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On a class of fractional q-Integral inequalities

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Abstract

In the present paper, we use the fractional q-calculus to generate some new integral inequalities for some monotonic functions. Other fractional q-integral results, using convex functions, are also presented.

Keywords: Convex function, fractional q-calculus, q-Integral inequalities.

2010 MSC: 26D15.

1 Introduction

The study of the q-integral inequalities play a fundamental role in the theory of differential equations. We refer the reader to [3, 8, 9, 14] for further information and applications. To motivate our work, we shall introduce some important results. The first one is given in [13], where Ngo et al. proved that for any positive continuous function f on $[0,1]$ satisfying $\int_x^1 f(\tau)d\tau \geq \int_x^1 \tau d\tau, x \in [0,1]$, and for $\delta > 0$, the inequalities

$$\int_0^1 f^{\delta+1}(\tau)d\tau \geq \int_0^1 \tau^\delta f(\tau)d\tau \tag{1.1}$$

and

$$\int_0^1 f^{\delta+1}(\tau)d\tau \geq \int_0^1 \tau f^\delta(\tau)d\tau \tag{1.2}$$

are valid.
In [11], W.J. Liu, G.S. Cheng and C.C. Li proved that

$$\int_a^b f^{\alpha+\beta}(\tau)d\tau \geq \int_a^b (\tau-a)^\alpha f^\beta(\tau)d\tau, \tag{1.3}$$

for any $\alpha > 0, \beta > 0$ and for any positive continuous function f on $[a,b]$, such that

$$\int_x^b f^\gamma(\tau)d\tau \geq \int_x^b (\tau-a)^\gamma d\tau;\ \gamma := min(1,\beta), x \in [a,b].$$

Recently, Liu et al. [12] proved another interesting form of integral result, and the following inequality

$$\frac{\int_a^b f^\beta(\tau)d\tau}{\int_a^b f^\gamma(\tau)d\tau} \geq \frac{\int_a^b (\tau-a)^\delta f^\beta(\tau)d\tau}{\int_a^b (\tau-a)^\delta f^\gamma(\tau)d\tau}, \beta \geq \gamma > 0, \delta > 0 \tag{1.4}$$

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(where f is a positive continuous and decreasing function on $[a, b]$), was proved in this paper. Several interesting inequalities can be found in [12].

Many researchers have given considerable attention to (1),(3) and (4) and a number of extensions and generalizations appeared in the literature (e.g. [4, 5, 6, 7, 10, 11, 15, 16]).

The main purpose of this paper is to establish some new fractional q-integral inequalities on the specific time scales $T_{t_0} = \{t : t = t_0 q^n, n \in N\} \cup \{0\}$, where $t_0 \in R$, and $0 < q < 1$. Other fractional q-integral results, involving convex functions, are also presented. Our results have some relationships with those obtained in [12].

2 Notations and Preliminaries

In this section, we provide a summary of the mathematical notations and definitions used in this paper. For more details, one can consult [1, 2].
Let $t_0 \in R$. We define

$$T_{t_0} := \{t : t = t_0 q^n, n \in N\} \cup \{0\}, 0 < q < 1. \tag{2.5}$$

For a function $f : T_{t_0} \to R$, the ∇ q-derivative of f is:

$$\nabla_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} \tag{2.6}$$

for all $t \in T \setminus \{0\}$ and its ∇q-integral is defined by:

$$\int_0^t f(\tau)\nabla\tau = (1-q)t \sum_{i=0}^{\infty} q^i f(tq^i) \tag{2.7}$$

The fundamental theorem of calculus applies to the q-derivative and q-integral. In particular, we have:

$$\nabla_q \int_0^t f(\tau)\nabla\tau = f(t). \tag{2.8}$$

If f is continuous at 0, then

$$\int_0^t \nabla_q f(\tau)\nabla\tau = f(t) - f(0). \tag{2.9}$$

Let T_{t_1}, T_{t_2} denote two time scales. Let $f : T_{t_1} \to R$ be continuous let $g : T_{t_1} \to T_{t_2}$ be q-differentiable, strictly increasing, and $g(0) = 0$. Then for $b \in T_{t_1}$, we have:

$$\int_0^b f(t)\nabla_q g(t)\nabla t = \int_0^{g(b)} (f \circ g^{-1})(s)\nabla s. \tag{2.10}$$

The q-factorial function is defined as follows:
If n is a positive integer, then

$$(t-s)^{\underline{(n)}} = (t-s)(t-qs)(t-q^2 s)...(t-q^{n-1}s). \tag{2.11}$$

If n is not a positive integer, then

$$(t-s)^{\underline{(n)}} = t^n \prod_{k=0}^{\infty} \frac{1-(\frac{s}{t})q^k}{1-(\frac{s}{t})q^{n+k}}. \tag{2.12}$$

The q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)^{\underline{(n)}} = \frac{1-q^n}{1-q}(t-s)^{\underline{(n-1)}}, \tag{2.13}$$

and the q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)^{\underline{(n)}} = -\frac{1-q^n}{1-q}(t-qs)^{\underline{(n-1)}}. \tag{2.14}$$

The q-exponential function is defined as

$$e_q(t) = \prod_{k=0}^{\infty}(1-q^k t), e_q(0) = 1 \tag{2.15}$$

The fractional q-integral operator of order $\alpha \geq 0$, for a function f is defined as

$$\nabla_q^{-\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-q\tau)^{\underline{\alpha-1}} f(\tau)\nabla\tau; \quad \alpha > 0, t > 0, \tag{2.16}$$

where $\Gamma_q(\alpha) := \frac{1}{1-q}\int_0^1 (\frac{u}{1-q})^{\alpha-1} e_q(qu)\nabla u$.

3 Main Results

Theorem 3.1. *Let f and g be two positive and continuous functions on T_{t_0} such that f is decreasing and g is increasing on T_{t_0}. Then for all $\alpha > 0, \beta \geq \gamma > 0, \delta > 0$, we have*

$$\frac{\nabla_q^{-\alpha}[f^\beta(t)]}{\nabla_q^{-\alpha}[f^\gamma(t)]} \geq \frac{\nabla_q^{-\alpha}[g^\delta f^\beta(t)]}{\nabla_q^{-\alpha}[g^\delta f^\gamma(t)]}, t > 0. \tag{3.17}$$

Proof. Let us consider

$$H(\tau,\rho) := \Big(g^\delta(\rho) - g^\delta(\tau)\Big)\Big(f^\beta(\tau)f^\gamma(\rho) - f^\gamma(\tau)f^\beta(\rho)\Big), \tau,\rho \in (0,t), t > 0. \tag{3.18}$$

We have

$$H(\tau,\rho) \geq 0. \tag{3.19}$$

Hence, we get

$$\begin{aligned}\int_0^t \frac{(t-q\tau)^{\underline{(\alpha-1)}}}{\Gamma_q(\alpha)} H(\tau,\rho)\nabla\tau &= g^\delta(\rho)f^\gamma(\rho)\nabla_q^{-\alpha}[f^\beta(t)] + f^\beta(\rho)\nabla_q^{-\alpha}[g^\delta(t)f^\gamma(t)] \\ &-f^\gamma(\rho)\nabla_q^{-\alpha}[g^\delta(t)f^\beta(t)] - g^\delta(\rho)f^\beta(\rho)\nabla_q^{-\alpha}[f^\gamma(t)] \geq 0.\end{aligned} \tag{3.20}$$

Consequently,

$$\begin{aligned}2^{-1}\int_0^t\int_0^t \frac{(t-q\rho)^{\underline{(\alpha-1)}}(t-q\tau)^{\underline{(\alpha-1)}}}{\Gamma_q^2(\alpha)} H(\tau,\rho)\nabla\tau\nabla\rho &= \nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\alpha}[g^\delta(t)f^\gamma(t)] \\ &-\nabla_q^{-\alpha}[f^\gamma(t)]\nabla_q^{-\alpha}[g^\delta(t)f^\beta(t)) \geq 0.\end{aligned} \tag{3.21}$$

Theorem 3.1 is thus proved.

□

Another result which generalizes Theorem 3.1 is described in the following theorem:

Theorem 3.2. *Suppose that f and g are two positive and continuous functions on T_{t_0}, such that f is decreasing and g is increasing on T_{t_0}. Then for all $\alpha > 0, \omega > 0, \beta \geq \gamma > 0, \delta > 0$, we have*

$$\frac{\nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\omega}[g^\delta f^\gamma(t)] + \nabla_q^{-\omega}[f^\beta(t)]\nabla_q^{-\alpha}[g^\delta f^\gamma(t)]}{\nabla_q^{-\alpha}[f^\gamma(t)]\nabla_q^{-\omega}[g^\delta f^\beta(t)] + \nabla_q^{-\omega}[f^\gamma(t)]\nabla_q^{-\alpha}[g^\delta f^\beta(t)]} \geq 1; t > 0. \tag{3.22}$$

Proof. The relation (3.20) allows us to obtain

$$\begin{aligned}\int_0^t\int_0^t \frac{(t-q\rho)^{\underline{(\omega-1)}}(t-q\tau)^{\underline{(\alpha-1)}}}{\Gamma_q(\omega)\Gamma_q(\alpha)} H(\tau,\rho)\nabla\tau\nabla\rho &= \nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\omega}[g^\delta f^\gamma(t)] \\ +\nabla_q^{-\omega}[f^\beta(t)]\nabla_q^{-\alpha}[g^\delta f^\gamma(t)] - \nabla_q^{-\alpha}[f^\gamma(t)]\nabla_q^{-\omega}[g^\delta f^\beta(t)) &- \nabla_q^{-\omega}[f^\gamma(t)]\nabla_q^{-\alpha}[g^\delta f^\beta(t)] \geq 0,\end{aligned} \tag{3.23}$$

for any $\omega > 0$.
Hence, we have (3.22).

□

Remark 3.1. *It is clear that Theorem [3.1] would follow as a special case of Theorem [3.2] for $\alpha = \omega$.*

The third result is given by the following theorem:

Theorem 3.3. *Let f and g be two positive continuous functions on T_{t_0}, such that*

$$\Big(f^{\delta}(\tau)g^{\delta}(\rho) - f^{\delta}(\rho)g^{\delta}(\tau)\Big)\Big(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\Big) \geq 0; \tau, \rho \in (0,t), t > 0. \tag{3.24}$$

Then we have

$$\frac{\nabla_q^{-\alpha}[f^{\delta+\beta}(t)]}{\nabla_q^{-\alpha}[f^{\delta+\gamma}(t)]} \geq \frac{\nabla_q^{-\alpha}[g^{\delta}f^{\beta}(t)]}{\nabla_q^{-\alpha}[g^{\delta}f^{\gamma}(t)]}, \tag{3.25}$$

for any $\alpha > 0, \beta \geq \gamma > 0, \delta > 0$.

Proof. We consider the quantity:

$$K(\tau,\rho) := \Big(f^{\delta}(\tau)g^{\delta}(\rho) - f^{\delta}(\rho)g^{\delta}(\tau)\Big)\Big(f^{\gamma}(\rho)f^{\beta}(\tau) - f^{\gamma}(\tau)f^{\beta}(\rho)\Big); \tau, \rho \in (0,t), t > 0$$

and we use the same arguments as in the proof of Theorem [3.1]. □

Using two fractional parameters, we obtain the following generalization of Theorem [3.3]:

Theorem 3.4. *Let f and g be two positive continuous functions on T_{t_0}, such that*

$$\Big(f^{\delta}(\tau)g^{\delta}(\rho) - f^{\delta}(\rho)g^{\delta}(\tau)\Big)\Big(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\Big) \geq 0; \tau, \rho \in (0,t), t > 0. \tag{3.26}$$

Then for all $\alpha > 0, \omega > 0, \beta \geq \gamma > 0, \delta > 0$, we have

$$\frac{\nabla_q^{-\alpha}[f^{\delta+\beta}(t)]\nabla_q^{-\omega}[g^{\delta}f^{\gamma}(t)] + \nabla_q^{-\omega}[f^{\delta+\beta}(t)]\nabla_q^{-\alpha}[g^{\delta}f^{\gamma}(t)]}{\nabla_q^{-\alpha}[f^{\gamma+\delta}(t)]\nabla_q^{-\omega}[g^{\delta}f^{\beta}(t)] + \nabla_q^{-\omega}[f^{\gamma+\delta}(t)]\nabla_q^{-\alpha}[g^{\delta}f^{\beta}(t)]} \geq 1. \tag{3.27}$$

Remark 3.2. *Applying Theorem [3.4], for $\alpha = \omega$, we obtain Theorem [3.3].*

Involving convex functions, we have the following result:

Theorem 3.5. *Let f and h be two positive continuous functions on T_{t_0} and $f \leq h$ on T_{t_0}. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty[$, then for any convex function $\phi; \phi(0) = 0$, the inequality*

$$\frac{\nabla_q^{-\alpha}(f(t))}{\nabla_q^{-\alpha}(h(t))} \geq \frac{\nabla_q^{-\alpha}(\phi(f(t)))}{\nabla_q^{-\alpha}(\phi(h(t)))}, t > 0, \alpha > 0 \tag{3.28}$$

is valid.

Proof. Using the fact that on T_{t_0}, $\frac{\phi(f(.))}{f(.)}$ is an increasing function and $\frac{f}{h}$ is a decreasing function, we can write

$$L(\tau,\rho) \geq 0, \tau, \rho \in (0,t), t > 0, \tag{3.29}$$

where

$$\begin{gathered} L(\tau,\rho) := \frac{\phi(f(\tau))}{f(\tau)}f(\rho)h(\tau) + \frac{\phi(f(\rho))}{f(\rho)}f(\tau)h(\rho) \\ -\tfrac{\phi(f(\rho))}{f(\rho)}f(\rho)h(\tau) - \tfrac{\phi(f(\tau))}{f(\tau)}f(\tau)h(\rho), \tau, \rho \in (0,t), t > 0. \end{gathered} \tag{3.30}$$

Multiplying both sides of (3.29) by $\frac{(t-q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)}$, then integrating the resulting inequality with respect to τ over $(0,t)$, yields

$$\begin{gathered} f(\rho)\nabla_q^{-\alpha}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] + \frac{\phi(f(\rho))}{f(\rho)}h(\rho)\nabla_q^{-\alpha}f(t) \\ -\tfrac{\phi(f(\rho))}{f(\rho)}f(\rho)\nabla_q^{-\alpha}h(t) - h(\rho)\nabla_q^{-\alpha}\Big[\tfrac{\phi(f(t))}{f(t)}f(t)\Big] \geq 0. \end{gathered} \tag{3.31}$$

With the same arguments as before, we obtain

$$\nabla_q^{-\alpha} f(t)\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] - \nabla_q^{-\alpha} h(t)\nabla_q^{-\alpha}\Big[\frac{\phi(f(t))}{f(t)}f(t)\Big] \geq 0. \tag{3.32}$$

On the other hand, we have

$$\frac{\phi(f(\tau))}{f(\tau)} \leq \frac{\phi(h(\tau))}{h(\tau)}, \tau \in (0,t), t > 0. \tag{3.33}$$

Therefore,

$$\frac{(t-q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(\tau)\frac{\phi(f(\tau))}{f(\tau)} \leq \frac{(t-q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(\tau)\frac{\phi(h(\tau))}{h(\tau)}, \tau \in (0,t), t > 0. \tag{3.34}$$

The inequality (3.34) implies that

$$\nabla_q^{-\alpha}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] \leq \nabla_q^{-\alpha}\Big[\frac{\phi(h(t))}{h(t)}h(t)\Big]. \tag{3.35}$$

Combining (3.32) and (3.35), we obtain (3.28). □

To finish, we present to the reader the following result which generalizes the previous theorem:

Theorem 3.6. *Let f and h be two positive continuous functions on on T_{t_0} and $f \leq h$ on T_{t_0}. If $\frac{f}{h}$ is decreasing and f is increasing on T_{t_0}, then for any convex function $\phi; \phi(0) = 0$, we have*

$$\frac{\nabla_q^{-\alpha}(f(t))\nabla_q^{-\omega}(\phi(h(t))) + \nabla_q^{-\omega}(f(t))\nabla_q^{-\alpha}(\phi(h(t)))}{\nabla_q^{-\alpha}(h(t))\nabla_q^{-\omega}(\phi(f(t))) + \nabla_q^{-\omega}(h(t))\nabla_q^{-\alpha}(\phi(f(t)))} \geq 1, \alpha > 0, \omega > 0, t > 0. \tag{3.36}$$

Proof. The relation (3.31) allows us to obtain

$$\begin{aligned} &\nabla_q^{-\omega} f(t)J^{\alpha}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] + \nabla_q^{-\omega}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big]\nabla_q^{-\alpha} f(t) \\ &-\nabla_q^{-\omega}\Big[\tfrac{\phi(f(t))}{f(t)}f(t)\Big]\nabla_q^{-\alpha} h(t) - \nabla_q^{-\omega} h(t)\nabla_q^{-\alpha}\Big[\tfrac{\phi(f(t))}{f(t)}f(t)\Big] \geq 0. \end{aligned} \tag{3.37}$$

On the other hand, we have:

$$\frac{(t-q\tau)^{(\omega-1)}}{\Gamma_q(\omega)}h(\tau)\frac{\phi(f(\tau))}{f(\tau)} \leq \frac{(t-q\tau)^{(\alpha-1)}}{\Gamma_q(\omega)}h(\tau)\frac{\phi(h(\tau))}{h(\tau)}, \tau \in [0,t], t > 0. \tag{3.38}$$

Integrating both sides of (3.38) with respect to τ over $(0,t)$, yields

$$\nabla_q^{-\omega}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] \leq \nabla_q^{-\omega}\Big[\frac{\phi(h(t))}{h(t)}h(t)\Big]. \tag{3.39}$$

By (3.35), (3.37) and (3.39), we get (3.36). □

Remark 3.3. *Applying Theorem [3.6], for $\alpha = \omega$, we obtain Theorem [3.5].*

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Solutions of fractional difference equations using S-transforms

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Abstract

In the present paper, we define the nabla discrete Sumudu transform (S-transform) and present some of its basic properties. We obtain the nabla discrete Sumudu transform of fractional sums and differences. We apply this transform to solve some fractional difference equations with initial value problems. Finally, using S-transforms, we prove that discrete Mittag-Leffler function is the eigen function of Caputo type fractional difference operator ∇^{α}.

Keywords: Difference equation, fractional difference, Caputo type, initial value problem, Sumudu transform.

2010 MSC: 39A10, 39A99.

1 Introduction

In the literature there are numerous integral transforms that are widely used in physics, astronomy, as well as engineering. In order to solve the differential equations, the integral transforms were extensively used and thus there are several works on the theory and application of integral transforms such as the Laplace, Fourier, Mellin, and Hankel, to name but a few. In the sequence of these transforms in early 90s, Watugala [13] introduced a new integral transform named the Sumudu transform and further applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of the functions

$$A = \{f(t) : \exists M, \tau_1, \tau_2, |f(t)| < Me^{\frac{t}{\tau_j}}, \quad \text{if} \quad t \in (-1)^j \times [0, \infty)\}$$

by the following formula

$$G(u) = S[f(t); u)] = \int_0^{\infty} f(ut)e^{-t}dt, \quad t \in (-\tau_1, \tau_2).$$

The existence and uniqueness and properties of the Sumudu transform and its derivatives were discussed in [2, 3, 4, 5, 8, 9, 11]. Although the Sumudu transform of a function has a deep connection to its Laplace transform, the main advantage of the Sumudu transform is the fact that it may be used to solve problems without resorting to a new frequency domain because it preserves scales and unit properties. By these properties, the Sumudu transform may be used to solve intricate problems in engineering and applied sciences that can hardly be solved when the Laplace transform is used. Moreover, some properties of the Sumudu transform make it more advantageous than the Laplace transform.

Fractional calculus has gained importance during the past three decades due to its applicability in diverse fields of science and engineering, such as, viscoelasticity, diffusion, neurology, control theory, and statistics. The analogous theory for discrete fractional calculus was initiated and properties of the theory of fractional sums and differences were established. Recently, a series of papers continuing this research has appeared in

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which G.V.S.R.Deekshitulu and J.Jagan Mohan discussed some basic inequalities, comparison theorems and qualitative properties of the solutions of fractional difference equations [6, 7].

Now, we introduce some basic definitions and results concerning nabla discrete fractional calculus. Throughout the article, for notations and terminology we refer [1]. Let $u(n):\mathbb{N}_0^+\to\mathbb{R}$ and $m-1<\alpha<m$ where $\alpha\in\mathbb{R}$ and $m\in\mathbb{Z}^+$.

Definition 1.1. *The fractional sum operator of order α is defined as*

$$\nabla^{-\alpha}u(n)=\sum_{j=0}^{n-1}\binom{j+\alpha-1}{j}u(n-j)=\sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}u(j). \tag{1.1}$$

Definition 1.2. *The Caputo type fractional difference operator of order α is defined as*

$$\nabla^{\alpha}u(n) = \nabla^{\alpha-m}[\nabla^m u(n)]=\sum_{j=0}^{n-1}\binom{j-\alpha+m-1}{j}\nabla^m u(n-j) \tag{1.2}$$

$$= \sum_{j=1}^{n}\binom{n-j-\alpha-1}{n-j}u(j)-\sum_{k=0}^{m-1}\binom{n+k-\alpha-1}{n-1}[\nabla^k u(j)]_{j=0}. \tag{1.3}$$

2 S-Transforms and Properties

Now we initiate the study of S-transforms in the present section. Let $u(n),v(n):\mathbb{N}_0^+\to\mathbb{R}$.

Definition 2.1. *The S-transform of $u(n)$ is defined as*

$$S[u(n)]=\frac{1}{z}\sum_{j=1}^{\infty}u(j)(1-\frac{1}{z})^{j-1} \tag{2.1}$$

for each $z\in\mathbb{C}\setminus\{0\}$ for which the series converges.

Definition 2.2. *A function $u(n)$ is of exponential order r, $r>0$ if there exists a constant $A>0$ such that $|u(n)|\le Ar^{-n}$ for sufficiently large n.*

The following lemma discusses the convergence of S-transform.

Lemma 2.1. *Suppose $u(n)$ is of exponential order r, $r>0$. Then $S[u(n)]$ exists for all $z\in\mathbb{C}\setminus\{0\}$ such that $|1-\frac{1}{z}|<r$.*

Now we derive some important properties of S-transforms.

Theorem 2.1. *(Linearity) For any constants a and b,*

$$S[au(n)+bv(n)]=aS[u(n)]+bS[v(n)]. \tag{2.2}$$

The following lemma relates the shifted S-transform to the original.

Lemma 2.2. *(Shifting Theorem) Let $k\in\mathbb{N}_0^+$ and let $u(n)$ and $v(n)$ are of exponential order r, $r>0$. Then for all $z\in\mathbb{C}\setminus\{0\}$ such that $|1-\frac{1}{z}|<r$,*

$$S[u(n-k)]=(1-\frac{1}{z})^k S[u(n)]. \tag{2.3}$$

and

$$S[u(n+k)]=(1-\frac{1}{z})^{-k}\frac{1}{z}\Big[zS[u(n)]-u(1)-(1-\frac{1}{z})^1u(2)-.....-(1-\frac{1}{z})^{k-1}u(k)\Big]. \tag{2.4}$$

Definition 2.3. *The convolution of $u(n)$ and $v(n)$ is defined as*

$$u(n)*v(n)=\sum_{m=1}^{n}u(m)v(n-m+1). \tag{2.5}$$

Lemma 2.3. *(Convolution Theorem) Let $u(n)$ and $v(n)$ are of exponential order r, $r > 0$. Then for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$,*

$$S[u(n) * v(n)] = zS[u(n)]S[v(n)]. \tag{2.6}$$

Proof. Consider

$$\begin{aligned} S[u(n) * v(n)] &= \frac{1}{z}\sum_{j=1}^{\infty}[u(j) * v(j)]\ (1-\frac{1}{z})^{j-1} \\ &= \frac{1}{z}\sum_{j=1}^{\infty}\Big[\sum_{m=1}^{j} u(m)v(j-m+1)\Big]\ (1-\frac{1}{z})^{j-1} \\ &= z\Big[\frac{1}{z}\sum_{m=1}^{\infty} u(m)(1-\frac{1}{z})^{m-1}\Big]\Big[\frac{1}{z}\sum_{j=1}^{\infty} v(j-m+1)(1-\frac{1}{z}z)^{j-m}\Big]. \end{aligned}$$

Take $j - m + 1 = i$ then i varies from 1 to ∞. Then

$$S[u(n) * v(n)] = z\Big[\frac{1}{z}\sum_{m=1}^{\infty} u(m)(1-z)^{m-1}\Big]\Big[\frac{1}{z}\sum_{i=1}^{\infty} v(i)(1-z)^{i-1}\Big] = zS[u(n)]S[v(n)].$$

□

Henry L Gray and Nien fan Zhang [10] defined the following function, which is very useful in solving initial value problems

Definition 2.4. *For any complex numbers α and β, $(\alpha)_\beta$ be defined as follows.*

$$(\alpha)_\beta = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} & \text{when } \alpha \text{ and } \alpha+\beta \text{ are neither zero nor negative integers,} \\ 1 & \text{when } \alpha = \beta = 0, \\ 0 & \text{when } \alpha = 0,\ \beta \text{ is neither zero nor negative integer,} \\ undefined & \text{otherwise.} \end{cases}$$

Remark 2.1. *It is clear from the above definition that*

$$(\alpha)_\beta = \Gamma(\beta+1)\binom{\alpha+\beta-1}{\alpha-1} = \Gamma(\beta+1)\binom{\alpha+\beta-1}{\beta}. \tag{2.7}$$

Lemma 2.4. *Let $a \in \mathbb{R} \setminus \{....., -2, -1\}$ and $n \in \mathbb{N}_0^+$. Then for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$,*

$$S[(n)_a] = \Gamma(a+1)z^a. \tag{2.8}$$

Proof. Consider

$$\begin{aligned} S[(n)_a] = S\Big[\frac{\Gamma(n+a)}{\Gamma(n)}\Big] &= \frac{1}{z}\sum_{j=1}^{\infty}\frac{\Gamma(j+a)}{\Gamma(j)}(1-\frac{1}{z})^{j-1} \\ &= \frac{\Gamma(a+1)}{z}\sum_{j=1}^{\infty}\binom{j+a-1}{j-1}(1-\frac{1}{z})^{j-1} \\ &= \frac{\Gamma(a+1)}{z}\Big[1 + (1+a) + \frac{(1+a)(2+a)}{2!}(1-\frac{1}{z})^1 + ...\Big] \\ &= \frac{\Gamma(a+1)}{z}[1-(1-\frac{1}{z})]^{-a-1} = \Gamma(a+1)z^a. \end{aligned}$$

□

Remark 2.2. *From the above lemma, we get*

$$S\Big[\binom{n+a-1}{n-1}\Big] = z^a. \tag{2.9}$$

Lemma 2.5. *Suppose $u(n)$ is of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$. Then for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$,*

$$S\Big[\nabla^{-\alpha} u(n)\Big] = z^{\alpha} S[u(n)]. \tag{2.10}$$

Proof. Consider

$$\begin{aligned} S\Big[\nabla^{-\alpha} u(n)\Big] = S\Big[\sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u(j)\Big] &= \frac{1}{\Gamma(\alpha)} S\Big[\sum_{j=1}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} u(j)\Big] \\ &= \frac{1}{\Gamma(\alpha)} S\Big[\sum_{j=1}^{n} (n-j+1)_{\alpha-1} u(j)\Big] \\ &= \frac{1}{\Gamma(\alpha)} S\Big[u(n) * (n)_{\alpha-1}\Big] \\ &= \frac{z}{\Gamma(\alpha)} S[u(n)] S\Big[(n)_{\alpha-1}\Big] = z^{\alpha} S[u(n)]. \end{aligned}$$

□

Lemma 2.6. *Suppose $u(n)$ is of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}^+$ such that $m - 1 < \alpha < m$. Then for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$,*

$$S\Big[\nabla^{\alpha} u(n)\Big] = z^{-\alpha} \Big[S[u(n)] - \sum_{k=0}^{m-1} z^k [\nabla^k u(j)]_{j=0}\Big]. \tag{2.11}$$

Proof. Consider

$$S\Big[\nabla^{\alpha} u(n)\Big] = S\Big[\sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0}\Big] = S_1 + S_2$$

where

$$S_1 = S\Big[\sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j)\Big] = z^{-\alpha} S[u(n)]$$

and

$$S_2 = S\Big[\sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0}\Big] = \sum_{k=0}^{m-1} S\Big[\binom{n+k-\alpha-1}{n-1}\Big] [\nabla^k u(j)]_{j=0}.$$

Now we consider

$$S\Big[\binom{n+k-\alpha-1}{n-1}\Big] = \frac{1}{\Gamma(k-\alpha+1)} S\Big[(n)_{k-\alpha}\Big] = z^{k-\alpha}.$$

Thus

$$S\Big[\nabla^{\alpha} u(n)\Big] = z^{-\alpha} \Big[S[u(n)] - \sum_{k=0}^{m-1} z^k [\nabla^k u(j)]_{j=0}\Big].$$

□

3 Solutions of fractional difference equations using S-transforms

In this section, we will illustrate the possible use of the S-transform by applying it to solve some fractional order initial value problems.

In 2003, Atsushi Nagai [12] defined the discrete Mittag-Leffler function

$$F_{\alpha}(a, n) = \sum_{j=0}^{\infty} \Big[a^j \binom{n+j\alpha-1}{n-j}\Big] \tag{3.1}$$

which is a generalization of nabla exponential function on the time scale of integers. He also proved that $F_{\alpha}(a, n)$ is an eigen function of Caputo type fractional difference operator defined in (1.3), that is,

$$\nabla^{\alpha} F_{\alpha}(a, n) = a F_{\alpha}(a, n). \tag{3.2}$$

Now we prove the same using S-transforms.

Example 3.1. *Let $u(n)$ is of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Then the solution of*

$$\nabla^\alpha u(n) = au(n), \tag{3.3}$$
$$u(0) = a_0 \tag{3.4}$$

is $F_\alpha(a,n)$, where a and a_0 are constants.

Solution: Taking S-transforms on both sides of (3.3), we have

$$\begin{aligned} S[\nabla^\alpha u(n)] &= aS[u(n)] \\ \text{or } z^{-\alpha}\Big[S[u(n)] - u(0)\Big] &= aS[u(n)] \\ \text{or } S[u(n)] &= a_0\Big[\frac{z^{-\alpha}}{z^{-\alpha} - a}\Big] \\ \text{or } S[u(n)] &= a_0\Big[1 + az^\alpha + a^2z^{2\alpha} + \ldots\ldots\ldots\ldots\Big]. \end{aligned}$$

Applying inverse S-transforms on both sides, we get

$$\begin{aligned} u(n) &= a_0 S^{-1}\Big[1 + az^\alpha + a^2z^{2\alpha} + \ldots\ldots\ldots\ldots\Big] \\ &= a_0\Big[S^{-1}\Big(1\Big) + aS^{-1}\Big(z^\alpha\Big) + a^2S^{-1}\Big(z^{2\alpha}\Big) + \ldots\ldots\ldots\ldots\Big] \\ &= a_0\Big[1 + a\binom{n+\alpha-1}{n-1} + a^2\binom{n+2\alpha-1}{n-2} + \ldots\ldots\ldots\ldots\Big] \\ \text{or } u(n) &= a_0\sum_{j=0}^{\infty}\Big[a^j\binom{n+j\alpha-1}{n-j}\Big] = a_0F_\alpha(a,n). \end{aligned}$$

Thus the solution of (3.3) is the discrete Mittag-Leffler function defined in (3.1).

Remark 3.3. *It is clear from the above example that*

$$S\Big[F_\alpha(a,n)\Big] = \frac{z^{-\alpha}}{z^{-\alpha} - a}. \tag{3.5}$$

Example 3.2. *Let $u(n)$ and $v(n)$ are of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Find the solution of*

$$\nabla^\alpha u(n) = av(n), \tag{3.6}$$
$$u(0) = a_0 \tag{3.7}$$

where a and a_0 are constants.

Solution: Taking S-transforms on both sides of (3.6), we have

$$\begin{aligned} z^{-\alpha}\Big[S[u(n)] - u(0)\Big] &= aS[v(n)] \\ \text{or } S[u(n)] &= a_0 + a\Big[S[v(n)] \times z^\alpha\Big]. \end{aligned}$$

Applying inverse S-transforms on both sides and applying convolution theorem, we get

$$\begin{aligned} u(n) &= a_0 + S^{-1}\Big[z \times S[v(n)] \times z^{\alpha-1}\Big] \\ &= a_0 + \Big[v(n) * \binom{n+\alpha-2}{n-1}\Big] \\ \text{or } u(n) &= a_0 + \sum_{j=1}^{n}\Big[v(j)\binom{n-j+\alpha-1}{n-j}\Big]. \end{aligned}$$

Example 3.3. *Let $u(n)$ and $v(n)$ are of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Find the solution of*

$$\nabla^\alpha u(n) = au(n) + bv(n), \tag{3.8}$$
$$u(0) = a_0 \tag{3.9}$$

where a, b and a_0 are constants.

Solution: Taking S-transforms on both sides of (3.8), we have

$$\begin{aligned} z^{-\alpha}\Big[S[u(n)] - u(0)\Big] &= aS[u(n)] + bS[v(n)] \\ or\ S[u(n)] &= a_0\Big[\frac{z^{-\alpha}}{z^{-\alpha} - a}\Big] + b\Big[S[v(n)] \times z^\alpha\Big]. \end{aligned}$$

Applying inverse S-transforms on both sides and applying convolution theorem, we get

$$\begin{aligned} u(n) &= a_0 S^{-1}\Big[\frac{z^{-\alpha}}{z^{-\alpha} - a}\Big] + bS^{-1}\Big[z \times S[v(n)] \times z^{\alpha-1}\Big] \\ &= a_0 F_\alpha(a, n) + b\sum_{j=1}^{n}\Big[v(j)\binom{n-j+\alpha-1}{n-j}\Big]. \end{aligned}$$

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Chaos and bifurcation of the Logistic discontinuous dynamical systems with piecewise constant arguments

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Abstract

In this paper we are concerned with the definition and some properties of the discontinuous dynamical systems generated by piecewise constant arguments. Then we study two discontinuous dynamical system of the Logistic equation as an example. The local stability at the fixed points is studied. The bifurcation analysis and chaos are discussed. In addition, we compare our results with the discrete dynamical systems of the Logistic equation.

Keywords: Discontinuous dynamical systems, piecewise constant argument, Logistic equation, fixed points, bifurcation, chaos.

2010 MSC: 39B05, 37N30, 37N20.

1 Introduction

The discontinuous dynamical systems generated by the retarded functional equations have been defined in [1]-[4]. The dynamical systems with piecewise constant arguments have been studied in [5]-[8] and the references therein. In this work we define the discontinuous dynamical systems generated by functional equations with piecewise constant arguments. The dynamic properties of two discontinuous dynamical systems of the Logistic equation will be discussed. Comparison with the corresponding discrete dynamical systems of the Logistic equation

$$x_n = \rho x_{n-1}(1 - x_{n-1}), \quad n = 1, 2, 3, ...,$$

and

$$x_{n+1} = \rho x_n(1 - x_{n-1}), \quad n = 1, 2, 3, ...,$$

will be given.

1.1 Piecewise constant arguments

Consider the problem of functional equation with piecewise constant arguments

$$x(t) = f(x(r[\frac{t}{r}])), \quad t > 0, r > 0. \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where [.] denotes the greatest integer function.
Let $n = 1, 2, 3, ...$ and $t \in [nr, (n + 1)r)$, then

$$x(t) = f(x(nr)), \quad t \in [nr, (n + 1)r).$$

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Let $r = 1$ and take the limit as $t \to n+1$, we get

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, ...$$

This shows that the discrete dynamical system

$$x_n = f(n, x_{n-1}), \quad n = 1, 2, 3, ..., T.$$
$$x(0) = x_o,$$

is a special case of the problem of functional equation with piecewise constant arguments (1.1)-(1.2). Now let $t \in [0, r)$, then $\frac{t}{r} \in [0, 1)$, $x(r[\frac{t}{r}]) = x(0)$ and the solution of (1.1)-(1.2) is given by

$$x(t) = x_1(r) = f(x(0)), \quad t \in [0, r),$$

with

$$x_1(r) = \lim_{t \to r^-} x(t) = f(x(0)).$$

For $t \in [r, 2r)$, then $\frac{t}{r} \in [1, 2)$, $x(r[\frac{t}{r}]) = x(r)$ and the solution of (1.1)-(1.2) is given by

$$x(t) = x_2(t) = f(x_1(r)), \quad t \in [r, 2r).$$

Repeating the process we can easily deduce that the solution of (1.1)-(1.2) is given by

$$x(t) = x_{(n+1)}(t) = f(x_n(nr)), \quad t \in [nr, (n+1)r),$$

which is continuous on each subinterval $[k, (k+1))$, $k = 1, 2, 3, ..., n$, but

$$\lim_{t \to kr^+} x_{(k+1)}(t) = f(x_k(kr)) \neq x_k(kr).$$

Hence the problem (1.1)-(1.2) is piecewise continuous which we call it "discontinuous" and we have proved the following theorem

Theorem 1.1. *The solution of the problem of functional equation with piecewise constant arguments (1.1)-(1.2) is discontinuous (sectionally continuous) even if the function f is continuous.*

Now let $f : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}$ and $r \in \mathbb{R}^+$. Then, the following definition can be given

Definition 1.1. *The discontinuous dynamical system generated by piecewise constant arguments is the problem*

$$x(t) = f(t, x(r[\frac{t}{r}]), x(r[\frac{t-1}{r}]), ..., x(r[\frac{t-n}{r}])), \quad t \in [0, T], \tag{1.3}$$

$$x(t) = x_0, \quad t \leq 0. \tag{1.4}$$

Definition 1.2. *The fixed points of the discontinuous dynamical system (1.3) and (1.4) are the solution of the equation*

$$x(t) = f(t, x, x, ..., x).$$

2 Main problems

Consider the discontinuous dynamical systems generated by piecewise constant arguments of Logistic equation

$$x(t) = \rho x(r[\frac{t}{r}])(1 - x(r[\frac{t}{r}])), \quad t, r > 0, \text{ and } x(0) = x_0. \tag{2.1}$$

and

$$x(t) = \rho x(r[\frac{t}{r}])(1 - x(r[\frac{t-r}{r}])), \quad t, r > 0, \text{ and } x(0) = x_0. \tag{2.2}$$

Here we study the stability at the fixed points. In order to study bifurcation and chaos we take firstly $r = 1$ and we compare the results with the results of the discrete dynamical systems of Logistic difference equation

$$x_{n+1} = \rho x_n(1 - x_n), \quad n = 1, 2, 3, ..., \text{ and } x_0 = x_o. \tag{2.3}$$

and

$$x_{n+1} = \rho x_n(1 - x_{n-1}) \quad n = 1, 2, 3, ..., \text{ and } x_0 = x_o. \tag{2.4}$$

Secondly, we take some other values of r and T and study some examples.

2.1 Fixed points and stability

As in the case of discrete dynamical systems, the fixed points of the dynamical systems (2.1) and (2.2) are the solution to the equation $f(x) = x$. Thus there are two fixed points which are

$$(x_{fixed})_1 = 0,$$

$$(x_{fixed})_2 = 1 - \frac{1}{\rho}.$$

To study the stability of these fixed points we take into account the following theorem.

Theorem 2.1. *[9] Let f be a smooth map on $\mathbb{R}$, and assume that x_0 is a fixed point of f.*
1. If $|f^{'}(x_0)| < 1$, then x_0 is stable.
2. If $|f^{'}(x_0)| > 1$, then x_0 is unstable.

Now since in our case $f(x) = \rho x(1-x)$, the first fixed point $(x_{fixed})_1 = 0$ is stable if

$$|\rho| < 1,$$

that is, $-1 < \rho < 1$.
The second fixed point $(x_{fixed})_2 = 1 - \frac{1}{\rho}$ is stable if

$$|2 - \rho| < 1,$$

that is, $1 < \rho < 3$.
Figures (1) and (2) show the trajectories of (2.1) and (2.2) when $r = 1$ respectively, while Figures (3) and (4) show the trajectories of (2.3) and (2.4), respectively.

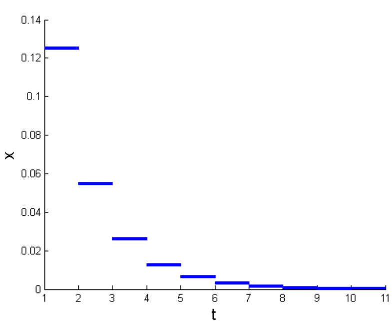

Figure 1: Trajectories of (2.1), r=1.

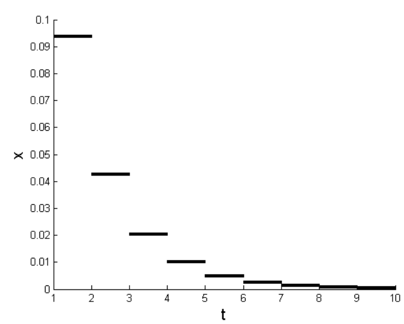

Figure 2: Trajectories of (2.2), r=1.

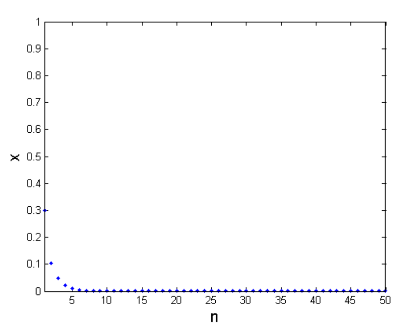

Figure 3: Trajectories of (2.3).

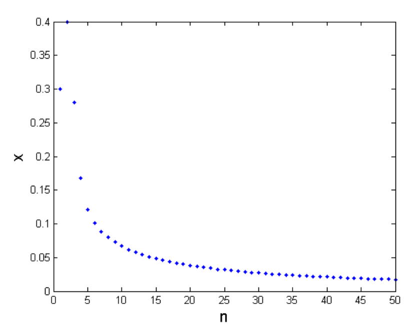

Figure 4: Trajectories of (2.4).

3 Bifurcation and Chaos

In this section, the numerical experiments show that the dynamical behaviors of the discontinuous dynamical systems (2.1) and (2.2) change when we change both r and T as follows

1. Take $r = 1$ and $t \in [0, 30]$, in this case the dynamical behaviors of the two dynamical systems (2.1 and (2.3) are identical (Figure 5).

2. Take $r = 1$ and $t \in [0, 30]$, in this case the dynamical behaviors of the two dynamical systems (2.2) and (2.4) are identical (Figure 6).

3. Take $r = 0.25$ and $t \in [0, 2]$ in the dynamical system (2.1) (Figure 7).

4. Take $r = 0.5$ and $t \in [0, 2]$ in the dynamical system (2.1) (Figure 8).

5. Take $r = 0.25$ and $t \in [0, 3]$ in the dynamical system (2.2) (Figure 9).

6. Take $r = 0.5$ and $t \in [0, 3]$ in the dynamical system (2.2) (Figure 10).

7. Take $r = 0.25$ and $T = N = 13$ in the dynamical system (2.1) (Figure 11).

8. Take $r = 0.5$ and $T = N = 35$ in the dynamical system (2.1) (Figure 12).

9. Take $r = 0.25$ and $T = N = 13$ in the dynamical system (2.2) (Figure 13).

10. Take $r = 0.5$ and $T = N = 35$ in the dynamical system (2.2) (Figure 14).

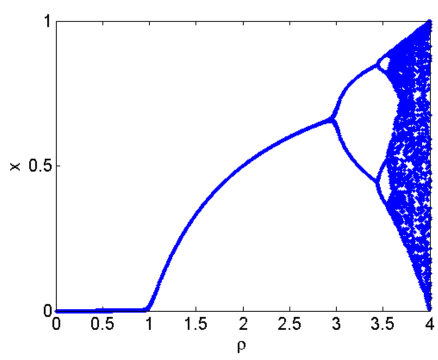

Figure 5: Bifurcation diagram of the dynamical systems (2.1) with $r = 1$ and (2.3) where $N = T = 70$.

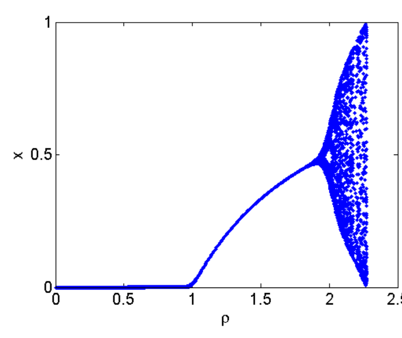

Figure 6: Bifurcation diagram of the dynamical systems (2.2) with $r = 1$ and (2.4) where $N = T = 70$.

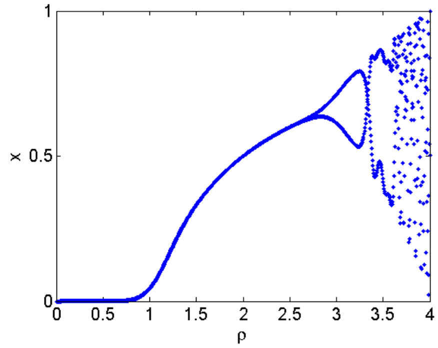

Figure 7: Bifurcation diagram for (2.1), $r = 0.25$, $t = [0, 3]$.

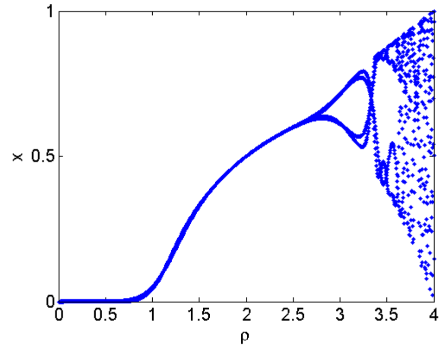

Figure 8: Bifurcation diagram for (2.1), $r = 0.5$, $t = [0, 3]$.

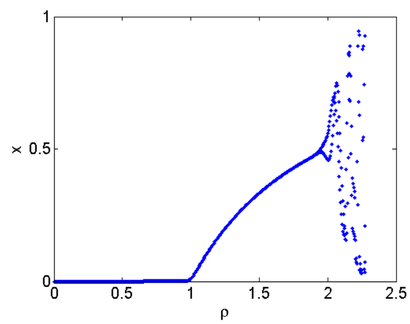

Figure 9: Bifurcation diagram for (2.2), $r = 0.25$, $t = [0, 3]$.

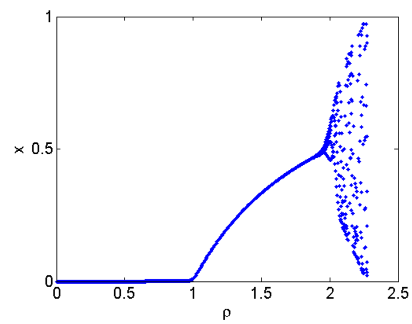

Figure 10: Bifurcation diagram for (2.2), $r = 0.5$, $t = [0, 3]$.

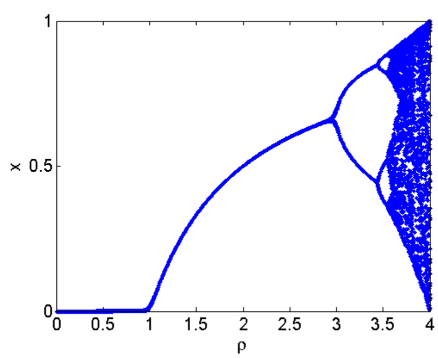

Figure 11: Bifurcation diagram for (2.1), $r = 0.25$, $T = N = 13$.

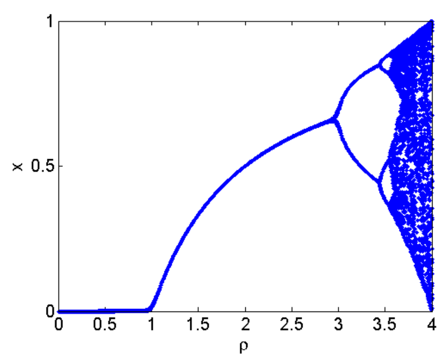

Figure 12: Bifurcation diagram for (2.1), $r = 0.5$, $T = N = 35$.

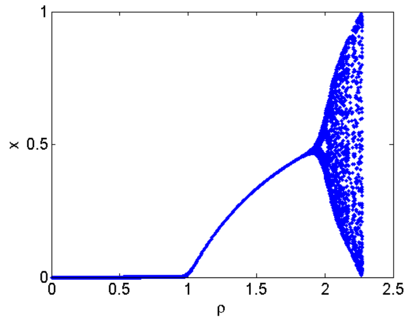

Figure 13: Bifurcation diagram for (2.2), $r = 0.25$, $T = N = 13$.

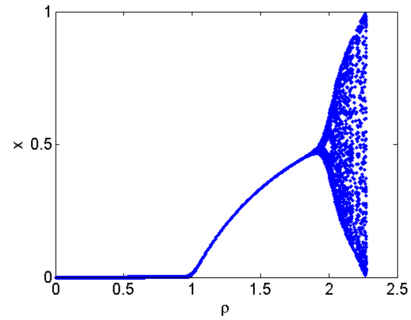

Figure 14: Bifurcation diagram for (2.2), $r = 0.5$, $T = N = 35$.

4 Conclusion

The discontinuous dynamical system models generated by piecewise constant arguments have the same behavior as its discrete version when $r = 1$.
On the other hand, changing the parameter r together with the time $t \in [0, T]$ affects the chaotic behavior of the dynamical system generated by the piecewise constant arguments model as it is shown clearly in the above figures.

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On some fractional q-Integral inequalities

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Abstract In this paper, using the Riemann-Liouville fractional q-integral, we establish some new results of the Gruss and Chebyshev q-integral inequalities.

Keywords: Integral inequalities, Riemann-Liouville fractional integral, Gruss inequality, Chebyshev inequality.

2010 MSC: 26D15, 26A33.

1 Introduction

Let us consider the functional (see[2]):

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \quad (1.1)$$

where f and g are two integrable functions on $[a,b]$.
In [11], Gruss proved the well known inequality:

$$|T(f,g)| \leq \frac{(\Phi-\varphi)(\Psi-\psi)}{4}, \quad (1.2)$$

where f and g are two integrable functions on $[a,b]$ satisfying the conditions

$$\varphi \leq f(x) \leq \Phi, \qquad \psi \leq g(x) \leq \Psi, \qquad \varphi,\Psi,\Phi,\psi \in \mathbb{R}, x \in [a,b]. \quad (1.3)$$

In the case of $f^{'}, g^{'} \in L_\infty(a,b)$, S. S. Dragomir (see[6]) proved that

$$|S(f,p,g)| \leq \|f^{'}\|_\infty \|g^{'}\|_\infty \left[\int_a^b p(x)dx \int_a^b x^2 p(x)dx - \left(\int_a^b xp(x)dx\right)^2\right], \quad (1.4)$$

where

$$S(p,f,g) := \frac{1}{2}T(f,g,p,q) = \int_a^b p(x)\int_a^b p(x)f(x)g(x)dx - \left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right). \quad (1.5)$$

If f is M-g-Lipschitzian on $[a,b]$: i.e.

$$|f(x)-f(y)| \leq M|g(x)-g(y)|; M>0, x,y \in [a,b], \quad (1.6)$$

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Dragomir [6] proved that

$$|S(f,p,g)| \leq M\left[\int_a^b p(x)dx\int_a^b p(x)g^2(x)dx - \left(\int_a^b g(x)p(x)dx\right)^2\right], \tag{1.7}$$

and if f is an L_1- lipschitzian function on $[a,b]$ and g is an L_2-lipschitzian function on $[a,b]$, the author proved that [6]

$$|S(p,f,g)| \leq L_1L_2\left(\int_a^b p(x)\int_a^b x^2p(x)dx - \left(\int_a^b xp(x)\right)^2\right). \tag{1.8}$$

Using the Riemann-Liouville fractional integral, many authors have studied the fractional integral inequalities and their applications(see[1, 3, 4, 5, 6]).
In [5], Dahmani et al. gave the following fractional integral inequalities, using the Riemann-Liouville fractional integral :
let f and g be two integrable functions on $[0,\infty[$ and p,q two positive functions, then for all $t>0, \alpha>0$,

$$\begin{aligned}|J^\alpha q(t)J^\alpha pfg(t) + J^\alpha p(t)J^\alpha qfg(t) - J^\alpha pf(t)J^\alpha qg(t) - J^\alpha qf(t)J^\alpha pg(t)|\\ \leq J^\alpha p(t)J^\alpha q(t)(\Phi-\varphi)(\Psi-\psi)\end{aligned}$$

Morever, if f and g are two lipschitzian functions on $[0,\infty[$, we have

$$\begin{aligned}|J^\alpha q(t)J^\alpha pfg(t) + J^\alpha p(t)J^\alpha qfg(t) - J^\alpha qf(t)J^\alpha pg(t) - J^\alpha pf(t)J^\alpha qg(t)|\\ \leq L_1L_2(J^\alpha q(t)J^\alpha t^2p(t) + J^\alpha p(t)J^\alpha t^2q(t) - J^\alpha tq(t)J^\alpha tp(t)).\end{aligned}$$

In [4], Dahmani established a new class of inequalities for the extended Chebyshev functional as follows:
let f and g two differentiable functions on $[0,\infty[$ and p,q two positive functions. If $f^{'}, g^{'} \in L_\infty([0,\infty[)$, then

$$\begin{aligned}&|J^\beta q(t)J^\alpha pfg(t) + J^\alpha p(t)J^\beta qfg(t) - J^\alpha pf(t)J^\beta qg(t) - J^\beta qf(t)J^\alpha pg(t)|\\ &\leq \|f^{'}\|_\infty\|g^{'}\|_\infty(J^\alpha p(t)J^\beta t^2q(t) + J^\beta q(t)J^\alpha t^2p(t) - 2(J^\alpha tp(t))(J^\beta tq(t)),\end{aligned}$$

for all $t>0, \alpha>0$, and $\beta>0$.
The main aim of this paper is to establish some generalization of these inequalities using q-fractional integrals.

2 Basic Definitions

Throughout this paper, we will fix $q \in (0,1)$. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [7] and [9] and [12]). We write for $a,b \in \mathbb{C}$,

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad (a-b)^{(\alpha)} = a^\alpha\frac{(\frac{b}{a};q)_\infty}{(q^\alpha\frac{b}{a};q)_\infty}.$$

The q-Jackson integral from 0 to a is defined by (see [8])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{2.1}$$

provided the sum converges absolutely.
The q-Jackson integral in a generic interval $[a,b]$ is given by (see [8])

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.2}$$

The fractional q-integral of the Riemann-Liouville type is (see [12])

$$\left(J_{q,a}^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)}\int_a^x (x-qt)^{(\alpha-1)}f(t)d_qt; \quad \alpha>0 \tag{2.3}$$

where

$$\Gamma_q(\alpha) = \frac{1}{1-q}\int_0^1 \left(\frac{u}{1-q}\right)^{\alpha-1} e_q(qu)d_qu, \qquad and \quad e_q(t) = \prod_{k=0}^{\infty}(1-q^k t).$$

The q-fractional integration has the following semi-group property

$$\left(J^{\beta}_{q,a}J^{\alpha}_{q,a}f\right)(x) = \left(J^{\alpha+\beta}_{q,a}f\right)(x); \quad \alpha > 0, \beta > 0. \tag{2.4}$$

Finally, for $b > 0$ and $a = bq^n$, $n = 1, 2, \dots, \infty$, we write

$$[a, \quad b]_q = \{bq^k : \quad 0 \le k \le n\}.$$

3 Main results

Theorem 3.1. *Let f and g be two functions defined on $[a, b]_q$ satisfying the condition (1.3) and let v, w be two positive functions on $[a, b]_q$.*
Then

$$\begin{aligned}|J^{\alpha}_{q,a}w(b)J^{\alpha}_{q,a}vfg(b) + J^{\alpha}_{q,a}v(b)J^{\alpha}_{q,a}wfg(b) - J^{\alpha}_{q,a}vf(b)J^{\alpha}_{q,a}wg(b) - J^{\alpha}_{q,a}wf(b)J^{\alpha}_{q,a}vg(b)|\\ \le J^{\alpha}_{q,a}v(b)J^{\alpha}_{q,a}w(b)(\Phi-\varphi)(\Psi-\psi).\end{aligned} \tag{3.1}$$

Proof. From the condition (1.3), we have

$$|f(\tau) - f(\rho)| \le \Phi - \varphi, \quad |g(\tau) - g(\rho)| \le \Psi - \psi, \quad \tau, \rho \in [a, b]_q, \tag{3.2}$$

which implies that

$$|\left(f(\tau) - f(\rho)\right)\left(g(\tau) - g(\rho)\right)| \le (\Phi - \varphi)(\Psi - \psi). \tag{3.3}$$

Define

$$H(\tau, \rho) = f(\tau)g(\tau) + f(\rho)g(\rho) - f(\tau)g(\rho) - f(\rho)g(\tau), \quad \tau, \rho \in [a, b]_q. \tag{3.4}$$

Multiplying (3.4) by $\frac{(b-q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)}v(\tau)$ and integrating with respect to τ from a to b, we get

$$\begin{aligned}&\frac{1}{\Gamma_q(\alpha)}\int_a^b (b-q\tau)^{(\alpha-1)}v(\tau)H(\tau,\rho)d_q\tau\\ &= J^{\alpha}_{q,a}vfg(b) + f(\rho)g(\rho)J^{\alpha}_{q,a}v(b) - g(\rho)J^{\alpha}_{q,a}vf(b) - f(\rho)J^{\alpha}_{q,a}vg(b).\end{aligned} \tag{3.5}$$

Now, multiplying (3.5) by $\frac{(b-q\rho)^{(\alpha-1)}}{(\Gamma_q(\alpha))}w(\rho)$ and integrating with respect to ρ from a to b, we can state that

$$\begin{aligned}&\frac{1}{(\Gamma_q(\alpha))^2}\int_a^b\int_a^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\alpha-1)}v(\tau)w(\rho)H(\tau,\rho)d_q\tau d_q\rho\\ &= J^{\alpha}_{q,a}w(b)J^{\alpha}_{q,a}vfg(b) + J^{\alpha}_{q,a}v(b)J^{\alpha}_{q,a}wfg(b) - J^{\alpha}_{q,a}vf(b)J^{\alpha}_{q,a}wg(b) - J^{\alpha}_{q,a}wf(b)J^{\alpha}_{q,a}vg(b).\end{aligned} \tag{3.6}$$

Using (3.3), we can estimate (3.6) as follows

$$\begin{aligned}&|\frac{1}{(\Gamma_q(\alpha))^2}\int_a^b\int_a^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\alpha-1)}v(\tau)w(\rho)H(\tau,\rho)d_q\tau d_q\rho|\\ \le\ &\frac{(\Phi-\varphi)(\Psi-\psi)}{(\Gamma_q(\alpha))^2}\int_a^b\int_a^b (b-q\tau)^{\alpha-1}(b-q\rho)^{\alpha-1}v(\tau)w(\rho)d_q\tau d_q\rho.\end{aligned} \tag{3.7}$$

Consequently,

$$\begin{aligned}&|\frac{1}{(\Gamma_q(\alpha))^2}\int_a^b\int_a^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\alpha-1)}v(\tau)w(\rho)H(\tau,\rho)d_q\tau d_q\rho|\\ \le\ &J^{\alpha}_{q,a}v(b)J^{\alpha}_{q,a}w(b)(\Phi-\varphi)(\Psi-\psi).\end{aligned}$$

Theorem (3.1) is thus proved. □

Theorem 3.2. *Let f and g be two functions defined on $[a,b]_q$ satisfying the condition (1.3) and let v,w be two positive functions on $[a,b]_q$.*
Then

$$|J_{q,a}^{\beta}w(b)J_{q,a}^{\alpha}vfg(b)+J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}wfg(b)-J_{q,a}^{\alpha}vf(b)J_{q,a}^{\beta}wg(b)-J_{q,a}^{\beta}wf(b)J_{q,a}^{\alpha}vg(b)| \\ \leq J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}w(b)(\Phi-\varphi)(\Psi-\psi). \quad (3.8)$$

Proof. Multiplying (3.5) by $\frac{(b-q\rho)^{(\beta-1)}}{\Gamma_q(\beta)}w(\rho)$ and integrating with respect to ρ from a to b, we get

$$\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_a^b\int_a^b(b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\beta-1)}v(\tau)w(\rho)H(\tau,\rho)d_q\tau d_q\rho \\ =J_{q,a}^{\beta}w(b)J_{q,a}^{\alpha}vfg(b)+J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}wfg(b)-J_{q,a}^{\alpha}vf(b)J_{q,a}^{\beta}wg(b)-J_{q,a}^{\beta}wf(b)J_{q,a}^{\alpha}vg(b). \quad (3.9)$$

On the other hand

$$\frac{(\Phi-\varphi)(\Psi-\psi)}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_a^b\int_a^b(b-q\tau)^{(\alpha-1)}(b-q\rho)^{\beta-1}v(\tau)w(\rho)d_q\tau d_q\rho \\ =J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}w(b)(\Phi-\varphi)(\Psi-\psi). \quad (3.10)$$

Hence

$$|\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_a^b\int_a^b(b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\beta-1)}v(\tau)w(\rho)H(\tau,\rho)d_q\tau d_q\rho| \\ \leq J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}w(b)(\Phi-\varphi)(\Psi-\psi). \quad (3.11)$$

This ends the proof. □

Remark 3.1. *Applying Theorem (3.2) for $\alpha=\beta$, we obtain Theorem (3.1).*

Theorem 3.3. *Let f and g be two functions defined on $[a,b]_q$ satisfying the condition (1.6) and let v,w be two positive functions on $[a,b]_q$. Then the inequality*

$$|J_{q,a}^{\beta}w(b)J_{q,a}^{\alpha}vfg(b)+J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}wfg(b)-J_{q,a}^{\alpha}vf(b)J_{q,a}^{\beta}wg(b)-J_{q,a}^{\beta}wf(b)J_{q,a}^{\alpha}vg(b)| \\ \leq M[J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}wg^2(b)+J_{q,a}^{\beta}w(b)J_{q,a}^{\alpha}vg^2(b)-2J_{q,a}^{\alpha}vg(b)J_{q,a}^{\beta}wg(b)] \quad (3.12)$$

is valid.

Proof. Multiplying (3.4) by $\frac{(b-q\tau)^{(\alpha-1)}v(\tau)}{\Gamma_q(\alpha)}$ and integrating the resulting identity with respect to τ from a to b, we obtain

$$\frac{1}{\Gamma_q(\alpha)}\int_a^b(b-q\tau)^{(\alpha-1)}v(\tau)H(\tau,\rho)d_q\tau \\ =J_{q,a}^{\alpha}vfg(b)-f(\rho)J_{q,a}^{\alpha}vg(b)-g(\rho)J_{q,a}^{\alpha}vf(b)+f(\rho)g(\rho)J_{q,a}^{\alpha}v(b). \quad (3.13)$$

Multiplying (3.13) by $\frac{(b-q\rho)^{(\beta-1)}w(\rho)}{\Gamma_q(\beta)}$ and integrating the resulting identity with respect to ρ from a to b, we get

$$\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_a^b\int_a^b(b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\beta-1)}v(\tau)w(\rho)H(\tau,\rho)d_q\tau d_q\rho \\ =J_{q,a}^{\beta}w(b)J_{q,a}^{\alpha}vfg(b)-J_{q,a}^{\beta}wf(b)J_{q,a}^{\alpha}vg(b)-J_{q,a}^{\alpha}vf(b)J_{q,a}^{\beta}wg(b)+J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}wfg(b). \quad (3.14)$$

On the other hand, we have

$$|f(\tau)-f(\rho)|\leq M|g(\tau)-g(\rho)|. \quad (3.15)$$

This implies that

$$|H(\tau,\rho)|\leq M(g(\tau)-g(\rho))^2, \quad \tau,\rho\in[a,b]_q. \quad (3.16)$$

Hence, it follows that

$$\frac{1}{\Gamma_q(\alpha)}\int_a^b (b-q\tau)^{(\alpha-1)}v(\tau)|H(\tau,\rho)|d_q\tau \\ \leq M\left(J_{q,a}^{\alpha}vg^2(b)-2g(\rho)J_{q,a}^{\alpha}vg(b)+g^2(\rho)J_{q,a}^{\alpha}v(b)\right). \quad (3.17)$$

Consequently,

$$\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_a^b\int_a^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\beta-1)}v(\tau)w(\rho)|H(\tau,\rho)|d_q\tau d_q\rho \\ \leq \frac{M}{\Gamma_q(\beta)}\int_a^b \left((b-q\rho)^{\beta-1}w(\rho)\left[J_{q,a}^{\alpha}vg^2(b)-2g(\rho)J_{q,a}^{\alpha}vg(b)+g^2(\rho)J_{q,a}^{\alpha}v(b)\right]\right)d_q\rho. \quad (3.18)$$

So,

$$\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_a^b\int_a^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\beta-1)}v(\tau)w(\rho)|H(\tau,\rho)|d_q\tau d_q\rho \\ \leq M[J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}wg^2(b)+J_{q,a}^{\beta}w(b)J_{q,a}^{\alpha}vg^2(b)-2J_{q,a}^{\alpha}vg(b)J_{q,a}^{\beta}wg(b)]. \quad (3.19)$$

Theorem (3.3) is thus proved. □

In the particular case $\beta=\alpha$, we have the following result.

Corollary 3.1. *Under the assumptions of Theorem 3.3, we have*

$$|J_{q,a}^{\alpha}w(b)J_{q,a}^{\alpha}vfg(b)+J_{q,a}^{\alpha}v(b)J_{q,a}^{\alpha}wfg(b)-J_{q,a}^{\alpha}vf(b)J_{q,a}^{\alpha}wg(b)-J_{q,a}^{\alpha}wf(b)J_{q,a}^{\alpha}vg(b)| \leq \\ M[J_{q,a}^{\alpha}v(b)J_{q,a}^{\alpha}wg^2(b)+J_{q,a}^{\alpha}w(b)J_{q,a}^{\alpha}vg^2(b)-2J_{q,a}^{\alpha}vg(b)J_{q,a}^{\alpha}wg(b)]. \quad (3.20)$$

Theorem 3.4. *Let f and g be two lipschitzian functions on $[a,b]_q$ with the constants L_1 and L_2 and let v,w be two positive functions on $[a,b]_q$. Then, the inequality*

$$|J_{q,a}^{\beta}w(b)J_{q,a}^{\alpha}vfg(b)+J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}wfg(b)-J_{q,a}^{\alpha}vf(b)J_{q,a}^{\beta}wg(b)-J_{q,a}^{\alpha}vf(b)J_{q,a}^{\beta}wg(b)| \\ \leq L_1L_2(J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}(\tau^2w)(b)+J_{q,a}^{\beta}w(b)J_{q,a}^{\alpha}(\tau^2v)(b)-2J_{q,a}^{\alpha}(\tau v)(b)J_{q,a}^{\beta}(\tau w)(b))$$

is valid.

Proof. For all $\tau,\rho\in[a,b]_q$, we have

$$|f(\tau)-f(\rho)|\leq L_1|\tau-\rho|, \quad |g(\tau)-g(\rho)|\leq L_2|\tau-\rho|. \quad (3.21)$$

Hence

$$|H(\tau,\rho)|\leq L_1L_2(\tau-\rho)^2. \quad (3.22)$$

Setting

$$R(\tau,\rho):=L_1L_2(\tau-\rho)^2, \quad (3.23)$$

then, multiplying (3.23) by $\frac{(b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\beta-1)}}{\Gamma_q(\alpha)\Gamma_q(\beta)}v(\tau)w(\rho)$ and integrating with respect to τ and ρ on $[a,b]_q^2$, we get

$$|\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_a^b\int_a^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\beta-1)}v(\tau)w(\rho)R(\tau,\rho)d_q\tau d_q\rho| \\ = L_1L_2\left(J_{q,a}^{\alpha}v(b)J_{q,a}^{\beta}(\tau^2w)(b)+J_{q,a}^{\beta}w(b)J_{q,a}^{\alpha}(\tau^2v)(b)-2J_{q,a}^{\alpha}(\tau v)(b)J_{q,a}^{\beta}(\tau w)(b)\right).$$

The result is thus proved. □

Theorem 3.5. *Let f and g be two lipschitzian functions on $[a,b]_q$ with the constants L_1 and L_2 and let v,w be two positive functions on $[a,b]_q$. The inequality*

$$|J_{q,a}^{\alpha}w(b)J_{q,a}^{\alpha}vfg(b)+J_{q,a}^{\alpha}v(b)J_{q,a}^{\alpha}wfg(b)-J_{q,a}^{\alpha}wf(b)J_{q,a}^{\alpha}vg(b)-J_{q,a}^{\alpha}vf(b)J_{q,a}^{\alpha}wg(b)| \\ \leq L_1L_2(J_{q,a}^{\alpha}w(b)J_{q,a}^{\alpha}(\tau^2v)(b)+J_{q,a}^{\alpha}v(b)J_{q,a}^{\alpha}(\tau^2w)(b)-J_{q,a}^{\alpha}(\tau w)(b)J_{q,a}^{\alpha}(\tau v)(b)). \quad (3.24)$$

is valid.

Proof. same approach, we take $\alpha = \beta$ in Theorem 3.4. □

Corollary 3.2. *Let f and g be two functions defined on $[a,b]_q$ and let v, w be two positive functions on $[a,b]_q$. Then, the inequality*

$$\begin{aligned}&|J^{\beta}_{q,a}w(b)J^{\alpha}_{q,a}vfg(b) + J^{\alpha}_{q,a}v(b)J^{\beta}_{q,a}wfg(b) - J^{\alpha}_{q,a}vf(b)J^{\beta}_{q,a}wg(b) - J^{\alpha}_{q,a}vf(b)J^{\beta}_{q,a}wg(b)|\\ &\leq \|D_qf\|_\infty\|D_qg\|_\infty(J^{\alpha}_{q,a}v(b)J^{\beta}_{q,a}(\tau^2w)(b) + J^{\beta}_{q,a}w(b)J^{\alpha}_{q,a}(\tau^2v)(b) - 2J^{\alpha}_{q,a}(\tau v)(b)J^{\beta}_{q,a}(\tau w)(b))\end{aligned}$$

is valid, where $\|D_qh\|_\infty = \sup_{x\in[a,b]_q}|D_qh(x)|$.

Proof. We have

$$f(\tau) - f(\rho) = \int_\rho^\tau D_qf(t)d_qt, \quad g(\tau) - g(\rho) = \int_\rho^\tau D_qg(t)d_qt$$

so

$$|f(\tau) - f(\rho)| \leq \|D_qf\|_\infty|\tau - \rho| \quad and \quad |g(\tau) - g(\rho)| \leq \|D_qg\|_\infty|\tau - \rho|$$

and the result follows from Theorem 3.4. □

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On mild solutions of nonlocal semilinear functional integro-differential equations

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Abstract

In the present paper, we investigate the existence, uniqueness and continuous dependence on initial data of mild solutions of first order nonlocal semilinear functional integro-differential equations of more general type with delay in Banach spaces. Our analysis is based on semigroup theory and modified version of Banach contraction theorem.

Keywords: Existence, delay, functional, integro-differential equation, fixed point, semigroup theory.

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1 Introduction

In the present paper we consider semilinear functional integro-differential equation of first order of the type:

$$x'(t) = Ax(t) + f(t, x_t, \int_0^t k(t,s)h(s,x_s)ds), \quad t \in [0,T], \tag{1.1}$$

$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \le t \le 0, \tag{1.2}$$

where $0 < t_1 < t_2 < ... < t_p \le T$, $p \in N$; A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t), t \ge 0$ on X; f, g, h, k and ϕ are given functions satisfying some assumptions and $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r, 0]$ and $t \in [0,T]$.

Equations of the form (1.1)-(1.2) or their special forms serve as an abstract formulation of partial integro-differential equations which arise in the problems with memory visco-elasticity and many other physical phenomena, see [1],[5],[8],[15] and the references given therein. The problems of existence, uniqueness and other qualitative properties of solutions for semilinear differential equations in Banach spaces has been studied extensively in the literature for last many years, see[1]-[11],[14],[15]. On the other hand, as nonlocal condition is more precise to describe natural phenomena than classical initial condition, the Cauchy problem with nonlocal condition also received much attention in recent years, see [2]-[4],[9],[10],[12],[17],[18].

L.Byzewski and H.Acka[3] studied existence,uniqueness and continuous dependence of a mild solution on initial data of problem (1.1)-(1.2) by Banach contraction theorem. The objective of this paper is to generalize and improve their results. We are achieving the same results with less restrictions by using modified version of Banach contraction principle.

The paper is organized as follows: Section 2 presents preliminaries and hypotheses. In section 3, we prove existence and uniqueness of solutions. Section 4, deals with continuous dependence on initial data of mild solutions . Finally in section 5, we give application based on our result.

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2 Preliminaries and Hypotheses

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = \mathcal{C}([-r, 0], X), 0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r, 0] \to X$ endowed with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(t)\| : -r \leq t \leq 0\}.$$

Let $B = \mathcal{C}([-r, T], X), T > 0$, be the Banach space of all continuous functions $x : [-r, T] \to X$ with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : -r \leq t \leq T\}$. For any $x \in B$ and $t \in [0, T]$, we denote x_t the element of C given by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]$ and ϕ is a given element of C.

In this paper, we assume that, there exist positive constant $K \geq 1$ such that $\|T(t)\| \leq K$, for every $t \in [0, T]$. Also $k : [0, T] \times [0, T] \to \mathbb{R}$ and since k is continuous on compact set $[0, T] \times [0, T]$, there is constant $L_1 > 0$ such that $|k(t, s)| \leq L_1$, for $0 \leq s \leq t \leq T$.

Definition 2.1. *A function $x \in B$ satisfying the equations:*

$$x(t) = T(t)\phi(0) - T(t)(g(x_{t_1}, ..., x_{t_p}))(0) + \int_0^t T(t-s) f(s, x_s, \int_0^s k(s, \tau) h(\tau, x_\tau) d\tau) ds, \quad t \in [0, T],$$
$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0,$$

is said to be the mild solution of the initial value problem (1.1)-(1.2).

The following Lemma is known as Pachpatte's inequality .

Lemma 2.1. *[13, p.33] Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}^+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t f(s) u(s) ds + \int_0^t f(s) (\int_0^s g(\sigma) u(\sigma) d\sigma) ds, \quad t \in \mathbb{R}^+,$$

holds, where u_0 is nonnegative constant. Then

$$u(t) \leq u_0 [1 + \int_0^t f(s) exp(\int_0^s [f(\sigma) + g(\sigma)] d\sigma) ds], \quad t \in \mathbb{R}^+$$

Our results are based on the modified version of Banach contraction principle.

Lemma 2.2. *[16, p.196] Let X be a Banach space. Let D be an operator which maps the elements of X into itself for which D^r is a contraction, where r is a positive integer. Then D has a unique fixed point.*

We list the following hypotheses for our convenience.

(H_1) Let $f : [0, T] \times C \times X \to X$ such that for every $w \in B$,$x \in X$ and $t \in [0, T]$, $f(., w_t, x) \in B$ and there exists a constant $L > 0$ such that

$$\|f(t, \psi, x) - f(t, \phi, y)\| \leq L(\|\psi - \phi\|_C + \|x - y\|), \qquad \phi, \psi \in C, \quad x, y \in X.$$

(H_2) Let $h : [0, T] \times C \to X$ such that for every $w \in B$ and $t \in [0, T]$, $h(., w_t) \in B$ and there exists a constant $H > 0$ such that

$$\|h(t, \psi) - h(t, \phi)\| \leq H \|\psi - \phi\|_C, \qquad \phi, \psi \in C.$$

(H_3) Let $g : C^p \to C$ such that exists a constant $G \geq 0$ satisfying

$$\|(g(x_{t_1}, x_{t_2}, ..., x_{t_p}))(t) - (g(y_{t_1}, y_{t_2}, ..., y_{t_p}))(t)\| \leq G \|x - y\|_B, \quad t \in [-r, 0].$$

3 Existence and Uniqueness

Theorem 3.1. *Suppose that the hypotheses (H_1) - (H_3) are satisfied. Then the initial-value problem (1.1)-(1.2) has a unique mild solution x on $[-r, T]$.*

Proof. Let $x(t)$ be a mild solution of the problem (1.1)-(1.2) then it satisfies the equivalent integral equation

$$x(t) = T(t)\phi(0) - T(t)(g(x_{t_1}, ..., x_{t_p}))(0) + \int_0^t T(t-s)f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau)ds, \quad t \in [0,T], \tag{3.1}$$

$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \le t \le 0. \tag{3.2}$$

Now, we rewrite solution of initial value problem (1.1)-(1.2) as follows: For $\phi \in C$, define $\widehat{\phi} \in B$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t) - (g(x_{t_1}, ..., x_{t_p}))(t) & \text{if } -r \le t \le 0 \\ T(t)[\phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0)] & \text{if } 0 \le t \le T \end{cases}$$

If $y \in B$ and $x(t) = y(t) + \widehat{\phi}(t)$, $t \in [-r, T]$, then it is easy to see that y satisfies

$$y(t) = 0; \quad -r \le t \le 0 \quad \text{and} \tag{3.3}$$

$$y(t) = \int_0^t T(t-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau\right)ds, \quad t \in [0,T] \tag{3.4}$$

if and only if $x(t)$ satisfies the equations (3.1)-(3.2).
We define the operator $F : B \to B$, by

$$(Fy)(t) = \begin{cases} 0 \quad \text{if} \quad -r \le t \le 0 \\ \int_0^t T(t-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau\right)ds & \text{if } t \in [0,T]. \end{cases} \tag{3.5}$$

From the definition of an operator F defined by the equation (3.5), it is to be noted that the equations (3.3)-(3.4) can be written as

$$y = Fy.$$

Now we show that F^n is a contraction on B for some positive integer n. Let $y, w \in B$ and using hypotheses (H_1) - (H_3) , we get,

$$\begin{aligned}
\|(Fy)(t) - (Fw)(t)\| &\le \int_0^t \|T(t-s)\| \|f(s, y_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau) \\
&\quad - f(s, w_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau)\|ds \\
&\le \int_0^t KL[\|(y_s + \widehat{\phi}_s) - (w_s + \widehat{\phi}_s)\|_C + L_1 \int_0^s \|h(\tau, y_\tau + \widehat{\phi}_\tau) - h(\tau, w_\tau + \widehat{\phi}_\tau)\|d\tau]ds \\
&\le KL \int_0^t \|y_s - w_s\|_C ds + KL \int_0^t L_1 H \int_0^s \|y_\tau - w_\tau\|_C d\tau ds \\
&\le KL \int_0^t \|y - w\|_B ds + KL \int_0^t L_1 H \int_0^s \|y - w\|_B d\tau ds \\
&\le KL\|y - w\|_B t + KLL_1H\|y - w\|_B \frac{t^2}{2} \\
&\le KL\|y - w\|_B t + KLL_1HT\|y - w\|_B \frac{t}{2} \\
&\le KL\|y - w\|_B t + KLL_1HT\|y - w\|_B t \\
&\le KL(1 + L_1HT)\|y - w\|_B t
\end{aligned}$$

$$
\begin{aligned}
&\|(F^2y)(t)-(F^2w)(t)\| \\
&=\|(F(Fy))(t)-(F(Fw))(t)\| \\
&=\|(F(y_1))(t)-(F(w_1))(t)\| \\
&\leq \int_0^t \|T(t-s)\|\|f(s,{y_1}_s+\widehat{\phi}_s,\int_0^s k(s,\tau)h(\tau,{y_1}_\tau+\widehat{\phi}_\tau)) \\
&\quad -f(s,{w_1}_s+\widehat{\phi}_s,\int_0^s k(s,\tau)h(\tau,{w_1}_\tau+\widehat{\phi}_\tau))\|ds \\
&\leq \int_0^t KL\|{y_1}_s-{w_1}_s\|_C+KL\int_0^t L_1H\|{y_1}_\tau-{w_1}_\tau\|_C d\tau ds \\
&\leq KL\int_0^t \|y_1-w_1\|_{\mathcal{C}([-r,s],X)}ds+KL\int_0^t L_1H\int_0^s \|y_1-w_1\|_{\mathcal{C}([-r,\tau],X)}d\tau ds \\
&\leq KL\int_0^t \sup_{\tau\in[-r,s]}\|y_1(\tau)-w_1(\tau)\|ds+KLL_1H\int_0^t\int_0^s \sup_{\eta\in[-r,\tau]}\|y_1(\eta)-w_1(\eta)\|d\tau ds \\
&\leq KL\int_0^t \sup_{\tau\in[-r,s]}\|Fy(\tau)-Fw(\tau)\|ds+KLL_1H\int_0^t\int_0^s \sup_{\eta\in[-r,\tau]}\|Fy(\eta)-Fw(\eta)\|d\tau ds \\
&\leq KL\int_0^t \sup_{\tau\in[-r,s]}(KL[1+L_1HT]\|y-w\|_B\tau)ds \\
&\quad +KLL_1H\int_0^t\int_0^s \sup_{\eta\in[-r,\tau]}(KL[1+L_1HT]\|y-w\|_B\eta)d\tau ds \\
&\leq K^2L^2[1+L_1HT]\|y-w\|_B[\int_0^t(\sup_{\tau\in[-r,s]}\tau)ds+\int_0^t L_1H\int_0^s(\sup_{\eta\in[-r,\tau]}\eta)d\tau ds] \\
&\leq K^2L^2[1+L_1HT]\|y-w\|_B[\int_0^t s ds+\int_0^t L_1H\int_0^s \tau d\tau ds] \\
&\leq K^2L^2[1+L_1HT]\|y-w\|_B[\frac{t^2}{2}+L_1H\frac{t^3}{3!}] \\
&\leq K^2L^2[1+L_1HT]^2\|y-w\|_B[\frac{t^2}{2}+L_1HT\frac{t^2}{3!}] \\
&\leq K^2L^2[1+L_1HT]^2\|y-w\|_B[\frac{t^2}{2!}+L_1HT\frac{t^2}{2!}] \\
&\leq \frac{(KL[1+L_1HT]t)^2}{2!}\|y-w\|_B
\end{aligned}
$$

Continuing in this way, we get,

$$\|(F^ny)(t)-(F^nw)(t)\|\leq \frac{(KL[1+L_1HT]t)^n}{n!}\|y-w\|_B.$$

For n large enough, $\frac{(KL[1+L_1HT]t)^n}{n!}<1$. Thus there exist a positive integer n such that F^n is a contraction in B. By virtue of Lemma 2.2, the operator F has a unique fixed point $\tilde{y}$ in B. Then $\tilde{x}=\tilde{y}+\widehat{\phi}$ is a solution of the Cauchy problem (1.1)-(1.2). This completes the proof. □

4 Continuous Dependence on Initial Data

Theorem 4.1. *Suppose that the functions f, h and g satisfies the hypotheses (H_1) - (H_3). Then for each $\phi_1,\phi_2\in C$ and for the corresponding mild solutions x_1,x_2 of the problems*

$$x'(t)=Ax(t)+f(t,x_t,\int_0^t k(s,t)h(t,x_t)dt),\quad t\in[0,T], \tag{4.1}$$

$$x(t)+(g(x_{t_1},...,x_{t_p}))(t)=\phi_i(t),\quad -r\leq t\leq 0,\quad (i=1,2) \tag{4.2}$$

the inequality

$$\|x_1-x_2\|_B\leq[K\|\phi_1-\phi_2\|_C+KG\|x_1-x_2\|_B]\left[1+KLTe^{(KL+L_1H)T}\right]. \tag{4.3}$$

is true.

Moreover, if $G = 0$, then it reduces to classical inequality

$$\|x_1 - x_2\|_B \leq K\left[1 + KLTe^{(KL+L_1H)T}\right]\|\phi_1 - \phi_2\|_C. \tag{4.4}$$

.

Proof. Let $\phi_i (i = 1, 2)$ be arbitrary functions in C and let $x_i (i = 1, 2)$ be the mild solutions of the problem (4.1)-(4.2).
Then for $t \in [-r, 0]$,

$$x_1(t) - x_2(t) = \phi_1(t) - (g(x_{1t_1}, ..., x_{1t_p}))(t) - \phi_2(t) + (g(x_{2t_1}, ..., x_{2t_p}))(t) \tag{4.5}$$

and for $t \in [0, T]$,

$$\begin{aligned} x_1(t) - x_2(t) =& T(t)[\phi_1(0) - \phi_2(0) - (g(x_{1t_1}, ..., x_{1t_p}))(0) + (g(x_{2t_1}, ..., x_{2t_p}))(0)] \\ &+ \int_0^t T(t-s)[f(s, x_{1s}, \int_0^s k(s,\tau)h(\tau, x_{1\tau})d\tau) \\ &- f(s, x_{2s}, \int_0^s k(s,\tau)h(\tau, x_{2\tau})d\tau)]ds \end{aligned} \tag{4.6}$$

From (4.6) and hypotheses $(H1)$ - $(H3)$, we get, for $t \in [0, t]$,

$$\begin{aligned} \|x_1(t) - x_2(t)\| =& \|T(t)\|\|\phi_1 - \phi_2\|_C + G\|T(t)\|\|x_1 - x_2\|_B \\ &+ \int_0^t \|T(t-s)\|\|f(s, x_{1s}, \int_0^s k(s,\tau)h(\tau, x_{1\tau})d\tau) \\ &- f(s, x_{2s}, \int_0^s k(s,\tau)h(\tau, x_{2\tau})d\tau)\|ds \\ &\leq K\|\phi_1 - \phi_2\|_C + KG\|x_1 - x_2\|_B \\ &+ \int_0^t KL\left[\|x_{1s} - x_{2s}\|_C + L_1H\int_0^s \|x_{1\tau} - x_{2\tau}\|_C d\tau\right] ds \end{aligned} \tag{4.7}$$

Define the function $z : [-r, T] \to \mathbb{R}$ by $z(t) = \sup\{\|x_1(s) - x_2(s)\| : -r \leq s \leq t\}, t \in [0, T]$. Let $t^* \in [-r, t]$ be such that $z(t) = \|x_1(t^*) - x_2(t^*)\|$. If $t^* \in [0, t]$, then from inequality (4.7),we have

$$\begin{aligned} z(t) =& \|x_1(t^*) - x_2(t^*)\| \leq K\|\phi_1 - \phi_2\|_C + KG\|x_1 - x_2\|_B \\ &+ \int_0^{t^*} KL\left[\|x_{1s} - x_{2s}\|_C + L_1H\int_0^s \|x_{1\tau} - x_{2\tau}\|_C d\tau\right] ds \\ &\leq K\|\phi_1 - \phi_2\|_C + KG\|x_1 - x_2\|_B \\ &+ \int_0^t KL\left[\|x_{1s} - x_{2s}\|_C + L_1H\int_0^s \|x_{1\tau} - x_{2\tau}\|_C d\tau\right] ds \\ &\leq K\|\phi_1 - \phi_2\|_C + KG\|x_1 - x_2\|_B + \int_0^t KL\left[z(s) + L_1H\int_0^s z(\tau)d\tau\right] ds \end{aligned} \tag{4.8}$$

If $t^* \in [-r, 0]$ then $z(t) \leq \|\phi_1 - \phi_2\|_C + G\|x_1 - x_2\|_B$ and since $K > 1$ the inequality (4.8) holds good. Thus $t^* \in [-r, T]$ the inequality (4.8) holds good. Thanks to Pachpatte's inequality given in Lemma 2.1 and applying it to inequality (4.8) we get,

$$\begin{aligned} z(t) &\leq [K\|\phi_1 - \phi_2\|_C + KG\|x_1 - x_2\|_B]\left[1 + \int_0^t KLe^{\int_0^s (KL+L_1H)d\tau}ds\right] \\ &\leq [K\|\phi_1 - \phi_2\|_C + KG\|x_1 - x_2\|_B]\left[1 + \int_0^t KLe^{(KL+L_1H)T}ds\right] \\ &\leq [K\|\phi_1 - \phi_2\|_C + KG\|x_1 - x_2\|_B]\left[1 + KLTe^{(KL+L_1H)T}\right] \end{aligned}$$

Consequently,

$$\|x_1 - x_2\|_B \leq [K\|\phi_1 - \phi_2\|_C + KG\|x_1 - x_2\|_B]\left[1 + KLTe^{(KL+L_1H)T}\right]. \tag{4.9}$$

Hence the inequality (4.3) holds. Finally inequality (4.4) is a consequence of the inequality (4.9). Hence the proof is complete. □

5 Applications

To illustrate the application of our result proved in section 3, consider the following semilinear partial functional differential equation of the form

$$\frac{\partial}{\partial t}w(u,t) = \frac{\partial^2}{\partial u^2}w(u,t) + H\left(t, w(u,t-r), \int_0^t k(t,s)P(s,w(s-r))ds\right), 0 \le u \le \pi, t \in [0,T] \tag{5.1}$$

$$w(0,t) = w(\pi,t) = 0, \quad 0 \le t \le T, \tag{5.2}$$

$$w(u,t) + \sum_{i=1}^{p} w(u,t_i+t) = \phi(u,t), \quad 0 \le u \le \pi, \quad -r \le t \le 0, \tag{5.3}$$

where $0 < t_1 \le t_2 \le t_p \le T$, the function $H : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous . We assume that the functions H and P satisfy the following conditions:
For every $t \in [0,T]$ and $u, v, x, y \in \mathbb{R}$, there exists a constant $l, p > 1$ such that

$$|H(t,u,x) - H(t,v,y)| \le l(|u-v| + |x-y|)$$
$$|P(t,u) - P(t,v)| \le p(|u-v|).$$

Let us take $X = L^2[0,\pi]$. Define the operator $A : X \to X$ by $Az = z^{''}$ with domain $D(A) = \{z \in X : z, \ \ z^{'}$ are absolutely continuous, $z^{''} \in X$ and $z(0) = z(\pi) = 0\}$. Then the operator A can be written as

$$Az = \sum_{n=1}^{\infty} -n^2(z, z_n)z_n, \quad z \in D(A)$$

where $z_n(u) = (\sqrt{2/\pi})\sin nu, \ n = 1, 2, ..$ is the orthogonal set of eigenvectors of A and A is the infinitesimal generator of an analytic semigroup $T(t), \ t \ge 0$ and is given by

$$T(t)z = \sum_{n=1}^{\infty} exp(-n^2 t)(z, z_n)z_n, \quad z \in X.$$

Now, the analytic semigroup $T(t)$ being compact, there exists constant K such that

$$|T(t)| \le K, \quad \text{for each} \quad t \in [0,T].$$

Define the function $f : [0,T] \times C \times X \to X$, as follows

$$f(t,\psi,x)(u) = H(t,\psi(-r)u, x(u)),$$
$$h(t,\phi)(u) = P(t,\phi(-r)u)$$

for $t \in [0,T], \psi, \phi \in C$, $x \in X$ and $0 \le u \le \pi$. With these choices of the functions the equations (5.1)-(5.3) can be formulated as an abstract integro-differential equation in Banach space X:

$$x'(t) = Ax(t) + f\Big(t, x_t, \int_0^t k(t,s)h(s,x_s)ds\Big), \quad t \in [0,T]$$
$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad t \in [-r, 0].$$

Since all the hypotheses of the theorem 3.1 are satisfied, the theorem 3.1, can be applied to guarantee the existence of mild solution $w(u,t) = x(t)u, \quad t \in [0,T], u \in [0,\pi]$, of the semilinear partial integro-differential equation (5.1)-(5.3).

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Some oscillation theorems for second order nonlinear neutral type difference equations

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Abstract

In this paper some new sufficient conditions for the oscillatory behavior of second order nonlinear neutral type difference equation of the form

$$\Delta\Big(a_n\Delta(x_n + p_n x_{n-k})\Big) + q_n f(x_{\sigma(n+1)}) = 0$$

where $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are real sequences, $\{\sigma(n)\}$ is a sequence of integers, k is a positive integer and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$ are established. Examples are provided to illustrate the main results.

Keywords: Second order, nonlinear, neutral type difference equation, oscillation.

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1 Introduction

In this paper we study the oscillatory behavior of second order neutral type difference equation of the form

$$\Delta\Big(a_n\Delta(x_n + p_n x_{n-k})\Big) + q_n f(x_{\sigma(n+1)}) = 0, \quad n \in \mathbb{N}(n_0) \tag{1.1}$$

where k is a positive integer, $\{a_n\}$, $\{p_n\}$, $\{q_n\}$ are real sequences defined on $\mathbb{N}(n_0)$ and $\sigma(n+1)$ is a sequence of integers. We assume the following conditions without further mention:

(H_1) $\{a_n\}$ is a positive real sequence with $\displaystyle\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$;

(H_2) $\{p_n\}$ is a real sequence with $p_n \geq 1$ for all $n \in \mathbb{N}(n_0)$;

(H_3) $\{q_n\}$ is a positive real sequence for all $n \in \mathbb{N}(n_0)$;

(H_4) $\{\sigma(n)\}$ is an increasing sequence of integers such that $\sigma(n) \leq n$ and $\sigma(n) \to \infty$ as $n \to \infty$;

(H_5) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a constant $L > 0$ such that $\frac{f(u)}{u^\alpha} \geq L$ for all $u \neq 0$, where α is a ratio of odd positive integers.

Let $\theta = max\Big\{k, min_{n\in\mathbb{N}(n_0)}\sigma(n)\Big\}$. By a solution of equation (1.1), we mean a nontrivial real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfying the equation (1.1) for all $n \in \mathbb{N}(n_0)$. A solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

In recent years, there has been much research concerning the oscillation of delay and neutral type difference equations. In most of the papers, the authors considered the case $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$ and either $-1 < p \leq p_n \leq 0$ or $0 \leq p_n \leq p < 1$, see for example [3-6, 9, 10, 13-16]. In [7, 8, 11, 12] the authors considered equation (1.1) under the assumptions $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$ and $0 \leq p_n \leq p < 1$ and established sufficient conditions for the

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oscillation of all solutions of equation (1.1).

Motivated by this observation in this paper we present some sufficient conditions for the oscillation of all solutions of equation (1.1) under the conditions $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$ and $p_n \geq 1$ for all $n \in \mathbb{N}(n_0)$. In Section 2, we present some preliminary lemmas, and in Section 3 we obtain some sufficient conditions for the oscillation of all solutions of equation (1.1). In Section 4, we provide some examples to illustrate the main results.

2 Some preliminary lemmas

Throughout this paper we use the following notation without further mention:

$$\begin{aligned} z_n &= x_n + p_n x_{n-k} \ , \\ A(n) &= a_{\sigma(n)} \sum_{s=n_0}^{n} \frac{1}{a_{\sigma(s)}} \ , \qquad R(n) = \sum_{s=n_0}^{n-1} \frac{1}{a_s} \ , \\ B(n) &= \frac{1}{p_{n+k}} \Big(1 - \frac{R(n+2k)}{R(n+k) p_{n+2k}} \Big) > 0 \ , \\ C(n) &= \frac{1}{p_{n+k}} \Big(1 - \frac{1}{p_{n+2k}} \Big) \ , \quad \text{and} \quad E(n) = \sum_{s=\tau(n)}^{\infty} \frac{1}{a_s} \ , \end{aligned}$$

where $\{\tau(n)\}$ is defined later. Note that from the assumptions it is enough to state and prove the lemmas and theorems for the case $\{x_n\}$ is eventually positive since the opposite case is proved similarly. To prove our main results we need the following lemmas.

Lemma 2.1. *Let $\{x_n\}$ be an eventually positive solution of equation (1.1). Then one of the following two cases holds for all sufficiently large n:*

(I) $z_n > 0, \quad a_n \Delta z_n > 0, \quad \Delta(a_n \Delta z_n) \leq 0;$

(II) $z_n > 0, \quad a_n \Delta z_n < 0, \quad \Delta(a_n \Delta z_n) \leq 0.$

Proof. The proof of the lemma can be found in [11]. □

Lemma 2.2. *Let $\{x_n\}$ be an eventually positive solution of equation (1.1) and suppose case (I) of Lemma 2.1 holds. Then there exists $N \in \mathbb{N}(n_0)$ such that*

$$x_n \geq B(n) z_n, \quad \textit{for all} \ \ n \geq N. \tag{2.1}$$

Proof. From the definition of z_n, we have

$$\frac{z_{n+k}}{p_{n+k}} = \frac{x_{n+k}}{p_{n+k}} + x_n$$

or

$$x_n = \frac{1}{p_{n+k}} (z_{n+k} - x_{n+k}). \tag{2.2}$$

On the other hand

$$z_n = z_{n_0} + \sum_{s=n_0}^{n-1} \frac{a_s \Delta z_s}{a_s} \ \geq \ a_n R(n) \Delta z_n$$

or

$$R(n) \Delta z_n - z_n \Delta R(n) \leq 0.$$

or

$$\frac{R(n) \Delta z_n - z_n \Delta R(n)}{R(n) R(n+1)} \leq 0.$$

or

$$\Delta \Big(\frac{z_n}{R(n)} \Big) \leq 0.$$

Thus z_n is increasing and $\frac{z_n}{R(n)}$ is nonincreasing. Further

$$x_{n+k} \leq \frac{1}{p_{n+2k}} R(n+2k) \frac{z_{n+2k}}{R(n+2k)} \leq \frac{R(n+2k)}{p_{n+2k}} \Big(\frac{z_{n+k}}{R(n+k)} \Big). \tag{2.3}$$

From (2.2) and (2.3) we obtain (2.1). This completes the proof. □

Lemma 2.3. *Let $\{x_n\}$ be an eventually positive solution of equation (1.1) and suppose case (II) of Lemma 2.1 holds. Then there exists $N \in \mathbb{N}(n_0)$ such that*

$$x_n \geq C(n) z_{n+k}, \quad \textit{for all} \quad n \geq N. \tag{2.4}$$

Proof. From the proof of Lemma 2.2, we have (2.2). From $\Delta z_n < 0$ we have

$$x_{n+k} \leq \frac{z_{n+2k}}{p_{n+2k}} \leq \frac{z_{n+k}}{p_{n+2k}}. \tag{2.5}$$

Using (2.5) in (2.2), we obtain (2.4). This completes the proof. □

Lemma 2.4. *Let $\{x_n\}$ be an eventually positive solution of equation (1.1) and suppose case (I) of Lemma 2.1 holds. Then there exists $N \in \mathbb{N}(n_0)$ such that*

$$z_{\sigma(n+1)} \geq A(n) \Delta z_{\sigma(n)}, \quad \textit{forall} \quad n \geq N. \tag{2.6}$$

Proof. Since $\Delta(a_n \Delta z_n) \leq 0$ and $\Delta\sigma(n) > 0$, we see that

$$z_{\sigma(n+1)} = z_{\sigma(N)} + \sum_{s=N}^{n} \Delta z_{\sigma(s)} \geq a_{\sigma(n)} \Delta z_{\sigma(n)} \sum_{s=N}^{n} \frac{1}{a_{\sigma(s)}}.$$

The proof is now complete. □

3 Oscillation results

In this section we obtain some new sufficient conditions for the oscillation of all solutions of equation (1.1).

Theorem 3.1. *Assume that $\alpha \geq 1$, and there exists a sequence of integers $\{\tau(n)\}$ such that $\tau(n) \geq n$, $\Delta\tau(n) > 0$ and $\sigma(n) \leq \tau(n) - k$. If there exists a positive increasing real sequence $\{\rho_n\}$ such that for all constants $M > 0$ and $D > 0$ one has*

$$\sum_{n=N}^{\infty} \left[L\rho_n q_n B^{\alpha}(\sigma(n+1)) - \frac{1}{4\alpha M^{\alpha-1}} \frac{(\Delta\rho_n)^2 a_{\sigma(n)}}{\rho_n} \right] = \infty \tag{3.1}$$

and

$$\sum_{n=N}^{\infty} \left[L q_n E^{\alpha}(n+1) C^{\alpha}(\sigma(n+1)) - \frac{\alpha}{D^{\alpha-1} E(n) a_{\tau(n)}} \right] = \infty \tag{3.2}$$

then every solution of equation (1.1) is oscillatory.

Proof. Assume to the contrary that there exists a nonoscillatory solution $\{x_n\}$ of equation (1.1). Without loss of generality we may assume that $x_{n-\theta} > 0$ for all $n \geq N \in \mathbb{N}(n_0)$, where N is chosen so that one of the cases of Lemma 2.1 hold for all $n \geq N$. We shall show that in each case we are led to a contradiction.
Case(I). From Lemma 2.2 and equation (1.1), we have

$$\Delta(a_n \Delta z_n) + L q_n B^{\alpha}(\sigma(n+1)) z^{\alpha}_{\sigma(n+1)} \leq 0, \quad n \geq N. \tag{3.3}$$

Define

$$w_n = \rho_n \frac{a_n \Delta z_n}{z^{\alpha}_{\sigma(n)}}, \quad n \geq N,$$

we have

$$\begin{aligned} \Delta w_n \;=\; & \frac{\rho_n \Delta(a_n \Delta z_n)}{z^{\alpha}_{\sigma(n+1)}} + \Delta\rho_n a_{\frac{n+1 \Delta z_{n+1}}{z^{\alpha}_{\sigma(n+1)}}} - \rho_n \frac{a_n \Delta z_n}{z^{\alpha}_{\sigma(n+1)} z^{\alpha}_{\sigma(n)}} \Delta z^{\alpha}_{\sigma(n)} - L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\ & - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z^{\alpha}_{\sigma(n)}}{z^{\alpha}_{\sigma(n)}} \end{aligned} \tag{3.4}$$

for $n \geq N$. By Mean value theorem

$$\Delta z^{\alpha}_{\sigma(n)} = \alpha t^{\alpha-1} \Delta z_{\sigma(n)},$$

where $z_{\sigma(n)} < t < z_{\sigma(n+1)}$. Since $\alpha \geq 1$, we have

$$\Delta z^{\alpha}_{\sigma(n)} \geq \alpha z^{\alpha-1}_{\sigma(n)} \Delta z_{\sigma(n)}. \tag{3.5}$$

Using (3.5) in (3.3) we obtain for $n \geq N$

$$\Delta w_n \leq -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha \rho_n}{\rho_{n+1}} w_{n+1} \frac{z^{\alpha-1}_{\sigma(n)} \Delta z_{\sigma(n)}}{z^{\alpha}_{\sigma(n)}}. \tag{3.6}$$

Since z_n increasing and $a_n \Delta z_n$ is nonincreasing we have from (3.6)

$$\Delta w_n \leq -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha \rho_n}{\rho^2_{n+1}} \frac{M^{\alpha-1}}{a_{\sigma(n)}} w^2_{n+1} \tag{3.7}$$

where $M = z_{\sigma(N)}$. Summing the last inequality from N to $n-1$ and using completing the square we have

$$0 < w_n \leq w_N - \sum_{s=N}^{n-1} \Big[L\rho_s q_s B^{\alpha}(\sigma(s+1)) - \frac{1}{4\alpha M^{\alpha-1}} \frac{(\Delta \rho_s)^2 a_{\sigma(s)}}{\rho_s} \Big].$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.1).

Case(II). Define

$$v_n = \frac{a_n \Delta z_n}{z^{\alpha}_{\sigma(n)}} \ , \quad n \geq N. \tag{3.8}$$

Then $v_n < 0$ for $n \geq N$. Since $\{a_n \Delta z_n\}$ is nonincreasing, we have

$$\Delta z_s \leq \frac{a_n \Delta z_n}{a_s} \ , \quad s \geq n.$$

Summing the last inequality from $\tau(n)$ to ∞, we obtain

$$z_{\infty} \leq z_{\tau(n)} + a_n \Delta z_n \sum_{s=\tau(n)}^{\infty} \frac{1}{a_s}.$$

Since $z_n > 0$ for all sufficiently large n we have

$$0 \leq z_{\infty} \leq z_{\tau(n)} + a_n \Delta z_n E(n) \ , \quad n \geq N,$$

or

$$\frac{a_n \Delta z_n E(n)}{z_{\tau(n)}} \geq -1 \ , \quad n \geq N.$$

Thus

$$- \frac{a_n \Delta z_n (-a_n \Delta z_n)^{\alpha-1}}{z^{\alpha}_{\tau(n)}} E^{\alpha}(n) \leq 1.$$

So, by $\Delta(-a_n \Delta z_n) > 0$ and (3.8), we have

$$-\frac{1}{D^{\alpha-1}} \leq v_n E^{\alpha}(n) \leq 0 \ , \quad n \geq N, \tag{3.9}$$

where $D = -a_N \Delta z_N$. From (3.8), we have

$$\Delta v_n = \frac{\Delta(a_n \Delta z_n)}{z^{\alpha}_{\tau(n+1)}} - \frac{a_n \Delta z_n}{z^{\alpha}_{\tau(n)} z^{\alpha}_{\tau(n+1)}} \Delta z^{\alpha}_{\tau(n)}.$$

By Mean Value Theorem,

$$\Delta z^{\alpha}_{\tau(n)} = \alpha t^{\alpha-1} \Delta z_{\tau(n)}$$

where $z_{\tau(n+1)} < t < z_{\tau(n)}$. Since $\alpha \geq 1$ and $\Delta z_{\tau(n)} < 0$, we have

$$\Delta z^{\alpha}_{\tau(n)} \leq \alpha z^{\alpha-1}_{\tau(n+1)} \Delta z_{\tau(n)}.$$

Therefore

$$\Delta v_n \le -\frac{Lq_n x^{\alpha}_{\sigma(n+1)}}{z^{\alpha}_{\tau(n+1)}} - \frac{\alpha a_n \Delta z_n}{z^{\alpha}_{\tau(n)} z_{\tau(n+1)}} \Delta z_{\tau(n)}. \tag{3.10}$$

From (2.4) and by $\sigma(n) \le \tau(n) - k$, we have

$$\frac{x^{\alpha}_{\sigma(n+1)}}{z^{\alpha}_{\tau(n+1)}} \ge C^{\alpha}(\sigma(n+1)). \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\Delta v_n + Lq_n C^{\alpha}(\sigma(n+1)) \le 0, \quad n \ge N. \tag{3.12}$$

Multiplying (3.12) by $E^{\alpha}(n+1)$ and then summing it from N to $n-1$, we have

$$\sum_{s=N}^{n-1} E^{\alpha}(s+1)\Delta v_s + \sum_{s=N}^{n-1} LE^{\alpha}(s+1) q_s C^{\alpha}(\sigma(s+1)) \le 0.$$

Summation by parts formula yields

$$\sum_{s=N}^{n-1} E^{\alpha}(s+1)\Delta v_s = E^{\alpha}(n)v_n - E^{\alpha}(N)v_N - \sum_{s=N}^{n-1} v_s \Delta E^{\alpha}(s).$$

Using Mean Value Theorem, we obtain

$$\Delta E^{\alpha}(s) \ge -\frac{\alpha E^{\alpha-1}(s)}{a_{\tau(s)}}.$$

Since $v_n < 0$, we have

$$\sum_{s=N}^{n-1} E^{\alpha}(s+1)\Delta v_s \ge E^{\alpha}(n)v_n - E^{\alpha}(N)v_N + \sum_{s=N}^{n-1} \frac{\alpha v_s E^{\alpha-1}(s)}{a_{\tau(s)}} \ ,$$

or

$$E^{\alpha}(n)v_n - E^{\alpha}(N)v_N + \sum_{s=N}^{n-1} \frac{\alpha v_s E^{\alpha-1}(s)}{a_{\tau(s)}} + \sum_{s=N}^{n-1} Lq_s E^{\alpha}(s+1) C^{\alpha}(\sigma(s+1)) \le 0. \tag{3.13}$$

Therefore, from (3.9) and (3.13), we obtain

$$-\frac{1}{D^{\alpha-1}} \quad \le \quad E^{\alpha}(n)v_n \le E^{\alpha}(N)v_N - \sum_{s=N}^{n-1} \Big[Lq_s E^{\alpha}(s+1) C^{\alpha}(\sigma(s+1)) - \frac{\alpha}{D^{\alpha-1} E(s) a_{\tau(s)}} \Big].$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.2). This completes the proof. $\square$

Theorem 3.2. *Assume that* $\alpha \ge 1$ *and there exists a sequence* $\{\tau(n)\}$ *of integers such that* $\tau(n) \ge n$, $\Delta\tau(n) > 0$ *and* $\tau(n) \le \sigma(n) - k$. *If there exists a positive increasing real sequence* $\{\rho_n\}$ *such that for every constants* $M > 0$, *and* $D > 0$, (3.1) *holds, and*

$$\sum_{n=N}^{\infty} \Big[q_n E^{\alpha+1}(n+1) C^{\alpha}(\sigma(n+1)) - \frac{\alpha+1}{D^{\alpha-1} a_{\tau(n)}} \Big] = \infty, \tag{3.14}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1, we see that Lemma 2.1 holds for $n \ge N \in \mathbb{N}(n_0)$.
Case(I). Proceeding as in the proof of Theorem 3.1(Case(I)) we obtain a contradiction to (3.1).
Case(II). Proceeding as in the proof of Theorem 3.1(Case(II)) we obtain (3.9) and (3.12). Multiplying (3.12) by $E^{\alpha+1}(n+1)$ and then summing it from N to $n-1$ we have

$$\sum_{s=N}^{n-1} E^{\alpha+1}(s+1)\Delta v_s + \sum_{s=N}^{n-1} Lq_s E^{\alpha+1}(s+1) C^{\alpha}(\sigma(s+1)) \le 0.$$

Using the summation by parts formula in the first term of the last inequality and then rearranging, we obtain

$$E^{\alpha+1}(n)v_n - E^{\alpha+1}(N)v_N + \sum_{s=N}^{n-1} \frac{(\alpha+1)v_s E^{\alpha}(s)}{a_{\tau(s)}} + \sum_{s=N}^{n-1} Lq_s E^{\alpha+1}(s+1)C^{\alpha}(\sigma(s+1)) \le 0. \quad (3.15)$$

In view of (3.9), we have $-v_n E^{\alpha+1}(n) \le \frac{1}{D^{\alpha-1}} E(n) < \infty$ as $n \to \infty$, and

$$\sum_{s=N}^{n-1} Lq_s E^{\alpha+1}(s+1)C^{\alpha}(\sigma(s+1)) \le E^{\alpha+1}(N)v_N - E^{\alpha+1}(n)v_n + \frac{(\alpha+1)}{D^{\alpha-1}} \sum_{s=N}^{n-1} \frac{1}{a_{\tau(s)}}.$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.14). This completes the proof. □

Theorem 3.3. *Assume that $\alpha \ge 1$, and there exists a sequence $\{\tau(n)\}$ of integers such that $\tau(n) \ge n$, $\Delta\tau(n) > 0$ and $\sigma(n) \le \tau(n) - k$. If there exists a positive increasing real sequence $\{\rho_n\}$ such that for every constant $M > 0$, (3.1) holds, and*

$$\sum_{n=N}^{\infty} \frac{1}{a_n} \sum_{s=N}^{n-1} q_s E^{\alpha}(s+1)C^{\alpha}(\sigma(s+1)) = \infty, \quad (3.16)$$

then every solution of equation (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 3.1, we see that Lemma 2.1 holds and Case(I) is eliminated by the condition (3.1).
Case(II) Proceeding as in the proof of Theorem 3.1(Case(II)), we have

$$z_{\tau(n)} \ge -a_n \Delta z_n E(n) \ge -a_N \Delta z_N E(n) = dE(n)$$

where $d = -a_N \Delta z_N$. From equation (1.1), we have

$$\Delta(-a_n \Delta z_n) \ge Lq_n x^{\alpha}_{\sigma(n+1)},$$

and

$$\frac{x_{\sigma(n+1)}}{z_{\tau(n+1)}} \ge C(\sigma(n+1)).$$

Hence

$$\Delta(-a_n \Delta z_n) \ge d^{\alpha} Lq_n C^{\alpha}(\sigma(n+1))E^{\alpha}(n+1).$$

Summing the last inequality from N to $n-1$, we obtain

$$\begin{aligned} -a_n \Delta z_n &\ge -a_N \Delta z_N + d^{\alpha} L \sum_{s=N}^{n-1} q_s C^{\alpha}(\sigma(s+1))E^{\alpha}(s+1) \\ &\ge Ld^{\alpha} \sum_{s=N}^{n-1} q_s C^{\alpha}(\sigma(s+1))E^{\alpha}(s+1). \end{aligned}$$

Again summing the last inequality from N to $n-1$, we have

$$z_N \ge z_N - z_n \ge Ld^{\alpha} \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=N}^{s-1} q_t C^{\alpha}(\sigma(t+1))E^{\alpha}(t+1).$$

Letting $n \to \infty$ in the above inequality, we obtain

$$Ld^{\alpha} \sum_{n=N}^{\infty} \frac{1}{a_n} \sum_{t=N}^{n-1} q_t C^{\alpha}(\sigma(t+1))E^{\alpha}(t+1) \le z_N$$

a contradiction to (3.16). This completes the proof. □

Next, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1) when $0 < \alpha \le 1$.

Theorem 3.4. *Assume that $0<\alpha\le 1$, and there exist a real sequence $\{\tau(n)\}$ of integers such that $\tau(n)\ge n$, $\Delta\tau(n)>0$ and $\sigma(n)\le\tau(n)-k$. If there exists a positive nondecreasing sequence $\{\rho_n\}$ such that for all constants $M_1>0$ and $M_2>0$, one has*

$$\sum_{n=N}^{\infty}\Big[L\rho_n q_n B^{\alpha}(\sigma(n+1)) - M_1^{1-\alpha}\frac{\Delta\rho_n a^{\alpha}_{\sigma(n)}}{A^{\alpha}(n)}\Big]=\infty \tag{3.17}$$

and

$$\sum_{n=N}^{\infty}\Big[LM_2^{\alpha-1}q_nE(n+1)C^{\alpha}(\sigma(n+1)) - \frac{1}{4a_{\tau(n)}E(n+1)}\Big]=\infty, \tag{3.18}$$

then every solution of equation (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 3.1, we see that Lemma 2.1 holds for all $n\ge N\in\mathbb{N}(n_0)$.
Case(I). Define

$$w_n=\rho_n\frac{a_n\Delta z_n}{z^{\alpha}_{\sigma(n)}},\quad n\ge N.$$

Then $w_n>0$ and from equation (1.1) and from (2.1), we have

$$\Delta w_n\le -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \Delta\rho_n\frac{a_{\sigma(n)}\Delta z_{\sigma(n)}}{z^{\alpha}_{\sigma(n+1)}},\quad n\ge N.$$

Using (2.6) in the last inequality, we obtain

$$\Delta w_n\le -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n a^{\alpha}_{\sigma(n)}}{A^{\alpha}(n)}(a_{\sigma(n)}\Delta z_{\sigma(n)})^{1-\alpha},\quad n\ge N.$$

From the monotonicity of $\{a_n\Delta z_n\}$ and $0<\alpha\le 1$, we have from the last inequality

$$\Delta w_n\le -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n a^{\alpha}_{\sigma(n)}}{A^{\alpha}(n)}M_1^{1-\alpha},\quad n\ge N, \tag{3.19}$$

where $M_1=a_{\sigma(N)}\Delta z_{\sigma(N)}$. Summing the inequality (3.19) from N to $n-1$, we obtain

$$0<w_n\le w_N-\sum_{s=N}^{n-1}\Big(L\rho_s q_s B^{\alpha}(\sigma(s+1)) - \frac{M_1^{1-\alpha}a^{\alpha}_{\sigma(s)}\Delta\rho_s}{A^{\alpha}(s)}\Big). \tag{3.20}$$

Letting $n\to\infty$ in (3.20), we obtain a contradiction to (3.17).
Case(II). Define

$$v_n=\frac{a_n\Delta z_n}{z_{\tau(n)}},\quad n\ge N. \tag{3.21}$$

Then $v_n<0$ for $n\ge N$. Further, we have

$$a_s\Delta z_s\le a_n\Delta z_n,\quad s\ge n.$$

Dividing the last inequality by a_s and then summing it from $\tau(n)$ to ℓ, we obtain

$$z_{\ell+1}-z_{\tau(n)}\le a_n\Delta z_n\sum_{s=\tau(n)}^{\ell}\frac{1}{a_s}.$$

Letting $\ell\to\infty$, we obtain

$$0\le z_{\tau(n)}+a_n\Delta z_nE(n)$$

or

$$-1\le v_nE(n),\quad n\ge N. \tag{3.22}$$

From (3.21) and equation (1.1), we have

$$\Delta v_n\le -\frac{Lq_nx^{\alpha}_{\sigma(n+1)}}{z_{\tau(n+1)}} - \frac{a_n\Delta z_n\Delta z_{\tau(n)}}{z_{\tau(n)}z_{\tau(n+1)}}.$$

Since $\tau(n) \geq n$ and $a_n\Delta z_n$ is negative and decreasing, we have

$$a_{\tau(n)}\Delta z_{\tau(n)} \leq a_n\Delta z_n.$$

Therefore

$$\Delta v_n \leq -Lq_n \frac{x^{\alpha}_{\sigma(n+1)}}{z_{\tau(n+1)}} - \frac{(a_n\Delta z_n)^2}{a_{\tau(n)}z_{\tau(n)}z_{\tau(n+1)}}, \quad n \geq N.$$

Since z_n is positive and decreasing, we have $z_{\tau(n+1)} \leq z_{\tau(n)}$ for $n \geq N$. Combining the last two inequalities, we obtain

$$\Delta v_n \leq -Lq_n \frac{x^{\alpha}_{\sigma(n+1)}}{z_{\tau(n+1)}} - \frac{v_n^2}{a_{\tau(n)}}, \quad n \geq N. \tag{3.23}$$

Now using (3.11) in (3.23), we have

$$\Delta v_n \leq -Lq_n \frac{C^{\alpha}(\sigma(n+1))}{M_2^{1-\alpha}} - \frac{v_n^2}{a_{\tau(n)}}$$

for some constant $M_2 = z_{\tau(N+1)} > 0$. That is,

$$\Delta v_n + LM_2^{\alpha-1}q_nC^{\alpha}(\sigma(n+1)) + \frac{v_n^2}{a_{\tau(n)}} \leq 0, \quad n \geq N. \tag{3.24}$$

Multiplying (3.23) by $E(n+1)$, and then summing it from N to $n-1$, we have

$$\sum_{s=N}^{n-1} E(s+1)\Delta v_s \quad + \quad \sum_{s=N}^{n-1} LM_2^{\alpha-1}q_sE(s+1)C^{\alpha}(\sigma(s+1)) + \sum_{s=N}^{n-1} \frac{E(s+1)v_s^2}{a_{\tau(s)}} \leq 0. \tag{3.25}$$

Using the summation by parts formula in the first term of (3.25) and then rearranging, we obtain

$$E(n)v_n - E(N)v_N \quad + \quad \sum_{s=N}^{n-1} LM_2^{\alpha-1}q_sE(s+1)C^{\alpha}(\sigma(s+1)) + \sum_{s=N}^{n-1}\Big(\frac{v_s}{a_{\tau(s)}} + \frac{v_s^2E(s+1)}{a_{\tau(s)}}\Big) \leq 0.$$

Using completing the square in the last term of the above inequality, we obtain

$$\begin{aligned} &E(n)v_n - E(N)v_N + \sum_{s=N}^{n-1} LM_2^{\alpha-1}q_sE(s+1)C^{\alpha}(\sigma(s+1)) \\ &+ \sum_{s=N}^{n-1}\frac{E(s+1)}{a_{\tau(s)}}\Big(v_s + \frac{1}{2E(s+1)}\Big)^2 - \sum_{s=N}^{n-1}\frac{1}{4a_{\tau(s)}E(s+1)} \leq 0 \end{aligned}$$

or

$$E(n)v_n \quad \leq \quad E(N)v_N - \sum_{s=N}^{n-1}\Big(LM_2^{\alpha-1}q_sE(s+1)C^{\alpha}(\sigma(s+1)) - \frac{1}{4a_{\tau(s)}E(s+1)}\Big).$$

Letting $n \to \infty$ in the last inequality and using (3.22), we obtain a contradiction to (3.18). The proof is now complete. □

4 Examples

In this section, we present some examples to illustrate the main results.
Example 4.1. Consider the neutral difference equation

$$\Delta\Big(2^{n+1}\Delta(x_n + 2x_{n-2})\Big) + 9 \times 2^{n+2}x_{n-1} = 0, \quad n \in \mathbb{N}(0). \tag{4.1}$$

Here $a_n = 2^{n+1}$, $p_n = 2$, $k = 2$, $\sigma(n+1) = n-1$, $\alpha = 1$, $q_n = 36(2^n)$ and $\tau(n) = n+2$. Then $R(n) = \frac{2^n-1}{2^n}$, $E(n) = \frac{1}{2^{n+2}}$, $C(n) = \frac{1}{4}$ and $B(n) = \frac{1}{16}\Big(\frac{4\ (2^{n+2})-7}{2^{n+2}-1}\Big)$. By taking $\rho_n = 1$, we see that all conditions of Theorem 3.1 are satisfied and hence every solution of equation (4.1) is oscillatory. In fact

$\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (4.1) since it satisfies the given equation.

Example 4.2. Consider the neutral difference equation

$$\Delta\Big(2^{n+1}\Delta(x_n + 2x_{n-3})\Big) + 1905 \times 2^{5(n-3)}x_{n-2}^3 = 0, \quad n \in \mathbb{N}(0). \tag{4.2}$$

Here $a_n = 2^{n+1}$, $p_n = 2$, $k = 3$, $\sigma(n+1) = n-2$, $\alpha = 3$, $q_n = \left(\frac{1905}{32768}\right)2^{5n}$, $L = 1$ and $\tau(n) = n+2$. Then $R(n) = \frac{2^n-1}{2^n}$, $E(n) = \frac{1}{2^{n+2}}$, $C(n) = \frac{1}{4}$ and $B(n) = \frac{1}{32}\left(16 - \frac{2^{n+6}-1}{2^{n+1}-1}\right)$. By taking $\rho_n = 1$, we see that all conditions of Theorem 3.2 are satisfied and hence every solution of equation (4.2) is oscillatory. In fact $\{x_n\} = \{\frac{(-1)^n}{4^n}\}$ is one such oscillatory solution of equation (4.2) since it satisfies the given equation.

Example 4.3. Consider the neutral difference equation

$$\Delta\Big((n+1)(n+2)\Delta(x_n + 3x_{n-1})\Big) + 8(n+2)^2 x_{n-2}^{1/3} = 0, \quad n \in \mathbb{N}(1). \tag{4.3}$$

Here $a_n = (n+1)(n+2)$, $p_n = 3$, $k = 1$, $\sigma(n+1) = n-2$, $\alpha = \frac{1}{3}$, $q_n = 8(n+2)^2$ and $\tau(n) = n$. Then $R(n) = \frac{n-1}{2(n+1)}$, $E(n) = \frac{1}{(n+1)}$, $C(n) = \frac{2}{9}$ and $B(n) = \frac{2}{9}\left(\frac{n^2+3n-1}{n(n+3)}\right)$. By taking $\rho_n = 1$, we see that all conditions of Theorem 3.4 are satisfied and hence every solution of equation (4.3) is oscillatory. In fact $\{x_n\} = \{(-1)^{3n}\}$ is one such oscillatory solution of equation (4.3) since it satisfies the given equation.
We conclude this paper with the following remark.

Remark 4.1. *The results obtained in this paper are new and complement to that of in [8, 11, 12].*

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Maximum principles for fourth order semilinear elliptic boundary value problems

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Dedicated to 61^{th} Birthday of Dr. D. B. Dhaigude

Abstract

The paper is devoted to prove maximum principles for the certain functionals defined on solution of the fourth order semilinear elliptic equation. The maximum principle so obtained is used to prove the non-existence of nontrivial solutions of the fourth order semilinear elliptic equation with some zero boundary conditions. Hopf's maximum principle is main ingredient.

Keywords: Maximum principles, fourth order elliptic equations, integral bounds.

2010 MSC: 35J40, 35J61.

1 Introduction

The 'P- function' technique for deducing maximum principle results for partial differential equations of order ≥ 2 is well known. For instance, [5] Miranda shows that the P- function

$$P = |\nabla u(x)|^2 - u\Delta u \tag{1.1}$$

is subharmonic, where u is a classical solution to the biharmonic equation $\Delta^2 u = 0$. Since, then many others have employed this technique on various classes of fourth order partial differential equations. In [7], for example, Schaefer utilizes auxiliary functions of type (1.1) to study semilinear equations of the form

$$\Delta^2 u + \rho(x, y)f(u) = 0,$$

in a plane domain. Still other types of functions have been employed in the pursuit of maximum principle results for fourth order differential equations [1, 2, 4]. Recently [3] Dhaigude and Gosavi extend a maximum principle for a class of fourth order semilinear elliptic equations due to Schaefer [7] to a more general fourth order semilinear elliptic equation of the form

$$\Delta^2 u + a(x, y)\Delta u + b(x, y)f(u) = 0.$$

In this paper, we study the existence problem for fourth order semilinear elliptic equation of the form

$$\Delta^2 u + a(x, y)\Delta u + b(x, y)f(u) = 0.$$

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For simplicity, we use the summation convention and denote partial derivatives $\dfrac{\partial u}{\partial x_i}$ by $u_{,i}$ and $\dfrac{\partial^2 u}{\partial x_i^2}$ by $u_{,ii}$.

This paper is organized as follows. In section 2 we develop a maximum principle for a class of fourth order semilinear elliptic equations. The maximum principle will be used to deduce the non-existence of nontrivial solutions of the boundary value problem under consideration in the last section of this paper.

2 Maximum principles

Suppose Ω is a plane domain bounded by a sufficiently smooth curve $\partial\Omega$. The following Lemma [8] is useful to prove our results.

Lemma 2.1. *For a sufficiently smooth function v the inequality*

$$N v_{,ik} v_{,ik} \geq (\Delta v)^2$$

holds in N dimensions.

Now, we prove the following maximum principles for the function P denoted by $P = |\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds$, which will be the main result of this paper.

Theorem 2.1. *Let $u \in C^4$ be a sufficiently smooth solution of*

$$\Delta^2 u + a(x,y)\Delta u + b(x,y)f(u) = 0 \tag{2.1}$$

where $a \leq 0, b > 0$ in Ω and

$$b(x,y)u(x,y)f(u) + a(x,y)|\nabla u|^2 \geq 0 \quad in \quad \Omega. \tag{2.2}$$

If φ satisfy

$$\varphi(s) \geq 0,\ \varphi^{'}(s) \geq 0\, for\ s \geq 0,\ \int_0^u \varphi(s)\,ds \leq 0 \tag{2.3}$$

then the function

$$P = |\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds \tag{2.4}$$

assumes its maximum on $\partial\Omega$.

Proof. We have, the function

$$P = |\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds.$$

By straightforward computations

$$P_{,k} = 2u_{,i}u_{,ik} - u_{,k}\Delta u - u(\Delta u)_{,k} + \varphi(u)\,u_{,k} \tag{2.5}$$

$$P_{,kk} = 2u_{,ik}u_{,ik} - (\Delta u)^2 - u\Delta^2 u + \varphi^{'}(u)|\nabla u|^2 + \varphi(u)\Delta u. \tag{2.6}$$

Using (2.1) in (2.6), we get

$$\Delta P = 2u_{,ik}u_{,ik} - (\Delta u)^2 + au\Delta u + buf + \varphi^{'}(u)|\nabla u|^2 + \varphi(u)\Delta u. \tag{2.7}$$

Using (2.4), we have

$$\begin{aligned}\Delta P + aP =& 2u_{,ik}u_{,ik} - (\Delta u)^2 + a|\nabla u|^2 + buf + \varphi^{'}(u)|\nabla u|^2 \\ &+ \varphi(u)\Delta u + a\int_0^u \varphi(s)\,ds.\end{aligned} \tag{2.8}$$

By Lemma 2.1 and assumption (2.2) and (2.3), we see that the right hand side of (2.8) is non-negative. Thus

$$\Delta P + aP \geq 0 \quad in \quad \Omega.$$

□

By maximum principle, the result follows.

Theorem 2.2. *Let $u \in C^4$ be a sufficiently smooth solution of*

$$\Delta^2 u + a(x,y)\Delta u + b(x,y)f(u) = 0 \tag{2.9}$$

where $a \leq 0, b > 0$ in Ω and

$$b(x,y)u(x,y)f(u) + a(x,y)|\nabla u|^2 \geq 0 \quad in \quad \Omega. \tag{2.10}$$

If φ satisfy

$$\varphi(s) \geq 0,\ \varphi^{'}(s) \geq 0\ for\, s \geq 0,\ \int_0^u \varphi(s)\,ds \leq 0 \tag{2.11}$$

then the function

$$P = \frac{1}{b}\left[|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds\right] \tag{2.12}$$

assumes its maximum on $\partial\Omega$ unless $P < 0$ in Ω.

Proof. We have, the function

$$P = \frac{1}{b}\left[|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds\right]$$

By straightforward computations

$$P_{,k} = \frac{1}{b}\left[\nabla(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds)\right] - \frac{b_{,k}}{b^2}\left(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds\right) \tag{2.13}$$

$$\begin{aligned} P_{,kk} = \frac{1}{b}\left[\Delta(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds)\right] - \frac{b_{,k}}{b^2}\Big[\nabla(|\nabla u(x)|^2 - u\Delta u + \\ \int_0^u \varphi(s)\,ds)\Big] - \frac{b_{,k}}{b^2}\left[\nabla(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds)\right] - \frac{b_{,kk}}{b^2}\Big(|\nabla u(x)|^2 - \\ u\Delta u + \int_0^u \varphi(s)\,ds\Big) + \frac{2b_{,k}b_{,k}}{b^3}\left(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds\right). \end{aligned} \tag{2.14}$$

Using (2.12) and after some rearrangements, we have

$$\begin{aligned} \Delta P - 2b\nabla\left(\frac{1}{b}\right)\nabla P + \frac{\Delta b}{b}P = \\ \frac{1}{b}\left[\Delta(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds)\right]. \end{aligned} \tag{2.15}$$

$$\begin{aligned} \Delta P - 2b\nabla\left(\frac{1}{b}\right)\nabla P + \left[\frac{\Delta b}{b} + a\right]P = \\ \frac{1}{b}\Big[2u_{,ik}u_{,ik} - (\Delta u)^2 + a|\nabla u|^2 + buf \\ + \varphi^{'}(u)|\nabla u|^2 + \varphi(u)\Delta u + a\int_0^u \varphi(s)\,ds\Big]. \end{aligned} \tag{2.16}$$

Then it follows from Lemma 2.1 and assumptions (2.10) and (2.11) that P satisfies

$$\Delta P - 2b\nabla\left(\frac{1}{b}\right)\nabla P + \left[\frac{\Delta b}{b} + a\right]P \geq 0 \quad in \quad \Omega.$$

By Hopf's maximum principle [6], the result follows. □

The next Lemma [7]is useful in proving the non-existence result in the last section of the paper.

Lemma 2.2. *If (2.2) is satisfied and if u is a C^4 solution of (2.1) which vanishes on $\partial\Omega$, then*

$$\int_\Omega |\nabla u(x)|^2 dxdy \leq \frac{1}{2}A|\nabla u(x)|^2_M$$

where A is the area of Ω.

3 Applications

In this section as an application of our maximum principle we prove non-existence of nontrivial solutions $u \in C^4$ of the following boundary value problem

$$\Delta^2 u + a(x,y)\Delta u + b(x,y) f(u) = 0, \quad \text{in} \quad \Omega \tag{3.1}$$

$$u(x,y) = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.2}$$

and

$$\Delta^2 u + a(x,y)\Delta u + b(x,y) f(u) = 0, \quad \text{in} \quad \Omega \tag{3.3}$$

$$u(x,y) = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.4}$$

Theorem 3.1. *If (2.2) is satisfied then no non-trivial solution of (3.1)-(3.2) exists.*

Proof. It is by contradiction. Assume on the contrary that a nontrivial solution u of the given BVP (3.1)-(3.2) exists. We have P as defined in (2.4). Now, Theorem 2.1 and boundary condition (3.2) gives

$$u_{,i}u_{,i} - u\Delta u + \int_0^u \varphi(s)\, ds \leq 0. \tag{3.5}$$

Further integrating (3.5) over Ω, we have

$$\int_\Omega \Big[u_{,i}u_{,i} - u\Delta u \Big] dxdy + \int_\Omega \left(\int_0^u \varphi(s)\, ds \right) \leq 0. \tag{3.6}$$

Using Green's first identity

$$\int_\Omega \Big[v\Delta u + \nabla v \cdot \nabla u \Big] dxdy = \int_{\partial\Omega} v \frac{\partial u}{\partial n} d\sigma, \quad \text{with} \quad v = u \tag{3.7}$$

and (3.2) in (3.7), we get

$$2\int_\Omega |\nabla u|^2 dxdy + \int_\Omega \left(\int_0^u \varphi(s)\, ds \right) \leq 0. \tag{3.8}$$

Consequently $|\nabla u| = 0$ in Ω and by continuity $u \equiv 0$ in $\Omega \cup \partial\Omega$. This is a contradiction. Hence there is no nontrivial solution of (3.1)-(3.2). □

Theorem 3.2. *If (2.2) is satisfied in a convex domain Ω then no nontrivial solution of (3.3)-(3.4) exists.*

Proof. It is by contradiction. Assume on the contrary that a nontrivial solution u of the given BVP (3.3) - (3.4) exists. We have P as defined in (2.4). Then by Theorem 2.1, P takes its maximum on the boundary $\partial\Omega$ at a point, say Q. By Hopf's second maximum principle, either $\frac{\partial P}{\partial n}(Q) > 0$ or P is constant in $\Omega \cup \partial\Omega$.

Case I. Suppose $\frac{\partial P}{\partial n}(Q) > 0$ holds. Differentiate P partially in the normal direction and use boundary condition (3.4) to get

$$\frac{\partial P}{\partial n}(Q) = 2\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2}. \tag{3.9}$$

We know the following relation from differential geometry,

$$\frac{\partial^2 u}{\partial n^2} + k\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial s^2} = u_{,ii} = \Delta u \quad (see [8], p.46) \tag{3.10}$$

where $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial s}$ are normal and tangential derivatives respectively. The tangential component $\frac{\partial^2 u}{\partial s^2}$ is zero. Equation (3.10) becomes

$$\frac{\partial^2 u}{\partial n^2} = \Delta u - k\frac{\partial u}{\partial n}. \tag{3.11}$$

Using (3.11) and (3.4) in (3.9), we get

$$\frac{\partial P}{\partial n}(Q) = -2k\left(\frac{\partial u}{\partial n}\right)^2. \tag{3.12}$$

Since Ω is convex, $k > 0$. So $\frac{\partial P}{\partial n}(Q) > 0$ is impossible. Therefore, in this case no nontrivial solution exists.

Case II. Suppose P is a constant say c in $\Omega \cup \partial\Omega$. Then we have

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial n}\right)^2 = c \quad \text{on} \quad \partial\Omega. \tag{3.13}$$

Now as $P = c$ in $\Omega \cup \partial\Omega$, we have $\frac{\partial P}{\partial n} = 0$ on $\partial\Omega$. But from (3.12) we have

$$\frac{\partial P}{\partial n}(Q) = -2kc.$$

For a bounded convex domain with a continuously turning tangent on the boundary, $k \neq 0$. Moreover $c \neq 0$, for if $c = 0$ then $|\nabla u|_M = 0$ and by Lemma 2.2 and reasoning as in Theorem 2.1 we are led to the conclusion that $u \equiv 0$ in Ω. Thus $P = c$ is impossible. As neither case is possible, we conclude that no nontrivial solution exists. □

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Triparametric self information function and entropy

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Abstract

In this paper we start with a triparametric self information function and triparametric entropy. Some familiar entropies are derived as particular cases. A measure called information deviation and some generalization of Kullback's information are obtained under some boundary conditions.

Keywords: Shannon entropy, Kullback's information, joint entropy, generalized inaccuracy, information deviation.

2010 MSC: 94A15, 94A24, 26D15.

1 Introduction

Shannon[10] first introduced the idea of self-information function in the form

$$f(x) = -\log_2 x, \quad 0 < x \leq 1. \tag{1.1}$$

In this paper we use the method of averaging self-informations introduced by Shannon. Like Shannon we introduce a triparametric self-information function defined by

$$f_3(x;\alpha,\beta,\gamma) = \frac{k(x^{\alpha/\gamma} - x^{\beta/\gamma})}{x}, \quad 0 < x \leq 1,\ \alpha \geq 0,\ \beta \geq 0,\ \ \gamma > 0,\ \alpha \neq \beta \neq \gamma \tag{1.2}$$

Where k is a constant, depending upon the real valued parameters $\alpha,\ \beta,\ \gamma$ and k is ascertained by a suitable pair $(x,\ f_3)$. We apply the following conditions on f_3:

(i) $f_3 \to\ 0$ as $x \to 0$.
(ii) $f_3 = 0$, when $x = 1$.
(iii) $f_3 = 1$, $x = \frac{1}{2}$, then $k = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1}$.

The function shows the following particular behaviors:

(I) If $\alpha,\ \beta$ are fixed, then for $x < \frac{1}{2}$, $f_3 \to \infty$ as $\gamma \to \infty$ and for $x > \frac{1}{2}$, $f_3 \to 0$ as $\gamma \to \infty$.
(II) For any fixed γ, $f_3 \to -(2x)^{\frac{\alpha-\gamma}{\gamma}} \log_2 x$ as $\alpha \to \beta$.
(III) If $\beta = \gamma$ and $\alpha \to \gamma(\alpha < \gamma)$, then $f_3 \to -\log_2 x$.

Self-information function is different from information function. Different authors, namely Darcozy [4], Aczel [1], Arndt [2], Chaundy and Mcleod [3], Havrda and Charvat [5], Kannapan [6], Sharma and Taneja [11], Mittal

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[9] and some others have solved some typical functional equations and have used their solutions as entropy, inaccuracy, directed divergence etc., In the capacity of finite measures only in complete probability distributions. The method of averaging self-informations includes the case of generalized probability distributions. Moreover, we have discussed in this paper, information measures in the capacity of even an infinite range, because a parameter can have negative values also corresponding to phenomenal circumstances. Further since it is uncertain and difficult to choose an arbitrary functional equation and to find its suitable solutions to be used as information measures, it becomes easier if we choose any suitable parametric self-information function that can satisfy a number of effective boundary conditions. We have given a most simple and general choice in (1.2).

Section 2 describes a triparametric entropy from which other familiar entropies have been deduced as particular cases. We have given a number of this entropy in section 3 as joint entropy, triparmetric information functions, generalized information function, generalized inaccuracy, a new information called information deviation and lastly generalizations of Kullback's information.

2 Triparametric entropy

Let $P = (p_1, p_2, ..., p_n)$ be a finite discrete probability distribution, where $0 < p_i \leq 1, \quad \sum_{i=1}^{n} p_i \leq 1$. Then, averaging the function $f_3(p_i; \alpha, \beta, \gamma)$ with respect to P, we define the triparametric entropy as

$$H(P;\alpha,\beta,\gamma) = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1} \left[\sum_{i=1}^{n}(p_i^{\alpha/\gamma} - p_i^{\beta/\gamma})\right] \Big/ \sum_{i=1}^{n} p_i, \tag{2.1}$$

where $\alpha, \beta, \gamma > 0, \ \alpha \neq \beta \neq \gamma$.
When P is complete, we have

$$H(P;\alpha,\beta,\gamma) = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1} \left[\sum_{i=1}^{n}(p_i^{\alpha/\gamma} - p_i^{\beta/\gamma})\right], \tag{2.2}$$

where $\alpha \geq 0, \ \beta \geq 0, \ \ \gamma > 0, \ \alpha \neq \beta \neq \gamma$.

2.1 Some familiar entropies

From (2.2), we get the following entropies as particular cases:
(i) $\gamma = 1$ gives Sharma and Taneja's entropy [11] of type (α,β) in the form

$$H(P;\alpha,\beta) = \left(2^{1-\alpha} - 2^{1-\beta}\right)^{-1} \left[\sum_{i=1}^{n}(p_i^{\alpha} - p_i^{\beta})\right], \quad \alpha \neq \beta \tag{2.3}$$

and

$$\underset{\alpha\to\beta}{Lt} H(P;\alpha,\beta) = \left(\sum_{i=1}^{n} p_i^{\beta} \log_2 \frac{1}{p_i}\right) 2^{\beta-1}.$$

(ii) Putting $\alpha = \gamma = 1$, we get Darcozy's entropy [4] of type β as

$$H(P;\beta) = \left(2^{1-\beta} - 1\right)^{-1} \left[\sum_{i=1}^{n}(p_i^{\beta} - 1)\right], \quad \beta > 0, \ \beta \neq 1 \tag{2.4}$$

(iii) When $\beta = \gamma$ and $\alpha \to \gamma$, then (2.2) reduces to

$$H(P) = \sum_{i=1}^{n} p_i \log_2 \frac{1}{p_i} \ , \tag{2.5}$$

which is Shannon entropy.
(iv) When $n > 2$, then $H \to \infty$ as $\gamma \to \infty$; when $n = 1$, then $H = 0, \ p_1 = 1$ and when $n = 2$, then $H = 1$.

3 Application of the entropy (2.2)

3.1 Joint entropy

For joint probability distribution, a relation similar to (2.2) also holds in the form

$$H(PQ;\alpha,\beta,\gamma)=\left(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1}\left[\sum_{k=1}^{n}\sum_{j=1}^{m}(p_{kj}^{\alpha/\gamma}-p_{kj}^{\beta/\gamma})\right], \tag{3.1}$$

$$0<p_{kj}\leq 1,\ \sum_{k=1}^{n}\sum_{j=1}^{m}p_{kj}=1;\ \alpha\geq 0,\ \beta\geq 0,\gamma>0,\ \alpha\neq\beta.$$

Theorem 3.1. *If $P=(p_1,p_2,...,p_n)$ be the distribution of input symbols of a source, $Q=(q_1,q_2,...,q_m)$ be that of output symbols and $PQ=(p_{k1},p_{k2},...,p_{km};\ \ k=1,2,...,n)$ be the joint distribution of input and output symbols; also*

$$R_k=\left(\frac{p_{k1}}{p_k},\frac{p_{k2}}{p_k},...,\frac{p_{km}}{p_k}\right)$$

be the conditional distribution of output symbols and

$$R_j=\left(\frac{p_{1j}}{q_j},\frac{p_{2j}}{q_j},...,\frac{p_{nj}}{q_j}\right)$$

be the conditional distribution of input symbols, where

$$p_{kj}/p_k=p_{j|k},(j=1,2,...,m);\ \ p_{kj}/q_j=p_{k|j},\ (k=1,2,...,n);$$

$$\sum_{j=1}^{m}p_{kj}=p_k\ \textit{and}\ \sum_{k=1}^{n}p_{kj}=q_j,$$

then

$$H(PQ;\alpha,\beta,\gamma)=\sum_{k=1}^{n}p_k^{\frac{\beta}{\gamma}}H(R_k;\alpha,\beta,\gamma)+\left(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\beta-\gamma}{\gamma}}\right)^{-1}\left[\sum_{k=1}^{n}\left(p_k^{\frac{\alpha}{\gamma}}-p_k^{\frac{\beta}{\gamma}}\right)\sum_{j=1}^{m}p_{j|k}^{\frac{\alpha}{\gamma}}\right]. \tag{3.2}$$

Putting $\alpha=\gamma=1$ and using $\sum_{j=1}^{m}p_{j|k}=1$ in (3.2), we have

$$H(PQ;\ \beta)=\sum_{k=1}^{n}p_k^{\beta}H_1(R_k;\beta)+H_1(P;\beta). \tag{3.3}$$

Theorem 3.2. *If $p_{kj}=p_kq_j$, then*

$$\begin{aligned}H(PQ;\ \alpha,\beta,\gamma)&=\sum_{k=1}^{n}p_k^{\frac{\alpha}{\gamma}}H(R_k;\alpha,\beta,\gamma)+\sum_{j=1}^{m}q_j^{\frac{\beta}{\gamma}}H(R_j;\alpha,\beta,\gamma)\\&=\sum_{k=1}^{n}p_k^{\frac{\alpha}{\gamma}}H(Q;\alpha,\beta,\gamma)+\sum_{j=1}^{m}q_j^{\frac{\beta}{\gamma}}H(P;\alpha,\beta,\gamma).\end{aligned} \tag{3.4}$$

3.2 Triparametric information function

With the help of equation (2.2), we define a triparametric information function in the form

$$F_3(x)=F_3(x;\alpha,\beta,\gamma)=\left(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1}(x^{\alpha/\gamma}-x^{\beta/\gamma}) \tag{3.5}$$

$\alpha\geq 0,\ \beta\geq 0,\ \ \gamma>0,\ \alpha\neq\beta\neq\gamma$ and $0<x\leq 1$.
Where $F_3(0)=0$, but $F_3(1)=0$ and $F_3\left(\frac{1}{2}\right)=\frac{1}{2}$ always.
Thus

$$H(P;\alpha,\beta,\gamma)=\sum_{k=1}^{n}F(p_k),\ \ 0<p_k\leq 1,\ \ \ \sum_{ki=1}^{n}p_k=1. \tag{3.6}$$

Putting $a=\alpha/\gamma,\ \ b=\beta/\gamma$ in (3.5), we have

$$F_3(x)=F(x;\alpha,\beta,\gamma)=\left(2^{1-a}-2^{1-b}\right)^{-1}(x^a-x^b),\ \ -\infty<a,b<\infty,\ \ a\neq b. \tag{3.7}$$

Now, from practical point of view, as far as an inaccuracy in a measure is concerned, a measure is associated with at least two probability distributions, corresponding to which at least two variables u and v are needed. This suggests the choice of at least four parameters a,b,c and d.

3.3 Generalized information function

Concerning an association of two variables u, v and four parameters a, b, c, d, an information measure similar to (3.7) is introduced by

$$F_4(u,v) = F(u,v;a,b,c,d) = G[u^a v^b - u^c v^d], \tag{3.8}$$

$$0 < u, v \leq 1, \quad -\infty < a, b, c, d < \infty, \quad a \neq b \neq c \neq d$$

as the generalized information function, which possesses the characteristic of becoming both bounded and unbounded.

3.3.1 Boundary conditions

(i) At $u = 1$, $v = \frac{1}{2}$, $F_4\left(1, \frac{1}{2}\right) = \frac{1}{2}$, so that $G = (2^{1-b} - 2^{1-d})^{-1}$, where $b \neq d$.

If $a + b = c + d$, where $a \neq c$, then $F_4\left(\frac{1}{2}, \frac{1}{2}\right) = 0$. Similarly at $u = \frac{1}{2}$, $v = 1$, $F_4\left(\frac{1}{2}, 1\right) = \frac{1}{2}$ so that $G = (2^{1-a} - 2^{1-c})^{-1}$, where $a \neq c$.

(ii) At $u = 1$, $v = \frac{1}{2}$, $F_4\left(1, \frac{1}{2}\right) = \frac{1}{2}$ so that $G = (2^{-b} - 2^{-d})^{-1}$, where $b \neq d$.
At $u = \frac{1}{2}$, $v = 1$, $F_4\left(\frac{1}{2}, 1\right) = 1$, so that $G = (2^{-a} - 2^{-c})^{-1}$, where $a \neq c$.

3.3.2 Generalize inaccuracy

Let $P = (p_1, p_2, ..., p_n)$ and $Q = (q_1, q_2, ..., q_n)$ be two discrete probability distributions concerned with (3.8), where $0 < p_i \leq 1$, $0 < q_i$, $\sum_{i=1}^n p_i = 1$, $\sum_{i=1}^n q_i = 1$, $(u, v) = (p_i, q_i)$ or (q_i, p_i); $i = 1, 2, ..., n$.
We may then define the generalized inaccuracies by

$$I_4(P \parallel Q) = \sum_{i=1}^n F_4(p_i, q_i) = (2^{1-b} - 2^{1-d})^{-1} \left[\sum_{i=1}^n p_i^a q_i^b - \sum_{i=1}^n p_i^c q_i^d\right], b \neq d, \tag{3.9}$$

$$I_4(Q \parallel P) = \sum_{i=1}^n F_4(q_i, p_i) = (2^{1-b} - 2^{1-d})^{-1} \left[\sum_{i=1}^n q_i^a p_i^b - \sum_{i=1}^n q_i^c p_i^d\right], b \neq d, \tag{3.10}$$

which follows from (3.8) and boundary condition 3.3.1(i).
Given P and Q, we see that
(i) $I_4(P \parallel Q) \to +\infty$ or $-\infty$, according as $a \to -\infty$ or $c \to -\infty$ for $b < d$; or as $c \to -\infty$ or $a \to -\infty$ for $b > d$.
(ii) If $d = 1$, $c = 0$, then $I_4(P \parallel Q) \to (1 - 2^{1-b})^{-1}$ as $a \to \infty$.
(iii) If $d = 1$, $c = 0$, then $I_4(P \parallel Q) \to 1$ as $b \to \infty$.
It is to be noted that when $d = 1$, $c = 0$, then

$$I_4(Q \parallel P) = (2^{1-b} - 1)^{-1} \left[\sum_{i=1}^n p_i^a q_i^b - 1\right]. \tag{3.11}$$

3.3.3 Information deviations

If $d = 1, c = 0, a + b = 1$, then we introduce the quantities

$$D(Q \parallel P \parallel Q) = \underset{b \to 1}{Lt}\, I_4(P \parallel Q) = H(Q) - H(Q \parallel P) \tag{3.12}$$

and

$$D(P \parallel Q \parallel P) = \underset{b \to 1}{Lt}\, I_4(Q \parallel P) = H(P) - H(P \parallel Q) \tag{3.13}$$

as the information deviation of Q from P and of P from Q respectively, where

$$H(P) = \sum_{k=1}^n p_k \log_2 \frac{1}{p_k}, \quad H(Q) = \sum_{k=1}^n q_k \log_2 \frac{1}{q_k}$$

are Shannon's [10] entropies and

$$H(Q\| P) = \sum_{k=1}^{n} q_k \log_2 \frac{1}{p_k}, \quad H(P\| Q) = \sum_{k=1}^{n} p_k \log_2 \frac{1}{q_k}$$

are Kerridge's [7] inaccuracies. Thus

$$D(Q\| P \|Q) = \sum_{k=1}^{n} q_k \log_2 \frac{p_k}{q_k}, \quad D(P\| Q \|P) = \sum_{k=1}^{n} p_k \log_2 \frac{q_k}{p_k} \tag{3.14}$$

3.3.4 Kullback's information and its generalizations

If we take the boundary conditions 3.3.1(ii), then

$$I_4(P \parallel Q) = \frac{1}{2} I_4^*(P \| Q), \tag{3.15}$$

where

$$I_4^*(P \parallel Q) = (2^{-b} - 2^{-d})^{-1} \left[\sum_{i=1}^{n} p_i^a q_i^b - \sum_{i=1}^{n} p_i^c q_i^d \right], \quad b \neq d. \tag{3.16}$$

Now if $d = 0,\ c = 1,\ a + b = 1$, then

$$\underset{b \to 0}{Lt}\, I_4(P \| Q) = \tfrac{1}{2}\, I(P \| P \| Q), \quad \underset{b \to 0}{Lt}\, I_4(Q \| P) = \tfrac{1}{2}\, I(Q \| Q \| P), \tag{3.17}$$

where

$$D(P\| P \|Q) = \sum_{k=1}^{n} p_k \log_2 \frac{p_k}{q_k} = H(P \|Q) - H(P) \tag{3.18}$$

and

$$D(Q\| Q \|P) = \sum_{k=1}^{n} q_k \log_2 \frac{q_k}{p_k} = H(Q \|P) - H(Q) \tag{3.19}$$

represents Kullback's [8] informations.

Information deviations and Kullback's informations are equal and opposite measures. The fact follows from

$$D(Q \parallel P \|Q) + I(Q \|Q \|P) \ = 0, D(P \parallel Q \|P) + I(P \|P \|Q) \ = 0 \tag{3.20}$$

It may be noted that information deviations and Kullback's informations become zero, if $p_k = q_k$ for $k = 1, 2, ..., n$.

3.3.5 Generalized Boundary Conditions

We shall now show that so far as our generalized inaccuracies (3.9) and (3.10) are concerned, there exist certain boundary conditions for which certain limiting functions of (3.9) and (3.10) may be taken as the generalized forms of Kullback's informations. For this, we generalized the boundary conditions in the following ways and get the results:

(i) Let $u = 1$, $v = \frac{1}{2}$, $F_4\left(1, \frac{1}{2}\right) = \frac{1}{2^m}$,

where m is real number ≥ 0. Then, we have for $d = 0,\ c = 1,\ a + b = 1$,

$$I^{(1)}(P, Q, m) = \underset{b \to 0}{Lt}\, I_4(P \| Q) = 2^{-m} \sum_{k=1}^{n} p_k \log_2 \frac{p_k}{q_k}, \quad m \geq 0 \tag{3.21}$$

to be called the first generalized Kullback's information. For $m = 0$ in (3.21), we get Kullback's information. The information (3.21) decreases as m increases.

(ii) let $u = 1$, $v = \frac{1}{2}$, $F_4\left(1, \frac{1}{2}\right) = \frac{1}{2^m}$, where m is real number ≥ 0. Also let $d = 0,\ c = 1 + m,\ a + b = 1 + m$, then we have

$$I^{(2)}(P, Q, m) = \underset{b \to 0}{Lt}\, I_4(P \| Q) = 2^{-m} \sum_{k=1}^{n} p_k^{m+1} \log_2 \frac{p_k}{q_k}, \quad m \geq 0 \tag{3.22}$$

to be called the second generalized Kullback's information. It is observed that $I^2(P,Q,m) \le I^1(P,Q,m)$.

For $m=0$ in (3.22), we get Kullback's information.

(iii) let $u=1$, $v=\frac{1}{2}$, $F_4\left(1,\frac{1}{2}\right)=2^{-1/m}$, where m is any positive real number. Then the values $d=0,\ c=1+1/m,\ a+b=1+1/m$, lead to the information

$$I^{(3)}(P,Q,m) = \underset{b\to 0}{Lt}\, I_4(P\,\|Q)\; = 2^{-1/m}\sum_{k=1}^{n} p_k^{1/m+1}\log_2\frac{p_k}{q_k}\ , \tag{3.23}$$

which may be called the third generalized Kullback's information. In this case

$\underset{m\to 0}{Lt}\, I^{(3)}(P,Q,m)=0$ and $\underset{m\to\infty}{Lt}\, I^{(3)}(P,Q,m)=I(P\,\|P\,\|Q)$.

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Total edge product cordial labeling of graphs

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Abstract

The total product cordial labeling is a variant of cordial labeling. We introduce an edge analogue product cordial labeling as a variant of total product cordial labeling and name it as total edge product cordial labeling. Unlike to total product cordial labeling the roles of vertices and edges are interchanged in total edge product cordial labeling. We investigate several results on this newly defined concept.

Keywords: Cordial labeling, product cordial labeling, edge product cordial labeling, total edge product cordial labeling.

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1 Introduction

We begin with simple, finite, connected and undirected graph G = (V(G), E(G)) with order p and size q. The members of $V(G)$ and $E(G)$ are commonly termed as graph elements while $|V(G)|$ and $|E(G)|$ denotes number of vertices and edges in graph G respectively. For all standard terminology and notations we follow West [10]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1. *A* graph labeling *is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of mapping is the set of vertices(edges) then the labeling is called* a vertex(an edge) labeling.

Most of the graph labeling techniques trace their origin to β - labelings introduced by Rosa [5] in 1967. This labeling was renamed as graceful labeling by Golomb [3] and it is now the popular term which is defined as follows.

Definition 1.2. *A graph $G = (V(G), E(G))$ of order p and size q is said to be* graceful *if there exists an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ such that the induced function $f^* : E(G) \rightarrow \{1, 2, \ldots, q\}$ defined by $f^*(e = uv) = |f(u) - f(v)|$ for each edge $e = uv$ is a bijection and f is said to be* graceful labeling *of G.*

Vast amount of literature is available on different types of graph labeling. Labeling of graphs is a potential area of research and more than 1500 research papers have been published so far in past five decades. For an extensive survey on graph labeling and bibliographic references we refer to Gallian [2].

Graham and Sloane [4] have introduced harmonious labeling during their study on modular versions of additive bases problems stemming from error correcting codes.

Definition 1.3. *A graph $G = (V(G), E(G))$ is said to be* harmonious *if there exists an injection $f : V(G) \rightarrow Z_q$ such that the induced function $f^* : E(G) \rightarrow Z_q$ defined by $f^*(e = uv) = (f(u) + f(v))$ (mod q) is a bijection and f is said to be* harmonious labeling *of G.*

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In 1987, Cahit [1] have introduced cordial labeling as a weaker version of graceful labeling and harmonious labeling which is defined as follows.

Definition 1.4. *For a graph G, a vertex labeling function $f : V(G) \to \{0,1\}$ induces an edge labeling function $f^* : E(G) \to \{0,1\}$ defined as $f^*(uv) = |f(u) - f(v)|$. Let $v_f(i)$ be the number of vertices of G having label i under f and $e_f(i)$ be the number of edges of G having label i under f^* for $i = 0, 1$. The function f is called* cordial labeling *of G if $|e_f(0) - e_f(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$. A graph is called* cordial *if it admits cordial labeling.*

In the same paper Cahit [1] proved many results on cordial labeling.

After this some labelings schemes like prime cordial labeling, A - cordial labeling, H-cordial labeling, product cordial labeling, etc. are also introduced as variants of cordial labeling.

The concept of E-cordial labeling was introduced by Yilmaz and Cahit [11] which is defined as follows.

Definition 1.5. *For a graph G, an edge labeling function $f^* : E(G) \to \{0,1\}$ induces a vertex labeling function $f : V(G) \to \{0,1\}$ defined as $f(v) = \sum\{f^*(uv)/uv \in E(G)\}$ (mod 2). The function f^* is called* E-cordial labeling *of G if $|e_f(0) - e_f(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$. A graph is called* E-cordial *if it admits E-cordial labeling.*

Definition 1.6. *For a graph G, an edge labeling function $f^* : E(G) \to \{0,1\}$ induces a vertex labeling function $f : V(G) \to \{0,1\}$ defined as $f(v) = \prod\{f^*(uv)/uv \in E(G)\}$. The function f^* is called* edge product cordial labeling *of G if $|e_f(0) - e_f(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$. A graph is called* edge product cordial *if it admits edge product cordial labeling.*

The concept of edge product cordial labeling is introduced in recent past by Vaidya and Barasara [7] and they have investigated several results on this newly defined concept in [7–9].

Definition 1.7. *For a graph G, a vertex labeling function $f : V(G) \to \{0,1\}$ induces an edge labeling function $f^* : E(G) \to \{0,1\}$ defined as $f^*(uv) = f(u)f(v)$. The function f is called* total product cordial labeling *of G if $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| \leq 1$. A graph is called* total product cordial *if it admits total product cordial labeling.*

In 2006, Sundaram *et al.* [6] have introduced total product cordial labeling and also proved some general results.

In this paper we introduce an edge analogue of total product cordial labeling which is defined as follows.

Definition 1.8. *For a graph G, an edge labeling function $f^* : E(G) \to \{0,1\}$ induces a vertex labeling function $f : V(G) \to \{0,1\}$ defined as $f(v) = \prod\{f^*(uv)/uv \in E(G)\}$. The function f^* is called a* total edge product cordial labeling *of G if $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| \leq 1$. A graph is called* total edge product cordial *if it admits total edge product cordial labeling.*

This work also rules out any possibility of forbidden subgraph characterizations for total edge product cordial labeling as it is established that for $n > 2$, K_n is total edge product cordial graph.

Definition 1.9. *Let $C_n^{(t)}$ denote the one-point union of t cycles of length n.*

Definition 1.10. *The* wheel *W_n is defined to be the join $C_n + K_1$. The vertex corresponding to K_1 is known as apex vertex, the vertices corresponding to cycle are known as rim vertices.*

Definition 1.11. *Let $e = uv$ be an edge of graph G and w is not a vertex of G. The edge e is* subdivided *when it is replaced by the edges $e' = uw$ and $e'' = wv$.*

Definition 1.12. *The gear graph G_n is obtained from the wheel W_n by subdividing each of its rim edge.*

Definition 1.13. *The* fan *f_n is the graph obtained by taking $n-2$ concurrent chords in cycle C_{n+1}. The vertex at which all the chords are concurrent is called the apex vertex. It is also given by $f_n = P_n + K_1$.*

Definition 1.14. *The* double fan *DF_n is defined as $P_n + 2K_1$.*

2 Main results

Theorem 2.1. *Every edge product cordial graph of either even order or even size admit total edge product cordial labeling.*

Proof. Let G be an edge product cordial graph with order p and size q. To prove our claim we consider following three cases.

Case 1: When p is even and q is even.

Since G is edge product cordial graph, $v_f(0) = v_f(1) = \frac{p}{2}$ and $e_f(0) = e_f(1) = \frac{q}{2}$. Therefore, $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| = 0$.

Case 2: When p is even and q is odd.

Since G is edge product cordial graph, $v_f(0) = v_f(1) = \frac{p}{2}$ and $|e_f(0) - e_f(1)| = 1$. Therefore, $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| = 1$.

Case 3: When p is odd and q is even.

Since G is edge product cordial graph, $e_f(0) = e_f(1) = \frac{q}{2}$ and $|v_f(0) - v_f(1)| = 1$. Therefore, $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| = 1$.

Thus in either case G satisfies the condition for total edge product cordial. i.e. G admits total edge product cordial labeling. □

Theorem 2.2. *The graph with degree sequences* $(1, 1), (2, 2, 2, 2)$ *or* $(3, 2, 2, 1)$ *are not total edge product cordial graphs.*

Proof. For the graph with degree sequence $(1, 1)$ has one edge and two vertices. If we label the edge with 1 or 0 then both the vertices will receive the same label. Consequently $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| = 3$.

For the graph with degree sequence $(2, 2, 2, 2)$ or $(3, 2, 2, 1)$ has four edges and four vertices. If we assign label 0 to any edge then two end vertices will receive label 0 then $v_f(0) + e_f(0) = 3$. If we assign label 0 to two incident edges then three vertices will receive label 0(including a common vertex and two remaining vertices) then $v_f(0) + e_f(0) = 5$. If we assign label 0 to two non-incident edges then four end vertices will receive label 0 consequently $v_f(0) + e_f(0) = 6$. Hence in all situations $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| > 2$.

Hence, the graph with degree sequences $(1, 1), (2, 2, 2, 2)$ or $(3, 2, 2, 1)$ are not total edge product cordial graphs. □

Theorem 2.3. *The cycle* C_n *is a total edge product cordial graph except for* $n \neq 4$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n. We will consider following two cases.

Case 1: When n is odd.

$$\begin{aligned} &f(v_i v_{i+1}) = 0; && 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ &f(v_i v_{i+1}) = 1; && \left\lceil \frac{n}{2} \right\rceil \leq i \leq n-1 \\ &f(v_1 v_n) = 1. && \end{aligned}$$

Case 2: When n is even and $n \neq 4$.

$$\begin{aligned} &f(v_i v_{i+1}) = 0; && 1 \leq i \leq \frac{n-4}{2} \\ &f(v_i v_{i+1}) = 1; && i = \frac{n-2}{2} \\ &f(v_i v_{i+1}) = 0; && i = \frac{n}{2} \\ &f(v_i v_{i+1}) = 1; && \frac{n}{2} + 1 \leq i \leq n-1 \\ &f(v_1 v_n) = 1. && \end{aligned}$$

In both the cases we have $v_f(0) + e_f(0) = n$ and $v_f(1) + e_f(1) = n$. So, $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| \leq 1$.

Hence, the cycle C_n is a total edge product cordial graph except for $n \neq 4$. □

Example 2.1. *The cycle* C_5 *and its total edge product cordial labeling is shown in figure 1.*

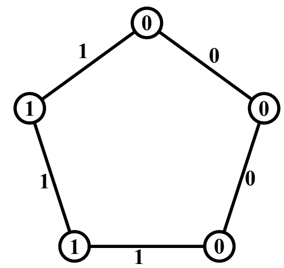

figure 1

Theorem 2.4. *The graph $C_n^{(t)}$ is a total edge product cordial graph.*

Proof. Let $v_{k,1}, v_{k,2}, \ldots, v_{k,n-1}$ be the vertices of k^{th} copy of cycle C_n and v be an common vertex of $C_n^{(t)}$. The vertices $v_{k,1}$ and $v_{k,n-1}$ of k^{th} copy of cycle C_n are adjacent to v. $|V(C_n^{(t)})| = t(n-1)+1$ and $|E(C_n^{(t)})| = tn$. We will consider following three cases.

Case 1: When t is even.

Here $C_n^{(t)}$ is of even size and it is edge product cordial graph as proved by Vaidya and Barasara [9]. Then by Theorem 2.1 result holds.

Case 2: When t and n both are odd.

$$\begin{aligned}
&f(v_{i,j}v_{i,j+1}) = 0; && 1 \le i \le \frac{t-1}{2} && \text{and} \quad 1 \le j \le n-2\\
&f(vv_{i,1}) = 0; && 1 \le i \le \frac{t-1}{2} &&\\
&f(vv_{i,n-1}) = 0; && 1 \le i \le \frac{t-1}{2} &&\\
&f(v_{i,j}v_{i,j+1}) = 1; && \frac{t+1}{2} \le i \le t-1 && \text{and} \quad 1 \le j \le n-2\\
&f(vv_{i,1}) = 1; && \frac{t+1}{2} \le i \le t-1 &&\\
&f(vv_{i,n-1}) = 1; && \frac{t+1}{2} \le i \le t-1 &&\\
&f(v_{t,i}v_{t,i+1}) = 0; && 1 \le i \le \frac{n-3}{2} &&\\
&f(vv_{t,1}) = 0; && &&\\
&f(v_{t,i}v_{t,i+1}) = 1; && \frac{n-1}{2} \le i \le n-2 &&\\
&f(vv_{t,n-1}) = 1. && &&
\end{aligned}$$

Case 3: When t is odd and n is even.

$$\begin{aligned}
&f(v_{i,j}v_{i,j+1}) = 0; && 1 \le i \le \frac{t-3}{2} && \text{and} \quad 1 \le j \le n-2\\
&f(vv_{i,1}) = 0; && 1 \le i \le \frac{t-3}{2} &&\\
&f(vv_{i,n-1}) = 0; && 1 \le i \le \frac{t-3}{2} &&\\
&f(v_{i,j}v_{i,j+1}) = 0; && i = \frac{t-1}{2} && \text{and} \quad 1 \le j \le n-2\\
&f(vv_{i,1}) = 0; && i = \frac{t-1}{2} &&\\
&f(vv_{i,n-1}) = 1; && i = \frac{t-1}{2} &&\\
&f(v_{i,j}v_{i,j+1}) = 0; && i = \frac{t+1}{2} && \text{and} \quad 1 \le j \le \frac{n-2}{2}\\
&f(vv_{i,1}) = 0; && i = \frac{t+1}{2} &&\\
&f(v_{i,j}v_{i,j+1}) = 1; && i = \frac{t+1}{2} && \text{and} \quad \frac{n}{2} \le j \le n-2\\
&f(vv_{i,n-1}) = 1; && i = \frac{t+1}{2} &&\\
&f(vv_{i,1}) = 1; && \frac{t+3}{2} \le i \le t &&\\
&f(v_{i,j}v_{i,j+1}) = 1; && \frac{t+3}{2} \le i \le t && \text{and} \quad 1 \le j \le n-2\\
&f(vv_{i,n-1}) = 1; && \frac{t+3}{2} \le i \le t &&
\end{aligned}$$

In case 2 and case 3 we have $v_f(0)+e_f(0)=\dfrac{2nt-t+1}{2}$ and $v_f(1)+e_f(1)=\dfrac{2nt-t+1}{2}$. Therefore $|\,(v_f(0)+e_f(0))-(v_f(1)+e_f(1))\,|\le 1$. Hence, the graph $C_n^{(t)}$ is a total edge product cordial graph. □

Example 2.2. *The graph $C_4^{(3)}$ and its total edge product cordial labeling is shown in figure 2.*

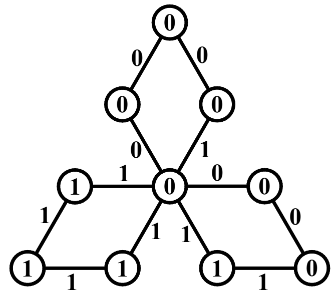

figure 2

Theorem 2.5. *The wheel W_n is a total edge product cordial graph.*

Proof. Let $v_1,v_2,\ldots,v_n$ be the rim vertices and v be an apex vertex of wheel W_n. To define $f:E(W_n)\to\{0,1\}$ we will consider following two cases.

Case 1: When n is odd.

$$\begin{aligned}
&f(v_{2i-1}v_{2i})=0; && 1\le i\le \frac{n-1}{2}\\
&f(v_1v_n)=0;\\
&f(v_{2i}v_{2i+1})=1; && 1\le i\le \frac{n-1}{2}\\
&f(vv_i)=1; && 1\le i\le n.
\end{aligned}$$

In view of the above defined labeling pattern we have $v_f(0)+e_f(0)=\dfrac{3n+1}{2}$ and $v_f(1)+e_f(1)=\dfrac{3n+1}{2}$.

Case 2: When n is even.

$$\begin{aligned}
&f(v_{2i-1}v_{2i})=0; && 1\le i\le \frac{n}{2}\\
&f(v_1v_n)=1;\\
&f(v_{2i}v_{2i+1})=1; && 1\le i\le \frac{n-2}{2}\\
&f(vv_i)=1; && 1\le i\le n.
\end{aligned}$$

In view of the above defined labeling pattern we have $v_f(0)+e_f(0)=\dfrac{3n}{2}$ and $v_f(1)+e_f(1)=\dfrac{3n}{2}+1$. Thus in both the cases we have $|\,(v_f(0)+e_f(0))-(v_f(1)+e_f(1))\,|\le 1$. Hence, the wheel W_n is a total edge product cordial graph. □

Example 2.3. *The wheel W_5 and its total edge product cordial labeling is shown in figure 3.*

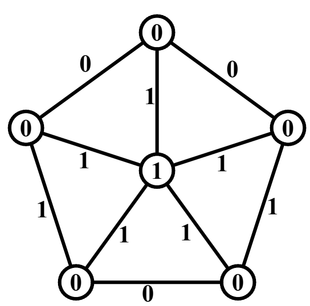

figure 3

Theorem 2.6. *The gear graph G_n is a total edge product cordial graph.*

Proof. Let $v_1,v_2,\ldots,v_{2n}$ be the rim vertices and v is apex vertex of gear graph G_n. To define $f:E(G_n)\to\{0,1\}$ we will consider following two cases.

Case 1: When n is odd.

$$\begin{aligned}
&f(v_iv_{i+1}) = 0; && 1 \le i \le n-1\\
&f(v_iv_{i+1}) = 1; && n \le i \le 2n-1\\
&f(v_1v_{2n}) = 1; &&\\
&f(vv_{2i-1}) = 0; && 1 \le i \le \left\lceil \frac{n}{2} \right\rceil\\
&f(vv_{2i-1}) = 1; && \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n.
\end{aligned}$$

In view of the above defined labeling patten we have $v_f(0) + e_f(0) = \frac{5n+1}{2}$ and $v_f(1) + e_f(1) = \frac{5n+1}{2}$.

Case 2: When n is even.

Subcase 1: When $n \equiv 0 \pmod 4$.

$$\begin{aligned}
&f(vv_{2i-1}) = 0; && 1 \le i \le n\\
&f(v_{2i-1}v_{2i}) = 0; && 1 \le i \le \frac{n}{4}\\
&f(v_1v_{2n}) = 1; &&\\
&f(v_{2i}v_{2i+1}) = 1; && 1 \le i \le \frac{n}{4}\\
&f(v_iv_{i+1}) = 1; && \frac{n}{4} + 1 \le i \le 2n-1.
\end{aligned}$$

Subcase 2: When $n \equiv 2 \pmod 4$.

$$\begin{aligned}
&f(vv_{2i-1}) = 0; && 1 \le i \le n\\
&f(v_{2i-1}v_{2i}) = 0; && 1 \le i \le \frac{n-2}{4}\\
&f(v_2v_3) = 0; &&\\
&f(v_{2i}v_{2i+1}) = 1; && 2 \le i \le \frac{n-2}{4}\\
&f(v_iv_{i+1}) = 1; && \frac{n+2}{4} \le i \le 2n-1\\
&f(v_1v_{2n}) = 1. &&
\end{aligned}$$

In subcase 1 and subcase 2 we have $v_f(0) + e_f(0) = \frac{5n}{2} + 1$ and $v_f(1) + e_f(1) = \frac{5n}{2}$. Thus in both the cases we have $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| \le 1$. Hence, the gear graph G_n is a total edge product cordial graph. □

Example 2.4. *The gear graph G_5 and its total edge product cordial labeling is shown in figure 4.*

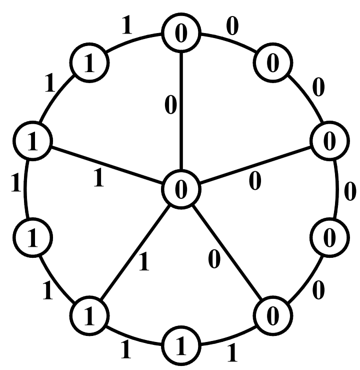

figure 4

Theorem 2.7. *The complete graph K_n admits total edge product cordial labeling for $n > 2$.*

Proof. For complete graph K_n, $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$.Hence total number of elements in K_n is $\frac{n(n+1)}{2}$. For $m < n$, K_m is a subgraph of K_n. Now we search for the smallest integer m for which $\left\lceil \frac{n(n+1)}{4} \right\rceil \le \frac{m(m+1)}{2}$. Denote $\frac{m(m+1)}{2} - \left\lceil \frac{n(n+1)}{4} \right\rceil$ by l and assign label 0 to $\frac{m(m-1)}{2} - l$ edges of subgraph K_m and assign label 1 to all the remaining edges of supergraph K_n. Then $v_f(0) = m$, $e_f(0) = \frac{m(m-1)}{2} - l$, $v_f(1) = n - m$ and $e_f(1) = \frac{n(n-1)}{2} - \frac{m(m-1)}{2} + l$. Thus

$|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))|$

$$= \left|\left(m + \frac{m(m-1)}{2} - l\right) - \left(n - m + \frac{n(n-1)}{2} - \frac{m(m-1)}{2} + l\right)\right|$$
$$= \left|\left(\frac{m(m+1)}{2} - l\right) - \left(\frac{n(n+1)}{2} - \frac{m(m+1)}{2} + l\right)\right|$$
$$= \left|\left\lceil\frac{n(n+1)}{4}\right\rceil - \left(\frac{n(n+1)}{2} - \left\lceil\frac{n(n+1)}{4}\right\rceil\right)\right|$$
$$= \left|\left\lceil\frac{n(n+1)}{4}\right\rceil - \left\lfloor\frac{n(n+1)}{4}\right\rfloor\right|$$
$$\leq 1.$$

Hence, K_n admits total edge product cordial labeling for $n > 2$. □

Example 2.5. *The complete graph K_5 and its total edge product cordial labeling is shown in figure 5. Here $m = 4$ and $l = 2$.*

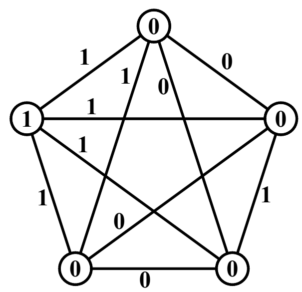

figure 5

Remark 2.1. *There is no possibility for any forbidden subgraph characterization for total edge product cordial labeling as K_n admits total edge product cordial labeling.*

Theorem 2.8. *The complete bipartite graph $K_{m,n}$ is a total edge product cordial graph except $K_{1,1}$ and $K_{2,2}$.*

Proof. For complete bipartite graph $K_{m,n}$, $|V(K_{m,n})| = m + n$ and $|E(K_{m,n})| = mn$. Therefore total number of elements in $K_{m,n}$ is $m + n + mn$. Without loss of generality assume that $m \leq n$. Let $v_1, v_2, \ldots, v_m$ be the vertices of one partite set and $u_1, u_2, \ldots, u_n$ be the vertices of other partite set. We will consider following two cases.

Case 1: When $m = 1$ and $n > 1$.

$K_{1,n}$ is a tree of either even order or even size. But Vaidya and Barasara [7] have proved that all trees of order greater than 2 are edge product cordial graph. Hence the result holds from Theorem 2.1.

Case 2: When $m > 2$.

For $l < n$, $K_{m,l}$ is a subgraph of $K_{m,n}$. Now we search for the largest integer l for which $m + l + ml \leq \left\lfloor\frac{m+n+mn}{2}\right\rfloor$. Let $r = \left\lfloor\frac{m+n+mn}{2}\right\rfloor - (m + l + ml)$. We define $f : E(K_{m,n}) \to \{0, 1\}$ as follows.

$$\begin{array}{lll} f(v_iu_j) = 0; & 1 \leq i \leq m & \text{and} \quad 1 \leq j \leq l \\ f(v_iu_{l+1}) = 0; & 1 \leq i \leq r-1 & \\ f(v_iu_{l+1}) = 1; & r \leq i \leq m & \\ f(v_iu_j) = 1; & 1 \leq i \leq m & \text{and} \quad l+2 \leq j \leq n. \end{array}$$

In view of the above defined labeling pattern we have $v_f(0) + e_f(0) = \left\lfloor\frac{m+n+mn}{2}\right\rfloor$ and $v_f(1) + e_f(1) = \left\lceil\frac{m+n+mn}{2}\right\rceil$. Therefore, $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| \leq 1$.

Hence, the complete bipartite graph $K_{m,n}$ is a total edge product cordial graph except $K_{1,1}$ and $K_{2,2}$. □

Example 2.6. *The complete bipartite graph $K_{3,4}$ and its total edge product cordial labeling is shown in figure 6. Here $m = 3$, $n = 4$. Hence $l = 1$ and $r = 2$. For which $v_f(0) = 5$, $e_f(0) = 4$, $v_f(1) = 2$ and $e_f(1) = 8$. Therefore, $|\,(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))\,| = 1$*

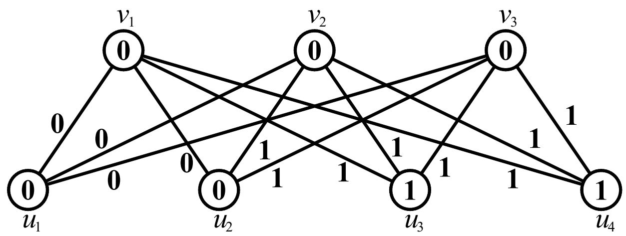

figure 6

Theorem 2.9. *The fan f_n is a total edge product cordial graph.*

Proof. Let v be an apex vertex and $v_1, v_2, \ldots, v_n$ be the other vertices of the fan f_n. To define $f : E(f_n) \to \{0, 1\}$ we will consider following two cases.

Case 1: When n is odd.

$$\begin{aligned}
f(v_iv_{i+1}) &= 0; \quad 1 \le i \le \frac{n-1}{2} \\
f(v_iv_{i+1}) &= 1; \quad \frac{n+1}{2} \le i \le n-1 \\
f(vv_i) &= 0; \quad 1 \le i \le \frac{n-1}{2} \\
f(vv_i) &= 1; \quad \frac{n+1}{2} \le i \le n.
\end{aligned}$$

In view of the above defined labeling pattern we have $v_f(0) + e_f(0) = \dfrac{3n+1}{2}$ and $v_f(1) + e_f(1) = \dfrac{3n-1}{2}$.

Case 1: When n is even.

$$\begin{aligned}
f(v_iv_{i+1}) &= 0; \quad 1 \le i \le \frac{n-2}{2} \\
f(v_iv_{i+1}) &= 1; \quad \frac{n}{2} \le i \le n-1 \\
f(vv_i) &= 0; \quad 1 \le i \le \frac{n}{2} \\
f(vv_i) &= 1; \quad \frac{n+2}{2} \le i \le n.
\end{aligned}$$

In view of the above defined labeling pattern we have $v_f(0) + e_f(0) = \dfrac{3n}{2}$ and $v_f(1) + e_f(1) = \dfrac{3n}{2}$.

Thus in both the cases we have $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \le 1$.

Hence, the fan f_n is a total edge product cordial graph. □

Example 2.7. *The fan f_4 and its total edge product cordial labeling is shown in figure 7.*

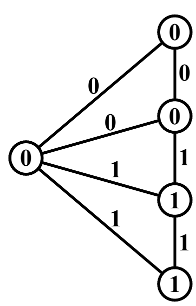

figure 7

Theorem 2.10. *The double fan Df_n is a total edge product cordial graph.*

Proof. Let v and u be vertices with degree $n-1$ and $v_1, v_2, \ldots, v_n$ be the other vertices of the double fan $Df(n)$. We define $f : E(Df_n) \to \{0, 1\}$ as follows.

$$\begin{aligned}
f(vv_i) &= 0; \quad 1 \le i \le n \\
f(v_iv_{i+1}) &= 1; \quad 1 \le i \le n-1 \\
f(uv_i) &= 1; \quad 1 \le i \le n.
\end{aligned}$$

In view of the above defined labeling pattern we have $v_f(0)+e_f(0) = 2n+1$ and $v_f(1)+e_f(1) = 2n$. Therefore, $|\,(v_f(0)+e_f(0)) - (v_f(1)+e_f(1))\,| \leq 1$.

Hence, the double fan Df_n is a total edge product cordial graph. □

Example 2.8. *The double fan Df_4 and its total edge product cordial labeling is shown in figure 8.*

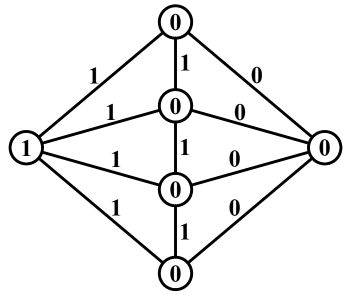

figure 8

3 Concluding remarks

Labeling of discrete structure is a potential area of research. We have introduced the concept of total edge product cordial labeling and derive several results on it. To investigate analogous results for various graphs as well as in the context of different graph labeling problems is an open area of research.

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Fuzzy boundedness and contractiveness on intuitionistic 2-fuzzy 2-normed linear space

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Abstract

The concepts of fuzzy boundedness, fuzzy continuity and intuitionistic fuzzy 2- contractive mapping on intutionistic 2-fuzzy 2-normed linear space are introduced. Using these concepts some theorems are proved.

Keywords: Intuitionistic 2-fuzzy 2-normed linear space, convergent and Cauchy sequences , intuitionistic 2-fuzzy 2-Banach space.

2010 MSC: 46B20, 46B99, 46A19, 46A99, 47H10.

1 Introduction

In 1965, the theory of fuzzy sets was introduced by L. Zadeh [9]. In 1964, a satisfactory theory of 2-norm on a linear space has been introduced and developed by Gahler [2]. In 2003, the concepts fuzzy norm and α-norm were introduced by Bag and Samanta [1]. Jialu Zhang [3] has defined fuzzy linear space in a different way. The notion of 2-fuzzy 2-normed linear space of the set of all fuzzy sets of a set was introduced by R.M. Somasundaram and Thangaraj Beaula [6]. The concept of intuitionistic 2fuzzy 2-normed linear space of the set of all fuzzy sets of a set was introduced by Thangaraj Beaula and Lilly Esthar Rani [7].

We have introduced the concepts of fuzzy boundedness, fuzzy continuity and intuitionistic fuzzy 2 contractive mapping on intutionistic 2-fuzzy 2-normed linear space. Using these concepts some theorems are proved.

2 Preliminaries

For the sake of completeness, we reproduce the following definitions due to Gahler [2], Bag and Samanta [1] and Jialu Zhang [3].

Definition 2.1. *[2] Let X be a real linear space of dimension greater than one and let $\|\cdot,\cdot\|$ be a real valued function on $X \times X$ satisfying the following conditions:*

1. $\|x, y\| = 0$ *if and only if x and y are linearly dependent,*
2. $\|x, y\| = \|y, x\|$,
3. $\|\alpha x, y\| = |\alpha|\ \|x, y\|$, *where α is real,*
4. $\|x, y+z\| \leq \|x, y\| + \|x, z\|$.

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$\|\cdot,\cdot\|$ is called a 2-norm on X and the pair (X, $\|\cdot,\cdot\|$) is called a 2-normed linear space.

Definition 2.2. *[1] Let X be a linear space over K(the field of real or complex numbers). A fuzzy subset N of $X \times R$ (R, the set of real numbers) is called a fuzzy norm on X if and only if for all x, $u \in X$ and $c \in K$.*

(N1) for all $t \in R$ with $t \leq 0$, N(x, t)=0.

(N2) for all $t \in R$ with $t > 0$, N(x, t)=1 if and only if $x = 0$.

(N3) for all $t \in R$ with $t > 0$, $N(cx, t) = N(x, \frac{t}{|c|})$, if $c \neq 0$.

(N4) for all s, $t \in R$, x, $u \in X$, $N(x+u, s+t) \geq \min\{N(x, s), N(u, t)\}$.

(N5) N(x, ·) is a non decreasing function of R and $\lim_{t\to\infty} N(x, t) = 1$.

The pair (X, N) will be referred to as a fuzzy normed linear space.

Definition 2.3. *[3] Let X be any non - empty set and F(X) be the set of all fuzzy sets on X. Then for U, V $\in$ F(X) and $k \in K$ the field of real numbers, define*

$$\begin{aligned} U + V &= \{ (x + y, \lambda \wedge \mu) | (x, \lambda) \in U, (y, \mu) \in V \} , \\ kU &= \{ (kx, \lambda | (x, \lambda) \in U \} . \end{aligned}$$

Definition 2.4. *[3] A fuzzy linear space $\tilde{X} = X \times (0,1]$ over the number field K, where the addition and scalar multiplication operation on $\tilde{X}$ are defined by*

$(x, \lambda)+(y, \mu)=(x+y, \lambda\wedge\mu)$, $k(x, \lambda) = (kx, \lambda)$

is a fuzzy normed space if to every $(x, \lambda) \in \tilde{X}$ there is associated a non-negative real number, $\|(x, \lambda)\|$, called the fuzzy norm of (x, λ) , in such a way that

1. *$\|(x, \lambda)\| = 0$ if and only if x=0 the zero element of X, $\lambda \in (0,1]$.*
2. *$\|k(x, \lambda)\| = |k| \, \|(x, \lambda)\|$ for all $(x, \lambda) \in \tilde{X}$ and all $k \in K$.*
3. *$\|(x, \lambda)+(y, \mu)\| \leq \|(x, \lambda \wedge \mu\| + \|(y, \lambda \wedge \mu)\|$ for all (x, λ) and $(y, \mu) \in \tilde{X}$.*
4. *$\|(x, \vee\lambda_t)\| = \vee\|(x, \lambda_t)\|$ for $\lambda_t \in (0,1]$.*

Definition 2.5. *[6] Let X be a non empty and F(X) be the set of all fuzzy sets in X. If $f \in F(X)$ then f={ (x, μ) | $x \in X$ and $\mu \in (0,1]$} . Clearly f is a bounded function for $|f(x)| \leq 1$. Let K be the space of real numbers, then F(X) is a linear space over the field K where the addition and scalar multiplication are defined by*

$$\begin{aligned} f + g &= \{ (x, \mu) + (y, \eta) = \{ (x + y, \mu \wedge \eta) | (x, \mu) \in f, \text{ and } (y, \eta) \in g \} \\ kg &= \{ (kf, \mu \mid (x, \mu) \in f \} \text{ where } k \in K. \end{aligned}$$

The linear space F(X) is said to be normed space if to every $f \in F(X)$, there is associated a non-negative real number $\|f\|$ called the norm of f in such a way that

1. $\|f\| = 0$ if and only if $f = 0$

For, $\|f\| = 0 \Leftrightarrow \{ \|(x, \mu)\| \mid (x,\mu) \in f \} = 0$
$\Leftrightarrow x = 0$, $\mu \in (0,1]$
$\Leftrightarrow f = 0$.

2. $\|kf\| = |k| \, \|f\|$, $k \in K$

For, $\|kf\| = \{ \|k(x, \mu)\| \mid (x, \mu) \in f, k \in K\}$
$= \{ |k| \, \|x, \mu\| \mid (x, \mu) \in f\}$
$= |k| \, \|f\|$.

3. $\|f+g\| \leq \|f\| + \|g\|$ for every f , $g \in F(X)$

For, $\|f+g\| = \{ \|(x, \mu) + (y, \eta)\| \mid x,y \in X, \mu, \eta \in (0,1]\}$
$= \{ \|(x+y), (\mu \wedge \eta)\| \mid x,y \in X, \mu, \eta \in (0,1]\}$
$\leq \{ \|x, \mu \wedge \eta\| + \|y, \mu \wedge \eta\| \mid (x, \mu) \in f$ and $(y, \eta) \in g\}$
$= \|f\| + \|g\|$.

And so (F(X), $\|\cdot\|$) is a normed linear space.

Definition 2.6. [6] *A 2-fuzzy set on X is a fuzzy set on F(X).*

Definition 2.7. [6] *Let F(X) be a linear space over the real field K. A fuzzy subset N of F(X)× R,(R, the set of real numbers) is called a 2-fuzzy 2-norm on X(or fuzzy 2-norm on F(X)) if and only if,*

(N1) for all $t \in R$ with $t \leq 0$, $N(f_1, f_2, t)=0$.

(N2) for all $t \in R$ with $t>0$, $N(f_1, f_2, t)=1$ if and only if f_1 and f_2 are linearly dependent.

(N3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2.

(N4) for all $t \in N$ with $t \geq 0$,
$N(f_1, cf_2, t) = N(f_1, f_2, \frac{t}{|c|})$ if $c \neq 0$, $c \in K$ (field).

(N5) for all $s, t \in R$, $N(f_1, f_2 + f_3, s+t) \geq min\{N(f_1, f_2, s), N(f_1, f_3, t)\}$.

(N6) $N(f_1, f_2, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

(N7) $\lim_{t\rightarrow\infty} N(f_1, f_2, t) = 1$.

Then the pair (F(X), N) is a fuzzy 2-normed linear space or (X, N) is a 2-fuzzy 2-normed linear space.

Definition 2.8. *A binary operation* $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ *is a continuous t-norm if it satisfies the following conditions:*

1. $*$ is commutative and associative.
2. $*$ is continuous.
3. $a * 1 = a$, for all $a \in [0, 1]$.
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.9. *A binary operation* $\diamondsuit : [0, 1] \times [0, 1] \rightarrow [0, 1]$ *is a continuous t - conorm if it satisfies the following conditions:*

1. $\diamondsuit$ is commutative and associative.
2. $\diamondsuit$ is continuous.
3. $a \diamondsuit 0 = a$, for all $a \in [0, 1]$.
4. $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

Remark 2.1. (1) For any $r_1, r_2 \in (0, 1)$ with $r_1 > r_2$ there exist $r_3, r_4 \in (0, 1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \diamondsuit r_2$.

(2) For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamondsuit r_7 \geq r_5$.

Definition 2.10. *An intuitionistic fuzzy 2- normed linear space (I-F-2-NLS) is of the form* $A = \{ F(X), N(f_1, f_2, t), M(f_1, f_2, t) \mid (f_1, f_2) \in F[(X)]^2\}$ *where F(X) is a linear space over a field K,* $*$ *is a continuous t-norm,* $\diamondsuit$ *is a continuous t-conorm, N and M are fuzzy sets on* $[F(X)]^2\times(0,\infty)$ *such that N denotes the degree of membership and M denotes the degree of non-membership of* $(f_1, f_2, t) \in [F(X)]^2\times(0,\infty)$ *satisfying the following conditions:*

(1) $N(f_1, f_2, t) + M(f_1, f_2, t) \leq 1$.

(2) $N(f_1, f_2, t) > 0$.

(3) $N(f_1, f_2, t) = 1$ if and only if f_1, f_2 are linearly dependent.

(4) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2.

(5) $N(f_1, f_2, t) : (0, \infty) \rightarrow [0,1]$ is continuous in t.

(6) $N(f_1, cf_2, t) = N(f_1, f_2, \frac{t}{|c|})$, if $c \neq 0$, $c \in K$.

(7) $N(f_1, f_2, s) * N(f_1, f_3, t) \leq N(f_1, f_2 + f_3, s + t)$.

(8) $M(f_1, f_2, t) > 0$.

(9) $M(f_1, f_2, t) = 0$ if and only if f_1, f_2 are linearly dependent.

(10) $M(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2.

(11) $M(f_1, cf_2, t) = M(f_1, f_2, \frac{t}{|c|})$ if $c \neq 0$, $c \in k$.

(12) $M(f_1, f_2, s) \Diamond M(f_1, f_3, t) \geq M(f_1, f_2 + f_3, s + t)$.

(13) $M(f_1, f_2, t) : (0, \infty) \rightarrow [0,1]$ is continuous in t.

3 Fuzzy boundedness and fuzzy continuity on intuitionistic fuzzy 2- normed linear space

Definition 3.1. *A sequence $\{f_n\}$ in an (IF 2-NLS) is said to converge to f if for given $r > 0$, $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that*

$N(f_n - f, g_1, t) > 1 - r$, $N(f_n - f, g_2, t) > 1 - r$

$M(f_n - f, g_1, t) < r$, $M(f_n - f, g_2, t) < r$

where g_1, g_2 are linearly independent (or) $N(f_n - f, g_i, t) \rightarrow 1$ as $n \rightarrow \infty$ for $i = 1, 2$ and $M(f_n - f, g_i, t) \rightarrow 0$ as $n \rightarrow \infty$ for $i = 1, 2$.

Definition 3.2. *A sequence $\{f_n\}$ is a cauchy sequence if for given $\in > 0$,*
$N(f_n - f_m, g_i, t) > 1 - \in$, $M(f_n - f_m, g_i, t) < \in$, $0 < \in < 1$, $t > 0$, g_i's are linearly independent, for $i = 1, 2$.

Definition 3.3. *Let $A = \{ (F(X), N(f_1, f_2, t), M(f_1, f_2, t) \mid (f_1, f_2) \in [F(X)]^2) \}$ be an intuitionistic fuzzy 2-normed linear space then*

$$N((f_1, f_2), (f_1', f_2'), t) = N((f_1 - f_1'), (f_2 - f_2'), t)$$

$$M((f_1, f_2), (f_1', f_2'), t) = M((f_1 - f_1'), (f_2 - f_2'), t)$$

*are intuitionistic 2-fuzzy metrics defined on A and (A, N, M, *) is an intuitionistic 2-fuzzy metric space (i-2-f-m-s).*

Definition 3.4. *Let (A, N, M, *) be an intuitionistic 2-fuzzy normed linear space. For $t > 0$, define the openball $B((f_1, f_2), r, t)$ with center $(f_1, f_2) \in A$ and radius $0< r<1$ as*

$$B((f_1, f_2), r, t) = \{ (g_1, g_2) \in A : N((f_1, g_1), (f_2, g_2), t) > 1 - r$$

$$M(f_1 - g_1), (f_2 - g_2) < r\}.$$

Definition 3.5. *A subset $G \subset A$ is said to be open if for each $(f_1, f_2) \in G$, there exists $t > 0$ and $0 < r < 1$ such that $B((f_1, f_2), r, t), r, t) \subset G$.*

Definition 3.6. *Let $\Im$ be the set of all open subsets of A, then it is called the intuitionistic 2-fuzzy topology induced by the intuitionistic 2-fuzzy norm.*

Definition 3.7. *Let (A, N, M, *) be an i-2-f-m-s then a subset D of A is said to be intuitionistic 2- fuzzy bounded if there exists* $t > 0$ *and* $0 < r < 1$ *such that*

$M((f_1, f_2), (g_1, g_2), t) > 1 - r$, $N((f_1, f_2), (g_1, g_2), t) < r$
for each
$((f_1, f_2), (g_1, g_2)) \in [F(X)]^2$.

Definition 3.8. *Let* $(A, N_1, M_1, *)$ $(B, N_2, M_2, *)$ *be an intuitionistic 2-fuzzy normed linear space , a mapping* $T : A \to B$ *is said to be an intuitionistic fuzzy 2 - bounded if there exist constants* $m_1, m_2 \in R^+$ *such that for every* $f \in A$ *and for each* $t > 0$,

$N_2(Tf, Tg, t) > N_1(f, g, \frac{t}{m_1})$

$M_2(Tf, Tg, t) > M_1(f, g, \frac{t}{m_2})$.

Definition 3.9. *Let* $T : A \to B$ *be a linear operator from IF 2-Banach Space A to IF 2- Banach space B. Then T is said to be an intuitionistic 2 -fuzzy continuous if for each* $\in$ *with* $0 < \in < 1$, *there exists* δ , $0 < \delta < 1$, *such that*

$N_1(f, g, t) \geq 1 - \delta$ *and* $M_1(f, g, t) \leq \delta$, *implies*

$N_2(Tf, Tg, t) \geq 1 - \in$ *and* $M_2(Tf, Tg, t) \leq \in$.

Theorem 3.1. *A linear operator* $T : (A, N_1, M_1, *) \to (B, N_2, M_2, *)$ *is an intuitionistic 2- fuzzy bounded if and only if it is an intuitionistic 2- fuzzy continuous.*

Proof. Assume $T : A \to B$ is an intuitionistic 2-fuzzy bounded. Then there exist constants $m_1, m_2 \in R^+$ such that for every $f \in A$ and for each $t > 0$,

$$N_2(Tf, Tg, t) \geq N_1(f, g, \frac{t}{m_1})$$
$$M_2(Tf, Tg, t) \leq M_1(f, g, \frac{t}{m_2}). \tag{3.1}$$

Suppose for $\in$, with $0 < \in < 1$, choose δ, with $0 < \delta < 1$

such that $N_1(f, g, t) \geq 1 - \delta$ and $M_1(f, g, t) \leq$ for any $t > 0$
and $N_1(f, g, \frac{t}{m_1}) \leq 1 - \in$

$$M_1(f, g, \frac{t}{m_2}) < \in \text{ (because } m_1, m_2 > 0). \tag{3.2}$$

Using (3.2) in (3.1) we get

$N_2(Tf, Tg, t) \geq 1 - \in$ and $M_2(Tf, Tg, t) \leq \in$

Hence T is an intuitionistic 2- fuzzy continuous.
Conversely,

Suppose T is an intuitionistic 2- fuzzy continuous.

For $\in$ with $0 < \in < 1$, there exists δ with $0 < \delta < 1$

such that $N_1(f, g, t) < 1 - \delta$, $M_1(f, g, t) < \delta$ implies that

$$N_2(Tf, Tg, t) > 1 - \in, M_2(Tf, Tg, t) < \in . \tag{3.3}$$

Choose m_1, $m_2 \in R^+$ such that

N_1 (f, g, $\frac{t}{m_1}$) $\leq$ 1 - $\in$ for given N_1(f, g, t)>1-δ and

$$M_1(f, g, \frac{t}{m_2}) \geq \in \text{ for given } \quad M_1(f, g, t) < \delta. \tag{3.4}$$

Then applying (3.4) on (3.3) we get

N_2(Tf, Tg, t) $>$ 1 - $\in \geq N_1$(f, g, $\frac{t}{m_1}$)

M_2 (Tf, Tg, t) $< \delta \leq M_1$ (f, g, $\frac{t}{m_2}$)

Therefore T is intuitionistic 2- fuzzy bounded. □

4 Intuitionistic 2-fuzzy contraction on intuitionistic 2-fuzzy metric space

Definition 4.1. *Let (A, N, M, *) be an intuitionistic 2-fuzzy metric space then*
$T : A \to A$ is said to be intuitionistic 2- fuzzy contraction if there exists $C \in (0, 1)$ such that CN_2 (Tf, Tg, t) $\geq N_1$ (f, g, t) and $\frac{1}{C}$ M_2 (Tf, Tg, t) $\leq M_1$ (f, g, t).

Theorem 4.1. *Let (A, N, M, *) be a intuitionistic 2-fuzzy metric space. If $T : A \to A$ is an intuitionistic 2-fuzzy contractive mapping then T is an intuitionistic 2- fuzzy uniformly continuous.*

Proof. Assume T : A $\to$ A is an intuitionistic 2- fuzzy contractive mapping. Then there exists C $\in$ (0, 1) such that CN_2(Tf, Tg, t) $\geq N_1$(f, g, t) and

$\frac{1}{C}$ M_2 (Tf, Tg, t) $\leq M_1$ (f, g, t) for every t $<$ 0

Assume for a given $\in$ with $0 < \in < 1$ there exists $0 < \delta < 1$ such that
N_1 (f, g, t) $\geq$ 1-δ and M_1 (f, g, t) $< \delta$

Then CN_2 (Tf, Tg, t) $\geq$ 1 -δ implies N_2 (Tf, Tg, t) $\geq \frac{1-\delta}{C}$ and M_2 (Tf, Tg, t)$\leq \delta$ implies M_2 (Tf, Tg, t) $\leq \delta C$

Choose C and δ in such a way that $\delta = \frac{1}{1+C}$.
Then we can define $\in$ so that it satisfies the relationship $\frac{1-\delta}{C} \geq 1- \in$ and $\delta C \leq \in$.
Thus N_2 (Tf, Tg, t)$\geq$ 1-$\in$ and M_2 (Tf, Tg, t)$\leq\in$. Therefore, T is an intuitionistic 2- fuzzy uniformly continuous. □

Definition 4.2. *Let (F(X), N, M) be an intuitionistic 2-fuzzy normed linear space. S is said to be is intuitionistic 2- fuzzy closed if and only if any sequence $\{ f_n \}$ in S converges to $f \in S$.*
(ie) $\lim_{n\to\infty}$ N (f_n - f, g_i, t) = 1 and $\lim_{n\to\infty}$ M (f_n-f, g_i, t) = 0 for i = 1, 2 implies $f \in S$.

Definition 4.3. *Let (F(X), N, M) be an intuitionistic 2 -fuzzy normed linear space. $\bar{B}$ (f, $\in$, t) = \{ g $\in$ F(X) | N (f, g, t) > 1 -$\in$, M (f, g, t) < $\in$\} is said to be a closed ball centered at f of radius $\in$ w.r.to t if and only if any sequence $\{ f_n \}$ in $\bar{B}$(f, $\in$, t) converges to g then $g \in \bar{B}$ (f, $\in$, t).*

Theorem 4.2. *Suppose A = (F(X), N, M) is an intuitionistic 2-fuzzy Banach space. Let $T : A \to A$ be an intuitionistic 2- fuzzy contractive mapping on $\bar{B}$(f, $\in$, t) with contraction constant C and C N (f, Tf, t) > 1 - $\in$ and $\frac{1}{C}$ M(f, Tf, t) < $\in$ Then there exists a sequence $\{ f_n \}$ in F(X) such that N(f, f_n, t) >1-$\in$ and M (f, f_n, t) < $\in$.*

Proof. Assume f_1 = T(f), f_2 = T(f_1) = T (T(f_1)) = T^2 (f_1)
therefore f_n = T(f_{n-1}) = T^n (f) for all n $\in$ N.
Then C N (f, Tf, t) > 1-$\in$ implies N (f, Tf, t)>$\frac{1-\in}{C}$ > 1 -$\in$

Therefore N (f, f_1, t) > 1 -$\in$

Also $\frac{1}{C}$M (f, Tf, t) $<\in$ implies M(f, Tf, t) $<$ C $\in <\in$

Thus M(f, Tf, t) $<\in$ and so f$\in \bar{B}$ (f,$\in$, t)

Now assume $f_1, f_2, ., f_{n-1} \in \bar{B}$ (f, $\in$, t)

Let us show that $f_n \in \bar{B}$ (f, $\in$, t)

C N (f_1, f_2, t) = C N (Tf, Tf_1, t)

$\geq$ N (f, f_1, t)

$> 1- \in$

So, N (f_1, f_2 t) $> \frac{1-\in}{C} > 1 - \in$

C N (f_2, f_3, t) = C N (T(f_1), T(f_2), t)

$\geq$ N (f_1, f_2, t)

$> 1- \in$

therefore N (f_2, f_3, t) $> \frac{1-\in}{C} > 1 - \in$

Again $\frac{1}{C}$M (f_1, f_2, t) = $\frac{1}{C}$ M (T(f), T(f_1), t) $\leq$ M (f, f_1, t)

Thus M(f_1, f_2, t) $\leq$ C M (f, f_1, t) $<$ C $\in <\in$

Again $\frac{1}{C}$M (f_2, f_3, t) = $\frac{1}{C}$ M(T(f_1), T(f_2), t) $\leq$ M(f_1, f_2, t)

So,
M(f_2, f_3, t) $\leq$ C M (f_1, f_2, t)$<$ C $\in < \in$

Thus we obtain
N(f_3, f_4, t) $>$ 1 -$\in$, M (f_3, f_4, t) $<\in$,..., N(f_{n-1}, f_n, t) $>$ 1 -$\in$ M (f_{n-1}, f_n, t) $<\in$

Thus we obtain N(f, f_n, t) $\geq$ N(f, f_1, $\frac{t}{n}$) * N(f_1, f_2, $\frac{t}{n}$) * ,..., N(f_{n-1}, f_n, $\frac{t}{n}$)

$> (1- \in) * (1- \in) * .... * (1- \in)$

$= 1- \in$

Therefore , N(f, f_n, t) $>$ 1 -$\in$

M(f, f_n, t) $\leq$ M (f,f_1,$\frac{t}{n}$) $\diamond$ ... $\diamond$ M(f_{n-1}, f_n, $\frac{t}{n}$)

= r $\diamond$ r $\diamond$ ... $\diamond$ r = r
Thus N (f, f_n, t) $>$ 1- $\in$ and M (f, f_n, t) $<\in$. □

Lemma 4.1. *Let (F(X), N, M, *) be an intuitionistic 2-fuzzy normed linear space. Let T : F(X) $\to$ F(X) be an intuitionistic 2- fuzzy continuous. If $f_n \to f$ then*
T(f_n) $\to$ T(f) as n$\to \infty$.

Proof. Given $f_n \to f$ in (F(X), N, M, *). Then for given $\in > 0$, $t > 0$, $0 < t < 1$ there exists an integer $n_0 \in N$ such that $N(f_n\text{-}f, g_i, t) > 1 - \in$ and $M(f_n\text{-}f, g_i, t) < \in$

where g_i's are linearly independent for all $n \geq n_0$, i = 1, 2.

Since T is is intuitionistic 2- fuzzy continuous,

$N(T(f_n - f), Tg_i, t) > 1 - \in$ and $M(T(f_n - f), Tg_i, t) < \in$ implies

$N(Tf_n - Tf, g_i', t) > 1 - \in$ and $M(Tf_n - Tf, g_i', t) < \in$

Thus $Tf_n \to Tf$ as $n \to \infty$. □

Lemma 4.2. *Let (F(X), N, M, *) be an intuitionistic 2-fuzzy normed linear space then N and M are jointly continuous.*

Proof. If $f_n \to f$ and $g_n \to g$ in (F(X), N, M, *)
we have to prove that $N(f_n\text{-}f, g_n\text{-} g, t) > 1 - \in$ and $M(f_n\text{-}f, g_n - g, t) < \in$ as $n \to \infty$.

We know that

$\lim_{n\to\infty} N(f_n\text{-}f, f_i', t) = 1$ or $> 1\text{-}\in$, $\lim_{n\to\infty} N(g_n\text{-}g, f_i', t) = 1 > 1 - \in$ and

$\lim_{n\to\infty} M(f_n - f, f_i', t) = 0 < \in$, $\lim_{n\to\infty} M(g_n\text{-}g, f_i', t) = 0 < \in$

$N(f_n\text{-}f, g_n\text{-}g, t) \geq N(f_n\text{-}f, f_i', \frac{t}{2}) * N(g_n\text{-}g, f_i', \frac{t}{2})$

$> (1- \in) * (1- \in)$

$= 1- \in$

And, $(f_n\text{-}f, g_n\text{-}g, t) \leq M(f_n\text{-}f, f_i', \frac{t}{2}) \diamond M(g_n\text{-}g, f_i', \frac{t}{2})$

$< \in \diamond \in = \in$. □

Definition 4.4. *Let (F(X), N, M, *) be an intuitionistic 2-fuzzy normed linear space. A subset A of F (X) is said to be an intuitionistic 2- fuzzy bounded if N(f, g, t) ≥ 1 - M and M(f, g,t) ≤ M where M is a positive constant.*

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