delta-1}}(A+B)^{\delta}. \tag{2.4}$$

*Proof.* Proof can be found in [21]. □

**Theorem 2.3.** *Assume that $0<\beta=\gamma \leq 1$ and $\sigma_2>\sigma_1>\max\{\tau_1,\tau_2\}$. If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\beta/\alpha}}{(1+b^{\beta}+c^{\beta})^{\beta/\alpha}}\ y^{\beta/\alpha}(t+\sigma_2-\sigma_1) \tag{2.5}$$

*has no positive increasing solution, and the second order differential inequality*

$$y''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{(1+b^{\beta}+c^{\beta})^{\beta/\alpha}}\ y^{\beta/\alpha}(t-\sigma_1+\tau_1) \tag{2.6}$$

*has no positive decreasing solution, then every solution of equation (1.1) is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of equation (1.1) for all $t \geq t_1 \geq t_0$. Without loss of generality, we may assume that $x(t)$ is a positive solution of equation (1.1) for all $t \geq\ t_1 \geq\ t_0$ (since the case $x(t)$ is negative is similar). Then there exists a $t_2 \geq t_1$ such that $x(t-\theta)>0$ for all $t \geq t_2$. By the definition of $z(t)$ we have, $z(t-\theta)>0$ for all $t \geq t_2$. Define a function $y(t)$ by

$$y(t)=z(t)+b^{\beta}z(t-\tau_1)+c^{\beta}z(t+\tau_2),\ \text{for all } t \geq t_2. \tag{2.7}$$

Then $y(t) > 0$ for all $t \geq t_2$, and

$$\begin{aligned} y'''(t) &= z'''(t) + b^\beta z'''(t-\tau_1) + c^\beta z'''(t+\tau_2) \\ &= q(t)x^\beta(t-\sigma_1) + p(t)x^\beta(t+\sigma_2) + b^\beta q(t-\tau_1)x^\beta(t-\tau_1-\sigma_1) + \\ &\quad b^\beta p(t-\tau_1)x^\beta(t-\tau_1+\sigma_2) + c^\beta q(t+\tau_2)x^\beta(t+\tau_2-\sigma_1) + \\ &\quad c^\beta p(t+\tau_2)x^\beta(t+\tau_2+\sigma_2) \\ &\geq Q(t)[x^\beta(t-\sigma_1) + b^\beta x^\beta(t-\tau_1-\sigma_1) + c^\beta x^\beta(t+\tau_2-\sigma_1)] + \\ &\quad P(t)[x^\beta(t+\sigma_2) + b^\beta x^\beta(t-\tau_1+\sigma_2) + c^\beta x^\beta(t+\tau_2+\sigma_2)]. \end{aligned}$$

Using (2.3) twice, the above inequality becomes

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1) + P(t)z^{\beta/\alpha}(t+\sigma_2). \tag{2.8}$$

Since $x(t)$ is a positive solution of equation (1.1), from Lemma 2.1 we have two cases for $z(t)$.
**Case (I):** In this case, we have $z'(t) > 0,\ z''(t) > 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then from (2.7), we have $y'(t) > 0,\ y''(t) > 0$ and $y'''(t) > 0$ for all $t \geq t_2$.

From the inequality (2.8), we have

$$y'''(t) \geq P(t)z^{\beta/\alpha}(t+\sigma_2). \tag{2.9}$$

Since $z'(t)$ is increasing, we have

$$\begin{aligned} y'(t) &= z'(t) + b^\beta z'(t-\tau_1) + c^\beta z'(t+\tau_2) \\ &\leq (1+b^\beta+c^\beta)z'(t+\tau_2) \text{ for all } t \geq t_0. \end{aligned} \tag{2.10}$$

Now

$$z(t+\sigma_1-\tau_2) - z(t) = \int_t^{t+\sigma_1-\tau_2} z'(s)\,ds$$

or

$$z(t+\sigma_1-\tau_2) \geq z'(t)(\sigma_1-\tau_2). \tag{2.11}$$

Using (2.10) and (2.11) in (2.9), we obtain

$$\begin{aligned} y'''(t) &\geq P(t)z^{\beta/\alpha}(t+\sigma_2) \\ &\geq P(t)(\sigma_1-\tau_2)^{\beta/\alpha}(z'(t+\sigma_2-\sigma_1+\tau_2))^{\beta/\alpha} \\ &\geq \frac{P(t)(\sigma_1-\tau_2)^{\beta/\alpha}}{(1+b^\beta+c^\beta)^{\beta/\alpha}}(y'(t+\sigma_2-\sigma_1))^{\beta/\alpha}, t \geq t_2. \end{aligned} \tag{2.12}$$

By setting $y'(t) = w(t)$, we see that $w(t) > 0$ and $w'(t) > 0$ for all $t \geq t_2$. Now inequality (2.9) becomes

$$w''(t) \geq \frac{P(t)}{(1+b^\beta+c^\beta)^{\beta/\alpha}}(\sigma_1-\tau_2)^{\beta/\alpha}w^{\beta/\alpha}(t+\sigma_2-\sigma_1), t \geq t_2. \tag{2.13}$$

That is, $w(t)$ is a positive increasing solution of the second order differential inequality (2.5), which is a contradiction.

**Case (II).** In this case, we have $z'(t) > 0,\ z''(t) < 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then $y'(t) > 0,\ y''(t) < 0$ for all $t \geq t_2$. From the inequality (2.8), we have

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1). \tag{2.14}$$

Since $z'(t)$ and $y'(t)$ are decreasing, we have

$$\begin{aligned} y'(t) &= z'(t) + b^\beta z'(t-\tau_1) + c^\beta z'(t+\tau_2) \\ &\leq (1+b^\beta+c^\beta)z'(t-\tau_1) \end{aligned}$$

or

$$y'(t-\sigma_1+\tau_1) \leq (1+b^\beta+c^\beta)z'(t-\sigma_1),\ t \geq t_2. \tag{2.15}$$

Now

$$z(t) - z(t - (\sigma_1 - \tau_1)) = \int_{t-(\sigma_1-\tau_1)}^{t} z'(s)\, ds$$

or

$$z(t) \geq z'(t)(\sigma_1 - \tau_1). \tag{2.16}$$

Using (2.15) and (2.16) in (2.14), we obtain

$$\begin{aligned} y'''(t) &\geq Q(t)z^{\beta/\alpha}(t - \sigma_1) \\ &\geq Q(t)(\sigma_1 - \tau_1)^{\beta/\alpha}(z'(t - \sigma_1))^{\beta/\alpha} \\ &\geq \frac{Q(t)(\sigma_1 - \tau_1)^{\beta/\alpha}}{(1 + b^\beta + c^\beta)^{\beta/\alpha}}(y'(t - \sigma_1 + \tau_1))^{\beta/\alpha}, t \geq t_2. \end{aligned}$$

By taking $y'(t) = w(t)$, we see that $w(t) > 0$ and $w'(t) < 0$. Thus, $w(t)$ is a positive decreasing solution of the second order differential inequality

$$w''(t) \geq \frac{Q(t)}{(1 + b^\beta + c^\beta)^{\beta/\alpha}}(\sigma_1 - \tau_1)^{\beta/\alpha} w^{\beta/\alpha}(t - \sigma_1 + \tau_1), \tag{2.17}$$

which is a contradiction to (2.6). This completes the proof. □

**Theorem 2.4.** *Assume that $\beta = \gamma \geq 1$ and $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1 - \tau_2)^{\beta/\alpha}}{4^{\beta-1}(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}} y^{\beta/\alpha}(t + \sigma_2 - \sigma_1) \tag{2.18}$$

*has no positive increasing solution, and the second order differential inequality*

$$y''(t) \geq \frac{Q(t)(\sigma_1 - \tau_1)^{\beta/\alpha}}{4^{\beta-1}(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}} y^{\beta/\alpha}(t + \tau_1 - \sigma_1) \tag{2.19}$$

*has no positive decreasing solution, then every solution of equation (1.1) is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of equation (1.1) for all $t \geq t_1 \geq t_0$. Without loss of generality, we may assume that $x(t)$ is a positive solution of equation (1.1). Then there exists a $t_2 \geq t_1$ such that $x(t-\theta) > 0$ for all $t \geq t_2$. By the definition of $z(t)$, we have $z(t - \theta) > 0$ for all $t \geq t_2$. Now define a function $y(t)$ by

$$y(t) = z(t) + b^\beta z(t - \tau_1) + \frac{c^\beta}{2^{\beta-1}} z(t + \tau_2), t \geq t_2. \tag{2.20}$$

Then, since $z(t) > 0$, we have $y(t) > 0$ and

$$\begin{aligned} y'''(t) &= z'''(t) + b^\beta z'''(t - \tau_1) + \frac{c^\beta}{2^{\beta-1}} z'''(t + \tau_2) \\ &= q(t)x^\beta(t - \sigma_1) + p(t)x^\beta(t + \sigma_2) + b^\beta q(t - \tau_1)x^\beta(t - \tau_1 - \sigma_1) + \\ &\quad b^\beta p(t - \tau_1)x^\beta(t - \tau_1 + \sigma_2) + \frac{c^\beta}{2^{\beta-1}} q(t + \tau_2)x^\beta(t + \tau_2 - \sigma_1) + \\ &\quad \frac{c^\beta}{2^{\beta-1}} p(t + \tau_2)x^\beta(t + \tau_2 + \sigma_2) \\ &\geq Q(t)[x^\beta(t - \sigma_1) + b^\beta x^\beta(t - \tau_1 - \sigma_1) + \frac{c^\beta}{2^{\beta-1}} x^\beta(t + \tau_2 - \sigma_1)] + \\ &\quad P(t)[x^\beta(t + \sigma_2) + b^\beta x^\beta(t - \tau_1 + \sigma_2) + \frac{c^\beta}{2^{\beta-1}} x^\beta(t + \tau_2 + \sigma_2)],\ t \geq t_2. \end{aligned}$$

Now using (2.4) twice in the last inequality, we obtain

$$y'''(t) \geq \frac{Q(t)}{4^{\beta-1}} z^{\beta/\alpha}(t - \sigma_1) + \frac{P(t)}{4^{\beta-1}} z^{\beta/\alpha}(t + \sigma_2)t \geq t_2. \tag{2.21}$$

Since $x(t)$ is a positive solution of equation (1.1), there are only two cases, as given in Lemma 2.1, for $z(t)$.

**Case (I):** In this case, we have $z(t) > 0,\ z'(t) > 0,\ z''(t) > 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then from (2.20), we have $y'(t) > 0,\ y''(t) > 0$ for all $t \geq t_2$. From the inequality (2.21), we have

$$y'''(t) \geq \frac{P(t)}{4^{\beta-1}} z^{\beta/\alpha}(t+\sigma_2), t \geq t_2. \tag{2.22}$$

Since $z'(t)$ is increasing, we have

$$\begin{aligned} y'(t) &= z'(t) + b^{\beta} z'(t-\tau_1) + \frac{c^{\beta}}{2^{\beta-1}} z'(t+\tau_2) \\ &\leq (1 + b^{\beta} + \frac{c^{\beta}}{2^{\beta-1}}) z'(t+\tau_2), t \geq t_2 \end{aligned}$$

or

$$y'(t) \leq (1 + b^{\beta} + \frac{c^{\beta}}{2^{\beta-1}}) z'(t+\sigma_2+\tau_2), t \geq t_2 \tag{2.23}$$

and

$$z(t+\sigma_1-\tau_2) - z(t) = \int_t^{t+\sigma_1-\tau_2} z'(s)\ ds \geq z'(t)(\sigma_1-\tau_2)$$

or

$$z(t+\sigma_1-\tau_2) \geq z'(t)(\sigma_1-\tau_2). \tag{2.24}$$

Now using (2.23) and (2.24) in (2.22), we have

$$\begin{aligned} y'''(t) &\geq \frac{P(t)}{4^{\beta-1}} z^{\beta/\alpha}(t+\sigma_2) \\ &\geq \frac{P(t)}{4^{\beta-1}} (\sigma_1-\tau_2)^{\beta/\alpha} (z'(t+\tau_2-\sigma_1+\sigma_2))^{\beta/\alpha} \end{aligned}$$

$$y'''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\beta/\alpha}(y'(t+\sigma_2-\sigma_1))^{\beta/\alpha}}{4^{\beta-1}(1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}})^{\beta/\alpha}}, t \geq t_2. \tag{2.25}$$

Setting $y'(t) = w(t)$, we see that $w(t) > 0$, $w'(t) = y''(t) > 0$ and

$$w''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\beta/\alpha} w(t+\sigma_2-\sigma_1)^{\beta/\alpha}}{4^{\beta-1}(1+b^{\beta}+c^{\beta})^{\beta/\alpha}}, t \geq t_2. \tag{2.26}$$

That is $w(t)$ is a positive increasing solution of the second order differential inequality (2.18), which is a contradiction.

**Case (II):** In this case we have $z'(t) > 0,\ z''(t) < 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then from (2.20), we obtain $y'(t) > 0$ and $y''(t) < 0$ for all $t \geq t_2$. From the inequality (2.21), we have

$$y'''(t) \geq \frac{Q(t)}{4^{\beta-1}} z^{\beta/\alpha}(t-\sigma_1), t \geq t_2. \tag{2.27}$$

Using the monotonicity of $z'(t)$ and $y'(t)$, we have

$$\begin{aligned} y'(t) &= z'(t) + b^{\beta} z'(t-\tau_1) + \frac{c^{\beta}}{2^{\beta-1}} z'(t+\tau_2) \\ &\leq (1 + b^{\beta} + \frac{c^{\beta}}{2^{\beta-1}}) z'(t-\tau_1) \end{aligned}$$

or

$$y'(t+\sigma_1) \leq (1 + b^{\beta} + \frac{c^{\beta}}{2^{\beta-1}}) z'(t+\sigma_1-\tau_1), t \geq t_2. \tag{2.28}$$

Also from the monotonicity of $z'(t)$ we have

$$z(t) - z(t-\sigma_1+\tau_1) = \int_{t-(\sigma_1-\tau_1)}^{t} z'(s)\ ds \geq z'(t)(\sigma_1-\tau_1)$$

or

$$z(t) \geq (\sigma_1 - \tau_1)z'(t). \tag{2.29}$$

Using (2.28) and (2.29) in (2.27), we get

$$\begin{aligned} y'''(t) &\geq \frac{Q(t)}{4^{\beta-1}} z^{\beta/\alpha}(t-\sigma_1) \\ &\geq \frac{Q(t)}{4^{\beta-1}}(\sigma_1-\tau_1)^{\beta/\alpha}(z'(t-\sigma_1))^{\beta/\alpha} \\ &\geq \frac{Q(t)}{4^{\beta-1}}(\sigma_1-\tau_1)^{\beta/\alpha}\frac{(y'(t-\sigma_1+\tau_1))^{\beta/\alpha}}{(1+b^\beta+\frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}} \end{aligned}$$

or

$$y'''(t) \geq \frac{Q(t)}{4^{\beta-1}}(\sigma_1-\tau_1)^{\beta/\alpha}\frac{(y'(t-\sigma_1+\tau_1))^{\beta/\alpha}}{(1+b^\beta+\frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}}, t \geq t_2.$$

Set $y'(t) = w(t)$. Then $w(t) > 0$ and $w'(t) = y''(t) < 0$ and the last inequality becomes

$$w''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha} w^{\beta/\alpha}(t-\sigma_1+\tau_1)}{4^{\beta-1}(1+b^\beta+\frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}}, t \geq t_2. \tag{2.30}$$

Thus, $w(t)$ is a positive decreasing solution of the second order differential inequality (2.19), which is a contradiction. Now the proof is complete. □

**Theorem 2.5.** *Assume that* $0 < \beta \leq 1,\ \gamma \geq 1,\ b \leq 1, c \leq 1$ *and* $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. *If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}}{4^{\gamma-1}(1+b^\beta+c^\beta)^{\gamma/\alpha}} y^{\beta/\alpha}(t+\sigma_2-\sigma_1) \tag{2.31}$$

*has no positive increasing solution, and the second order differential inequality*

$$y''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{(1+b^\beta+c^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1+\tau_1) \tag{2.32}$$

*has no positive decreasing solution, then every solution of equation (1.1) is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of equation (1.1) for all $t \geq t_1 \geq t_0$. Let us assume that $x(t)$ is a positive solution of (1.1) for all $t \geq t_1 \geq t_0$.. Then there exists a $t_2 \geq t_1$ such that $x(t-\theta) > 0$ for all $t \geq t_2$. By the definition of $z(t)$, we have $z(t-\theta) > 0$ for all $t \geq t_2$. Set

$$y(t) = z(t) + b^\beta z(t-\tau_1) + c^\beta z(t+\tau_2) \text{ for all } t \geq t_2. \tag{2.33}$$

Then, $y(t) > 0$, and

$$\begin{aligned} y'''(t) &= z'''(t) + b^\beta z'''(t-\tau_1) + c^\beta z'''(t+\tau_2) \\ &= q(t)x^\beta(t-\sigma_1) + p(t)x^\gamma(t+\sigma_2) + b^\beta q(t-\tau_1)x^\beta(t-\tau_1-\sigma_1) + \\ &\quad b^\beta p(t-\tau_1)x^\gamma(t-\tau_1+\sigma_2) + c^\beta q(t+\tau_2)x^\beta(t+\tau_2-\sigma_1) + \\ &\quad c^\beta p(t+\tau_2)x^\gamma(t+\tau_2+\sigma_2) \\ &\geq Q(t)[x^\beta(t-\sigma_1) + b^\beta x^\beta(t-\tau_1-\sigma_1) + c^\beta x^\beta(t+\tau_2-\sigma_1)] + \\ &\quad P(t)[x^\gamma(t+\sigma_2) + b^\beta x^\gamma(t-\tau_1+\sigma_2) + c^\beta x^\gamma(t+\tau_2+\sigma_2)], t \geq t_2. \end{aligned}$$

Using (2.3) twice in the first part of righthand side of the last inequality, we have

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1) + P(t)[x^\gamma(t+\sigma_2) + b^\beta x^\gamma(t-\tau_1+\sigma_2) + c^\beta x^\gamma(t+\tau_2+\sigma_2)], t \geq t_2. \tag{2.34}$$

Using the fact that $b \leq 1,\ c \leq 1,\ \gamma \geq 1,$ and $0 < \beta \leq 1$, we have

$$\begin{aligned} & x^\gamma(t+\sigma_2) + b^\beta x^\gamma(t-\tau_1+\sigma_2) + c^\beta x^\gamma(t+\tau_2+\sigma_2) \\ \geq\ & x^\gamma(t+\sigma_2) + b^\gamma x^\gamma(t-\tau_1+\sigma_2) + c^\gamma x^\gamma(t+\tau_2+\sigma_2) \end{aligned}$$

$$\geq \quad x^{\gamma}(t+\sigma_2) + b^{\gamma}x^{\gamma}(t-\tau_1+\sigma_2) + \frac{c^{\gamma}}{2^{\gamma-1}}x^{\gamma}(t+\tau_2+\sigma_2),$$

and applying (2.4) twice and simplifying, we obtain

$$x^{\gamma}(t+\sigma_2) + b^{\beta}x^{\gamma}(t-\tau_1+\sigma_2) + c^{\beta}x^{\gamma}(t+\tau_2+\sigma_2) \;\geq\; \frac{1}{4^{\gamma-1}}z^{\frac{\gamma}{\alpha}}(t+\sigma_2). \tag{2.35}$$

Substituting (2.35) in (2.34), we get

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1) + \frac{P(t)}{4^{\gamma-1}}z^{\gamma/\alpha}(t+\sigma_2), t \geq t_2. \tag{2.36}$$

Now we consider the following two cases for $z(t)$ as in Lemma 2.1.

**Case (I):** In this case we have $z'(t) > 0,\ z''(t) > 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then from (2.33), we have $y'(t) > 0,\ y''(t) > 0$ and $y'''(t) > 0$ for all $t \geq t_2$.

From the inequality (2.36), we have

$$y'''(t) \geq \frac{P(t)}{4^{\gamma-1}}z^{\gamma/\alpha}(t+\sigma_2), t \geq t_2. \tag{2.37}$$

Using the monotonicity of $z'(t)$, we get

$$\begin{aligned} y'(t) &= z'(t) + b^{\beta}z'(t-\tau_1) + c^{\beta}z'(t+\tau_2) \\ &\leq (1+b^{\beta}+c^{\beta})z'(t+\tau_2), t \geq t_2. \end{aligned} \tag{2.38}$$

Again using the monotonicity of $z'(t)$, we obtain

$$z(t+\sigma_1-\tau_2) - z(t) = \int_{t}^{t+\sigma_1-\tau_2} z'(s)\ ds \geq z'(t)(\sigma_1-\tau_2),$$

or

$$z(t+\sigma_1-\tau_2) \geq (\sigma_1-\tau_2)z'(t). \tag{2.39}$$

Now using (2.38) and (2.39) in (2.37), we obtain

$$\begin{aligned} y'''(t) &= \frac{P(t)}{4^{\gamma-1}}z^{\gamma/\alpha}(t+\sigma_2) \\ &\geq \frac{P(t)}{4^{\gamma-1}}(\sigma_1-\tau_2)^{\gamma/\alpha}(z'(t+\sigma_2-\sigma_1+\tau_2))^{\gamma/\alpha} \\ &\geq \frac{P(t)}{4^{\gamma-1}}\frac{(\sigma_1-\tau_2)^{\gamma/\alpha}(y'(t+\sigma_2-\sigma_1))^{\gamma/\alpha}}{(1+b^{\beta}+c^{\beta})^{\gamma/\alpha}}, t \geq t_2. \end{aligned}$$

By setting $y'(t) = w(t)$, we see that $w(t) = y'(t) > 0$, $w'(t) = y''(t) > 0$ and it satisfies

$$w''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}}{4^{\gamma-1}(1+b^{\beta}+c^{\beta})^{\gamma/\alpha}}w^{\gamma/\alpha}(t+\sigma_2-\sigma_1), t \geq t_2.$$

Thus, $w(t)$ is a positive increasing solution of the second order differential inequality (2.31), which is a contradiction.

**Case (II):** In this case we have $z'(t) > 0,\ z''(t) < 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Therefore $y'(t) > 0,\ y''(t) < 0$ and $y'''(t) > 0$ for all $t \geq t_2$. From the inequality (2.36) we have

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1), t \geq t_2. \tag{2.40}$$

Since $z''(t) < 0$, we have $z'(t)$ is decreasing and therefore

$$\begin{aligned} y'(t) &= z'(t) + b^{\beta}z'(t-\tau_1) + c^{\beta}z'(t+\tau_2) \\ &\leq (1+b^{\beta}+c^{\beta})z'(t-\tau_1), \end{aligned} \tag{2.41}$$

or

$$y'(t-\sigma_1) \leq (1+b^{\beta}+c^{\beta})z'(t-\sigma_1-\tau_1), t \geq t_2.$$

Again using the monotonicity of $z'(t)$, we see that

$$z(t) - z(t-\sigma_1+\tau_1) = \int_{t-(\sigma_1-\tau_1)}^{t} z'(s)\, ds \geq (\sigma_1-\tau_1)z'(t),$$

or

$$z(t) \geq (\sigma_1-\tau_1)z'(t). \tag{2.42}$$

Substituting (2.41) and (2.42) in (2.40), we obtain

$$y'''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{(1+b^\beta+c^\beta)^{\beta/\alpha}}(y'(t-\sigma_1+\tau_1))^{\beta/\alpha}, t \geq t_2. \tag{2.43}$$

By setting $y'(t) = w(t)$, we see that $w(t)$ is a positive decreasing solution of

$$w''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{(1+b^\beta+c^\beta)^{\beta/\alpha}} w^{\beta/\alpha}(t-\sigma_1+\tau_1), t \geq t_2, \tag{2.44}$$

which is a contradiction to (2.32). This completes the proof. □

**Theorem 2.6.** *Assume that* $0 < \gamma \leq 1,\ \beta \geq 1,\ b \leq 1, c \leq 1$ *and* $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. *If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}}{(1+b^\beta+c^\beta)^{\gamma/\alpha}} y^{\beta/\alpha}(t+\sigma_2-\sigma_1) \tag{2.45}$$

*has no positive increasing solution and the second order differential inequality*

$$y''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{4^{\beta-1}(1+b^\beta+c^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1+\tau_1) \tag{2.46}$$

*has no positive decreasing solution, then equation* (1.1) *is oscillatory.*

*Proof.* The proof is similar to that of Theorem 2.5 and hence the details are omitted. □

**Theorem 2.7.** *Assume that* $\beta \geq 1,\ 0 < \gamma \leq 1,\ b \geq 1, c \geq 1$ *and* $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. *If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}}{(1+b^\beta+c^\beta)^{\gamma/\alpha}} y^{\gamma/\alpha}(t+\sigma_2-\sigma_1) \tag{2.47}$$

*has no positive increasing solution, and the second order differential inequality*

$$y''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{4^{\beta-1}(1+b^\beta+c^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t+\tau_1-\sigma_1) \tag{2.48}$$

*has no positive decreasing solution, then equation* (1.1) *is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of equation (1.1) for all $t \geq t_1 \geq t_0$. Without loss of generality, let us assume that $x(t)$ is a positive solution of equation (1.1) for all $t \geq\ t_1 \geq\ t_0$. Then there exists a $t_2 \geq t_1$ such that $x(t-\theta) > 0$ for all $t \geq t_2$. By the definition of $z(t)$ we have $z(t-\theta) > 0$ for all $t \geq t_2$. Set

$$y(t) = z(t) + b^\beta z(t-\tau_1) + \frac{c^\beta}{2^{\gamma-1}} z(t+\tau_2) \text{ for all } t \geq t_2. \tag{2.49}$$

Then, $y'(t) > 0$, and using the fact $b \geq 1,\ c \geq 1,\ \gamma \leq 1,\ \beta \geq 1$, we have

$$\begin{aligned} y'''(t) &= z'''(t) + b^\beta z'''(t-\tau_1) + \frac{c^\beta}{2^{\gamma-1}} z'''(t+\tau_2) \\ &\geq Q(t)[x^\beta(t-\sigma_1) + b^\beta x^\beta(t-\tau_1-\sigma_1) + \frac{c^\beta}{2^{\beta-1}} x^\beta(t+\tau_2-\sigma_1)] + \\ &\quad P(t)[x^\gamma(t+\sigma_2) + b^\beta x^\gamma(t-\tau_1+\sigma_2) + \frac{c^\beta}{2^{\gamma-1}} x^\gamma(t+\tau_2+\sigma_2)] \\ &\geq Q(t)[x^\beta(t-\sigma_1) + b^\beta x^\beta(t-\tau_1-\sigma_1) + \frac{c^\beta}{2^{\beta-1}} x^\beta(t+\tau_2+\sigma_2)] \end{aligned}$$

$$+P(t)[x^\gamma(t+\sigma_2)+b^\gamma x^\gamma(t-\tau_1+\sigma_2)+c^\gamma x^\gamma(t+\tau_2+\sigma_2)], t\geq t_2$$

Now applying (2.4) and (2.3) twice in first and second part of right hand side of last inequality, we get

$$y'''(t)\geq \frac{Q(t)}{4^{\beta-1}}z^{\beta/\alpha}(t-\sigma_1)+P(t)z^{\gamma/\alpha}(t+\sigma_2), t\geq t_2. \tag{2.50}$$

Now we consider the following two cases for $z(t)$ as given in Lemma 2.1.

**Case (I):** In this case we have $z'(t)>0,\ z''(t)>0$ and $z'''(t)>0$ and therefore $y'(t)>0,\ y''(t)>0$ and $y'''(t)>0$ for all $t\geq t_2$. From the inequality (2.50), we have

$$y'''(t)\geq P(t)z^{\gamma/\alpha}(t+\sigma_2), t\geq t_2. \tag{2.51}$$

Applying monotonicity of $z'(t)$, we get

$$\begin{aligned} y'(t) &= z'(t)+b^\beta z'(t-\tau_1)+c^\beta z'(t+\tau_2) \\ y'(t) &\leq (1+b^\beta+c^\beta)z'(t+\tau_2), t\geq t_2. \end{aligned} \tag{2.52}$$

Also using the monotonicity of $z'(t)$, we get

$$z(t+\sigma_1-\tau_2)-z(t)=\int\limits_t^{t+\sigma_1-\tau_2} z'(s)\ ds > z'(t)(\sigma_2-\tau_2)$$

$$z(t+\sigma_1-\tau_2)\geq z'(t)(\sigma_1-\tau_2). \tag{2.53}$$

Combining (2.51), (2.52) and (2.53), we obtain

$$\begin{aligned} y'''(t) &= P(t)z^{\gamma/\alpha}(t+\sigma_2) \\ &\geq P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}z'(t+\tau_2) \\ &\geq \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}(y'(t+\sigma_2-\sigma_1))^{\gamma/\alpha}}{(1+b^\beta+c^\beta)^{\gamma/\alpha}}, t\geq t_2. \end{aligned}$$

By putting $y'(t)=w(t)$, we see that $w(t)$ is a positive increasing solution of

$$w''(t)\geq \frac{P(t)(\sigma_2-\tau_2)^{\gamma/\alpha}}{(1+b^\beta+c^\beta)^{\gamma/\alpha}}w^{\gamma/\alpha}(t+\sigma_2-\sigma_1), t\geq t_2$$

which is a contradiction (2.47).

**Case (II):** In this case we have $z''(t)<0$ for all $t\geq t_2$. Therefore $z'(t)$ is decreasing, for all $t\geq t_2$. Since $z'(t)$ is decreasing we have

$$\begin{aligned} y'(t) &= z'(t)+b^\beta z'(t-\tau_1)+c^\beta z'(t+\tau_2) \\ &\leq (1+b^\beta+c^\beta)z'(t-\tau_1), t\geq t_2. \end{aligned} \tag{2.54}$$

Also using the monotonicity of $z'(t)$, we get

$$z(t)-z(t-\sigma_1+\tau_1)=\int\limits_{t-(\sigma_1-\tau_1)}^{t} z'(s)\ ds\geq (\sigma_1-\tau_1)z'(t)$$

or

$$z(t)\geq(\sigma_1-\tau_1)z'(t). \tag{2.55}$$

From (2.50), we have

$$y'''(t)\geq \frac{Q(t)}{4^{\beta-1}}z^{\beta/\alpha}(t-\sigma_1), t\geq t_2. \tag{2.56}$$

Combining (2.54), (2.55) and (2.56), we obtain

$$y'''(t)\geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{4^{\beta-1}(1+b^\beta+c^\beta)^{\beta/\alpha}}(y'(t-\sigma_1+\tau_1))^{\beta/\alpha}, t\geq t_2. \tag{2.57}$$

By taking $y'(t) = w(t)$, we see that $w(t)$ is a positive decreasing solution of

$$w''(t) \geq \frac{(\sigma_1 - \tau_1)^{\beta/\alpha} Q(t)}{4^{\beta-1}(1 + b^\beta + c^\beta)^{\beta/\alpha}} (w(t - \sigma_1 + \tau_1))^{\beta/\alpha}, t \geq t_2, \tag{2.58}$$

which is a contradiction to (2.48). This completes the proof. □

**Theorem 2.8.** *Assume that $\gamma \geq 1,\ 0 < \beta \leq 1,\ b \geq 1, c \geq 1$ and $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. If the second order differential inequality*

$$y''(t) \geq \frac{P(t) y^{\gamma/\alpha}(t + \sigma_2 - \sigma_1)(\sigma_1 - \tau_2)^{\gamma/\alpha}}{4^{\gamma-1}(1 + b^\beta + c^\beta)^{\gamma/\alpha}} \tag{2.59}$$

*has no positive increasing solution and the second order differential inequality*

$$y''(t) \geq \frac{Q(t) y^{\beta/\alpha}(t - \sigma_1 + \tau_1)(\sigma_1 - \tau_1)^{\beta/\alpha}}{(1 + b^\beta + c^\beta)^{\beta/\alpha}} \tag{2.60}$$

*has no positive decreasing solution, then equation (1.1) is oscillatory.*

*Proof.* The proof is similar to that of Theorem 2.7 and hence the details are omitted. □

**Corollary 2.9.** *Assume that $\alpha = \beta = \gamma \geq 1$, $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. If*

$$\limsup_{t\to\infty} \int_t^{t+\sigma_2-\tau_2-2} (t + \sigma_2 - \tau_2 - s - 1) P(s)\ ds \geq 4^{\alpha-1}(1 + a^\alpha + \frac{b^\alpha}{2^{\alpha-1}}) \tag{2.61}$$

*and*

$$\limsup_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} (t - s + 1) Q(s)\ ds \geq 4^{\alpha-1}(1 + a^\alpha + \frac{b^\alpha}{2^{\alpha-1}}) \tag{2.62}$$

*then every solution of equation (1.1) is oscillatory.*

*Proof.* Condition (2.61) and (2.62) imply that the differential inequalities (2.59) and (2.60) have no positive increasing and no positive decreasing solutions respectively see [12, 16]. Now the result follows from Theorem 2.8. □

**Corollary 2.10.** *Let $\beta < \gamma,\ b \leq 1,\ c \leq 1,\ \sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. If*

$$\int_{t_0}^{\infty} \Big( \int_t^{t+\sigma_1-\tau_1} Q(s)\ ds \Big)\ dt\ =\ \infty \tag{2.63}$$

$$\int_{t_0}^{\infty} \Big( \int_{t-\sigma_2+\tau_2+1}^{t} P(s)\ ds \Big)\ dt\ =\ \infty \tag{2.64}$$

*then every solution of equation (1.1) is oscillatory.*

*Proof.* Conditions (2.63) and (2.64) imply that the differential inequalities (2.31) and (2.32) have no positive increasing and no positive decreasing solutions respectively [12, 16]. Now the result follows from Theorem 2.5. □

## 3 Examples

In this section, we shall see some examples to illustrate main results.

**Example 3.1.** *Consider the third order differential equation*

$$((x(t) + 2x(t-1) + 3x(t+2))^3)''' = (t+1)x^3(t-3) + tx^3(t+5),\ t \geq 1 \tag{3.1}$$

*Here $b(t) = 2,\ c(t) = 3,\ \tau_1 = 1,\ \tau_2 = 2,\ \sigma_1 = 3,\ \sigma_2 = 5,\ q(t) = t+1,\ p(t) = t$ and $\alpha = \beta = \gamma = 3$. Then $Q(t) = t,\ P(t) = t - 1$ and we can easily see that all the conditions of Corollary* 2.9 *are satisfied. Therefore all the solutions of equation (3.1) are oscillatory.*

**Example 3.2.** *Consider the third order differential equation*

$$((x(t)+\frac{1}{2}x(t-1)+\frac{1}{3}x(t+2))^3)'''=(t+1)x(t-3)+(t+2)^2x^3(t+4),\ t\geq 1 \tag{3.2}$$

*Here* $b(t)=\frac{1}{2},\ c(t)=\frac{1}{3},\ \tau_1=1,\ \tau_2=2,\ \sigma_1=3,\ \sigma_2=4,\ \alpha=1,\ \beta=1,\ \gamma=3,\ q(t)=t+1,\ p(t)=(t+2)^2$. *Then* $Q(t)=t,\ P(t)=t^2$ *and we can easily see that all the conditions of Corollary* 2.10 *are satisfied. Therefore all the solutions of equation* (3.2) *are oscillatory.*

## References


[1] R. P. Agarwal, M. F. Aktas, and A. Tiryaki, On oscillation criteria for third order nonlinear delay differential equations, *Arch. Math.,* 45(2009), 1-18.

[2] R. P. Agarwal, B. Baculáková, J. Džurina and T.Li, Oscillation of third-order nonlinear functional differential equations with mixed arguments, *Acta Math. Hungar.,* 134(2011), 54-67.

[3] R. P. Agarwal, S. R. Grace and D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equation,* Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[4] B. Baculíková and J. Džurina, Oscillation of third-order functional differential equations, *Elec. J. Qual. Theo. Diff. Equ.,* 43(2010), 1-10.

[5] B. Baculíková and J. Džurina, Oscillation of third-order neutral differential equations, *Math. Comput. Modelling,* 52(2010), 215-226.

[6] B. Baculíková and J. Džurina, On the aymptotic behavior of a class of third order nonlinear neutral differential equations, *Cent. European J. Math.,* 8(2010), 1091-1103.

[7] B. Baculíková and J. Džurina, Oscillation of third-order nonlinear differential equations, *Appl. Math. Lett.,* (2011), 466-470.

[8] B. Baculíková, E. M. Elabbasy, S. H. Saker, and J. Džurina, Oscillation criteria for third-order nonlinear differential euqations, *Math. Slovaca,* 58(2008), 201-220.

[9] T. Candan and R. S. Dahiya, Functional differential equations of third order, *Elec. J. Diff. Equ.,* 12(2005), 47-56.

[10] P. Das, Oscillation criteria for odd order neutral equations, *J. Math. Anal. Appl.,* 188(1994), 245-257.

[11] K. Gopalsamy, B. S. Lalli and B. G. Zhang, Oscillation of odd order neutral differential equations, *Czech. Math. J.,* 42(1992), 313-323.

[12] S.R.Grace, Oscillation criteria for nth order neutral functional differential equations, *J. Math. Anal. Appl.,* 184(1994), 44-55.

[13] S. R. Grace, On oscillation of mixed neutral equations, *J. Math. Anal. Appl.,* 194(1995), 377-388.

[14] S. R. Grace, Oscillation of mixed neutral fractional differential equations, *Appl. Math. Comp.,* 68(1995), 1-13.

[15] S. R. Grace, R. P. Agarwal, R. Pavani, and E. Thandapani, On the oscillation of certain third order nonlinear functional differential equations, *Appl. Math. Comput.,* 202(2008), 102-112.

[16] S.R.Grace and B.S.Lalli, Oscillation theorems for certain neutral functional differential equations with periodic cofficients, *Dynam. Systems Appl.,* 3(1994), 85-93.

[17] J. R. Graef, R. Savithri, and E. Thandapani, Oscillatory properties of third order neutral delay differential equations, *Disc. Cont. Dyn. Sys. A,* 2003, 342-350.

[18] T. Li and E. Thandapani, Oscillation of solutions to odd-order nonlinear neutral functional differential equations, *Elec. J. Diff. Equ.,* 23(2011), 1-12.

[19] T.Li, C.Zhang and G.Xing, Oscillation of third order neutral delay differential equations, *Abstract and Applied Analysis*, 2012(2012), 1-11.

[20] S. H. Saker and J. Džurina, On the oscillation of certain class of third-order nonlinear delay differential equations, *Math. Bohemica,* 135(2010), 225-237.

[21] E. Thandapani and T. Li, On the oscillation of third-order quasi-linear neutral functional differentiall equations, *Arch. Math.,* 47(2011), 181-199.

[22] J. R. Yau, Oscillation of higher order neutral differential equations of mixed type, *Israel J. Math.,* 115(2000), 125-136.

[23] C. Zhang, T. Li, B. Sun and E. Thandapani, On the oscillation of higher-order half-linear delay differential equations, *Appl. Math. Lett.,* 24(2011), 1618-1621.

*Received*: November 11, 2012; *Accepted*: January 7, 2013

UNIVERSITY PRESS

Malaya Journal of Matematik 1(1)(2013) 50–56

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

# Discontinuous dynamical system represents the Logistic retarded functional equation with two different delays

A. M. A. El-Sayed,[a,*] and M. E. Nasr[b]

[a] Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt

[b] Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt.

**Abstract**

In this work we are concerned with the discontinuous dynamical system representing the problem of the logistic retarded functional equation with two different delays,

$$x(t) = \rho x(t-r_1)[1-x(t-r_2)], \quad t \in (0,T],$$
$$x(t) = x_0, \quad t \le 0.$$

The existence of a unique solution $x \in L^1[0,T]$ which is continuously dependence on the initial data, will be proved. The local stability at the equilibrium points will be studied. The bifurcation analysis and chaos will be discussed.

*Keywords:* Logistic functional equation, existence, uniqueness, equilibrium points, local stability, Chaos and Bifurcation.

*2010 MSC:* 39A05, 39A28, 39A30.



## 1 Introduction

Let $R_+$ be the set of positive real numbers and let $r \in R_+$. Consider the problem of retarded functional equation

$$x(t) = f(t, x(t-r)), \quad t \in (0,T] \tag{1.1}$$

$$x(t) = x_o, \quad t \le 0. \tag{1.2}$$

Now, if $T$ be positive integer, $r = 1$, and $t = n = 1,2,3,\cdots T$, then the problem (1.1)-(1.2) will be the discrete dynamical system

$$x_n = f(n, x_{n-1}), \quad n = 1,2,3,\cdots T \tag{1.3}$$

$$x_0 = x_o, \quad t \le 0. \tag{1.4}$$

This shows that the discrete dynamical system (1.3)-(1.4) is a special case of the problem of retarded functional equation (1.1)-(1.2).

## 2 Discontinuous dynamical systems

The discontinuous dynamical systems have been studied, recently, in [3]-[5]. The results in [4] and [5] shows the richness of the models of discontinuous dynamical systems.

Consider the problem of retarded functional equation

$$x(t) = f(x(t-r)), \quad t \in (0,T] \tag{2.5}$$

*Corresponding author.

*E-mail addresses*: amasayed@gmail.com (A. M. A. El-Sayed) and moh_nasr_2000@yahoo.com (M. E. Nasr).

$$x(t) = x_o, \quad t \leq 0.$$

Let $t \in (0, r]$, then $t - r \in (-r, 0]$ and the solution of (1.1) − (1.2) is given by

$$x(t) = x_r(t) = f(x_o), \quad t \in (0, r].$$

For $t \in (r, 2r]$, we find that $t - r \in (0, r]$ and the solution of (1.1)-(1.2) is given by

$$x(t) = x_{2r}(t) = f(x_r(t)) = f(f(x_o)) = f^2(x_o), \quad t \in (r, 2r].$$

Repeating the process we can deduce that the solution of the problem (1.1)-(1.2) is given by

$$x(t) = x_{nr}(t) = f^n(x_o), \quad t \in ((n-1)r, nr],$$

which is continuous on each subinterval $((k-1)r, kr)$, $k = 1, 2, \cdots, n$, but

$$\lim_{t \to kr^+} x_{(k+1)r}(t) = f^{k+1}(x_o) \neq x_{kr}(t),$$

which implies that the solution of the problem (1.1)-(1.2) is discontinuous (sectionally continuous) on $(0, T]$ and we have proved the following theorem

**Theorem 2.1.** *The solution of the problem of retarded functional equation (1.1)-(1.2) is discontinuous (sectionally continuous) even the function $f$ is continuous.*

Now, let $f : [0, T] \times R^n \to R^n$ and $r_1, r_2, ..., r_n \in R_+$. Then we can give the following definition

**Definition 2.1.** *The discontinuous dynamical system is the problem of retarded functional equation*

$$x(t) = f(t, x(t - r_1), x(t - r_2), \cdots, x(t - r_n)), \quad t \in (0, T], \tag{2.6}$$

$$x(t) = x_0, \quad t \leq 0 \tag{2.7}$$

**Definition 2.2.** *The equilibrium points of the discontinuous dynamical system (2.6)-(2.7) is the solutions of the equation,*

$$x(t) = f(t, x, x, \cdots, x).$$

Consider now the discontinuous dynamical system of the Logistic retarded functional equation with two different delays $r_1, r_2 > 0$

$$x(t) = \rho x(t - r_1)[1 - x(t - r_2)], \quad t \in (0, T], \tag{2.8}$$

$$x(t) = x_0, \quad t \leq 0. \tag{2.9}$$

We study here the existence of a unique continuously dependent solution $x \in L^1[0, T]$ of the problem (2.8)−(2.9). The asymptotic stability (see [1]- [9]) at the equilibrium points will be studied. We study the chaos and bifurcation for different values of $r_1$, $r_2$ and $T$ and we compare the results with the results of the discrete dynamical system of the Logistic difference equations,

$$x_n = \rho\, x_{n-1}(1 - x_{n-1}), \quad n = 1, 2, \cdots. \tag{2.10}$$

and

$$x_n = \rho\, x_{n-1}(1 - x_{n-2}), \quad n = 1, 2, \cdots. \tag{2.11}$$

## 3 Existence and Uniqueness

Let $L^1 = L^1[0,T], \quad T < \infty$ be the class of Lebesgue integrable functions on $[0,T]$ with norm

$$\|f\| = \int_0^T |f(t)|\,dt, \qquad f \in L^1.$$

Let $D = \{x \in R : 0 \le x(t) \le 1,\ t \in (0,T] \quad and \quad x(0) = x_0,\ t \le 0\}$.

**Definition 3.3.** *By a solution of the problem (2.8) – (2.9) we mean a function $x \in L^1$ satisfying the conditions (2.8) – (2.9).*

**Theorem 3.2.** *The problem (2.8) – (2.9) has a unique solution $x \in L^1$.*

*Proof.* Define, on $D$, the operator $F : L^1 \longrightarrow L^1$ by

$$Fx(t) = \rho x(t-r_1)[1 - x(t-r_2)].$$

The operator $F$ makes sense, indeed for $x \in D$ we have

$$|Fx(t)| \le \rho\,|x(t-r_1)|$$

and

$$\|Fx\| \le \rho(x_0 r_1 + \|x\|).$$

Now for $x, y \in D$,we can obtain

$$\begin{aligned}|Fx - Fy| &= |\rho x(t-r_1)(1 - x(t-r_2)) - \rho y(t-r_1)(1 - y(t-r_2))| \\ &\le \rho\,|x(t-r_1) - y(t-r_1)| + \rho\,|x(t-r_2) - y(t-r_2)|\end{aligned}$$

which implies that

$$\begin{aligned}\|Fx - Fy\| &\le \rho\int_0^T |x(t-r_1) - y(t-r_1)|\,dt + \rho\int_0^T |x(t-r_2) - y(t-r_2)|\,dt = \\ &= \rho\left[\int_0^{r_1} |x(t-r_1) - y(t-r_1)|\,dt + \int_{r_1}^T |x(t-r_1) - y(t-r_1)|\,dt + \right. \\ &\left. + \int_0^{r_2} |x(t-r_2) - y(t-r_2)|\,dt + \int_{r_2}^T |x(t-r_2) - y(t-r_2)|\,dt\right] = \\ &= \rho\left[\int_{r_1}^T |x(t-r_1) - y(t-r_1)|\,dt + \int_{r_2}^T |x(t-r_2) - y(t-r_2)|\,dt\right] \\ &\le \rho\left[\int_0^{T-r_1} |x(\theta) - y(\theta)|\,d\theta + \int_0^{T-r_2} |x(\varphi) - y(\psi)|\,d\varphi\right] \\ &\le \rho\left[\int_0^T |x(\theta) - y(\theta)|\,d\theta + \int_0^T |x(\varphi) - y(\psi)|\,d\varphi\right] \\ &\le 2\rho\,\|x - y\|\,.\end{aligned}$$

If $\rho < \frac{1}{2}$ we deduce that

$$\|Fx - Fy\| < \|x - y\|$$

and then the problem (2.8) – (2.9) has, on $D$, a unique solution $x \in L^1$. □

## 4 Continuous dependence on initial conditions

Consider the problem

$$x(t) = \rho x(t-r_1)[1 - x(t-r_2)], \quad t \in (0,T],$$

$$x(t) = x_0^*, \quad t \le 0. \tag{4.12}$$

For the continuous dependence of The solution of (2.8) – (2.9) on the initial data we have the following theorem.

**Theorem 4.3.** *The solution of the discontinuous dynamical system represents the problem of the logistic retarded functional equation with two different delays is continuously dependent on the initial data.*

*Proof.* Let $x(t)$ and $x^*(t)$ be the solution of the two problems (2.8) – (2.9) and (2.8) – (4.12) respectively, then

$$|x(t) - x^*(t)| \le \rho\,|x(t-r_1) - x^*(t-r_1)| + \rho\,|x(t-r_2) - x^*(t-r_2)|$$

which implies that

$$\begin{aligned} \|x(t) - x^*(t)\| &\le \rho \int_0^T |x(t-r_1) - x^*(t-r_1)|\,dt + \rho \int_0^T |x(t-r_2) - x^*(t-r_2)|\,dt = \\ &= \rho \left[ \int_0^{r_1} |x(t-r_1) - x^*(t-r_1)|\,dt + \int_{r_1}^T |x(t-r_1) - x^*(t-r_1)|\,dt + \right. \\ &+ \left. \int_0^{r_2} |x(t-r_2) - x^*(t-r_2)|\,dt + \int_{r_2}^T |x(t-r_2) - x^*(t-r_2)|\,dt \right] = \\ &= \rho \left[ |x_0 - x_0^*| \int_0^{r_1} dt + \|x - x^*\| + |x_0 - x_0^*| \int_0^{r_2} dt + \|x - x^*\| \right] \\ &\le \rho(r_1 + r_2)\,|x_0 - x_0^*| + 2\rho\,\|x - x^*\| \end{aligned}$$

which implies

$$\|x - x^*\| \le \frac{\rho(r_1 + r_2)}{1 - 2\rho}\,|x_0 - x_0^*|$$

and prove that

$$|x_0 - x_0^*| \le \delta \qquad \Rightarrow \qquad \|x - x^*\| \le \varepsilon = \frac{\rho(r_1 + r_2)}{1 - 2\rho}\delta$$

and the theorem is proved. □

## 5 Equilibrium Points and their asymptotic stability

The equilibrium points of (2.8) are the solution of the equation

$$\rho\, x_{eq}\,(1 - x_{eq}) = x_{eq}$$

which are

$$\begin{aligned} (x_{eq})_1 &= 0, \\ (x_{eq})_2 &= 1 - \frac{1}{\rho}. \end{aligned}$$

The equilibrium point of (2.8) is locally asymptotically sable if all the roots $\lambda$ of the equation,

$$1 = \rho\left[(1 - x_{eq})\,\lambda^{-r_1} - x_{eq}\,\lambda^{-r_2}\right], \tag{5.13}$$

satisfy $|\lambda| < 1$ (see [10]).
Then the equilibrium point $x_{eq} = 0$ is locally asymptotically sable if $\rho < 1$, while the second equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$ is locally asymptotically sable if all the roots $\lambda$ of the equation,

$$\lambda^{r_2} - \lambda^{r_2 - r_1} + (\rho - 1) = 0. \tag{5.14}$$

satisfy $|\lambda| < 1$.
The equilibrium point $x_{eq} = 0$ is locally asymptotically sable if $\rho < 1$, which is the same as in the discrete case (2.10). Also, when $r_2 = r_1 = 1$, we deduce that the equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$, $\rho > 1$ is locally asymptotically sable if $1 < \rho < 3$, which is the same as in the discrete case (2.10).
In studying (2.8) – (2.9) it may be useful to study the difference equations (2.10) and (2.11).

## 6 Bifurcation and Chaos

In this section, some numerical simulations results are presented to show that dynamics behaviors of the discontinuous dynamical system (2.8) − (2.9) change for different values of $r_1$, $r_2$ and $T$ . To do this, we will use the bifurcation diagrams as follow:-

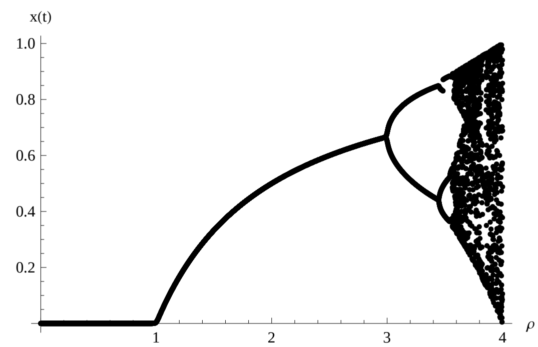


Figure 6.1
Bifurcation diagram of map (2.8)-(2.9) with respect to $\rho$, $r_1 = r_2 = 1$ and $t \in [0, 200]$.

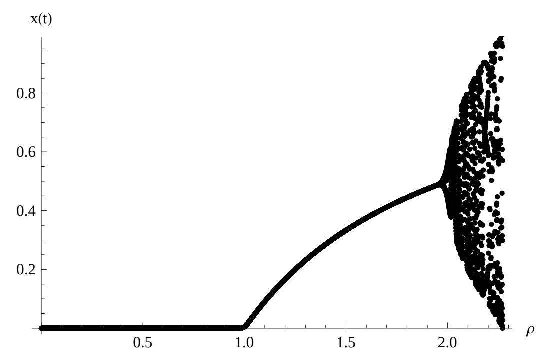


Figure 6.2
Bifurcation diagram of map (2.8)-(2.9) with respect to $\rho$, $r_1 = 1, r_2 = 2$ and $t \in [0, 200]$.

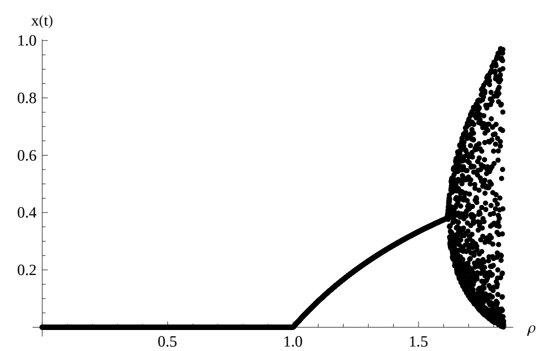


Figure 6.3
Bifurcation diagram of map (2.8)-(2.9) with respect to $\rho$, $r_1 = 0.1, r_2 = 0.3$ and $t \in [0, 200]$.

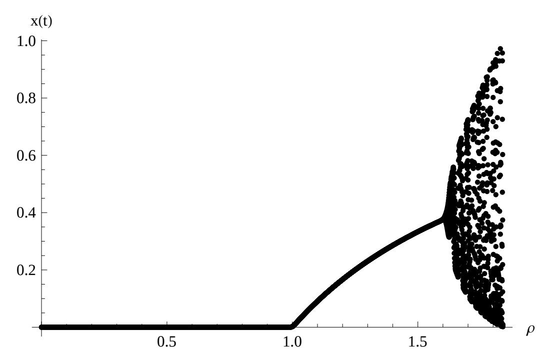


Figure 6.4
Bifurcation diagram of map (2.8)-(2.9) with respect to $\rho$, $r_1 = 0.25, r_2 = 0.75$ and $t \in [0, 200]$.

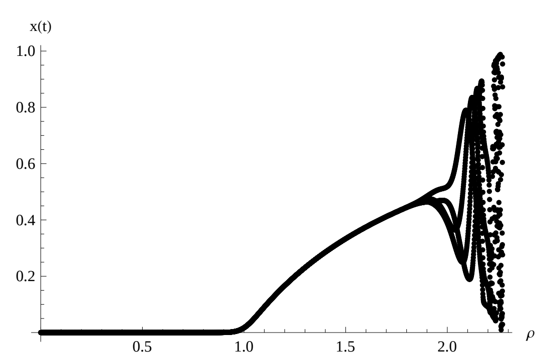


Figure 6.5
Bifurcation diagram of map (2.8)-(2.9) with respect to $\rho$, $r_1 = 1, r_2 = 2$ and $t \in [0, 50]$.

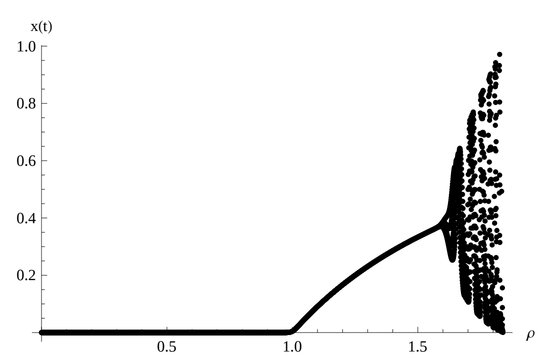


Figure 6.6
Bifurcation diagram of map (2.8)-(2.9) with respect to $\rho$, $r_1 = 0.25, r_2 = 0.75$ and $t \in [0, 50]$.

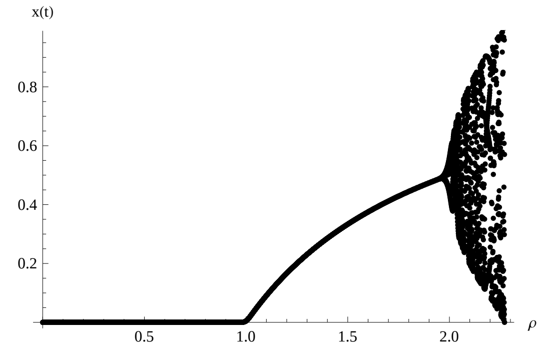


Figure 6.7
Bifurcation diagram of map (2.8)-(2.9) with respect to $\rho$, $r_1 = 0.5, r_2 = 1$ and $t \in [0, 100]$.

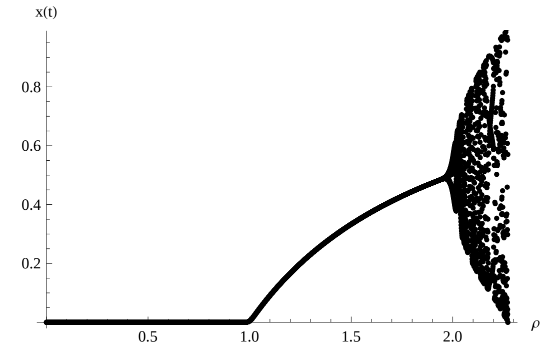


Figure 6.8
Bifurcation diagram of map (2.8)-(2.9) with respect to $\rho$, $r_1 = 0.1, r_2 = 0.2$ and $t \in [0, 20]$.

From Figures (6.1-6.8) we deduce that the change of $r_1$, $r_2$ and $T$ effect of stability of the Logistic equation model, occurs of a bifurcation point, parameter sets for which aperiodic behavior occur and parameter sets for which a chaotic behavior occur.

## 7 Conclusions

Discrete dynamical system of the Logistic equation model describes the dynamical properties for the case $r_1 = r_2$ and the time is discrete $t = 1, 2, 3, 4, \cdots$.
On the other hand, discontinuous dynamical system of the Logistic equation model describes the dynamical properties for different values of the delayed parameters $r_1$ and $r_2$ and the time is continuous. Figures (6.1),(6.2) agrees with standard results. This confirms the correctness of our computation. The results of the other figures are new behavior (there is no analytic explanation for this behavior). From figures (6.2),(6.7) and (6.8), it locks like that there is a scale that gives identical chaos behavior.
This shows the richness of the models of discontinuous dynamical systems.

# References


[1] L. Berezansky and E. Braverman, On stability of some linear and nonlinear delay differential equations, *J. Math. Anal. Appl.*, 314(2006), 9-15.

[2] A. El-Sayed, A. El-Mesiry and H. EL-Saka, On the fractional-order logistic equation, *Applied Mathematics Letters*, 20(2007), 817-823.

[3] A. M. A. El-Sayed and M. E. Nasr, Existence of uniformly stable solutions of nonautonomous discontinuous dynamical systems, *J. Egypt Math. Soc.*,19(1)(2011),10-16.

[4] A. M. A. El-Sayed and M. E. Nasr, On some dynamical properties of discontinuous dynamical systems, *American Academic & Scholarly Research Journal*, 2(1)(2012), 28-32.

[5] A. M. A. El-Sayed and M. E. Nasr, Dynamic properties of the predator-prey discontinuous dynamical system, *J. Z. Naturforsch. A*, 67a(2012), 57-60.

[6] H. EL-Saka, E. Ahmed, M. I. Shehata, A. M. A. El-Sayed, On stability, persistence, and Hopf bifurcation in fractional order dynamical systems, *Nonlinear Dyn*, 56(2009), 121-126.

[7] J. Hale and S.M. Verduyn Lunel, *Introduction to functional differential equations*, springrverlag, New york, 1993.

[8] Y. Hamaya and A. Redkina, On global asymptotic stability of nonlinear stochastic difference equations with delays, *Internat. J.Diff.*, 1(2006), 101-118.

[9] Y. Kuang, *Dely differential equations with applications in population dynamics*, Academic Press, New York, 1993.

[10] J.D. Murray, *Mathematical Biology I: An Introduction*, Springer-Verlag, Berlin, 3rd edition, 2002.

*Received*: October 16, 2012; *Accepted*: January 12, 2013

UNIVERSITY PRESS

Malaya Journal of Matematik 1(1)(2013) 57–69

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

# On extended $M-$ series

Dharmendra Kumar Singh,*

*Department of Mathematics*

*University Institute of Engineering and Technology*

*Chhatrapati Shahu Ji Maharaj University, Kanpur (U.P.) India.*

**Abstract**

This paper deals with extended M-series, which is extension of the generalized M-series [12]. Mittag-Leffler function, $\omega-$ hypergeometric function, generalized $\omega-$ Gauss hypergeometric function, $\omega-$ confluent hypergeometric function, Bessel-Maitland function can be deduced as special cases of our finding. Moreover, we obtain some theorem for extended M-series by using fractional calculus operators and many results associated with Riemann-Liouville, Weyl and Erdelyi-Kober operators. We begin our study from the following definitions.

*Keywords:* Extended M-series, Saigo- Meada operators, Pathway fractional integral operator.

*2010 MSC:* 26A33, 44A15, 33C60, 33E12. 

## 1 Introduction

Fractional calculus operators $(I_{0+}^{\alpha,\beta,\eta}f)(x), (I_{-}^{\alpha,\beta,\eta}f)(x), (D_{0+}^{\alpha,\beta,\eta}f)(x)$ and $(D_{-}^{\alpha,\beta,\eta}f)(x)$ be defined for and complex $\alpha, \beta, \eta \in C$ and $x \in \Re_+$; by Saigo [10].

$$\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt \tag{1.1}$$

$(\Re(\alpha) > 0)$;

$$= \frac{d^n}{dx^n}\left(I_{0+}^{\alpha+n,\beta-n,\eta-n}f\right)(x) \tag{1.2}$$

$(\Re(\alpha) \leq 0; n = [\Re(-\alpha)] + 1)$;

$$\left(I_{-}^{\alpha,\beta,\eta}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)dt \tag{1.3}$$

$(\Re(\alpha) > 0)$;

$$= (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{\alpha+n,\beta-n,\eta}f\right)(x) \tag{1.4}$$

$(\Re(\alpha) \leq 0; n = [\Re(-\alpha)] + 1)$ and

$$\left(D_{0+}^{\alpha,\beta,\eta}f\right)(x) = \left(I_{0+}^{-\alpha,-\beta,\alpha+\eta}f\right)(x) = \frac{d^n}{dx^n}\left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n}f\right)(x) \tag{1.5}$$

$(\Re(\alpha) > 0; n = [\Re(\alpha)] + 1)$;

$$\left(D_{-}^{\alpha,\beta,\eta}f\right)(x) = \left(I_{-}^{-\alpha,-\beta,\alpha+\eta}f\right)(x) = (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta}f\right)(x) \tag{1.6}$$

*Corresponding author.

*E-mail addresses*: drdksinghabp@gmail.com (D.K. Singh)

$(\Re(\alpha) > 0; n = [\Re(\alpha)] + 1)$.
When $\beta = -\alpha$, (1.1) and (1.3) coincide with the classical Riemann-Liouville and Weyl fractional integral of order $\alpha \in C$ shown below

$$\left(R_{0,x}^{\alpha} f\right)(x) = \left(I_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, (\Re(\alpha) > 0); \tag{1.7}$$

$$= \frac{d^n}{dx^n} \left(R_{0,x}^{\alpha+n} f\right)(x) \tag{1.8}$$

$(0 < \Re(\alpha) + n \leq 1; n = 1, 2, ...)$;

$$\left(W_{x,\infty}^{\alpha} f\right)(x) = \left(I_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\,dt \tag{1.9}$$

$(\Re(\alpha) > 0)$;

$$= (-1)^n \frac{d^n}{dx^n} \left(W_{x,\infty}^{\alpha+n} f\right)(x) \tag{1.10}$$

$(0 < \Re(\alpha) + n \leq 1; n = 1, 2, ...)$;
and equation (1.5) and (1.6) coincide with Riemann- Liouville fractional derivative of order $\alpha > 0$ is defined by

$$\left(D_{0+}^{\alpha} f\right)(x) = \left(D_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}} \tag{1.11}$$

$(n = [\Re(\alpha)] + 1)$;

$$\left(D_{-}^{\alpha} f\right)(x) = \left(D_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^\infty \frac{f(t)dt}{(t-x)^{\alpha-n+1}} \tag{1.12}$$

$(n = [\Re(\alpha)] + 1)$.
While for $\beta = 0$, (1.1) and (1.3) coincide with the Erdelyi- Kober fractional calculus operators of order $\alpha \in C$

$$\left(E_{0,x}^{\alpha,\eta} f\right)(x) = \left(I_{0+}^{\alpha,0,\eta} f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\,dt \tag{1.13}$$

$(\Re(\alpha) > 0)$;

$$\left(K_{x,\infty}^{\alpha,\eta} f\right)(x) = \left(I_{-}^{\alpha,0,\eta} f\right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\,dt \tag{1.14}$$

$(\Re(\alpha) > 0)$.
Now here the definition of the following generalized fractional integration and differentiation operators of any complex order involving Appell function $F_3(.)$ due to Saigo and Meada [11, p. 393, Eqs. (4.12) and (4.13)] in the kernal in the following form.

Let $\alpha, \alpha', \beta, \beta', \gamma \in C, x > 0$, then the generalized fractional calculus operators involving the Appell function $F_3$ are defined by the following equations:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x t^{-\alpha'} (x-t)^{\gamma-1}$$

$$\times F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\,dt, (\Re(\gamma) > 0); \tag{1.15}$$

$$= \frac{d^n}{dx^n} \left(I_{0+}^{\alpha,\alpha',\beta+n,\beta',\gamma+n} f\right)(x) \tag{1.16}$$

$(\Re(\gamma) \leq 0; n = [-\Re(\gamma)] + 1)$;

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty t^{-\alpha} (t-x)^{\gamma-1}$$

$$\times F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\,dt, (\Re(\gamma) > 0); \tag{1.17}$$

$$= (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{\alpha,\alpha',\beta,\beta'+n,\gamma+n} f\right)(x) \tag{1.18}$$

$(\Re(\gamma) \leq 0; n = [-\Re(\gamma)] + 1)$ and

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \tag{1.19}$$

$$= \frac{d^n}{dx^n}\left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n} f\right)(x); \tag{1.20}$$

$(\Re(\gamma) > 0; n = [\Re(\gamma)] + 1)$;

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \tag{1.21}$$

$$= (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n} f\right)(x) \tag{1.22}$$

$(\Re(\gamma) > 0; n = [\Re(\gamma)] + 1)$.

These operators reduce to that in (1.15)-(1.22) as the following.

$$\left(I_{0+}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{\gamma,\alpha-\gamma,-\beta} f\right)(x)(\gamma \in C); \tag{1.23}$$

$$\left(I_{-}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{\gamma,\alpha-\gamma,-\beta} f\right)(x)(\gamma \in C); \tag{1.24}$$

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x)(\Re(\gamma) > 0); \tag{1.25}$$

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x)(\Re(\gamma) > 0). \tag{1.26}$$

Our results are based on a preliminary assertion giving composition formulas of generalized fractional integrals (1.15) and (1.17) with a power function established by Saigo and Meada [11, p. 394, eqs. (4.18) and (4.19)], we also have

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\,\Gamma\left(\rho+\gamma-\alpha-\alpha'-\beta\right)\Gamma\left(\rho+\beta'-\alpha'\right)}{\Gamma(\rho+\gamma-\alpha-\alpha')\,\Gamma(\rho+\gamma-\alpha'-\beta)\,\Gamma(\rho+\beta')} \times x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{1.27}$$

where $\Re(\gamma) > 0, \Re(\rho) > \max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')]$, and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x) = \frac{\Gamma\left(1+\alpha+\alpha'-\gamma-\rho\right)\Gamma\left(1+\alpha+\beta'-\gamma-\rho\right)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\,\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\,\Gamma(1+\alpha-\beta-\rho)} \times x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{1.28}$$

where $\Re(\gamma) > 0, \Re(\rho) < 1 + \min[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)]$.

For fractional integrals (1.1) and (1.3) with a power function established by Saigo [10], given below

(a) If $\alpha, \beta, \eta, \rho \in C$ are such that

$$\Re(\alpha) > 0, \Re(\rho) > \max[0, \Re(\beta-\eta)], \tag{1.29}$$

then

$$\left(I_{0+}^{\alpha,\beta,\eta} x^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\beta)}{\Gamma(\rho-\beta)\Gamma(\rho+\alpha+\eta)} x^{\rho-\beta-1} (x > 0). \tag{1.30}$$

(b) If $\alpha, \beta, \eta, \rho \in C$ are such that

$$\Re(\alpha) > 0, \Re(\rho) > -\min[\Re(\beta), \Re(\eta)], \tag{1.31}$$

then

$$\left(I_{-}^{\alpha,\beta,\eta} x^{-\rho}\right)(x) = \frac{\Gamma(\rho+\beta)\Gamma(\rho+\eta)}{\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\eta)} x^{-\rho-\beta} (x > 0). \tag{1.32}$$

## 2 Extended M-Series

Extended $M$-series is the Special case of the generalized Wright function [9] as remarked by Saxena [16]. Since

$$
\begin{aligned}
{}_{p+2}\overset{\omega}{M}_{q+2}\left[\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu) \end{array} \mid z\right] &= \kappa_{p+2}\Psi_{q+2}\left[\begin{array}{c} (a_1,1),...,(a_p,1),(1,1),(\tau,\omega) \\ (b_1,1),...,(b_q,1),(\delta,\omega),(\xi,\mu) \end{array} \mid z\right] \\
&= \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{z^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}z^k, \qquad (2.1)
\end{aligned}
$$

where ${}_{p+2}\overset{\omega}{M}_{q+2}(.)$ is called omega M-series ($\omega - M$ series) and $\kappa = \frac{\Pi_{j=1}^{q}\Gamma(b_j)_k}{\Pi_{j=1}^{p}\Gamma(a_j)_k}; \tau,\xi,\mu,\delta \in C, \Re(\mu) > 0, \Re(\omega) > 0, p \le q+1$.

## 3 Special Cases

**(i)** If $\delta = \tau$ then equation (2.1) can be written in the following form

$$
{}_{p}\overset{\xi,\mu}{M}_{q}(z) = \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k\Gamma(\xi+\mu k)}z^k, \qquad (3.1)
$$

where z, $\xi,\mu \in C, \Re(\mu) > 0, p \le q+1$ is known as generalized M-Series [12].
**(ii)** If we put $\xi = 1$ then from the above equation (3.1) called the M-series [12].

$$
{}_{p}\overset{\mu}{M}_{q}(z) = \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k\Gamma(1+\mu k)}z^k, \qquad (3.2)
$$

where $\mu \in C, p \le q+1$.
**(iii)** The $\omega$-confluent hypergeometric function [13, 14]: when $p = q = 0$ and $\xi = \mu = 1$, we have

$$
\frac{\Gamma(\tau)}{\Gamma(\delta)}{}_{1}\overset{\omega}{\Phi}_{1}(\tau;\delta;z) = \sum_{k=0}^{\infty}\frac{\Gamma(\tau+\omega k)}{\Gamma(\delta+\omega k)\Gamma(1+k)}z^k = \sum_{k=0}^{\infty}\frac{\Gamma(\tau+\omega k)}{\Gamma(\delta+\omega k)k!}z^k, \qquad (3.3)
$$

where $|z| < \infty, \omega > 0, (\delta+\omega k) \neq 0,-1,-2,...$ .
**(iv)** The $\omega$-hypergeometric function [14]: For $p = 1, q = 0,\ \xi = \mu = 1$, we have

$$
\frac{\Gamma(\tau)}{\Gamma(\delta)}{}_{2}\overset{\omega}{R}_{1}(a,\tau;\delta;z) = \sum_{k=0}^{\infty}\frac{(a)_k(1)_k\Gamma(\tau+\omega k)}{\Gamma(\delta+\omega k)\Gamma(1+k)}\frac{z^k}{k!} = \sum_{k=0}^{\infty}\frac{(a)_k\Gamma(\tau+\omega k)}{\Gamma(\delta+\omega k)k!}z^k, \qquad (3.4)
$$

where $|z| < 1, \omega > 0$.
**(v)** The generalized $\omega$-Gauss hypergeometric function [21]: If we take $p = 2, q = 1,\ \xi = \mu = 1$, then we have

$$
\frac{\Gamma(\tau)}{\Gamma(\delta)}{}_{3}\overset{\omega}{R}_{2}(a_1,\underline{a_2},\tau;\underline{b_1},\delta;z) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\Gamma(\tau+\omega k)}{(b_1)_k\Gamma(\delta+\omega k)k!}z^k, \qquad (3.5)
$$

where $\underline{a}$ is defined to be $\frac{\Gamma(a+\omega k)}{\Gamma(a)}$ and $|z| < 1$.
**(vi)** When $p = 0, q = 1,\ \tau = \delta, b = 1, \xi = \xi+1$ and z is replaced by -z, the function $\phi(\mu,\xi+1;-z)$ is denoted by $J_{\eta}^{\mu}(z)$ :

$$
J_{\xi}^{\mu}(z) \equiv \phi(\mu,\xi+1;-z) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\xi+1+\mu k)}\frac{(-z)^k}{k!} \qquad (3.6)
$$

and such a function is known as the Bessel-Maitland function, or the Wright generalized Bessel function, See [20, p. 352] and [15, 8.3].

**(vii)** If we put $p=1, q=1$ and $\tau=\delta$, $b=1$ in (2.1), then we have

$$E_{\xi,\mu}^{a}(z)=\sum_{k=0}^{\infty}\frac{(a)_k}{\Gamma(\xi+\mu k)}\frac{z^k}{k!}, \tag{3.7}$$

where $\xi,\mu\in C, \Re(\xi)>0, \Re(\mu)>0$ and $|z|<1$ is called generalized Mitteg- leffer function introduced by Prabhakar [19] and studied by Killbas. et. al. [1] and [3].

**(viii)** For $\xi=\mu=1$ and $\tau=\delta$, we obtain

$${}_pF_q\left[\begin{array}{c} a_1,...,a_p \\ b_1,...,b_q \end{array} \mid z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k\Gamma(1+k)}z^k, \tag{3.8}$$

where $p\leq q+1$ and $|z|<1$ and ${}_pF_q(.)$ is known as generalized hypergeometric function [3].

**(ix)** H-Function [2, 4, 8]: $\omega-M$ series can be represented as a special case of the Fox H-function

$${}_{p+2}\overset{\omega}{M}_{q+2}\left[\begin{array}{c} a_1...a_p,(1,1),(\tau,\omega) \\ b_1...b_q,(\delta,\omega),(\xi,\mu) \end{array} \mid z\right]$$

$$=kH_{p+2,q+2}^{1,n+2}\left[\begin{array}{c} (1-a_1,1),...,(1-a_p,1),(0,1),(1-\tau,\omega) \\ (1-b_1,1),...,(1-b_q,1),(0,1),(1-\delta,\omega),(1-\xi,\mu) \end{array} \mid (-z)\right], \tag{3.9}$$

where $k=\frac{\prod_{j=1}^{q}\Gamma(b_j)_r}{\Gamma_{j=1}^{p}\Pi(a_j)_r}$.

# 4 Left-Side Generalized Fractional Integration and Differentiation of Extended M-Series

**Theorem 4.1.** *Let $\alpha,\alpha^{'},\beta,\beta^{'},\gamma\in C$ be a complex number such that $\Re(\gamma)>0$ and let $\rho,\delta,\xi,\tau,\mu\in C, \Re(\rho)>0, p\leq q+1$ and $|x|<1$. If the condition*

$$\Re(\rho)>\max\left[0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\right]$$

*is satisfied then*

$$\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)=x^{\rho+\gamma-\alpha-\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\gamma-\alpha-\alpha',1), \end{array}\right.$$

$$\left.\begin{array}{c} (\rho+\gamma-\alpha-\alpha'-\beta,1),(\rho+\beta'-\alpha',1) \\ (\rho+\gamma-\alpha'-\beta,1),(\rho+\beta',1) \end{array};x\right). \tag{4.1}$$

*Proof.* From the equations (1.15) and (2.1), we have

$$\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)k!}\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}(t^{\rho+k-1})\right](x). \tag{4.2}$$

Now using equation (1.27), we obtained

$$=x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\rho+k)}{(b_1)_k...(b_q)_k\Gamma(\rho+\gamma-\alpha-\alpha'+k)}$$

$$\times\frac{\Gamma(\rho+\gamma-\alpha-\alpha'-\beta+k)\Gamma(\rho+\beta'-\alpha'+k)\Gamma(\tau+\omega k)}{\Gamma(\rho+\gamma-\alpha'-\beta+k)\Gamma(\rho+\beta'+k)\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{x^k}{k!}, \tag{4.3}$$

which is the required result. □

**Corollary 4.1.** *Let $\alpha, \beta, \eta, \rho \in C$ be such that condition (1.29) is satisfied, and further let $\delta, \xi, \tau, \mu \in C, \Re(\rho) > 0$ and $|x| < 1$. Then by relation (1.23) and (1.30) there hold the formula*

$$\left[I_{0+}^{\alpha,\beta,\eta}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\rho-\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1),(\rho-\beta+\eta,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-\beta,1),(\rho+\alpha+\eta,1)\end{array};x\right). \tag{4.4}$$

**Corollary 4.2.** *Let $\alpha, \rho \in C$ be such that $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Further let $\delta, \xi, \tau, \mu \in C$, and $|x| < 1$ then the relation (1.7) indicates that equation (4.4) reduces to the following result*

$$\left[R_{0,x}^{\alpha}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\rho+\alpha-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\alpha,1)\end{array};x\right). \tag{4.5}$$

**Corollary 4.3.** *Let $\alpha, \eta, \rho \in C$ be such that $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Further let $\delta, \xi, \tau, \mu \in C$, and $|x| < 1$ then the relation (1.13) indicates that equation (4.4) reduces to the following result*

$$\left[E_{0,x}^{\alpha,\eta}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\rho-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho+\eta,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\alpha+\eta,1)\end{array};x\right). \tag{4.6}$$

**Theorem 4.2.** *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$ be a complex number such that $\Re(\gamma) > 0$ and let $\rho, \delta, \xi, \tau, \mu \in C$, $\Re(\rho) > 0, p \leq q+1$ and $|x| < 1$. If the condition*

$$\Re(\rho) > \max\left[0, \Re(\gamma-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\right]$$

*is satisfied then*

$$\left[D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x) = x^{\rho-\gamma+\alpha+\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{p+5}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-\gamma+\alpha+\alpha',1),\end{array}\right.$$

$$\left.\begin{array}{c}(\rho-\gamma+\alpha+\alpha'+\beta',1),(\rho-\beta+\alpha,1) \\ (\rho-\gamma+\alpha+\beta',1),(\rho-\beta,1)\end{array};x\right). \tag{4.7}$$

*Proof.* By using equations (1.20) and (2.1), we have

$$\left[D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= \left(\frac{d}{dx}\right)^m \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)k!}$$

$$\times\left[I_{0+}^{-\alpha',-\alpha,-\beta'+m,-\beta,-\gamma+m}(t^{\rho+k-1})\right](x). \tag{4.8}$$

Now using equation (1.27), we obtained

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{1}{k!}$$

$$\times\frac{\Gamma(\rho+k)\Gamma(\rho+k-\gamma+\alpha+\alpha'+\beta')\Gamma(\rho+k-\beta+\alpha)}{\Gamma(\rho+k-\gamma+m+\alpha+\alpha')\Gamma(\rho+k-\gamma+\alpha+\beta')\Gamma(\rho+k-\beta)}$$

$$\times\left(\frac{d}{dx}\right)^m x^{\rho+k-\gamma+m+\alpha+\alpha'-1}.$$

Using the formula $\frac{d^m x^n}{dx^m} = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m}, n \geq m$, we have

$$= x^{\rho-\gamma+\alpha+\alpha'-1} \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k (1)_k \Gamma(\rho+k)}{(b_1)_k...(b_q)_k \Gamma(\rho-\gamma+\alpha+\alpha'+k)}$$

.

$$\times \frac{\Gamma(\rho-\gamma+\alpha+\alpha'+\beta'+k)\Gamma(\rho-\beta+\alpha+k)\Gamma(\tau+\omega k)}{\Gamma(\rho-\gamma+\alpha+\beta'+k)\Gamma(\rho-\beta+k)\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)} \frac{x^k}{k!}. \tag{4.9}$$

Which is the required result. □

If we set $\alpha = 0$ in (4.7) we arrive at

**Corollary 4.4.** *Let $\alpha, \beta, \eta, \rho \in C$ be such that $\Re(\alpha) \geq 0$,*

$$\Re(\rho) > -\min\left[0, \Re(\alpha+\beta+\eta)\right],$$

*and further let $\delta, \xi, \tau, \mu \in C, \Re(\rho) > 0$. Then by the relation (1.25) there hold the formula*

$$\left[D_{0+}^{\alpha,\beta,\eta}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\rho+\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1),(\rho+\alpha+\beta+\eta,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\beta,1),(\rho+\eta,1) \end{array};x\right). \tag{4.10}$$

**Corollary 4.5.** *Let $\alpha, \rho \in C$ be such $\Re(\alpha) \geq 0$, and further let $\delta, \xi, \tau, \mu \in C$, $\Re(\rho) > 0$. Then by the relation (1.11) there hold the formula*

$$\left[D_{0+}^{\alpha}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\rho-\alpha-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-\alpha,1) \end{array};x\right). \tag{4.11}$$

# 5 Right -Side Generalized Fractional Integration and Differentiation of Extended M-Series

**Theorem 5.1.** *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$ be a complex number such that $\Re(\gamma) > 0$ and further let $\tau, \delta, \xi, \mu \in C, \Re(\rho) > 0, p \leq q+1$ and $|x| < 1$. If the condition*

$$\Re(\rho) < 1 + \min\left[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\right]$$

*is satisfied then*

$$\left[I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\tfrac{1}{t}\right)\right)\right](x) = x^{-\rho+\gamma-\alpha-\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho-\gamma+\alpha+\alpha',1), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\rho-\beta,1),(1+\rho-\gamma+\beta'+\alpha,1) \\ (1+\rho+\alpha-\beta,1),(1+\rho+\alpha+\alpha'+\beta'-\gamma,1) \end{array};\frac{1}{x}\right). \tag{5.1}$$

*Proof.* Proof of the theorem is similar to that of Theorem 1. □

**Corollary 5.1.** *Let $\alpha, \beta, \eta, \rho \in C$ be such that condition (1.31) is satisfied, and further let $\delta, \xi, \tau, \mu \in C, \Re(\rho) > 0$ and $|x| < 1$. Then by relation (1.24) and (1.32) there hold the formula*

$$\left[I_{-}^{\alpha,\beta,\eta}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho-\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\rho+\beta,1),(1+\rho+\eta,1) \\ (1+\rho,1),(1+\rho+\alpha+\beta+\eta,1) \end{array};\frac{1}{x}\right). \tag{5.2}$$

**Corollary 5.2.** *Let $\alpha, \rho \in C$ be such that $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Further let $\delta, \xi, \tau, \mu \in C$ and $|x| < 1$ then the relation (1.9) indicates that equation (5.2) reduces to the following result*

$$\left[W^{\alpha}_{x,\infty}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho+\alpha-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho-\alpha,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1) \end{array};\frac{1}{x}\right). \tag{5.3}$$

**Corollary 5.3.** *Let $\alpha, \eta, \rho \in C$ be such that $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Further let $\delta, \xi, \tau, \mu \in C$, and $|x| < 1$ then the relation (1.14) indicates that equation (5.2) reduces to the following result*

$$\left[K^{\alpha,\eta}_{x,\infty}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho+\eta,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho+\alpha+\eta,1) \end{array};\frac{1}{x}\right). \tag{5.4}$$

**Theorem 5.2.** *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$ be a complex number such that $\Re(\gamma) > 0$ and further let $\rho, \delta, \tau, \xi, \mu \in C, \Re(\rho) > 0, p \leq q+1$ and $|x| < 1$. If the condition*

$$\Re(\rho) < 1 + \min\left[\Re(\beta'), \Re(\gamma-\alpha-\alpha'-k), \Re(\gamma-\alpha'-\beta)\right]$$

*is satisfied then*

$$\left[D^{\alpha,\alpha',\beta,\beta',\gamma}_{-}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x) = x^{-\rho+\gamma-\alpha-\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\zeta,\omega),(1+\rho+\gamma-\alpha-\alpha',1), \\ b_1,...,b_q,(\delta,\omega),(\eta,\mu),(1+\rho,1), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\rho+\beta',1),(1+\rho+\gamma-\alpha'-\beta,1) \\ (1+\rho+\beta'-\alpha',1),(1+\rho-\alpha-\alpha'-\beta+\gamma,1) \end{array};\frac{1}{x}\right). \tag{5.5}$$

*Proof.* It is similar to the previous Theorem. □

**Corollary 5.4.** *Let $\alpha, \beta, \eta, \rho \in C$ be such $\Re(\alpha) \geq 0$,*

$$\Re(\rho) > -\min\left[\Re(-\beta-n), \Re(\alpha+\eta)\right],$$

*$n = [\Re(\alpha)] + 1$, and further let $\delta, \xi, \tau, \mu \in C, \Re(\rho) > 0$. Then by the relation (1.26) there hold the formula*

$$\left[D^{\alpha,\beta,\eta}_{-}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho+\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\rho-\beta,1),(1+\rho+\alpha+\eta,1) \\ (1+\rho,1),(1+\rho+\eta-\beta,1) \end{array};\frac{1}{x}\right). \tag{5.6}$$

**Corollary 5.5.** *Let $\alpha, \rho \in C$ be such $\Re(\alpha) \geq 0$ and further let $\delta, \xi, \tau, \mu \in C$, $\Re(\rho) > 0$. Then by the relation (1.12) there hold the formula*

$$\left[D^{\alpha}_{-}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho-\alpha-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho+\alpha,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1) \end{array};\frac{1}{x}\right). \tag{5.7}$$

## 6 Fractional Integro-Differentiation of Extended M Series

**Theorem 6.1.** *Let $\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma \in C$ be a complex number such that $\Re(\gamma) > 0$ and let $\rho, \delta, \xi, \tau, \mu \in C, \Re(\rho) > 0, p \leq q+1$ and $|x| < 1$. If the condition*

$$\Re(\rho) > \max\left[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\right]$$

*is satisfied then*

$$\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x) = x^{\rho+\gamma-\alpha-\alpha'-1}$$

$$\times {}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\gamma-\alpha-\alpha',1), \end{array}\right.$$

$$\left.\begin{array}{c} (\rho+\gamma-\alpha-\alpha'-\beta,1),(\rho+\beta'-\alpha',1) \\ (\rho+\gamma-\alpha'-\beta,1),(\rho+\beta',1) \end{array} ;x\right). \tag{6.1}$$

*Proof.* To prove (6.1) using equation (1.16) which represent integro-differentiation operator and applying the same reasoning similar to the Theorem 1. Therefore we omit detail. □

If we take $\alpha^{'} = 0$ (6.1), we arrive at

**Corollary 6.1.** *Let $\alpha, \beta, \eta, \rho \in C$ be such that condition (1.29) is satisfied, and further let $\delta, \xi, \tau, \mu \in C, \Re(\rho) > 0$ and $|x| < 1$. Then by relation (1.2) and (1.30) there hold the formula*

$$\left[I_{0+}^{\alpha,\beta,\eta}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\rho-\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1),(\rho-\beta+\eta,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-\beta,1),(\rho+\alpha+\eta,1) \end{array};x\right). \tag{6.2}$$

**Theorem 6.2.** *Let $\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma \in C$ be a complex number such that $\Re(\gamma) > 0$ and further let $\tau, \delta, \xi, \mu \in C, \Re(\rho) > 0, p \leq q+1$ and $|x| < 1$. If the condition*

$$\Re(\rho) < 1 + \min\left[\Re(-\beta), \Re(\alpha+\alpha^{'}-\gamma), \Re(\alpha+\beta^{'}-\gamma)\right]$$

*is satisfied then*

$$\left[I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x) = x^{-\rho+\gamma-\alpha-\alpha'-1}$$

$$\times {}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho-\gamma+\alpha+\alpha',1), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\rho-\beta,1),(1+\rho-\gamma+\beta'+\alpha,1) \\ (1+\rho+\alpha-\beta,1),(1+\rho+\alpha+\alpha'+\beta'-\gamma,1) \end{array};\frac{1}{x}\right). \tag{6.3}$$

*Proof.* In view of (1.18) and (2.1), we have

$$\left[I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= (-1)^n\frac{d^n}{dx^n}x^{-\alpha-\alpha'+\gamma+n-1}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\zeta+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\eta+\mu k)k!}$$

$$\times\left[I_{0+}^{\alpha,\alpha',\beta,\beta'+n,\gamma+n}\left(t^{1+\alpha+\alpha'-\gamma-n+\rho+k-1}\right)\right]\left(\frac{1}{x}\right). \tag{6.4}$$

With the help of equation (1.27) we arrive at

$$= \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k}{(b_1)_k...(b_q)_k}\frac{1}{k!}$$

$$\times\frac{\Gamma(1+\rho+k+\alpha+\alpha'-\gamma-n)\Gamma(1+\rho+k-\beta)\Gamma(1+\rho+k+\alpha+\beta'-\gamma)\Gamma(\zeta+\omega k)}{\Gamma(1+\rho+k)\Gamma(1+\rho+k+\alpha-\beta)\Gamma(1+\rho+\alpha+\alpha'-\gamma+\beta')\Gamma(\delta+\omega k)\Gamma(\eta+\mu k)}$$

$$\times\left(1+\rho+k+\alpha+\alpha^{'}-\gamma-n\right)_n x^{-\rho-k-\alpha-\alpha^{'}+\gamma+n-1}.$$

Finally using formula $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}, a\neq 0$, the above expression becomes

$$=x^{-\rho+\gamma-\alpha-\alpha'-1}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k}{(b_1)_k...(b_q)_k}\frac{\Gamma(1+\rho+k+\alpha+\alpha'-\gamma)}{\Gamma(1+\rho+k)\Gamma(1+\rho+k+\alpha-\beta)}$$

$$\times\frac{\Gamma(1+\rho+k-\beta)\Gamma(1+\rho+k+\alpha+\beta'-\gamma)\Gamma(\zeta+\omega k)}{\Gamma(1+\rho+\alpha+\alpha'-\gamma+\beta')\Gamma(\delta+\omega k)\Gamma(\eta+\mu k)}\frac{x^{-k}}{k!}, \tag{6.5}$$

which is the required result. □

If we take $\alpha'=0$ in (6.3), then the following result holds:

**Corollary 6.2.** *Let $\alpha,\beta,\eta,\rho\in C$ be such that condition (1.31) is satisfied, and further let $\delta,\xi,\tau,\mu\in C,\Re(\rho)>0$ and $|x|<1$. Then by relation (1.4) there hold the formula*

$$\left[I_{-}^{\alpha,\beta,\eta}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$=x^{-\rho-\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho+\beta,1),(1+\rho+\eta,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1),(1+\rho+\alpha+\beta+\eta,1)\end{array};\frac{1}{x}\right). \tag{6.6}$$

## 7 Usual Differentiation of the Extended M-Series

It is know that for the natural $\alpha=m\in N$, the Riemann- Liouville fractional derivative (1.11) is the usual derivative of order m, while (1.12) coincides with the usual derivative of order m with exactness to the multiplier $(-1)^m$ for example see [18, section 2 and 5]:

$$\left(D_{0+}^m f\right)(x)=\left(\frac{d}{dx}\right)^m f(x),$$

$$\left(D_{-}^m f\right)(x)=(-1)^m\left(\frac{d}{dx}\right)^m f(x)\,(x>0);$$

There hold the following result.

**Theorem 7.1.** *Let $m\in N$ and let $\delta,\xi,\tau,\mu\in C,\rho>0$. Then for $z\in C(z\neq 0)$ there hold the formula*

$$\left(\frac{d}{dx}\right)^m\left(z^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(z)\right)$$

$$=z^{\rho-m-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-m,1)\end{array};z\right), \tag{7.1}$$

*and*

$$\left(\frac{d}{dx}\right)^m\left(z^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{z}\right)\right)$$

$$=(-1)^m z^{-\rho-m-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho+m,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1)\end{array};\frac{1}{z}\right). \tag{7.2}$$

*Proof.* With the help of corollaries 4.5 and 5.5 we deduce the differentiation formulas for the extended M-series (2.1). Therefore these relations can be extended from $x>0$ to any complex $z\in C$, expect $z=0$, and the condition for their validity can be omitted. □

# 8 Pathway Fractional Integration of Extended M-Series

The Pathway model is introduced by Mathai [5] and studied further by Mathai and Haubold [6], [7] and Seema S. Nair [17]. Let $f(x) \in L(a,b), \eta \in C, \Re(\eta) > 0, a > 0$ and let us take a pathway parameter $\alpha < 1$. Then the pathway fractional integration operator, as an extension of (1.7) is defined as follows:

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a\,(1-\alpha)\,t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)dt. \tag{8.1}$$

**Theorem 8.1.** *Let f(x)$\in L(a,b), \eta, \rho \in C, \Re(\eta) > 0, \Re(\rho) > 0, a > 0$ and pathway parameter $\alpha < 1$. Further let $\tau, \delta, \xi, \mu \in C, p \leq q+1$. Then for the pathway fractional integral $P_{0,+}^{(\eta,\alpha)}$ the following formula holds for the image of extended M series*

$$\left[P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= \frac{x^{\rho+\eta}}{(a\,(1-\alpha))^{\rho}}\,{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{matrix} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\frac{\eta}{(1-\alpha)}+1,1)\end{matrix}; \frac{x}{a\,(1-\alpha)}\right) \tag{8.2}$$

*Proof.* From Equation (8.1) and (2.1) we have

$$\left[P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\eta} \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a\,(1-\alpha)\,t}{x}\right]^{\frac{\eta}{(1-\alpha)}}$$

$$\times \left(\sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{t^{\rho+k-1}}{k!}\right)dt.$$

Interchanging the order of integration and summation which is permissible under the condition and which is stated with the above theorem

$$= x^{\eta}\sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{1}{k!}\int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a\,(1-\alpha)\,t}{x}\right]^{\frac{\eta}{(1-\alpha)}} t^{\rho+k-1}dt.$$

If we substitute $\frac{a((1-\alpha)t)}{x} = u$ in the above integral, and using Type-1 beta family i.e. B(m, n), it reduced to

$$\left[P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= \frac{x^{\rho+\eta}}{(a\,(1-\alpha))^{\rho}}\sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)\Gamma(\rho+k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)\Gamma\left(\rho+\frac{\eta}{(1-\alpha)}+1+k\right)}$$

$$\times \frac{x^k}{(a\,(1-\alpha))^k}\frac{1}{k!}, \tag{8.3}$$

which is the required result. □

**Remark 8.1.** *When $\alpha = 0, a = 1$, then replacing $\eta$ by $\eta - 1$ in (8.3) the integral operator get the form of equation (4.5).*

**Acknowledgment**

The author is very thankful to the referee for his/her valuable comments and suggestion.

# References


[1] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, North Holland Math. Studies, 204, Amsterdam etc, 2006.

[2] A. M. Mathai and R. K. Saxena, *The H-function with Applications in Statistics and other Disciplines*, Halsted Press, New York-London-Sydney-Toronto, 1978.

[3] A. M. Mathai and H. J. Habould, *Special Functions for Applied Scientists*, Springer, New York, 2008.

[4] A. M. Mathai, R. K. Saxena and H. J. Haubold, *The H-function: Theory and Applications*, Springer, New York, 2010.

[5] A. M. Mathai, A pathway to matrix-variate gamma and normal densities, *Linear Algebra and Its Applications*, 396(2005), 317-328.

[6] A. M. Mathai and H. J. Haubold, On generalized distributions and path-ways, *Physics Letter*, 372(2008), 2109-2113.

[7] A. M. Mathai and H. J. Haubold, Pathway model, superstatistics, Tsallis statistics and a generalize measure of entropy, *Physica A*, 375(2007), 110-122.

[8] C. Fox, The G and H-functions as symmetrical Fourier Kernels, *Trans. Amer. Math. Soc.,* 98(1961), 395-429.

[9] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, *J. London Math. Soc.,* 10(1935), 287-293.

[10] M. Saigo, A Remark on integral operators involving the Gauss Hypergeometric functions, *Math. Rep. College General Ed. Kyushu Univ.,* 11(1978), 135-143.

[11] M. Saigo and N. Meada, More generalization of fractional calculus, In: Transform Methods and Special Functions (*Eds: P. Ruse v, I. Dimovski, V. Kiryakova, Proc. 2nd Int. workshop, Varna, 1996*), Sofia, 1998, 386-400.

[12] M. Sharma and Renu Jain, A note on generalized M-Series as a special function of fractional calculus, *Fract. Calc. Appl. Anal.,* 12(4)(2009), 449-452.

[13] N. A. Virchenko, S. L. Kalla and A. Al-Zamel, Some results on a generalized hypergeometric function, Integral Transform Spec. Funct., 12(1)(2001), 89-100.

[14] N. A. Virchenko, On some generalizations of the functions of hypergeometric type, *Fract. Calc. Appl. Anal.,* 2(3)(1999), 233-244.

[15] O. I. Marichev, *Handbook of Integral Transforms and Higher Transcendental Functions. Theory and Algorithmic Tables*, Ellis Horwood, Chichester [John Wiley and Sons], New York, 1983.

[16] R. K. Saxena, A remark on a paper on M-series, *Fract. Calc. Appl. Anal.,* 12(1)(2009), 109-110.

[17] Seema S. Nair, Pathway fractional integration operator, *Fract. Calc. Appl. Anal.,* 12(3)(2009), 237-252.

[18] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, 1993.

[19] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math. J.,* 19(1971), 7-15.

[20] V. S. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics Series No. 301, Longman and J. Wiley, Harlow and New York, 1994.

[21] Y. Ben Nakhi and S. L. Kalla, A generalized beta function and associated probability density, *Int. J. Math. Math. Sci.,* 30(8)(2002), 467-478.

*Received*: October 16, 2012; *Accepted*: January 17, 2012

UNIVERSITY PRESS

Malaya Journal of Matematik 1(1)(2013) 70–85

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

# Existence and controllability results for damped second order impulsive neutral functional differential systems with state-dependent delay in Banach spaces

N.Y. Nadaf[a] and M. Mallika Arjunan[b,*]

[a] *Department of Mathematics, Anjuman Institute of Technology and Management, Bhatkal–581320, Karnataka, India.*

[b] *Department of Mathematics, C. B. M. College, Kovaipudur, Coimbatore–641 042, Tamil Nadu, India.*

**Abstract**

In this paper, we investigate the existence and controllability of mild solutions for a damped second order impulsive neutral functional differential equation with state-dependent delay in Banach spaces. The results are obtained by using Sadovskii's fixed point theorem combined with the theories of a strongly continuous cosine family of bounded linear operators. Finally, an example is provided to illustrate the main results.

*Keywords:* Damped second order differential equations, impulsive neutral differential equations, controllability, state-dependent delay, cosine function, mild solution, fixed point.

*2010 MSC:* 34A37, 93B05. 

## 1 Introduction

In this paper, we are interested to study the existence and controllability of mild solutions for a damped second order impulsive neutral functional differential equation with state-dependent delay in Banach spaces. First, we consider the following class of damped second order impulsive neutral functional differential equation with state-dependent delay in the form:

$$\frac{d}{dt}[x'(t) - g(t, x_t)] = Ax(t) + \mathcal{D}x'(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = \eta \in X, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, \dots, n, \tag{1.3}$$

$$\Delta x'(t_i) = J_i(x_{t_i}), \quad i = 1, 2, \dots, n, \tag{1.4}$$

where $A$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operator $(C(t))_{t\in\mathbb{R}}$ defined on a Banach space $X$; the function $x_s : (-\infty, 0] \to X$, $x_s(\theta) = x(s + \theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically; $\mathcal{D}$ is a bounded linear operator on a Banach space $X$; $0 < t_1 < \cdots < t_n < a$ are prefixed numbers; $f, g : I \times \mathcal{B} \to X$, $\rho : I \times \mathcal{B} \to (-\infty, a]$, $I_i(\cdot) : \mathcal{B} \to X$, $J_i(\cdot) : \mathcal{B} \to X$ are appropriate functions and the symbol $\Delta\xi(t)$ represents the jump of the function $\xi(\cdot)$ at $t$, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the total duration of the process, such changes can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. These process tend to be more suitably modeled by impulsive differential equations, which

*Corresponding author.

*E-mail addresses*: nadaf_nabisab@yahoo.com (N. Y. Nadaf) and arjunphd07@yahoo.co.in (M. Mallika Arjunan).

allow for discontinuities in the evolution of the state. For more details on this theory and on its applications, we refer to the monographs of Lakshmikantham et al. [1], Samoilenko and Perestyuk [2], Bainov and Simeonov [3], and the papers of [4, 5, 6, 7, 8, 9, 10, 11] and the references therein. Ordinary differential equations of first and second order with impulses have been treated in several works, see for instance [12, 13]. Abstract partial differential equations with impulses have been studied by Liu [9], Rogovchenko [10, 11], Chang et al. [4, 43], and Hernández et al. [27, 28].

In control theory, one of the most important qualitative properties of dynamical systems is controllability. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to final state, where the initial and final states may vary over the entire space. Many authors has been studied the controllability of nonlinear systems with and without impulses, see for instance [14, 15, 16, 17, 18, 19, 20]. In dynamical systems damping is another important issue; it may be mathematically modelled as a force synchronous with the velocity of the object but opposite in direction to it. Concerning first and second order differential equations with damped term we cite [21, 22, 23, 24, 25] among some works.

On the other hand, functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received much attention in the recent years. The reader is referred to [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42] and the references therein for some examples and applications. The literature related to second order impulsive differential system with state-dependent delay is very limited, and related to this matter we only cite [43, 44]. To the best of our knowledge, the study of the existence and controllability system described in the abstract form (1.1)-(1.4) is an untreated problem, and this fact is the main motivation of this paper.

This paper is organized as follows. In Section 2, we recall some notations, definitions and preliminary facts which will be used throughout this paper. In Section 3, we establish sufficient conditions for the existence of mild solutions for the problem (1.1)-(1.4) by using Sadovskii's fixed point theorem combined with the theories of a cosine family of bounded linear operators. In Section 4, we study controllability results for the problem (1.1)-(1.4). In Section 5, we present some examples to show the application of the results.

## 2 Preliminaries

In this section, we recall briefly some notations, definitions and lemmas needed to establish our main results.

Throughout this paper, $A$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on Banach space $(X, \|\cdot\|)$.

**Definition 2.1.** *A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space $X$ into itself is called a strongly continuous cosine family iff*

*(i) $C(s+t)+C(s-t)=2C(s)C(t)$ for all $s,t\in\mathbb{R}$,*

*(ii) $C(0)=I$;*

*(iii) $C(t)x$ is continuous in $t$ on $\mathbb{R}$ for each fixed $x\in X$.*

We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by $S(t)x=\int_0^t C(s)xds, x\in X, t\in\mathbb{R}$ and we always assume that $N$ and $\overline{N}$ are positive constants such that $\|C(t)\|\leq N$ and $\|S(t)\|\leq\overline{N}$, for every $t\in I$. The infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ is the operator $A: X\to X$ defined by

$$Ax=\frac{d^2}{dt^2}C(t)x|_{t=0}, \quad x\in D(A),$$

where $D(A)=\{x\in X: C(t)x \text{ is twice differentiable in } t\}$. Define $E=\{x\in X: C(t)x \text{ is once continuously differentiable in } t\}$.

The following properties are well known [45]:

(i) If $x\in X$ then $S(t)x\in E$ for every $t\in\mathbb{R}$.

(ii) If $x\in E$ then $S(t)x\in D(A)$, $(\frac{d}{dt})C(t)x=AS(t)x$ and $(\frac{d^2}{dt^2})S(t)x=S(t)x$ for every $t\in\mathbb{R}$.

(iii) If $x \in D(A)$ then $C(t)x \in D(A)$, and $(\frac{d^2}{dt^2})C(t)x = AC(t)x = C(t)Ax$ for every $t \in \mathbb{R}$.

(iv) If $x \in D(A)$ then $S(t)x \in D(A)$, and $(\frac{d^2}{dt^2})S(t)x = AS(t)x = S(t)Ax$ for every $t \in \mathbb{R}$.

In this paper, $[D(A)]$ stands for the domain of the operator $A$ endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$. Moreover, in this work, $E$ is the space formed by the vectors $x \in X$ for which $C(\cdot)x$ is of class $C^1$ on $\mathbb{R}$. It was proved by Kisinsky [46] that $E$ endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0 \le t \le 1} \|AS(t)x\|, \quad x \in E, \tag{2.1}$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows from this that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0, \ t \to 0,$ for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is a locally integrable function, then $z(t) = \int_0^t S(t-s)x(s)ds$ defines an $E$-valued continuous function. This is a consequence of the fact that

$$\int_0^t G(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\, ds, & \int_0^t C(t-s)x(s)\, ds \end{bmatrix}^T$$

defines an $E \times X$-valued continuous function.

The existence of solutions for the second order abstract Cauchy problem

$$x''(t) = Ax(t) \ + \ h(t), \ \ 0 \le t \le a, \tag{2.2}$$

$$x(0) = z, \ \ x'(0) = w, \tag{2.3}$$

where $h : I \to X$ is an integrable function has been discussed in [45]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [47]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)z + S(t)w + \int_0^t S(t-s)h(s)ds, \quad 0 \le t \le a, \tag{2.4}$$

is called mild solution of (2.2)-(2.3), and that when $z \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)z + C(t)w + \int_0^t C(t-s)h(s)ds, \quad 0 \le t \le a. \tag{2.5}$$

For additional details about cosine function theory, we refer to the reader to [45, 47].

To consider the impulsive conditions (1.3)-(1.4), it is convenient to introduce some additional concepts and notations.

A function $u : [\sigma, \tau] \to X$ is said to be a normalized piecewise continuous function on $[\sigma, \tau]$ if $u$ is piecewise continuous and left continuous on $(\sigma, \tau]$. We denote by $\mathcal{PC}([\sigma, \tau], X)$ the space of normalized piecewise continuous functions from $[\sigma, \tau]$ into $X$. In particular, we introduce the space $\mathcal{PC}$ formed by all normalized piecewise continuous functions $u : [0, a] \to X$ such that $u$ is continuous at $t \neq t_i$, $i = 1, \ldots, n$. It is clear that $\mathcal{PC}$ endowed with the norm $\| u \|_{\mathcal{PC}} = \sup_{s \in I} \| u(s) \|$ is a Banach space.

In what follows, we set $t_0 = 0$, $t_{n+1} = a$, and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_i$, for $i = 0, 1, ..., n-1$, the function $\tilde{u}_i \in C([t_i, t_{i+1}]; X)$ given by $\widetilde{u}_i(t) = u(t)$ for $t \in (t_i, t_{i+1}]$ and $\widetilde{u}_i(t_i) = \lim_{t \to t_i^+} u(t)$. Moreover, for a set $B \subseteq \mathcal{PC}$, we denote by $\widetilde{B}_i$, for $i = 0, 1, ..., n-1$, the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$.

**Lemma 2.1.** *[48] A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, each set $\widetilde{B}_i$, $i = 0, 1, \cdots, n-1$, is relatively compact in $C([t_i, t_{i+1}], X)$.*

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$, which has been used in [48] and suitably modified to treat retarded impulsive differential equations. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into $X$ endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$ and we will assume that $\mathcal{B}$ satisfies the following axioms:

**(A)** If $x : (-\infty, \sigma + b] \to X$, $b > 0$, is such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b], X)$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:

(i) $x_t$ is in $\mathcal{B}$,

(ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$,

(iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K$ is continuous, $M$ is locally bounded, and $H, K, M$ are independent of $x(\cdot)$.

**(B)** The space $\mathcal{B}$ is complete.

For more details about phase space axioms and examples, we refer the reader to [40].

Additional terminologies and notations used in the sequel are standard in functional analysis. In particular, for Banach spaces $(Z, \| \cdot \|_Z)$, $(W, \| \cdot \|_W)$, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from $Z$ into $W$ and we abbreviate to $\mathcal{L}(Z)$ whenever $Z = W$. Additionally, $B_r(x, Z)$ denotes the closed ball with center at $x$ and radius $r > 0$ in $Z$.

Our main results are based upon the following well-known result.

**Lemma 2.2.** *[49, Sadovskii's Fixed Point Theorem] Let $G$ be a condensing operator on a Banach space $X$. If $G(S) \subset S$ for a convex, closed and bounded set $S$ of $X$, then $G$ has a fixed point in $S$.*

## 3 Existence Results

In this section we discuss the existence of mild solutions for the abstract system (1.1)-(1.4). We also suppose that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is a continuous function. Additionally, we introduce following conditions.

$(H_\varphi)$ Let $\mathcal{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in I \times \mathcal{B},\ \rho(s, \psi) \leq 0\}$. The function $t \to \varphi_t$ is well defined from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\| \varphi_t \|_{\mathcal{B}} \leq J^\varphi(t) \| \varphi \|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

$(H_1)$ The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions:

(i) Let $x : (-\infty, a] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. The function $t \to f(t, x_{\rho(t,x_t)})$ is measurable on $I$ and the function $t \to f(s, x_t)$ is continuous on $\mathcal{R}(\rho^-) \cup I$ for every $s \in I$.

(ii) For each $t \in I$, the function $f(t, \cdot) : \mathcal{B} \to X$ is continuous.

(iii) There exist an integrable function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that for every $(t, \psi) \in I \times \mathcal{B}$

$$\| f(t, \psi) \| \quad \leq \quad m(t) W(\| \psi \|_{\mathcal{B}}),\ \liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} = \Lambda < \infty.$$

$(H_2)$ The function $g : I \times \mathcal{B} \to X$ is continuous and there exists $L_g > 0$ such that

$$\|g(t, \psi_1) - g(t, \psi_2)\| \leq L_g \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad (t, \psi_i) \in I \times \mathcal{B},\ i = 1, 2.$$

$(H_3)$ There exist positive constants $c_1, c_2$ such that $\|g(t, \psi)\| \leq c_1 \|\psi\|_{\mathcal{B}} + c_2$, for every $(t, \psi) \in I \times \mathcal{B}$.

$(H_4)$ There are positive constants $L_{I_i}, L_{J_i}$ such that

$$\|I_i(\psi_1) - I_i(\psi_2)\| \leq L_{I_i} \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad \psi_j \in \mathcal{B}, \quad i = 1, 2, \ldots, n,$$
$$\|J_i(\psi_1) - J_i(\psi_2)\| \leq L_{J_i} \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad \psi_j \in \mathcal{B}, \quad i = 1, 2, \ldots, n.$$

$(H_5)$ The maps $I_i, J_i : \mathcal{B} \to X$, $i = 1, 2, \ldots, n$ are completely continuous and there exist continuous nondecreasing functions $\lambda_i, \mu_i : [0, \infty) \to (0, \infty)$, $i = 1, 2, \ldots, n$, such that

$$\|I_i(\psi)\| \leq \lambda_i(\|\psi\|_{\mathcal{B}}), \quad \liminf_{\zeta \to +\infty} \frac{\lambda_i(\zeta)}{\zeta} = \zeta_i < \infty, \quad \text{and}$$

$$\|J_i(\psi)\| \leq \mu_i(\|\psi\|_{\mathcal{B}}), \quad \liminf_{\zeta \to +\infty} \frac{\mu_i(\zeta)}{\zeta} = \eta_i < \infty.$$

**Remark 3.1.** The condition $H_\varphi$ is frequently satisfied by functions that are continuous and bounded. In fact, assume that the space of continuous and bounded functions $C_b((-\infty,0],X)$ is continuously included in $\mathcal{B}$. Then, there exists $L>0$ such that

$$\| \varphi_t \|_{\mathcal{B}} \leq L\frac{\sup_{\theta\leq 0}\| \varphi(\theta) \|}{\| \varphi \|_{\mathcal{B}}} \| \varphi \|_{\mathcal{B}}, \quad t\leq 0,\ \varphi\neq 0,\ \varphi\in C_b((-\infty,0]:X).$$

It is easy to see that the space $C_b((-\infty,0],X)$ is continuously included in $\mathcal{PC}_g(X)$ and $\mathcal{PC}_g^0(X)$. Moreover, if $g(\cdot)$ verifies (g-5)-(g-6) in [? ] and $g(\cdot)$ is integrable on $(-\infty,-r]$, then the space $C_b((-\infty,0],X)$ is also continuously included in $\mathcal{PC}_r\times L^p(g;X)$. For complementary details related this matter, see Proposition 7.1.1 and Theorems 1.3.2 and 1.3.8 in [50].

If $x(\cdot)$ is a solution of (1.1)-(1.4), then from (2.4), we adopt the following concept of mild solution,

$$x(t) = C(t)\varphi(0)+S(t)[\eta-g(0,\varphi)]+\int_0^t C(t-s)g(s,x_s)ds+\int_0^t S(t-s)[\mathcal{D}x'(s)+f(s,x_{\rho(s,x_s)})]ds$$
$$+\sum_{0<t_i<t} C(t-t_i)I_i(x_{t_i})+\sum_{0<t_i<t} S(t-t_i)J_i(x_{t_i}),\quad t\in I.$$

Inspired from the above expression, we present the following definition.

**Definition 3.1.** *A function $x:(-\infty,a]\to X$ is called a mild solution of the abstract Cauchy problem (1.1)-(1.4) if $x_0=\varphi, x_{\rho(s,x_s)}\in\mathcal{B}$ for every $s\in I; x(\cdot)|_I\in\mathcal{PC}$ and*

$$x(t) = C(t)\varphi(0)+S(t)[\eta-g(0,\varphi)]+\int_0^t C(t-s)g(s,x_s)ds+\sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}x(t_{i+1}^-)-S(t-t_i)\mathcal{D}x(t_i^+)]$$
$$-S(t-t_j)\mathcal{D}x(t_j^+)+\int_0^c C(t-s)\mathcal{D}x(s)ds+\int_0^t S(t-s)f(s,x_{\rho(s,x_s)})ds$$
$$+\sum_{0<t_i<t} C(t-t_i)I_i(x_{t_i})+\sum_{0<t_i<t} S(t-t_i)J_i(x_{t_i}),\quad t\in I.$$

**Remark 3.2.** In the rest of this paper, $y:(-\infty,a]\to X$ is the function defined by $y(t)=\varphi(t)$ on $(-\infty,0]$ and $y(t)=C(t)\varphi(0)+S(t)\zeta$ for $t\in I$. In addition, $\| y \|_a$, $M_a$, $K_a$, and $J_0^\varphi$ are the constants defined by $\| y \|_a=\sup\limits_{s\in[0,a]}\| y(s) \|$, $M_a=\sup\limits_{s\in[0,a]}M(s)$, $K_a=\sup\limits_{s\in[0,a]}K(s)$ and $J_0^\varphi=\sup\limits_{t\in\mathcal{R}(\rho^-)}J^\varphi(t)$.

**Lemma 3.1.** *[51, Lemma 2.1] Let $x:(-\infty,a]\to X$ be a function such that $x_0=\varphi$ and $x|_I\in\mathcal{PC}$. Then*

$$\| x_s \|_{\mathcal{B}} \leq (M_a+J_0^\varphi)\| \varphi \|_{\mathcal{B}}+K_a\sup\{\| x(\theta) \|;\theta\in[0,\max\{0,s\}]\},\quad s\in\mathcal{R}(\rho^-)\cup I.$$

**Theorem 3.1.** *Let conditions $(H_\varphi),(H_1)-(H_4)$ be hold and assume that $S(t)$ is compact for every $t\in\mathbb{R}$. If*

$$K_a\Big[aNL_g+\frac{1}{K_a}(3\overline{N}+aN)\|\mathcal{D}\|+\overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds+\sum_{i=1}^n(NL_{I_i}+\overline{N}L_{J_i})\Big]<1,$$

*then the problem (1.1)-(1.4) has at least one mild solution on $(-\infty,a]$.*

*Proof.* On the space $Y=\{x\in\mathcal{PC}: u(0)=\varphi(0)\}$ endowed with the uniform convergence topology, we define the operator $\Gamma:Y\to Y$ by

$$\begin{aligned}\Gamma x(t) &= C(t)\varphi(0)+S(t)[\eta-g(0,\varphi)]+\int_0^t C(t-s)g(s,\bar{x}_s)ds+\sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-)-S(t-t_i)\mathcal{D}\bar{x}(t_i^+)]\\ &\quad -S(t-t_j)\mathcal{D}\bar{x}(t_j^+)+\int_0^c C(t-s)\mathcal{D}\bar{x}(s)ds+\int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds+\sum_{0<t_i<t}C(t-t_i)I_i(\bar{x}_{t_i})\\ &\quad +\sum_{0<t_i<t}S(t-t_i)J_i(\bar{x}_{t_i}),\quad t\in I,\end{aligned}$$

where $\bar{x}:(-\infty,a]\to X$ is such that $\bar{x}_0=\varphi$ and $\bar{x}=x$ on $I$. From the axiom $(A)$ and our assumptions on $\varphi$, we infer that $\Gamma x\in\mathcal{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(y_{|I}, Y)) \subseteq B_r(y_{|I}, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(y_{|I}, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - y(t^r)\|$. Then, from Lemma 3.1, we get

$$
\begin{aligned}
r &< \|\Gamma x^r(t^r) - y(t^r)\| \\
&\leq NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
&\quad + 3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds \\
&\quad + \sum_{i=1}^{n} N(L_{I_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|I_i(y_{t_i})\|) + \sum_{i=1}^{n}\overline{N}(L_{J_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|J_i(y_{t_i})\|) \\
&\leq NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a r + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds + 3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r \\
&\quad + \overline{N}W((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a r + K_a \parallel y \parallel_a)\int_0^a m(s)ds \\
&\quad + \sum_{i=1}^{n} N(L_{I_i}K_a r + \|I_i(y_{t_i})\|) + \sum_{i=1}^{n}\overline{N}(L_{J_i}K_a r + \|J_i(y_{t_i})\|),
\end{aligned}
$$

and hence

$$
1 \leq K_a\Big[aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^{n}(NL_{I_i} + \overline{N}L_{J_i})\Big],
$$

which is contrary to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(y_{|I}, Y)) \subset B_r(y_{|I}, Y)$. In order to prove that $\Gamma$ is a condensing map on $B_r(y_{|I}, Y)$ into $B_r(y_{|I}, Y)$. We introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$
\begin{aligned}
\Gamma_1 x(t) &= C(t)\varphi(0) + S(t)[\eta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \sum_{i=0}^{j-1}[S(t - t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(t - t_i)\mathcal{D}\bar{x}(t_i^+)] \\
&\quad - S(t - t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^a C(t-s)\mathcal{D}\bar{x}(s)ds + \sum_{0<t_i<t} C(t - t_i)I_i(\bar{x}_{t_i}) + \sum_{0<t_i<t} S(t - t_i)J_i(\bar{x}_{t_i}). \\
\Gamma_2 x(t) &= \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds.
\end{aligned}
$$

From the proof of [39, Theorem 3.4], we conclude that $\Gamma_2$ is completely continuous. Moreover, from the estimate

$$
\begin{aligned}
\|\Gamma_1 x - \Gamma_1 z\|_{\mathcal{PC}} &\leq aNL_gK_a\|x - z\|_{\mathcal{PC}} + 3\overline{N}\|\mathcal{D}\|\|x - z\|_{\mathcal{PC}} + aN\|\mathcal{D}\|\|x - z\|_{\mathcal{PC}} \\
&\quad + K_a\sum_{i=1}^{n}(NL_{I_i} + \overline{N}L_{J_i})\|x - z\|_{\mathcal{PC}} \\
&\leq K_a[aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \sum_{i=1}^{n}(NL_{I_i} + \overline{N}L_{J_i})]\|x - z\|_{\mathcal{PC}}.
\end{aligned}
$$

It follows that $\Gamma_1$ is contraction on $B_r(y_{|I}, Y)$, which implies that $\Gamma$ is a condensing operator on $B_r(y_{|I}, Y)$.

Finally, from Lemma 2.2, we infer that there exists a mild solution of (1.1)-(1.4). The completes the proof. □

**Theorem 3.2.** *Let conditions $(H_\varphi), (H_1) - (H3)$ and $(H5)$ be hold and assume that $S(t)$ is compact for every $t \in \mathbb{R}$. If*

$$
K_a\Big[aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^{n}(N\zeta_i + \overline{N}\eta_i)\Big] < 1,
$$

*then there exists a mild solution of (1.1)-(1.4).*

*Proof.* On the space $Y = \{x \in \mathcal{PC} : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology, we define the operator $\Gamma : Y \to Y$ by

$$\begin{aligned}\Gamma x(t) &= C(t)\varphi(0) + S(t)[\eta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] \\ &\quad -S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^c C(t-s)\mathcal{D}\bar{x}(s)ds + \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) \\ &\quad + \sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t \in I,\end{aligned}$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on $I$. From axiom $(A)$ and our assumptions on $\varphi$, we infer that $\Gamma x \in \mathcal{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(y_{|I}, Y)) \subseteq B_r(y_{|I}, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(y_{|I}, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - y(t^r)\|$. Then, from Lemma 3.1 we get

$$\begin{aligned} r &< \|\Gamma x^r(t^r) - y(t^r)\| \\ &\leq NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\ &\quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + N\sum_{i=1}^n\|I_i(\bar{x}_{t_i})\| + \overline{N}\sum_{i=1}^n\|J_i(\bar{x}_{t_i})\| \\ &\leq NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\ &\quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}W((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_ar + K_a \parallel y \parallel_a)\int_0^a m(s)ds \\ &\quad +N\sum_{i=1}^n\lambda_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}) + \overline{N}\sum_{i=1}^n\mu_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}).\end{aligned}$$

Since $\lambda_i$ and $\mu_i$ are nondecreasing operators, we have

$$\begin{aligned} r &< NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\ &\quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}W((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_ar + K_a \parallel y \parallel_a)\int_0^a m(s)ds \\ &\quad +N\sum_{i=1}^n\lambda_i(r^*) + \overline{N}\sum_{i=1}^n\mu_i(r^*),\end{aligned}$$

where $\|\bar{x}_{t_i}\|_{\mathcal{B}} \leq r^* = (M_a + J_0^{\phi})\|\varphi\|_{\mathcal{B}} + K_a(r + \|y\|_a)$
and hence

$$1 \leq K_a\Big[aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^n(N\zeta_i + \overline{N}\eta_i)\Big],$$

which contradicts to our assumption.

Arguing as in the proof of Theorem 3.1, we can prove that $\Gamma(\cdot)$ is a condensing map on $B_r(y|_I, Y)$ and, from Lemma 2.2, we conclude that there exists a mild solution $x(\cdot)$ for (1.1)-(1.4). The proof is now complete. □

## 4 Controllability results

In this section, we shall establish sufficient conditions for the controllability of mild solutions for a damped second order impulsive neutral functional differential equation with state-dependent delay. More precisely, we consider the following abstract control system in the form:

$$\frac{d}{dt}[x'(t) - g(t,x_t)] = Ax(t) + \mathcal{D}x'(t) + Bu(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0,a], \tag{4.1}$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = \eta \in X, \tag{4.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, \ldots, n, \tag{4.3}$$

$$\Delta x'(t_i) = J_i(x_{t_i}), \quad i = 1, 2, \ldots, n, \tag{4.4}$$

where $A, \mathcal{D}, f, I_i$ and $J_i$ are defined as in equations (1.1)-(1.4), the control function $u(\cdot)$ given in $L^2(I, U)$, a Banach space of admissible control functions with $U$ as a Banach space and $B : U \to X$ is a bounded linear operator on a Banach space X with $D(\mathcal{D}) \subset D(A)$.

Furthermore, we assume the following conditions:

$(H_1)'$ The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions:

(i) The function $f : I \times \mathcal{B} \to X$ is completely continuous.

(ii) For every positive constant $r$, there exists an $\alpha_r \in L^1(r)$ such that

$$\sup_{\|\psi\| \le r} \|f(t, \psi)\| \le \alpha_r(t).$$

$(H_6)$ $B$ is continuous operator from $U$ to $X$ and the linear operator $W : L^2(I, U) \to X$, defined by

$$Wu = \int_0^a S(a - s)Bu(s)ds,$$

has a bounded invertible operator, $W^{-1}$ which takes the values in $L^2(I, U)/KerW$ such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$ for some positive integers $M_1, M_2$.

**Definition 4.1.** *The system (4.1)-(4.4) is said to be controllable on the interval* $[0, a]$ *if for every* $x_0 = \varphi \in \mathcal{B}, x'(0) = \eta \in X$ *and* $x_1 \in X$*, there exists a control* $u \in L^2(J, U)$ *such that the mild solution* $x(t)$ *of (4.1)-(4.4) satisfies* $x(a) = x_1$.

**Definition 4.2.** *A functions* $x : (-\infty, a] \to X$ *is called a mild solution of the abstract Cauchy problem (4.1)-(4.4) if* $x_0 = \varphi, x_{\rho(s,x_s)} \in \mathcal{B}$ *for every* $s \in I; x(\cdot)|_I \in \mathcal{PC}$ *and*

$$\begin{aligned} x(t) = {} & C(t)\varphi(0) + S(t)[\eta - g(0, \varphi)] + \int_0^t C(t - s)g(s, x_s)ds + \sum_{i=0}^{j-1} [S(t - t_{i+1})\mathcal{D}x(t_{i+1}^-) - S(t - t_i)\mathcal{D}x(t_i^+)] \\ & - S(t - t_j)\mathcal{D}x(t_j^+) + \int_0^t C(t - s)\mathcal{D}x(s)ds + \int_0^t S(t - s)\big[Bu(s) + f(s, x_{\rho(s,x_s)})\big]ds \\ & + \sum_{0<t_i<t} C(t - t_i)I_i(x_{t_i}) + \sum_{0<t_i<t} S(t - t_i)J_i(x_{t_i}), \quad t \in I. \end{aligned}$$

**Theorem 4.1.** *Let conditions* $(H_\varphi), (H_1) - (H_6)$ *and* $(H_1)'$ *be hold. Then the system (4.1)-(4.4) is controllable on* $(-\infty, a]$ *provided that*

$$(1 + a\overline{N}M_1M_2)\Big[K_a\Big(aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi \to \infty} \frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^n (NL_{I_i} + \overline{N}L_{J_i})\Big)\Big] < 1,$$

*Proof.* Consider the space $Y = \{x \in \mathcal{PC}; u(0) = \varphi(0)\}$ endowed with the uniform convergence topology. Using the assumption $(H_6)$, for an arbitrary function $x(\cdot)$, we define the control

$$\begin{aligned} u(t) = W^{-1}\Big[ & x_1 - C(a)\varphi(0) - S(t)[\eta - g(0, \varphi)] - \int_0^a C(a - s)g(s, x_s)ds - \sum_{i=0}^{j-1}[S(a - t_{i+1})\mathcal{D}x(t_{i+1}^-) \\ & - S(a - t_i)\mathcal{D}x(t_i^+)] + S(a - t_j)\mathcal{D}x(t_j^+) - \int_0^a C(a - s)\mathcal{D}x(s)ds - \int_0^a S(a - s)f(s, x_{\rho(s,x_s)})ds \\ & - \sum_{0<t_i<a} C(a - t_i)I_i(x_{t_i}) - \sum_{0<t_i<a} S(a - t_i)J_i(x_{t_i})\Big](t). \end{aligned}$$

Using this control, we shall show that the operator $\Gamma : Y \to Y$ defined by

$$\Gamma x(t) \;=\; C(t)\varphi(0) + S(t)[\eta - g(0, \varphi)] + \int_0^t C(t - s)g(s, \bar{x}_s)ds + \sum_{i=0}^{j-1}[S(t - t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-)$$

$$-S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] - S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^t C(t-s)\mathcal{D}\bar{x}(s)ds + \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds$$
$$+\int_0^t S(t-\xi)BW^{-1}\Big[x_1 - C(a)\varphi(0) - S(a)[\eta - g(0,\varphi)] - \int_0^a C(a-s)g(s,\bar{x}_s)ds$$
$$-\sum_{i=0}^{j-1}[S(a-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(a-t_i)\mathcal{D}\bar{x}(t_i^+)] + S(a-t_j)\mathcal{D}\bar{x}(t_j^+) - \int_0^a C(a-s)\mathcal{D}\bar{x}(s)ds$$
$$-\sum_{0<t_i<a} C(a-t_i)I_i(\bar{x}_{t_i}) - \sum_{0<t_i<a} S(a-t_i)J_i(\bar{x}_{t_i})\Big](\xi)d\xi + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i})$$
$$+\sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t \in I,$$

has a fixed point $x(\cdot)$. This fixed point $x(\cdot)$ is then a mild solution of the system (4.1)-(4,4). Clearly, $(\Gamma x)(a) = x_1$, which means that the control $u$ steers the system from the initial state $\varphi$ to $x_1$ in time $a$, provided we obtain a fixed point of the operator which implies that the system is controllable. Here $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From the axiom (A) and our assumptions on $\varphi$, we infer that $\Gamma x \in \mathcal{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(y_{|I}, Y)) \subseteq B_r(y_{|I}, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(y_{|I}, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - y(t^r)\|$. Then, from Lemma 3.1, we get

$$\begin{aligned}
r &< \|\Gamma x^r(t^r) - y(t^r)\| \\
&\leq NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
&\quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + \overline{N}M_1M_2\int_0^{t^r}\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} \\
&\quad +\overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
&\quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + \sum_{i=1}^n N(L_{I_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|I_i(y_{t_i})\|) \\
&\quad +\sum_{i=1}^n \overline{N}(L_{J_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|J_i(y_{t_i})\|)\Big] + \sum_{i=1}^n N(L_{I_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|I_i(y_{t_i})\|) \\
&\quad +\sum_{i=1}^n \overline{N}(L_{J_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|J_i(y_{t_i})\|) \\
r &\leq NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
&\quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}W\Big((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_ar + K_a\|y\|_a\Big)\int_0^a m(s)ds \\
&\quad +\overline{N}M_1M_2\int_0^{t^r}\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds \\
&\quad +N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds + 3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds \\
&\quad +\sum_{i=1}^n N(L_{I_i}K_ar + \|I_i(y_{t_i})\|) + \sum_{i=1}^n \overline{N}(L_{J_i}K_ar + \|J_i(y_{t_i})\|)\Big] \\
&\quad +\sum_{i=1}^n N(L_{I_i}K_ar + \|I_i(y_{t_i})\|) + \sum_{i=1}^n \overline{N}(L_{J_i}K_ar + \|J_i(y_{t_i})\|),
\end{aligned}$$

and hence

$$1 \leq (1 + a\overline{N}M_1M_2)\Big[K_a\Big(aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^n(NL_{I_i} + \overline{N}L_{J_i})\Big)\Big],$$

which contradicts to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(y_{|I}, Y)) \subset B_r(y_{|I}, Y)$. In order to prove that $\Gamma$ is a condensing map on $B_r(y_{|I}, Y)$ into $B_r(y_{|I}, Y)$. We introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned}\Gamma_1 x(t) &= C(t)\varphi(0) + S(t)[\eta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(t-t_i)\mathcal{D}\bar{x}(t_i^+)]\\ &\quad -S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^a C(t-s)\mathcal{D}\bar{x}(s)ds + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) + \sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}).\\ \Gamma_2 x(t) &= \int_0^t S(t-s)\Big[f(s,\bar{x}_{\rho(s,\bar{x}_s)}) + Bu(s)\Big]ds.\end{aligned}$$

Now

$$\begin{aligned}\|Bu(s)\| &\le \|B\|\|W^{-1}\|\Big[\|x_1\| + \|C(a)\|\|\varphi(0)\| + \|S(t)\|\big[\|\eta\| + \|g(0,\varphi)\|\big] + \int_0^a \|C(a-s)\|\|g(s,\bar{x}_s)ds\|\\ &\quad + \sum_{i=0}^{j-1}[\|S(a-t_{i+1})\|\|\mathcal{D}\|\|\bar{x}(t_{i+1}^-)\| + \|S(a-t_i)\|\|\mathcal{D}\|\|\bar{x}(t_i^+)]\| + \|S(a-t_j)\|\|\mathcal{D}\|\|\bar{x}(t_j^+)\|\\ &\quad + \int_0^a \|C(a-s)\|\|\mathcal{D}\|\|\bar{x}(s)\|ds + \int_0^a \|S(a-s)\|\|f(s,\bar{x}_{\rho(s,\bar{x}_s)})\|ds + \sum_{0<t_i<a}\|C(a-t_i)\|\|I_i(\bar{x}_{t_i})\|\\ &\quad + \sum_{0<t_i<a}\|S(a-t_i)\|\|J_i(\bar{x}_{t_i})\|\Big]\\ &\le M_1M_2\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \overline{N}\big[\|\eta\| + c_1\|\varphi\| + c_2\big] + N\int_0^a (c_1\|\bar{x}_s\| + c_2)ds + 3\overline{N}\|D\|r + aN\|\mathcal{D}\|r\\ &\quad + \overline{N}\int_0^a \alpha_r(s)ds + N\sum_{i=1}^n \lambda_i\|\bar{x}_{t_i}\| + \sum_{i=1}^n \mu_i\|\bar{x}_{t_i}\|\Big]\\ &\le M_1M_2\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \overline{N}\big[\|\eta\| + c_1\|\varphi\| + c_2\big] + aN(c_1r + c_2) + 3\overline{N}\|D\|r + aN\|\mathcal{D}\|r\\ &\quad + \overline{N}\int_0^a \alpha_r(s)ds + \sum_{i=1}^n r(N\lambda_i + \overline{N}\mu_i)\Big] = A_0.\end{aligned}$$

Here by applying the same technique that is used in the proof of [16, Lemma 3.1], we arrived that $\Gamma_2$ is completely continuous.

Next, we show that $\Gamma_1$ is contraction on $B_r(y|_I, Y)$. Indeed, $x, z \in B_r(y|_I, Y)$, we have

$$\begin{aligned}\|\Gamma_1 x - \Gamma_1 z\|_{\mathcal{PC}} &\le aN\|\mathcal{D}\|\|x-z\|_{\mathcal{PC}} + aNL_gK_a\|x-z\|_{\mathcal{PC}} + 3\overline{N}\|\mathcal{D}\|\|x-z\|_{\mathcal{PC}} + \sum_{i=1}^n NL_{I_i}K_a\|x-z\|_{\mathcal{PC}}\\ &\quad + \sum_{i=1}^n \overline{N}L_{J_i}K_a\|x-z\|_{\mathcal{PC}}\\ &\le K_a\Big[aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \sum_{i=1}^n (NL_{I_i} + \overline{N}L_{J_i})\Big]\|x-z\|_{\mathcal{PC}}.\end{aligned}$$

It follows that $\Gamma_1$ is a contraction on $B_r(y|_I, Y)$ which implies that $\Gamma$ is a condensing operator on $B_r(y|_I, Y)$.

Finally, from the Sadovskii's Fixed Point Theorem, $\Gamma$ has a fixed point on $Y$. This means that any fixed point of $\Gamma$ is a mild solution of the problem (4.1)-(4.4). This completes the proof. □

**Theorem 4.2.** *Let conditions $(H_\varphi), (H_1) - (H_3), (H_5)$ and $(H_1)'$ be hold. Then the system (4.1)-(4.4) is controllable on $(-\infty, a]$ provided that*

$$(1 + a\overline{N}M_1M_2)\Big[K_a\Big(aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^n (N\zeta_i + \overline{N}\eta_i)\Big)\Big] < 1.$$

*Proof.* Consider the space $Y = \{x \in \mathcal{PC}; u(0) = \varphi(0)\}$ endowed with the uniform convergence topology. Using the assumption $(H_6)$, for an arbitrary function $x(\cdot)$, we define the control

$$u(t) = W^{-1}\Big[x_1 - C(a)\varphi(0) - S(t)[\eta - g(0,\varphi)] - \int_0^a C(a-s)g(s,x_s)ds - \sum_{i=0}^{j-1}[S(a-t_{i+1})\mathcal{D}x(t_{i+1}^-)$$

$$- S(a-t_i)\mathcal{D}x(t_i^+)] + S(a-t_j)\mathcal{D}x(t_j^+) - \int_0^a C(a-s)\mathcal{D}x(s)ds - \int_0^a S(a-s)f(s,x_{\rho(s,x_s)})ds$$
$$- \sum_{0<t_i<a} C(a-t_i)I_i(x_{t_i}) - \sum_{0<t_i<a} S(a-t_i)J_i(x_{t_i})\Big](t).$$

Using this control, we shall show that the operator $\Gamma : Y \to Y$ defined by

$$\begin{aligned}
\Gamma x(t) \;=\; & C(t)\varphi(0) + S(t)[\eta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) \\
& -S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] - S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^t C(t-s)\mathcal{D}\bar{x}(s)ds + \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds \\
& + \int_0^t S(t-\xi)BW^{-1}\Big[x_1 - C(a)\varphi(0) - S(a)[\eta - g(0,\varphi)] - \int_0^a C(a-s)g(s,\bar{x}_s)ds \\
& - \sum_{i=0}^{j-1}[S(a-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(a-t_i)\mathcal{D}\bar{x}(t_i^+)] + S(a-t_j)\mathcal{D}\bar{x}(t_j^+) - \int_0^a C(a-s)\mathcal{D}\bar{x}(s)ds \\
& - \sum_{0<t_i<a} C(a-t_i)I_i(\bar{x}_{t_i}) - \sum_{0<t_i<a} S(a-t_i)J_i(\bar{x}_{t_i})\Big](\xi)d\xi + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) \\
& + \sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t \in I,
\end{aligned}$$

has a fixed point $x(\cdot)$. This fixed point $x(\cdot)$ is then a mild solution of the system (4.1)-(4,4). Clearly, $(\Gamma x)(a) = x_1$, which means that the control $u$ steers the system from the initial state $\varphi$ to $x_1$ in time $a$, provided we obtain a fixed point of the operator which implies that the system is controllable. Here $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From the axiom (A) and our assumptions on $\varphi$, we infer that $\Gamma x \in \mathcal{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(y_{|I}, Y)) \subseteq B_r(y_{|I}, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(y_{|I}, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - y(t^r)\|$. Then, from Lemma 3.1, we get

$$\begin{aligned}
& r < \|\Gamma x^r(t^r) - y(t^r)\| \\
& \quad \le NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
& \quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + \overline{N}M_1M_2\int_0^{t^r}\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} \\
& \quad +\overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
& \quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + N\sum_{i=1}^n\lambda_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}) \\
& \quad +\overline{N}\sum_{i=1}^n\mu_i(\|\bar{x}_{t_i}\|_{\mathcal{B}})\Big] + N\sum_{i=1}^n\lambda_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}) + \overline{N}\sum_{i=1}^n\mu_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}).
\end{aligned}$$

Since $\lambda_i$ and $\mu_i$ are non-decreasing operators, we have

$$\begin{aligned}
r \;\le\; & NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
& +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}W\Big((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_ar + K_a\|y\|_a\Big)\int_0^a m(s)ds \\
& +\overline{N}M_1M_2\int_0^{t^r}\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds \\
& +N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds + 3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + N\sum_{i=1}^n\lambda_i(r^*) \\
& +\overline{N}\sum_{i=1}^n\mu_i(r^*)\Big] + N\sum_{i=1}^n\lambda_i(r^*) + \overline{N}\sum_{i=1}^n\mu_i(r^*),
\end{aligned}$$

where $\|\bar{x}_{t_i}\|_{\mathcal{B}} \leq r^* = (M_a + J_0^{\phi})\|\varphi\|_{\mathcal{B}} + K_a(r + \|y\|_a)$
and hence

$$1 \leq (1 + a\overline{N}M_1M_2)\Big[K_a\Big(aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^{n}(N\zeta_i + \overline{N}\eta_i)\Big)\Big],$$

which contradicts to our assumption.

Arguing as in the proof of Theorem 4.1, we can prove that $\Gamma(\cdot)$ is a condensing map on $B_r(y|_I, Y)$ and, from Lemma 2.2, we conclude that there exists a mild solution $x(\cdot)$ for (4.1)-(4.4). The proof is now completed.

# 5 An example

In this section, we consider an application of our abstract results. We choose the space $X = L^2([0,\pi]), \mathcal{B} = \mathcal{PC}_0 \times L^2(g, X)$ is the space introduced in [50] and $A : D(A) \subset X \to X$ is the operator defined by $Au = u''$ with domain $D(A) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}$. It is well-known that $A$ is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on $X$. Furthermore, $A$ has a discrete spectrum, the eigenvalues are $-n^2$, for $n \in \mathbb{N}$, with corresponding eigenvectors $z_n(\tau) = \big(\frac{2}{\pi}\big)^{1/2}\sin(n\tau)$, and the following properties hold.

(a) The set of functions $\{z_n : n \in \mathbb{N}\}$ forms an orthonormal basis of $X$.

(b) If $x \in D(A)$, then $Ax = -\sum_{n=1}^{\infty} n^2\langle x, x_n\rangle x_n$, for $\varphi \in D(A)$.

(c) For $x \in X$, $C(t)x = \sum_{n=1}^{\infty}\cos(nt)\langle x, x_n\rangle x_n$ and the associated sine family is

$$S(t)x = \sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle x, x_n\rangle x_n,$$

which implies that the operator $S(t)$ is compact, for all $t \in \mathbb{R}$ and that $\|C(t)\| = \|S(t)\| = 1$, for all $t \in \mathbb{R}$.

(d) If $G$ is the group of translations on $X$ defined by $G(t)x(\zeta) = \tilde{x}(\zeta + t)$, where $\tilde{x}(\cdot)$ is the extension of $x(\cdot)$ with period $2\pi$, then $C(t) = \frac{1}{2}\big[\Phi(t) + \Phi(-t)\big]$ and $A = B^2$, where $B$ is the infinitesimal generator of $\Phi$ and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$ ( see [52] for details).

**5.1 Second order neutral system**

Consider the following second order damped impulsive neutral differential system with state-dependent delay

$$\frac{\partial}{\partial t}\Big[\frac{\partial}{\partial t}w(t,\zeta) + \int_{-\infty}^{t}\int_0^{\pi} b(t-s,\eta,\zeta)w(s,\eta)d\eta ds\Big] = \frac{\partial^2}{\partial\zeta^2}w(t,\zeta) + \alpha\frac{\partial}{\partial t}w(t,\zeta) + \int_0^{\pi}\beta(s)\frac{\partial}{\partial t}w(t,s)ds$$
$$+\int_{-\infty}^{t} k(s-t)w(s-\rho_1(t)\rho_2(\|w(t)\|),\zeta)ds, \quad t \in I,\ \zeta \in [0,\pi] \tag{5.1}$$

$$w(t,0) = w(t,\pi) = 0, \quad t \in I \tag{5.2}$$

$$\frac{\partial}{\partial t}w(0,\zeta) = \zeta(\pi), \tag{5.3}$$

$$w(\tau,\zeta) = \varphi(\tau,\zeta), \quad \tau \leq 0, \quad 0 \leq \zeta \leq \pi \tag{5.4}$$

$$\triangle w(t_i)(\zeta) = \int_{-\infty}^{t_i} b_i(t_i - s)w(s,\zeta)ds, \quad i = 1,2,\ldots,n, \tag{5.5}$$

$$\triangle w'(t_i)(\zeta) = \int_{-\infty}^{t_i} \tilde{b}_i(t_i - s)w(s,\zeta)ds, \quad i = 1,2,\ldots,n, \tag{5.6}$$

where we assume that $\varphi \in \mathcal{B}$ with the identity $\varphi(s)(\zeta) = \varphi(s,\zeta), \varphi(0,\cdot) \in H^1([0,\pi])$ and $0 < t_1 < t_2 < \cdots < a$. Here $\alpha$ is a prefixed real number and $\beta \in L^2([0,\pi])$.

Let the functions $\rho_i : [0,\infty) \to [0,\infty), i = 1,2,;\ k : \mathbb{R} \to \mathbb{R}$ are continuous, $L_f = \Big(\int_{-\infty}^{0}\frac{(a^2(s))}{g(s)}ds\Big)^{\frac{1}{2}} < \infty$, and that the following conditions hold:

(a) The functions $b_i, \tilde{b}_i \in C(\mathbb{R},\mathbb{R})$ and $L_{I_i} := \left(\int_{-\infty}^{0} \frac{b_i^2(s)}{g(s)} ds\right)^{\frac{1}{2}}$, $L_{J_i} := \left(\int_{-\infty}^{0} \frac{\tilde{b}_i^2(s)}{g(s)} ds\right)^{\frac{1}{2}}$, $i = 1, \cdots, n$, are finite.

(b) The functions $b(s,\eta,\zeta)$, $\frac{\partial b(s,\eta,\zeta)}{\partial \zeta}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ and

$$L_g = \max\left\{\left(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{g(s)}\left(\frac{\partial^i b(s,\eta,\zeta)}{\partial \zeta^i}\right)^2 d\eta ds d\zeta\right)^{\frac{1}{2}} : i = 0,1\right\} < \infty.$$

Define the functions $\mathcal{D} : X \to X, g, f : J \times \mathcal{B} \to X,\ \rho : I \times \mathcal{B} \to X,\ I_i : \mathcal{B} \to X$ and $J_i : \mathcal{B} \to X$ by

$$\begin{aligned}
\mathcal{D}\psi(\zeta) &= \alpha\psi(t,\zeta) + \int_0^{\pi} \beta(s)\psi(t,s)ds,\\
f(\psi)(\zeta) &= \int_{-\infty}^{0} k(s)\psi(s,\zeta)ds,\\
g(\psi)(\zeta) &= \int_{-\infty}^{0}\int_0^{\pi} b(s,\nu,\zeta)\psi(s,\nu)d\nu ds,\\
\rho(s,\psi) &= s - \rho_1(s)\rho_2(\|\psi(0)\|),\\
I_i(\psi)(\zeta) &= \int_{-\infty}^{0} b_i(-s)\psi(s,\zeta)ds, \quad i = 1,2,\ldots,n,\\
J_i(\psi)(\zeta) &= \int_{-\infty}^{0} \tilde{b}_i(-s)\psi(s,\zeta)ds, \quad i = 1,2,\ldots,n.
\end{aligned}$$

With the choice of $A, \mathcal{D}, f, g, \rho, I_i$ and $J_i$, the system (1.1)-(1.4) is the abstract formulation of (5.1)-(5.6). Moreover, the maps $\mathcal{D}, g,\ f,\ I_i,\ J_i,\ i = 1,2,\ldots,n$ are bounded linear operators with

$$\|\mathcal{D}\|_{\mathcal{L}(X)} \le |\alpha| + \|\beta\|_{L^2(0,a)},\ \|g(t,\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \le L_g,\ \|f(t,\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \le L_f,\ \|I_i\|_{\mathcal{L}(\mathcal{B},X)} \le L_{I_i}, \|J_i\|_{\mathcal{L}(\mathcal{B},X)} \le L_{J_i}.$$

□

# References


[1] V. Lakshmikantham, D. Bainov, P.S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

[2] A. M. Samoilenko, N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[3] D. Bainov, P. Simeonov, *Impulsive differential equations*, John Wiley & Sons, New York, 1993.

[4] Y. -K. Chang, A. Anguraj, M. Mallika Arjunan, Existence results for non-densely defined neutral impulsive differential inclusions with nonlocal conditions, *J. Appl. Math. Comp.,* 28(2008), 79–91.

[5] Y. -K. Chang, A. Anguraj, M. Mallika Arjunan, Existence results for impulsive neutral functional differential equations with infinite delay, *Nonlinear Anal. Hybrid Systems*, 2(2008), 209–218.

[6] E. Hernandez, A second order impulsive Cauchy problem, *Int. J. Math. Math. Sci.,* 31(8)(2002), 45–461.

[7] E. Hernandez, H. Henriquez, Impulsive partial neutral differential equations, *Appl. Math. Lett.,* 19(3)(2006), 215–222.

[8] E. Hernandez, M. Rabello, H. Henriquez, Existence of solutions for impulsive partial neutral functional differential equations, *J. Math. Anal. Appl.,* 331(2007), 1135–1158.

[9] J. H. Liu, Nonlinear impulsive evolution equations, *Dynam. Contin. Discrete Impuls. Systems*, 6(1)(1999), 77–85.

[10] Y. Rogovchenko, Impulsive evolution systems: main results and new trends, *Dynam. Contin. Discrete Impuls. Systems*, 3(1)(1997), 57–88.

[11] Y. Rogovchenko, Nonlinear impulse evolution systems and applications to population models, *J. Math. Anal. Appl.,* 207(2)(1997), 300–315.

[12] A. Cabada, J. Nieto, D. Franco and S.J. Trofimchuk, A generalization of the monotone method for second order periodic boundary value problem with impulses at fixed points, *Dynam. Contin. Discrete Impuls. System*, 7 (1)(2000), 145–158.

[13] M. Benchohra, J. Henderson and S.K. Ntouyas, *Impulsive differential equations and inclusions*, Hindawi Publishing Corporation, New York, 2006.

[14] K. Balachandran, S. M. Anthoni, Controllability of second-order semilinear neutral functional differential systems in Banach spaces, *Comput. Math. Appl.,* 41(2001), 1223– 1235.

[15] J. M. Jeong, H. G. Kim, Controllability for semilinear functional integrodifferential equations, *Bull. Korean Math. Soc.,* 46(2009), 463–475.

[16] S. K. Ntouyas, D. O'Regan, Some remarks on controllability of evolution equations in Banach spaces, *Electron. J. Differential Equations*, 79(2009), 1–6.

[17] J. Y. Park, H. K. Han, Controllability for some second order differential equations, *Bull. Korean Math. Soc.,* 34 (1997), 411–419.

[18] J. K. Park, S. H. Park, Y. H. Kang, Controllability of second-order impulsive neutral functional differential inclusions in Banach spaces, *Math. Methods Appl. Sci.,* 33(3)(2010), 249–262.

[19] B. Radhakrishnan, K. Balachandran, Controllability of neutral evolution integrodiffe- rential systems with state-dependent delay, *J. Optim. Theory Appl.,* 153(1)(2012), 85–97.

[20] R. Sakthivel, N. I. Mahmudov, J. H. Kim, On controllability of second-order nonlinear impulsive differential systems, *Nonlinear Anal.,* 71(2009), 45–52.

[21] G. Arthi, K. Balachandran, Controllability of damped second-order impulsive neutral functional differential systems with infinite delay, *J. Optim. Theory Appl.,* 152(2012), 799–813.

[22] G. Arthi, K. Balachandran, Controllability of damped second-order neutral functional differential systems with impulses, *Taiwan. J. Math.,* 16(2012), 89–106.

[23] E. Hernandez, K. Balachandran, N. Annapoorani, Existence results for a damped second order abstract functional differential equation with impulses, *Math. Comput. Model.,* 50(2009), 1583–1594.

[24] Y. Lin, N. Tanaka, Nonlinear abstract wave equations with strong damping, *J. Math. Anal. Appl.,* 225(1998), 46–61.

[25] G. F. Webb, Existence and asymptotic behaviour for a strongly damped nonlinear wave equations, *Can. J. Math.,* 32(1980), 631–643.

[26] J. K. Hale, S.M.V. Lunel, *Introduction to Functional Differential Equations*: Applied Mathematical Sciences, Vol. 99, Springer-Verlag, New York, (1993).

[27] E. Hernandez, H. Henriquez, Existence of periodic solutions of partial neutral functional differential equations with unbounded delay, *J. Math. Anal. Appl.,* 221(2)(1998), 499–522.

[28] E. Hernandez, H. Henriquez, Existence results for partial neutral functional differential equations with unbounded delay, *J. Math. Anal. Appl.,* 221(2)(1998), 452–475.

[29] W. G. Aiello, H. I. Freedman, J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay, *SIAM J. Appl. Math.,* 52(3)(1992), 855–869.

[30] A. Anguraj, M. Mallika Arjunan, E. Hernandez, Existence results for an impulsive neutral functional differential equation with state-dependent delay, *Appl. Anal.,* 86(2007), 861–872.

[31] M. Bartha, Periodic solutions for differential equations with state-dependent delay and positive feedback, *Nonlinear Anal. TMA*, 53(6(2003), 839–857.

[32] Y. Cao, J. Fan, T. C. Gard, The effects of state-dependent time delay on a stagestructured population growth model, *Nonlinear Anal. TMA*, 19(2)(1992), 95–105.

[33] F. Chen, D. Sun, J. Shi, Periodicity in a food-limited population model with toxicants and state-dependent delays, *J. Math. Anal. Appl.,* 288(2003), 136–146.

[34] A. Domoshnitsky, M. Drakhlin, E. Litsyn, On equations with delay depending on solution, *Nonlinear Anal. TMA*, 49(5)(2002), 689–701.

[35] F. Hartung, Parameter estimation by quasilinearization in functional differential equations with state-dependent delays: a numerical study. Proceedings of the Third World Congress of Nonlinear Analysts, Part 7 (Catania, 2000), *Nonlinear Anal. TMA*, 47(7)(2001), 4557–4566.

[36] F. Hartung, T. L. Herdman, J. Turin, Parameter identification in classes of neutral differential equations with state-dependent delays, *Nonlinear Anal. TMA*, 39(3)(2000), 305–325.

[37] F. Hartung, T. Krisztin, H. -O. Walther, J. Wu, Functional Differential Equations with State-Dependent Delays: Theory and Applications, in: A. Canada, P. Drabek, A. Fonda (Eds.), Handbook of Differential Equations : Ordinary Differential Equations, *Elsevier B. V*, 2006.

[38] F. Hartung, J. Turi, Identification of parameters in delay equations with state-dependent delays, *Nonlinear Anal. TMA*, 29(11)(1997), 1303–1318.

[39] E. Hernandez, Existence of solutions for a second order abstract functional differential equation with state-dependent delay, *Electron. J. Differential Equations*, 21(2007), 1–10.

[40] E. Hernandez, R. Sakthivel, S. Tanaka, Existence results for impulsive evolution differential equations with state-dependent delay, *Electron. J. Differential Equations*, 28(2008), 1–11.

[41] E. Hernandez, M. A. McKibben, H. Henriquez, Existence results for partial neutral functional differential equations with state-dependent delay, *Math. Comput. Modelling*, 49(2009), 1260–1267.

[42] M. Mallika Arjunan, V. Kavitha, Existence results for impulsive neutral functional differential equations with statedependent delay, *Electron. J. Qual. Theory Differ. Equ.,* 26(2009), 113.

[43] Y. -K. Chang, M. Mallika Arjunan, V. Kavitha, Existence results for a second order impulsive functional differential equation with statedependent delay, *Differ. Equ. Appl.,* 1(3)(2009), 325–339.

[44] E. Hernandez, A. Anguraj, M. Mallika Arjunan, Existence results for an impulsive second order differential equation with state-dependent delay, *Dynam. Contin. Discrete Impuls. Systems*, 17(2010), 287–301.

[45] C. C. Travis, G. F. Webb, Compactness, regularity, and uniform continuity properties of strongly continuous cosine families, *Houston J. Math.,* 3(4)(1977), 555–567.

[46] J. Kisynski, On cosine operator functions and one parameter group of operators, *Studia Math.,* 49(1972), 93–105.

[47] C. C. Travis, G. F. Webb, Cosine families and abstract nonlinear second order differential equations, *Acta Math. Acad. Sci. Hungaricae*, 32(1978), 76–96.

[48] E. Hernandez, M. Pierri, G. Uniao, Existence results for a impulsive abstract partial differential equation with state-dependent delay, *Comput. Math. Appl.,* 52(2006), 411–420.

[49] B. N. Sadovskii, On a fixed point principle, *Funct. Anal. Appl.,* 1(1967), 74–76.

[50] Y. Hino, S. Murakami, T. Naito, *Functional-Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, 1473. Springer-Verlag, Berlin, 1991.

[51] E. Hernandez, A. Prokopczyk, L. A. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlinear Anal. RWA*, 7(2006), 510–519.

[52] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North- Holland Mathematics Studies, Vol. 108, North-Holland, Amsterdam, 1985.

*Received*: October 11, 2012; *Accepted*: January 7, 2013

UNIVERSITY PRESS

Malaya Journal of Matematik 1(1)(2013) 86–97

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

# Relative controllability of fractional stochastic dynamical systems with multiple delays in control

Toufik Guendouzi[a,*] and Iqbal Hamada[b]

[a,b] *Laboratory of Stochastic Models, Statistic and Applications, Tahar Moulay University, 20000 Saida, Algeria.*

**Abstract**

This paper is concerned with the global relative controllability of fractional stochastic dynamical systems with multiple delays in control for finite dimensional spaces. Sufficient conditions for controllability results are obtained using Banach fixed point theorem and the controllability Grammian matrix which is defined by the Mittag-Leffler matrix function. An example is provided to illustrate the theory.

*Keywords:* Control delay, Relative controllability, Stochastic systems, Fractional differential equations, Mittag-Leffler matrix function.

*2010 MSC:* 34G20, 34G60, 34A37.



## 1 Introduction

Control theory is an important area of application oriented mathematics which deals with the design and analysis of control systems. In particular, the concept of controllability plays an important role in both the deterministic and the stochastic control theory. In recent years, controllability problems for various types of nonlinear dynamical systems in infinite dimensional spaces by using different kinds of approaches have been considered in many publications. An extensive list of these publications can be found (see [2, 3, 6, 17] and the references therein). Moreover, the exact controllability enables to steer the system to arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. Klamka [8] derived a set of sufficient conditions for the exact controllability of semilinear systems. Further, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications. The approximate controllability of systems represented by nonlinear evolution equations has been investigated by several authors [9, 13, 14, 18], in which the authors effectively used the fixed point approach. Fu and Mei [6] studied the approximate controllability of semilinear neutral functional differential systems with finite delay. The conditions are established with the help of semigroup theory and fixed point technique under the assumption that the linear part of the associated nonlinear system is approximately controllable.

Stochastic differential equations have many applications in economics, ecology and finance. In recent years, the controllability problems for stochastic differential equations have become a field of increasing interest, (see [10, 11, 19] and references therein). The extensions of deterministic controllability concepts to stochastic control systems have been discussed only in a limited number of publications.

We would like to mention that controllability and approximate controllability of fractional dynamical systems with or without delay in control have been considered by a few authors (see, for instance [1, 5, 20]). As for the stochastic systems, there are less number of papers on the controllability and the approximate controllability of fractional stochastic dynamical systems with delay in control. Recently, Sakthivel et al. [16] established a set of sufficient conditions for obtaining the approximate controllability of semilinear fractional differential systems in

*Corresponding author.

*E-mail addresses*: tf.guendouzi@gmail.com (Toufik Guendouzi) and iqbalhamada@gmail.com (Iqbal Hamada).

Hilbert spaces. The same author in [15] prove the approximate controllability of nonlinear fractional stochastic control system under the assumptions that the corresponding linear system is approximately controllable. More recently, the approximate controllability of neutral stochastic fractional integro-differential equation with infinite delay in a Hilbert space by using Krasnoselskii's fixed point theorem and stochastic analysis theory has been discussed in [18]. The authors derived a new set of sufficient conditions for the approximate controllability of nonlinear fractional stochastic system under the assumption the corresponding linear system is approximately controllable. Shen [21] studied the relative controllability of stochastic nonlinear systems with delay in control. However, to the best of our knowledge, there are no relevant reports on the relative controllability of fractional stochastic dynamical systems with multiple delay in control as treated in the current paper. Motivated by this consideration, in this article we will study the global relative controllability problem for fractional stochastic dynamical systems with multiple delays in control variables for finite dimensional spaces. Sufficient conditions for the controllability results are obtained by using the Banach fixed point theorem and fractional calculus. The paper is organized as follows: In Section 2, some well known fractional operators and special functions, along with a set of properties are defined and the solution representation of linear fractional stochastic differential equations are derived using Laplace transform for further discussion. In Section 3, the linear and nonlinear stochastic fractional dynamical systems with multiple delays in control are proposed and the controllability condition is established using the controllability Grammian matrix which is defined by means of the Mittag-Leffler matrix function. In Section 4, example is discussed to illustrate the effectiveness of our results. Finally, concluding remarks are given in Section 5.

## 2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Let $\alpha, \beta > 0$, with $n-1 < \alpha < n$, $n-1 < \beta < n$ and $n \in \mathbb{N}$, $D$ is the usual differential operator. Let $\mathbb{R}^m$ be the $m$-dimensional Euclidean space, $\mathbb{R}_+ = [0, \infty)$, and suppose $f \in L^1(\mathbb{R}_+)$. The following definitions and properties are well known, for $\alpha, \beta > 0$ and $f$ as a suitable function (see, for instance, [7]):

(a) Riemann-Liouville fractional operators:

$$\begin{aligned} (I^{\alpha}_{0+}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \\ (D^{\alpha}_{0+}f)(x) &= D^n(I^{n-\alpha}_{a+}f)(x). \end{aligned}$$

(b) Caputo fractional derivative:

$$({}^cD^{\alpha}_{0+}f)(x) = (I^{n-\alpha}_{0+}D^n f)(x),$$

in particular $I^{\alpha}_{0+}\,{}^cD^{\alpha}_{0+}f(t) = f(t) - f(0)$, $(0 < \alpha < 1)$.

The following is a well known relation, for finite interval $[a, b] \in \mathbb{R}_+$

$$(D^{\alpha}_{a+}f)(x) = ({}^cD^{\alpha}_{a+}f)(x) + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha}, \quad n = \mathcal{R}(\alpha) + 1.$$

The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\{{}^cD^{\alpha}_{0+}f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+)s^{\alpha-1-k}.$$

The Riemann-Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann-Liouville sense require initial conditions of special form lacking physical interpretation. To overcome this difficulty Caputo introduced a new definition of fractional derivative but in general, both the Riemann-Liouville and the Caputo fractional operators possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. Due to this fact, the concept of sequential fractional differential equations are discussed in [7].

(c) Linear Sequential Derivative:

For $n \in \mathbb{N}$ the sequential fractional derivative for suitable function $f$ is defined by

$$f^{(k\alpha)} := (\mathbf{D}^{k\alpha} f)(x) = (\mathbf{D}^{\alpha}\mathbf{D}^{(k-1)\alpha} f)(x),$$

where $k = 1, \ldots, n$, $(\mathbf{D}^{\alpha} f)(x) = f(x)$, and $\mathbf{D}^{\alpha}$ is any fractional differential operator, here we mention it as ${}^cD^{\alpha}_{0+}$.

(d) Mittag-Leffler Function

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k\alpha+\beta)}, \quad \alpha, \beta > 0.$$

The general Mittag-Leffler function satisfies

$$\int_0^{\infty} e^{-t} t^{\beta-1} E_{\alpha,\beta}(t^{\alpha} y)dt = \frac{1}{1-y}, \quad |y| < 1.$$

The Laplace transform of $E_{\alpha,\beta}(y)$ follows from the integral

$$\int_0^{\infty} e^{-st} t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha})dt = \frac{s^{\alpha-\beta}}{(s \mp a)}.$$

That is

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha})\} = \frac{s^{\alpha-\beta}}{(s \mp a)},$$

for $\mathcal{R}(s) > |a|^{1/\alpha}$ and $\mathcal{R}(\beta) > 0$. In particular, for $\beta = 1$,

$$E_{\alpha,1}(\lambda y^{\alpha}) = E_{\alpha}(\lambda y^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k y^{k\alpha}}{\Gamma(\alpha k+1)}, \quad \lambda, y \in \mathbf{C}$$

have the interesting property ${}^cD^{\alpha} E_{\alpha}(\lambda t^{\alpha}) = \lambda E_{\alpha}(\lambda t^{\alpha})$ and

$$\mathcal{L}\{E_{\alpha}(\pm a t^{\alpha})\} = \frac{s^{\alpha-1}}{(s \mp a)}, \quad \text{for } \beta = 1.$$

For brevity of notation let us take $I^q_{0+}$ as $I^q$ and ${}^cD^q_{0+}$ as ${}^cD^q$ and the fractional derivative is taken as Caputo sense.

Let us consider the linear fractional stochastic differential equation of the form

$$\begin{aligned} {}^cD^q x(t) &= Ax(t) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in [0, T], \\ x(0) &= x_0, \end{aligned} \tag{2.1}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$, $A$ is an $n \times n$ matrix, $w(t)$ is a given $l$-dimensional Wiener process with the filtration $\mathcal{F}_t$ generated by $w(s)$, $0 \le s \le t$ and $\sigma : [0, T] \to \mathbb{R}^{n \times l}$ is appropriate function. In order to find the solution, apply Laplace transform on both sides and use the Laplace transform of Caputo derivative, we get

$$s^q X(s) - s^{q-1}x(0) = AX(s) + \Sigma(s)\frac{dw(s)}{ds}.$$

Apply inverse Laplace transform on both sides (see [4]) we have

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{s^{q-1}(s^q I - A)^{-1}\}x_0 + \mathcal{L}^{-1}\{\Sigma(s)\tfrac{dw(s)}{ds}\} * \mathcal{L}^{-1}\{(s^q I - A)^{-1}\}.$$

Finally, substituting Laplace transformation of the Mittag-Leffler function, we get the solution of the given system

$$x(t) = E_q(At^q)x_0 + \int_0^t (t-s)^{q-1}\left(\int_0^{\tau} \sigma(\theta)dw(\theta)\right) E_{q,q}(A(t-s)^q)ds$$

where $E_q(At^q)$ is the matrix extension of the mentioned Mittag-Leffler functions with the following representation:

$$E_q(At^q) = \sum_{k=0}^{\infty} \frac{A^k t^{kq}}{\Gamma(1+kq)}$$

with the property ${}^cD^q E_q(At^q) = AE_q(At^q)$.

## 3 Controllability results

Let $L^2_{\mathcal{F}_t}(J \times \Omega, \mathbb{R}^n)$ be the Banach space of all $\mathcal{F}_t$-measurable square integrable processes $x(t)$ with norm $\|x\|^2_{L^2} = \sup_{t\in J} \mathbb{E}\|x(t)\|^2$, where $\mathbb{E}(.)$ denotes the expectation with respect to the measure $\mathbb{P}$. Let $C = C([0,T]; L^2_{\mathcal{F}_t})$ be the Banach space of continuous maps from $[0,T]$ into $L^2_{\mathcal{F}_t}(J \times \Omega, \mathbb{R}^n)$ satisfying $\sup_{t\in J} \mathbb{E}\|x(t)\|^2 < \infty$. Consider the linear fractional stochastic dynamical system with multiple delays in control represented by the fractional stochastic differential equation of the form

$$\begin{aligned} {}^cD^q x(t) &= Ax(t) + \sum_{k=1}^{M} B_k u(h_k(t)) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in J := [0,T] \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^l$, $A$ is an $n \times n$ matrix, $B_k$ are $n \times l$ matrices, for $k = 0, 1, \dots, M$, $w(t)$ is a given $l$-dimensional Wiener process with the filtration $\mathcal{F}_t$ generated by $w(s)$, $0 \leq s \leq t$ and $\sigma : [0,T] \to \mathbb{R}^{n\times l}$ is appropriate function.

Let us assume the following assumptions:

(**i**) Assume the function $h_k : J \to \mathbb{R}$, $k = 0, 1, \dots, M$ are twice continuously differentiable and strictly increasing in $J$. Moreover,

$$h_k(t) \leq t \quad \text{for } t \in J, i = 0, 1, \dots, M. \tag{3.2}$$

(**ii**) Introduce the time lead functions $r_k(t) : [h_k(0), h_k(T)] \to J$, $k = 0, 1, \dots, M$ such that $r_k(h_k(t)) = t$ for $t \in J$. Further assume that $h_0(t) = t$ and for $t = T$, the following inequalities hold

$$h_M(T) \leq h_{M_1}(T) \leq \dots h_{M_{m+1}}(T) \leq 0 = h_m(T) < h_{m-1}(T) = \dots h_1(T) = h_0(T) = T. \tag{3.3}$$

(**iii**) let $h > 0$ be given. For functions $u : [-h, T] \to \mathbb{R}^l$ and $t \in J$, we use the symbol $u_t$ to denote the function on $[-h, 0]$, defined by $u_t(s) = u(t+s)$ for $s \in [-h, 0)$.

The following definitions of complete state of the system (2) at time $t$ and relative controllability are assumed.

**Definition 3.1.** *The set $\phi(t) = \{x(t), u_t\}$ is the complete state of the system* (2) *at time $t$.*

**Definition 3.2.** *System* (2) *is said to be globally relatively controllable on $J$ if for every complete state $\phi(0)$ and every vector $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on $J$ such that the corresponding trajectory of the system* (2) *satisfies $x(T) = x_1$.*

Note that the solution of system (2) ca be expressed in the following form

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \sum_{k=0}^{M} B_k u(h_k(s)) ds \\ &+ \int_0^t (t-s)^{q-1} \left( \int_0^\tau \sigma(\theta) dw(\theta) \right) E_{q,q}(A(t-s)^q) ds. \end{aligned}$$

Taking into account the time lead functions $r_k(t)$, this solution can be further changed into

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^{M} \int_{h_k(0)}^{h_k(t)} (t - r_k(s))^{q-1} E_{q,q}(A(t - r_k(s))^q) B_k r_k'(s) u(s) ds \\ &+ \int_0^t (t-s)^{q-1} \left( \int_0^\tau \sigma(\theta) dw(\theta) \right) E_{q,q}(A(t-s)^q) ds. \end{aligned} \tag{3.4}$$

Using the inequalities (4), the above equation becomes,

$$\begin{aligned} x(t) &= E_q(At^q)x_0 + \sum_{k=0}^{m}\int_{h_k(0)}^{0}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \sum_{k=0}^{m}\int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u(s)ds \\ &+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds. \end{aligned} \tag{3.5}$$

For brevity, let us introduce the following notation:

$$\begin{aligned} \varphi(t) &= \sum_{k=0}^{m}\int_{h_k(0)}^{0}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \end{aligned} \tag{3.6}$$

and

$$\chi(t) = \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds.$$

Recall the controllability Grammian matrix

$$\psi_0^T = \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)][E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)]^\star ds$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and the $\star$ denotes the matrix transpose.

**Theorem 3.3.** *The linear stochastic control system* (2) *is relatively controllable on* $[0,T]$ *if and only if the controllability Grammian matrix* $\psi_0^T$ *is positive definite for some* $T>0$.

*Proof.* Since $\psi$ is positive definite, it is non-singular and therefore its inverse is well defined. Define the control function as,

$$u(t) = [B_k^\star E_{q,q}(A^\star(T-r_k(t))^q)r_k'(t)]\psi^{-1}[x_1 - E_q(At^q)x_0 - \varphi(T) - \chi(T)], \quad k=0,1,\ldots,m \tag{3.7}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily. Inserting (8) in (6) and using (7) we get

$$\begin{aligned} x(T) &= E_q(At^q)x_0 + \varphi(T) + \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)] \\ &\times [B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)]\psi^{-1}[x_1 - E_q(AT^q)x_0 - \varphi(T) - \chi(T)]ds \\ &+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\ &= x_1. \end{aligned}$$

Thus the control $u(t)$ transfers the initial state $\phi(0)$ to the desired vector $x_1 \in \mathbb{R}^n$ at time $T$. Hence the system (2) is controllable.

On the other hand, if it is not positive definite, there exists a nonzero $\phi$ such that $\phi^\star\psi\phi = 0$, that is

$$\begin{aligned} &\phi^\star \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)][E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)]^\star \phi ds = 0 \\ &\phi^\star \sum_{k=0}^{m}(T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)] = 0, \end{aligned}$$

on $[0,T]$. Let $x_0 = [E_q(AT^q)]^{-1}\phi$. By assumption, there exists a control u such that it steers the complete initial state $\phi(0) = \{x(0), u_0(s)\}$ to the origin in the interval $[0,T]$. It follows that

$$\begin{aligned}
x(T) &= E_q(At^q)x_0 + \varphi(T) + \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)] \\
&\times [B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)]\psi^{-1}[x_1 - E_q(AT^q)x_0 - \varphi(T) - \chi(T)]ds \\
&+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\
&= \phi + \varphi(T) + \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)] \\
&\times [B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)]\psi^{-1}[x_1 - E_q(AT^q)x_0 - \varphi(T) - \chi(T)]ds \\
&+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\
&= 0.
\end{aligned}$$

Thus,

$$0 = \phi^\star\phi + \sum_{k=0}^{m}\int_0^T \phi^\star(T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)]u(s)ds + \phi^\star(\varphi(T) + \chi(T)).$$

But the second and third term are zero leading to the conclusion $\phi^\star\phi = 0$. This is a contradiction to $\phi \neq 0$. Thus $\psi$ is positive definite. Hence the desired result. □

Consider a nonlinear fractional stochastic dynamical system with multiple delays in control represented by the fractional stochastic differential equation of the form

$$\begin{aligned}
{}^cD^q x(t) &= Ax(t) + \sum_{k=1}^{M} B_k u(h_k(t)) + f(t,x(t)) + \sigma(t,x(t))\frac{dw(t)}{dt}, \quad t \in J := [0,T] \\
x(0) &= x_0,
\end{aligned} \tag{3.8}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^l$, $A, B_k$ are defined as above and $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : J \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are appropriate functions. Then the solution of the system (9) ca be expressed in the following form

$$\begin{aligned}
x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\sum_{k=0}^{M} B_k u(h_k(s))ds \\
&+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds + \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds.
\end{aligned}$$

Using the time lead functions $r_k(t)$ the solution becomes,

$$\begin{aligned}
x(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^{M}\int_{h_k(0)}^{h_k(t)} (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u(s)ds \\
&+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds + \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds.
\end{aligned} \tag{3.9}$$

Now using the inequalities (4), the above equation for $t = T$ can be expressed as

$$\begin{aligned}
x(T) &= E_q(A(T)^q)x_0 + \sum_{k=0}^{m}\int_{h_k(0)}^{0} (T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\
&+ \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u(s)ds \\
&+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)} (T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\
&+ \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\
&+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds.
\end{aligned} \tag{3.10}$$

For brevity, let us introduce the following notation using (7)

$$\begin{aligned}\Upsilon(\phi(0),x_1;x) &= x_1 - E_q(A(T)^q)x_0 - \varphi(T) - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\ &- \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds.\end{aligned} \tag{3.11}$$

Now let us define the controllability Grammian matrix and the control function

$$\psi_0^T = \sum_{k=0}^m \int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)][E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)]^\star ds \tag{3.12}$$

$$u(t) = [B_k^\star E_{q,q}(A^\star(T-r_k(t))^q)r_k'(s)]\psi^{-1}\Upsilon(\phi(0),x_1;x), \quad \text{for } k=0,1,\dots,m \tag{3.13}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and $\star$ denotes the matrix transpose. Inserting (14) in (11) by using (12) and (13), it is easy to verify that the control $u(t)$ transfers the initial complete state $\phi(0)$ to the desired vector $x_1$ at time $T$ for each fixed $x$. Now observing (12) and substituting (14) in (10), we have

$$\begin{aligned}x(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^m \int_{h_k(0)}^0 (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \sum_{k=0}^m \int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s) \\ &\times B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)\psi^{-1}\Upsilon(\phi(0),x_1;x)ds \\ &+ \sum_{k=m+1}^M \int_{h_k(0)}^{h_k(t)} (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds.\end{aligned} \tag{3.14}$$

Now, we impose the following conditions on data of the problem:

(**iv**) The linear fractional stochastic dynamical system (2) is globally relatively controllable.
(**v**) $f$ and $\sigma$ satisfy Lipschitz and linear growth conditions. That is, there exists some constants $N, \tilde{N}, L, \tilde{L} > 0$ such that

$$\begin{aligned}\|f(t,x)-f(t,y)\|^2 &\leq N\|x-y\|^2, \quad \|f(t,x)\|^2 \leq \tilde{N}(1+\|x\|^2) \\ \|\sigma(t,x)-\sigma(t,y)\|^2 &\leq L\|x-y\|^2, \quad \|\sigma(t,x)\|^2 \leq \tilde{L}(1+\|x\|^2).\end{aligned}$$

For our convenience, let us introduce the following notations.

$$\begin{aligned}a_1 &= \max\{\|E_q(At^q)\|^2; t\in J\}, \quad a_2 = \max\{\|u_0(t)\|^2; t\in J\}, \quad r_k = \max\{\|r_k'(t)\|^2; t\in J\} \\ b_k &= \max\{\|E_{q,q}(A(t-r_k(s))^q)\|^2; s\in[0,T]\}, \quad c_k = \int_0^T (T-r_k(s))^{2(q-1)}ds \\ \tilde{c}_k &= \int_{h_k(0)}^0 (T-r_k(s))^{2(q-1)}ds, \quad \hat{c}_k = \int_{h_k(0)}^{h_k(T)} (T-r_k(s))^{2(q-1)}ds\end{aligned}$$

We claim that if (**iv**) holds, the operator $\psi_0^T$ is strictly positive definite and thus the inverse linear operator $(\psi_0^T)^{-1}$ is bounded, say, by $l$, (see [10] for more details).

**Theorem 3.4.** *Under the conditions* (**iv**) *and* (**v**), *the nonlinear system (9) is globally relatively controllable on* $J$.

*Proof.* Firstly, from the definition (14) we can write the control function $u$ as

$$\begin{aligned} u(t) &= B_k^\star E_{q,q}(A^\star(T-r_k(t))^q)r_k'(t)\psi^{-1} \\ &\times \Bigg[x_1 - E_q(A(T)^q)x_0 - \sum_{k=0}^{m}\int_{h_k(0)}^{0}(T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &- \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &- \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\ &- \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds\Bigg]. \end{aligned}$$

Secondly, we define the operator $\mathcal{P}: C \to C$ by

$$\begin{aligned} \mathcal{P}(x)(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^{m}\int_{h_k(0)}^{0}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \sum_{k=0}^{m}\int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s) \\ &\times B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)\psi^{-1}\Upsilon(\phi(0),x_1;x)ds \\ &+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds. \end{aligned}$$

In order to prove the global relative controllability of the system (9) it is enough to show that $\mathcal{P}$ has a fixed point in $C$. To do this, we can employ the contraction mapping principle. To apply the principle, first we show that $\mathcal{P}$ maps $C$ into itself. We have

$$\begin{aligned} \mathbf{E}\|\mathcal{P}(x)(t)\|^2 &\leq 6a_1\mathbf{E}\|x_0\|^2 + 6\sum_{k=0}^{m}\mathbf{E}\left\|\int_{h_k(0)}^{0}(T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds\right\|^2 \\ &+ 6\sum_{k=0}^{m}\mathbf{E}\Bigg\|\int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s) \\ &\times B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)\psi^{-1}\Upsilon(\phi(0),x_1;x)ds\Bigg\|^2 \\ &+ 6\sum_{k=m+1}^{M}\mathbf{E}\left\|\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds\right\|^2 \\ &+ 6\mathbf{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds\right\|^2 \\ &+ 6\mathbf{E}\left\|\int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds\right\|^2. \end{aligned}$$

It follows from Lemma 2.5, in [15], and the above notation that:

$$\begin{aligned} \mathbf{E}\|\mathcal{P}(x)(t)\|^2 &\leq 6a_1\mathbf{E}\|x_0\|^2 + 6a_2\left(\sum_{k=0}^{m}\tilde{c}_k b_k r_k\|B_k\|^2 + \sum_{k=m+1}^{M}\hat{c}_k b_k r_k\|B_k\|^2\right) \\ &+ 6b\frac{t^{2q-1}}{2q-1}\int_0^t \mathbf{E}\|f(s,x(s))\|^2 ds + 6l^2\sum_{k=0}^{m}c_k b_k^2 r_k^2\|B_k\|^4\int_0^t \mathbf{E}\|\Upsilon(\phi(0),x_1;x)\|^2 ds \\ &+ 6L_\sigma b\frac{t^{2q-1}}{2q-1}\int_0^t\left(\int_0^\tau \mathbf{E}\|\sigma(\theta,x(\theta))\|^2 d\theta\right)ds. \end{aligned}$$

Thus we have

$$\begin{aligned}\mathbf{E}\|\mathcal{P}(x)(t)\|^2 &\leq 6a_1\mathbf{E}\|x_0\|^2 + 6a_2\beta + 6b\frac{t^{2q-1}}{2q-1}\tilde{N}\int_0^t (1+\mathbf{E}\|x(s)\|^2)ds \\ &+ 6l^2\eta\Bigg[\mathbf{E}\|x_1\|^2 + a_1\mathbf{E}\|x_0\|^2 + a_2\beta + b\frac{T^{2q-1}}{2q-1}\tilde{N}\int_0^T (1+\mathbf{E}\|x(s)\|^2)ds \\ &+ L_\sigma b\frac{T^{2q-1}}{2q-1}\tilde{L}\int_0^T \left(\int_0^\tau (1+\mathbf{E}\|x(\theta)\|^2)d\theta\right)ds\Bigg] \\ &+ 6L_\sigma b\frac{t^{2q-1}}{2q-1}\tilde{L}\int_0^t \left(\int_0^\tau (1+\mathbf{E}\|x(\theta)\|^2)d\theta\right)ds.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{E}\|\mathcal{P}(x)(t)\|^2 &\leq 6l^2\eta\mathbf{E}\|x_1\|^2 + 6a_1\mathbf{E}\|x_0\|^2(1+l^2\eta) + 6a_2\beta(1+l^2\eta) \\ &+ 6b\frac{T^{2q-1}}{2q-1}\tilde{N}(1+l^2\eta)(1+\|x\|^2_{L^2}) + 6L_\sigma\tilde{L}b\frac{T^{2q-1}}{2q-1}(1+l^2\eta)(1+T\|x\|^2_{L^2}).\end{aligned}$$

It follows from from the above inequality and the condition (**v**) that there exists $c>0$ such that

$$\mathbf{E}\|\mathcal{P}(x)(t)\|^2 \leq c(1+\|x\|^2_{L^2}).$$

Therefore $\mathcal{P}$ maps $C$ into itself.

Secondly, we claim that $\mathcal{P}$ is a contraction mapping on $C$. For $x, y \in C$,

$$\begin{aligned}&\mathbf{E}\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\|^2 \\ \leq\ & 3\sum_{k=0}^m \mathbf{E}\Bigg\|\int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s) \\ \times\ & B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)\psi^{-1}[\Upsilon(\phi(0),x_1;x)-\Upsilon(\phi(0),x_1;y)]ds\Bigg\|^2 \\ +\ & 3\mathbf{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)(f(s,x(s))-f(s,y(s)))ds\right\|^2 \\ +\ & 3\mathbf{E}\left\|\int_0^t (t-s)^{q-1}\left(\int_0^\tau (\sigma(\theta,x(\theta))-\sigma(\theta,y(\theta)))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds\right\|^2.\end{aligned}$$

Using Lemma 2.5, in [15], condition (**v**), and the above notations we get

$$\begin{aligned}&\mathbf{E}\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\|^2 \\ \leq\ & 3l^2\frac{T^{2q}}{2q-1}b\sum_{k=0}^m c_k b_k^2 r_k^2\|B_k\|^4\Bigg[\int_0^T \mathbf{E}\|f(s,y(s))-f(s,x(s))\|^2 ds \\ +\ & L_\sigma\int_0^\tau \mathbf{E}\|\sigma(\theta,y(\theta))-\sigma(\theta,x(\theta))\|^2 d\theta\Bigg] \\ +\ & 3\frac{T^{2q-1}}{2q-1}b\int_0^t \mathbf{E}\|f(s,x(s))-f(s,y(s))\|^2 ds \\ +\ & 3\frac{T^{2q-1}}{2q-1}bL_\sigma\int_0^t\left(\int_0^\tau \mathbf{E}\|\sigma(\theta,x(\theta))-\sigma(\theta,y(\theta))\|^2 d\theta\right)ds. \\ \leq\ & 3l^2 b\eta\frac{T^{2q-1}}{2q-1}[N+LL_\sigma]\int_0^T \mathbf{E}\|x(s)-y(s)\|^2 ds \\ +\ & 3b\frac{T^{2q-1}}{2q-1}[N+TLL_\sigma]\int_0^T \mathbf{E}\|x(s)-y(s)\|^2 ds.\end{aligned}$$

It results that

$$\sup_{t\in[0,T]}\mathbf{E}\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\|^2 \leq \left[3l^2b\eta\frac{T^{2q-1}}{2q-1}[N+LL_\sigma]+3b\frac{T^{2q-1}}{2q-1}[N+TLL_\sigma]\right]\sup_{t\in[0,T]}\mathbf{E}\|x(t)-y(t)\|^2.$$

Therefore we conclude that if $3l^2b\eta\frac{T^{2q-1}}{2q-1}[N+LL_\sigma]+3b\frac{T^{2q-1}}{2q-1}[N+TLL_\sigma]<1$, then $\mathcal{P}$ is a contraction mapping on $C$, implies that the mapping $\mathcal{P}$ has a unique fixed point $x(\cdot)\in C$. Hence we have

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0+\sum_{k=0}^{m}\int_{h_k(0)}^{0}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_kr_k'(s)u_0(s)ds \\ &+ \sum_{k=0}^{m}\int_0^t(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_kr_k'(s)u(s)ds \\ &+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_kr_k'(s)u_0(s)ds \\ &+ \int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+ \int_0^t(t-s)^{q-1}\left(\int_0^\tau\sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds. \end{aligned}$$

Thus $x(t)$ is the solution of the system (9), and it is easy to verify that $x(T)=x_1$. Further the control function u(t) steers the system (9) from initial complete state $\phi(0)$ to $x_1$ on $J$. Hence the system (9) is globally relatively controllable on $J$.

□

## 4 An example

In this section, we apply the results obtained in the previous section for the following stochastic fractional dynamical systems with multiple delays in control which involves sequential Caputo derivative

$$\begin{aligned} {}^cD^qx(t) &= Ax(t)+B_1u(t)+B_2u(t-h)+f(t,x(t))+\sigma(t,x(t))\frac{dw(t)}{dt};\ 0<q<1, t\in[0,T] \\ x(0) &= x_0, \end{aligned} \tag{4.1}$$

where

$$A=\begin{pmatrix}-1 & 0\\ 3 & -2\end{pmatrix},\quad B_1=B_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$f(t,x(t))=\begin{pmatrix}x_1(t)\cos x_2(t)+3x_2(t)\\ x_2(t)\sin x_1(t)+2x_1(t)\end{pmatrix},\sigma(t,x(t))=\begin{pmatrix}(2t^2+1)x_1(t)e^{-t} & 0\\ 0 & x_2(t)e^{-t}\end{pmatrix}.$$

Let us introduce the variables $x_1(t)=x(t)$ and $x_2(t)={}^cD^{\frac{q}{2}}x_1(t)$. Then ${}^cD^{\frac{q}{2}}x_1(t)={}^cD^{\frac{q}{2}}x(t)=x_2$.

The Mittag-Leffler matrix of the given system is given by

$$\begin{pmatrix}E_q(-t^q) & 0\\ 3E_q(-t^q)-3E_q(-2t^q) & E_q(-2t^q)\end{pmatrix}.$$

Further

$$E_{q,q}(A(T-s)^q)=\begin{pmatrix}E_{q,q}(-(T-s)^q) & 0\\ 3E_{q,q}(-(T-s)^q)-3E_{q,q}(-2(T-s)^q) & E_{q,q}(-2(T-s)^q)\end{pmatrix},$$

$$E_{q,q}(A(T-(s+h))^q)=\begin{pmatrix}E_{q,q}(-(T-(s+h))^q) & 0\\ 3E_{q,q}(-(T-(s+h))^q)-3E_{q,q}(-2(T-(s+h))^q) & E_{q,q}(-2(T-(s+h))^q)\end{pmatrix}.$$

By simple matrix calculation one can see that the controllability matrix

$$\begin{aligned} \psi_0^T &= \sum_{k=0}^{m}\int_0^T(T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_kr_k'(s)][E_{q,q}(A(T-r_k(s))^q)B_kr_k'(s)]^\star ds \\ &= \int_0^T\left[(T-s)^{q-1}\begin{pmatrix}a^2 & ac\\ ac & b^2+c^2\end{pmatrix}+(T-(s+h))^{q-1}\begin{pmatrix}\bar{a}^2 & \bar{a}\bar{c}\\ \bar{a}\bar{c} & \bar{b}^2+\bar{c}^2\end{pmatrix}\right]ds. \end{aligned}$$

is positive definite for any $T > h$, where

$$\begin{aligned} a &= E_{q,q}(-(T-s)^q), \quad b = E_{q,q}(-2(T-(s+h))^q), \\ c &= 3E_{q,q}(-(T-s)^q) - 3E_{q,q}(-2(T-s)^q), \quad \bar{a} = E_{q,q}(-(T-(s+h))^q) \\ \bar{b} &= E_{q,q}(-2(T-(s+h))^q), \quad \bar{c} = 3E_{q,q}(-(T-(s+h))^q) - 3E_{q,q}(-2(T-(s+h))^q). \end{aligned}$$

Further the functions $f(t, x(t))$ and $\sigma(t, x(t))$ satisfies the hypothesis mentioned in Theorem 3.4., and so the fractional system (16) is globally relatively controllable on [0,T].

## 5 Conclusion

The article contains some controllability results for global relative controllability for the linear and nonlinear fractional stochastic dynamical systems with multiple delays in control function. The result shows that the Banach fixed point theorem can effectively be used to study the control problems for establishing sufficient conditions. Here it is proved that under some hypotheses together with the assumption that the linear stochastic system is globally relatively controllable, the nonlinear fractional stochastic system is also globally relatively controllable. An example is also included to illustrate the importance of the results obtained.

## 6 Acknowledgment

The work of the first author is supported by The National Agency of Development of University Research (ANDRU), Algeria (PNR-SMA 2011-2014).

# References

[1] J.L. Adams, T.T. Hartley, Finite time controllability of fractional order systems, *J. Comput. Nonlinear Dyn.,* 3(2008), 021402-1-021402-5.

[2] R.P. Agarwal, S. Baghli, M. Benchohra, Controllability for semilinear functional and neutral functional evolution equations with infinite delay in Frchet spaces, *Applied Mathematics and Optimization*, 60(2009), 253-274.

[3] K. Balachandran, S. Karthikeyan, J.Y. Park, Controllability of stochastic systems with distributed delays in control, *Internat. J. Control*, 82(2009), 1288-1296.

[4] K. Balachandran, J. Kokila, On the controllability of fractional dynamical systems, *Int. J. Appl. Math. Comput. Sci.,* 22(3)(2012), 523-531.

[5] Y.Q. Chen, H.S. Ahn, D. Xue, Robust controllability of interval fractional order linear time invariant systems, *Signal Process*, 86(2006), 2794-2802.

[6] X. Fu, K. Mei, Approximate controllability of semilinear partial functional differential systems, *Journal of Dynamical and Control Systems*, 15(2009), 425-443.

[7] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[8] J. Klamka, Constrained controllability of semilinear systems with delays, *Nonlinear Dynamics*, 56(2009), 169-177.

[9] J. Klamka, Constrained approximate controllability, *IEEE Transactions on Automatic Control*, 45(2000), 1745-1749.

[10] J. Klamka, Stochastic controllability of linear systems with delay in control, *Bulletin of the Polish Academy of Sciences: Technical Sciences*, 55(2007), 23-29.

[11] N.I. Mahmudov, S. Zorlu, Controllability of non-linear stochastic systems, *Internat. J. Control*, 76(2003), 95-104.

[12] Y. Ren, L. Hu, R. Sakthivel, Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay, *Journal of Computational and Applied Mathematics*, 235(2011), 2603-2614.

[13] R. Sakthivel, E.R. Anandhi, Approximate controllability of impulsive differential equations with state-dependent delay, *International Journal of Control*, 83(2010), 387-393.

[14] R. Sakthivel, Y. Ren, N.I. Mahmudov, Approximate controllability of second order stochastic differential equations with impulsive effects, *Modern Physics Letters B* , 24(2010), 1-14.

[15] R. Sakthivel, S. Suganya, S.M. Anthoni, Approximate controllability of fractional stochastic evolution equations, *Computers & Mathematics with Applications*, 63(2012), 660-668.

[16] R. Sakthivel, Y. Ren, N.I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, *Computers and Mathematics with Applications*, 62(2011), 1451-1459.

[17] R. Sakthivel, N.I. Mahmudov, J.J. Nieto, Controllability for a class of fractional-order neutral evolution control systems, *Applied Mathematics and Computation*, 218(2012), 10334-10340.

[18] R. Sakthivel, R. Ganesh, S. Suganya, Approximate controllability of fractional neutral stochastic system with infinite delay, *Reports on Mathematical Physics*, 70(2012), 291-311.

[19] R. Sakthivel, Y. Ren, Complete controllability of stochastic evolution equations with jumps, *Reports on Mathematical Physics*, 68(2011), 163-174.

[20] A.B. Shamardan, M.R.A. Moubarak, Controllability and observability for fractional control systems, *J. Fract. Cal.,* 15(1999), 2534.

[21] L. Shen, J. Sun, Relative controllability of stochastic nonlinear systems with delay in control, *Nonlinear Analysis: RWA,* 13(2012), 2880-2887.


*Received*: December 14, 2012; *Accepted*: January 5, 2013

UNIVERSITY PRESS

Malaya Journal of Matematik 1(1)(2013) 98–105

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

# Global stability of mutualistic interactions among three species population model with continuous time delay

A. B. Munde[a] and M. B. Dhakne[b,*]

[a,b] Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431 004, India.

**Abstract**

This paper deals with the study on a mathematical model consisting of mutualistic interactions among three species with continuous time delay. The delay kernels are being convex combinations of suitable nonnegative and normalized functions, the linear chain trick gives an expanded system of ordinary differential equations with the same stability properties as the original integro-differential system. Global stability is discussed by constructing Lyapunov function. It has been shown that equilibrium state of the model is globally stable. Finally, numerical simulations supporting our theoretical results are also included.

*Keywords:* Mutualism model, local and global stabilities, Lyapunov function, population dynamics, time delay.

*2010 MSC:* 92B05, 92D25, 93A30, 93C15 93D99. 

## 1 Introduction

The study of equations describing population growth is very interesting and challenging mathematically as well as biologically to discuss the problems on global stability. In the biological process of evolution, the population of one species does not respond instantaneously to interact with other species. To incorporate this role in a modeling approach, time delay models have been developed. Gopalsamy K. [5] and Kuang Y. [9] discussed the necessity of delay differential equation models, see also Beretta E. and Takeuchi [1], Busekros A. W. [2], Cushing J. M. [3], Gopalsamy K. [6], Hale J. K. and Waltman P. [7], Harlan S. W. [8], Mc Donald N. [10], and Solimano F. and Beretta E. [13]. Relatively less attention has been given to the study of three species model with continuous time delay and their dynamical behavior. This motivates the authors to study mutualistic interactions among three species population model with continuous time delay.

The main purpose of this paper is to establish global stability of three species mutualistic system with continuous time delay. In section 2, we introduce our mathematical model. In section 3, we discuss global stability about the biologically feasible equilibrium point of the model by constructing a Lyapunov functional. In section 4, we illustrate our results by some examples. We conclude with a short discussion in section 5.

## 2 Mathematical Model

In this section, we consider a mathematical model for three mutually interacting species with continuous time delay is given by the following integro-differential equations:

$$\frac{dN_1}{dt} = N_1\left(a_1 - \alpha_{11}N_1 + \alpha_{12}\int_{-\infty}^{t} k_2(t-s)N_2(s)ds + \alpha_{13}\int_{-\infty}^{t} k_3(t-s)N_3(s)ds\right),$$
$$\frac{dN_2}{dt} = N_2\left(a_2 - \alpha_{22}N_2 + \alpha_{21}\int_{-\infty}^{t} k_1(t-s)N_1(s)ds + \alpha_{23}\int_{-\infty}^{t} k_3(t-s)N_3(s)ds\right),$$

*Corresponding author.

*E-mail addresses*: mashok.math@gmail.com (A. B. Munde) and mbdhkane@yahoo.com (M.B. Dhakne)

$$\frac{dN_3}{dt} = N_3\Big(a_3 - \alpha_{33}N_3 + \alpha_{31}\int_{-\infty}^{t} k_1(t-s)N_1(s)ds + \alpha_{32}\int_{-\infty}^{t} k_2(t-s)N_2(s)ds\Big), \tag{2.1}$$

where $N_i, i = 1, 2, 3$ represent the population density of first, second and third species respectively, $a_i$ represent the intrinsic growth rate of first, second and third species respectively, $\alpha_{ii}, i = 1, 2, 3$ represent the rate of decrease of first, second and third species due to limited resources, $\alpha_{12}$ is the mutual coefficient of second species to first species, $\alpha_{13}$ is the mutual coefficient of third species to first species, $\alpha_{21}$ is the mutual coefficient of first species to second species, $\alpha_{23}$ is the mutual coefficient of third species to second species, $\alpha_{31}$ is the mutual coefficient of first species to third species, $\alpha_{32}$ is the mutual coefficient of second species to third species, $k_i(t)$ called the delay kernels, are weighting factors which indicating how much emphasis should be given to the size of the population at earlier times to determine the present effect on resources availability. Here $a_i, \alpha_{ii}, i = 1, 2, 3,$ and $\alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{23}, \alpha_{31}, \alpha_{32}$ are assumed to be nonnegative constants. Usually the delay kernels are normalized so that

$$\int_0^{\infty} k_i(u)du = 1 \qquad i = 1, 2, 3.$$

We assume that every kernel $k_i$ appearing in system (2.1) is a normalized convex combination of functions

$$k(u) = \frac{\beta^n u^{n-1} e^{-\beta u}}{(n-1)!} \qquad n = 1, 2, ..$$

with $\beta > 0$ is a constant, $n$ an integer. When $n = 1$, the kernel is $k(u) = \beta e^{-\beta u}$. Therefore, the system (2.1) becomes

$$\begin{aligned}
\frac{dN_1}{dt} &= N_1\Big(a_1 - \alpha_{11}N_1 + \alpha_{12}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_2(s)ds + \alpha_{13}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_3(s)ds\Big),\\
\frac{dN_2}{dt} &= N_2\Big(a_2 - \alpha_{22}N_2 + \alpha_{21}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_1(s)ds + \alpha_{23}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_3(s)ds\Big),\\
\frac{dN_3}{dt} &= N_3\Big(a_3 - \alpha_{33}N_3 + \alpha_{31}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_1(s)ds + \alpha_{32}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_2(s)ds\Big),
\end{aligned} \tag{2.2}$$

where using linear chain trick, define

$$\begin{aligned}
P_1(t) &= \int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_1(s)ds,\\
P_2(t) &= \int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_2(s)ds,\\
P_3(t) &= \int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_3(s)ds.
\end{aligned}$$

Therefore, the system (2.2) is equivalent to the following system of six ordinary differential equations.

$$\begin{aligned}
\frac{dN_1}{dt} &= N_1(a_1 - \alpha_{11}N_1 + \alpha_{12}P_2 + \alpha_{13}P_3),\\
\frac{dN_2}{dt} &= N_2(a_2 - \alpha_{22}N_2 + \alpha_{21}P_1 + \alpha_{23}P_3),\\
\frac{dN_3}{dt} &= N_3(a_3 - \alpha_{33}N_3 + \alpha_{31}P_1 + \alpha_{32}P_2),\\
\frac{dP_1}{dt} &= \beta(N_1 - P_1),\\
\frac{dP_2}{dt} &= \beta(N_2 - P_2),\\
\frac{dP_3}{dt} &= \beta(N_3 - P_3).
\end{aligned} \tag{2.3}$$

## 3 Stability Analysis

In this section, the existence of the unique positive biologically feasible equilibrium point of the system (2.3) and local and global stabilities are investigated. The equilibrium point $E_1(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*)$ exists if and only if there is a unique positive solution to the following equations.

$$\begin{aligned} -\alpha_{11}N_1 + \alpha_{12}P_2 + \alpha_{13}P_3 &= -a_1, \\ -\alpha_{22}N_2 + \alpha_{21}P_1 + \alpha_{23}P_3 &= -a_2, \\ -\alpha_{33}N_3 + \alpha_{31}P_1 + \alpha_{32}P_2 &= -a_3, \\ \beta(N_1 - P_1) &= 0, \\ \beta(N_2 - P_2) &= 0, \\ \beta(N_3 - P_3) &= 0, \end{aligned}$$

provided that the four conditions

$(C_1)$ $a_1\alpha_{22}\alpha_{33} + a_2(\alpha_{12}\alpha_{33} + \alpha_{13}\alpha_{32}) + a_3(\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{22}) > a_1\alpha_{23}\alpha_{32}$,

$(C_2)$ $a_1(\alpha_{21}\alpha_{33} + \alpha_{23}\alpha_{31}) + a_2\alpha_{11}\alpha_{33} + a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21}) > a_2\alpha_{13}\alpha_{31}$,

$(C_3)$ $a_1(\alpha_{22}\alpha_{31} + \alpha_{21}\alpha_{32}) + a_2(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31}) + a_3\alpha_{11}\alpha_{22} > a_3\alpha_{12}\alpha_{21}$,

$(C_4)$ $\alpha_{11}\alpha_{22}\alpha_{33} > \alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{22}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32}$,

hold, where

$$\begin{aligned} N_1^* = P_1^* = &\Big[a_1(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + a_2(\alpha_{12}\alpha_{33} + \alpha_{13}\alpha_{32} + a_3(\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{22})\Big]\Big/\Big[\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} \\ &- \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}\Big], \\ N_2^* = P_2^* = &\Big[a_1(\alpha_{21}\alpha_{33} + \alpha_{23}\alpha_{31}) + a_2(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}) + a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21})\Big]\Big/\Big[\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} \\ &- \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}\Big], \\ N_3^* = P_3^* = &\Big[a_1(\alpha_{22}\alpha_{31} + \alpha_{21}\alpha_{32}) + a_2(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31}) + a_3(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\Big]\Big/\Big[\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} \\ &- \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}\Big]. \end{aligned}$$

We note that the equilibrium point $E_1$ of the system (2.3) is also an equilibrium point of the system (2.1) with the kernel $\beta e^{-\beta u}$. To discuss the local stability of the system (2.3), we compute variational matrix about equilibrium point $E_1$ as

$$J_1(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*) = \begin{bmatrix} -\alpha_{11}N_1^* & 0 & 0 & 0 & \alpha_{12}N_1^* & \alpha_{13}N_1^* \\ 0 & -\alpha_{22}N_2^* & 0 & \alpha_{21}N_2^* & 0 & \alpha_{23}N_2^* \\ 0 & 0 & -\alpha_{33}N_3^* & \alpha_{31}N_3^* & \alpha_{32}N_3^* & 0 \\ \beta & 0 & 0 & -\beta & 0 & 0 \\ 0 & \beta & 0 & 0 & -\beta & 0 \\ 0 & 0 & \beta & 0 & 0 & -\beta \end{bmatrix}$$

The characteristic equation of the above variational matrix about equilibrium point $E_1$ is

$$\lambda^6 + k_1\lambda^5 + k_2\lambda^4 + k_3\lambda^3 + k_4\lambda^2 + k_5\lambda + k_6 = 0,$$

where,

$$\begin{aligned} k_1 =& 3\beta + \alpha_{11}N_1^* + \alpha_{22}N_2^* + \alpha_{33}N_3^* \\ k_2 =& 3\beta^2 + 3\Big(\alpha_{11}N_1^* + \alpha_{22}N_2^* + \alpha_{33}N_3^*\Big)\beta + \alpha_{11}\alpha_{22}N_1^*N_2^* + \alpha_{22}\alpha_{33}N_2^*N_3^* + \alpha_{11}\alpha_{33}N_1^*N_3^* \end{aligned}$$

$$\begin{aligned}
k_3 =& \beta^3 + 3\Big(\alpha_{11}N_1^* + \alpha_{22}N_2^* + \alpha_{33}N_3^*\Big)\beta^2 + 3\Big(\alpha_{11}\alpha_{22}N_1^*N_2^* + \alpha_{22}\alpha_{33}N_2^*N_3^* + \alpha_{11}\alpha_{33}N_1^*N_3^*\Big)\beta \\
&+ \alpha_{11}\alpha_{22}\alpha_{33}N_1^*N_2^*N_3^* \\
k_4 =& \Big(\alpha_{11}N_1^* + \alpha_{22}N_2^* + \alpha_{33}N_3^*\Big)\beta^3 + \Big[\Big(3\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\Big)N_1^*N_2^* + \Big(3\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}\Big)N_2^*N_3^* \\
&+ \Big(3\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\Big)N_1^*N_3^*\Big]\beta^2 + 3\alpha_{11}\alpha_{22}\alpha_{33}N_1^*N_2^*N_3^*\beta \\
k_5 =& \Big[\Big(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\Big)N_1^*N_2^* + \Big(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}\Big)N_2^*N_3^* + \Big(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\Big)N_1^*N_3^*\Big]\beta^3 \\
&+ \Big(3\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} - \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{13}\alpha_{22}\alpha_{31}\Big)N_1^*N_2^*N_3^*\beta^2 \\
k_6 =& \Big(\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} - \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}\Big)N_1^*N_2^*N_3^*\beta^3.
\end{aligned}$$

It is very difficult to find the roots or apply Routh-Hurwitz criteria. Therefore, we conclude that if all the roots have negative real part then the system (2.3) is stable (see numerical examples in Section 4).
Now we establishes the global stability of the system (2.3) by constructing a suitable Lyapunov function in the following theorem.

**Theorem 3.1.** *The positive equilibrium point $E_1(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*)$ of the system (2.3) is globally stable, if*

$$\begin{aligned}
2\alpha_{11} &> \alpha_{12}^2 + \alpha_{13}^2 + 4 \\
2\alpha_{22} &> \alpha_{21}^2 + \alpha_{23}^2 + 4 \\
2\alpha_{33} &> \alpha_{31}^2 + \alpha_{32}^2 + 4
\end{aligned}$$

*holds.*

*Proof.* The proof can be reached by using a Lyapunov stability theorem which gives a sufficient condition. Now, let us consider a positive definite function

$$V(N_1, N_2, N_3) = V_1(N_1) + V_2(N_2) + V_3(N_3) + V_4(P_1) + V_5(P_2) + V_6(P_3)$$

where,

$$\begin{aligned}
V_1(N_1) &= 2\Big(N_1 - N_1^* - N_1^* \ln \frac{N_1}{N_1^*}\Big), \\
V_2(N_2) &= 2\Big(N_2 - N_2^* - N_2^* \ln \frac{N_2}{N_2^*}\Big), \\
V_3(N_3) &= 2\Big(N_3 - N_3^* - N_3^* \ln \frac{N_3}{N_3^*}\Big), \\
V_4(P_1) &= \frac{2}{\beta}(P_1 - P_1^*)^2, \\
V_5(P_2) &= \frac{2}{\beta}(P_2 - P_2^*)^2, \\
V_6(P_3) &= \frac{2}{\beta}(P_3 - P_3^*)^2,
\end{aligned}$$

on $H = \{(N_1, N_2, N_3, P_1, P_2, P_3) \mid \quad N_1 > 0, N_2 > 0, N_3 > 0, P_1 > 0, P_2 > 0, P_3 > 0\}$. It is obvious that $V(N_1, N_2, N_3, P_1, P_2, P_3) \in C^1(H, R)$ and $V(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*) = 0$. The function $V(N_1, N_2, N_3, P_1, P_2, P_3)$ satisfies

$$V(N_1, N_2, N_3, P_1, P_2, P_3) > V(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*) = 0$$

which holds for all $V(N_1, N_2, N_3, P_1, P_2, P_3) \in H - \{E_1\}$. Then the time derivative of $V(N_1, N_2, N_3, P_1, P_2, P_3)$ computed along the solution of the system (2.3) is

$$\frac{dV}{dt} = 2\Big[ - \alpha_{11}(N_1 - N_1^*)^2 - \alpha_{22}(N_2 - N_2^*)^2 - \alpha_{33}(N_3 - N_3^*)^2$$

$$
\begin{aligned}
&+\alpha_{12}(N_1-N_1^*)(P_2-P_2^*)+\alpha_{13}(N_1-N_1^*)(P_3-P_3^*)\\
&\quad+\alpha_{21}(N_2-N_2^*)(P_1-P_1^*)+\alpha_{23}(N_2-N_2^*)(P_3-P_3^*)\\
&\quad\quad+\alpha_{31}(N_3-N_3^*)(P_1-P_1^*)+\alpha_{32}(N_3-N_3^*)(P_2-P_2^*)\Big]\\
&+4\Big[(N_1-N_1^*)(P_1-P_1^*)+(N_2-N_2^*)(P_2-P_2^*)\\
&\quad+(N_3-N_3^*)(P_3-P_3^*)-(P_1-P_1^*)^2-(P_2-P_2^*)^2-(P_3-P_3^*)^2\Big]\\
=&-(P_1-P_1^*)^2-(P_2-P_2^*)^2-(P_3-P_3^*)^2-\Big[2\alpha_{11}-\alpha_{12}^2\\
&\quad-\alpha_{13}^2-4\Big](N_1-N_1^*)^2-\Big[2\alpha_{22}-\alpha_{21}^2-\alpha_{23}^2-4\Big](N_2-N_2^*)^2\\
&\quad-\Big[2\alpha_{33}-\alpha_{31}^2-\alpha_{32}^2-4\Big](N_3-N_3^*)^2-\Big[\alpha_{12}(N_1-N_1^*)-(P_2-P_2^*)\Big]^2\\
&\quad-\Big[\alpha_{13}(N_1-N_1^*)-(P_3-P_3^*)\Big]^2-\Big[\alpha_{21}(N_2-N_2^*)-(P_1-P_1^*)\Big]^2\\
&\quad-\Big[\alpha_{23}(N_2-N_2^*)-(P_3-P_3^*)\Big]^2-\Big[\alpha_{31}(N_3-N_3^*)-(P_1-P_1^*)\Big]^2\\
&\quad-\Big[\alpha_{32}(N_3-N_3^*)-(P_2-P_2^*)\Big]^2\\
<&\quad 0
\end{aligned}
$$

This shows that $\frac{dV}{dt}<0$ on $H$. Therefore, the function $V$ is a Lyapunov function with respect to $E_1$. Hence, the equilibrium point $E_1$ is globally asymptotically stable on $H$. □

Consequently, we have the following result.

**Theorem 3.2.** *The equilibrium point* $(N_1^*, N_2^*, N_3^*)$ *of the system (2.1) with a kernel* $k(u)=\beta e^{-\beta u}$ *is globally stable.*

# 4 Numerical Simulations

To check the feasibility of our analysis regarding stability conditions, we have conducted some numerical computation by choosing the following set of parameters values in model system (2.3) as

$$
\begin{aligned}
&a_1=1,\quad a_2=0.5,\quad a_3=2,\quad \alpha_{11}=1,\quad \alpha_{12}=0.1,\quad \alpha_{13}=0.3,\quad \alpha_{21}=0.2,\\
&\alpha_{22}=1.5,\quad \alpha_{23}=0.3,\quad \alpha_{31}=0.4\quad \alpha_{32}=0.6\quad \alpha_{33}=1.3,\quad \beta=8
\end{aligned}
$$

With the above parameter values, it follows that the system (2.3) is locally stable as shown in Figure 1. However, even if these parameter do not satisfy the conditions of Theorem 3.1, Figure 2 exhibits that the system (2.3) seems to be globally stable.
Consider the another set of parameters values in system (2.3) as

$$
\begin{aligned}
&a_1=2,\quad a_2=4,\quad a_3=3,\quad \alpha_{11}=2.5,\quad \alpha_{12}=0.1,\quad \alpha_{13}=0.3,\quad \alpha_{21}=0.2,\\
&\alpha_{22}=3.5,\quad \alpha_{23}=0.3,\quad \alpha_{31}=0.4\quad \alpha_{32}=0.6\quad \alpha_{33}=3.3
\end{aligned}
$$

From Theorem 3.1, under these parameters values the system (2.3) is globally stable as shown in the figure 3.

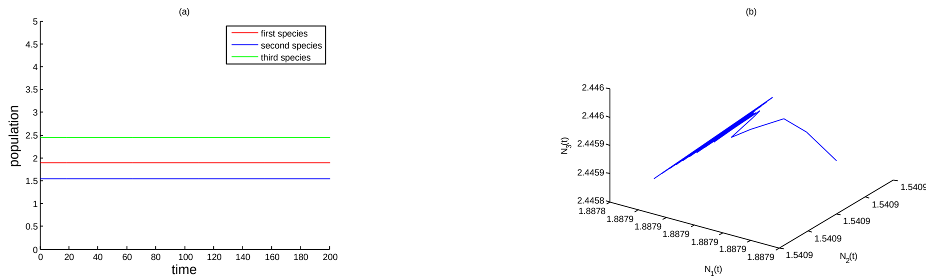


**Figure 1:** (a) Time series for $N_1(t), N_2(t)$ and $N_3(t)$. (b) The phase graph with initial condition $(1.8879, 1.5409, 2.4459, 1.8879, 1.5409, 2.4459)$.

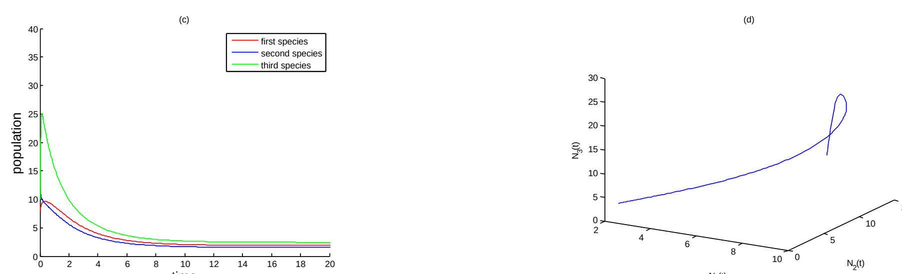


**Figure 2:** (c) Time series for $N_1(t), N_2(t)$ and $N_3(t)$. (d) The phase graph with initial condition $(8, 12, 10, 40, 30, 20)$.

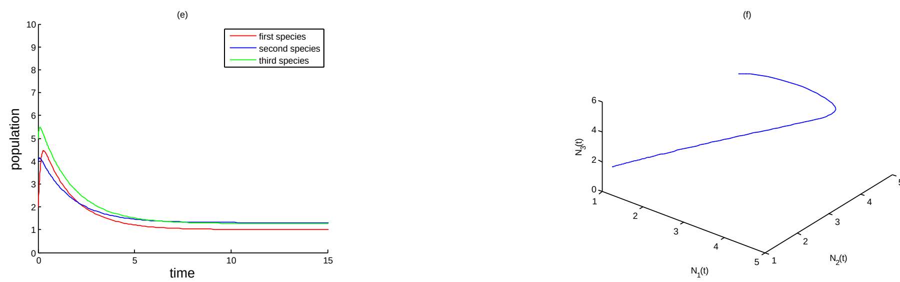


**Figure 3:** (e) Time series for $N_1(t), N_2(t)$ and $N_3(t)$. (f) The phase graph with initial condition $(2, 4, 5, 10, 20, 30)$.

## 5 Discussion

In this article, local and global stabilities of the three mutually interacting species with continuous time delay has been investigated. Our numerical simulation shows that even if time delay parameter vary for large value the system (2.3) remains stable. The approach of study in this article differs from Feng C. H. and Chao P. H. [4], Mukherjee D. [11], Shukla V. P. [12] and Xia Y. [14] in the sense that it studies two species mutualistic system with discrete delay. To the best of our knowledge, this paper is the first time to deal with the research for system (2.1) which belongs to a three species mutualism model with continuous time delay. There is a lot of work to do in this area. For example it would be interesting to see what the behavior of the model (2.1) would be when several delays occurs in this system. However less attention has been given to the study of mutualism as compared to the prey-predator and competition. Thus the present article contributes a few more results on mutualism model.

## References

[1] E. Beretta and Y. Takeuchi, Global stability of single species diffusion Volterra models continuous time delays, *Bulletin of Mathematical Biology*, 49(4)(1987), 431-448.

[2] A.W. Busekros, Global stability in ecological systems with continuous time delay, *SIAM J. Appl. Math.*, 35(1)(1978), 123-134.

[3] J.M. Cushing, Integro-differential equations and delay models in population dynamics, *Lecture-Notes in Biomathematics*, No. 20, Springer-Verlag, Berlin, 1977.

[4] C.H. Feng and P.H. Chao, Global stability for the Lotka Volterra mutualistic system with time delay, *Tunghai Science*, 8(2006), 81-107.

[5] K. Gopalsamy, *Stability on the oscillations in delay differential equations of population dynamics*, Academic Press, New York, 1993.

[6] K. Gopalsamy, Global asymptotic stability in Volterra's population systems, *J. Math. Biology*, 19(1984), 157-168.

[7] J.K. Hale and P. Waltman, Persistence in finite-dimensional systems, *SIAM J. Math. Anal.*, 20(1989), 388-395.

[8] S.W. Harlan, The effect of time lags on the stability of the equilibrium state of a population growth equation, *J. Math. Biology*, 5(1978), pp. 115-120.

[9] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, San Diego, 1993.

[10] N. Mc Donald, Time lags in biological models, *Lecture Notes in Biomathematics*, No. 27, Springer-Verlag, Berlin 1978.

[11] D. Mukherjee, Permanence and global attractivity for facultative mutualism system with delay, *Math. Meth. Appl. Sci.*, 26(2003), 1-9.

[12] V.P. Shukla, Condition for global stability of two species population models with discrete time delay, *Bull. Math. Biology*, 45(5)(1983), 793-805.

[13] F. Solimano and E. Beretta, Existence of a globally asymptotically stable equilibrium in Volterra models with continuous time delay, *J. Math. Biology*, 18(1983), 93-102.

[14] Y. Xia, Existence of positive solutions of mutualism systems with several delays, *Advances in Dynamical Systems and Applications*, 1(2)(2006), 209-217.

*Received*: October 19, 2012; *Accepted*: February 3, 2013

UNIVERSITY PRESS