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Properties of some nonlinear partial dynamic equations on time scales

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Abstract

The aim of the present paper is to study some basic qualitative properties of solutions of certain nonlinear partial dynamic equations on time scales. The tools employed are application of Banach fixed point theorem and a variant of certain fundamental integral inequality with explicit estimates on time scales.

Keywords: Dynamic equations, time scales, qualitative properties, inequalities with explicit estimates.

2010 MSC: 26E70, 34N05.

1 Introduction

During the past few years many authors have obtained the time scale analogue of well known partial dynamic equations see [1, 6, 10, 11]. Recently in [2, 3, 9] authors have obtained inequalities on two independent variables on time scales. In the present paper we establish some basic qualitative properties of solutions of certain partial dynamic equation on time scales. We use certain fundamental integral inequalities with explicit estimates to establish our results. We assume understanding of time scales and its notation. Excellent information about introduction to time scales can be found in [4, 5].

In what follows $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ the set of integers and $\mathbb{T}$ denotes arbitrary time scales. Let C_{rd} be the set of all rd continuous functions. We assume $\mathbb{T}_1$ and $\mathbb{T}_2$ are two time scales and $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$. The delta partial derivative of a function $z(x,y)$ for $(x,y) \in \Omega$ with respect to x, y and xy are denoted by $z^{\Delta_1}(x,y)$, $z^{\Delta_2}(x,y)$ and $z^{\Delta_1\Delta_2}(x,y) = z^{\Delta_2\Delta_1}(x,y)$ with the given boundary conditions.

$$u^{\Delta_2\Delta_1}(x,y) = f\left(x, y, u(x,y), u^{\Delta_1}(x,y)\right) \tag{1.1}$$

with the given initial boundary conditions

$$u(x,y_0) = \alpha(x), u(x_0,y) = \beta(y), u(x_0,y_0) = 0, \tag{1.2}$$

for $(x,y) \in \Omega$, where $f \in C_{rd}(\Omega \times R^n \times R^n, R_+)$, $\alpha, \beta \in (R_+, R^n)$.

In this paper we study the existence, uniqueness and other properties of the solutions of (1.1)-(1.2) under some suitable conditions on the functions involved in (1.1)-(1.2). The main tool employed is based on application of Banach fixed point theorem [8] coupled with Bielecki-type norm [7] and suitable integral inequality with explicit estimate.

2 Preliminaries and Basic Inequality

For a function $u(x,y)$ and its delta derivative $u^{\Delta_1}(x,y)$ in $C_{rd}(\Omega, R^n)$ we denote by $|u(x,y)|_W = |u(x,y)| + \left|u^{\Delta_1}(x,y)\right|$. For $(t,s) \in \Omega$ the notation $a(t,s) = O(b(t,s))$ then there exists a constant $q > 0$ such that

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$\left|\frac{a(t,s)}{b(t,s)}\right| \leq q$ right-hand neighborhood. Let G be the space whose functions $\left(\phi(x,y), \phi^{\Delta}(x,y)\right) \in W$ which are rd-continuous for $(x,y) \in \Omega$ and satisfy the condition

$$|\phi(x,y)|_W = O\left(e_\lambda(x,y)\right), \tag{2.1}$$

for $(x,y) \in \Omega$, where $\lambda > 0$ is a constant. In space G we define the norm

$$|\phi|_G = \sup_{(x,y)\in\Omega}\left[|\phi(x,y)|_W\, e_{\ominus\lambda}(x,y)\right]. \tag{2.2}$$

It is easy to see that G with norm defined in (2.2) is a Banach space. The condition (2.1) implies that there is a constant $N \geq 0$ such that

$$|\phi(x,y)|_W \leq N e_\lambda(x,y). \tag{2.3}$$

Using the fact that in (2.2) we observe that

$$|\phi(x,y)|_W \leq N. \tag{2.4}$$

By a solution of $(1.1)-(1.2)$ we means a function $u(x,y) \in C_{rd}(\Omega, R^n)$ which satisfies the equation $(1.1)-(1.2)$. It is easy to see that solution $u(x,y)$ of $(1.1)-(1.2)$ satisfies the following equation

$$u^{\Delta_1}(x,y) = \alpha^{\Delta}(x) + \int_{y_0}^{y} f\left(x,t,u(x,t),u^{\Delta_1}(x,t)\right)\Delta t, \tag{2.5}$$

and

$$u(x,y) = \alpha(x) + \beta(y) + \int_{x_0}^{x}\int_{y_0}^{y} f\left(s,t,u(s,t),u^{\Delta_1}(s,t)\right)\Delta t\Delta s, \tag{2.6}$$

for $(x,y) \in \Omega$.

We require the following integral inequality.

Lemma 2.1. *Let $u, a, b \in C_{rd}(\Omega, R^n)$*

$$u(x,y) \leq c + \int_{y_0}^{y} a(x,t)u(x,t)\Delta t + \int_{x_0}^{x}\int_{y}^{y} b(s,t)u(s,t)\,\Delta t\Delta s, \tag{2.7}$$

for $(x,y) \in \Omega$, then

$$u(x,y) \leq cH(x,y)\, e_{\int_{y_0}^{y} b(s,t)H(s,t)\Delta t}(x,x_0), \tag{2.8}$$

for $(x,y) \in \Omega$, where

$$H(x,y) = e_a(y,y_0), \tag{2.9}$$

for $(x,y) \in \Omega$.

Proof. Define a function $n(x,y)$ by

$$n(x,y) = c + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)u(s,t)\,\Delta t\Delta s, \tag{2.10}$$

then (2.7) can be restated as

$$u(x,y) \leq n(x,y) + \int_{y_0}^{y} a(x,t)u(x,t)\Delta t. \tag{2.11}$$

Clearly $n(x,y)$ is nonnegative for $(x,y) \in \Omega$ and nondecreasing for x. Treating (2.11) as a one-dimensional integral inequality and a suitable application of inequality given in Theorem 3.1 [12] yields

$$u(x,y) \leq n(x,y)H(x,y), \tag{2.12}$$

where $H(x,y)$ is given by (2.9). From (2.10) and (2.12) we have

$$n(x,y) \leq c + \int_{x_0}^{x} \int_{y_0}^{y} b(s,t)H(s,t)n(s,t)\,\Delta t \Delta s. \tag{2.13}$$

Now a suitable application of Theorem 2.1 [9] to (2.13) yields

$$n(x,y) \leq c e_{\int_{y_0}^{y} b(s,t)H(s,t)\Delta t}(x,x_0). \tag{2.14}$$

Using (2.14) into (2.12) we get the required inequality in (2.8). □

3 Main results

Our main results are given in the following theorems.

Theorem 3.1. *Suppose that*
(i) the function f in (1.1) satisfies the condition

$$|f(x,y,u,v) - f(x,y,\overline{u},\overline{v})| \leq p(x,y)\left[|u-\overline{u}| + |v-\overline{v}|\right], \tag{3.1}$$

where $p \in C_{rd}(\Omega, R^n)$,
(ii) for λ as in (2.1)
(a) there exists a nonnegative constant γ such that $\gamma < 1$ and

$$\int_{y_0}^{y} p(x,t)e_\lambda(x,t)\,\Delta t + \int_{x_0}^{x} \int_{y_0}^{y} p(s,t)e_\lambda(s,t)\,\Delta t\Delta s \leq \gamma e_\lambda(x,y), \tag{3.2}$$

(b) there exists a nonnegative constant η such that

$$\begin{aligned} &|\alpha(x)| + |\beta(y)| + \left|\alpha^{\Delta}(x)\right| + \int_{y_0}^{y} |f(x,t,0,0)|\Delta t \\ &+ \int_{x_0}^{x} \int_{y_0}^{y} |f(s,t,0,0)|\,\Delta t\Delta s \leq \eta e_\lambda(x,y), \end{aligned} \tag{3.3}$$

for $(x,y) \in \Omega$ where α, β are the functions given in (1.2). Then (1.1) – (1.2) has a unique solution in G.

Proof. Let $u(x,y) \in G$ and define operator S by

$$(Su)(x,y) = |\alpha(x)| + |\beta(y)| + \int_{x_0}^{x} \int_{y_0}^{y} f\left(s,t,u(s,t),u^{\Delta_1}(s,t)\right)\Delta t\Delta s. \tag{3.4}$$

Delta differentiating both sides of (3.4) with respect to x we get

$$(Su)^{\Delta_1}(x,y) = \left|\alpha^{\Delta}(x)\right| + \int_{y_0}^{y} f\left(x,t,u(x,t),u^{\Delta_1}(x,t)\right)\Delta t. \tag{3.5}$$

First we show that Su maps G into itself. Evidently (Su) is rd-continuous . We verify that (2.1) is fulfilled.

From (3.4) and (3.5) using the hypotheses and (2.3) we have

$$
\begin{aligned}
&|(Su)(x,y)| + \left|(Su)^{\Delta_1}(x,y)\right| \\
&\le |\alpha(x)| + |\beta(y)| + \left|\alpha^{\Delta}(x)\right| + \int_{x_0}^{x}\int_{y_0}^{y} \left|f(s,t,u(s,t),u^{\Delta_1}(s,t)) - f(s,t,0,0)\right|\Delta t\Delta s \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} |f(s,t,0,0)|\Delta t\Delta s + \int_{y_0}^{y} \left|f(x,t,u(s,t),u^{\Delta_1}(x,t)) - f(x,t,0,0)\right|\Delta t \\
&+ \int_{y_0}^{y} |f(x,t,0,0)|\Delta t \\
&\le \eta e_\lambda(x,y) + \int_{y_0}^{y} p(x,t)\, e_\lambda(x,t)\, |u(x,t)|_G\, e_{\ominus\lambda}(x,t)\,\Delta t \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} p(s,t) e_\lambda(s,t)\, |u(s,t)|_G\, e_{\ominus\lambda}(s,t)\,\Delta t\Delta s \\
&\le \eta e_\lambda(x,y) + |u|_s \left[\int_{y_0}^{y} p(x,t)\, e_\lambda(x,t)\,\Delta t + \int_{x_0}^{x}\int_{y_0}^{y} p(s,t) e_\lambda(s,t)\,\Delta t\Delta s\right] \\
&\le [\eta + N\gamma]\, e_\lambda(x,y)\,.
\end{aligned} \tag{3.6}
$$

From (3.6) it follows that $Su \in G$. This proves S maps G into itself.

Now we verify S is a contraction map. Let $u(x,y), v(x,y) \in G$. From (3.4) and (3.5) and using hypotheses we have

$$
\begin{aligned}
&|(Su)(x,y) - (Sv)(x,y)| + \left|(Su)^{\Delta_1}(x,y) - (Sv)^{\Delta_1}(x,y)\right| \\
&\le \int_{x_0}^{x}\int_{y_0}^{y} \left|f\left(s,t,u(s,t),u^{\Delta_1}(s,t)\right) - f\left(s,t,v(s,t),v^{\Delta_1}(s,t)\right)\right|\Delta t\Delta s \\
&+ \int_{y_0}^{y} \left|f\left(x,t,u(x,t),u^{\Delta_1}(x,t)\right) - f\left(x,t,v(x,t),v^{\Delta_1}(x,t)\right)\right|\Delta t \\
&\le \int_{y_0}^{y} p(x,t)\, e_\lambda(x,t)\, |u(x,t) - v(x,t)|_G\, e_{\ominus\lambda}(x,t)\,\Delta t \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} p(s,t) e_\lambda(s,t)\, |u(s,t) - v(s,t)|_G\, e_{\ominus\lambda}(s,t)\,\Delta t\Delta s \\
&\le |u - v|_G\, \gamma e_\lambda(x,y)\,.
\end{aligned} \tag{3.7}
$$

Consequently from (3.7) we have

$$
|Su - Sv|_G \le \gamma\, |u - v|_G\,.
$$

Since $\gamma < 1$, it follows from Banach fixed point theorem that S has a unique fixed point in G. The fixed point of S is however solution of (1.1) – (1.2). The proof is complete. □

Now we give theorem concerning the uniqueness of solutions of (1.1) – (1.2) without existence.

Theorem 3.2. *Assume that the function f in* (1.1) *satisfies the condition* (3.1). *Then* (1.1) – (1.2) *has at most one solution on* Ω.

Proof. Let $u_1(x,y)$ and $u_2(x,y)$ be two solutions of (1.1) – (1.2). Then by hypotheses we have

$$
\begin{aligned}
&|u_1(x,y)-u_2(x,y)|+\left|u_1^{\Delta_1}(x,y)-u_2^{\Delta_1}(x,y)\right| \\
&\le \int_{x_0}^{x}\int_{y_0}^{y}\left|f\left(s,t,u_1(s,t),u_1^{\Delta_1}(s,t)\right)-f\left(s,t,u_2(s,t),u_2^{\Delta_1}(s,t)\right)\right|\Delta t\Delta s \\
&+\int_{y_0}^{y}\left|f\left(x,t,u_1(x,t),u_1^{\Delta_1}(x,t)\right)-f\left(x,t,u_2(x,t),u_2^{\Delta_1}(x,t)\right)\right|\Delta t \\
&\le \int_{y_0}^{y}p(x,t)\left[|u_1(x,t)-u_2(x,t)|+\left|u_1^{\Delta_1}(x,t)-u_2^{\Delta_1}(x,t)\right|\right]\Delta t \\
&+\int_{x_0}^{x}\int_{y_0}^{y}p(s,t)\left[|u_1(s,t)-u_2(s,t)|+\left|u_1^{\Delta_1}(s,t)-u_2^{\Delta_1}(s,t)\right|\right]\Delta t\Delta s. \qquad (3.8)
\end{aligned}
$$

Now an application of Lemma 2.1 (with $a(x,y)=b(x,y)=p(x,y)$ and $c=0$) to (3.8) yields.

$$|u_1(x,y)-u_2(x,y)|+\left|u_1^{\Delta_1}(x,y)-u_2^{\Delta_1}(x,y)\right|\le 0, \qquad (3.9)$$

which implies $u_1(x,y)=u_2(x,y)$ for $(x,y)\in\Omega$. Thus there is at most one solution of (1.1) – (1.2) on Ω. □

4 Boundedness and continuous dependence

In this section we study the boundedness of solution of (1.1) – (1.2) and the continuous dependence of solutions of equation (1.1) on the given initial boundary values, the function f involved therein and also the continuous dependence of solutions of equations of the (1.1) on parameters.

The following theorem concerning the estimate on the solution (1.1) – (1.2) holds.

Theorem 4.1. *Assume that*

$$|\alpha(x)|+|\beta(y)|+\left|\alpha^{\Delta}(x)\right|\le k, \qquad (4.1)$$

$$|f(x,y,u,v)|\le r(x,y)\left[|u|+|v|\right], \qquad (4.2)$$

where $k\ge 0$ is a constant $r\in C_{rd}(\Omega,R^n)$. If $u(x,y),(x,y)\in\Omega$ is any solution of (1.1) – (1.2) *then*

$$|u(x,y)|+\left|u^{\Delta_1}(x,y)\right|\le kq(x,y)\,e_{\int_{y_0}^{y} r(s,t)q(s,t)\Delta t}(x,x_0), \qquad (4.3)$$

for $(x,y)\in\Omega$, where

$$q(x,y)=e_r(y,y_0), \qquad (4.4)$$

for $(x,y)\in\Omega$.

Proof. Using the fact that $u(x,y)$ is a solution of (1.1) – (1.2) and conditions (4.1) – (4.2) we have

$$
\begin{aligned}
&|u(x,y)|+\left|u^{\Delta_1}(x,y)\right| \\
&\le|\alpha(x)|+|\beta(y)|+\left|\alpha^{\Delta}(x)\right| \\
&+\int_{x_0}^{x}\int_{y_0}^{y}f\left(s,t,u(s,t),u^{\Delta_1}(s,t)\right)\Delta t\Delta s+\int_{y_0}^{y}f\left(x,t,u(x,t),u^{\Delta_1}(x,t)\right)\Delta t \\
&\le k+\int_{y_0}^{y}r(x,t)\left[|u(x,t)|+\left|u^{\Delta_1}(x,t)\right|\right]\Delta t \\
&+\int_{x_0}^{x}\int_{y_0}^{y}r(s,t)\left[|u(s,t)|+\left|u^{\Delta_1}(s,t)\right|\right]\Delta t\Delta s. \qquad (4.5)
\end{aligned}
$$

Now a suitable application of Lemma 2.1 to (4.5) yields (4.3). □

Remark 4.1. *The estimates obtained in* (4.3) *yields not only the bound on the solution* $u(x,y)$ *of* $(1.1)-(1.2)$ *but also the bound on* $u^{\Delta_1}(x,y)$. *If the estimate on the right side in* (4.3) *is bounded then the solution of* $u(x,y)$ *of* $(1.1)-(1.2)$ *and also* $u^{\Delta_1}(x,y)$ *are bounded on* Ω.

Our next result deals with the continuous dependence of solutions of equation (1.1) and given initial boundary conditions.

Theorem 4.2. *Let* $u_1(x,y)$ *and* $u_2(x,y)$ *be the solution of equation* (1.1) *and given initial boundary conditions*

$$u_1(x,y_0)=\alpha_1(x), \quad u_1(x_0,y)=\beta_1(y), \quad u_1(x_0,y_0)=0, \tag{4.6}$$

$$u_2(x,y_0)=\alpha_2(x), \quad u_2(x_0,y)=\beta_2(y), \quad u_2(x_0,y_0)=0, \tag{4.7}$$

respectively where $\alpha_1,\alpha_2,\beta_1,\beta_2\in(R_+,R^n)$. *Suppose that the function* f *in equation* (1.1) *satisfies the condition* (3.1) *and*

$$|\alpha_1(x)+\beta_1(y)-\alpha_2(x)-\beta_2(y)|+\left|\alpha_1^{\Delta}(x)-\alpha_2^{\Delta}(x)\right|\leq d, \tag{4.8}$$

for $(x,y)\in\Omega$, *where* $d\geq 0$ *is a constant. Then*

$$|u_1(x,y)-u_2(x,y)|+\left|u_1^{\Delta_1}(x,y)-u_2^{\Delta_2}(x,y)\right|\leq d\overline{q}(x,y)\,e_{\int_{y_0}^{y}p(s,t)\overline{q}(s,t)\Delta t\Delta s}(x,x_0), \tag{4.9}$$

for $(x,y)\in\Omega$, *where* $\overline{q}(x,y)$ *is defined by the right hand side of (4.4) replacing* $r(x,y)$ *by* $p(x,y)$ *for* $(x,y)\in\Omega$.

Proof. From the hypotheses it is easy to observe that

$$\begin{aligned}
&|u_1(x,y)-u_2(x,y)|+\left|u_1^{\Delta_1}(x,y)-u_2^{\Delta_2}(x,y)\right| \\
&\leq |\alpha_1(x)+\beta_1(y)-\alpha_2(x)-\beta_2(y)|+\left|\alpha_1^{\Delta}(x)-\alpha_2^{\Delta}(x)\right| \\
&+\int_{x_0}^{x}\int_{y_0}^{y}\left|f\left(s,t,u_1(s,t),u_1^{\Delta_1}(s,t)\right)-f\left(s,t,u_2(s,t),u_2^{\Delta_1}(s,t)\right)\right|\Delta t\Delta s \\
&+\int_{y_0}^{y}\left|f\left(x,t,u_1(x,t),u_1^{\Delta_1}(x,t)\right)-f\left(x,t,u_2(x,t),u_2^{\Delta_1}(x,t)\right)\right|\Delta t \\
&\leq d+\int_{y_0}^{y}p(x,t)\left[|u_1(x,t)-u_2(x,t)|-\left|u_1^{\Delta_1}(x,t)-u_2^{\Delta_1}(x,t)\right|\right]\Delta t \\
&+\int_{x_0}^{x}\int_{y_0}^{y}p(s,t)\left[|u_1(s,t)-u_2(s,t)|+\left|u_1^{\Delta_1}(s,t)-u_2^{\Delta_1}(s,t)\right|\right]\Delta t\Delta s.
\end{aligned} \tag{4.10}$$

Now a suitable application of Lemma 2.1 to (4.10) yields the bound in (4.9), which shows dependency on solution of equation (1.1) on given initial boundary values. □

Consider the initial boundary value problem (1.1)-(1.2) and the corresponding initial boundary value problem

$$v^{\Delta_2\Delta_1}(x,y)=F\left(x,y,v(x,y),v^{\Delta_1}(x,y)\right), \tag{4.11}$$

$$v(x,y_0)=\overline{\alpha}(x), \ v(x_0,y)=\overline{\beta}(y), \ v(x_0,y_0)=0, \tag{4.12}$$

for $(x,y)\in\Omega$ where $v\in C_{rd}(\Omega,R_+)$, $\overline{\alpha},\overline{\beta}\in(R_+,R^n)$ and $F\in C_{rd}(\Omega\times R^n\times R^n,R^n)$.

The next theorem deals with continuous dependence solutions of initial boundary value problem (1.1)-(1.2) on the functions involved therein.

Theorem 4.3. *Assume that the function f in equation* (1.1) *satisfies the condition (3.1). Let $v(x,y)$ for $(x,y)\in\Omega$ be a solution of initial boundary value problem* (4.11) − (4.12) *and*

$$\begin{aligned}&\left|\alpha(x)+\beta(y)-\overline{\alpha}(x)-\overline{\beta}(y)\right|+\left|\alpha^{\Delta}(x)-\overline{\alpha}^{\Delta}(x)\right|\\&+\int_{y_0}^{y}\left|f\left(x,t,v(x,t),v^{\Delta_1}(x,t)\right)-F\left(x,t,v(x,t),v^{\Delta_1}(x,t)\right)\right|\Delta t\\&+\int_{x_0}^{x}\int_{y_0}^{y}\left|f\left(s,t,v(s,t),v^{\Delta_1}(s,t)\right)-F\left(s,t,v(s,t),v^{\Delta_1}(s,t)\right)\right|\Delta t\Delta s\leq\epsilon,\end{aligned}\tag{4.13}$$

for $(x,y)\in\Omega$ where $\overline{\alpha},\overline{\beta},F$ are functions involved in initial boundary value problem (1.1) − (1.2) *and initial boundary value problem* (4.11) − (4.12) *and $\epsilon\geq 0$ is a constant. Then the solution $u(x,y)$ on initial boundary value problem* (1.1) − (1.2) *depends continuously on the functions involved therein.*

Proof. Since $u(x,y)$ and $v(x,y)$ are solutions of initial boundary value problem (1.1)-(1.2) and initial boundary value problem (4.11) − (4.12) and the conditions (3.1), (4.13) we get

$$\begin{aligned}&|u(x,y)-v(x,y)|+\left|u^{\Delta_1}(x,y)-v^{\Delta_1}(x,y)\right|\\&\leq\left|\alpha(x)+\beta(y)-\overline{\alpha}(x)-\overline{\beta}(y)\right|+\left|\alpha(x)-\overline{\alpha}^{\Delta}(x)\right|\\&+\int_{x_0}^{x}\int_{y_0}^{y}\left|f\left(s,t,u(s,t),u^{\Delta_1}(s,t)\right)-f\left(s,t,v(s,t),v^{\Delta_1}(s,t)\right)\right|\Delta t\Delta s\\&+\int_{x_0}^{x}\int_{y_0}^{y}\left|f\left(s,t,v(s,t),v^{\Delta_1}(s,t)\right)-F\left(s,t,v(s,t),v^{\Delta_1}(s,t)\right)\right|\Delta t\Delta s\\&+\int_{y_0}^{y}\left|f\left(x,t,u(x,t),u^{\Delta_1}(x,t)\right)-f\left(x,t,v(x,t),v^{\Delta_1}(x,t)\right)\right|\Delta t\\&+\int_{y_0}^{y}\left|f\left(x,t,v(x,t),v^{\Delta_1}(x,t)\right)-F\left(x,t,v(x,t),v^{\Delta_1}(x,t)\right)\right|\Delta t\\&\leq\varepsilon+\int_{y_0}^{y}p(x,t)\left[|u(x,t)-v(x,t)|+\left|u^{\Delta_1}(x,t)-v^{\Delta_1}(x,t)\right|\right]\Delta t\\&+\int_{x_0}^{x}\int_{y_0}^{y}p(x,t)\left[|u(s,t)-v(x,t)|+\left|u^{\Delta_1}(s,t)-v^{\Delta_1}(s,t)\right|\right]\Delta t\Delta s.\end{aligned}\tag{4.14}$$

Now a suitable application of Lemma 2.1 to (4.14) yields

$$|u(x,y)-v(x,y)|+\left|u^{\Delta_1}(x,y)-v^{\Delta_1}(x,y)\right|\leq\varepsilon\overline{q}(x,y)\,e_{\int_{y_0}^{y}p(s,t)\overline{q}(s,t)}(x,x_0).\tag{4.15}$$

□

Now we consider following equation

$$z^{\Delta_2\Delta_1}(x,y)=f\left(x,t,z(x,t),z^{\Delta_1}(x,t),\mu\right),\tag{4.16}$$

$$z^{\Delta_2\Delta_1}(x,y)=f\left(x,t,z(x,t),z^{\Delta_1}(x,t),\mu_0\right),\tag{4.17}$$

with the given initial boundary conditions

$$z(x,y_0)=\tau(x),\ \ z(x_0,y)=\psi(y),\ \ z(x_0,y_0)=0,\tag{4.18}$$

where $z\in C_{rd}(\Omega,R_+)$, $\tau,\psi\in(R_+,R^n)$, $f\in C_{rd}(\Omega\times R^n\times R^n\times R,R^n)$ and μ,μ_0 are real parameters.

Finally, we present following theorem which deals with continuous dependency of solutions of initial boundary value problem (4.16) − (4.18) and initial boundary value problem (4.17) − (4.18) on parameters.

Theorem 4.4. *Assume that the function f in* (4.16) *and* (4.17) *satisfy the conditions*

$$|f(x,y,u,v,\mu) - f(x,y,\overline{u},\overline{v},\mu)| \leq h(x,y)\left[|u-\overline{u}| + |v-\overline{v}|\right], \tag{4.19}$$

$$|f(x,y,u,v,\mu) - f(x,y,u,v,\mu_0)| \leq m(x,y)\,|\mu-\mu_0|, \tag{4.20}$$

for $(x,y) \in \Omega$ *where* $n, m \in C_{rd}(\Omega, R_+)$ *and*

$$\int_{y_0}^{y} m(x,y)\Delta t + \int_{x_0}^{x}\int_{y_0}^{y} m(s,t)\,\Delta t\Delta s \leq \delta, \tag{4.21}$$

where $\delta \geq 0$ *is a constant. Let* $z_1(x,y)$ *and* $z_2(x,y)$ *be the solutions of initial boundary value problem* (4.16) $-$ (4.18) *and initial boundary value problem* (4.17) $-$ (4.18) *respectively. Then*

$$\begin{aligned} &|z_1(x,y) - z_2(x,y)| + \left|z_1^{\Delta_1}(x,y) - z_2^{\Delta_1}(x,y)\right| \\ &\leq |\mu-\mu_0|\,\delta Q(x,y)\, e_{\int_{y_0}^{y} h(s,t)Q(s,t)\Delta t}(x,x_0), \end{aligned} \tag{4.22}$$

for $x, y \in \Omega$, *where*

$$Q(x,y) = \int_{y_0}^{y} h(x,t)\Delta t, \tag{4.23}$$

for $x, y \in \Omega$.

Proof. Since $z_1(x,y)$ and $z_2(x,y)$ be the solutions of initial boundary value problem (4.16) $-$ (4.18) and and initial boundary value problem (4.17) $-$ (4.18) and conditions (4.19) $-$ (4.21) we have

$$\begin{aligned} &|z_1(x,y) - z_2(x,y)| + \left|z_1^{\Delta_1}(x,y) - z_2^{\Delta_1}(x,y)\right| \\ &\leq \int_{x_0}^{x}\int_{y_0}^{y} \left|f\left(s,t,z_1(s,t), z_1^{\Delta_1}(s,t),\mu\right) - f\left(s,t,z_2(s,t), z_2^{\Delta_1}(s,t),\mu\right)\right|\Delta t\Delta s \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} \left|f\left(s,t,z_2(s,t), z_2^{\Delta_1}(s,t),\mu\right) - f\left(s,t,z_2(s,t), z_2^{\Delta_1}(s,t),\mu_0\right)\right|\Delta t\Delta s \\ &+ \int_{y_0}^{y} \left|f\left(x,t,z_1(x,t), z_1^{\Delta_1}(x,t),\mu\right) - f\left(s,t,z_2(x,t), z_2^{\Delta_1}(x,t),\mu\right)\right|\Delta t \\ &+ \int_{y_0}^{y} \left|f\left(x,t,z_1(x,t), z_1^{\Delta_1}(x,t),\mu\right) - f\left(s,t,z_2(x,t), z_2^{\Delta_1}(x,t),\mu_0\right)\right|\Delta t \\ &\leq \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)\left[|z_1(s,t) - z_2(s,t)| + \left|z_1^{\Delta_1}(s,t) - z_2^{\Delta_1}(s,t)\right|\right]\Delta t\Delta s \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} m(s,t)\left[|\mu-\mu_0|\right]\Delta t\Delta s \\ &+ \int_{y_0}^{y} h(x,t)\left[|z_1(x,t) - z_2(x,t)| + \left|z_1^{\Delta_1}(x,t) - z_2^{\Delta_1}(x,t)\right|\right] \\ &+ \int_{y_0}^{y} m(x,t)\left[|\mu-\mu_0|\right]\Delta t \\ &\leq |\mu-\mu_0|\,\delta + \int_{y_0}^{y} h(x,t)\left[|z_1(x,t) - z_2(x,t)| + \left|z_1^{\Delta_1}(x,t) - z_2^{\Delta_1}(x,t)\right|\right]\Delta t \end{aligned}$$

$$+\int_{x_0}^{x}\int_{y_0}^{y} h(s,t)\left[\left|z_1(s,t)-z_2(s,t)\right|+\left|z_1^{\Delta_1}(s,t)-z_2^{\Delta_1}(s,t)\right|\right]\Delta t\Delta s. \tag{4.24}$$

Now a suitable application of Lemma to (4.24) yields (4.22) which shows the dependency of solutions of initial boundary value problem (4.16) – (4.17) and initial boundary value problem (4.17) – (4.18) on parameters. □

Remark 4.2. *The results obtained above can be very easily extended to the following partial dynamic equation on time scales*

$$u^{\Delta_2\Delta_1}(x,y) = f\left(x,y,u(x,y),u^{\Delta_2}(x,y)\right) \tag{1.1}$$

with the initial boundary conditions in (1.2) *by modifying suitably the inequality given in Lemma* 2.1

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New existence and uniqueness results for an α order boundary value problem

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Abstract

This paper is concerned with the existence of solutions for a non local fractional boundary value problem with integral conditions. New existence and uniqueness results are established using Banach fixed point theorem. Other existence results are obtained using Schauder and Krasnoselskii theorems. As an application, we give an example to illustrate our results.

Keywords: Caputo derivative, fixed point theorem, boundary value problem.

2010 MSC: 26A33, 34B15.

1 Introduction

Differential equations of fractional order occur more frequently in different research areas such as engineering, physics, chemistry, economics, etc. Indeed, we can find numerous applications in visco-elasticity, electrochemistry control, porous media, electromagnetic and signal processing, etc. [3, 4, 5]. For an extensive collection of results about this type of equations, we refer the reader to [1, 2, 9, 11] and the references therein. In this paper, we are concerned with the following fractional differential problem

$$\begin{gathered} D^{\alpha}u(t) = f(t, u(t), u'(t)),\ t \in J, 2 < \alpha < 3, \\ u(0) = 0, au'(0) - bu''(0) = \int_0^1 u(t)A(t)dt := \delta\left[u\right], \\ cu'(1) + du''(1) = \int_0^1 u(t)B(t)dt := \beta\left[u\right] \end{gathered} \tag{1.1}$$

where, A, B are two continuous functions on $J := [0,1]$, $A_1 = \sup_{t\in J}|A(t)|, B_1 = \sup_{t\in J}|B(t)|$, $f \in C\left(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right)$, and a, b, c, d are nonnegative constants with $\rho := -2\left(ac + ad + bc\right)$.

2 Notations and Preliminaries

In the following, we give the necessary notation and basic definitions which will be used in this paper:

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as*

$$J^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \alpha > 0, \\ \\ f(t), \alpha = 0, \end{cases} \tag{2.1}$$

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where
$\Gamma(\alpha) = \int\limits_0^{+\infty} e^{-t}t^{\alpha-1}dt.$

Definition 2.2. *The fractional derivative of $f \in C^n([0,\infty[)$ in the sense of Caputo is defined as*

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, n-1 < \alpha < n, n \in \mathbb{N} \\ \\ \frac{d^n}{dt^n} f(t), \alpha = n. \end{cases} \tag{2.2}$$

Details on Caputo's derivative can be found in [8, 10].
We give also the following lemmas [5, 7].

Lemma 2.1. *The general solution of the fractional differential equation*

$$D^\alpha x(t) = 0, \alpha > 0 \tag{2.3}$$

is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + ...t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, ...n-1, n = [\alpha] + 1$.

Lemma 2.2. *Let $\alpha > 0$, then*

$$J^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + t^{n-1} \tag{2.5}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ...n-1, n = [\alpha] + 1$.

Let us now introduce the space

$$\widetilde{C}(J, \mathbb{R}) = \{u \in C(J, \mathbb{R}), u' \in C(J, \mathbb{R})\} \tag{2.5}$$

On $\widetilde{C}(J, \mathbb{R})$, we define the norm

$$\|u\|_1 := \max(\|u\|, \|u'\|); ||u|| = \sup_{t \in J} |u(t)|, ||u'|| = \sup_{t \in J} |u'(t)|. \tag{2.6}$$

It is clear that $\left(\widetilde{C}(J, \mathbb{R}), \|.\|_1\right)$ is a Banach space.

The following lemma is crucial to prove our results.

Lemma 2.3. *Let $2 < \alpha < 3$. The unique solution of the problem (1.1) is given by:*

$$u(t) = J^\alpha f(t, u(t), u'(t)) - c_0 - c_1 t - c_2 t^2, t \in J, \tag{2.7}$$

where

$$\begin{aligned} c_0 &= 0, \quad J^\alpha f(1, u(1), u'(1)) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, u(\tau), u'(\tau))d\tau, \\ c_1 &= \frac{2(c+d)\delta[u] - 2b\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta[u]\right]}{\rho}, \\ c_2 &= \frac{-a\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta[u]\right] - c\delta[u]}{\rho}. \end{aligned} \tag{2.8}$$

Proof. Let $u \in \widetilde{C}(J, \mathbb{R})$, then we have

$$D^\alpha u(t) = f(t, u(t), u'(t)), t \in J. \tag{2.9}$$

Applying J^{α} for both sides of (2.9), and using the identity

$$J^{\alpha}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2, t \in J, \tag{2.10}$$

then using the initial conditions of (1.1), we obtain:

$$\begin{aligned} u(t) &= J^{\alpha}f(t,u(t),u'(t)) \\ &+ \frac{-2\,(c+d)\,\delta\,[u] + 2b\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta\,[u]\right]}{\rho}t \\ &+ \frac{a\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta\,[u]\right] + c\delta\,[u]}{\rho}t^2. \end{aligned} \tag{2.11}$$

□

Now, let us define the operator $T : \widetilde{C}\,(J,\mathbb{R}) \to \widetilde{C}\,(J,\mathbb{R})$ as follows:

$$\begin{aligned} Tu(t) &= J^{\alpha}f(t,u(t),u'(t)) \\ &+ \frac{-2\,(c+d)\,\delta\,[u] + 2b\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta\,[u]\right]}{\rho}t \\ &+ \frac{a\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta\,[u]\right] + c\delta\,[u]}{\rho}t^2. \end{aligned} \tag{2.12}$$

It is clear that

$$\begin{aligned} (Tu)'\,(t) &= J^{\alpha-1}f(t,u(t),u'(t)) \\ &+ \frac{-2\,(c+d)\,\delta\,[u] + 2b\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta\,[u]\right]}{\rho} \\ &+ \frac{2a\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta\,[u]\right] + 2c\delta\,[u]}{\rho}t. \end{aligned} \tag{2.13}$$

3 Main Results

The following conditions are essential to prove our results:

(H_1) : Suppose that $|f(t,u_1,v_1) - f(t,u_2,v_2)| \le k\max(|u_1-u_2|,|v_1-v_2|)$, for all $t \in J$, and $u_1,v_1,u_2,v_2 \in \mathbb{R}$.

(H_2) : The function f is continuous on $J \times \mathbb{R} \times \mathbb{R}$.

(H_3) : There exists a positive constant N, such that $|f(t,u,v)| \le N$, for all $t \in J, u, v \in \mathbb{R}$.

Our first result is based on the Banach fixed point theorem. We have:

Theorem 3.1. *Suppose that the condition (H_1) is satisfied. If*

$$\frac{|\rho|\,k + \left[(4c+2d)A_1 + 2(a+b)B_1\right]\Gamma(\alpha) + 2(a+b)\alpha k\,(c+d(\alpha-1))}{|\rho|\,\Gamma(\alpha)} < 1, \tag{3.1}$$

then the boundary value problem (1.1) has a unique solution on J.

Proof. To prove this theorem, we need to prove that the operator T has a fixed point on $\widetilde{C}\,(J,\mathbb{R})$. So, we shall prove that T is a contraction mapping on $\widetilde{C}\,(J,\mathbb{R})$.

Let $u,v \in \widetilde{C}(J,\mathbb{R})$. Then for all $t \in J$, we can write

$$\begin{aligned}
|Tu(t)-Tv(t)| &= |\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&-\frac{2(c+d)t}{\rho}\int_0^1 A(\tau)\left(u(\tau)-v(\tau)\right)d\tau \\
&+\frac{2bct}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&+\frac{2bdt}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&-\frac{2bt}{\rho}\int_0^1 B(\tau)\left(u(\tau)-v(\tau)\right)d\tau \qquad (3.2)\\
&+\frac{act^2}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&+\frac{adt^2}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&-\frac{at^2}{\rho}\int_0^1 B(\tau)\left(u(\tau)-v(\tau)\right)d\tau+\frac{ct^2}{\rho}\int_0^1 A(\tau)\left(u(\tau)-v(\tau)\right)d\tau|.
\end{aligned}$$

Thanks to (H_1), we obtain

$$\begin{aligned}
\|Tu-Tv\| &\leq \frac{k}{\Gamma(\alpha+1)}\|u-v\|_1+\left[\frac{(3c+2d)A_1+(a+2b)\,B_1}{|\rho|}\right]\|u-v\| \\
&+\frac{(a+2b)\,k}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\|u-v\|_1. \qquad (3.3)
\end{aligned}$$

Since$\|u-v\|\leq\|u-v\|_1$, then we get

$$\|Tu-Tv\|\leq\frac{|\rho|\,k+\left[(3c+2d)A_1+(a+2b)B_1\right]\Gamma(\alpha+1)+(a+2b)k\alpha\left[c+d(\alpha-1)\right]}{|\rho|\,\Gamma(\alpha+1)}\|u-v\|_1. \qquad (3.4)$$

On the other hand, we have

$$\left\|(Tu)'-(Tv)'\right\|\leq\frac{|\rho|\,k+\left[(4c+2d)A_1+2(a+b)B_1\right]\Gamma(\alpha)+2(a+b)k\alpha\left[c+d(\alpha-1)\right]}{|\rho|\,\Gamma(\alpha)}\|u-v\|_1. \qquad (3.5)$$

By the condition (3.1), we conclude that T is a contraction mapping. Hence, by Banach fixed point theorem, there exists a unique fixed point $u\in\widetilde{C}(J,\mathbb{R})$ which is a solution of the problem (1.1). □

Our second result is the following:

Theorem 3.2. *Suppose that the conditions (H_2) and (H_3) are satisfied. If*

$$|\rho|>(4c+2d)A_1+2\,(a+b)\,B_1, \qquad (3.6)$$

then the problem (1.1) has at least a solution in $\widetilde{C}\,(J,\mathbb{R})$.

Proof. We use Schaefer's fixed point theorem to prove that T has a fixed point on $\widetilde{C}\,(J,\mathbb{R})$.

Let us first choose ν such that

$$\nu\geq\max\left(\frac{|\rho|\,N+N(a+2b)\alpha\left[c+d(\alpha-1)\right]}{\Gamma(\alpha+1)\left(|\rho|-\left[(3c+2d)A_1+(a+2b)B_1\right]\right)},\frac{|\rho|\,N+2N(a+b)\left[c+d(\alpha-1)\right]}{\Gamma(\alpha)\left(|\rho|-\left[(4c+2d)A_1+2(a+b)B_1\right]\right)}\right) \qquad (3.7)$$

and set $\widetilde{C}_\nu = \{u \in C(J,\mathbb{R}), ||u||_1 \le \nu\}$. It is clear that $\widetilde{C}_\nu$ is a closed and convex subset.

Step1: T is continuous:

Let $(u_n)_n$ be a sequence such that $u_n \to u$, $n \to +\infty$ in $\widetilde{C}(J,\mathbb{R})$. For each $t \in J$, we have

$$\begin{aligned} |Tu_n(t) - Tu(t)| &\le \frac{t^\alpha}{\Gamma(\alpha+1)} |f(t,u_n(t),u_n'(t)) - f(t,u(t),u'(t))| \\ &+ \left[\frac{2(c+d)A_1 + 2bB_2}{|\rho|}\right] |u_n(t) - u(t)|t \\ &+ \frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] |f(t,u_n(t),u_n'(t)) - f(t,u(t),u'(t))|\, t \\ &+ \left[\frac{aB_1 + cA_1}{|\rho|}\right] |u_n(t) - u(t)|t^2 \\ &+ \frac{a}{|\rho|}\left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] |f(t,u_n(t),u_n'(t)) - f(t,u(t),u'(t))|\, t^2 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \left|(Tu_n)'(t) - (Tu)'(t)\right| &\le \frac{t^\alpha}{\Gamma(\alpha)} |f(t,u_n(t),u_n'(t)) - f(t,u(t),u'(t))| \\ &+ \left[\frac{2(c+d)A_1 + 2bB_1}{|\rho|}\right] |u_n(t) - u(t)| \\ &+ \frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] |f(t,u_n(t),u_n'(t)) - f(t,u(t),u'(t))| \\ &+ \left[\frac{2aB_1 + 2cA_1}{|\rho|}\right] |u_n(t) - u(t)|t \\ &+ \frac{2a}{|\rho|}\left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] |f(t,u_n(t),u_n'(t)) - f(t,u(t),u'(t))|\, t. \end{aligned} \tag{3.9}$$

Since f is a continuous function, the right-hand sides of (3.8) (3.9) tend to zero as n tends to $+\infty$. Then

$$||T(u_n) - T(u)||_1 \to 0,\ \ n \to +\infty. \tag{3.10}$$

Step2: We shall prove that $T(\widetilde{C}_\nu) \subset \widetilde{C}_\nu$:

Let us take $u \in \widetilde{C}_\nu$. Then for each $t \in J$, we have

$$\begin{aligned} |Tu(t)| &\le \frac{1}{\Gamma(\alpha+1)} \sup_{t\in J} |f(t,u(t),u'(t))| \\ &+ \left[\frac{(3c+2d)A_1 + (a+2b)B_1}{|\rho|}\right] \|u\| \\ &+ \frac{(a+2b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] \sup_{t\in J} |f(t,u(t),u'(t))| \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \left|(Tu)'(t)\right| &\le \frac{1}{\Gamma(\alpha)} \sup_{t\in J} |f(t,u(t),u'(t))| \\ &+ \left[\frac{(4c+2d)A_1 + 2(a+b)B_1}{|\rho|}\right] \|u\| \\ &+ \frac{2(a+b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] \sup_{t\in J} |f(t,u(t),u'(t))|\,. \end{aligned} \tag{3.12}$$

By (H_3), we obtain

$$\begin{aligned} |\,|Tu|| &\le \frac{N}{\Gamma(\alpha+1)} + \left[\frac{(3c+2d)A_1 + (a+2b)B_1}{|\rho|}\right]\nu + \frac{(a+2b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] N \\ &\le \nu \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\left\|(Tu)'\right\| &\leq \frac{N}{\Gamma(\alpha)}+\left[\frac{(4c+2d)A_1+2\left(a+b\right)B_1}{|\rho|}\right]\nu+\frac{2N\left(a+b\right)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\\ &\leq \nu. \end{aligned} \tag{3.14}$$

Consequently,

$$\|Tu\|_1 \leq \nu. \tag{3.15}$$

Step3: **T maps bounded sets into equi-continuous sets of** $\widetilde{C}(J,\mathbb{R})$:

Let $t_1,\ t_2 \in J,\ t_1 < t_2, u \in \widetilde{C}_\nu$. Then, we can write

$$\begin{aligned}|Tu(t_2)-Tu(t_1)| &\leq \frac{1}{\Gamma(\alpha+1)}\left(t_2^\alpha-t_1^\alpha\right)\sup_{t\in J}|f(t,u(t),u'(t))|+\left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u\|\left(t_2-t_1\right)\\ &+\frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2-t_1\right)\sup_{t\in J}|f(t,u(t),u'(t))|\\ &+\left[\frac{aB_1+cA_1}{|\rho|}\right]\|u\|\left(t_2^2-t_2^2\right)\\ &+\frac{a}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2^2-t_2^2\right)\sup_{t\in J}|f(t,u(t),u'(t))|. \end{aligned} \tag{3.16}$$

Using (H_3), we obtain the following result

$$\begin{aligned}|Tu(t_2)-Tu(t_1)| &\leq \frac{N}{\Gamma(\alpha+1)}\left(t_2^\alpha-t_1^\alpha\right)+\nu\left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\left(t_2-t_1\right)\\ &+\frac{2bN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2-t_1\right)+\nu\left[\frac{aB_1+cA_1}{|\rho|}\right]\left(t_2^2-t_2^2\right)\\ &+\frac{aN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2^2-t_2^2\right), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\left|(Tu)'(t_2)-(Tu)'(t_1)\right| &\leq \frac{N}{\Gamma(\alpha)}\left(t_2^\alpha-t_1^\alpha\right)+2\nu\left[\frac{aB_1+cA_1}{|\rho|}\right]\left(t_2-t_2\right)\\ &+\frac{2aN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2-t_2\right). \end{aligned} \tag{3.18}$$

As $t_2 \to t_1$, the right-hand sides of (3.17) and (3.18) tend to zero. Then, as a consequence of Steps $1,2,3$ together with the Arzela-Ascoli theorem, we conclude that T is completely continuous.

Step4: The set B is bounded:

Now, we prove that the set $B=\{u\in \widetilde{C}(J,\mathbb{R}), u=\lambda T(u), 0<\lambda<1\}$ is bounded.

Let $u \in B$, then $u=\lambda T(u)$, for some $0<\lambda<1$. Hence, for each $t\in J$, we have

$$\begin{aligned}\frac{|u(t)|}{\lambda} &\leq \frac{Nt^\alpha}{\Gamma(\alpha+1)}+\|u\|\left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]t+\frac{2bN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]t\\ &+\|u\|\left[\frac{aB_1+cA_1}{|\rho|}\right]t^2+\frac{aN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]t^2. \end{aligned} \tag{3.19}$$

Since $t\in J$, hence we can write

$$|u(t)| \leq \frac{\lambda N}{|\rho|-\lambda\left[(3c+2d)A_1+(a+2b)\,B_1\right]}\left[\frac{|\rho|}{\Gamma(\alpha+1)}+(a+2b)\left(\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right)\right] \tag{3.20}$$

and

$$|u'(t)| \leq \frac{\lambda N}{|\rho|-\lambda\left[(4c+2d)A_1+2\left(a+b\right)B_1\right]}\left[\frac{|\rho|}{\Gamma(\alpha)}+2\left(a+b\right)\left(\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right)\right]. \tag{3.21}$$

Thanks to 3.6, we get

$$\|u\|_1 < \infty. \tag{3.22}$$

This shows that the set is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that T has a fixed point which is a solution of the problem (1.1). □

We state a result due to Krasnoselskii [6] which is needed to prove the existence of at least one solution of the problem (1.1).

Theorem 3.3. *(Krasnoselskii fixed point theorem) Let S be a closed convex and nonempty subset of a Banach space X. Let P, Q be the operators such that*
(i) $Px + Qy \in S$; whenever $x, y \in S$
(ii) P is compact and continuous;
(iii) Q is a contraction mapping. Then there exists x^* such that $x^* = Px^* + Qx^*$.

We have:

Theorem 3.4. *Suppose that there exist ω and θ two positives real numbers such that $0 < \omega < 1, \theta > 0$. If the following conditions are satisfied*

$$\frac{N}{\Gamma(\alpha+1)} + \frac{(a+2b)\,N}{|\rho|}\left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] \;\leq\; (1-\omega)\theta, \tag{3.23}$$

$$\left[\frac{(3c+2d)A_1 + (a+2b)\,B_1}{|\rho|}\right] \;\leq\; \omega, \tag{3.24}$$

and

$$\frac{|\rho|\,k + [2(c+d)A_1 + 2bB_1]\,\Gamma(\alpha) + 2bk\,[c + d(\alpha-1)]}{|\rho|\,\Gamma(\alpha)} < 1, \tag{3.25}$$

then (1.1) has a solution u such that $\|u\|_1 \leq \theta$.

Proof. Let $B_\theta = \{u \in C(J,\mathbb{R}), ||u||_1 \leq \theta\}$. We define the operator R as follows:

$$\begin{aligned} Ru(t) \;:\; &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,u(\tau),u'(\tau))d\tau + (\frac{-2(c+d)}{\rho}\int_0^1 A(t)\,(u(t)-v(t))\,dt \\ &+\frac{2bc}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f(\tau,u(\tau),u'(\tau))d\tau \\ &+\frac{2bd}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f(\tau,u(\tau),u'(\tau))d\tau - \frac{2b}{\rho}\int_0^1 B(t)\,(u(t)-v(t))\,dt)t, \end{aligned} \tag{3.26}$$

It is clear that

$$\begin{aligned} (Ru)'(t) \;:\; &= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2} f(\tau,u(\tau),u'(\tau))d\tau - \frac{2(c+d)}{\rho}\int_0^1 A(t)u(t)dt \\ &+\frac{2bc}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f(\tau,u(\tau),u'(\tau))d\tau + \frac{2bd}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f(\tau,u(\tau),u'(\tau))d\tau \\ &-\frac{2b}{\rho}\int_0^1 B(\tau)u(\tau)d\tau, \end{aligned} \tag{3.27}$$

We also define the operator S by:

$$\begin{aligned} Su(t) \;:\; &= \frac{act^2}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f(\tau,u(\tau),u'(\tau))d\tau + \frac{adt^2}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f(\tau,u(\tau),u'(\tau))d\tau \\ &-\frac{at^2}{\rho}\int_0^1 B(\tau)u(\tau)d\tau + \frac{ct^2}{\rho}\int_0^1 A(\tau)u(\tau)d\tau. \end{aligned} \tag{3.28}$$

Then,

$$(Su)'(t) := \frac{2act}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f(\tau,u(\tau),u'(\tau))d\tau + \frac{2adt}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f(\tau,u(\tau),u'(\tau))d\tau - \frac{2at}{\rho}\int_0^1 B(\tau)u(\tau)d\tau + \frac{2ct}{\rho}\int_0^1 A(\tau)u(\tau)d\tau. \quad (3.29)$$

(1*) Let $u,v\in B_\theta$. We have

$$|Ru(t)+Sv(t)| \leq \frac{1}{\Gamma(\alpha+1)}|f(t,u(t),u'(t))| + \left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u\| + \left[\frac{cA_1+aB_1}{|\rho|}\right]\|v\| + \frac{(a+2b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]|f(t,u(t),u'(t))|, \quad (3.30)$$

and

$$\left|(Ru)'(t)+(Sv)'(t)\right| \leq \frac{1}{\Gamma(\alpha)}|f(t,u(t),u'(t))| + \left[\frac{2(c+d)A_1+bB_1}{|\rho|}\right]\|u\| + \left[\frac{2cA_1+2aB_1}{|\rho|}\right]\|v\| + \frac{2(a+b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]|f(t,u(t),u'(t))|. \quad (3.31)$$

Thanks to (3.23) and (3.24), we can write

$$\|Ru+Sv\| \leq \frac{N}{\Gamma(\alpha+1)} + \theta\left[\frac{(3c+2d)A_1+(2b+a)B_1}{|\rho|}\right] + \frac{(a+2b)N}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right] \leq \omega\theta+(1-\omega)\theta=\theta. \quad (3.32)$$

Consequently,

$$Ru+Sv\in B_\theta. \quad (3.33)$$

(2*) Now we prove the contraction of R.

$$|Ru(t)-Rv(t)| \leq \frac{1}{\Gamma(\alpha+1)}|f(t,u(t),u'(t))-f(t,v(t),v'(t))| + \left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u-v\| + \frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]|f(t,u(t),u'(t))-f(t,v(t),v'(t))|, \quad (3.34)$$

and

$$\left|(Ru)'(t)-(Rv)'(t)\right| \leq \frac{1}{\Gamma(\alpha)}|f(t,u(t),u'(t))-f(t,v(t),v'(t))| + \left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u-v\| + \frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]|f(t,u(t),u'(t))-f(t,v(t),v'(t))|, \quad (3.35)$$

By the hypothesis (H_1), we have

$$|Ru-Rv| \leq \frac{k}{\Gamma(\alpha+1)}\|u-v\|_1 + \left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u-v\| + \frac{2bk}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\|u-v\|_1, \quad (3.36)$$

and

$$\left|(Ru)'(t)-(Rv)'(t)\right| \leq \frac{k}{\Gamma(\alpha)}\|u-v\|_1 + \left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u-v\| + \frac{2bk}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\|u-v\|_1, \quad (3.37)$$

Hence, by (3.25), R is a contraction mapping.

(3*) The Continuity of f implies that the operator S is continuous.

(4*) The compactness of S :

Let us take $u \in B_\theta, t_1, t_2 \in J, t_1 < t_2$. We have

$$|Su(t_1) - Su(t_2)| \leq \nu \left[\frac{aB_1 + cA_1}{|\rho|}\right] \left(t_2^2 - t_1^2\right) + \frac{aN}{|\rho|} \left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] \left(t_2^2 - t_1^2\right), \tag{3.38}$$

and

$$\left|(Su)'(t_1) - (Su)'(t_2)\right| \leq 2\nu \left[\frac{aB_1 + cA_1}{|\rho|}\right] (t_2 - t_1) + \frac{2aN}{|\rho|} \left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] (t_2 - t_1). \tag{3.39}$$

The right hand side of (3.38) and (3.39) are independent of u. Hence S is equicontinuous. And as $t_1 \to t_2$, the left hand sides of (3.38) and (3.39) tend to 0; so $S(B_\theta)$ is relatively compact and then by Ascolli-Arzella theorem, the operator S is compact. Finally, by Krasnoselskii theorem, we conclude that there exists a solution to (1.1). Theorem 3.4 is thus proved. □

4 Example

Consider the three-point BVP

$$\begin{cases} D^{\frac{5}{2}}u(t) = \frac{u(t)+u'(t)}{64}e^{-t^2} + \frac{1}{1+t^2}, t \in [0,1], \\ u(0) = 0, u'(0) - u''(0) = \int_0^1 u(t)\frac{e^{-t}}{64}dt, \\ 2u'(1) + 2u''(1) = \int_0^1 u(t)\frac{e^{-t^2}}{64}dt, \end{cases} \tag{4.1}$$

In this example, we have $a = b = 1, c = d = 2$, $A(t) = \frac{e^{-t}}{64}, B(t) = \frac{e^{-t^2}}{64}, N = \frac{1}{64}, A_1 = B_1 = k.$

The condition (3.1) is given by

$$\frac{|\rho|\, k + [(4c+2d)A_1 + 2(a+b)B_1]\, \Gamma(\alpha) + 2(a+b)\alpha k \left(c + d(\alpha-1)\right)}{|\rho|\, \Gamma(\alpha)} = \frac{31 + 6\sqrt{\pi}}{288\sqrt{\pi}} < 1.$$

Then, the problem (4.1) has a solution on $[0,1]$.

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Time dependent solution of Non-Markovian queue with two phases of service and general vacation time distribution

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Abstract

We consider an $M^{[x]}/G/1$ queue with two phases of service, with different general (arbitrary) service time distributions. The first phase of service is essential for all customers, as soon as the first service of a customer is completed, then with probability θ, he may opt for the second service or else with probability $(1-\theta)$, he leaves the system. At each service completion, the server will take compulsory vacation. The vacation period of the server has two heterogeneous phases. Phase one is compulsory and phase two follows the phase one vacation in such a way that the server may take phase two vacation with probability p or return back to the system with probability $(1-p)$. The service and vacation periods are assumed to be general. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly. Also the average number of customers in the queue and the waiting time are also derived.

Keywords: Batch arrival, optional service, second optional vacation, stability condition, average queue size, average waiting time.

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1 Introduction

The modelling analysis for the queueing systems with vacations has been done by a considerable amount of work in the past and successfully used in various applied problems such as production/inventory system, communication systems, computer networks etc. A comprehensive and excellent study on the vacation models can be found in Levy and Yechiali [11], Doshi [6], Takagi [15], Lee et al. [10], Krishna Reddy et al. [9], Hur and Paik [7] and others. Batch arrival $M^{[x]}/G/1$ queueing systems with multiple vacations were first studied by Baba [1]. Badamchi Zadeh [2] studied a batch arrival queuening system with two phases of heterogenous service with optional second service and restricted admissibility with single vacation policy.

Recently, there have been several contributions considering queueing system of M/G/1 type in which the server may provide a second phase service. Such queueing situations occur in day-to-day life, for example in many applications such as hospital services, production systems, bank services, computer and communication networks there is two phase of services such that the first phase is essential for all customers, but as soon as the essential services completed, it may leave the system or may immediately go for the second phase of service. One may refer to Medhi [14], Krishnakumar et al. [8], Choudhury [4], Madan and Choudhury [13], Choudhury and Paul [5]. Badamchi and Shankar [3] have also studied a single server queue with two phase queueing system with Bernoulli feedback and Bernoulli schedule server vacation. Madan and Choudhury [12] proposed an queueing system with restricted admissibilty of arriving batches and Bernoulli schedule server vacation.

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In this paper we consider batch arrival queue with two phases of service and optional server vacation. The first phase of service is essential for all customers, as soon as the first service of a customer is completed, then with probability θ, he may opt for the second service or else with probability $(1-\theta)$, he leaves the system. After completion of each service, the server will take compulsory vacation. The vacation periods of the server has two heterogeneous phases. However, after returning from phase one compulsory vacation the server may take one more optional vacation with probability p or return back to the system with probability $(1-p)$.

This paper is organized as follows. The mathematical description of our model is given in section 2. Definitions and equations governing the system are given in section 3. The time dependent solution have been obtained in section 4 and corresponding steady state results have been derived explicitly in section 5. Average queue size and average waiting time are computed in section 6.

2 Mathematical description of the model

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ $(i = 1, 2, \ldots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) There is a single server who provides the first phase of essential service for all customers, as soon as the essential service of a customer is completed, then with probability θ, he may opt for the second service or else with probability $(1-\theta)$, he leave the system.

c) The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x+dx]$, given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}, \quad i = 1, 2,$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i = 1, 2.$$

d) After completion of each service, the server will take compulsory vacation of random length. The vacation time has two phases with phase one is compulsory. However, after phase one vacation, the server takes phase two optional vacation with probability p or may return back to the system with probability $(1-p)$.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V_i(t)$ and density function $v_i(t)$. Let $\gamma_i(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x, so that

$$\gamma_i(x) = \frac{v_i(x)}{1-V_i(x)}, \quad i = 1, 2,$$

and therefore,

$$v_i(t) = \gamma_i(t) e^{-\int_0^t \gamma_i(x)dx} \quad i = 1, 2.$$

f) Various stochastic processes involved in the system are assumed to be independent of each other.

3 Definitions and equations governing the system

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing essential service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x.

Consequently $P_n^{(1)}(t) = \int\limits_0^\infty P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the essential service irrespective of the value of x.
$P_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing second optional service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $P_n^{(2)}(t) = \int\limits_0^\infty P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the second optional service irrespective of the value of x.

$V_n^{(1)}(x,t)$ = Probability that at time t, the server is under phase one compulsory vacation with elapsed vacation time x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(1)}(t) = \int\limits_0^\infty V_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase one compulsory vacation irrespective of the value of x.

$V_n^{(2)}(x,t)$ = Probability that at time t, the server is under phase two optional vacation with elapsed vacation time x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(2)}(t) = \int\limits_0^\infty V_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase two optional vacation irrespective of the value of x.
Q(t) = Probability that at time t, there are no customers in the queue and the server is idle but available in the system.

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + [\lambda + \mu_1(x)]P_0^{(1)}(x,t) = 0 \tag{3.1}$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + [\lambda + \mu_1(x)]P_n^{(1)}(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}^{(1)}(x,t), \ n \geq 1 \tag{3.2}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial t}P_0^{(2)}(x,t) + [\lambda + \mu_2(x)]P_0^{(2)}(x,t) = 0 \tag{3.3}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + [\lambda + \mu_2(x)]P_n^{(2)}(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}^{(2)}(x,t), \quad n \geq 1 \tag{3.4}$$

$$\frac{\partial}{\partial x}V_0^{(1)}(x,t) + \frac{\partial}{\partial t}V_0^{(1)}(x,t) + [\lambda + \gamma_1(x)]V_0^{(1)}(x,t) = 0 \tag{3.5}$$

$$\frac{\partial}{\partial x}V_n^{(1)}(x,t) + \frac{\partial}{\partial t}V_n^{(1)}(x,t) + [\lambda + \gamma_1(x)]V_n^{(1)}(x,t) = \lambda\sum_{k=1}^{n} c_k V_{n-k}^{(1)}(x,t), \quad n \geq 1 \tag{3.6}$$

$$\frac{\partial}{\partial x}V_0^{(2)}(x,t) + \frac{\partial}{\partial t}V_0^{(2)}(x,t) + [\lambda + \gamma_2(x)]V_0^{(2)}(x,t) = 0 \tag{3.7}$$

$$\frac{\partial}{\partial x}V_n^{(2)}(x,t) + \frac{\partial}{\partial t}V_n^{(2)}(x,t) + [\lambda + \gamma_2(x)]V_n^{(2)}(x,t) = \lambda\beta\sum_{k=1}^{n} c_k V_{n-k}^{(2)}(x,t), \ n \geq 1 \tag{3.8}$$

$$\frac{d}{dt}Q(t) + \lambda Q(t) = (1-p)\int_0^\infty \gamma_1(x)V_0^{(1)}(x,t)dx + \int_0^\infty \gamma_2(x)V_0^{(2)}(x,t)dx \tag{3.9}$$

The above equations are to be solved subject to the following boundary conditions:

$$P_n^{(1)}(0,t) = \alpha\lambda C_{n+1}Q(t) + (1-p)\int_0^\infty \gamma_1(x)V_{n+1}^{(1)}(x,t)dx + \int_0^\infty \gamma_2(x)V_{n+1}^{(2)}(x,t)dx, \quad n \geq 0 \tag{3.10}$$

$$P_n^{(2)}(0,t) = \theta\int_0^\infty \mu_1(x)P_n^{(1)}(x,t)dx, \quad n \geq 0 \tag{3.11}$$

$$V_n^{(1)}(0,t) = (1-\theta)\int_0^\infty \mu_1(x)P_n^{(1)}(x,t)dx + \int_0^\infty \mu_2(x)P_n^{(2)}(x,t)dx, \quad n \geq 0 \tag{3.12}$$

$$V_n^{(2)}(0,t) = p\int_0^\infty \gamma_1(x)V_n^{(1)}(x,t)dx, \quad n \geq 0 \tag{3.13}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$\begin{aligned} &V_0^{(i)}(0) = V_n^{(i)}(0) = 0, \quad Q(0) = 1 \text{ and} \\ &P_n^{(i)}(0) = 0 \text{ for } n = 0,1,2,..., \quad i = 1,2. \end{aligned} \tag{3.14}$$

4 Generating functions of the queue length: The time-dependent solution

In this section we obtain the transient solution for the above set of dfferential-difference equations.

Theorem 4.1. *The system of differential difference equations to describe an $M^{[x]}/G/1$ queue with first essential service, second optional service, first phase of vacation and optional vacation are given by equations (3.1) to (3.13) with initial conditions (3.14) and the generating functions of transient solution are given by equation (4.50) to (4.53).*

Proof. We define the probability generating functions ,

$$P^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x,t); \quad P^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(t), \text{for } i = 1,2. \tag{4.1}$$

$$V^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n V_n^{(i)}(x,t); V^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n V_n^{(i)}(t), C(z) = \sum_{n=1}^{\infty} c_n z^n \tag{4.2}$$

which are convergent inside the circle given by z $\leq$ 1 and define the Laplace transform of a function f(t) as

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt, \quad \Re(s) > 0 \tag{4.3}$$

Taking the Laplace transform of equations (3.1) to (3.13) and using (3.14), we obtain

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s) + (s+\lambda+\mu_1(x))\bar{P}_0^{(1)}(x,s) = 0 \tag{4.4}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + (s+\lambda+\mu_1(x))\bar{P}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(1)}(x,s), n \geq 1 \tag{4.5}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s) + (s+\lambda+\mu_2(x))\bar{P}_0^{(2)}(x,s) = 0 \tag{4.6}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + (s+\lambda+\mu_2(x))\bar{P}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(2)}(x,s), n \geq 1 \tag{4.7}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(1)}(x,s) + (s+\lambda+\gamma_1(x))\bar{V}_0^{(1)}(x,s) = 0 \tag{4.8}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,s) + (s+\lambda\beta+\gamma_1(x))\bar{V}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(1)}(x,s), n \geq 1 \tag{4.9}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(2)}(x,s) + (s+\lambda+\gamma_2(x))\bar{V}_0^{(2)}(x,s) = 0 \tag{4.10}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,s) + (s+\lambda+\gamma_2(x))\bar{V}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(2)}(x,s), n \geq 1 \tag{4.11}$$

$$(s+\lambda)\bar{Q}(s) = 1 + (1-p)\int_0^\infty \gamma_1(x)\bar{V}_0^{(1)}(x,s)dx + \int_0^\infty \gamma_2(x)\bar{V}_0^{(2)}(x,s)dx \tag{4.12}$$

$$\bar{P}_n^{(1)}(0,s) = \alpha\lambda c_{n+1}\bar{Q}(s) + (1-p)\int_0^\infty \gamma_1(x)\bar{V}_{n+1}^{(1)}(x,s)dx + \int_0^\infty \gamma_2(x)\bar{V}_{n+1}^{(2)}(x,s)dx, \quad n \geq 0 \tag{4.13}$$

$$\bar{P}_n^{(2)}(0,s) = \theta\int_0^\infty \mu_1(x)\bar{P}_n^{(1)}(x,s)dx, \quad n \geq 0 \tag{4.14}$$

$$\bar{V}_n^{(1)}(0,s) = (1-\theta)\int_0^\infty \mu_1(x)\bar{P}_n^{(1)}(x,s)dx + \int_0^\infty \mu_2(x)\bar{P}_n^{(2)}(x,s)dx, \ n \geq 0 \tag{4.15}$$

$$\bar{V}_n^{(2)}(0,s) = p\int_0^\infty \gamma_1(x)\bar{V}_n^{(1)}(x,s)dx, \quad n \geq 0 \tag{4.16}$$

Now multiplying equations (4.5), (4.7), (4.9) and (4.11) by z^n and summing over n from 1 to ∞, adding to equation (4.4), (4.6), (4.8) and (4.10) using the generating functions defined in (4.1) and (4.2) we get

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,z,s) + [s+\lambda-\lambda C(z)+\mu_1(x)]\bar{P}^{(1)}(x,z,s) = 0 \tag{4.17}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,z,s) + [s+\lambda(1-C(z))+\mu_2(x)]\bar{P}^{(2)}(x,z,s) = 0 \tag{4.18}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,z,s) + [s+\lambda-\lambda C(z)+\gamma_1(x)]\bar{V}^{(1)}(x,z,s) = 0 \tag{4.19}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,z,s) + [s+\lambda-\lambda C(z)+\gamma_2(x)]\bar{V}^{(2)}(x,z,s) = 0 \tag{4.20}$$

For the boundary conditions, we multiply both sides of equation (4.13) by z^n sum over n from 0 to ∞, and use the equation (4.12), we get

$$\begin{aligned} z\bar{P}^{(1)}(0,z,s) &= [1-s\bar{Q}(s)] + \lambda[C(z)-1]\bar{Q}(s) \\ &+(1-p)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx + \int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx \end{aligned} \tag{4.21}$$

Performing similar operation on equations (4.14) to (4.16) we get,

$$\bar{P}^{(2)}(0,z,s) = \theta\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx \tag{4.22}$$

$$\bar{V}^{(1)}(0,z,s) = (1-\theta)\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx + \int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx \tag{4.23}$$

$$\bar{V}^{(2)}(0,z,s) = p\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \tag{4.24}$$

Integrating equation (4.17) between 0 to x, we get

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \mu_1(t)dt} \tag{4.25}$$

where $P^{(1)}(0,z,s)$ is given by equation (4.21).
Again integrating equation (4.25) by parts with respect to x yields,

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{4.26}$$

where

$$\bar{B}_1(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB_1(x) \tag{4.27}$$

is the Laplace-Stieltjes transform of the first phase of essential service time $B_1(x)$. Now multiplying both sides of equation (4.25) by $\mu_1(x)$ and integrating over x we obtain

$$\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda(1-c(z))] \tag{4.28}$$

Similarly, on integrating equations (4.18) to (4.20) from 0 to x, we get

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \mu_2(t)dt} \tag{4.29}$$

$$\bar{V}^{(1)}(x,z,s) = \bar{V}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \gamma_1(t)dt} \tag{4.30}$$

$$\bar{V}^{(2)}(x,z,s) = \bar{V}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \gamma_2(t)dt} \tag{4.31}$$

where $\bar{P}^{(2)}(0,z,s)$, $\bar{V}^{(1)}(0,z,s)$ and $\bar{V}^{(2)}(0,z,s)$ are given by equations (4.22) to (4.24). Again integrating equations (4.29) to (4.31) by parts with respect to x yields,

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda c(z)}\right] \tag{4.32}$$

$$\bar{V}^{(1)}(z,s) = \bar{V}^{(1)}(0,z,s)\left[\frac{1-\bar{V}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{4.33}$$

$$\bar{V}^{(2)}(z,s) = \bar{V}^{(2)}(0,z,s)\left[\frac{1-\bar{V}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{4.34}$$

where

$$\bar{B}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB_2(x) \tag{4.35}$$

$$\bar{V}_1(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV_1(x) \tag{4.36}$$

$$\bar{V}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV_2(x) \tag{4.37}$$

are the Laplace-Stieltjes transform of the second optional service time $B_2(x)$, first phase of vacation time $V_1(x)$ and second optional vacation $V_2(x)$ respectively. Now multiplying both sides of equation (4.29) to (4.31) by $\mu_2(x)$, $\gamma_1(x)$ and $\gamma_2(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)] \tag{4.38}$$

$$\int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx = \bar{V}^{(1)}(0,z,s)\bar{V}_1[s+\lambda-\lambda C(z)] \tag{4.39}$$

$$\int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx = \bar{V}^{(2)}(0,z,s)\bar{V}_2[s+\lambda-\lambda C(z)] \tag{4.40}$$

Using equations (4.28) and (4.38), we can write equation (4.23) as

$$\bar{V}^{(1)}(0,z,s) = (1-\theta)\bar{B}_1(R)\bar{P}^{(1)}(0,z,s) + \bar{B}_2(R)\bar{P}^{(2)}(0,z,s) \tag{4.41}$$

Using equation (4.28) in (4.22), we get

$$\bar{P}^{(2)}(0,z,s) = \theta\bar{P}^{(1)}(0,z,s)\bar{B}_1(R) \tag{4.42}$$

By using equation (4.42), equation (4.41) reduces to

$$\bar{V}^{(1)}(0,z,s) = \bar{B}_1(R)[1-\theta+\theta\bar{B}_2(R)]\bar{P}^{(1)}(0,z,s) \tag{4.43}$$

Using equations (4.39) and (4.43) in (4.24), we get

$$\bar{V}^{(2)}(0,z,s) = p\bar{B}_1(R)\bar{V}_1(R)[1-\theta+\theta\bar{B}_2(R)]\bar{P}^{(1)}(0,z,s) \tag{4.44}$$

Similarly using equations (4.39), (4.40), (4.43) and (4.44) in (4.21), we get

$$\bar{P}^{(1)}(0,z,s) = \frac{[1-s\bar{Q}(s)]+\lambda[(C(z)-1]\bar{Q}(s)}{DR} \tag{4.45}$$

where

$$DR = z - \bar{B}_1(R)\bar{V}_1(R)[1-\theta+\theta\bar{B}_2(R)](1-p+p\bar{V}_2(R)), \tag{4.46}$$

$R = s+\lambda-\lambda C(z)$. Substituting the value of $\bar{P}^{(1)}(0,z,s)$ from equation (4.45) into equations (4.42), (4.43) and (4.44), we get

$$\bar{P}^{(2)}(0,z,s) = \theta\frac{\bar{B}_1(R)[(1-s\bar{Q}(s))+\lambda\alpha(C(z)-1)\bar{Q}(s)]}{DR} \tag{4.47}$$

$$\bar{V}^{(1)}(0,z,s) = \bar{B}_1(R)(1-\theta+\theta\bar{B}_2(R))\frac{[(1-s\bar{Q}(s))+\lambda\alpha(C(z)-1)\bar{Q}(s)]}{DR} \tag{4.48}$$

$$\bar{V}^{(2)}(0,z,s) = p\bar{B}_1(R)(1-\theta+\theta\bar{B}_2(R))\bar{V}_1(R)\frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{DR} \tag{4.49}$$

Using equations (4.45), (4.47), (4.48) and (4.49) in (4.26), (4.32), (4.33) and (4.34), we get

$$\bar{P}^{(1)}(z,s) = \frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{DR}\frac{[1-\bar{B}_1(R)]}{R} \tag{4.50}$$

$$\bar{P}^{(2)}(z,s) = \frac{\theta\bar{B}_1(R)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{DR}\frac{[1-\bar{B}_2(R)]}{R} \tag{4.51}$$

$$\bar{V}^{(1)}(z,s) = \frac{[1-\theta+\theta\bar{B}_2(R)]\bar{B}_1(R)}{DR} [(1-s\bar{Q}(s))(\lambda C(z)-\lambda)\bar{Q}(s)]\frac{[1-\bar{V}_1(R)]}{R} \tag{4.52}$$

$$\bar{V}^{(2)}(z,s) = p\bar{B}_1(R)\bar{V}_1(R)\frac{[1-\theta+\theta\bar{B}_2(R)]}{DR} [(1-s\bar{Q}(s))+(\lambda C(z)-\lambda)\bar{Q}(s)]\frac{[1-\bar{V}_2(R)]}{R} \tag{4.53}$$

where DR is given by equation (60). Thus $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ are completely determined from equations (4.50) to (4.53) which completes the proof of the theorem. □

5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \tag{5.1}$$

In order to determine $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (4.50) to (4.53). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1)+P^{(2)}(1)+V^{(1)}(1)+V^{(2)}(1)+Q = 1 \tag{5.2}$$

Theorem 5.1. *The steady state probabilities for an $M^{[x]}/G/1$ first essential service, second optional service, first phase of vacation and optional vacation are given by*

$$P^{(1)}(1) = \frac{\lambda E(I)E(B_1)Q}{dr} \tag{5.3}$$

$$P^{(2)}(1) = \frac{\theta\lambda E(I)E(B_2)Q}{dr} \tag{5.4}$$

$$V^{(1)}(1) = \frac{\lambda E(I)E(V_1)Q}{dr} \tag{5.5}$$

$$V^{(2)}(1) = \frac{p\lambda E(I)E(V_2)Q}{dr} \tag{5.6}$$

where

$$dr = 1 - \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)], \tag{5.7}$$

$P^{(1)}(1)$, $P^{(2)}(1)$, $V^{(1)}(1)$, $V^{(2)}(1)$ and Q are the steady state probabilities that the server is providing first essential service, second optional service, server under first phase of vacation, optional vacation, server under idle respectively without regard to the number of customers in the system.

Proof. Multiplying both sides of equations (4.50) to (4.53) by s, taking limit as $s \to 0$, applying property (5.1) and simplifying, we obtain

$$P^{(1)}(z) = \frac{[\bar{B}_1(T) - 1]Q}{D(z)} \tag{5.8}$$

$$P^{(2)}(z) = \frac{\theta\bar{B}_1(T)[\bar{B}_2(T) - 1]Q}{D(z)} \tag{5.9}$$

$$V^{(1)}(z) = \frac{\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)][\bar{V}_1(T) - 1]Q}{D(z)} \tag{5.10}$$

$$V^{(2)}(z) = \frac{p\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)]\bar{V}_1(f_3(z))[\bar{V}_2(T) - 1]Q}{D(z)} \tag{5.11}$$

where

$$D(z) = z - \bar{V}_1(T)\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)][1 - p + p\bar{V}_2(T)], \tag{5.12}$$

$T = \lambda - \lambda C(z)$.
Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (5.8) to (5.11) we obtain

$$W_q(z) = P^{(1)}(z) + P^{(2)}(z) + V^{(1)}(z) + V^{(2)}(z)$$

$$\begin{aligned} W_q(z) = {} & \frac{[\bar{B}_1(T) - 1]Q}{D(z)} \\ & + \frac{\theta\bar{B}_1(T)[\bar{B}_2(T) - 1]Q}{D(z)} \\ & + \frac{\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)][\bar{V}_1(T) - 1]Q}{D(z)} \\ & + \frac{p\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)]\bar{V}_1(T)[\bar{V}_2(T) - 1]Q}{D(z)} \end{aligned} \tag{5.13}$$

where C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers,
$-\bar{B}_i'(0) = E(B_i), -\bar{V}_i'(0) = E(V_i)$, for i = 1, 2.
In order to find Q, we use the normalization condition $W_q(1)$ + Q = 1. We see that for z=1, $W_q(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying we get,

$$W_q(1) = \frac{\alpha\lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1 + pE(V_2))]}{1 - \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)]} Q \tag{5.14}$$

Therefore adding Q to equation (5.14), equating to 1 and simplifying, we get

$$Q = 1 - \rho \tag{5.15}$$

and hence the utilization factor ρ of the system is given by

$$\rho = \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)] \tag{5.16}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (5.15) gives the probability that the server is idle. Substituting Q from (5.15) into (5.13), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size. □

6 The mean queue size and the mean system size

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz} W_q(z) \text{ at } z = 1$$

Since this formula gives 0/0 form, then we write $W_q(z)$ given in (5.13) as $W_q(z) = \frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are numerator and denominator of the right hand side of (5.13) respectively. Then we use

$$L_q = \lim_{z \to 1} \frac{d}{dz} W_q(z) = \lim_{z \to 1} \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right] \tag{6.1}$$

where primes and double primes in (6.1) denote first and second derivative at $z = 1$, respectively. Carrying out the derivative at $z = 1$ we have

$$N'(1) = \lambda\alpha\beta E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)]Q \tag{6.2}$$

$$\begin{aligned} N''(1) = [&\lambda^2 (E(I))^2 [E(B_1^2) + \theta E(B_2^2) + E(V_1^2) + pE(V_2^2))] \\ &+ \lambda E(I(I-1))[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2] \\ &+ 2\lambda^2 (E(I))^2 [E(B_1)E(V_1) + p\theta E(B_2)E(V_2)] \\ &+ 2\lambda^2 (E(I))^2 (E(B_1) + E(V_1))(\theta E(B_2) + pE(V_2))]Q \end{aligned} \tag{6.3}$$

$$D'(1) = 1 - \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)] \tag{6.4}$$

$$\begin{aligned} D''(1) = &-\lambda^2 (E(I))^2 [E(B_1) + \theta E(B_2) + E(V_1 + pE(V_2] \\ &-2\lambda^2 (E(I))^2 [E(B_1)E(V_1) + p\theta E(B_2)E(V_2)] \\ &+2\lambda^2 (E(I))^2 [E(B_1) + E(V_1)][\theta E(B_2) + pE(V_2)] \\ &+\lambda E(I(I-1))[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)], \end{aligned} \tag{6.5}$$

where $E(V^2)$ are the second moment of the vacation time. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (6.2) to (6.5) into equation (6.1) we obtain L_q in the closed form.

Further, we find the mean system size L using Little's formula. Thus we have

$$L = L_q + \rho \tag{6.6}$$

where L_q has been found by equation (6.1) and ρ is obtained from equation (5.16).

7 The average waiting time

Let W_q and W denote the mean waiting time in the queue and in the system respectively. Then using Little's formula, we obtain,

$$W_q = \frac{L_q}{\lambda} \tag{7.1}$$

$$W = \frac{L}{\lambda}, \tag{7.2}$$

where L_q and L have been found in equations (6.1) and (6.6).

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Existence of mild solutions for impulsive fractional stochastic equations with infinite delay

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Abstract

This paper is mainly concerned with the existence of mild solutions for a class of fractional stochastic differential equations with impulses in Hilbert spaces. A new set of sufficient conditions are formulated and proved for the existence of mild solutions by means of Sadovskii's fixed point theorem. An example is given to illustrate the theory.

Keywords: Existence result, fractional stochastic differential equation, fixed point technique, infinite delay, resolvent operators.

2010 MSC: 34K30, 34K50, 26A33.

1 Introduction

The stochastic differential equations have been widely applied in science, engineering, biology, mathematical finance and in almost all applied sciences. In the present literature, there are many papers on the existence and uniqueness of solutions to stochastic differential equations (see [1, 2, 8] and references therein). More recently, Chang et al. [4] investigated the existence of square-mean almost automorphic mild solutions to nonautonomous stochastic differential equations in Hilbert spaces by using semigroup theory and fixed point approach. Fu and Liu [8] discussed the existence and uniqueness of square-mean almost automorphic solutions to some linear and nonlinear stochastic differential equations and in which they studied the asymptotic stability of the unique square-mean almost automorphic solution in the square-mean sense.

Recently, fractional differential equations have found numerous applications in various fields of science and engineering [11]. The existence of solutions for nonlinear fractional stochastic differential equations have been studied by few authors [9, 18].

On the other hand, the theory of impulsive differential equations is emerging as an active area of investigation due to the application in area such as mechanics, electrical engineering, medicine biology, and ecology, see Benchohra and Henderson [3], Hernández et al. [10], Lin and Hu [13], Prato and Zabczyk [14]. As an adequate model, impulsive differential equations are used to study the evolution of processes that are subject to sudden changes in their states. However, to the best of our knowledge, it seems that little is known about impulsive fractional stochastic equations with infinite delay and the aim of this paper is to fill this gap. We refer the interested reader, for instance, to [18] and references therein for impulsive fractional stochastic equations.

Inspired by the mentioned work [18] in this paper, we are interested in studying the existence of mild solutions

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of the following impulsive fractional stochastic differential equations with infinite delay in the form

$$\begin{cases} {}^cD_t^\alpha[x(t)+g(t,x_t)] &=& A\Big[x(t)+g(t,x_t)\Big]+f(t,x_t,B_1x(t))+\sigma(t,x_t,B_2x(t))\dfrac{dw(t)}{dt}, \\ && t\in J:=[0,T],\quad T>0,\quad t\neq t_k, \\ \Delta x(t_k) &=& I_k(x(t_k^-)),\quad k=1,2,\dots,m, \\ x(t) &=& \phi(t),\quad \phi(t)\in\mathcal{B}_h, \end{cases} \tag{1.1}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order α, $0<\alpha<1$; $x(.)$ takes the value in the separable Hilbert space $\mathcal{H}$; $A:\mathcal{D}(A)\subset\mathcal{H}\to\mathcal{H}$ is the infinitesimal generator of an α-resolvent family $S_\alpha(t)_{t\geq 0}$. The history $x_t:(-\infty,0]\to\mathcal{H}$, $x_t(\theta)=x(t+\theta)$, $\theta\leq 0$, belongs to an abstract phase space $\mathcal{B}_h$, which will be described axiomatically in Section 2; $g:J\times\mathcal{B}_h\to\mathcal{H}$, $f:J\times\mathcal{B}_h\times\mathcal{H}\to\mathcal{H}$ and $\sigma:J\times\mathcal{B}_h\times\mathcal{H}\to\mathcal{L}_2^0$ are appropriate functions to be specified later; $I_k:\mathcal{B}_h\to\mathcal{H}$, $k=1,2,\dots,m$, are appropriate functions. The terms $B_1x(t)$ and $B_2x(t)$ are given by $B_1x(t)=\int_0^t K(t,s)x(s)ds$ and $B_2x(t)=\int_0^t P(t,s)x(s)ds$ respectively, where $K,P\in\mathcal{C}(\mathcal{D},\mathbb{R}^+)$ are the set of all positive continuous functions on $\mathcal{D}=\{(t,s)\in\mathbb{R}^2:0\leq s\leq t\leq T\}$. Here $0=t_0<t_1<\dots<t_m<t_{m+1}=T$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, $x(t_k^+)=\lim_{h\to 0}x(t_k+h)$ and $x(t_k^-)=\lim_{h\to 0}x(t_k-h)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively. The initial data $\phi=\{\phi(t),t\in(-\infty,0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued random variable independent of w with finite second moments.

The paper is organized as follows. In section 2, we briefly present some basic notations and preliminaries. In section 3, is devoted to the development of our main existence results and our basic tool include Sadovskii's fixed point theorem. Finally, the paper is conclude with an example to illustrate the obtained results.

2 Preliminaries and basic properties

Let $\mathcal{H},\mathcal{K}$ be two separable Hilbert spaces and $\mathcal{L}(\mathcal{K},\mathcal{H})$ be the space of bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$. For convenience, we will use the same notation $\|.\|$ to denote the norms in $\mathcal{H},\mathcal{K}$ and $\mathcal{L}(\mathcal{K},\mathcal{H})$, and use $(.,.)$ to denote the inner product of $\mathcal{H}$ and $\mathcal{K}$ without any confusion. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $w=(w_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ with the covariance operator Q such that $trQ<\infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathcal{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k=\lambda_k e_k$, $k=1,2,\dots$ and a sequence $\{\beta_k\}_{k\geq 1}$ of independent Brownian motions such that

$$(w(t),e)_{\mathcal{K}}=\sum_{k=1}^{\infty}\sqrt{\lambda_k}(e_k,e)_{\mathcal{K}}\beta_k(t),\quad e\in\mathcal{K},t\in[0,b].$$

Let $\mathcal{L}_2^0=\mathcal{L}_2(Q^{1/2}\mathcal{K},\mathcal{H})$ be the space of all HilbertSchmidt operators from $Q^{1/2}\mathcal{K}$ into $\mathcal{H}$ with the inner product $\langle\psi,\pi\rangle_{\mathcal{L}_2^0}=tr[\psi Q\pi^\star]$.

Assume that $h:(-\infty,0]\to(0,\infty)$ with $l=\int_{-\infty}^0 h(t)dt<\infty$ a continuous function. We define the abstract phase space $\mathcal{B}_h$ by

$$\begin{aligned}\mathcal{B}_h = \Big\{&\phi:(-\infty,0]\to\mathcal{H},\ \text{for any } a>0, (\mathbb{E}|\phi(\theta)|^2)^{1/2} \text{ is bounded and measurable} \\ &\text{function on } [-a,0] \text{ with } \phi(0)=0 \text{ and } \int_{-\infty}^0 h(s)\sup_{s\leq\theta\leq 0}(\mathbb{E}|\phi(\theta)|^2)^{1/2}ds<\infty\Big\}.\end{aligned}$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h}=\int_{-\infty}^0 h(s)\sup_{s\leq\theta\leq 0}(\mathbb{E}|\phi(\theta)|^2)^{1/2}ds,\quad \phi\in\mathcal{B}_h,$$

then $(\mathcal{B}_h,\|.\|_{\mathcal{B}_h})$ is a Banach space [5].

We consider the space

$$\begin{aligned}\mathcal{B}_b = \Big\{&x:(-\infty,T]\to\mathcal{H} \text{ such that } x|_{J_k}\in\mathcal{C}(J_k,\mathcal{H}) \text{ and there exist} \\ &x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k)=x(t_k^-),\ x_0=\phi\in\mathcal{B}_h,\ k=1,2,\dots,m\Big\},\end{aligned}$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$. the function $\|.\|_{\mathcal{B}_h}$ to be a seminorm in $\mathcal{B}_b$, it is defined by

$$\|x\|_{\mathcal{B}_b} = \|\phi\|_{\mathcal{B}_h} + \sup_{0\le s\le T} (\mathbb{E}\|x(s)\|^2)^{1/2}, \quad x \in \mathcal{B}_b$$

Lemma 2.1 ([16]). *Assume that $x \in B_h$; then for $t \in J$, $x_t \in \mathcal{B}_h$. Moreover,*

$$l(\mathbb{E}\|x(t)\|^2)^{1/2} \le l \sup_{0\le s\le T} (\mathbb{E}\|x(s)\|^2)^{1/2} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$.

Let us recall the following known definitions. For more details see [12].

Definition 2.1. *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.2. *Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, n-1 < \alpha < n. \tag{2.2}$$

Definition 2.3. *The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^c D^\alpha f(t) = {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \right), \quad t > 0, n-1 < \alpha < n. \tag{2.3}$$

If $f(t) \in C^n[0, \infty)$, then

$$^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds = I^{n-\alpha} f^n(s), \quad t > 0, n-1 < \alpha < n$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{^c D^\alpha f(t); s\} = s^\alpha \hat{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \quad n-1 \le \alpha < n.$$

Definition 2.4. *A two parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \quad \alpha, \beta \in C, \mathcal{R}(\alpha) > 0,$$

where C is a contour which starts and ends at $-\infty$ end encircles the disc $|\mu| \le |z|^{1/2}$ counter clockwise.

For short, $E_\alpha(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(z^2) = \cos h(z)$, $E_2(-z^2) = \cos(z)$, and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad Re\lambda > \omega^{\frac{1}{\alpha}}, \omega > 0,$$

and for more details see [12].

Definition 2.5 ([23]). *A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied:*

i. $\rho(A) \subset \Sigma_{\theta,\omega} = \{\lambda \in C : \lambda \ne \omega, |arg(\lambda - \omega)| < \theta\}$, *ii.* $\|R(\lambda, A)\| = \|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda-\omega|}$, $\lambda \in \Sigma_{\theta,\omega}$.

Definition 2.6. *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space H. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(H)$, where $L(H)$ is a Banach space of all bounded linear operators from H into H and the corresponding norm is denoted by $\|.\|$, such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)x dt, \quad Re\lambda > \omega, x \in H, \tag{2.4}$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.7. *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space H and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(H)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)x dt, \quad Re\lambda > \omega, x \in H, \tag{2.5}$$

where $S_\alpha(t)$ is called the solution operator generated by A.

The concept of the solution operator is closely related to the concept of a resolvent family. For more details on α-resolvent family and solution operators, we refer the reader to [12].

Lemma 2.2 ([6]). *If f satisfies the uniform Hölder condition with the exponent $\beta \in (0, 1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem*

$$\begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t) + f(t, x_t, Bx(t)), \quad t > t_0, t_0 \geq 0, 0 < \alpha < 1, \\ x(t) &= \phi(t), \quad t \leq t_0, \end{aligned} \tag{2.6}$$

is given by

$$x(t) = T_\alpha(t - t_0)(x(t_0^+)) + \int_{t_0}^t S_\alpha(t - s)f(s, x_s, Fx(s))ds, \tag{2.7}$$

where

$$T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda, \tag{2.8}$$

$$S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{1}{\lambda^\alpha - A}d\lambda, \tag{2.9}$$

here $\widehat{B}_r$ denotes the Bromwich path; $S_\alpha(t)$ is called the α-resolvent family and $T_\alpha(t)$ is the solution operator generated by A.

The following result on the operator $S_\alpha(t)$ appeared and proved in [23].

Theorem 2.1. *If $\alpha \in (0, 1)$ and $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$ is a sectorial operator, then for any $x \in \mathcal{H}$ and $t > 0$, we have*

$$\|S_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1}), \qquad t > 0, \omega > \omega_0,$$

where C is a constant depending only on θ and ω.

At the end of this section, we recall the fixed point theorem of Sadovskii [17] which is used to establish the existence of the mild solution to the impulsive fractional system (1.1).

Theorem 2.2 ([17]). *Let Φ be a condensing operator on a Banach space $\mathcal{H}$, that is, Φ is continuous and takes bounded sets into bounded sets, and $\mu(\Phi(B)) \leq \mu(B)$ for every bounded set B of $\mathcal{H}$ with $\mu(B) > 0$. If $\Phi(N) \subset N$ for a convex, closed and bounded set N of $\mathcal{H}$, then Φ has a fixed point in $\mathcal{H}$ (where $\mu(.)$ denotes Kuratowski's measure of noncompactness).*

3 The mild solution and existence

In this section, we consider the fractional impulsive system (1.1). We first present the definition of mild solutions for the system based on the paper [7].

Definition 3.1. *An $\mathcal{H}$-valued stochastic process $\{x(t), t \in (-\infty, T]\}$ is said to be a mild solution of the system (1.1) if $x_0 = \phi \in \mathcal{B}_h$ satisfying $x_0 \in \mathcal{L}_2^0(\Omega, \mathcal{H})$ and the following conditions hold.*

i. $x(t)$ is $\mathcal{F}_t$ adapted and measurable, $t \geq 0$;

ii. x_t is $\mathcal{B}_h$-valued and the restriction of $x(.)$ to the interval $(t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$ is continuous;

iii. for each $t \in J$, $x(t)$ satisfies the following integral equation

$$x(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] - g(t, x_t) + \displaystyle\int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds \\ \quad + \displaystyle\int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) dw(s), \quad t \in [0, t_1], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + T_\alpha(t-t_1) I_1(x(t_1^-)) - g(t, x_t) \\ \quad + T_\alpha(t-t_1)[g(t_1, x_{t_1} + I_1(x_{t_1^-})) - g(t_1, x_{t_1})] \\ \quad + \displaystyle\int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + \displaystyle\sum_{k=1}^m T_\alpha(t-t_k) I_k(x(t_k^-)) - g(t, x_t) \\ \quad + \displaystyle\sum_{k=1}^m T_\alpha(t-t_k)[g(t_k, x_{t_k} + I_k(x_{t_k^-})) - g(t_k, x_{t_k})] \\ \quad + \displaystyle\int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) dw(s), \quad t \in (t_m, T]. \end{cases} \tag{3.1}$$

iv. $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \ldots, m$ the restriction of $x(.)$ to the interval $[0, T) \setminus \{t_1, \ldots, t_m\}$ is continuous.

In order to explain our theorem, we need the following assumptions.

$(H1)$: If $\alpha \in (0,1)$ and $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$, then for $x \in \mathcal{H}$ and $t > 0$ we have $\|T_\alpha(t)\| \leq Me^{\omega t}$ and $\|S_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1})$, $\omega > \omega_0$. Thus we have

$$\|T_\alpha(t)\| \leq \widetilde{M}_T \quad \text{and} \quad \|S_\alpha(t)\| \leq t^{\alpha-1}\widetilde{M}_S,$$

where $\widetilde{M}_T = \sup_{0 \leq t \leq T} \|T_\alpha(t)\|$, and $\widetilde{M}_S = \sup_{0 \leq t \leq T} Ce^{\omega t}(1 + t^{1-\alpha})$ (fore more details, see [23]).

$(H2)$: The function $g : J \times \mathcal{B}_h \to \mathcal{H}$ is continuous and there exists some constant $M_g > 0$ such that

$$\mathbb{E}\|g(t, \psi_1) - g(t, \psi_2)\|_{\mathcal{H}}^2 \leq M_g \|\psi_1 - \psi_2\|_{\mathcal{B}_h}^2, \quad (t, \psi_i) \in J \times \mathcal{B}_h, \quad i = 1, 2,$$

$$\mathbb{E}\|g(t, \psi)\|_{\mathcal{H}}^2 \leq M_g \Big(\|\psi\|_{\mathcal{B}_h}^2 + 1\Big).$$

$(H3)$: The function $f : J \times \mathcal{B}_h \times \mathcal{H} \to \mathcal{H}$ satisfies the following properties:

i. $f(t, \cdot, \cdot) : \mathcal{B}_h \times \mathcal{H} \to \mathcal{H}$ is continuous for each $t \in J$ and for each $(\psi, x) \in \mathcal{B}_h \times \mathcal{H}$, $f(\cdot, \psi, x) : J \to \mathcal{H}$ is strongly measurable;

ii. there exist two positive integrable functions $\mu_1, \mu_2 \in L^1([0, T])$ and a continuous nondecreasing function $\Xi_f : [0, \infty) \to (0, \infty)$ such that for every $(t, \psi, x) \in J \times \mathcal{B}_h \times \mathcal{H}$, we have

$$\mathbb{E}\|f(t, \psi, x)\|_{\mathcal{H}}^2 \leq \mu_1(t)\Xi_f\Big(\|\psi\|_{\mathcal{B}_h}^2\Big) + \mu_2(t)\mathbb{E}\|x\|_{\mathcal{H}}^2, \qquad \liminf_{q \to \infty} \frac{\Xi_f(q)}{q} = \Lambda < \infty.$$

iii. there exist two positive integrable functions $\mu_1, \mu_2 \in L^1([0,T])$ such that

$$\mathbb{E}\|f(t,\psi,x) - f(t,\varphi,y)\|^2_{\mathcal{H}} \leq \mu_1(t)\|\psi-\varphi\|^2_{\mathcal{B}_h} + \mu_2(t)\mathbb{E}\|x-y\|^2_{\mathcal{H}},$$

for every (t,ψ,x) and $(t,\varphi,y) \in J \times \mathcal{B}_h \times \mathcal{H}$.

$(H4)$: The function $\sigma : J \times \mathcal{B}_h \times \mathcal{H} \to \mathcal{L}_2^0$ satisfies the following properties:

i. $\sigma(t,\cdot,\cdot) : \mathcal{B}_h \times \mathcal{H} \to \mathcal{L}_2^0$ is continuous for each $t \in J$ and for each $(\psi,x) \in \mathcal{B}_h \times \mathcal{H}$, $\sigma(\cdot,\psi,x) : J \to \mathcal{L}_2^0$ is strongly measurable;

ii. there exist two positive integrable functions $\nu_1, \nu_2 \in L^1([0,T])$ and a continuous nondecreasing function $\Xi_\sigma : [0,\infty) \to (0,\infty)$ such that for every $(t,\psi,x) \in J \times \mathcal{B}_h \times \mathcal{H}$, we have

$$\mathbb{E}\|\sigma(t,\psi,x)\|^2_{\mathcal{L}_2^0} \leq \nu_1(t)\Xi_\sigma\left(\|\psi\|^2_{\mathcal{B}_h}\right) + \nu_2(t)\mathbb{E}\|x\|^2_{\mathcal{H}}, \qquad \liminf_{q\to\infty} \frac{\Xi_\sigma(q)}{q} = \Upsilon < \infty.$$

iii. there exist two positive integrable functions $\nu_1, \nu_2 \in L^1([0,T])$ such that

$$\mathbb{E}\|\sigma(t,\psi,x) - \sigma(t,\varphi,y)\|^2_{\mathcal{L}_2^0} \leq \nu_1(t)\|\psi-\varphi\|^2_{\mathcal{B}_h} + \nu_2(t)\mathbb{E}\|x-y\|^2_{\mathcal{H}},$$

for every (t,ψ,x) and $(t,\varphi,y) \in J \times \mathcal{B}_h \times \mathcal{H}$.

$(H5)$: The function $I_k : \mathcal{H} \to \mathcal{H}$ is continuous and there exists $\Theta > 0$ such that

$$\Theta = \max_{1\leq k\leq m,\ \ x\in B_q} \{\mathbb{E}\|I_k(x)\|^2_{\mathcal{H}}\},$$

where $B_q = \{y \in \mathcal{B}_b^0, \|y\|^2_{\mathcal{B}_b^0} \leq q,\ \ q>0\}$.
The set B_q is clearly a bounded closed convex set in $\mathcal{B}_b^0$ for each q and for each $y \in B_q$. From Lemma 2.1, we have

$$\begin{aligned}
\|y_t + \bar{z}_t\|^2_{\mathcal{B}_h} &\leq 2(\|y_t\|^2_{\mathcal{B}_h} + \|\bar{z}_t\|^2_{\mathcal{B}_h}) \\
&\leq 4\left(l^2 \sup_{0\leq t\leq T} \mathbb{E}\|y(t)\|^2_{\mathcal{H}} + \|y_0\|^2_{\mathcal{B}_h}\right) + 4\left(l^2 \sup_{0\leq t\leq T} \mathbb{E}\|y(t)\|^2_{\mathcal{H}} + \|\bar{z}_0\|^2_{\mathcal{B}_h}\right) \qquad (3.2)\\
&\leq 4(\|\phi\|^2_{\mathcal{B}_h} + l^2 q).
\end{aligned}$$

The main object of this paper is to explain and prove the following theorem.

Theorem 3.1. *Assume that the assumptions $(H1)$-$(H5)$ hold. Then the impulsive stochastic fractional system (1.1) has a mild solution on $(-\infty,T]$ provided that*

$$\widetilde{C} + 16M_g l^2 + 7\widetilde{M}_S^2 T^{2\alpha}\Big[\frac{\eta_1}{\alpha^2} + \frac{\eta_2}{T(2\alpha-1)}\Big] < 1 \qquad (3.3)$$

and

$$l^2 M_g + \widetilde{M}_S^2 T^{2\alpha}\Big[\frac{\vartheta_1}{\alpha^2} + \frac{\vartheta_2}{T(2\alpha-1)}\Big] < 1, \qquad (3.4)$$

$\widetilde{C}$ is a positive constant depending only on $\widetilde{M}_T, M_g$ and l.

Proof. Consider the operator $\mathcal{P}:\mathcal{B}_b\to\mathcal{B}_b$ defined by

$$\mathcal{P}(t)=\begin{cases}\phi(t), \quad t\in(-\infty,0],\\ T_\alpha(t)[\phi(0)+g(0,\phi)]-g(t,x_t)+\int_0^t S_\alpha(t-s)f(s,x_s,B_1x(s))ds\\ \quad+\int_0^t S_\alpha(t-s)\sigma(s,x_s,B_2x(s))dw(s), \quad t\in[0,t_1],\\ T_\alpha(t)[\phi(0)+g(0,\phi)]+T_\alpha(t-t_1)I_1(x(t_1^-))-g(t,x_t)\\ \quad+T_\alpha(t-t_1)[g(t_1,x_{t_1}+I_1(x_{t_1^-}))-g(t_1,x_{t_1})]\\ \quad+\int_0^t S_\alpha(t-s)f(s,x_s,B_1x(s))ds+\int_0^t S_\alpha(t-s)\sigma(s,x_s,B_2x(s))dw(s), \quad t\in(t_1,t_2],\\ \vdots\\ T_\alpha(t)[\phi(0)+g(0,\phi)]+\sum_{k=1}^m T_\alpha(t-t_k)I_k(x(t_k^-))-g(t,x_t)\\ \quad+\sum_{k=1}^m T_\alpha(t-t_k)[g(t_k,x_{t_k}+I_k(x_{t_k^-}))-g(t_k,x_{t_k})]\\ \quad+\int_0^t S_\alpha(t-s)f(s,x_s,B_1x(s))ds+\int_0^t S_\alpha(t-s)\sigma(s,x_s,B_2x(s))dw(s), \quad t\in(t_m,T].\end{cases} \tag{3.5}$$

We shall show that $\mathcal{P}$ has a fixed point, which is then a mild solution for the impulsive system (1.1). For $\phi\in\mathcal{B}_h$, define

$$\bar z(t)=\begin{cases}\phi(t), & t\in(-\infty,0];\\ 0, & t\in J.\end{cases}$$

Then $\bar z\in\mathcal{B}_b$. Let $x(t)=y(t)+\bar z(t)$, $t\in(-\infty,T]$. It is easy to check that x satisfies (1.1) if and only if $y_0=0$ and

$$y(t)=\begin{cases}T_\alpha(t)[\phi(0)+g(0,\phi)]-g(t,y_t+\bar z_t)+\int_0^t S_\alpha(t-s)f(s,y_s+\bar z_s,B_1(y(s)+\bar z(s)))ds\\ \quad+\int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar z_s,B_2(y(s)+\bar z(s)))dw(s), \quad t\in[0,t_1],\\ T_\alpha(t)[\phi(0)+g(0,\phi)]+T_\alpha(t-t_1)I_1(y(t_1^-))-g(t,y_t+\bar z_t)\\ \quad+T_\alpha(t-t_1)[g(t_1,y_{t_1}+\bar z_{t_1}+I_1(y_{t_1^-}+\bar z_{t_1^-}))-g(t_1,y_{t_1}+\bar z_{t_1})]\\ \quad+\int_0^t S_\alpha(t-s)f(s,y_s+\bar z_s,B_1(y(s)+\bar z(s)))ds\\ \quad+\int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar z_s,B_2(y(s)+\bar z(s)))dw(s), \quad t\in(t_1,t_2],\\ \vdots\\ T_\alpha(t)[\phi(0)+g(0,\phi)]+\sum_{k=1}^m T_\alpha(t-t_k)I_k(y(t_k^-))-g(t,y_t+\bar z_t)\\ \quad+\sum_{k=1}^m T_\alpha(t-t_k)[g(t_k,y_{t_k}+\bar z_{t_k}+I_k(y_{t_k^-}+\bar z_{t_k^-}))-g(t_k,y_{t_k}+\bar z_{t_k})]\\ \quad+\int_0^t S_\alpha(t-s)f(s,y_s+\bar z_s,B_1(y(s)+\bar z(s)))ds\\ \quad+\int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar z_s,B_2(y(s)+\bar z(s)))dw(s), \quad t\in(t_m,T].\end{cases}$$

Set

$$\mathcal{B}_b^0=\{y\in\mathcal{B}_b, y_0=0\in\mathcal{B}_h\}.$$

Thus, for any $y\in\mathcal{B}_b^0$ we have

$$\|y\|_b=\|y_0\|_{\mathcal{B}_h}+\sup_{0\le s\le T}\left(\mathbb{E}\|y(s)\|^2\right)^{\frac12}=\sup_{0\le s\le T}\left(\mathbb{E}\|y(s)\|^2\right)^{\frac12}.$$

Therefore, $(\mathcal{B}_b^0,\|\cdot\|_b)$ is a Banach space.

Consider the map Π on $\mathcal{B}_b^0$ defined by

$$(\Pi y)(t) = \begin{cases} T_\alpha(t)[\phi(0)+g(0,\phi)] - g(t,y_t+\bar{z}_t) + \int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in[0,t_1], \\ T_\alpha(t)[\phi(0)+g(0,\phi)] + T_\alpha(t-t_1)I_1(y(t_1^-)) - g(t,y_t+\bar{z}_t) \\ \quad +T_\alpha(t-t_1)[g(t_1,y_{t_1}+\bar{z}_{t_1}+I_1(y_{t_1^-}+\bar{z}_{t_1^-})) - g(t_1,y_{t_1}+\bar{z}_{t_1})] \\ \quad + \int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in(t_1,t_2], \\ \vdots \\ T_\alpha(t)[\phi(0)+g(0,\phi)] + \sum_{k=1}^m T_\alpha(t-t_k)I_k(y(t_k^-)) - g(t,y_t+\bar{z}_t) \\ \quad +\sum_{k=1}^m T_\alpha(t-t_k)[g(t_k,y_{t_k}+\bar{z}_{t_k}+I_k(y_{t_k^-}+\bar{z}_{t_k^-})) - g(t_k,y_{t_k}+\bar{z}_{t_k})] \\ \quad + \int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in(t_m,T]. \end{cases}$$

It is clear that the operator $\mathcal{P}$ has a fixed point if and only if Π has a fixed point. So let us prove that Π has a fixed point. Now, we decompose Π as $\Pi = \Pi_1 + \Pi_2$, where

$$(\Pi_1 y)(t) = \begin{cases} 0, \quad t\in[0,t_1], \\ T_\alpha(t-t_1)I_1(y(t_1^-)) \\ \quad +T_\alpha(t-t_1)[g(t_1,y_{t_1}+\bar{z}_{t_1}+I_1(y_{t_1^-}+\bar{z}_{t_1^-})) - g(t_1,y_{t_1}+\bar{z}_{t_1})], \quad t\in(t_1,t_2], \\ \vdots \\ \sum_{k=1}^m T_\alpha(t-t_k)I_k(y(t_k^-)) \\ \quad +\sum_{k=1}^m T_\alpha(t-t_k)[g(t_k,y_{t_k}+\bar{z}_{t_k}+I_k(y_{t_k^-}+\bar{z}_{t_k^-})) - g(t_k,y_{t_k}+\bar{z}_{t_k})], \quad t\in(t_m,T], \end{cases}$$

$$\begin{aligned} (\Pi_2 y)(t) &= T_\alpha(t)g(0,\phi) - g(t,y_t+\bar{z}_t) + \int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds \\ &\quad + \int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in J. \end{aligned}$$

In order to use Theorem 2.2 we will verify that Π_1 is compact and continuous while Π_2 is a contraction operator. For the sake of convenience, we divide the proof into several steps.

Step1. We show that there exists a positive number q such that $\Pi(B_q)\subset B_q$. If this is not true, then for each $q>0$, there exists a function $y^q(\cdot)\in B_q$, but $\Pi(y^q)\notin B_q$, that is $\mathbb{E}\|(\Pi y^q)(t)\|_{\mathcal{H}}^2 > q$. An elementary inequality can show that, for $t\in[0,t_1]$

$$\begin{aligned} q &\le \mathbb{E}\|\Pi(y^q)(t)\|_{\mathcal{H}}^2 \\ &\le 4\mathbb{E}\|T_\alpha(t)g(0,\phi)\|_{\mathcal{H}}^2 + 4\mathbb{E}\|g(t,y_t^q+\bar{z}_t\|_{\mathcal{H}}^2 + 4\mathbb{E}\left\|\int_0^t S_\alpha(t-s)f(s,y_s^q+\bar{z}_s,B_1(y^q(s)+\bar{z}(s)))ds\right\|_{\mathcal{H}}^2 \\ &\quad +4\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s,y_s^q+\bar{z}_s,B_2(y^q(s)+\bar{z}(s)))dw(s)\right\|_{\mathcal{H}}^2 \\ &= 4\sum_{i=1}^4 I_i. \end{aligned} \tag{3.6}$$

Let us now estimate each term above I_i, $i = 1, \ldots, 4$. By Lemma 2.1 and assumptions $(H1)$-$(H2)$, we have

$$I_1 \leq \widetilde{M}_T^2 \mathbb{E}\|g(0,\phi)\|_{\mathcal{H}}^2 \leq \widetilde{M}_T^2 M_g(\|\phi\|_{\mathcal{B}_h}^2 + 1), \tag{3.7}$$

$$I_2 \leq M_g\Big(\|y_t^q + \bar{z}_t\|_{\mathcal{B}_h}^2 + 1\Big) \leq M_g\Big[4\Big(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q\Big) + 1\Big]. \tag{3.8}$$

Together with assumption $(H3)$ and (3.2), we have

$$\begin{aligned}
I_3 &\leq \int_0^t \|S_\alpha(t-s)\| ds \int_0^t \|S_\alpha(t-s)\| \mathbb{E}\|f(s, y_s^q + \bar{z}_s, B_1(y^q(s) + \bar{z}(s)))\|_{\mathcal{H}}^2 ds \\
&\leq \widetilde{M}_S^2 \int_0^t (t-s)^{\alpha-1} ds \int_0^t (t-s)^{\alpha-1} \Big[\mu_1(s)\Xi_f\Big(\|y_s^q + \bar{z}_s\|_{\mathcal{B}_h}^2\Big) + \mu_2(s)\mathbb{E}\|B_1(y^q(s) + \bar{z}(s))\|_{\mathcal{H}}^2\Big] ds \\
&\leq \widetilde{M}_S^2 \frac{T^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1} \Big[\Xi_f\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\mu_1^* + B_1^* \mu_2^* \sup_{0\leq s\leq T} \mathbb{E}\|y^q(s) + \bar{z}(s)\|_{\mathcal{H}}^2\Big] ds \\
&\leq \widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2} \Big[\Xi_f\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\mu_1^* + B_1^* \mu_2^* q\Big],
\end{aligned} \tag{3.9}$$

where $B_1^* = \sup_{t\in[0,T]} \int_0^t K(t,s) ds < \infty$, $\mu_1^* = \sup_{s\in[0,t]} \mu_1(s)$, $\mu_2^* = \sup_{s\in[0,t]} \mu_2(s)$.
A similar argument involves assumption $(H4)$, we obtain

$$\begin{aligned}
I_4 &\leq \int_0^t \|S_\alpha(t-s)\|^2 \mathbb{E}\|\sigma(s, y_s^q + \bar{z}_s, B_2(y^q(s) + \bar{z}(s)))\|_{\mathcal{L}_2^0}^2 ds \\
&\leq \widetilde{M}_S^2 \int_0^t (t-s)^{2(\alpha-1)} \Big[\Xi_\sigma\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\nu_1^* + B_2^* \nu_2^* \sup_{0\leq s\leq T} \mathbb{E}\|y^q(s) + \bar{z}(s)\|_{\mathcal{H}}^2\Big] ds \\
&\leq \widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1} \Big[\Xi_\sigma\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\nu_1^* + B_2^* \nu_2^* q\Big],
\end{aligned} \tag{3.10}$$

where $B_2^* = \sup_{t\in[0,T]} \int_0^t P(t,s) ds < \infty$, $\nu_1^* = \sup_{s\in[0,t]} \nu_1(s)$, $\nu_2^* = \sup_{s\in[0,t]} \nu_2(s)$.
Combining these estimates (3.6)-(3.10) yields

$$\begin{aligned}
q &\leq \mathbb{E}\|\Pi(y^q)(t)\|_{\mathcal{H}}^2 \\
&\leq L_0 + 16 M_g l^2 q + 4\widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2} \Big[\Xi_f\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\mu_1^* + B_1^* \mu_2^* q\Big] \\
&\quad + 4\widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1} \Big[\Xi_\sigma\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\nu_1^* + B_2^* \nu_2^* q\Big],
\end{aligned} \tag{3.11}$$

where

$$L_0 = 4\widetilde{M}_T^2 M_g\Big(\|\phi\|_{\mathcal{B}_h}^2 + 1\Big) + 4M_g\Big(1 + 4\|\phi\|_{\mathcal{B}_h}^2\Big).$$

Dividing both sides of (3.11) by q and taking $q \to \infty$, we obtain

$$\begin{aligned}
&16 M_g l^2 + 4\widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2} \Big[4\Lambda \mu_1^* + B_1^* \mu_2^*\Big] + 4\widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1} \Big[4\Upsilon \nu_1^* + B_2^* \nu_2^*\Big] \\
&= 16 M_g l^2 + 4\widetilde{M}_S^2 T^{2\alpha} \Big[\frac{\eta_1}{\alpha^2} + \frac{\eta_2}{T(2\alpha-1)}\Big] \geq 1,
\end{aligned}$$

which is a contradiction to our assumption in (3.3).

For $t \in (t_1, t_2]$, we have

$$\begin{aligned}
q &\leq \mathbb{E}\|\Pi(y^q)(t)\|_{\mathcal{H}}^2 \\
&\leq 7\|T_\alpha(t-t_1)\|^2 \mathbb{E}\|I_1(y^q(t_1^-))\|_{\mathcal{H}}^2 + 7\|T_\alpha(t-t_1)\|^2 \mathbb{E}\|g(t_1, y_{t_1}^q + \bar{z}_{t_1} + I_1(y_{t_1^-}^q + \bar{z}_{t_1^-}))\|_{\mathcal{H}}^2 \\
&\quad + 7\|T_\alpha(t-t_1)\|^2 \mathbb{E}\|g(t_1, y_{t_1}^q + \bar{z}_{t_1})\|_{\mathcal{H}}^2 + 7\mathbb{E}\|T_\alpha(t) g(0,\phi)\|_{\mathcal{H}}^2 + 7\mathbb{E}\|g(t, y_t^q + \bar{z}_t\|_{\mathcal{H}}^2 \\
&\quad + 4\mathbb{E}\left\|\int_0^t S_\alpha(t-s) f(s, y_s^q + \bar{z}_s, B_1(y^q(s) + \bar{z}(s))) ds\right\|_{\mathcal{H}}^2 \\
&\quad + 7\mathbb{E}\left\|\int_0^t S_\alpha(t-s) \sigma(s, y_s^q + \bar{z}_s, B_2(y^q(s) + \bar{z}(s))) dw(s)\right\|_{\mathcal{H}}^2.
\end{aligned} \tag{3.12}$$

Using assumptions $(H1)$-$(H5)$ we obtain

$$\begin{aligned}
&\mathbb{E}\|\Pi(y^q)(t)\|^2_{\mathcal{H}}\\
\leq\ & L_1 + 70\widetilde{M}_T^2 M_g l^2 q + 28 M_g l^2 q + 7\widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2}\Big[\Xi_f\Big(4(\|\phi\|^2_{\mathcal{B}_h} + l^2 q)\Big)\mu_1^* + B_1^*\mu_2^* q\Big]\\
&+7\widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Big[\Xi_\sigma\Big(4(\|\phi\|^2_{\mathcal{B}_h} + l^2 q)\Big)\nu_1^* + B_2^*\nu_2^* q\Big],
\end{aligned}$$

where

$$L_1 = 7\widetilde{M}_T^2\Big(\Theta + M_g\Big[1 + 6(\|\phi\|^2_{\mathcal{B}_h} + l^2\Theta)\Big]\Big) + 7\widetilde{M}_T^2 M_g\Big(1 + \|\phi\|^2_{\mathcal{B}_h}\Big) + 7M_g\Big(1 + 4\|\phi\|^2_{\mathcal{B}_h}\Big).$$

A Similar argument gives

$$\begin{aligned}
&70\widetilde{M}_T^2 M_g l^2 + 28 M_g l^2 + 7\widetilde{M}_S^2\frac{T^{2\alpha}}{\alpha^2}\Big[4\Lambda\mu_1^* + B_1^*\mu_2^*\Big] + 7\widetilde{M}_S^2\frac{T^{2\alpha-1}}{2\alpha-1}\Big[4\Upsilon\nu_1^* + B_2^*\nu_2^*\Big]\\
&= 70\widetilde{M}_T^2 M_g l^2 + 28 M_g l^2 + 7\widetilde{M}_S^2 T^{2\alpha}\Big[\frac{\eta_1}{\alpha^2} + \frac{\eta_2}{T(2\alpha-1)}\Big] \geq 1,
\end{aligned}$$

which is a contradiction to our assumption in (3.3).

Similarly for $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we obtain

$$\begin{aligned}
&\widetilde{C} + 16 M_g l^2 + 7\widetilde{M}_S^2\frac{T^{2\alpha}}{\alpha^2}\Big[4\Lambda\mu_1^* + B_1^*\mu_2^*\Big] + 7\widetilde{M}_S^2\frac{T^{2\alpha-1}}{2\alpha-1}\Big[4\Upsilon\nu_1^* + B_2^*\nu_2^*\Big]\\
&= \widetilde{C} + 16 M_g l^2 + 7\widetilde{M}_S^2 T^{2\alpha}\Big[\frac{\eta_1}{\alpha^2} + \frac{\eta_2}{T(2\alpha-1)}\Big] \geq 1,
\end{aligned}$$

with $\eta_1 = 4\Lambda\mu_1^* + B_1^*\mu_2^*$, $\eta_2 = 4\Upsilon\nu_1^* + B_2^*\nu_2^*$ and $\widetilde{C}$ is a positive constant depending only on $\widetilde{M}_T, M_g$ and l. This is a contradiction to our assumption in (3.3).
Thus, for some positive number q, $\Pi(B_q) \subset B_q$.

Step 2. The map Π_1 is continuous on B_q.

Let $\{y^n\}_{n=1}^\infty$ be a sequence in B_q with $\lim y^n \to y \in B_q$. Then for $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned}
&\mathbb{E}\|(\Pi_1 y^n)(t) - (\Pi_1 y)(t)\|\\
\leq\ & 3\sum_{k=1}^{i}\|T_\alpha(t-t_k)\|^2\Bigg[\mathbb{E}\|I_k(y^n(t_k^-)) - I_k(y(t_k^-))\|^2_{\mathcal{H}} +\\
&\mathbb{E}\|g(t_k, y^n_{t_k} + \bar{z}_{t_k} + I_k(y^n_{t_k^-} + \bar{z}_{t_k^-})) - g(t_k, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-}))\|^2_{\mathcal{H}}\\
&+\mathbb{E}\|g(t_k, y^n_{t_k} + \bar{z}_{t_k}) - g(t_k, y_{t_k} + \bar{z}_{t_k})\|^2_{\mathcal{H}}\Bigg].
\end{aligned}$$

Since the functions g, I_i, $i = 1, 2, \ldots, m$ are continuous, hence $\lim_{n\to\infty}\mathbb{E}\|\Pi_1 y^n - \Pi_1 y\|^2 = 0$ which implies that the mapping Π_1 is continuous on B_q.

Step 3. Π_1 maps bounded sets into bounded sets in B_q.

Let us prove that for $q > 0$ there exists a $\delta > 0$ such that for each $y \in B_q$, we have $\mathbb{E}\|(\Pi_1 y)(t)\|^2_{\mathcal{H}} \leq \delta$ for $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$. We have

$$\begin{aligned}
\mathbb{E}\|(\Pi_1 y)(t)\|^2_{\mathcal{H}} \ &\leq\ 3\sum_{k=1}^{i}\|T_\alpha(t-t_k)\|^2\Bigg[\mathbb{E}\|I_k(y(t_k^-))\|^2_{\mathcal{H}} + \mathbb{E}\|g(t_k, y_{t_k} + \bar{z}_{t_k})\|^2_{\mathcal{H}}\\
&\quad +\mathbb{E}\|g(t_k, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-}))\|^2_{\mathcal{H}}\Bigg]\\
&\leq\ 3m\widetilde{M}_T^2\Bigg[\Theta\Big(1 + 6M_g l^2\Big) + 2M_g + 10M_g\Big(\|\phi\|^2_{\mathcal{B}_h} + l^2 q\Big)\Bigg]\\
&:=\ \delta,
\end{aligned}$$

which proves the desired result.

Step 4. The set $\{\Pi_1 y, y \in B_q\}$ is an equicontinuous family of functions on J.
Let $u, v \in (t_i, t_{i+1}]$, $t_i \leq u < v \leq t_{i+1}$, $i = 0, 1, \ldots, m$, $y \in B_q$. We have

$$\begin{aligned}
&\mathbb{E}\|(\Pi_1 y)(v) - (\Pi_1 y)(u)\|^2_{\mathcal{H}} \\
\leq\ & 3\sum_{k=1}^{i} \|T_\alpha(v-t_k) - T_\alpha(u-t_k)\|^2 \Bigg[\mathbb{E}\|I_k(y(t_k^-))\|^2_{\mathcal{H}} + \mathbb{E}\|g(t_k, y_{t_k} + \bar{z}_{t_k})\|^2_{\mathcal{H}} \\
&+\mathbb{E}\|g(t_k, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-}))\|^2_{\mathcal{H}}\Bigg] \\
\leq\ & 3\left[\Theta\Big(1 + 6M_g l^2\Big) + 2M_g + 10M_g\Big(\|\phi\|^2_{\mathcal{B}_h} + l^2 q\Big)\right]\sum_{k=1}^{i}\|T_\alpha(v-t_k) - T_\alpha(u-t_k)\|^2.
\end{aligned}$$

Since T_α is strongly continuous and it allows us to conclude that $\lim_{u\to v}\|T_\alpha(v-t_k) - T_\alpha(u-t_k)\|^2 = 0$ for all $k = 1, 2, \ldots, m$, which implies that the set $\{\Pi_1 y, y \in B_q\}$ is equicontinuous. Finally, combining Step 1 to Step 4 together with Ascoli's theorem, we conclude that the operator Π_1 is compact.

Step 5. Π_2 is contractive. Let $y, y^* \in B_q$ and $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$. Then

$$\begin{aligned}
&\mathbb{E}\|(\Pi_2 y)(t) - (\Pi_2 y^*)(t)\|^2_{\mathcal{H}} \\
\leq\ & 3\|g(t, y_t + \bar{z}_t) - g(t, y^*_t + \bar{z}_t)\|^2_{\mathcal{H}} \\
&+3\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\Big[f(s, y_s + \bar{z}_s, B_1(y(s) + \bar{z}(s))) - f(s, y^*_s + \bar{z}_s, B_1(y^*(s) + \bar{z}(s)))\Big]ds\right\|^2_{\mathcal{H}} \\
&+3\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\Big[\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))) - \sigma(s, y^*_s + \bar{z}_s, B_2(y^*(s) + \bar{z}(s)))\Big]dw(s)\right\|^2_{\mathcal{H}} \\
\leq\ & 3\|g(t, y_t + \bar{z}_t) - g(t, y^*_t + \bar{z}_t)\|^2_{\mathcal{H}} + 3\int_0^t \|S_\alpha(t-s)\|ds\int_0^t \|S_\alpha(t-s)\| \\
&\times\mathbb{E}\|f(s, y_s + \bar{z}_s, B_1(y(s) + \bar{z}(s))) - f(s, y^*_s + \bar{z}_s, B_1(y^*(s) + \bar{z}(s)))\|^2_{\mathcal{H}}ds \\
&+3\int_0^t \|S_\alpha(t-s)\|^2\mathbb{E}\|\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))) - \sigma(s, y^*_s + \bar{z}_s, B_2(y^*(s) + \bar{z}(s)))\|^2_{\mathcal{L}_2^0}ds \\
\leq\ & 3M_g\|y_t - y^*_t\|^2_{\mathcal{B}_h} + 3\widetilde{M}^2_S\int_0^t (t-s)^{\alpha-1}ds\int_0^t (t-s)^{\alpha-1} \\
&\times\Big[\mu_1(s)\|y_s - y^*_s\|^2_{\mathcal{B}_h} + \mu_2(s)\mathbb{E}\|B_1(y(s) + \bar{z}(s)) - B_1(y^*(s) + \bar{z}(s))\|^2_{\mathcal{H}}\Big]ds \\
&+3\widetilde{M}^2_S\int_0^t (t-s)^{2(\alpha-1)}\Big[\nu_1(s)\|y_s - y^*_s\|^2_{\mathcal{B}_h} + \nu_2(s)\mathbb{E}\|B_2(y(s) + \bar{z}(s)) - B_2(y^*(s) + \bar{z}(s))\|^2_{\mathcal{H}}\Big]ds \\
\leq\ & 3M_g\|y_t - y^*_t\|^2_{\mathcal{B}_h} + 3\widetilde{M}^2_S\frac{T^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1} \\
&\times\Big[\mu_1^* l^2\sup\mathbb{E}\|y(s) - y(s)^*\|^2_{\mathcal{H}} + \mu_2^* B_1^*\sup\mathbb{E}\|y(s) - y(s)^*\|^2_{\mathcal{H}}\Big]ds \\
&+3\widetilde{M}^2_S\int_0^t (t-s)^{2(\alpha-1)}\Big[\nu_1^* l^2\sup\mathbb{E}\|y(s) - y(s)^*\|^2_{\mathcal{H}} + \nu_2^* B_2^*\sup\mathbb{E}\|y(s) - y(s)^*\|^2_{\mathcal{H}}\Big]ds \\
\leq\ & 3\left(l^2 M_g + \widetilde{M}^2_S T^{2\alpha}\left[\tfrac{1}{\alpha^2}(\mu_1^* l^2 + \mu_2^* B_1^*) + \tfrac{1}{T(2\alpha-1)}(\nu_1^* l^2 + \nu_2^* B_2^*)\right]\right)\|y - y^*\|^2_{\mathcal{B}_b^0} \\
=\ & 3\left(l^2 M_g + \widetilde{M}^2_S T^{2\alpha}\left[\tfrac{\vartheta_1}{\alpha^2} + \tfrac{\vartheta_2}{T(2\alpha-1)}\right]\right)\|y - y^*\|^2_{\mathcal{B}_b^0}.
\end{aligned}$$

So Π_2 is a contraction by our assumption in (3.4). Hence, by Sadovskii's fixed point theorem we can conclude that the problem (1.1) has at least one solution on $(-\infty, T]$. This completes the proof of the theorem. □

4 An example

In this section, we consider an example to illustrate our main theorem. We examine the existence of solutions for the following fractional stochastic partial differential equation of the form

$$\begin{aligned} & D_t^q[u(t,x) + \int_{-\infty}^t a(t,x,s-t)Q_1(u(s,x))ds] = \frac{\partial^2}{\partial x^2}[u(t,x) + \int_{-\infty}^t a(t,x,s-t)Q_1(u(s,x))ds] \\ & \quad + \int_{-\infty}^t H(t,x,s-t)Q_2(u(s,x))ds + \int_0^t k(s,t)e^{-u(s,x)}ds \\ & \quad + \left[\int_{-\infty}^t V(t,x,s-t)Q_3(u(s,x))ds + \int_0^t p(s,t)e^{-u(s,x)}ds\right]\frac{d\beta(t)}{dt}, \\ & \quad x \in [0,\pi], \quad t \in [0,b], \quad t \neq t_k \\ & u(t,0) = 0 = u(t,\pi), \quad t \geq 0 \\ & u(t,x) = \phi(t,x), \quad t \in (-\infty,0], \quad x \in [0,\pi], \\ & \Delta u(t_i)(x) = \int_{-\infty}^t q_i(t_i - s)u(s,x)ds, \quad x \in [0,\pi], \end{aligned} \tag{4.1}$$

where $\beta(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$; D_t^q is the Caputo fractional derivative of order $0 < q < 1$; $0 < t_1 < t_2 < \ldots < t_n = T$ are prefixed numbers; a, Q_1, H, Q_2, V, Q_3 are continuous; $\phi \in \mathcal{B}_h$.

Let $\mathcal{H} = L^2([0,\pi])$ with the norm $\|\cdot\|$. Define $A : \mathcal{H} \to \mathcal{H}$ by $Ay = y''$ with the domain

$$\mathcal{D}(A) = \{y \in \mathcal{H}; y, y' \text{ are absolutely continuous, } y'' \in \mathcal{H} \text{ and } y(0) = y(\pi) = 0\}.$$

Then, $Ay = \sum_{n=1}^{\infty} n^2(y, y_n)y_n$, $y \in \mathcal{D}(A)$, where $y_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n = 1, 2, \ldots$, is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $\mathcal{H}$ is given by

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}(y, y_n)y_n, \quad \text{for all } y \in \mathcal{H}, t > 0.$$

It follows from the above expressions that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that, $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$.

Let $h(s) = e^{2s}$, $s < 0$, then $l = \int_{-\infty}^0 h(s)ds = \frac{1}{2}$ and define

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} \left(\mathbb{E}|\phi(\theta)|^2\right)^{1/2} ds.$$

Hence for $(t, \phi) \in [0, T] \times \mathcal{B}_h$, where $\phi(\theta)(y) = \phi(\theta, y)$, $(\theta, y) \in (-\infty, 0] \times [0, \pi]$. Set $u(t)(x) = u(t, x)$,

$$\begin{aligned} g(t,\phi)(x) &= \int_{-\infty}^0 a(t,x,\theta)Q_1(\phi(\theta)(x))d\theta, \\ f(t,\phi,B_1u(t))(x) &= \int_{-\infty}^0 H(t,x,\theta)Q_2(\phi(\theta)(x))d\theta + B_1u(t)(x), \\ \sigma(t,\phi,B_2u(t))(x) &= \int_{-\infty}^0 V(t,x,\theta)Q_3(\phi(\theta)(x))d\theta + B_2u(t)(x), \\ I_i(\phi)(x) &= \int_{-\infty}^0 q_i(-\theta)\phi(\theta)(x)d\theta, \end{aligned}$$

where $B_1u(t) = \int_0^t k(s,t)e^{-u(s,x)}ds$ and $B_2u(t) = \int_0^t p(s,t)e^{-u(s,x)}ds$. Then with these settings the equations in (4.1) can be written in the abstract form of Eq. (1.1). All conditions of Theorem 3.1 are now fulfilled, so we deduce that the system (4.1) has a mild solution on $(-\infty, T]$.

5 Conclusion

We have studied the existence of mild solutions for a class of impulsive fractional stochastic differential equations in Hilbert spaces, which is new and allow us to develop the existence of various fractional differential equations and stochastic fractional differential equations. An example is provided to illustrate the applicability of the new result. The results presented in this paper extend and improve the corresponding ones announced by Dabas et al [6], Dabas and Chauhan [7], Shu et al [23], Sakthivel et al [18] and others.

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Fitting ellipsoids to objects by the first order polarization tensor

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Abstract

This article present the manual to determine ellipsoids that has the same first order polarization tensor to any conducting objects included in electrical field. Given the first order polarization tensor for an object at specified conductivity, the analytical formula of the first order polarization tensor for ellipsoid in the integral form is firstly expressed as system of nonlinear equation by the trapezium rule. We will then discuss how the derived equations are simultaneously solved by appropriated numerical method to uniquely compute all semi principal axes of the ellipsoid. Few examples to use the proposed technique in this study are also provided in three different situations. In each case, the first order polarization tensor for the obtained ellipsoid can be calculated back from the analytical formula to examine the effectiveness of the method.

Keywords: Integral operator, multi-indices, matrices, numerical integration, eigenvalues.

2010 MSC: 65D30, 34A34, 05B20.

1 Introduction

There has been many efforts for examples in [3, 6, 10, 11, 12] done to describe and present shape of objects theoretically as they are very essential in science and engineering applications. Most shapes discussed are very unique and special and has their own mathematical properties. However, certain properties of two different shapes sometimes can be mathematically related to each other. For example, the length of semi principal axes of an ellipsoid each can be equal to half of the length of every side of a cuboid. This suggest that both cuboid and ellipsoid are physically similar in some sense.

Extensive studies by [3, 12, 14] indicate that shape of conducting objects included in an electrical field can be recognized and described through their first order polarization tensor. In electrical imaging especially, instead of fully reconstructing the shape, fitting the shape with its first order polarization tensor could also be very useful to describe the shape as it offers lower computational cost. Indeed, it is also almost impossible to mathematically and correctly obtain the first order polarization tensor for most shapes unless by using well established methodology such as proposed by Pólya and Szegő [12] or Ammari and Kang [3].

Furthermore, for some applications [14, 16], it might be essential to know an ellipsoid which have identical first order polarization tensor with the other shape. This is possible to achieve as the analytical formula of the first order polarization tensor for the ellipsoid in these applications exists and clearly explained in [3]. Therefore, the main purpose of this paper is to discuss procedures to determine such ellipsoid with examples for future relevant applications.

Basically, this ellipsoid is obtained after all of its semi principal axes which are included in the analytical formula of its first order polarization tensor are determined. Thus, we will derive three nonlinear equations with three unknowns from the analytical formula after setting the formula equal to the first order polarization tensor for a known object and then simultaneously solve them. Since these equations can not be directly and

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analytically obtained, some ideas and numerical properties to derive the equations will become the main focus of this study.

For convenience, this paper is organized into six sections. Section 2 mentions mathematical background about the first order polarization tensor and some of its applications. Section 3 then will provide framework and method on how to mathematically obtain the elliposid which have similar first order polarization tensor with other shape. After that, few numerical examples for this purpose will be included in Section 4. Finally, discussion and conlusions about this study are stated in the last two sections of this paper.

2 First order PT

The concept of polarization tensor (PT) that arise from a transmission problem discussed by many literatures will be firstly stated here. Following [3], consider a Lipschitz bounded domain B in $\mathbb{R}^3$ such that the origin O is in B and let the conductivity of B be equal to k where $0 < k \neq 1 < +\infty$. Suppose that H be a harmonic function in $\mathbb{R}^3$ and u be the solution to the following problem

$$\begin{cases} \text{div}(1 + (k-1)\chi(B)\text{grad}(u)) = 0 \text{ in } \mathbb{R}^3 \\ u(x) - H(x) = O(1/|x|^2) \text{ as } |x| \to \infty \end{cases} \tag{2.1}$$

where χ denotes the characteristic function of B. This mathematical formulation (2.1) actually appears in many industrial applications such as medical imaging, landmine detector and material sciences [1, 3, 8, 12]. The PT is then defined through the following far-field expansion of u by [3] as

$$(u - H)(x) = \sum_{|i|,|j|=1}^{+\infty} \frac{(-1)^{|i|}}{i!j!} \partial_x^i \Gamma(x) M_{ij}(k, B) \partial^j H(0) \text{ as } |x| \to +\infty \tag{2.2}$$

for $i = (i_1, i_2, i_3)$, $j = (j_1, j_2, j_3)$ multi indices, Γ is a fundamental solution of the Laplacian and $M_{ij}(k, B)$ is the generalized polarization tensor (GPT).

Generally, the GPT is referred as the dipole in electromagnetic applications by physicists probably because it shows the conductivity distribution of the object. Furthermore, the definition of GPT in (2.2) is extended by Ammari and Kang [3] through an integral operator over the boundary of B by

$$M_{ij} = \int_{\partial B} y^j \phi_i(y) d\sigma(y) \tag{2.3}$$

where $\phi_i(y)$ is given by

$$\phi_i(y) = (\lambda I - \mathcal{K}_B^*)^{-1}(v_x \cdot \nabla x^i)(y) \tag{2.4}$$

for $x, y \in \partial B$ with v_x is the outer unit normal vector to the boundary ∂B at x and λ is defined by $\lambda = (k+1)/2(k-1)$. $\mathcal{K}_B^*$ is a singular integral operator defined with Cauchy principal value $P.V.$ by

$$\mathcal{K}_B^* \phi(x) = \frac{1}{4\pi} P.V. \int_{\partial B} \frac{\langle x - y, v_x \rangle}{|x - y|^3} \phi(y) d\sigma(y). \tag{2.5}$$

Consequently, the first order PT can be evaluated by using (2.3) for i, j=$(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ only so that $|i| = i_1 + i_2 + i_3 = 1 = |j|$. By combining all possible values of i and j, the first order PT of an object B is a real 3×3 matrix in the form

$$M = \begin{bmatrix} M_{(1,0,0)(1,0,0)} & M_{(1,0,0)(0,1,0)} & M_{(1,0,0)(0,0,1)} \\ M_{(0,1,0)(1,0,0)} & M_{(0,1,0)(0,1,0)} & M_{(0,1,0)(0,0,1)} \\ M_{(0,0,1)(1,0,0)} & M_{(0,0,1)(0,1,0)} & M_{(0,0,1)(0,0,1)} \end{bmatrix}. \tag{2.6}$$

Furthermore, if B is an ellipsoid centered at origin in the Cartesian coordinate system represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where a, b and c each is the length of semi principal axes of B, the first order PT of B when the conductivity is k is given by [3] as

$$M(k, B) = (k-1)|B| \begin{bmatrix} \frac{1}{(1-P)+kP} & 0 & 0 \\ 0 & \frac{1}{(1-Q)+kQ} & 0 \\ 0 & 0 & \frac{1}{(1-R)+kR} \end{bmatrix} \tag{2.7}$$

where $|B|$ is the volume of B, P, Q and R are constants defined by

$$\begin{aligned} P &= \frac{bc}{a^2}\int_1^{+\infty} \frac{1}{t^2\sqrt{t^2-1+(\frac{b}{a})^2}\sqrt{t^2-1+(\frac{c}{a})^2}}dt, \\ Q &= \frac{bc}{a^2}\int_1^{+\infty} \frac{1}{(t^2-1+(\frac{b}{a})^2)^{\frac{3}{2}}\sqrt{t^2-1+(\frac{c}{a})^2}}dt, \\ R &= \frac{bc}{a^2}\int_1^{+\infty} \frac{1}{\sqrt{t^2-1+(\frac{b}{a})^2}(t^2-1+(\frac{c}{a})^2)^{\frac{3}{2}}}dt. \end{aligned} \tag{2.8}$$

Generally, if given any object then the first order PT of the object can be obtained by formula (2.3) - (2.6) while the first order PT for an ellipsoid can be alternatively calculated by using simpler formula (2.7) - (2.8). Both approaches are possible only by using numerical method as the integrand in (2.3), (2.5) and (2.8) can not be analytically determined. In this study however, our purpose is to find an ellipsoid by using (2.7) and (2.8) if given to us the first order PT of any object in the form of (2.6).

3 Mathematical properties of the first order PT for ellipsoid

This section review some mathematical properties of the formula (2.7) which are very useful in determining the ellipsoid when the first order PT is given. For this purpose, every integrand in (2.8) is firstly denoted as function $f_1(t,a,b,c)$, $f_2(t,a,b,c)$ and $f_3(t,a,b,c)$ such that

$$\begin{aligned} P &= \frac{bc}{a^2}\int_1^{+\infty} f_1(t,a,b,c)dt, \\ Q &= \frac{bc}{a^2}\int_1^{+\infty} f_2(t,a,b,c)dt, \\ R &= \frac{bc}{a^2}\int_1^{+\infty} f_3(t,a,b,c)dt. \end{aligned}$$

Lemma 3.1. *For any $t > 0$,*

1. *if $0 < a \le b \le c$ then $f_1(t,a,b,c) \le 1/t^4$.*
2. *if $0 < c \le b \le a$ then there exist positive constant K so that $f_1(t,a,b,c) \le K/t^4$.*

Proof. In order to prove (1), starting from $a \le b$, it is easy to show that for any $t > 0$

$$t \le \sqrt{t^2-1+(b/a)^2}$$

while $a \le c$ implies

$$t \le \sqrt{t^2-1+(c/a)^2}$$

for any $t > 0$. Multiplying both inequalities yield to

$$t^2 \le \sqrt{t^2-1+(b/a)^2}\sqrt{t^2-1+(c/a)^2}$$

and hence $t^4 \le (f_1(t,a,b,c))^{-1}$. This completes the proof of (1).

Similarly, $b \le a$ and $c \le a$ imply that for any $t > 0$

$$\sqrt{t^2-1+(b/a)^2} \le t \text{ and } \sqrt{t^2-1+(c/a)^2} \le t \text{ respectively.}$$

This leads to

$$\sqrt{t^2-1+(b/a)^2}\sqrt{t^2-1+(c/a)^2} \le t^2 \text{ and } 1/t^4 \le f_1(t,a,b,c).$$

As $0 < 1/t^4 \le f_1(t,a,b,c)$, multiply the right hand-sided of $1/t^4 \le f_1(t,a,b,c)$ with a positive constant K to complete the proof of the lemma. □

The above lemma explains that $f_1(t,a,b,c)$ is bounded for both $0 < a \leq b \leq c$ and $0 < c \leq b \leq a$ and is also true if $f_1(t,a,b,c)$ is replaced by $f_2(t,a,b,c)$ and $f_3(t,a,b,c)$. This lemma is important to derive three nonlinear equations with three variables from (2.7) and (2.8) by appropriate numerical method. Next, we prove the following lemma which might be essential to solve the obtained system of nonlinear equations later on.

Lemma 3.2. *Let the constants $C_1, C_2, C_3 > 1$. For $a,b,c > 0$ and $i = 1,2,3$, the function $f_i(t,a,b,c)$ is continuous in the interval $1 < t < C_i$.*

Proof. The continuity of $f_1(t,a,b,c)$ is firstly shown. Notice that for $C_1 > 1$,

$$\int_1^{+\infty} f_1(t,a,b,c)dt = \int_1^{C_1} f_1(t,a,b,c)dt + \int_{C_1}^{+\infty} f_1(t,a,b,c)dt.$$

Since $\int_1^{+\infty} f_1(t,a,b,c)dt$ exist for any $a,b,c > 0$ by formula (2.7), then $\int_1^{C_1} f_1(t,a,b,c)dt$ also exist by [9] for $a,b,c > 0$. This concludes that $f_1(t,a,b,c)$ is continuous in the interval $1 < t < C_1$. Similar steps can be repeated to show that each $f_2(t,a,b,c)$ and $f_3(t,a,b,c)$ is continuous for $1 < t < C_2$ and $1 < t < C_3$ respectively. □

4 Numerical setup and methodology

The discussion in this section based on the previous properties is divided into two parts. Firstly, this section explains the numerical integration method used to derive three nonlinear equations with three unknowns from (2.7). After that, the procedure to solve these equations is briefly discussed.

4.1 Formulating the nonlinear equations

The discussion begins with the explanation on how integral equations in (2.7) are estimated. Since all integrals involve infinite interval, a common approach to estimate them is by truncating the limits where the infinite range is replaced by a sufficiently large value L so that the integrals becomes finite and then can be approximated by a standard numerical integration method of finite interval [5]. Therefore, it is neccessary to properly choose L to avoid inaccurate result if L is underestimated or expending needless effort if L is overestimated.

Before proceeding further, consider the problem to numerically determine

$$I = \int_1^{\infty} (K/t^4)dt \text{ for } K > 0. \tag{4.9}$$

By following [5],

$$I = \int_C^{+\infty} (K/t^4)dt \tag{4.10}$$

where the constant $C > 1$ is firstly estimated. For $t \geq C$ then $t^4 \geq Ct^3$. Hence,

$$\int_C^{+\infty} (K/t^4)dt \leq \int_C^{+\infty} (K/Ct^3)dt = K/2C^3. \tag{4.11}$$

Thus, if $K = 20$ and $C = 100$ for example then $K/2C^3 \approx 10^{-5}$ so (4.9) can be approximated to four figures of computation by $I = \int_1^{100}(20/t^4)dt$ with $K = 20$.

Suppose that now we want to approximate $\int_1^{+\infty} f_1(t,a,b,c)dt$ by sufficiently $\int_1^{C_1} f_1(t,a,b,c)dt$. Since $\int f_1(t,a,b,c)dt$ can not be analytically integrated then it is impossible to investigate C_1. However, by using Lemma 3.1, we have

$$\int_1^{C_1} f_1(t,a,b,c)dt \leq \int_1^{C_1} (1/t^4)dt \tag{4.12}$$

for $0 < a \leq b \leq c$ and

$$\int_1^{C_1} f_1(t,a,b,c)dt \leq \int_1^{C_1} (K/t^4)dt \tag{4.13}$$

where K positive constant for $0 < c \leq b \leq a$. Furthermore, $\int_1^{C_1}(1/t^4)dt \leq \int_1^{C_1}(K/t^4)dt$. Therefore, we may refer C in (4.10) to guess C_1 when doing computation for both cases.

Similar approach is also used to approximate $\int_1^{+\infty} f_2(t,a,b,c)dt$ and $\int_1^{+\infty} f_3(t,a,b,c)dt$ by $\int_1^{C_2} f_2(t,a,b,c)dt$ and $\int_1^{C_3} f_3(t,a,b,c)dt$ respectively. This means our problem now becomes to derive the nonlinear equations from

$$\begin{aligned}\int_1^{C_1} f_1(t,a,b,c)dt &= \hat{f}_1(a,b,c),\\ \int_1^{C_2} f_2(t,a,b,c)dt &= \hat{f}_2(a,b,c),\\ \int_1^{C_3} f_3(t,a,b,c)dt &= \hat{f}_3(a,b,c).\end{aligned} \tag{4.14}$$

In this study, the trapezoidal rule is implemented to achieve this such that

$$\begin{aligned}\hat{f}_1(a,b,c) &= h_1\left[\frac{f_1(1,a,b,c)+f_1(C_1,a,b,c)}{2}+\sum_{k=1}^{n_1-1} f_1(1+kh_1,a,b,c)+R_1\right],\\ \hat{f}_2(a,b,c) &= h_2\left[\frac{f_2(1,a,b,c)+f_2(C_2,a,b,c)}{2}+\sum_{k=1}^{n_2-1} f_2(1+kh_2,a,b,c)+R_2\right],\\ \hat{f}_3(a,b,c) &= h_3\left[\frac{f_3(1,a,b,c)+f_3(C_3,a,b,c)}{2}+\sum_{k=1}^{n_3-1} f_3(1+kh_3,a,b,c)+R_3\right].\end{aligned} \tag{4.15}$$

where for $i=1,2,3$, R_i is the small error in the approximation and $h_i=(C_i-1)/n_i$ must be very small to increase the accuracy of the computation [5]. Then, if given the first order PT of any shape S at conductivity k denoted by $M(k,S)$ as diagonal matrix of size 3, the desired system of nonlinear equation is obtained by comparing $\hat{f}_1(a,b,c)$, $\hat{f}_2(a,b,c)$ and $\hat{f}_3(a,b,c)$ with the appropriate diagonal element of $M(k,S)$. Finally, by using the original formula (2.7) and (2.8) with (4.14) and rearranged, the system will be in the form

$$m_{ii}+(k-1)\left[m_{ii}\frac{bc}{a^2}\hat{f}_i(a,b,c)-|B|\right]=0 \tag{4.16}$$

where m_{ii} is the diagonal element of $M(k,S)$ for $i=1,2,3$.

4.2 Solving system of nonlinear equations

The system of nonlinear equations (4.16) are solved in order to obtain the ellipsoid that has same first order PT with an object S at conductivity k. This ellipsoid must be unique based on formula (2.7) and (2.8). According to [2], at least n nonlinear equations are needed if we want to find a unique solution for n independent variables provided the solution exists. This seems to be true if the system is solved by analytical techniques but our system can only be solved by numerical method. In addition, some authors such as [4] and [7] claims that there is no guarantee to find the unique or all solutions for the system of nonlinear equations by numerical method because of the the difficulties in analyzing the existence and uniqueness of solutions to such system.

Therefore, following the claim by Press et. al [13] that there is no particular 'good method' in solving the system of nonlinear equations, only the standard method in the function $fsolve.m$ of MatLab is used to solve (4.16) with initial value $a=b=c=0$ in this study. This method is chosen because we believe we can obtain approximately correct unique solutions for the system due to (4.16) satisfies the following criteria :

1. The system has three equations with three unknown variables.
2. All variables are strictly posivite real number.
3. Every equation is continous with respect to every variable.

In this case, condition (1) is obvious, condition (2) is based on the fact that the system is derived based on original definition of the variables in (2.7) and (2.8) while condition (3) is explained in the next lemma.

Lemma 4.3. *Let* $g_i(a,b,c)=m_{ii}+(k-1)\left[m_{ii}\frac{bc}{a^2}\hat{f}_i(a,b,c)-|B|\right]$ *for* $i=1,2,3$. *Every* $g_i(a,b,c)$ *is continuous for* $a,b,c>0$.

Proof. Since any other terms in $g_i(a,b,c)$ are just a real-valued constants for all i, it is suffiecient to prove that $g_i(a,b,c)$ continuous by showing $\hat{f}_i(a,b,c)$ is continuous whenever $a,b,c>0$. According to (4.14), $\hat{f}_i(a,b,c)$ is approximated by the summation of $f_i(t_i,a,b,c)$ for some t_i such that $1<t_i<C_i$. The continuity of $g_i(a,b,c)$ for every i whenever $a,b,c>0$ arrives from Lemma 3.2 and by the property of summation between continuous function. □

5 Examples and applications

In order to demonstrate some examples and applications of the previous explanation, the constants in (4.14) are set to be equal such that $C_1=C_2=C_3=50$ and $h_1=h_2=h_3=0.00001$ during every computation throughout this section. Furthermore, all semi axes of every ellipsoid are calculated to only four decimal places. We divide the implementation of the discussed technique to determine the ellipsoid in this paper into three cases and each of them will be discussed next.

5.1 Objects with similar first order PT

An ellipsoid can be constructed for any conducting object so that both of them has similar first order PT as diagonal matrix of size 3. It is assumed that the first order PT of any object is diagonal when the object is centered at the origin. This claim can be numerically seen by using previous proposed method for example in [15]. Thus, based on the first order PT of any other shape, every semi axes of the desired ellipsoid can be computed by solving (4.16). Table 1 shows few common shapes and respective ellipsoids which are obtained by using the method discussed in Section 4 according to the first order PT of every object at the same level of conductivity k.

5.2 Eigenvalues of the first order PT

Sometimes, it is more useful to characterize the first order PT of any object S at conductivity k denoted by $M(k,S)$ by the three eigenvalues of $M(k,S)$ especially when it is difficult to locate the center of the object because of complicated shape. This approach is actually applicable to any shape whether the shape is centered at the origin or not because the eigenvalues preserve regardless the position of the object. As the eigenvalues of the first order PT of ellipsoids in (2.7) are just the diagonal terms, we can then determine an ellipsoid centered at origin from any eigenvalues of $M(k,S)$ by solving (4.16) equal to the corresponds eigenvalues (see Table 2).

5.3 First order PT for several ellipsoids

Suppose that we want to investigate several ellipsoids where their first order PT are related. Let the first order PT of the ellipsoid A, $(x/2)^2+(y/2)^2+(z)^2=1$ at conductivity 2.75 be

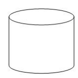
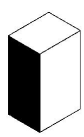
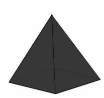
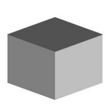
$$M_A=\begin{bmatrix}20.74 & 0.00 & 0.00\\ 0.00 & 20.74 & 0.00\\ 0.00 & 0.00 & 15.25\end{bmatrix}.$$

If M_A satisfies $M_AM_B=I$ where $I=M_C$ is the identity matrix, B and C at the same conductivity with A can be obtained by solving (4.16) equal to $M_B=M_A^{-1}$ and I respectively where they are obtained as the ellipsoid with semi principal axes 0.1738, 0.1738, 0.3782 and sphere of radius 0.6.

6 Discussion and conclusion

A system of nonlinear equation has being developed and solved in this study to determine an ellipsoid that share similar mathematical description with other object which is the first order PT. A few mathematical properties are discussed to provide framework for this purpose. By achieving this, an ellipsoid can be constructed from the same material with the appropriated object as the conductivity for both are equally fixed to deeper investigate other relevant properties for both objects.

Furthermore, the trapezoidal rule from numerical integration technique of finite interval is used to develop the system of nonlinear equations from the analytical formula of the first order PT for the ellipsoid. These

Object	k	First Order PT	Ellipsoid $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$
 **Cylinder** $d = 3$, $h = 3$	5×10^{-5}	$\begin{bmatrix} -33.81 & 0.00 & 0.00 \\ 0.00 & -33.81 & 0.00 \\ 0.00 & 0.00 & -33.53 \end{bmatrix}$	$a = 1.7427$ $b = 1.7427$ $c = 1.7671$
 **Hemisphere** $d = 3$	1×10^{-2}	$\begin{bmatrix} -9.70 & 0.00 & 0.00 \\ 0.00 & -9.70 & 0.00 \\ 0.00 & 0.00 & -15.41 \end{bmatrix}$	$a = 1.5235$ $b = 1.5235$ $c = 0.7703$
 **Cuboid** $2 \times 4 \times 1$	1.5	$\begin{bmatrix} 27.95 & 0.00 & 0.00 \\ 0.00 & 29.92 & 0.00 \\ 0.00 & 0.00 & 25.03 \end{bmatrix}$	$a = 2.5182$ $b = 4.3458$ $c = 1.3990$
 **Pyramid** $2 \times 2 \times 2$	500	$\begin{bmatrix} 12.85 & 0.00 & 0.00 \\ 0.00 & 12.85 & 0.00 \\ 0.00 & 0.00 & 8.51 \end{bmatrix}$	$a = 1.0773$ $b = 1.0773$ $c = 0.7576$
 **Cube** $2 \times 2 \times 2$	10000	$\begin{bmatrix} 28.90 & 0.00 & 0.00 \\ 0.00 & 28.90 & 0.00 \\ 0.00 & 0.00 & 28.90 \end{bmatrix}$	$a = 1.3201$ $b = 1.3201$ $c = 1.3201$

note : d = diameter, h = height

Table 1: Ellipsoid and Object with Similar First Order PT

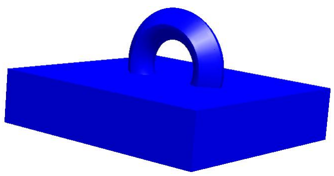
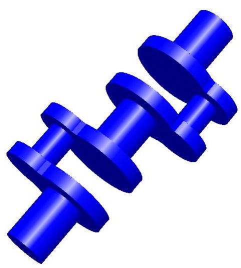
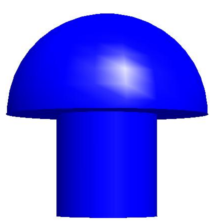
Object	k	Eigenvalues of First Order PT	Ellipsoid $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$
 **half-Ring on Cuboid**	1.5	56.50 56.50 46.62	$a = 4.4860$ $b = 4.4860$ $c = 1.4607$
 **Shaft**	500	30.90×10^5 12.42×10^5 8.40×10^5	$a = 84.8042$ $b = 41.8165$ $c = 29.2011$
 **Mushroom**	10000	87.42 79.00 79.00	$a = 1.9735$ $b = 1.8144$ $c = 1.8144$

Table 2: The Similar Eigenvalues of First Order PT

nonlinear equations are derived step by step while the convergence of each equation is also discussed here. This method is choosen since we want to develop the simplest set of equations with the hope that they can be easily and directly solved by any existing method.

Finally, by solving the developed system of nonlinear equations, some examples of ellipsoids which are determined from the first order PT for related object are also given. These examples are categorized into three different situations and can be further explored for relevant applications in the future. In addition, the first order PT for every ellipsoid determined in Section 5.1 will give the similar first order PT for the other related objects while ellipsoids from Section 5.2 will give the same eigenvalues of the first order PT for the other related objects accurate to three decimal places if both of them are calculated back by using formula (2.7).

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A Variant of Jensen's Inequalities

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Abstract

In this paper, we give an estimate from below and from above of a variant of Jensen's Inequalities for convex functions in the discrete and continuous cases.

Keywords: Convex functions, Jensen inequalities, Integral inequalities.

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1 Introduction and main results

Throughout this note, we write I and $\mathring{I}$ for the intervals $[a,b]$ and (a,b) respectively. A function f is said to be convex on I if $\lambda f(x) + (1-\lambda) f(y) \geqslant f(\lambda x + (1-\lambda) y)$ for all $x, y \in I$ and $0 \leqslant \lambda \leqslant 1$. Conversely, if the inequality always holds in the opposite direction, the function is said to be concave on the interval. A function f that is continuous function on I and twice differentiable on $\mathring{I}$ is convex on I if $f^{''}(x) \geqslant 0$ for all $x \in \mathring{I}$ (concave if the inequality is flipped).

Let $x_1 \leqslant x_2 \leqslant ... \leqslant x_n$ be real numbers and λ_k $(1 \leqslant k \leqslant n)$ be positive weights associated with x_k and whose sum is unity. Then the famous Jensen's discrete and continuous inequalities [2] read:

Theorem A. [2] *If φ is a convex function on an interval containing the x_k, then*

$$\varphi\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leqslant \sum_{k=1}^{n} \lambda_k \varphi(x_k). \tag{1.1}$$

Theorem B. [2] *Let φ be a convex function on $I \subset \mathbb{R}$, let $f : [c,d] \longrightarrow I$ and $p : [c,d] \longrightarrow (0,+\infty)$ be continuous functions on $[c,d]$. Then*

$$\varphi\left(\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right) \leqslant \frac{\int_c^d p(x)\,\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}. \tag{1.2}$$

If φ is strictly convex, then inequality in (1.2) *is strict.*

In [3], S. M. Malamud gave some complements to the Jensen and Chebyshev inequalities and in [1], I. Budimir, S. S. Dragomir, J. E. Pečarić obtained some results which counterpart Jensen's discret inequality. Recently, A. McD. Mercer [4] studied a variant of the inequality (1.1) and have obtained:

Theorem C. [4] *If φ is a convex function on an interval of positive real numbers containing the x_k, then*

$$\varphi\left(x_1 + x_n - \sum_{k=1}^{n} \lambda_k x_k\right) \leqslant \varphi(x_1) + \varphi(x_n) - \sum_{k=1}^{n} \lambda_k \varphi(x_k). \tag{1.3}$$

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Our purpose in this paper is to give an estimate, from below and from above, of a variant of Jensen's discrete and continuous cases inequalities for convex functions. We obtain the following results:

Theorem 1.1. *Assume that φ is a convex function on I containing the x_k and λ_k $(1 \leqslant k \leqslant n)$ are positive weights associated with x_k and whose sum is unity. Then*

$$2\varphi\left(\frac{a+b}{2}\right) - \sum_{k=1}^{n} \lambda_k \varphi(x_k) \leqslant \varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right)$$

$$\leqslant \varphi(a) + \varphi(b) - \sum_{k=1}^{n} \lambda_k \varphi(x_k). \tag{1.4}$$

If φ is strictly convex, then inequalities in (1.4) *are strict.*

Remark 1.1. *If $[a, b] = [x_1, x_n]$, then the result of Theorem C is given by the right-hand of inequalities* (1.4).

Theorem 1.2. *Let φ be a convex function on $I \subset \mathbb{R}$, let $f : [c, d] \longrightarrow I$ and $p : [c, d] \longrightarrow (0, +\infty)$ be continuous functions on $[c, d]$. Then*

$$2\varphi\left(\frac{a+b}{2}\right) - \frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx} \leqslant \varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$

$$\leqslant \varphi(a) + \varphi(b) - \frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}. \tag{1.5}$$

If φ is strictly convex, then inequalities in (1.5) *are strict.*

Corollary 1.1. *Under the hypotheses of Theorem 1.1, we have*

$$\left|\varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right) + \sum_{k=1}^{n}\lambda_k\varphi(x_k)\right|$$

$$\leqslant \max\left\{2\left|\varphi\left(\frac{a+b}{2}\right)\right|, |\varphi(a)+\varphi(b)|\right\}. \tag{1.6}$$

Corollary 1.2. *Under the hypotheses of Theorem 1.2, we have*

$$\left|\varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right) + \frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}\right|$$

$$\leqslant \max\left\{2\left|\varphi\left(\frac{a+b}{2}\right)\right|, |\varphi(a)+\varphi(b)|\right\}. \tag{1.7}$$

In [5], S. Simić have obtained an upper global bound without a differentiability restriction on f. Namely, he proved the following:

Theorem D. [5] *If φ is a convex function on I containing the x_k and λ_k $(1 \leqslant k \leqslant n)$ are positive weights associated with x_k and whose sum is unity, then*

$$\sum_{k=1}^{n}\lambda_k\varphi(x_k) - \varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right) \leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right). \tag{1.8}$$

In the following, we improve this result by proving:

Theorem 1.3. *If φ is a convex function on I containing the x_k and λ_k $(1 \leqslant k \leqslant n)$ are positive weights associated with x_k and whose sum is unity, then*

$$0 \leqslant \left| \sum_{k=1}^{n} \lambda_{\sigma(k)} \varphi(a+b-x_k) - \varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right) \right.$$
$$\left. + \sum_{k=1}^{n} \lambda_{\sigma(k)} \varphi(x_k) - \varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right) \right| \leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right) \tag{1.9}$$

holds for all permutation $\sigma(k)$ of $\{1, 2, ..., n\}$.

Theorem 1.4. *Let φ be a convex function on $I \subset \mathbb{R}$, let $f : [c, d] \longrightarrow I$ and $p : [c, d] \longrightarrow (0, +\infty)$ be continuous functions on $[c, d]$. Then*

$$0 \leqslant \frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx} - \varphi\left(\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$
$$\leqslant \frac{\int_c^d p(x)\varphi(a+b-f(x))\,dx}{\int_c^d p(x)\,dx} - \varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$
$$+\frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx} - \varphi\left(\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right) \leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right). \tag{1.10}$$

Corollary 1.3. *If φ is a convex function on $I \subset \mathbb{R}$, $f : [0, 1] \longrightarrow I$ is a continuous function on $[0, 1]$, then*

$$0 \leqslant \int_0^1 \varphi(f(x))\,dx - \varphi\left(\int_0^1 f(x)\,dx\right)$$
$$\leqslant \varphi\left(a+b-\int_0^1 f(x)\,dx\right) - \int_0^1 \varphi(a+b-f(x))\,dx + \int_0^1 \varphi(f(x))\,dx$$
$$-\varphi\left(\int_0^1 f(x)\,dx\right) \leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right). \tag{1.11}$$

Corollary 1.4. *If φ is a convex function on I containing the x_k and λ_k $(1 \leqslant k \leqslant n)$ are positive weights associated with x_k and whose sum is unity, then*

$$0 \leqslant \sum_{k=1}^{n} \lambda_k \varphi(x_k) - \varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right)$$
$$\leqslant \sum_{k=1}^{n} \lambda_k \varphi(a+b-x_k) - \varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right) + \sum_{k=1}^{n} \lambda_k \varphi(x_k) - \varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right)$$
$$\leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right). \tag{1.12}$$

Remark 1.2. *If φ is a concave function, then the above inequalities are opposite.*

2 Lemma

Towards proving these theorems we shall need the following lemma.

Lemma 2.1. *Let φ be convex function on $I = [a, b]$. Then, we have*

$$2\varphi\left(\frac{a+b}{2}\right) \leqslant \varphi(a+b-x) + \varphi(x) \leqslant \varphi(a) + \varphi(b). \tag{2.1}$$

Proof. Let φ be a convex function on I. Then, we have

$$\varphi\left(\frac{a+b}{2}\right)=\varphi(\frac{a+b-x+x}{2})\leqslant\frac{1}{2}\left(\varphi(a+b-x)+\varphi(x)\right). \tag{2.2}$$

If we choose $x=\lambda a+(1-\lambda)\,b$ $(0\leqslant\lambda\leqslant1)$ in (2.2), then we obtain

$$\begin{aligned}&\frac{1}{2}\left(\varphi(a+b-x)+\varphi(x)\right)\\&=\frac{1}{2}\left(\varphi(a+b-(\lambda a+(1-\lambda)\,b))+\varphi(\lambda a+(1-\lambda)\,b)\right)\\&=\frac{1}{2}\left(\varphi\left(\lambda b+(1-\lambda)\,a\right)+\varphi\left(\lambda a+(1-\lambda)\,b\right)\right).\end{aligned} \tag{2.3}$$

By using the convexity of φ, we get

$$\frac{1}{2}\left(\varphi\left(\lambda b+(1-\lambda)\,a\right)+\varphi\left(\lambda a+(1-\lambda)\,b\right)\right)\leqslant\frac{1}{2}(\varphi(a)+\varphi(b)). \tag{2.4}$$

Thus, by (2.2), (2.3) and (2.4), we obtain

$$\varphi\left(\frac{b+a}{2}\right)\leqslant\frac{1}{2}\left(\varphi(a+b-x)+\varphi(x)\right)\leqslant\frac{1}{2}(\varphi(a)+\varphi(b)). \tag{2.4}$$

3 Proof of Theorems

Proof of Theorem 1.1. Let φ be a convex function and let λ_k $(0\leqslant k\leqslant n)$ be positive weights associated with x_k and whose sum is unity. Then, by using inequality (1.1), we have

$$\begin{aligned}\varphi\left(a+b-\sum_{k=1}^{n}\lambda_kx_k\right)&=\varphi\left(\sum_{k=1}^{n}\lambda_k\left(a+b\right)-\sum_{k=1}^{n}\lambda_kx_k\right)\\&=\varphi\left(\sum_{k=1}^{n}\lambda_k\left(a+b-x_k\right)\right)\leqslant\sum_{k=1}^{n}\lambda_k\varphi\left(a+b-x_k\right).\end{aligned} \tag{3.1}$$

By Lemma 2.1, we get

$$\begin{aligned}\varphi\left(\sum_{k=1}^{n}\lambda_k\left(a+b-x_k\right)\right)&\leqslant\sum_{k=1}^{n}\lambda_k\left(\varphi\left(a\right)+\varphi\left(b\right)-\varphi\left(x_k\right)\right)\\&=\varphi\left(a\right)+\varphi\left(b\right)-\sum_{k=1}^{n}\lambda_k\varphi\left(x_k\right).\end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\varphi\left(a+b-\sum_{k=1}^{n}\lambda_kx_k\right)\leqslant\varphi\left(a\right)+\varphi\left(b\right)-\sum_{k=1}^{n}\lambda_k\varphi\left(x_k\right),$$

which is the right-hand of inequalities in (1.4). Now, using Lemma 2.1, we obtain

$$2\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\sum_{k=1}^{n}\lambda_kx_k\right)\leqslant\varphi\left(a+b-\sum_{k=1}^{n}\lambda_kx_k\right). \tag{3.3}$$

Since φ is a convex function, then from (3.3) and inequality (1.1), we deduce that

$$\begin{aligned}2\varphi\left(\frac{a+b}{2}\right)-\sum_{k=1}^{n}\lambda_k\varphi\left(x_k\right)&\leqslant2\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\sum_{k=1}^{n}\lambda_kx_k\right)\\&\leqslant\varphi\left(a+b-\sum_{k=1}^{n}\lambda_kx_k\right),\end{aligned}$$

which is the left-hand of inequalities in (1.4). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let φ be a convex function. Then, by using inequality (1.2), we have

$$\varphi\left(a+b-\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)}\right)=\varphi\left(\frac{\int_c^d p(x)\,(a+b-f(x))\,dx}{\int_c^d p(x)}\right)$$
$$\leqslant\frac{\int_c^d p(x)\,\varphi(a+b-f(x))\,dx}{\int_c^d p(x)\,dx}. \tag{3.4}$$

By Lemma 2.1, we get

$$\frac{\int_c^d p(x)\,\varphi(a+b-f(x))\,dx}{\int_c^d p(x)\,dx}\leqslant\frac{\int_c^d p(x)\,(\varphi(a)+\varphi(b)-\varphi(f(x)))\,dx}{\int_c^d p(x)\,dx}$$
$$=\varphi(a)+\varphi(b)-\frac{\int_c^d p(x)\,\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}. \tag{3.5}$$

From (3.4) and (3.5) , we obtain

$$\varphi\left(a+b-\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)}\right)\leqslant\varphi(a)+\varphi(b)-\frac{\int_c^d p(x)\,\varphi(f(x))\,dx}{\int_c^d p(x)\,dx},$$

which is the right-hand inequalities in (1.5). Using now Lemma 2.1, we obtain

$$2\varphi\left(\frac{a+b}{2}\right)\leqslant\varphi\left(\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right)+\varphi\left(a+b-\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right). \tag{3.6}$$

This implies

$$2\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right)\leqslant\varphi\left(a+b-\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right). \tag{3.7}$$

Since φ is a convex function, then from (3.7) and inequality (1.2), we deduce that

$$2\varphi\left(\frac{a+b}{2}\right)-\frac{\int_c^d p(x)\,\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}\leqslant 2\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$
$$\leqslant\varphi\left(a+b-\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right).$$

The left-hand of inequalities in (1.5) is proved. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. By using Lemma 2.1, we obtain for all $x_k \in I$

$$2\varphi\left(\frac{a+b}{2}\right)\leqslant\varphi(a+b-x_k)+\varphi(x_k)\leqslant\varphi(a)+\varphi(b). \tag{3.8}$$

Multiplying (3.8) by $\lambda_{\sigma(k)}$ and adding, we get

$$2\varphi\left(\frac{a+b}{2}\right)\leqslant\sum_{k=1}^{n}\lambda_{\sigma(k)}\varphi(a+b-x_k)+\sum_{k=1}^{n}\lambda_{\sigma(k)}\varphi(x_k)\leqslant\varphi(a)+\varphi(b). \tag{3.9}$$

On other hand by Lemma 2.1, we have

$$2\varphi\left(\frac{a+b}{2}\right)\leqslant\varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right)+\varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right)\leqslant\varphi(a)+\varphi(b).$$

This implies

$$-(\varphi(a)+\varphi(b)) \leqslant -\varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right)-\varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right)$$
$$\leqslant -2\varphi\left(\frac{a+b}{2}\right). \tag{3.10}$$

By addition from (3.9) and (3.10), we get our result.

Proof of Theorem 1.4. By using Lemma 2.1, we obtain for all $f(x) \in I$

$$2\varphi\left(\frac{a+b}{2}\right) \leqslant \varphi(a+b-f(x))+\varphi(f(x)) \leqslant \varphi(a)+\varphi(b). \tag{3.11}$$

Multiplying (3.11) by $p(x)$ and integrating over $[c,d]$, we get

$$2\varphi\left(\frac{a+b}{2}\right) \leqslant \frac{\int_c^d p(x)\,\varphi(a+b-f(x))\,dx}{\int_c^d p(x)\,dx}+\frac{\int_c^d p(x)\,\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}$$
$$\leqslant \varphi(a)+\varphi(b). \tag{3.12}$$

On other hand by Lemma 2.1, we have

$$2\varphi\left(\frac{a+b}{2}\right) \leqslant \varphi\left(a+b-\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right)+\varphi\left(\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$
$$\leqslant \varphi(a)+\varphi(b). \tag{3.13}$$

This implies

$$-(\varphi(a)+\varphi(b)) \leqslant -\varphi\left(a+b-\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right)-\varphi\left(\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$
$$\leqslant -2\varphi\left(\frac{a+b}{2}\right). \tag{3.14}$$

By addition from (3.13) and (3.14), we get our result.

4 Applications

Let $x_k \in [a,b]$ $(b>a>0)$, $\lambda_k \in [0,1]$ such that $\sum_{k=1}^{n}\lambda_k = 1$. Then, by Theorem 1.1 and Theorem 1.3 for $\varphi(x) = -\ln x$, we obtain respectively

$$\sqrt{ab} \leqslant \sqrt{\frac{A'G+AG'}{2}} \leqslant \frac{a+b}{2}$$

and

$$1 \leqslant \sqrt{\frac{A}{G}\frac{A'}{G'}} \leqslant \frac{\frac{a+b}{2}}{\sqrt{ab}},$$

where $A=\sum_{k=1}^{n}\lambda_k x_k$, $G=\prod_{k=1}^{n}x_k^{\lambda_k}$, $A'=a+b-\sum_{k=1}^{n}\lambda_k x_k$ and $G'=\prod_{k=1}^{n}(a+b-x_k)^{\lambda_k}$.

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On nonlinear Volterra-Fredholm functional integrodifferential equations with nonlocal condition in Banach spaces

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Abstract

In this paper we study the existence, uniqueness and continuous dependence of solutions of nonlinear Volterra-Fredholm functional integrodifferential equations with nonlocal condition in Banach space by using the Hausdorff's measure of noncompactness and Darbo-Sadovskii fixed point theorem. An application is provided to illustrate the theory.

Keywords: Volterra-Fredholm functional integrodifferential equation, nonlocal condition, Hausdorff's measure of noncompactness, Darbo-Sadovskii fixed point theorem.

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1 Introduction

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = C\big([-r,0],\, X\big),\ 0 < r < \infty,$ be the Banach space of all continuous functions $x : [-r,0] \to X$ with the supremum norm

$$\big\|x\big\|_C = sup\big\{\|x(t)\| : -r \le t \le 0\big\}.$$

We denote the Banach space of all continuous functions $y : [-r,T] \to X$ with the supremum norm

$$\big\|y\big\|_B = sup\big\{\|y(t)\| : -r \le t \le T\big\}$$

by $B = C\big([-r,T],\, X\big)$. For any $y \in B$ and $t \in [0,T]$ we denote by y_t the element of $C = C\big([-r,0],\, X\big)$ given by $y_t(\theta) = y(t+\theta)$ for $\theta \in [-r,0]$. Consider the nonlinear Volterra-Fredholm functional integrodifferential equations with nonlocal condition of the type

$$x^{'}(t) + Ax(t) = f\left(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^T b(t,s)k(s,x_s)ds\right),\ \ t \in [0,T], \tag{1.1}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t),\quad t \in [-r,0], \tag{1.2}$$

and

$$\frac{d}{dt}[x(t) - w(t,x_t)] + Ax(t) = f\left(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^T b(t,s)k(s,x_s)ds\right),\ \ t \in [0,T], \tag{1.3}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t),\quad t \in [-r,0], \tag{1.4}$$

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where $0 < t_1 < ... < t_p \leq T$, $p \in N$, $f : [0,T] \times C \times X \times X \to X$, $a, b : [0,T] \times [0,T] \to \mathbb{R}$, $w, h, k : [0,T] \times C \to X$ are continuous functions, $g : C^p \to C$ is given, ϕ is a given element of C. $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)$ in X.

The study of Cauchy problems with nonlocal conditions is of great significance. It has many applications in physics and other areas of applied mathematics. Q.Dong et al [14] studied the existence of the nonlocal neutral functional differential and integrodifferential equations of the form

$$\frac{d}{dt}[x(t) - g(t, x(t), x_t)] = Ax(t) + f\big(t, x(t), x_t\big), \;\; t \in [0, b],$$
$$x_0 = \phi + h(x)$$

and

$$\frac{d}{dt}[x(t) - g(t, x(t), x_t)] = Ax(t) + \int_0^t K(t,s) f\big(s, x(s), x_s\big), \;\; t \in [0, b],$$
$$x_0 = \phi + h(x)$$

using the Hausdorff's measure of noncompactness. Many authors have investigated the existence, uniqueness and other properties of solutions of the nonlocal Cauchy problems for functional differential equations with delay, see [3, 6, 7, 8] and the references cited therein. Balachandran and Park in [3] established existence, continuous dependence and controllabilty for the functional integrodifferential equation with nonlocal condition of the form

$$\frac{du(t)}{dt} + Au(t) = f\Big(t, u_t, \int_0^t k(t, \tau, u_\tau) d\tau\Big), \;\; t \in [0, a],$$
$$u(s) + \big(g(u_{t_1}, ..., u_{t_p})\big)(s) = \phi(s), \quad s \in [-r, 0],$$

using the Banach fixed point principle. In the present paper we prove the existence, uniqueness, continuous dependence and controllabilty of the mild solutions of the more general nonlocal problems (1.1)-(1.2) and (1.3)-(1.4) using the Hausdorff's measure of noncompactness and the Darbo-Sadovskii's fixed point theorem.

The paper is organised as follows. In section 2 we present the preliminaries. Section 3 deals with the existence of mild solutions of the nonlocal problems (1.1)-(1.2) and (1.3)-(1.4). In section 4 we establish sufficient conditions for the continuous dependence and uniqueness of mild solutions of the nonlocal problems. In section 5 an application is provided to illustrate the theory.

2 Preliminaries

We setforth some preliminaries from [4, 12] and hypotheses that will be used in our further discussions.

The functions a, b being continuous on compact domains, there are constants λ and μ such that

$$\big|a(t,s)\big| \leq \lambda \text{ and } \big|b(t,s)\big| \leq \mu, \text{ for } s, t \in [0, T]. \tag{2.1}$$

Since the operator $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)$ in X we have $\|T(t)\| \leq U$ where $U \geq 1$. Let $0 \in \rho(A)$. It is now possible to define the fractional power $A^\alpha, 0 < \alpha \leq 1$ as a closed linear operator on its domain $D(A^\alpha)$. Further, $D(A^\alpha)$ is dense in X and the expression $\|x\|_\alpha = \|A^\alpha x\|$ defines a norm on $D(A^\alpha)$. If X_α is the space $D(A^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$ then X_α is a Banach space and therefore the following Lemma 2.1 obviously holds.

Lemma 2.1. *[12] Let $0 < \alpha \leq \beta \leq 1$. Then the following properties hold:*

(1) X_β is a Banach space and $X_\beta \hookrightarrow X_\alpha$ is continuous.

(2) The function $s \mapsto (A)^\alpha T(s)$ is continuous in the uniform operator topology on $(0, \infty)$ and there exists a positive constant C_α such that $\|A^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}$ for every $t > 0$.

Definition 2.2. *The Hausdorff's measure of noncompactness χ_Y is defined by $\chi_Y(S) = \inf\{r > 0, S$ can be covered by finite number of balls with radii $r\}$ for bounded set S in any Banach space Y.*

Lemma 2.3. [4] *Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, then the following properties are satisfied:*
(1) B is precompact if and only if $\chi_{Y}(B) = 0$;
(2) $\chi_{Y}(B) = \chi_{Y}(\bar{B}) = \chi_{Y}(convB)$ where $\bar{B}$ and $convB$ mean the closure and convex hull of B respectively;
(3) $\chi_{Y}(B) \leq \chi_{Y}(C)$ when $B \subseteq C$;
(4) $\chi_{Y}(B + C) \leq \chi_{Y}(B) + \chi_{Y}(C)$ where $B + C = \{x + y; x \in B, y \in C\}$;
(5) $\chi_{Y}(B \cup C) \leq \max\{\chi_{Y}(B), \chi_{Y}(C)\}$;
(6) $\chi_{Y}(\lambda B) = |\lambda|\, \chi_{Y}(B)$ for any $\lambda \in R$;
(7) If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k then $\chi_{Z}(Q(B)) \leq k\, \chi_{Y}(B)$ for any bounded set $B \subseteq D(Q)$, where Z is a Banach space;
(8) $\chi_{Y}(B) = \inf\{d_Y(B, C); C \subseteq Y$ be precompact$\} = \inf\{d_Y(B, C); C \subseteq Y$ be finite valued$\}$, where $d_Y(B, C)$ means the nonsymmetric (or symmetric) Hausdorff distance between B and C in Y;
(9) If $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded, closed nonempty subsets of Y and $\lim_{n\to+\infty} \chi_{Y}(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.

Definition 2.4. *The map $Q : W \subseteq Y \to Y$ is said to be a χ_{Y}-contraction if there exists a positive constant $k < 1$ such that $\chi_{Y}(Q(S)) \leq k\, \chi_{Y}(S)$ for any bounded closed subset $S \subseteq W$ where Y is a Banach space.*

The following lemma known as Darbo-Sadovskii fixed point theorem given in [4] plays a crucial role in our subsequent discussions.

Lemma 2.5. [4] *If $W \subseteq Y$ is bounded, closed and convex, the continuous map $Q : W \to W$ is a $\chi_{Y}-$contraction, then the map Q has atleast one fixed point in W.*

In this paper we use the notations χ and χ_{B} to denote the Hausdorff's measure of noncompactness of the Banach space X and that of the Banach space $B = C\big([-r, T], X\big)$ respectively.

Lemma 2.6. [4] *If $W \subseteq C([a, b], X)$ is bounded, then*

$$\chi(W(t)) \leq \chi_{C}(W)$$

for all $t \in [a, b]$, where $W(t) = \{u(t); u \in W\} \subseteq X$. Furthermore if W is equicontinuous on $[a, b]$, then $\chi(W(t))$ is continuous on $[a, b]$ and

$$\chi_{C}(W) = \sup\{\chi(W(t)), t \in [a, b]\}.$$

Lemma 2.7. [4] *If $W \subseteq C([a, b]; X)$ is bounded and equicontinuous, then $\chi(W(s))$ is continuous and*

$$\chi\Big(\int_a^t W(s)ds\Big) \leq \int_a^t \chi(W(s))ds$$

for all $t \in [a, b]$, where $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$.

Definition 2.8. *The C_0 semigroup $T(t)$ is said to be equicontinuous if $t \to \{T(t)x : x \in S\}$ is equicontinuous for $t > 0$ for all bounded set S in X.*

We know that the analytic semigroup is equicontinuous. The following lemma is obivious.

Lemma 2.9. *If the semigroup $T(t)$ is equicontinuous and $\eta \in L(0, b; R^{+})$, then the set $\big\{ \int_0^t T(t-s)u(s)ds, \|u(s)\| \leq \eta(s)$ for a.e. $s \in [0, b]\big\}$ is equicontinuous for $t \in [0, b]$.*

Definition 2.10. *Let $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X. A function $x \in C\big([-r, T], X\big)$ is said to be a mild solution of the nonlocal problem (1.1)-(1.2) if it satisfies the following:*

$$(i)\ \ x(t) = T(t)\big[\phi(0) - \big(g(x_{t_1}, ..., x_{t_p})\big)(0)\big] + \int_0^t T(t-s) f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau, x_\tau)d\tau, \int_0^T b(s,\tau)k(\tau, x_\tau)d\tau\Big)ds, t \in [0, T] \quad (2.2)$$

$$(ii)\ \ x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t), \ \ t \in [-r, 0]. \quad (2.3)$$

Definition 2.11. *Let $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X. A function $x \in C\big([-r,T],X\big)$ is said to be a mild solution of the nonlocal problem(1.3)-(1.4) if for each $t \in [0,T]$, the function $AT(t-s)w(s,x_s)$, $s \in [0,t)$ is integrable and satisfies the following:*

$$\begin{aligned}
&(i)\ \ x(t) = T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0) - w(0,x_0)\big] \\
&+ w(t,x_t) + \int_0^t AT(t-s)w(s,x_s)ds \\
&+ \int_0^t T(t-s)f\bigg(s,x_s,\int_0^s a(s,\tau)h(\tau,x_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,x_\tau)d\tau\bigg)ds, t\in[0,T] \quad (2.4)\\
&(ii)\ \ x(t) + \big(g(x_{t_1},...,x_{t_p})\big)(t) = \phi(t),\ \ t \in [-r,0]. \quad (2.5)
\end{aligned}$$

We shall make use of the following hypotheses to prove our main results :

(H_1) There exists a continuous function $l : [0,T] \to \mathbb{R}_+ = [0,\infty)$ such that

$$\big\|f(t,\psi,x,y)\big\| \le l(t)\big(\|\psi\|_C + \|x\| + \|y\|\big)$$

for every $t \in [0,T], \psi \in C$ and $x, y \in X$.

(H_2) There exists a continuous function $p : [0,T] \to \mathbb{R}_+$ such that

$$\big\|h(t,\psi)\big\| \le p(t)\,H\big(\|\psi\|_C\big)$$

for every $t \in [0,T], \psi \in C$ where $H : \mathbb{R}_+ \to (0,\infty)$ is a continuous nondecreasing function.

(H_3) There exists a continuous function $q : [0,T] \to \mathbb{R}_+$ such that

$$\big\|k(t,\psi)\big\| \le q(t)\,K\big(\|\psi\|_C\big)$$

for every $t \in [0,T], \psi \in C$ where $K : \mathbb{R}_+ \to (0,\infty)$ is a continuous nondecreasing function.

(H_4) For each $t \in [0,T]$ the function $f(t,.,.,.) : C\times X\times X \to X$ is continuous and for each $(\psi,x,y) \in C\times X\times X$ the function $f(.,\psi,x,y) : [0,T] \to X$ is strongly measurable.

(H_5) For each $t \in [0,T]$ the functions $h(t,.), k(t,.) : C \to X$ are continuous and for each $\psi \in C$ the functions $h(.,\psi), k(.,\psi) : [0,T] \to X$ are strongly measurable.

(H_6) There exists a constant $\rho > 0$ such that

$$\big\|\big(g(u_{t_1},...,u_{t_p})\big)(s) - \big(g(v_{t_1},...,v_{t_p})\big)(s)\big\| \le \rho\,\big\|u-v\big\|_B$$

for $u, v \in B,\ s \in [-r,0]$.

(H_7) There exist constant G such that

$$G = \max_{y\in B}\big\|g(y_{t_1},...,y_{t_p})\big\|, \tag{2.6}$$

(H_8) There exists $0 < \beta < 1$ such that w is $X_\beta-$valued, $A^\beta w(\cdot)$ is continuous and there exist positive constants c_1, c_2 and V such that

$$\begin{aligned}
&\|A^\beta w(t,\psi)\| \le c_1\|\psi\| + c_2, && (2.7)\\
&\|A^\beta\big[w(t,\psi_1) - w(t,\psi_2)\big]\| \le V\|\psi_1 - \psi_2\|_C && (2.8)
\end{aligned}$$

for $t \in [0,T]$ and $\psi, \psi_1, \psi_2 \in C$.

(H_9) $U_1 + UM^*T\Big\{1 + M^*T\Big[\liminf_{m\to\infty}\Big(\frac{H(m)}{m} + \frac{K(m)}{m}\Big)\Big]\Big\} < 1$
where

$$U_1 = \Big\{\big[U+1\big]\big\|A^{-\beta}\big\| + C_{1-\beta}\frac{T^\beta}{\beta}\Big\}\, c_1 \tag{2.9}$$

$$M^* = \sup\{M(t),\ t\in[0,T]\} \quad \text{and} \tag{2.10}$$

$$M(t) = \max\{l(t),\ \lambda p(t),\ \mu q(t)\} \text{ for each } t\in[0,T]. \tag{2.11}$$

(H_{10}) $UM^*T\Big\{1 + M^*T\Big[\liminf_{m\to\infty}\Big(\frac{H(m)}{m} + \frac{K(m)}{m}\Big)\Big]\Big\} < 1$

(H_{11}) There exists integrable functions $\eta, \eta_1, \eta_2 : [0,T] \to [0,\infty)$ such that for any bounded set $W \subset C([-r,T],\, X)$ and $s\in[0,T]$ we have

$$\chi\big(T(t-s)\, f\Big(s, W_s, \int_0^s a(s,\tau)h(\tau,W_\tau)d\tau, \int_0^T b(s,\tau)k(\tau,W_\tau)d\tau\Big)\big)$$
$$\le \eta(s)\Big(\sup_{-r\le\theta\le 0}\chi\big(W(s+\theta)\big) + \int_0^s \big|a(s,\tau)\big|\,\eta_1(\tau)\sup_{-r\le\theta\le 0}\chi\big(W(\tau+\theta)\big)d\tau$$
$$+ \int_0^T \big|b(s,\tau)\big|\,\eta_2(\tau)\sup_{-r\le\theta\le 0}\chi\big(W(\tau+\theta)\big)d\tau\Big)$$

3 Existence of mild solutions

Theorem 3.1. *Suppose that the hypotheses (H_1)-(H_7), (H_{10}) and (H_{11}) holds. Then the nonlocal problem (1.1)-(1.2) has a mild solution x on $[-r,T]$ if*

$$\big\{U\rho + \int_0^T \eta(s)\,\big[1 + \int_0^s \lambda\,\eta_1(\tau)d\tau + \int_0^T \mu\,\eta_2(\tau)d\tau\big]ds\big\} < 1. \tag{3.1}$$

Theorem 3.2. *Suppose that the hypotheses (H_1)-(H_9) and (H_{11}) holds. Then the nonlocal problem (1.3)-(1.4) has a mild solution x on $[-r,T]$ if*

$$\big\{\rho_1 + \int_0^T \eta(s)\,\big[1 + \int_0^s \lambda\,\eta_1(\tau)d\tau + \int_0^T \mu\,\eta_2(\tau)d\tau\big]ds\big\} < 1 \tag{3.2}$$

where the constant term

$$\rho_1 = \Big\{U\,\rho + V\,\big\|A^{-\beta}\big\|\,\big(U+1\big) + V\,C_{1-\beta}\,\frac{T^\beta}{\beta}\Big\}. \tag{3.3}$$

Proof. The proofs of the Theorems 3.1 - 3.2 resemble one another. Therefore, we give the details of Theorem 3.2 only and the proof of Theorem 3.1 can be completed by closely looking at the proof of the Theorem 3.2 with slight modifications.

We prove the existence of mild solution of nonlinear mixed integrodifferential equations (1.3)-(1.4), by using the Darbo-Sadovskii fixed point theorem and the Hausdorff's measure of noncompactness. Consider the bounded set $B_m = \{y\in B : \|y\| \le m\}$ for each $m\in N$ (the set of all positive integers).
Define an operator $F : B = C\big([-r,T],\, X\big) \to B$ by $F = F_1 + F_2$

$$(F_1x)(t) = \begin{cases} \phi(t) - \big(g(x_{t_1},...,x_{t_p})\big)(t), & -r\le t\le 0 \\ T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0) - w(0,x_0)\big] \\ +w(t,x_t) + \int_0^t AT(t-s)w(s,x_s)ds & 0\le t\le T \end{cases} \tag{3.4}$$

$$(F_2x)(t) = \begin{cases} 0, & -r\le t\le 0 \\ \int_0^t T(t-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau,x_\tau)d\tau, \\ \int_0^T b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds & 0\le t\le T \end{cases} \tag{3.5}$$

Using lemma 2.1, hypothesis (H_8) and the fact that $\|x_s\|_C \leq \|x\|_B$ for $s \in (0,t)$ and $x \in B_m$ we have,

$$\begin{aligned}\big\|AT(t-s)w(s,x_s)\big\| &= \big\|A^{1-\beta}T(t-s)A^{\beta}w(s,x_s)\big\| \\ &\leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}}(c_1\|x_s\|_C + c_2) \\ &\leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}}(c_1\|x\|_B + c_2) \\ &\leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}}(c_1 m + c_2).\end{aligned}$$

Using this and the fact that the function $s \to AT(t-s)$ is continuous in the uniform operator topology on $(0,t)$ we conclude that $AT(t-s)w(s,x_s)$ is integrable on $(0,t)$ for every $t \in (0,T]$ and $x \in B_m$. Therefore F is well-defined and with values in B.

From the definition of F it follows that the fixed point of F is the mild solution of the nonlocal problem (1.3)-(1.4). We first show that $F : B \to B$ is continuous. Let $\{u_n\}$ be a sequence of elements of B converging to u in B. Consider the case when $t \in [-r,0]$, then using hypothesis (H_6) we have

$$\begin{aligned}\big\|(Fu_n)(t) - (Fu)(t)\big\| &= \big\|\phi(t) - \big(g(u_{n_{t_1}},...,u_{n_{t_p}})\big)(t) - \phi(t) + \big(g(u_{t_1},...,u_{t_p})\big)(t)\big\| \\ &= \big\|\big(g(u_{t_1},...,u_{t_p})\big)(t) - \big(g(u_{n_{t_1}},...,u_{n_{t_p}})\big)(t)\big\| \\ &\leq \rho\,\big\|u - u_n\big\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.6}$$

Now let $t \in [0,T]$ then using hypotheses (H_4) and (H_5) we have

$$\begin{aligned}&f\bigg(t, u_{n_t}, \int_0^t a(t,s)h(s,u_{n_s})ds, \int_0^T b(t,s)k(s,u_{n_s})\bigg) \\ &\quad\to f\bigg(t, u_t, \int_0^t a(t,s)h(s,u_s)ds, \int_0^T b(t,s)k(s,u_s)\bigg).\end{aligned}$$

Using the dominated convergence theorem, hypotheses (H_6), (H_8) and lemma 2.1 we have for $t \in (0,T]$,

$$\begin{aligned}&\big\|(Fu_n)(t) - (Fu)(t)\big\| \\ &= \big\|T(t)\big[\phi(0)\big] - T(t)\big[\big(g(u_{n_{t_1}},...,u_{n_{t_p}})\big)(0)\big] - T(t)\big[w(0,u_{n_0})\big] \\ &+ w(t,u_{n_t}) + \int_0^t AT(t-s)w(s,u_{n_s})ds \\ &+ \int_0^t T(t-s)f\bigg(s, u_{n_s}, \int_0^s a(s,\tau)h(\tau,u_{n_\tau})d\tau, \int_0^T b(s,\tau)k(\tau,u_{n_\tau})d\tau\bigg)ds \\ &- T(t)\big[\phi(0)\big] + T(t)\big[\big(g(u_{t_1},...,u_{t_p})\big)(0)\big] + T(t)\big[w(0,u_0)\big] \\ &- w(t,u_t) - \int_0^t AT(t-s)w(s,u_s)ds \\ &- \int_0^t T(t-s)f\bigg(s, u_s, \int_0^s a(s,\tau)h(\tau,u_\tau)d\tau, \int_0^T b(s,\tau)k(\tau,u_\tau)d\tau\bigg)ds\big\| \\ &= \big\|T(t)\big[\big(g(u_{t_1},...,u_{t_p})\big)(0) - \big(g(u_{n_{t_1}},...,u_{n_{t_p}})\big)(0)\big] \\ &+ T(t)\big[w(0,u_0) - w(0,u_{n_0})\big] + w(t,u_{n_t}) - w(t,u_t) \\ &+ \int_0^t AT(t-s)\big[w(s,u_{n_s}) - w(s,u_s)\big]ds \\ &+ \int_0^t T(t-s)\bigg[f\bigg(s, u_{n_s}, \int_0^s a(s,\tau)h(\tau,u_{n_\tau})d\tau, \int_0^T b(s,\tau)k(\tau,u_{n_\tau})d\tau) \\ &- f\bigg(s, u_s, \int_0^s a(s,\tau)h(\tau,u_\tau)d\tau, \int_0^T b(s,\tau)k(\tau,u_\tau)d\tau\bigg)\bigg]ds\big\|\end{aligned}$$

$$\leq \Big\{ V\|A^{-\beta}\| \Big(U \|u_0 - u_{n0}\|_C + \|u_{n_t} - u_t\|_C \Big) + V\, C_{1-\beta} \|u_n - u\|_B \frac{t^\beta}{\beta}$$
$$+ U\Big[\rho\|u - u_n\| + \int_0^t \|f\Big(s, u_{n_s}, \int_0^s a(s,\tau)h(\tau, u_{n_\tau})d\tau, \int_0^T b(s,\tau)k(\tau, u_{n_\tau})d\tau\Big)$$
$$- f\Big(s, u_s, \int_0^s a(s,\tau)h(\tau, u_\tau)d\tau, \int_0^T b(s,\tau)k(\tau, u_\tau)d\tau\Big)\|ds\Big]\Big\} \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Since$\|(Fu_n) - (Fu)\|_B = \sup_{t\in[-r,T]} \|(Fu_n)(t) - (Fu)(t)\|$, inequalities (3.6) and (3.7) imply $Fu_n \to Fu$ in B as $u_n \to u$ in B. Therefore F is continuous.

We shall show that F is a χ_B- contraction on some bounded closed convex subset $B_m \subseteq B = (C[-r,T], X)$. And then by using Darbo-Sadovskii's fixed point theorem we get a fixed point of F.

Firstly by using the method of contradiction we obtain a $m \in N$ such that $F_{B_m} \subseteq B_m$. Suppose that for each $m \in N$ there is a $y^m \in B_m$ and $t^m \in [-r,T]$ such that $\|(Fy^m)(t^m)\| > m$. If $t^m \in [-r,0]$ then using hypothesis (H_7) we obtain

$$\begin{aligned} m < \|(Fy^m)(t^m)\| &\leq \|\phi(t^m)\| + \|\big(g(y^m_{t_1},...,y^m_{t_p})\big)(t^m)\| \\ &\leq c + G. \end{aligned} \tag{3.8}$$

where c denotes $\|\phi\|_C$. Also we know that if $\|y^m\|_B \leq m$ then

$$\|y^m_t\|_C \leq m \text{ for all } t \in [0,T] \tag{3.9}$$

Using hypotheses $(H_1) - (H_3)$ and conditions (2.1), (2.6), (2.7), (2.11),(2.10) and (3.9) for the case when $t^m \in [0,T]$ we obtain

$$\begin{aligned} m &< \|(Fy^m)(t^m)\| \\ &\leq U\left[\|\phi(0)\| + \|\big(g(y^m_{t_1},...,y^m_{t_p})\big)(0)\|\right] + \|T(t^m)w(0,y^m_0)\| \\ &+ \|w(t^m, y^m_{t^m})\| + \int_0^{t^m} \|AT(t^m - s)w(s, y^m_s)\|ds \\ &+ \int_0^{t^m} U\|f\Big(s, y^m_s, \int_0^s a(s,\tau)h(\tau, y^m_\tau)d\tau, \int_0^T b(s,\tau)k(\tau, y^m_\tau)d\tau\Big)\|ds \\ &\leq \|A^{-\beta}\, T(t^m)\, A^\beta\, w(0, y^m_0)\| + \|A^{-\beta}\, A^\beta\, w(t^m, y^m_{t^m})\| \\ &+ \int_0^{t^m} \|A^{1-\beta}\, T(t^m - s)\|\, \|A^\beta w(s, y^m_s)\|ds + U\Big[c + G \\ &+ \int_0^{t^m} \|f\Big(s, y^m_s, \int_0^s a(s,\tau)h(\tau, y^m_\tau)d\tau, \int_0^T b(s,\tau)k(\tau, y^m_\tau)d\tau\Big)\|ds\Big] \\ &\leq U\,\|A^{-\beta}\|\,(c_1 m + c_2) + \|A^{-\beta}\|\,(c_1 m + c_2) \\ &+ C_{1-\beta}\,(c_1 m + c_2) \int_0^{t^m} (t^m - s)^{\beta-1}ds + U\Big[c + G + \int_0^{t^m} l(s) \\ &\Big(\|y^m_s\|_C + \int_0^s |a(s,\tau)|\,\|h(\tau, y^m_\tau)\|\,d\tau + \int_0^T |b(s,\tau)|\,\|k(\tau, y^m_\tau)\|\,d\tau\Big)ds\Big] \\ &\leq [U+1]\,\|A^{-\beta}\|\,(c_1 m + c_2) + C_{1-\beta}\,(c_1 m + c_2)\,\frac{(t^m)^\beta}{\beta} + U\Big[c + G \\ &+ \int_0^{t^m} M(s)\Big(m + \int_0^s M(\tau)H(m)d\tau + \int_0^T M(\tau)K(m)d\tau\Big)ds\Big] \\ &\leq \Big\{[U+1]\,\|A^{-\beta}\| + C_{1-\beta}\,\frac{T^\beta}{\beta}\Big\}\,(c_1 m + c_2) + U\Big[c + G \\ &+ \int_0^T M(s)\Big(m + \int_0^s M(\tau)H(m)d\tau + \int_0^T M(\tau)K(m)d\tau\Big)ds\Big] \end{aligned}$$

$$\leq \left\{ [U+1] \, \|A^{-\beta}\| + C_{1-\beta} \, \frac{T^\beta}{\beta} \right\} (c_1 m + c_2) + U \Bigg[c + G$$
$$+ \int_0^T M^* \Big(m + M^* H(m) T + M^* K(m) T \Big) ds \Bigg] \tag{3.10}$$

Thus using the fact that $U \geq 1$ we combine (3.8) and (3.10) so that we obtain

$$m < \left\{ [U+1] \, \|A^{-\beta}\| + C_{1-\beta} \, \frac{T^\beta}{\beta} \right\} (c_1 m + c_2) + U \, [c+G]$$
$$+ U M^* T \Big(m + M^* H(m) T + M^* K(m) T \Big) \tag{3.11}$$

Dividing by m on both sides of (3.11) we obtain

$$1 < \left\{ [U+1] \, \|A^{-\beta}\| + C_{1-\beta} \, \frac{T^\beta}{\beta} \right\} \Big(c_1 + \frac{c_2}{m} \Big) + U \left[\frac{c+G}{m} \right]$$
$$+ U M^* T \Big(1 + M^* T \, \frac{H(m)}{m} + M^* T \, \frac{K(m)}{m} \Big) \tag{3.12}$$

Now taking $\liminf$ as $m \to \infty$ on both sides of (3.12) we get

$$1 < U_1 + U M^* T \Bigg\{ 1 + M^* T \Bigg[\liminf_{m \to \infty} \Big(\frac{H(m)}{m} + \frac{K(m)}{m} \Big) \Bigg] \Bigg\}.$$

which contradicts the hypothesis (H_9). Thus there is a $m \in N$ such that $F_{B_m} \subseteq B_m$. Hereafter we will consider the restriction of F on this B_m.

Now we show that F_1 is Lipschitz continuous. Let $x, y \in B_m$ then using hypothesis (H_6) we have for $t \in [-r, 0]$

$$\big\|(F_1 x)(t) - (F_1 y)(t)\big\| \leq \big\| \big(g(x_{t_1}, ..., x_{t_p})\big)(t) - \big(g(y_{t_1}, ..., y_{t_p})\big)(t) \big\|$$
$$\leq \rho \big\| x - y \big\|_B \tag{3.13}$$

Now using hypothesis (H_6), condition (2.8) and lemma 2.1 for $t \in [0, T]$ we have

$$\begin{aligned}
\big\|(F_1 x)(t) - (F_1 y)(t)\big\| &= \big\| T(t)\big[\phi(0)\big] - T(t)\big[\big(g(x_{t_1}, ..., x_{t_p})\big)(0)\big] - T(t)\big[w(0, x_0)\big] \\
&+ w(t, x_t) + \int_0^t AT(t-s) w(s, x_s) ds \\
&- T(t)\big[\phi(0)\big] + T(t)\big[\big(g(y_{t_1}, ..., y_{t_p})\big)(0)\big] + T(t)\big[w(0, y_0)\big] \\
&- w(t, y_t) - \int_0^t AT(t-s) w(s, y_s) ds \big\| \\
&= \big\| T(t)\big[\big(g(y_{t_1}, ..., y_{t_p})\big)(0) - \big(g(x_{t_1}, ..., x_{t_p})\big)(0)\big] \\
&+ T(t)\big[w(0, y_0) - w(0, x_0)\big] + w(t, x_t) - w(t, y_t) \\
&+ \int_0^t AT(t-s)\big[w(s, x_s) - w(s, y_s)\big] ds \big\| \\
&\leq U \, \rho \, \big\| y - x \big\| + \big\| A^{-\beta} \, T(t) \, A^\beta \, \big[w(0, y_0) - w(0, x_0)\big] \big\| \\
&+ \big\| A^{-\beta} \, A^\beta \, \big[w(t, x_t) - w(t, y_t)\big] \big\| \\
&+ \int_0^t \big\| A^{1-\beta} \, T(t-s) \, A^\beta \, \big[w(s, x_s) - w(s, y_s)\big] \big\| \, ds \\
&\leq U \, \rho \, \big\| y - x \big\| + V \, \big\| A^{-\beta} \big\| \, \Big[U \big\| y_0 - x_0 \big\|_C + \big\| x_t - y_t \big\|_C \Big] \\
&+ V \, C_{1-\beta} \, \big\| x - y \big\|_B \int_0^t (t-s)^{\beta - 1} ds \\
&\leq U \, \rho \, \big\| y - x \big\|_B + V \, \big\| A^{-\beta} \big\| \, (U+1) \, \big\| y - x \big\|_B \\
&+ V \, C_{1-\beta} \, \big\| x - y \big\|_B \, \frac{t^\beta}{\beta} \\
&\leq \Big\{ U \, \rho + V \, \big\| A^{-\beta} \big\| \, (U+1) + V \, C_{1-\beta} \, \frac{T^\beta}{\beta} \Big\} \big\| x - y \big\|_B
\end{aligned} \tag{3.14}$$

Since $U \geq 1$ and $\rho > 0$ we have $U\rho\|x-y\|_B \geq \rho\|x-y\|_B$ and so in view of (3.13) and (3.14) we obtain

$$\|(F_1x)(t)-(F_1y)(t)\| \leq \left\{U\,\rho + V\,\|A^{-\beta}\|(U+1) + V\,C_{1-\beta}\,\frac{T^\beta}{\beta}\right\}\|x-y\|_B.$$

for all $t \in [-r,T]$ and $x, y \in B_m$. Consequently using (3.3) we get

$$\|(F_1x)-(F_1y)\| \leq \rho_1\,\|x-y\|_B$$

Thus F_1 is Lipschitzian with Lipschitz constant ρ_1. Hence using lemma 2.3(7) we now have

$$\chi_B(F_1W) \leq \rho_1 \chi_B(W) \tag{3.15}$$

for any bounded set $W \subseteq B_m$.

Further let W be any bounded subset of B_m. We first show that F_2W is bounded. Let $y \in W \subseteq B_m$ then $\|y\|_B \leq m$ and so $\|y_t\|_C \leq m,\ t \in [0,T]$. Let $t \in [-r,0]$ and $y \in B_m$ then from the definition of F_2 we have

$$\|(F_2y)(t)\| = 0$$

Now for $t \in [0,T]$ and $y \in B_m$ we get

$$\begin{aligned}
\|(F_2y)(t)\| &\leq \int_0^t U\Big\|f\Big(s, y_s, \int_0^s a(s,\tau)h(\tau,y_\tau)d\tau, \int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big)\Big\|ds\\
&\leq U\Big[\int_0^t l(s)\Big(\|y_s\|_C + \int_0^s |a(s,\tau)|\,\|h(\tau,y_\tau)\|d\tau\\
&+ \int_0^T |b(s,\tau)|\,\|k(\tau,y_\tau)\|d\tau\Big)ds\Big]\\
&\leq U\int_0^t M(s)\Big(m + \int_0^s M(\tau)H(m)d\tau + \int_0^T M(\tau)K(m)d\tau\Big)ds\\
&\leq U\int_0^t M^*\Big(m + M^*H(m)\,s + M^*K(m)\,T\Big)ds\\
&\leq U\,T\,M^*\Big(m + \frac{TM^*H(m)}{2} + T\,M^*K(m)\Big)
\end{aligned} \tag{3.16}$$

The R.H.S. of the inequality (3.16) being constant we conclude that the set $\{(F_2y)(t) : y \in W, -r \leq t \leq T\}$ is bounded in X and hence F_2W is bounded in B . Now we prove that F_2W is equicontinuous. For this let $y \in W,\ s_1, s_2 \in [-r,T]$ and consider the following cases :

Case:1 Suppose $0 \leq s_1 \leq s_2 \leq T$ then using hypothesis $(H_1)-(H_3)$ and conditions (2.11), (2.10) and (3.9), we get

$$\begin{aligned}
&\|(F_2y)(s_2)-(F_2y)(s_1)\|\\
&= \Big\|\int_0^{s_2} T(s_2-s)f\Big(s, y_s, \int_0^s a(s,\tau)h(\tau,y_\tau)d\tau, \int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big)ds\\
&- \int_0^{s_1} T(s_1-s)f\Big(s, y_s, \int_0^s a(s,\tau)h(\tau,y_\tau)d\tau, \int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big)ds\Big\|\\
&\leq \int_0^{s_1} \|T(s_2-s)-T(s_1-s)\|\\
&\Big[l(s)\Big(\|y_s\|_C + \Big\|\int_0^s a(s,\tau)h(\tau,y_\tau)d\tau\Big\| + \Big\|\int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big\|\Big)\Big]ds\\
&+ \int_{s_1}^{s_2} \|T(s_2-s)\|\\
&\Big[l(s)\Big(\|y_s\|_C + \Big\|\int_0^s a(s,\tau)h(\tau,y_\tau)d\tau\Big\| + \Big\|\int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big\|\Big)\Big]ds
\end{aligned}$$

$$\leq \int_0^{s_1} \big\|\big[T(s_2-s)-T(s_1-s)\big]\big\|$$
$$\Big[M(s)\Big(m+\int_0^s M(\tau)H(m)d\tau+\int_0^T M(\tau)K(m)d\tau\Big)\Big]ds$$
$$+U\int_{s_1}^{s_2} M(s)\Big(m+\int_0^s M(\tau)H(m)d\tau+\int_0^T M(\tau)K(m)d\tau\Big)ds$$
$$\leq \int_0^{s_1} \big\|\big[T(s_2-s)-T(s_1-s)\big]\big\|\Big[M^*\Big(m+M^*H(m)s+M^*K(m)T\Big)\Big]ds$$
$$+U\int_{s_1}^{s_2} M^*\Big(m+M^*H(m)\int_0^s d\tau+M^*K(m)\int_0^T d\tau\Big)ds$$
$$\leq \gamma\int_0^{s_1} \big\|T(s_2-s)-T(s_1-s)\big\|ds+U\gamma\big|s_2-s_1\big|\Big\}$$
$$\to 0 \ \text{ as } s_2\to s_1,$$

where $\gamma=\Big[M^*\Big(m+M^*H(m)T+M^*K(m)T\Big)\Big]$. The compactness of T(t) for $t>0$ implies the continuity in the uniform operator topology. Therefore the right hand side of above equation tends to zero as $s_2 \to s_1$.

Case:2 Suppose $-r\leq s_1\leq 0\leq s_2\leq T$ then we get

$$\big\|(F_2y)(s_2)-(F_2y)(s_1)\big\|$$
$$=\Big\|\int_0^{s_2} T(s_2-s)f\Big(s,y_s,\int_0^s a(s,\tau)h(\tau,y_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big)ds\Big\|$$

Now proceeding as in Case 1 for the integral on the right hand side of above inequality we further obtain

$$\big\|(F_2y)(s_2)-(F_2y)(s_1)\big\|\leq U\gamma\big|s_2-s_1\big|\to 0 \ \text{ as } s_2\to 0_+ \text{ and } s_1\to 0_-.$$

Case:3 Suppose $-r\leq s_1\leq s_2\leq 0$. In this case we have

$$\big\|(F_2y)(s_2)-(F_2y)(s_1)\big\|=0 \tag{3.17}$$

Thus cases (1)-(3) imply that $\|(F_2y)(s_2)-(F_2y)(s_1)\|\to 0$ as $s_1\to s_2$, for all $s_1,s_2\in[-r,T]$. Thus we conclude that F_2W is an equicontinuous family of functions.

Further for a bounded subset W of B_m we define the notations $W(t)=\{x(t);x\in W\}\subseteq X$ and $W_t=\{x_t;x\in W\}\subseteq C\big([-r,0],X\big)$. Now using lemma 2.3, lemma 2.6-2.7, lemma 2.9 and hypothesis (H_{11}) we obtain

$$\chi_B(F_2W)=\sup_{-r\leq t\leq T}\chi(F_2W(t))$$
$$=\sup_{0\leq t\leq T}\chi(F_2W(t))$$
$$=\sup_{0\leq t\leq T}\chi\Big(\int_0^t T(t-s)f\Big(s,W_s,\int_0^s a(s,\tau)h(\tau,W_\tau)d\tau,$$
$$\int_0^T b(s,\tau)k(\tau,W_\tau)d\tau\Big)ds\Big)$$
$$=\sup_{0\leq t\leq T}\int_0^t \chi\Big(T(t-s)f\Big(s,W_s,\int_0^s a(s,\tau)h(\tau,W_\tau)d\tau,$$
$$\int_0^T b(s,\tau)k(\tau,W_\tau)d\tau\Big)\Big)ds$$
$$\leq \sup_{0\leq t\leq T}\int_0^t \eta(s)\Big(\sup_{-r\leq\theta\leq 0}\chi(W(s+\theta))$$
$$+\int_0^s \big|a(s,\tau)\big|\,\eta_1(\tau)\sup_{-r\leq\theta\leq 0}\chi(W(\tau+\theta))d\tau$$
$$+\int_0^T \big|b(s,\tau)\big|\,\eta_2(\tau)\sup_{-r\leq\theta\leq 0}\chi(W(\tau+\theta))d\tau\Big)ds$$

$$
\begin{aligned}
&\leq \sup_{0\leq t\leq T}\int_0^t \eta(s)\Big(\sup_{s-r\leq s+\theta\leq s}\chi(W(s+\theta))\\
&+\int_0^s \lambda\,\eta_1(\tau)\sup_{\tau-r\leq \tau+\theta\leq \tau}\chi(W(\tau+\theta))d\tau\\
&+\int_0^T \mu\,\eta_2(\tau)\sup_{\tau-r\leq \tau+\theta\leq \tau}\chi(W(\tau+\theta))d\tau\Big)ds\\
&=\sup_{0\leq t\leq T}\int_0^t \eta(s)\Big(\sup_{s-r\leq s_1\leq s}\chi(W(s_1))+\int_0^s \lambda\,\eta_1(\tau)\sup_{\tau-r\leq \tau_1\leq \tau}\chi(W(\tau_1))d\tau\\
&+\int_0^T \mu\,\eta_2(\tau)\sup_{\tau-r\leq \tau_1\leq \tau}\chi(W(\tau_1))d\tau\Big)ds\\
&\leq \sup_{0\leq t\leq T}\int_0^t \eta(s)\Big(\sup_{-r\leq s_1\leq T}\chi(W(s_1))+\int_0^s \lambda\,\eta_1(\tau)\sup_{-r\leq \tau_1\leq T}\chi(W(\tau_1))d\tau\\
&+\int_0^T \mu\,\eta_2(\tau)\sup_{-r\leq \tau_1\leq T}\chi(W(\tau_1))d\tau\Big)ds\\
&\leq \sup_{0\leq t\leq T}\int_0^t \Big(\chi_B(W)+\int_0^s \lambda\,\eta_1(\tau)\chi_B(W)d\tau+\int_0^T \mu\,\eta_2(\tau)\chi_B(W)d\tau\Big)ds\\
&\leq \chi_B(W)\sup_{0\leq t\leq T}\int_0^t \eta(s)\Big(1+\int_0^s \lambda\,\eta_1(\tau)d\tau+\int_0^T \mu\,\eta_2(\tau)d\tau\Big)ds\\
&\leq \chi_B(W)\int_0^T \eta(s)\Big(1+\int_0^s \lambda\,\eta_1(\tau)d\tau+\int_0^T \mu\,\eta_2(\tau)d\tau\Big)ds \qquad (3.18)
\end{aligned}
$$

Therefore using (3.2), (3.15) and (3.18) we obtain

$$
\begin{aligned}
\chi_B(FW) &\leq \chi_B(F_1W)+\chi_B(F_2W)\\
&\leq \Big(\rho_1+\int_0^T \eta(s)\big[1+\int_0^s \lambda\,\eta_1(\tau)d\tau+\int_0^T \mu\,\eta_2(\tau)d\tau\big]ds\Big)\chi_B(W)\\
&< \chi_B(W) \qquad (3.19)
\end{aligned}
$$

for any bounded subset W of B_m.

Hence F is a χ_B- contraction. Now applying lemma 2.5 we get a fixed point x of F in B_m. This x is a mild solution of (1.3)-(1.4). The proof of the theorem is complete. □

4 Continuous dependence of mild solution

Theorem 4.1. *Suppose that the functions f, g, h, k, w satisfy the hypotheses (H_1)-(H_9) and (H_{11}). Also suppose that*

(H_{12}) *there exist a constant N such that*

$$\|f(t,x_t,z_1,z_2)-f(t,y_t,z_3,z_4)\|\leq N\big[\|x_t-y_t\|_C+\|z_1-z_3\|+\|z_2-z_4\|\big] \qquad (4.1)$$

for $t\in[0,T]$, $x,y\in B$ and $z_1,z_2,z_3,z_4\in X$.

(H_{13}) *there exist a constant P such that*

$$\|h(t,x_t)-h(t,y_t)\|\leq P\|x_t-y_t\|_C \qquad (4.2)$$

for $t\in[0,T]$ and $x_t,y_t\in C$.

(H_{14}) *there exist a constant Q such that*

$$\|k(t,x_t)-k(t,y_t)\|\leq Q\|x_t-y_t\|_C \qquad (4.3)$$

for $t \in [0,T]$ *and* $x_t, y_t \in C$.

Then for each $\phi_1, \phi_2 \in C$ *and for the corresponding mild solutions* x_1, x_2 *of the problems*

$$\frac{d}{dt}[x(t) - w(t,x_t)] + Ax(t) = f\Big(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^T b(t,s)k(s,x_s)ds\Big), \quad t \in [0,T], \tag{4.4}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi_i(t), \quad t \in [-r,0], \quad (i = 1,2), \tag{4.5}$$

the following inequality

$$\|x_1 - x_2\|_B \le U\|\phi_1 - \phi_2\|_C + \big[U\,\rho + (U+1)V\|A^{-\beta}\| + V\,C_{1-\beta}\frac{T^\beta}{\beta} + U N\,\{T + (\lambda\,P + \mu\,Q)T^2\}\big]\|x_1 - x_2\|_B \tag{4.6}$$

holds and if

$$U^* = \big[U\,\rho + (U+1)V\|A^{-\beta}\| + V\,C_{1-\beta}\frac{T^\beta}{\beta} + U N\,\{T + (\lambda\,P + \mu\,Q)T^2\}\|\big] < 1 \tag{4.7}$$

then

$$\|x_1 - x_2\|_B \le \frac{U}{1 - U^*}\|\phi_1 - \phi_2\|_C \tag{4.8}$$

Proof. Let $\phi_i\,(i = 1,2)$ be arbitrary functions of C and let $x_i\,(i = 1,2)$ be mild solutions of the nonlocal problems (1.3)-(1.4). Then using the hypothesis (H_6) and the fact that $U \ge 1$ we have for $t \in [-r,0]$,

$$\begin{aligned}
\|x_1(t) - x_2(t)\| &= \|\phi_1(t) - \big(g((x_1)_{t_1}, ..., (x_1)_{t_p})\big)(t) - \phi_2(t) + \big(g((x_2)_{t_1}, ..., (x_2)_{t_p})\big)(t)\| \\
&\le \|\phi_1(t) - \phi_2(t)\| \\
&\quad + \|\big(g((x_2)_{t_1}, ..., (x_2)_{t_p})\big)(t) - \big(g((x_1)_{t_1}, ..., (x_1)_{t_p})\big)(t)\| \\
&\le \|\phi_1 - \phi_2\|_C + \rho\|x_2 - x_1\|_B \\
&\le U\|\phi_1 - \phi_2\|_C + U\rho\|x_2 - x_1\|_B.
\end{aligned} \tag{4.9}$$

Now using hypotheses $(H_6), (H_8), (H_{12}) - (H_{14})$, lemma 2.1 and condition (2.1) we have for $t \in [0,T]$,

$$\begin{aligned}
\|x_1(t) - x_2(t)\| &\le \|T(t)\big[\phi_1(0) - \phi_2(0)\big]\| \\
&\quad + \|T(t)\big[\big(g((x_2)_{t_1}, ..., (x_2)_{t_p})\big)(0) - \big(g((x_1)_{t_1}, ..., (x_1)_{t_p})\big)(0)\big]\| \\
&\quad + \|T(t)\big[w(0,(x_2)_0) - w(0,(x_1)_0)\big]\| + \|w(t,(x_1)_t) - w(t,(x_2)_t)\| \\
&\quad + \|\int_0^t AT(t-s)\big[w(s,(x_1)_s) - w(s,(x_2)_s)\big]ds\| \\
&\quad + \|\int_0^t T(t-s)\Big[f\Big(s, (x_1)_s, \int_0^s a(s,\tau)h(\tau,(x_1)_\tau)d\tau, \\
&\quad \int_0^T b(s,\tau)k(\tau,(x_1)_\tau)d\tau\Big) - f\Big(s, (x_2)_s, \int_0^s a(s,\tau)h(\tau,(x_2)_\tau)d\tau, \\
&\quad \int_0^T b(s,\tau)k(\tau,(x_2)_\tau)d\tau\Big)\Big]ds\|
\end{aligned}$$

$$
\begin{aligned}
&\leq U\|\phi_1-\phi_2\|_C+U\rho\|x_2-x_1\|_B\\
&+U\|A^{-\beta}A^{\beta}[w(0,(x_2)_0)-w(0,(x_1)_0)]\\
&+\|A^{-\beta}A^{\beta}[w(t,(x_1)_t)-w(t,(x_2)_t)]\|\\
&+\int_0^t\|A^{1-\beta}T(t-s)\|\|A^{\beta}[w(s,(x_1)_s)-w(s,(x_2)_s)]\|ds\\
&+U\int_0^t N\Big[\|(x_1)_s-(x_2)_s\|_C+\|\int_0^s a(s,\tau)h(\tau,(x_1)_\tau)d\tau\\
&-\int_0^s a(s,\tau)h(\tau,(x_2)_\tau)d\tau\|\\
&+\|\int_0^T b(s,\tau)k(\tau,(x_1)_\tau)d\tau-\int_0^T b(s,\tau)k(\tau,(x_2)_\tau)d\tau\|\Big]ds\\
&\leq U\|\phi_1-\phi_2\|_C+U\rho\|x_2-x_1\|_B+UV\|A^{-\beta}\|\|x_2-x_1\|_B\\
&+V\|A^{-\beta}\|\,\|x_2-x_1\|_B+VC_{1-\beta}\frac{t^{\beta}}{\beta}\|x_2-x_1\|_B\\
&+UN\int_0^t\Big[\|x_1-x_2\|_B+\lambda P\int_0^s\|(x_1)_\tau)-(x_2)_\tau)\|_C d\tau\\
&+\mu Q\int_0^T\|(x_1)_\tau)-(x_2)_\tau)\|_C d\tau\Big]ds\\
&\leq U\|\phi_1-\phi_2\|_C+\big[U\rho+(U+1)V\|A^{-\beta}\|+VC_{1-\beta}\frac{T^{\beta}}{\beta}\\
&+UN\{T+(\lambda P+\mu Q)T^2\}\big]\|x_2-x_1\|_B. \qquad (4.10)
\end{aligned}
$$

Thus in view of inequality (4.9) and (4.10) we get

$$
\begin{aligned}
\|x_1(t)-x_2(t)\|\leq U\|\phi_1-\phi_2\|_C&+\big[U\rho+(U+1)V\|A^{-\beta}\|+VC_{1-\beta}\frac{T^{\beta}}{\beta}\\
&+UN\{T+(\lambda P+\mu Q)T^2\}\big]\|x_1-x_2\|_B, \qquad t\in[-r,T] \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
\|x_1-x_2\|_B\leq U\|\phi_1-\phi_2\|_C&+\big[U\rho+(U+1)V\|A^{-\beta}\|+VC_{1-\beta}\frac{T^{\beta}}{\beta}\\
&+UN\{T+(\lambda P+\mu Q)T^2\}\big]\|x_1-x_2\|_B. \qquad (4.12)
\end{aligned}
$$

Using (4.7) we get

$$
\|x_1-x_2\|_B\leq\frac{U}{1-U^*}\|\phi_1-\phi_2\|_C.
$$

Hence the proof is complete. □

Remark 4.2. *We remark that the uniqueness of the solution of the nonlocal problem* (1.3)-(1.4) *follows from the above continuous dependence theorem.*

5 Application

As an application of the Theorem 3.1, we consider the system (1.1)-(1.2) with control parameter

$$
x'(t)+Ax(t)=Ez(t)+f\Big(t,x_t,\int_0^t a(t,s)h(s,x_s)ds,\int_0^{\zeta}b(t,s)k(s,x_s)ds\Big),\quad t\in[0,\zeta], \qquad (5.1)
$$

$$
x(t)+\big(g(x_{t_1},...,x_{t_p})\big)(t)=\phi(t),\quad t\in[-r,0], \qquad (5.2)
$$

where E is a bounded linear operator from a Banach space Z to X and $z\in L^2([0,\zeta],Z)$. In this case the mild solution is given by

$$(i)\ \ x(t) = T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big] + \int_0^t T(t-s)Ez(s)ds$$
$$+ \int_0^t T(t-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau, x_\tau)d\tau, \int_0^\zeta b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds, t\in[0,\zeta] \tag{5.3}$$
$$(ii)\ \ x(t) + \big(g(x_{t_1},...,x_{t_p})\big)(t) = \phi(t),\ \ t\in[-r,0]. \tag{5.4}$$

We say that the system (5.1)-(5.2) is controllable to the origin if for any given initial function $\phi \in C$ there exists a control $z \in L^2([0,\zeta], Z)$ such that the mild solution $x(t)$ of (5.1)-(5.2) satisfies $x(\zeta) = 0$.

Note:We note that in this section the interval $[0,T]$ is replaced by $[0,\zeta]$ for notational convenience.
To derive the result we need the following additional hypotheses:

(H_{15}) The linear operator Ψ from $L^2([0,\zeta], Z)$ into X, defined by

$$\Psi z = \int_0^\zeta T(\zeta - s)Ez(s)ds$$

has an inverse operator Ψ^{-1} which takes values in $\frac{L^2([0,\zeta],Z)}{ker\Psi}$ such that the operator $E\Psi^{-1}$ is bounded.

(H_{16}) $UM^*\zeta\big[1+U\,\|E\Psi^{-}\|\zeta\big]\Big\{1+M^*\zeta\Big[\liminf_{m\to\infty}\Big(\frac{H(m)}{m}+\frac{K(m)}{m}\Big)\Big]\Big\} < 1$

Theorem 5.1. *If the hypotheses (H_1)-(H_7), (H_{11})-$(H_{14})$and (H_{15})-(H_{16}) are satisfied, then the system (5.1) with (5.2) is controllable if*

$$\big\{U\rho_2 + \int_0^\zeta \eta(s)\,\big[1+\int_0^s \lambda\,\eta_1(\tau)d\tau + \int_0^\zeta \mu\,\eta_2(\tau)d\tau\big]ds\big\} < 1. \tag{5.5}$$

where the constant term

$$\rho_2 = U\rho + U\,\|E\Psi^{-1}\|\,\zeta\,\big[U\rho + \zeta UN\big\{1+\lambda P\zeta + \mu Q\zeta\big\}\big]. \tag{5.6}$$

Proof. Using hypothesis (H_{15}) for an arbitrary function $x(\cdot)$, define the control

$$z(t) = -\,\Psi^{-1}\Big[T(\zeta)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big]$$
$$+ \int_0^\zeta T(\zeta-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau,x_\tau)d\tau, \int_0^\zeta b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds\Big](t) \tag{5.7}$$

for $t\in[0,\zeta]$. Using this control define an operator Γ as

$$(\Gamma x)(t) = \begin{cases} T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big] + \int_0^t T(t-s)Ez(s)ds \\ + \int_0^t T(t-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau,x_\tau)d\tau, \\ \int_0^\zeta b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds & 0\le t\le\zeta \\ \\ \phi(t) - \big(g(x_{t_1},...,x_{t_p})\big)(t), & -r\le t\le 0 \end{cases} \tag{5.8}$$

Substituting $z(s)$ in (5.8), we get

$$(\Gamma x)(t) = \begin{cases} T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big] \\ -\int_0^t T(t-s)E\Psi^{-1}\Big[T(\zeta)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big] + \int_0^\zeta T(\zeta-\eta)\cdot \\ f\Big(\eta, x_\eta, \int_0^\eta a(\eta,\tau)h(\tau,x_\tau)d\tau, \int_0^\zeta b(\eta,\tau)k(\tau,x_\tau)d\tau\Big)d\eta\Big](s)ds \\ + \int_0^t T(t-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau,x_\tau)d\tau, \\ \int_0^\zeta b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds & 0\le t\le\zeta \\ \\ \phi(t) - \big(g(x_{t_1},...,x_{t_p})\big)(t), & -r\le t\le 0 \end{cases} \tag{5.9}$$

Clearly $(\Gamma x)(\zeta) = 0$, which means that the control z steers the system from the initial function ϕ to the origin in time ζ if we can obtain a fixed point of the operator Γ. The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted. Thus the system (5.1) with (5.2) is controllable. □

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Contractive modulus and common fixed point for three asymptotically regular and weakly compatible self-maps

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Abstract

A common fixed point theorem for three self-maps on a metric space has been proved through the notions of orbital completeness, asymptotic regularity and weak compatibility. Our result generalizes those of Singh and Mishra, and the first author.

Keywords: Orbit, asymptotic regularity, weakly compatible self-maps, common fixed point.

2010 MSC: 52H25.

1 Introduction

Throughout this paper, (X,d) denotes a metric space, Sx the image of $x \in X$ under a self-map S on X and SA, the composition of self-maps S and A on X.

Definition 1.1. Self-maps S and A on X are compatible [1] if

$$\lim_{n\to\infty} d(SAx_n, ASx_n) = 0 \tag{1.1}$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = p \quad \text{for some} \quad p \in X. \tag{1.2}$$

If $x_n = x$ for all n, compatibility of (S, A) implies that $SAx = ASx$ whenever $Ax = Sx$. Self-maps which commute at their coincidence points are weakly compatible [2].

Definition 1.2. Let $\psi \equiv \psi : [0,\infty) \to [0,\infty)$ be a contractive modulus [3] with the choice $\psi(0) = 0$ and $\psi(t) < t$ for $t > 0$. A contractive modulus ψ is upper semicontinuous (abbriviated as usc) if and only if $\limsup_{n\to\infty} \psi(t_n) \le \psi(t_0)$ for all $t = t_0$ and all $\langle t_n \rangle_{n=1}^{\infty} \subset [0,\infty)$ with $\lim_{n\to\infty} t_n = t_0$.

Using these ideas, Singh and Mishra [5] proved the following result:

Theorem 1.1. *Let S, T and A be self-maps on X satisfying the inclusions*

$$S(X) \subset A(X) \quad \textit{and} \quad T(X) \subset A(X) \tag{1.3}$$

and the contractive-type condition

$$d(Sx,Ty) \le \psi\left(\max\left\{d(Ax,Ay), d(Sx,Ax), d(Ty,Ay), \frac{d(Ty,Ax)+d(Sx,Ay)}{2}\right\}\right) \textit{ for all } x,y \in X, \tag{1.4}$$

where ψ is an usc contractive modulus. Suppose that

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(a) *one of* $S(X), T(X)$ *and* $A(X)$ *is a complete subspace of* X,

(b) (A, S) *and* (A, T) *are weakly compatible.*

Then the three maps S, T *and* A *will have a unique common fixed point.*

In this paper, we generalize Theorem 1.1 by using the notion of asymptotic regularity (cf. Section 2) and by weakening the condition (b) under a weaker form of the inequality (1.4), when the contractive modulus ψ is nondecreasing. Our result also generalizes a result of the first author under an alternate condition.

2 Main Result

We need the following definitions from [4]:

Definition 2.1. Given $x_0 \in X$ and f, g and h self-maps on X, if we can find points $x_1, x_2, ..., x_n, ...$, then the associated sequence $\langle y_n \rangle_{n=1}^{\infty}$ with the choice

$$y_{2n-1} = Sx_{2n-2} = Ax_{2n-1}, y_{2n} = Tx_{2n-1} = Ax_{2n}, \text{ for } n = 1, 2, 3, ... \tag{2.1}$$

is called an (S, T, A)-orbit or simply an orbit $O(x_0)$ at x_0.

Definition 2.2. The space X is (S, T, A)-orbitally complete or orbitally complete at x_0 if every Cauchy sequence in some orbit $O(x_0)$ converges in X.

Definition 2.3. The pair (S, T) is asymptotically regular (abbriviated as a.r.) at x_0 with respect to A if there is an orbit $O(x_0)$ with the choice (2.1) such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Remark 2.1. Every complete metric space is orbitally complete at each of its points. However the converse of this statement is not true as in the following corrected form of the example from [4]:

Example 2.1. Let $X = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}_+, p \le q, q > 0 \right\}$ with $d(x, y) = |x - y|$ for all $x, y \in X$, where $\mathbb{Z}_+ = \{0, 1, 2, 3, ...\}$. Define S, T and $A : X \to X$ by

$$Sx = \tfrac{x}{3},\ Tx = \tfrac{x}{2} \text{ and } Ax = \begin{cases} \frac{2x}{3} & \text{if } x < 1 \\ \frac{3}{4} & \text{if } x = 1. \end{cases}$$

Then X is incomplete. For instance, the sequence 0.7, 0.705, 0.707, 0.7071, 0.707105, 0.7071065, ... is Cauchy which converges to $\frac{1}{\sqrt{2}} \notin X$. Given $x_0 \in X$, we choose $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that $x_n = \left(\frac{1}{2}\right)\left(\frac{3}{8}\right)^{\frac{n-1}{2}} x_0$ or $\left(\frac{2}{3}\right)\left(\frac{3}{8}\right)^{\frac{n}{2}} x_0$ according as n is odd or even. Then $O(x_0) = \left\{ \left(\frac{1}{3}\right)x_0, \left(\frac{1}{6}\right)x_0, \left(\frac{1}{8}\right)x_0, \left(\frac{1}{16}\right)x_0, ... \right\}$ with $y_n = Ax_n = \left(\frac{1}{3}\right)\left(\frac{3}{8}\right)^{\frac{n-1}{2}} x_0$ or $\left(\frac{1}{6}\right)\left(\frac{3}{8}\right)^{\frac{n}{2}-1} x_0$ according as n is odd or even and $O(x_0)$ converges to $0 \in X$. Thus X is orbitally complete at x_0.

The following is our result, which was presented in the National Conference on Applications of Mathematics in Engineering, Physical and Life Sciences, Tirupaty (7-9 December, 2012):

Theorem 2.1. *Let* S, T *and* A *be self-maps on* X *satisfying the inclusions* (1.3) *and the inequality*

$$d(Sx, Ty) \le \psi(d(Ax, Ay), d(Ax, Sx), d(Ay, Ty), d(Ax, Ty), d(Ay, Sx)) \quad \text{for all} \quad x, y \in X, \tag{2.2}$$

where ψ *is a nondecreasing and usc contractive modulus.*
Given $x_0 \in X$, *suppose that*

(c) *the pair* (S, T) *is a.r. at* x_0 *with respect to* A

(d) *any one of* $S(X), T(X)$ *and* $A(X)$ *is orbitally complete at* x_0.

Then S, T *and* A *will have a common coincidence point. Further, if*

(e) *either* (A, S) *or* (A, T) *is a weakly compatible pair,* *then A, S and T will have a unique common fixed point.*

Proof. Let $x_0 \in X$. Using the inclusions (1.3), we can inductively find points $x_1, x_2, ..., x_n, ...$ in X to define $O(x_0) = \langle y_n \rangle_{n=1}^{\infty}$ with the choice (2.1).

We show that $\langle y_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence. Suppose that it is not Cauchy. Then for some $\epsilon > 0$, we choose sequences $\langle 2m_k \rangle_{k=1}^{\infty}$ and $\langle 2m_k \rangle_{k=1}^{\infty}$ of even integers such that $d(y_{2m_k}, y_{2n_k}) \geq \epsilon$ for $2m_k > 2n_k > k$ for all k. Let $2m_k$ be the smallest even integer with this property so that $d(y_{2m_k-2}, y_{2n_k}) \leq \epsilon$.

Using the triangle inequality of d and asymptotic regularity (c), above inequalities give

$$\lim_{n\to\infty} d(y_{2m_k}, y_{2n_k}) = \lim_{n\to\infty} d(y_{2m_k}, y_{2n_k+1}) = \lim_{n\to\infty} d(y_{2m_k+1}, y_{2n_k+1}) = \lim_{n\to\infty} d(y_{2m_k+1}, y_{2n_k+2}). \tag{2.3}$$

Since ψ is nondecreasing, from the inequality (2.2) we have

$$d(Sx_{2m_k}, T_{2n_k+1}) \leq \psi(\max\{d(Ax_{2m_k}, A_{2n_k+1}), d(Ax_{2m_k}, Sx_{2m_k}), d(Ax_{2n_k+1}, Tx_{2n_k+1}), d(Ax_{2m_k}, Tx_{2n_k+1}), d(Ax_{2n_k+1}, Sx_{2m_k})\}).$$

Proceeding the limit as $k \to \infty$ in this, then using (c), (2.3) and the upper semicontinuity of ψ, we get $0 < \epsilon \leq \psi(\max\{0+\epsilon, 0, 0, 0+\epsilon, \epsilon\}) = \psi(\epsilon) < \epsilon$. This contradiction establishes that $\langle y_n \rangle_{n=1}^{\infty}$ must be a Cauchy sequence and its subsequences $\langle y_{2n} \rangle_{n=1}^{\infty}$ and $\langle y_{2n+1} \rangle_{n=1}^{\infty}$ are also Cauchy.

Case (i): $A(X)$ is orbitally complete at x_0.
Then

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = z = Au \quad \text{for some} \quad u \in X. \tag{2.4}$$

Thus $\langle y_{2n} \rangle_{n=1}^{\infty}$ is a subsequence of the Cauchy sequence $\langle y_n \rangle_{n=1}^{\infty}$ converging to z. Hence $\langle y_n \rangle_{n=1}^{\infty}$ also converges to $z = Au$.

But then, (2.2) with $x = u$ and $y = x_{2n+1}$ gives

$$d(Su, Tx_{2n+1}) \leq \psi(\max\{d(Au, A_{2n+1}), d(Au, Su), d(Ax_{2n+1}, T_{2n+1}), d(Au, Tx_{2n+1}), d(Ax_{2n+1}, Su)\}).$$

Since ψ is usc, applying the limit as $n \to \infty$, this implies

$$d(Su, z) \leq \psi(\max\{d(Au, z), d(Au, Su), 0, 0, d(Su, z)\}) = \psi(d(Su, z))$$

or $Sz = z$. That is

$$Su = Au = z. \tag{2.5}$$

Writing $x = y = z$ in (2.2) and using (2.5), it follows that

$$d(Su, Tu) \leq \psi(\max\{d(Au, Su), d(Au, Tu)\}) = \psi(d(Su, Tu))$$

or $d(Su, Tu) = 0$ so that $Su = Tu$. Thus

$$Su = Au = Tu = z. \tag{2.6}$$

Thus u is a common coincidence point for A, S and T and z, their common point of coincidence.

Now with $x = y = z$, (2.2) again implies

$$d(Sz, Tz) \leq \psi(\max\{d(Az, Sz), d(Az, Tz)\}). \tag{2.7}$$

If (A, S) is weakly compatible, from (2.6) we get $Az = Sz$ and hence (2.7) yields $d(Sz, Tz) = \psi(d(Sz, Tz))$ or $d(Sz, Tz) = 0$ so that $Sz = Tz$.

Similarly if (A, T) is weakly compatible, from (2.6) we get $Az = Tz$, which together with (2.7) implies that $Sz = Tz$. Thus

$$Sz = Az = Tz, \tag{2.8}$$

whenever (e) holds good.

Finally, writing $x = z$ and $y = x_{2n-1}$ in (2.2), we see that

$$d(Sz, Tx_{2n-1}) \leq \psi(\max\{d(Az, Ax_{2n-1}), d(Az, Sz), d(Ax_{2n-1}, Tx_{2n-1}), d(Az, Tx_{2n-1}), d(Ax_{2n-1}, Sz)\}).$$

In the limit as $n \to \infty$, this along with (2.8) will give

$$d(Sz, z) \leq \psi(\max\{d(Sz, z), 0, 0, d(Sz, z), d(z, Sz)\}) = \psi(d(z, Sz))$$

so that $d(Sz, z) = 0$ or $Sz = z$. This again in view of (2.8) reveals that z is a common fixed point of A, S and T.

Case (ii): Let $S(X)$ be orbitally complete at x_0. Then $\langle y_n \rangle_{n=1}^{\infty}$ converges to $z \in S(X) \subset A(X)$. The conclusion follows from Case (i).

Case (iii): Let $T(X)$ be orbitally complete at x_0. Then $\langle y_n \rangle_{n=1}^{\infty}$ converges to $z \in T(X)$ and hence $z \in A(X)$, in view of (1.3). Again the conclusion follows from Case (i).

Uniqueness of the common fixed point follows directly from (2.2). □

Remark 2.2. Let $x_0 \in X$ be arbitrary and $r_n = d(y_{n-1}, y_n)$ for $n \geq 2$.
We now show that (1.4) of Theorem 1.1 implies the condition (c) of Theorem 2.1.

In fact, by a routine procure, it follows that

$$r_n \leq \psi(\max\{r_{n-1}, r_n\}) \quad \text{for} \quad n \geq 2. \tag{2.9}$$

If $r_m > r_{m-1}$ for some $m \geq 2$, then the choice of ψ and (2.9) would give a contradiction that $0 < r_m \leq \psi(r_m) < r_m$. Therefore $r_n \leq r_{n-1}$ for all $n \geq 2$. Using this again in (2.9), we get

$$r_n \leq \psi(r_{n-1}) \quad \text{for} \quad n = 2, 3, 4, ... \tag{2.10}$$

Repeated application of (2.10) and the choice of ψ will imply that $r_1 \geq r_2 \geq r_3 \geq ... \geq r_{n-1} \geq r_n \geq ...$, where $r_n \geq 0$ for all n. Hence $\lim\limits_{n\to\infty} r_n = a$ for some $a \geq 0$. Then employing the limit as $n \to \infty$ in (2.10) and the upper semicontinuity of ψ, we get $a \leq \psi(a)$ so that $a = 0$, which the condition (c).

Further if ψ is nondcreasing, we see that the right hand side of (1.4) is less than or equal to the the right hand side of (2.2) due to the fact that $\frac{a+b}{2} \leq \max\{a, b\}$ for any $a \geq 0$ and $b \geq 0$. That is, (2.2) is weaker than (1.4) if ψ is nondcreasing.

Moreover, (a) of Theorem 1.1 implies (d) of Theorem 2.1, in vew of Remark 2.1. Therefore, a unique common fixed point of S, T and A can be ensured by Theorem 2.1. Thus Theorem 1.1 follows as a particular case of Theorem 2.1, when ψ is nondecreasing.

Our proof requires weak compatibility of only one of the pairs (A, S) and (A, T), where as Theorem 1.1 required weak compatibility of both the pairs.

Corollary 2.1. *Let S, T and A be self-maps on X satisfying inclusions (1.3) and the inequality*

$$d(Sx, Ty) \leq \omega(d(Ax, Ay), d(Ax, Sx), d(Ay, Ty), d(Ax, Ty), d(Ay, Sx)) \quad \textit{for all} \quad x, y \in X, \tag{2.11}$$

where $\omega : [0, \infty)^5 \to [0, \infty)$ is nondecreasing and usc in each coordinate variable with $\omega(t, t, t, t, t) < t$ for $t > 0$. Given $x_0 \in X$, suppose that (c) holds good and

(f) X is orbitally complete at x_0,

(g) any one of S, T and A is onto.

If either (A, S) or (A, T) is compatible, then S, T and A will have a unique common fixed point.

Proof. We write $\psi(t) = \omega(t, t, t, t, t)$ for $t \geq 0$. Then (2.11) is a particular case of (2.2), and the conditions (f) and (g) imply (d). Also every compatible pair is weakly compatible. Therefore A, S and T will have a unique common fixed point, by Theorem 2.1. □

Remark 2.3. *When (g) is replaced by the condition that A is orbitally continuous at x_0 in the sense that A is continuous at every point of some $O(x_0)$, Corollary 2.1 gives Theorem B of the first author ([4], p.46).*

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Periodic solutions of nonlinear finite difference systems with time delays

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Abstract

In this paper a coupled system of nonlinear finite difference equations corresponding to a class of periodic-parabolic systems with time delays and with nonlinear boundary conditions in a bounded domain is investigated. Using the method of upper-lower solutions two monotone sequences for the finite difference system are constructed. Existence of maximal and minimal periodic solutions of coupled system of finite difference equations with nonlinear boundary conditions is also discussed. The proof of existence theorem is based on the method of upper-lower solutions and its associated monotone iterations. It is shown that the sequence of iterations converges monotonically to unique solution of the nonlinear finite difference system with time delays under consideration.

Keywords: Periodic solution, periodic parabolic system, finite difference equation, upper and lower solution, quasimonotone nondecreasing function.

2010 MSC: 35B10, 35K61, 65N06.

1 Introduction

Many researchers investigated periodic solutions of parabolic boundary value problems. Their work is either on scalar periodic-parabolic boundary value problem [1,2] or to system of reaction-diffusion type of equations with specific reaction functions [3,4]. Much of the work for parabolic initial boundary value problem with time delay [5,6,7] and without time delay [1,2,4,10] is found in literature. The recent work of [12] is on the periodic parabolic system with time delays under linear boundary conditions. Also Pao [8,9] has discussed system of periodic parabolic equations with nonlinear boundary conditions with and without time delays. Recently Pao [7] investigated some numerical aspect of the class of coupled nonlinear systems with time delays. Most of the works in the literature are devoted to the qualitative analysis of the system and dynamics of the system [6]. In this paper we give a treatment to a coupled system of finite difference equations of periodic-parabolic system with time delay and with nonlinear boundary conditions and obtain the results which are motivated by earlier results of Pao [6,7,8].

2 Finite Difference Equations

Consider the system which consists of an arbitrary number of parabolic equations in a bounded domain Ω in $\Re^p$ (p = 1,2,3, . . .) with boundary $\partial\Omega$ and with fixed period T > 0 in the form.

$$\text{(2.1)} \qquad \begin{cases} \frac{\partial u^{(l)}}{\partial t} - L^{(l)} u^{(l)} = f^{(l)}(x,t,\mathbf{u},\mathbf{u}_\tau) \ , \quad x \in \Omega \ , \quad t > 0 \\ B^{(l)} u^{(l)} = g^{(l)}(x,t,\mathbf{u}), \qquad x \in \partial\Omega \ , \quad t > 0 \\ u^{(l)}(x,t) = u^{(l)}(x,t+T), \qquad x \in \Omega, -\tau_l \le t \le 0, \end{cases}$$

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where $\mathbf{u} = \mathbf{u}(x,t) = \big(u^{(1)}(x,t), u^{(2)}(x,t), ..., u^{(N)}(x,t)\big)$

$$\mathbf{u}_\tau = \mathbf{u}_\tau(x,t) = \Big(u^{(1)}(x,t-\tau_1), u^{(2)}(x,t-\tau_2), ..., u^{(N)}(x,t-\tau_N)\Big)$$

for some time delays $\tau_1, \tau_2, ..., \tau_N > 0$ and for each l = 1,2, . . ., N ; $L^{(l)}u^{(l)}$ and $B^{(l)}u^{(l)}$ are given by

$$L^{(l)}u^{(l)} = \nabla \cdot (D^{(l)} \nabla u^{(l)}) + V^{(l)} \cdot \nabla u^{(l)} \quad , \quad B^{(l)}u^{(l)} = \alpha^{(l)} \frac{\partial u^{(l)}}{\partial \nu} + \beta^{(l)} u^{(l)}$$

with $\frac{\partial}{\partial \nu}$ denoting outward normal derivative on $\partial\Omega$. It is assumed that the diffusion coefficient $D^{(l)} = D^{(l)}(x,t) > 0$ and the convection coefficient $V^{(l)} = (V_1^{(l)}, V_2^{(l)}, ..., V_p^{(l)})$ *of* $L^{(l)}$ where $V_\nu^{(l)} = V_\nu^{(l)}(x,t)$ *for* ν = 1,2,. . .,p are continuous on $\overline{D}_T = \overline{\Omega} \times [0,T]$ for every finite T > 0. The coefficients $\alpha^{(l)}$ and $\beta^{(l)} = \beta^{(l)}(x,t)$ *of* $B^{(l)}$ are continuous on $S_T = \partial\Omega \times [0,T]$ with either $\alpha^{(l)} = 0$, $\beta^{(l)} > 0$ (Dirichlet condition) or $\alpha^{(l)} = 1$, $\beta^{(l)} \geq 0$ (Neumann or Robin condition) where $\overline{\Omega} = \Omega \cup \partial\Omega$.

It is assumed that $f^{(l)}$, $g^{(l)}$ *and* $u^{(l)}$ are continuous functions in their respective domains and $f^{(l)}(\cdot, \mathbf{u}, \mathbf{u}_\tau)$, $g^{(l)}(\cdot, \mathbf{u})$ are in general nonlinear in $\mathbf{u}$ and $\mathbf{u}_\tau$; and satisfy the conditions in hypothesis (H_2) of Section 3.

Let $\mathbf{x}_j = (x_{j_1}, x_{j_2}, ..., x_{j_p})$ be an arbitrary mesh point in $\overline{\Omega}$, where j = $(j_1, j_2, ..., j_p)$ is a multiple index with j_ν = 1,2, . . ., M_ν and for each ν = 1,2, . . . , p , M_ν is the total number of mesh points in the x_ν direction. Denote by Ω_p, Λ_p and $Q_0^{(l)}$ the sets of mesh points in $\Omega, \Omega \times (0,\infty)$ *and* $\Omega \times [-\tau_l, 0]$ respectively. Similarly denote by $\partial\Omega_p, S_p$ and $Q_p^{(l)}$ the sets of mesh points in $\partial\Omega, \partial\Omega \times [0,\infty)$ *and* $\overline{\Omega} \times [-\tau_l, \infty)$ respectively. Further let $Q_p = Q_p^{(1)} \times Q_p^{(2)} \times ... \times Q_p^{(N)}$. The set of all mesh points in $\overline{\Omega}$ and $\overline{\Omega} \times [0,\infty)$ are denoted by $\overline{\Omega_p}$ and $\overline{\Lambda}_p$ respectively. It is assumed that, the domain Ω is connected. Let $k_n = t_n - t_{n-1}$ be the time increment and h_ν the spatial increment in the x_ν direction. For each l = 1, 2, . . . , N we choose k_n such that $\tau_l = k_1 + k_2 + ... + k_{s_l}$ for some integer $s_l > 0$.

Define

$$u_{j,n}^{(l)} = u^{(l)}(x_j, t_n) \qquad , \quad \mathbf{u}_{j,n} = (u_{j,n}^{(1)}, u_{j,n}^{(2)}, ..., u_{j,n}^{(N)}),$$
$$u_{j,n-s_l}^{(l)} = u^{(l)}(x_j, t_{n-s_l}) \qquad , \quad \mathbf{u}_{j,n-s} = (u_{j,n-s_1}^{(1)}, u_{j,n-s_2}^{(2)}, ..., u_{j,n-s_N}^{(N)}),$$
$$f^{(l)}(\mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) = f^{(l)}(\mathbf{x}_j, t_n, \mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) \quad , \quad g^{(l)}(\mathbf{u}_{j,n}) = g^{(l)}(\mathbf{x}_j, t_n, \mathbf{u}_{j,n}).$$

Define the standard central difference operators as follows:

$$\triangle_{u_{j,n}}^{(\nu)} = h_\nu^{-2} \left[u(\mathbf{x}_j + h_\nu e_\nu, t_n) - 2u(\mathbf{x}_j, t_n) + u(\mathbf{x}_j - h_\nu e_\nu, t_n)\right]$$
$$\delta_{u_{j,n}}^{(\nu)} = 2h_\nu^{-1} \left[u(\mathbf{x}_j + h_\nu e_\nu, t_n) - u(\mathbf{x}_j - h_\nu e_\nu, t_n)\right]$$

where e_ν is the unit vector in $\Re^p$ with ν^{th} component 1 and zero elsewhere. Approximating the parabolic system in (2.1) by the nonlinear finite difference system, we have

$$(2.2) \qquad \begin{cases} k_n^{-1}(u_{j,n}^{(l)} - u_{j,n-1}^{(l)}) - L^{(l)}u_{j,n}^{(l)} = f^{(l)}(\mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) \quad in \ \Lambda_p \\ \\ B^{(l)}[u_{j,n}^{(l)}] = g^{(l)}(\mathbf{u}_{j,n}) \qquad on \ S_p, \\ \\ u_{j,n}^{(l)} = u_{j,n+k}^{(l)} \qquad in \ Q_0^{(l)} \ \ , \ l = 1,2,,...,N \end{cases}$$

where

$$L^{(l)}u_{j,n}^{(l)} = \sum_{\nu=1}^p \left(D_{j,n}^{(l)} \triangle_{u_j,n}^{(\nu)} + (V_{j,n}^{(l)})_\nu \delta^{(\nu)} u_{j,n}^{(l)}\right), D_{j,n}^{(l)} = D^{(l)}(x,t) \ , \ \ (V_{j,n}^{(l)})_\nu = (V^{(l)}(x,t))_\nu$$

$$B^{(l)}[u_{j,n}^{(l)}] = \alpha^{(l)}(\mathbf{x}_j) \, |\mathbf{x}_j - \hat{x_j}|^{-1} \left[u^{(l)}(\mathbf{x}_j, t_n) - u^{(l)}(\hat{x_j}, t_n)\right] + \beta^{(l)}(\mathbf{x}_j, t_n) u^{(l)}(\mathbf{x}_j, t_n),$$

and $u_{j,n+k}^{(l)} = u^{(l)}(\mathbf{x}_j, t_{n+T}),\ T > 0$, In the boundary conditions $\hat{x_j}$ is a suitable point in Ω_p and $|x_j - \hat{x_j}|$ is the distance between x_j and $\hat{x_j}$. We define upper and lower solutions for the discrete problem (2.2) in the following section.

3 Upper and Lower Solutions

Definition 3.1. *A function $\tilde{\mathbf{u}}_{j,n} \equiv (\tilde{u}^{(1)}_{j,n}, \tilde{u}^{(2)}_{j,n}, ..., \tilde{u}^{(N)}_{j,n})$ in Q_p is called an upper solution of (2.2) if*

$$(3.1) \qquad \begin{cases} k_n^{-1}(\tilde{u}^{(l)}_{j,n} - \tilde{u}^{(l)}_{j,n-1}) - L^{(l)}\tilde{u}^{(l)}_{j,n} \geq f^{(l)}_{j,n}(\tilde{\boldsymbol{u}}_{j,n}, \tilde{\boldsymbol{u}}_{j,n-s}) & in\ \Lambda_p\ , \\ B^{(l)}[\tilde{u}^{(l)}_{j,n}] \geq g^{(l)}_{j,n}(\tilde{\boldsymbol{u}}_{j,n}) \quad on\ S_p\ , \\ \tilde{u}^{(l)}_{j,n} \geq \tilde{u}^{(l)}_{j,n+k} \quad in\ Q^{(l)}_0\ . \end{cases}$$

Similarly $\hat{\boldsymbol{u}}_{j,n} \equiv (\hat{u}^{(1)}_{j,n}, \hat{u}^{(2)}_{j,n}, ..., \hat{u}^{(N)}_{j,n})$ in Q_p is called a lower solution of (2.2) if it satisfies the inequalities in (3.1) in reverse order.

Suppose $\tilde{\mathbf{u}}_{j,n}, \hat{\mathbf{u}}_{j,n}$ exist and $\tilde{\mathbf{u}}_{j,n} \geq \hat{\mathbf{u}}_{j,n}$.
Define

$$\mathscr{S}^{(1)} = \{\mathbf{u}_{j,n} \in Q_p\ ,\ \hat{\mathbf{u}}_{j,n} \leq \mathbf{u}_{j,n} \leq \tilde{\mathbf{u}}_{j,n}\}\ ,$$

$$\mathscr{S}^{(2)} = \{\mathbf{v}_{j,n} \in Q_p\ ,\ \hat{\mathbf{u}}_{j,n-s} \leq \mathbf{v}_{j,n-s} \leq \tilde{\mathbf{u}}_{j,n-s}\}\ ,$$

$$\mathscr{S} = \{(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \in Q_p \times Q_p\ ;\ \hat{\mathbf{u}}_{j,n} \leq \mathbf{u}_{j,n} \leq \tilde{\mathbf{u}}_{j,n}\ ,\ \hat{\mathbf{u}}_{j,n-s} \leq \mathbf{v}_{j,n-s} \leq \tilde{\mathbf{u}}_{j,n-s}\}\ .$$

Also define

$$\mathbf{f}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) = \Big(f^{(1)}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n})\ ,\ f^{(2)}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n})\ ,\ ...\ ,\ f^{(N)}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n})\Big)\ ,$$

$$\mathbf{g}_{j,n}(\mathbf{u}_{j,n}) = \Big(g^{(1)}_{j,n}(\mathbf{u}_{j,n})\ ,\ g^{(2)}_{j,n}(\mathbf{u}_{j,n})\ ,\ ...\ ,\ g^{(N)}_{j,n}(\mathbf{u}_{j,n})\Big)\ .$$

Definition 3.2. *A function $\boldsymbol{f}_{j,n}(\boldsymbol{u}_{j,n}, \boldsymbol{v}_{j,n})$ is said to be quasi-monotone nondecreasing in $\mathscr{S}$ if for each* l *and each $(\boldsymbol{u}_{j,n}, \boldsymbol{v}_{j,n}) \in \mathscr{S}$, $f^{(l)}_{j,n}(\boldsymbol{u}_{j,n}, \boldsymbol{v}_{j,n})$ is nondecreasing in $\boldsymbol{u}_{j,n} = (u^{(1)}_{j,n}, u^{(2)}_{j,n}\ ,\ ...\ ,\ u^{(N)}_{j,n})$ for all $u^{(l)}_{j,n} \neq u^{(m)}_{j,n}$ and nondecreasing in $\boldsymbol{v}_{j,n} = (v^{(1)}_{j,n}, v^{(2)}_{j,n}\ ,\ ...\ ,\ v^{(N)}_{j,n})$ for all $v^{(m)}_{j,n}$, m = 1,2 , . . . , N .*

We now make the following hypothesis

(H_1) For each $l = 1\ ,\ 2\ ,\ ...\ ,$ N the coefficients $D^{(l)}$, $V^{(l)}$ of $L^{(l)}$ and the functions $f^{(l)}_{j,n}(\cdot)$, $g^{(l)}_{j,n}(\cdot)$ and $\beta(x_j, t_n)$ are all k-periodic in n.

(H_2) $\mathbf{f}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n})$ and $\mathbf{g}_{j,n}(\mathbf{u}_{j,n})$ are quasi-monotone nondecreasing C^1 -functions of $(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \in \mathscr{S}$ and $\mathbf{u}_{j,n} \in \mathscr{S}^{(l)}$ respectively.

The hypothesis (H_2) is equivalent to the condition

$\frac{\partial f^{(l)}_{j,n}}{\partial u^{(m)}}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \geq 0$ for $m \neq l$, $\quad \frac{\partial f^{(l)}_{j,n}}{\partial v^{(m)}}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \geq 0$ for m = 1 ,2 , .., N .

where $(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \in \mathscr{S}$.

and $\frac{\partial g^{(l)}_{j,n}}{\partial v^{(m)}}(\mathbf{u}_{j,n} \geq 0$ for $m \neq l$ where $\mathbf{u}_{j,n} \in \mathscr{S}^{(1)}$ for l , m = 1 ,2,.. , N .

The subsets $\mathscr{S}$, $\mathscr{S}^{(1)}$ *and* $\mathscr{S}^{(2)}$ are the sectors between the pairs of upper and lower solutions.

Assume that the quasi-monotone condition in (H_2) holds in the above subsets $\mathscr{S}$ *and* $\mathscr{S}^{(1)}$. Let

$$(3.2)\qquad \begin{cases} \gamma_{j,n}^{(l)} \geq Max\left\{ -\frac{\partial f_{j,n}^{(l)}}{\partial \mathbf{u}^{(l)}}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n})\ ,\ (j,n) \in \overline{\Lambda}_p\ ,\ (\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \in \mathscr{S} \right\} \\ \sigma_{j,n}^{(l)} \geq Max\left\{ -\frac{\partial g_{j,n}^{(l)}}{\partial u^{(l)}}(\mathbf{u}_{j,n})\ ,\ (j,n) \in \overline{\Lambda}_p\ ,\ \mathbf{u}_{j,n} \in \mathscr{S}^{(1)} \right\} \end{cases}$$

Define

$$(3.3)\qquad \begin{cases} \mathscr{L}^{(l)}[u_{j,n}^{(l)}] = k_n^{-1}(u_{j,n}^{(l)} - u_{j,n-1}^{(l)}) - L^{(l)} u_{j,n}^{(l)} + \gamma_{j,n}^{(l)} u_{j,n}^{(l)} \\ \mathscr{B}^{(l)}[u_{j,n}^{(l)}] = B^{(l)}[u_{j,n}^{(l)}] + \sigma_{j,n}^{(l)} u_{j,n}^{(l)} \\ F_{j,n}^{(l)}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) = \gamma_{j,n}^{(l)} u_{j,n}^{(l)} + f_{j,n}^{(l)}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \\ G_{j,n}^{(l)}(\mathbf{u}_{j,n}) = \sigma_{j,n}^{(l)} u_{j,n}^{(l)} + g_{j,n}^{(l)}(\mathbf{u}_{j,n})\ ,\ l = 1, 2, ..., N. \end{cases}$$

By hypothesis (H_2) , $F^{(l)}$ *and* $G^{(l)}$ possess the property,

$$(3.4)\qquad \begin{cases} F_{j,n}^{(l)}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \geq F_{j,n}^{(l)}(\mathbf{u}'_{j,n}, \mathbf{v}'_{j,n}) \\ \qquad when\ \ (\hat{\mathbf{u}}_{j,n}, \hat{\mathbf{v}}_{j,n}) \leq (\mathbf{u}'_{j,n}, \mathbf{v}'_{j,n}) \leq (\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \leq (\tilde{\mathbf{u}}_{j,n}, \tilde{\mathbf{v}}_{j,n})\ , \\ G_{j,n}^{(l)}(\mathbf{u}_{j,n}) \geq G_{j,n}^{(l)}(\mathbf{u}'_{j,n})\ , \\ \qquad when\ \ \hat{\mathbf{u}}_{j,n} \leq \mathbf{u}'_{j,n} \leq \mathbf{u}_{j,n} \leq \tilde{\mathbf{u}}_{j,n}\ , l = 1, 2, ..., N. \end{cases}$$

Using either $\mathbf{u}_{j,n}^{(0)} = \tilde{\mathbf{u}}_{j,n}$ *or* $\mathbf{u}_{j,n}^{(0)} = \hat{\mathbf{u}}_{j,n}$ as the initial iteration we construct a sequence $\left\{\mathbf{u}_{j,n}^{(m)}\right\} = \left\{(u_{j,n}^{(1)})^m\ ,\ (u_{j,n}^{(2)})^m\ ,\ .\ .\ .\ , (u_{j,n}^{(N)})^m\right\}$ from the linear discrete system

$$(3.5)\qquad \begin{cases} \mathscr{L}^{(l)}[u_{j,n}^{(l)}]^m = F_{j,n}^{(l)}(\mathbf{u}_{j,n}^{(m-1)}, \mathbf{u}_{j,n-s}^{(m-1)}) & in\ \ \Lambda_p \\ \mathscr{B}^{(l)}[u_{j,n}^{(l)}]^m = G_{j,n}^{(l)}(\mathbf{u}_{j,n}^{(m-1)}) & on\ \ S_p \\ [u_{j,n}^{(l)}]^m = (u_{j,n+k}^{(l)})^{m-1} & in\ \ Q_0^{(l)}\ , \end{cases}$$

where n = 0 , -1 , -2, . . . , - s_l , $l = 1$,2 , . . . , N , k > 0 , and m = 1 ,2 , . . .

From the above , it is clear that, the sequence $\left\{\mathbf{u}_{j,n}^{(m)}\right\}$ is well defined. Denote this sequence by $\left\{\overline{\mathbf{u}}_{j,n}^{(m)}\right\}$ *if* $\mathbf{u}_{j,n}^{(0)} = \tilde{\mathbf{u}}_{j,n}$ *and by* $\left\{\underline{\mathbf{u}}_{j,n}^{(m)}\right\}$ *if* $\mathbf{u}_{j,n}^{(0)} = \hat{\mathbf{u}}_{j,n}$.

Now we prove the monotone property of these sequences.

Lemma 3.1. *The sequences* $\left\{\overline{\boldsymbol{u}}^{(m)}_{j,n}\right\}$, $\left\{\underline{\boldsymbol{u}}^{(m)}_{j,n}\right\}$ *possess the monotone property,*

$$\hat{\boldsymbol{u}}_{j,n} \leq \underline{\boldsymbol{u}}^{(m)}_{j,n} \leq \underline{\boldsymbol{u}}^{(m+1)}_{j,n} \leq \overline{\boldsymbol{u}}^{(m+1)}_{j,n} \leq \overline{\boldsymbol{u}}^{(m)}_{j,n} \leq \tilde{\boldsymbol{u}}_{j,n} \text{ on } Q_p \tag{3.6}$$

where $m = 1, 2, \ldots.$

Proof. Let $\left[\overline{w}^{(l)}_{j,n}\right]^{(0)} = \left[\overline{u}^{(l)}_{j,n}\right]^{(0)} - \left[\overline{u}^{(l)}_{j,n}\right]^{(1)}$ where $\left[\overline{u}^{(l)}_{j,n}\right]^{(0)} = \tilde{u}^{(l)}_{j,n}$.

By (3.3), (3.5) and (3.1) we have

$$\mathscr{L}^{(l)}\left[\overline{w}^{(l)}_{j,n}\right]^{(0)} = k_n^{-1}(\tilde{u}^{(l)}_{j,n} - \tilde{u}^{(l)}_{j,n-1}) - L^{(l)}\tilde{u}^{(l)}_{j,n} + \gamma^{(l)}_{j,n}\tilde{u}^{(l)}_{j,n} - \left[\gamma^{(l)}_{j,n}\left(\overline{u}^{(l)}_{j,n}\right)^{(0)} + f^{(l)}_{j,n}\left(\overline{\mathbf{u}}^{(0)}_{j,n}, \overline{\mathbf{u}}^{(0)}_{j,n-s}\right)\right]$$

$$= k_n^{-1}(\tilde{u}^{(l)}_{j,n} - \tilde{u}^{(l)}_{j,n-1}) - L^{(l)}\tilde{u}^{(l)}_{j,n} - f^{(l)}_{j,n}(\tilde{\mathbf{u}}^{(l)}_{j,n}\ ,\ \tilde{\mathbf{u}}^{(l)}_{j,n-s}) \geq 0 \ \text{ in } \Lambda_p$$

$$\mathscr{B}^{(l)}[\overline{w}^{(l)}_{j,n}]^{(0)} = \left[B^{(l)}[\tilde{u}^{(l)}_{j,n}] + \tilde{u}^{(l)}_{j,n}\sigma^{(l)}_{j,n}\right] - \left[\sigma^{(l)}_{j,n}\left(\overline{u}^{(l)}_{j,n}\right)^{(0)} + g^{(l)}_{j,n}\left(\overline{\mathbf{u}}^{(0)}_{j,n}\right)\right]$$

$$= B^{(l)}[\tilde{u}^{(l)}_{j,n}] - g^{(l)}_{j,n}\left(\tilde{\mathbf{u}}^{(0)}_{j,n}\right) \geq 0 \ \text{ on } S_p$$

$$[\overline{w}^{(l)}_{j,n}]^{(0)} = \tilde{u}^{(l)}_{j,n} - \tilde{u}^{(l)}_{j,n+k} \geq 0 \ \text{ in } Q^{(l)}_0.$$

By positivity Lemma of [11] for finite difference equations of parabolic initial boundary value problem

$$[\overline{w}^{(l)}_{j,n}]^{(0)} \geq 0 \ \text{ on } Q^{(l)}_p$$

Thus $(\overline{u}^{(l)}_{j,n})^{(0)} \geq [\overline{u}^{(l)}_{j,n}]^{(1)}$ *on* $Q^{(l)}_p$. This yields $\overline{\mathbf{u}}^{(0)}_{j,n} \geq \overline{\mathbf{u}}^{(1)}_{j,n}$ *on* Q_p.

A similar argument using the property of a lower solution gives $\underline{\mathbf{u}}^{(1)}_{j,n} \geq \underline{\mathbf{u}}^{(0)}_{j,n}$
Put $[\overline{w}^{(l)}_{j,n}]^{(1)} = [\overline{u}^{(l)}_{j,n}]^{(1)} - [\underline{u}^{(l)}_{j,n}]^{(1)}$. Then by (3.4) and (3.5) , we have

$$\mathscr{L}^{(l)}[\overline{w}^{(l)}_{j,n}]^{(1)} = F^{(l)}_{j,n}\left(\overline{\mathbf{u}}^{(0)}_{j,n}, \overline{\mathbf{u}}^{(0)}_{j,n-s}\right) - F^{(l)}_{j,n}\left(\underline{\mathbf{u}}^{(0)}_{j,n}, \underline{\mathbf{u}}^{(0)}_{j,n-s}\right) \geq 0 \ \text{ in } \Lambda_p$$

$$\mathscr{B}^{(l)}[\overline{w}^{(l)}_{j,n}]^{(1)} = G^{(l)}_{j,n}\left(\overline{\mathbf{u}}^{(0)}_{j,n}\right) - G^{(l)}_{j,n}\left(\underline{\mathbf{u}}^{(0)}_{j,n}\right) \geq 0 \ \text{ on } S_p$$

$$[\overline{w}^{(l)}_{j,n}]^{(1)} = [\overline{u}^{(l)}_{j,n+k}]^{(0)} - [\underline{u}^{(l)}_{j,n+k}]^{(0)} \geq 0 \ \text{ on } Q^{(l)}_0.$$

It follows again from positivity lemma of [11]that $[\overline{w}^{(l)}_{j,n}]^{(1)} \geq 0.$

i.e. $(\overline{u}^{(l)}_{j,n})^{(1)} \geq (\underline{u}^{(l)}_{j,n})^{(1)}$ *on* $Q^{(l)}_p$. This gives $\overline{\mathbf{u}}^{(1)}_{j,n} \geq \underline{\mathbf{u}}^{(1)}_{j,n}$.

The above conclusions show that

$$\overline{\mathbf{u}}^{(0)}_{j,n} \geq \overline{\mathbf{u}}^{(1)}_{j,n} \geq \underline{\mathbf{u}}^{(1)}_{j,n} \geq \underline{\mathbf{u}}^{(0)}_{j,n} \ \text{ on } Q_p$$

The monotone property (3.6)follows by an induction argument as in [11]. □

It is clear from the monotone property (3.6) that the point-wise limits

$$\lim_{m\to\infty} \overline{\mathbf{u}}^{(m)}_{j,n} = \overline{\mathbf{u}}_{j,n} \text{ and } \lim_{m\to\infty} \underline{\mathbf{u}}^{(m)}_{j,n} = \underline{\mathbf{u}}_{j,n} \tag{3.7}$$

exist and satisfy the relation

(3.8) $\quad \hat{\mathbf{u}}_{j,n} \le \underline{\mathbf{u}}^{(m)}_{j,n} \le \underline{\mathbf{u}}^{(m+1)}_{j,n} \le \underline{\mathbf{u}}_{j,n} \le \overline{\mathbf{u}}_{j,n} \le \overline{\mathbf{u}}^{(m+1)}_{j,n} \le \overline{\mathbf{u}}^{(m)}_{j,n} \le \tilde{\mathbf{u}}_{j,n}$ *on* Q_p

Now we show that $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ are respectively maximal and minimal k-periodic solutions of (2.2).

Theorem 3.1. *Let $\hat{\mathbf{u}}_{j,n}$ and $\tilde{\mathbf{u}}_{j,n}$ be ordered lower and upper solutions of (2.2) and let hypothesis $(H_1),(H_2)$ be satisfied. Then the problem (2.2) has a maximal k-periodic solution $\overline{\mathbf{u}}_{j,n}$ and a minimal k-periodic solution $\underline{\mathbf{u}}_{j,n}$ in $\mathscr{S}^{(1)}$. Moreover the sequences $\left\{\overline{\mathbf{u}}^{(m)}_{j,n}\right\}$ and $\left\{\underline{\mathbf{u}}^{(m)}_{j,n}\right\}$ converge monotonically to $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ respectively and satisfy the relation (3.8). If in addition $\overline{\mathbf{u}}_{j,0} = \underline{\mathbf{u}}_{j,0}$ then $\overline{\mathbf{u}}_{j,n} = \underline{\mathbf{u}}_{j,n}$ $(= \mathbf{u}^*_{j,n})$, and $\mathbf{u}^*_{j,n}$ is the unique solution of (2.2) in $\mathscr{S}^{(1)}$.*

Proof. The sequence $\left\{\mathbf{u}^{(m)}_{j,n}\right\}$ constructed from the linear system (3.5) with initial iteration either upper or lower solution of (2.2) converge to $\overline{\mathbf{u}}_{j,n}$or $\underline{\mathbf{u}}_{j,n}$ according to initial iteration as $\tilde{\mathbf{u}}_{j,n}$ or $\hat{\mathbf{u}}_{j,n}$ respectively and using (3.7) it shows that both $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ satisfy the equations

$$(3.9)\qquad \begin{cases} \mathscr{L}^{(l)}[u^{(l)}_{j,n}] = F^{(l)}_{j,n}(\mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) & in \ \ \Lambda_p \ , \\ \mathscr{B}^{(l)}[u^{(l)}_{j,n}] = G^{(l)}_{j,n}(\mathbf{u}_{j,n},) & on \ \ S_p \ , \\ u^{(l)}_{j,n} = u^{(l)}_{j,n+k} & in \ \ \ Q^{(l)}_0 \ . \end{cases}$$

In view of (3.3) $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ are solutions of (2.2).

To show that $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ are k-periodic solutions we let $w^{(l)}_{j,n} = u^{(l)}_{j,n} - u^{(l)}_{j,n+k}$,

where $u^{(l)}_{j,n}$ stands for either $\overline{u}^{(l)}_{j,n}$ or $\underline{u}^{(l)}_{j,n}$, l =1, 2 , . . . , N.

By Hypothesis (H_1) and mean value theorem, we have

$$\begin{aligned} k_n^{-1}(w^{(l)}_{j,n} - w^{(l)}_{j,n-1}) - L^{(l)}_n w^{(l)}_{j,n} &= k_n^{-1}(u^{(l)}_{j,n} - u^{(l)}_{j,n-1}) - L^{(l)}_n u^{(l)}_{j,n} \\ &\quad - \left[k_n^{-1}(u^{(l)}_{j,n+k} - u^{(l)}_{j,n+k-1}) - L^{(l)}_{n+k} u^{(l)}_{j,n+k}\right] \\ &= f^{(l)}_{j,n}(\mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) - f^{(l)}_{j,n}(\mathbf{u}_{j,n+k}, \mathbf{u}_{j,n+k-s}) \\ &= \sum_{m=1}^{N} \left(\frac{\partial f^{(l)}_{j,n}}{\partial u^{(m)}}(\xi,\eta)\right) w^{(m)}_{j,n} + \sum_{m=1}^{N} \left(\frac{\partial f^{(l)}_{j,n}}{\partial v^{(m)}}(\xi,\eta)\right) w^{(m)}_{j,n-s} \ \ in \ \Lambda_p \end{aligned}$$

$$(3.10)\qquad \begin{aligned} B^{(l)}[w^{(l)}_{j,n}] &= B^{(l)}_n[u^{(l)}_{j,n}] - B^{(l)}_{n+k}[u^{(l)}_{j,n+k}] \\ &= g^{(l)}_{j,n}(\mathbf{u}_{j,n}) - g^{(l)}_{j,n}(\mathbf{u}_{j,n+k}) \\ &= \sum_{m=1}^{N} \left(\frac{\partial g^{(l)}_{j,n}}{\partial u^{(m)}}(\xi')\right) w^{(m)}_{j,n} \qquad \text{on} \quad S_p \ , \end{aligned}$$

and $\quad w^{(l)}_{j,n} = u^{(l)}_{j,n} - u^{(l)}_{j,n+k} = 0 \quad$ in $\quad Q^{(l)}_0$, l = 1,2, ..., N .

where $\xi = \xi_{j,n}$, $\xi' = \xi'_{j,n}$ and $\eta = \eta_{j,n}$ are some intermediate values in $\mathscr{S}^{(1)}$ and $\mathscr{S}^{(2)}$ respectively and

$$(3.11)\qquad w^{(l)}_{j,n-s} = u^{(l)}_{j,n-s_l} - u^{(l)}_{j,n+k-s_l}.$$

Let $s = min\{s_l \ , \ s_l > 0 \ \ , \ l = 1,2,...,N\} \ > \ 0$ and consider the system (3.10) in the domain $\Lambda_s = \Omega_p \times$ [0 , s] . From (3.10) and (3.11) we have $w^{(l)}_{j,n-s} = 0$ on Λ_s .

This implies that

$$(3.12)\qquad \begin{cases} k_n^{-1}(w_{j,n}^{(l)} - w_{j,n-1}^{(l)}) - L_n^{(l)} w_{j,n}^{(l)} = \sum_{m=1}^{N} b_{j,n}^{lm} w_{j,n}^{(m)} & in\ \Lambda_s\ , \\ B^{(l)}\left[w_{j,n}^{(l)}\right] = \sum_{m=1}^{N} c_{j,n}^{lm} w_{j,n}^{(m)} & on\ S_s\ , \\ w_{j,0}^{(l)} = 0 & in\ \Omega_p\ , \end{cases}$$

where

$$S_s = \partial\Omega_p \times [0,s]\ ,$$

$$b_{j,n}^{lm} = \frac{\partial f_{j,n}^{(l)}}{\partial u^{(m)}}(\xi_{j,n}, \eta_{j,n})\ ,$$

and $c_{j,n}^{lm} = \frac{\partial g_{j,n}^{(l)}}{\partial u^{(m)}}(\xi'_{j,n})$.

From the hypothesis (H_2) it is clear that $b^{lm} \geq 0$ and $c^{lm} \geq 0$ on Λ_s when $m \neq l$.
By Lemma 10.9.1 of [10] we obtain $w_{j,n}^{(l)} = 0$ *on* $\overline{\Lambda}_s = \overline{\Omega}_p \times [0,s]$. This shows that $w_{j,n-s}^{(l)} = 0$ *on* $\overline{\Lambda}_{2s} = \overline{\Omega}_p \times [0,2s]$ and so $w_{j,n}^{(l)}$ satisfies the equations in (3.12), in the domain Λ_{2s}. It follows again from Lemma 10.9.1 of [10] that $w_{j,n}^{(l)} = 0$ *on* $\overline{\Lambda}_{2s}$.

A continuation of the similar argument shows that $w_{j,n}^{(l)} = 0$ *on* $\overline{\Omega}_p \times [0, Ms]$ for every positive integer M. This proves the periodic property $\mathbf{u}_{j,n} = \mathbf{u}_{j,n+k}$ *on* Q_p.

Since by definition every k-periodic solution $\mathbf{u}^*_{j,n}$ of (2.2) is an upper solution as well as a lower solution, the consideration of $\left(\mathbf{u}^*_{j,n}, \hat{\mathbf{u}}_{j,n}\right)$ and $\left(\tilde{\mathbf{u}}_{j,n}, \mathbf{u}^*_{j,n}\right)$ as the pair of upper and lower solutions in the above argument, leads to the relation $\underline{\mathbf{u}}_{j,n} \leq \mathbf{u}^*_{j,n} \leq \overline{\mathbf{u}}_{j,n}$ *on* Q_p. This ensures the maximal and minimal property of $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$. Finally if $\overline{\mathbf{u}}_{j,0} = \underline{\mathbf{u}}_{j,0}\ (\equiv \mathbf{u}_{j,0})$ then by considering problem (2.2) as an initial boundary value problem with condition $\mathbf{u}_{j,0} = \mathbf{u}_j$, the standard existence-uniqueness theorem for finite difference system of initial boundary value problem of parabolic type ensures that $\overline{\mathbf{u}}_{j,n} = \underline{\mathbf{u}}_{j,n}$ on Q_p. This completes the proof. □

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On $\widetilde{g}_\alpha$-sets in bitopological spaces

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Abstract

The paper introduces $\widetilde{g}_\alpha$-closed sets in bitopological spaces and establishes the relationship between other existing sets. As an application $(i,j)\widetilde{g}_\alpha$-closure is introduced to define a new topology. We also derive a new decomposition of continuity.

Keywords: τ_j-open set, $\widetilde{g}_\alpha$-closed set, $\widetilde{g}_\alpha$-open set, $^{\#}gs$-open set.

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1 Introduction

The notion of generalised closed sets introduced by Levine[7] plays a significant role in general topology. A number of generalised closed sets have been introduced and their properties are investigated. Only a few of the class of generalised closed sets form a topology. The class of $\widetilde{g}_\alpha$-closed sets[4] is one among them. Kelly[5] introduced the concepts of bitopological spaces. Many topologists have introduced different types of sets in bitopological spaces. We have introduced $\widetilde{g}_\alpha$-closed sets in bitopological spaces and discussed their basic properties. We have introduced $(i,j)\widetilde{g}_\alpha$-closure and defined a new topology. We also introduced $(i,j)T_{\widetilde{g}_\alpha}, (i,j)^{\#}T_{\widetilde{g}_\alpha}$-spaces and derived a new decomposition of continuity in bitopological spaces.

2 Preliminaries

We list some definitions which are useful in the following sections. The interior and the closure of a subset A of (X,τ) are denoted by $Int(A)$ and $Cl(A)$, respectively. Throughout the paper, (X,τ_1,τ_2), (Y,σ_1,σ_2) and (Z,η_1,η_2) (or simply X, Y and Z) represent bitopological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, A^c denote the complement of A.

Definition 2.1. *A subset A of a topological space (X,τ) is called*

(i) an ω-closed set [10] if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is semi-open in (X,τ),

(ii) a $^{*}g$-closed set [11] if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is ω-open in (X,τ) ,

(iii) a $^{\#}g$-semi-closed set[13](briefly $^{\#}gs$-closed)[12] if $sCl(A)\subseteq U$ whenever $A\subseteq U$ and U is $^{*}g$-open in (X,τ) and

(iv) $\widetilde{g}_\alpha$ closed set[4] if $\alpha Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is $^{\#}gs$-open in (X,τ)

The complement of ω-closed(resp $^{*}g$-closed,$^{\#}gs$-closed,$\widetilde{g}_\alpha$-closed)set is said to be ω-open(resp $^{*}g$-open,$^{\#}gs$-open,$\widetilde{g}_\alpha$-open)

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Definition 2.2. *A subset A of a bitopological space (X, τ_1, τ_2) is called a*

(i)(τ_i, τ_j)-g-closed set[2] if τ_j-Cl(A)$\subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$,

(ii)(τ_i, τ_j)-gp-closed set[1] if τ_j-pCl(A)$\subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$,

(iii)(τ_i, τ_j)-gpr-closed set[3] if τ_j-pCl(A)$\subseteq U$ whenever $A \subseteq U$ and U is regular open in τ_i,

(iv)(τ_i, τ_j)-ω-closed set[3] if τ_j-Cl(A)$\subseteq U$ whenever $A \subseteq U$ and U is semi-open in τ_i,

(v)$(\tau_i, \tau_j)g^*$-closed set[9] if τ_j-Cl(A)$\subseteq U$ whenever $A \subseteq U$ and U is g-open in τ_i.

The family of all (τ_i, τ_j)-g-closed (resp (τ_i, τ_j)-gp-closed,(τ_i, τ_j)-gpr-closed and (τ_i, τ_j)-ω-closed) subsets of a botopological space (X, τ_1, τ_2) is denoted by $D(\tau_i, \tau_j)$ (resp $GPC(\tau_i, \tau_j)$,$\zeta(\tau_i, \tau_j)$,$C(\tau_i, \tau_j)$ and $D^*(\tau_i, \tau_j)$).

Definition 2.3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be*

(i) (τ_i, τ_j)-gp-σ_k-continuous[1] if the inverse image of every σ_k-closed set in (Y, σ_1, σ_2) is (τ_i, τ_j)-gp-closed in (X, τ_1, τ_2),

(ii)$\zeta(\tau_i, \tau_j)$-σ_k-continuous[3] if the inverse image of every σ_k-closed set in (Y, σ_1, σ_2) is (τ_i, τ_j)-gpr-closed in (X, τ_1, τ_2),

(iii)$D^*(\tau_i, \tau_j)$-σ_k-continuous[9] if the inverse image of every σ_k-closed set in (Y, σ_1, σ_2) is (τ_i, τ_j)-g^*-closed in (X, τ_1, τ_2),

(iv)$D(\tau_i, \tau_j)$-σ_k-continuous[2] if the inverse image of every σ_k-closed set in (Y, σ_1, σ_2) is (τ_i, τ_j)-g-closed in (X, τ_1, τ_2),

(v)τ_j-σ_k-continuous[8] if the inverse image of every σ_k-closed set in (Y, σ_1, σ_2) is τ_j-closed in (X, τ_1, τ_2),

(vi) bi-continuous[8] if f is τ_1-σ_1-continuous and τ_2-σ_2-continuous,

(v) strongly bi-continuous[8] if f is bi-continuous,τ_1-σ_2-continuous and τ_2-σ_1-continuous.

Definition 2.4. *A topological space X is called a*

(i) $T_{\widetilde{g}_\alpha}$-space[4] if every $\widetilde{g}_\alpha$-closed set in it is α-closed.

(ii)$^{\#}T_{\widetilde{g}_\alpha}$-space[4] if every $\widetilde{g}_\alpha$-closed set in it is closed.

3 (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed sets

Definition 3.1. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed if τ_j-$\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $^{\#}gs$-open in τ_i .*
The collection of all (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed sets is denoted by $D\widetilde{G}_\alpha(\tau_i, \tau_j)$.

Remark 3.2. *If $\tau_1 = \tau_2$ then (τ_i, τ_j)-$\widetilde{g}_\alpha$ set reduces to $\widetilde{g}_\alpha$-closed set in a topological space.*

Proposition 3.3. *(i)Every τ_j-closed set is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed.*

(ii) Every τ_j-α-closed set is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed.

Proof. (i) Let A be a τ_j-closed set and U be any $^{\#}gs$-open set in (X, τ_i) containing A. Then τ_j-$\alpha Cl(A) \subseteq \tau_j$-$Cl(A) = A \subseteq U$. Hence A is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed.
(ii)Let Let A be a τ_j-α-closed set and U be any $^{\#}gs$-open set in (X, τ_i) containing A. Then τ_j-$\alpha Cl(A) = A \subseteq U$. Hence A is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed. □

Remark 3.4. *The converse of the proposition 3.3 need not be true.*

Example 3.5. *Let $X = \{a, b, c\}$,$\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$, $D\widetilde{G}_\alpha(\tau_i, \tau_j) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\},$*
The set $\{b\}$ is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed but not τ_j-α-closed and τ_j-closed.

Proposition 3.6. *Every (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed set is $(\tau_i, \tau_j)gp$-closed and $(\tau_i, \tau_j)gpr$-closed.*

Proof. (i)Let A be a (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed set and U be any τ_i-open set in X. Then τ_j-$pCl(A) \subseteq \tau_j$-$\alpha Cl(A) \subseteq U$. Since every τ_i-open set is $^{\#}gs$-open in (X, τ_i)[12]. Hence A is $(\tau_i, \tau_j)gp$-closed.
(ii)Let A be a (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed set and U be any regular open set in (X, τ_i)-open set in X. Then τ_j-$pCl(A) \subseteq \tau_j$-$\alpha Cl(A) \subseteq U$. Since every regular open set is $^{\#}gs$-open in (X, τ_i).Hence A is $(\tau_i, \tau_j)gp$-closed. □

Remark 3.7. *The converse of the proposition 3.6 need not be true.*

Example 3.8. *Let $X = \{a, b, c, d\}$,$\tau_1 = \{\phi, X, \{b, c\}, \{b, c, d\}, \{a, b, c\}\}$,*
$\tau_2 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}\}$,
$D\widetilde{G}_\alpha(\tau_i, \tau_j) = \{\phi, X, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$,
$GPC(\tau_i, \tau_j) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.
The set $\{a, c\}$ is (τ_i, τ_j)-gp-closed but not (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed.

Example 3.9. *Let $X = \{a, b, c, d\}$,$\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}\}$, $D\widetilde{G}_\alpha(\tau_i, \tau_j) =$ $\{\phi, X, \{b\}, \{a, b\}, \{b, c, d\}\}$,*
$\zeta(\tau_i, \tau_j) = \{\phi, X, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}$.
The set $\{c\}$ is (τ_i, τ_j)-gpr-closed but not $(\tau_i, \tau_j)\widetilde{g}_\alpha$-closed.

Remark 3.10. *(τ_i, τ_j)-$\widetilde{g}_\alpha$-closed sets are independent of (τ_i, τ_j)-ω-closed sets,(τ_i, τ_j)-g-closed sets and (τ_i, τ_j)-g^*-closed sets.*

Example 3.11. *Let $X = \{a, b, c, d\}$,$\tau_1 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}\}$, $\tau_2 = \{\phi, X, \{a\}\}$, $D\widetilde{G}_\alpha(\tau_i, \tau_j) = \{\phi, X, \{b\}, \{c\}, \{d\}, \{b, c$*
$C(\tau_i, \tau_j) = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$.
$D(\tau_i, \tau_j) = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$
,$D^*(\tau_i, \tau_j = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$
The set $\{b\}$ is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed but not (τ_i, τ_j)-ω-closed. The set $\{c\}$ is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed but not (τ_i, τ_j)-g-closed. The set $\{a, b\}$ is $(\tau_i, \tau_j$-ω-closed, (τ_i, τ_j)-g-closed and $(\tau_i, \tau_j$-g^*-closed but not (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed.

Proposition 3.12. *Union of two (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed sets is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed.*

Proof. Let A and B are (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed sets and U be any $^{\#}gs$-open set in (X, τ_i) containing A and B. Then τ_j-$\alpha Cl(A) \subseteq U$,τ_j-$\alpha Cl(A) \subseteq U$, τ_j-$\alpha Cl(A \cup B) = \tau_j$-$\alpha Cl(A) \cup \tau_j$-$\alpha Cl(B) \subseteq U$. Hence $A \cup B$ is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed. □

Remark 3.13. *Intersection of two (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed sets need not be a (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed set. In example 3.11 $\{a, b, c\}$ and $\{a, b, d\}$ are (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed but their intersection $\{a, b\}$ is not (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed.*

Remark 3.14. *In general $D\widetilde{G}_\alpha(\tau_1, \tau_2)$ is not equal to $D\widetilde{G}_\alpha(\tau_2, \tau_1)$.*
In example 3.8
$D\widetilde{G}_\alpha(\tau_1, \tau_2) = \{\phi, X, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$,
$D\widetilde{G}_\alpha(\tau_2, \tau_1) = \{\phi, X, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}\}$. *Hence they are not equal.*

Proposition 3.15. *If $\tau_1 \subseteq \tau_2$ in (X, τ_1, τ_2) then $D\widetilde{G}_\alpha(\tau_1, \tau_2) \supseteq D\widetilde{G}_\alpha(\tau_2, \tau_1)$.*

Proof. Let A be $(\tau_2, \tau_1)\widetilde{g}_\alpha$-closed and U be any τ_1-$^{\#}gs$-open set containing A. Since $\tau_1 \subseteq \tau_2$, U is τ_2-$^{\#}gs$-open, τ_2-$\alpha Cl(A) \subseteq \tau_1$-$\alpha Cl(A) \subseteq U$. Hence A is (τ_1, τ_2)-$\widetilde{g}_\alpha$-closed. □

Theorem 3.16. *For each point x of (X, τ_1, τ_2), $\{x\}$ is τ_i-$^{\#}gs$-closed or $\{x\}^c$ is (τ_1, τ_2)-$\widetilde{g}_\alpha$-closed for each fixed integer i,j of $\{1, 2\}$.*

Proof. Suppose $\{x\}$ is not τ_i-$^{\#}gs$-closed. Then $\{x\}^c$ is not τ_i-$^{\#}gs$-open. Then the only τ_i-$^{\#}gs$-open set containing $\{x\}$ is the set X. τ_j-$\alpha Cl(\{x\}^c) \subseteq X$. Hence $\{x\}^c$ is (τ_1, τ_2)-$\widetilde{g}_\alpha$-closed. □

Proposition 3.17. *If a set A is a (τ_1, τ_2)-$\widetilde{g}_\alpha$-closed set in (X, τ_1, τ_2) then τ_j-$\alpha Cl(A) - A$ contains no non empty τ_i-$^{\#}gs$-closed set.*

Proof. Let A be a (τ_1, τ_2)-$\widetilde{g}_\alpha$-closed and F be a τ_i-$^{\#}gs$-closed set contained in $\tau_j \alpha Cl(A) - A$. Since A is (τ_1, τ_2)-$\widetilde{g}_\alpha$-closed τ_j-$\alpha Cl(A) \subseteq F^c$. Hence $F \subseteq$
τ_j-$\alpha Cl(A) \cap (\tau_j$-$\alpha Cl(A))^c = \phi$. □

Remark 3.18. *The converse of the proposition 3.17 need not be true.*

Example 3.19. *Let* $X = \{a,b,c,d\}, \tau_1 = \{\phi, X, \{a,b\}\}, \tau_2 = \{\phi, X, \{a,b\}, \{c,d\}\}$. ${}^{\#}GSO(X,\tau_1) = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,b$
$D\widetilde{G}_\alpha(\tau_1,\tau_2) = \{\phi, X, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}\}$,
If $A = \{b\}, \tau_j\text{-}\alpha Cl(A) - A = \{a,b\} - \{b\} = \{a\}$. *But* $\{b\}$ *is not* (τ_1,τ_2)-$\widetilde{g}_\alpha$*-closed.*

Proposition 3.20. *If A is* (τ_i,τ_j)-$\widetilde{g}_\alpha$*-closed in* (X,τ_1,τ_2) *then A is* τ_j-α*-closed if and only if* $\tau_j\alpha Cl(A) - A$ *is* $\tau_i -^{\#} gs$*-closed.*

Proof. Necessity implies τ_j-$\alpha Cl(A) - A = \phi$ and hence τ_j-$\alpha Cl(A) - A$ is
$\tau_i -^{\#} gs$-closed.
Sufficiency. If τ_j-$\alpha Cl(A) - A$ is $\tau_i -^{\#} gs$-closed then by proposition 3.17
τ_j-$\alpha Cl(A) - A = \phi$. Therefore A is τ_j-α-closed. □

Proposition 3.21. *If A is* (τ_i,τ_j)-$\widetilde{g}_\alpha$*-closed then* τ_i-$\alpha Cl(\{x\}) \cap A \neq \phi$ *for each* $x \in \tau_j$-$\alpha Cl(A)$.

Proof. If τ_i-$\alpha Cl(\{x\}) \cap A = \phi$ for each $x \in \tau_j$-$\alpha Cl(A)$ then $A \subseteq (\tau_i \alpha Cl(\{x\}))^c$. Since A is (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed, we have τ_j-$\alpha Cl(A) \subseteq (\tau_i$-$\alpha Cl(\{x\})^c$.[Since $(\tau_j$-$\alpha Cl(\{x\})^c$ is τ_i-α-open and therefore τ_i-semi-open, τ_i-${}^{\#}$gs-open]. This gives $x \notin \tau_j \alpha Cl(A)$. A contradiction. □

Proposition 3.22. *If A is* (τ_i,τ_j)-$\widetilde{g}_\alpha$*-closed* $A \subseteq B \subseteq \tau_j$-$\alpha Cl(A)$ *then B is* (τ_i,τ_j)-$\widetilde{g}_\alpha$*-closed.*

Proof. Let $B \subseteq U$ where U is τ_i-${}^{\#}gs$-open. Then $A \subseteq B \subseteq U$.Since A is (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed τ_j-$\alpha Cl(A) \subseteq U$. Therefore τ_j-$\alpha Cl(B) \subseteq \tau_j$-$\alpha Cl(\tau_j$-$\alpha Cl(A)) =$
τ_j-$\alpha Cl(A)$. Thus τ_j-$\alpha Cl(B) \subseteq U$. Therefore B is (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed. □

Proposition 3.23. *If* $A \subseteq Y \subseteq X$ *and A is* (τ_i,τ_j)-$\widetilde{g}_\alpha$*-closed then A is* (τ_i,τ_j)-$\widetilde{g}_\alpha$*-closed relative to Y.*

Proof. Let S be any τ_i-${}^{\#}gs$-open set in Y such that $A \subseteq S$. Then $S = U \cap Y$ for some $U \in^{\#} GSO(X,\tau_i)$. Thus $A \subseteq U \cap Y$ and $A \subseteq U$. Since A is
(τ_i,τ_j)-$\widetilde{g}_\alpha$-closed in X, τ_j-$\alpha Cl(A) \subseteq U$. Therefore $Y \cap \tau_j$-$\alpha Cl(A) \subseteq Y \cap U$.That is τ_j-$\alpha Cl_Y(A) \subseteq S$. Since τ_j-$\alpha Cl_Y(A) = Y \cap \tau_j$-$\alpha Cl(A)$. Hence A is
(τ_i,τ_j)-$\widetilde{g}_\alpha$-closed. □

Definition 3.24. *A subset A of a bitopological space* (X,τ_1,τ_2) *is said to be* (τ_i,τ_j)-$\widetilde{g}_\alpha$*-open if its complement is* (τ_i,τ_j)-$\widetilde{g}_\alpha$*-closed in X.*

Theorem 3.25. *In a bitopological space* (X,τ_1,τ_2)

(i) Every τ_j*-open set is* $(\tau_i,\tau_j)\widetilde{g}_\alpha$*-open.*

(ii)Every τ_j-α*-open set is* $(\tau_i,\tau_j)\widetilde{g}_\alpha$*-open.*

(iii)Every (τ_i,τ_j)-$\widetilde{g}_\alpha$*-open set is* $(\tau_i,\tau_j)gp$*-open.*

(iv)Every (τ_i,τ_j)-$\widetilde{g}_\alpha$*-open set is* $(\tau_i,\tau_j)gpr$*-open.*

Remark 3.26. *The converse of the theorem 3.25 need not be true.*

Theorem 3.27. *A subset A of a bitopological space* (X,τ_1,τ_2) *is* (τ_i,τ_j)-$\widetilde{g}_\alpha$*-open if and only if* $F \subseteq \tau_j$-$\alpha Int(A)$ *whenever F is* τ_i-${}^{\#}gs$*-closed and* $F \subseteq A$.

Proof. Necessity. Let A be a (τ_i,τ_j)-$\widetilde{g}_\alpha$-open set in X and F be a τ_i-${}^{\#}gs$-closed set such that $F \subseteq A$. Then $A^c \subseteq F^c$ where F^c is τ_i-${}^{\#}gs$-open and A^c is (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed implies τ_j-$\alpha Cl(A^c) \subseteq (F^c)$,$\tau_j$-$(\alpha Int(A))^c \subseteq F^c$. Hence $F \subseteq \tau_j$-$\alpha Int(A)$.
Sufficiency.Let $F \subseteq \tau_j$-$\alpha Int(A)$ where F is ${}^{\#}gs$-closed and $F \subseteq A$. Then $A^c \subseteq F^c = G$ and G is τ_i-${}^{\#}gs$-open. Then $G^c \subseteq A$ implies $G^c \subseteq \tau_j$-$\alpha Int(A)$ or τ_j-$\alpha Cl(A^c)$= $(\tau_j$-$\alpha Int(A))^c \subseteq G$. Thus A^c is (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed or A is
(τ_i,τ_j)-$\widetilde{g}_\alpha$-open. □

4 Applications

Definition 4.1. *A bitopological space (X, τ_1, τ_2) is said to be a $(i, j)T_{\widetilde{g}_\alpha}$-space if every (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed set in it is τ_j-α-closed.*

Remark 4.2. *If $\tau_1 = \tau_2$ then $(i, j)T_{\widetilde{g}_\alpha}$-space becomes a $T_{\widetilde{g}_\alpha}$-space.*

Theorem 4.3. *A bitopological space (X, τ_1, τ_2) is a $(i, j)T_{\widetilde{g}_\alpha}$-space if and only if each $\{x\}$ is τ_j-α-open or τ_i-$^{\#}gs$-closed for each $x \in X$.*

Proof. Suppose that $\{x\}$ is not τ_i-$^{\#}gs$-closed then by theorem 3.16, $\{x\}^c$ is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed. Since X is a $(i, j)T_{\widetilde{g}_\alpha}$-space, $\{x\}^c$ is τ_j-α-closed. Therefore $\{x\}$ is τ_j-α-open in X.
Conversely let F be a (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed set. By assumption $\{x\}$ is τ_j-α-open or τ_i-$^{\#}gs$-closed for any $x \in \tau_j$-$\alpha Cl(F)$.
case(i) $\{x\}$ is τ_j-α-open. Since $x \in \tau_j$-$\alpha Cl(F)$, $\{x\} \cap F \neq \phi$. Hence $x \in F$.
case(ii) Suppose $\{x\}$ is τ_i-$^{\#}gs$-closed. If $x \notin F$, then $\{x\} \subseteq \tau_j$-$\alpha Cl(F) - F$ which is a contradiction by proposition 3.20. Therefore $x \in F$. Hence F is a $\tau_j\alpha$-closed subset of X. □

Definition 4.4. *A bitopological space (X, τ_1, τ_2) is said to be $(i, j)^{\#}T_{\widetilde{g}_\alpha}$-space if every (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed set in it is τ_j-closed.*

Proposition 4.5. *Every $(i, j)^{\#}T_{\widetilde{g}_\alpha}$-space is a $(i, j)T_{\widetilde{g}_\alpha}$-space*

Proof. Since every τ_j-closed set is τ_j-α-closed, the proposition is valid. □

Remark 4.6. *The converse of the proposition 4.5 need not be true. Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a, \}, \{b, c\}\}, \tau_2 = \{\phi, X, \{a\}\}$. The space X is a $(i, j)T_{\widetilde{g}_\alpha}$-space but not a $(i, j)^{\#}T_{\widetilde{g}_\alpha}$-space.Since the set $\{b\}$ is not τ_j-closed.*

Definition 4.7. *For a subset A of a bitopological space (X, τ_1, τ_2), the $(i, j)\widetilde{g}_\alpha$-$Cl(A)$ is defined as $(i, j)\widetilde{g}_\alpha$-$Cl(A) = \bigcap\{F : A \subseteq F, F \in D\widetilde{G}_\alpha(\tau_i, \tau_j)\}$.*

Proposition 4.8. *Let A and B be two subsets of (X, τ_i, τ_j).*

(i) If $A \subseteq B$ then $(i, j)\widetilde{g}_\alpha$-$Cl(A) \subseteq (i, j)\widetilde{g}_\alpha$-$Cl(B)$,

(ii)If $\tau_1 \subseteq \tau_2$ then $(1, 2)\widetilde{g}_\alpha$-$Cl(A) \subseteq (2, 1)\widetilde{g}_\alpha$-$Cl(B)$.

Proof. It follows from proposition 3.15. □

Proposition 4.9. *For a subset A of (X, τ_1, τ_2),*

(i) $A \subseteq (i, j)\widetilde{g}_\alpha$-$Cl(A) \subseteq \tau_j$-$Cl(A)$,

(ii)If A is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed then $(i, j)\widetilde{g}_\alpha$-$Cl(A) = A$.

Proof. (i) By definition $A \subseteq (i, j)\widetilde{g}_\alpha$-$Cl(A)$. By Proposition 3.3 τ_j-$Cl(A)$ is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed. Therefore $A \subseteq (i, j)\widetilde{g}_\alpha$-$Cl(A) \subseteq \tau_j$-$Cl(A)$.
(ii) Follows from (i) and the definition 4.5. □

Remark 4.10. *The converse of the proposition 4.9 need not be true. In example 3.11 if $A = \{a, b\}$ then $(i, j)\widetilde{g}_\alpha$-$Cl(A) = \{a, b\}$. But A is not (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed.*

Theorem 4.11. *The closure operator $(i, j)\widetilde{g}_\alpha$-closure is the Kuratowski closure operator.*

Proof. (i)From proposition 3.3 and proposition 4.9 (ii) it follows that $(i, j)\widetilde{g}_\alpha Cl(\phi) = \phi$.

(ii) From proposition 4.9 (i) $A \subseteq (i, j)\widetilde{g}_\alpha$-$Cl(A)$.

(iii) If A and B are two subsets of X then $(i, j)\widetilde{g}_\alpha$-$Cl(A) \cup (i, j)\widetilde{g}_\alpha$-$Cl(B) \subseteq (i, j)\widetilde{g}_\alpha$-$Cl(A \cup B)$. If x does not belong to $(i, j)\widetilde{g}_\alpha$-$Cl(A) \cup (i, j)\widetilde{g}_\alpha$-$Cl(B)$, then there exist $C, D \in D\widetilde{G}_\alpha(\tau_i, \tau_j)$ such that $A \subseteq C, x \notin C, B \subseteq D, x \notin D$. Hence $A \cup B \subseteq C \cup D$ and $x \notin C \cup D$. Since $C \cup D$ is (τ_i, τ_j)-$\widetilde{g}$-closed by proposition 3.12, x does not belong to $(i, j)\widetilde{g}_\alpha$-$Cl(C \cup D)$. Hence $(i, j)\widetilde{g}_\alpha$-$Cl(A \cup B) \subseteq (i, j)\widetilde{g}_\alpha$-$Cl(A) \cup (i, j)\widetilde{g}_\alpha$-$Cl(B)$.Thus $(i, j)\widetilde{g}_\alpha$-$Cl(A \cup B) = (i, j)\widetilde{g}_\alpha$-$Cl(A) \cup (i, j)\widetilde{g}_\alpha$-$Cl(B)$.

(iv) Let E be a subset of X and A be a (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed set containing E, We have $(i,j)\widetilde{g}_\alpha$-$Cl(E) \subseteq A$, we have $(i,j)\widetilde{g}_\alpha$-$Cl(E) \supseteq (i,j)\widetilde{g}_\alpha$-$Cl((i,j)\widetilde{g}_\alpha$-$Cl(E))$. Conversely $(i,j)\widetilde{g}_\alpha$-$Cl(E) \subseteq (i,j)\widetilde{g}_\alpha$-$Cl((i,j)\widetilde{g}_\alpha$-$Cl(E))$.(By proposiotion 4.8 (i)). Hence $(i,j)\widetilde{g}_\alpha$-$Cl(E) = (i,j)\widetilde{g}_\alpha$-$Cl((i,j)\widetilde{g}_\alpha$-$Cl(E)$.

□

Definition 4.12. *$(i,j)\widetilde{g}_\alpha$-closure defines a new topology on X. The topology defined by $(i,j)\widetilde{g}_\alpha$-closure is defined as and denoted as $\widetilde{G}_\alpha(\tau_i,\tau_j) = \{E \subseteq X : (i,j)\widetilde{g}_\alpha\text{-}Cl(E^c) = E^c\}$.*

Example 4.13. *Let $X = \{a,b,c,d\}, \tau_1 = \{\phi, X, \{b,c\}, \{b,c,d\}, \{a,b,c\}\}, \tau_2 = \{\phi, X, \{a\}, \{c,d\}, \{a,c,d\}\}$.*
$D\widetilde{G}_\alpha(\tau_i,\tau_j) = \{\phi, X, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.
$\widetilde{G}_\alpha(\tau_i,\tau_j) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$.

Proposition 4.14. *Let $i,j \in \{1,2\}$ be two fixed integers.*

(i)$\tau_j \subseteq \widetilde{G}_\alpha(\tau_i,\tau_j)$.

(ii) If a subset A of X is (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed then A is $\widetilde{G}_\alpha(\tau_i,\tau_j)$-closed.

Proof. (i) Let U be any τ_j-open set, by proposition 3.3 and proposition 4.9 (ii) $(i,j)\widetilde{g}_\alpha$-$Cl(U^c) = U^c$. Hence $U \in \widetilde{G}_\alpha(\tau_i,\tau_j)$.

(ii) It follows from proposition 4.9(ii). If A is (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed then $(i,j)\widetilde{g}_\alpha$-$Cl(A) = A$, this implies $A^c \in \widetilde{G}_\alpha(\tau_i,\tau_j)$ or A is $\widetilde{G}_\alpha(\tau_i,\tau_j)$-closed.

□

Proposition 4.15. *If $\tau_1 \subseteq \tau_2$ then $\widetilde{G}_\alpha(\tau_1,\tau_2) \supseteq \widetilde{G}_\alpha(\tau_2,\tau_1)$.*

Proof. Let $A \in \widetilde{G}_\alpha(\tau_2,\tau_1)$ then $(2,1)\widetilde{g}_\alpha$-$Cl(A^c) = A^c$. Since $\tau_1 \subseteq \tau_2$, $(1,2)\widetilde{g}_\alpha$-$Cl(A) \subseteq \widetilde{g}_\alpha$-$Cl(A)$, $A^c \subseteq (1,2)\widetilde{g}_\alpha$-Cl$(A^c) \subseteq (2,1)\widetilde{g}_\alpha$-Cl$(A^c) = A^c$. Thus $A^c = (1,2)\widetilde{g}_\alpha$-$Cl(A^c)$. Therefore $A \in \widetilde{G}_\alpha(1,2)$. □

5 $\widetilde{G}_\alpha(\tau_i,\tau_j)$-Continuity

Definition 5.1. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called $D\widetilde{G}_\alpha(\tau_i,\tau_j)$-$\sigma_k$-continuous if the inverse image of every σ_k-closed set in (Y,σ_1,σ_2) is a (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed set in (X,τ_1,τ_2).*

Remark 5.2. *If $\tau_1 = \tau_2 = \tau$ and $\sigma_1 = \sigma_2 = \sigma$ then the $D\widetilde{G}_\alpha(\tau_i,\tau_j)$-$\sigma_k$-continuous function coincides with $\widetilde{g}_\alpha$-continuous function.[6]*

Definition 5.3. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called τ_j-α-σ_k-continuous if the inverse image of every σ_k-closed set in (Y,σ_1,σ_2) is a τ_j-α-closed set in (X,τ_1,τ_2).*

Proposition 5.4. *If a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is τ_j-σ_k(resp τ_j-α-σ_k-continuous) continuous then it is $D\widetilde{G}_\alpha(\tau_i,\tau_j)$-$\sigma_k$-continuous.*

Proof. Let V be any σ_k-closed set in (Y,σ_1,σ_2) then $f^{-1}(V)$ is τ_j-closed (resp τ_j-α-closed). Since every τ_j-closed (resp τ_j-α-closed) set is (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed. Hence f is $D\widetilde{G}_\alpha(\tau_i,\tau_j)$-$\sigma_k$-continuous. □

Remark 5.5. *The converse of the proposition 5.4 need not be true.*

Example 5.6. *Let $X = Y = \{a,b,c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{b\}\}, \sigma_1 = \{\phi, Y, \{a,b\}\}, \sigma_2 = \{\phi, Y, \{a\}\}$. The function f is defined as $f(a) = b, f(b) = a, f(c) = c$. Here f is $D\widetilde{G}_\alpha(\tau_i,\tau_j)$-$\sigma_k$-continuous but not τ_j-σ_k(resp τ_j-α-σ_k)continuous.*

Proposition 5.7. *If a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is $D\widetilde{G}_\alpha(\tau_i,\tau_j)$-$\sigma_k$-continuous then it is (τ_i,τ_j)-gp-σ_k continuous and $\zeta(\tau_i,\tau_j)$-σ_k continuous.*

Proof. Let V be any σ_k-closed set in (Y,σ_1,σ_2) then $f^{-1}(V)$ is (τ_i,τ_j)-$\widetilde{g}_\alpha$-closed which is (τ_i,τ_j)-gp-closed and (τ_i,τ_j)-gpr-closed. Hence f is (τ_i,τ_j)-gp-σ_k continuous and $\zeta(\tau_i,\tau_j)$-σ_k continuous. □

Remark 5.8. *The converse of the proposition 5.7 need not be true.*

Example 5.9. *Let* $X = Y = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$, $\sigma_1 = \{\phi, Y, \{a, b\}\}, \sigma_2 = \{\phi, Y, \{a, c\}\}$. *The function f is the identity map. Here f is* (τ_i, τ_j)*-gp-*σ_k*-continuous but not* $D\widetilde{G}_\alpha(\tau_i, \tau_j)$*-*$\sigma_k$*-continuous.*

Example 5.10. *Let* $X = Y = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{a, b\}\}$, $\sigma_1 = \{\phi, Y, \{a, b\}\}, \sigma_2 = \{\phi, Y, \{a, c\}\}$. *The function f is the identity map. Here f is* $\zeta(\tau_i, \tau_j)$*-*σ_k*-continuous but not* $D\widetilde{G}_\alpha(\tau_i, \tau_j)$*-*$\sigma_k$*-continuous.*

Remark 5.11. $D\widetilde{G}_\alpha(\tau_i, \tau_j)$*-*$\sigma_k$*-coninuity is independent of* $D(\tau_i, \tau_j)$*-*σ_k*-coninuity and* $C(\tau_i, \tau_j)$*-*σ_k*-continuity.*

Example 5.12. *Let* $X = Y = \{a, b, c\}, \tau_1 = \{\phi, X, \{a, b\}\}, \tau_2 = \{\phi, X, \{a\}\}$, $\sigma_1 = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}, \sigma_2 = \{\phi, Y, \{a, c\}\}$. *The function f is the identity map. Here f is* $D\widetilde{G}_\alpha(\tau_i, \tau_j)$*-*$\sigma_k$*-continuous but not* $D(\tau_i, \tau_j)$*-*σ_k*-continuous and* $C(\tau_i, \tau_j)$*-gp-*σ_k*-continuous.*

Example 5.13. *Let* $X = Y = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}, \{b, c\}\}, \tau_2 = \{\phi, X, \{a, b\}\}$, $\sigma_1 = \{\phi, Y, \{b\}\}, \sigma_2 = \{\phi, Y, \{c\}\}$. *The function f is the identity map. Here f is* $D(\tau_i, \tau_j)$*-*σ_k*-continuous and* $C(\tau_i, \tau_j)$*-*σ_k*-continuous but not* $D\widetilde{G}_\alpha(\tau_i, \tau_j)$*-*$\sigma_k$*-continuous.*

Definition 5.14. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called* $\widetilde{g}_\alpha$*-bi continuous if f is both* $D\widetilde{G}_\alpha(2, 1)$*-*$\sigma_1$*-continuous and* $D\widetilde{G}_\alpha(1, 2)$*-*$\sigma_2$*-continuous.*

Definition 5.15. *A function* $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called strongly* $\widetilde{g}_\alpha$*-bi continuous if it is* $\widetilde{g}_\alpha$*-bi continuous,*$D\widetilde{G}_\alpha(2, 1)$*-*$\sigma_2$*-continuous and* $D\widetilde{G}_\alpha(1, 2)$*-*$\sigma_1$*-continuous.*

Proposition 5.16. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a function.*
(i) If f is bi-continuous then it is $\widetilde{g}_\alpha$*-bi continuous.*
(ii) If f is strongly -bi-continuous then it is strongly $\widetilde{g}_\alpha$*-bi continuous.*

Proof. (i)Let f be bi-continuous. Then f is τ_1-σ_1-continuous and τ_2-σ_2-continuous. By proposition 5.4 f is $\widetilde{g}_\alpha$-bi continuous.
(ii)Let f be strongly bi-continuous then f is τ_1-σ_2-continuous and τ_2-σ_1-continuous. Then by proposition 5.4 f is strongly $\widetilde{g}_\alpha$-bi continuous. □

Example 5.17. *Let* $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{b, c\}\}$, $Y = \{p, q\}, \sigma_1 = \{\phi, Y, \{p\}\}, \sigma_2 = \{\phi, Y, \{p\}, \{q\}\}$.*The function f is defined as* $f(a) = p, f(b) = f(c) = q$. *Here f is* $\widetilde{g}_\alpha$*-bi continuous and strongly* $\widetilde{g}_\alpha$*-bi continuous but not bi-continuous and strongly bi-continuous.*

Proposition 5.18. *If* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is strongly* $\widetilde{g}_\alpha$*-bi continuous then it is* $\widetilde{g}_\alpha$*-bi continuous.*

Proof. It follows from the definitions. □

Example 5.19. *Let* $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, $Y = \{p, q\}, \sigma_1 = \{\phi, Y, \{p\}\}, \sigma_2 = \{\phi, Y, \{q\}\}$.*The function f is defined as* $f(a) = p, f(b) = f(c) = q$. *Here f is* $\widetilde{g}_\alpha$*-bi continuous but not strongly* $\widetilde{g}_\alpha$*-bi continuous.*

Proposition 5.20. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is* $D\widetilde{G}_\alpha(\tau_i, \tau_j)\sigma_k$*-continuous if and only if* $f^{-1}(U)$ *is* (τ_i, τ_j)*-*$\widetilde{g}_\alpha$*-open in* (X, τ_1, τ_2) *for every* σ_k*-open set in* (Y, σ_1, σ_2)*.*

Proof. Let f be $D\widetilde{G}_\alpha(\tau_i, \tau_j)\sigma_k$-continuous and U be a σ_k-open set in (Y, σ_1, σ_2).Then $f^{-1}(U^c)$ is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed. But $f^{-1}(U^c) = (f^{-1}(U))^c$ and so $f^{-1}(U)$ is (τ_i, τ_j)-$\widetilde{g}_\alpha$-open.
Conversely let for any σ_k-open set U in (Y, σ_1, σ_2) then $f^{-1}(U)$ be (τ_i, τ_j)-$\widetilde{g}_\alpha$-open in (X, τ_1, τ_2).Let F be a σ_k-closed set in (Y, σ_1, σ_2) But $f^{-1}(F^c) = (f^{-1}(F))^c$ and so $f^{-1}(F)$ is $(\tau_i, \tau_j)\widetilde{g}_\alpha$-closed. Hence f is $D\widetilde{G}_\alpha(\tau_i, \tau_j)$-$\sigma_k$-continuous. □

Theorem 5.21. *If a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is* $D\widetilde{G}_\alpha(\tau_i, \tau_j)$*-*$\sigma_k$*-continuous and if* (X, τ_1, τ_2) *is a* $(i, j)^{\#}T_{\widetilde{g}_\alpha}$*-space then f is*
(i) τ_j*-*σ_k*-continuous.*
(ii)$D(\tau_i, \tau_j)$*-*σ_k*-continuous.*
(iii)$C(\tau_i, \tau_j)$*-*σ_k*-continuous.*

Proof. Let V be any σ_k-closed set in Y then $f^{-1}(V)$ is (τ_i, τ_j)-$\widetilde{g}_\alpha$-closed in X. Since X is a $(i, j)^{\#}T_{\widetilde{g}_\alpha}$-space $f^{-1}(V)$ is τ_j-closed in X. Hence f is τ_j-σ_k-continuous. Since τ_j-closed set is τ_j-g-closed and τ_j-ω-closed, f is $D(\tau_i, \tau_j)$-σ_k-continuous and $C(\tau_i, \tau_j)$-σ_k-continuous. □

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Sinc-collocation solution for nonlinear two-point boundary value problems arising in chemical reactor theory

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Abstract

Numerical solution of nonlinear second order two-point boundary value problems based on Sinc-collocation method, developed in this work. We first apply the method to the class of nonlinear two-point boundary value problems in general and specifically solved special problem that is arising in chemical reactor theory. Properties of the Sinc-collocation method are utilized to reduce the solution of nonlinear two-point boundary value problem to some nonlinear algebraic equations. By solving such system we can obtain the numerical solution. We compared the obtained numerical result with the previous methods so far, such as Adomiad method, shooting method, Sinc Galerkin method and contraction mapping principle method.

Keywords: Sinc-collocation, nonlinear, boundary value problems, chemical.

2010 MSC: 65L60, 34B15, 65l10.

1 Introduction

Boundary value problems arise in a variety of the fields in applied mathematics, theoretical physics, chemical reaction and engineering. These categories of the problems have been handled by a reasonable number of researches who are working both numerically and analytically. The majority of these problems cannot be solved analytically, so we have to use the numerical methods, but there is not unified method to handle all types of problems.

In this paper, we consider the nonlinear differential equations:

$$au''(x) + bu'(x) + F(u(x), x) = 0, \tag{1.1}$$

with boundary conditions:

$$\begin{cases} a_0u(0) + a_1u'(0) = 0, \\ b_0u(1) + b_1u'(1) = 0, \end{cases} \tag{1.2}$$

where $a_0, a_1, b_0, b_1, b, a \neq 0$ are given constant, F is analytic function in $(0, 1)$ and may be singular in 0 or 1 or both. The special class of (1.1)-(1.2) arises in chemical reactor theory. This differential equation is the mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction. For steady state solutions this model can be reduce to the following ordinary differential equation given in [7] by

$$u''(x) - \lambda u'(x) + F\Big(u(x), \mu, \beta, \lambda\Big) = 0, \tag{1.3}$$

with boundary conditions

$$u'(0) = \lambda u(0), u'(1) = 0, \tag{1.4}$$

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where $F\Big(u(x), \mu, \beta, \lambda\Big) = \lambda\mu(\beta - u)\exp(u)$.

The unknown u represents the steady state temperature of the reaction, and the parameters λ, μ and β represent the Peclet number, the Damkohler number and the dimensionless adiabatic temperature rise , respectively. The existence solution of this problem has been considered in [4]. This problem has been studied by several authors [5], [7], [8], [11] who have demonstrated numerically the existence of solutions for particular ranges.

Recently equation (1.3) with conditions (1.4) have been consider by Modbouly et. al [7], first by using Green function technique, converted the problem into a Hammerestein integral equation and then solve the problem by Adomian׳s method. The Galerkin method, based on Sinc function is reported in [11].

Sinc methods have increasingly been recognized as powerful tools for problems in applied physics, chemistry and engineering. The Sinc methods are easily implemented and given good accuracy for problem not only in regular equations but also for problem with singularities. Approximation by Sinc functions are typified by errors of the form $\mathrm{O}(\exp(-\frac{c}{h}))$ where $c > 0$ is constant and h is an step size [13], [14]. This property is good reason for many authors to use these approximation for solving problems. Numerical solutions of boundary value problems by using Sinc functions have been studied first by Frank Stenger more than thirty years ago [12]. The efficiency of the method has been formally proved by many researchers. Bialecki [1] used Sinc-collocation method to solve a linear two point boundary value problems. Lund [6] applied symmetrization Sinc-Galerkin for boundary value problems. Dehghan and Saadatmandi [2] used Sinc-collocation method for solving nonlinear system of second order. El-Gamel [3] solving a class of linear and nonlinear two point boundary value problem by Sinc-Galerkin method. The books by Stenger [13], [14] and by Lund and Bowers [6] provide excellent over wive of existing methods based on Sinc function for solving integral equations, ordinary and partial differential equations. In our pervious work we applied Sinc collocation method for solution of linear and nonlinear integral equations [9] [10] .

In this study, we first apply Sinc-collocation method for solving (1.1)-(1.2) and also (1.3)-(1.4). Our method reduce the solution (1.1)-(1.2) to a set of nonlinear algebraic equations. By solving this algebraic equations we can find the approximation solution based on Sinc function.

In section 2 we give the relevant properties of Sinc function such as definitions, notations and some theorems. In section 3 we start the treatment on the boundary conditions and then used Sinc-collocation method find approximation solution for (1.1)-(1.2). In section 4 we solve one test example with various parameters and demonstrate the accuracy of the proposed numerical scheme by considering this special example.

2 Preliminaries and Fundamentals

In this section , some definitions, notations and theorems from [6] are presented. The Sinc function is defined on the whole real line $-\infty < x < \infty$ by

$$Sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

For $h > 0$ and $k = 0, \pm 1, \pm 2, ...$ the translated Sinc function with evenly space nodes are given as follow

$$S(k,h)(x) = Sinc\Big(\frac{x - kh}{h}\Big) = \begin{cases} \frac{\sin((\pi/h)(x-kh))}{(\pi/h)(x-kh)}, & x \neq kh \\ 1, & x = kh. \end{cases}$$

The Sinc function form for the interpolating points $x_j = jh$ is given by

$$S(k,h)(jh) = \delta^{(0)}_{kj} = \begin{cases} 0, & k \neq j \\ 1, & k = j. \end{cases}$$

If a function f(x) is defined on the real line, then for $h > 0$ the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) Sinc(\frac{x - kh}{h}),$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of Whittaker cardinal expansion have been extensively studied on [15].
These properties are derived in the infinite strip D_d of the complex w-plane, where for $d > 0$

$$D_d = \{w = \xi + i\eta : |\eta| < d \leq \frac{\pi}{2}\}.$$

Many problems that arise in applied mathematics do not have the whole real line as their natural domain. There are two point of view. One is to change variables in the problem so that, in the new variables, the problem has a domain corresponding to that of the numerical process. A second procedures is to move the numerical process and to study it on the new domain. The latter approach is the method chosen here. The development for transform Sinc method from one domain to another is accomplished via conformal mapping. Approximation can be constructed for infinite,semi-infinite and finite interval.

Definition 2.1. *Let D be a domain in the $w = u + iv$ plane with boundary points $a \neq b$, let $z = \phi(w)$ be a one-to-one conformal map of D onto the infinite strip D_d where $\phi(a) = -\infty$, $\phi(b) = +\infty$. Denote by $w = \psi(z)$ inverse of the mapping ϕ and let*

$$\Gamma \equiv \Big\{w\epsilon C : w = \psi(x), x\epsilon(-\infty,\infty)\Big\} = \psi((-\infty,\infty)),$$

$$\Gamma_a \equiv \Big\{w\epsilon\Gamma : w = \psi(x) \ \ , \ x\epsilon(-\infty,0)\Big\} = \psi((-\infty,0)),$$

$$\Gamma_b \equiv \Big\{w\epsilon\Gamma : w = \psi(x) \ \ , \ x\epsilon[0,\infty)\Big\} = \psi([0,\infty)).$$

Definition 2.2. *Let B(D) denote the class of analytic functions* F *in D which satisfy for some constant α with $0 \leq \alpha < 1$,*

$$\int_{\psi(x+\Gamma)} |F(w)dw| = O(|x|^{\alpha}) \qquad x \to \pm\infty,$$

where $\Gamma = \{iy : |y| < d\}$ and for γ a simple closed contour in D

$$N(F,D) \equiv \lim_{\gamma\to\partial D} \int_{\gamma} |F(w)dw| < \infty.$$

Further , for $h > 0$, define the nodes

$$w_k = \psi(kh) \qquad k = 0, \pm1, \pm2, \pm3, ... \tag{2.1}$$

Theorem 2.1. *[6] Let $\phi' F \ \epsilon \ B(D)$ and $h > 0$. Let ϕ be a one-to-one conformal map of the domain D onto D_d. Let $\psi = \varphi^{-1}$, $w_k = \psi(kh)$, furthermore assume that there are positive constant α, β and C so that*

$$|F(\zeta)| \leq C \begin{cases} \exp(-\alpha|\phi(\zeta)|), & \zeta \ \epsilon \ \Gamma_a \\ \exp(-\beta|\phi(\zeta)|), & \zeta \ \epsilon \ \Gamma_b \end{cases}$$

If the selections

$$N = [|\frac{\alpha}{\beta}M + 1|], \tag{2.2}$$

$$h = \Big(\frac{\pi d}{\alpha M}\Big)^{1/2}, \tag{2.3}$$

are made, then, for all $\zeta \ \epsilon \ \Gamma$

$$\varepsilon = F(\zeta) - \sum_{k=-M}^{N} F(w_k) Sinc(\frac{\phi(\zeta) - kh}{h}),$$

is bounded by

$$\|\varepsilon\|_\infty \leq K M^{1/2} \exp(-(\pi d\alpha M)^{1/2}),$$

and K is a constant depending on F, d, ϕ and D.

To construct approximation on the interval $(0,1)$, which is used in this paper, consider the conformal map:

$$\phi(z) = \log\Big(\frac{z}{1-z}\Big). \tag{2.4}$$

The map ϕ carries the eye-shaped region

$$D = \Big\{z = x + iy : |\arg(\frac{z}{1-z})| < d \leq \frac{\pi}{2}\Big\},$$

onto the infinite strip D_d. The basis function on the interval $\Gamma = (0,1)$ for $z\epsilon\ D$ are derived from the composite translated Sinc functions

$$S_k(z) = S(k,h)o\phi(z) = Sinc\Big(\frac{\phi(z)-kh}{h}\Big), \qquad z\ \epsilon D. \tag{2.5}$$

The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \psi(w) = \frac{\exp(w)}{1+\exp(w)}. \tag{2.6}$$

The collocation method requires derivatives of composite Sinc function evaluated at the node so that we need to use the following lemma.

Lemma 2.1. *[6] Let ϕ be the conformal one-to-one mapping of the simply connected domain D onto D_d given by* (8) *then*

$$\delta^{(0)}_{jk} = S(j,h)o\phi(z_k) = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}$$

$$\delta^{(1)}_{jk} = \frac{d}{d\phi}[S(j,h)o\phi(z)](z_k) = \frac{1}{h}\begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases}$$

$$\delta^{(2)}_{jk} = \frac{d^2}{d\phi^2}[S(j,h)o\phi(z)](z_k) = \frac{1}{h^2}\begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases}$$

For the assembly of discrete system, it is convenient to define the following matrices:

$$I^{(r)} = [\delta^{(r)}_{jk}] \qquad r = 0,1,2 \ , \tag{2.7}$$

where $\delta^{(r)}_{jk}$ denotes the (j,k)th element of the matrix $I^{(r)}$.

3 Discretization of Problem by Sinc-Collocation Method:

For discretization of the following problem

$$au''(x) + bu'(x) + F\Big(u(x), x\Big) = 0, \tag{3.1}$$

with boundary conditions:

$$\begin{cases} a_0u(0) + a_1u'(0) = 0, \\ b_0u(1) + b_1u'(1) = 0. \end{cases} \tag{3.2}$$

Since $\frac{d}{dx}[S(k,h)o\phi(x)]$ is undefined at $x=0$ and $x=1$ (by $\phi(x)$ in (2.4)) and because the mixed boundary conditions must be handled at endpoints by the approximation solution, thus we consider the approximation solution as follows

$$u_m(x) = c_{-M-1}w_a(x) + \frac{1}{\phi'(x)}\sum_{k=-M}^{N} c_kS_k(x) + c_{N+1}w_b(x), \qquad m = M+N+1. \tag{3.3}$$

In (3.3) $S_k(x)$ is from (2.5) and the boundary bases function w_a, w_b are cubic Hermit functions given by

$$w_a(x) = a_0 x(1-x)^2 - a_1(2x+1)(1-x)^2,$$

and

$$w_b(x) = b_1(-2x+3)x^2 + b_0(1-x)x^2.$$

These boundary functions interpolate the boundary conditions in (3.2) via the identities

$$w_a(0) = -a_1, w_a'(0) = a_0,$$

$$w_b(1) = b_1, w_b'(1) = -b_0,$$

and

$$w_a(1) = w_a'(1) = 0,$$
$$w_b(0) = w_b'(0) = 0.$$

For the purpose of illustrating the exposition of our method we define

$$Lu(x) = au''(x) + bu'(x), \tag{3.4}$$

then, (3.1) is now given by

$$Lu(x) = -F(x, u(x)). \tag{3.5}$$

In (3.3) the $M+N+3$ coefficients $\{c_k\}_{-M-1}^{N+1}$, are determined by substituting $u_m(x)$ into equation (3.1) and evaluating the result at the Sinc points

$$x_j = \frac{\exp(jh)}{1+\exp(jh)} \qquad j = -M-1, -M, ..., N, N+1, \tag{3.6}$$

where h is defined in (2.3).
For evaluating the result we need to evaluate first and second derivatives from (14), so we first differentiate from $\frac{1}{\phi'(x)} S(k,h)o\phi(x)$ as

$$\frac{d}{dx}\Big(\frac{1}{\phi'(x)} S(k,h)o\phi(x)\Big) = \Big(\frac{1}{\phi'}\Big)' S_k(x) + \frac{d}{d\phi} S_k(x), \tag{3.7}$$

so the first derivative of $u_m(x)$ is

$$u_m'(x) = c_{-M-1} w_a'(x) + \sum_{k=-M}^{N} c_k \Big[\Big(\frac{1}{\phi'}\Big)' S_k(x) + \frac{d}{d\phi} S_k(x)\Big] + c_{N+1} w_b'(x), \tag{3.8}$$

similarly by taking the second derivative from $\frac{1}{\phi'(x)} S(k,h)o\phi(x)$ we have

$$u_m''(x) = c_{-M-1} w_a''(x) + \sum_{k=-M}^{N} c_k \Big[\Big(\frac{1}{\phi'}\Big)'' S_k(x) + \phi'\Big(\frac{1}{\phi'}\Big)' \frac{d}{d\phi} S_k(x) + (\phi')\frac{d^2}{d\phi^2} S_k(x)\Big] + c_{N+1} w_b''(x), \tag{3.9}$$

and we know that

$$u_m(x_j) = \begin{cases} c_{-M-1} w_a(x_j) + \frac{c_j}{\phi'(x_j)} + c_{N+1} w_b(x_j), & j = -M, ..., N, \\ c_{-M-1} w_a(x_j) + c_{N+1} w_b(x_j), & j = -M-1, N+1. \end{cases}$$

Substituting (3.3), (3.8), (3.9) in (3.1) and multiplying the resulting equation by $\{\frac{1}{\phi'}\}$ and then setting $x = x_j$ as collocation points in (3.6), finally by using of $u_m(x_j)$ definition above, we obtain the following nonlinear system

$$\begin{aligned} c_{-M-1} \frac{Lw_a(x_j)}{\phi'(x_j)} &+ \sum_{k=-M}^{N} c_k \Big[a\frac{d^2}{d\phi^2} S_k(x_j) + g_1(x_j) \frac{d}{d\phi} S_k(x_j) + g_2(x_j) S_k(x_j)\Big] \\ &+ c_{N+1} \frac{Lw_b(x_j)}{\phi'(x_j)} = -\frac{F(u_m(x_j), x_j)}{\phi'(x_j)}, \qquad j = -M-1, \cdots, N+1, \end{aligned} \tag{3.10}$$

where

$$Lw_a = aw_a'' + bw_a', \tag{3.11}$$

$$Lw_b = aw_b'' + bw_b', \tag{3.12}$$

$$g_1(x) = a\left(\frac{1}{\phi'(x)}\right)' + b\left(\frac{1}{\phi'(x)}\right) = a\ (1-2x) + b\ x\ (1-x), \tag{3.13}$$

$$g_2(x) = \left(\frac{1}{\phi'(x)}\right)\left[a\left(\frac{1}{\phi'(x)}\right)'' + b\left(\frac{1}{\phi'(x)}\right)'\right] = x(1-x)\ (-2a + b\ (1-2x)). \tag{3.14}$$

Equations (3.10) gives $M+N+3$ nonlinear algebraic equations which can be solved for the unknown coefficients c_k by using Newton's method. Consequently, $u_m(x)$ given in (3.3) can be calculated.

4 Numerical Results

To validate the application of Sinc-collocation method to equation (1.1) with conditions (1.2) we camper the solution with numerical results by some classical techniques. We consider the special problem (1.3) with conditions (1.4) that occurs in adiabatic tubular chemical reactor. For this problem, the entire discrete system in (3.10) is constructed as follows.
Define the components of the $(M+N+3)\times 1$ columns vectors by

$$(\overrightarrow{a}_{-M-1})_j = \frac{Lw_a(x_j)}{\phi'(x_j)}, \qquad (\overrightarrow{b}_{-M-1})_j = w_a(x_j), \tag{4.1}$$

$$(\overrightarrow{a}_{N+1})_j = \frac{Lw_b(x_j)}{\phi'(x_j)}, \qquad (\overrightarrow{b}_{N+1})_j = w_b(x_j), \tag{4.2}$$

$$(\overrightarrow{f})_j = \frac{\lambda\mu}{\phi'(x_j)}, \qquad \mathbf{1} = (1,1,...,1,1)^T, \tag{4.3}$$

then the discrete system (3.10), by using lemma 2.1 can be represented by

$$AC + D(f)\Big(diag(\beta.\mathbf{1} - BC)diag(\exp(BC))\Big)\,\mathbf{1} = 0, \tag{4.4}$$

where the $(M+N+3)\times(M+N+3)$ matrix A is the border matrix

$$A = \Big[\overrightarrow{a}_{-M-1}\ \mid\ A_s\ \mid\ \overrightarrow{a}_{N+1}\Big], \tag{4.5}$$

such that

$$A_s = I^{(2)} + D(g_1)I^{(1)} + D(g_2)I^{(0)} \tag{4.6}$$

where $\overrightarrow{a}_{-M-1}, \overrightarrow{a}_{N+1}$ are vectors defined in (4.1), (4.2) and $I^{(r)}$ is $(M+N+3)\times(M+N+1)$ matrices which are defined in (2.7) and $D(g(x_j))$ denote the $(M+N+3)\times(M+N+3)$ diagonal matrix with

$$D\Big(g(x_j)\Big) = \begin{cases} g(x_i), & j = i, \\ 0, & j \neq i, \end{cases}$$

the $(M+N+3)\times(M+N+3)$ matrix B is the border matrix

$$B = \Big[\overrightarrow{b}_{-M-1}\ \mid\ B_s\ \mid\ \overrightarrow{b}_{N+1}\Big], \tag{4.7}$$

where $\overrightarrow{b}_{-M-1}$ and $\overrightarrow{b}_{N+1}$ are vectors defined in (4.1), (4.2) and

$$B_s = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ (\frac{1}{\phi'})(x_{-M}) & 0 & 0 & \dots & 0 \\ 0 & (\frac{1}{\phi'})(x_{-M+1}) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & & (\frac{1}{\phi'})(x_N) \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Table 1: Comparative of the result from existing methods

x	(CMP) [7]	(SM) [7]	(AM) [7]	(SGM) [2] (N=20)
0.0	0.006079	0.006048	0.006048	0.006048
0.2	0.018224	0.018192	0.018192	0.018192
0.4	0.030456	0.030424	0.030424	0.030424
0.6	0.042701	0.042669	0.042669	0.042669
0.8	0.054401	0.054371	0.054371	0.054371
1.0	0.061459	0.061458	0.061458	0.061458

Table 2: Result of the Sinc-collocation method

x	N=10	N=15	N=20
0.0	0.006045	0.006048	0.006047
0.2	0.018189	0.018192	0.018192
0.4	0.030422	0.030424	0.030424
0.6	0.042666	0.042668	0.042669
0.8	0.054367	0.054370	0.054371
1.0	0.061441	0.061455	0.061458

and $C = (c_{-M-1}, c_{-M}, ..., c_N, c_{N+1})^T$ and $exp(C) = (exp(c_{-M-1}), exp(c_{-M}), ..., exp(c_{N+1}))^T$, furthermore $diag(C)$ is the $(M+N+3)\times(M+N+3)$ diagonal matrix with

$$diag(C) = \begin{cases} c_i, & j = i, \\ 0, & j \neq i, \end{cases}$$

For Sinc-collocation method we use $\alpha = 1$, $d = \frac{\pi}{2}$ and also truncate the numerical results after the sixth decimal points.

Case 1

In this case, in problem (1.3)-(1.4) we use particular value of the parameters, $\lambda = 10$, $\beta = 3, \mu = 0.02$. For such value of the parameters, a unique solution is guaranteed by the contraction mapping principle [7] and several authoress solve this problem by these parameters, we compared our results with them. Table (1) gives a comparison of the results from the contraction mapping principle (CMP) [7], the shooting method (SM) [7], the Adomian's method (AM) [7] and the Sinc-Galerkin method (SGM) [2] but in Table (2) numerical results based on our method with N=10, 15, 20 are reported.

As shown in Table (1) and (2), the results using our method with N=20 agree with those of the shooting method, Adomian's method and Sinc-Galerkin method up to sixth decimal place. For this problem the results from the contraction mapping principle agree to at least three decimal place, because by using this method, the required integrations can not be done analytically so are evaluated numerically by using the trapezoidal rule which will involve errors.

Case 2

In problem (1.3)-(1.4) we use another values of the parameters for λ, β, μ that for them the existence of solution is guaranteed by [4]. In Table (3) the numerical results for $\lambda = 5$, $\beta = 0.53, \mu = 0.05$, in Table (4) for $\lambda = 5$,

Table 3: Numerical results for $\lambda = 5$, $\beta = 0.53, \mu = 0.05$ by Sinc-collocation method

x	N=10	N=15	N=20	N=30
0.0	0.005214	0.005216	0.005216	0.005216
0.1	0.007814	0.007816	0.007816	0.007816
0.2	0.010394	0.010395	0.010395	0.010395
0.3	0.012943	0.012944	0.012944	0.012944
0.4	0.015446	0.015448	0.015448	0.015448
0.5	0.017879	0.017881	0.017881	0.017881
0.6	0.020199	0.020200	0.020201	0.020201
0.7	0.022336	0.022338	0.022338	0.022338
0.8	0.024175	0.024177	0.024178	0.024178
0.9	0.025527	0.025530	0.025530	0.025530
1.0	0.026076	0.026080	0.026081	0.026081

$\mu = 0.7$, $\beta = 0.8$ and in Table (5) for $\lambda = 0.05$, $\mu = 0.5$, $\beta = 0.6$ are reported.

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Table 4: Numerical results for $\lambda = 5$, $\mu = 0.7$, $\beta = 0.8$ by Sinc-collocation method

x	N=10	N=15	N=20	N=30
0.0	0.101630	0.101642	0.101645	0.101646
0.1	0.151591	0.151604	0.151607	0.151607
0.2	0.199674	0.199687	0.199690	0.199690
0.3	0.245631	0.245644	0.245647	0.245647
0.4	0.289179	0.289192	0.289195	0.289195
0.5	0.329957	0.329970	0.329973	0.329973
0.6	0.367448	0.367462	0.367465	0.367466
0.7	0.400840	0.400856	0.400860	0.400860
0.8	0.428768	0.428789	0.428794	0.428794
0.9	0.448864	0.448893	0.448900	0.448902
1.0	0.456946	0.456991	0.457001	0.457005

Table 5: Numerical results for $\lambda = 0.05$, $\mu = 0.5$, $\beta = 0.6$ by Sinc-collocation method

x	N=10	N=15	N=20	N=30
0.0	0.2268657	0.2268659	0.2268659	0.2268659
0.1	0.2279443	0.2279445	0.2279445	0.2279445
0.2	0.2289112	0.2289114	0.2289114	0.2289114
0.3	0.2297660	0.2297662	0.2297662	0.2297662
0.4	0.2305083	0.2305085	0.2305085	0.2305085
0.5	0.2311377	0.2311379	0.2311379	0.2311379
0.6	0.2316538	0.2316540	0.2316540	0.2316540
0.7	0.2320561	0.2320563	0.2320563	0.2320563
0.8	0.2323441	0.2323443	0.2323443	0.2323443
0.9	0.2325173	0.2325175	0.2325175	0.2325175
1.0	0.2325752	0.2325754	0.2325754	0.2325754

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Anti-periodic boundary value problems involving nonlinear fractional q-difference equations

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Abstract

In this paper, we consider a class of anti-periodic boundary value problems involving nonlinear fractional q-difference equations. Some existence and uniqueness results are obtained by applying some standard fixed point theorems. As applications, some examples are presented to illustrate the main results.

Keywords: Fractional q-difference equations, anti-periodic boundary conditions, existence and uniqueness, fixed point theorem.

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1 Introduction

Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes and have recently received considerable attention. For examples and details of anti-periodic boundary conditions, see [2, 3, 4, 5, 7, 10] and the references therein.

The q-difference calculus or quantum calculus is an old subject that was initially developed by Jackson [17, 18], basic definitions and properties of q-difference calculus can be found in the book mentioned in [19].

The fractional q-difference calculus had its origin in the works by Al-Salam [8] and Agarwal [1]. More recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus were made, for example, q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform, q-Taylor's formula, Mittage-Leffler function [9, 22, 23], just to mention some.

Recently, boundary value problems of nonlinear fractional q-difference equations have aroused considerable attention. Many people pay attention to the existence and multiplicity of solutions or positive solutions for boundary value problems of nonlinear fractional q-difference equations by means of some fixed point theorems, such as the Krasnosel'skii fixed-point theorem, the Leggett-Williams fixed-point theorem, and the Schauder fixed-point theorem, For examples, see [11, 12, 20, 21, 26, 27, 28] and the references therein. Graef and Kong [16] investigated the uniqueness, existence, and nonexistence of positive solutions for the boundary value problem with fractional q-derivatives in terms of different ranges of λ. Ahmad et al. [6] studied the following nonlinear fractional q-difference equation with nonlocal boundary conditions by applying some well-known tools of fixed point theory such as Banach contraction principle, Krasnoselskiis fixed point theorem, and the Leray-Schauder nonlinear alternative. Zhao et al. [29] considered some existence results of positive solutions to nonlocal q-integral boundary value problem of nonlinear fractional q-derivatives equation using the generalized Banach contraction principle, the monotone iterative method, and Krasnoselskii's fixed point theorem.

El-Shahed and Hassan [13] studied the existence of positive solutions of the q-difference boundary value problem

$$\begin{cases} -(D_q^2 u)(t) = a(t)f(u(t)), \quad 0 \le t \le 1, \\ \alpha u(0) - \beta D_q u(0) = 0, \quad \gamma u(1) - \delta D_q u(1) = 0. \end{cases}$$

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Ferreira [14] and [15] considered the existence of positive solutions to nonlinear q-difference boundary value problems

$$\begin{cases} -(D_q^\alpha u)(t) = -f(t, u(t)), & 0 \le t \le 1, \quad 1 < \alpha \le 2 \\ u(0) = u(1) = 0, \end{cases}$$

and

$$\begin{cases} (D_q^\alpha u)(t) = -f(t, u(t)), & 0 \le t \le 1, \quad 2 < \alpha \le 3, \\ u(0) = (D_q u)(0) = 0, \ (D_q u)(1) = \beta \ge 0, \end{cases}$$

respectively. By applying a fixed point theorem in cones, sufficient conditions for the existence of nontrivial solutions were enunciated.

In this paper, we investigate the existence and uniqueness results for anti-periodic boundary value problems involving nonlinear fractional q-difference equations given by

$$\begin{cases} ({}^cD_q^\alpha u)(t) = f(t, u(t)), & t \in [0, 1], \quad 1 < \alpha \le 2, \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1), \end{cases} \tag{1.1}$$

where ${}^cD_q^\alpha$ denotes the Caputo fractional q-derivative of order α, and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. Our results are based on some standard fixed point theorems.

2 Preliminaries

For theconvenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory to facilitate analysis of problem (1.1). These details can be found in the recent literature; see [19] and references therein. Let $q \in (0, 1)$ and define

$$[a]_q = \frac{q^a - 1}{q - 1}, \quad a \in \mathbb{R}.$$

The q-analogue of the power $(a - b)^n$ with $n \in \mathbb{N}_0$ is

$$(a - b)^{(0)} = 1, \ (a - b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k), \ n \in \mathbb{N}, \ a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},$$

and satisfies $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \ \text{ and } \ (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1 - q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \ \text{ and } \ (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [19]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \quad {}_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)},$$

$$\left({}_xD_q \int_0^x f(x,t)d_qt\right)(x) = \int_0^x {}_xD_q f(x,t)d_qt + f(qx,x).$$

We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$ [14].

Definition 2.1 ([24]). Let $\alpha \ge 0$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0 f(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)d_qt, \quad \alpha > 0, x \in [0,1].$$

Definition 2.2 ([24]). The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by $D_q^0 f(x) = f(x)$ and

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Definition 2.3 ([24]). The fractional q-derivative of the Caputo type of order $\alpha \ge 0$ is defined by

$$({}^cD_q^\alpha f)(x) = (I_q^{m-\alpha} D_q^m f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 2.1 ([14]). *Let $\alpha, \beta \ge 0$ and f be a function defined on $[0,1]$. Then the next formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = I_q^{\alpha+\beta} f(x)$,

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2.2 ([14]). *Let $\alpha > 0$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then the following equality holds:*

$$(I_q^{\alpha c} D_q^\alpha f)(x) = f(x) - \sum_{k=0}^{m-1} \frac{x^k}{\Gamma_q(k+1)} (D_q^k f)(0),$$

where m is the smallest integer greater than or equal to α.

Lemma 2.3. *For any $y \in C[0,1]$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} ({}^cD_q^\alpha u)(t) = y(t), \quad t \in [0,1], \quad 1 < \alpha \le 2, \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1), \end{cases} \tag{2.2}$$

is given by

$$u(t) = \int_0^1 G(t,qs)y(s)d_qs,$$

where

$$G(t,s)=\begin{cases}\dfrac{2(t-s)^{(\alpha-1)}-(1-s)^{(\alpha-1)}}{2\Gamma_q(\alpha)}+\dfrac{(1-2t)(1-s)^{(\alpha-2)}}{4\Gamma_q(\alpha-1)}, & 0\le s\le t\le 1,\\ -\dfrac{(1-s)^{(\alpha-1)}}{2\Gamma_q(\alpha)}+\dfrac{(1-2t)(1-s)^{(\alpha-2)}}{4\Gamma_q(\alpha-1)}, & 0\le t\le s\le 1.\end{cases} \tag{2.3}$$

Proof. We may apply Lemma 2.1 and Lemma 2.2; we see that

$$u(t)=\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}y(s)d_qs+c_1+c_2t. \tag{2.4}$$

Differentiating both sides of (2.4), we obtain

$$(D_qu)(t)=\frac{1}{\Gamma_q(\alpha-1)}\int_0^t (t-qs)^{(\alpha-2)}y(s)d_qs+c_2.$$

Applying the boundary conditions for the problem (2.2), we find that

$$c_1=\frac{1}{2\Gamma(\alpha)}\int_0^1(1-s)^{(\alpha-1)}y(s)d_qs-\frac{1}{4\Gamma(\alpha-1)}\int_0^1(1-s)^{(\alpha-2)}y(s)d_qs,$$
$$c_2=\frac{1}{2\Gamma_q(\alpha-1)}\int_0^1(1-s)^{(\alpha-2)}y(s)d_qs.$$

Thus, the unique solution of (2.2) is

$$\begin{aligned}u(t) &= \int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}y(s)d_qs-\frac{1}{2}\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}y(s)d_qs-\frac{1-2t}{4}\int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}y(s)d_qs\\ &= \int_0^1 G(t,qs)y(s)d_qs,\end{aligned}$$

where $G(t,s)$ is given by (2.3). This completes the proof. □

3 Main results

In this section, we establish some sufficient conditions for the existence and uniqueness of solutions for boundary value problem (1.1).

Let $\mathbb{C}=C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with the norm defined by $\|u\|=\sup\{|u(t)|,t\in[0,1]\}$.

Now we state some known fixed point theorems which are needed to prove the existence of solutions for (1.1).

Theorem 3.1 ([25]). *Let X be a Banach space. Assume that $T:X\to X$ is a completely continuous operator and the set $V=\{u\in X|u=\mu Tu, 0<\mu<1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.2 ([25]). *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta\in\Omega$ and let $T:\overline{\Omega}\to X$ be a completely continuous operator such that*

$$\|Tu\|\le\|u\|,\quad \forall u\in\partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

We define, in relation to (1.1), an operator $T:\mathbb{C}\to\mathbb{C}$ as follows

$$\begin{aligned}(Tu)(t) &= \int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs-\frac{1}{2}\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs\\ &\quad -\frac{1-2t}{4}\int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs,\quad t\in[0,1].\end{aligned} \tag{3.1}$$

From Lemma 2.3, we observe that the problem (3.1) has a solution if and only if the operator T has a fixed point.

Theorem 3.3. *Assume that there exists a positive constant M such that $|f(t,u)| \le M$ for $t \in [0,1]$ and $u \in \mathbb{C}$. Then the problem (1.1) has at least one solution.*

Proof. We show, as a first step, that the operator T is completely continuous. Clearly, continuity of the operator T follows from the continuity of f. Let $\Omega \in \mathbb{C}$ be bounded. Then, $u \in \Omega$ together with the assumption $|f(t,u)| \le M$, we get

$$\begin{aligned}
|(Tu)(t)| &\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs \\
&\quad + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs \\
&\le M\left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs\right) \\
&\le \frac{M(3\Gamma_q(\alpha)+\Gamma_q(\alpha+1))}{2\Gamma_q(\alpha)\Gamma_q(\alpha+1)} = M_2,
\end{aligned}$$

which implies that $\|(Tu)(t)\| \le M_2$. Furthermore,

$$\begin{aligned}
|D_q(Tu)(t)| &\le \int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs \\
&\le M\left(\int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs\right) \\
&\le \frac{3M}{2\Gamma_q(\alpha)} = M_3,
\end{aligned}$$

Hence, for $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| \le \int_{t_1}^{t_2} |D_q(Tu)(s)|d_qs \le M_3(t_2 - t_1).$$

This implies that T is equicontinuous on $[0,1]$. Thus, by the Arzela-Ascoli theorem, the operator $T : \mathbb{C} \to \mathbb{C}$ is completely continuous.

Next, we consider the set $V = \{u \in X | u = \mu Tu, 0 < \mu < 1\}$, and show that the set V is bounded. Let $u \in V$; then $u = \mu Tu$, $0 < \mu < 1$. For any $t \in [0,1]$, we have

$$|u(t)| = \mu|(Tu)(t)| \le |(Tu)(t)| = M_2.$$

Thus, $\|u\| \le M_2$ for any $t \in [0,1]$. So, the set V is bounded. Thus, by the conclusion of Theorem 3.1, the operator T has at least one fixed point, which implies that (1.1) has at least one solution. The proof is complete. □

Theorem 3.4. *Let $\lim_{u\to 0} f(t,u)/u = 0$. Then the problem (1.1) has at least one solution.*

Proof. Since $\lim_{u\to 0} f(t,u)/u = 0$, there therefore exists a constant $r > 0$ such that $|f(t,u)| \le \delta|u|$ for $0 < |u| < r$, where $\delta > 0$ is such that $M_2\delta < 1$.

Define $\Omega = \{u \in \mathbb{C} | \|u\| < r\}$ and take $u \in \mathbb{C}$ such that $\|u\| = r$, that is, $u \in \partial\Omega$. As before, it can be shown that T is completely continuous and $|(Tu)(t)| \le M_2\delta\|u\|$, which, in view of $M_2\delta < 1$, yields $\|Tu\| \le \|u\|$, $u \in \partial\Omega$. Therefore, by Theorem 3.2, the operator T has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution. □

Theorem 3.5. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function satisfying*

$$|f(t,u) - f(t,v)| \le L|u - v|, \quad \forall t \in [0,1], u, v \in \mathbb{R}$$

with

$$L \le \frac{\Gamma_q(\alpha)\Gamma_q(\alpha+1)}{3\Gamma_q(\alpha) + \Gamma_q(\alpha+1)}.$$

Then the problem (1.1) has a unique solution.

Proof. Defining $\sup_{t\in[0,1]}|f(t,0)| = K < \infty$ and selecting

$$r \geq \frac{K(3\Gamma_q(\alpha)+\Gamma_q(\alpha+1))}{\Gamma_q(\alpha)\Gamma_q(\alpha+1)},$$

we show that $TB_r \subset B_r$, where $B_r = \{u \in \mathbb{C} : \|u\| \leq r\}$. For $u \in B_r$, we have

$$\begin{aligned}
&|(Tu)(t)| \\
\leq\ & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs \\
&+\frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs \\
\leq\ & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}(|f(s,u(s))-f(s,0)|+|f(s,0)|)d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}(|f(s,u(s)) \\
&-f(s,0)|+|f(s,0)|)d_qs + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(|f(s,u(s))-f(s,0)|+|f(s,0)|)d_qs \\
\leq\ & (Lr+K)\left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs\right) \\
\leq\ & (Lr+K)\frac{3\Gamma_q(\alpha)+\Gamma_q(\alpha+1)}{2\Gamma_q(\alpha)\Gamma_q(\alpha+1)} \leq r.
\end{aligned}$$

Taking the maximum over the interval $[0,1]$, we get $\|(Tu)(t)\| \leq r$. Now, for $u, v \in \mathbb{C}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
&\|(Tu)(t)-(Tv)(t)\| \\
\leq\ & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))-f(s,v(s))|d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))-f(s,v(s))|d_qs \\
&+\frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))-f(s,v(s))|d_qs \\
\leq\ & L\|u-v\|\left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs\right) \\
\leq\ & \frac{L(3\Gamma_q(\alpha)+\Gamma_q(\alpha+1))}{2\Gamma_q(\alpha)\Gamma_q(\alpha+1)}\|u-v\| = \Lambda_{L,\alpha}\|u-v\|,
\end{aligned}$$

where

$$\Lambda_{L,\alpha} = \frac{L(3\Gamma_q(\alpha)+\Gamma_q(\alpha+1))}{2\Gamma_q(\alpha)\Gamma_q(\alpha+1)},$$

which depends only on the parameters involved in the problem. As $\Lambda_{L,\alpha} < 1$, T is therefore a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (the Banach fixed point theorem). □

4 Some examples

Example 4.1. Consider the anti-periodic fractional q-difference boundary value problem

$$\begin{cases} ({}^cD_q^\alpha u)(t) = \dfrac{e^{-\cos^2 u(t)}[5+\cos 2t + 4\ln(5+2\sin^2 u(t))]}{2+\sin^2 u(t)}, \quad t\in[0,1], \quad 1<\alpha\leq 2, \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1). \end{cases} \tag{4.1}$$

Clearly, $M = 3 + 2\ln 7$, and the hypothesis of Theorem 3.3 holds. Therefore, the conclusion of Theorem 3.3 implies that the problem (4.1) has at least one solution.

Example 4.2. Consider the anti-periodic fractional q-difference boundary value problem

$$\begin{cases} ({}^cD_q^\alpha u)(t) = (16 + u^3(t))^{\frac{1}{2}} + 2(t^2+1)(\tan u(t) - u(t)) - 4, & t \in [0,1], \quad 1 < \alpha \le 2, \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1). \end{cases} \tag{4.2}$$

It can easily be verified that all the assumptions of Theorem 3.4 holds. Consequently, the conclusion of Theorem 3.4 implies that the problem (4.2) has at least one solution.

Example 4.3. Consider the anti-periodic fractional q-difference boundary value problem

$$\begin{cases} ({}^cD_q^\alpha u)(t) = \dfrac{e^{-\pi t}|u(t)|}{(5+e^{-\pi t})(1+|u(t)|)}, & t \in [0,1], \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1), \end{cases} \tag{4.3}$$

where $\alpha = 1.5$ and $q = 0.5$. Let

$$f(t,u) = \frac{e^{-\pi t}|u|}{(5+e^{-\pi t})(1+|u|)}.$$

Clearly, $L = 1/5$ as $|f(t,u) - f(t,v)| \le 1/5|u-v|$. Further,

$$\frac{L(3\Gamma_q(\alpha) + \Gamma_q(\alpha+1))}{\Gamma_q(\alpha)\Gamma_q(\alpha+1)} = \frac{3\Gamma_{0.5}(1.5) + \Gamma_{0.5}(2.5)}{5\Gamma_{0.5}(1.5)\Gamma_{0.5}(2.5)} \approx 0.721135 < 1.$$

Thus, all the assumptions of Theorem 3.5 are satisfied. Therefore, the conclusion of Theorem 3.5 implies that the problem (4.3) has a unique solution.

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Error detection of irreducible cyclic codes on q-ary symmetric channel

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Abstract

Irreducible cyclic codes are well-known classes of block codes. These codes have wide range of applications specifically in deep space. Their weight distribution of Irreducible cyclic codes is known in only a few cases specifically they are known for binary cyclic codes. Previously, it has been shown that irreducible binary cyclic codes of even dimension and their duals are either proper or not good for error detection. In this correspondence it has been established that irreducible cyclic codes in number of cases are proper when transmitted over q-ary symmetric channel. The nonzero weights of the codes treated with in this paper vary between one and four.

Keywords: binary cyclic codes, irreducible cyclic codes, weight distribution, probability of undetected error.

2010 MSC: 68P30, 94A24.

1 Introduction

Irreducible cyclic codes are the most useful block codes. These codes have numerous applications in space such as (32, 6) first-order binary Reed-Muller code was used on Mariner flight projects and the (24, 12) binary Golay code which has been proposed for a Mariner Jupiter/Saturn 1977 (MJS'77) are both (essentially) irreducible cyclic codes. These missions are part of the Mars Exploration Program of NASA. Non-binary irreducible cyclic codes could be used to conserve bandwidth for low rate, deep-space telemetry.

Irreducible cyclic codes are binary and non-binary block codes whose encoders are linear feedback shift registers, such that the polynomial that represents the feedback logic is irreducible. The weight enumerator of a block code of length n is the polynomial

$$A(x) = \sum_{i=0}^{n} A_i x^i \tag{1.1}$$

where A_i denotes the number of words of weight i in the code. The enumerator $A(Z)$ provides valuable information about the performance of the code, and is needed to compute the error probability associated with proposed decoding algorithms.

C is called an (n, k) irreducible cyclic code over F_p. It had been supposed that $q = p^s$ and $r = q^m$, where p is a prime, s and m are positive integers. A linear $[n, m, d]$ code over $GF(q)$ is a m-dimensional subspace of $GF(q)^n$ with minimum (Hamming) distance d. Let $N > 1$ be an integer dividing $r - 1$, and put

$$n = \frac{(r-1)}{N}$$

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Let ω be primitive element of $GF(r)$ and $\epsilon = \omega^N$. The set

$$C(r,N) = \{Tr_{\frac{r}{q}}(\alpha), Tr_{\frac{r}{q}}(\alpha\epsilon), Tr_{\frac{r}{q}}(\alpha\epsilon^{n-1}) : \alpha \in GF(R)\}$$

is called an irreducible cycle code $[n, M]$ over $GF(r)$, where $Tr_{\frac{r}{q}}$ is trace function from $GF(r)$ onto $GF(q)$ and M divides m.

Baumert and Mykkeltveit (1974) allowed to compute the weight enumerator of all $(n,k)p$-ary irreducible codes for which the integer $N = \frac{(p^k-1)}{n}$ is a prime congruent to 3 (mod 4) for which p has order $\frac{(N-1)}{2}$.

Properness of a linear error detecting code is a property which in a certain sense makes the code more appropriate for error detection over a symmetric memoryless channel than a non-proper one. This property is related to the undetected error probability of the code, which is a function of the channel symbol error probability, involving the code weight distribution. Number of authors (Leung and Hellman, 1976; Wolf et al 1982; Kasami et al 1983 and Kasami and Lin, 1984) had discussed the undetected error probability, $P_u(\epsilon)$, of linear $[n, k]$ block codes used solely for error detection on a binary symmetric channel (BSC) with bit error rate ϵ. Most of the work reported in the literature regarding the undetected error probability is restricted to binary linear codes. Although research related to the undetected error probability on the binary symmetric channel is very important but its practical value is restricted by the fact that the binary symmetric channel does not always adequately describe real communication channels (Kana and Sastry, 1978).

Error detection is used extensively in communication and computer systems to combat noise. Detection is accomplished by examining the received word. If it is a codeword, the word is accepted as error-free. If it is not a codeword, the word is rejected as being erroneous. The undetected error occurs if an error-detecting scheme fails to detect an error i.e. if the received word is a codeword different from the transmitted codeword. The probability of undetected error is given by (MacWilliams and Sloane, 1977)

$$P_u(\epsilon) = \sum_{i=1}^{n} A_i \big(\frac{\epsilon}{q-1}\big)(1-\epsilon)^{n-i} \tag{1.2}$$

where $0 \le \epsilon \le \frac{q-1}{q}$, A_i is the number of code words of weight i in code. For $i = 0, A_0 = 1$.

Also the weight enumerator given in (1.1) can be written as

$$A(x) - 1 = \sum_{i=0}^{n} A_i x^i.$$

Probability of undetected error $P_u(\epsilon)$ (1.2) of linear (n, k) code can be expressed as

$$\begin{aligned} P_u(\epsilon) &= \sum_{i=1}^{n} A_i \big(\frac{\epsilon}{q-1}\big)(1-\epsilon)^{n-i} \\ &= (1-\epsilon)^n \Big[A\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big) - 1\Big]. \end{aligned}$$

Code C is called good if

$$P_u(\epsilon) \le P_u\Big(\frac{q-1}{q}\Big) = \frac{(M-1)}{q^n} \tag{1.3}$$

for all $\epsilon \in \Big[0, \frac{q-1}{q}\Big]$, where M is number of information and a code is proper if $P_u(\epsilon)$ is an increasing function for $\epsilon \in \Big[0, \frac{q-1}{q}\Big]$. Proper codes are fine for error detection. If

$$P_u(\epsilon) \le q^{(-(n-k))} \, for \, 0 \le \epsilon \le \frac{q-1}{q}, \tag{1.4}$$

the code is called satisfying $q^{-(n-k)}$ bound. The code not satisfying $q^{-(n-k)}$ bound is not fine for error detection (Kasami et al 1983). Upper bound on undetected error probability for optimal linear codes is also studied by Wolf et al (1982) and Klove, (1984).

Earlier it was believed that this upper bound holds for all codes since it was assumed that $P_u(\epsilon)$ is increasing for $\epsilon \in \Big[0, \frac{q-1}{q}\Big]$ and $P_u(\epsilon)$ attains its maximum value at $\epsilon = \frac{q-1}{q}$. However, this assumption was shown to be wrong by some codes that do not obey the upper bound (Leung and Hellman, 1976) some classes of codes are known to obey this bound.

To classify codes as proper, non proper but good, or not good, often turns out to be complicated, and such a classification has been done so far for relatively few codes. Many codes which are known to be optimal or close to optimal in one sense or other, turn out to be proper, such as Maximum Distance Separable (MDS) codes, the Hamming codes, the Maximum Minimum Distance codes and their duals etc (Dodunekova et al 2008).

Our study of the codes $C(r, N)$ in this manuscript will reveal that the following irreducible cyclic codes are proper

whose length is $\frac{q^m-1}{N}$ and for any q satisfying the condition $q-1=\frac{N}{2}(modN)$.
whose length is $\frac{q^m-1}{3}$ and for any q satisfying the condition $q=1(mod3)$.
whose length is $\frac{q^m-1}{4}$ and for any q satisfying the condition $q=3(mod4)$.

In this correspondence, we study the error-detecting performance of the irreducible binary cyclic codes $C(r, N)$ introduced by Delsarte and Goethals (1970) by calculating probability of undetected error $P_u(\epsilon)$. The probability has been evaluated by using weight distribution of irreducible cyclic codes derived by Ding (2009). First, we derive a new formula on the probability of undetected error for irreducible cyclic codes. Second, using this new formula, we calculated the table which shows that the $P_u(\epsilon)$ is monotonic function w.r.t error rate ϵ. The rest of this correspondence is organized as follows. In Section 2, we review some basic properties of the weight distribution of irreducible cyclic codes. In Section 3, we derive a new formula on the probability of undetected error for irreducible cyclic code. This formula plays an important role in establishing that irreducible cyclic are proper for error detection.

2 Weight distribution of irreducible cyclic codes

Determining the weight distribution of the irreducible cyclic codes in general is difficult. However, in certain special cases the weight distribution is known. Delsarte and Goethals (1970) and Baumert and McEliece (1972) have determined this polynomial in many of the simpler cases. In particular, when $k=\frac{\phi(N)}{2}$ they indicate methods that can be used to solve the problem (at least for those cases with

$$\frac{(p^k-1)}{(p-1)}=0 \tag{2.1}$$

modulo N, as it always is for $p=2$). Here, when N is a prime number of the form $4t+1$ the code weight distributions are particularly nice. When N is a prime of the form $4t+3$, things are a bit more difficult.

Baumert and Mykkeltveit (1973) determined the weight distribution for prime values of N with $N = 3(mod4)$ and $ord_q(N)=\frac{N-1}{2}$.

McEliece and Rumsey (1972) also generalized these results and showed that the weights of an irreducible cyclic code can be expressed as a linear combination of Gauss sums via the Fourier transform. Helleseht, et al (1977) investigated the weight distribution of some irreducible cyclic codes. Schmidt and White (2002) gave a characterization of irreducible cyclic codes with at most two weights. Aubry and Langevin (2005) studied the divisibility of weights in binary irreducible cyclic codes. Segal and Ward computed the weight distributions of some irreducible cyclic codes Segal and Ward (1986). Moisio and Vaananen (1999) developed two recursive algorithms for computing the weight distribution of certain irreducible cyclic codes. Van der Vlugt (1995) investigated the weight hierarchy of irreducible cyclic codes.

However, weight distribution of only a few classes of irreducible cyclic codes is known. In contrast, little has been done on the determination of the weight distribution of the duals of irreducible cyclic codes. Ding et al (2002) determine the minimum distance and some weights of the duals of certain classes of binary irreducible cyclic codes. Ding et al (2002) show that the weight distribution of the duals of binary irreducible cyclic codes is totally determined by the cyclotomic numbers of certain order. Prior to this Niederreiter, (1977) determined the weight distribution by applying the semiprimitive cases, cyclotomy and exponential sums. Numerical examples of the weight distribution of certain minimal cyclic codes are given by MacWilliams and Seery (1981). In the semiprimitive cases and several special cases, the weight distribution of irreducible cyclic codes has been determined (Baumert and McEliece, (1972), Delsarte and Goethals, (1970), Helleseth et al (1977).

Ding (2009) described the weight distribution of the irreducible cyclic codes for all N with $2 \leq N \leq 4$ and a few other cases. The number of distinct nonzero weights in the irreducible cyclic codes dealt with in this paper varies between one and four.

3 Undetected error probability for irreducible cyclic codes

This section analyzes the properness of irreducible cyclic codes by finding undetected error probability over q-ary symmetric channel. The probability has been found from the weight distribution of irreducible cyclic codes which is given by Ding (2009).

Theorem 3.1. *Let gcd(n,N)=1, where N is even. If* $q-1=\frac{N}{2}(modeN)$ *and* $gcd\Big(\frac{r-1}{q-1}modN, N\Big)=2$, *then the set* $C(r,N)$ *is a* $\Big[\frac{q^m-1}{N}, m, \frac{(q-1)(r-\sqrt{r})}{Nq}\Big]$ *two-weight code with weight distribution*

$$A(x)=1+\frac{r-1}{2}x^{\frac{(q-1)(r-\sqrt{r})}{Nq}}+\frac{r-1}{2}x^{\frac{(q-1)(r-\sqrt{r})}{Nq}}. \tag{3.1}$$

Theorem 3.2. *Irreducible cyclic codes* $C(r,N)$ *of length* $\frac{q^m-1}{N}$ *are proper for any* q *satisfying the condition* $q-1=\frac{N}{2}(modN)$.

Proof. $P_u(\epsilon)=(1-\epsilon)^n\Big[A\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)-1\Big]$. Using the weight distribution by (3.1), we get

$$P_u(\epsilon)=(1-\epsilon)^n\Big[1+\frac{r-1}{2}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r-\sqrt{r})}{Nq}}+\frac{r-1}{2}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r-\sqrt{r})}{Nq}}-1\Big] \tag{3.2}$$

$$=(1-\epsilon)^n\Big[\frac{r-1}{2}\Big\{\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r-\sqrt{r})}{Nq}}+\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r-\sqrt{r})}{Nq}}\Big\}\Big]$$

$$=\frac{r-1}{2}\Big(\frac{1}{1-\epsilon}\Big)^{\frac{(q-1)(r-\sqrt{r})}{q(r-1)}}\Big(\frac{\epsilon}{q-1}\Big)^{\frac{(q-1)(r-\sqrt{r})}{Nq}}\Big[1+\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{2\sqrt{r}(q-1)}{qN}}\Big] \tag{3.3}$$

Following tables has been derived by putting different values of r, ϵ, q and N in (3.3).

Table 1 and Table 2 shows that undetected error probability $P_u(\epsilon)$ is a monotonic function and increases with the error rate ϵ and it should obey the $q^{-(n-k)}$ bound, which proves that the codes are irreducible cyclic codes are proper for error detection. □

Theorem 3.3. *Let* $q=1(mod3), p=2(mod3)$, *and* $m=0(mod3)$. *Let* $r-1=nN$, *where* $N=3$. *If* $sm=0(mod4)$, *then* $C(r,3)$ *is an* $[(r-1)/3, m, ((q-1)(r-\sqrt{r}))/3q]$ *code over* $GF(q)$ *with weight distribution*

$$A(x)=1+\frac{2(r-1)}{3}x^{\frac{(q-1)(r-\sqrt{r})}{3q}}+\frac{r-1}{3}x^{\frac{(q-1)(r+2\sqrt{r})}{3q}} \tag{3.4}$$

If $sm=2(mod4)$ *then* $C(r,3)$ *is an* $[(r-1)/3, m, ((q-1)(r-2\sqrt{r}))/3q]$ *code over* $GF(q)$ *with weight distribution*

$$A(x)=1+\frac{(r-1)}{3}x^{\frac{(q-1)(r-2\sqrt{r})}{3q}}+\frac{2(r-1)}{3}x^{\frac{(q-1)(r+\sqrt{r})}{3q}}. \tag{3.5}$$

Theorem 3.4. *Irreducible cyclic codes* $C(r,3)$ *of length* $\frac{q^m-1}{3}$ *are proper for any* q *satisfying the condition* $q=1(mod3)$.

Proof. Proof of this theorem is on the same pattern as that of Theorem 3.2.
Case I: If $sm=2(mod4)$,

$$P_u(\epsilon)=(1-\epsilon)^n\Big[1+\frac{2(r-1)}{3}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r-\sqrt{r})}{3q}}+\frac{r-1}{3}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r+2\sqrt{r})}{3q}}-1\Big] \tag{3.6}$$

$$=\frac{r-1}{3}\Big(\frac{1}{1-\epsilon}\Big)^{\frac{(q-1)(r-\sqrt{r})}{q(r-1)}}\Big(\frac{\epsilon}{q-1}\Big)^{\frac{(q-1)(r-\sqrt{r})}{3q}}\Big[2+\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{\sqrt{r}(q-1)}{q}}\Big]$$

Case II: If $sm=2(mod4)$,

$$P_u(\epsilon)=(1-\epsilon)^n\Big[1+\frac{(r-1)}{3}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r-2\sqrt{r})}{3q}}+\frac{2(r-1)}{3}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r+\sqrt{r})}{3q}}-1\Big] \tag{3.7}$$

$$=\frac{r-1}{3}\Big(\frac{1}{1-\epsilon}\Big)^{\frac{(q-1)(r-2\sqrt{r})}{q(r-1)}}\Big(\frac{\epsilon}{q-1}\Big)^{\frac{(q-1)(r-2\sqrt{r})}{3q}}\Big[1+2\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{\sqrt{r}(q-1)}{q}}\Big]$$

The values listed in Table 3 and Table 4 illustrates that undetected error probability $P_u(\epsilon)$ increases with error rate ϵ. This proves are theorem that irreducible cyclic codes $C(r,3)$ are proper. □

Theorem 3.5. *Let $q = 3(mod4)$, and let $r - 1 = nN$, where $N = 4$. If $m = 0(mod4)$, then $C(r,4)$ is an $[(r-1)/4, m, ((q-1)(r-\sqrt{r}))/4q]$ code over $GF(q)$ with weight distribution*

$$A(x) = 1 + \frac{3(r-1)}{4}x^{\frac{(q-1)(r-\sqrt{r})}{4q}} + \frac{r-1}{4}x^{\frac{(q-1)(r+3\sqrt{r})}{4q}} \tag{3.8}$$

If $m = 2(mod4)$ then $C(r,4)$ is an $[(r-1)/4, m, ((q-1)(r-3\sqrt{r}))/4q]$ code over $GF(q)$ with weight distribution

$$A(x) = 1 + \frac{(r-1)}{4}x^{\frac{(q-1)(r-3\sqrt{r})}{4q}} + \frac{3(r-1)}{4}x^{\frac{(q-1)(r+\sqrt{r})}{4q}}. \tag{3.9}$$

Theorem 3.6. *Irreducible cyclic codes $C(r,4)$ of length $\frac{q^m-1}{4}$ are proper for any q satisfying the condition $q = 3(mod4)$.*

Proof. Case I: If $m = 0(mod4)$,

$$P_u(\epsilon) = (1-\epsilon)^n\Big[1 + \frac{3(r-1)}{4}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r-\sqrt{r})}{4q}} + \frac{r-1}{4}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r+3\sqrt{r})}{4q}} - 1\Big] \tag{3.10}$$

$$= \frac{r-1}{4}\Big(\frac{1}{1-\epsilon}\Big)^{\frac{(q-1)(r-\sqrt{r})}{q(r-1)}}\Big(\frac{\epsilon}{q-1}\Big)^{\frac{(q-1)(r-\sqrt{r})}{4q}}\Big[3 + \Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{\sqrt{r}(q-1)}{q}}\Big]$$

Case II: If $m = 2(mod4)$,

$$P_u(\epsilon) = (1-\epsilon)^n\Big[1 + \frac{(r-1)}{4}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r-3\sqrt{r})}{4q}} + \frac{3(r-1)}{4}\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{(q-1)(r+\sqrt{r})}{4q}} - 1\Big] \tag{3.7}$$

$$= \frac{r-1}{4}\Big(\frac{1}{1-\epsilon}\Big)^{\frac{(q-1)(r-3\sqrt{r})}{q(r-1)}}\Big(\frac{\epsilon}{q-1}\Big)^{\frac{(q-1)(r-3\sqrt{r})}{4q}}\Big[1 + 3\Big(\frac{\epsilon}{(1-\epsilon)(q-1)}\Big)^{\frac{\sqrt{r}(q-1)}{q}}\Big]$$

The values listed in Table 5 and Table 6 illustrates that undetected error probability $P_u(\epsilon)$ increases with error rate ϵ. This proves are theorem that irreducible cyclic codes $C(r,3)$ are proper. □

4 Conclusion

Irreducible cyclic codes are of practical interest as they have been used in transmission of data. In this work we had examined the performance of these codes in terms of probability of undetected error when codes are transmitted through q-ary symmetric channel. It has been substantiated that $C(r,N)$ for $2 \leq N \leq 4$ irreducible cyclic codes are proper codes.

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Table 1: Undetected Error Probability $P_u(\epsilon)$ of $[39, 2, 36]$ cyclic irreducible code

r	ϵ	q	N	$P_u(\epsilon)$
625	0.1	25	16	7.05972E-84
625	0.2	25	16	5.40861E-73
625	0.3	25	16	1.3363E-66
625	0.4	25	16	4.84733E-62
625	0.5	25	16	1.76756E-58
625	0.6	25	16	1.53968E-55
625	0.7	25	16	5.16578E-53
625	0.8	25	16	9.22568E-51
625	0.9	25	16	1.27255E-48

Table 2: Undetected Error Probability $P_u(\epsilon)$of $[21, 2, 18]$ cyclic irreducible code

r	ϵ	q	N	$P_u(\epsilon)$
169	0.1	13	8	3.45333E-36
169	0.2	13	8	1.00144E-30
169	0.3	13	8	1.65956E-27
169	0.4	13	8	3.35991E-25
169	0.5	13	8	2.18149E-23
169	0.6	13	8	7.04171E-22
169	0.7	13	8	1.45259E-20
169	0.8	13	8	2.34179E-19

Table 3: Undetected Error Probability $P_u(\epsilon)$of $[21, 3, 14]$ cyclic irreducible code

r	ϵ	q	$P_u(\epsilon)$
64	0.2	4	1.67E-15
64	0.3	4	5.33E-13
64	0.4	4	3.31E-11
64	0.5	4	8.51E-10
64	0.6	4	1.28E-08
64	0.7	4	1.48E-07
64	0.8	4	4.3E-06
64	0.9	4	0.003408

Table 4: Undetected Error Probability $P_u(\epsilon)$of $[21, 3, 12]$ cyclic irreducible code

r	ϵ	q	$P_u(\epsilon)$
64	0.2	4	1.83866E-13
64	0.3	4	5.32745E-13
64	0.4	4	3.31373E-11
64	0.5	4	8.51366E-10
64	0.6	4	1.27745E-08
64	0.7	4	1.47605E-07

Table 5: Undetected Error Probability $P_u(\epsilon)$of $[20, 4, 12]$ cyclic irreducible code

r	ϵ	q	$P_u(\epsilon)$
81	0.1	3	8.66909E-20
81	0.2	3	6.85958E-11
81	0.3	3	4.33931E-12
81	0.4	3	4.7521E-10
81	0.5	3	1.89687E-08
81	0.6	3	4.33238E-07

Table 6: Undetected Error Probability $P_u(\epsilon)$of $[12, 4, 6]$ cyclic irreducible code

r	ϵ	q	$P_u(\epsilon)$
49	0.1	7	1.59094E-20
49	0.2	7	7.11832E-17
49	0.3	7	1.02086E-14
49	0.4	7	3.61778E-13
49	0.5	7	6.03604E-12
49	0.6	7	6.36294E-11
49	0.7	7	5.0283E-10
49	0.8	7	3.52614E-09

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An Intuitionistic fuzzy count and cardinality of Intuitionistic fuzzy sets

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Abstract

The notion of Intuitonistic fuzzy sets was introduced by Atanassov [1] as an extension of the concept of fuzzy sets introduced by Zadeh such that it is applicable to more real life situations. In order to measure the cardinality of fuzzy sets several attempts have been made [4,6,8]. However, there are no such measures for intuitionistic fuzzy sets. In this paper we define the sigma count and relative sigma count for intuitionistic fuzzy sets and establish their properties. Also, we illustrate the genration of quantification rules.

Keywords: Fuzzy set, Intuitionistic Fuzzy set, Intuitionistic fuzzy count, Relative Intuitionistic fuzzy count.

2010 MSC: 57E05.

1 Introduction

The introduction of the fuzzy concept by Zadeh [7] is considered as a paradigm shift [5]. It introduces the concept of graded membership of elements instead of the binary membership used in Aristotelian logic. It is a very powerful modeling language that can cope with a large fraction of uncertainties of real life situations. Because of its generality it can be well adapted to different circumstances and contexts.

The cardinality of a set in the crisp sense plays an important role in Mathematics and its applications. Similarly it is worthwhile to think of cardinality of fuzzy sets, which is a measure. The concept of cardinality of a fuzzy set is an extension of the count of elements of a crisp set. A simple way of extending the concept of cardinality was suggested by Deluca and Termini [4]. This concept is related to the notion of the probability measure of a fuzzy set introduced by Zadeh [8] and is termed as the sigma count or the non-fuzzy cardinality of a set.

According to fuzzy set theory, the non-membership value of an element is one's complement of its membership value. However, in practical cases it is observed that this happens to be a serious constraint. So, Atanassov [1] introduced the notion of intuitionistic fuzzy sets as a generalisation of the concept of fuzzy sets which does not have the deficiency mentioned above. Unlike, the cardinality of a fuzzy set ([4],[6],[8]) there are no definitions of the cardinality of an intuitionistic fuzzy set in the literature. In this paper we introduce the sigma count as an extension of the notion of the corresponding notion for fuzzy sets and establish many properties. Also, we introduce the notion of relative sigma count and establish some properties. Finally we illustrate the generation of quantification rules.

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2 Definitions and Notations

In this section we shall provide some definitions and notations to be used in this paper. First we introduce the notion of a fuzzy set.

Definition 2.1. *Let X be a universal set. Then a fuzzy set A on X is defined through a membership function associated with A and denoted by μ_A as*

$$\mu_A : X \rightarrow [0,1], \tag{2.1}$$

such that every $x\epsilon X$ is associated with its membership value $\mu_A(x)$ lying in the interval $[0,1]$.

Clearly, the fuzzy set A is completely characterized by the set of points $\{(x, \mu_A(x)) : x\epsilon X\}$.

Definition 2.2. *For any two fuzzy sets A and B in X, we define the relationships between A and B as*

$$A = B \ iff \ \mu_A(x) = \mu_B(x), \forall x\epsilon X \tag{2.2}$$

$$A \subseteq B \ iff \ \mu_A(x) \leq \mu_B(x), \forall x\epsilon X \tag{2.3}$$

$$B \supseteq A \ iff \ A \subseteq B \tag{2.4}$$

Definition 2.3. *The union of the two fuzzy sets A and B is given by its membership function $\mu_{A\cup B}(x)$ defined by*

$$\mu_{A\cup B}(x) = max\{\mu_A(x), \mu_B(x)\}, \forall x\epsilon X. \tag{2.5}$$

Definition 2.4. *The intersection of the two fuzzy sets A and B is given by its membership function $\mu_{A\cap B}(x)$ defined by*

$$\mu_{A\cap B}(x) = min\{\mu_A(x), \mu_B(x)\}, \forall x\epsilon X. \tag{2.6}$$

Definition 2.5. *The complement $\bar{A}$ of the fuzzy set A with respect to universal set X is given by its membership function $\mu_{\bar{A}}(x)$ defined by*

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x), \forall x\epsilon X. \tag{2.7}$$

Definition 2.6. *Let X be an universal set. An intiitionistic fuzzy set or IFS A on X is defined through two functions μ_A and ν_A, called the membership and non-membership functions of A defined as*

$$\mu_A : X \rightarrow [0,1] \ and \ \nu_A : X \rightarrow [0,1] \tag{2.8}$$

such that every $x\epsilon X$ is associated with its membership value $\mu_A(x)$ and non-membership value $\nu_A(x)$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.7. *If A and B are two IFSs of the set X, then*

$$A \subset B \ iff \ \forall x\epsilon X, \mu_A(x) \leq \mu_B(x) \ and \ \nu_A(x) \geq \nu_B(x) \tag{2.9}$$

$$A \subset B \ iff \ B \supset A \tag{2.10}$$

$$A = B \ iff \ \forall x\epsilon X, [\mu_A(x) = \mu_B(x) \ and \ \nu_A(x) = \nu_B(x)] \tag{2.11}$$

$$\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x\epsilon X\} \tag{2.12}$$

$$A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x))\rangle : x\epsilon X\} \tag{2.13}$$

$$A \cup B = \{\langle x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x))\rangle : x\epsilon X\} \tag{2.14}$$

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x)\rangle : x\epsilon X\} \tag{2.15}$$

$$A \cdot B = \{\langle x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)\rangle : x \epsilon X\} \tag{2.16}$$

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \epsilon X\} \tag{2.17}$$

$$*A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \epsilon X\} \tag{2.18}$$

$$C(A) = \{\langle x, K, L\rangle : x \epsilon X\}, \; where \; K = \max_{x \epsilon X} \mu_A(x) \; and \; L = \min_{x \epsilon X} \nu_A(x) \tag{2.19}$$

$$I(A) = \{\langle x, k, l\rangle : x \epsilon X\}, \; where \; k = \min_{x \epsilon X} \mu_A(x) \; and \; l = \max_{x \epsilon X} \nu_A(x) \tag{2.20}$$

3 Cardinality of Intuitionistic Fuzzy Sets

In this section, we introduce the cardinality of intuitionistic fuzzy sets and establish some properties.

3.1 Definitions

The measure of fuzzy set is the form of its Σ count (sigma count) was introduced by Deluca and Termini [4] as a simple extension of the concept of cardinality of crisp sets. As mentioned above Intuitionistic fuzzy sets have better modeling power than those of fuzzy sets, by the way introducing the hesitation part. Here we define the cardinality of an Intuitionistic fuzzy set by extending the notion of Σ count stated above. Also, we establish some of their properties, and provide certain examples and application of these results.

Definition 3.1. *A fuzzy set A on X to be finite if $\mu_A(x) \neq 0$ for only a finite number of elements of X.*

Definition 3.2. *For any finite fuzzy set A on X, the sigma count of A, denoted by Σ count (A) is given by*

$$\Sigma \quad count \quad (A) = \sum_{x \epsilon X} \mu_A(x) \tag{3.21}$$

Definition 3.3. *For any IFS A on X we define cardinality of A (denoted by $\Sigma count(A)$) as*

$$\Sigma \quad count \quad (A) = \left[\sum_{i=1}^{n} \mu_A(x_i), \sum_{i=1}^{n} 1 - \nu_A(x_i)\right] \tag{3.22}$$

$$= [\Sigma \; count \; \Box A, \Sigma \; count \; * A] \tag{3.23}$$

It may be noted that when A is a fuzzy set on $X, \nu_A(x) = 1 - \mu_A(x)$, for all $x \epsilon X$, so that

$$\Sigma \quad count \quad (A) = \left[\sum_{i=1}^{n} \mu_A(x_i), \sum_{i=1}^{n} \mu_A(x_i)\right] = \sum_{i=1}^{n} \mu_A(x_i)$$

which is the definition of Σ count of a fuzzy set A defined above.

3.2 Properties of Σ count

We establish some properties of σ count of IFSs in this section.

Theorem 3.1. *For any two IFSs A and B on X*

(i) Σ count $(A \cup B) + \Sigma$ count $(A \cap B) = \Sigma$ count $(A) + \Sigma$ count(B)

(ii) Σ count $(A + B) + \Sigma$ count $(A \cdot B) = \Sigma$ count $(A) + \Sigma$ count(B)

Proof. We have

$$\Sigma \ \ count \ \ (A \cup B) = \left[\sum_{i=1}^{n}(\mu_A(x_i) \vee \mu_B(x_i)), \sum_{i=1}^{n} 1 - (\nu_A(x_i) \wedge \nu_B(x_i))\right]$$

$$= \left[\sum_{i=1}^{n}(\mu_A(x_i) \vee \mu_B(x_i)), \sum_{i=1}^{n}\{(1 - \nu_A(x_i)) \vee (1 - \nu_B(x_i))\}\right]$$

and

$$\Sigma \ \ count \ \ (A \cap B) = \left[\sum_{i=1}^{n}(\mu_A(x_i) \wedge \mu_B(x_i)), \sum_{i=1}^{n} 1 - (\nu_A(x_i) \vee \nu_B(x_i))\right]$$

$$= \left[\sum_{i=1}^{n}(\mu_A(x_i) \wedge \mu_B(x_i)), \sum_{i=1}^{n}\{(1 - \nu_A(x_i)) \wedge (1 - \nu_B(x_i))\}\right]$$

So,

$$\Sigma \ \ count \ \ (A \cup B) + \Sigma \ \ count \ \ (A \cap B) = \left[\sum_{i=1}^{n}(\mu_A(x_i) + \mu_B(x_i)), \sum_{i=1}^{n}\{(1 - \nu_A(x_i)) + (1 - \nu_B(x_i))\}\right]$$

$$= \left[\sum_{i=1}^{n}\mu_A(x_i), \sum_{i=1}^{n} 1 - \nu_A(x_i)\right] + \left[\sum_{i=1}^{n}\mu_B(x_i), \sum_{i=1}^{n} 1 - \nu_B(x_i)\right]$$

$$\Sigma \ \ count \ \ (A) + \Sigma \ \ count \ \ (B)$$

The proof of (ii) is similar to that of (i). □

Theorem 3.2. *For any two IFSs A and B on X*
(i) Σ *count* $\overline{(A \cup B)} + \Sigma$ *count* $(\overline{A \cap B}) = \Sigma$ *count* $(\bar{A}) + \Sigma$ *count*$(\bar{B})$
(ii) Σ *count* $(\overline{A + B}) + \Sigma$ *count* $(\overline{A \cdot B}) = \Sigma$ *count* $(\bar{A}) + \Sigma$ *count*$(\bar{B})$

Proof.

$$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \epsilon X\}$$

$$\overline{A \cup B} = \{\langle x, \nu_A(x) \wedge \nu_B(x), \mu_A(x) \vee \mu_B(x)\rangle : x \epsilon X\}$$

So,

$$\Sigma \ \ count \ \ (\overline{A \cup B}) = \left[\sum_{i=1}^{n}(\nu_A(x_i) \wedge \nu_B(x_i)), \sum_{i=1}^{n} 1 - (\mu_A(x_i) \vee \mu_B(x_i))\right]$$

$$= \left[\sum_{i=1}^{n}(\nu_A(x_i) \wedge \nu_B(x_i)), \sum_{i=1}^{n}\{(1 - \mu_A(x_i)) \wedge (1 - \mu_B(x_i))\}\right]$$

similarly,

$$A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \epsilon X\}$$

$$\overline{A \cap B} = \{\langle x, \nu_A(x) \vee \nu_B(x), \mu_A(x) \wedge \mu_B(x)\rangle : x \epsilon X\}$$

and

$$\Sigma \ \ count \ \ (\overline{A \cap B}) = \left[\sum_{i=1}^{n}(\nu_A(x_i) \vee \nu_B(x_i)), \sum_{i=1}^{n} 1 - (\mu_A(x_i) \wedge \mu_B(x_i))\right]$$

$$= \left[\sum_{i=1}^{n}(\nu_A(x_i) \vee \nu_B(x_i)), \sum_{i=1}^{n}\{(1 - \mu_A(x_i)) \vee (1 - \mu_B(x_i))\}\right]$$

Hence,

$$\Sigma \ \ count \ \ (\overline{A \cup B}) + \Sigma \ \ count \ \ (\overline{A \cap B}) = \left[\sum_{i=1}^{n}(\nu_A(x_i) + \nu_B(x_i)), \sum_{i=1}^{n}\{(1-\mu_A(x_i)) + (1-\mu_B(x_i))\}\right]$$

$$= \left[\sum_{i=1}^{n}\nu_A(x_i), \sum_{i=1}^{n}1-\mu_A(x_i)\right] + \left[\sum_{i=1}^{n}\nu_B(x_i), \sum_{i=1}^{n}1-\mu_B(x_i)\right]$$

$$= \Sigma \ \ count \ \ (\bar{A}) + \Sigma \ \ count \ \ (\bar{B})$$

The proof of (ii) is similar to that of (i) □

Next, by using the results of Atanassov [2,3], the following properties of Σ count of IFSs can be obtained.

Theorem 3.3. *For any IFSs A, we have:*
(i) Σ *count* $\Box A = \Sigma$ *count* $(\overline{*\bar{A}})$
(ii) Σ *count* $*A = \Sigma$ *count* $(\overline{\Box\bar{A}})$
(iii) Σ *count* $\Box\Box A = \Sigma$ *count* $\Box A$
(iv) Σ *count* $\Box * A = \Sigma$ *count* $*A$
(v) Σ *count* $*\Box A = \Sigma$ *count* $\Box A$
(vi) Σ *count* $** A = \Sigma$ *count* $*A$
(vii) Σ *count* $*\bar{A} = \Sigma$ *count* $\overline{\Box A}$
(viii) Σ *count* $\Box\bar{A} = \Sigma$ *count* $\overline{*A}$
(ix) Σ *count* $\bar{A} = \Sigma$ *count* A

It may be noted that from the definition of Σ *count of an IFS, it can be obtained directly that if* $A \subseteq B$ *then it is not always true that* Σ *count* $A \leq \Sigma$ *count* B. *Also*
(x) $\Sigma count \ \Box A \leq \Sigma count {*A}$

Proof. We have

$$\Sigma \ \ count \ \ \Box A = \left[\sum_{i=1}^{n}\mu_A(x_i), \sum_{i=1}^{n}1-(1-\mu_A(x_i))\right] = \sum_{i=1}^{n}\mu_A(x_i)$$

and

$$\Sigma \ \ count \ \ *A = \left[\sum_{i=1}^{n}(1-\nu_A(x_i)), \sum_{i=1}^{n}(1-\nu_A(x_i))\right] = \sum_{i=1}^{n}(1-\nu_A(x_i))$$

□

Also, by the definition of an IFS, $\mu_A(x_i) \leq 1-\nu_A(x_i), i = 1, 2, \ldots, n$. So the claim follows.

Theorem 3.4. *For any two IFSs A and B,*
(i) $\Sigma count \ \Box(A \cup B) = \Sigma count \ (\Box A \cup \Box B)$
(ii) $\Sigma count \ *(A \cup B) = \Sigma count(*A \cup *B)$
(iii) $\Sigma count \ \Box(A \cap B) = \Sigma count(\Box A \cap \Box B)$
(iv) $\Sigma count \ *(A \cap B) = \Sigma count(*A \cap *B)$
(v) $\Sigma count \ (\overline{A \cup B}) = \Sigma count(\bar{A} \cap \bar{B})$
(vi) $\Sigma count \ (\overline{A \cap B}) = \Sigma count(\bar{A} \cup \bar{B})$

Theorem 3.5. *For any two IFSs A and B on X,*
(i) $\Sigma count \ \Box(A \cup B) + \Sigma count \ \Box(A \cap B) = \Sigma count \ \Box A + \Sigma count \ \Box B$
(ii) $\Sigma count \ *(A \cup B) + \Sigma count \ *(A \cap B) = \Sigma count \ *A + \Sigma count \ *B$
(iii) $\Sigma count \ \Box(A + B) + \Sigma count \ \Box(A \cdot B) = \Sigma count \ \Box A + \Sigma count \ \Box B$
(iv) $\Sigma count \ *(A + B) + \Sigma count \ *(A \cdot B) = \Sigma count \ *A + \Sigma count \ *B$

Proof. (i)Σcount$\Box(A \cup B) = \Sigma$count $(\Box A \cup \Box B)$ and Σcount$\Box(A \cap B) = \Sigma$count $(\Box A \cap \Box B)$ so,

$$\Sigma\ count\ \Box(A \cup B) + \Sigma\ count\ \Box(A \cap B) = \Sigma\ count\ (\Box A \cup \Box B) + \Sigma\ count\ (\Box A \cap \Box B)$$

$$= \Sigma\ count\ \Box A + \Sigma\ count\ \Box B$$

□

Similarly (ii) can be established.
(iii) Σcount $\Box(A + B) + \Sigma$count $\Box(A \cdot B)$

Proof.

$$= \sum_{i=1}^{n} (\mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i) \cdot \mu_B(x_i)) + \sum_{i=1}^{n} \mu_A(x_i) \cdot \mu_B(x_i)$$

$$= \sum_{i=1}^{n} \mu_A(x_i) + \sum_{i=1}^{n} \mu_B(x_i) = \Sigma\ count\ \Box A + \Sigma\ count\ \Box B$$

□

Similarly (iv) can be established.

Note 3.2.1. By using Theorems 3.1, 3.2 and 3.5, we have the following

(i) $\Sigma\text{count}(A \cup B) + \Sigma\text{count} A \cap B = \Sigma\text{count}(A + B) + \Sigma\text{count}(A \cdot B)$

$$= \Sigma\ count\ A + \Sigma\ count\ B$$

(ii) $\Sigma\text{count}(\overline{A \cup B}) + \Sigma\text{count}\overline{A \cap B} = \Sigma\text{count}(\overline{A + B}) + \Sigma\text{count}(\overline{A \cdot B})$

$$= \Sigma\ count\ \bar{A} + \Sigma\ count\ \bar{B}$$

(iii) $\Sigma\text{count}\Box(A \cup B) + \Sigma\text{count}\Box(A \cap B) = \Sigma\text{count}\Box(A + B) + \Sigma\text{count}\Box(A \cdot B)$

$$= \Sigma\ count\ \Box A + \Sigma\ count\ \Box B$$

(iv) $\Sigma\text{count}{*}(A \cup B) + \Sigma\text{count}{*}(A \cap B) = \Sigma\text{count}{*}(A + B) + \Sigma\text{count}{*}(A \cdot B)$

$$= \Sigma\ count\ *A + \Sigma\ count\ *B$$

3.3 Relative Σ count

The notion of relative Σ count for fuzzy sets has been introduced by Zadeh [9].

Definition 3.4. *If A and B are two fuzzy sets, then we define the relative sigma count of A with respect to B as rel Σcount $(A/B) = (\Sigma count(A \cap B)) \cdot (\Sigma count(B))^{-1}$, if A and b are two IFSs, then*

$$\Sigma\ \ count\ \ (A \cap B) = \left[\sum_{i=1}^{n} \mu_A(x_i) \wedge \mu_b(x_i), \sum_{i=1}^{n} 1 - (\nu_A(x_i) \vee \nu_B(x_i))\right]$$

$$(\Sigma\ \ count\ \ (B))^{-1} = \left[\sum_{i=1}^{n} \mu_B(x_i), \sum_{i=1}^{n} 1 - \nu_B(x_i)\right]^{-1} = \left[\frac{1}{\sum_{i=1}^{n}(1 - \nu_B(x_i))}, \frac{1}{\sum_{i=1}^{n} \mu_B(x_i)}\right]$$

Consequently,

$$(\Sigma\ count\ (A \cap B)) \cdot (\Sigma\ count\ (B))^{-1} = \left[\frac{\sum_{i=1}^{n} \mu_A(x_i) \wedge \mu_B(x_i)}{\sum_{i=1}^{n}(1 - \nu_B(x_i))}, \frac{\sum_{i=1}^{n} 1 - (\nu_A(x_i) \vee \nu_B(x_i))}{\sum_{i=1}^{n} \mu_B(x_i)}\right]$$

$$= \left[\frac{\sum_{i=1}^{n} \mu_A(x_i) \wedge \mu_B(x_i)}{\sum_{i=1}^{n}(1 - \nu_B(x_i))}, \frac{\sum_{i=1}^{n}\{1 - \nu_A(x_i)) \wedge (1 - \nu_B(x_i))\}}{\sum_{i=1}^{n} \mu_B(x_i)}\right]$$

It may be noted that the right hand expression of the above interval may be greater than 1. For example, taking $X = \{x_1, x_2\}$ and A and B two IFSs over X defined by $A = \{(.8, .1)/x_1, (.2, .6)/x_2\}, B = \{(.6, .2)/x_1, (.3, .6)/x_2\}$. Then $(1-\nu_A(x_1)) \wedge (1-\nu_B(x_1)) + (1-\nu_A(x_2)) \wedge (1-\nu_B(x_2)) = .8 + .4 = 1.2$.

In view of the above remark, we define rel Σ count (A/B) for intuitionistic fuzzy sets as

$$rel\Sigma count(A/B) = \left[\frac{\sum_{i=1}^n \mu_A(x_i) \wedge \mu_B(x_i)}{\sum_{i=1}^n (1-\nu_B(x_i))}, min\left(1, \frac{\sum_{i=1}^n \{(1-\nu_A(x_i)) \wedge (1-\nu_B(x_i))\}}{\sum_{i=1}^n \mu_B(x_i)}\right)\right]$$

3.3.1 Some Pathological Cases

Case I: Suppose A and B are fuzzy sets. Then $A = \Box A, B = \Box B, 1-\nu_A = \mu_A$ and $1-\nu_B = \mu_B$. So

$$rel\Sigma count(A/B) = \frac{\sum_{i=1}^n \mu_A(x_i) \wedge \mu_B(x_i)}{\sum_{i=1}^n (1-\nu_B(x_i))}$$

which is same as the Prop (A/B) introduced by Zadeh.

Case II: Suppose A is an IFS and B is a fuzzy set. Then,$1-\nu_B = \mu_B$. So that

$$rel\Sigma count(A/B) = \left[\frac{\sum_{i=1}^n (\mu_A(x_i) \wedge \mu_B(x_i))}{\sum_{i=1}^n \mu_B(x_i)}, \frac{\sum_{i=1}^n ((1-\nu_A(x_i)) \wedge \mu_B(x_i)}{\sum_{i=1}^n \mu_B(x_i)}\right]$$

$$\frac{1}{\sum_{i=1}^n \mu_B(x_i)} = \left[\sum_{i=1}^n (\mu_A(x_i) \wedge \mu_B(x_i)), \sum_{i=1}^n ((1-\nu_A(x_i) \wedge \mu_B(x_i))\right]$$

Also, in this case the right hand limit of the interval is less than or equal to 1. So, we need not impost this additional restriction.

Case III: If a is a fuzzy set and B is a crisp set, then

$$rel\ \Sigma count(A/B) = \left[\frac{\sum_{i=1}^n \mu_A(x_i)}{Card(B)}, min\left(1, \frac{\sum_{i=1}^n \mu_A(x_i)}{Card(B)}\right)\right]$$

In particular when $B = X = \{x_1, x_2, \ldots x_n\}$, we get

$$rel\ \Sigma count(A/B) = \left[\frac{\sum_{i=1}^n \mu_A(x_i)}{n}, min\left(1, \frac{\sum_{i=1}^n \mu_A(x_i)}{n}\right)\right]$$

$$= \frac{\sum_{i=1}^n \mu_A(x_i)}{n} = \frac{1}{n}\sum_{i=1}^n \mu_A(x_i)$$

4 Some Applications

Definition 4.1. *Let A and B be two IFSs on X. Then the rel $\sum count(A/B)$ is defined by the interval $[e_1, e_2]$, where*

$$e_1 = \frac{\sum_{i=1}^n (\mu_A(x_i) \wedge \mu_B(x_i))}{\sum_{i=1}^n (1-\nu_B(x_i))} = min\left(1, \frac{\sum_{i=1}^n ((1-\nu_A(x_i)) \wedge (1-\nu_B(x_i))}{\sum_{i=1}^n \mu_B(x_i)}\right)$$

Here e_1 indicates the minimum amount of similarity between A and B and e_2 indicates the maximum amount of similarity between a and B.

Clearly, rel $\Sigma count(A/B) \subseteq [0,1]$ and rel $\Sigma count(A/B) \neq rel\Sigma count(B/A)$ in general.

$$rel\Sigma count(A/A) = \left[\frac{\sum_{i=1}^n \mu_A(x_i)}{\sum_{i=1}^n (1-\nu_A(x_i))}, 1\right].$$

Definition 4.2. *For a given class $\{A_i\}i\epsilon\lambda$ of IFSs on X, the IFS $'S'$ on X is said to be the super IFS if $S = \{< x, \mu_s(x), \nu_s(x) >: x\epsilon X\}$ where*

$$\mu_S(x) = \sup_{i\epsilon\lambda} \mu_{A_i}(x) \ \ and \ \ \nu_S(x) = \inf_{i\epsilon\lambda} \nu_{A_i}(x).$$

Definition 4.3. *Let A and B two IFSs on X. Then we say A dominates B if*

$$mid\ \ value(rel\ \ \Sigma\ \ count(A/S)) \geq mid\ \ value\ \ (rel\ \ \Sigma\ \ count\ \ (B/S))$$

	C_1	C_2	C_3	C_4	C_5	C_6
A	(.2, .7)	(.5, .2)	(.8, .1)	(.6, .3)	(.4, .5)	(.3, .6)
B	(.6, .2)	(.2, .7)	(.7, .3)	(.8, .2)	(.5, .3)	(.9, .1)
C	(.2, .7)	(.4, .5)	(.8, .2)	(.9, .1)	(.6, .3)	(.5, .2)
D	(.5, .4)	(.3, .5)	(.6, .3)	(.5, .3)	(.7, .2)	(.9, .0)

S	(.6, .2)	(.5, .2)	(.8, .1)	(.9, .1)	(.7, .2)	(.9, .0)

4.1 Case Studies

Case Study 1: Consider the problem of gradation of students of a class. The Characteristics, which are to determine the gradation, may be some characteristics as

- Skill
- Knowledge
- Discipline in the school
- Punctually
- Efficiency in extracurricular activities
- Age

A selector may have to use the above characteristics and make their evaluation for each student in a class, considering all the information. The gradation list can be prepared basing upon the evaluation and some technique. We may use the technique of dominance as defined in definition 4.3 as the factor of gradation.

To make a case study, we assume that the number of characteristics be six. On the basis of these six characteristics which we denote by C_1, C_2, C_3, C_4, C_5 and C_6, suppose there are four students with the characteristics as mentioned above in the form of a matrix:

The super IFS $'S'$ will be given above in the form of matrix:

$$rel\Sigma count(A/S) = \left[\frac{\sum_{i=1}^{n}(\mu_A(x_i) \wedge \mu_S(x_i))}{\sum_{i=1}^{n}(1-\nu_s(x_i))}, min\left(\frac{1, \sum_{i=1}^{n}((1-\nu_A(x_i)) \wedge (1-\nu_B(x_i)))}{\sum_{i=1}^{n}\mu_S(x_i)}\right)\right]$$

$$= \left[\frac{2.8}{5.2}, min\left(1, \frac{3.6}{4.4}\right)\right] = \left[\frac{7}{13}, \frac{9}{11}\right] = [.54, .82]$$

$$rel\Sigma count(B/S) = \left[\frac{3.7}{5.2}, min\left(1, \frac{4.2}{4.4}\right)\right] = \left[\frac{37}{52}, \frac{21}{22}\right] = [.71, .95]$$

$$rel\Sigma count(C/S) = \left[\frac{3.4}{5.2}, min\left(1, \frac{4}{4.4}\right)\right] = \left[\frac{17}{26}, \frac{10}{11}\right] = [.65, .9]$$

$$rel\Sigma count(D/S) = \left[\frac{3.5}{5.2}, min\left(1, \frac{4.3}{4.4}\right)\right] = \left[\frac{35}{52}, \frac{43}{44}\right] = [.67, .98]$$

The mid values of rel$\Sigma count(A/S)$, rel$\Sigma count(B/S)$, rel$\Sigma count(C/S)$ and rel$\Sigma count(D/S)$ are .68,.83,.75, .825 respectively. So, the grading is B, D, C, A.

Definition 4.4. *Let A be an IFS on $X = \{x_1, x_2, \ldots, x_n\}$. Then depth of A denoted by depth (A) is given by*

$$depth(A) = [n, n] - \Sigma count A = [n, n] - [a_1, a_2]$$

where

$$a_1 = \sum_{i=1}^{n}\mu_A(x_i) \quad and \quad a_2 = \sum_{i=1}^{n}(1-\nu_A(x_i))) = [n-a_2, n-a_1]$$

Clearly, $depth(X) = 0$ and depth $(\phi) = n$.

Definition 4.5. *Let A_1 and A_2 be two IFSs over X, then we say A_2 is a better representative of X than A_1 denoted by $A_2 \supset A_1$, if and only if*

$$| \, depth(A_2) \, | < | \, depth(A_1) \, |$$

where $| \, [a, b] \, |$ is given by $max(| \, a \, |, | \, b \, |)$.

Using the above definitions a grading of IFSs defined over a set X can be made. The ordering being the A_i comes higher in the order the A_k if A_k is a better representative of X than A_i. We explain this by a case study.

Case Study 2: Consider four IFSs A_1, A_2, A_3 and A_4 defined over the finite set $X = \{x_1, x_2\}$ given by

$$A_1 = \{(.5, .4)/x_1, (.2, .8)/x_2\}$$

$$A_2 = \{(.1, .8)/x_1, (.9, 0)/x_2\}$$

$$A_3 = \{(.1, .9)/x_1, (0, 1)/x_2\} \ \ and$$

$$A_4 = \{(.2, .5)/x_1, (.1, .7)/x_2\} \ \ and$$

Here

$$|depth(A_1)| = |[2 - .8, 2 - .7]| = |[1.2, 1.3]| = 1.3$$

$$|depth(A_2)| = |[2 - 1.2, 2 - .1]| = |[.8, 1]| = 1$$

$$|depth(A_3)| = |[2 - .1, 2 - .1]| = |[1.9, 1.9]| = 1.9$$

$$|depth(A_4)| = |[2 - .9, 2 - .3]| = |[1.1, 1.7]| = 1.7$$

So, $A_2 \supset A_1 \supset A_4 \supset A_3$. Thus A_2 is the best representative of x.

5 Quantification Rules

If "$x \ is \ A$" be a proposition, then the proposition is modified by the modifier by $'m'$ as not , very, fairly etc. Hence the modifier proposition be "x is mA".

Similarly proposition may be quantified by intuitionistic fuzzy quantifiers such as usually, frequently, most etc. Quantifiers are denoted by Q. So, "$Qx's$ are $A's$" is a quantified proposition and "$QA's$ are $B's$" is known as extended quantified propositions. For example, 'most cars are fast' is a quantified proposition, where 'most fast cars are dangerous' is an extended quantified proposition.

The extended quantified proposition as "$QA's$ are $B's$", where Q is a intuitionistic fuzzy quantifier with membership function $\mu_Q(x)$ and the non-membership function $\nu_Q(x)$ and the IFSs A and B have membership and non-membership functions with the same argument on $x \epsilon U, (\mu_A(x), \nu_A(x))$ and$(\mu_B(x), \nu_B(x))$ correspondingly.

We have to find out the truth of the above quantified propositions.

Let A and B are two IFSs on a finite universe of discourse $U = \{x_1, x_2, \ldots, x_n\}$ then

$$\Sigma count A = \left[\sum_{i=1}^{n} \mu_A(x_i), \sum_{i=1}^{n} (1 - \nu_A(x_i)) \right]$$

$$\Sigma count B = \left[\sum_{i=1}^{n} \mu_B(x_i), \sum_{i=1}^{n} (1 - \nu_B(x_i)) \right]$$

In particular Σ count $X = [n, n] = n$, where each $x_i, i = 1, 2, \ldots n$ has a membership value $'1'$ and non-membership value $'0'$.

The truth value of the proposition in a finite universe U is determined by truth $(QAs\ are\ Bs) = (\mu_Q(r), \nu_Q(r))$, where the value of $'r'$ is

$$r = rel\Sigma count(B/A) = \frac{\Sigma count(A \cap B)}{\Sigma count(A)}$$

$$= \left[\frac{\sum_{i=1}^{n}(\mu_A(x_i) \wedge \mu_B(x_i))}{\sum_{i=1}^{n}(1-\nu_s(x_i))}, min\left(1, \frac{\sum_{i=1}^{n}((1-\nu_A(x_i)) \wedge (1-\nu_B(x_i))}{\sum_{i=1}^{n}\mu_A(x_i)}\right)\right]$$

The meaning of the coefficient $r = rel\Sigma count(B/A)$ is that it expresses the proportion of B in A.

In particular case, when A and B are fuzzy sets instead of IFSs, then the proposition "QA are B" reduced to the concept of Zadeh's sense.

Also, in the case, "Qxs are Bs" that is, when instead of an IFS A, we have a crisp set $\{x_i\} = U$, then truth value of "Qxs are B" be

$$truth(Qx's\ are\ B) = (\mu_Q(r_0), \nu_Q(r_0))$$

where

$$r_0 = rel\Sigma count(B/U) = \left[\frac{\sum_{i=1}^{n}\mu_B(x_i)}{n}, min\left(1, \frac{\sum_{i=1}^{n}(1-\nu_B(x_i))}{n}\right)\right]$$

Example 5.1. *Consider the proposition "most cars are fast". Assume that cars, fast and most are defined as*

$$cars\underline{\Delta}y = \{y_1, y_2, y_3\}, U = \{y_1, y_2, y_3\}$$

$$cars\underline{\Delta}B = (.1, .8)/y_1 + (.6, .2)/y_2 + (.8, .2)/y_3$$

and most = Q, where

$$\mu_Q(x) = \begin{cases} 0 & 0 \le x \le .3; \\ 1 - \{1 + (2x - 0.6)^2\}^{-1} & .3 \le x \le .7; \\ 1 & .7 \le x. \end{cases}$$

and

$$\nu_Q(x) = \begin{cases} 1 & 0 \le x \le .4; \\ \{1 + (2x - 0.8)^2\}^{-1} & .4 \le x \le .8; \\ 0 & .8 \le x. \end{cases}$$

$$r_0 = rel\Sigma count(B/U) = \left[\frac{\sum_{i=1}^{n}\mu_B(x_i)}{n}, min\left(1, \frac{\sum_{i=1}^{n}(1-\nu_B(x_i))}{n}\right)\right]$$

$$= \left[\frac{1.5}{3}, min\left(1, \frac{1.8}{3}\right)\right] = [.5, .6]$$

mid value $(r_0) = .55$, which is the average of the degree of car speed.

Now substituting $r_0 = .55$ for $'x'$, we have

$$\mu_Q(.55) = .2 \text{ and } \nu_Q(.55) = .53$$

The truth value depends on how both the quantifiers Q(most) and the set B(fast) are defined.

Example 5.2. *Let us consider the more general proposition, 'Most fast cars are dangerous', using the data in the above example for cars, fast and most.*

In addition, let dangerous be defined as

$$dangerous\ \underline{\Delta}A = (.2, .7)/x_1 + (.5, .4)/x_2 + (.6, .4)/x_3$$

to define $'r'$ *we have to calculate*

$$r = rel\Sigma count(B/A) = \left[\frac{1.2}{1.5}, min\left(1, \frac{1.4}{1.3}\right)\right] = \left[\frac{4}{5}, min\left(1, \frac{14}{13}\right)\right] = \left[\frac{4}{5}, 1\right]$$

$$mid\ value\ (r) = \frac{\frac{4}{5}+1}{2} = \frac{4+5}{5} \times \frac{1}{2} = \frac{9}{10} = .9$$

which represent the proportion of B *in* A.

Finally, substituting $'r'$ *for* $'x'$, *we have*

$$\mu_Q(.9) = 1\ and\ \nu_Q(.9) = 0.$$

6 Conclusion

In this paper a measure of cardinality of IFS, called $\Sigma count$ which generalizes the notion of Σcount of fuzzy sets introduced [4] has been put forth and studied. Many results involving Σ count of transformed IFSs by using modal operations have been established. A notion called relative Σ count is defined and as an application, a case study is made. Intuitionistic fuzzy quantifiers are discussed and illustrated by taking examples.

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Embedding in distance degree regular and distance degree injective graphs

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Abstract

The eccentricity $e(u)$ of a vertex u is the maximum distance of u to any other vertex of G.The distance degree sequence (dds) of a vertex u in a graph $G = (V, E)$ is a list of the number of vertices at distance 1, 2,..., $e(u)$ in that order, where $e(u)$ denotes the eccentricity of u in G. Thus the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}, \ldots)$ is the dds of the vertex v_i in G where d_{i_j} denotes number of vertices at distance j from v_i. A graph is distance degree regular (DDR) graph if all vertices have the same dds. A graph is distance degree injective (DDI) graph if no two vertices have the same dds.

In this paper, we consider the construction of a DDR graph having any given graph G as its induced subgraph. Also we consider construction of some special class of DDI graphs.

Keywords: Distance degree sequence, Distance degree regular (DDR) graphs, Almost DDR graphs, Distance degree injective(DDI) graphs.

2010 MSC: 05C12.

1 Introduction

Unless mentioned otherwise for terminology and notation the reader may refer Buckley and Harary [6], new ones will be introduced as and when found necessary.

In this paper we consider simple undirected graphs without self-loops.

The distance $d(u, v)$ from a vertex u of G to a vertex v is the length of a shortest u to v path. The eccentricity $e(v)$ of v is the distance to a farthest vertex from v. If $dist(u, v) = e(u), (v \neq u)$, we say that v is an eccentric vertex of u.

The distance degree sequence (dds) of a vertex v in a graph $G = (V, E)$ is a list of the number of vertices at distance 1, 2,...,$e(v)$ in that order, where $e(v)$ denotes the eccentricity of v in G. Thus, the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ \ldots, \ d_{i_j}, \ldots)$ is the dds of the vertex v_i in G where, d_{i_j} denotes number of vertices at distance j from v_i. The concept of distance degree regular (DDR) graphs was introduced by G. S. Bloom et.al.[3], as the graphs for which all vertices have the same dds. For example, the three dimensional cube $Q_3 = K_2 \times K_2 \times K_2$,cycles,complete graphs are all DDR graphs. By definition it is clear that the DDR graphs must be regular but not conversely. The DDR graphs are studied by Bloom et.al [3], [4]. In [9] Halberstam et.al. have dealt the problem of path degree sequence and distance degree sequence using algorithms. All properties of cubic graphs up to a specified order are listed by Bussemaker et,al [7].The cubic graph generation is looked into by Brinkmann [5]. In [3], Bloom et.al have proved a result which states that "every regular graph with diameter at most two is DDR".This result shows that getting a DDR graph of higher diameter is challenging. In [12] Itagi Huilgol et.al. have listed all DDR graphs of diameter three with extremal degree regularity. But, the question of characterizing DDR graphs of diameter greater than two still remains open. In [12] Itagi Huilgol et.al. have shown the existence of a diameter three DDR graph of arbitrary regularity. In [13] Itagi Huilgol et.

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al. have constructed some more DDR graphs of higher diameter and considered the behavior of DDR graphs under other graph binary operations. In [14], Itagi Huilgol et.al.have constructed higher order DDR graphs by considering the simplest of the products viz.the cartesian and normal product. Another famous product is the lexicographic product of graphs. The lexicographic product is defined as follows. Given graphs G and H, the lexicographic product $G[H]$ has vertex set $\{(g,h) : g \in V(G), h \in V(H)\}$ and two vertices (g,h), (g',h') are adjacent if and only if either gg' is an edge of G or $g = g'$ and hh' is an edge of H.

The other extreme of DDR graphs is DDI graphs. The concept of DDI graphs was introduced in [4]. A graph G is said to be DDI graph if no two of its vertices have same distance degree sequence. In literature, in comparison to DDR graphs the number of DDI graphs is very less. So construction of new DDI graphs is also a challenging one. In [14], Itagi Huilgol et. al. have constructed higher order DDI graphs by using the products. A question was posed on the existence of r-regular DDI graphs by Bloom et. al. in [4]. In [9], Halberstam and Quintas showed the existence of a cubic DDI graph of diameter 10 and order 24. It was reduced to order 22 and diameter 8 by Martinez and Quintas in [15]. They also constructed a general cubic DDI graph with 22+2k points and diameter 8+k. It was further reduced to order 18 and diameter 7 by J. Volf in [19].

Characterization of graphs with a given property in terms of other properties is very common. A trend has been developed in characterizing the graphs with given property in terms of certain class of graphs which are not induced subgraphs of graphs with the property considered, that is, in terms of the " forbidden subgraphs". The first and foremost such characterization was given by Kuratowski [6] in case of planar graphs. From the definition it is clear that the study of planar graphs necessarily involves the topology of the plane. In general, the notion of embedding is extended to other surfaces too, viz, mobius band, torous. The problem gets interesting as we know that not all graphs can be embedded in the plane, or any other surface. In recent years, this type of characterization is considered as a "good characterization". Such a characterization has been used by many researchers. To quote a few Bieneke [6] in case of line graphs, Cook [8] for the graphs corresponding to $(0,1)$- matrices, Berge [2] for perfect graph conjecture.

In this paper, we consider the embedding of a graph in a DDR graph and/or DDI graphs. As mentioned above, the DDR and DDI graphs are quite different. We relax a condition to introduce the concepts of almost DDR or ADDR, in short and almost DDI or ADDI in short. Here, we have also considered the embedding into ADDR and ADDI graphs.

2 Embeddings

DDR graphs exhibit high regularity in terms of the vertices and their distance distribution. If we relax for only one vertex to have different dds, then we can call the graph to be almost DDR, or in short ADDR. Similarly, we can define almost DDI graphs or ADDI in short.

Definition 2.1. *A graph G of order p is said to be almost DDR if $p-1$ vertices have same dds and one vertex with different dds.*

Definition 2.2. *A graph G of order p is said to be almost DDI if $p-2$ vertices have different dds and two vertices with same dds.*

In [17], Nandakumar et. al have proved that " For each vertex u with $e(u) > r(G)$, one of its neighbors v satisfies $e(v) = e(u) - 1$", which we are using to prove the following result.

Theorem 2.1. *If G is almost DDR, then $r(G) \leq diam(G) \leq r(G) + 1$.*

Proof. Let G be almost DDR. The left hand inequality follows from the definition of radius and diameter. Suppose on contrary, if $diam(G) \geq r(G) + 2$. Let u be a vertex with $e(u) = r(G) + 2$. From Nandakumar [17], there exists a vertex v adjacent to u with $e(v) = e(u) - 1$. Hence there exist three vertices having distinct eccentricities, a contradiction. Hence $diam(G) \leq r(G) + 1$. □

Remark 2.1. *Let G be a DDI graph having a vertex v such that $|d_2(v) - d_2(v_i)| \neq 1$, for all $v_i \in V(G)$ then adding a vertex u and making it adjacent with all the neighbors of v we get an almost DDI graph.*

Proposition 2.1. *Any path can be embedded in an almost DDI graph.*

Proof. Consider a DDI graph G as in [4] on $p+1$ vertices having path on p vertices as its induced subgraph as shown in Figure(1) below. Now adding an edge $(v, 3)$ in G, we obtain an almost DDI graph in which $dds(u) = dds(v)$ as shown in the Figure(2). □

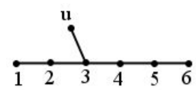

Figure 1: DDI graph on p+1 vertices

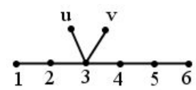

Figure 2: Embedding of P_p in an almost DDI graph

Theorem 2.2. *Any graph can be embedded in a DDR graph.*

Proof. First we prove that any regular graph can be embedded in a DDR graph. Let G be any regular graph of order k, the generalized lexicographic product $C_p[G, G, G, \ldots, G]$, is a DDR graph with diameter $\lfloor \frac{p}{2} \rfloor$ and having the dds of each vertex as $dds(v) = (1, 2k + r, 3k - r - 1, 2k, 2k, \ldots, 2k)$ if p is odd and $dds(v) = (1, 2k + r, 3k - r - 1, 2k, 2k, \ldots, k)$ if p is even. It is clear that G is an induced subgraph of $C_p[G, G, G, \ldots, G]$. Hence, any regular graph can be embedded in a DDR graph.

Next, we prove that any non-regular graph G can be embedded in a regular graph of regularity $\Delta(G)$, where $\Delta(G)$ is the maximum degree of G. Then by embedding it in a DDR graph we achieve the result.

Let G be any non regular graph. Let $t = \sum_{i=1}^{l} (\Delta - deg(v_i))$, where Δ and l are the maximum degree and order of G, respectively. Here two cases arise,

Case(i): $t = n\Delta$, for some $n \geq 1$,

Case(ii): $t = n\Delta + s$, $s < \Delta$, $n \geq 1$

Case(i): If $t = n\Delta$, consider $\overline{K}_n$ and add all $n\Delta$ edges such that (i) every edge has one end in $\overline{K}_n$ and the other end in G, (ii) every vertex of $\overline{K}_n$ receives exactly Δ edges, (iii) degree of every vertex in G becomes Δ. The resulting graph G' is a regular graph having G as its induced subgraph.

Case(ii): $t = n\Delta + s$, $s < \Delta$. Here we need to consider four subcases,

Case(a): s even and Δ even. Consider $\overline{K}_{n+s}$. Let S_1 and S_2 be the partition of $n + s$ vertices, such that $|S_1| = n$ and $|S_2| = s$. Add $n\Delta$ edges such that (i) every edge has one end in S_1 and the other end in G, (ii) every vertex of S_1 receives exactly Δ edges and add the remaining s edges such that (i) every edge has one end in S_2 and the other end in G, (ii) every vertex of S_2 receives exactly 1 edge. To make the vertices of S_2, Δ - regular we need exactly $\Delta - 1$ edges incident to each vertex of S_2. For this, take a complete graph $K_{\Delta+1}$ on $\Delta + 1$ vertices. Now we add the edges between S_2 and $K_{\Delta+1}$ preserving the regularity of $K_{\Delta+1}$. Suppose v and w are any two vertices in S_2, remove an edge (u_i, u_j) from $K_{\Delta+1}$ and add two edges (v, u_i) and (v, u_j) continuing the process of removing and adding the edges, we can make the degree of v equal to $\Delta - 1$ as Δ is odd. We have to add one more edge e_v(say) to v so that degree of v becomes Δ. Adding the edges to w as above, we can make the degree of w equal to $\Delta - 1$. We have to add one more edge e_w(say) to w so that degree of w becomes Δ, for that remove an edge (u_i', u_j') and add the edges (v, u_i') and (v, u_j'). In this way we can make degree of every vertex in S_2 equal to Δ as $|S_2|$ is even.

Similarly we can do it for the other cases given as below

Case(b): s odd and Δ even. This case is similar to Case(a) when we replace s by $s - 1$. Since $\Delta - 1$ is odd, there exists a vertex u_1 in S_2 with degree $\Delta - 1$ after removing and adding the edges as in above case. To make the degree of u_1 equal to Δ, take an isomorphic copy(G_2') of the above resulting graph(G_1') and make u_1 adjacent with its mirror image u_1' in G_2'.

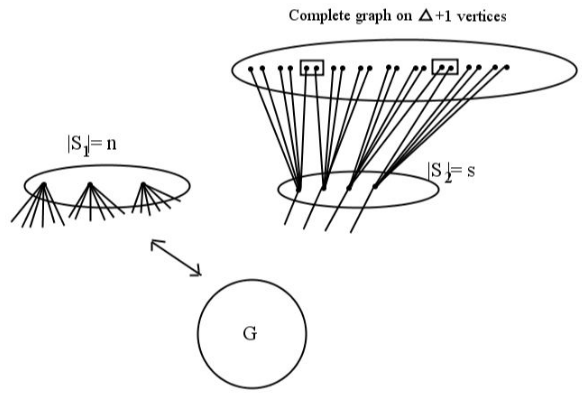

Figure 3: Case(a)

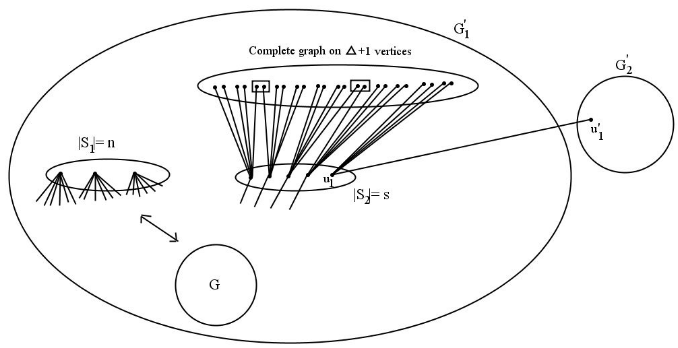

Figure 4: Case(b)

Case(c): s odd and Δ odd. As in case(a), we can make degree of every vertex of S_1 equal to Δ and since $\Delta-1$ is even, it is possible to add $\Delta-1$ edges to each vertex of S_2 by removing $\frac{\Delta-1}{2}$ number of edges from $K_{\Delta+1}$.

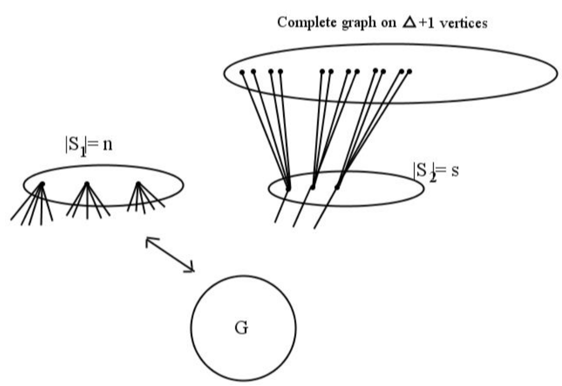

Figure 5: Case(c)

Case(d): s even and Δ odd. Proof of this case is similar to the proof of Case(c). Hence the proof. □

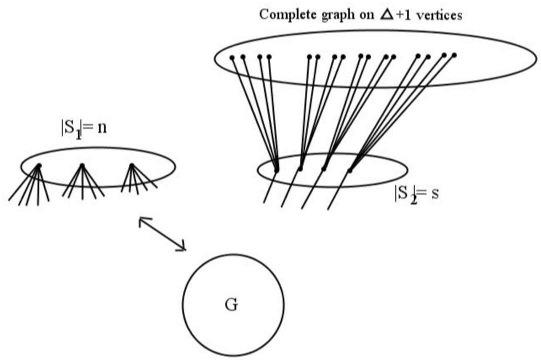

Figure 6: Case(d)

Theorem 2.3. *Any connected/disconnected graph G can be embedded in an almost DDR graph.*

Proof. First we prove any regular graph can be embedded in an almost DDR graph. Let G be any regular graph of regularity r. Add a vertex v to G and make it adjacent to all the vertices of G. The resulting graph is an almost DDR graph with the dds of all p vertices of G as $(1, r+1, p-r-1)$ and $dds(v) = (1, p)$ and having G as its induced subgraph. We know from the above theorem that any graph can be embedded in a regular graph. Hence any graph G can be embedded in an almost DDR graph. □

Ex:

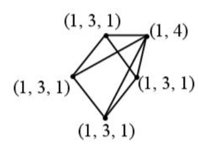

Figure 7: Embedding of C_4 in an almost DDR graph

Theorem 2.4. *Every cycle can be embedded in a DDI graph.*

Proof. Let the vertices of a cycle C_p be labeled as $1, 2, 3, \ldots, p$ and $P_1, P_2, P_3, \ldots, P_p$ be paths of lengths $1, 2, 3, \ldots, p$, respectively. Concatinating a pendent vertex of each P_i with a vertex i on C_p, the resulting graph G is shown to be a DDI graph. Now we prove this by showing no two vertices have same dds. Here three cases arise, Case(i): No two vertices on each path will have same dds.
Case(ii): No two vertices on the cycle C_p will have same dds.
Case(iii): No two vertices from two different paths will have same dds.
Case(i): Eccentricity of every vertex on a path is different as eccentricity increases by one as we move one step towards the pendent vertex of that path. Hence no two vertices on each path will have same dds.
Case(ii): Number of vertices at distance i from a vertex i, where $2 \leq i \leq p$ is always greater than the number of vertices at distance $i, 2 \leq i \leq p$ from a vertex $j, 1 \leq j \leq i-1$. Hence no two vertices on the cycle C_p have same dds.
Case(iii): No two vertices from two different paths have same dds as these vertices lie on the paths having different lengths. □

Note: Adding a pendent vertex at p^{p-1} in above DDI graph, we get an almost DDI graph.

Lemma 2.1. *If a graph G containing two vertices u and v which are the only eccentric vertices of each other with eccentricities equal to three and $deg(u) = deg(v)$ then G is non DDI.*

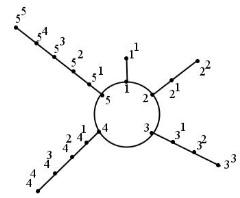

Figure 8: Embedding of C_p in a DDI graph

Proof. Let G be a graph containing two vertices u and v with eccentricities equal to three such that u is the only eccentric vertex of v and v is the only eccentric vertex of u and $deg(u) = deg(v) = k$. Let S_1, S_2 be vertex sets at distance one and two respectively from u. Every vertex in S_1 is adjacent to at least one vertex of S_2 otherwise, the eccentricity of v will be more than three, a contradiction. Now for S_2 two cases arise.
Case(i): $|S_2| = k$ and Case(ii): $|S_2| > k$.
Case(i): $|S_2| = k$, clearly $dds(u) = (1, k, k, 1) = dds(v)$.
Case(ii): $|S_2| > k$. Let $|S_2| - k = t$ be the number of vertices non adjacent to v. Here consider two cases,
Case(a): Suppose there is a vertex w in $S_2 - N(v)$ non adjacent with any of the vertices in $N(v)$, then w is at distance three from v, a contradiction. Hence this case is excluded.
Case(b): Suppose every vertex in $S_2 - N(v)$ is adjacent to at least one vertex in $N(v)$, then every vertex in $S_2 - N(v)$ is at distance two from both u and v and hence $dds(u) = (1, k, k + |S_2 - N(v)|, 1) = dds(v)$. Hence G is non DDI. □

Lemma 2.2. *There exists no regular self centered DDI graph of diameter three.*

Proof. Let G be a regular self centered graph of radius three. The dds of any vertex is given by $dds(v) = (1, k, d_2, d_3)$. Since the graph G is self centered, d_2 and d_3 will satisfy the following $d_2 + d_3 = p - k - 1$, $1 \leq d_2 \leq p - k - 2$ and $1 \leq d_3 \leq p - k - 2$. Therefore the number of vertices having distinct dds is atmost $p - k - 2$. Any set containing at least $p - k - 1$ contains at least two vertices having same dds. Hence G is non DDI. □

The u.e.n graphs are defined by Nandakumar et. al. as follows:
Definition[17]:A graph G is said to be unique eccentric node (u.e.n) graph if every vertex has a unique eccentric vertex.

Corollary 2.1. *There exist no regular u.e.n. DDI graph with diameter three.*

Proof. Let G be a regular u.e.n. graph with diameter three. There exist two vertices u and v at distance three from each other. Let $dds(u) = (1, k, d_2, 1)$ and $dds(v) = (1, k, d_2', 1)$. Comparing the dds of u and v, we get $d_2 = d_2'$. Hence $dds(u) = dds(v)$, implying G is not DDI. □

Corollary 2.2. *There exists no regular self centered u.e.n. DDI graph with diameter four.*

Proof. Let G be a regular self centered u.e.n. graph of diameter four. The dds of any vertex is given by $dds(v) = (1, k, d_2, d_3, 1)$. It is clear from lemma[2.2] that there exist at least two vertices having same dds. Hence G is non DDI. □

Note: Combining the above two results we can say "There exists no regular u.e.n. DDI graph with diameter atmost four".

Lemma 2.3. *If G is a graph with radius two containing at least two central vertices having same degree, then G is non DDI.*

Proof. Let G be a graph with radius two containing at least two central vertices u and v having same degree, then their dds are given by $dds(u) = (1, k, p - k - 1)$ and $dds(v) = (1, k, p - k - 1)$, i.e., $dds(u) = dds(v)$. Hence G is non DDI. □

Remark 2.2. *If a regular self centered graph G with radius three has at least two vertices having same number of vertices at distance two or three then G is non DDI.*

Lemma 2.4. *There exists no regular self centered DDI graph whose complement is also DDI.*

Proof. In [11], If $d(G) \geq 3$, then $d(\overline{G}) \leq 3$, but in lemma[2.2] we have proved that there exists no regular self centered DDI graph of radius three and also it is clear that there exists no self centered DDI graph of radius two. Hence there exists no regular self centered DDI graph whose complement is also DDI. □

Combining the results in [4], "If both G and $\overline{G}$ are DDI then both G and $\overline{G}$ are of diameter three" and in [6], "If G is regular with diameter 3, then $d(\overline{G}) = 2$", we have the following remark.

Remark 2.3. *There exists no regular DDI graph whose complement is also DDI.*

Remark 2.4. *A graph G having two vertices u and v such that $N(u) \cap N(v) = N(u) = N(v)$ is non DDI, where $N(u)$ and $N(v)$ are sets of vertices adjacent to u and v respectively.*

Lemma 2.5. *There exists a u.e.n. DDI graph having diameter $d = 2n + 1$,where $n \geq 3$*

Proof. Let $P_p : 1, 2, 3, \ldots, p-2, p-1, p(\geq 6)$ be a path on p vertices and $P_{p-2}, P_{p-3}, P_{p-4}, \ldots, P_2$ be paths on $p-2, p-3, p-4, \ldots, 2$ respectively. The graph obtained by concatination of a pendant vertex of each path P_{p-i} with a vertex i, where $2 \leq i \leq p-2$ on the above said path P_p is a u.e.n. DDI graph having diameter $d = 2n + 1$,where $n \geq 3$. □

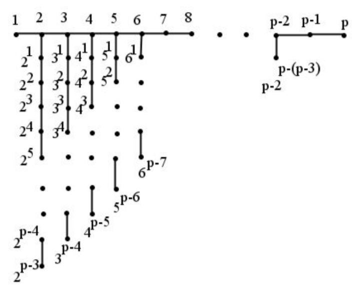

Figure 9: A u.e.n. DDI graph having diameter $d = 2n + 1$,where $n \geq 3$

Lemma 2.6. *There are at least $p - 5$ non-isomorphic DDI graphs of order p,where $p \geq 7$.*

Proof. Let $v_1, v_2, v_3, \ldots, v_{p-1}$ be a path on $p-1$ vertices and v_p be a vertex to be made adjacent to a vertex of above said path on $p-1$ vertices such that resulting graph is DDI. Amongst $p-1$ points on the path(of length $p-2$) we can not join the vertex to the end vertices v_1 and v_{p-1}, otherwise, the induced graph would be a path and hence is not DDI. We also can not join the vertex to either v_2 or v_{p-2}, since v_1 and v_p or v_{p-2} and v_p would have the same dds, contradicting to the fact that G is DDI. Now, if the path induced by $p-1$ vertices is of odd length, then we can join a vertex at any of the vertices $v_3, v_4, \ldots, v_{p-3}$, without any repetition of dds. Hence we can join a vertex at $p-5$ vertices to get different DDI graphs. If the path induced by $p-1$ vertices is of even length then, we can join v_p at the vertices $v_3, v_4, \ldots, v_{\frac{p-1}{2}-1}, v_{\frac{p-1}{2}+1}, \ldots, v_{p-3}$; i.e., except at the central vertex of the path. So we can join at $p-6$ vertices to get different DDI graphs. □

We conclude this paper with a couple of open problems.
<u>**Problem 1 :**</u> Characterize DDR graphs of diameter higher than 3.
<u>**Problem 2 :**</u> Can any graph be embedded in a DDI graph?
<u>**Problem 3 :**</u> Does there exist DDI r-regular graph for $r \geq 4$?

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Oscillatory properties of third-order quasilinear difference equations

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Abstract

Some new oscillation criteria are obtained for the third-order quasilinear difference equation $\Delta^2\left(p_n\left(\Delta x_n\right)^{\alpha}\right)-q_n\left(\Delta x_n\right)^{\alpha}+r_nf(x_n)=0$, $n=0,1,2,...$, where $\alpha>0$ is the ratio of odd positive integers. The method uses techniques based on Schwarz's inequality. Example is inserted to illustrate the result.

Keywords: Oscillation, third order, quasilinear, difference equation, Schwarz's inequality.

2010 MSC: 39A11.

1 Introduction

The notion of nonlinear difference equations was studied intensively by R.P.Agarwal [1]. Recently there has been a lot of interest in the study of oscillatory behavior of solutions of nonlinear difference equations. Motivated by the references [1]- [28], in this paper, we have considered the oscillatory properties of third-order quasilinear difference equation of the form

$$\Delta^2(p_n\left(\Delta x_n\right)^{\alpha})-q_n\left(\Delta x_n\right)^{\alpha}+r_nf(x_n)=0, n=0,1,2,..., \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$, provided the following conditions are assumed to hold:

(C1) $\alpha>0$ is the ratio of odd positive integer,

(C2) $\{p_n\},\{q_n\},\{r_n\}$ are real positive sequences,

(C3) $f:R\to R$ is a continuous function and $xf(x)>0$ for all $x\neq 0$,

(C4) there exists a real valued function g such that
$f(u)-f(v)=g(u,v)(u-v)$ for all $u\neq 0, v\neq 0$ and
$g(u,v)\geq L>0\in R$,

(C5) $\displaystyle\sum_{n=M}^{\infty} {p_n}^2<\infty$ for $M\geq 0$,

(C6) $\displaystyle\sum_{n=M}^{\infty}\left(\Delta\left(p_{n+1}\left(\Delta x_{n+1}\right)^{\alpha}\right)\right)^2<\infty$ for $M\geq 0$,

(C7) $\displaystyle\sum_{n=M}^{\infty}\frac{1}{p_n^{\frac{1}{\alpha}}}=\infty$ for $M\geq 0$,

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(C8) $\sum_{n=M}^{\infty} {q_n}^2 < \infty$ for $M \geq 0$,

(C9) $\sum_{n=M}^{\infty} (n+1)\, r_n = \infty$ for $M \geq 0$,

Our objective here is to proceed further in this direction to obtain the oscillation of all solutions of equation (1.1) which include and generalize some earlier results cited there in references.

By a solution of equation (1.1) we mean a real sequence $\{x_n\}$, $n = 0, 1, 2, ...$, which satisfies equation (1.1) for all $n > n_0$, where $n_0 \geq 0$. We recall that a nontrivial solution of equation (1.1) is said to be oscillatory if for every $M > 0$ there exists an integer $n \geq M$ such that $x_n x_{n+1} \leq 0$; otherwise it is said to be nonoscillatory. Thus, a nonoscillatory solution is either eventually positive or eventually negative.

2 Main Result

In this section, we present some sufficient conditions for the oscillatory properties of all solutions of equation (1.1).

Theorem 2.1. *If the conditions (C1), (C2), (C3), (C4), (C5), (C6), (C7), (C8) and (C9) hold, then every solution of equation (1.1) is oscillatory.*

Proof. Without loss of generality we may assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1) such that $x_n > 0$ for all $n \geq M$, $M \geq 0$ is an integer.
From equation (1.1), we have

$$\Delta\left(p_{n+1}\left(\Delta x_{n+1}\right)^{\alpha}\right) - \Delta\left(p_n\left(\Delta x_n\right)^{\alpha}\right) - q_n\left(\Delta x_n\right)^{\alpha} + r_n f\left(x_n\right) = 0. \tag{2.1}$$

Multiplying equation (2.1) by $\frac{n+1}{f\left(x_n\right)}$ and summing from M to $n-1$, we obtain

$$\begin{aligned} \sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} \Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right) - \sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} \Delta\left(p_s\left(\Delta x_s\right)^{\alpha}\right) \\ - \sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} q_s\left(\Delta x_s\right)^{\alpha} + \sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} r_s f\left(x_s\right) = 0. \end{aligned} \tag{2.2}$$

Consider the first summation from equation (2.2),

$$\begin{aligned} \sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} \Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right) = &\frac{n+1}{f\left(x_n\right)} p_{n+1}\left(\Delta x_{n+1}\right)^{\alpha} \\ &- \frac{M+1}{f\left(x_M\right)} p_{M+1}\left(\Delta x_{M+1}\right)^{\alpha} \\ &- \sum_{n=M}^{n-1} p_{s+2}\left(\Delta x_{s+2}\right)^{\alpha} \\ &\quad \left(\frac{f\left(x_s\right) \Delta(s+1) - (s+1) \Delta f\left(x_s\right)}{f\left(x_s\right) f\left(x_{s+1}\right)}\right). \end{aligned}$$

That is,

$$\begin{aligned}\sum_{n=M}^{n-1}\frac{s+1}{f(x_s)}\Delta\left(p_{s+1}(\Delta x_{s+1})^{\alpha}\right) = &\frac{n+1}{f(x_n)}p_{n+1}(\Delta x_{n+1})^{\alpha}\\ &-\frac{M+1}{f(x_M)}p_{M+1}(\Delta x_{M+1})^{\alpha}\\ &-\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^{\alpha}}{f(x_{s+1})}\\ &+\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^{\alpha}(s+1)}{f(x_s)f(x_{s+1})}\\ &\quad g(x_{s+2},x_{s+1})\Delta x_{s+1}.\end{aligned} \tag{2.3}$$

Now consider the second summation from equation (2.2) and similarly we obtain

$$\begin{aligned}\sum_{n=M}^{n-1}\frac{s+1}{f(x_s)}\Delta\left(p_s(\Delta x_s)^{\alpha}\right) = &\frac{n+1}{f(x_n)}p_n(\Delta x_n)^{\alpha}\\ &-\frac{M+1}{f(x_M)}p_M(\Delta x_M)^{\alpha}\\ &-\sum_{n=M}^{n-1}\frac{p_{s+1}(\Delta x_{s+1})^{\alpha}}{f(x_{s+1})}\\ &+\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^{\alpha+1}(s+1)}{f(x_s)f(x_{s+1})}g(x_{s+2},x_{s+1}).\end{aligned} \tag{2.4}$$

Substituting equations (2.3) and (2.4) in equation (2.2), we have

$$\begin{aligned}&\frac{n+1}{f(x_n)}p_{n+1}(\Delta x_{n+1})^{\alpha}-\frac{M+1}{f(x_M)}p_{M+1}(\Delta x_{M+1})^{\alpha}-\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^{\alpha}}{f(x_{s+1})}\\ &+\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^{\alpha}(s+1)}{f(x_s)f(x_{s+1})}g(x_{s+2},x_{s+1})\Delta x_{s+1}+\frac{n+1}{f(x_n)}p_n(\Delta x_n)^{\alpha}\\ &-\frac{M+1}{f(x_M)}p_M(\Delta x_M)^{\alpha}-\sum_{n=M}^{n-1}\frac{p_{s+1}(\Delta x_{s+1})^{\alpha}}{f(x_{s+1})}\\ &+\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^{\alpha+1}(s+1)}{f(x_s)f(x_{s+1})}g(x_{s+2},x_{s+1})-\sum_{n=M}^{n-1}\frac{s+1}{f(x_s)}q_s(\Delta x_s)^{\alpha}\\ &+\sum_{n=M}^{n-1}(s+1)r_s=0.\end{aligned}$$

That is,

$$\begin{aligned}&\frac{n+1}{f(x_n)}\Delta\left(p_n(\Delta x_n)^{\alpha}\right)-\sum_{n=M}^{n-1}\frac{1}{f(x_{s+1})}\Delta\left(p_{s+1}(\Delta x_{s+1})^{\alpha}\right)\\ &-\sum_{n=M}^{n-1}\frac{(-s-1)g(x_{s+2},x_{s+1})\Delta x_{s+1}}{f(x_s)f(x_{s+1})}\Delta\left(p_{s+1}(\Delta x_{s+1})^{\alpha}\right)\\ &-\sum_{n=M}^{n-1}\frac{s+1}{f(x_s)}q_s(\Delta x_s)^{\alpha}\\ &\qquad=\frac{M+1}{f(x_M)}\Delta\left(p_M(\Delta x_M)^{\alpha}\right)-\sum_{n=M}^{n-1}(s+1)r_s.\end{aligned} \tag{2.5}$$

By Schwarz's inequality we obtain the following:

$$\sum_{n=M}^{n-1} \frac{1}{f\left(x_{s+1}\right)} \Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right) \leq\left(\sum_{n=M}^{n-1} \frac{1}{f^{2}\left(x_{s+1}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1}\left(\Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right)\right)^{2}\right)^{\frac{1}{2}}, \tag{2.6}$$

$$\sum_{n=M}^{n-1} \frac{(-s-1) g\left(x_{s+2}, x_{s+1}\right) \Delta x_{s+1}}{f\left(x_{s}\right) f\left(x_{s+1}\right)} \Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right) \leq\left(\sum_{n=M}^{n-1} \frac{(s+1)^{2} g^{2}\left(x_{s+2}, x_{s+1}\right)\left(\Delta x_{s+1}\right)^{2}}{f^{2}\left(x_{s}\right) f^{2}\left(x_{s+1}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1}\left(\Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right)\right)^{2}\right)^{\frac{1}{2}} \tag{2.7}$$

and

$$\sum_{n=M}^{n-1} \frac{s+1}{f\left(x_{s}\right)} q_{s}\left(\Delta x_{s}\right)^{\alpha} \leq\left(\sum_{n=M}^{n-1} \frac{(s+1)^{2}\left(\Delta x_{s+1}\right)^{2}}{f^{2}\left(x_{s}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1} q_{s}^{2}\right)^{\frac{1}{2}}. \tag{2.8}$$

In view of the above inequalities (2.6), (2.7) and (2.8), the summations in (2.5) are bounded.Therefore, equation (2.5) becomes

$$\begin{aligned}
&\frac{n+1}{f\left(x_{n}\right)} \Delta\left(p_{n}\left(\Delta x_{n}\right)^{\alpha}\right)-\left(\sum_{n=M}^{n-1} \frac{1}{f^{2}\left(x_{s+1}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1}\left(\Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right)\right)^{2}\right)^{\frac{1}{2}} \\
&-\left(\sum_{n=M}^{n-1} \frac{(s+1)^{2} g^{2}\left(x_{s+2}, x_{s+1}\right)\left(\Delta x_{s+1}\right)^{2}}{f^{2}\left(x_{s}\right) f^{2}\left(x_{s+1}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1}\left(\Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right)\right)^{2}\right)^{\frac{1}{2}} \\
&-\left(\sum_{n=M}^{n-1} \frac{(s+1)^{2}\left(\Delta x_{s+1}\right)^{2}}{f^{2}\left(x_{s}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1} q_{s}^{2}\right)^{\frac{1}{2}} \\
&\leq \frac{M+1}{f\left(x_{M}\right)} \Delta\left(p_{M}\left(\Delta x_{M}\right)^{\alpha}\right)-\sum_{n=M}^{n-1}(s+1) r_{s} .
\end{aligned} \tag{2.9}$$

In view of the conditions (C1),(C2),(C3),(C4),(C5),(C6),(C8),(C9) and from the above inequality (2.9), we obtain

$$\frac{n+1}{f\left(x_{n}\right)} \Delta\left(p_{n}\left(\Delta x_{n}\right)^{\alpha}\right) \rightarrow \infty \text{ as } n \rightarrow \infty .$$

Hence there exists an integer $M_1 > 0$ such that

$$\Delta\left(p_{n}\left(\Delta x_{n}\right)^{\alpha}\right)<0 \text { for } n \geq M_{1} .$$

Summing the above inequality from M_1 to $n-1$, we have

$$p_{n}^{\frac{1}{\alpha}} \Delta x_{n}<p_{M_{1}}^{\frac{1}{\alpha}} \Delta x_{M_{1}} . \tag{2.10}$$

Hence there exists a real number $K > 0$ such that $p_{M_1}^{\frac{1}{\alpha}} \Delta x_{M_1} < -K$.
Therefore, from equation (2.10), we have

$$p_{n}^{\frac{1}{\alpha}} \Delta x_{n}<-K .$$

$$\text { i.e., } \quad \Delta x_{n}<-\frac{K}{p_{n}^{\frac{1}{\alpha}}} .$$

Summing the above inequality from M_1 to $n-1$, we have

$$x_{n}<x_{M_{1}}-K \sum_{n=M_{1}}^{n-1} \frac{1}{p_{n}^{\frac{1}{\alpha}}} . \tag{2.11}$$

In view of the condition (C7), from the above inequality (2.11) we find that $x_n \to -\infty$ as $n \to \infty$. This is a contradiction to the fact that $x_n > 0$ for all $n \geq M \geq 0$.

The proof is similar to the case when $x_n < 0$ for all $n \geq M$, $M \geq 0$ is an integer.

Hence the theorem is completely proved. □

3 Example

Example 3.1. *Consider the difference equation*

$$\Delta^2\left(\frac{1}{n}\left(\Delta x_n\right)^3\right) - \frac{4\left(2n^2+4n+1\right)}{n\left(n+1\right)\left(n+2\right)}\left(\Delta x_n\right)^3 + \frac{8\left(2n^2+4n+1\right)}{n\left(n+1\right)\left(n+2\right)}x_n = 0,\ n > 0. \tag{3.1}$$

$$\left[\text{Here } p_n = \frac{1}{n},\ q_n = \frac{4\left(2n^2+4n+1\right)}{n\left(n+1\right)\left(n+2\right)},\ r_n = \frac{8\left(2n^2+4n+1\right)}{n\left(n+1\right)\left(n+2\right)},\ \text{and } f\left(x_n\right) = x_n\right]$$

All conditions of Theorem (2.1) are satisfied.
Hence all solutions of equation (3.1) are oscillatory.
In fact, $\{x_n\} = \{(-1)^n\}$ is such a solution of equation (3.1).

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Prime cordial labeling of some wheel related graphs

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Abstract

A *prime cordial labeling* of a graph G with the vertex set $V(G)$ is a bijection $f : V(G) \rightarrow \{1, 2, 3, \ldots, |V(G)|\}$ such that each edge uv is assigned the label 1 if $gcd(f(u), f(v)) = 1$ and 0 if $gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits prime cordial labeling is called *prime cordial graph*. In this paper we prove that the gear graph G_n admits prime cordial labeling for $n \geq 4$. We also show that the helm H_n for every n, the closed helm CH_n (for $n \geq 5$) and the flower graph Fl_n (for $n \geq 4$) are prime cordial graphs.

Keywords: Prime cordial labeling, gear graph, helm, closed helm, flower graph.

2010 MSC: 05C78.

1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. For standard terminology and notations we follow Gross and Yellen [5]. We will provide brief summary of definitions and other information which are necessary for the present investigations.

Definition 1.1. *If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.*
Any graph labeling will have following three common characteristics:
1. a set of numbers from which vertex labels are chosen;
2. a rule that assigns a value to each edge;
3. a condition that this value has to satisfy.

According to Beineke and Hegde [1] graph labeling serves as a frontier between number theory and structure of graphs. Graph labelings have many applications within mathematics as well as to several areas of computer science and communication networks. According to Graham and Sloane [4] the harmonious labellings are closely related to problems in error correcting codes while odd harmonious labeling is useful to solve undetermined equations as described by Liang and Bai [6]. The optimal linear arrangement concern to wiring network problems in electrical engineering and placement problems in production engineering can be formalised as a graph labeling problem as stated by Yegnanaryanan and Vaidhyanathan [13]. The watershed transform is an important morphological tool used for image segmentation. An improved algorithm using Graceful labeling for watershed image segmentation is also proposed by Sridevi *et al.*[7]. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [3].

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Definition 1.2. *A mapping $f : V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

Definition 1.3. *If for an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Then*

$$\left.\begin{aligned} v_f(i) &= \text{ number of vertices of } G \text{ having label } i \text{ under } f \\ e_f(i) &= \text{ number of edges of } G \text{ having label } i \text{ under } f^* \end{aligned}\right\} \text{where } i = 0 \text{ or } 1$$

Definition 1.4. *A binary vertex labeling f of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.*

The concept of cordial labeling was introduced by Cahit [2]. Some labeling schemes are also introduced with minor variations in cordial theme. Product cordial labeling, total product cordial labeling and prime cordial labeling are among mention a few. The present work is focused on prime cordial labeling.

Definition 1.5. *A prime cordial labeling of a graph G with vertex set $V(G)$ is a bijection $f : V(G) \rightarrow \{1, 2, 3, \ldots, |V(G)|\}$ and the induced function $f^* : E(G) \rightarrow \{0, 1\}$ is defined by*

$$\begin{aligned} f^*(e = uv) &= 1, \quad \text{if } gcd(f(u), f(v)) = 1; \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

satisfies the condition $|e_f(0) - e_f(1)| \leq 1$. A graph which admits prime cordial labeling is called a prime cordial graph.

The concept of prime cordial labeling was introduced by Sundaram *et al.*[8] and in the same paper they have investigated several results on prime cordial labeling. Vaidya and Vihol [9] as well as Vaidya and Shah [12] have discussed prime cordial labeling in the context of some graph operations. Prime cordial labeling for some cycle related graphs have been discussed by Vaidya and Vihol in [10]. Vaidya and Shah [11] have investigated many results on prime cordial labeling. Same authors in [12] have proved that the wheel graph W_n admits prime cordial labeling for $n \geq 8$. The present work is aimed to investigate some new results on prime cordial labeling for some wheel related graphs.

Definition 1.6. *The wheel W_n is defined to be the join $K_1 + C_n$. The vertex corresponding to K_1 is known as apex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges. We continue to recognize apex of wheel as the apex of respective graphs corresponding to definitions 1.6 to 1.9.*

Definition 1.7. *The gear graph G_n is obtained from the wheel by subdividing each of its rim edge.*

Definition 1.8. *The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n pendant vertices.*

Definition 1.9. *The closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n vertices degree 3.*

Definition 1.10. *The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm. It contains three types of vertices: an apex of degree $2n$, n vertices of degree 4 and n vertices of degree 2.*

2 Main Results

Theorem 2.1. *Gear graph G_n is a prime cordial graph for $n \geq 4$.*

Proof. Let W_n be the wheel with apex vertex v and rim vertices $v_1, v_2, \ldots, v_n$. To obtain the gear graph G_n subdivide each rim edge of wheel by the vertices $u_1, u_2, \ldots, u_n$. Where each u_i is added between v_i and v_{i+1} for $i = 1, 2, \ldots, n-1$ and u_n is added between v_1 and v_n. Then $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$. To define $f : V(G) \rightarrow \{1, 2, 3, \ldots, 2n+1\}$, we consider following four cases.

Case 1: $n = 3$

In G_3 to satisfy the edge condition for prime cordial labeling it is essential to label four edges with label 0 and five edges with label 1 out of nine edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most three edges and 1 labels for at least six edges. That is, $|e_f(0) - e_f(1)| = 3 > 1$. Hence, G_3 is not prime cordial graph.

Case 2: $n = 4$ to $9, 11, 14, 19$

For $n = 4$, $f(v) = 6$, $f(v_1) = 3, f(v_2) = 9, f(v_3) = 4, f(v_4) = 8$ and $f(u_1) = 1, f(u_2) = 7, f(u_3) = 2, f(u_4) = 5$. Then $e_f(0) = 6 = e_f(1)$.

For $n = 5$, $f(v) = 6$, $f(v_1) = 9, f(v_2) = 5, f(v_3) = 4, f(v_4) = 8, f(v_5) = 3$ and $f(u_1) = 7, f(u_2) = 10, f(u_3) = 2, f(u_4) = 1, f(u_5) = 11$. Then $e_f(0) = 8, e_f(1) = 7$.

For $n = 6$, $f(v) = 6$, $f(v_1) = 9, f(v_2) = 8, f(v_3) = 4, f(v_4) = 11, f(v_5) = 1, f(v_6) = 10$ and $f(u_1) = 12, f(u_2) = 2, f(u_3) = 13, f(u_4) = 5, f(u_5) = 7, f(u_6) = 3$. Then $e_f(0) = 9 = e_f(1)$.

For $n = 7$, $f(v) = 2$, $f(v_1) = 7, f(v_2) = 4, f(v_3) = 6, f(v_4) = 12, f(v_5) = 8, f(v_6) = 10, f(v_7) = 14$ and $f(u_1) = 5, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 1, f(u_6) = 11, f(u_7) = 13$. Then $e_f(0) = 10, e_f(1) = 11$.

For $n = 8$, $f(v) = 2$, $f(v_1) = 1, f(v_2) = 4, f(v_3) = 6, f(v_4) = 12, f(v_5) = 8, f(v_6) = 10, f(v_7) = 14, f(v_8) = 16$ and $f(u_1) = 7, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 5, f(u_6) = 11, f(u_7) = 13, f(u_8) = 17$. Then $e_f(0) = 12 = e_f(1)$.

For $n = 9$, $f(v) = 2$, $f(v_1) = 4, f(v_2) = 5, f(v_3) = 6, f(v_4) = 12, f(v_5) = 18, f(v_6) = 8, f(v_7) = 10, f(v_8) = 14, f(v_9) = 16$ and $f(u_1) = 7, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 1, f(u_6) = 11, f(u_7) = 13, f(u_8) = 17, f(u_9) = 19$. Then $e_f(0) = 13, e_f(1) = 14$.

For $n = 11$, $f(v) = 2$, $f(v_1) = 4, f(v_2) = 5, f(v_3) = 6, f(v_4) = 12, f(v_5) = 18, f(v_6) = 8, f(v_7) = 10, f(v_8) = 14, f(v_9) = 16, f(v_{10}) = 20, f(v_{11}) = 22$ and $f(u_1) = 7, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 21, f(u_6) = 11, f(u_7) = 13, f(u_8) = 17, f(u_9) = 19, f(u_{10}) = 23, f(u_{11}) = 1$. Then $e_f(0) = 16, e_f(1) = 17$.

For $n = 14$, $f(v) = 2$, $f(v_1) = 4, f(v_2) = 5, f(v_3) = 6, f(v_4) = 12, f(v_5) = 18, f(v_6) = 24, f(v_7) = 8, f(v_8) = 10, f(v_9) = 14, f(v_{10}) = 16, f(v_{11}) = 20, f(v_{12}) = 22, f(v_{13}) = 26, f(v_{14}) = 28$ and $f(u_1) = 7, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 21, f(u_6) = 27, f(u_7) = 11, f(u_8) = 13, f(u_9) = 17, f(u_{10}) = 19, f(u_{11}) = 23, f(u_{12}) = 25, f(u_{13}) = 29, f(u_{14}) = 1$. Then $e_f(0) = 21 = e_f(1)$.

For $n = 19$, $f(v) = 2$, $f(v_1) = 4, f(v_2) = 5, f(v_3) = 6, f(v_4) = 12, f(v_5) = 18, f(v_6) = 24, f(v_7) = 30, f(v_8) = 36, f(v_9) = 8, f(v_{10}) = 10, f(v_{11}) = 14, f(v_{12}) = 16, f(v_{13}) = 20, f(v_{14}) = 22, f(v_{15}) = 26, f(v_{16}) = 28, f(v_{17}) = 32, f(v_{18}) = 34, f(v_{19}) = 38$ and $f(u_1) = 3, f(u_2) = 7, f(u_3) = 9, f(u_4) = 15, f(u_5) = 21, f(u_6) = 27, f(u_7) = 33, f(u_8) = 39, f(u_9) = 11, f(u_{10}) = 13, f(u_{11}) = 17, f(u_{12}) = 19, f(u_{13}) = 23, f(u_{14}) = 25, f(u_{14}) = 29, f(u_{14}) = 31, f(u_{14}) = 35, f(u_{14}) = 37, f(u_{14}) = 1$. Then $e_f(0) = 29, e_f(1) = 28$.

Now for the remaining two cases let,

$s = \left\lfloor \frac{n}{3} \right\rfloor, k = \left\lfloor \frac{2n+1}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor, t = \left(n + \left\lfloor \frac{2n+1}{3} \right\rfloor - 2\right) - \left\lceil \frac{3n}{2} \right\rceil$,

$m = \left\lfloor \frac{2n+1}{2} \right\rfloor - (2 + s + t)$, h_e = largest even number not divisible by $3 \leq 2n$,

h_o =largest odd number not divisible by $3 \leq 2n + 1$.

$f(v) = 2$,

$f(v_1) = 4, \quad f(v_2) = 5$,

$f(v_{2+i}) = 6i; \quad 1 \leq i \leq s$

$f(u_1) = 7$,

$f(u_{1+i}) = 3 + 6(i - 1); \quad 1 \leq i \leq k$

If $k = s$, then $f(u_{k+2}) = 1$ or $f(u_n) = 1$.

Case 3: $t = 0(n = 10, 12, 13, 15, 17)$

For m odd, consider $x_1 = \left\lceil \frac{m}{2} \right\rceil, x_2 = x_3 = x_4 = \left\lfloor \frac{m}{2} \right\rfloor$ and for m even consider, $x_1 = x_2 = x_3 = x_4 = \frac{m}{2}$

$f(v_{s+1+2i}) = 8 + 6(i - 1); \quad 1 \leq i \leq x_1$

$f(v_{s+2+2i}) = 10 + 6(i - 1), \quad 1 \leq i \leq x_3$

$f(u_{s+1+2i}) = 11 + 6(i - 1); \quad 1 \leq i \leq x_2$

$f(u_{s+2+2i}) = 13 + 6(i - 1); \quad 1 \leq i \leq x_4$

which assigns all the vertex labels for case 3.

Case 4: $t \geq 1(n = 16, 18, n \geq 20)$

For m odd, consider $x_1 = x_2 = x_3 = \left\lfloor \frac{m}{2} \right\rfloor, x_4 = \frac{m-3}{2}$ and for m even consider, $x_1 = \frac{m}{2}, x_2 = x_3 = x_4 =$

$\dfrac{m-2}{2}$.

$f(v_{s+1+2i}) = 8 + 6(i-1); \quad 1 \le i \le x_1$
$f(v_{s+2+2i}) = 10 + 6(i-1), \quad 1 \le i \le x_3$
$f(u_{s+1+2i}) = 11 + 6(i-1); \quad 1 \le i \le x_2$
$f(u_{s+2+2i}) = 13 + 6(i-1); \quad 1 \le i \le x_4$

For the vertices $u_{n-1}, u_{n-2}, \ldots, u_{n-(t+1)}$ we assign even numbers (not congruent 0 mod 3) in descending order starting from h_e respectively while for u_n,$v_n, v_{n-1}, v_{n-2}, \ldots, v_{n-t}$ we assign odd numbers(not congruent 0 mod 3) in descending order starting from h_0 respectively such that $f^*(v_j u_{j-i})$ or $f^*(v_j u_{j+i})$ do not generate edge label 0. Which assigns all the vertex labels for case 4.

In view of the above defined labeling pattern for cases 3 and 4, we have

$e_f(0) = \left\lceil \dfrac{3n}{2} \right\rceil$ and $e_f(1) = \left\lfloor \dfrac{3n}{2} \right\rfloor$.

Thus, we have $|e_f(0) - e_f(1)| \le 1$.

Hence, G_n is a prime cordial graph for $n \ge 4$. □

Example 2.1. *For the graph G_{20}, $|V(G_{20})| = 41$ and $|E(G_{20})| = 60$. In accordance with Theorem 2.1 we have $s = 6, k = 7, t = 1, m = 11, x_1 = x_2 = x_3 = 5, x_4 = 4$ and using the labeling pattern described in case 4. The corresponding prime cordial labeling is shown in Fig. 1. It is easy to visualise that $e_f(0) = 30 = e_f(1)$.*

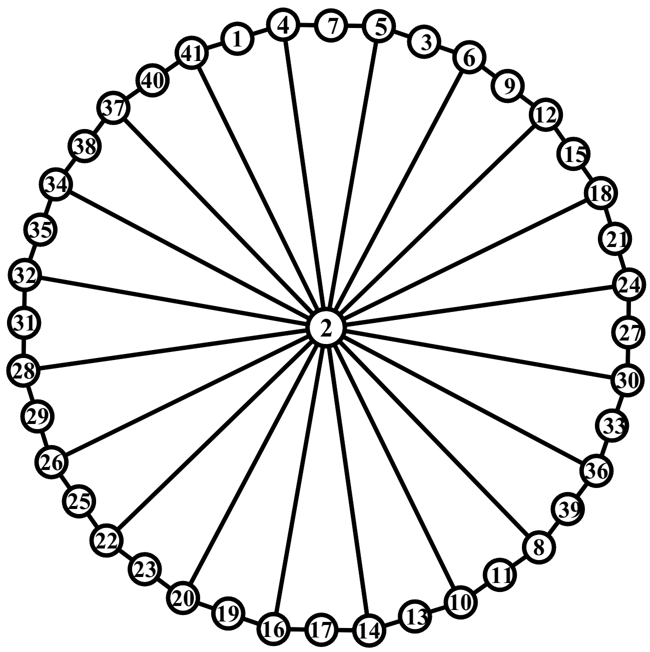

Fig. 1

Theorem 2.2. *Helm graph H_n is a prime cordial graph for every n.*

Proof. Let v be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the pendant vertices of H_n. Then $|V(H_n)| = 2n + 1$ and $|E(H_n)| = 3n$. To define $f : V(G) \to \{1, 2, 3, \ldots, 2n + 1\}$, we consider following three cases.

Case 1: $n = 3$ to 9

For $n = 3$, $f(v) = 6$, $f(v_1) = 2, f(v_2) = 4, f(v_3) = 3$ and $f(u_1) = 1, f(u_2) = 7, f(u_3) = 5$. Then $e_f(0) = 4, e_f(1) = 5$.

For $n = 4$, $f(v) = 6$, $f(v_1) = 3, f(v_2) = 2, f(v_3) = 4, f(v_4) = 8$ and $f(u_1) = 1, f(u_2) = 9, f(u_3) = 5, f(u_4) = 7$. Then $e_f(0) = 6 = e_f(1)$.

For $n = 5$, $f(v) = 6$, $f(v_1) = 3, f(v_2) = 2, f(v_3) = 5, f(v_4) = 8, f(v_5) = 10$ and $f(u_1) = 11, f(u_2) = 1, f(u_3) = 5, f(u_4) = 7, f(u_5) = 9$. Then $e_f(0) = 8, e_f(1) = 7$.

For $n = 6$, $f(v) = 2$, $f(v_1) = 3, f(v_2) = 6, f(v_3) = 8, f(v_4) = 10, f(v_5) = 7, f(v_6) = 11$ and $f(u_1) = 1, f(u_2) = 4, f(u_3) = 12, f(u_4) = 5, f(u_5) = 9, f(u_6) = 13$. Then $e_f(0) = 9 = e_f(1)$.

For $n = 7$, $f(v) = 2$, $f(v_1) = 3, f(v_2) = 6, f(v_3) = 4, f(v_4) = 10, f(v_5) = 5, f(v_6) = 11, f(v_7) = 1$ and

$f(u_1)=9, f(u_2)=8, f(u_3)=12, f(u_4)=14, f(u_5)=7, f(u_6)=13, f(u_7)=15$. Then $e_f(0)=11, e_f(1)=10$.
For $n=8$, $f(v)=2,\ f(v_1)=6, f(v_2)=8, f(v_3)=4, f(v_4)=10, f(v_5)=5, f(v_6)=9, f(v_7)=13, f(v_8)=17$ and $f(u_1)=3, f(u_2)=12, f(u_3)=14, f(u_4)=16, f(u_5)=7, f(u_6)=11, f(u_7)=15, f(u_8)=1$. Then $e_f(0)=12=e_f(1)$.
For $n=9$, $f(v)=2,\ f(v_1)=3, f(v_2)=6, f(v_3)=8, f(v_4)=4, f(v_5)=10, f(v_6)=5, f(v_7)=9, f(v_8)=13, f(v_9)=17$ and $f(u_1)=1, f(u_2)=12, f(u_3)=14, f(u_4)=16, f(u_5)=18, f(u_6)=7, f(u_7)=11, f(u_8)=15, f(u_9)=19$. Then $e_f(0)=17, e_f(1)=16$.

Case 2: n is even, $n \geq 10$

$f(v)=2,$ $\quad f(v_1)=10,$
$f(v_2)=4,$ $\quad f(v_3)=8,$
$f(v_{3+i})=12+2(i-1);$ $\quad 1 \leq i \leq \frac{n}{2}-4$
$f\left(v_{\frac{n}{2}}\right)=6,$ $\quad f\left(v_{\frac{n}{2}+1}\right)=1,$
$f\left(u_{\frac{n}{2}}\right)=3,$ $\quad f\left(u_{\frac{n}{2}+1}\right)=2n+1,$
$f(u_i)=2n-2(i-1);$ $\quad 1 \leq i \leq \frac{n}{2}-1$
$f(v_{n-i})=5+4i;$ $\quad 0 \leq i \leq \frac{n}{2}-1$
$f(u_{n-i})=7+4i;$ $\quad 0 \leq i \leq \frac{n}{2}-1$

Case 3: n is odd, $n \geq 11$

$f(v)=2,$ $\quad f(v_1)=10,$
$f(v_2)=4,$ $\quad f(v_3)=8,$
$f(v_{3+i})=12+2(i-1);$ $\quad 1 \leq i \leq \frac{n-1}{2}-4$
$f\left(v_{\frac{n-1}{2}}\right)=6,$ $\quad f\left(v_{\frac{n+1}{2}}\right)=3,$
$f\left(u_{\frac{n+1}{2}}\right)=1,$
$f(u_i)=2n-2(i-1);$ $\quad 1 \leq i \leq \frac{n-1}{2}$
$f(v_{n-i})=5+4i;$ $\quad 0 \leq i < \frac{n-1}{2}$
$f(u_{n-i})=7+4i;$ $\quad 0 \leq i < \frac{n-1}{2}$

In view of the above defined labeling pattern for cases 2 and 3,
If $2n-1 \equiv 0 (mod\ 3)$ then $e_f(0)=\left\lceil \frac{3n}{2} \right\rceil$ and $e_f(1)=\left\lfloor \frac{3n}{2} \right\rfloor$,
otherwise $e_f(0)=\left\lfloor \frac{3n}{2} \right\rfloor$ and $e_f(1)=\left\lceil \frac{3n}{2} \right\rceil$.
Thus, we have $|e_f(0)-e_f(1)| \leq 1$.
Hence, H_n is a prime cordial graph for every n. □

Example 2.2. *The graph H_{13} and its prime cordial labeling is shown in Fig. 2.*

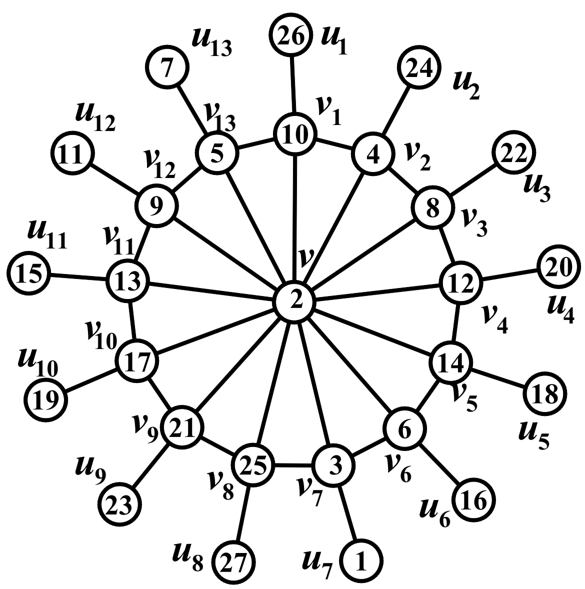

Fig. 2

Theorem 2.3. *Closed helm CH_n is a prime cordial graph for $n \geq 5$.*

Proof. Let v be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the vertices of degree 3 of CH_n. Then $|V(CH_n)| = 2n+1$ and $|E(CH_n)| = 4n$. To define $f : V(G) \rightarrow \{1, 2, 3, \ldots, 2n+1\}$, we consider following three cases.

Case 1: $n = 3, 4$

In CH_3 to satisfy the edge condition for prime cordial labeling it is essential to label six edges with label 0 and six edges with label 1 out of twelve edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most four edges and 1 labels for at least eight edges. That is, $|e_f(0) - e_f(1)| = 4 > 1$. Hence, CH_3 is not prime cordial graph.

In CH_4 to satisfy the edge condition for prime cordial labeling it is essential to label eight edges with label 0 and eight edges with label 1 out of sixteen edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most seven edges and 1 labels for at least nine edges. That is, $|e_f(0) - e_f(1)| = 2 > 1$. Hence, CH_4 is not prime cordial graph.

Case 2: $n = 5, 6$

For $n = 5$, $f(v) = 6$, $f(v_1) = 2, f(v_2) = 4, f(v_3) = 3, f(v_4) = 9, f(v_5) = 11$ and $f(u_1) = 10, f(u_2) = 8, f(u_3) = 1, f(u_4) = 7, f(u_5) = 5$. Then $e_f(0) = 10 = e_f(1)$.

For $n = 6$, $f(v) = 6$, $f(v_1) = 1, f(v_2) = 5, f(v_3) = 10, f(v_4) = 4, f(v_5) = 12, f(v_6) = 3$ and $f(u_1) = 11, f(u_2) = 13, f(u_3) = 8, f(u_4) = 2, f(u_5) = 9, f(u_6) = 7$. Then $e_f(0) = 12 = e_f(1)$.

Case 3: $n \geq 7$

$f(v) = 2,$ $f(v_1) = 4,$

$f(v_2) = 6,$ $f(v_3) = 3,$

$f(v_4) = 12,$ $f(v_5) = 8,$

$f(v_6) = 10,$

$f(v_{6+i}) = 14 + 2(i-1);$ $1 \leq i \leq n-6$

$f(u_1) = 1,$ $f(u_2) = 5,$

$f(u_3) = 7,$ $f(u_4) = 9,$

$f(u_5) = 13,$ $f(u_6) = 11,$

$f(u_{6+i}) = 15 + 2(i-1);$ $1 \leq i \leq n-6$

In view of the above defined labeling pattern we have $e_f(0) = 2n = e_f(1)$.

Thus, we have $|e_f(0) - e_f(1)| \leq 1$.

Hence, CH_n is a prime cordial graph for $n \geq 5$. □

Example 2.3. *The graph CH_{10} and its prime cordial labeling is shown in Fig. 3.*

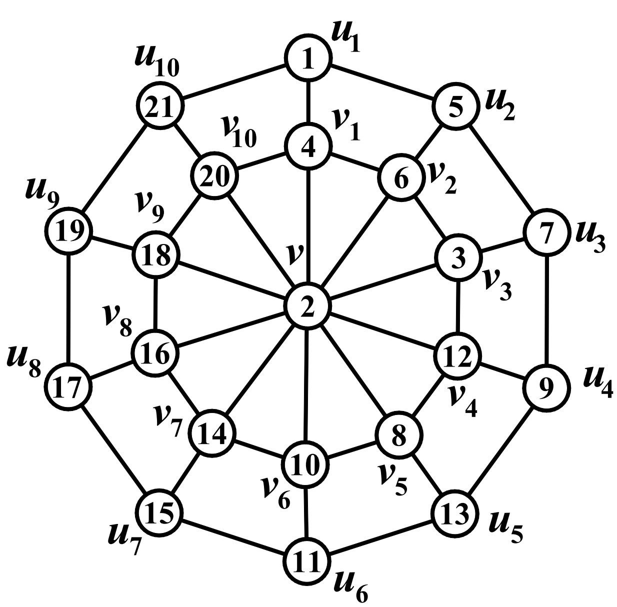

Fig. 3

Theorem 2.4. *Flower graph Fl_n is a prime cordial graph for $n \geq 4$.*

Proof. Let v be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 of Fl_n. Then $|V(Fl_n)| = 2n+1$ and $|E(Fl_n)| = 4n$. To define $f : V(G) \to \{1, 2, 3, \ldots, 2n+1\}$, we consider following four cases.

Case 1: $n = 3$
In Fl_3 to satisfy the edge condition for prime cordial labeling it is essential to label six edges with label 0 and six edges with label 1 out of twelve edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most four edges and 1 labels for at least eight edges. That is, $|e_f(0) - e_f(1)| = 4 > 1$. Hence, Fl_3 is not prime cordial graph.
Case 2: $n = 4$ to 9.
For Fl_4, $f(v) = 6$, $f(v_1) = 4, f(v_2) = 2, f(v_3) = 9, f(v_4) = 3$ and $f(u_1) = 7, f(u_2) = 8, f(u_3) = 5, f(u_4) = 1$. Then $e_f(0) = 8 = e_f(1)$.
For Fl_5, $f(v) = 6$, $f(v_1) = 2, f(v_2) = 4, f(v_3) = 8, f(v_4) = 10, f(v_5) = 3$ and $f(u_1) = 11, f(u_2) = 7, f(u_3) = 5, f(u_4) = 1, f(u_5) = 9$. Then $e_f(0) = 10 = e_f(1)$.
For Fl_6, $f(v) = 6$, $f(v_1) = 2, f(v_2) = 4, f(v_3) = 8, f(v_4) = 10, f(v_5) = 12, f(v_6) = 3$ and $f(u_1) = 5, f(u_2) = 7, f(u_3) = 9, f(u_4) = 11, f(u_5) = 13, f(u_6) = 1$. Then $e_f(0) = 12 = e_f(1)$.
For Fl_7, $f(v) = 2$, $f(v_1) = 3, f(v_2) = 12, f(v_3) = 10, f(v_4) = 8, f(v_5) = 14, f(v_6) = 4, f(v_7) = 6$ and $f(u_1) = 1, f(u_2) = 5, f(u_3) = 11, f(u_4) = 7, f(u_5) = 13, f(u_6) = 15, f(u_7) = 9$. Then $e_f(0) = 14 = e_f(1)$.
For Fl_8, $f(v) = 2$, $f(v_1) = 3, f(v_2) = 6, f(v_3) = 4, f(v_4) = 8, f(v_5) = 10, f(v_6) = 14, f(v_7) = 16, f(v_8) = 12$ and $f(u_1) = 1, f(u_2) = 9, f(u_3) = 7, f(u_4) = 5, f(u_5) = 11, f(u_6) = 13, f(u_7) = 15, f(u_8) = 17$. Then $e_f(0) = 16 = e_f(1)$.
For Fl_9, $f(v) = 2$, $f(v_1) = 3, f(v_2) = 12, f(v_3) = 4, f(v_4) = 8, f(v_5) = 10, f(v_6) = 14, f(v_7) = 16, f(v_8) = 18, f(v_9) = 6$ and $f(u_1) = 17, f(u_2) = 1, f(u_3) = 5, f(u_4) = 7, f(u_5) = 11, f(u_6) = 13, f(u_7) = 15, f(u_8) = 19, f(u_9) = 9$. Then $e_f(0) = 18 = e_f(1)$.
Case 3: n is even, $n \geq 10$

$$\begin{array}{ll} f(v) = 2, & f(v_1) = 10, \\ f(v_2) = 4, & f(v_3) = 8, \\ f(v_{3+i}) = 12 + 2(i-1); & 1 \leq i \leq \frac{n}{2} - 4 \\ f\left(v_{\frac{n}{2}}\right) = 6, & \\ f(v_{n-i}) = 5 + 4i; & 0 \leq i < \frac{n}{2} - 2 \\ f(u_i) = 2n - 2(i-1); & 1 \leq i \leq \frac{n}{2} - 1 \\ f\left(u_{\frac{n}{2}}\right) = 3, & \\ f(u_{n-i}) = 7 + 4i; & 0 \leq i < \frac{n}{2} - 2 \end{array}$$

For $2n + 1 \equiv 0(mod\ 3)$

$$\begin{array}{ll} f\left(v_{\frac{n}{2}+i}\right) = 2n + 1 - 4(i-1); & 1 \leq i \leq 2 \\ f\left(u_{\frac{n}{2}+1}\right) = 1, & f\left(u_{\frac{n}{2}+2}\right) = 2n - 1 \end{array}$$

For $2n + 1 \equiv 1(mod\ 3)$

$$\begin{array}{ll} f\left(v_{\frac{n}{2}+1}\right) = 2n - 3, & f\left(v_{\frac{n}{2}+2}\right) = 1, \\ f\left(u_{\frac{n}{2}+1}\right) = 2n + 1, & f\left(u_{\frac{n}{2}+2}\right) = 2n - 1 \end{array}$$

For $2n + 1 \equiv 2(mod\ 3)$

$$\begin{array}{ll} f\left(v_{\frac{n}{2}+1}\right) = 2n - 1, & f\left(v_{\frac{n}{2}+2}\right) = 2n - 3, \\ f\left(u_{\frac{n}{2}+1}\right) = 2n + 1, & f\left(u_{\frac{n}{2}+2}\right) = 1 \end{array}$$

Case 4: n is odd, $n \geq 11$

$$\begin{array}{ll} f(v) = 2, & f(v_1) = 10, \\ f(v_2) = 4, & f(v_3) = 8, \\ f(v_{3+i}) = 12 + 2(i-1); & 1 \leq i \leq \frac{n-1}{2} - 4 \\ f\left(v_{\frac{n-1}{2}}\right) = 6, & f\left(v_{\frac{n+1}{2}}\right) = 3, \\ f(v_{n-i}) = 5 + 4i; & 0 \leq i < \frac{n-1}{2} \\ f(u_i) = 2n - 2(i-1); & 1 \leq i \leq \frac{n-1}{2} \\ f(u_{n-i}) = 7 + 4i; & 0 \leq i < \frac{n-5}{2} \end{array}$$

For $2n + 1 \equiv 0(mod\ 3)$

$$\begin{array}{ll} f\left(u_{\frac{n+1}{2}}\right) = 2n + 1, & f\left(u_{\frac{n+1}{2}+1}\right) = 1, \\ f\left(u_{\frac{n+1}{2}+2}\right) = 2n - 3, & \end{array}$$

For $2n + 1 \equiv 1(mod\ 3)$

$f\left(u_{\frac{n+1}{2}}\right) = 2n-3, \quad f\left(u_{\frac{n+1}{2}+1}\right) = 2n+1,$
$f\left(u_{\frac{n+1}{2}+2}\right) = 1,$
For $2n+1 \equiv 2(mod\ 3)$
$f\left(u_{\frac{n+1}{2}}\right) = 1,$
$f\left(u_{\frac{n+1}{2}+i}\right) = 2n+1-4(i-1); \quad 1 \leq i \leq 2$
In view of the above defined labeling pattern we have $e_f(0) = 2n = e_f(1)$.
Thus, we have $|e_f(0) - e_f(1)| \leq 1$.
Hence, Fl_n is a prime cordial graph for $n \geq 4$. □

Example 2.4. *The graph Fl_{11} and its prime cordial labeling is shown in Fig. 4.*

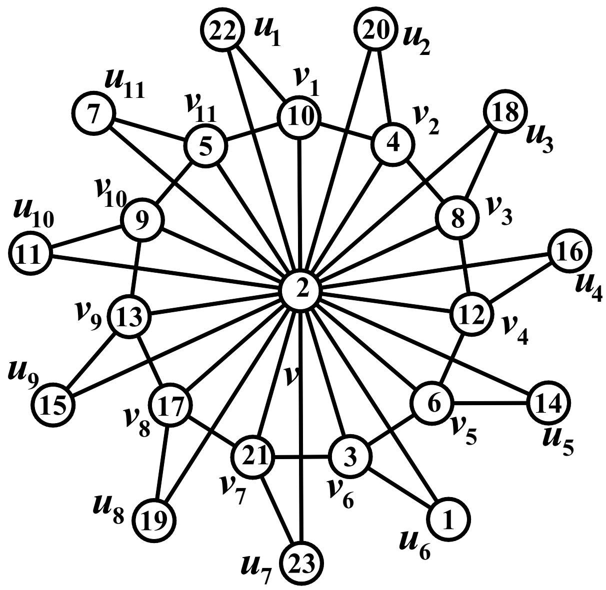

Fig. 4

3 Open problems

- To investigate necessary and sufficient conditions for a graph to admit a prime cordial labeling.
- To investigate some new graph or graph families which admit prime cordial labeling.
- To obtain forbidden subgraph(s) characterisation for prime cordial labeling.

4 Conclusion

As all the graphs are not prime cordial graphs it is very interesting and challenging as well to investigate prime cordial labeling for the graph or graph families which admit prime cordial labeling. Here we have contributed some new results by investigating prime cordial labeling for some wheel related graphs.

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Strongly unique best simulations approximation in linear 2-normed spaces

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Abstract

In this paper we established some basic properties of the set of strongly unique best simultaneous approximation in the context of linear 2-normed space.

Keywords: Linear 2-normed space, strongly unique best approximation, best simultaneous approximation and strongly unique best simultaneous approximation.

2010 MSC: 41A50, 41A52, 41A99, 41A28.

1 Introduction

The problem of simultaneous approximation was studied by several authors. Diaz and McLaughlin [2,3], Dunham [4] and Ling, et al.[8] have considered the simultaneous approximation of two real-valued functions defined on a closed interval $[a, b]$. Several results related with best simultaneous approximation in the context of normed linear space under different norms were obtained by Goel, et al. [5,6], Phillips, et al. [11], Dunham [4] and Ling, et al. [8]. Strongly unique best simultaneous approximation are investigated by Laurent, et al. [7]. Pai, et al. [9,10] studied the characterization and unicity of strongly unique best simultaneous approximation in normed linear spaces. The notion of strongly unique best simultaneous approximation in the context of linear 2-normed spaces is introduced in this paper. Section 2 gives some important definitions and results that are used in the sequel. Some fundamental properties of the set of strongly unique best simultaneous approximation with respect to 2-norm are established in Section 3.

2 Preliminaries

Definition 2.1. *Let X be a linear space over real numbers with dimension greater than one and let $\|.,.\|$ be a real-valued function on $X \times X$ satisfying the following properties for all x, y, z in X.*

(i) $\|x, y\| = 0$ *if and only if x and y are linearly dependent,*

(ii) $\|x, y\| = \|y, x\|$,

(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, *where α is a real number,*

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

Then $\|.,.\|$ is called a 2-norm and the linear space X equipped with the 2-norm is called a linear 2-normed space. It is clear that 2-norm is non-negative.

The following important property of 2-norm was established by Cho [1].

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Theorem 2.1. *[1] For any points $x, y \in X$ and any $\alpha \in \mathbb{R}$,*

$$\|x, y\| = \|x, y + \alpha x\|.$$

Definition 2.2. *Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a strongly unique best approximation to $x \in X$ from G, if there exists a constant $t > 0$ such that for all $g \in G$,*

$$\|x - g_0, k\| \leq \|x - g, k\| - t||g - g_0, k||, \quad \text{for all } k \in X \backslash [G, x].$$

Definition 2.3. *Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a best simultaneous approximation to $x_1, \cdots, x_n \in X$ from G if for all $g \in G$,*

$$\max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \leq \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\},$$

$$\text{for all } k \in X \backslash [G, x_1, \cdots, x_n].$$

The definition of strongly unique best simultaneous approximation in the context of linear 2-normed space is introduced here for the first time as follows:

Definition 2.4. *Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a strongly unique best simultaneous approximation to $x_1, \cdots, x_n \in X$ from G, if there exists a constant $t > 0$ such that for all $g \in G$,*

$$\max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \leq \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_0, k||,$$

$$\text{for all } k \in X \backslash [G, x_1, \cdots, x_n],$$

where $[G, x_1, \cdots, x_n]$ represents a linear space spanned by elements of G and $x_1, \cdots, x_n$. Let $Q_G(x_1, \cdots, x_n)$ denote the set of all elements of strongly unique best simultaneous approximations to $x_1, \cdots, x_n \in X$ from G. The subset G is called an existence set if $Q_G(x_1, \cdots, x_n)$ contains at least one element for every $x \in X$. G is called a uniqueness set if $Q_G(x_1, \cdots, x_n)$ contains at most one element for every $x \in X$. G is called an existence and uniqueness set if $Q_G(x_1, \cdots, x_n)$ contains exactly one element for every $x \in X$.

3 Some fundamental properties of $Q_G(x_1, \cdots, x_n)$

Some basic properties of strongly unique best simultaneous approximation are obtained in the following Theorems.

Theorem 3.1. *Let G be a non-empty subset of a linear 2-normed space X and $x_1, \cdots, x_n \in X$. Then the following statements hold.*

(i) $Q_G(x_1, \cdots, x_n)$ is closed if G is closed.

(ii) $Q_G(x_1, \cdots, x_n)$ is convex if G is convex.

(iii) $Q_G(x_1, \cdots, x_n)$ is bounded.

Proof. (i). Let G be closed.

Let $\{g_m\}$ be a sequence in $Q_G(x_1, \cdots, x_n)$ such that $g_m \to \tilde{g}$.

To prove that $Q_G(x_1, \cdots, x_n)$ is closed, it is enough to show that $\tilde{g} \in Q_G(x_1, \cdots, x_n)$.

Since G is closed, $\{g_m\} \in G$ and $g_m \to \tilde{g}$, we have $\tilde{g} \in G$. Since $\{g_m\} \in Q_G(x_1, \cdots, x_n)$, we have for all $k \in X \backslash [G, x_1, \cdots, x_n], g \in G$ and for some $t > 0$ that

$$\begin{aligned} & \max\{\|x_1 - g_m, k\|, \cdots, \|x_n - g_m, k\|\} \leq \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_m, k||. \\ \Rightarrow \quad & \max\{\|x_1 - \tilde{g}, k\| - \|g_m - \tilde{g}, k\|, \cdots, \|x_n - \tilde{g}, k\| - \|g_m - \tilde{g}, k\|\} \\ & \quad \leq \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_m, k|| \end{aligned} \tag{3.1}$$

Since $g_m \to \tilde{g}$, $g_m - \tilde{g} \to 0$. So $\|g_m - \tilde{g}, k\| \to 0$, since 0 and k are linearly dependent.

Therefore, it follows from (3.1) that

$$\max\{\|x_1-\tilde{g},k\|,\cdots,\|x_n-\tilde{g},k\|\}\le \max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}-t||g-\tilde{g},k||,$$
$$\text{for all } g\in G \text{ and } k\in X\backslash[G,x_1,\ldots,x_n], \quad \text{when } m\to\infty.$$

Thus $\tilde{g}\in Q_G(x_1,\cdots,x_n)$. Hence $Q_G(x_1,\cdots,x_n)$ is closed.
(ii). Let G be a convex set, $g_1,g_2\in Q_G(x_1,\cdots,x_n)$ and $0<\alpha<1$. To show that $\alpha g_1+(1-\alpha)g_2\in Q_G(x_1,\cdots,x_n)$, let $k\in X\setminus[G,x_1,\cdots,x_n]$.
Then

$\max\{\|x_1-(\alpha g_1+(1-\alpha)g_2),k\|,\cdots,\|x_n-(\alpha g_1+(1-\alpha)g_2),k\|\}$
$\le \max\{\alpha||x_1-g_1,k||+(1-\alpha)||x_1-g_2,k||,\cdots,\alpha\|x_n-g_1,k\|+(1-\alpha)||x_n-g_2,k\|\}$
$\le \max\{\alpha\|x_1-g_1,k\|,\cdots,\alpha\|x_n-g_1,k\|\}+\max\{(1-\alpha)\|x_1-g_2,k\|,\cdots,$
$(1-\alpha)\|x_n-g_2,k\|\}$
$\le \alpha(\max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}-t||g-g_1,k||)$
$+(1-\alpha)(\max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}-t||g-g_2,k||)$, for all $g\in G$ and for some $t>0$.
$=\max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}-t(\|\alpha g-\alpha g_1,k\|+\|(1-\alpha)g-(1-\alpha)g_2,k\|)$
$\le \max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}-t\|\alpha g-\alpha g_1+(1-\alpha)g-(1-\alpha)g_2,k\|$
$=\max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}-t||g-(\alpha g_1+(1-\alpha)g_2),k\|$.

Thus $\alpha g_1+(1-\alpha)g_2\in Q_G(x_1,\cdots,x_n)$. Hence $Q_G(x_1,\cdots,x_n)$ is convex.
(iii). To prove that $Q_G(x_1,\cdots,x_n)$ is bounded, it is enough to prove for arbitrary $g_0,\tilde{g_0}\in Q_G(x_1,\cdots,x_n)$ that $\|g_0-\tilde{g_0},k\|<C$ for some $C>0$, since $\|g_0-\tilde{g_0},k\|<C$ implies that $\sup\limits_{g_0,\tilde{g_0}\in Q_G(x_1,\ldots,x_n)}\|g_0-\tilde{g_0},k\|$ is finite and hence the diameter of $Q_G(x_1,\cdots,x_n)$ is finite.
Let $g_0,\tilde{g_0}\in Q_G(x_1,\cdots,x_n)$. Then there exists a constant $t>0$ such that for all $g\in G$ and $k\in X\setminus[G,x_1,\cdots,x_n]$,
$\max\{\|x_1-g_0,k\|,\cdots,\|x_n-g_0,k\|\}\le\max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}-t||g-g_0,k||$
and
$\max\{\|x_1-\tilde{g_0},k\|,\cdots,\|x_n-\tilde{g_0},k\|\}\le\max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}-t||g-\tilde{g_0},k||$.
Now,

$$\begin{aligned}\|g-g_0,k\| &\le \|x_1-g,k\|+\|x_1-g_0,k\|\\ &\le 2\max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}-t||g-g_0,k||.\end{aligned}$$

Thus $\|g-g_0,k\|\le\frac{2}{1+t}\max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}$ for all $g\in G$.

Hence $\|g-g_0,k\|\le\frac{2}{1+t}d$,
where $d=\inf\limits_{g\in G}\max\{\|x_1-g,k\|,\cdots,\|x_n-g,k\|\}$.

Similarly, $\|g-\tilde{g_0},k\|\le\frac{2}{1+t}d$.
Therefore, it follows that

$$\begin{aligned}\|g_0-\tilde{g_0},k\| &\le \|g_0-g,k\|+\|g-\tilde{g_0},k\|\\ &\le \frac{4}{1+t}d\\ &= C.\end{aligned}$$

Hence $Q_G(x_1,\cdots,x_n)$ is bounded.

Let X be a linear 2-normed space, $x\in X$ and $[x]$ denote the set of all scalar multiplications of x.

$$\text{i.e.,}\qquad [x]=\{\alpha x:\alpha\in\mathbb{R}\}.$$

□

Theorem 3.2. *Let G be a subset of a linear 2-normed space $X,x_1,\cdots,x_n\in X$ and $k\in X\setminus[G,x_1,\cdots,x_n]$. Then the following statements are equivalent for all $y\in[k]$.*

(i) $g_0\in Q_G(x_1,\cdots,x_n)$.

(ii) $g_0\in Q_G(x_1+y,\cdots,x_n+y)$.

(iii) $g_0\in Q_G(x_1-y,\cdots,x_n-y)$.

(iv) $g_0 + y \in Q_G(x_1 + y, \cdots, x_n + y)$.

(v) $g_0 + y \in Q_G(x_1 - y, \cdots, x_n - y)$.

(vi) $g_0 - y \in Q_G(x_1 + y, \cdots, x_n + y)$.

(vii) $g_0 - y \in Q_G(x_1 - y, \cdots, x_n - y)$.

(viii) $g_0 + y \in Q_G(x_1, \cdots, x_n)$.

(ix) $g_0 - y \in Q_G(x_1, \cdots, x_n)$.

Proof. The proof follows immediately by using Theorem 2.1. □

Theorem 3.3. *Let G be a subspace of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then*

$$g_0 \in Q_G(x_1, \cdots, x_n) \Leftrightarrow g_0 \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0),$$
$$\text{for all } \alpha \in \mathbb{R} \text{ and } m = 0, 1, 2, \cdots$$

Proof. Claim:

$$g_0 \in Q_G(x_1, \cdots, x_n) \Leftrightarrow g_0 \in Q_G(\alpha x_1 + (1-\alpha)g_0, \cdots, \alpha x_n + (1-\alpha)g_0), \text{ for all } \alpha \in \mathbb{R}.$$

Let $g_0 \in Q_G(x_1, \cdots, x_n)$. Then

$$\begin{aligned}
& \max\{\|x_1 - g_0, k\|, \cdots, \| x_n - g_0, k \|\} \\
& \leq \max\{\| x_1 - g, k \|, \cdots, \|x_n - g, k\|\} - t\|g - g_0, k \|, \text{ for all } g \in G \text{ and for some } t > 0. \\
\Rightarrow\ & \max\{\| \alpha x_1 - \alpha g_0, k \|, \cdots, \| \alpha x_n - \alpha g_0, k \|\} \\
& \leq \max\{\| \alpha x_1 - \alpha g, k \|, \cdots, \| \alpha x_n - \alpha g, k \|\} - t \| \alpha g - \alpha g_0, k \|\}, \text{ for all } g \in G. \\
\Rightarrow\ & \max\{\| \alpha x_1 - \alpha g_0, k \|, \cdots, \| \alpha x_n - \alpha g_0, k \|\}
\end{aligned}$$

$$\begin{aligned}
& \leq \max\left\{\| \alpha x_1 - \alpha\left(\frac{(\alpha-1)g_0 + g}{\alpha}\right), k \|, \cdots, \| \alpha x_n - \alpha\left(\frac{(\alpha-1)g_0 + g}{\alpha}\right), k \|\right\} \\
& \quad -t \| \alpha\left(\frac{(\alpha-1)g_0 + g}{\alpha}\right) - \alpha g_0, k \|, \text{ for all } g \in G \text{ and } \alpha \neq 0, \text{ since } \frac{(\alpha-1)g_0 + g}{\alpha} \in G. \\
\Rightarrow\ & \max\{\| \alpha x_1 + (1-\alpha)g_0 - g_0, k \|, \cdots, \| \alpha x_n + (1-\alpha)g_0 - g_0, k \|\} \\
& \leq \max\{\| \alpha x_1 + (1-\alpha)g_0 - g, k\|, \cdots, \|\alpha x_n + (1-\alpha)g_0 - g, k \|\} - t||g - g_0, k||.
\end{aligned}$$

$\Rightarrow g_0 \in Q_G(\alpha x_1 + (1-\alpha)g_0, \cdots, \alpha x_n + (1-\alpha)g_0)$, when $\alpha \neq 0$.
If $\alpha = 0$, then it is clear that $g_0 \in Q_G(\alpha x_1 + (1-\alpha)g_0, \cdots, \alpha x_n + (1-\alpha)g_0)$.

The converse is obvious by taking $\alpha = 1$. Hence the claim is true. □

Corollary 3.1. *Let G be a subspace of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then the following statements are equivalent for all $y \in [k], \alpha \in \mathbb{R}$ and $m = 0, 1, 2, \cdots$*

(i) $g_0 \in Q_G(x_1, \cdots, x_n)$.

(ii) $g_0 \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(iii) $g_0 \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(iv) $g_0 + y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(v) $g_0 + y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(vi) $g_0 - y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(vii) $g_0 - y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(viii) $g_0 + y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.

(ix) $g_0 - y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.

Proof. The proof follows immediately from simple application of Theorem 2.2 and Theorem 3.3. □

Theorem 3.4. *Let G be a subset of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then*

$$g_0 \in Q_G(x_1, \cdots, x_n) \Leftrightarrow g_0 \in Q_{G+[k]}(x_1, \cdots, x_n).$$

Proof. The proof follows from a simple application of Theorem 3.2. □

A corollary similar to that of Corollary 3.4 is established next as follows:

Corollary 3.2. *Let G be a subspace of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then the following statements are equivalent for all $y \in [k], \alpha \in \mathbb{R}$ and $m = 0, 1, 2, \cdots$*

(i) $g_0 \in Q_{G+[k]}(x_1, \cdots, x_n)$.

(ii) $g_0 \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(iii) $g_0 \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(iv) $g_0 + y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(v) $g_0 + y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(vi) $g_0 - y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(vii) $g_0 - y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(viii) $g_0 + y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.

(ix) $g_0 - y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.

Proof. The proof easily follows from Theorem 3.5 and Corollary 3.4. □

Proposition 3.1. *Let G be a subset of a linear 2-normed space X, $x_1, \cdots, x_n \in X$, $k \in X \setminus [G, x_1, \cdots, x_n]$ and $0 \in G$.* If *$g_0 \in Q_G(x_1, \cdots, x_n)$, then there exists a constant $t > 0$ such that*
$\max\{\| x_1 - g_0, k \|, \cdots, \| x_n - g_0, k \|\} \leq \max\{\| x_1, k \|, \cdots, \| x_n, k \|\} - t||g_0, k||$.

Proof. The proof is obvious. □

Proposition 3.2. *Let G be a subset of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. If $g_0 \in Q_G(x_1, \cdots, x_n)$, then there exists a constant $t > 0$ such that for all $g \in G$,*
$||g - g_0, k|| \leq 2\max\{||x_1 - g, k||, \cdots, ||x_n - g, k||\} - t||g - g_0, k||$.

Proof. The proof is trivial. □

Theorem 3.5. *Let G be a subspace of a linear 2-normed space X and $x_1, \cdots, x_n \in X$. Then the following statements hold.*
(i) $Q_G(x_1 + g, \cdots, x_n + g) = Q_G(x_1, \cdots, x_n) + g$, for all $g \in G$.
(ii) $Q_G(\alpha x_1, \cdots, \alpha x_n) = \alpha Q_G(x_1, \cdots, x_n)$, for all $\alpha \in \mathbb{R}$.

Proof. (i). Let $\tilde{g}$ be an arbitrary but fixed element of G.
Let $g_0 \in Q_G(x_1, \cdots, x_n)$. It is clear that $g_0 + \tilde{g} \in Q_G(x_1, \cdots, x_n) + \tilde{g}$.
To prove that $Q_G(x_1, \cdots, x_n) + \tilde{g} \subseteq Q_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$, it is enough to show that $g_0 + \tilde{g} \in Q_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$.
Now,

$$\begin{aligned}
& \max\{\|x_1 + \tilde{g} - g_0 - \tilde{g}, k\|, \cdots, \|x_n + \tilde{g} - g_0 - \tilde{g}, k\|\} \\
\leq\ & \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_0, k||, \\
& \text{for all } g \in G \text{ and for some } t > 0. \\
\Rightarrow\ & \max\{\|x_1 + \tilde{g} - (g_0 + \tilde{g}), k\|, \cdots, \|x_n + \tilde{g} - (g_0 + \tilde{g}), k\|\} \\
\leq\ & \max\{\|x_1 + \tilde{g} - g, k\|, \cdots, \|x_n + \tilde{g} - g, k\|\} - t||g - (g_0 + \tilde{g}), k||, \\
& \text{for all } g \in G \text{ and for some } t > 0, \text{ since } g - \tilde{g} \in G.
\end{aligned}$$

Thus $g_0 + \tilde{g} \in Q_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$.

Conversely, let $g_0+\tilde{g} \in Q_G(x_1+\tilde{g}, \cdots, x_n+\tilde{g})$. To prove that $Q_G(x_1+\tilde{g}, \cdots, x_n+\tilde{g}) \subseteq Q_G(x_1, \cdots, x_n)+\tilde{g}$, it is enough to show that $g_0 \in Q_G(x_1, \cdots, x_n)$. Let $k \in X \setminus [G, x_1, \cdots, x_n]$.

$$
\begin{aligned}
\text{Then} \quad & \max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \\
= \quad & \max\{\|x_1 + \tilde{g} - (g_0 + \tilde{g}), k\|, \cdots, \|x_n + \tilde{g} - (g_0 + \tilde{g}), k\|\} \\
\leq \quad & \max\{\|x_1 + \tilde{g} - g, k\|, \cdots, \|x_n + \tilde{g} - g, k\|\} - t||g - (g_0 + \tilde{g}), k||, \\
& \qquad \text{for all } g \in G \text{ and for some } t > 0. \\
\Rightarrow \quad & \max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \\
& \leq \max\{\|x_1 + \tilde{g} - (g + \tilde{g}), k\|, \cdots, \|x_n + \tilde{g} - (g + \tilde{g}), k\|\} \\
& \qquad -t||(g + \tilde{g}) - (g_0 + \tilde{g}), k||, \\
& \qquad \text{for all } g \in G \text{ and for some } t > 0, \text{ since } g + \tilde{g} \in G. \\
\Rightarrow \quad & g_0 \in Q_G(x_1, \cdots, x_n). \text{ Thus the result follows.}
\end{aligned}
$$

(ii). The proof is similar to that of (i). □

Remark 3.1. *Theorem 3.9 can be restated as*

$$Q_G(\alpha x_1 + g, \cdots, \alpha x_n + g) = \alpha Q_G(x_1, \cdots, x_n) + g, \text{ for all } g \in G.$$

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Variational homotopy perturbation method for the approximate solution of the foam drainage equation with time and space fractional derivatives

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Abstract

In this paper, variational homotopy perturbation method (VHPM) is applied for solving the foam drainage equation with time and space-fractional derivatives. Numerical solutions are obtained for various values of the time and space-order derivative in (0,1]. For the first-order time and space derivative, compared with the exact solution, the result showed that the proposed method could be used as an alternative method for obtaining an analytic and approximate solution for different types of differential equations.

Keywords: Caputo fractional derivative, variational homotopy perturbation method, foam drainage equation, fractional differential equations.

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1 Introduction

Fractional models have been shown by many scientists to adequately describe the operation of variety of physical and biological processes and systems [13]. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively. Many powerful methods have been presented for solving such kind of problems. Among them, the Adomian decomposition method [1, 2] (ADM), the variational iteration method (VIM) [7], and the homotopy perturbation method (HPM) [8].

In this paper, we consider the following foam drainage equation with time and space fractional derivatives of the form

$$\begin{cases} {}^cD_t^{\alpha}u = \frac{1}{2}uu_{xx} - 2u^2\ {}^cD_x^{\beta}u + ({}^cD_x^{\beta}u)^2 & ; \quad 0 < \alpha,\, \beta \le 1 \\ u\,(x,0) = g\,(x)\ . \end{cases} \tag{1.1}$$

When $\alpha = \beta = 1$, this fractional equation is reduced to the foam drainage equation of the form

$$u_t = \frac{1}{2}uu_{xx} - 2u^2\ u_x + (u_x)^2. \tag{1.2}$$

Notice that Eq. (1.2) is the reduced form obtained by putting $\Psi(x,t) = u^2(x,t)$ in the original one [15] defined as

$$\frac{\partial \Psi}{\partial t} + \frac{\partial}{\partial x}(\Psi^2 - \frac{\sqrt{\Psi}}{2}\frac{\partial \Psi}{\partial x}) = 0. \tag{1.3}$$

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The term foam drainage originally described the process by which fluid flows out of a foam, such as liquid draining out of a soap froth [4, 17]. Since then many technological applications have been developed for foams, which include cleansing, water purification, and minerals extraction as well as production of cushions, food stuffs, and ultra-lightweight structural materials [14]. Foams are metastable dispersions of gas in liquid that are evolving in time, which complicates precise measurements and obfuscates experimental trends.

Here α and β are the parameters standing for the order of the fractional time and space derivatives, and they satisfy $0 < \alpha, \beta \leq 1$. In fact, different response equations can be obtained when at least one of the parameters varies. In recent years, Eq. (1.1), has attracted many authors and has been studied from various methods. For example, Dahmani et al. [5] used the ADM method for solving Eq. (1.1) and then the VIM method for solving the same equation (see [6]). Later Yildirim and Koçak [18] employed VIM and HPM method for solving Eq. (1.1). The result is obtained in the form of power series convergent to the exact solution.

In what follows, we will use the variational homotopy perturbation method (VHPM) [9, 11], to solve the foam drainage equation of the form (1.1). The main characteristic of this proposed method is to avoid calculating the Adomian polynomials as in [5], and instead of using separately VIM and HPM as in [18], we will use the combined one between VIM and HPM. The result is obtained in the form of convergent series, and this method will be proved to be very useful to accelerate the convergence. Furthermore, the exact solution for $\alpha = \beta = 1$ will be used to compare those obtained by the VHPM method.

2 Basic Definitions

There are several definitions of a fractional derivative of order $\alpha > 0$ (see [13]). The most commonly used definitions are the Riemann–Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper .

Definition 2.1. *A real function $f(t), t > 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1 \in C[0, \infty)$, and it is said to be in the space $C_\mu^n, n \in \mathbb{N}$, if $f^{(n)} \in C_\mu$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as*

$$\begin{aligned} (J^\alpha f)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ \alpha > 0, t > 0 \\ (J^0 f)(t) &= f(t). \end{aligned} \tag{2.4}$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 2.3. *Let $u \in C_{-1}^n$, $n \in \mathbb{N}^*$. Then the (left sided) Caputo fractional derivative of u is defined as*

$$^cD_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,t)}{\partial t^n} d\tau, \ n-1 < \alpha < n, \ n \in \mathbb{N}^*, t > 0, \\ \frac{\partial^n u(x,t)}{\partial t^n}, \ \alpha = n \in \mathbb{N}. \end{cases} \tag{2.5}$$

According to (2.5), we can obtain:

$^cD^\alpha K = 0$, K is a constant, and $^cD^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \beta > \alpha - 1 \\ 0, \ \beta \leq \alpha - 1. \end{cases}$

3 Variational Homotopy Perturbation Method

To illustrate the basic idea of the VHPM, we consider the following general differential equation [9, 11]:

$$^cD_t^\alpha u(x,t) + R[u(x,t)] + N[u(x,t)] = g(x,t), \tag{3.6}$$

where $^cD_t^\alpha$ is the Caputo fractional derivative, R is a linear operator, N is a nonlinear operator, $g(x,t)$ is an in homogeneous term, and $m - 1 < \alpha \leq m$, $m \in \mathbb{N}^*$. According to the variational iteration method [16], we can construct a correct functional as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau) \left\{ \frac{\partial_{\tau}^{m} u_n}{\partial \tau^m}(x,\tau) + R[u_n(x,\tau)] + N[\tilde{u}_n(x,\tau)] - g(x,\tau) \right\} d\tau, \tag{3.7}$$

where λ is a general Lagrange multiplier. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n(t) = 0$ and (3.7) is called a correct functional. Now, we apply the homotopy perturbation method [3, 12]

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + p \int_0^t \lambda(\tau) \left\{ \sum_{i=0}^{\infty} p^i \frac{\partial_{\tau}^{\alpha} u_i(x,\tau)}{\partial \tau^{\alpha}} + R\left[\sum_{i=0}^{\infty} p^i u_i(x,\tau)\right] + N\left[\sum_{i=0}^{\infty} p^i u_i(x,\tau)\right] - g(x,\tau)) \right\} ds, \tag{3.8}$$

which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and He's polynomials [9]. The embedding parameter $p \in [0,1]$ can be considered as an expanding parameter. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain

$$f = \sum_{i=0}^{+\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + ... \tag{3.9}$$

If $p \to 1$, then (3.9) becomes the approximate solution of the form

$$u = \lim_{p \longrightarrow 1} f = u_0 + u_1 + u_2 + ... \tag{3.10}$$

A comparison of like powers of p gives solutions of various orders.

4 Application of the VHPM for the time-fractional derivative

We consider the foam drainage equation with time fractional derivative:

$${}^{c}D_t^{\alpha} u = \frac{1}{2} u u_{xx} - 2u^2\, u_x + (u_x)^2; \quad 0 < \alpha \le 1, \tag{4.11}$$

subject to the initial condition

$$u(x,0) = g(x) = -\sqrt{c} \tanh \sqrt{c}(x), \tag{4.12}$$

where c is the velocity of wavefront [15].

The exact solution of (4.11) for the special case $\alpha = 1$ is

$$\begin{cases} u(x,t) = -\sqrt{c}\tanh\ [\sqrt{c}(x - ct)] & ;\ x \le ct. \\ 0 & ;\ x > ct. \end{cases} \tag{4.13}$$

According to the VIM method, the correction variational functional of equation (4.11) can be expressed as follows

$$u_{k+1} = u_k + \int_0^t \lambda(\tau) \left\{ \frac{\partial_{\tau}^{\alpha} u_k}{\partial \tau^{\alpha}} - \frac{1}{2} u_k\, (u_k)_{xx} + 2u_k^2\, (u_k)_x - ((u_k)_x)^2 \right\} d\tau. \tag{4.14}$$

Since $\alpha \in (0,1]$, the calcul of the Lagrange multiplier optimally via variational theory yields the stationary conditions $\left\{ \begin{array}{c} \lambda' = 0 \\ \lambda + 1 = 0 \end{array} \right\}$, and hence, the general Lagrange multiplier can be readily identified as $\lambda = -1$. Substituting this value of the Lagrangian multiplier into functional (4.14) gives the iteration formula

$$u_{k+1} = u_k - \int_0^t \left\{ \frac{\partial_{\tau}^{\alpha} u_k}{\partial \tau^{\alpha}} - \frac{1}{2} u_k\, (u_k)_{xx} + 2u_k^2\, (u_k)_x - ((u_k)_x)^2 \right\} d\tau. \tag{4.15}$$

While applying the variational homotopy perturbation method, one obtains

$$u_0+pu_1+p^2u_2+... = u_0-p\int_0^t \left\{ \begin{array}{c} \frac{\partial\,{}_\tau^\alpha}{\partial\,\tau^\alpha}\left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x,\tau)\right) - \frac{1}{2}\left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x,\tau)\right)\frac{\partial^2}{\partial^2 x}\left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x,\tau)\right) \\ +2\left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x,\tau)\right)^2\frac{\partial}{\partial x}\left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x,\tau)\right) - \left(\frac{\partial}{\partial x}\left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x,\tau)\right)\right)^2 \end{array} \right\} d\tau. \tag{4.16}$$

Comparing the coefficients of like powers of p one obtains the following set of linear partial differential equations

$$p^0 \quad : \quad u_0\left(x,t\right) = g(x) \tag{4.17}$$

$$p^1 \quad : \quad u_1\left(x,t\right) = \int_0^t \left\{ \begin{array}{c} -\frac{\partial\,{}_\tau^\alpha u_0}{\partial \tau^\alpha}(x,\tau) + \frac{1}{2}u_0(x,\tau)\frac{\partial^2 u_0}{\partial^2 x}(x,\tau) \\ -2u_0^2\ \frac{\partial u_0}{\partial x}(x,\tau) + \left(\frac{\partial u_0}{\partial x}(x,\tau)\right)^2 \end{array} \right\} d\tau$$

$$p^2 \quad : \quad u_2\left(x,t\right) = \int_0^t \left\{ \begin{array}{c} -\frac{\partial\,{}_\tau^\alpha u_1}{\partial\,\tau^\alpha} + \frac{1}{2}u_0\frac{\partial^2 u_1}{\partial^2 x} + \frac{1}{2}u_1\frac{\partial^2 u_0}{\partial^2 x} \\ -2u_0^2\ \frac{\partial u_1}{\partial x} - 4u_0u_1\ \frac{\partial u_0}{\partial x} + 2\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} \end{array} \right\} d\tau$$

$$p^3 \quad : \quad u_3\left(x,t\right) = \int_0^t \left\{ \begin{array}{c} -\frac{\partial\,{}_\tau^\alpha u_2}{\partial\,\tau^\alpha} + \frac{1}{2}u_1\frac{\partial^2 u_1}{\partial^2 x} + \frac{1}{2}u_2\frac{\partial^2 u_0}{\partial^2 x} + \frac{1}{2}u_0\frac{\partial^2 u_2}{\partial^2 x} \\ -4u_0u_2\ \frac{\partial u_0}{\partial x} - 4u_0u_1\ \frac{\partial u_1}{\partial x} + 2\frac{\partial u_0}{\partial x}\frac{\partial u_2}{\partial x} \\ -2(u_1)^2\frac{\partial u_0}{\partial x} - 2(u_0)^2\frac{\partial u_2}{\partial x} + (\frac{\partial u_1}{\partial x})^2 \end{array} \right\} d\tau, \tag{4.18}$$

and so on, in the same manner the rest of components can be obtained using the Maple package. Consequently, while taking the initial value $u_0(x,t) = -\sqrt{c}\tanh\sqrt{c}(x)$, and according to Eqs. (4.17)– (4.18), the first few components of the variational homotopy perturbation solution for Eq. (4.11) are derived as follows

$$\begin{aligned} u_0(x,t) &= -\sqrt{c}\tanh\sqrt{c}(x), \\ u_1(x,t) &= \frac{c^2}{\cosh(\sqrt{c}x)^2}t, \\ u_2(x,t) &= -c^{7/2}\left(-1+\left(\tanh\left(\sqrt{c}x\right)\right)^2\right)t^2\tanh\left(\sqrt{c}x\right) + \frac{c^2\left(-1+\left(\tanh\left(\sqrt{c}x\right)\right)^2\right)t^{2-\alpha}}{\Gamma\left(2-\alpha\right)\left(2-\alpha\right)}, \\ u_3(x,t) &= \frac{c^2}{\left(\cosh\left(\sqrt{c}x\right)\right)^2\Gamma\left(4-2\,\alpha\right)}t^{3-2\alpha} - 4\,\frac{c^{7/2}\sinh\left(\sqrt{c}x\right)}{\left(\cosh\left(\sqrt{c}x\right)\right)^3\Gamma\left(4-\alpha\right)}t^{3-\alpha} \\ &\quad + \frac{1}{3}\frac{\left(2\ \left(\cosh\left(\sqrt{c}x\right)\right)^2-3\right)c^5}{\left(\cosh\left(\sqrt{c}x\right)\right)^4}t^3 \\ &\quad .... \end{aligned}$$

The other components of the (VHPM) can be determined in a similar way. Finally, the approximate solution of Eq. (4.11) in a series form is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...$$

Consequently, the third-order approximation solution of Eq. (4.11) is given by

$$\begin{aligned} u(x,t) \quad : \quad &= c^2 t\mathrm{Sech}\left[\sqrt{c}x\right]^2 + \frac{c^2t^{3-2\alpha}\mathrm{Sech}\left[\sqrt{c}x\right]^2}{\Gamma[4-2\alpha]} + \frac{c^2t^{2-\alpha}\mathrm{Sech}\left[\sqrt{c}x\right]^2}{(-2+\alpha)\Gamma[2-\alpha]} \\ &+\frac{1}{3}c^5t^3\left(-3+2\mathrm{Cosh}\left[\sqrt{c}x\right]^2\right)\mathrm{Sech}\left[\sqrt{c}x\right]^4 - \sqrt{c}\mathrm{Tanh}\left[\sqrt{c}x\right] \\ &+c^{7/2}t^2\mathrm{Sech}\left[\sqrt{c}x\right]^2\mathrm{Tanh}\left[\sqrt{c}x\right] - \frac{4c^{7/2}t^{3-\alpha}\mathrm{Sech}\left[\sqrt{c}x\right]^2\mathrm{Tanh}\left[\sqrt{c}x\right]}{\Gamma[4-\alpha]} \end{aligned} \tag{4.19}$$

4.1 Numerical results

For $\alpha = 1$ and $c = \frac{1}{5}$, while inserting in (4.19), one obtains the approximation

$$u(x,t) = \frac{1}{25}t\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 + \frac{t^3\left(-3+2\text{Cosh}\left[\frac{x}{\sqrt{5}}\right]^2\right)\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^4}{9375} - \frac{\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{\sqrt{5}} + \frac{t^2\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{125\sqrt{5}}.$$

Now, an expansion of the exact solution (4.13) in Taylor series over $t = 0$ to order 3 gives:

$$u(x,t) = -\frac{\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{\sqrt{5}} + \frac{1}{25}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 t + \frac{\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]t^2}{125\sqrt{5}} \tag{4.20}$$

$$+\frac{\left(-2+\text{Cosh}\left[\frac{2x}{\sqrt{5}}\right]\right)\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^4 t^3}{9375} + O[t]^4 \tag{4.21}$$

This confirms the accuracy of the method.

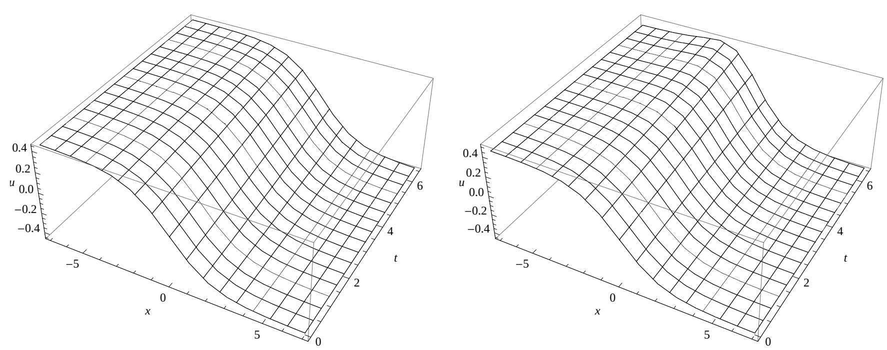

Figure 1: (Left): Exact solution (4.13) for Eqs. (4.11)-(4.12); (Right): Series approximation solution of Eqs. (4.11)-(4.12) by VHPM method for $\alpha = 1$ with four terms.

So, for $\alpha = \frac{1}{2}$ and $c = \frac{1}{5}$, while inserting in (4.19), one obtains the approximation

$$u(x,t) = \frac{1}{25}t\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 - \frac{4t^{3/2}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2}{75\sqrt{\pi}} + \frac{1}{50}t^2\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 + \frac{t^3\left(-3+2\text{Cosh}\left[\frac{x}{\sqrt{5}}\right]^2\right)\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^4}{9375} - \frac{\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{\sqrt{5}} + \frac{t^2\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{125\sqrt{5}} - \frac{32t^{5/2}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{1875\sqrt{5\pi}}$$

For $\alpha = 0.9$ and $c = \frac{1}{5}$, while inserting in (4.19), one obtains the approximation

$$u_{0.9}(x,t) = \frac{1}{25}t\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 - 0.0382232t^{1.1}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 + 0.0363041t^{1.2}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 + \frac{t^3\left(-3+2\text{Cosh}\left[\frac{x}{\sqrt{5}}\right]^2\right)\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^4}{9375} - \frac{\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{\sqrt{5}} + \frac{t^2\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{125\sqrt{5}} - 0.00651197t^{2.1}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]$$

These figures represent the graphs of Eq. (1.1) for various values of α. For example, Fig. 1 (left) represents the graph of the exact solution (4.13) of the initial value problem (4.11)–(4.12). Fig. 1 (right) is the graph

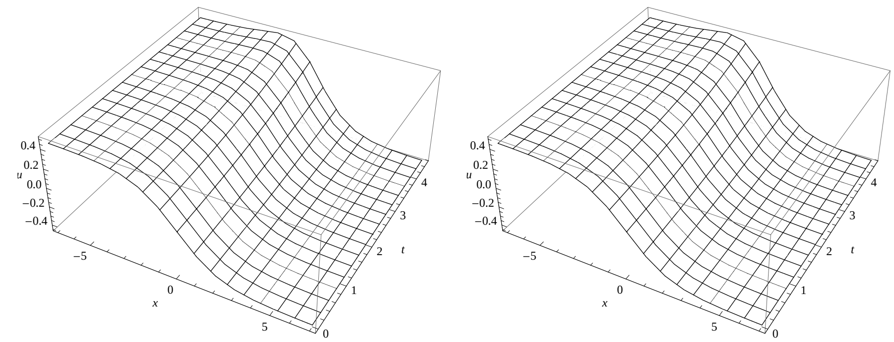

Figure 2: Series approximation solution of Eqs. (4.11)-(4.12) by VHPM method with four terms (Right: $\alpha = 1/2$; Left: $\alpha = 0.9$).

of the numerical one obtained by VHPM for $\alpha = 1$ with four terms in the series solution. One observes that there is a similarity between the two figures and this leads to say that the method employed could be used as an alternative method for obtaining an analytic and approximate solution for different types of differential equations. In Fig. 2, one has represented the graphs of Eq. (1.1) for $\alpha = 1/2$ (left) and $\alpha = 0.9$ (right) respectively.

5 Application of the VHPM for the space-fractional derivative

We next consider the following space-fractional foam drainage equation

$$\begin{cases} u_t = \frac{1}{2} u u_{xx} - 2u^2 \; {}^cD_x^\beta u + ({}^cD_x^\beta u)^2 \quad ; \quad 0 < \beta \quad \leq 1 \\ u\,(x, 0) = x^2. \end{cases} \tag{5.22}$$

This initial condition is taken as polynomial to avoid heavy calculations of fractional differentiation. According to the VHPM method, one obtains

$$u_0 + pu_1 + p^2 u_2 + ... = u_0 - p \int_0^t \left\{ \begin{array}{c} \frac{\partial_{\ \tau}}{\partial\ \tau} \left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x, \tau) \right) \\ -\frac{1}{2} \left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x, \tau) \right) \frac{\partial^2}{\partial^2 x} \left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x, \tau) \right) \\ +2 \left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x, \tau) \right)^2 \frac{\partial^\beta}{\partial x^\beta} \left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x, \tau) \right) \\ - \left(\frac{\partial^\beta}{\partial x^\beta} \left(\sum\limits_{i=0}^{\infty} p^i\ u_i(x, \tau) \right) \right)^2 \end{array} \right\} d\tau.$$

Comparing the coefficients of like powers of p one obtains the following set of linear partial differential equations

$$\frac{\partial u_0}{\partial t} = \frac{\partial v_0}{\partial t}, u_0(x, 0) = x^2 \tag{5.23}$$

$$\frac{\partial u_1}{\partial t} = -\frac{\partial u_0}{\partial t} + \frac{1}{2} u_0 \frac{\partial^2 u_0}{\partial x^2} - 2{u_0}^2 \frac{\partial^\beta u_0}{\partial x^\beta} + \left(\frac{\partial^\beta u_0}{\partial x^\beta} \right)^2, u_1(x, 0) = 0 \tag{5.24}$$

$$\begin{aligned} \frac{\partial u_2}{\partial t} = \tfrac{1}{2}(u_0 \frac{\partial^2 u_1}{\partial x^2} + u_1 \frac{\partial^2 u_0}{\partial x^2}) - \frac{\partial u_1}{\partial t} - 2\frac{\partial^\beta u_1}{\partial x^\beta}({u_0}^2 \frac{\partial^\beta u_1}{\partial x^\beta} - \frac{\partial^\beta u_0}{\partial x^\beta} \frac{\partial^\beta u_1}{\partial x^\beta}) \\ -4u_0 u_1 \frac{\partial^\beta u_0}{\partial x^\beta}, u_2(x, 0) = 0 \end{aligned} \tag{5.25}$$

$$\begin{aligned} \frac{\partial u_3}{\partial t} = \tfrac{1}{2} u_2 \frac{\partial^2 u_0}{\partial x^2} - 4u_0 u_1 \frac{\partial^\beta u_1}{\partial x^\beta} + \left(\frac{\partial^\beta u_1}{\partial x^\beta} \right)^2 + \tfrac{1}{2} u_0 \frac{\partial^2 u_2}{\partial x^2} + \tfrac{1}{2} u_1 \frac{\partial^2 u_1}{\partial x^2} \\ -4u_0 u_2 \frac{\partial^\beta u_0}{\partial x^\beta} - 2u_0^2 \frac{\partial^\beta u_2}{\partial x^\beta} + 2\frac{\partial^\beta u_0}{\partial x^\beta} \frac{\partial^\beta u_2}{\partial x^\beta} - 2u_1^2 \frac{\partial^\beta u_0}{\partial x^\beta} - 2\frac{\partial u_2}{\partial t}, u_2(x, 0) = 0. \end{aligned} \tag{5.26}$$

Selecting the initial value $u(x,0)=x^2$ and using equations (5.23)-(5.26) one obtains the following successive approximations

$$u_0(x,t)=x^2 \tag{5.27}$$

$$u_1(x,t)=\frac{t\left(x^2\left(\Gamma\left(3-\beta\right)\right)^2-4\,x^{6-\beta}\Gamma\left(3-\beta\right)+4\,x^{4-2\,\beta}\right)}{\left(\Gamma\left(3-\beta\right)\right)^2}, \tag{5.28}$$

and so on, in the same manner the rest of components can be obtained using the iteration formula (5.23)-(5.26). Hence, for $\beta=1$, and the initiale condition $u(x,0)=x^2$, one obtains the third order approximation of the initial value problem (5.22) as

$$u(x,t)=x^2+20tx^2\left(2-5x^3+2x^6\right)+t^2x^2\left(5-42x^3+16x^6\right). \tag{5.29}$$

When $\beta=1/2$, one obtains the approximate solution for the initial value problem (5.22) as

$$\begin{aligned} u(x,t) \;=\;& x^2+\frac{1}{2160\pi^2}t^2x^2\begin{pmatrix} 2160\pi^2+30720\pi x-88560\pi^{3/2}x^{7/2} \\ -81920\sqrt{\pi}x^{9/2}+61440\pi x^7 \end{pmatrix} \\ &+\frac{1}{2160\pi^2}tx^2\begin{pmatrix} 30720\pi x+262144x^2-11520\pi^{3/2}x^{7/2} \\ -98304\sqrt{\pi}x^{9/2}-83160\pi^{3/2}x^{9/2}+31185\pi^2x^7 \end{pmatrix} \end{aligned}$$

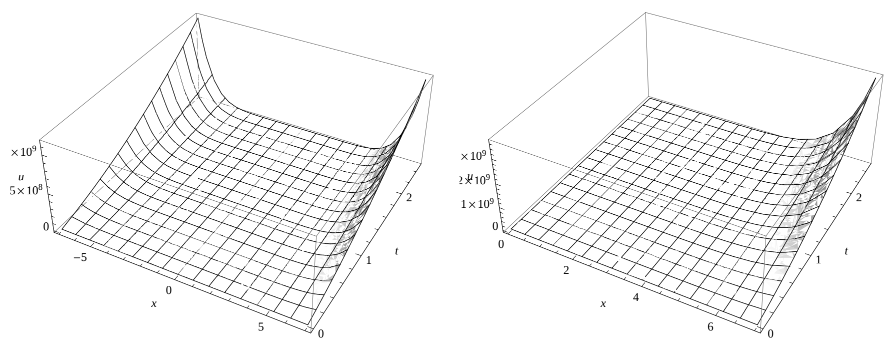

Figure 3: Series approximation solution of Eq. (5.22) by VHPM method with four terms for $\beta=1$ (Left), and $\beta=1/2$ (Right).

6 Conclusion

In this paper, based on the VIM and HPM, the variational homotopy perturbation method VHPM is considered for solving the time and space-fractional foam drainage partial differential equation. The numerical results obtained with different values of the time and space derivatives showed that the VHPM is a powerful and reliable method for finding the approximate analytical solutions of the time and space-fractional foam drainage. The current work illustrates that the VHPM is indeed a powerful analytical technique for most types of nonlinear problems and several such problems in scientific studies and engineering may be solved by this method.

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(λ, μ)-Fuzzy quasi-ideals and (λ, μ)-fuzzy bi-ideals in ternary semirings

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Abstract

In this paper we introduce the notion of (λ, μ)-Fuzzy quasi ideals and (λ, μ)-Fuzzy bi-ideals in ternary semirings which can be regarded as the generalization of fuzzy quasi ideals and fuzzy bi-ideals in ternary semirings.

Keywords: (λ, μ)-Fuzzy ternary subsemirings, (λ, μ)-Fuzzy ideal, (λ, μ)-Fuzzy quasi-ideal, (λ, μ)-Fuzzy bi-ideal.

2010 MSC: 08A72, 16Y60, 20M12, 17A40.

1 Introduction

The notion of ternary algebraic system was introduced by Lehmer [12] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [13] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [1] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings [1, 2, 3, 4, 5, 6, 7, 8]. The theory of fuzzy sets was first studied by Zadeh [16] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Kavikumar et al.[9] and [10] studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ternary semirings. Ronnason Chinram et al. [14] studied L-fuzzy ideals in ternary semirings. In [11], we introduce the notion of (λ, μ)-Fuzzy ideals in ternary semirings. In this paper we introduce the notion of (λ, μ)-Fuzzy quasi ideals and (λ, μ)-Fuzzy bi-ideals in ternary semirings which can be regarded as the generalization of fuzzy quasi ideals and fuzzy bi-ideals in ternary semirings.

2 Preliminaries

Definition 2.1. *A nonempty set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) is called a semiring if $(S, +)$ is a commutative semigroup, $(S, \cdot)$ is a semigroup and multiplicative distributes over addition both from the left and the right, i.e., $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.*

Definition 2.2. *A nonempty set S together with a binary operation called, addition $+$ and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S, +)$ is a commutative semigroup satisfying the following conditions:*
(i) $(abc)de = a(bcd)e = ab(cde)$,

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(ii) $(a+b)cd = acd + bcd$,
(iii) $a(b+c)d = abd + acd$
and (iv) $ab(c+d) = abc + abd$ *for all* $a, b, c, d, e \in S$.

We can see that any semiring can be reduced to a ternary semiring. However, a ternary semiring does not necessarily reduce to a semiring by this example. We consider Z_0^-, the set of all non-positive integers under usual addition and multiplication, we see that Z_0^- is an additive semigroup which is closed under the triple multiplication but is not closed under the binary multiplication. Moreover, Z_0^- is a ternary semiring but is not a semiring under usual addition and multiplication.
Throughout this paper S denotes a ternary semiring with zero.

Definition 2.3. *Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then 0 is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.*

Definition 2.4. *An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.*

Definition 2.5. *An additive subsemigroup I of S is called a left [resp. right, lateral] ideal of S if $s_1s_2i \in I$ [resp. $is_1s_2 \in I, s_1is_2 \in I$] for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S, then I is called an ideal of S.*

Definition 2.6. *An additive subsemigroup $(Q,+)$ of a ternary semiring S is called a quasi-ideal of S if $QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q$.*

Definition 2.7. *An additive subsemigroup $(Q,+)$ of a ternary semiring S is called a bi-ideal of S if $QSQSQ \subseteq Q$.*

It is obvious that every ideal of a ternary semiring with zero contains the zero element.

Definition 2.8. *Let X be a non-empty set. A map $A : X \to [0,1]$ is called a fuzzy set in X.*

Definition 2.9. *Let A, B and C be any three fuzzy subsets of a ternary semiring S. Then $A\cap B$, $A\cup B$, $A+B$, $A\cdot B\cdot C$ are fuzzy subsets of S defined by*

$$(A\cap B)(x) = min\{A(x),\ B(x)\}$$
$$(A\cup B)(x) = max\{A(x),\ B(x)\}$$

$$(A+B)(x) = \begin{cases} sup\{min\{A(y), B(z)\}\} & if \quad x = y+z \\ 0 & otherwise \end{cases}$$

$$(A\cdot B\cdot C)(x) = \begin{cases} sup\{min\{A(u), B(v), C(w)\}\} & if \quad x = uvw, \\ 0 & otherwise \end{cases}$$

Definition 2.10. *Let X be a nonempty set and let A be a fuzzy subset of X. Let $0 \le t \le 1$. Then the set $A_t = \{x \in X \ /\ A(x) \ge t\}$ is called a level set of X with respect to A.*

Definition 2.11. *Let A be a fuzzy set of a ternary semiring S. Then A is called a fuzzy ternary subsemiring of S if*
1. $A(x+y) \ge min\{A(x), A(y)\}$
2. $A(xyz) \ge min\{A(x), A(y), A(z)\}$ *for all* $x, y, z \in S$.

Definition 2.12. *A fuzzy set A of a ternary semiring S is called a fuzzy ideal of S if*
(i) $A(x+y) \ge min\{A(x), A(y)\}$
(ii) $A(xyz) \ge A(x)$
(iii) $A(xyz) \ge A(z)$ *and*
(iv) $A(xyz) \ge A(y)$ *for all* $x, y, z \in S$.
A fuzzy subset A with conditions (i) and (ii) is called a fuzzy right ideal of S. If A satisfies (i)and(iii), then it is called a fuzzy left ideal of S. Also if A satisfies (i)and(iv), then it is called a fuzzy lateral ideal of S. It is clear that A is a fuzzy ideal of a ternary semiring S if and only if $A(xyz) \ge max\{A(x), A(y), A(z)\}$ for all $x, y, z \in S$.

Definition 2.13. *A fuzzy ternary subsemiring A of a ternary semiring S is called a fuzzy quasi-ideal of S if for all $x \in S$,*
$A(x) \geq \{(A \cdot \boldsymbol{S} \cdot \boldsymbol{S}) \cap (\boldsymbol{S} \cdot A \cdot \boldsymbol{S} + \boldsymbol{S} \cdot \boldsymbol{S} \cdot A \cdot \boldsymbol{S} \cdot \boldsymbol{S}) \cap (\boldsymbol{S} \cdot \boldsymbol{S} \cdot A)\}(x)$.

Definition 2.14. *A fuzzy ternary subsemiring A of S is called a fuzzy bi ideal of S if for all $x \in S$, $A(x) \geq (A \cdot \boldsymbol{S} \cdot A \cdot \boldsymbol{S} \cdot A)(x)$.*

Lemma 2.1. *For any non-empty subsets A, B and C of S,*
(1) $f_A \cdot f_B \cdot f_C = f_{ABC}$
(2) $f_A \cap f_B \cap f_C = f_{A \cap B \cap C}$
(3) $f_A + f_B = f_{A+B}$.

Lemma 2.2. *[10] Let Q be an additive subsemigroup of S.*
(1) Q is a quasi-ideal of S if and only if f_Q is a fuzzy quasi-ideal of S.
(2) Q is a bi-ideal of S if and only if f_Q is a fuzzy bi-ideal of S.

Theorem 2.1. *[9] Let A be a fuzzy subset of S. Then A is a fuzzy quasi-ideal of S, if and only if A_t is a quasi-ideal of S, for all $t \in Im(A)$.*

Theorem 2.2. *[10] Let A be a fuzzy subset of S. Then A is a fuzzy bi-ideal of S, if and only if A_t is a bi-ideal of S, for all $t \in Im(A)$.*

3 (λ, μ)- Fuzzy quasi ideals

Based on the concept of (λ, μ)-fuzzy subrings and (λ, μ)-fuzzy ideals introduced by B.Yao [15], we introduce the following concepts which are the generalization of fuzzy sets. Throughout this paper λ and μ $(0 \leq \lambda < \mu \leq 1)$, are arbitrary, but fixed. In this section we introduce the notion of (λ, μ)-fuzzy quasi ideals in ternary semirings.

Definition 3.1. *[11] Let A be a fuzzy set of S. Then A is called a (λ, μ)-fuzzy ternary subsemiring of S if*
1. $A(x + y) \vee \lambda \geq min\{A(x), A(y), \mu\}$
2. $A(xyz) \vee \lambda \geq min\{A(x), A(y), A(z), \mu\}$ for all $x, y, z \in S$.

Definition 3.2. *[11] Let A be a fuzzy set of a ternary semiring S. A is called a (λ, μ)-fuzzy right(resp. left, lateral) ideal of S if*
1. $A(x + y) \vee \lambda \geq min\{A(x), A(y), \mu\}$
2. $A(xyz) \vee \lambda \geq min\{A(x), \mu\}$ [resp. $A(xyz) \vee \lambda \geq min\{A(z), \mu\}$, $A(xyz) \vee \lambda \geq min\{A(y), \mu\}$] for all $x, y, z \in S$.

Definition 3.3. *A (λ, μ)-fuzzy ternary subsemiring A of S is called a (λ, μ)-fuzzy quasi-ideal of S if for all $x \in S$,*
$A(x) \vee \lambda \geq min\{[(A \cdot \boldsymbol{S} \cdot \boldsymbol{S}) \cap (\boldsymbol{S} \cdot A \cdot \boldsymbol{S} + \boldsymbol{S} \cdot \boldsymbol{S} \cdot A \cdot \boldsymbol{S} \cdot \boldsymbol{S}) \cap (\boldsymbol{S} \cdot \boldsymbol{S} \cdot A)](x), \mu\}$.

Remark 3.1. *Every fuzzy quasi ideal is a (λ, μ)-fuzzy quasi ideal of S by taking $\lambda = 0$ and $\mu = 1$. But the converse need not be true.*

Example 3.1. *Consider the set of integer modulo 6, non-positive integer $Z_6^- = \{0, -1, -2, -3, -4, -5\}$ with the usual addition and ternary multiplication,*
we have

+	0	-1	-2	-3	-4	-5
0	0	-1	-2	-3	-4	-5
-1	-1	-2	-3	-4	-5	0
-2	-2	-3	-4	-5	0	-1
-3	-3	-4	-5	0	-1	-2
-4	-4	-5	0	-1	-2	-3
-5	-5	0	-1	-2	-3	-4

$\cdot$	0	-1	-2	-3	-4	-5
0	0	0	0	0	0	0
-1	0	1	2	3	4	5
-2	0	2	4	0	2	4
-3	0	3	0	3	0	3
-4	0	4	2	0	4	2
-5	0	5	4	3	2	1

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
-1	0	-1	-2	-3	-4	-5
-2	0	-2	-4	0	-2	-4
-3	0	-3	0	-3	0	-3
-4	0	-4	-2	0	-4	-2
-5	0	-5	-4	-3	-2	-1

Then $(Z_6^-,+,\cdot)$ is a ternary semiring. Let a fuzzy subset $A : Z_6^- \to [0,1]$ be defined by

$$A(x) = \begin{cases} 0.8 & if \quad x = 0 \\ 0.9 & if \quad x = -3 \\ 0.2 & \quad otherwise \end{cases}$$

Clearly A is a $(0.3, 0.8)$- fuzzy quasi ideal of S. But A is not a fuzzy quasi ideal of S, since $A_{0.9}$ is not a quasi ideal.

Lemma 3.3. *Let A and B be any two (λ,μ)-fuzzy ternary subsemirings of S. Then $A\cap B$ is a (λ,μ)-fuzzy ternary subsemiring of S.*

Proof. Let $x, y, z \in S$.

i) $(A\cap B)(x+y)\vee\lambda = min\{A(x+y), B(x+y)\}\vee\lambda$
$\geq min\{A(x+y)\vee\lambda, B(x+y)\vee\lambda\}$
$\geq min\{min\{A(x), A(y), \mu\}, min\{B(x), B(y), \mu\}\}$
$= min\{min\{A(x), B(x)\}, min\{A(y), B(y)\}, \mu\}$
$= min\{(A\cap B)(x), (A\cap B)(y), \mu\}$.

ii)$(A\cap B)(xyz)\vee\lambda = min\{A(xyz), B(xyz)\}\vee\lambda$
$\geq min\{A(xyz)\vee\lambda, B(xyz)\vee\lambda\}$
$\geq min\{min\{A(x), A(y), A(z), \mu\}, min\{B(x), B(y), B(z), \mu\}\}$
$\geq min\{min\{A(x), B(x)\}, min\{A(y), B(y)\}, min\{A(z), B(z)\}, \mu\}$
$= min\{(A\cap B)(x), (A\cap B)(y), (A\cap B)(z), \mu\}$.

Thus $A\cap B$ is a (λ,μ)-fuzzy ternary subsemiring of S. □

Lemma 3.4. *Let Q be any nonempty subset of S. Then Q is a quasi-ideal in S if and only if f_Q is a (λ,μ)-fuzzy quasi-ideal of S.*

Proof. Let Q be a quasi-ideal in S. By Lemma 2.2, f_Q is a fuzzy quasi-ideal of S and by Remark 3.1, f_Q is a (λ,μ)-fuzzy quasi-ideal of S. Conversely, let f_Q be a (λ,μ)-fuzzy quasi-ideal of S with $0\leq\lambda<\mu\leq 1$. Let $x\in QSS\cap(SQS+SSQSS)\cap SSQ$. Then we have

$f_Q(x)\vee\lambda \geq min\{(f_Q\cdot f_S\cdot f_S)(x), (f_S\cdot f_Q\cdot f_S + f_S\cdot f_S\cdot f_Q\cdot f_S\cdot f_S)(x), (f_S\cdot f_S\cdot f_Q)(x), \mu\}$
$= min\{f_{(QSS\cap(SQS+SSQSS)\cap SSQ)}(x), \mu\} = min\{1, \mu\} = \mu$.

Thus $f_Q(x)\vee\lambda\geq\mu>\lambda$ and hence $f_Q(x)\geq\mu$.
This implies that $x\in Q$ and so $QSS\cap(SQS+SSQSS)\cap SSQ\subseteq Q$. This means that Q is a quasi-ideal of S. □

Theorem 3.3. *Let A be a fuzzy set of a ternary semiring S. If A is a (λ,μ)-fuzzy lateral(right, left) ideal of S then A is a (λ,μ)-fuzzy quasi-ideal of S.*

Proof. Let A be a (λ,μ)-fuzzy lateral ideal of S. Let $x\in S$.
Suppose $x = as_1s_2 = s_3(b+s_4cs_5)s_6 = s_7s_8d$ where $a, b, c, d, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \in S$.
We have $\{A\cdot\mathbf{S}\cdot\mathbf{S}\cap(\mathbf{S}\cdot A\cdot\mathbf{S}+\mathbf{S}\cdot\mathbf{S}\cdot A\cdot\mathbf{S}\cdot\mathbf{S})\cap\mathbf{S}\cdot\mathbf{S}\cdot A\}(x)$
$= min\{A\cdot\mathbf{S}\cdot\mathbf{S}(x), (\mathbf{S}\cdot A\cdot\mathbf{S}+\mathbf{S}\cdot\mathbf{S}\cdot A\cdot\mathbf{S}\cdot\mathbf{S})(x), \mathbf{S}\cdot\mathbf{S}\cdot A(x)\}$
$= min\{\sup\limits_{x=as_1s_2}\{A(a)\}, \sup\limits_{x=s_3(b+s_4cs_5)s_6}\{min\{A(b), A(c)\}\}, \sup\limits_{x=s_7s_8d}\{A(d)\}\}$
Now since A is a (λ,μ)-fuzzy lateral ideal,
$A(s_3(b+s_4cs_5)s_6)\vee\lambda\geq min\{A(b+s_4cs_5), \mu\}\geq min\{A(b), A(c), \mu\}$.
We have $min\{(A\cdot\mathbf{S}\cdot\mathbf{S}\cap(\mathbf{S}\cdot A\cdot\mathbf{S}+\mathbf{S}\cdot\mathbf{S}\cdot A\cdot\mathbf{S}\cdot\mathbf{S})\cap\mathbf{S}\cdot\mathbf{S}\cdot A)(x), \mu\}$

$= min\{min\{supA(a), sup\{min\{A(b), A(c)\}\}, supA(d)\}, \mu\}$
$\leq min\{min\{1, sup\{min\{A(b), A(c)\}\}, 1\}, \mu\}$
$= min\{sup\{min\{A(b), A(c)\}\}, \mu\}$
$\leq \sup_{x=s_3(b+s_4cs_5)s_6} \{min\{A(b), A(c), \mu\}\}$
$\leq \sup A(s_3(b + s_4cs_5)s_6) \vee \lambda = A(x) \vee \lambda.$
We remark that if x is not expressed as $x = as_1s_2 = s_3(b+s_4cs_5)s_6 = s_7s_8d$ then $(A \cdot \mathbf{S} \cdot \mathbf{S} \cap (\mathbf{S} \cdot A \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot \mathbf{S}) \cap \mathbf{S} \cdot \mathbf{S} \cdot A)(x) = 0 \leq A(x) \vee \lambda$.
Thus $A(x) \vee \lambda \geq min\{(A \cdot \mathbf{S} \cdot \mathbf{S} \cap (\mathbf{S} \cdot A \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot \mathbf{S}) \cap \mathbf{S} \cdot \mathbf{S} \cdot A)(x), \mu\}$ for all $x \in S$.
Hence A is a (λ, μ)-fuzzy quasi-ideal of S. □

Theorem 3.4. *Let A be a fuzzy set of a ternary semiring S. A is a (λ, μ)-fuzzy quasi-ideal of S if and only if A_t is a quasi-ideal in S for all $t \in (\lambda, \mu]$ whenever nonempty.*

Proof. Let A be a (λ, μ)-fuzzy quasi-ideal of S. Let $x, y \in S$. Suppose $x, y \in A_t$, $t \in (\lambda, \mu]$ then $A(x) \geq t, A(y) \geq t$ and $\mu \geq t$. This implies that $min\{A(x), A(y), \mu\} \geq t$. Since A is (λ, μ)-fuzzy quasi-ideal, $A(x + y) \vee \lambda \geq min\{A(x), A(y), \mu\} \geq t > \lambda$. This implies that $A(x + y) \geq t$. Hence $x + y \in A_t$. Next, let $x \in A_tSS \cap (SA_tS + SSA_tSS) \cap SSA_t$. Then there exist $a, b, c, d \in A_t$ and $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \in S$ such that $x = as_1s_2 = s_3(b + s_4cs_5)s_6 = s_7s_8d$. Thus $A(a) \geq t, A(b) \geq t, A(c) \geq t, A(d) \geq t$.
Then $(A \cdot \mathbf{S} \cdot \mathbf{S} \cap (\mathbf{S} \cdot A \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot \mathbf{S}) \cap \mathbf{S} \cdot \mathbf{S} \cdot A)(x)$
$= min\{A \cdot \mathbf{S} \cdot \mathbf{S}(x), (\mathbf{S} \cdot A \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot \mathbf{S})(x), \mathbf{S} \cdot \mathbf{S} \cdot A(x)\}$
$= min\{\sup_{x=as_1s_2} \{A(a)\}, \sup_{x=s_3(b+s_4cs_5)s_6} \{min\{A(b), A(c)\}\}, \sup_{x=s_7s_8d} \{A(d)\}\} \geq t.$
Now $min\{(A \cdot \mathbf{S} \cdot \mathbf{S} \cap (\mathbf{S} \cdot A \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot \mathbf{S}) \cap \mathbf{S} \cdot \mathbf{S} \cdot A)(x), \mu\} \geq min\{t, \mu\} = t$.
Since A is a (λ, μ)-fuzzy quasi-ideal of S, $A(x) \vee \lambda \geq t > \lambda$. Then $A(x) \geq t$ and $x \in A_t$ and hence A_t is a quasi-ideal in S.
Conversely let us assume that A_t, $t \in (\lambda, \mu]$ is a quasi-ideal in S. By Theorem:2.1, A is a fuzzy quasi-ideal of S. By remark:3.1, A is a (λ, μ)-fuzzy quasi-ideal of S. □

Theorem 3.5. *Let A and B be any two (λ, μ)-fuzzy quasi-ideals of S. Then $A \cap B$ is also a (λ, μ)-fuzzy quasi-ideal of S.*

Proof. Let A and B be (λ, μ)-fuzzy quasi-ideals of S. By Lemma:3.3, $A \cap B$ is a (λ, μ)-fuzzy ternary subsemiring of S. Let $x \in S$.
Suppose $x = as_1s_2 = s_3(b + s_4cs_5)s_6 = s_7s_8d$ where $a, b, c, d, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \in S$. Since A and B are (λ, μ)-fuzzy quasi-ideals of S, we have
$A(x) \vee \lambda \geq min\{\sup\{A(a), min\{A(b), A(c)\}, A(d)\}, \mu\}$ and
$B(x) \vee \lambda \geq min\{\sup\{B(a), min\{B(b), B(c)\}, B(d)\}, \mu\}$.
Consider $min\{[((A \cap B) \cdot \mathbf{S} \cdot \mathbf{S}) \cap (\mathbf{S} \cdot (A \cap B) \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot (A \cap B) \cdot \mathbf{S} \cdot \mathbf{S}) \cap (\mathbf{S} \cdot \mathbf{S} \cdot (A \cap B))](x), \mu\}$
$= min\{min\{((A \cap B) \cdot \mathbf{S} \cdot \mathbf{S})(x), (\mathbf{S} \cdot (A \cap B) \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot (A \cap B) \cdot \mathbf{S} \cdot \mathbf{S})(x), (\mathbf{S} \cdot \mathbf{S} \cdot (A \cap B))(x)\}, \mu\}$
$= min\{min\{\sup_{x=as_1s_2} \{(A \cap B)(a)\},$
$\sup_{x=s_3(b+s_4cs_5)s_6} \{min\{(A \cap B)(b), (A \cap B)(c)\}\}, \sup_{x=s_7s_8d} \{(A \cap B)(d)\}\}, \mu\}$
$= min\{min\{\sup_{x=as_1s_2} \{min\{A(a), B(a)\}\},$
$\sup_{x=s_3(b+s_4cs_5)s_6} \{min\{min\{A(b), B(b)\}, min\{A(c), B(c)\}\}\}, \sup_{x=s_7s_8d} \{min\{A(d), B(d)\}\}\}, \mu\}$
$= min\{min\{\sup\{min\{A(a), B(a)\}\},$
$\sup\{min\{min\{A(b), A(c)\}, min\{B(b), B(c)\}\}\}, \sup\{min\{A(d), B(d)\}\}\}, \mu\}$
$\leq min\{min\{min\{\sup\{A(a), B(a)\}\},$
$min\{\sup\{min\{A(b), A(c)\}, min\{B(b), B(c)\}\}\}, min\{\sup\{A(d), B(d)\}\}\}, \mu\}$
$= min\{min\{min\{\sup\{A(a), min\{A(b), A(c)\}, A(d)\}\},$
$min\{\sup\{B(a), min\{B(b), B(c)\}, B(d)\}\}\}, \mu\}$
$= min\{min\{\sup\{A(a), min\{A(b), A(c)\}, A(d)\}, \mu\},$
$min\{\sup\{B(a), min\{B(b), B(c)\}, B(d)\}, \mu\}\}$
$\leq min\{[A(x) \vee \lambda], [B(x) \vee \lambda]\} \leq min\{A(x), B(x)\} \vee \lambda = (A \cap B)(x) \vee \lambda$. Thus $A \cap B$ is a (λ, μ)-fuzzy quasi-ideal of S. □

4 (λ, μ)- Fuzzy Bi ideals

Definition 4.1. *A (λ,μ)-fuzzy ternary subsemiring A of S is called a (λ,μ)-fuzzy bi ideal of S if for all $x \in S$, $A(x) \vee \lambda \geq min\{(A \cdot \boldsymbol{S} \cdot A \cdot \boldsymbol{S} \cdot A)(x), \mu\}$.*

Remark 4.1. *Every fuzzy bi ideal is a (λ,μ)-fuzzy bi ideal of S by taking $\lambda = 0$ and $\mu = 1$. But the converse need not be true.*

Example 4.1. *Consider the ternary semiring $(Z_6^-, +, \cdot)$ as defined in Example 3.1. Let a fuzzy subset $A : Z_6^- \to [0,1]$ be defined by $A(0) = 0.9$, $A(-1) = 0$, $A(-2) = 0.9$, $A(-3) = 0$, $A(-4) = 0.8$ and $A(-5) = 0$. Clearly A is a $(0.3, 0.8)$- fuzzy bi-ideal. But A is not a fuzzy bi-ideal, since $A_{0.9}$ is not a fuzzy bi-ideal.*

Lemma 4.1. *Let Q be any nonempty subset of S. Then Q is a bi-ideal in S if and only if f_Q is a (λ,μ)-fuzzy bi-ideal of S.*

Proof. Let Q be a bi-ideal of S. By Lemma 2.2, f_Q is a fuzzy bi-ideal of S and by Remark 4.1, f_Q is a (λ,μ)-fuzzy bi-ideal of S. Conversely, let f_Q be a (λ,μ)-fuzzy bi-ideal of S. Let $x \in QSQSQ$. Then we have $f_Q(x) \vee \lambda \geq min\{(f_Q \cdot f_S \cdot f_Q \cdot f_S \cdot f_Q)(x), \mu\} = min\{f_{(QSQSQ)}(x), \mu\} = min\{1, \mu\} = \mu$. Thus $f_Q(x) \vee \lambda \geq \mu$. Hence $x \in Q$ and so $QSQSQ \subseteq Q$. This means that Q is a bi-ideal of S. □

Lemma 4.2. *Any (λ,μ)-fuzzy quasi-ideal of S is a (λ,μ)-fuzzy bi-ideal of S.*

Proof. Let A be a (λ,μ)-fuzzy quasi-ideal of S. Then we have
$A \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot A \subseteq A \cdot (\mathbf{S} \cdot \mathbf{S} \cdot \mathbf{S}) \cdot \mathbf{S} \subseteq A \cdot \mathbf{S} \cdot \mathbf{S}$,
$A \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot A \subseteq \mathbf{S} \cdot (\mathbf{S} \cdot \mathbf{S} \cdot \mathbf{S}) \cdot A \subseteq \mathbf{S} \cdot \mathbf{S} \cdot A$,
$A \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot A \subseteq \mathbf{S} \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot \mathbf{S}$ and taking $\{0\} \subseteq \mathbf{S} \cdot A \cdot \mathbf{S}$.
So $A \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot A \subseteq (\mathbf{S} \cdot A \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot \mathbf{S})$.
Hence we have $A \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot A \subseteq A \cdot \mathbf{S} \cdot \mathbf{S} \cap (\mathbf{S} \cdot A \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot \mathbf{S}) \cap \mathbf{S} \cdot \mathbf{S} \cdot A \subseteq A$.
Let $x \in S$. Now $min\{(A \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot A)(x), \mu\} \leq min\{(A \cdot \mathbf{S} \cdot \mathbf{S} \cap (\mathbf{S} \cdot A \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot \mathbf{S}) \cap \mathbf{S} \cdot \mathbf{S} \cdot A)(x), \mu\} \leq A(x) \vee \lambda$.
It follows that A is a (λ,μ)-fuzzy bi-ideal of S. □

Theorem 4.1. *Let A be a fuzzy set of a ternary semiring S. If A is a (λ,μ)-fuzzy lateral(right, left) ideal of S then A is a (λ,μ)-fuzzy bi-ideal of S.*

Proof. By Theorem:3.3 , every (λ,μ)-fuzzy lateral ideal of S is a (λ,μ)-fuzzy quasi ideal of S and by Lemma:4.2 , it is a (λ,μ)-fuzzy bi-ideal of S. □

Theorem 4.2. *Let A be a fuzzy set of a ternary semiring S. A is a (λ,μ)-fuzzy bi-ideal of S if and only if A_t is a bi-ideal in S for all $t \in (\lambda,\mu]$ whenever nonempty.*

Proof. Let A be a (λ,μ)-fuzzy bi-ideal of S. Let $x, y \in S$. Suppose $x, y \in A_t$, $t \in (\lambda,\mu]$. Then $A(x) \geq t, A(y) \geq t$ and $\mu \geq t$ and hence $min\{A(x), A(y), \mu\} \geq t$. Since A is a (λ,μ)-fuzzy bi-ideal, $A(x+y) \vee \lambda \geq min\{A(x), A(y), \mu\} \geq t$. This implies that $A(x+y) \geq t$. Thus $x + y \in A_t$. Let $u \in S$. Suppose $u \in A_tSA_tSA_t$ then there exist $x, y, z \in A_t$ and $s_1, s_2 \in S$ such that $u = xs_1ys_2z$. Thus $A(x) \geq t, A(y) \geq t, A(z) \geq t$. Now $(A \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot A)(u) = sup\{min\{A(x), A(y), A(z)\}\} \geq t$. Since A is a (λ,μ)-fuzzy bi-ideal of S, $A(u) \vee \lambda \geq min\{(A \cdot \mathbf{S} \cdot A \cdot \mathbf{S} \cdot A)(u), \mu\} \geq min\{t, \mu\} = t$. Thus $A(u) \geq t$ and $u \in A_t$. Hence A_t is a bi-ideal in S.
Conversely let us assume that A_t, $t \in (\lambda,\mu]$ is a bi-ideal in S. By Theorem:2.2, A is a fuzzy bi-ideal of S. By Remark:4.1, A is a (λ,μ)-fuzzy bi-ideal of S. □

Theorem 4.3. *Let A and B be any two (λ,μ)-fuzzy bi-ideals of S. Then $A \cap B$ is also a (λ,μ)-fuzzy bi-ideal of S.*

Proof. Let A and B be (λ,μ)-fuzzy bi-ideals of S. By Lemma:3.3, $A \cap B$ is a (λ,μ)-fuzzy ternary subsemiring of S. Let $a \in S$ and $s_1, s_2, x, y, z \in S$ such that $a = xs_1ys_2z$. Since A and B are (λ,μ)-fuzzy bi-ideals of S, we have $A(a) \vee \lambda \geq min\{\sup\limits_{a=xs_1ys_2z}\{A(x), A(y), A(z)\}, \mu\}$ and
$B(a) \vee \lambda \geq min\{\sup\limits_{a=xs_1ys_2z}\{B(x), B(y), B(z)\}, \mu\}$.
Consider $min\{((A \cap B) \cdot \mathbf{S} \cdot (A \cap B) \cdot \mathbf{S} \cdot (A \cap B))(a), \mu\}$

$$= min\{\sup_{a=xs_1ys_2z}\{min\{(A\cap B)(x), \mathbf{S}(s_1), (A\cap B)(y), \mathbf{S}(s_2), (A\cap B)(z)\}\}, \mu\}$$
$$= min\{\sup_{a=xs_1ys_2z}\{min\{(A\cap B)(x), (A\cap B)(y), (A\cap B)(z)\}\}, \mu\}$$
$$\leq min\{min\{\sup_{a=xs_1ys_2z}\{min\{A(x), B(x)\}, min\{A(y), B(y)\}, min\{A(z), B(z)\}\}\}, \mu\}$$
$$\leq min\{min\{min\{\sup_{a=xs_1ys_2z}\{A(x), A(y), A(z)\}, \sup_{a=xs_1ys_2z}\{B(x), B(y), B(z)\}\}\}, \mu\}$$
$$= min\{min\{\sup_{a=xs_1ys_2z}\{A(x), A(y), A(z)\}\}, \mu\}, min\{\sup_{a=xs_1ys_2z}\{B(x), B(y), B(z)\}\}, \mu\}\}$$
$$\leq min\{\{A(a)\vee\lambda\}, \{B(a)\vee\lambda\}\}$$
$$\leq min\{A(a), B(a)\}\vee\lambda = (A\cap B)(a)\vee\lambda.$$ Thus $A\cap B$ is a (λ, μ)-fuzzy bi-ideal of S. □

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