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Oscillation criteria for third order neutral difference equations with distributed delay

R. Arul[a] and G. Ayyappan[b,*]

[a,b] *Department of Mathematics, Kandaswami Kandar's College, Velur - 638 182, Tamil Nadu, India.*

Abstract

In this paper we study the oscillatory behavior of third order neutral difference equation of the form

$$\Delta\Big(r(n)\Delta^2 z(n)\Big) + \sum_{s=c}^{d} q(n,s) f(x(n+s-\sigma)) = 0, n \geq n_0 \geq 0, \tag{0.1}$$

where $z(n) = x(n) + \sum_{s=a}^{b} p(n,s)x(n+s-\tau)$. We establish some sufficient conditions which ensure that every solution of the equation (0.1) oscillates or converges to zero by using a generalized Ricaati transformation and Philos - type technique. An example is given to illustrate the main result.

Keywords: Third order, oscillation, neutral difference equations, Philos - type.

2010 MSC: 39A10.

1 Introduction

In this paper we consider the oscillatory behavior of third order neutral difference equation of the form

$$\Delta\Big(r(n)\Delta^2 z(n)\Big) + \sum_{s=c}^{d} q(n,s) f(x(n+s-\sigma) = 0, n \in \mathbb{N}_0 \tag{1.1}$$

where

$$z(n) = x(n) + \sum_{s=a}^{b} p(n,s)x(n+s-\tau), \tag{1.2}$$

Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, $\mathbb{N}_0 = \{n_0,\ n_0+1,\ n_0+2,\ \ldots\}$, n_0 is a nonnegative integer, and $a,\ b,\ c,\ d \in \mathbb{N}_0$ subject to the following conditions:

(C_1) $\{r(n)\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{r(n)} = \infty$;

(C_2) $\{q(n,s)\}$ and $\{p(n,s)\}$ are nonnegative real sequences with $0 \leq p(n) \equiv \sum_{s=a}^{b} p(n,s) \leq P < 1$;

(C_3) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\frac{f(u)}{u} \geq L > 0$, for $u \neq 0$.

By a solution of equation (1.1) we mean a real sequence $\{x(n)\}$ and satisfying equation (1.1) for all $n \in \mathbb{N}_0$. We consider only those solution $\{x(n)\}$ of equation (1.1) which satisfy $sup\{|x(n)| : n \geq N\} > 0$ for all $N \in \mathbb{N}_0$. A solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years there is a great interest in studying the oscillatory behavior of third order difference equations, see for example [1–5, 7–14] and the references cited therein. Motivated by this observation, in this paper

*Corresponding author.
E-mail addresses: rarulkkc@gmail.com (R. Arul) and ayyapmath@gmail.com (G. Ayyappan).

we obtain some sufficient conditions for the oscillation of all solution of equation (1.1).

In Section 2, we present some preliminary lemmas and in Section 3, we establish some sufficient conditions which ensure that all solutions of equation (1.1) are either oscillatory or converges to zero. An example is given to illustrate the main result.

2 Preliminary Lemmas

In this section, we present some lemmas which will be useful to prove our main results.

Lemma 2.1. *Let* $\{x(n)\}$ *be a positive solution of equation* (1.1) *and* $\{z(n)\}$ *be defined as in* (1.2). *Then* $\{z(n)\}$ *satisfies only of the following two cases eventually*

(*I*) $z(n) > 0, \quad \Delta z(n) > 0, \quad \Delta^2 z(n) > 0;$

(*II*) $z(n) > 0, \quad \Delta z(n) < 0, \quad \Delta^2 z(n) > 0.$

Proof. Assume that $\{x(n)\}$ is a positive solution of equation (1.1). By definition of $\{z(n)\}$ we have $z(n) > x(n) > 0$ for all $n \geq n_0$. From the equation (1.1), we have

$$\Delta(r(n)\Delta^2 z(n)) = -\sum_{s=c}^{d} q(n,s) f(x(n+s-\sigma) < 0.$$

Thus $r(n)\Delta^2 z(n)$ is a nonincreasing function and therefore eventually of one sign. So $\Delta^2 z(n)$ is either eventually positive or eventually negative for $n \geq n_1 \geq n_0$. If $\Delta^2 z(n) < 0$, then there is constant $M > 0$ such that

$$r(n)\Delta^2 z(n) \leq -M < 0, \ n \geq n_1.$$

Summing the last inequality from n_1 to $n-1$, we obtain

$$\Delta z(n) \leq \Delta z(n_1) - M \sum_{s=n_1}^{n-1} \frac{1}{r(s)}.$$

Letting $n \to \infty$, then using condition (C_1), we have $\Delta z(n) \to -\infty$, and therefore $\Delta z(n) < 0$. Since $\Delta^2 z(n) < 0$ and $\Delta z(n) < 0$, we have $z(n) < 0$, which is a contradiction to our assumption. This proves that $\Delta^2 z(n) > 0$ and we have only Case (I) or (II) for $\{z(n)\}$. This completes the proof. □

Lemma 2.2. *Let* $\{x(n)\}$ *be a positive solution of equation* (1.1), *and let the corresponding function* $\{z(n)\}$ *satisfies the Case* (II) *of Lemma 2.1. If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \left[\frac{1}{r(s)} \sum_{t=s}^{\infty} \sum_{j=c}^{d} q(t,j)\right] = \infty, \tag{2.1}$$

then $\lim_{n\to\infty} x(n) = \lim_{n\to\infty} z(n) = 0$.

Proof. Let $\{x(n)\}$ be a positive solution of equation (1.1), and $\{z(n)\}$ satisfies Case (II) of Lemma 2.1. Then there exists $\ell \geq 0$ such that $\lim_{n\to\infty} z(n) = \ell$. We shall prove that $\ell = 0$. Assume that $\ell > 0$, then we have $\ell + \epsilon < z(n) < \ell$ for all $\epsilon > 0$ and $n \geq n_1 \geq n_0$. Choosing $0 < \epsilon < \frac{\ell(1-P)}{P}$. From (1.2), we have

$$\begin{aligned} x(n) &= z(n) - \sum_{s=a}^{b} p(n,s) x(n+s-\tau) \\ &> \ell - \sum_{s=a}^{b} p(n,s) z(n+s-\tau) \\ &> \ell - P(\ell+\epsilon) \\ &= \frac{\ell - P(\ell+\epsilon)}{\ell+\epsilon}(\ell+\epsilon) \\ &> kz(n) \end{aligned} \tag{2.2}$$

where $k = \frac{\ell - P(\ell+\epsilon)}{\ell+\epsilon}$. From the equation (1.1), we have

$$\begin{aligned}\Delta(r(n)\Delta^2 z(n)) &= -\sum_{s=c}^{d} q(n,s) f(x(n+s-\sigma) \\ &\leq -\sum_{s=c}^{d} q(n,s) L x(n+s-\sigma).\end{aligned}$$

Now using (2.2), we obtain

$$\Delta(r(n)\Delta^2 z(n)) \leq -kL \sum_{s=c}^{d} q(n,s) z(n+s-\sigma).$$

Summing the last inequality from n to ∞, we have

$$-r(n)\Delta^2 z(n) \leq -kL \sum_{t=n}^{\infty} \sum_{s=c}^{d} q(t,s) z(t+s-\sigma)$$

or

$$\Delta^2 z(n) \geq kL\ell \frac{1}{r(n)} \sum_{t=n}^{\infty} \sum_{s=c}^{d} q(t,s).$$

Summing again from n to ∞, we have

$$-\Delta z(n) \geq kL\ell \sum_{s=n}^{\infty} \Big[\frac{1}{r(s)} \sum_{t=s}^{\infty} \sum_{j=c}^{d} q(t,j)\Big].$$

Summing the last inequality from n_1 to ∞, we obtain

$$z(n_1) \geq kL\ell \sum_{n=n_1}^{\infty} \sum_{s=n}^{\infty} \Big[\frac{1}{r(s)} \sum_{t=s}^{\infty} \sum_{j=c}^{d} q(t,j)\Big]$$

which contradicts condition (2.1). Thus $\ell = 0$. Moreover, the inequality $0 < x(n) \leq z(n)$ implies that $\lim_{n\to\infty} x(n) = 0$. The proof is now complete. □

Lemma 2.3. *Assume that $y(n) > 0$, $\Delta y(n) \geq 0$, $\Delta^2 y(n) \leq 0$ for all $n \geq n_0$. Then for each $\alpha \in (0,1)$ there exists a $N \in \mathbb{N}_0$ such that*

$$\frac{y(n-\sigma)}{n-\sigma} \geq \alpha \frac{y(n+1)}{n+1} \quad \textit{for all } n \geq N. \tag{2.3}$$

Proof. From the monotonicity property of $\{\Delta y(n)\}$, we have

$$y(n+1) - y(n-\sigma) = \sum_{s=n-\sigma}^{n} \Delta y(s) \leq (\sigma+1)\Delta y(n-\sigma)$$

or

$$\frac{y(n+1)}{y(n-\sigma)} \leq 1 + \frac{(\sigma+1)\Delta y(n-\sigma)}{y(n-\sigma)}. \tag{2.4}$$

Also,

$$y(n-\sigma) \geq y(n-\sigma) - y(n_0) \geq (n-\sigma-n_0)\Delta y(n-\sigma).$$

So, for each $\alpha \in (0,1)$, there is a $N \in \mathbb{N}_0$ such that

$$\frac{y(n-\sigma)}{\Delta y(n-\sigma)} \geq \alpha(n-\sigma), \quad n \geq N. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\frac{y(n+1)}{y(n-\sigma)} \leq 1 + \frac{(\sigma+1)}{\alpha(n-\sigma)} \leq \frac{\alpha n - \alpha\sigma + \sigma + 1}{\alpha(n-\sigma)}$$

or

$$\frac{y(n+1)}{y(n-\sigma)} \leq \frac{(n+1)}{\alpha(n-\sigma)}.$$

This completes the proof. □

Lemma 2.4. *Assume that* $z(n) > 0,\ \Delta z(n) > 0,\ \Delta^2 z(n) > 0,\ \Delta^3 z(n) \le 0$ *for all* $n \ge N$. *Then*

$$\frac{z(n)}{\Delta z(n)} \ge \frac{n-N}{2} \text{ for all } n \ge N. \tag{2.6}$$

Proof. From the monotonicity property of $\{\Delta^2 z(n)\}$, we have

$$\Delta z(n) = \Delta z(N) + \sum_{s=N}^{n-1} \Delta^2 z(n) \ge (n-N)\Delta^2 z(n).$$

Summing from N to $n-1$, we obtain

$$\begin{aligned} z(n) &\ge z(N) + \sum_{s=N}^{n-1} (s-N)\Delta^2 z(s) \\ &= z(N) + (n-N)\Delta z(n) - z(n+1) + z(N). \end{aligned}$$

Hence $z(n) \ge \frac{(n-N)}{2}\Delta z(n),\ \ n \ge N$. This completes the proof. □

3 Main Results

In this section, we obtain new oscillation criteria for the equation (1.1) by using the generalized Riccati transformation and Philos type technique.

Theorem 3.1. *Assume that condition* (2.1) *holds. If there exists a positive nondecreasing real sequence* $\{\rho(n)\}$ *such that*

$$\lim_{n\to\infty} \sum_{s=N}^{n-1} \Big[Q(s) - \frac{(\Delta\rho(s))^2}{4\rho(s+1)r(s)} \Big] = \infty \tag{3.1}$$

where

$$Q(n) = \rho(n) q_1(n) \frac{\alpha(n-\sigma)(n+c-\sigma-N)}{2(n+1)}, \tag{3.2}$$

and

$$q_1(n) = L(1-P)\sum_{s=c}^{d} q(n,s), \tag{3.3}$$

then every solution of equation (1.1) *is either oscillatory or converges to zero.*

Proof. Assume that $\{x(n)\}$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(n) > 0,\ x(n+s-\tau) > 0$ for $n \ge n_1 \ge n_0 \in \mathbb{N}_0$ and $\{z(n)\}$ is defined as in (1.2). Then $\{z(n)\}$ satisfies two cases of Lemma 2.1.
Case(I). Let $\{z(n)\}$ satisfies Case (I) of Lemma 2.1. From (1.2), we have

$$\begin{aligned} x(n) &\ge z(n) - \sum_{s=a}^{b} p(n,s) z(n+s-\tau) \\ &\ge \Big(1 - \sum_{s=a}^{b} p(n,s)\Big) z(n) \\ &\ge (1-P)z(n). \end{aligned} \tag{3.4}$$

Using condition (C_3) in equation (1.1), we have

$$\Delta(r(n)\Delta^2 z(n)) \le -\sum_{s=c}^{d} q(n,s) L x(n+s-\sigma). \tag{3.5}$$

Now using (3.4) in inequality (3.5), we obtain

$$\Delta(r(n)\Delta^2 z(n)) \le -L(1-P)\sum_{s=c}^{d} q(n,s) z(n+s-\sigma)$$

$$\leq \quad -q_1(n)z(n+c-\sigma). \tag{3.6}$$

Define

$$w(n) = \rho(n)\frac{r(n)\Delta^2 z(n)}{\Delta z(n)}, \quad n \geq n_1. \tag{3.7}$$

Then $w(n) > 0$ for all $n \geq n_1$ and from (3.6), we have

$$\begin{aligned}
\Delta w(n) \quad &\leq \quad -\rho(n)\frac{q_1(n)z(n+c-\sigma)}{\Delta z(n+1)} + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\
&\qquad - w(n+1)\frac{\Delta^2 z(n)}{\Delta z(n)} \\
&\leq \quad -\rho(n)\frac{q_1(n)z(n+c-\sigma)}{\Delta z(n+1)} + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\
&\qquad - \frac{w^2(n+1)}{\rho(n+1)r(n)}.
\end{aligned} \tag{3.8}$$

By Lemma 2.3 with $y(n) = \Delta z(n)$, we have

$$\frac{1}{\Delta z(n+1)} \leq \frac{\alpha(n-\sigma)}{n+1}\frac{1}{\Delta z(n-\sigma)} \quad for\ all \quad n \geq N. \tag{3.9}$$

Unsing (3.9) in (3.8), we obtain

$$\begin{aligned}
\Delta w(n) \quad &\leq \quad -\rho(n)q_1(n)\frac{\alpha(n-\sigma)}{n+1}\frac{z(n+c-\sigma)}{\Delta z(n-\sigma)} + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\
&\qquad - \frac{w^2(n+1)}{\rho(n+1)r(n)}.
\end{aligned}$$

Now applying Lemma 2.4 in the last inequality, we obtain

$$\begin{aligned}
\Delta w(n) \quad &\leq \quad -\rho(n)q_1(n)\frac{\alpha(n-\sigma)}{n+1}\frac{(n+c-\sigma-N)}{2} \\
&\qquad + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) - \frac{w^2(n+1)}{\rho(n+1)r(n)} \\
&\leq \quad -Q(n) + A(n)w(n+1) - B(n)w^2(n+1)
\end{aligned}$$

or

$$Q(n) \leq -\Delta w(n) + A(n)w(n+1) - B(n)w^2(n+1) \tag{3.10}$$

where

$$A(n) = \frac{\Delta\rho(n)}{\rho(n+1)}, \qquad B(n) = \frac{1}{\rho(n+1)r(n)}.$$

Now, using completing the square, we have

$$Q(n) - \frac{(A(n))^2}{4B(n)} \leq -\Delta w(n).$$

Summing the last inequality from N to $n-1$, we have

$$\sum_{s=N}^{n-1}\left(Q(s) - \frac{(\Delta\rho(s))^2}{4\rho(s+1)r(s)}\right) \leq w(N) - w(n) \leq w(N).$$

Letting $n \to \infty$, we obtain a contradiction to (3.1).
If $\{z(n)\}$ satisfies Case (II) of Lemma 2.1, then by condition (2.1) we have $\lim_{n\to\infty} x(n) = 0$. This completes the proof. □

Before stating the next theorem, we define functions $h,\ H : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ such that

(i) $H(n,n) = 0$ for $n \geq n_0 \geq 0$;

(ii) $H(n,s) > 0$ for $n > s \geq n_0$;

(iii) $\Delta_2 H(n,s) = H(n,s+1) - H(n,s) \leq 0$ for $n > s \geq n_0$ and there exists a positive real sequence $\{\rho(n)\}$ such that

$$\Delta_2 H(n,s) + \frac{\Delta\rho(s)}{\rho(s+1)} H(n,s) = -h(n,s)\sqrt{H(n,s)}$$

for $n > s \geq n_0$.

Theorem 3.2. *Assume that* (2.1) *holds. If there exists a positive real sequence* $\{\rho(n)\}$ *such that*

$$\lim_{n\to\infty} sup \frac{1}{H(n,n_0)} \sum_{s=n_0}^{n-1} \Big[H(n,s)Q(s) - \frac{1}{4}\rho(s+1)r(s)h^2(n,s)\Big] = \infty, \tag{3.11}$$

then every solution of equation (1.1) *is either oscillatory or converges to zero.*

Proof. Assume that $\{x(n)\}$ is a nonoscillatory solution of equation (1.1). Proceeding as the proof of Theorem 3.1, we have (3.10). Now multiplying the inequality (3.10) by $H(n,s)$, then summing the resulting inequality from n_2 to $n-1$ for all $n \geq n_2 \geq n_0$, we have

$$\sum_{s=n_2}^{n-1} H(n,s)Q(s) \leq -\sum_{s=n_2}^{n-1} \Delta w(s)H(n,s) + \sum_{s=n_2}^{n-1} (A(s)w(s+1) - B(s)w^2(s+1))H(n,s).$$

By summation by parts, we obtain

$$\begin{aligned}\sum_{s=n_2}^{n-1} H(n,s)Q(s) &\leq H(n,n_2)w(n_2) + \sum_{s=n_2}^{n-1} w(s+1)\Delta_2 H(n,s) \\ &\quad + \sum_{s=n_2}^{n-1} A(s)w(s+1)H(n,s) - \sum_{s=n_2}^{n-1} B(s)w^2(s+1)H(n,s) \\ &\leq H(n,n_2)w(n_2) + \sum_{s=n_2}^{n-1} \Big[\Delta_2 H(n,s) + \frac{\Delta\rho(s)}{\rho(s+1)} H(n,s)\Big] \times \\ &\quad w(s+1) - \sum_{s=n_2}^{n-1} B(s)w^2(s+1)H(n,s). \end{aligned}\tag{3.12}$$

Using completing the square in the last inequality, we obtain

$$\sum_{s=n_2}^{n-1} \Big[H(n,s)Q(s) - \frac{1}{4}\rho(s+1)r(s)h^2(n,s)\Big] \leq H(n,n_2)w(n_2)$$

or

$$\frac{1}{H(n,n_2)} \sum_{s=n_2}^{n-1} \Big[H(n,s)Q(s) - \frac{1}{4}\rho(s+1)r(s)h^2(n,s)\Big] \leq w(n_2).$$

Letting $n \to \infty$, we obtain a contradiction to (3.1).
If $\{z(n)\}$ satisfies Case (II) of Lemma 2.1, then by condition (2.1) we have $\lim_{n\to\infty} x(n) = 0$. This completes the proof. □

Corollary 3.1. *If* $H(n,s) = (n-s)^\beta$ *for all* $n \geq s \geq 0$ *and*

$$\lim_{n\to\infty} sup \frac{1}{n^\beta} \sum_{s=n_0}^{n-1} \Big[(n-s)^\beta Q(s) - \frac{1}{4}\rho(s+1)r(s)(n-s)^{\beta-2}\Big] = \infty, \tag{3.13}$$

for every $\beta \geq 1$*, then every solution of equation* (1.1) *is oscillatory.*

Corollary 3.2. *If* $H(n,s) = \left(log\frac{n+1}{s+1}\right)^{\beta}$ *for all* $n \geq s \geq 0$ *and*

$$\lim_{n\to\infty} sup(log(n+1))^{-\beta}\frac{1}{n^{\alpha}}\sum_{s=n_0}^{n-1}\Big[\Big(log\frac{n+1}{s+1}\Big)^{\beta}Q(s) - \frac{1}{4(s+1)^2}\rho(s+1)r(s)\Big(log\frac{n+1}{s+1}\Big)^{\beta-2}\Big] = \infty, \tag{3.14}$$

for every $\beta \geq 1$, *then every solution of equation* (1.1) *is oscillatory.*

The proof of Corollary 3.1 and 3.2 follows from Theorem 3.2 and hence the details are omitted.

Theorem 3.3. *Assume that all conditions of Theorem 3.2 are satisfied except condition* (3.11). *Also let*

$$0 < inf_{s\geq n_0}\Big[\lim_{n\to\infty} inf\frac{H(n,s)}{H(n,n_0)}\Big] \leq \infty \tag{3.15}$$

and

$$\lim_{n\to\infty} sup\frac{1}{H(n,n_0)}\sum_{s=n_0}^{n-1}\rho(s+1)r(s)h^2(n,s) < \infty \tag{3.16}$$

hold. If there exists a positive sequence $\{\psi(n)\}$ *such that*

$$\lim_{n\to\infty} sup\sum_{s=n_0}^{n-1}\frac{(\psi(n))^2}{\rho(s+1)r(s)} = \infty \tag{3.17}$$

and

$$\lim_{n\to\infty} sup\frac{1}{H(n,N)}\sum_{s=N}^{n-1}\Big[H(n,s)Q(s) - \frac{1}{4}\rho(s+1)r(s)h^2(n,s)\Big] \geq \psi(N), \tag{3.18}$$

then every solution of equation (1.1) *is either oscillatory or converges to zero.*

Proof. Proceeding as in the proof of Theorem 3.2, we obtain (3.12). Using completing the square in (3.12) and rearranging we obtain

$$\lim_{n\to\infty} sup\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\Big[H(n,s)Q(s) - \frac{h^2(n,s)}{4B(s)}\Big] \leq w(n_2) - \lim_{n\to\infty} inf\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\Big[\sqrt{H(n,s)B(s)}w(s+1) + \frac{h(n,s)}{2\sqrt{B(s)}}\Big]^2$$

for $n \geq n_2$. It follow from (3.18) that

$$w(n_2) \geq \psi(n_2) + \lim_{n\to\infty} inf\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\Big[\sqrt{H(n,s)B(s)}w(s+1) + \frac{h(n,s)}{2\sqrt{B(s)}}\Big]^2,$$

which means that,

$$w(n_2) \geq \psi(n_2) \quad for \quad n \geq N \tag{3.19}$$

and

$$\lim_{n\to\infty} inf\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\Big[\sqrt{H(n,s)B(s)}w(s+1) + \frac{h(n,s)}{2\sqrt{B(s)}}\Big]^2 < \infty.$$

Therefore

$$\lim_{n\to\infty} inf\Big[\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}H(n,s)B(s)w^2(s+1)$$

$$+\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}h(n,s)\sqrt{H(n,s)}w(s+1)$$
$$+\frac{1}{4}\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\frac{h^2(n,s)}{B(s)}\Big] < \infty.$$

Then

$$\lim_{n\to\infty} inf\Big[\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}H(n,s)B(s)w^2(s+1) + \frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}h(n,s)\sqrt{H(n,s)}w(s+1)\Big] < \infty. \tag{3.20}$$

Define the functions

$$U(n) = \frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}H(n,s)B(s)w^2(s+1)$$
$$V(n) = \frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\sqrt{H(n,s)}h(n,s)w(s+1)$$

Then, the inequality (3.20), implies that

$$\lim_{n\to\infty} inf[U(n)+V(n)] < \infty. \tag{3.21}$$

The rest of the proof is similar to that of Theorem 2 of [6], and hence the details are omitted.
If $\{z(n)\}$ satisfies Case (II) of Lemma 2.1, then by condition (2.1) we have $\lim_{n\to\infty} x(n) = 0$. This completes the proof. □

Theorem 3.4. *Assume that all conditions of Theorem 3.3 are satisfied except condition* (3.16). *Also let*

$$\lim_{n\to\infty} inf\frac{1}{H(n,n_0)}\sum_{s=n_0}^{n-1}H(n,s)Q(s) < \infty \tag{3.22}$$

and

$$\lim_{n\to\infty} inf\frac{1}{H(n,N)}\sum_{s=N}^{n-1}\Big[H(n,s)Q(s)-\frac{1}{4}\rho(s+1)r(s)h^2(n,s)\Big] \geq \psi(N) \tag{3.23}$$

then every solution of equation (1.1) *is either oscillatory or converges to zero.*

Proof. The proof is similar to that of Theorem 3.3 and hence the details are omitted. □

Now, let us define

$$H(n,s) = (n-s)^\beta, \quad n \geq s \geq 0,$$

where $\beta \geq 1$ is a constant. Then $H(n,n) = 0$, for $n \geq 0$ and $H(n,s) > 0$ for $n > s \geq 0$. Clearly $\Delta_2 H(n,s) \leq 0$ for $n > s \geq 0$ and

$$h(n,s) = [(n-s)^\beta - (n-s-1)^\beta](n-s)^{-(\beta/2)} \leq \beta(n-s)^{(\beta-2)/2},$$

for $n > s \geq 0$. We see that (3.15) holds,

$$\lim_{n\to\infty}\frac{H(n,s)}{H(n,n_0)} = \lim_{n\to\infty}\frac{(n-s)^\beta}{n^\beta} = 1.$$

Hence, by Theorems 3.3 and 3.4, we have the following two corollaries.

Corollary 3.3. *Let $\beta \geq 1$ be a constant, and suppose that*

$$\lim_{n\to\infty} sup \frac{1}{n^\beta} \sum_{s=n_0}^{n-1} \beta\rho(s+1)r(s)(n-s)^{\beta-2} < \infty. \tag{3.24}$$

If there is a sequence $\{\psi(n)\}$ satisfying (3.17) *and*

$$\lim_{n\to\infty} sup \frac{1}{(n-N)^\beta} \sum_{s=N}^{n-1} \left[(n-s)^\beta Q(s) - \frac{\beta^2}{4}\rho(s+1)r(s)(n-s)^{\beta-2}\right] \geq \psi(N) \tag{3.25}$$

then every solution of equation (1.1) *is oscillatory or converges to zero.*

Proof. The proof follows from Theorem 3.3 and hence the details are omitted. □

Corollary 3.4. *Let $\beta \geq 1$ be a constant, and suppose that*

$$\lim_{n\to\infty} inf \frac{1}{n^\beta} \sum_{s=n_0}^{n-1} (n-s)^\beta Q(s) < \infty. \tag{3.26}$$

If there is a sequence $\{\psi(n)\}$ satisfying (3.17) *and*

$$\lim_{n\to\infty} inf \frac{1}{(n-N)^\beta} \sum_{s=N}^{n-1} \left[(n-s)^\beta Q(s) - \frac{\beta^2}{4}\rho(s+1)r(s)(n-s)^{\beta-2}\right] \geq \psi(N) \tag{3.27}$$

then every solution of equation (1.1) *is oscillatory or converges to zero.*

Proof. The proof follows from Theorem 3.4 and hence the details are omitted. □

We conclude this paper with the following example.

4 An example

Consider the difference equation

$$\Delta\Big(n\Delta^2\Big(x(n) + \sum_{s=1}^{2} \frac{1}{2}x(n+s-1)\Big)\Big) + \sum_{s=1}^{2}\Big(4n + \frac{4}{3}s\Big)x(n+s-1) = 0. \tag{4.1}$$

Here $r(n) = n,\ p(n,s) = \frac{1}{2},\ q(n,s) = 4n + \frac{4}{3}s,\ \sigma = \tau = 1,\ a = 1,\ b = 2,\ c = 1$ and $d = 2$. It is easy to see that all conditions of Theorem 3.1 are satisfied. Hence every solution of equation (4.1) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (4.1).

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Difference entire sequence spaces of fuzzy numbers

Sunil K. Sharma[a,*] and Kuldip Raj[b]

[a] *Department of Mathematics, Model Institute of Engineering & Technology, Kot Bhalwal -181122, J&K, India.*

[b] *Department of Mathematics, School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India.*

Abstract

In the present paper we introduce difference entire sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions. We also make an effort to study some topological properties and inclusion relations between these spaces.

Keywords: Double sequences, P-convergent, modulus function, paranorm space.

2010 MSC: 42B15, 40C05.

1 Introduction and Preliminaries

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [16] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [11] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. For more details about sequence spaces of fuzzy numbers see ([1], [4], [7], [12], [13], [14], [15]) and references therein.

The notion of difference sequence spaces was introduced by Kızmaz [8], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [6] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let r, s be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_s^r) = \{x = (x_k) \in w : (\Delta_s^r x_k) \in Z\},$$

where $\Delta_s^r x = (\Delta_s^r x_k) = (\Delta_s^{r-1} x_k - \Delta_s^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_s^r x_k = \sum_{v=0}^{r} (-1)^v \begin{pmatrix} r \\ v \end{pmatrix} x_{k+sv}.$$

Taking $s = 1$, we get the spaces which were introduced and studied by Et and Çolak [6]. Taking $r = s = 1$, we get the spaces which were introduced and studied by Kızmaz [8].

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

*Corresponding author.

E-mail addresses: sunilksharma42@yahoo.co.in (Sunil K. Sharma) and kuldipraj68@gmail.com (Kuldip Raj).

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [9] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

Let D be the set of all bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line $\mathbb{R}$. For $A, B \in D$, define $A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$, $d(A, B) = \max\{\underline{A} - \underline{B}, \overline{A} - \overline{B}\}$.
Then it can be easily see that d defines a metric on D and (D, d) is complete metric space (see [5]).

A fuzzy number is fuzzy subset of the real line $\mathbb{R}$ which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semi continuous and have compact support, i.e. if $X \in L(\mathbb{R})$ then for any $\alpha \in [0, 1]$, X^α is compact where

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha, & \text{if} \quad 0 < \alpha \leq 1, \\ t : X(t) > 0, & \text{if} \ \ \alpha = 0. \end{cases}$$

For each $0 < \alpha \leq 1$, the α-level set X^α is a non-empty compact subset of $\mathbb{R}$. The linear structure of $L(\mathbb{R})$ includes addition $X + Y$ and scalar multiplication λX, (λa scalar) in terms of α-level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda [X]^\alpha,$$

for each $0 \leq \alpha \leq 1$.

Define a map $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

For $X, Y \in L(\mathbb{R})$ define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0, 1]$. It is known that $(L(\mathbb{R}, \bar{d}))$ is a complete metric space (see [11]).

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(\mathbb{R})$. The fuzzy number X_n denotes the value of the function at $n \in \mathbb{N}$ and is called the n^{th} term of the sequence.

In this paper we define difference entire sequence spaces of fuzzy numbers by using regular matrices $A = (a_{nk}), (n, k = 1, 2, 3, \cdots)$. By the regularity of A we mean that the matrix which transform convergent sequence into a convergent sequence leaving the limit (see [10]). We denote by $w(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

Let $X = (X_k)$ be a sequence of fuzzy numbers, $A = (a_{nk})$ $n, k = 1, 2, 3, \cdots$ be a non-negative regular matrix and $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. Now, we define the following sequence spaces in this paper :
$\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r) =$

$$\Big\{X = (X_k) : \sum_k a_{nk}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0\Big)\Big)\Big]^{p_k} \to 0 \text{ as } k \to \infty, \ \text{ for some } \ \rho > 0\Big\}$$

and
$\Lambda_{\mathcal{M}}(F, A, p, \Delta_s^r) =$

$$\left\{X = (X_k) : \sup_n \left(\sum_k a_{nk} \left[\bar{d}\left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

If $A = I$, the unit matrix, we get the above spaces as follows :
$\Gamma_{\mathcal{M}}(F, p, \Delta_s^r) =$

$$\left\{X = (X_k) : \left[\bar{d}\left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0 \right\}$$

and
$\Lambda_{\mathcal{M}}(F, p, \Delta_s^r) =$

$$\left\{X = (X_k) : \sup_n \left[\bar{d}\left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

If we take $\mathcal{M}(x) = x$, we get
$\Gamma(F, A, p, \Delta_s^r) =$

$$\left\{X = (X_k) : \sum_k a_{nk} \left[\bar{d}\left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right]^{p_k} \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0 \right\}$$

and
$\Lambda(F, A, p, \Delta_s^r) =$

$$\left\{X = (X_k) : \sup_n \left(\sum_k a_{nk} \left[\bar{d}\left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right]^{p_k} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

If we take $p = (p_k) = 1 \ \forall \ k$, we get
$\Gamma_{\mathcal{M}}(F, A, \Delta_s^r) =$

$$\left\{X = (X_k) : \sum_k a_{nk} \left[\bar{d}\left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right] \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0 \right\}$$

and
$\Lambda_{\mathcal{M}}(F, A, \Delta_s^r) =$

$$\left\{X = (X_k) : \sup_n \left(\sum_k a_{nk} \left[\bar{d}\left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right] \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

If $A = (a_{nk})$ is a Cesaro matrix of order 1, i.e.

$$a_{nk} = \begin{cases} \frac{1}{n}, & k \le n, \\ 0, & k > n \end{cases}$$

then we get
$\Gamma_{\mathcal{M}}(F, p, \Delta_s^r) =$

$$\left\{X = (X_k) : \frac{1}{n} \sum_{k=1}^{n} \left[\bar{d}\left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0 \right\}$$

and
$\Lambda_{\mathcal{M}}(F, p, \Delta_s^r) =$

$$\left\{X = (X_k) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[\bar{d}\left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

The space Γ is defined as follows:

$$\Gamma = \left\{ X = (X_k) : \frac{1}{n}\sum_{k=1}^{n} |X_k|^{\frac{1}{k}} \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0 \right\}.$$

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$ and let $K = \max\{1, 2^{H-1}\}$. Then for sequences $\{a_k\}$ and $\{b_k\}$ in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}).$$

The main purpose of this paper is to study difference entire sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions. We also studied some topological properties and interesting inclusion relations between the above defined sequence spaces.

2 Main Results

Proposition 2.1. *If $\bar{d}$ is a translation invariant metric on $L(\mathbb{R})$ then*

(i) $\bar{d}(X + Y, 0) \leq \bar{d}(X, 0) + \bar{d}(Y, 0)$,

(ii) $\bar{d}(\lambda X, 0) \leq |\lambda| \bar{d}(X, 0), |\lambda| > 1$.

Proof. It is easy to prove so we omit the details. □

Theorem 2.2. *If $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, then $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ is a complete metric space under the metric*

$$d(X, Y) = \sup_n \left[\frac{1}{n}\sum_{k=1}^{n} \bar{d}\Big(M_k \Big(\frac{|\Delta_s^r (X_k - Y_k)|^{\frac{1}{k}}}{\rho}, 0 \Big) \Big) \right]^{p_k}.$$

Proof. Let $X = (X_k),\ \ Y = (Y_k) \in \Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$. Let $\{X^{(n)}\}$ be a Cauchy sequence in $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$. Then given any $\epsilon > 0$ there exists a positive integer N depending on ϵ such that $d(X^{(n)}, X^{(m)}) < \epsilon$, for all $n, m \geq N$. Hence

$$\sup_{(n)} \left[\frac{1}{n}\sum_{k=1}^{n} \bar{d}\Big(M_k \Big(\frac{|\Delta_s^r X_k^{(n)} - \Delta_s^r X_k^{(m)}|^{\frac{1}{k}}}{\rho}, 0 \Big) \Big) \right]^{p_k} < \epsilon \quad \forall m, n \geq N.$$

Consequently $\{X_k^{(n)}\}$ is a Cauchy sequence in the metric space $L(\mathbb{R})$. But $L(\mathbb{R})$ is complete. So, $X_k^{(n)} \to X_k$ as $n \to \infty$. Hence there exists a positive integer n_0 such that

$$\left[\frac{1}{n}\sum_{k=1}^{n} \bar{d}\Big(M_k \Big(\frac{|\Delta_s^r X_k^{(n)} - \Delta_s^r X_k^{(m)}|^{\frac{1}{k}}}{\rho}, 0 \Big) \Big) \right]^{p_k} < \epsilon \ \forall n > n_0.$$

In particular, we have

$$\left[\frac{1}{n}\sum_{k=1}^{n} \bar{d}\Big(M_k \Big(\frac{|\Delta_s^r X^{(n_0)} - \Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \Big) \Big) \right]^{p_k} < \epsilon.$$

Now

$$\begin{aligned} \left[\frac{1}{n}\sum_{k=1}^{n} \bar{d}\Big(M_k \Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \Big) \Big) \right]^{p_k} &\leq \left[\frac{1}{n}\sum_{k=1}^{n} \bar{d}\Big(M_k \Big(\frac{|\Delta_s^r X_k - \Delta_s^r X_k^{(n_0)}|^{\frac{1}{k}}}{\rho}, 0 \Big) \Big) \right]^{p_k} \\ &\quad + \left[\frac{1}{n}\sum_{k=1}^{n} \bar{d}\Big(M_k \Big(\frac{|\Delta_s^r X_k^{(n_0)}|^{\frac{1}{k}}}{\rho}, 0 \Big) \Big) \right]^{p_k} \\ &\leq \epsilon + 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus

$$\left(\frac{1}{n}\sum_{k=1}^{n} \bar{d}\Big(M_k \Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \Big) \Big) \right)^{p_k} < \epsilon \ \text{ as } \ n \to \infty.$$

This implies that $(X_k) \in \Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$. Hence $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ is a complete metric space. This completes the proof. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $p = (p_k)$ be a bounded sequence of positive real numbers, the space $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ is a linear over the field of complex numbers $\mathbb{C}$.*

Proof. Let $X = (X_k),\ Y = (Y_k) \in \Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist some positive numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_1}, 0\Big)\Big)\Big]^{p_k} \to 0 \ \text{ as } \ k \to \infty$$

and

$$\sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r Y_k|^{\frac{1}{k}}}{\rho_2}, 0\Big)\Big)\Big]^{p_k} \to 0 \ \text{ as } \ k \to \infty.$$

Let $\frac{1}{\rho_3} = \min\Big\{\frac{1}{|\alpha|^p}\frac{1}{\rho_1}, \frac{1}{|\beta|^p}\frac{1}{\rho_2}\Big\}$. Since M is non-decreasing and convex so by using inequality (1.1), we have

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r(\alpha X_k + \beta Y_k)|^{\frac{1}{k}}}{\rho_3}, 0\Big)\Big)\Big]^{p_k}\\
&\le \sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r \alpha X_k|^{\frac{1}{k}}}{\rho_3} + \frac{|\Delta_s^r \beta Y_k|^{\frac{1}{k}}}{\rho_3}, 0\Big)\Big)\Big]^{p_k}\\
&\le \sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\alpha|^{\frac{1}{k}}|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_3} + \frac{|\beta|^{\frac{1}{k}}|\Delta_s^r \alpha Y_k|^{\frac{1}{k}}}{\rho_3}, 0\Big)\Big)\Big]^{p_k}\\
&\le \sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\alpha||\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_3} + \frac{|\beta||\Delta_s^r \alpha Y_k|^{\frac{1}{k}}}{\rho_3}, 0\Big)\Big)\Big]^{p_k}\\
&\le \sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_1} + \frac{|\Delta_s^r \alpha Y_k|^{\frac{1}{k}}}{\rho_2}, 0\Big)\Big)\Big]^{p_k}\\
&\le K\sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_1}, 0\Big)\Big)\Big]^{p_k} + K\sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r Y_k|^{\frac{1}{k}}}{\rho_2}, 0\Big)\Big)\Big]^{p_k}\\
&\to 0 \ \text{ as } \ k \to \infty.
\end{aligned}$$

Hence $\sum_{k=1}^{n} \frac{1}{n}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r(\alpha X_k + \beta Y_k)|^{\frac{1}{k}}}{\rho_3}, 0\Big)\Big)\Big]^{p_k} \to 0$ as $k \to \infty$. Hence $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ is a linear space. This completes the proof. □

Theorem 2.4. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the space $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ is complete with respect to the paranorm defined by*

$$g(X) = \sup_{(k)}\Big(\sum a_{nk}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{H}},$$

where $H = \max\{1, \sup_k(p_k/H)\}$ and $\bar{d}$ is translation metric.

Proof. Clearly, $g(0) = 0$, $g(-x) = g(x)$. It can also be seen easily that $g(x+y) \le g(x)+g(y)$ for $X = (X_k)$, $Y = (Y_k)$ in $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$, since $\bar{d}$ is translation invariant. Now for any scalar λ, we have $|\lambda|^{\frac{p_k}{H}} < \max\{1, \sup|\lambda|\}$, so that $g(\lambda x) < \max\{1, \sup|\lambda|\}$, λ fixed implies $\lambda x \to 0$. Now, let $\lambda \to 0$, X fixed for $\sup|\lambda| < 1$, we have

$$\Big[\sum a_{nk}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0\Big)\Big)\Big]^{p_k}\Big]^{\frac{1}{H}} < \epsilon \ \text{ for } \ N > N(\epsilon).$$

Also for $1 \le n \le N$, since

$$\Big[\sum a_{nk}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0\Big)\Big)\Big]^{p_k}\Big]^{\frac{1}{H}} < \epsilon,$$

there exists m such that

$$\Big[\sum_{k=m}^{\infty} a_{nk}\Big[\bar{d}\Big(M_k\Big(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0\Big)\Big)\Big]^{p_k}\Big]^{\frac{1}{H}} < \epsilon.$$

Taking λ small enough, we have

$$\left[\sum_{k=m}^{\infty} a_{nk}\left[\bar{d}\left(M_k\left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho},0\right)\right)\right]^{p_k}\right]^{\frac{1}{H}} < 2\epsilon \text{ for all } k.$$

Since $g(\lambda X) \to 0$ as $\lambda \to 0$. Therefore g is a paranorm on $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$.
To show the completeness, let $(X^{(i)})$ be a Cauchy sequence in $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$. Then for a given $\epsilon > 0$ there is $r \in N$ such that

$$\left[\sum a_{nk}\left[\bar{d}\left(M_k\left(\frac{|\Delta_s^r (X^{(i)} - X^{(j)})|^{\frac{1}{k}}}{\rho},0\right)\right)\right]^{p_k}\right]^{\frac{1}{H}} < \epsilon \text{ for all } i,j > r. \tag{2.1}$$

Since $\bar{d}$ is a translation, so equation (2.1) implies that

$$\left[\sum a_{nk}\left[\bar{d}\left(M_k\left(\frac{|\Delta_s^r (X_k^{(i)} - X_k^{(j)})|^{\frac{1}{k}}}{\rho},0\right)\right)\right]^{p_k}\right]^{\frac{1}{H}} < \epsilon \text{ for all } i,j > r \text{ and each } n. \tag{2.2}$$

Hence

$$\left[\bar{d}\left(M_k\left(\frac{|\Delta_s^r (X_k^{(i)} - X_k^{(j)})|^{\frac{1}{k}}}{\rho},0\right)\right)\right] < \epsilon \text{ for all } i,j > r.$$

Therefore $(X^{(i)})$ is a Cauchy sequence in $L(\mathbb{R})$. Since $L(\mathbb{R})$ is complete, $\lim_{j\to\infty} X_k^j = X_k$. Fixing $r_0 \geq r$ and letting $j \to \infty$, we obtain (2.2) that

$$\left[\sum a_{nk}\left[\bar{d}\left(M_k\left(\frac{|\Delta_s^r (X_k^{(i)} - X_k)|^{\frac{1}{k}}}{\rho},0\right)\right)\right]\right] < \epsilon \text{ for all } r_0 > r, \tag{2.3}$$

since $\bar{d}$ is a translation invariant. Hence

$$\left[\sum a_{nk}\left[\bar{d}\left(M_k\left(\frac{|\Delta_s^r (X^{(i)} - X)|^{\frac{1}{k}}}{\rho},0\right)\right)\right]^{p_k}\right]^{\frac{1}{H}} < \epsilon$$

i.e. $X^{(i)} \to X$ in $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$. It is easy to see that $X \in \Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$. Hence $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ is complete. This completes the proof. □

Theorem 2.5. *Let $A = (a_{nk})$ $(n, k = 1, 2, 3, \cdots)$ be an infinite matrix with complex entries. Then $A \in \Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ if and only if given $\epsilon > 0$ there exists $N = N(\epsilon) > 0$ such that $|a_{nk}| < \epsilon^n N^k$ $(n, k = 1, 2, 3, \cdots)$.*

Proof. Let $X = (X_k) \in \Gamma$ and let $Y_n = \left(\sum_{k=1}^{\infty} a_{nk}\bar{d}\left(M_k\left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho},0\right)\right)^{p_k}\right)$, $(n = 1, 2, 3, \cdots)$. Then $(Y_n) \in \Gamma$ if and only if given any $\epsilon > 0$ there exists $N = N(\epsilon) > 0$ such that $|a_{nk}| < \epsilon^n N^k$ by using Theorem 4 of [3]. Thus $A \in \Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ if and only if the condition holds. □

Theorem 2.6. *If $A = (a_{nk})$ transforms Γ into $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ then $\lim_{n\to\infty}(a_{nk})q^n = 0$ for all integers $q > 0$ and each fixed $k = 1, 2, 3, \cdots$, where $X = (X_k)$ be a sequence of fuzzy numbers and $\bar{d}$ is translation invariant.*

Proof. Let $Y_n = \left[\sum_{k=1}^{\infty} a_{nk}\bar{d}\left(M_k\left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho},0\right)\right)^{p_k}\right]$ $(n = 1, 2, 3, \cdots)$. Let $(X_k) \in \Gamma$ and $(Y_n) \in \Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$. Take $(X_k) = \delta^k = (0, 0, 0, \cdots, 1, 0, 0, \cdots)$, 1 in the k^{th} place and zero's elsewhere, then $(X_k) \in \Gamma$. Hence $\sum_{k=1}^{\infty} |a_{nk}|q^n < \infty$ for every positive q. In particular $\lim_{n\to\infty}(a_{nk})q^n = 0$ for all positive integers q and each fixed $k = 1, 2, 3, \cdots$. This completes the proof. □

Theorem 2.7. *If $A = (a_{nk})$ transforms $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ into Γ, then $\lim_{n\to\infty}(a_{nk})q^n = 0$ for all integers $q > 0$ and each fixed $k = 1, 2, 3, \cdots$, where $X = (X_k)$ be a sequence of fuzzy numbers and $\bar{d}$ is translation invariant.*

Proof. Let

$$t_n = \left[\sum_{k=1}^{\infty} \bar{d}\left(M_k\left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0\right)\right)^{p_k}\right] \in \Gamma.$$

Let

$$s_n = \left[\sum_{k=1}^{\infty} \bar{d}\left(M_k\left(\frac{|0|^{\frac{1}{k}}}{\rho}, 0\right)\right)^{p_k}\right] \in \Gamma.$$

Then $Y_n = (t_n - s_n) = \left[\sum_{k=1}^{\infty} a_{nk}\bar{d}\left(M_k\left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0\right)\right)^{p}\right]$ and $\bar{d}\left(M_k\left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0\right)^{p_k}\right) \in \Gamma$. Hence $(Y_n) \in \Gamma$. Therefore $(a_{nk})q^n \to 0$ as $n \to \infty$ $\forall k$. This completes the proof. □

Theorem 2.8. *If $A = (a_{nk})$ transforms $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ into $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$, then $\lim_{n\to\infty}(a_{nk})q^n = 0$ for all integers $q > 0$ and each fixed $k = 1, 2, 3, \cdots$, where $X = (X_k)$ be a sequence of fuzzy numbers and $\bar{d}$ is translation invariant.*

Proof. The proof of the Theorem follows from Theorem 2.6 and Theorem 2.7. □

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A new classes of open mappings

M. Lellis Thivagar[a] and M. Anbuchelvi[b,*]

[a] School of Mathematics, Madurai Kamaraj University, Madurai-625021, Tamil Nadu, India.

[b] Department of Mathematics, V.V. Vannniaperumal College for Women, Virudhunagar-626001, Tamil Nadu, India.

Abstract

The aim of this paper is to introduce new classes of mappings namely $\hat{\Omega}$-open mappings, somewhat $\hat{\Omega}$ open functions and hardly $\hat{\Omega}$-open mappings by utilizing $\hat{\Omega}$-closed sets. Also investigate some of their properties.

Keywords: $\hat{\Omega}$-closed sets, $\hat{\Omega}$ dense sets, $\hat{\Omega}$-open mappings, somewhat $\hat{\Omega}$ open mappings, hardly $\hat{\Omega}$-open mappings.

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1 Introduction

In 1969, Karl R. et al.[6] introduced the concept of Somewhat continuous and somewhat open function and investigated their properties.These functions are nothing but Frolik functions such that the condition onto was just dropped. These notions are also related to the idea of weakly equivalent topologies which was first introduced by Yougslova [11]. In this paper we study the concept of somewhat $\hat{\Omega}$ continuous and somewhat $\hat{\Omega}$ open function and investigated their properties by giving suitable examples on it. More over,we introduce and study two more kinds of open mappings via $\hat{\Omega}$-closed sets. Also we investigate their properties.

2 Preliminaries

Throughout this paper (X, τ) (or briefly X) represent a topological space with no separation axioms assumed unless otherwise explicitly stated. For a subset A of (X, τ) , we denote the closure of A, the interior of A and the complement of A as $cl(A), int(A)$ and A^c respectively. The following notations are used in this paper. The family of all open (resp.δ-open,$\hat{\Omega}$-open) sets on X are denoted by $O(X)$ (resp.$\delta O(X), \hat{\Omega}O(X)$). The family of all $\hat{\Omega}$-closed sets on X are denoted by $\hat{\Omega}C(X)$.

- $O(X, x) = \{U \in X \diagup x \in U \in O(X)\}$
- $\delta O(X, x) = \{U \in X \diagup x \in U \in \delta O(X)\}$
- $\hat{\Omega}O(X, x) = \{U \in X \diagup x \in U \in \hat{\Omega}O(X)\}$

Let us sketch some existing definitions,which are useful in the sequel as follows.

Definition 2.1. *[5] A subset A of Xis called δ-closed in a topological space (X, τ) if $A = \delta cl(A)$, where $\delta cl(A) = \{x \in X : int(cl(U)) \cap A \neq \emptyset, U \in O(X, x)\}$.The complement of δ-closed set in (X, τ) is called δ-open set in (X, τ). From [5],lemma 3, $\delta cl(A) = \cap\{F \in \delta C(X) : A \subseteq F\}$ and from corollary 4, $\delta cl(A)$ is a δ-closed for a subset A in a topological space (X, τ).*

Definition 2.2. *A subset A of a topological space (X, τ) is called*

*Corresponding author.
E-mail addresses: mlthivagar@yahoo.co.in (M. Lellis Thivagar) and rsanbuchelvi@gmail.com (M. Anbuchelvi).

(i) *semiopen set in* (X, τ) *if* $A \subseteq cl(int(A))$.

(ii) $\hat{\Omega}$*-closed set [7] if* $\delta cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi open in* (X, τ).

The complement of $\hat{\Omega}$*-closed set is called* $\hat{\Omega}$*-open.*

Definition 2.3. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called*

(i) *somewhat open [6] if* $U \in \tau$ *and* $U \neq \emptyset$, *then there exists* $V \in \sigma$ *such that* $V \neq \emptyset$ *and* $U \subseteq f(U)$.

(ii) *somewhat b open [3] if* $U \in \tau$ *and* $U \neq \emptyset$, *then there exists a b-open set* $V \in \sigma$ *such that* $V \neq \emptyset$ *and* $U \subseteq f(U)$.

(iii) *somewhat sg open [2] if* $U \in \tau$ *and* $U \neq \emptyset$, *then there exists a sg-open set* $V \in \sigma$ *such that* $V \neq \emptyset$ *and* $U \subseteq f(U)$.

(iv) *perfectly continuous [10] if the inverse image of open set in* Y *is clopen set in* X.

(v) *completely continuous [1] if the inverse image of open set in* Y *is regular open set in* X.

(vi) *super continuous [9] if the inverse image of open set in* Y *is* δ *open set in* X.

(vii) *somewhat continuous [6] if* $U \in \sigma$ *and* $f^{-1}(U) \neq \emptyset$, *then there exists a non empty set*$V \in \tau$ *such that* $V \subseteq f^{-1}(U)$.

(viii) *somewhat b continuous [3] if* $U \in \sigma$ *and* $f^{-1}(U) \neq \emptyset$, *then there exists a non empty b-open set*V *in* (X, τ) *such that* $V \subseteq f^{-1}(U)$.

(ix) *somewhat sg continuous [2] if* $U \in \sigma$ *and* $f^{-1}(U) \neq \emptyset$, *then there exists a non empty sg-open set*V *in* (X, τ) *such that* $V \subseteq f^{-1}(U)$.

Definition 2.4. *A space* (X, τ) *is said to be* $T_{\frac{3}{4}}$ *[4] if every* δg*-open set is* δ*-open set in* X.

Definition 2.5. *A space* (X, τ) *is said to be* T_1 *if for every two different point* x *and* y, *there exists open sets* U *and* V *such that* $x \in U, y \notin U$ *and* $y \in V, x \notin V$. *Also every proper set is contained in a proper open set.*

Theorem 2.6. *[8] A space* (X, τ) *is* ${}_{\omega}T_{\hat{\Omega}}$*-space if and only if every closed set is* $\hat{\Omega}$*-closed in* (X, τ).

Theorem 2.7. *[8] A space* (X, τ) *is semi-*$T_{\frac{1}{2}}$ *if and only if every* $\hat{\Omega}$*-open set is open in* (X, τ).

3 $\hat{\Omega}$-open mappings

Definition 3.1. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* $\hat{\Omega}$*-open function if the image of every open set in* X *is* $\hat{\Omega}$*-open set in* Y.

Example 3.2. *Let* $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}, \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. *Define* $f : (X, \tau) \to (Y, \sigma)$ *by* $f(a) = a, f(b) = a, f(c) = b$. *Then* f *is* $\hat{\Omega}$*-open function.*

Remark 3.3. *The notion of* $\hat{\Omega}$*-open function and open mappings are independent from the following examples.*

Example 3.4. *Let* $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, , X\}, \sigma = \{\emptyset, \{a, b\}, Y\}$. *Define* $f : (X, \tau) \to (Y, \sigma)$ *by* $f(a) = a, f(b) = b, f(c) = c$. *Then* f *is* $\hat{\Omega}$*-open but not open function.*

Example 3.5. *Let* $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b, c\}, X\}, \sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. *Define* $f : (X, \tau) \to (Y, \sigma)$ *by* $f(a) = a, f(b) = a, f(c) = b$. *Then* f *is open but not* $\hat{\Omega}$*-open function.*

Let us characterize $\hat{\Omega}$-open function in the following theorems.

Theorem 3.6. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is* $\hat{\Omega}$*-open function if and only if for any subset* A *of* Y *and for any closed set* F *in* X *such that* $f^{-1}(A) \subseteq F$, *there exists a* $\hat{\Omega}$*-closed set* F^1 *in* Y *such that* $A \subseteq F^1$ *and* $f^{-1}(F^{-1}) \subseteq F$.

Proof. **Necessity-** Let A be any subset in Y and F be any closed set in X such that $f^{-1}(A) \subseteq F$. Then $(X \setminus F)$ is open in X. By hypothesis, $f((X \setminus F))$ is $\hat{\Omega}$-open in Y and hence $Y \setminus f((X \setminus F))$ is $\hat{\Omega}$-closed in Y. Since $f^{-1}(A) \subseteq F, (X \setminus F) \subseteq (X \setminus f^{-1}(A)) = f^{-1}(Y \setminus A)$. Therefore, $f(X \setminus F) \subseteq (Y \setminus A)$ and hence $A \subseteq (Y \setminus f(X \setminus F))$. Now $f^{-1}(Y \setminus f(X \setminus F)) = (X \setminus f^{-1}f(X \setminus F)) \subseteq F$. If we take $F^1 = (Y \setminus f(X \setminus F))$,then F^1 is a $\hat{\Omega}$-closed set in Y such that $f^{-1}(F^1) \subseteq F$.
Sufficiency- Suppose that U is any open set in X. Then $(X \setminus U)$ is closed in X and $f^{-1}(Y \setminus f(U)) \subseteq (X \setminus U)$ By hypothesis, there exists $\hat{\Omega}$-closed set F in Y such that $(Y \setminus f(U)) \subseteq F$ and $f^{-1}(F) \subseteq (X \setminus U)$. Therefore, $(Y \setminus F) \subseteq f(U)$ and $U \subseteq (X \setminus f^{-1}(F)) = f^{-1}(Y \setminus F)$. Therefore,$(Y \setminus F) \subseteq f(U) \subseteq (Y \setminus F)$ and hence $(Y \setminus F) = f(U)$. Thus $f(U)$ is $\hat{\Omega}$-open set in Y. □

Theorem 3.7. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is $\hat{\Omega}$-open function if and only if for any subset B of Y,$f^{-1}(\hat{\Omega}cl(B)) \subseteq cl(f^{-1}(B))$.*

Proof. **Necessity-** For any subset B of $Y, f^{-1}(B) \subseteq cl(f^{-1}(B))$. By theorem 3.6,there exists a $\hat{\Omega}$-closed set A in Y such that $B \subseteq A$ and $f^{-1}(A) \subseteq cl(f^{-1}(B))$. By[7] the definition of $\hat{\Omega}$ closure, $\hat{\Omega}cl(B) \subseteq A$. Then $f^{-1}(\hat{\Omega}cl(B)) \subseteq f^{-1}(A) \subseteq cl(f^{-1}(B))$. Thus, $f^{-1}(\hat{\Omega}cl(B)) \subseteq cl(f^{-1}(B))$.
Sufficiency- Let A be any set in Y and F be any closed set in X such that $f^{-1}(A) \subseteq F$. If $F^1 = \hat{\Omega}cl(A)$,then [7] theorem 5.3, F^1 is $\hat{\Omega}$-closed set in Y containing A. By hypothesis, $f^{-1}(F^1) = f^{-1}(\hat{\Omega}cl(A)) \subseteq cl(f^{-1}(A)) \subseteq cl(F) \subseteq F$. By theorem 3.6, f is $\hat{\Omega}$-open function. □

Theorem 3.8. *For any function $f\colon (X,\tau) \to (Y,\sigma)$, the following statements are true.*

(i) f is $\hat{\Omega}$-open mapping.

(ii) $f(\delta int(A)) \subseteq \hat{\Omega}int(f(A))$ for any subset A in X.

(iii) For every $x \in X$ and for every δ-open set U in X containing x,there exists a $\hat{\Omega}$-open set W in Y containing $f(x)$ such that $W \subseteq f(U)$.

Proof. $(i) \Rightarrow (ii)$ Suppose that A is any subset of X.Then $\delta int(A)$ is open in X and $\delta int(A) \subseteq A$. By hypothesis, $f(\delta int(A))$ is $\hat{\Omega}$-open set in Y and $f(\delta int(A)) \subseteq f(A)$. By the definition of $\hat{\Omega}$ interior,$\hat{\Omega}int(f(A))$ is the largest $\hat{\Omega}$-open set contained in $f(A)$. Therefore, $f(\delta int(A)) \subseteq \delta int(f(A))$.

$(ii) \Rightarrow (iii)$ Let $x \in X$ and U be any δ-open set in X containing x. Then there exists δ-open set V in X such that $x \in V \subseteq U$. By hypothesis, $f(V) = f(\delta int(V)) \subseteq \hat{\Omega}int(f(V))$. Then $f(V)$ is $\hat{\Omega}$-open in Y containing $f(x)$ such that $f(V) \subseteq f(U)$. If we take $W = f(V)$,then W satisfies our requirement.

$(iii) \Rightarrow (i)$ Suppose that U is any δ-open set in X and y is any point in $f(U)$. By hypothesis, there exists an $\hat{\Omega}$-open set W_y in Y containing y such that $W_y \subseteq f(U)$.Therefore,$f(U) = \bigcup\{W_y : y \in f(U)\}$. By [7] theorem 4.16,$f(U)$ is $\hat{\Omega}$-open set in Y. □

Theorem 3.9. *A surjective function $f\colon (X,\tau) \to (Y,\sigma)$ is $\hat{\Omega}$-open function if and only if $f^{-1}\colon Y \to X$ is $\hat{\Omega}$-continuous.*

Proof. **Necessity-** If U is any open set in X then by hypothesis, $(f^{-1})^{-1}(U) = f(U)$ is $\hat{\Omega}$-open in Y.Hence $f^{-1}\colon Y \to X$ is $\hat{\Omega}$-continuous.
Sufficiency- If U is any open set in X, then by hypothesis,$f(U) = (f^{-1})^{-1}(U)$ is $\hat{\Omega}$-open in Y. Hence $f\colon (X,\tau) \to (Y,\sigma)$ is $\hat{\Omega}$-open function. □

Remark 3.10. *In general, composition of any two $\hat{\Omega}$-open functions is not a $\hat{\Omega}$-open function from the following example.*

Example 3.11. *$X = Y = \{a,b,c,d\}$ and $Z = \{a,b,c\}, \tau = \{\emptyset,\{b,c\},\{a,b,c\},\{b,c,d\},X\}$, $\sigma = \{\emptyset,\{a\},\{b,c,d\},Y\}, \eta = \{\emptyset,\{a\},\{b\},\{a,b\},Z\}$. Then $\hat{\Omega}O(Y) = P(X), \hat{\Omega}O(Z) = \eta$. If $f\colon (X,\tau) \to (Y,\sigma)$ is defined by $f(a) = a, f(b) = c, f(c) = d, f(d) = a$. Then f is $\hat{\Omega}$-open function. If $g\colon (Y,\sigma) \to (Z,\eta)$ is defined by $g(a) = a, g(b) = a, g(c) = b, g(d) = c$. Then g is $\hat{\Omega}$-open function.But $g \circ f\colon (X,\tau) \to (Z,\eta)$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is not $\hat{\Omega}$-open function because $(g \circ f)(\{b,c\}) = \{b,c\}$ not belongs to $\hat{\Omega}O(Z)$.*

Theorem 3.12. *If $f\colon (X,\tau) \to (Y,\sigma)$ is open function and $g\colon (Y,\sigma) \to (Z,\eta)$ is $\hat{\Omega}$-open function, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is $\hat{\Omega}$-open function.*

Proof. It follows from their definitions. □

Theorems on Composition

Theorem 3.13. *Let (Y,σ) be a semi-$T_{\frac{1}{2}}$-space. If $f\colon (X,\tau) \to (Y,\sigma)$ and $g\colon (Y,\sigma) \to (Z,\eta)$ are $\hat{\Omega}$-open functions, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is $\hat{\Omega}$-open function.*

Proof. It follows from their definitions. □

Theorem 3.14. *If $f\colon (X,\tau) \to (Y,\sigma)$ and $g\colon (Y,\sigma) \to (Z,\eta)$ are any two functions such that $g \circ f\colon (X,\tau) \to (Z,\eta)$ is $\hat{\Omega}$-open function then,*

(i) f is $\hat{\Omega}$-open mapping if g is $\hat{\Omega}$-irresolute and injective.

(ii) g is $\hat{\Omega}$-open mapping if f is continuous and surjective.

Proof. (i) If U is any open set in X, $g(f(U))$ is $\hat{\Omega}$-open in Z. Since g is $\hat{\Omega}$-irresolute, $g^{-1}(g(f(U)))$ is $\hat{\Omega}$-open in Y. Since g is injective, $g^{-1}(g(f(U))) = f(U)$ is $\hat{\Omega}$-open in Y. Thus f is $\hat{\Omega}$-open mapping.

(ii) If U is any open set in Y, then $f^{-1}(U)$ is open set in X. Since $g \circ f\colon (X,\tau) \to (Z,\eta)$ is $\hat{\Omega}$-open function, $g(f(f^{-1}(U)))$ is $\hat{\Omega}$-open in Z. Since f is surjective, $g(f(f^{-1}(U))) = g(U)$ is $\hat{\Omega}$-open in Z.

□

4 Somewhat $\hat{\Omega}$-open, Hardly $\hat{\Omega}$-open mappings

Definition 4.1. *A subset A of a space X is said to be $\hat{\Omega}$-dense in X if $\hat{\Omega}cl(A) = X$. Or, there is no $\hat{\Omega}$-closed between A and X.*

Example 4.2. *Let $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$. Then $\hat{\Omega}$-dense sets in X are $\{\{a\},\{a,b\},\{a,c\}\}$.*

Definition 4.3. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be somewhat $\hat{\Omega}$-open if for each non empty set $U \in O(X)$, there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U)$.*

Example 4.4. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is somewhat $\hat{\Omega}$-open mapping.*

Theorem 4.5. *Every somewhat $\hat{\Omega}$-open mapping is somewhat b-(resp.sg-) open mapping.*

Proof. Assume that $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-open mapping and suppose that U is any non empty set in X. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U)$. By [7] remark 3.13, V is b-(resp.sg-)open set in Y. Hence f is somewhat b open mapping. □

Remark 4.6. *The following example shows that the reversible implication is not true in general.*

Example 4.7. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is somewhat b-(resp.sg-) open mapping but not somewhat $\hat{\Omega}$-open mapping.*

Remark 4.8. *The notions, somewhat open (resp.somewhat semi open)mapping and somewhat $\hat{\Omega}$-open mapping are independent from the following examples.*

Question: Is there any example on a mapping which is somewhat open but not somewhat $\hat{\Omega}$-open?

Example 4.9. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, and $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$.Then f is somewhat $\hat{\Omega}$-open mapping but not somewhat open mapping.*

Theorem 4.10. *If (Y,σ) is semi-$T_{\frac{1}{2}}$, then every somewhat $\hat{\Omega}$-open mapping $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat open mapping.*

Proof. Let $U \in O(X)$ be any non empty set in X. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U)$. Since in a *semi*-$T_{\frac{1}{2}}$ space, every $\hat{\Omega}$-open set is open,$V \in O(Y)$. Hence f is somewhat open mapping. □

Theorem 4.11. *If (Y, σ) is $_{\omega}T_{\hat{\Omega}}$, then every somewhat open mapping $f \colon (X, \tau) \to (Y, \sigma)$ is somewhat $\hat{\Omega}$-open mapping.*

Proof. Let $U \in O(X)$ be any non empty set in X. By hypothesis,there exists a non empty set $V \in O(Y)$ such that $V \subseteq f(U)$.Since in a $_{\omega}T_{\hat{\Omega}}$ space,every open set is $\hat{\Omega}$-open, $V \in \hat{\Omega}O(Y)$. Hence f is somewhat $\hat{\Omega}$-open mapping. □

Let us prove a characterization of somewhat $\hat{\Omega}$ open mapping.

Theorem 4.12. *A mapping $f \colon (X, \tau) \to (Y, \sigma)$ is somewhat $\hat{\Omega}$ open if and only if inverse image of a $\hat{\Omega}$-dense set in Y is dense in X.*

Proof. **Necessity-** Suppose that D is $\hat{\Omega}$-dense set in Y and suppose $f^{-1}(D)$ is not dense in X. Therefore, there exists a proper closed set F in X such that $f^{-1}(D) \subseteq F \subseteq X$. Then $X \setminus F$ is a non empty open set in X. By hypothesis, there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(X \setminus F)$ or $Y \setminus f(X \setminus F) \subseteq Y \setminus V$. Moreover, $X \setminus F \subseteq X \setminus f^{-1}(D) = f^{-1}(Y \setminus D)$ implies that $f(X \setminus F) \subseteq Y \setminus D$. Then $D \subseteq Y \setminus f(X \setminus F) \subseteq Y \setminus V$. We have some proper $\hat{\Omega}$-closed set $Y \setminus V$ in Y such that $D \subseteq Y \setminus V \subseteq Y$ a contradiction to D is $\hat{\Omega}$-dense set in Y. Therefore,$f^{-1}(D)$ is dense in X.

Sufficiency- If f is not somewhat $\hat{\Omega}$-open mapping,for every non empty open set U in X, no non empty $\hat{\Omega}$-open set in Y is such that $V \subseteq f(U)$. Then no proper $\hat{\Omega}$ closed set $Y \setminus V$ is such that $Y \setminus f(U) \subseteq Y \setminus V \subseteq Y$. Therefore, $Y \setminus f(U)$ is $\hat{\Omega}$-dense in Y. By hypothesis, $f^{-1}(Y \setminus f(U))$ is dense in X or $X \setminus (f^{-1}(f(U))$ is dense in X. Therefore,$cl(X \setminus (f^{-1}(f(U))) = X$. Moreover, $U \subseteq (f^{-1}(f(U))$ implies that $X \setminus (f^{-1}(f(U)) \subseteq X \setminus U$. Then $X = cl(X \setminus (f^{-1}(f(U))) \subseteq cl(X \setminus U) = X \setminus \setminus int(U)$ and hence $int(U) = \emptyset$, a contradiction to U is a non empty set in X. □

Theorem 4.13. *Suppose that $f \colon (X, \tau) \to (Y, \sigma)$ is a bijective mapping. f is somewhat $\hat{\Omega}$-open mapping if and only if for every closed set F in X such that $f(F) \neq Y$, there exists a proper set $D \in \hat{\Omega}C(X)$ such that $f(F) \subseteq D$.*

Proof. **Necessity-** Suppose that F is any closed set in X such that $f(F) \neq Y$. Then $X \setminus F$ is a non empty open set in X.By hypothesis, there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(X \setminus F)$ or $Y \setminus f(X \setminus F) \subseteq Y \setminus V$. Since f is bijective,$f(F) \subset Y \setminus V$. If we define, $D = Y \setminus V$,then $D \neq \emptyset, D \in \hat{\Omega}C(Y)$ such that $f(F) \subseteq D$.

Sufficiency- Suppose that U is any non empty open set in X.Then $X \setminus U$ is a proper closed set in X. If $f(X \setminus U) = Y$,then it is easily seen that $U = \emptyset$, a contradiction.Therefore,$f(X \setminus U) \neq Y$. By hypothesis,there exists a proper $\hat{\Omega}$-closed set D in Y such that $f(X \setminus U) \subseteq D$. That is,$Y \setminus D \subseteq Y \setminus f(X \setminus U) = f(U)$, where $Y \setminus D \neq \emptyset, Y \setminus D \in \hat{\Omega}O(Y)$.Thus f is somewhat $\hat{\Omega}$-open mapping. □

Theorem 4.14. *Suppose that A is any open set in a topological space (X, τ). If $f \colon (X, \tau) \to (Y, \sigma)$ is somewhat $\hat{\Omega}$-open mapping, then $f \mid_A \colon (A, \tau \mid_A) \to (Y, \sigma)$ is also somewhat $\hat{\Omega}$-open mapping on the subspace $(A, \tau \mid_A)$.*

Proof. Suppose that $U \in \tau \mid_A, U \neq \emptyset$. Since U is open in $(A, \tau \mid_A)$ and A is open in X, U is open in X. By hypothesis,there exists a non empty $\hat{\Omega}$-open set V in Y such that $V \subseteq f(U)$. Therefore,$f \mid_A$ is somewhat $\hat{\Omega}$-open mapping. □

Theorem 4.15. *Suppose that (X, τ) and (Y, σ) are any two topological spaces and suppose $X = A \cup B$, where A and B are open in X. If $f \colon (X, \tau) \to (Y, \sigma)$ is any function such that $f \mid_A$ and $f \mid_B$ are somewhat $\hat{\Omega}$-open mappings,then f is a somewhat $\hat{\Omega}$-open mapping.*

Proof. Let U be any open set in X. Then $U \cap A$ and $U \cap B$ are open sets in the subspaces $(A, \tau \mid_A)$ and $(A, \tau \mid_B)$ respectively. Since $X = A \cup B$, either $A \cap U \neq \emptyset$ or $B \cap U \neq \emptyset$ or both $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$.
case(i) If $U \cap A \neq \emptyset$.
Since $f \mid_A$ is somewhat $\hat{\Omega}$-open mapping,there exists a non empty $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U \cap A) \subseteq f(U)$.

It follows that f is somewhat $\hat{\Omega}$-open mapping.
case(ii) If $U \cap B \neq \emptyset$.
Since $f\mid_B$ is somewhat $\hat{\Omega}$-open mapping, there exists a non empty $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U \cap B) \subseteq f(U)$.
It follows that f is somewhat $\hat{\Omega}$-open mapping.
case(iii) If both $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$. It follows from case(i) or case(ii). □

Remark 4.16. *Composition of two somewhat $\hat{\Omega}$-open mappings is not always somewhat $\hat{\Omega}$-open mapping from the following example.*

Example 4.17. *$X = Y = \{a, b, c, d\}$ and $Z = \{a, b, c\}, \tau = \{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}, \sigma = \{\emptyset, \{a\}, \{b, c, d\}, Y\}$, $\eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Z\}$. Then $\hat{\Omega}O(Y) = P(X), \hat{\Omega}O(Z) = \eta$. If $f : (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = a, f(b) = c, f(c) = d, f(d) = a$. Then f is somewhat $\hat{\Omega}$-open function. If $g : (Y, \sigma) \to (Z, \eta)$ is defined by $g(a) = a, g(b) = a, g(c) = b, g(d) = c$. Then g is somewhat $\hat{\Omega}$-open function.But $g \circ f : (X, \tau) \to (Z, \eta)$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is not a somewhat $\hat{\Omega}$-open function because $(g \circ f)(\{b, c\}) = \{b, c\}$ does not contain any $\hat{\Omega}$-open set in Z.*

The following theorem states the condition under which the composition of two somewhat $\hat{\Omega}$-open mappings is again a somewhat $\hat{\Omega}$-open mappings.

Theorem 4.18. *If $f : (X, \tau) \to (Y, \sigma)$ is an open mapping and $g : (Y, \sigma) \to (Z, \eta)$ is a somewhat $\hat{\Omega}$-open mapping,then $g \circ f : (X, \tau) \to (Z, \eta)$ is a somewhat $\hat{\Omega}$-open mapping.*

Proof. Suppose $U \in O(X)$ is any non empty set in X. Since f is an open mapping, $f(U)$ is an open set in Y. Since g is somewhat $\hat{\Omega}$-open mapping,there exists a non empty set $V \in \hat{\Omega}O(Z)$ such that $V \subseteq g(f(U)) = g \circ f(U)$. Hence $g \circ f$ is somewhat $\hat{\Omega}$-open mapping. □

Definition 4.19. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be hardly $\hat{\Omega}$-open if for each $\hat{\Omega}$ dense set A in Y that is contained in a proper $\hat{\Omega}$-open set in $Y, f^{-1}(A)$ is $\hat{\Omega}$-dense in X.*

Example 4.20. *Let $X = Y = \{a, b, c\}, \tau = \sigma = \{\emptyset, \{a\}, X\}$. Then $\hat{\Omega}$ dense sets in X are $\{\{a\},\{a,b\},\{a,c\},X\}$.Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a, f(b) = a$ and $f(c) = c$. Then f is hardly $\hat{\Omega}$-open mapping.*

Theorem 4.21. *Let Y be a T_1 space. A function $f : (X, \tau) \to (Y, \sigma)$ is hardly open function if and only if for each $\hat{\Omega}$-dense set A in $Y, f^{-1}(A)$ is $\hat{\Omega}$-dense in X.*

Proof. Since in a T_1 space,every set is properly contained in a proper open set,it follows. □

Theorem 4.22. *[4] A topological space is $T_{\frac{3}{4}}$ if and only if $\{x\}$ is either δ-open or closed.*

Theorem 4.23. *If Y is a $T_{\frac{3}{4}}$ space,then $f : (X, \tau) \to (Y, \sigma)$ is hardly $\hat{\Omega}$-open function if and only if for each $\hat{\Omega}$-dense set D in $Y, f^{-1}(D)$ is $\hat{\Omega}$-dense in X.*

Proof. **Necessity-** Assume that f is hardly $\hat{\Omega}$-open function and D is any $\hat{\Omega}$-dense set in Y. Let $y \in Y \setminus D$ be an arbitrary point. Since D is $\hat{\Omega}$-dense in $Y, \hat{\Omega}cl(D) = Y$. That is,$Y \setminus \hat{\Omega}cl(D) = \emptyset$. By [7] theorem 5.3 (vii),$Y \setminus \delta cl(D) = \emptyset$ or $\delta int(Y \setminus D) = \emptyset$. Therefore,$\{y\}$ is not a δ open in a $T_{\frac{3}{4}}$ space Y.By the theorem 4.22, $\{y\}$ is a closed set in Y and hence $Y \setminus \{y\}$ is a proper open set in Y. Therefore,D is contained in a proper open set $Y \setminus \{y\}$. By hypothesis,$f^{-1}(D)$ is $\hat{\Omega}$-dense in X.

Sufficiency- From the given hypothesis, f is hardly $\hat{\Omega}$-open function. □

Theorem 4.24. *$f : (X, \tau) \to (Y, \sigma)$ is hardly $\hat{\Omega}$-open function if and only if $\hat{\Omega}int(f^{-1}(A)) = \emptyset$ for each subset A in Y such that $\hat{\Omega}int(A) = \emptyset$ and A contains a nonempty closed set.*

Proof. **Necessity-** Assume that f is hardly $\hat{\Omega}$-open function and $A \subseteq Y$ such that $\hat{\Omega}int(A) = \emptyset$ and F, a nonempty closed set in Y such that $F \subseteq A$. Then, $\hat{\Omega}cl(Y \setminus A) = Y \setminus \hat{\Omega}int(A) = Y$. Since $F \subseteq A, Y \setminus A \subseteq Y \setminus F \neq Y$. Therefore, $Y \setminus A$ is a $\hat{\Omega}$-dense in Y which is contained in a proper open set $Y \setminus F$. By hypothesis,$f^{-1}(Y \setminus A)$ is $\hat{\Omega}$-dense in X. Therefore, $X = \hat{\Omega}cl(f^{-1}(Y \setminus A)) = X \setminus \hat{\Omega}int(f^{-1}(A))$. Thus, $X \setminus \hat{\Omega}int(f^{-1}(A)) = X$ and hence $\hat{\Omega}int(f^{-1}(A)) = \emptyset$.

Sufficiency- Suppose that D is any $\hat{\Omega}$-dense in Y such that it is contained in a proper open set U. Since $U \neq \emptyset, Y \setminus U$ is a non empty closed set contained in $Y \setminus D$. By hypothesis, $\hat{\Omega}int(f^{-1}(Y \setminus D)) = \emptyset$.Then,$X \setminus \hat{\Omega}cl(f^{-1}(D)) = \emptyset$ and hence $\hat{\Omega}cl(f^{-1}(D)) = X$. Thus,$f^{-1}(D)$ is $\hat{\Omega}$ dense in X. □

Theorem 4.25. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function. If $\hat{\Omega}int(f(A)) \neq \emptyset$ for every subset A of X having the property that $\hat{\Omega}int(A) \neq \emptyset$ and there exists a non empty closed set F in X such that $f^{-1}(F) \subseteq A$,then f is hardly $\hat{\Omega}$-open function.*

Proof. Suppose that D is any $\hat{\Omega}$-dense in Y which is contained in a proper open set U. Since $U \neq \emptyset, Y \setminus U \neq \emptyset$ and hence $Y \setminus U$ is a non empty closed set contained in $Y \setminus D$.If we define $A = f^{-1}(Y \setminus D), F = Y \setminus U$, then $f^{-1}(F) \subseteq A$. Moreover, $\hat{\Omega}int(f(A)) = \hat{\Omega}int(f(f^{-1}(Y \setminus D))) \subseteq \hat{\Omega}int(Y \setminus D) = \emptyset$. By hypothesis, we should have $\hat{\Omega}int(A) = \emptyset$. That is, $\hat{\Omega}int(f^{-1}(Y \setminus D)) = \emptyset$. Therefore,$X \setminus \hat{\Omega}cl(f^{-1}(D)) = \emptyset$ and hence $\hat{\Omega}cl(f^{-1}(D)) = X$. Thus, $f^{-1}(D)$ is $\hat{\Omega}$ dense in X. Therefore, f is hardly $\hat{\Omega}$-open function. □

Theorem 4.26. *If $f\colon (X,\tau) \to (Y,\sigma)$ is hardly $\hat{\Omega}$-open function, then $\hat{\Omega}int(f(A)) \neq \emptyset$ for every subset A of X having the property that $\hat{\Omega}int(A) \neq \emptyset$ and $f(A)$ contains a non empty closed set.*

Proof. Suppose that A is any set in X such that $\hat{\Omega}int(A) \neq \emptyset$ and F is any non empty closed set in Y such that $F \subseteq f(A)$. If $\hat{\Omega}int(f(A)) = \emptyset$,then $Y \setminus f(A)$ is $\hat{\Omega}$-dense in Y such that $Y \setminus f(A)$ is contained in a proper open set $Y \setminus F$. Since f is hardly $\hat{\Omega}$-open function, $f^{-1}(Y \setminus f(A))$ is $\hat{\Omega}$ dense in X. That is, $\hat{\Omega}cl(f^{-1}(Y \setminus f(A))) = X$ or $X \setminus \hat{\Omega}int(f^{-1}(f(A))) = X$. Then, $\hat{\Omega}int(f^{-1}(f(A))) = \emptyset$ and hence $\hat{\Omega}int(A) = \emptyset$, a contradiction.Therefore,our assumption is wrong and thus $\hat{\Omega}int(f(A)) \neq \emptyset$. □

Theorem 4.27. *If $f\colon (X,\tau) \to (Y,\sigma)$ is surjective,then the following statements are true.*

(i) *f is hardly $\hat{\Omega}$-open function.*

(ii) *$\hat{\Omega}int(f(A)) \neq \emptyset$ for every subset A of X having the property that $\hat{\Omega}int(A) \neq \emptyset$ and there exists a non empty closed set F in Y such that $F \subseteq f(A)$*

(iii) *$\hat{\Omega}int(f(A)) \neq \emptyset$ for every subset A of X having the property that $\hat{\Omega}int(A) \neq \emptyset$ and there exists a non empty closed set F in Y such that $f^{-1}(F) \subseteq A$*

Proof. $(i) \Rightarrow (ii)$ It's nothing but the theorem 4.10.

$(ii) \Rightarrow (iii)$ Since f is surjective,$f^{-1}(F) \subseteq f^{-1}(f(A)) = A$. Hence it holds.

$(iii) \Rightarrow (i)$ It follows from the theorem 4.9.

□

5 Somewhat $\hat{\Omega}$-Continuous functions

Definition 5.1. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be somewhat $\hat{\Omega}$-continuous if for each non empty set $U \in O(Y)$ and $f^{-1}(U) \neq \emptyset$, there exists a non empty set $V \in \hat{\Omega}O(X)$ such that $V \subseteq f^{-1}(U)$.*

Example 5.2. *Let $X = Y = \{a,b,c\}, \tau = \{\emptyset, \{b,c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = a, f(c) = a$. Then f is somewhat $\hat{\Omega}$-continuous.*

Theorem 5.3. *Every somewhat $\hat{\Omega}$-continuous is somewhat b (resp.sg)continuous*

Proof. Assume that $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-continuous and suppose that U is any non empty set in Y such that $f^{-1}(U) \neq \emptyset$. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f^{-1}(U)$. By [7] figure-1,V is b (resp.sg)open set in Y.Hence f is somewhat b continuous. □

Remark 5.4. *The following example shows that the reversible implication is not true in general.*

Example 5.5. *Let $X = Y = \{a,b,c,d\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = c, f(b) = a, f(c) = a, f(d) = a$. Then f is both somewhat b continuous and somewhat sg continuous but not somewhat $\hat{\Omega}$-continuous.*

Remark 5.6. *The notions, somewhat continuous and somewhat $\hat{\Omega}$-continuous are independent from the following examples.*

Example 5.7. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. If $f\colon (X,\tau) \to (Y,\sigma)$ is defined by $f(a) = b, f(b) = a, f(c) = c$ then f is somewhat $\hat{\Omega}$-continuous but not somewhat continuous.*

Question 2:Is there any example on a mapping which is somewhat continuous but not $\hat{somewhat}\Omega$-continuous?

Example 5.8. *There is no example on another one.*

Theorem 5.9. *If (Y,σ) is semi-$T_{\frac{1}{2}}$, then every somewhat $\hat{\Omega}$-continuous $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat continuous.*

Proof. Let $U \in O(X)$ be any non empty set in X. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U)$. Since in a *semi-*$T_{\frac{1}{2}}$ space, every $\hat{\Omega}$-open set is open,$V \in O(Y)$.Hence f is somewhat continuous. □

Let us prove a characterization of somewhat $\hat{\Omega}$-continuous.

Theorem 5.10. *Let (X,τ) and (Y,σ) be any two topological spaces. Then the following are equivalent statements.*

(i) f is somewhat $\hat{\Omega}$-continuous.

(ii) If F is a closed subset of Y such that $f^{-1}(F) \neq X$, then there exists a proper set $G \in \hat{\Omega}C(X)$, such that $f^{-1}(F) \subseteq G$.

(iii) Image of a $\hat{\Omega}$-dense set in X is dense in Y.

Proof. (**i**) ⇒ (**ii**). Suppose that F is any closed set in Y such that $f^{-1}(F) \neq X$. Then $Y \setminus F$ is a non empty open set in Y such that $f^{-1}(F^c) = (f^{-1}(F))^c \neq \emptyset$. By hypothesis, there exists a non empty set $V \in \hat{\Omega}O(X)$ such that $V \subseteq f^{-1}(F^c) = (f^{-1}(F))^c$. Then,$f^{-1}(F) \subseteq V^c$. If we define, $G = V^c$, then $G \neq \emptyset, G \in \hat{\Omega}C(X)$ such that $f^(F) \subseteq G$.

(**ii**) ⇒ (**i**). Suppose that U is any non empty open set in Y such that $f^{-1}(U) \neq \emptyset$. Then $Y \setminus U$ is a proper closed set in Y such that $f^{-1}(U^c) = (f^{-1}(U))^c \neq X$. By hypothesis,there exists a proper set $G \in \hat{\Omega}C(X)$ such that $f^{-1}(U^c) = (f^{-1}(U))^c \subseteq G$. Then,$G^c \neq \emptyset, G^c \in \hat{\Omega}O(X)$ and $G^c \subseteq f^{-1}(U)$. Therefore, f is somewhat $\hat{\Omega}$-continuous.

(**ii**) ⇒ (**iii**). Suppose that D is any $\hat{\Omega}$-dense set in X and assume that $f(D)$ is not dense in Y. Then, there exists a proper closed set F in Y such that $f(D) \subseteq F \subseteq Y$. Since $F \neq Y, f^{-1}(F) \neq f^{-1}(Y) \neq X$. By hypothesis,there exists a proper set $G \in \hat{\Omega}C(X)$ such that $f^{-1}(F) \subseteq G$. Therefore, $D \subseteq f^{-1}f(D) \subseteq f^{-1}(F) \subseteq G$. We have a proper $\hat{\Omega}$-closed set G in X such that $D \subseteq G \subseteq X$,a contradiction to D is $\hat{\Omega}$-dense in X. Therefore,$f(D)$ is dense in Y.

(**iii**) ⇒ (**ii**). If (ii) not holds, then there exists a closed set F in Y such that $f^{-1}(F) \neq X$and there is no proper set $G \in \hat{\Omega}C(X)$, such that $f^{-1}(F) \subseteq G \subseteq X$. Then, $f^{-1}(F)$ is $\hat{\Omega}$-dense in X and hence by hypothesis, $f(f^{-1}(F))$ is $\hat{\Omega}$-dense in Y. Moreover, F vis dense in Y,a contraction to the choice of F. □

Remark 5.11. *The following example reveals that composition of two somewhat $\hat{\Omega}$-continuous functions is not always the somewhat $\hat{\Omega}$-continuous.*

Example 5.12. *$X = Y = Z = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$, $\eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Z\}$. If $f\colon (X,\tau) \to (Y,\sigma)$ is defined by $f(a) = a, f(b) = b, f(c) = c$ and $g\colon (Y,\sigma) \to (Z,\eta)$ is defined by $g(a) = b, g(b) = c, g(c) = a$. Then $f$and g are somewhat $\hat{\Omega}$-continuous functions. But $g \circ f\colon (X,\tau) \to (Z,\eta)$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is not a somewhat $\hat{\Omega}$-continuous because $(g \circ f)(\{a\}) = \{c\}$ is not containing any non empty $\hat{\Omega}$-open set in X.*

Composition Theorems

Theorem 5.13. *Suppose that $(X,\tau), (Y,\sigma)$ and (Z,η) are three topological spaces.*

(i) If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous function and $g\colon (Y,\sigma) \to (Z,\eta)$ is surjective continuous, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is a somewhat $\hat{\Omega}$-continuous mapping.

(ii) If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous function and $g\colon (Y,\sigma) \to (Z,\eta)$ is surjective super continuous, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is a somewhat $\hat{\Omega}$-continuous.

(iii) If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous function and $g\colon (Y,\sigma) \to (Z,\eta)$ is surjective completely continuous, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is a somewhat $\hat{\Omega}$-continuous.

(iv) If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous function and $g\colon (Y,\sigma) \to (Z,\eta)$ is surjective perfectly continuous, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is a somewhat $\hat{\Omega}$-continuous.

Proof. (i) Suppose that U is any open set in Z such that $(g \circ f)^{-1}(U) \neq \emptyset$.Since g is surjective continuous, $g^{-1}(U)$ is a non empty open set in Y. Since f is a somewhat $\hat{\Omega}$-continuous,there exists a non empty $\hat{\Omega}$-open set Vin X such that $V \subseteq (g \circ f)^{-1}(U)$. Therefore, $g \circ f$ is somewhat $\hat{\Omega}$-continuous.

(ii) Suppose that U is any open set in Z such that $(g\circ f)^{-1}(U) \neq \emptyset$. Since g is surjective super continuous,$g^{-1}(U)$ is a non empty δ-open and hence open set in Y. Since f is a somewhat $\hat{\Omega}$-continuous, there exists a non empty $\hat{\Omega}$-open set V in X such that $V \subseteq (g \circ f)^{-1}(U)$. Therefore, $g \circ f$ is somewhat $\hat{\Omega}$-continuous.

- •The proofs of **(iii)** and **(iv)** are similar to **(ii)**.

□

Theorem 5.14. *Suppose that A is any open pre closed and $\hat{\Omega}$-dense set in a topological space (X,τ). If $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-continuous, then $f \mid A : (A, \tau\mid_A) \to (Y,\sigma)$ is also somewhat $\hat{\Omega}$-continuous on the subspace $(A, \tau \mid A)$.*

Proof. Suppose that $U \in O(Y)$ such that $(f \mid A)^{-1}(U) \neq A$.If $f^{-1}(U) = X$,then $f^{-1}(U) \cap A = X \cap A = A$,a contradiction to $(f \mid A)^{-1}(U) \neq A$. Therefore, $f^{-1}(U) \neq X$. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(X)$ such that $V \subseteq f^{-1}(U)$. Then, $V \cap A \subseteq f^{-1}(U) \cap A = (f \mid A)^{-1}(U)$. Since A is $\hat{\Omega}$-dense set in $X, A \cap V \neq \emptyset$. By[7] theorem 6.8, $A \cap V$ is $\hat{\Omega}$-open in the subspace $(A, \tau \mid A)$. Therefore,$f \mid A$ is somewhat $\hat{\Omega}$-continuous on the subspace $(A, \tau \mid A)$. □

Theorem 5.15. *Suppose that (X,τ) and (Y,σ) are any two topological spaces and suppose $X = A \cup B$, where A and B are both δ open and pre closed in X. If $f\colon (X,\tau) \to (Y,\sigma)$ is any function such that $f \mid A$ and $f \mid B$ are somewhat $\hat{\Omega}$-continuous functions,then f is a somewhat $\hat{\Omega}$-continuous.*

Proof. Let U be any open set in Y such that $f^{-1}(U) \neq \emptyset$. If both $(f\mid_A)^{-1}(U) = f^{-1}(U) \cap A, (f\mid_B)^{-1}(U) = f^{-1}(U) \cap B$ are empty, then $f^{-1}(U) = \emptyset$, a contradiction.Therefore, the possible cases are either $f^{-1}(U) \cap A \neq \emptyset$ or $f^{-1}(U) \cap B \neq \emptyset$ or both $f^{-1}(U) \cap A$ and $f^{-1}(U) \cap B$ are nonempty. It is enough to prove only for the case either $f^{-1}(U) \cap A \neq \emptyset$ or $f^{-1}(U) \cap B \neq \emptyset$.Then automatically second one follows.
Suppose that either $f^{-1}(U) \cap A \neq \emptyset$ or $f^{-1}(U) \cap B \neq \emptyset$. If $f^{-1}(U) \cap A \neq \emptyset$, by hypothesis,there exists a non empty $\hat{\Omega}$-open set $V \in (A, \tau\mid_A)$ such that $V \subseteq f^{-1}(U) \cap A \subseteq f^{-1}(U)$. By[7] theorem 6.9,$V$ is $\hat{\Omega}$-open in X.Therefore.f is a somewhat $\hat{\Omega}$-continuous. □

Definition 5.16. *Let τ and σ are two topologies on a set X. Then τ is said to be equivalent (resp.$\hat{\Omega}$-equivalent) to σ if for every non empty $U \in \tau$ there exists an non empty open (resp.$\hat{\Omega}$-open) set V in (X,σ) such that $V \subseteq U$ and if for every non empty $U \in \sigma$ there exists an non empty open (resp.$\hat{\Omega}$-open) set V in (X,τ) such that $V \subseteq U$.*

Theorem 5.17. *Let τ^* be a topology on X which is $\hat{\Omega}$-equivalent to a topology τ on X. If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat continuous, then $f : (X,\tau^*) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-continuous.*

Proof. Suppose that U is any open set in (X,σ) such that $f^{-1}(U) \neq \emptyset$. Since f is somewhat continuous, there exists a non empty open set V in (X,τ) such that $V \subseteq f^{-1}(U)$. Since τ^* $\hat{\Omega}$-equivalent to τ, there exists $\hat{\Omega}$-open set V_1 in (X,τ^*) such that $V_1 \subseteq f^{-1}(U)$. Hence f is somewhat $\hat{\Omega}$-continuous. □

Theorem 5.18. *Let σ^* be a topology on Y which is equivalent to a topology σ on Y. If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous surjective function, then $f : (X,\tau) \to (Y,\sigma^*)$ is somewhat $\hat{\Omega}$-continuous.*

Proof. Suppose that U is any open set in (Y,σ^*) such that $f^{-1}(U) \neq \emptyset$. Since σ^* is equivalent to σ, there exists a non empty open set V in (Y,σ) such that $V \subseteq U$. Since $f\colon (X,\tau) \to (Y,\sigma)$ is surjective, $f^{-1}(V) \neq \emptyset$.Since $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-continuous, there exists a non empty $\hat{\Omega}$-open set G in (X,τ) such that $G \subseteq f^{-1}(V)$. Hence $f : (X,\tau) \to (Y,\sigma^*)$ is somewhat $\hat{\Omega}$-continuous. □

6 $\hat{\Omega}$-irresolvable spaces

In this section we establish the definition of $\hat{\Omega}$-resolvable spaces and it's properties.

Definition 6.1. *A space (X,τ) is said to be $\hat{\Omega}$-resolvable,if there exists a subset A of X such that both A and A^c are $\hat{\Omega}$-dense in X.Otherwise it is known as $\hat{\Omega}$-irresolvable space.*

Example 6.2. *Example 3.2 is $\hat{\Omega}$-irresolvable space.*

Let us prove a characterization of $\hat{\Omega}$-resolvable space.

Theorem 6.3. *A space (X,τ) is $\hat{\Omega}$-resolvable if and only if it has a pair of disjoint $\hat{\Omega}$ dense sets in X.*

Proof. **Necessity-** Suppose that X is $\hat{\Omega}$-resolvable. Therefore,there exists a subset A of X such that both A and A^c are $\hat{\Omega}$-dense in X.If we define $B = A^c$, then we get a pair of disjoint $\hat{\Omega}$-dense sets in X.

Sufficiency- By hypothesis,we can choose a disjoint pair of $\hat{\Omega}$-dense sets namely A and B in X. Then $\hat{\Omega}cl(A) = \hat{\Omega}cl(B) = X$ such that $A \subseteq B^c$ or $B \subseteq A^c$. If $A \subseteq B^c$, then by [7] theorem 5.3 (ii), $\hat{\Omega}cl(A) \subseteq \hat{\Omega}cl(B^c)$. Then $X \subseteq \hat{\Omega}cl(B^c)$ and hence $X = \hat{\Omega}cl(B^c)$. Therefore,we have a subset B in X such that B and B^c are both $\hat{\Omega}$-dense in X.If $B \subseteq A^c$, then $\hat{\Omega}cl(B) \subseteq \hat{\Omega}cl(A^c)$. Then $X \subseteq \hat{\Omega}cl(B^c)$ and hence $X = \hat{\Omega}cl(B^c)$. Therefore,we have a subset A in X such that A and A^c are both $\hat{\Omega}$-dense in X. Therefore, X is $\hat{\Omega}$-resolvable. □

Theorem 6.4. *A space (X,τ) is $\hat{\Omega}$-irresolvable if and only if $\hat{\Omega}int(A) \neq \emptyset$ for every $\hat{\Omega}$-dense set A in X.*

Proof. **Necessity-** Suppose that A is any $\hat{\Omega}$-dense set in X.By hypothesis,$\hat{\Omega}cl(A^c) \neq X$ and hence $(\hat{\Omega}int(A))^c \neq \emptyset^c$. Therefore, $\hat{\Omega}int(A) \neq \emptyset$.

Sufficiency- Suppose that X is $\hat{\Omega}$-resolvable. Then,there exists a subset A of X such that both A and A^c are $\hat{\Omega}$-dense in X. Then $\hat{\Omega}cl(A^c) = X$ and hence $[\hat{\Omega}int(A)]^c = [\emptyset]^c$. Therefore, $\hat{\Omega}int(A) = \emptyset$,a contradiction. □

Theorem 6.5. *If $X = A \cup B$, where A and B are such that $\hat{\Omega}int(A) = \emptyset, \hat{\Omega}int(B) = \emptyset$. Then X is $\hat{\Omega}$-resolvable.*

Proof. Given that $X = A \cup B, A$ and B are such that $\hat{\Omega}int(A) = \emptyset, \hat{\Omega}int(B) = \emptyset$. Therefore,$\hat{\Omega}cl(A^c) = X, \hat{\Omega}cl(B^c) = X$.Moreover,$X \setminus (A \cup B) = \emptyset$, or $[X \setminus A] \cap [X \setminus B] = \emptyset$. Then $X \setminus A \subseteq [X \setminus B]^c$. Therefore, $\hat{\Omega}cl(A^c) \subseteq \hat{\Omega}cl(B)$ and hence $X \subseteq \hat{\Omega}cl(B)$.Thus we get a subset B in X such that both B and B^c are $\hat{\Omega}$-dense in X.Therefore, X is $\hat{\Omega}$-resolvable. □

Remark 6.6. *The above theorem can be extended to any finite number. That is, if $X = \bigcup_{i=1}^{i=n} A_i$ for any finite number of empty $\hat{\Omega}$ interior sets $A_1, A_2, ...A_n$, then X is $\hat{\Omega}$-resolvable.*

Theorem 6.7. *If $f: (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-open mapping on a irresolvable space X, then Y is $\hat{\Omega}$ irresolvable space.*

Proof. Suppose that A is any non empty $\hat{\Omega}$ dense set in Y. Assume that $\hat{\Omega}(int(A)) = \emptyset$. Then $\hat{\Omega}cl(Y \setminus A) = Y$. Since f is somewhat $\hat{\Omega}$-open by theorem 4.13,$f^{-1}(Y \setminus A)$ is dense in X. Then,$cl(f^{-1}(Y \setminus A)) = X$ and hence $cl(X \setminus f^{-1}(A)) = X$. Thus,$int(f^{-1}(A)) = \emptyset$.Again by hypothesis, $f^{-1}(A)$ is a dense set in X with a empty interior, a contradiction to X is irresolvable. Therefore, our assumption is wrong and hence $\hat{\Omega}(int(A)) \neq \emptyset$. By theorem 6.4,$Y$ is a $\hat{\Omega}$ irresolvable space. □

Theorem 6.8. *Let Y be irresolvable space.If $f: (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous bijective mapping,then X is $\hat{\Omega}$ irresolvable space.*

Proof. Suppose that A is any non empty $\hat{\Omega}$ dense set in X. Assume that $\hat{\Omega}(int(A)) = \emptyset$. Then $\hat{\Omega}cl(X \setminus A) = X$. Since f is somewhat $\hat{\Omega}$-continuous by theorem 5.10 (ii), $f(X \setminus A)$ is dense in Y. Then,$cl(f(X \setminus A)) = Y$. Since f is bijective, $cl(Y \setminus f(A)) = Y$. Thus, $int(f(A)) = \emptyset$. Again by hypothesis, $f(A)$ is a dense set in Y with a empty interior, a contradiction to Y is irresolvable. Therefore, our assumption is wrong and hence $\hat{\Omega}(int(A)) \neq \emptyset$. By theorem 6.4, X is a $\hat{\Omega}$ irresolvable space. □

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Radio Number for Strong Product $P_2 \boxtimes P_n$

S. K. Vaidya[a,*] and D. D. Bantva[b]

[a] *Department of Mathematics, Saurashtra University, Rajkot-360 005, Gujarat, India.*

[b] *Department of Mathematics, L. E. College, Morvi-363 642, Gujarat, India.*

Abstract

A radio labeling of a graph G is a function f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(u) - f(v)| \geq diam(G) + 1 - d_G(u,v)$, where $diam(G)$ and $d_G(u,v)$ are diameter and distance between u and v in graph G respectively. The radio number $rn(G)$ of G is the smallest number k such that G has radio labeling with $\max\{f(v) : v \in V(G)\} = k$. We investigate radio number for strong product of P_2 and P_n.

Keywords: Interference, channel assignment, radio labeling, radio number, strong product.

2010 MSC: 05C15, 05C78.

1 Introduction

In 1980, Hale[5] initiated the problem to determine the minimum number of channels in a given network which is now popular as a channel assignment problem. He classified transmitter as *very close* and *close* transmitter according to the interference between them. He called *very close* transmitters if a pair of transmitters has major interference and called *close* transmitters if a pair of transmitters has minor interference. Hale[5] gave the graphical representation for the channel assignment problem wherein he represented transmitters by vertices and interference between a pair of transmitters by edges. Two transmitters are joined by an edge if major interference occurs between them and minor interference is taken as vertices at distance two in a graph.

In 1991, Roberts[10] suggested a solution for channel assignment problem and proposed that a pair of transmitters having minor interference must receive different channels and a pair of transmitters having major interference must receive channels that are at least two apart. Motivated through this Griggs and Yeh[4] introduced the distance two labeling which is defined as follows:

A distance two labeling (or $L(2,1)$-labeling) of a graph $G = (V(G), E(G))$ is a function f from vertex set $V(G)$ to the set of nonnegative integers such that the following conditions are satisfied:

(1) $|f(u) - f(v)| \geq 2$ if $d(u,v) = 1$.

(2) $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$.

The difference between the largest and the smallest label assigned by f is called the span of f and the minimum span over all $L(2,1)$-labeling of G is called the λ-number of G, denoted by $\lambda(G)$. The $L(2,1)$-labeling has been explored in past two decades by many researchers like Yeh[17, 18], Georges and Mauro[3], Sakai[11], Chang and Kuo[1], Wang[15], Vaidya and Bantva[12] and Vaidya et al.[13].

But as time passed, practically it has been observed that the interference among transmitters might go beyond two levels. Radio labeling extends the number of interference level considered in $L(2,1)$-labeling from two to the largest possible - the diameter of G. The diameter of G is denoted by $diam(G)$ or simply by d is the maximum distance among all pairs of vertices in G. Motivated through the problem of channel assignment of FM radio stations Chartrand et. al[2] introduced the concept of radio labeling of graph as follows.

*Corresponding author.

E-mail addresses: samirkvaidya@yahoo.co.in (S. K. Vaidya) and devsi.bantva@gmail.com (D. D. Bantva).

A radio labeling of a graph G is an injective function $f : V(G) \rightarrow \{0, 1, 2, ...\}$ such that the following is satisfied for all u, $v \in V(G)$:

$$|f(u) - f(v)| \geq diam(G) + 1 - d_G(u, v).$$

The radio number denoted by $rn(G)$ is the minimum span of a radio labeling for G. Note that when $diam(G)$ is two then radio labeling and distance two labeling are identical. The radio labeling is studied in the past decade by many researchers like Liu[6], Liu and Xie[7, 8], Liu and Zhu[9] and Vaidya and Vihol[14].

In this paper, we completely determine the radio number of strong product of P_2 with P_n. Through out this discussion, the order of $P_2 \boxtimes P_n$ is p and we consider $n \geq 3$ as $P_2 \boxtimes P_2$ is simply K_4 for which $L(2, 1)$-labeling and radio labeling coincide. Moreover terms not defined here are used in the sense of West[16].

2 Main results

The strong product $G \boxtimes H$ of G and H is the graph in which the vertex (u, v) is adjacent to the vertex (u', v') if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$, or $uu' \in E(G)$ and $vv' \in E(H)$.

For $P_2 \boxtimes P_{2k+1}$, let v_0 and v'_0 be the centers. Let v_{L1}, v_{L2}, ... ,v_{Lk} be the vertices on the left side and v_{R1}, v_{R2}, ... ,v_{Rk} be the vertices on the right side with respect to center v_0 and v'_{L1}, v'_{L2}, ... ,v'_{Lk} be the vertices on the left side and v'_{R1}, v'_{R2}, ... ,v'_{Rk} be the vertices on the right side with respect to center v'_0.

For $P_2 \boxtimes P_{2k}$, let v_{L0} and v_{R0}, v'_{L0} and v'_{R0} be the centers. Let v_{L1}, v_{L2}, ... ,$v_{L(k-1)}$ be the vertices on the left side and v_{R1}, v_{R2}, ... ,$v_{R(k-1)}$ be the vertices on the right side with respect to centers v_{L0} and v_{R0} and v'_{L1}, v'_{L2}, ... ,$v'_{L(k-1)}$ be the vertices on the left side and v'_{R1}, v'_{R2}, ... ,$v'_{R(k-1)}$ be the vertices on the right side with respect to centers v'_{L0} and v'_{R0}.

Let for $P_2 \boxtimes P_{2k+1}$, $V(P_2 \boxtimes P_{2k+1}) = V_L \cup V_R \cup V'_L \cup V'_R$

$V_L = \{v_0, v_{L1}, v_{L2}, ... ,v_{Lk}\}$

$V_R = \{v_0, v_{R1}, v_{R2}, ... ,v_{Rk}\}$

$V'_L = \{v'_0, v'_{L1}, v'_{L2}, ... ,v'_{Lk}\}$

$V'_R = \{v'_0, v'_{R1}, v'_{R2}, ... ,v'_{Rk}\}$

Let for $P_2 \boxtimes P_{2k}$, $V(P_2 \boxtimes P_{2k}) = V_L \cup V_R \cup V'_L \cup V'_R$

$V_L = \{v_{L0}, v_{L1}, v_{L2}, ... ,v_{L(k-1)}\}$

$V_R = \{v_{R0}, v_{R1}, v_{R2}, ... ,v_{R(k-1)}\}$

$V'_L = \{v'_{L0}, v'_{L1}, v'_{L2}, ... ,v'_{L(k-1)}\}$

$V'_R = \{v'_{R0}, v'_{R1}, v'_{R2}, ... ,v'_{R(k-1)}\}$

In $P_2 \boxtimes P_n$, we say two vertices u and v are on opposite side if $u \in V_L$ or V'_L and $v \in V_R$ or V'_R.

We define the level function on $V(P_2 \boxtimes P_n)$ to the set of whole numbers W from a center vertex w by

$L(u) = \{d(u, w) : w$ is a center vertex $\}$, for any $u \in V(P_2 \boxtimes P_n)$.

In $P_2 \boxtimes P_n$, the maximum level is k if $n = 2k + 1$ and $k - 1$ if $n = 2k$.

Observation 2.1. *For $P_2 \boxtimes P_n$,*

$$(1) \quad |V(P_2 \boxtimes P_n)| = \begin{cases} 4k + 2 & if \quad n = 2k + 1 \\ 4k & if \quad n = 2k \end{cases}$$

$$(2)\quad d(u,v) \quad \leq \quad \begin{cases} L(u)+L(v) & if \quad n=2k+1 \\ L(u)+L(v)+1 & if \quad n=2k \end{cases}$$

(3) If u_i, $u_{i+1} \in V(P_2 \boxtimes P_n)$, $1 \leq i \leq p-1$ are on opposite side and $d(u_i, u_{i+1}) = d(u_{i+1}, u_{i+2})$ or $d(u_i, u_{i+1}) = d(u_{i+1}, u_{i+2}) \pm 1$ then $d(u_i, u_{i+2}) = 1$.

Theorem 2.2. *Let $P_2 \boxtimes P_n$ be a strong product of P_2 and P_n and $k = \lfloor \frac{n}{2} \rfloor$ then*

$$rn(P_2 \boxtimes P_n) \quad \geq \quad \begin{cases} 2k(2k+1)+1 & if \quad n=2k+1 \\ 2k(2k-1)+1 & if \quad n=2k \end{cases}$$

Moreover, the equality holds if and only if there exist a radio labeling f with ordering $\{u_1, u_2, \ldots, u_p\}$ of vertices of $P_2 \boxtimes P_n$ such that $f(u_1) = 0 < f(u_2) < f(u_3) < \ldots < f(u_p)$, where all the following holds (for all $1 \leq i \leq p-1$):
(1) u_i and u_{i+1} are on opposite side,
(2) $\{u_1, u_p\} = \{w_1, w_2\}$ where w_1, w_2 are center vertex.

Proof. Let f be an optimal radio labeling for $P_2 \boxtimes P_n$, where $f(u_1) = 0 < f(u_2) < f(u_3) < \ldots < f(u_p)$. Then $f(u_{i+1}) - f(u_i) \geq (d+1)$ - $d(u_i, u_{i+1})$, for all $1 \leq i \leq p-1$. Summing these $p-1$ inequalities we get

$$rn(P_2 \boxtimes P_n) = f(u_p) \geq (p-1)(d+1) - \sum_{i=1}^{p-1} d(u_i, u_{i+1}) \tag{2.1}$$

Case - 1 : n is odd.

For $P_2 \boxtimes P_{2k+1}$, we have

$$\sum_{i=1}^{p-1} d(u_i, u_{i+1}) \leq \sum_{i=1}^{p-1} [L(u_i) + L(u_{i+1})]$$
$$= 2 \sum_{u \in V(G)} L(u) - L(u_1) - L(u_p)$$
$$= 2 \sum_{u \in V(G)} L(u) \tag{2.2}$$

Substituting (2.2) in (2.1), we get

$$rn(P_2 \boxtimes P_n) = f(u_p) \geq (p-1)(d+1) - 2 \sum_{u \in V(G)} L(u)$$

For $P_2 \boxtimes P_{2k+1}$, $p = 4k+2$, $d = 2k$ and $\sum_{u \in V(G)} L(u) = 2k(k+1)$

$$rn(P_2 \boxtimes P_n) = f(u_p) \geq (4k+2-1)(2k+1) - 4(k(k+1))$$

$$= (4k+1)(2k+1) - 4k(k+1)$$

$$= 8k^2 + 4k + 2k + 1 - 4k^2 - 4k$$

$$= 4k^2 + 2k + 1$$

$$= 2k(2k+1) + 1$$

Case - 2 : n is even.

For $P_2 \boxtimes P_{2k}$, we have

$$\sum_{i=1}^{p-1} d(u_i, u_{i+1}) \leq \sum_{i=1}^{p-1} [L(u_i) + L(u_{i+1}) + 1]$$

$= 2\sum_{u\in V(G)} L(u)$ - $L(u_1)$ - $L(u_p) + (p-1)$

$$= 2\sum_{u\in V(G)} L(u) + (p-1) \tag{2.3}$$

Substituting (2.3) in (2.1), we get

$rn(P_2 \boxtimes P_n) = f(u_p) \geq (p-1)(d+1)$ - $2\sum_{u\in V(G)} L(u)$ - $(p-1)$

For $P_2 \boxtimes P_{2k}$, $p = 4k$, $d = 2k-1$ and $\sum_{v\in V(G)} L(u) = 2k(k-1)$

$rn(P_2 \boxtimes P_n) = f(u_p) \geq (4k-1)(2k-1+1)$ - $4(k(k-1))$ - $(4k-1)$

$= 8k^2$ - $2k$ - $4k^2 + 1$

$= 4k^2$ - $2k + 1$

$= 2k(2k-1) + 1$

Thus, from Case - 1 and Case - 2, we have

$$rn(P_2 \boxtimes P_n) \quad \geq \quad \begin{cases} 2k(2k+1)+1 & \text{if} \quad n = 2k+1 \\ 2k(2k-1)+1 & \text{if} \quad n = 2k \end{cases}$$

□

Theorem 2.3. *Let f be an assignment of distinct non-negative integers to $V(P_2 \boxtimes P_n)$ and $\{u_1, u_2, u_3, ..., u_p\}$ be the ordering of $V(P_2 \boxtimes P_n)$ such that $f(u_i) < f(u_{i+1})$ defined by $f(u_1) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - d(u_i, u_{i+1})$. Then f is a radio labeling if for any $1 \leq i \leq p-2$ and $k = \lfloor \frac{n}{2} \rfloor$ the following holds.*
(1) $d(u_i, u_{i+1}) \leq k+1$ if n is odd,
(2) $d(u_i, u_{i+1}) \leq k+1$ and $d(u_i, u_{i+1}) \neq d(u_{i+1}, u_{i+2})$ if n is even.

Proof. Let $f(u_1) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - d(u_i, u_{i+1})$, for any $1 \leq i \leq p-1$ and $k = \lfloor \frac{n}{2} \rfloor$.

For each $i = 1, 2, ..., p-1$, let $f_i = f(u_{i+1}) - f(u_i)$. Now we want to prove that f is a radio labeling if (1) and (2) holds. *i.e.* for any $i \neq j$, $|f(u_j) - f(u_i)| \geq d + 1 - d(u_i, u_j)$

Case - 1 : n is odd.

If $n = 2k+1$ then $d = 2k$ and let (1) holds.

Let $j > i$ then $f(u_j) - f(u_i) = f_i + f_{i+1} + ... + f_{j-1}$

$= (j-i)(d+1)$ - $d(u_i, u_{i+1})$ - $d(u_{i+1}, u_{i+2})$ - ... - $d(u_{j-1}, u_j)$

$\geq (j-i)(d+1) - (j-i)(k+1)$ as $d(u_i, u_{i+1}) \leq k+1$

$= (j-i)(2k+2) - (j-i)(k+1)$

$= (j-i)(2k+2-k-1)$

$= (j-i)(k+1)$

$\geq d + 1 - d(u_i, u_j)$.

Case - 2 : n is even.

If $n = 2k$ then $d = 2k - 1$ and let (2) holds.

Let $j > i$ then $f(u_j) - f(u_i) = f_i + f_{i+1} + ... + f_{j-1}$

$= (j-i)(d+1)$ - $d(u_i, u_{i+1})$ - $d(u_{i+1}, u_{i+2})$ - ... - $d(u_{j-1}, u_j)$

If $j - i =$ even then

$\geq (j-i)(d+1) - \frac{j-i}{2}(k+1) - \frac{j-i}{2}(k)$

$= (j-i)(2k) - (j-i)(k) - \frac{j-i}{2}$

$= (j-i)(k) - \frac{j-i}{2}$

$\geq d + 1 - d(u_i, u_j)$

If $j - i =$ odd then

$\geq (j-i)(d+1) - \frac{j-i+1}{2}(k+1) - \frac{j-i-1}{2}(k)$

$\geq d + 1 - d(u_i, u_j)$

Thus, in both the cases f is a radio labeling and hence the result. □

Theorem 2.4. *Let $P_2 \boxtimes P_n$ be a strong product of P_2 and P_n and $k = \lfloor \frac{n}{2} \rfloor$ then*

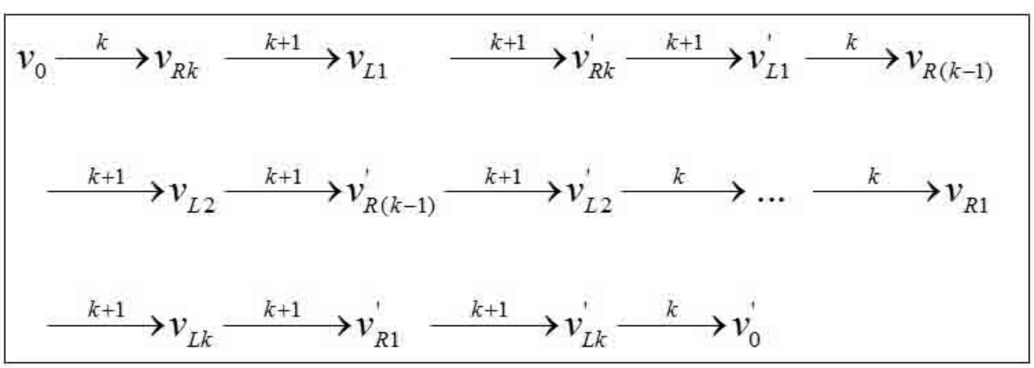
$$rn(P_2 \boxtimes P_n) \leq \begin{cases} 2k(2k+1)+1 & \text{if } n = 2k+1 \\ 2k(2k-1)+1 & \text{if } n = 2k \end{cases}$$

Proof. Here we consider following two cases.

Case - 1 : n is odd.

For $P_2 \boxtimes P_{2k+1}$, define f : V$(P_2 \boxtimes P_{2k+1}) \rightarrow \{0,1,2, ... ,2k(2k+1)+1\}$ by $f(u_{i+1}) = f(u_i) + d + 1$ - $L(u_i)$ - $L(u_{i+1})$ as per ordering of vertices shown in Table 1:

Table 1



Case - 2 : n is even.

For $P_2 \boxtimes P_{2k}$, define $f : V(P_2 \boxtimes P_{2k}) \rightarrow \{0,1,2, \ldots ,2k(2k-1)+1\}$ by $f(u_{i+1}) = f(u_i) + d - L(u_i) - L(u_{i+1})$ as per ordering of vertices shown in Table 2:

Table 2

$$v_{L0} \xrightarrow{k} v_{R(k-1)} \xrightarrow{k+1} v_{L1} \xrightarrow{k} v'_{R(k-2)} \xrightarrow{k+1} v'_{L2} \xrightarrow{k} v_{R(k-3)}$$

$$\xrightarrow{k+1} \ldots \xrightarrow{k+1} v_{L(k-1)} \xrightarrow{k} v_{R0} \xrightarrow{1} v'_{L0} \xrightarrow{k} v'_{R(k-1)}$$

$$\xrightarrow{k+1} v'_{L1} \xrightarrow{k} v'_{R(k-2)} \xrightarrow{k+1} v'_{L2} \xrightarrow{k} v'_{R(k-3)} \xrightarrow{k+1} \ldots$$

$$\xrightarrow{k+1} v'_{L(k-1)} \xrightarrow{k} v'_{R0}$$

Thus in Case - 1 and Case - 2, it is possible to assign labeling to the vertices of $P_2 \boxtimes P_n$ with span equal to the lower bound satisfying the condition of Theorem 2.3. Hence f is a radio labeling. □

Theorem 2.5. *Let $P_2 \boxtimes P_n$ be a strong product of P_2 and P_n and $k = \lfloor \frac{n}{2} \rfloor$ then*

$$rn(P_2 \boxtimes P_n) = \begin{cases} 2k(2k+1)+1 & \text{if } n = 2k+1 \\ 2k(2k-1)+1 & \text{if } n = 2k \end{cases}$$

Proof. The proof follows from Theorem 2.2 and Theorem 2.4. □

Example 2.1. *In Figure 1, ordering of the vertices and optimal radio labeling of $P_2 \boxtimes P_9$ is shown.*

$v_0 \rightarrow v_{R4} \rightarrow v_{L1} \rightarrow v'_{R4} \rightarrow v'_{L1} \rightarrow v_{R3} \rightarrow v_{L2} \rightarrow v'_{R3} \rightarrow v'_{L2} \rightarrow v_{R2} \rightarrow v_{L3} \rightarrow$

$v'_{R2} \rightarrow v'_{L3} \rightarrow v_{R1} \rightarrow v_{L4} \rightarrow v'_{R1} \rightarrow v'_{L4} \rightarrow v'_0 = rn(P_2 \boxtimes P_9)$

Example 2.2. *In Figure 2, ordering of the vertices and optimal radio labeling of $P_2 \boxtimes P_{10}$ is shown.*

$v_{L0} \rightarrow v_{R4} \rightarrow v_{L1} \rightarrow v_{R3} \rightarrow v_{L2} \rightarrow v_{R2} \rightarrow v_{L3} \rightarrow v_{R1} \rightarrow v_{L4} \rightarrow v_{R0} \rightarrow v'_{L0} \rightarrow$

$v'_{R4} \rightarrow v'_{L1} \rightarrow v'_{R3} \rightarrow v'_{L2} \rightarrow v'_{R2} \rightarrow v'_{L3} \rightarrow v'_{R1} \rightarrow v'_{L4} \rightarrow v'_{R0} = rn(P_2 \boxtimes P_{10})$

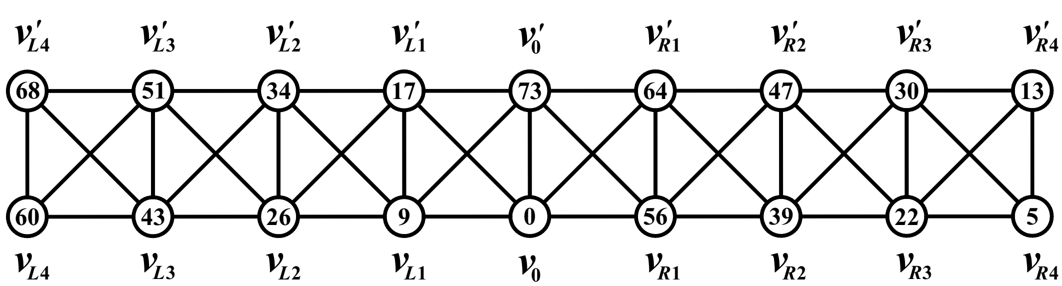


Figure 1. $rn(P_2 \boxtimes P_9) = \mathbf{73}$

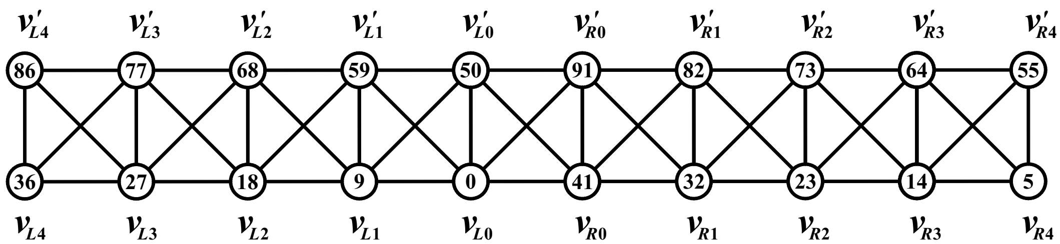


Figure 2. $rn(P_2 \boxtimes P_{10}) = \mathbf{91}$

3 Concluding Remarks

The assignment of channels is of great importance for the establishment of transmitter network which is free of interference. The radio labeling is an intelligent move in this direction because the level of interference is maximum at diametrical distance. We take up this problem in the context of strong product of P_2 and P_n and determine radio number for the same. To derive similar results for other graph families is an open area of research.

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On degree of approximation of conjugate series of a Fourier series by product summability

Mahendra Misra[a], B. P. Padhy[b], Dattaram Bisoyi[c], and U. K. Misra[d,*]

[a] P. G. Department of Mathematics, N. C. College(Autonomous), Jajpur, Odisha, India.

[b] Department of Mathematics, Roland Institute of Technology, Golanthara-761008, Odisha, India.

[c] Department of Mathematics, L. N. Mahavidyalaya, Kodala, Ganjam, Odisha, India.

[d] Department of Mathematics, National Institute of Science and Technology, Palur Hills, Golanthara-761008, Odisha, India.

Abstract

In this paper a theorem on degree of approximation of a function $f \in Lip(\alpha, r)$ by product summability $(E, q)(\bar{N}, p_n)$ of conjugate series of Fourier series associated with f has been proved.

Keywords: Degree of Approximation, $Lip(\alpha, r)$ class of function, (E, q) mean, $(\bar{N}, p_n)$ mean, $(E, q)(\bar{N}, p_n)$ product mean, Fourier series, conjugate of the Fourier series, Lebesgue integral.

2010 MSC: 42B05, 42B08.

1 Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \longrightarrow \infty, as\ n \longrightarrow \infty, (P_{-i} = p_{-i} = 0, i \geq 0). \tag{1.1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\upsilon=0}^{n} p_\upsilon s_\upsilon, \tag{1.2}$$

defines the sequence $\{t_n\}$ of the $(\bar{N}, p_n)$-mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \longrightarrow s,\ as\ n \longrightarrow \infty, \tag{1.3}$$

then the series $\sum a_n$ is said to be $(\bar{N}, p_n)$ summable to s.
The conditions for regularity of $(\bar{N}, p_n)$-summability are easily seen to be [1]

$$\begin{cases} (i) P_n \to \infty, & as \quad n \to \infty, \\ (ii) \sum_{i=0}^{n} p_i \leq C \mid P_n \mid, & as \quad n \to \infty. \end{cases} \tag{1.4}$$

The sequence-to-sequence transformation, [1]

$$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon, \tag{1.5}$$

*Corresponding author.
E-mail addresses:mahendramisra@2007.gmail.com (Mahendra Misra), iraady@gmail.com (B. P. Padhy), dbisoyi2@gmail.com (Dattaram Bisoyi) and umakanta_misra@yahoo.com (U. K. Misra).

defines the sequence $\{T_n\}$ of the (E,q) mean of the sequence $\{s_n\}$. If

$$T_n \to s, \quad as \quad n \to \infty, \tag{1.6}$$

then the series Σa_n is said to be (E,q) summable to s. Clearly (E,q) method is regular. Further, the (E,q) transformation of the $(\bar{N},p_n)$ transform of $\{s_n\}$ is defined by

$$\begin{aligned}\tau_n &= \frac{1}{(1+q)^n}\Sigma_{k=0}^{n}\binom{n}{k}q^{n-k}T_k \\ &= \frac{1}{(1+q)^n}\Sigma_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\Sigma_{\upsilon=0}^{k}p_\upsilon s_\upsilon\right\}\end{aligned} \tag{1.7}$$

If

$$\tau_n \to s, \; as \; n \to \infty, \tag{1.8}$$

then $\sum a_n$ is said to be $(E,q)(\bar{N},p_n)$-summable to s.

Let $f(t)$ be a periodic function with period 2π and L-integrable over $(-\pi,\pi)$. The Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\ nx + b_n sin\ nx) \equiv \sum_{n=0}^{\infty} A_n(x), \tag{1.9}$$

and the conjugate series of the Fourier Series (1.9) is

$$\sum_{n=1}^{\infty}(b_n cos\ nx - a_n sin\ nx) \equiv \sum_{n=0}^{\infty} B_n(x). \tag{1.10}$$

Let $\bar{s}_n(f:x)$ be the n-th partial sum of (1.10). The L_∞-norm of a function $f: R \to R$ is defined by

$$\| f \|_\infty = sup\{|f(x)| : x \in R\} \tag{1.11}$$

and the L_υ-norm is defined by

$$\| f \|_\upsilon = \left(\int_0^{2\pi} |f(x)|^\upsilon\right)^{\frac{1}{\upsilon}}, \upsilon \geq 1. \tag{1.12}$$

The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\| \cdot \|_\infty$ is defined by [5]

$$\| P_n - f \|_\infty = sup\{|p_n(x) - f(x)| : x \in R\} \tag{1.13}$$

and the degree of approximation $E_n(f)$ a function $f \in L_\upsilon$ is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_\upsilon. \tag{1.14}$$

A function f is said to satisfy Lipschitz condition (here after we write $f \in$ Lip α) if

$$|f(x+1) - f(x)| = O(|t|^\alpha), 0 < \alpha \leq 1. \tag{1.15}$$

and $f(x)\epsilon Lip(\alpha, r)$, for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = 0(|t|^\alpha), 0 < \alpha \leq 1, r \geq 1, t > 0. \tag{1.16}$$

For a given positive increasing function $\xi(t)$, the function $f(x) \in$ Lip $(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = 0(\xi(t)), r \geq 1, t > 0. \tag{1.17}$$

We use the following notation throughout this paper:

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}, \tag{1.18}$$

and

$$\overline{K}_n(t) = \frac{1}{\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon \frac{\cos\frac{t}{2} - \cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}.$$

Further, the method $(E,q)(\overline{N},P_n)$ is assumed to be regular.

2 Known Theorems

Dealing with the degree of approximation by the product Misra et. al. [2] proved the following theorem using $(E,q)(\overline{N},p_n)$-mean of Conjugate Series of Fourier series:

Theorem 2.1. *If f is 2π-periodic function of class $Lip\alpha$, then degree of approximation by the product $(E,q)(\bar{N},p_n)$ summability mean of the conjugate series (1.10) of the Fourier Series (1.9) is given by* $\|\tau_n - f|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right)$, $0<\alpha<1$, *where τ_n is as defined in (1.7).*

Very recently Paikray et. al [3] established a theorem on degree of approximation by the product mean $(E,q)(\bar{N},p_n)$ of the Conjugate Series of fourier Series of a function of class $Lip(\alpha,r)$. They proved:

Theorem 2.2. *If f is a 2π-Periodic function of class $Lip(\alpha,r)$, then degree of approximation by the product $(E,q)(\bar{N},p_n)$ summability means on on he Conjugate Series (1.10) of the Fourier series (1.9) is given by* $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{r}}}\right), 0<\alpha<1,\ r\geq 1$, *where τ_n is as defined in (1.7).*

3 Main Theorem

In this paper, we have proved a theorem on degree of approximation by the product mean $(E,q)(\bar{N},p_n)$ of the conjugate series of the Fourier series of a function of class $Lip(\xi(t),r)$. We prove:

Theorem 3.3. *Let $\xi(t)$ be a positive increasing function and f a 2π- periodic function of the class $Lip(\xi(t),r), r\geq 1, t>0$. Then degree of approximation by the product $(E,q)(\bar{N},p_n)$ summability means on the Conjugate Series (1.10) of the Fourier series (1.9) is given by* $\|\ \tau_n - f\ \|_\infty = O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right), r\geq 1$, *where τ_n is as defined in (1.7).*

4 Required Lemmas

We require the following Lemmas to prove the theorem.

Lemma 4.1.

$$|\bar{K}_n(t)| = O(n), 0\leq t\leq \frac{1}{n+1}.$$

Proof. For $0\leq t\leq \frac{1}{n+1}$, we have $sin\ nt \leq n\ sin\ t$ then

$$\begin{aligned}
|\bar{K}_n(t)| &= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\upsilon t\cdot\cos\frac{t}{2}+\sin\upsilon t\cdot\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(\frac{cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)}{\sin\frac{t}{2}}+\sin\upsilon t\right)\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(O\left(2\sin\upsilon\frac{t}{2}\sin\upsilon\frac{t}{2}\right)+\upsilon\sin t\right)\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon(O(\upsilon)+O(\upsilon))\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{O(k)}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right| \\
&= O(n).
\end{aligned}$$

This proves the lemma. □

Lemma 4.2.

$$|\bar{K}_n(t)| = O\left(\frac{1}{t}\right), for \frac{1}{n+1} \leq t \leq \pi.$$

Proof. For $\frac{1}{n+1} \leq t \leq \pi$, by Jordan's lemma, we have $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$.Then

$$\begin{aligned}
|\bar{K}_n(t)| &= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\upsilon\frac{t}{2}\cdot\cos\frac{t}{2}+\sin\upsilon\frac{t}{2}\cdot\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}\frac{\pi}{2t}p_\upsilon\left(\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)+\sin\upsilon\frac{t}{2}\cdot\sin\frac{t}{2}\right)\right\}\right| \\
&\leq \frac{\pi}{2\pi(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right\}\right| = \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right\}\right| \\
&= \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right| \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

This proves the lemma. □

5 Proof of Theorem 3.1

Using Riemann-Lebesgue theorem, we have for the n-th partial sum $\bar{s}_n(f:x)$ of the conjugate Fourier series (1.10) of $f(x)$, following Titchmarch [4]

$$\bar{s}_n(f:x) - f(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\bar{K}_n dt,$$

the (N, p_n) transform of $\bar{s}_n(f:x)$ using (1.2) is given by

$$t_n - f(x) = \frac{2}{\pi P_n}\int_0^\pi \psi(t)\sum_{k=0}^{n}p_k\frac{\cos\frac{t}{2}-\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt,$$

denoting the $(E,q)(N,p_n)$ transform of $\bar{s}_n(f:x)$ by τ_n, we have

$$\begin{aligned}
\|\tau_n - f\| &= \frac{1}{\pi(1+q)^n}\int_0^\pi \psi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\sin\left(\upsilon+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right\}dt \\
&= \int_0^\pi \psi(t)\bar{K}_n(t)dt \\
&= \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right\}\psi(t)\bar{K}_n(t)dt \\
&= I_1 + I_2, \quad \text{say.} \qquad (5.1)
\end{aligned}$$

Now

$$
\begin{aligned}
|I_1| &= \frac{2}{\pi(1+q)^n}\left|\int_0^{\frac{1}{n+1}}\psi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right\}dt\right| \\
&= \left|\int_0^{\frac{1}{n+1}}\psi(t)\bar{K}_n(t)dt\right| \\
&= \left(\int_0^{\frac{1}{n+1}}\left(\frac{\psi(t)}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{n+1}}(\xi(t)\bar{K}_n(t))^s dt\right)^{\frac{1}{s}}, \quad \text{using Holder's inequality} \\
&= O(1)\left(\int_0^{\frac{1}{n+1}}\xi(t)n^s dt\right)^{\frac{1}{s}} \\
&= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left(\frac{n^s}{n+1}\right)^{\frac{1}{s}} \\
&= O\left(\xi\left(\frac{1}{n+1}\right)\frac{1}{(n+1)^{\frac{1}{s}-1}}\right) \\
&= O\left(\xi\left(\frac{1}{n+1}\right)\frac{1}{(n+1)^{-\frac{1}{r}}}\right) \\
&= O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right).
\end{aligned} \tag{5.2}
$$

Next

$$
\begin{aligned}
|I_2| &\le \left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\phi(t)}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{1}{n+1}}^{\pi}(\xi(t)\bar{K}_n(t))^s dt\right)^{\frac{1}{s}}, \text{ using Holder's inequality} \\
&= O(1)\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t}\right)^s dt\right)^{\frac{1}{s}}, \quad \text{using Lemma 4.1} \\
&= O(1)\left(\int_{\frac{1}{\pi}}^{n+1}\left(\frac{\xi\left(\frac{1}{y}\right)}{\frac{1}{y}}\right)^s\frac{dy}{y^2}\right)^{\frac{1}{s}}.
\end{aligned} \tag{5.3}
$$

Since $\xi(t)$ is a positive increasing function, so is $\xi(1/y)/(1/y)$. Using second mean value theorem we get

$$
= O\left((n+1)\xi\left(\frac{1}{n+1}\right)\right)\left(\int_\delta^{n+1}\frac{dy}{y^2}\right)^{\frac{1}{s}}, \quad \text{for some} \quad \frac{1}{\pi}\le\delta\le n+1
$$

$$
= O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right)
$$

Then from (5.2) and (5.3), we have

$$
|\ \tau_n - f(x)\ | = O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right), for \quad r\ge 1.
$$

$$
\|\ \tau_n - f(x)\ \|_\infty = \sup_{-\pi<x<\pi}|\tau_n - f(x)| = O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right), r\ge 1.
$$

This completes the proof of the theorem.

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Nonlocal impulsive fractional semilinear differential equations with almost sectorial operators

M.C. Ranjini[a] and A. Anguraj[b,*]

[a,b] Department of Mathematics, P.S.G. College of Arts and Science, Coimbatore-641 014, Tamil Nadu, India.

Abstract

This paper is concerned with the existence and uniqueness of mild solutions for a class of impulsive fractional semilinear differential equations with nonlocal condition in a Banach space by using the concepts of almost sectorial operators. The results are established by the application of the Banach fixed point theorem and Krasnoselskii's fixed point theorem.

Keywords: Fractional differential equations, impulses, nonlocal condition, almost sectorial operator, semigroup of growth γ, mild solution

2010 MSC: 34A37, 34K37, 47A60.

1 Introduction

Sectorial operators, that is, linear operators A defined in Banach spaces, whose spectrum lies in a sector

$$S_w = \{\lambda \in \mathbb{C}/\{0\} \mid |arg\lambda| \leq w\} \cup \{0\} \quad for\ some \quad 0 \leq w \leq \frac{\pi}{2}$$

and whose resolvent satisfies an estimate

$$||(\lambda - A)^{-1}|| \quad \leq \quad C|\lambda|^{-1}, \qquad \forall \quad \lambda \in \mathbb{C}\backslash S_w, \tag{1.1}$$

have been studied extensively during the last 40 years, both in abstract settings and for their applications to partial differential equations. Many important elliptic differential operators belong to the class of sectorial operators, especially when they are considered in the Lebesgue spaces or in spaces of continuous functions (see [1] and [[2], chapter 3]). However, if we look at spaces of more regular functions such as the spaces of Holder continuous functions, we find that these elliptic operators do no longer satisfy the estimate (1.1) and therefore are not sectorial as was pointed out by Von Wahl (see [[3], Ex.3.1.33], see [4]).

Neverthless, for these operators estimates such as

$$||(\lambda - A)^{-1}|| \leq \frac{C}{|\lambda|^{1-\gamma}}, \qquad \lambda \in \sum_{w,v} = \{\lambda \in \mathbb{C} : |arg(\lambda - w)| < v\} \tag{1.2}$$

where $\gamma \in (0, 1)$, $w \in \mathbb{R}$ and $v \in (\frac{\pi}{2}, \pi)$, can be obtained, (see[4]) which allows to define an associated "analytic semigroup" by means of the Dunford Integral

$$T(t) \quad = \quad \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda - A)^{-1}d\lambda, \qquad t > 0 \tag{1.3}$$

where $\Gamma_\theta = \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$.

*Corresponding author.

E-mail addresses: angurajpsg@yahoo.com (A.Anguraj) and ranjiniprasad@gmail.com (M.C.Ranjini).

In the literature, a linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ which satisfy the condition (1.2) is called almost sectorial and the operator family $\{T(t), T(0) = I, t \geq 0\}$ is said the "semigroup of growth γ" generated by A. The operator family $T(t)_{t\geq 0}$ has properties similar at those of analytic semigroup which allow to study some classes of partial differential equations via the usual methods of semigroup theory. Concerning almost sectorial operators, semigroups of growth γ and applications to partial differential equations, we refer the reader to [4, 5, 6, 7, 8] and the references there in.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc., involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. Though the concepts and the calculus of fractional derivative are few centuries old, it is realized only recently that these derivatives form an excellent framework for modeling real world problems.

In the consequence, fractional differential equations have been of great interest. For details, see the monographs of Kilbas et al.[9], Lakshimkantham et al.[10], Miller and Ross [11], Podlubny [12] and the papers in [13, 14, 15, 16] and the references therein.

On the otherhand, the theory of impulsive differential equations has undergone rapid development over the years and played a very imortant role in modern applied mathematical models of real processes arising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics. See, the monographs of Bainov and Simeonov [17], Benchohra et al.[18], Lakshmikantham et al.[19], Samoilenko and Perestyuk[20], A. Anguraj et al.[21, 22] and the references therein. However impulsive fractional differential equations have been studied by the authors, see for instance [23, 24, 25].

We have also seen articles dealing with nonlocal conditions. That is a classical initial condition $x(0) = x_0$ is extended to the following nonlocal condition $x(0) + g(x(.)) = x_0$, where x(.) is a solution and g is a mapping defined on some function space into $\mathbb{X}$. Such nonlocal conditions were first used by K. Deng, in [26]. In his paper, Deng indicated that the diffusion phenomenon of a small amount of gas in a transparent tube can give a better result than using the usual local condition. For the importance of nonlocal conditions in different fields, we refer the reader to [27, 28, 29, 30] and the references contained therein.

Very recently, Rong-Nian Wang et al.[31], studied the classical and mild solutions of abstract fractional cauchy problems using almost sectorial operators and in [32], A.N. Carvalho et al. established the existence of mild solutions for cauchy problem for non-autonomous evolution equation, in which the operator in the linear part depends on time t and for each t, it is almost sectorial.To the best of our knowledge, much less is known about the nonlocal impulsive fractional differential equations with almost sectorial operators. Using the concepts of the above mentioned papers, we proved the existence and uniqueness of mild solutions of the nonlocal impulsive fractional differential equations with almost sectorial operators.

Here, we consider the semilinear impulsive fractional differential equations with nonlocal conditions in the following form.

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f(t, x(t)), & t \in I = [0, T], \quad t \neq t_k \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), & t = t_k, \quad k = 1, 2, ..., m. \\ x(0) + g(x) = x_0 \end{cases} \tag{1.4}$$

where ${}^cD^\alpha$ is the standard Caputo's fractional derivative of order α, $0 < \alpha < 1$ and $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is an almost sectorial operator on a Banach space $\mathbb{X}$. Here, $0 < t_1 < t_2 < ... < t_m = T$, $I_k \in C(\mathbb{X}, \mathbb{X})$, $k = 1, 2, ..., m$. Let $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively. The nonlocal condition

$$g(x) = \sum_{i=1}^{n} c_i x(s_i)$$

where c_i, i = 1,2,...n, are given constants and $0 < s_1 < s_2... < s_n \leq T$.

2 Preliminaries

In this section, we recall some notations, properties of $T(t)$ and the definition of a mild solution of (1.4) by investigating the Classical solutions of the system (1.4).

Proposition 2.1. *([5, 6]). Let A be the almost sectorial operator satisfying the conditions (1.2) and (1.3). Then the following properties are satisfied.*

(i) The operator A is closed, $T(t+s) = T(t)T(s)$ *and* $AT(t)x = T(t)Ax, \forall\ t, s \in [0, \infty)$ *and each* $x \in D(A)$.

(ii) $\frac{d}{dt}T(t) = AT(t)$.

(iii) There exists a constant $C_0 > 0$ *such that* $||A^n T(t)|| \le C_n t^{-(n+\gamma)}\ \ (t > 0)$.

Now, we state the necessary notions and facts on fractional calculus.

Definition 2.1. *([9]) The Riemann-Liouville fractional integral operator of order* $q > o$ *with the lower limit* t_0 *for a function* f *is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds, \quad t > t_0$$

provided the right-hand side is pointwise defined on $[t_0, \infty)$*, where* Γ *is the gamma function.*

Definition 2.2. *([9]) The Riemann-Liouville (R-L) derivative of order* $q > 0$ *with the lower limit* t_0 *for a function* $f : [t_0, \infty) \to \mathbb{R}$ *can be written as*

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{(n-q-1)} f(s) ds, \quad t > t_0, \quad n-1 < q < n.$$

Definition 2.3. *([9]). The Caputo fractional derivative of order* $q > 0$ *with the lower limit* t_0 *for a function* $f : [t_0, \infty) \to \mathbb{R}$ *can be written as*

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{(n-q-1)} f^{(n)}(s) ds = I^{(n-q)} f^{(n)}(t), \quad t > t_0, \quad n-1 < q < n.$$

Denote $E_{\alpha,\beta}$ the generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{\wp} \frac{\lambda^{\alpha-\beta} e^{\lambda}}{\lambda^{\alpha} - z} d\lambda, \quad \alpha, \beta > 0, \quad z \in \mathbb{C}$$

where $\wp$ is a contour which starts and ends at $-\infty$.

Throughout this section we let A be an almost sectorial operator with semigroup of growth γ, where $0 < \gamma < 1$. In the sequel, we will define two families of operators based on the generalized Mittag-Leffler-type functions and the resolvent operators associated with A. They will be two families of linear and bounded operators.

Next, we consider the definition of mild solution of (1.4).

Consider, the following cauchy problem,

$$\begin{cases} {}^cD^{\alpha} x(t) = Ax(t) + f(t, x(t)), \quad 0 < \alpha < 1, \\ x(0) + g(x) = x_0 \in X \end{cases} \tag{2.5}$$

where f is an abstract function defined on $[0, \infty)$ and with values in X, A is almost sectorial operator.

Using Mittag-Leffler function, the Classical solution of the system (2.5) is given by,

$$x(t) = [x_0 - g(x)] E_{\alpha,1}(At^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) f(s, x(s)) ds. \tag{2.6}$$

Denote the operators $P_{\alpha}(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha})$ and $S_{\alpha}(t) = E_{\alpha,1}(At^{\alpha})$. Then x(t) can be expressed as

$$x(t) = S_{\alpha}(t)[x_0 - g(x)] + \int_0^t P_{\alpha}(t-s) f(s, x(s)) ds. \tag{2.7}$$

where $S_\alpha(t)$ and $P_\alpha(t)$ can be expressed as

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda.$$
$$P_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda^\alpha - A)^{-1}d\lambda.$$

where $\Gamma_\theta = \{\mathbb{R}_+e^{i\theta}\} \cup \{\mathbb{R}_+e^{-i\theta}\}$, is oriented counter-clockwise.

Lemma 2.1. *For each fixed $t > 0$, $S_\alpha(t)$ and $P_\alpha(t)$ are linear and bounded operators on $\mathbb{X}$. Moreover, there exist constants $C_s = C(\alpha,\gamma) > 0$, $C_p = C(\alpha,\gamma) > 0$ such that for all $t > 0$,*

$$||S_\alpha(t)|| \le C_s t^{-\alpha\gamma}, \qquad ||P_\alpha(t)|| = C_p t^{\alpha(1-\gamma)-1}, \qquad \text{where } \ 0 < \gamma < 1.$$

Proof. Since, $t > 0$, $0 < \gamma < 1$, there exists a constant $C > 0$ such that

$$||(\lambda - A)^{-1}|| \le \frac{C}{|\lambda|^{1-\gamma}} \qquad , \lambda \in \sum_{w,v}$$

From [32], observe that $\frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda - A)^{-1}d\lambda$ converge in the uniform operator topology for all $t > 0$ and by (1.3), we have that

$$\begin{aligned}
||S_\alpha(t)|| &\le \left\|\frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda\right\| \\
&\le \frac{1}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\lambda|t}|\lambda|^{\alpha-1}||(\lambda^\alpha - A)^{-1}||d|\lambda| \\
&\le \frac{1}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\lambda|t}|\lambda|^{\alpha-1}\frac{C}{|\lambda|^{\alpha(1-\gamma)}}d|\lambda| \\
&\le \frac{C\ t^{-\alpha\gamma}}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\mu|}|\mu|^{\alpha\gamma-1}d\mu \\
&\le C_s\ t^{-\alpha\gamma}
\end{aligned}$$

Also, we have

$$\begin{aligned}
||P_\alpha(t)|| &\le \left\|\frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda^\alpha - A)^{-1}d\lambda\right\| \\
&\le \frac{1}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\lambda|t}||(\lambda^\alpha - A)^{-1}||d|\lambda| \\
&\le \frac{1}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\lambda|t}\frac{C}{|\lambda|^{\alpha(1-\gamma)}}d|\lambda| \\
&\le \frac{C\ t^{\alpha(1-\gamma)-1}}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\mu|}|\mu|^{-\alpha(1-\gamma)}d\mu \\
&\le C_s\ t^{\alpha(1-\gamma)-1}
\end{aligned}$$

□

Lemma 2.2. *([31]) For $t > 0$, $S_\alpha(t)$ and $P_\alpha(t)$ are continuous in the uniform operator topology. Moreover, for every $r > 0$, the continuity is uniform on $[r,\infty)$.*

Theorem 2.1. *If f satisfies the uniform Holder condition with exponent $\beta \in (0,1]$ and A is an almost sectorial operator, then any solution of the Cauchy problem (1.4) is a fixed point of the operator given below*

$$\Gamma x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)] + \int_0^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in [0,t_1]; \\ S_\alpha(t-t_1)\big(x(t_1^-) + I_1(x(t_1^-))\big) + \int_{t_1}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_1,t_2]; \\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) + \int_{t_m}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_m,T]. \end{cases}$$

In fact, from(2.7) it is easy to see that Theorem [2.1] holds, so the proof is omitted.

Now let us consider the set of functions $PC(I,\mathbb{X}) = \{x : I \to \mathbb{X} : x \in C((t_k,t_{k+1}],\mathbb{X}),\ \ k = 1,2,...,m$ and there exist $x(t_k^+)$ and $x(t_k^-),\ \ k = 1,2,...m$ with $x(t_k^-) = x(t_k)\}$
endowed with the norm $||x||_{PC} = \sup_{t\in I}||x(t)||$.
From Theorem[2.1], we can define the mild solution of the system (1.4) as follows:

Definition 2.4. *A function $x : I \to \mathbb{X}$ is called a mild solution of a system (1.4), if $x \in PC(I,\mathbb{X})$ and satisfies the following equation,*

$$x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)] + \int_0^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in [0,t_1]; \\ S_\alpha(t-t_1)\big(x(t_1^-) + I_1(x(t_1^-))\big) + \int_{t_1}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_1,t_2]; \\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) + \int_{t_m}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_m,T]. \end{cases}$$

Remark 2.1. *It is easy to verify that a classical solution of (1.4) is a mild solution of the same system.*

3 Existence Results

In this section, we give the main results on the existence of mild solutions of the system (1.4).
To establish our results, we introduce the following hypotheses.

(H_1) $f : I \times \mathbb{X} \to \mathbb{X}$ is continuous and there exists a constant $M > 0$ such that

$$\begin{aligned} ||f(t,x) - f(t,y)|| &\le M||x-y||, \quad \forall\ t \in I, x, y \in \mathbb{X} \\ ||f(t,0)|| &\le k_1, \end{aligned}$$

where k_1 is a constant.

(H_2) $g : PC(I,\mathbb{X}) \to \mathbb{X}$ is continuous and there exists a constant b such that

$$\begin{aligned} ||g(x) - g(y)|| &\le b||x-y||_{PC}, \quad \forall\ t \in I, x, y \in PC(I,\mathbb{X}) \\ ||g(0)|| &\le k_2, \end{aligned}$$

where k_2 is a constant.

(H_3) for each $k = 1,2,...,m$, there exists $\rho_k > 0$ such that

$$\begin{aligned} ||I_k(x) - I_k(y)|| &\le \rho_k||x-y||, \quad \forall x, y \in \mathbb{X} \\ ||I_k(0)|| &\le k_3, \end{aligned}$$

where k_3 is a constant.

(H_4) For each $x_0 \in \mathbb{X}$, there exists a constant $r > 0$ such that
$r \ge \max_{1\le i\le m}\Big\{C_sT^{-\alpha\gamma}\big[||x_0|| + r(\rho_i + b + 1) + k_2 + k_3)\big] + C_p(Mr + k_1)\frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big\}$

Theorem 3.2. *Under the assumptions $(H_1) - (H_3)$, the system (1.4) has a unique mild solution $x \in PC(I,\mathbb{X})$ if*

$$N = \max_{1\le i\le m}\Big\{C_sT^{-\alpha\gamma}\big[b + 1 + \rho_i\big] + C_pM\frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big\} < 1 \tag{3.8}$$

Proof. Define $\Gamma : PC(I,\mathbb{X}) \to PC(I,\mathbb{X})$ by

$$\Gamma x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)] + \int_0^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in [0,t_1]; \\ S_\alpha(t-t_1)\big(x(t_1^-) + I_1(x(t_1^-))\big) + \int_{t_1}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_1,t_2]; \\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) + \int_{t_m}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_m,T]. \end{cases}$$

Clearly, the fixed points of the operator Γ are the solutions of the problem (1.4). We shall use the Banach contracton principle to prove that Γ has a fixed point.
We shall show that Γ is a contraction.
Let $x, y \in PC(I, \mathbb{X})$. Then for each $t \in [0, t_1]$ and by the lemma (2.1), we have

$$\begin{aligned} ||\Gamma x(t) - \Gamma y(t)|| &\leq ||S_\alpha(t)||\ ||g(x) - g(y)|| + \int_0^t ||P_\alpha(t-s)||\ ||f(s,x(s)) - f(s,y(s))||ds \\ &\leq C_s t^{-\alpha\gamma}\ b\ ||x-y|| + C_p \int_0^t (t-s)^{\alpha(1-\gamma)-1}\ M||x(s)-y(s)||ds \\ &\leq \left[C_s T^{-\alpha\gamma}\ b + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\right] ||x-y||_{PC} \end{aligned}$$

For $t \in (t_1, t_2]$,

$$\begin{aligned} ||\Gamma x(t) - \Gamma y(t)|| &\leq ||S_\alpha(t-t_1)||\big[||x(t_1^-) - y(t_1^-)|| + ||I_1(x(t_1^-)) - I_1(y(t_1^-))||\big] \\ &\quad + \int_{t_1}^t ||P_\alpha(t-s)||\ ||f(s,x(s)) - f(s,y(s)||ds \\ &\leq C_s(t-t_1)^{-\alpha\gamma}\big[||x(t_1^-) - y(t_1^-)|| + \rho_1||x(t_1^-) - y(t_1^-)||\big] \\ &\quad + C_p \int_{t_1}^t (t-s)^{\alpha(1-\gamma)-1}\ M||x(s)-y(s)||ds \\ &\leq \left[C_s\ T^{-\alpha\gamma}(\rho_1+1) + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\right] ||x-y||_{PC} \end{aligned}$$

Similarly, for all $t \in (t_i + t_{i+1}]$,

$$||\Gamma x(t) - \Gamma y(t)|| \leq \left[C_s\ T^{-\alpha\gamma}(\rho_i+1) + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\right] ||x-y||_{PC}$$

and for $t \in (t_m, T]$,

$$||\Gamma x(t) - \Gamma y(t)|| \leq \left[C_s\ T^{-\alpha\gamma}(\rho_m+1) + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\right] ||x-y||_{PC}$$

Thus, for all $t \in [0, T]$,

$$\begin{aligned} ||\Gamma x(t) - \Gamma y(t)|| &\leq \max_{1\leq i\leq m} \left\{C_s\ T^{-\alpha\gamma}(b+\rho_i+1) + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\right\} ||x-y||_{PC} \\ &\leq N\ ||x-y||_{PC} \end{aligned}$$

Thus, by the equation (3.8), Γ is a contraction mapping. As a consequence of Banach fixed point theorem, we deduce that Γ has a unique fixed point $x_0 \in PC(I, \mathbb{X})$ which is a solution of the problem (1.4). □

Our next result is based on Krasnoselskii's fixed point theorem.

Lemma 3.3. *(Krasnoselskii's Fixed point theorem)([14]). Let $\mathbb{X}$ be a Banach space, let E be a bounded closed convex subset of $\mathbb{X}$ and let Γ_1, Γ_2 be maps of E into $\mathbb{X}$ such that $\Gamma_1 x + \Gamma_2 y \in E$ for every pair $x, y \in E$. If Γ_1 is a contraction and Γ_2 is completely continuous, then the equation $\Gamma_1 x + \Gamma_2 x = x$ has a solution on E.*

Theorem 3.3. *Assume that the hypothesis $(H_1)-(H_4)$ are satisfied, then the system has atleast one mild solution on I.*

Proof. Define operator $\Gamma : PC(I,\mathbb{X}) \to PC(I,\mathbb{X})$, as in Theorem 3.2 by

$$\Gamma x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)] + \int_0^t P_\alpha(t-s)\ f(s,x(s))ds, & t\in[0,t_1];\\ S_\alpha(t-t_1)\big(x(t_1^-)+I_1(x(t_1^-))\big) + \int_{t_1}^t P_\alpha(t-s)\ f(s,x(s))ds, & t\in(t_1,t_2];\\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-)+I_m(x(t_m^-))\big) + \int_{t_m}^t P_\alpha(t-s)\ f(s,x(s))ds, & t\in(t_m,T]. \end{cases}$$

Define B_r as $B_r = \Big\{x \in PC(I,\mathbb{X}) : ||x||_{PC} \le r\Big\}$. Then, B_r is a closed, bounded and convex subset of $PC(I,\mathbb{X})$. On B_r, we define the operators Γ_1 and Γ_2 as follows.

$$\Gamma_1 x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)], & t\in[0,t_1];\\ S_\alpha(t-t_1)\big(x(t_1^-)+I_1(x(t_1^-))\big), & t\in(t_1,t_2];\\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-)+I_m(x(t_m^-))\big), & t\in(t_m,T]. \end{cases}$$

and

$$\Gamma_2 x(t) = \begin{cases} \int_0^t P_\alpha(t-s)\ f(s,x(s))ds, & t\in[0,t_1];\\ \int_{t_1}^t P_\alpha(t-s)\ f(s,x(s))ds, & t\in(t_1,t_2];\\ . \\ . \\ . \\ \int_{t_m}^t P_\alpha(t-s)\ f(s,x(s))ds, & t\in(t_m,T]. \end{cases}$$

Now, we show that $\Gamma_1+\Gamma_2$ has a fixed point in B_r. The proof is divided into three steps.
Step 1: $\Gamma_1 x + \Gamma_2 y \in B_r$, for every pair $x,y\in B_r$.
Consider for any $x,y\in B_r$ and for $t\in[0,t_1]$, we have

$$\begin{aligned} ||\Gamma_1 x(t)+\Gamma_2 y(t)|| &\le ||S_\alpha(t)||\ \big[||x_0|| + ||g(x)-g(0)|| + ||g(0)||\big] \\ &\quad + \int_0^t ||P_\alpha(t-s)||\ [||f(s,y(s))-f(s,0)|| + ||f(s,0)||]ds \\ &\le C_s\ t^{-\alpha\gamma}\ \big[||x_0|| + b||x|| + k_2\big] + C_p \int_0^t (t-s)^{\alpha(1-\gamma)-1}\ (M||y||+k_1)ds \\ &\le C_s\ T^{-\alpha\gamma}\ \big[||x_0|| + br + k_2\big] + C_p\ (Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)} \end{aligned}$$

For $t\in(t_1,t_2]$, we have

$$\begin{aligned} ||\Gamma_1 x(t)+\Gamma_2 y(t)|| &\le ||S_\alpha(t-t_1)||\big[||x(t_1^-)|| + ||I_1(x(t_1^-))||\big] + \int_{t_1}^t ||P_\alpha(t-s)||\ ||f(s,y(s))||ds \\ &\le C_s\ (t-t_1)^{-\alpha\gamma}\ \big[r + (\rho_1 r + k_3)\big] + C_p \int_{t_1}^t (t-s)^{\alpha(1-\gamma)-1}\ (Mr+k_1)ds \\ &\le C_s\ T^{-\alpha\gamma}\ \big[r(1+\rho_1) + k_3\big] + C_p\ (Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)} \end{aligned}$$

Similarly, we have

$$||\Gamma_1 x(t) + \Gamma_2 y(t)|| \leq C_s\ T^{-\alpha\gamma}\ \big[r(1+\rho_i) + k_3\big] + C_p\ (Mr + k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}, \quad \forall\ \ t \in (t_i, t_{i+1}]$$

and

$$||\Gamma_1 x(t) + \Gamma_2 y(t)|| \leq C_s\ T^{-\alpha\gamma}\ \big[r(1+\rho_m) + k_3\big] + C_p\ (Mr + k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}, \quad \forall\ \ t \in (t_m, T]$$

Thus, for all $t \in [0, T]$ and by (H_4), we have

$$\begin{aligned} ||\Gamma_1 x(t) + \Gamma_2 y(t)|| &\leq \max_{1\leq i\leq m} \Big\{C_s\ T^{-\alpha\gamma}\ \big[||x_0|| + r(1+\rho_i+b) + k_2 + k_3\big] + C_p\ (Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big\} \\ &\leq r \end{aligned}$$

which means that $\Gamma_1 x + \Gamma_2 y \in B_r$ for any $x, y \in B_r$.

Step 2: Γ_1 is contraction on B_r.
Let $x, y \in B_r$. By (H_2) and (H_3) , for each $t \in [0, t_1]$,

$$\begin{aligned} ||\Gamma_1 x(t) - \Gamma_1 y(t)|| &\leq ||S_\alpha(t)||\ ||g(x) - g(y)|| \\ &\leq C_s t^{-\alpha\gamma}\ b\ ||x-y|| \\ &\leq b\ C_s T^{-\alpha\gamma}\ ||x-y|| \end{aligned}$$

For $t \in (t_1, t_2]$,

$$\begin{aligned} ||\Gamma_1 x(t) - \Gamma_1 y(t)|| &\leq ||S_\alpha(t-t_1)||\big[||x(t_1^-) - y(t_1^-)|| + ||I_1(x(t_1^-)) - I_1(y(t_1^-))||\big] \\ &\leq C_s T^{-\alpha\gamma}\ [1+\rho_1]||x-y|| \end{aligned}$$

Similarly, for all $t \in (t_i, t_{i+1}]$,

$$||\Gamma_1 x(t) - \Gamma_1 y(t)|| \leq C_s\ T^{-\alpha\gamma}(\rho_i + 1)\ ||x-y||$$

and therefore for all $t \in (t_m, T]$,

$$||\Gamma_1 x(t) - \Gamma_1 y(t)|| \leq C_s\ T^{-\alpha\gamma}(\rho_m + 1)\ ||x-y||$$

Thus, for all $t \in [0, T]$,

$$\begin{aligned} ||\Gamma_1 x(t) - \Gamma_1 y(t)|| &\leq \max_{1\leq i\leq m} \Big\{C_s\ T^{-\alpha\gamma}(b+\rho_i+1)\Big\}\ ||x-y|| \\ &\leq N||x-y|| \end{aligned}$$

Thus, from equation (3.8), Γ_1 is contraction on B_r.
Step 3: Now, we show that Γ_2 is a completely continuous operator.
For that consider, for any $t \in [0, t_1]$, we have

$$\begin{aligned} ||\Gamma_2 x(t)|| &\leq \int_0^t ||P_\alpha(t-s)||\ ||f(s, x(s))||ds \\ &\leq C_p \int_0^t (t-s)^{\alpha(1-\gamma)-1}\ (M||x|| + k_1)ds \\ &\leq C_p(Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)} \end{aligned}$$

Similarly, for all $t \in (t_i, t_{i+1}]$,

$$||\Gamma_2 x(t)|| \leq C_p(Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}$$

Thus, from the above inequalities, $\Big\{\Gamma_2 x : x \in B_r\Big\}$ is uniformly bounded for every $t \in [0,T]$.

Next, we will prove that $\Big\{\Gamma_2 x : x \in B_r\Big\}$ is equicontinuous.
Let, $s_1, s_2 \in [0,t_1]$, with $s_1 < s_2$, then $\forall s_1, s_2$, we have

$$\begin{aligned}
||(\Gamma_2 x)(s_2) - (\Gamma_2 x)(s_1)|| &\leq \int_0^{s_2} ||P_\alpha(s_2 - s)||\; ||f(s,x(s))||ds - \int_0^{s_1} ||P_\alpha(s_1 - s)||\; ||f(s,x(s))||ds \\
&\leq C_p\Big[\int_0^{s_1} \big[(s_2 - s)^{\alpha(1-\gamma)-1} - (s_1 - s)^{\alpha(1-\gamma)-1}\big]\; ||f(s,x(s))||ds \\
&\quad + \int_{s_1}^{s_2} (s_2 - s)^{\alpha(1-\gamma)-1}\; ||f(s,x(s))||ds\Big] \\
&\leq \frac{C_p\;(Mr + k_1)}{\alpha(1-\gamma)}\Big[s_2^{\alpha(1-\gamma)} - s_1^{\alpha(1-\gamma)}\Big]
\end{aligned}$$

Similarly, $\forall s_1, s_2 \in (t_i, t_{i+1}]$, with $s_1 < s_2$, $i = 1, 2, ..., m$, we have

$$||(\Gamma_2 x)(s_2) - (\Gamma_2 x)(s_1)|| \leq \frac{C_p\;(Mr + k_1)}{\alpha(1-\gamma)}\Big[(s_2 - t_i)^{\alpha(1-\gamma)} - (s_1 - t_i)^{\alpha(1-\gamma)}\Big]$$

Thus, from the above inequalities, we have $\lim_{s_2 \to s_1} ||(\Gamma_2 x)(s_2) - (\Gamma_2 x)(s_1)|| = 0$. So, Γ_2 is equicontinuous. Moreover, it is clear that from the lemma (2.2), Γ_2 is continuous. So, Γ_2 is a completely continuous operator.

Therefore, Krasnoselskii's fixed point theorem shows that $\Gamma = \Gamma_1 + \Gamma_2$ has a fixed point on B_r and hence the system (1.4) has a solution on I. □

4 Example

Let $\hat{A} = (-i\Delta + \sigma)^{\frac{1}{2}}$, $D(\hat{A}) = W^{1,3}(\mathbb{R}^2)$ (a sobolev space)
be as in example 6.3([31]), in which the authors demonstrate that $\hat{A}$ is an almost sectorial operator for some $0 < w < \frac{\pi}{2}$ and $\gamma = \frac{1}{6}$. We denote the semigroup associated with $\hat{A}$ by T(t) and $||T(t)|| \leq C_0 t^{-\frac{1}{6}}$, where C_0 is a constant.
Let $\mathbb{X} = L^3(\mathbb{R}^2)$, we consider the following problem.

$$\begin{cases} {}^c D^{\frac{1}{2}} x(t) = \hat{A}x(t) + \frac{\cos t}{(t+6)^2}\,\frac{|x(t)|}{1+|x(t)|}, & t \in I = [0,1], \quad t \neq \frac{1}{2} \\ \Delta x(\frac{1}{2}) = \frac{|x(\frac{1}{2}^-)|}{15 + |x(\frac{1}{2}^-)|}, & t = \frac{1}{2} \\ x(0) + \frac{1}{2}x(\frac{1}{5}) = x(1) \end{cases}$$

where

$$\begin{aligned}
f(t,x(t)) &= \frac{\cos t}{(t+6)^2}\,\frac{|x(t)|}{1+|x(t)|}, \\
I_1(x) &= \frac{|x|}{15 + |x|}, \\
g(x) &= \frac{1}{2}x(\frac{1}{5}).
\end{aligned}$$

By direct computations, we see that

$$\begin{aligned}
\|f(t,x(t)) - f(t,y(t))\| &= \left|\frac{\cos t}{(t+6)^2}\right|\;\left\|\frac{|x(t)|}{1+|x(t)|} - \frac{|y(t)|}{1+|y(t)|}\right\| \leq \frac{1}{36}\|x(t) - y(t)\| \\
\|I_1(x) - I_1(y)\| &\leq \frac{1}{15}\|x - y\| \\
\|g(x) - g(y)\| &\leq \frac{1}{2}\|x - y\|
\end{aligned}$$

So,it is clear that the functions f, g and I_k satisfy the assumptions (H_1), (H_2) and (H_3) with M $= \frac{1}{36}$ b $= \frac{1}{2}$, and $\rho_1 = \frac{1}{15}$. Then, choosing for instance $\alpha = \frac{1}{2}$ and T = 1, we have from the equation (3.8),

$$N = C_s\Big[\frac{7}{4} + \frac{1}{15}\Big] + C_p\frac{1}{36}\frac{12}{5} < 1$$

for the suitable values of the constants C_s and C_p. Moreover the assumption (H_4) is also satisfied. Thus, all the assumptions of Theorem 3.2 and Theorem 3.3 are satisfied and hence by the conclusion of the Theorems 3.2 and 3.3, the nonlocal impulsive fractional problem (1.4) has a unique solution on [0,1].

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Application of random fixed point theorems in solving nonlinear stochastic integral equation of the Hammerstein type

Debashis Dey[a,*] and Mantu Saha[b]

[a] Koshigram Union Institution, Koshigram-713150, Burdwan , West Bengal, India.

[b] Department of Mathematics, The University of Burdwan, Burdwan-713104, West Bengal, India.

Abstract

In the present paper, we apply random analogue Kannan fixed point theorem [10] to analyze the existence of a solution of a nonlinear stochastic integral equation of the Hammerstein type of the form

$$x(t;\omega) = h(t;\omega) + \int_S k(t,s;\omega) f(s, x(s;\omega)) d\mu(s)$$

where $t \in S$, a σ-finite measure space with certain properties, $\omega \in \Omega$, the supporting set of a probability measure space (Ω, β, μ) and the integral is a Bochner integral.

Keywords: random fixed point, Kannan operator, stochastic integral equation.

2010 MSC: 47H10, 60H25.

1 Introduction

The importance of random fixed point theory lies in its vast applicability in probabilistic functional analysis and various probabilistic models. The introduction of randomness however leads to several new questions of measurability of solutions, probabilistic and statistical aspects of random solutions. It is well known that random fixed point theorems are stochastic generalization of classical fixed point theorems what we call as determinstic results. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Špaček [18] and Hanš (see [6]-[7]). The survey article by Bharucha-Reid [4] in 1976 attracted the attention of several mathematicians and gave wings to this theory. Itoh [8] extended Špaček's and Hanš's theorems to multivalued contraction mappings. Random fixed point theorems with an application to Random differential equations in Banach spaces are obtained by Itoh [8]. Sehgal and Waters [17] had obtained several random fixed point theorems including random analogue of the classical results due to Rothe [13]. In recent past, several fixed point theorems including Kannan type [10] Chatterjeea [5] and Zamfirescu type [20] have been generalized in stochastic version (see for detail in Joshi and Bose [9], Saha et al. ([14], [15]).

On the otherhand, Padgett [12] used the random analogue of Banach fixed point theorem [3] to analyze the existence and uniqueness of random solution of a nonlinear stochastic integral equation of the Hammerstein type of the form

$$x(t;\omega) = h(t;\omega) + \int_S k(t,s;\omega) f(s, x(s;\omega)) d\mu(s)$$

and proved several theorems. Achari [1] and Saha et al.[16] continued to work on this application for more generalized random nonlinear contraction operators.

*Corresponding author.

E-mail addresses: debashisdey@yahoo.com (Debashis Dey) and mantusaha@yahoo.com (Mantu Saha).

In the following section, we study on application of two basic random fixed point theorems of importance, one - is Kannan fixed point theorem [10] and the other one is - Chatterjea fixed point theorem [5] to analyze the existence of solution for such integral equation.

2 Preliminaries

Let (X, β_X) be a separable Banach space where β_X is a σ-algebra of Borel subsets of X, and let (Ω, β, μ) denote a complete probability measure space with measure μ, and β be a σ-algebra of subsets of Ω. For more details one can see Joshi and Bose [9].

Theorem 2.1. (Joshi and Bose [9]) *Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator satisfying*

$$\begin{aligned} \|T(\omega, x_1) - T(\omega, x_2)\| \quad \leq \quad & k_1(\omega)\,[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] \\ & + k_2(\omega)\,[\|x_1 - T(\omega, x_2)\| + \|x_2 - T(\omega, x_1)\|] \\ & + k_3\,\|x_1 - x_2\| \end{aligned} \tag{2.1}$$

for all $\omega \in \Omega$ and $x_1, x_2 \in X$, $k_i(\omega) \geq 0$; $1 \leq i \leq 3$. are real valued random variables with $2k_1(\omega) + 2k_2(\omega) + k_3(\omega) < 1$ almost surely. Then there exists a unique random fixed point of T.

Remark 2.1. (I) *In the above theorem, setting $k_2(\omega) = k_3(\omega) = 0$, one can find random analogue of kannan fixed point theorem [10] and in that case the operator $T : \Omega \times X \to X$ takes the form:*

$$\|T(\omega, x_1) - T(\omega, x_2)\| \quad \leq \quad k_1(\omega)\,[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] \tag{2.2}$$

for all $\omega \in \Omega$ and $x_1, x_2 \in X$, $k_1(\omega) \geq 0$ is real valued random variables with $k_1(\omega) < \frac{1}{2}$ almost surely.

(II) *Setting $k_1(\omega) = k_3(\omega) = 0$, one can find random analogue of Chatterjea fixed point theorem [5] and in that case the operator $T : \Omega \times X \to X$ takes the form:*

$$\|T(\omega, x_1) - T(\omega, x_2)\| \quad \leq \quad k_2(\omega)\,[\|x_1 - T(\omega, x_2)\| + \|x_2 - T(\omega, x_1)\|] \tag{2.3}$$

for all $\omega \in \Omega$ and $x_1, x_2 \in X$, $k_2(\omega) \geq 0$ is real valued random variables with $k_2(\omega) < \frac{1}{2}$ almost surely.

Remark 2.2. *Note that neither Kannan operator nor Chatterjea operator is continuous in general. So random fixed point theorems for these two operators are slightly different from their deterministic approach.*

3 Application to a random nonlinear integral equation

We now show an application of stochastic version of Kannan fixed point theorem in solving nonlinear stochastic integral equation of the Hammerstein type of the form:

$$x(t;\omega) = h(t;\omega) + \int_S k(t, s;\omega) f(s, x(s;\omega)) d\mu_0(s) \tag{3.4}$$

where
(i)S is a locally compact metric space with metric d on $S \times S$, μ_0 is a complete σ-finite measure defined on the collection of Borel subsets of S;
(ii) $\omega \in \Omega$, where ω is a supporting set of probability measure space (Ω, β, μ);
(iii) $x(t;\omega)$ is the unknown vector-valued random variables for each $t \in S$.
(iv) $h(t;\omega)$ is the stochastic free term defined for $t \in S$;
(v) $k(t, s;\omega)$ is the stochastic kernel defuned for t and s in S and
(vi) $f(t, x)$ is vector-valued function of $t \in S$ and x
and the integral in equation (3.4) is a Bochner integral.
We will further assume that S is the union of a countable family of compact sets $\{C_n\}$ having the properties that $C_1 \subset C_2 \subset ...$ and that for any other compact set S there is a C_i which contains it (see [2]).

Definition 3.1. *We define the space $C(S, L_2(\Omega, \beta, \mu))$ to be the space of all continuous functions from S into $L_2(\Omega, \beta, \mu)$ with the topology of uniform convergence on compacta i.e. for each fixed $t \in S$, $x(t;\omega)$ is a vector valued random variable such that*

$$\|x(t;\omega)\|^2_{L_2(\Omega,\beta,\mu)} = \int_\Omega |x(t;\omega)|^2 \, d\mu(\omega) < \infty$$

It may be noted that $C(S, L_2(\Omega, \beta, \mu))$ is locally convex space (see [19]) whose topology is defined by a countable family of seminorms given by

$$\|x(t;\omega)\|_n = \sup_{t\in C_n} \|x(t;\omega)\|_{L_2(\Omega,\beta,\mu)}, n = 1, 2, ...$$

Moreover $C(S, L_2(\Omega, \beta, \mu))$ is complete relative to this topology since $L_2(\Omega, \beta, \mu)$ is complete.

We further define $BC = BC(S, L_2(\Omega, \beta, \mu))$ to be the Banach space of all bounded continuous functions from S into $L_2(\Omega, \beta, \mu)$ with norm

$$\|x(t;\omega)\|_{BC} = \sup_{t\in S} \|x(t;\omega)\|_{L_2(\Omega,\beta,\mu)}$$

The space $BC \subset C$ is the space of all second order vector-valued stochastic process defined on S which are bounded and continuous in mean square.
We will consider the function $h(t;\omega)$ and $f(t, x(t;\omega))$ to be in the space $C(S, L_2(\Omega, \beta, \mu))$ with respect to the stochastic kernel. We assume that for each pair (t, s), $k(t, s;\omega) \in L_\infty(\Omega, \beta, \mu)$ and denote the norm by

$$\|k(t, s;\omega)\| = \|k(t, s;\omega)\|_{L_\infty(\Omega,\beta,\mu)} = \mu - ess \sup_{\omega\in\Omega} |k(t, s;\omega)|.$$

Also we will suppose that $k(t, s;\omega)$ is such that $|||k(t, s;\omega)|||\,.\,\|x(s;\omega)\|_{L_2(\Omega,\beta,\mu)}$ is μ_0-integrable with respect to s for each $t \in S$ and $x(s;\omega)$ in $C(S, L_2(\Omega, \beta, \mu))$ and there exists a real valued function G defined μ_0-a.e. on S, so that $G(S)\,\|x(s;\omega)\|_{L_2(\Omega,\beta,\mu)}$ is μ_0-integarable and for each pair $(t, s) \in S \times S$,

$$|||k(t, u;\omega) - k(s, u;\omega)|||\,.\,\|x(u,\omega)\|_{L_2(\Omega,\beta,\mu)} \le G(u)\,\|x(u,\omega)\|_{L_2(\Omega,\beta,\mu)}$$

μ_0-a.e. Further, for allmost all $s \in S$, $k(t, s;\omega)$ will be continuous in t from S into $L_\infty(\Omega, \beta, \mu)$.
We now define the random integral operator T on $C(S, L_2(\Omega, \beta, \mu))$ by

$$(Tx)(t;\omega) = \int_S k(t, s;\omega)x(s;\omega)d\mu_0(s) \tag{3.5}$$

where the integral is a Bochner integral. Moreover, we have that for each $t \in S$, $(Tx)(t;\omega) \in L_2(\Omega, \beta, \mu)$ and that $(Tx)(t;\omega)$ is continuous in mean square by Lebesgue's dominated convergence theorem. So $(Tx)(t;\omega) \in C(S, L_2(\Omega, \beta, \mu))$.

Definition 3.2. (see [1], [11]) *Let B and D be Banach spaces. The pair (B, D) is said to be admissible with respect to a random operator $T(\omega)$ if $T(\omega)(B) \subset D$.*

Lemma 3.1. (see [12]) *The linear operator T defined by* (3.5) *is continuous from $C(S, L_2(\Omega, \beta, \mu))$ into itself.*

Lemma 3.2. (see [12], [11]) *If T is a continuous linear operator from $C(S, L_2(\Omega, \beta, \mu))$ into itself and $B, D \subset C(S, L_2(\Omega, \beta, \mu))$ are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ such that (B, D) is admissible with respect to T, then T is continuous from B into D.*

Remark 3.3. (see [12]) *The operator T defined by* (3.5) *is a bounded linear operator from B into D.*
It is to be noted that by a random solution of the equation (3.4) *we will mean a function $x(t;\omega)$ in $C(S, L_2(\Omega, \beta, \mu))$ which satisfies the equation* (3.4) *μ-a.e.*

We are now in a state to prove the following theorem.

Theorem 3.2. *We consider the stochastic integral equation* (3.4) *subject to the following conditions:*
(a) *B and D are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ such that (B, D) is admissible with respect to*

the integral operator defined by (3.5)*;*
(b) $x(t;\omega) \to f(t,x(t;\omega))$ *is an operator from the set*

$$Q(\rho) = \{x(t;\omega) : x(t;\omega) \in D, \|x(t;\omega)\|_D \le \rho\}$$

into the space B *satisfying*

$$\begin{aligned} \|f(t,x_1(t;\omega)) - f(t,x_2(t;\omega))\|_B \quad &\le \quad \lambda(\omega)\left[\|x_1(t;\omega) - f(t,x_1(t;\omega))\|_D \right. \\ &\quad \left. + \|x_2(t;\omega) - f(t,x_2(t;\omega))\|_D\right] \end{aligned} \tag{3.6}$$

for $x_1(t;\omega), x_2(t;\omega) \in Q(\rho)$*, where* $0 \le \lambda(\omega) < \frac{1}{2}$ *is a real valued random variable almost surely,*
(c) $h(t;\omega) \in D$*.*
Then there exists a unique random solution of (3.4) *in* $Q(\rho)$*, provided*
$\lambda(\omega)\,(1 + c(\omega)) < \frac{1}{2}$ *and*

$$\|h(t;\omega)\|_D + \frac{1+\lambda(\omega)}{1-\lambda(\omega)} c(\omega)\,\|f(t;0)\|_B \le \rho\left(1 - \frac{c(\omega)\lambda(\omega)}{1-\lambda(\omega)}\right)$$

where $c(\omega)$ *is the norm of* $T(\omega)$*.*

Proof. Define the operator $U(\omega)$ from $Q(\rho)$ into D by

$$(Ux)(t;\omega) = h(t;\omega) + \int_S k(t,s;\omega) f(s,x(s;\omega)) d\mu_0(s)$$

Now

$$\begin{aligned} \|(Ux)(t;\omega)\|_D &\le \|h(t;\omega)\|_D + c(\omega)\,\|f(t,x(t;\omega))\|_B \\ &\le \|h(t;\omega)\|_D + c(\omega)\,\|f(t;0)\|_B + c(\omega)\,\|f(t,x(t;\omega)) - f(t;0)\|_B \end{aligned}$$

Then from the condition (3.6) of this theorem

$$\begin{aligned} \|f(t,x(t;\omega)) - f(t;0)\|_B &\le \lambda(\omega)[\|x(t;\omega) - f(t,x(t;\omega))\|_D + \|f(t;0)\|_D] \\ &\le \lambda(\omega)\,[\|x(t;\omega)\|_D + \|f(t,x(t;\omega))\|_D + \|f(t;0)\|_D] \\ &\le \lambda(\omega)\,[\|x(t;\omega)\|_D + \|f(t,x(t;\omega)) - f(t;0)\|_D + 2\,\|f(t;0)\|_D] \end{aligned}$$

implies

$$\|f(t,x(t;\omega)) - f(t;0)\|_B \le \frac{\lambda(\omega)}{1-\lambda(\omega)}\rho + \frac{2\lambda(\omega)}{1-\lambda(\omega)}\,\|f(t;0)\|_B \tag{3.7}$$

Therefore by (3.7), we have

$$\begin{aligned} \|(Ux)(t;\omega)\|_D &\le \|h(t;\omega)\|_D + c(\omega)\,\|f(t;0)\|_B \\ &\quad + c(\omega)\left[\frac{\lambda(\omega)\rho}{1-\lambda(\omega)} + \frac{2\lambda(\omega)}{1-\lambda(\omega)}\,\|f(t;0)\|_B\right] \\ &= \|h(t;\omega)\|_D + \frac{c(\omega)\lambda(\omega)}{1-\lambda(\omega)}\rho + \frac{1+\lambda(\omega)}{1-\lambda(\omega)} c(\omega)\,\|f(t;0)\|_B \\ &< \rho \end{aligned}$$

Hence $(Ux)(t;\omega) \in Q(\rho)$. Then for $x_1(t;\omega), x_2(t;\omega) \in Q(\rho)$, we have by condition (b)

$$\begin{aligned} \|(Ux_1)(t;\omega) - (Ux_2)(t;\omega)\|_D &= \left\|\int_S k(t,s;\omega)[f(s,x_1(s;\omega)) - f(s,x_2(s;\omega))] d\mu_0(s)\right\|_D \\ &\le c(\omega)\,\|f(t,x_1(t;\omega)) - f(t,x_2(t;\omega))\|_B \\ &\le c(\omega)\lambda(\omega)\,[\|x_1(t;\omega) - f(t,x_1(t;\omega))\|_D \\ &\quad + \|x_2(t;\omega) - f(t,x_2(t;\omega))\|_D] \end{aligned}$$

since $c(\omega)\lambda(\omega) < \frac{1}{2}$, $U(\omega)$ is a Kannan contraction on $Q(\rho)$. Hence, by Theorem 2.1 and Remark 2.1(I), there exists a unique $x^*(t,\omega) \in Q(\rho)$, which is a fixed point of U, that is $x^*(t,\omega)$ is the unique random solution of the equation (3.4). □

A similar theorem can be obtained using random analogue of Chatterjea fixed point theorem [5].

Theorem 3.3. *Assume that the stochastic integral equation* (3.4) *subject to the following conditions:*
$(a^{'})$ *Same as* (a) of Theorem 3.2*;*
$(b^{'})$ $x(t;\omega) \to f(t, x(t;\omega))$ *is an operator from the set*

$$Q(\rho) = \{x(t;\omega) : x(t;\omega) \in D, \|x(t;\omega)\|_D \leq \rho\}$$

into the space B *satisfying*

$$\begin{aligned} \|f(t, x_1(t;\omega)) - f(t, x_2(t;\omega))\|_B \quad &\leq \quad \lambda(\omega)\,[\|x_1(t;\omega) - f\,(t, x_2(t;\omega))\|_D \\ &\quad + \|x_2(t;\omega) - f(t, x_1(t;\omega))\|_D] \end{aligned} \tag{3.8}$$

for $x_1(t;\omega), x_2(t;\omega) \in Q(\rho)$*, where* $0 \leq \lambda(\omega) < \frac{1}{2}$ *is a real valued random variable almost surely,*
$(c^{'})$ $h(t;\omega) \in D$*.*
Then there exists a unique random solution of (3.4) *in* $Q(\rho)$*, provided*
$\lambda(\omega)\,(1 + c(\omega)) < \frac{1}{2}$ *and*

$$\|h(t;\omega)\|_D + \frac{1 + \lambda(\omega)}{1 - \lambda(\omega)} c(\omega)\,\|f(t;0)\|_B \leq \rho \left(1 - \frac{c(\omega)\lambda(\omega)}{1 - \lambda(\omega)}\right)$$

where $c(\omega)$ *is the norm of* $T(\omega)$*.*

Proof. The proof is similar to that of Theorem 3.2. So we avoid repeatation. □

The following example illustrates the strength of our main result-Theorem 3.2.

Example 3.1. *Consider the following nonlinear stochastic integral equation:*

$$x(t;\omega) = \int_0^\infty \frac{e^{-t-s}}{8(1 + |x(s;\omega)|)} ds$$

Comparing with (3.4)*, we see that*

$$h(t,\omega) = 0, k(t,s;\omega) = \frac{1}{2} e^{-t-s}, f(s, x(s;\omega)) = \frac{1}{4(1 + |x(s;\omega)|)}$$

Then one can check that equation (3.6) *is satisfied with* $\lambda(\omega) = \frac{1}{3}$*.*

Comparing with integral operator equation (3.5)*, we see that the norm of* $T(\omega)$ *is* $c(\omega) = \frac{1}{4}$ *satisfying* $\lambda(\omega)(1 + c(\omega)) < \frac{1}{2}$*. So, all the conditions of* Theorem 3.2 *are satisfied and hence there exists a random fixed point of the integral operator* T *satisfying* (3.5)*.*

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Existence of positive periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale

Abdelouaheb Ardjouni[a,*] and Ahcene Djoudi[b]

[a,b] *Department of Mathematics, Faculty of Sciences, University of Annaba, P.O. Box 12 Annaba, Algeria.*

Abstract

Let $\mathbb{T}$ be a periodic time scale. The purpose of this paper is to use Krasnoselskii's fixed point theorem to prove the existence of positive periodic solutions on time scale of the nonlinear neutral dynamic equation with variable delay

$$(x(t) - g(t, x(t - \tau(t))))^{\triangle} = r(t)x(t) - f(t, x(t - \tau(t))).$$

We invert this equation to construct a sum of a contraction and a compact map which is suitable for applying the Krasnoselskii's theorem. The results obtained here extend the works of Raffoul [17] and Ardjouni and Djoudi [3].

Keywords: Positive periodic solutions, nonlinear neutral dynamic equations, fixed point theorem, time scales.

2010 MSC: Primary 34K13, 34A34; Secondary 34K30, 34L30.

1 Introduction

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of dynamic equations. Motivated by the papers [1]–[6], [9]–[17] and the references therein, we consider the following nonlinear neutral dynamic equation with variable delay

$$(x(t) - g(t, x(t - \tau(t))))^{\triangle} = r(t)x(t) - f(t, x(t - \tau(t))). \tag{1.1}$$

Throughout this paper we assume that $\tau : \mathbb{T} \to \mathbb{R}$ and that $id - \tau : \mathbb{T} \to \mathbb{T}$ is strictly increasing so that the function $x(t - \tau(t))$ is well defined over $\mathbb{T}$. Our purpose here is to use the Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions on time scales for equation (1.1). To reach our desired end we have to transform (1.1) into an integral equation written as a sum of two mapping; one is a contraction and the other is compact. After that, we use Krasnoselskii's fixed point theorem, to show the existence of a positive periodic solution for equation (1.1). In the special case $\mathbb{T} = \mathbb{R}$, in [3] we show that (1.1) has a positive periodic solution by using Krasnoselskii's fixed point theorem.

The organization of this paper is as follows. In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [18]. In Section 3, we present our main results on existence of positive periodic solutions of (1.1). The results presented in this paper extend the main results in [3, 17].

2 Preliminaries

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1], [2], [4]–[8], [14], [15] and

*Corresponding author.

E-mail addresses: abd_ardjouni@yahoo.fr (Abdelouaheb Ardjouni) and adjoudi@yahoo.com (Ahcene Djoudi).

papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [7] and [8] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Atici et al. [5] and Kaufmann and Raffoul [14]. The following two definitions are borrowed from [5] and [14].

Definition 2.1. *We say that a time scale $\mathbb{T}$ is periodic if there exist a $\omega > 0$ such that if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive ω is called the period of the time scale.*

Below are examples of periodic time scales taken from [14].

Example 2.1. *The following time scales are periodic.*
(1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih]$, $h > 0$ *has period* $\omega = 2h$.
(2) $\mathbb{T} = h\mathbb{Z}$ *has period* $\omega = h$.
(3) $\mathbb{T} = \mathbb{R}$.
(4) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ *where,* $0 < q < 1$ *has period* $\omega = 1$.

Remark 2.1 ([14]). *All periodic time scales are unbounded above and below.*

Definition 2.2. *Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scales with the period ω. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = n\omega$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.*

Remark 2.2 ([14]). *If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function μ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period ω.*

Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales ([7], Theorem 1.93).

Theorem 2.1 (Chain Rule). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our second theorem is the substitution rule ([7], Theorem 1.98).

Theorem 2.2 (Substitution). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t) \nu^{\triangle}(t) \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s) \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0$ for all $t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} Log(1 + \mu(z) p(z)) \triangle z\right). \tag{2.2}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t) y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.1 ([7]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t,s)$;

(iii) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1 + \mu(t) p(t)}$;

(iv) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s) e_p(s,r) = e_p(t,r)$;

(vi) $e_p^{\triangle}(.,s) = p e_p(.,s)$ *and* $\left(\dfrac{1}{e_p(.,s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(.,s)}$.

Theorem 2.3 ([6], Theorem 2.1). *Let $\mathbb{T}$ be a periodic time scale with period $\omega > 0$. If $p \in C_{rd}(\mathbb{T})$ is a periodic function with the period $T = n\omega$, then*

$$\int_{a+T}^{b+T} p(u) \triangle u = \int_a^b p(u) \triangle u, \;\; e_p(b+T, a+T) = e_p(b,a) \;\; \text{if } p \in \mathcal{R},$$

and $e_p(t+T, t)$ is independent of $t \in \mathbb{T}$ whenever $p \in \mathcal{R}$.

Lemma 2.2 ([1]). *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u) \triangle u\right), \;\; \forall t \in \mathbb{T}.$$

Corollary 2.1 ([1]). *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u) \triangle u\right) < 1.$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1.1). For its proof we refer the reader to [18].

Theorem 2.4 (Krasnoselskii). *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{D}$, *implies* $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,

(ii) $\mathcal{A}$ *is compact and continuous,*

(iii) $\mathcal{B}$ *is a contraction mapping.*

Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3 Existence of positive periodic solutions

We will state and prove our main result in this section. After we provide an example to illustrate our results. Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and (a,b) are defined similarly.

Define $P_T = \{\varphi : \mathbb{T} \to \mathbb{R} \mid \varphi \in C \text{ and } \varphi(t+T) = \varphi(t)\}$ where C is the space of continuous real-valued functions on $\mathbb{T}$. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|\varphi\| = \sup_{t \in \mathbb{T}} |\varphi(t)| = \sup_{t \in [0,T]} |\varphi(t)|.$$

We will need the following lemma whose proof can be found in [14].

Lemma 3.3. *Let $x \in C_T$. Then $\|x^\sigma\| = \|x \circ \sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

In this paper we assume that $r \in \mathcal{R}^+$ is continuous and for all $t \in \mathbb{T}$,

$$r(t+T) = r(t), \;\; (id - \tau)(t+T) = (id - \tau)(t), \tag{3.3}$$

where id is the identity function on $\mathbb{T}$. Also, we assume

$$\int_0^T r(s) \triangle s > 0. \tag{3.4}$$

We also assume that the functions $g(t,x)$ and $f(t,x)$ are continuous in their respective arguments and periodic in t with period T, that is,

$$g(t+T,x) = g(t,x), \ f(t+T,x) = f(t,x). \tag{3.5}$$

The following lemma is fundamental to our results.

Lemma 3.4. *Suppose (3.3)–(3.5) hold. If $x \in P_T$, then x is a solution of equation (1.1) if and only if*

$$x(t) = g(t, x(t-\tau(t))) + \int_t^{t+T} G(t,s) [f(s, x(s-\tau(s))) - r(s) g(s, x(s-\tau(s)))] \triangle s, \tag{3.6}$$

where

$$G(t,s) = \frac{e_r(t,\sigma(s))}{1 - e_{\ominus r}(t+T,t)}. \tag{3.7}$$

Proof. Let $x \in P_T$ be a solution of (1.1). First we write this equation as

$$\begin{aligned}&(x(t) - g(t,x(t-\tau(t))))^{\triangle} - r(t)(x(t) - g(t,x(t-\tau(t))))\\ &= -f(t,x(t-\tau(t))) + r(t) g(t,x(t-\tau(t))).\end{aligned}$$

Multiply both sides of the above equation by $e_{\ominus r}(\sigma(t),0)$ we get

$$\begin{aligned}&\left\{(x(t) - g(t,x(t-\tau(t))))^{\triangle} - r(t)(x(t) - g(t,x(t-\tau(t))))\right\} e_{\ominus r}(\sigma(t),0)\\ &= \{-f(t,x(t-\tau(t))) + r(t) g(t,x(t-\tau(t)))\} e_{\ominus r}(\sigma(t),0).\end{aligned}$$

Since $e_{\ominus r}(t,0)^{\triangle} = -r(t) e_{\ominus r}(\sigma(t),0)$ we find

$$\begin{aligned}&[(x(t) - g(t,x(t-\tau(t)))) e_{\ominus r}(t,0)]^{\triangle}\\ &= \{-f(t,x(t-\tau(t))) + r(t) g(t,x(t-\tau(t)))\} e_{\ominus r}(\sigma(t),0).\end{aligned}$$

Taking the integral from t to $t+T$, we obtain

$$\begin{aligned}&\int_t^{t+T} [(x(s) - g(s,x(s-\tau(s)))) e_{\ominus r}(s,0)]^{\triangle} \triangle s\\ &= \int_t^{t+T} \{-f(s,x(s-\tau(s))) + r(s) g(s,x(s-\tau(s)))\} e_{\ominus r}(\sigma(s),0) \triangle s.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&(x(t+T) - g(t+T, x(t+T-\tau(t+T)))) e_{\ominus r}(t+T,0)\\ &- (x(t) - g(t,x(t-\tau(t)))) e_{\ominus r}(t,0)\\ &= \int_t^{t+T} \{-f(s,x(s-\tau(s))) + r(s) g(s,x(s-\tau(s)))\} e_{\ominus r}(\sigma(s),0) \triangle s.\end{aligned}$$

Dividing both sides of the above equation by $e_{\ominus r}(t,0)$ and using the fact that $x(t+T) = x(t)$, (3.3), (3.5) and

$$\frac{e_{\ominus r}(t+T,0)}{e_{\ominus r}(t,0)} = e_{\ominus r}(t+T,t), \ \frac{e_{\ominus r}(\sigma(s),0)}{e_{\ominus r}(t,0)} = e_r(t,\sigma(s)),$$

we obtain

$$\begin{aligned}&x(t) - g(t,x(t-\tau(t)))\\ &= \int_t^{t+T} \frac{e_r(t,\sigma(s))}{1 - e_{\ominus r}(t+T,t)} \{f(s,x(s-\tau(s))) - r(s) g(s,x(s-\tau(s)))\} \triangle s.\end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. □

To simplify notation, we let

$$m=\frac{\exp\left(-\int_0^{2T}|r(u)|\triangle u\right)}{1-e_{\ominus r}(T,0)},\ M=\frac{\exp\left(\int_0^{2T}|r(u)|\triangle u\right)}{1-e_{\ominus r}(T,0)}.$$

It is easy to see that for all $(t,s)\in[0,2T]\times[0,2T]$,

$$m\leq G(t,s)\leq M, \tag{3.8}$$

and from Lemma 2.1 and Theorem 2.3, we have for all $t,s\in\mathbb{R}$,

$$G(t+T,s+T)=G(t,s). \tag{3.9}$$

To apply Theorem 2.4, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is compact. So, we let $(\mathbb{B},\|\cdot\|)=(P_T,\|\cdot\|)$ and $\mathbb{D}=\{\varphi\in\mathbb{B}:L\leq\varphi\leq K\}$, where L is non-negative constant and K is positive constant. We express equation (3.6) as

$$\varphi(t)=(\mathcal{B}\varphi)(t)+(\mathcal{A}\varphi)(t):=(H\varphi)(t),$$

where $\mathcal{A},\mathcal{B}:\mathbb{D}\to\mathbb{B}$ are defined by

$$(\mathcal{A}\varphi)(t)=\int_t^{t+T}G(t,s)\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\}\triangle s, \tag{3.10}$$

and

$$(\mathcal{B}\varphi)(t)=g(t,\varphi(t-\tau(t))). \tag{3.11}$$

In this section, we obtain the existence of a positive periodic solution of (1.1) by considering the two cases; $g(t,x)\geq 0$ and $g(t,x)\leq 0$ for all $t\in\mathbb{R}$, $x\in\mathbb{D}$. We assume that function $g(t,x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant k such that

$$|g(t,x)-g(t,y)|\leq k\|x-y\|,\ \text{for all } t\in[0,T],\ x,y\in\mathbb{D}. \tag{3.12}$$

In the case $g(t,x)\geq 0$, we assume that there exist a non-negative constant k_1 and positive constant k_2 such that

$$k_1x\leq g(t,x)\leq k_2x,\ \text{for all } t\in[0,T],\ x\in\mathbb{D}, \tag{3.13}$$

$$k_2<1, \tag{3.14}$$

and for all $t\in[0,T]$, $x\in\mathbb{D}$

$$\frac{L(1-k_1)}{mT}\leq f(t,x)-r(t)g(t,x)\leq\frac{K(1-k_2)}{MT}. \tag{3.15}$$

Lemma 3.5. *For $\mathcal{A}$ defined in (3.10), Suppose that the conditions (3.3)–(3.5) and (3.13)–(3.15) hold. Then $\mathcal{A}:\mathbb{D}\to\mathbb{B}$ is compact.*

Proof. We first show that $\mathcal{A}:\mathbb{D}\to\mathbb{B}$. Clearly, if φ is continuous, then $\mathcal{A}\varphi$ is. Evaluating (3.10) at $t+T$ gives

$$(\mathcal{A}\varphi)(t+T)=\int_{t+T}^{t+2T}G(t+T,s)\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\}\triangle s.$$

Use Theorem 2.2 with $u=s-T$ to get

$$\begin{aligned}(\mathcal{A}\varphi)(t+T)=\int_t^{t+T}&G(t+T,u+T)\{f(u+T,\varphi(u+T-\tau(u+T)))\\&-r(u+T)g(u+T,\varphi(u+T-\tau(u+T)))\}\triangle u.\end{aligned}$$

From (3.3), (3.4) and (3.9), we obtain

$$\begin{aligned}(\mathcal{A}\varphi)(t+T)&=\int_t^{t+T}G(t,u)\{f(u,\varphi(u-\tau(u)))-r(u)g(u,\varphi(u-\tau(u)))\}\triangle s\\&=(\mathcal{A}\varphi)(t).\end{aligned}$$

That is, $\mathcal{A}:\mathbb{D}\to\mathbb{B}$.

We show that $\mathcal{A}(\mathbb{D})$ is uniformly bounded. For $t\in[0,T]$ and for $\varphi\in\mathbb{D}$, we have

$$\begin{aligned}|(\mathcal{A}\varphi)(t)| &\leq \left|\int_t^{t+T} G(t,s)\left[f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\right]\triangle s\right| \\ &\leq MT\frac{K(1-k_2)}{MT}=K(1-k_2).\end{aligned}$$

by (3.8) and (3.15). Thus from the estimation of $|(\mathcal{A}\varphi)(t)|$ we arrive

$$\|\mathcal{A}\varphi\|\leq K(1-k_2).$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded.

It remains to show that $\mathcal{A}(\mathbb{D})$ is equicontinuous. Let $\varphi_n\in\mathbb{D}$, where n is a positive integer. Next we calculate $(\mathcal{A}\varphi_n)^{\triangle}(t)$ and show that it is uniformly bounded. By making use of (3.3) and (3.5) we obtain by taking the derivative in (3.3) that

$$\begin{aligned}(\mathcal{A}\varphi_n)^{\triangle}(t) &= \left[G(t,t+T)-G(t,t)\right]\left\{f(t,\varphi_n(t-\tau(t)))-r(t)g(t,\varphi_n(t-\tau(t)))\right\} \\ &\quad + r(t)(\mathcal{A}\varphi_n)^{\sigma}(t).\end{aligned}$$

Consequently, by invoking (3.8), (3.15) and Lemma 3.3, we obtain

$$\left|(\mathcal{A}\varphi_n)^{\triangle}(t)\right|\leq\frac{K(1-k_2)}{MT}+\|r\|K(1-k_2)\leq D,$$

for some positive constant D. Hence the sequence $(\mathcal{A}\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}\varphi_{n_k})$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. □

Lemma 3.6. *Suppose that (3.12) holds. If $\mathcal{B}$ is given by (3.11) with*

$$k<1, \tag{3.16}$$

then $\mathcal{B}:\mathbb{D}\to\mathbb{B}$ is a contraction.

Proof. Let $\mathcal{B}$ be defined by (3.4). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t+T)=(\mathcal{B}\varphi)(t)$. So, for any $\varphi,\psi\in\mathbb{D}$, we have

$$\begin{aligned}|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)| &\leq |g(t,\varphi(t-\tau(t)))-g(t,\psi(t-\tau(t)))| \\ &\leq k\|\varphi-\psi\|.\end{aligned}$$

Then $\|\mathcal{B}\varphi-\mathcal{B}\psi\|\leq k\|\varphi-\psi\|$. Thus $\mathcal{B}:\mathbb{D}\to\mathbb{B}$ is a contraction by (3.16). □

Theorem 3.5. *Suppose (3.3)–(3.5) and (3.12)–(3.16) hold. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

Proof. By Lemma 3.5, the operator $\mathcal{A}:\mathbb{D}\to\mathbb{B}$ is compact and continuous. Also, from Lemma 3.6, the operator $\mathcal{B}:\mathbb{D}\to\mathbb{B}$ is a contraction. Moreover, if $\varphi,\psi\in\mathbb{D}$, we see that

$$\begin{aligned}&(\mathcal{B}\psi)(t)+(\mathcal{A}\varphi)(t) \\ &= g(t,\psi(t-\tau(t))) \\ &+\int_t^{t+T} G(t,s)\left\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\right\}\triangle s \\ &\leq k_2K+M\int_t^{t+T}\left\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\right\}\triangle s \\ &\leq k_2K+MT\frac{K(1-k_2)}{MT}=K.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t) \\
&= g(t, \psi(t-\tau(t))) \\
&+ \int_t^{t+T} G(t,s)\{f(s,\varphi(s-\tau(s))) - r(s)\, g(s,\varphi(s-\tau(s)))\} \triangle s \\
&\geq k_1 L + m \int_t^{t+T} \{f(s,\varphi(s-\tau(s))) - r(s)\, g(s,\varphi(s-\tau(s)))\} \triangle s \\
&\geq k_1 L + mT \frac{L(1-k_1)}{mT} = L.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}x + \mathcal{B}x$. By Lemma 3.4 this fixed point is a solution of (1.1) and the proof is complete. □

Remark 3.3. *When $\mathbb{T} = \mathbb{R}$, Theorem 3.5 reduces to Theorem 3.1 of [3].*

In the case $g(t,x) \leq 0$, we substitute conditions (3.13)–(3.15) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3 x \leq g(t,x) \leq k_4 x, \text{ for all } t \in [0,T],\ x \in \mathbb{D}, \tag{3.17}$$

$$-k_3 < 1, \tag{3.18}$$

and for all $t \in [0,T]$, $x \in \mathbb{D}$

$$\frac{L - k_3 K}{mT} \leq f(t,x) - r(t)\, g(t,x) \leq \frac{K - k_4 L}{MT}. \tag{3.19}$$

Theorem 3.6. *Suppose (3.3)–(3.5), (3.12) and (3.16)–(3.19) hold. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

The proof follows along the lines of Theorem 3.5, and hence we omit it.

Remark 3.4. *When $\mathbb{T} = \mathbb{R}$, Theorem 3.6 reduces to Theorem 3.2 of [3].*

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Transient solution of an $M^{[X]}/G/1$ queuing model with feedback, random breakdowns and Bernoulli schedule server vacation having general vacation time distribution

G. Ayyappan[a,*] and S.Shyamala[b]

[a] *Department of Mathematics, Pondicherry Engineering College, Pondicherry, India.*

[b] *Department of Mathematics, Arunai Engineering College, Thiruvannamalai, Tamil Nadu, India.*

Abstract

This paper analyze an $M^{[X]}/G/1$ queue with feedback, random server breakdowns and Bernoulli schedule server vacation with general(arbitrary) distribution. Customers arrive in batches with compound Poisson process and are served one by one with first come first served basis. Both the service time and vacation time follow general (arbitrary) distribution. After completion of a service the may go for a vacation with probability θor continue staying in the system to serve a next customer, if any with probability $1-\theta$. With probability p, the customer feedback to the tail of original queue for repeating the service until the service be successful. With probability $1-p=q$, the customer departs the system if service be successful. The system may breakdown at random following Poisson process, whereas the repair time follows exponential distribution. We obtain the time dependent probability generating function in terms of their Laplace transforms and the corresponding steady state results explicitly. Also we derive the system performance measures like average number of customers in the queue and the average waiting time in closed form.

Keywords: $M^{[X]}/G/1$ queue, Poisson arrival, probability generating function, Bernoulli schedule, steady state, mean queue size, mean waiting time.

2010 MSC: 60K25, 60K30.

1 Introduction

Due to a lot of significance in the decision making process, the research on queuing theory has been extensively increased. Queuing theory has made a revolution in industry and logistics sector apart from its immense applications in many other areas like air traffic, bio-sciences, population studies, health sectors, manufacturing and production sections etc. According to the prevailing demands or situations in real life scenario, queuing models have been encountered enormously, in research perspective.

Most recently research studies on queues with server breakdown have been attracted, as an important area of queuing theory and have been studied extensively and successfully due to their various applications in production, communication systems. Mostly in the queuing literature, the server may be considered as an reliable one,such that service station never fails. But in real situations mostly the servers are unreliable, we often encounter the cases where service stations may fail which can be repaired. Similarly, many phenomena always occur in the area of computer communication networks and flexible manufacturing system etc. Since the performance of such a system may be heavily affected by server breakdowns, followed by a repair immediately, such systems with a repairable service stations are well worth investigating from the queuing theory point of view as well as reliability point of view.

*Corresponding author.
E-mail addresses: ayyappanpec@hotmail.com (G. Ayyappan) and subramaniyanshyamala@gmail.com (S.Shyamala).

Recently, there have been several contributions in considering non-Markovian single server queuing system, in which the server may experience with break downs and repairs, such system with repairable server has been studied as queuing models by many authors including Avi-Itzhak and Naor [2], Graver [6], Takine and Sengupta [21], Wang [24], Tang [22] Assani and Artalejo [1] etc.

Another feature in queuing theory is the study of queuing models with vacations. when the system is empty, the server becomes idle and this idle time may be utilized by the server for being engaged for other purposes. Thus, the non-availability of the server in the system is known as vacation. During the last three or four decades, queuing theorists are interested in the study of queuing models with vacations immensely, because of their applicability and theoretical structures in real life situations such as manufacturing and production systems, computer and communication systems, service and distribution systems, etc.

The most remarkable works have been done in recent past by some researchers on vacation models including Choudhary [3], Keilson and servi [9], Krishna Kumar [13], Levy and Yechiali [10], Wang [24], Madan [15, 16, 17, 18, 19], Thangaraj [23] etc. The details about vacation queues can be found in the survey of Doshi [5].

Transient state measures, which are very important to track down the functioning of the system at any instant of time. In this paper we present an analysis of the transient state behavior of a queuing system where breakdowns may occur at random, and once the system breaks down, it enters a repair process and the customer whose service is interrupted goes back to the head of the queue. At the same time the server may go on vacation. The vacations follow a Bernoulli distribution, that is, after a service completion, the server may go for a vacation with probability p $(0 \leq p \leq 1)$ or may continue to serve the next customer, if any, with probability $1 - p$. The service time and the vacation time are generally distributed, while the repair time is exponentially distributed. The customers arrive in batches to the system and served one by one on a “first come - first served” basis.

The rest of the paper has been organized as follows: in section 2, the mathematical description of our model has been found, in section 3, the transient solution of the system has been derived, in section 4, the steady state analysis has been discussed.

2 Mathematical description of the queuing model

To describe the required queuing model, we assume the following.

• Let $\lambda c_i dt; i = 1, 2, 3...$be the first order probability of arrival of 'i' customers in batches in the system during a short period of time(t,t+dt)where $0 \leq c_i \leq 1, \sum_{i=1}^{\infty} c_i = 1, \lambda > 0$ is the mean arrival rate of batches.

• There is a single server which provides service following a general(arbitrary) distribution with distribution function B(v)and density function b(v). Let$\mu(x)dx$ be the conditional probability density function of service completion during the interval (x,x+dx] given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)} \tag{2.1}$$

and therefore

$$b(v) = \mu(v)e^{-\int_0^v \mu(x)} dx \tag{2.2}$$

• After completion of service, if the customer is not satisfied with the service for certain reason or if customer received unsuccessful service, the customer may immediately join the tail of the original queue as a feedback customer for receiving another regular service with probability $p(0 < p < 1)$. Otherwise the customer may depart forever from the system with probability $q(= 1 - p)$. The service discipline for feedback and newly customers are first come first served. Also service time for a feedback customer is independent of its previous service times.

• As soon as a service is completed , the server may take a vacation of random length with probability θ (or) he may stay in the system providing service with probability $1 - \theta$, where $0 \leq \theta \leq 1$.

• The vacation time of the server follows a general (arbitrary)with distribution function V(s)and the density function v(s). Let $\nu(x)$dx be the conditional probability of a completion of a vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x so that

$$\nu(x) = \frac{v(x)}{1 - V(x)} \tag{2.3}$$

and therefore

$$v(s) = \nu(s)e^{-\int_0^s \nu(x)dx} \tag{2.4}$$

• The system may breakdown at random and the breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breakdown, the customer whose service is interrupted comes back to the head of queue.

• Once the system breaks down it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate $\beta > 0$.

• Various stochastic processes involved in the queuing system are assumed to be independent of each other.

3 Definitions and Equations governing the system

We let,

(i) $P_n(x,t)$ = Probability that at time 't' the server is active providing service and there are 'n' $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $p_n(t)$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in service irrespective of the value of x.

(ii) $V_n(x,t)$ = Probability that at time 't', the server is on vacation with elapsed vacation time x, and there are 'n' $(n \geq 0)$ customers waiting in the queue for service. Consequently $V_n(t)$ denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.

(iii) $R_n(t)$ = Probability that at time t, the server is inactive due to breakdown and the system is under repair while there are'n' $(n \geq 0)$ customers in the queue.

(iv) Q(t)= Probability that at time 't' there are no customers in the system and the server is idle but available in the system.

The model is then, governed by he following set of differential-difference equations.

$$\frac{\partial}{\partial t}P_n(x,t) + \frac{\partial}{\partial x}P_n(x,t) + (\lambda + \mu(x) + \alpha)P_n(x,t) = \lambda \sum_{i=1}^{n-1} c_i P_{n-i}(x,t); n \geq 1 \tag{3.1}$$

$$\frac{\partial}{\partial t}P_0(x,t) + \frac{\partial}{\partial x}P_0(x,t) + (\lambda + \mu(x) + \alpha)P_0(x,t) = 0 \tag{3.2}$$

$$\frac{\partial}{\partial t}V_n(x,t) + \frac{\partial}{\partial x}V_n(x,t) + (\lambda + \nu(x))V_n(x,t) = \lambda \sum_{i=1}^{n-1} c_i V_{n-i}(x,t); n \geq 1 \tag{3.3}$$

$$\frac{\partial}{\partial t}V_0(x,t) + \frac{\partial}{\partial x}V_0(x,t) + (\lambda + \nu(x))V_0(x,t) = 0 \tag{3.4}$$

$$\frac{d}{dt}R_n(t) = -(\lambda + \beta)R_n(t) + \lambda \sum_{i=1}^{n-1} c_i R_{n-i}(x,t) + \alpha \int_0^\infty P_{n-1}(x,t)dx \tag{3.5}$$

$$\frac{d}{dt}R_0(t) = -(\lambda + \beta)R_0(t) \tag{3.6}$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t) + \beta R_0(t) + \int_0^\infty V_0(x,t)\nu(x)dx + (1-\theta)q\int_0^\infty P_0(x,t)\mu(x)dx \tag{3.7}$$

The above equations are to be solved subject to the following boundary conditions

$$P_n(0,t) = (1-\theta)\left[p\int_0^\infty P_n(x,t)\mu(x)dx + q\int_0^\infty P_{n+1}(x,t)\mu(x)dx\right] + \int_0^\infty V_{n+1}(x,t)\nu(x)dx + \beta R_{n+1}(t) + \lambda c_{n+1}Q(t); n \geq 0 \tag{3.8}$$

$$V_n(0,t) = \theta \int_0^\infty P_n(x,t)\mu(x)dx; n \geq 0 \tag{3.9}$$

Assuming there are no customers in the system initially so that the server is idle.

$$V_0(0) = 0; V_n(0) = 0; Q(0) = 1; P_n(0) = 0, n = 0, 1, 2, ... \tag{3.10}$$

Generating functions of the queue length. The time dependent solution.

We define the probability generating functions
$P_q(x,z,t) = \sum_{n=0}^{\infty} z^n P_n(x,t)$

$P_q(z,t) = \sum_{n=0}^{\infty} z^n P_n(t)$

$V_q(x,z,t) = \sum_{n=0}^{\infty} z^n V_n(x,t)$

$V_q(z,t) = \sum_{n=0}^{\infty} z^n V_n(t)$

$R_q(z,t) = \sum_{n=0}^{\infty} z^n R_n(t)$

$$C(z) = \sum_{n=1}^{\infty} c_n z^n \tag{3.11}$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function f(t) as

$$\overline{f}(s) = \int_0^\infty f(t)e^{-st}dt. \tag{3.12}$$

Taking Laplace transforms of equations (3.1) to (3.9) and using the probability generating function defined above.

$$\frac{\partial}{\partial x}\bar{P}_n(x,s) + (s+\lambda+\mu(x)+\alpha)\bar{P}_n(x,s) = \lambda \sum_{i=1}^{n-1} c_i \bar{P}_{n-i}(x,s) \tag{3.13}$$

$$\frac{\partial}{\partial x}\bar{P}_0(x,s) + (s+\lambda+\mu(x)+\alpha)\bar{P}_0(x,s) = 0 \tag{3.14}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s) + (s+\lambda+\nu(x))\bar{V}_n(x,s) = \lambda \sum_{i=1}^{n-1} c_i \bar{V}_{n-i}(x,s) \tag{3.15}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s) + (s+\lambda+\nu(x))\bar{V}_0(x,s) = 0 \tag{3.16}$$

$$(s+\lambda+\beta)\bar{R}_n(s) = \lambda \sum_{i=1}^{n-1} c_i \bar{R}_{n-i}(s) + \alpha \int_0^\infty \bar{P}_{n-1}(x,s)dx \tag{3.17}$$

$$(s+\lambda+\beta)\bar{R}_0(s) = 0 \tag{3.18}$$

$$(s+\lambda)\bar{Q}(s) = 1 + \beta\bar{R}_0(s) + \int_0^\infty \bar{V}_0(x,s)\nu(x)dx + (1-\theta)q\int_0^\infty \bar{P}_0(x,s)\mu(x)dx \tag{3.19}$$

for boundary conditions,

$$\bar{P}_n(0,s) = (1-\theta)\left[p\int_0^\infty \bar{P}_n(x,s)\mu(x)dx + q\int_0^\infty \bar{P}_{n+1}(x,s)\mu(x)dx\right] + \int_0^\infty \bar{V}_{n+1}(x,s)\nu(x)dx + \beta\bar{R}_{n+1}(s) + \lambda c_{n+1}\bar{Q}(s); n \geq 0 \tag{3.20}$$

$$\bar{V}_n(0,s) = \theta \int_0^\infty \bar{P}_n(x,s)\mu(x)dx; n \geq 0 \tag{3.21}$$

multiply equation (3.13) by z^n and add (3.14) implies

$$\frac{\partial}{\partial x}\bar{P}_q(x,z,s) + (s+\lambda-\lambda C(z)+\mu(x)+\alpha)\bar{P}_q(x,z,s) = 0 \tag{3.22}$$

performing similar operations to equations (3.15) to (3.18).

$$\frac{\partial}{\partial x}\bar{V}_q(x,z,s) + (s+\lambda-\lambda C(z)+\nu(x))\bar{V}_q(x,z,s) = 0 \tag{3.23}$$

$$(s+\lambda-\lambda C(z)+\beta)\bar{R}_q(z,s) = \alpha z \int_0^\infty \bar{P}_q(x,z,s)dx \tag{3.24}$$

For the boundary conditions, we multiply equation(3.20)by z^{n+1}, sum over n from 0 to ∞ and use generating function defined above, we get

$$z\bar{P}_q(0,z,s) = (1-\theta)(pz+q)\int_0^\infty \bar{P}_q(x,z,s)\mu(x)dx$$
$$+ \int_0^\infty \bar{V}_q(x,z,s)\nu(x)dx + \beta\bar{R}_q(z,s) + (1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s) \tag{3.25}$$

Similarly multiply equation (3.21) by $z^n$and sum over n from 0 to ∞ and use generating function defined above

$$\bar{V}_q(0,z,s) = \theta\int_0^\infty \bar{P}_q(x,z,s)\mu(x)dx \tag{3.26}$$

Integrating equation(3.22) from 0 to x yields

$$\bar{P}_q(x,z,s) = \bar{P}_q(0,z,s)e^{-(s+\lambda-\lambda C(z)+\alpha)x - \int_0^x \mu(t)dt} \tag{3.27}$$

where $\bar{P}_q(0,z,s)$ is given by equation(3.25). Again integrating equation (3.27) by parts with respect to x yields

$$\bar{P}_q(z,s) = \bar{P}_q(0,z,s)\left[\frac{1-\bar{B}(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \tag{3.28}$$

where

$$\bar{B}(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}dB(x) \tag{3.29}$$

is Laplace - Stieltjes transform of the service time B(x). Now multiplying both sides of equation (3.27) by $\mu(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}_q(x,z,s)\mu(x)dx = \bar{P}_q(0,z,s)\bar{B}(s+\lambda-\lambda C(z)+\alpha) \tag{3.30}$$

Using equation (3.30) equation (3.26) becomes

$$\bar{V}_q(0,z,s) = \theta\bar{P}_q(0,z,s)\bar{B}(s+\lambda-\lambda C(z)+\alpha) \tag{3.31}$$

Similarly integrate equation (3.23) from 0 to x, we get

$$\bar{V}_q(x,z,s) = \bar{V}_q(0,z,s)e^{-(s+\lambda-\lambda C(z))x - \int_0^x \nu(t)dt} \tag{3.32}$$

substituting by the value of $\bar{V}_q(0,z,s)$ from (3.31), in equation (3.33) we get

$$\bar{V}_q(x,z,s) = \theta\bar{P}_q(0,z,s)\bar{B}(s+\lambda-\lambda(C(z))+\alpha)e^{-(s+\lambda-\lambda z)x - \int_0^x \nu(t)dt} \tag{3.33}$$

Again integrating equation (3.33) by parts with respect to x

$$\bar{V}_q(z,s) = \theta \bar{P}_q(0,z,s)\bar{B}(s+\lambda-\lambda C(z)+\alpha)\left[\frac{1-\bar{V}(s+\lambda-\lambda(C(z)))}{(s+\lambda-\lambda(C(z)))}\right] \tag{3.34}$$

where

$$\bar{V}(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-(s+\lambda-\lambda C(z))x} dV(x) \tag{3.35}$$

is Laplace - Stieltjes transform of the vacation time V(x). Now multiplying both sides of equation(3.33) by $\nu(x)$ and integrating over x, we get

$$\int_0^\infty \bar{V}_q(x,z,s)\nu(x)dx = \theta \bar{P}_q(0,z,s)\bar{B}(s+\lambda-\lambda C(z)+\alpha)\bar{V}(s+\lambda-\lambda C(z)) \tag{3.36}$$

Using equation (3.28), equation (3.24) becomes

$$\bar{R}_q(z,s) = \frac{\alpha z \bar{P}_q(0,z,s)[1-\bar{B}(s+\lambda-\lambda C(z)+\alpha)]}{[s+\lambda-\lambda C(z)+\beta][s+\lambda-\lambda C(z)+\alpha]} \tag{3.37}$$

Now using (3.30), (3.36) and (3.37) in equation (3.25) and solving for $\bar{P}_q(0,z,s)$ we get

$$\bar{P}_q(0,z,s) = \frac{f_1(z)f_2(z)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \tag{3.38}$$

where Dr $= f_1(z)f_2(z)\{z-(1-\theta)(pz+q)\bar{B}[f_1(z)] - \theta\bar{V}(s+\lambda-\lambda C(z))\bar{B}[f_1(z)]\}$
$-\alpha\beta z\{1-\bar{B}[f_1(z)]\}$

$f_1(z) = s+\lambda-\lambda C(z)+\alpha$

$f_2(z) = s+\lambda-\lambda C(z)+\beta$

substituting the value of $\bar{P}_q(0,z,s)$ from equation (3.38)in to equations (3.26), (3.34) and (3.37)

$$\bar{P}_q(z,s) = \frac{f_2(z)1-\bar{B}[f_1(z)]][(1-s\bar{Q}(s))+\lambda(C((z))-1)\bar{Q}(s)]}{Dr} \tag{3.39}$$

$$\bar{V}_q(z,s) = \frac{\theta f_1(z)f_2(z)\bar{B}[f_1(z)][(1-s\bar{Q}(s))+\lambda(C((z))-1)\bar{Q}(s)]\left[\frac{1-\bar{V}(s+\lambda-\lambda C(z))}{(s+\lambda-\lambda C(z))}\right]}{Dr} \tag{3.40}$$

$$\bar{R}_q(z,s) = \frac{\alpha z[1-\bar{B}[f_1(z)]][(1-s\bar{Q}(s))+\lambda(C((z))-1)\bar{Q}(s)]}{Dr} \tag{3.41}$$

where Dr is given in the above.

4 The steady state analysis

In this section we shall derive the steady state probability distribution for our queuing model. To define the steady state probabilities, suppress the argument't' where ever it appears in the time dependent analysis. By using well known Tauberian property,

$$Lt_{s\to 0}s\bar{f}(s) = Lt_{t\to\infty}f(t) \tag{4.1}$$

multiplying both sides of equation (3.39),(3.40),(3.41) and applying equation(4.1) and simplifying, we get

$$P_q(z) = \frac{f_2(z)(1-\bar{B}[f_1(z)])[\lambda(C((z))-1)Q]}{Dr} \tag{4.2}$$

$$V_q(z) = \frac{pf_1(z)f_2(z)\bar{B}[f_1(z)][\bar{V}(\lambda-\lambda C((z))-1)]Q}{Dr} \tag{4.3}$$

$$R_q(z) = \frac{\theta\lambda\alpha z[1-\bar{B}[f_1(z)]][(C((z))-1)Q]}{Dr} \tag{4.4}$$

where Dr and $f_1(z)$ and $f_2(z)$ are given by in previous section. Let$W_q(z)$ denotes the probability generating function of queue size irrespective of the state of the system. Then adding (4.2),(4.3) and (4.4), we get

$$W_q(z) = P_q(z) + V_q(z) + R_q(z) \tag{4.5}$$

$$W_q(z) = \frac{f_2(z)[1-B[f_1(z)]][\lambda(C((z))-1)Q]}{Dr} + \frac{\theta f_1(z) f_2(z) B[f_1(z)][V(\lambda-\lambda C((z))-1)]Q}{Dr}$$
$$+\frac{\lambda\alpha z[1-B[f_1(z)]][(C((z))-1)Q]}{Dr} \tag{4.6}$$

In order to obtain Q, we use the normalization condition, as follows

$$W_q(1) + Q = 1 \tag{4.7}$$

We see that at z=1, $W_q(z)$ is indeterminate of the form 0/0. We apply L'Hospital rule in equation (4.6)

$$W_q(1) = \frac{\lambda Q E(I)(\alpha+\beta)[1-\bar{B}(\alpha)] + \theta\alpha\beta B(\alpha)E[V]}{(q+p\theta)\alpha\beta\bar{B}(\alpha) - \lambda(\alpha+\beta)(1-\bar{B}(\alpha))E(I) - \theta\lambda\alpha\beta\bar{B}(\alpha)E(I)E[V]} \tag{4.8}$$

where $\bar{B}(0)=1, \bar{V}(0)=1, -V'(0)=E[V]$ the mean vacation time. Using equation (4.8) in equation (4.7)

$$Q = 1 - \frac{\lambda E(I)}{q+p\theta}\left\{\frac{1}{\beta\bar{B}(\alpha)} + \frac{1}{\alpha\bar{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V)\right\} \tag{4.9}$$

and the the utilization factorρ of the system is given by

$$\rho = \frac{\lambda E(I)}{q+p\theta}\left\{\frac{1}{\beta\bar{B}(\alpha)} + \frac{1}{\alpha\bar{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V)\right\} \tag{4.10}$$

where $\rho < 1$ is the stability condition under which the steady state exists, equation(4.9) gives the probability that the server is idle. Substitute Q from equation (4.9) in equation (4.6) $W_q(z)$ have been completely and explicitly determined which is the the probability generating function of the queue size.

The average queue size and average waiting time

Let $L_q$denote the mean number of customers in the queue under the steady state, then $L_q = \frac{d}{dz}W_q(z)\mid_{z=1}$,since this formula gives 0/0 form, then we write $W_q(z) = \frac{N(z)}{D(z)}$ where N(z) and D(z) are the numerator and denominator of the right hand side of equation (4.5) respectively, then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2} \tag{4.11}$$

where primes and double primes in equation (4.11) denote first and second derivation at z=1 respectively. Carrying out the derivatives at z=1, we have

$$N'(1) = \lambda E(I)Q[(\alpha+\beta) - \bar{B}(\alpha)(\theta\alpha\beta E(V) - \alpha - \beta)] \tag{4.12}$$

$$N''(1) = 2Q[\lambda E(I)]^2\Big\{(\tfrac{\alpha}{\lambda E(I)} - 1) + \bar{B}(\alpha)[1 - \tfrac{\alpha}{\lambda E(I)} - \theta\alpha E(V) - \theta\beta E(V)]$$
$$+\tfrac{1}{2}\theta\alpha\beta E(V^2) + \bar{B}'(\alpha)(\alpha+\beta-\theta\alpha\beta E(V))\Big\}$$
$$+\lambda Q E(I(I-1))\left\{(\alpha+\beta) + \bar{B}(\alpha)(\theta\alpha\beta E(V) - \alpha - \beta)\right\} \tag{4.13}$$

$$D'(1) = -\lambda E(I)(\alpha+\beta) + \bar{B}(\alpha)\left\{\alpha\beta(q+p\theta) + \lambda E(I)(\alpha+\beta) - \theta\alpha\beta E(V)\right\} \tag{4.14}$$

$$D''(1) = 2[\lambda E(I)]^2\Big\{(1 - \tfrac{\alpha+\beta}{\lambda E(I)}) + \bar{B}(\alpha)\left[-(q+p\theta) + \theta\alpha E(V) + \theta\beta E(V) - \tfrac{1}{2}\alpha\beta\theta E(V^2)\right]$$
$$+\bar{B}'(\alpha)[-(q+p\theta)(\alpha+\beta) - \tfrac{\alpha\beta}{\lambda E(I)} + \alpha\beta\theta E(V)]\Big\}$$
$$+\lambda E(I(I-1))\left\{-(\alpha+\beta) + \bar{B}(\alpha)(\alpha+\beta-\theta\alpha\beta E(V))\right\} \tag{4.15}$$

where $E(V^2)$ is the second moment of the vacation time and Q has been found in equation (4.9). Then if we substitute the values of $N'(1), N''(1), D'(1)$ and $D''(1)$ from equations (4.12), (4.13), (4.14) and (4.15) in to equation (4.11), we obtain L_q in a closed form.
Mean waiting time of a customer could be found, as follows

$$W_q = \frac{L_q}{\lambda} \tag{4.16}$$

by using Little's formula.

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Orthonormal series expansion and finite spherical Hankel transform of generalized functions

S.K. Panchal[a,*]

[a] *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431 004, (M.S.) India.*

Abstract

The finite spherical Hankel transformation is extended to generalized functions by using orthonormal series expansion of generalized functions. A complete orthonormal family of spherical Bessel functions is derived and certain spaces of testing functions and generalized functions are defined. The inversion and uniqueness theorems are obtained. The operational transform formula is derived and is applied to solve the problem of the propagation of heat released from a spherically symmetric point heat source.

Keywords: Finite spherical Hankel transform, orthonormal series expansion of generalized functions.

2010 MSC: 46F12, 44A15, 46F10, 41A58.

1 Introduction

Several authors treated the problem of expanding the elements of a distribution space using different orthonormal systems. Zemanian [2], [5] constructed the testing function space A for suitable complete orthonormal system $\{\Psi_n\}$ of eigenfunctions of the differential operator η. The elements of the dual space A' are generalized functions, each of which can be expanded into a series of eigenfunctions Ψ_n. As a special case of his general theory he defined the finite Fourier, Hermite, Jacobi and finite Hankel transformations of generalized functions where the inverse transformations are obtained by using orthonormal series expansions of generalized functions.

Bhosale and More[3] and Panchal and More[4] extended certain finite integral transformations to generalized functions by using the method of Zemanian. In this paper the variant of finite spherical Hankel transformation introduced by Chen I.I.H.[1] is extended it to certain space of generalized functions whose inverse is obtained in terms of Fourier-spherical Bessel series.

2 Preliminary Results, Notations and Terminology

Let I=$\{x/0 \leq x \leq a < \infty\}$ and $N_0 = \mathrm{N} \cup \{0\}$, where N is the set of natural numbers. Consider the self adjoint differential operator

$$L_0 = (x^{-1}D_x x^2 D_x x^{-1})$$

denoting the conventional or generalized partial differential operators, where $D_x = \frac{\partial}{\partial x}$. Let $J_{\frac{1}{2}}(x)$ and $j_0(x)$ be the Bessel function of the first kind of order 1/2, and spherical Bessel function of order zero respectively. Consider the eigenfunction system $\{\psi_n(x)\}_{n=1}^{\infty}$ corresponding to the differential operator L_0 where $\psi_n(x) = C_n x j_0(\lambda_n x)$, $C_n = \frac{2}{a[J'_{\frac{1}{2}}(\lambda_n a)]}\sqrt{\frac{\lambda_n}{\pi}}$, and the corresponding eigenvalues $\lambda_n, n = 1, 2, 3, ...$ are the positive roots of $j_0(\lambda z) = 0$ arranged in ascending order of magnitude.We see that,

$$L_0\psi_n(x) = -\lambda_n^2\psi_n(x). \tag{2.1}$$

*Corresponding author.

E-mail address: drpanchalsk@gmail.com (S.K. Panchal).

Let $L_2(I)$ be the linear space of functions that are absolutely square integrable on I and $< f, g >$ denote the inner product defined by,

$$< f, \overline{g} >= (f, g) = \int_I f(x)\overline{g}(x)dx. \tag{2.2}$$

Thus,

$$||f||_2^2 =< f, \overline{f} >= (f, f) = \int_I |f(x)|^2 dx \tag{2.3}$$

is the norm on $L_2(I)$. Hence

$$(\psi_m(x), \psi_n(x)) = \begin{cases} 1 & \text{m=n} \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

and $\int_I |\psi_n(x)|^2 dx = 1$ implies that $\psi_n(x) \in L_2(I)$ for every $n \in N_0$.

We define the finite spherical Hankel transform of $f(x) \in L_2(I)$ denoted by $SH[f(x)](n) = F_{SH}(n)$ as,

$$F_{SH}(n) = (f(x), \psi_n(x)) = \int_I f(x)\psi_n(x)dx. \tag{2.5}$$

The following theorem provides the inversion of the transformation defined in (2.5).

Theorem 2.1. *Every $f(x) \in L_2(I)$ admits the Fourier-spherical Bessel series expansion*

$$f(x) = \sum_{n=1}^{\infty} (f(x), \psi_n(x))\psi_n(x) \tag{2.6}$$

where the series converges point-wise on I.

3 Testing Function Space $S - H(I)$

For $n \in N_0$ we denote by $S - H(I)$ the space all complex valued smooth functions $\phi(x)$ defined on I such that for each non negative integers n *and* k.

i)

$$\eta^k(\phi) = \eta^0(L_0^k \phi) = \left\{ \int_I [L_0^k \phi(x)]^2 dx \right\}^{\frac{1}{2}} < \infty \tag{3.1}$$

ii)

$$(L_0^k \phi, \psi_n(x)) = (\phi, L_0^k \psi_n(x)) \tag{3.2}$$

Obviously $L_2(I) \subset S - H(I)$. The space $S - H(I)$ is a linear space and η^k is a seminorm on $S - H(I)$. Moreover η^0 is a norm on $S - H(I)$. Thus $\eta^k, k \in N_0$ is a countable multi-norm on $S - H(I)$. Also $S - H(I)$ is complete and hence a Frechet space. Thus $S - H(I)$ turns out to be a testing function space.

Lemma 3.1. *Every $\psi_n(x)$ is a member of $S - H(I)$.*

Proof. For each $k \in N_0$, from equations (2.1) and (3.1) we have

$$\begin{aligned} |\eta^k[\psi_n(x)]|^2 &\leq \int_I |L_0^k \psi_n(x)|^2 dx \\ &\leq |\lambda_n|^{2k} \int_I |\psi_n(x)|^2 dx \\ &= |\lambda_n|^{2k} < \infty. \end{aligned}$$

Next since λ_n are real then for $m \neq n$, we have

$$\begin{aligned} (L_0^k \psi_n(x), \psi_m(x)) &= \lambda_n^k (\psi_n(x), \psi_m(x)) \\ &= 0 = \lambda_m^k (\psi_n(x), \psi_m(x)) = (\psi_n(x), \lambda_m^k \psi_m(x)) \\ &= (\psi_n(x), L_0^k \psi_m(x)) \end{aligned}$$

and for $m = n$

$$(L_0^k \psi_n(x), \psi_n(x)) = (\lambda_n^k \psi_n(x), \psi_n(x)) = (\psi_n(x), \lambda_n^k \psi_n(x)) = (\psi_n(x), L_0^k \psi_n(x)).$$

Hence $\psi_n(x) \in$ S-H(I) for all $n \in N_0$. □

Lemma 3.2. *Every $\phi(x) \in S - H(I)$ can be expanded into the series*

$$\phi(x) = \sum_{n=0}^{\infty} (\phi(x), \psi_n(x))\psi_n(x) \tag{3.3}$$

where the series converges in $S - H(I)$.

Proof. Let $\phi(x) \in S - H(I)$, then $L_0^k \phi(x) \in L_2(I)$ and from theorem (2.1), we have

$$\begin{aligned} Ł_0^k \phi(x) &= \sum_{n=0}^{\infty} (L_0^k \phi(x), \psi_n(x))\psi_n(x) \\ &= \sum_{n=0}^{\infty} (\phi, L_0^k \psi_n(x))\psi_n(x) \\ &= \sum_{n=0}^{\infty} (\phi, \psi_n(x))\lambda_n^k \psi_n(x) \\ &= \sum_{n=0}^{\infty} (\phi, \psi_n(x))L_0^k \psi_n(x) \end{aligned}$$

which implies that $\eta^k[\phi(x) - \sum_{n=0}^{N} (\phi(x), \psi_n(x))\psi_n(x)] \to 0$ as $N \to \infty$ independently. Thus the series in (3.3) converges to $\phi(x)$ in $S - H(I)$. □

4 The Generalized Function Space $S - H'(I)$

The space of all continuous linear functions on $S-H(I)$, denoted by $S-H'(I)$ is called the dual of $S-H(I)$ and members of $S-H'(I)$ are called generalized functions on I. The number that $f \in S-H'(I)$ assigns to $\phi \in S-H(I)$ is denoted by $< f, \phi >$. Since the testing function space $S-H(I)$ is complete, so also is $S-H'(I)$[5]. Let $f(x)$ be a real valued continuous function locally integrable on I such that

$$\int_I |f(x)|^2 dx < \infty,$$

then f generates a member of $S - H'(I)$ through the definition

$$< f, \phi >= \int_I f(x)\phi(x)dx. \tag{4.1}$$

Clearly (4.1) defines a linear function f on $S-H(I)$ and the continuity of f can be verified by using Schwarz's inequality. Such members of $S-H'(I)$ are called regular generalized functions in $S-H'(I)$. All other generalized functions in $S-H'(I)$ are called singular generalized functions. Now we define a generalized differential operator L_0 on $S - H'(I)$ through the relationship

$$(f, L_0\phi) =< f, \overline{L_0\phi} >=< \overline{L_0}' f, \overline{\phi} >= (\overline{L_0}' f, \phi) \tag{4.2}$$

where $\overline{L_0}'$ is obtained from L_0 by reversing the order in which the differentiation and multiplication by smooth functions occurring in L_0, replacing each D_x by $-D_x$ and then taking the complex conjugate of the result. But this is precisely the same expression for L_0 [[5], sec 9.2, eq 4]. Thus $L_0 = \overline{L_0}'$ is defined as the generalized differential operator on $S - H'(I)$ through the equation

$$< L_0 f, \phi >=< f, L_0\phi >, \tag{4.3}$$

where $f \in S - H'(I), \phi \in S - H(I)$.

Some properties of $S - H(I)$ and $S - H'(I)$

I) $\mathscr{D}(I) \subset S-H(I) \subset \mathscr{E}(I)$ and since $\mathscr{D}(I)$ is dense in $\mathscr{E}(I), S-H(I)$ is also dense in $\mathscr{E}(I)$. It follows $\mathscr{E}'(I)$ is a subspace of $S - H'(I)$. The convergence of a sequence in $\mathscr{D}(I)$ implies its convergence in $S - H(I)$. The restriction of any $f \in S - H'(I)$ to $\mathscr{D}(I)$ is in $\mathscr{D}'(I)$. Moreover the convergence in $S - H'(I)$ implies convergence in $\mathscr{D}'(I)$.

II) For each $f \in S - H'(I)$ there exists a non negative integer r and a positive constant C such that

$$|<f,\phi>| \leq C \max_{0\leq k\leq r} \eta^k(\phi)$$

for every $\phi \in S - H(I)$. Here r and C depends on f but not on ϕ.

III) The mapping $\phi \rightarrow L_0\phi$ is continuous linear mapping of $S - H(I)$ into itself. It follows that $f \rightarrow L_0 f$ is also a continuous linear mapping of $S - H'(I)$ whenever f is a regular generalized function in $S - H'(I)$.

5 Finite Spherical Hankel transformation of generalized functions

We define the finite spherical Hankel transform of generalized function $f \in L - H'(I)$, denoted by $\mathcal{SH}[f] = \mathcal{F}_{SH}(n)$ as,

$$\mathcal{SH}[f(x)](n) = \mathcal{F}_{SH}(n) = (f(x), \psi_n(x)) \tag{5.1}$$

where $\psi_n(x) \in S - H(I)$ for $n \in N_0$. We see that $\mathcal{SH}$ is a linear and continuous mapping on S-H'(I), which maps $f \in S - H'(I)$ into a function $\mathcal{F}_{SH}(n)$ defined on N_0. The following theorem provides the inversion of the transformation defined in (5.1).

Theorem 5.1. *Let $f \in$ S-H'(I), then the series*

$$\sum_{n=0}^{\infty}(f(x), \psi_n(x))\psi_n(x) \tag{5.2}$$

converges to f in $S - H'(I)$.

Proof. From lemma 3.2 we have for every $\phi \in S - H(I)$, the series $\sum_{n=0}^{\infty}(\phi, \psi_n(x))\psi_n(x)$ converges to ϕ in $S - H(I)$, then for $f \in S - H'(I)$, we write

$$\begin{aligned}
(f, \phi) &= (f, \sum_{n=0}^{\infty}(\phi, \psi_n(x)), \psi_n(x)) \\
&= \sum_{n=0}^{\infty} \overline{(\phi, \psi_n(x))}(f, \psi_n(x)) \\
&= \sum_{n=0}^{\infty}(f, \psi_n(x))(\psi_n(x), \phi(x)) \\
&= \sum_{n=0}^{\infty}((f, \psi_n(x))\psi_n(x), \phi(x)) \\
&= \left(\sum_{n=0}^{\infty}(f, \psi_n(x))\psi_n(x), \phi(x)\right).
\end{aligned}$$

Thus the series $\sum_{n=0}^{\infty}(f, \psi_n(x))\psi_n(x)$ converges weakly to f in S-H'(I).

The above theorem lead to define the inverse of the finite spherical Hankel transformation of $f \in S - H'(I)$, denoted by $\mathcal{SH}^{-1}\mathcal{F}_{SH}(n) = f(x)$, as

$$\begin{aligned}
\mathcal{SH}^{-1}\mathcal{F}_{SH}(n) = f(x) &= \sum_{n=0}^{\infty}\mathcal{F}_{SH}(n)\psi_n(x) \\
&= \sum_{n=0}^{\infty}(f(x), \psi_n(x))\psi_n(x).
\end{aligned} \tag{5.3}$$

□

Theorem 5.2. *(Uniqueness Theorem): Let $f, g \in$ S-H'(I) are such that $\mathcal{SH}[f](n) = \mathcal{F}_{SH}(n) = \mathcal{G}_{SH}(n) = \mathcal{SH}[g](n)$ for every $n \in N_0$, then $f = g$ in the sense of equality in $S - H'(I)$.*

Proof. Let $\phi \in S-H(I)$, and $f,g \in S-H'(I)$ then

$$\begin{aligned} <f,\phi> - <g,\phi> &= <\sum_{n=0}^{\infty}(f,\psi_n(x))\psi_n(x),\phi(x)> \\ &- <\sum_{n=0}^{\infty}(g,\psi_n(x))\psi_n(x),\phi(x)> \\ &= <\sum_{n=0}^{\infty}\mathcal{F}_{SH}(n)\psi_n(x),\phi(x)> \\ &- <\sum_{n=0}^{\infty}\mathcal{G}_{SH}(n)\psi_n(x),\phi(x)> \\ &= <\sum_{n=0}^{\infty}[\mathcal{F}_{SH}(n)-\mathcal{G}_{SH}(n)]\psi_n(x),\phi(x)> \\ &= 0 \end{aligned}$$

for all n $\in N_0$. Hence $f=g$ in $S-H'(I)$. □

6 An Operational Calculus

Let $f(x) \in S-H'(I), \psi_n(x) \in S-H(I)$ and since the differential operator L_0 is a continuous linear mapping of $S-H'(I)$ into itself, then from equation (4.3), we have

$$\begin{aligned} \mathcal{SH}[L_0^k f](n) = <L_0^k f,\psi_n(x)> &= <f, L_0^k\psi_n(x)> \\ &= <f,-\lambda_n^{2k}\psi_n(x)> \\ &= -\lambda_n^{2k}<f,\psi_n(x)> \\ &= -\lambda_n^{2k}\mathcal{SH}[f](n) \\ &= -\lambda_n^{2k}\mathcal{F}_{SH}(n). \end{aligned} \tag{6.1}$$

We can use this fact to solve the distributional differential equations of the form

$$P(L_0)u = g \tag{6.2}$$

where P is a polynomial and the given g and unknown u are the generalized functions in S-H'(I). Applying the finite spherical Hankel transformation defined in (5.1) to the differential equation (6.2), we get

$$P(-\lambda_n^2)\mathcal{SH}[u](n) = \mathcal{SH}[g](n), \;\; n \in N_0. \tag{6.3}$$

If $P(-\lambda_n^2) \neq 0$ for all $n \in N_0$, we divide (6.3) by $P(-\lambda_n^2)$ and apply inverse finite spherical Hankel transform defined in (5.3), and get

$$u(x) = \sum_{n=0}^{\infty}\frac{\mathcal{SH}[g](n)}{P(-\lambda_n^2)}\psi_n(x) \tag{6.4}$$

where the series converges in $S-H'(I)$. In view of Theorem (5.1) and (5.2) the solution $u(x)$ in $S-H'(I)$ exists and is unique.

7 Application of finite spherical Hankel transform

The propagation of heat released from a spherically symmetric point heat source is governed by the heat conduction equation of the form

$$x^{-1}\frac{\partial^2(xu)}{\partial x^2} = k^{-1}\frac{\partial u}{\partial t} \tag{7.1}$$

where $k = K/\rho C_\nu$ is the thermal diffusivity for conductivity K, ρ is density, and C_ν is the heat capacity, respectively. We consider the folllowing initial and boundary conditions:

$$u(x,t) = f(x) \ \ when \quad t = 0 \quad at \ \ x = 0;$$
$$u(x,t) = 0 \ \ at \ \ x = a, \quad t > 0. \tag{7.2}$$

We now find the generalized solution $u(x,t)$ of this problem in the space $S - H'(I)$. Multiplying equation (7.1) by x^2, substituting $u = x^{-1}v(x,t)$ and then multiplying by x^{-1} we get

$$x^{-1}(x^2\frac{\partial^2}{\partial x^2} + 2x\frac{\partial}{\partial t})(x^{-1}v) = (1/k)\frac{\partial v}{\partial t} \tag{7.3}$$

Now applying the finite spherical Hankel transform defined in (5.1) to (7.3) we get

$$\frac{d\mathcal{V}_{SH}}{dt} + \lambda_n^2 k\mathcal{V}_{SH} = 0, \tag{7.4}$$

where $\mathcal{V}_{SH}$ is a finite spherical Hankel transform of $v(x,t)$. The solution of this equation is given by

$$\mathcal{V}_{SH}(\lambda_n, t) = Cexp(-\lambda_n^2 kt) \tag{7.5}$$

where the constant C can determined from the inial and boundary conditions given in (7.2) . Hence we have

$$\mathcal{V}_{SH}(\lambda_n, t) = \mathcal{F}_{SH}(n)exp(-\lambda_n^2 kt) \tag{7.6}$$

where $\mathcal{F}_{SH}(n)$ is the finite spherical Hankel transform of $f(t)$. Applying inverse finite spherical Hankel transform defined in (5.3), we get

$$v(x) = \sum_{n=0}^{\infty} \mathcal{F}_{SH}(n)exp(-\lambda_n^2 kt)\psi_n(x) \tag{7.7}$$

where the series converges in $S - H'(I)$. In view of Theorem (5.1) and (5.2) the solution $v(x)$ in $S - H'(I)$ exists and is unique. Thus $u(x,t) = x^{-1}v(x,t)$ is the required solution.

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