ISSN 2319-3786
VOLUME 2, ISSUE 1, JANUARY 2014

MJM

Malaya Journal of Matematik

an international journal of mathematical sciences

UNIVERSITY PRESS
5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),
Coimbatore- 641114, Tamil Nadu, India.
www.malayajournal.org

The Malaya Journal of Matematik is published quarterly in single volume annually and four issues constitute one volume appearing in the months of January, April, July and October.

Subscription

The subscription fee is as follows:

USD 350.00 For USA and Canada

Euro 190.00 For rest of the world

Rs. 4000.00 In India. (For Indian Institutions in India only)

Prices are inclusive of handling and postage; and issues will be delivered by Registered Air-Mail for subscribers outside India.

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Coimbatore- 641114, Tamil Nadu, India.

Contact No. : +91-9585408402

E-mail : info@mkdpress.com; editorinchief@malayajournal.org; publishingeditor@malayajournal.org

Website : https://mkdpress.com/index.php/index/index

Vol. 2 No. 01 (2014): Malaya Journal of Matematik (MJM)

Malaya J. Mat. 2(1)(2014) 1-9

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Robinson-Schensted correspondence for party algebras

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Abstract

In this paper, we construct a bijective proof of the identity $n^k = \sum_{[\tilde{\lambda}] \in \Lambda_n^k} f^{[\tilde{\lambda}]} m_k^{[\tilde{\lambda}]}$, where $m_k^{[\tilde{\lambda}]}$ is the multiplicity of the irreducible representation of $\mathbb{Z}_r \wr S_n$ module indexed by $[\tilde{\lambda}] \in \Lambda_n^k$, $f^{[\tilde{\lambda}]}$ is the degree of the corresponding representation indexed by $[\tilde{\lambda}] \in \Lambda_n^k$ and $\Lambda_n^k = \{[\tilde{\lambda}] \vdash n | \sum_{i=1}^k i|\lambda^{(i)}| = k\}$. We give the proof of Robinson-Schensted correspondence for the party algebras which gives the bijective proof of party diagrams and the pairs of vacillating tableaux.

Keywords: Partition, Bratteli diagram, Robinson-Schensted correspondence.

2010 MSC: 05E10, 05A05, 20C99.

1 Introduction

Let G be the group of linear transformations on a n-dimensional vector space V. Suppose that G acts diagonally on the k-fold tensor space $V^{\otimes k}$. Then the k-fold tensor space $V^{\otimes k}$ decomposes into irreducible representation of G as centraliser algebra $End_G\left(V^{\otimes k}\right)$. This work was successfully done in Partition algebra $End_{S_n}\left(V^{\otimes k}\right)$, Brauer algebra $End_{O(n)}\left(V^{\otimes k}\right)$ where $O(n)$ is the orthogonal group of degree n and so on.

The party algebra $\mathbb{C}L_k$ is the subalgebra of the partition algebra which is generated by S_n and the diagram corresponding to the set partition $\{\{1,2,1',2'\}\{3,3'\}\ldots\{k,k'\}\}$

Masashi Kosuda defined the irreducible representations of the party algebras. There exists a surjective homomorphism from $\mathbb{C}L_k$ to $End_{G(r,1,n)}\left(V^{\otimes k}\right)$. Moreover if $n \geq k$ and $r > n$, this homomorphism is injective and thus forms an irreducible representations of party algebras.

The number of standard Young tableaux of shape $[\lambda]$ is $f^{[\lambda]}$ which is the degree of the corresponding representation of the group $G(r,1,n)$. In this paper, we develop a Robinson-Schensted correspondence for the party algebras which gives the bijection between the diagrams in $\mathbb{C}L_k$ and the pairs of vacillating tableaux $\left(P_{[\lambda]}, Q_{[\lambda]}\right)$ in Γ_k. We also develop the bijection proof for the identity $n^k = \sum_{[\tilde{\lambda}] \in \Lambda_n^k} f^{[\tilde{\lambda}]} m_k^{[\tilde{\lambda}]}$, where $m_k^{[\tilde{\lambda}]}$ is the multiplicity of the irreducible representation of $\mathbb{Z}_r \wr S_n$ module indexed by $[\tilde{\lambda}]$, by constructing a bijection between the sequences $(i_1, i_2, \ldots, i_k)$, $1 \leq i_j \leq n$ and the pair $\left(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]}\right)$ where $T_{[\tilde{\lambda}]}$ is a standard tableau of shape $[\tilde{\lambda}]$ and $P_{[\tilde{\lambda}]}$ is the vacillating tableaux of shape $[\tilde{\lambda}]$.

2 Preliminaries

Definition 2.1. [7] *A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a non-increasing sequence of positive integers, that is $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that $|\lambda_1| + |\lambda_2| + \ldots + |\lambda_k| = n$. It is denoted by $\lambda \vdash n$.*

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Definition 2.2. *[4] A multipartition* $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ *such that each* $\lambda^{(i)}$ *is a partition and* $\sum_i |\lambda^{(i)}| = n$. *We say that* $\lambda^{(i)}$ *is the* $i-$*th component of* $[\lambda]$.

Definition 2.3. *[4] A diagram of a partition* λ *is an array of boxes in which first row contains* λ_1 *number of boxes, second row contains* λ_2 *number of boxes and so on.*

Definition 2.4. *[4] Let* $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ *be a multipartition of n. A* $[\lambda]$ *-tableau* $t = (t^{(1)}, t^{(2)}, \ldots, t^{(k)})$ *is obtained by filling the boxes of the diagram from* $\{1, 2, ..., n\}$.

1. *A* $[\lambda]-$*tableau t is said to be row standard if the entries in each row of each component is strictly increasing*

2. *A* $[\lambda]-$*tableau t is said to be standard if the entries in each row and in each column of each component is strictly increasing.*

Definition 2.5. *[7] A rim hook is a connected skew shape containing no* 2×2 *square.*

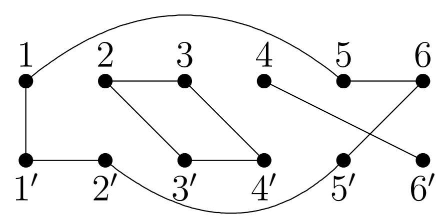
2.1 Party algebras

For $k \in \mathbb{Z}$, let

$$
\begin{aligned}
A_k &= \{ \text{ set partitions of } \{1, 2, \ldots, k, 1^{'}, 2^{'}, \ldots, k^{'}\}\} \text{ and} \\
A_{k+\frac{1}{2}} &= \{d \in A_{k+1} | (k+1) \text{ and } (k+1)^{'} \text{ are in the same block}\}.
\end{aligned}
$$

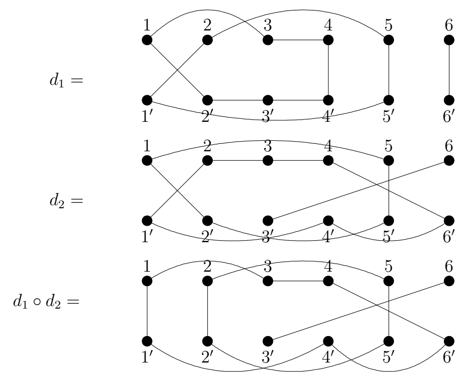
For a set partition $d = \{B_1, B_2, \ldots, B_s\} \in A_k$ and $B_i \in d$, let $N(B_i) = \#(B_i \cap \underline{k})$ and $M(B_i) = \#(B_i \cap \underline{k}')$.

For $k > 0$, let $L_k = \{d \in A_k | N(B_i) = M(B_i) \text{ for all } B_i \in d\}$. Represent $d \in L_k$ as a graph with two rows of k vertices, the first row of k vertices is labeled by $1, 2, \ldots, k$ and the second row of k vertices is labeled by $1', 2', \ldots, k'$. For example,



Definition 2.6. *Let* $d_1, d_2 \in L_k$*, the multiplication of diagrams* $d_1 \circ d_2$ *is obtained by placing* d_2 *below* d_1 *and identifying each vertex* i' *in the bottom row of* d_1 *with the each vertex* i *in the top row of* d_2 *and by removing any component that lie entirely in the middle row.*

For example,



For $k \in \mathbb{N}$, the party algebra $\mathbb{C}L_k$ is an associative subalgebra of the partition algebra $\mathbb{C}A_k$ with basis L_k.

3 Schur Weyl Duality between $\mathbb{Z}_r \wr S_n$ and $\mathbb{C}L_k$

The irreducible representations of $\mathbb{Z}_r \wr S_n$ are indexed by the multi partition $[\lambda]$ of n. If $[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ where each $\lambda^{(i)}$ is the partition of the $i-$ th component and $\sum_{i=0}^{r-1} |\lambda^{(i)}| = n$. Let V be the n dimensional representation of the group $\mathbb{Z}_r \wr S_n$. Consider $S_{n-1} \subseteq S_n$ as the subgroup of permutations that fix n. Let $V^{\otimes k}$ be the k fold tensor representation of V. Let $V^{[\lambda]}$ be the irreducible representation of $\mathbb{Z}_r \wr S_n$ indexed by $[\lambda] \vdash n$ where $[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r-1)})$ and $\sum_{i=0}^{r-1} |\lambda^{(i)}| = n$. The induction and restriction rules for $\mathbb{Z}_r \wr S_n$ are as follows:

If $[\lambda] \vdash n$, $Res^{\mathbb{Z}_r \wr S_n}_{\mathbb{Z}_r \wr S_{n-1}} V^{[\lambda]}$ denotes the irreducible representation obtained from restricting the multi partition $[\lambda] \vdash n$ to the multi partition $[\mu] \vdash n-1$ by removing a box from any one of the residues in $[\lambda]$. $Ind^{\mathbb{Z}_r \wr S_{n-1}}_{\mathbb{Z}_r \wr S_n} V^{[\mu]}$ denotes the irreducible representation obtained from inducing the multi partition $[\mu] \vdash n-1$ to the multi partition $[\lambda] \vdash n$ by adding a box in the $\lambda^{(l+1)}$ if the box is removed from $\mu^{(l)}$ while restriction.

$$Res^{\mathbb{Z}_r \wr S_n}_{\mathbb{Z}_r \wr S_{n-1}} V^{[\lambda]} \cong \bigoplus_{[\mu] \vdash n-1, [\mu] \subseteq [\lambda]} V^{[\mu]},$$

for $[\lambda] \vdash n$.

$$Ind^{\mathbb{Z}_r \wr S_{n-1}}_{\mathbb{Z}_r \wr S_n} V^{[\mu]} \cong \bigoplus_{[\mu] \vdash n, [\lambda] \subseteq [\mu]} V^{[\lambda]}.$$

for $[\mu] \vdash n-1$.

Suppose that $[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r-1)})$ and $[\mu] = (\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r-1)})$ are multi partitions of n. we say that $[\mu] \subseteq [\lambda]$ if,

$$\sum_{i=1}^{m-1} |\mu^i| + \sum_{i=1}^{j} \mu_j^{(m)} \leq \sum_{i=1}^{m-1} |\lambda^{(i)}| + \sum_{i=1}^{j} \lambda_i^{(m)}.$$

Starting with the trivial representation $\underbrace{(n, \emptyset, \ldots, \emptyset)}_{r\text{tuples}}$ and iterating the restriction and induction rules. We see the irreducible $\mathbb{Z}_r \wr S_n$ representation that appears in $V^{\otimes k}$ are labeled by the partition in Λ_n^k. If $[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)} \vdash n$ and if $[\tilde{\lambda}] = (n-t, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}, \emptyset^{(k+1)}, \ldots, \emptyset^{(r-1)})$, where $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}) \vdash t$, $1 \leq t \leq k$ and $r > k$.

$$\Lambda_n^k = \{[\tilde{\lambda}] \vdash n | \sum_{i=1}^{k} i|\lambda^{(i)}| = k\}$$

and the irreducible $\mathbb{Z}_r \wr S_{n-1}$ representation that appear in $V^{\otimes k}$ are labeled by the partitions in Λ_{n-1}^k. If $[\tilde{\lambda}] \vdash n-1, 0 \leq t \leq k$.

$$\Lambda_{n-1}^k = \{[\tilde{\lambda}] \vdash n-1 | \sum_{i=1}^{k} i|\lambda^{(i)}| \leq k\}.$$

There is an action of $\mathbb{C}L_k$ on $V^{\otimes k}$ [6] that commutes with $\mathbb{Z}_r \wr S_n$ and maps surjectively onto centralizer of $End_{\mathbb{Z}_r \wr S_n} V^{\otimes k}$. Furthermore when $n \geq k$ and $r > k$ we have

$$\mathbb{C}L_k \cong End_{\mathbb{Z}_r \wr S_n} V^{\otimes k} \text{ and } \mathbb{C}L_{k+\frac{1}{2}} \cong End_{\mathbb{Z}_r \wr S_{n-1}} V^{\otimes k}.$$

The Bratelli diagram for $\mathbb{C}L_k$ consists of rows of vertices with the rows labeled by $0, \frac{1}{2}, \ldots, k$ such that the vertices in row i are labeled by Λ_n^i and the vertices in row $i + \frac{1}{2}$ are Λ_{n-1}^i. Two vertices are connected by an edge if they are in consecutive rows and they differ by exactly one box. The irreducible representations of $\mathbb{C}L_k$ are indexed by Λ_n^k, so we let $M_k^{[\tilde{\lambda}]}$ denote the irreducible representation of $\mathbb{C}L_k$ indexed by $[\tilde{\lambda}] \in \Lambda_n^k$.

The decomposition of $V^{\otimes k}$ as an $\mathbb{Z}_r \wr S_n \times \mathbb{C}L_k$ bimodule is given by

$$V^{\otimes k} \cong \bigoplus_{[\tilde{\lambda}] \in \Lambda_n^k} V^{[\lambda]} \otimes M_k^{[\tilde{\lambda}]}. \tag{3.1}$$

The dimension of $M_k^{[\tilde{\lambda}]}$ equals the multiplicity of $V^{[\lambda]}$ in $V^{\otimes k}$.

$$m_k^{[\tilde{\lambda}]} = \dim(M_k^{[\tilde{\lambda}]}) = \{\text{the number of paths from the top of the Bratelli diagram to } [\tilde{\lambda}]\}.$$

4 Vacillating Tableaux

Let $[\lambda] \in \Lambda_n^k$. A vacillating tableaux of shape λ and length $2k$ is a sequence of partitions,

$$\Big((n, \emptyset, \ldots, \emptyset) = [\lambda]^{(0)}, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, [\lambda]^{1(\frac{1}{2})}, \ldots, [\lambda]^{(k-\frac{1}{2})}, [\lambda]^{(k)} = [\lambda]\Big),$$

satisfying, for each i,

1. $[\lambda]^{(i)} \in \Lambda_n^i$, $[\lambda]^{(i+\frac{1}{2})} \in \Lambda_{n-1}^i$,
2. $[\lambda]^{(i)} \supseteq [\lambda]^{(i+\frac{1}{2})}$ and $|[\lambda]^{(i)}/[\lambda]^{(i+\frac{1}{2})}| = 1$,
3. $[\lambda]^{(i+\frac{1}{2})} \subseteq [\lambda]^{(i+1)}$ and $|[\lambda]^{(i+1)}/[\lambda]^{(i+\frac{1}{2})}| = 1$.

The vacillating tableaux of shape $[\lambda]$ correspond exactly with paths from the top of the Brattelli diagram to $[\lambda]$.

For $n \geq k$ and $r > n$, the sets

$$\Lambda_n^k = \{[\tilde{\lambda}] \vdash n | \sum_{i=1}^{k} i|\lambda^{(i)}| = k\}$$

and if $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$,

$$\Gamma_k = \{[\lambda] \vdash t | \sum_{i=1}^{k} i|\lambda^{(i)}| = k \text{ and } 0 \leq t \leq k\}$$

are in bijection with one another. For example, the following sequence represents the same vacillating tableaux $P_{[\lambda]}$.

$P_{[\lambda]}$=$((\Box\Box\Box, \emptyset, \emptyset, \emptyset), (\Box\Box, \emptyset, \emptyset, \emptyset), (\Box\Box, \Box, \emptyset, \emptyset), (\Box\Box, \emptyset, \emptyset, \emptyset), (\Box\Box, \emptyset, \Box, \emptyset), (\Box, \emptyset, \Box, \emptyset), (\Box, \Box, \Box, \emptyset))$

$((\emptyset, \emptyset, \emptyset), (\emptyset, \emptyset, \emptyset), (\Box, \emptyset, \emptyset), (\emptyset, \emptyset, \emptyset), (\emptyset, \Box, \emptyset), (\emptyset, \Box, \emptyset), (\Box, \Box, \emptyset))$

Bratelli diagram for $End_{\mathbb{Z}_5 \wr S_4}(V^{\otimes 3})$

Thus, if we let $VT_k([\tilde{\lambda}])$ denote the set of vacillating tableaux of shape $[\tilde{\lambda}]$ and length k, then

$$m_k^{[\tilde{\lambda}]} = \dim(M_k^{[\tilde{\lambda}]}) = |VT_k([\tilde{\lambda}])|.$$

5 A Bijective Proof of $n^k = \sum_{[\tilde{\lambda}] \in \Lambda_n^k} f^{[\tilde{\lambda}]} m_k^{[\tilde{\lambda}]}$

Comparing dimensions on both sides of equation 3.1, gives

$$n^k = \sum_{[\tilde{\lambda}] \in \Lambda_n^k} f^{[\tilde{\lambda}]} m_k^{[\tilde{\lambda}]}$$

where $f^{[\tilde{\lambda}]}$ is the number of standard Young tableaux of shape $[\tilde{\lambda}]$.

Bratelli diagram for $\mathbb{C}L_k$

We now give the combinatorial proof of the above equality. Let $T_{[\tilde{\lambda}]}$ be the standard Young tableau of shape $[\tilde{\lambda}]$ and $P_{[\tilde{\lambda}]}$ be the vacillating tableau of shape $[\tilde{\lambda}]$. Let $SYT([\tilde{\lambda}])$ be the set of all standard Young tableau of shape $[\tilde{\lambda}]$.

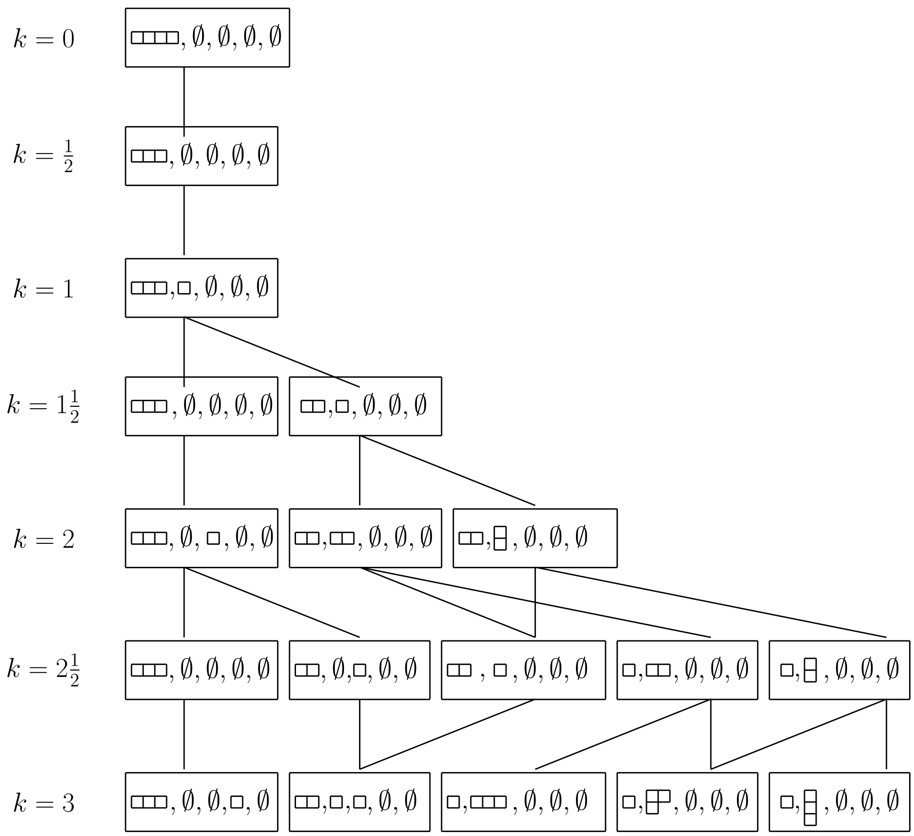
Theorem 5.1. *The map* $(i_1, i_2, \ldots, i_k) \mapsto (T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$ *is bijective where* $\{(i_1, i_2, \ldots, i_k) || 1 \leq i_j \leq n\}$ *and the pair* $(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]}) \in SYT([\tilde{\lambda}]) \times VT_k([\tilde{\lambda}])$.

Proof. To prove $(i_1, i_2, \ldots, i_k) \mapsto (T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$, we first initiate

$$T^{(0)} = \left(\boxed{1 \mid 2 \mid \cdots \mid n}^{(0)}, \emptyset^{(1)}, \emptyset^{(2)}, \ldots, \emptyset^{(r-1)} \right)$$

Then recursively define standard tableaux $T^{j+\frac{1}{2}}$ and T^{j+1} as

$$T^{(j+\frac{1}{2})} = \left(i_{j+1} \longleftarrow T^{(j)} \right)$$



$$T^{(j+1)} = \left(i_{j+1} \longrightarrow T^{(j+\frac{1}{2})}\right)$$

$i_{j+1} \longleftarrow T^{(j)}$ means delete i_{j+1} using Jeu-de-taquin [see [7], p.113] from the corresponding residue where it lies. $i_{j+1} \longrightarrow T^{(j+\frac{1}{2})}$ means insert i_{j+1} in the $\lambda^{(l+1)}$ using RSK insertion [see [7], p.92] if i_{j+1} is removed from $\lambda^{(l)}$.

Let $[\tilde{\lambda}]^{(j)} \in \Lambda_n^j$ be the shape of $T^{(j)}$ and $[\tilde{\lambda}]^{(j+\frac{1}{2})} \in \Lambda_{n-1}^j$ be the shape of $T^{(j+\frac{1}{2})}$. Then let

$$P_{[\tilde{\lambda}]} = \left([\tilde{\lambda}]^{(0)}, [\tilde{\lambda}]^{\frac{1}{2}}, \ldots, [\tilde{\lambda}]^{(k-\frac{1}{2})}, [\tilde{\lambda}]^{(k)}\right)$$

and $T_{[\tilde{\lambda}]} = T^{(k)}$.

This insertion and deletion process produces the vacillating tableaux $P_{[\tilde{\lambda}]}$ of shape $[\tilde{\lambda}] = [\tilde{\lambda}]^{(k)} \in \Lambda_n^k$ and the standard tableau $T_{[\tilde{\lambda}]}$ of the same shape $[\tilde{\lambda}]$. Hence $(i_1, i_2, \ldots i_k) \mapsto (T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$.

To prove $(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]}) \longleftarrow (i_1, i_2, \ldots, i_k)$.

Given $(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$ of shape $[\tilde{\lambda}] = [\tilde{\lambda}]^{(k)} \in \Lambda_n^k$. We use RSK reverse insertion [see [7], p.94] to obtain the sequence $(i_1, i_2, \ldots, i_k)$ from the given pair $(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$. □

For example, consider the sequence (i_1, i_2, i_3) as $(4, 2, 3)$.

j	i_j		$T^{(j)}$
0			$\left(\boxed{1\,2\,3\,4}^{(0)}, \emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$\frac{1}{2}$	4	$\longleftarrow$	$\left(\boxed{1\,2\,3}^{(0)}, \emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
1	4	$\longrightarrow$	$\left(\boxed{1\,2\,3}^{(0)}, \boxed{4}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$

j	i_j		$T^{(j)}$
$1\frac{1}{2}$	2	$\longleftarrow$	$\left(\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline\end{array}^{(0)}, \begin{array}{\|c\|}\hline 4 \\ \hline\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
2	2	$\longrightarrow$	$\left(\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline\end{array}^{(0)}, \begin{array}{\|c\|}\hline 2 \\ \hline 4 \\ \hline\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$2\frac{1}{2}$	3	$\longleftarrow$	$\left(\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}^{(0)}, \begin{array}{\|c\|}\hline 2 \\ \hline 4 \\ \hline\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
3	3	$\longrightarrow$	$\left(\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}^{(0)}, \begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$

6 $R - S$ correspondence for Party algebra

Represent $d \in L_k$ in a single row with vertices labeled by $1, 2, \ldots, 2k$ where we relate vertex j' with the label $2k - j + 1$. Connect vertices i and j in $d \in L_k$ as the graph represented in a single row by the standard representation with $i \leq j$ if and only if i and j are related in d and there does not exist k related to i and j with $i < k < j$. Each vertex is connected only to its nearest neighbours in its block.

We label each edge e of the diagram d represented in the standard representation by $2k + 1 - l$ where l is the right vertex of e.

Define the insertion sequence of a diagram to be the sequence $E = \{E_j\}$ indexed by j in the sequence $\frac{1}{2}, 1, 1\frac{1}{2}, \ldots, 2k$ where

$$E_j = \begin{cases} a, & \text{if vertex } j \text{ is the left endpoint of edge } a \\ \emptyset, & \text{if vertex } j \text{ is not a left endpoint of edge } a. \end{cases}$$

$$E_{j-\frac{1}{2}} = \begin{cases} a, & \text{if vertex } j \text{ is the right endpoint of edge } a \\ \emptyset, & \text{if vertex } j \text{ is not a right endpoint of edge } a. \end{cases}$$

For example, the standard representation and insertion sequence of $d \in L_k$ is as

j	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
E_j	$\emptyset$	2	$\emptyset$	6	6	3	$\emptyset$	4	4	$\emptyset$	3	1	2	$\emptyset$	1	$\emptyset$

For a given $d \in L_k$, with insertion sequence $\{E_j\}$, we will produce a pair of vacillating tableaux $(P_{[\lambda]}, Q_{[\lambda]})$ of shape $[\lambda] \in \Gamma_k$. Begin with the empty tableau,

$$T^{(0)} = (\emptyset, \emptyset, \ldots, \emptyset)$$

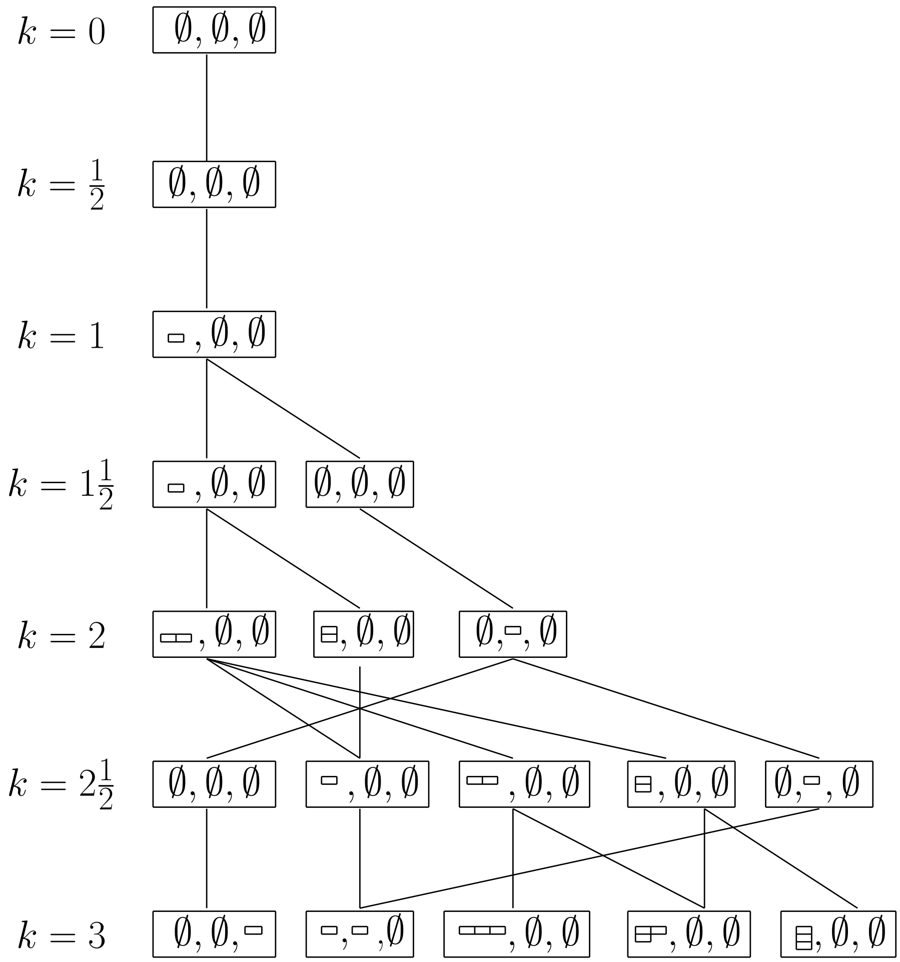
Deleting and inserting, $E_j, j = 1, \ldots, 2k$ in the corresponding residues $1, \ldots, k$, are the two procedures involved in this algorithm, then we successively deleting $E_{j-\frac{1}{2}}$ and inserting E_j as follows.

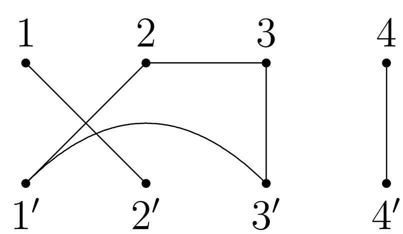
$$T^{(0)} = (\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)})$$

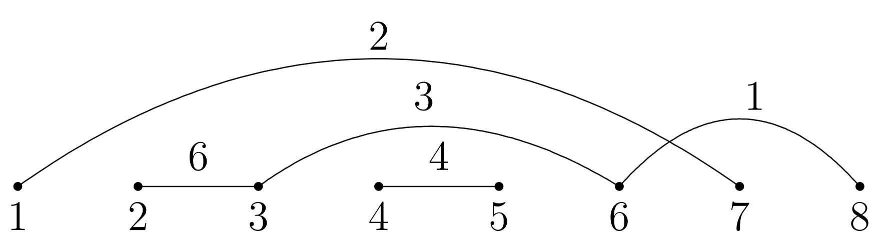
$$T^{(j-\frac{1}{2})} = \begin{cases} T^{(j-1)} & \text{if } E_{j-\frac{1}{2}} = \emptyset, \\ E_{j-\frac{1}{2}} \longleftarrow T^{(j-1)} & \text{if } E_{j-\frac{1}{2}} \neq \emptyset, \end{cases}$$

$E_{j-\frac{1}{2}} \longleftarrow T^{(j-1)}$ means that, delete $E_{j-\frac{1}{2}}$ in $T^{(j-1)}$ using jeu-de-taquin from where it lies.

$$T^{(j)} = \begin{cases} T^{(j-\frac{1}{2})} & \text{if } E_j = \emptyset, \\ E_j \longrightarrow T^{(j-\frac{1}{2})} & \text{if } E_j \neq \emptyset, \end{cases}$$







$E_j \longrightarrow T^{(j-\frac{1}{2})}$ means that the insertion of E_j into $T^{(j-\frac{1}{2})}$ in the following way:

1. For $j = 1, \ldots, k$, insert E_j into $\lambda^{(1)}$ if $E_{j-\frac{1}{2}} = \emptyset$, else insert E_j in $\lambda^{(l+1)}$ if $E_{j-\frac{1}{2}}$ is deleted from $\lambda^{(l)}$.

2. For $j = k+1, \ldots, 2k$, insert E_j into the $\lambda^{(1)}$ if $E_{j-\frac{1}{2}} = \emptyset$, else insert E_j in $\lambda^{(l-1)}$ if $E_{j-\frac{1}{2}}$ is deleted from $\lambda^{(l)}$.

Let $[\lambda]^{(i)}$ be the shape of $T^{(i)}$, let $[\lambda]^{(i+\frac{1}{2})}$ be the shape of $T^{(i+\frac{1}{2})}$ and let $[\lambda] = [\lambda]^{(k)}$. Define

$$Q_{[\lambda]} = \left(\emptyset, [\lambda]^{(\frac{1}{2})}, \ldots, [\lambda]^{(k-\frac{1}{2})}, [\lambda]^{(k)}\right)$$

.

$$P_{[\lambda]} = \left([\lambda]^{(2k)}, [\lambda]^{(2k-\frac{1}{2})}, \ldots, [\lambda]^{(k)}\right).$$

In this insertion process, every edge of the diagram is inserted when we come to its left endpoint and deleted when we come to its right endpoint. so the final shape is $[\lambda]^{(2k)} = \emptyset$. So we associate a pair of vacillating tableaux $(P_{[\lambda]}, Q_{[\lambda]})$ to $d \in L_k$. Denote this process by $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$.

j	E_j		$T^{(j)}$
0			$\left(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$\frac{1}{2}$	$\emptyset$	$\longleftarrow$	$\left(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
1	2	$\longrightarrow$	$\left(\boxed{2}^{\,(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$1\frac{1}{2}$	$\emptyset$	$\longleftarrow$	$\left(\boxed{2}^{\,(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
2	6	$\longrightarrow$	$\left(\boxed{2}\boxed{6}^{\,(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$2\frac{1}{2}$	6	$\longleftarrow$	$\left(\boxed{2}^{\,(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
3	3	$\longrightarrow$	$\left(\boxed{2}^{\,(1)}, \boxed{3}^{\,(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$3\frac{1}{2}$	$\emptyset$	$\longleftarrow$	$\left(\boxed{2}^{\,(1)}, \boxed{3}^{\,(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
4	4	$\longrightarrow$	$\left(\boxed{2}\boxed{4}^{\,(1)}, \boxed{3}^{\,(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$4\frac{1}{2}$	4	$\longleftarrow$	$\left(\boxed{2}^{\,(1)}, \boxed{3}^{\,(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
5	$\emptyset$	$\longrightarrow$	$\left(\boxed{2}^{\,(1)}, \boxed{3}^{\,(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$5\frac{1}{2}$	3	$\longleftarrow$	$\left(\boxed{2}^{\,(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
6	1	$\longrightarrow$	$\left(\begin{matrix}\boxed{1}\\ \boxed{2}\end{matrix}^{\,(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$6\frac{1}{2}$	2	$\longleftarrow$	$\left(\boxed{1}^{\,(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
7	$\emptyset$	$\longrightarrow$	$\left(\boxed{1}^{\,(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$7\frac{1}{2}$	1	$\longleftarrow$	$\left(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
8	$\emptyset$	$\longrightarrow$	$\left(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$

Theorem 6.2. *The map $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$ provides a bijection between the set of $d \in L_k$ and the pairs of vacillating tableaux of shape $[\lambda]$, $[\lambda] \in \Gamma_k$,*

Proof. From the above example it is clear that, for a given $d \in L_k$ we can construct a pair of vacillating tableau $(P_{[\lambda]}, Q_{[\lambda]})$ of shape $[\lambda]$. We prove the theorem by constructing the inverse of $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$. First we use $Q_{[\lambda]}$ followed by $P_{[\lambda]}$ in reverse order to construct the sequence $\left[\lambda^{(\frac{1}{2})}\right], \left[\lambda^{(1)}\right], \ldots, \left[\lambda^{(2k)}\right]$. We initialize $T^{(2k)} = \emptyset$.

We now show how to construct $T^{(i)}$ and E_{i+1} so that $T^{(i+1)} = (\ E_{i+1} \longrightarrow T^{(i)}\ )$. If $[\lambda]^{(i+1)}/[\lambda]^{(i)}$ is a box containing a, and we use reverse algorithm [see [7], p.94] on the value in the box containing a to produce $T^{(i)}$ and $I_{i+1}^{(d,n)}$ such that $T^{(i+1)} = (\ E_{i+1} \longrightarrow T^{(i)}\ )$. Since we remove the value in position a by using reverse RS insertion [see [7], p.92], we know that $T^{(i)}$ has shape $[\lambda]^{(i)}$.

We then show how to construct $T^{(i)}$ and E_{i+1} so that $T^{(i+1)} = (\ E_{i+1} \longleftarrow T^{(i)}\ )$. If $[\lambda]^{(i)}/[\lambda]^{(i+1)}$ is a box containing a. Let $T^{(i)}$ be the tableau of shape $\lambda^{(i)}$ with the same entries as $T^{(i+1)}$ and having the entry $2k - i$ in box containing a. Let $E_{i+1} = 2k - i$. At any given step i, $2k - i$ is the largest entry added to the tableau thus far, so $T^{(i)}$ is standard. Furthermore, $T^{(i+1)} = (\ E_{i+1} \longleftarrow T^{(i)}\ )$, since $E_{i+1} = 2k - i$ is already in the rim hook and thus simply delete it.

Proceeding in this manner, we will produce $E_{2k}, E_{2k-1}, \ldots, E_1$ which completely determines d. By the way we have constructed d, we have $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$. □

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Received: March 1, 2013; *Accepted*: October 11, 2013

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(1)(2014) 10–15

Malaya Journal of Matematik

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www.malayajournal.org

ISSN : 2319-3786

Curvature tensor of almost $C(\lambda)$ manifolds

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Abstract

The present paper deals with certain characterization of curvature conditions on Pseudo-projective and Quasi-conformal curvature tensor on almost $C(\lambda)$ manifolds. The main object of the paper is to study the flatness of the Pseudo-projective, Quasi-conformal curvature tensor, ξ-Pseudo-projective, ξ-Quasi-conformal curvature tensor on almost $C(\lambda)$ manifolds.

Keywords: Almost $C(\lambda)$ manifolds, Pseudo-projective curvature tensor, Quasi-conformal curvature tensor, ξ-Pseudo-projectively flat, ξ-Quasi-conformally flat, η-Einstein.

2010 MSC: 53C15, 53C20, 53C21, 53C25, 53D10.

1 Introduction

In 1981, D. Janssen and L. Vanhecke [6] have introduced the notion of almost $C(\lambda)$ manifolds. Further Z. Olszak and R. Rosca [11] investigated such manifolds. Again S. V. Kharitonava [8] studied conformally flat almost $C(\lambda)$ manifolds. In the paper [2] the author studied Ricci tecsor and quasi-conformal curvature tensor of almost $C(\lambda)$ manifolds. In the paper [1] the authors have studied on quasi-conformally flat spaces. Also in paper [4] the authors have studied on pseudo projective curvature tensor on a Riemannian manifold and in the paper [3] the authors are studied on the Conharmonic and Concircular curvature tensors of almost $C(\lambda)$ manifolds. Our present work is motivated by these works.

2 Preliminaries

Let M be a n-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M such that [5].

$$\eta(\xi) = 1, \tag{2.1}$$
$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$
$$g(X, \xi) = \eta(X), \tag{2.4}$$
$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.5}$$
$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.6}$$

If an almost contact Riemannian manifold M satisfies the condition

$$S = ag + b\eta \otimes \eta \tag{2.7}$$

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for some functions a and b in $C^\infty(M)$ and S is the Ricci tensor, then M is said to be an $\eta-$Einstein manifold. If, in particular, $a = 0$ then this manifold will be called a special type of $\eta-$Einstein manifold.
An almost contact manifold is called an almost $C(\lambda)$ manifold if the Riemannian curvature R satisfies the following relation [8]

$$R(X,Y)Z = R(\phi X,\phi Y)Z - \lambda[Xg(Y,Z) - g(X,Z)Y - \phi Xg(\phi Y,Z) + g(\phi X,Z)\phi Y] \tag{2.8}$$

where, $X,Y,Z \in TM$ and λ is a real number.

Remark 2.1. *A C(l)-curvature tensor is a Sasakian curvature tensor, a C(O)-curvature tensor is a co-Kahler or CK-curvature tensor and a C(l)-curvature tensor is a Kenmotsu curvature tensor.*

From [9] we have,

$$R(X,Y)\xi = R(\phi X,\phi Y)\xi - \lambda[X\eta(Y) - \eta(X)Y] \tag{2.9}$$

On an almost $C(\lambda)$ manifold, we also have [2]

$$QX = AX + B\eta(X)\xi. \tag{2.10}$$

wher, $A = -\lambda(n-2)$, $B = -\lambda$ and Q is the Ricci-operator.

$$\eta(QX) = (A+B)\eta(X), \tag{2.11}$$
$$S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y), \tag{2.12}$$
$$r = -\lambda(n-1)^2, \tag{2.13}$$
$$S(X,\xi) = (A+B)\eta(X), \tag{2.14}$$
$$S(\xi,\xi) = (A+B), \tag{2.15}$$
$$g(QX,Y) = S(X,Y). \tag{2.16}$$

3 Quasi-conformally flat almost $C(\lambda)$ manifolds

Definition 3.1. *The Quasi-conformal curvature tensor $\tilde{C}$ of type $(1,3)$ on a Riemannian manifold (M,g) of dimension n is defined by [1]*

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z + b(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY) \\ &\quad -\frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{3.17}$$

for all $X,Y,Z \in \chi(M)$, where Q is the Ricci-operator.

If $\tilde{C}$ vanishes identically then we say that the manifold is Quasi-conformally flat, where $a,b \neq 0$ are constants.
Thus for a Quasi-conformally flat $C(\lambda)$ manifold, we get from (3.17)

$$\begin{aligned}aR(X,Y)Z &= \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y] \\ &\quad -b(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY).\end{aligned} \tag{3.18}$$

By virtue of(2.10) and (2.12), (3.18) takes the form

$$\begin{aligned}aR(X,Y)Z &= \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y] - b[Ag(Y,Z)X + B\eta(Y)\eta(Z)X \\ &\quad -Ag(X,Z)YB\eta(X)\eta(Z)Y + Ag(Y,Z)X + B\eta(X)g(Y,Z) \\ &\quad -Ag(X,Z)Y - B\eta(Y)g(X,Z)].\end{aligned} \tag{3.19}$$

In view of (2.8) we get from (3.19)

$$\begin{aligned}aR(\phi X,\phi Y)Z &= \lambda a[Xg(Y,Z) - g(X,Z)Y - \phi Xg(\phi Y,Z) + g(\phi X,Z)\phi Y] \\ &+ \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y] - b[Ag(Y,Z)X + B\eta(Y)\eta(Z)X \\ &- Ag(X,Z)YB\eta(X)\eta(Z)Y + Ag(Y,Z)X + B\eta(X)g(Y,Z) \\ &- Ag(X,Z)Y - B\eta(Y)g(X,Z)]\end{aligned} \tag{3.20}$$

Putting $Y = \xi$ and using the value of A and B in (3.20) we get

$$\left[\lambda a + \lambda b + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right) + 2\lambda b(n-2)\right][X\eta(Z) - g(X,Z)\xi] = 0. \tag{3.21}$$

Taking inner product of (3.21) with a vector field ξ, we get

$$\left[\lambda a + \lambda b + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right) + 2\lambda b(n-2)\right][\eta(X)\eta(Z) - g(X,Z)] = 0. \tag{3.22}$$

Putting $X = QX$ in (3.22) we get

$$\left[\lambda a + \lambda b + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right) + 2\lambda b(n-2)\right][\eta(QX)\eta(Z) - g(QX,Z)] = 0. \tag{3.23}$$

Using (2.15),(2.11) and by the virtue of (2.13) in (3.24) we get

$$\left[\lambda a + \lambda b - \frac{\lambda(n-1)^2}{n}\left(\frac{a}{n-1} + 2b\right) + 2\lambda b(n-2)\right][(A+B)\eta(X)\eta(Z) - S(X,Z)] = 0. \tag{3.24}$$

Therefore, either

$$\lambda = 0 \ \ (or) \ \ S(X,Z) = A + B\eta(X)\eta(Y) \tag{3.25}$$

Thus we can state the following:

Theorem 3.1. *For a Quasi-conformally flat almost $C(\lambda)$ manifold, either $\lambda = 0$ i.e. $C(\lambda)$ is cosymplectic. or the manifold is special type of $\eta-$Einstein.*

Proof. Follows form (3.25) and remark (2.1). □

4 ξ-Quasi-conformally flat almost $C(\lambda)$ manifolds

Definition 4.1. *The Quasi-conformal curvature tensor $\tilde{C}$ of type $(1,3)$ on a Riemannian manifold (M,g) of dimension n will be defined as ξ-quasi-conformally flat [1] if $\tilde{C}(X,Y)\xi$=0 for all $X, Y \in TM$.*
Thus for a ξ-quasi-conformally flat almost $C(\lambda)$ manifolds we get from (3.17)

$$\begin{aligned} aR(X,Y)\xi &= \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[\eta(Y)X - \eta(X)Y] \\ &- b(S(Y,\xi)X - S(X,\xi)Y + \eta(Y)QX - \eta(X)QY) \end{aligned} \tag{4.26}$$

In the view of (2.9). Taking $Y = \xi$, by virtue of (2.10), (2.14)and (2.15), putting the value A and B taking inner product with respect to vector field V we get from (4.26).

$$\left[\lambda a + \lambda b - \frac{\lambda(n-1)^2}{n}\left(\frac{a}{n-1} + 2b\right) + 2b\lambda(n-2)\right][g(X,V) - \eta(X)\eta(V)] = 0 \tag{4.27}$$

Taking $X = QX$ in (4.27) we get

$$\left[\lambda a + \lambda b - \frac{\lambda(n-1)^2}{n}\left(\frac{a}{n-1} + 2b\right) + 2b\lambda(n-2)\right][g(QX,V) - \eta(QX)\eta(V)] = 0 \tag{4.28}$$

Using (2.11) and (2.16) in (4.28)

$$\left[\lambda a + \lambda b - \frac{\lambda(n-1)^2}{n}\left(\frac{a}{n-1} + 2b\right) + 2b\lambda(n-2)\right][S(X,V) - (A+B)\eta(X)\eta(V)] = 0 \tag{4.29}$$

Therefore, either

$$\lambda = 0 \ \ (or) \ \ S(X,Z) = A + B\eta(X)\eta(Y) \tag{4.30}$$

Thus we can state the following:

Theorem 4.1. *For a ξ-Quasi-conformally flat almost $C(\lambda)$ manifold, either $\lambda = 0$ i.e. $C(\lambda)$ is cosymplectic. or the manifold is special type of $\eta-$Einstein.*

Proof. Follows form (4.30) and remark (2.1). □

5 Pseudo-projectively curvature flat almost $C(\lambda)$ manifolds

Definition 5.1. *The Pseudo-projective curvature tensor $\tilde{P}$ of type $(1,3)$ on a Riemannian manifold (M,g) of dimension n is defined by [4]*

$$\begin{aligned}\tilde{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{5.31}$$

for all $X,Y,Z \in \chi(M)$. If $\tilde{p}$ vanishes identically then we say that the manifold is Pseudo-projectively curvature flat, where $a,b \neq 0$ are constants.

Thus for a Pseudo-projectively curvature flat $C(\lambda)$ manifold, we get from(5.31)

$$aR(X,Y)Z = \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X - g(X,Z)Y] - b[S(Y,Z)X - S(X,Z)Y]. \tag{5.32}$$

By virtue of (2.12), (5.32) takes the form

$$\begin{aligned}aR(X,Y)Z &= \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X - g(X,Z)Y] \\ &- b[Ag(Y,Z)X + B\eta(Y)\eta(Z)X - Ag(X,Z)Y - B\eta(X)\eta(Z)Y]. \end{aligned} \tag{5.33}$$

In view of (2.8) we get from (5.33)

$$\begin{aligned}aR(\phi X,\phi Y)Z &= \lambda a[Xg(Y,Z) - g(X,Z)Y - \phi Xg(\phi Y,Z) + g(\phi X,Z)\phi Y] \\ &+ \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X - g(X,Z)Y] \\ &- b[Ag(Y,Z)X + B\eta(Y)\eta(Z)X - Ag(X,Z)Y - B\eta(X)\eta(Z)Y]. \end{aligned} \tag{5.34}$$

Putting $Y = \xi$, using the value of A and B, taking inner product with a vector field ξ in (5.34)we get

$$\left[\lambda a + \lambda b + \frac{r}{n}\left(\frac{a}{n-1}+b\right) + b\lambda(n-2)\right][\eta(X)\eta(Z) - g(X,Z)] = 0. \tag{5.35}$$

Taking $X = QX$ and by the virtue of (2.13) in (5.35) we get

$$\left[\lambda a + \lambda b + \frac{r}{n}\left(\frac{a}{n-1}+b\right) + b\lambda(n-2)\right][\eta(QX)\eta(Z) - g(QX,Z)] = 0. \tag{5.36}$$

Using (2.15) and (2.11) in (5.36) we get

$$\left[\lambda a + \lambda b + \frac{r}{n}\left(\frac{a}{n-1}+b\right) + b\lambda(n-2)\left[(A+B)\eta(X)\eta(Z) - S(X,Z)\right] = 0. \tag{5.37}$$

Therefore, either

$$\lambda = 0 \ \ (or) \ \ S(X,Z) = A + B\eta(X)\eta(Y) \tag{5.38}$$

Thus we can state the following:

Theorem 5.1. *For a Pseudo-projectively curvature flat almost $C(\lambda)$ manifold, either $\lambda = 0$ i.e. $C(\lambda)$ is cosymplectic. or the manifold is special type of $\eta-$Einstein.*

Proof. Follows form (5.38) and remark (2.1). □

6 $\xi-$Pseudo-projectively curvature flat almost $C(\lambda)$ manifolds

Definition 6.1. *The Pseudo-projectively curvature tensor $\tilde{P}$ of type $(1,3)$ on a Riemannian manifold (M,g) of dimension n will be defined as ξ-Pseudo-projectively flat [4] if $\tilde{P}(X,Y)\xi=0$ for all $X,Y \in TM$.*
Thus for a ξ-Pseudo-projectively flat almost $C(\lambda)$ manifolds we get from (5.31)

$$\begin{aligned} aR(X,Y)\xi &= \frac{r}{n}\left(\frac{a}{n-1}+2b\right)[\eta(Y)X-\eta(X)Y] \\ &- b(S(Y,\xi)X-S(X,\xi)Y+\eta(Y)QX-\eta(X)QY) \end{aligned} \quad (6.39)$$

In the view of (2.9). Taking $Y=\xi$, by virtue of (2.10), (2.14)and (2.15), putting the value A and B taking inner product with respect to vector field V we get from (6.39)

$$\left[\lambda a+\lambda b+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)+b\lambda(n-2)\right][g(X,V)-\eta(X)\eta(V)]=0. \quad (6.40)$$

Taking $X=QX$ in (6.40) we get

$$\left[\lambda a+\lambda b+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)+b\lambda(n-2)\right][g(QX,V)-\eta(QX)\eta(V)]=0. \quad (6.41)$$

Using (2.11),(2.16) and by the virtue of (2.13) in (6.41)

$$\left[\lambda a+\lambda b+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)+b\lambda(n-2)\right][S(X,V)-(A+B)\eta(X)\eta(V)]=0. \quad (6.42)$$

Therefore, either

$$\lambda=0 \ \ (or) \ \ S(X,Z)=A+B\eta(X)\eta(Y) \quad (6.43)$$

Thus we can state the following:

Theorem 6.1. *For a ξ-Pseudo-projectively flat almost $C(\lambda)$ manifold, either $\lambda=0$ i.e. $C(\lambda)$ is cosymplectic. or the manifold is special type of $\eta-$Einstein.*

Proof. Follows form (6.43) and remark (2.1). □

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Received: September 14, 2013; *Accepted*: October 31, 2013

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Malaya J. Mat. 2(1)(2014) 16–23

Malaya Journal of Matematik

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www.malayajournal.org

ISSN : 2319-3786

Existence results for fractional differential equations with infinite delay and interval impulsive conditions

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Abstract

This paper is mainly concerned with the existence and uniqueness of mild solutions for nonlocal fractional infinite delay differential equations with interval impulses. The results are obtained by using fixed point theorem.

Keywords: Fractional derivative, fractional differential equations, impulsive conditions, fixed point theorem.

2010 MSC: 26A33, 34A08, 35R12, 47H10.

1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc., involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. Though the concepts and the calculus of fractional derivative are few centuries old, it is realized only recently that these derivatives form an excellent framework for modeling real world problems. In the consequence, fractional differential equations have been of great interest. For details, see the monographs of Kilbas et al. [7], Lakshmikantham et al. [8], Miller and Ross [11], Podlubny [12], Anguraj et al. [1], [2] and the references there in.

On the other hand, the theory of impulsive differential equations is also an important area of research which has been investigated in the last few years by great number of mathematicians. We recall that the impulsive differential equations may better model phenomena and dynamical processes subject to a great changes in short times issued for instance in biotechnology, automatics, population dynamics, economics and robotics. To learn more about this kind of problems, we refer to the books [9], [13].

Recently, the study of impulsive differential equations has attracted a great deal of attention in fractional dynamics and its theory has been treated in several works [5], [13]. Balachandran and Trujillo [3], [4] investigated the non-local cauchy poblem for non-linear fractional integro differential equations in Banach spaces. Xianmin Zhang [14] studied the existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay. In most of the impulsive differential equations studied so for, the impulses occur instantaneously. But there are some situations in which the impulsive action starts abruptly and stays active on a finite time interval. Eduardo Hernandez and Donal O'Regan [6] established on a new class of abstract impulsive differential equations for which the impulses are not instantaneous.

Motivated by [6], we consider the following fractional infinite delay differential equations with interval

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impulsive

$$D_t^q x(t) = f(t, x_t), \qquad t \in (s_i, t_{i+1}],\ i = 0, 1, ..., N, \tag{1.1}$$
$$x(t) = g_i(t, x(t)), \qquad t \in (t_i, s_i],\ i = 1, 2, ..., N, \tag{1.2}$$
$$x(0) + k(x) = \phi, \qquad \phi \in B_\vartheta, \tag{1.3}$$

where $0 < q < 1$ and the state $x(.)$ belongs to Banach space X endowed with the norm $\| \cdot \|$. D_t^q is the Caputo fractional derivative and f is a suitable function. $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < ...t_N \leq s_N \leq t_{N+1} = b$ are pre-fixed numbers, $g_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, 2, ..., N$. Let $x_t(.)$ denote $x_t(\theta) = x(t+\theta), \theta \in (-\infty, 0]$. The impulses starts abruptly at the point t_i and their action continue on the interval $[t_i, s_i]$.

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we study the existence and the uniqueness of solutions for the impulsive fractional system (1.1)-(1.3).

2 Preliminaries

In this section, we shall introduce some basic definitions, notations, lemmas and theorem which are used throughout this paper.

Assume that $\vartheta : (-\infty, 0] \to (0, +\infty)$ is a continuous function satisfying $\ell = \int_{-\infty}^0 \vartheta(t)dt < +\infty$. The Banach space $(B_\vartheta, ||.||_{B_\vartheta})$ induced by the function ϑ is defined as follows

$$B_\vartheta = \begin{cases} \varphi : (-\infty, 0] \longrightarrow X : \ for\ any\ c > 0,\ \varphi(\theta)\ is\ a\ bounded\ and \\ measurable\ function\ on\ [-c, 0]\ and\ \int_{-\infty}^0 \vartheta(t) sup_{t \leq \theta \leq 0} \|\varphi(\theta)\|\, dt < +\infty \end{cases}$$

endowed with the norm $\|\varphi\|_{B_\vartheta} = \int_{-\infty}^0 \vartheta(s) sup_{s \leq \theta \leq 0} \|\varphi(\theta)\|\, ds.$
Let us define the space

$$B'_\vartheta = \begin{cases} \varphi : (-\infty, b] \to X : \varphi_k \in C(J_k, X),\ k = 0, 1, 2, ..., N\ and\ there\ exist \\ \varphi(t_k^-) and\ \varphi(t_k^+)\ with\ \varphi(t_k) = \varphi(t_k^-),\ \varphi(t) = g_k(t, x(t)),\ t \in (t_k, s_k], \\ k = 1,\ 2, ..., N,\ \varphi_0 = \varphi(0) + k(\varphi) = \phi \in B_\vartheta \end{cases}$$

where φ_k is the restriction of φ to J_k, $J_0 = [0, t_1]$, $J_k = [s_k, t_{k+1}]$, $k = 1, 2, ..., N$.
Denote by $||.||_{B'_\vartheta}$, a seminorm in the space B'_ϑ, which is defined by
$||\varphi||_{B'_\vartheta} = ||\varphi||_{B_\vartheta} + max\left\{||\varphi_k||_{J_k},\ k = 1, 2, ..., N\right\}$ where $||\varphi_k||_{J_k} = sup_{s \in J_k} ||\varphi_k(s)||$.

For the impulsive conditions, we consider the space $PC(X)$ which is formed by all the functions $x : [0, b] \to X$ such that $x(.)$ is continuous at $t \neq t_i$, $x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exists for all $i = 1, 2, ..., N$, endowed with the uniform norm on $[0, b]$ denoted by $\|x\|_{PC(X)}$. It is easy to see that $PC(X)$ is a Banach space. For a function $x \in PC(X)$ and $i \in \{0, 1, ..., N\}$, we introduce the function $\tilde{x}_i \in C([t_i, t_{i+1}]; X)$ given by

$$\tilde{x}_i(t) = \begin{cases} x(t), & for\ t \in (t_i, t_{i+1}] \\ x(t_i^+), & for\ t = t_i. \end{cases} \tag{2.1}$$

In addition, for $B \subseteq PC(X)$ we use the notation $\tilde{B}_i$ for the set $\tilde{B}_i = \left\{x_i : x \in B\right\}$ and $i \in \{0, 1, ..., N\}$.

Lemma 2.1. *A set $B \subseteq PC(X)$ is relatively compact in $PC(X)$ if and only if each set $\tilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}], X)$.*

Theorem 2.1. *(Schauder's Theorem) Suppose that D is a closed bounded convex subset of the Banach space X and A is completely continuous function from D into D. Then there is a point $z \in D$ such that $Az = z$.*

Definition 2.1. *A function $x : (-\infty, b] \to X$ is called a mild solution of the problem $(1.1) - (1,3)$ if $x(0) + k(x) = \phi \in B_\vartheta$, $x(t) = g_i(t, x(t))$ for all $t \in (t_i, s_i]$, each $i = 1, 2, ..., N$, the restriction of $x(.)$ to the interval $J_k (k = 0, 1, 2, ..., N)$ is continuous, and the following integral equation holds*

$$x(t) = \phi(0) - k(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_s) ds, \qquad for\ all\ \ t \in [0, t_1]\ \ and$$
$$x(t) = g_i(s_i, x(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, x_s) ds,\ for\ all\ t \in [s_i, t_{i+1}]\ and\ every\ i = 1, 2, ..., N.$$

Definition 2.2. *The Riemann - Liouville fractional integral operator of order $q \geq 0$ of function $f \in L^1(R^+)$ is defined as*

$$I^q_{0^+} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \ \ t > 0$$

where Γ(.) is the Euler gamma function.

Definition 2.3. *The Caputo fractional derivative of order $q \geq 0$, $n-1 < q < n$, is defined as*

$$D^q_{0^+} f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{(n-q-1)} f^{(n)}(s) ds, \quad t > 0$$

where the function f(t) has absolutely continuous derivatives up to order (n-1).

If $0 < q < 1$, then

$$D^q_{0^+} f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{(-q)} f^{(1)}(s) ds$$

where $f^{(1)}(s) = Df(s) = \frac{df(s)}{ds}$ and f is an abstract function with values in X.

Lemma 2.2. *Assume that $x \in B'_\vartheta$ then, for $t \in [0, b]$, $x_t \in B_\vartheta$. Moreover*

$$\ell ||x(t)|| \leq ||x_t||_{B_\vartheta} \leq ||\phi||_{B_\vartheta} + \ell \ \ sup_{s \in [0,t]} ||x(s)||.$$

3 Main results

For $\phi \in B_\vartheta$, we define $\hat{\phi}$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ \phi(0), & t \in [0, t_1] \\ 0, & t \in (t_1, b] \end{cases}$$

then $\hat{\phi} \in B'_\vartheta$.

Let $x(t) = y(t) + \hat{\phi}(t), \ -\infty < t < b$. It is evident that y satisfies $y_0 = 0, \ t \in (-\infty, 0]$,

$$y(t) = -k(y+\hat{\phi}) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds, \qquad forall \ \ t \in [0, t_1],$$

$$y(t) = g_i(t, (y+\hat{\phi})(t)), \qquad for\ all\ t \in (t_i, s_i]\ and\ each\ i = 1, 2, ..., N,$$

and

$$y(t) = g_i(s_i, (y+\hat{\phi})(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds,$$
$$for\ all\ t \in [s_i, t_{i+1}]\ and\ every\ i = 1, 2, ..., N$$

if and only if x satisfies

$$x(t) = \phi(t), \qquad t \in (-\infty, 0],$$

$$x(t) = \phi(0) - k(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_s) ds, \qquad for\ all \ \ t \in [0, t_1],$$

$$x(t) = g_i(t, x(t)), \qquad for\ all\ t \in (t_i, s_i]\ and\ each\ i = 1, 2, ..., N,$$

and

$$x(t) = g_i(s_i, x(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, x_s) ds,$$
$$for\ all\ t \in [s_i, t_{i+1}]\ and\ every\ i = 1, 2, ..., N.$$

To prove our main results, we introduce the following conditions:

(H_1) $f : [0,b] \times B_\vartheta \to X$ is continuous and there exist two positive constants $K_1,\ K_2$ such that $||f(t,\phi_1) - f(t,\phi_2)|| \leq K_1||\phi_1 - \phi_2||_{B_\vartheta},\ K_2 = sup_{t\in[0,b]}||f(t,0)||$.

(H_2) The functions $g_i : (t_i, s_i] \times X \to X$ are continuous and there are positive constants L_{g_i} such that $\|g_i(t,x) - g_i(t,y)\| \leq L_{g_i}\|x-y\|$ for all $x,\ y \in X, t \in (t_i,s_i]$ and each $i = 0,1,...,N$.

(H_3) $k : B'_\vartheta \to X$ is continuous and there exist some positive constant$\delta_1,\ \delta_2$ such that $||k(x) - k(y)|| \leq \delta_1||x-y||_{B'_\vartheta}$ and $||k(x)|| \leq \delta_1||x||_{B'_\vartheta} + \delta_2$.

(H_4) $\Omega = max_i\left\{L_{g_i} + \frac{K_1\ell b^q}{\Gamma(q+1)} + \delta_1\right\} < 1,\ i = 1,2,...,N.$

Theorem 3.1. *Suppose that the conditions* $(H_1) - (H_4)$ *are satisfied then there exists a unique mild solution of the problem (1.1)-(1.3).*

Proof. Define $\Theta : B'_\vartheta \to B'_\vartheta$ by

$$\Theta y(t) = 0, \qquad t \in (-\infty, 0]$$

$$\Theta y(t) = -k(y+\hat{\phi}) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, y_s + \hat{\phi}_s)ds, \qquad for\ all\ \ t \in [0,t_1],$$

$$\Theta y(t) = g_i(t,(y+\hat{\phi})(t)), \qquad for\ all\ t \in (t_i,s_i]\ and\ each\ i = 1,2,...,N,$$

and

$$\Theta y(t) = g_i(s_i,(y+\hat{\phi})(s_i)) + \frac{1}{\Gamma(q)}\int_{s_i}^t (t-s)^{q-1} f(s,y_s+\hat{\phi}_s)ds, \qquad for\ all\ t \in [s_i,t_{i+1}]\ and\ every\ i = 1,2,...,N.$$

Clearly, y is a fixed point of Θ then $y + \hat{\phi}$ is a solution of the system (1.1)-(1.3). We shall show that Θ satisfies the hypotheses of Theorem 2.1.

Define the Banach space $(B''_\vartheta, ||.||_{B'_\vartheta})$ induced by B'_ϑ ,

$B''_\vartheta = \{y \in B'_\vartheta : y_0 = 0 \in B_\vartheta\}$ with norm $||y||_{B'_\vartheta} = sup\{||y(s)|| : s \in [0,b]\}$, set $B_r = \{y \in B''_\vartheta : ||y||_{B'_\vartheta} \leq r\}$ for some $r > 0$.

For any $y \in B_r$, $t \in [0,b]$ and by Lemma 2.2, we have

$$\begin{aligned} |y_t + \hat{\phi}_t||_{B_\vartheta} &\leq ||\phi||_{B_\vartheta} + \ell[r + \|\phi(0)\|], \\ ||y + \hat{\phi}||_{B'_\vartheta} &\leq r + ||\phi||_{B_\vartheta} + \|\phi(0)\|. \end{aligned}$$

From the assumption it is easy to see that Θ is well defined. Moreover, for $y_1, y_2 \in B'_\vartheta$, i $\in \{1,2,...,N\}$, and t $\in [s_i, t_{i+1}]$ we get

$$\begin{aligned} ||\Theta y_1(t) - \Theta y_2(t)|| &\leq ||g_i(s_i,(y_1+\hat{\phi})(s_i)) - g_i(s_i,(y_2+\hat{\phi})(s_i))|| \\ &\quad + \frac{1}{\Gamma(q)}\int_{s_i}^t (t-s)^{q-1}||f(s,y_{1_s}+\hat{\phi}_s) - f(s,y_{2_s}+\hat{\phi}_s)||ds \\ &\leq L_{g_i}||y_1-y_2||_{B'_\vartheta} + \frac{1}{\Gamma(q)}\int_{s_i}^t (t-s)^{q-1}k_1\ell||y_1-y_2||_{B'_\vartheta}ds \\ &\leq [L_{g_i} + \frac{K_1\ell b^q}{\Gamma(q+1)}]||y_1-y_2||_{B'_\vartheta} \end{aligned}$$

hence

$$\|\Theta y_1 - \Theta y_2\|_{C([s_i,t_{i+1}];X)} \leq \Omega\,\|y_1 - y_2\|_{B'_\vartheta},\ \ i = 1,2,...,N.$$

proceeding the same manner for the interval $[0,t_1]$, we obtain that

$$\begin{aligned} ||\Theta y_1(t) - \Theta y_2(t)|| &\leq ||-k(y_1+\hat{\phi}) + k(y_2+\hat{\phi})|| \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}||f(s,y_{1_s}+\hat{\phi}_s) - f(s,y_{2_s}+\hat{\phi}_s)||ds \\ &\leq [\delta_1 + \frac{K_1\ell b^q}{\Gamma(q+1)}]||y_1-y_2||_{B'_\vartheta} \end{aligned}$$

hence

$$\|\Theta y_1 - \Theta y_2\|_{C([0,t_1];X)} \leq \Omega\,\|y_1-y_2\|_{B'_\vartheta}$$

Moreover, for $t \in (t_i, s_i]$ we have

$$||\Theta y_1(t) - \Theta y_2(t)|| \leq L_{g_i}||y_1-y_2||_{B'_\vartheta}$$

hence

$$\|\Theta y_1 - \Theta y_2\|_{C((t_i,s_i];X)} \leq \Omega\,\|y_1-y_2\|_{B'_\vartheta},\ \ i=1,2,...,N$$

From the above we have that $\|\Theta y_1 - \Theta y_2\| \leq \Omega\,\|y_1-y_2\|_{B'_\vartheta}$. Therefore Θ is a contraction and there exists a unique mild solution of (1.1)-(1.3). This completes the proof. □

Next, we establish the existence of a mild solution using a fixed point criteria for completely continuous maps.

Theorem 3.2. *Assume the hypotheses $(H_1)-(H_4)$ are satisfied and the functions $g_i(.,0)$ are bounded then the system (1.1)-(1.3) has a mild solution.*

Proof. We divide the proof into five steps.
Step 1: To prove $\Theta B_r \subset B_r$.

There exists a positive integer r such that B_r is clearly a closed bounded convex set in B'_ϑ. If $\Theta B_r \subset B_r$ is not true then for each positive integer r, there exist y $\in B_r$ and $t \in (-\infty, b]$ such that $||\Theta(y)(t)|| > r$, where t is depending upon r.
However, on the other hand for i ≥ 1, let y $\in B_r$ and t $\in (t_i, s_i]$ we have

$$\begin{aligned} r &< ||\Theta y(t)|| \\ &\leq ||g_i(t,(y+\hat{\phi})(t))|| \\ &\leq L_{g_i}||y+\hat{\phi}||_{B'_\vartheta} + ||g_i(t,0)|| \\ &\leq L_{g_i}(r+||\phi||_{B_\vartheta}) + ||g_i(.,0)||_{C((t_i,s_i];X)} \end{aligned}$$

Dividing on both sides by r and taking the lower limit as $r \to +\infty$, weget $1 \leq L_{g_i}$. This is a contradiction to (H_4). Therefore $||\Theta y||_{C((t_i,s_i];X)} \leq r$ for $i \geq 1$.
Proceeding as above for $t \in [s_i, t_{i+1}]$ and $t \in [0,t_1]$, $i \geq 1$ we obtain that
$1 \leq L_{g_i} + \frac{K_1\ell b^q}{\Gamma(q+1)}$ and $1 \leq \delta_1 + \frac{K_1\ell b^q}{\Gamma(q+1)}$, which gives a contradiction to (H_4). Hence, for some positive integer r, $\Theta B_r \subset B_r$.

Next, we introduce the decomposition $\Theta = \Theta_1 + \Theta_2 = \sum_{i=0}^N \Theta_i^1 + \sum_{i=0}^N \Theta_i^2$ where $\Theta_i^j : B_r \to B_r, i = 1,2,...,N$, $j=1,2$ are given by

$$\Theta_i^1 y(t) = \begin{cases} 0, & for\ \ t \in (-\infty,0], \\ -k(y+\hat{\phi}), & for\ \ t \in [0,t_1], \\ g_i(t,(y+\hat{\phi})(t)), & for\ \ t \in (t_i,s_i],\ \ i \geq 1, \\ g_i(s_i,(y+\hat{\phi})(s_i)), & for\ \ t \in (s_i,t_{i+1}],\ \ i \geq 1, \\ 0, & for\ \ t \notin (t_i,t_{i+1}],\ \ i \geq 0. \end{cases}$$

$$\Theta_i^2 y(t) = \begin{cases} 0, & for\ \ t \in (-\infty, 0], \\ \frac{1}{\Gamma(q)} \int_{s_i}^{t} (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds, & for\ \ t \in (s_i, t_{i+1}], \quad i \geq 0, \\ 0, & for\ \ t \notin (s_i, t_{i+1}], \quad i \geq 0. \end{cases}$$

Step 2: The map $\Theta_1 = \sum_{i=0}^{N} \Theta_i^1$ is a contraction on B_r.
Take $y_1, y_2 \in B_r$ arbitrarily. Then, for each $t \in (-\infty, b]$ and from (H_2) to (H_4), we have

$$||\Theta_i^1 y_1(t) - \Theta_i^1 y_2(t)|| \leq \delta_1 ||y_1 - y_2||_{B'_\vartheta} + L_{g_i} ||y_1 - y_2||_{B'_\vartheta}$$

Which implies that $\left\| \sum_{i=0}^{N} \Theta_i^1 y_1 - \sum_{i=0}^{N} \Theta_i^1 y_2 \right\| \leq \Omega \left\| y_1 - y_2 \right\|_{B'_\vartheta}$.

This proves that Θ_1 is a contraction on B_r.

Next, we use the notation $\Theta_i^2 B_r(t) = \left\{ \Theta_i^2 y(t) : B_r \right\}$.

Step 3: For $i = 0, 1, ..., N$ and $s_i < s < t \leq t_{i+1}$, the set $\cup_{\tau \in [s,t]} \Theta_i^2 B_r(\tau)$ is relatively compact in B'_ϑ. Let $s_i < \mu < s$. For $\epsilon > 0$ we choose $0 < \lambda < \frac{s-\mu}{2}$ such that $\frac{\lambda^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] \leq \epsilon$ for all interval $I \subset [0, b]$ with $Diam(I) \leq \lambda$.

Then, for $\tau \in [s, t]$ and $y \in B_r$ we get

$$\begin{aligned} \Theta_i^2 y(\tau) &= \frac{1}{\Gamma(q)} \int_{s_i}^{\tau-\lambda} (\tau - \lambda - s)^{q-1} f(s, y_s + \hat{\phi}_s) ds + \frac{1}{\Gamma(q)} \int_{\tau-\lambda}^{\tau} (\tau - s)^{q-1} f(s, y_s + \hat{\phi}_s) ds \\ &\in B_{r_1} + B_{r_1, \epsilon}, \end{aligned}$$

where $r_1 = \frac{b^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2]$, $r_1, \epsilon = \frac{\lambda^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2]$, which implies that $\cup_{\theta \in [s,t]} \Theta_i^2 B_r(\theta) \subset B_{r_1} + B_{r_1,\epsilon}$. Since B_{r_1} is relatively compact and $Diam(B_\epsilon) \to 0$ as $\epsilon \to 0$, it follows that $\cup_{\theta \in [s,t]} \Theta_i^2 B_r(\theta)$ is relatively compact in B'_ϑ.

In the next step we use the notation introduced in (2.1).

Step 4: The set of functions $\left\{ \Theta_i^2 \widetilde{B_r} \right\}_i$, $i = 0, 1, ..., N$, is a equicontinuous subset of $C([t_i, t_{i+1}]; X)$.

It is clear that $\left\{ \Theta_i^2 \widetilde{B_r} \right\}_i$ is right equicontinuous on $[t_i, s_i)$ and left equicontinuous on $(t_i, s_i]$. Let $t \in (s_i, t_{i+1})$, since the set $\Theta_i^2 B_r(t)$ is relatively compact in B'_ϑ. Then, for $y \in B_r$ and $0 < h < \lambda < t_{i+1} - t$ we get

$$\begin{aligned} \left\| \tilde{\Theta_i^2} y(t+h) - \tilde{\Theta_i^2} y(t) \right\| &= \left\| \Theta_i^2 y(t+h) - \Theta_i^2 y(t) \right\| \\ &= || \frac{1}{\Gamma(q)} \int_{s_i}^{t+h} (t+h-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds \\ &\quad - \frac{1}{\Gamma(q)} \int_{s_i}^{t} (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds || \\ &\leq \frac{1}{\Gamma(q)} \int_{t}^{t+h} (t+h-s)^{q-1} ||f(s, y_s + \hat{\phi}_s)|| ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{s_i}^{t} ||(t+h-s)^{q-1} - (t-s)^{q-1}|| ||f(s, y_s + \hat{\phi}_s)|| ds \\ &\leq \frac{h^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] \\ &\quad + \frac{1}{\Gamma(q)} \int_{s_i}^{t} ||(t+h-s)^{q-1} - (t-s)^{q-1}|| [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] ds \end{aligned}$$

The right-hand side is independent of $y \in B_r$ and tends to zero as $h \to 0$. This shows that $\left\{ \widetilde{\Theta_i^2 B_r} \right\}_i$ is right equicontinuous at t.

In the simillar manner we proceed for $t = s_i$ and $h > 0$ with $s_i + h < t_{i+1}$we have that

$$\begin{aligned} \left\| \tilde{\Theta_i^2} y(s_i + h) - \tilde{\Theta_i^2} y(s_i) \right\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_{s_i}^{s_i+h} (t+h-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds \right\| \\ &\leq \frac{h^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] \end{aligned}$$

which implies that $\left\{\widetilde{\Theta_i^2 B_r}\right\}_i$ is right equicontinuous at s_i.

Now for $t \in (s_i, t_{i+1}]$. Let $\mu \in (s_i, t]$. Since $\cup_{s\in[\mu,t]}\Theta_i^2 B_r(s)$ is relatively compact in B'_ϑ, we select $0 < \lambda < \frac{t-\mu}{2}$ then for $0 < h \leq \lambda$ and $y \in B_r$ we get,

$$\begin{aligned}
\left\|\tilde{\Theta_i^2} y(t-h) - \tilde{\Theta_i^2} y(t)\right\| &= \left\|\Theta_i^2 y(t-h) - \Theta_i^2 y(t)\right\| \\
&\leq \frac{1}{\Gamma(q)}\int_{t-h}^{t}(t-s)^{q-1}||f(s, y_s + \hat{\phi}_s)||ds \\
&\quad + \frac{1}{\Gamma(q)}\int_{s_i}^{t-h}||(t-s)^{q-1} - (t-h-s)^{q-1}||||f(s, y_s + \hat{\phi}_s)||ds \\
&\leq \frac{h^q}{\Gamma(q+1)}[k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] \\
&\quad + \frac{1}{\Gamma(q)}\int_{s_i}^{t}||(t-s)^{q-1} - (t-h-s)^{q-1}||[k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2]ds
\end{aligned}$$

which shows that $\{\widetilde{\Theta_i^2 B_r}\}_i$ is left equicontinuous at $t \in (s_i, t_{i+1}]$.

This completes the proof that the set $\{\widetilde{\Theta_i^2 B_r}\}_i$ is equicontinuous.

Step 5: For $i \neq j$, the set $\{\widetilde{\Theta_i^2 B_r}\}_j$ is a equicontinuous subset of $C([t_j, t_{j+1}]; X)$.

From the above steps and Lemma 2.1 it follows that, the map Θ_1 is a contraction and the maps Θ_2 are completely continuous. Thus, $\Theta = \Theta_1 + \Theta_2$ is a condensing operator. Finally, from [[10], Theorem 4.3.2]. we assert that there exists a mild solution of (1.1)-(1.3). □

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Received: October 24, 2013; *Accepted*: November 25, 2013

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(1)(2014) 24-34

Malaya Journal of Matematik

www.malayajournal.org

MJM

an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Generalization for character formulas in terms of continued fraction identities

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Abstract

In recent work, Folsom discussed character formulas for classical mock theta functions of Ramanujan. Here, we suggest representations for character formulas in terms of continued fraction identities or in more precise language, we can say an applications of continued fraction identities to character formulas. As a consequence, we obtain fourteen new results.

Keywords: Character formulas, q-product identities, continued fractions identities.

2010 MSC: 11F27, 11F37, 11F50, 17B67, 33D15.

1 Introduction and Basic Terminology

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n) \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{(n-1)}) \tag{1.2}$$

$$(a_1, a_2, a_3, ..., a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty (a_3;q)_\infty ... (a_k;q)_\infty \tag{1.3}$$

Ramanujan has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

Jacobi's triple product identity [1, p.35] is given, as

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\Phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty \tag{1.6}$$

$$\Psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

$$f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

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Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty \tag{1.9}$$

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty \cdots (q^t;q^n)_\infty = (q^a,q^b,q^c \cdots q^t;q^n)_\infty \tag{1.10}$$

$$(q^a;q^n)_\infty(q^a;q^n)_\infty(q^c;q^n)_\infty \cdots (q^t;q^n)_\infty = (q^a,q^a,q^c \cdots q^t;q^n)_\infty \tag{1.11}$$

Computation of q-product identities:
In [1], Chaudhary has computed several q-product identities. Here we are giving some identities from [1], and some new identities have been computed, are useful for next section of this paper, as given below

$$(q^2;q^2)_\infty = \prod_{n=0}^{\infty}(1-q^{2n+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{2(4n)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+1)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+2)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{8n+2}) \times \prod_{n=0}^{\infty}(1-q^{8n+4}) \times \prod_{n=0}^{\infty}(1-q^{8n+6}) \times \prod_{n=0}^{\infty}(1-q^{8n+8})$$

$$= (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty$$

$$= (q^2,q^4,q^6,q^8;q^8)_\infty \tag{1.12}$$

$$(q^4;q^4)_\infty = \prod_{n=0}^{\infty}(1-q^{4n+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{4(3n)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+1)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{12n+4}) \times \prod_{n=0}^{\infty}(1-q^{12n+8}) \times \prod_{n=0}^{\infty}(1-q^{12n+12})$$

$$= (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty$$

$$= (q^4,q^8,q^{12};q^{12})_\infty \tag{1.13}$$

$$(q^4;q^{12})_\infty = \prod_{n=0}^{\infty}(1-q^{12n+4}) = \prod_{n=0}^{\infty}(1-q^{12(5n)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+1)+4}) \times$$

$$\times \prod_{n=0}^{\infty}(1-q^{12(5n+2)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+3)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+4)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{60n+4}) \times \prod_{n=0}^{\infty}(1-q^{60n+16}) \times \prod_{n=0}^{\infty}(1-q^{60n+28}) \times$$

$$\times \prod_{n=0}^{\infty}(1-q^{60n+40}) \times \prod_{n=0}^{\infty}(1-q^{60n+52})$$

$$= (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty$$

$$= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \tag{1.14}$$

Similarly we can compute following, as

$$(q^1;q^1)_\infty = (q^1;q^2)_\infty(q^2;q^2)_\infty$$

$$= (q^1,q^2;q^2)_\infty \tag{1.15}$$

$$(q^2;q^2)_\infty = (q^2;q^4)_\infty(q^4;q^4)_\infty$$

$$= (q^2, q^4; q^4)_\infty \tag{1.16}$$

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty\\ &= (q^2,q^4,q^6,q^8;q^8)_\infty\end{aligned} \tag{1.17}$$

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^{12})_\infty(q^4;q^{12})_\infty(q^6;q^{12})_\infty(q^8;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty\\ &= (q^2,q^4,q^6,q^8,q^{10},q^{12};q^{12})_\infty\end{aligned} \tag{1.18}$$

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^{16})_\infty(q^4;q^{16})_\infty(q^6;q^{16})_\infty(q^8;q^{16})_\infty(q^{10};q^{16})_\infty\times\\ &\quad\times(q^{12};q^{16})_\infty(q^{14};q^{16})_\infty(q^{16};q^{16})_\infty\\ &= (q^2,q^4,q^6,q^8,q^{10},q^{12},q^{14},q^{16};q^{16})_\infty\end{aligned} \tag{1.19}$$

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^{20})_\infty(q^4;q^{20})_\infty(q^6;q^{20})_\infty(q^8;q^{20})_\infty(q^{10};q^{20})_\infty(q^{12};q^{20})_\infty\times\\ &\quad\times(q^{14};q^{20})_\infty(q^{16};q^{20})_\infty(q^{18};q^{20})_\infty(q^{20};q^{20})_\infty\\ &= (q^2,q^4,q^6,q^8,q^{10},q^{12},q^{14},q^{16},q^{18},q^{20};q^{20})_\infty\end{aligned} \tag{1.20}$$

$$\begin{aligned}(q^3;q^3)_\infty &= (q^3;q^6)_\infty(q^6;q^6)_\infty\\ &= (q^3,q^6;q^6)_\infty\end{aligned} \tag{1.21}$$

$$\begin{aligned}(q^4;q^4)_\infty &= (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty\\ &= (q^4,q^8,q^{12};q^{12})_\infty\end{aligned} \tag{1.22}$$

$$\begin{aligned}(q^4;q^4)_\infty &= (q^4;q^{16})_\infty(q^8;q^{16})_\infty(q^{12};q^{16})_\infty(q^{16};q^{16})_\infty\\ &= (q^4,q^8,q^{12},q^{16};q^{16})_\infty\end{aligned} \tag{1.23}$$

$$\begin{aligned}(q^4;q^4)_\infty &= (q^4;q^{20})_\infty(q^8;q^{20})_\infty(q^{12};q^{20})_\infty(q^{16};q^{20})_\infty(q^{20};q^{20})_\infty\\ &= (q^4,q^8,q^{12},q^{16},q^{20};q^{20})_\infty\end{aligned} \tag{1.24}$$

$$\begin{aligned}(q^4;q^4)_\infty &= (q^4;q^{24})_\infty(q^8;q^{24})_\infty(q^{12};q^{24})_\infty(q^{16};q^{24})_\infty(q^{20};q^{24})_\infty(q^{24};q^{24})_\infty\\ &= (q^4,q^8,q^{12},q^{16},q^{20},q^{24};q^{24})_\infty\end{aligned} \tag{1.25}$$

$$\begin{aligned}(q^4;q^{12})_\infty &= (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty\\ &= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty\end{aligned} \tag{1.26}$$

$$\begin{aligned}(q^6;q^6)_\infty &= (q^6;q^{12})_\infty(q^{12};q^{12})_\infty\\ &= (q^6,q^{12};q^{12})_\infty\end{aligned} \tag{1.27}$$

$$\begin{aligned}(q^6;q^6)_\infty &= (q^6;q^{24})_\infty(q^{12};q^{24})_\infty(q^{18};q^{24})_\infty(q^{24};q^{24})_\infty\\ &= (q^6,q^{12},q^{18},q^{24};q^{24})_\infty\end{aligned} \tag{1.28}$$

$$\begin{aligned}(q^6;q^{12})_\infty &= (q^6;q^{60})_\infty(q^{18};q^{60})_\infty(q^{30};q^{60})_\infty(q^{42};q^{60})_\infty(q^{54};q^{60})_\infty\\ &= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty\end{aligned} \tag{1.29}$$

$$(q^8;q^8)_\infty = (q^8;q^{24})_\infty(q^{16};q^{24})_\infty(q^{24};q^{24})_\infty$$
$$= (q^8,q^{16},q^{24};q^{24})_\infty \quad (1.30)$$

$$(q^8;q^8)_\infty = (q^8;q^{48})_\infty(q^{16};q^{48})_\infty(q^{24};q^{48})_\infty(q^{32};q^{48})_\infty(q^{40};q^{48})_\infty(q^{48};q^{48})_\infty$$
$$= (q^8,q^{16},q^{24},q^{32},q^{40},q^{48};q^{48})_\infty \quad (1.31)$$

$$(q^8;q^{12})_\infty = (q^8;q^{60})_\infty(q^{20};q^{60})_\infty(q^{32};q^{60})_\infty(q^{44};q^{60})_\infty(q^{56};q^{60})_\infty$$
$$= (q^8,q^{20},q^{32},q^{44},q^{56};q^{60})_\infty \quad (1.32)$$

$$(q^8;q^{16})_\infty = (q^8;q^{48})_\infty(q^{24};q^{48})_\infty(q^{40};q^{48})_\infty$$
$$= (q^8,q^{24},q^{40};q^{48})_\infty \quad (1.33)$$

$$(q^{10};q^{20})_\infty = (q^{10};q^{60})_\infty(q^{30};q^{60})_\infty(q^{50};q^{60})_\infty$$
$$= (q^{10},q^{30},q^{50};q^{60})_\infty \quad (1.34)$$

$$(q^{12};q^{12})_\infty = (q^{12};q^{24})_\infty(q^{24};q^{24})_\infty$$
$$= (q^{12},q^{24};q^{24})_\infty \quad (1.35)$$

$$(q^{12};q^{12})_\infty = (q^{12};q^{60})_\infty(q^{24};q^{60})_\infty(q^{36};q^{60})_\infty(q^{48};q^{60})_\infty(q^{60};q^{60})_\infty$$
$$= (q^{12},q^{24},q^{36},q^{48},q^{60};q^{60})_\infty \quad (1.36)$$

$$(q^{16};q^{16})_\infty = (q^{16};q^{48})_\infty(q^{32};q^{48})_\infty(q^{48};q^{48})_\infty$$
$$= (q^{16},q^{32},q^{48};q^{48})_\infty \quad (1.37)$$

$$(q^{20};q^{20})_\infty = (q^{20};q^{60})_\infty(q^{40};q^{60})_\infty(q^{60};q^{60})_\infty$$
$$= (q^{20},q^{40},q^{60};q^{60})_\infty \quad (1.38)$$

The outline of this paper is as follows. In sections 2, we have recorded some well known results on continued fraction identities and recent results on character formulas for mock theta functions of Ramanujan given by Folsom [2], those are useful to the rest of the paper. In section 3, we obtain fourteen new results.

2 Preliminaries and Statement of Results

The famous Rogers-Ramanujan continued fraction identity [3, (1.6)], is

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\vdots}}}}} \quad (2.1)$$

In 1983 Denis [5], has introduced following continued fraction identity

$$(q^2;q^2)_\infty(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1+\vdots}}}}}}} \tag{2.2}$$

another well known continued fraction identity due to Ramanujan [4, (4.21)], is

$$\frac{(-q^3;q^4)_\infty}{(-q;q^4)_\infty} = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^3+q^2}{1+\cfrac{q^5}{1+\cfrac{q^7+q^4}{1+\cfrac{q^9}{1+\cfrac{q^{11}+q^6}{1+\vdots}}}}}}} \tag{2.3}$$

One of the most celebrated continued fractional identities associated with Ramanujan's academic career, given by Rogers-Ramanujan, is

$$C(q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\vdots}}}}} \tag{2.4}$$

Folsom in Table 1 [2; p. 450], recorded character formulas for order 3 mock θ - functions. We are giving below only those functions, which having terms of q-product identities, as

$$f(-q) = -\,4q.\widehat{\Theta}_{12}^{-1}(q^{\frac{1}{2}}.tr_{L(\wedge_{(1)};\,13)}q^{L_0} + q^{-\frac{1}{2}}.tr_{L(\wedge_{(5)};\,13)}q^{L_0})+$$
$$+\,4q\widehat{\beta}_{12,1}(\tau) + \frac{(q^2;q^2)_\infty^7}{(q;q)_\infty^3(q^4;q^4)_\infty^3} \tag{2.5}$$

$$\phi(q) = -\,2q.\widehat{\Theta}_{12}^{-1}(q^{\frac{1}{2}}.tr_{L(\wedge_{(1)};\,13)}q^{L_0} + q^{-\frac{1}{2}}.tr_{L(\wedge_{(5)};\,13)}q^{L_0})+$$
$$+\,2q\widehat{\beta}_{12,1}(\tau) + \frac{(q^2;q^2)_\infty^7}{(q;q)_\infty^3(q^4;q^4)_\infty^3} \tag{2.6}$$

$$\chi(-q) = -\,q.\widehat{\Theta}_{12}^{-1}(q^{\frac{1}{2}}.tr_{L(\wedge_{(1)};\,13)}q^{L_0} + q^{-\frac{1}{2}}.tr_{L(\wedge_{(5)};\,13)}q^{L_0})+$$
$$+\,q\widehat{\beta}_{12,1}(\tau) + \frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^{12};q^{12})_\infty^2} \tag{2.7}$$

$$v(q) = -\,q.\widehat{\Theta}_{12}^{-1}(tr_{L(\wedge_{(-2)};\,13)}q^{L_0} + tr_{L(\wedge_{(2)};\,13)}q^{L_0})+$$
$$+\,q\widehat{\beta}_{12,-2}(\tau) + \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} \tag{2.8}$$

$$\rho(q) = -\,\frac{1}{2}.\widehat{\Theta}_6^{-1}(tr_{L(\wedge_{(-1)};\,7)}q^{L_0} + tr_{L(\wedge_{(1)};\,7)}q^{L_0})+$$

$$+\frac{1}{2}\widehat{\beta}_{6,-1}(\tau)+\frac{3}{2}\frac{(q^6;q^6)^4_\infty}{(q^2;q^2)_\infty(q^3;q^3)^2_\infty} \tag{2.9}$$

$$\sigma(-q)=q^2.\widehat{\Theta}^{-1}_{36}(q^{\frac{3}{2}}.tr_{L(\wedge_{(3)};\,37)}q^{L_0}+q^{-\frac{3}{2}}.tr_{L(\wedge_{(15)};\,37)}q^{L_0})-$$

$$-q^2\widehat{\beta}_{36,3}(\tau)+\frac{(q^2;q^2)^3_\infty(q^{12};q^{12})^3_\infty}{(q;q)_\infty(q^4;q^4)^2_\infty(q^6;q^6)^2_\infty} \tag{2.10}$$

Folsom in Table 4 [2; p. 452], recorded character formulas for order 2 mock θ - functions. We are giving below only those functions, which having terms of q-product identities, as

$$A(q^2)=q.\widehat{\Theta}^{-1}_8 tr_{L(\wedge_{(2)};\,9)}q^{L_0}-q\widehat{\eta}_{8,2}(\tau)-q(-q^2;q^2)_\infty(-q^4;q^4)^2_\infty(q^8;q^8)_\infty \tag{2.11}$$

$$\mu(q^4)=-2q.\widehat{\Theta}^{-1}_4 tr_{L(\wedge_{(0)};\,5)}q^{L_0}+2q\widehat{\eta}_{4,0}(\tau)+$$

$$+\frac{(q^2;q^2)_\infty(q^4;q^4)^3_\infty(q^8;q^8)_\infty}{(q;q)^2_\infty(q^{16};q^{16})^2_\infty}\times 12 \tag{2.12}$$

Folsom in Table 5 [2; p. 452], recorded character formulas for order 6 mock θ - functions. We are giving below both the functions, which having terms of q-product identities, as

$$\phi(q^4)=-2q.\widehat{\Theta}^{-1}_{12} tr_{L(\wedge_{(4)};\,13)}q^{L_0}+2q\widehat{\eta}_{12,4}(\tau)+$$

$$+\frac{(q^2;q^2)^3_\infty(q^3;q^3)^2_\infty(q^{12};q^{12})^3_\infty}{(q;q)^2_\infty(q^6;q^6)^3_\infty(q^8;q^8)_\infty(q^{24};q^{24})_\infty} \tag{2.13}$$

$$\psi(q^4)=-q^3.\widehat{\Theta}^{-1}_{12} tr_{L(\wedge_{(0)};\,13)}q^{L_0}+q^3\widehat{\eta}_{12,0}(\tau)+$$

$$+q^3\frac{(q^2;q^2)^2_\infty(q^4;q^4)_\infty(q^{24};q^{24})^2_\infty}{(q;q)_\infty(q^3;q^3)_\infty(q^8;q^8)^2_\infty} \tag{2.14}$$

Folsom in Table 7 [2; p. 453], recorded character formulas for order 10 mock θ - functions. We are giving below all the four functions, which having terms of q-product identities, as

$$\phi(q)=2q.\widehat{\Theta}^{-1}_{10} tr_{L(\wedge_{(1)};\,11)}q^{L_0}-2q\widehat{\eta}_{10,1}(\tau)+\frac{(q^{10};q^{10})^2_\infty j(-q^2;q^5)}{(q^5;q^5)_\infty j(q^2;q^{10})} \tag{2.15}$$

$$\psi(q)=2q.\widehat{\Theta}^{-1}_{10} tr_{L(\wedge_{(3)};\,11)}q^{L_0}-2q\widehat{\eta}_{10,3}(\tau)-q\frac{(q^{10};q^{10})^2_\infty j(-q;q^5)}{(q^5;q^5)_\infty j(q^4;q^{10})} \tag{2.16}$$

$$X(-q^2)=-2q.\widehat{\Theta}^{-1}_{40} tr_{L(\wedge_{(18)};\,41)}q^{L_0}+2q.\widehat{\Theta}^{-1}_{40} tr_{L(\wedge_{(2)};\,41)}q^{L_0}+$$

$$+2q\widehat{\eta}_{40,18}(\tau)-2q\widehat{\eta}_{40,2}(\tau)+$$

$$+\frac{(q^4;q^4)^2_\infty(j(-q^2,q^{20})j(q^{12},q^{40})+2q(q^{40};q^{40})^3_\infty)}{(q^2;q^2)_\infty(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^8,q^{40})} \tag{2.17}$$

Here, we have written above equations with minor corrections, which was occurred in original published paper may be due to printing error.

$$\chi(-q^2)=-2q^3.\widehat{\Theta}^{-1}_{40} tr_{L(\wedge_{(14)};\,41)}q^{L_0}-2q^5.\widehat{\Theta}^{-1}_{40} tr_{L(\wedge_{(6)};\,41)}q^{L_0}+$$

$$+2q^3\widehat{\eta}_{40,14}(\tau)+2q^5\widehat{\eta}_{40,6}(\tau)+$$

$$+q^2\frac{(q^4;q^4)^2_\infty(2q(q^{40};q^{40})^3_\infty-j(-q^6,q^{20})^2 j(q^4,q^{40}))}{(q^2;q^2)_\infty(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^{16},q^{40})} \tag{2.18}$$

3 Main Results

In this section, we obtain representations for character formulas in terms of continued fraction identities or in more precise language, we can say an applications of continued fraction identities to character formulas given by Folsom [2]. We obtain fourteen new results parallel to character formulas (2.5) to (2.18), which are recorded in [2, pp. 450, and 452-453], using q-product identities given in (2.12) to (1.38).

$$f(-q) = -4q.\widehat{\Theta}_{12}^{-1}(q^{\frac{1}{2}}.tr_{L(\bigwedge_{(1)};\,13)}q^{L_0} + q^{-\frac{1}{2}}.tr_{L(\bigwedge_{(5)};\,13)}q^{L_0}) + 4q\widehat{\beta}_{12,1}(\tau)+$$

$$+\frac{(q^2;q^2)_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty(q^2,q^6,q^8,q^{10},q^{14};q^{16})^2_\infty}{(q^4,q^{16};q^{20})_\infty(q^{16};q^{16})^2_\infty}\times$$

$$\times\left[\cfrac{1}{1-\cfrac{q^8}{1+\cfrac{q^8(1-q^8)}{1-\cfrac{q^{24}}{1+\cfrac{q^{16}(1-q^{16})}{1-\cfrac{q^{40}}{1+\cfrac{q^{24}(1-q^{24})}{1+\vdots}}}}}}}\right]^2\times\left[\cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1+\vdots}}}}}}}\right]^3\times$$

$$\times\cfrac{1}{1+\cfrac{q^4}{1+\cfrac{q^8}{1+\cfrac{q^{12}}{1+\cfrac{q^{16}}{1+\cfrac{q^{20}}{1+\cfrac{q^{24}}{1+\vdots}}}}}}} \tag{3.1}$$

$$\phi(q) = -2q.\widehat{\Theta}_{12}^{-1}(q^{\frac{1}{2}}.tr_{L(\bigwedge_{(1)};\,13)}q^{L_0} + q^{-\frac{1}{2}}.tr_{L(\bigwedge_{(5)};\,13)}q^{L_0}) + 2q\widehat{\beta}_{12,1}(\tau)+$$

$$+\frac{(q^2;q^2)_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty(q^2,q^6,q^8,q^{10},q^{14};q^{16})^2_\infty}{(q^4,q^{16};q^{20})_\infty(q^{16};q^{16})^2_\infty}\times$$

$$\times\left[\cfrac{1}{1-\cfrac{q^8}{1+\cfrac{q^8(1-q^8)}{1-\cfrac{q^{24}}{1+\cfrac{q^{16}(1-q^{16})}{1-\cfrac{q^{40}}{1+\cfrac{q^{24}(1-q^{24})}{1+\vdots}}}}}}}\right]^2\times\left[\cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1+\vdots}}}}}}}\right]^3\times$$

$$\times\cfrac{1}{1+\cfrac{q^4}{1+\cfrac{q^8}{1+\cfrac{q^{12}}{1+\cfrac{q^{16}}{1+\cfrac{q^{20}}{1+\cfrac{q^{24}}{1+\vdots}}}}}}} \tag{3.2}$$

$$\chi(-q) = -\, q.\widehat{\Theta}_{12}^{-1}(q^{\frac{1}{2}}.tr_{L(\wedge_{(1)};\, 13)}q^{L_0} + q^{-\frac{1}{2}}.tr_{L(\wedge_{(5)};\, 13)}q^{L_0})+$$

$$+\, q\widehat{\beta}_{12,1}(\tau) + \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty}{(q^{12};q^{12})^2_\infty(q^2,q^6,q^{14},q^{18},q^{20};q^{20})^2_\infty}\times$$

$$\times\cfrac{1}{1-\cfrac{q^3}{1+\cfrac{q^3(1-q^3)}{1-\cfrac{q^9}{1+\cfrac{q^6(1-q^6)}{1-\cfrac{q^{15}}{1+\cfrac{q^9(1-q^9)}{1+\vdots}}}}}}}\times\left[\cfrac{1}{1-\cfrac{q^{10}}{1+\cfrac{q^{10}(1-q^{10})}{1-\cfrac{q^{30}}{1+\cfrac{q^{20}(1-q^{20})}{1-\cfrac{q^{50}}{1+\cfrac{q^{30}(1-q^{30})}{1+\vdots}}}}}}}\right]^2\times$$

$$\times\left[\cfrac{1}{1+\cfrac{q^4}{1+\cfrac{q^8}{1+\cfrac{q^{12}}{1+\cfrac{q^{16}}{1+\cfrac{q^{20}}{1+\cfrac{q^{24}}{1+\vdots}}}}}}}\right]^2\times\left[1+\cfrac{q^4}{1+\cfrac{q^8}{1+\cfrac{q^{12}}{1+\cfrac{q^{16}}{1+\cfrac{q^{20}}{1+\cfrac{q^{24}}{1+\cfrac{q^{28}}{1+\vdots}}}}}}}\right]^2 \tag{3.3}$$

$$v(q) = -\, q.\widehat{\Theta}_{12}^{-1}(tr_{L(\wedge_{(-2)};\, 13)}q^{L_0} + tr_{L(\wedge_{(2)};\, 13)}q^{L_0}) +\ q\widehat{\beta}_{12,-2}(\tau)+$$

$$+\frac{(q^4;q^4)_\infty}{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})^2_\infty}\times\left[\cfrac{1}{1-\cfrac{q^{10}}{1+\cfrac{q^{10}(1-q^{10})}{1-\cfrac{q^{30}}{1+\cfrac{q^{20}(1-q^{20})}{1-\cfrac{q^{50}}{1+\cfrac{q^{30}(1-q^{30})}{1+\vdots}}}}}}}\right]^2\times$$

$$\times\left[\cfrac{1}{1+\cfrac{q^4}{1+\cfrac{q^8}{1+\cfrac{q^{12}}{1+\cfrac{q^{16}}{1+\cfrac{q^{20}}{1+\cfrac{q^{24}}{1+\vdots}}}}}}}\right]^2\times\left[1+\cfrac{q^4}{1+\cfrac{q^8}{1+\cfrac{q^{12}}{1+\cfrac{q^{16}}{1+\cfrac{q^{20}}{1+\cfrac{q^{24}}{1+\cfrac{q^{28}}{1+\vdots}}}}}}}\right]^2 \tag{3.4}$$

$$\rho(q) = -\,\frac{1}{2}.\widehat{\Theta}_6^{-1}(tr_{L(\wedge_{(-1)};\, 7)}q^{L_0} + tr_{L(\wedge_{(1)};\, 7)}q^{L_0}) + \frac{1}{2}\widehat{\beta}_{6,-1}(\tau) + \frac{3}{2}\frac{1}{(q^2;q^2)_\infty}\times$$

$$\times\left[\cfrac{1}{1-\cfrac{q^3}{1+\cfrac{q^3(1-q^3)}{1-\cfrac{q^9}{1+\cfrac{q^6(1-q^6)}{1-\cfrac{q^{15}}{1+\cfrac{q^9(1-q^9)}{1+\vdots}}}}}}}\right]^2 \tag{3.5}$$

$$\sigma(-q)=q^2.\widehat{\Theta}_{36}^{-1}(q^{\frac{3}{2}}.tr_{L(\wedge_{(3)};\,37)}q^{L_0}+q^{-\frac{3}{2}}.tr_{L(\wedge_{(15)};\,37)}q^{L_0})-\;q^2\widehat{\beta}_{36,3}(\tau)+$$
$$+\frac{(q^2,q^{10};q^{12})_\infty^2(q^6;q^{12})_\infty}{(q;q^2)_\infty}\times\cfrac{1}{1-\cfrac{q^6}{1+\cfrac{q^6(1-q^6)}{1-\cfrac{q^{18}}{1+\cfrac{q^{12}(1-q^{12})}{1-\cfrac{q^{30}}{1+\cfrac{q^{18}(1-q^{18})}{1+\vdots}}}}}}} \tag{3.6}$$

$$A(q^2)=q.\widehat{\Theta}_8^{-1}tr_{L(\wedge_{(2)};\,9)}q^{L_0}-q\widehat{\eta}_{8,2}(\tau)-q(-q^2;q^2)_\infty(-q^4;q^4)_\infty\times$$
$$\times\cfrac{1}{1-\cfrac{q^4}{1+\cfrac{q^4(1-q^4)}{1-\cfrac{q^{12}}{1+\cfrac{q^8(1-q^8)}{1-\cfrac{q^{20}}{1+\cfrac{q^{12}(1-q^{12})}{1+\vdots}}}}}}} \tag{3.7}$$

$$\mu(q^4)=-\,2q.\widehat{\Theta}_4^{-1}tr_{L(\wedge_{(0)};\,5)}q^{L_0}+2q\widehat{\eta}_{4,0}(\tau)+\frac{(q^8;q^8)_\infty}{(q;q)_\infty(q^2;q^4)_\infty}\times$$
$$\times\cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1+\vdots}}}}}}}\times 12 \tag{3.8}$$

$$\phi(q^4)=-\,2q.\widehat{\Theta}_{12}^{-1}tr_{L(\wedge_{(4)};\,13)}q^{L_0}+2q\widehat{\eta}_{12,4}(\tau)+\frac{(q^2,q^4,q^6;q^8)_\infty(q^{12};q^{24})_\infty(q^3;q^6)_\infty^2}{(q;q)_\infty}\times$$
$$\times\cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1+\vdots}}}}}}}\times\cfrac{1}{1-\cfrac{q^6}{1+\cfrac{q^6(1-q^6)}{1-\cfrac{q^{18}}{1+\cfrac{q^{12}(1-q^{12})}{1-\cfrac{q^{30}}{1+\cfrac{q^{18}(1-q^{18})}{1+\vdots}}}}}}} \tag{3.9}$$

$$\psi(q^4) = -q^3.\widehat{\Theta}_{12}^{-1} tr_{L(\bigwedge_{(0)};\,13)} q^{L_0} + q^3 \widehat{\eta}_{12,0}(\tau) + \frac{(q^4, q^{12}, q^{20}, q^{24}; q^{24})_\infty}{(q^8, q^{16}; q^{24})_\infty} \times$$

$$\times \frac{q^3}{(q^3;q^3)_\infty} \times \cfrac{1}{1 - \cfrac{q}{1 + \cfrac{q(1-q)}{1 - \cfrac{q^3}{1 + \cfrac{q^2(1-q^2)}{1 - \cfrac{q^5}{1 + \cfrac{q^3(1-q^3)}{1 + \vdots}}}}}}} \tag{3.10}$$

$$\phi(q) = 2q.\widehat{\Theta}_{10}^{-1} tr_{L(\bigwedge_{(1)};\,11)} q^{L_0} - 2q\widehat{\eta}_{10,1}(\tau) + \frac{j(-q^2;q^5)}{j(q^2;q^{10})} \times$$

$$\times \cfrac{1}{1 - \cfrac{q^5}{1 + \cfrac{q^5(1-q^5)}{1 - \cfrac{q^{15}}{1 + \cfrac{q^{10}(1-q^{10})}{1 - \cfrac{q^{25}}{1 + \cfrac{q^{15}(1-q^{15})}{1 + \vdots}}}}}}} \tag{3.11}$$

$$\psi(q) = 2q.\widehat{\Theta}_{10}^{-1} tr_{L(\bigwedge_{(3)};\,11)} q^{L_0} - 2q\widehat{\eta}_{10,3}(\tau) - q\frac{j(-q;q^5)}{j(q^4;q^{10})} \times$$

$$\times \cfrac{1}{1 - \cfrac{q^5}{1 + \cfrac{q^5(1-q^5)}{1 - \cfrac{q^{15}}{1 + \cfrac{q^{10}(1-q^{10})}{1 - \cfrac{q^{25}}{1 + \cfrac{q^{15}(1-q^{15})}{1 + \vdots}}}}}}} \tag{3.12}$$

$$X(-q^2) = -2q.\widehat{\Theta}_{40}^{-1} tr_{L(\bigwedge_{(18)};\,41)} q^{L_0} + 2q.\widehat{\Theta}_{40}^{-1} tr_{L(\bigwedge_{(2)};\,41)} q^{L_0} + 2q\widehat{\eta}_{40,18}(\tau) -$$

$$-2q\widehat{\eta}_{40,2}(\tau) + \frac{(j(-q^2,q^{20})j(q^{12},q^{40}) + 2q(q^{40};q^{40})_\infty^3)}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^8,q^{40})} \times$$

$$\times \cfrac{1}{1 - \cfrac{q^2}{1 + \cfrac{q^2(1-q^2)}{1 - \cfrac{q^6}{1 + \cfrac{q^4(1-q^4)}{1 - \cfrac{q^{10}}{1 + \cfrac{q^6(1-q^6)}{1 + \vdots}}}}}}} \tag{3.13}$$

$$\chi(-q^2) = -2q^3.\widehat{\Theta}_{40}^{-1} tr_{L(\bigwedge_{(14)};\,41)} q^{L_0} - 2q^5.\widehat{\Theta}_{40}^{-1} tr_{L(\bigwedge_{(6)};\,41)} q^{L_0} + 2q^3\widehat{\eta}_{40,14}(\tau) +$$

$$+2q^5\widehat{\eta}_{40,6}(\tau) + q^2\frac{(2q(q^{40};q^{40})_\infty^3 - j(-q^6,q^{20})^2 j(q^4,q^{40}))}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^{16},q^{40})} \times$$

$$\times \cfrac{1}{1-\cfrac{q^2}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^6}{1+\cfrac{q^4(1-q^4)}{1-\cfrac{q^{10}}{1+\cfrac{q^6(1-q^6)}{1+\vdots}}}}}}} \tag{3.14}$$

Proof of (3.1): To prove this result, we start with (2.5), as given below

$$f(-q) = -4q.\widehat{\Theta}_{12}^{-1}(q^{\frac{1}{2}}.tr_{L(\bigwedge_{(1)};\,13)}q^{L_0} + q^{-\frac{1}{2}}.tr_{L(\bigwedge_{(5)};\,13)}q^{L_0})\,4q\widehat{\beta}_{12,1}(\tau)+$$
$$+\frac{(q^2;q^2)_\infty^7}{(q;q)_\infty^3(q^4;q^4)_\infty^3}$$

now, we make suitable rearrangement only in the part related to q-product identities, and keep rest part is unchanged in the above expression, as

$$f(-q) = -4q.\widehat{\Theta}_{12}^{-1}(q^{\frac{1}{2}}.tr_{L(\bigwedge_{(1)};\,13)}q^{L_0} + q^{-\frac{1}{2}}.tr_{L(\bigwedge_{(5)};\,13)}q^{L_0})\,4q\widehat{\beta}_{12,1}(\tau)+$$
$$+\frac{(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^2}\times\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^3}\times\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty}\times(q^2;q^2)_\infty$$

further, using q-product identities given in (1.19),(1.20),(1.23) and (1.24), and further applying continued fractional identities given in (2.1) and (2.2), after little algebra we get (3.1).

Proof of (3.2): Proof of (3.2) is similar as (3.1), as q-products identities are same in both expression.

Proofs of (3.3) to (3.14): Proofs for (3.3) to (3.14), can be obtain on similar lines as (3.1) and (3.2) by using suitable q-product identities listed in section 2 of this paper. We are leaving it for the readers.

4 Acknowledgments

The author would like to thank to the Department of Science and Technology, Government of India, New Delhi, for the financial assistance and Centre for Mathematical Sciences for providing necessary facilities.

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Received: October 10, 2013; *Accepted*: November 11, 2013

Malaya J. Mat. 2(1)(2014) 35–42

Malaya
Journal of
Matematik

www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Note on nonparametric M-estimation for spatial regression

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Abstract

In this paper, we investigate a nonparametric robust estimation for spatial regression. More precisely, given a strictly stationary random field $Z_{\mathbf{i}} = (X_{\mathbf{i}}, Y_{\mathbf{i}})$, $\mathbf{i} \in \mathbb{N}^N$, we consider a family of robust nonparametric estimators for a regression function based on the kernel method. We establish a p-mean consistency results of the kernel estimator under some conditions.

Keywords: Quadratic error, p-mean consistency, Nonparametric regression, Spatial process, Robust estimation.

2010 MSC: 26A33, 34A08, 35R12, 47H10.

1 Introduction

In the last years spatial statistics has been widely applied in diverse areas such as climatology, epidemiology, agronomy, meteorology, econometrics, image processing, etc. There is a vast literature on spatial models, see, for example, the books by Cressie (1991), Guyon (1995), Anselin and Florax (1995), Banerjee, Carlin and Gelfand (2004), Gelfand et al. (2010) and Cressie and Wikle (2011) for broad discussion and applications. However, the nonparametric treatment of such data has so for being limited. The first results were obtained by Tran (1990). For relevant works on the nonparametric modelization of spatial data, see Biau and Cadre (2004), Carbon et al. (2007), Li et al. (2009). In this paper, we consider the problem of the estimation of the regression function as the analysis tool of such kind of data. Noting that, this model is very interesting in practice. It is used as an alternative approach to classical methods, in particular when the data are affected by the presence of outliers. There is an extensive literature on robust estimation (see, for instance Huber (1964), Robinson (1984), Collomb and Härdle (1986), Fan et al. (1994) for previous results and Boente et al. (2009) for recent advances and references). The first results concerning the nonparametric robust estimation in functional statistic were obtained by Azzedine et al. (2008). They studied the almost complete convergence of robust estimators based on a kernel method, considering independent observations. Crambes et al. (2008) stated the convergence in L_p norm in both cases (i.i.d and strong mixing). While the asymptotic normality of these estimators is proved by Attouch et al. (2010). The main goal of this paper is to study the robust nonparametric, we study L_p mean consistency results of a nonparametric estimation of the spatial regression by using the robust approach.

The paper is organized as follows. We present our model and estimator in Section 2. Section 3 is devoted to assumptions. The p-mean consistency of the robust nonparametric estimators is stated in Section 4. Proofs are provided in the appendix.

2 The model

Consider $Z_{\mathbf{i}} = (X_{\mathbf{i}}, Y_{\mathbf{i}})$, $\mathbf{i} \in \mathbb{N}^N$ be a $\mathbb{R}^d \times \mathbb{R}$-valued measurable and strictly stationary spatial process, defined on a probability space $(\Omega, \mathcal{A},)$. We assume that the process under study $(Z_{\mathbf{i}})$ is observed over a

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rectangular domain $I_{\mathbf{n}} = \{\mathbf{i} = (i_1, ..., i_N) \in \mathbb{N}^N\ ,\ 1 \leq i_k \leq n_k,\ k = 1, ..., N\}$, $\mathbf{n} = (n_1, ..., n_N) \in \mathbb{N}^N$. A point $\mathbf{i}$ will be referred to as a *site*. We will write $\mathbf{n} \to \infty$ if $\min\{n_k\} \to \infty$ and $|\frac{n_j}{n_k}| < C$ for a constant C such that $0 < C < \infty$ for all j, k such that $1 \leq j, k \leq N$. For $\mathbf{n} = (n_1, ..., n_N) \in \mathbb{N}^N$, we set $\hat{\mathbf{n}} = n_1 \times ... \times n_N$. The nonparametric model studied in this paper, denoted by θ_x, is implicitly defined, for all vectors $x \in \mathbb{R}^d$, as a zero with respect to (w.r.t.) $t \in \mathbb{R}$ of the equation

$$\Psi(x, t) = [\psi_x(Y_{\mathbf{i}}, t) | X_{\mathbf{i}} = x] = 0$$

where ψ_x is a real-valued integrable function satisfying some regularity conditions to be stated below. In what follows, we suppose that, for all $x \in \mathbb{R}^d$, θ_x exists and is unique (see, for instance, Boente and Fraiman (1989)).

For all $(x, t) \in \mathbb{R}^{d+1}$, we propose a nonparametric estimator of $\Psi(x, t)$ given by

$$\hat{\Psi}(x, t) = \frac{\sum_{\mathbf{i} \in \mathcal{I}_{\mathbf{n}}} K(h^{-1}(x - X_{\mathbf{i}}))\psi_x(Y_{\mathbf{i}}, t)}{\sum_{\mathbf{i} \in \mathcal{I}_{\mathbf{n}}} K(h^{-1}(x - X_{\mathbf{i}}))},$$

with the convention $\frac{0}{0} = 0$, where K is a kernel and $h = h_n$ is a sequence of positive real numbers. A natural estimator $\hat{\theta}_x$ of θ_x is a zero w.r.t. t of the equation

$$\hat{\Psi}(x, t) = 0.$$

In this work, we will assume that the random filed $(Z_{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^N)$ satisfies the following mixing condition:

$$\begin{cases} \text{There exists a function } \varphi\,(t) \downarrow 0 \text{ quand } t \to \infty, \text{ such that} \\ \qquad \forall\, E,\ E' \text{ subsets of } \mathbb{N}^N \text{with finite cardinals} \\ \alpha\left(\mathcal{B}\,(E)\,, \mathcal{B}\left(E'\right)\right) = \sup\limits_{B \in \mathcal{B}(E), C \in \mathcal{B}\left(E'\right)} |\,(B \cap C) - \,(B)\,\,(C)| \\ \qquad \leq s\left(\text{Card}\,(E)\,, \text{Card}\left(E'\right)\right) \varphi\left(\text{dist}\left(E, E'\right)\right), \end{cases} \tag{2.1}$$

where $\mathcal{B}\,(E)$(*resp.* $\mathcal{B}\left(E'\right)$) denotes the Borel σ-field generated by $(Z_{\mathbf{i}},\ \mathbf{i} \in E)$ (*resp.* $\left(Z_{\mathbf{i}},\ \mathbf{i} \in E'\right)$), Card$(E)$ (*resp.* Card$\left(E'\right)$) the cardinality of E (*resp.* E'), dist$\left(E, E'\right)$ the Euclidean distance between E and E' and $s : \mathbb{N}^2 \to \mathbb{R}^+$ is a symmetric positive function nondecreasing in each variable such that either

$$s\,(n, m) \leq C \min\,(n, m)\,, \quad \forall n, m \in \mathbb{N} \tag{2.2}$$

or

$$s\,(n, m) \leq C\,(n + m + 1)^{\widetilde{\beta}}\,, \quad \forall n, m \in \mathbb{N} \tag{2.3}$$

for some $\widetilde{\beta} \geq 1$ and some $C > 0$.We assume also that the process satisfies a polynomial mixing condition:

$$\varphi(t) \leq Ct^{-\theta}, \quad \theta > 0, \quad t \in \mathbb{R}. \tag{2.4}$$

3 Assumptions

From now on, let x stand for a fixed point in $\mathbb{R}^d$ and we assume that the $Z_{\mathbf{i}}$'s have the same distribution with (X, Y). Moreover, we set $f(\cdot)$ to be the density of X and $h(\cdot|x)$ the conditional density of Y given $X = x$. Consider the following hypotheses.

(H1) The functions f and h such that:

(i) The density $f(.)$ has continuous derivative in the neighborhood of x with $f(x) > 0$

(ii) For all $t \in \mathbb{R}$, the function $h(t/.)$ has continuous derivative in the neighborhood of x.

(iii) The following functions, defined for $(u,t) \in \mathbb{R}^{d+1}$ by

$$\begin{cases} g(u,t) &= (\psi_x(Y,t)/X = u)f(u) \\ \lambda(u,t) &= (\psi_x^2(Y,t)/X = u)f(u) \quad \text{and} \\ \Gamma(u,t) &= (\frac{\partial \psi_x}{\partial t}(Y,t)/X = u)f(u) \end{cases} \tag{3.5}$$

have, also, continuous derivative w.r.t. the first component.

(H2) The function ψ_x is continuous, differentiable, strictly monotone bounded w.r.t. the second component and its derivative $\dfrac{\partial \psi_x(y,t)}{\partial t}$ is bounded and continuous at θ_x uniformly in y.

(H3) The joint probability density $f_{\mathbf{i},\mathbf{j}}$ of $X_{\mathbf{i}}$ and $X_{\mathbf{j}}$ exists and satisfies

$$|f_{\mathbf{i},\mathbf{j}}(u,v) - f(u)f(v)| \leq C$$

for some constant C and for all u, v,$\mathbf{i}$ and $\mathbf{j}$.

(H4) The mixing coefficient defined in (2.2) satisfies, for some $q > 2$ and some integer $r \geq 1$

$$\lim_{T\to\infty} T^a \sum_{i=T}^{\infty} t^{Nr-1}(\varphi(t))^{qr-2/qr} = 0,$$

for some $a \geq (rq-2)Nr/(2+rq-4r)$ with $q > (4r-2)/r$.

(H5) The probability kernel function K is a symmetric and bounded density function on $\mathbb{R}^d$ with compact support, C_K, and finite variance such that
$|K(x) - K(y)| \leq M\|x-y\|$ for $x, y \in C_K$ and $0 < M < \infty$.

(H6) The individual h_n satisfy

$$\lim_{n\to\infty} h = 0 \quad \text{and} \liminf_{n\to\infty} \hat{\mathbf{n}} h^{(2(r-1)a+N(qr-2))/(a+N)q} > 0.$$

4 Main results

Let $p \in [1,+\infty[$. In this section we state a pointwise p-mean consistency result for the estimator $\hat{\theta}_x$. We start with the case where $p = 2$, we give precise asymptotic evaluations of the quadratic error of this estimator.

4.1 Mean square error

Theorem 4.1. *If the assumptions (H1)-(H6) are satisfied and if* $\Gamma(x,\theta_x) \neq 0$ *then*

$$(\hat{\theta}_x - \theta_x)^2 = B^2(x,\theta_x)h^4 + \frac{A(x,\theta_x)}{\hat{\mathbf{n}}h^d} + O\left(\frac{1}{\hat{\mathbf{n}}h^d}\right)$$

where $A(x,\theta_x) = \dfrac{\lambda(x,\theta_x)}{\Gamma(x,\theta_x)} \int K^2(t)\mathrm{d}t$ *and* $B(x,\theta_x) = \dfrac{g^{(2)}(x,\theta_x)}{\Gamma(x,\theta_x)} \int t^2 K^2(t)\mathrm{d}t$

Before giving the proof, let us introduce some notation. For $y \in \mathbb{R}$, let

$$\hat{g}(x,t) = \frac{1}{\hat{\mathbf{n}}h^d} \sum_{\mathbf{i}\in\mathcal{I}_\mathbf{n}} K(h^{-1}(x - X_{\mathbf{i}}))\psi_x(Y_{\mathbf{i}},t) \quad \text{and} \quad \hat{f}(x) = \frac{1}{\hat{\mathbf{n}}h^d} \sum_{\mathbf{i}\in\mathcal{I}_\mathbf{n}} K(h^{-1}(x - X_{\mathbf{i}})).$$

So that if $\hat{f}(x) \neq 0$ we have

$$\hat{\Psi}(x,t) = \frac{\hat{g}(x,t)}{\hat{f}(x)}$$

Proof.
A Taylor expansion of the function $\hat{g}(x,.)$ in a neighborhood of θ_x gives:

$$\hat{g}(x,\hat{\theta}_x) = \hat{g}(x,\theta_x) + (\hat{\theta}_x - \theta_x)\frac{\partial \hat{g}}{\partial t}(x,\theta^*_{x\mathbf{n}})$$

where $\theta^*_{x\mathbf{n}}$ is between $\hat{\theta}_x$ and θ_x such that

$$\hat{g}(x,\hat{\theta}_x) = g(x,\theta_x) = 0.$$

Thus, we have under the case where $\hat{f}(x) \neq 0$

$$\hat{\theta}_x - \theta_x = \frac{-\hat{g}(x,\theta_x)}{\frac{\partial \hat{g}}{\partial t}(x,\theta^*_{x\mathbf{n}})}.$$

We have by lemma 6 of Gheriballah et al (2010),

$$\frac{\partial \hat{g}}{\partial t}(x,\theta^*_{x\mathbf{n}}) - \Gamma(x,\theta_x) \rightarrow 0 \quad \text{almost completely (a.co.)}$$

It follows that

$$(\hat{\theta}_x - \theta_x)^2 = \frac{1}{\Gamma(x,\theta_x)}[(\hat{g}(x,\theta_x))^2] + o([(\hat{g}(x,\theta_x))^2]) + (\hat{f}(x) = 0)$$

Now, Theorem 4.1 is a consequence of the following intermediate results, whose proofs are given in the Appendix.

Lemma 4.1. *Under Hypotheses (H1) and (H2)-(H4), we have,*

$$Var\left[\left(\hat{g}(x,\theta_x)\right)\right] = \frac{A(x,\theta_x)}{\hat{\mathbf{n}}h^d} + O\left(\frac{1}{\hat{\mathbf{n}}h^d}\right).$$

Lemma 4.2. *Under Hypotheses (H1) and (H2)-(H4), we have,*

$$\left[\left(\hat{g}(x,\theta_x)\right)\right] = B(x,\theta_x)h^2 + o(h^2).$$

Lemma 4.3. *Under the conditions of Theorem 4.1, we have*

$$(\hat{f}(x) = 0) = O\left(\frac{1}{\hat{\mathbf{n}}h^d}\right)^p.$$

4.2 Convergence in L_p norm

Theorem 4.2. *Under conditions (H1)-(H6) and if* $\Gamma(x,\theta_x) \neq 0$ *we get*

$$\|\hat{\theta}_x - \theta_x\|_p = O\left(h^2\right) + O\left(\frac{1}{\hat{\mathbf{n}}h^d}\right)^{1/2}$$

where $\|.\|_p$ *is the norm* L_p

Proof. We prove the case where $p = 2r$ (for all $r \in \mathbb{N}^*$) and we use the Holder inequality for lower values of p. Moreover, we use the same analytical arguments as those used in previous theorem, we have

$$\|\hat{\theta}_x - \theta_x\|_{2r} \leq C\|\hat{g}(x,\theta_x)\|_{2r} + \|o(\,[\hat{g}(x,\theta_x)])\|_{2r}. \tag{4.6}$$

Furthermore, we write

$$\|\hat{g}(x,\theta_x)\|_{2r} = \frac{1}{\hat{\mathbf{n}}h^d}\left(\left[\left(\sum_{\mathbf{i}\in\mathcal{I}_\mathbf{n}} \xi_\mathbf{i}\right)^{2r}\right]\right)^{1/2r}.$$

where $\xi_\mathbf{i} = K_\mathbf{i}\psi_x(Y_\mathbf{i},\theta_x) = K_\mathbf{i}[\psi_x(Y_\mathbf{i},\theta_x) - [\psi_x(Y_\mathbf{i},\theta_x)/X_\mathbf{i} = x]]$ with $K_\mathbf{i} = K(h^{-1}(x - X_\mathbf{i}))$.
Therefore, the first term of (4.6) is a consequence of the application of Theorem 2.2 of (Gao et al. 2008, P. 689) on $\xi_\mathbf{i}$ while the second one is given in Lemma 4.2.

5 Appendix

Proof of Lemma 4.1. Let $\Delta_{\mathbf{i}}(x) = K(h^{-1}(x - X_{\mathbf{i}}))\psi_x(Y_{\mathbf{i}}, \theta_x)$, then

$$
\begin{aligned}
\operatorname{Var}\left(\hat{g}(x,\theta_x)\right) &= \operatorname{Var}\left(\frac{1}{\hat{\mathbf{n}}h^d}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\left(K_{\mathbf{i}}\psi_x(Y_{\mathbf{i}},\theta_x)\right)\right)\\
&= \frac{1}{\hat{\mathbf{n}}^2h^{2d}}\operatorname{Var}\left(\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\Delta_{\mathbf{i}}\right)\\
&= \frac{1}{\hat{\mathbf{n}}h^{2d}}\operatorname{Var}(\Delta_{\mathbf{i}}) + \frac{1}{\hat{\mathbf{n}}^2h^{2d}}\sum_{\mathbf{i}\neq\mathbf{j}\in\mathcal{I}_{\mathbf{n}}}\left|\operatorname{Cov}(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}})\right|
\end{aligned}
$$

Concerning the variance term, we have

$$
\operatorname{Var}(\Delta_{\mathbf{i}}) = [\Delta_{\mathbf{i}}]^2 - {}^2[\Delta_{\mathbf{i}}]
$$

By the stationarity of the observations $(X_{\mathbf{i}}, Y_{\mathbf{i}})$ we get, firstly,

$$
\begin{aligned}
[\Delta_{\mathbf{i}}] &= [K(h^{-1}(x - X_{\mathbf{1}}))\psi_x(Y_{\mathbf{1}},\theta_x)]\\
&= [K(h^{-1}(x - X_{\mathbf{1}}))(\psi_x(Y_{\mathbf{1}},\theta_x)|X = X_{\mathbf{1}})]\\
&= \int_{\mathbb{R}^d} K(h^{-1}(x - z))(\psi_x(Y_{\mathbf{1}},\theta_x)|X = X_{\mathbf{1}})f(z)\mathrm{d}z.
\end{aligned}
$$

Next, by a classical change of variables, $u = h^{-1}(x - z)$ we write

$$
[\Delta_{\mathbf{i}}] = h^d \int_{\mathbb{R}^d} K(u)g(x - hu, \theta_x)du
$$

and by the Taylor expansion, under (H1), we obtain

$$
[\Delta_{\mathbf{i}}] = h^d g(x,\theta_x)\int_{\mathbb{R}^d} K(u)du + o(h^d) = o(h^d) \quad \text{since} \qquad g(x,\theta_x) = 0
$$

Secondly, by a similar arguments, we have

$$
\begin{aligned}
[\Delta_{\mathbf{i}}]^2 &= [K^2(h^{-1}(x - X_{\mathbf{1}}))\psi_x^2(Y_{\mathbf{1}},\theta_x)]\\
&= [K^2(h^{-1}(x - X_{\mathbf{1}}))(\psi_x^2(Y_{\mathbf{1}},\theta_x)|X = X_{\mathbf{1}})]\\
&= \int_{\mathbb{R}^d} K^2(h^{-1}(x - z))(\psi_x^2(Y_{\mathbf{1}},\theta_x)|X_{\mathbf{1}} = z)f(z)\mathrm{d}z\\
&= h^d\lambda(x,\theta_x)\int_{\mathbb{R}^d} K^2(u)\mathrm{d}u + o(h^d).
\end{aligned}
$$

Hence,

$$
\operatorname{Var}(\Delta_{\mathbf{i}}) = A(x,\theta_x)h^d + o(h^d). \tag{5.7}
$$

Now, to evaluate the second part, denoted by $R_{\mathbf{n}} = \sum_{\mathbf{i}\neq\mathbf{j}\in\mathcal{I}_{\mathbf{n}}}\left|\operatorname{Cov}(\Delta_{\mathbf{i}}(x),\Delta_{\mathbf{j}}(x))\right|$, we divide the rectangular region $\mathcal{I}_{\mathbf{n}}$ into two sets.

$$
S_1 = \{\mathbf{i},\mathbf{j} \in \mathcal{I}_{\mathbf{n}} : 0 < \|\mathbf{i} - \mathbf{j}\| \leq c_{\mathbf{n}}\},\ S_2 = \{\mathbf{i},\mathbf{j} \in \mathcal{I}_{\mathbf{n}} : \|\mathbf{i} - \mathbf{j}\| > c_{\mathbf{n}}\},
$$

where $c_{\mathbf{n}}$ is a real sequence that converges to infinity and will be made precise later:

$$
\begin{aligned}
R_{\mathbf{n}} &= \sum_{(\mathbf{i},\mathbf{j})\in S_1}\left|\operatorname{Cov}\left(\Delta_{\mathbf{i}}(x),\Delta_{\mathbf{j}}(x)\right)\right| + \sum_{(\mathbf{i},\mathbf{j})\in S_2}\left|\operatorname{Cov}\left(\Delta_{\mathbf{i}}(x),\Delta_{\mathbf{j}}(x)\right)\right|\\
&= R_{\mathbf{n}}^1 + R_{\mathbf{n}}^2.
\end{aligned}
$$

On the one hand, on S_1 we have, under (H2):

$$\begin{aligned} \left|Cov\left(\Delta_{\mathbf{i}}(x),\Delta_{\mathbf{j}}(x)\right)\right| &\leq C\left|\left[K_{\mathbf{i}}K_{\mathbf{j}}\right]-\left[K_{\mathbf{i}}\right]\left[K_{\mathbf{j}}\right]\right| \\ &\leq C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(h^{-1}(x-v))|f_{\mathbf{ij}}(u,v)-f(u,v)|dudv \\ &\leq C\left(\int_{\mathbb{R}^d} K(h^{-1}(x-v))du\right)^2 \\ &\leq Ch^{2d}. \end{aligned}$$

$$\begin{aligned} R_{\mathbf{n}}^1 &= \sum_{(\mathbf{i},\mathbf{j})\in S_1}\left|\left[K_{\mathbf{i}}K_{\mathbf{j}}\right]-\left[K_{\mathbf{i}}\right]\left[K_{\mathbf{j}}\right]\right| \\ &\leq C\hat{\mathbf{n}}c_{\mathbf{n}}^N h^{2d}. \end{aligned}$$

On the other hand, on S_2 we apply Lemma 2.1(ii) of Tran(1990) and we deduce that

$$\left|Cov\left(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}\right)\right| \leq C\varphi\left(\|\mathbf{i}-\mathbf{j}\|\right). \tag{5.8}$$

By (5.8) we have for some $v>N$

$$\begin{aligned} R_{\mathbf{n}}^2=\sum_{(\mathbf{i},\mathbf{j})\in S_2}\left|Cov\left(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}\right)\right| &\leq C\sum_{(\mathbf{i},\mathbf{j})\in S_2}\varphi\left(\|\mathbf{i}-\mathbf{j}\|\right)\leq C\hat{\mathbf{n}}\sum_{\mathbf{i}:\|\mathbf{i}\|\geq c_{\mathbf{n}}}\varphi\left(\|\mathbf{i}\|\right) \\ &\leq C\hat{\mathbf{n}}c_{\mathbf{n}}^{-v}\sum_{\mathbf{i}:\|\mathbf{i}\|\geq c_{\mathbf{n}}}\|\mathbf{i}\|^v\varphi\left(\|\mathbf{i}\|\right). \end{aligned}$$

Taking $c_{\mathbf{n}}=h^{-d/v}$, we see

$$\begin{aligned} R_{\mathbf{n}}^2 &\leq C\hat{\mathbf{n}}h^d\sum_{\mathbf{i}:\|\mathbf{i}\|\geq c_{\mathbf{n}}}\|\mathbf{i}\|^v\varphi\left(\|\mathbf{i}\|\right) \\ &\leq C\hat{\mathbf{n}}h^d\sum_{\mathbf{i}:\|\mathbf{i}\|\geq c_{\mathbf{n}}}\|\mathbf{i}\|^v\varphi\left(\|\mathbf{i}\|\right). \end{aligned}$$

Employing (H4) and the fact that $a^{-1}>2$ choose v positive numbers such that

$$\sum_{\mathbf{i}}\|\mathbf{i}\|^v\varphi\left(\|\mathbf{i}\|\right)<\infty \quad \text{and} \quad v>N.$$

Which allows us to write

$$R_{\mathbf{n}}^2=o(\hat{\mathbf{n}}h^d) \quad \text{and} \quad R_{\mathbf{n}}^1=o(\hat{\mathbf{n}}h^d)$$

Finally

$$R_{\mathbf{n}}=o\left(\frac{1}{\hat{\mathbf{n}}h^d}\right).$$

Combining the last result together with equation (5.7) we derive

$$Var\left(\hat{g}(x,\theta_x)\right)=\frac{A(x,\theta_x)}{\hat{\mathbf{n}}h^d}+o\left(\frac{1}{\hat{\mathbf{n}}h^d}\right)$$

Proof of Lemma 4.2. Keeping the notation of previous lemma, we write

$$\begin{aligned} \left[\hat{g}(x,\theta_x)\right] &= \left[\frac{1}{\hat{\mathbf{n}}h^d}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\left(K_{\mathbf{i}}\psi_x(Y_{\mathbf{i}},\theta_x)\right)\right] \\ &= \frac{1}{h^d}\left[\Delta_{\mathbf{i}}\right] \\ &= \frac{1}{h^d}\int_{\mathbb{R}^d} K(h^{-1}(x-z))(\psi_x(Y_{\mathbf{1}},\theta_x)/X_{\mathbf{1}}=z)f(z)dz. \end{aligned}$$

Next, by the classical change of variables, $u = h^{-1}(x - z)$ we have

$$\frac{1}{h^d}[\Delta_{\mathbf{i}}] = \int_{\mathbb{R}^d} K(u) g(x - hu, \theta_x) du$$

Using a Taylor expansion of order two , under (H1), we obtain

$$\left[\hat{g}(x, \theta_x)\right] = B(x, \theta_x) h^2 + o(h^d) = o(h^d).$$

Proof of Lemma 4.3. We have

$$\begin{array}{rcl} (\hat{f}(x) = 0) & = & (\hat{f}(x) \leq f(x) - \epsilon) \\ & \leq & (|\hat{f}(x) - f(x)| \geq \epsilon) \end{array}$$

The Markov's inequality allows to get, for any $p > 0$,

$$(\hat{f}(x) = 0) \leq \frac{(|\hat{f}(x) - f(x)|)^p}{\epsilon^p}$$

This yields the proof.

6 Acknowledgment

The authors wish to thank the referee for helpful comments.

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Received: February 28, 2013; *Accepted*: November 25, 2013

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(1)(2014) 43–48

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

An application of Lauricella hypergeometric functions to the generalized heat equations

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Abstract

In the recent paper, we give a formal solution of a certain one dimensional time fractional homogeneous conduction heat equation. This equation and its solution impose a rise to new forms of generalized fractional calculus. The new solution involves the Lauricella hypergeometric function of the third type. This type of functions is utilized to explain the probability of thermal transmission in random media. We introduce the analytic form of the thermal distribution related to such Lauricella function.

Keywords: Fractional calculus, fractional differential equations, analytic function

2010 MSC: 26A33, 34A08 .

1 Introduction

The fractional heat conduction equation is studied by Povstenko in 2004 [1]. He proposed a quasi-static uncoupled theory of thermoelasticity based on the heat conduction equation with a time-fractional derivative. Later he focused on the heat conduction with time and space fractional derivatives and on the theory of thermal stresses based on this equation [2, 3]. Recently, Povstenko [4] obtained a solution of these equations by applying Laplace and Weber integral transforms. Furthermore, he formulated fundamental solutions to the central symmetric space-time fractional heat conduction equation and associated thermal stresses [5].

Newly, Li et. al., described heat conduction in fractal media, such as polar bear hair, wool fibers and goose down. By employing the modified Riemann-Liouville derivative, a fractional complex transform is used to convert time-fractional heat conduction equations into ordinary differential equations, therefore, precise solutions can be easily obtained [6]. At the same time, the authors generalized the fractional complex transform to obtain accurate solutions for time-fractional differential equations with the modified Riemann-Liouville derivative [7]. Yang and Baleanu posed a local fractional variational iteration method for processing the local fractional heat conduction equation accruing in fractal heat transfer [8]. Sherief and Latief created the problem for a half-space formed of a material with variable thermal conductivity [9].

In this work, we utilize a Lauricella type function to describe the time evolution of the fractional heat equation. We find the analytic form of these equations related to such Lauricella function. The fractional calculus is taken in sense of the Caputo derivative. The advantage of Caputo fractional derivative is that the derivative of a constant is zero, whereas for the Riemann- Liouville is not. Moreover, Caputo's derivative requests higher conditions of regularity for differentiability which allows us to geometrize various physical problems with fractional order.

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Finally, the advantage of Caputo fractional derivative is that the fractional differential equations with Caputo fractional derivative use the initial conditions (including the mixed boundary conditions) on the same character as for the integer-order differential equations [10].

2 Calculus of arbitrary order

This section concerns with some preliminaries and notations regarding the Caputo operator. The Caputo fractional derivative strongly poses the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations.

Definition 2.1 The fractional order integral of the function h of order $\alpha > 0$ is defined by

$$I_a^\alpha h(t) = \int_\wp^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau) d\tau.$$

When $\wp = 0$, we write $I_\wp^\alpha h(t) = h(t) * \psi_\alpha(t)$, where $(*)$ denoted the convolution product, $\psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\psi_\alpha(t) = 0,\ t \le 0$ and $\psi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2 The Riemann-Liouville fractional order derivative of the function h of order $0 \le \alpha < 1$ is defined by

$$D_\wp^\alpha h(t) = \frac{d}{dt} \int_\wp^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} h(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} h(t).$$

Remark 2.1 From Definition 2.1 and Definition 2.2, $\wp = 0$, we have

$$D^\alpha t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha},\ \ \nu > -1;\ 0 < \alpha < 1$$

and

$$I^\alpha t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} t^{\nu+\alpha},\ \ \nu > -1;\ \alpha > 0.$$

Definition 2.3 The Caputo fractional derivative of order $\alpha > 0$ is defined, for a analytic function $h(t)$ by

$${}^c D^\alpha h(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(\zeta)}{(t-\zeta)^{\alpha-n+1}} d\zeta,$$

where $n = [\alpha] + 1$, (the notation $[\alpha]$ stands for the largest integer not greater than α). In the sequel, we shall use the notation

$${}^c D^\alpha h(t) := \frac{\partial^\alpha h(t)}{\partial t^\alpha}.$$

3 Generalized heat equation

Consider the two dimensional time fractional homogeneous heat conduction equation of the form

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \delta(T_{xx} + T_{yy}) \tag{3.1}$$

$$(\delta > 0,\ t > 0,\ 0 < x, y < 1,\ 0 < \alpha \le 1),$$

where:

- $T = T(x, y, t)$ is temperature as a function of space and time
- $\frac{\partial^\alpha T}{\partial t^\alpha}$ is the rate of change of temperature at a point over time
- δ is the thermal diffusivity

By using the fractional complex transform [7],

$$\zeta = \frac{\phi t^{\alpha}}{\Gamma(1+\alpha)} + \psi x + \kappa y,$$

it was shown that the exact solution of (3.1) can be expressed as

$$T(x,y,t) = c_1 + c_2 \exp\big(\frac{\phi\psi x}{\delta(\psi^2+\kappa^2)} + \frac{\phi\kappa y}{\delta(\psi^2+\kappa^2)} + \frac{\phi^2 t^{\alpha}}{\delta\Gamma(1+\alpha)(\psi^2+\kappa^2)}\big).$$

For special case we may have a solution of the form

$$T(x,y,t) = \exp\big(-\frac{\phi\psi x}{\delta(\psi^2+\kappa^2)} - \frac{\phi\kappa y}{\delta(\psi^2+\kappa^2)} - \frac{\phi^2 t^{\alpha}}{\delta\Gamma(1+\alpha)(\psi^2+\kappa^2)}\big). \tag{3.2}$$

In [11], the authors generalized the fractional probability of extinction, by applying the Caputo fractional derivative of one parameter as follows:

$$P_{\mu}(k,z) = \frac{(\nu z^{\mu})^k}{k!}\sum_{n=0}^{\infty}\frac{(n+k)!}{n!}\frac{(-\nu z^{\mu})^n}{\Gamma(\mu(n+k)+1)} \tag{3.3}$$

and the probability of transmission

$$P_{\mu}(0,z) = \sum_{n=0}^{\infty}\frac{(-\nu z)^n}{\Gamma(\mu n+1)} = E_{\mu}(-\nu z^{\mu}),$$

where

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+n\alpha)}$$

is the Mittag-Leffler function and its popularity increased significantly due to its important role in applications and fractional of arbitrary orders related differential and integral equations of fractional order, solutions of problems of control theory, fractional viscoelastic models, diffusion theory, continuum mechanics and fractals [10].

Newly, numerical routines for Mittag-Leffler functions have been developed, e.g., by Freed et al. [12], Gorenflo et al. [13] (with MATHEMATICA), Podlubny [14] (with MATLAB), Seybold and Hilfer [15].

Here, we generalize probability of extinction, using the fractional Poisson process of three variables as follows:

$$\begin{aligned} P_{\mu,\beta,\sigma}(k,z,w,u) =& \frac{(\nu z)^k}{k!}\frac{(\rho w)^k}{k!}\frac{(\sigma u)^k}{k!} \\ &\times \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(n+k)!}{n!}\frac{(j+k)!}{j!} \\ &\times \frac{(m+k)!}{m!}\frac{(-\nu z^{\mu})^n}{\Gamma(\mu(n+k)+1)}\frac{(-\rho w^{\beta})^j}{\Gamma(\beta(j+k)+1)} \\ &\times \frac{(-\sigma u^{\gamma})^m}{\Gamma(\gamma(m+k)+1)}; \end{aligned} \tag{3.4}$$

thus the 3-D probability of transmission becomes

$$P_{\mu,\beta,\sigma}(0,z,w,u) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\nu z^{\mu})^n}{\Gamma(\mu n+1)}\frac{(-\rho w^{\beta})^j}{\Gamma(\beta j+1)}\frac{(-\sigma u^{\gamma})^m}{\Gamma(\gamma m+1)}$$

$$= E_{\mu,\beta,\gamma}(-\nu z^{\mu}, -\rho w^{\beta}, -\sigma u^{\gamma}),$$

where $E_{\mu,\beta,\gamma}(-\nu z^{\mu}, -\rho w^{\beta}, -\sigma u^{\gamma})$ is a multi-index Mittag-Leffler function, which can be found in [10].

Our approach depends on the Lauricella hypergeometric function of third type of three variables, which can be defined by

$$F_A^{(3)}(a,b_1,b_2,b_3,c_1,c_2,c_3;x_1,x_2,x_3)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{(a)_{i_1+i_2+i_3}(b_1)_{i_1}(b_2)_{i_2}(b_3)_{i_3}}{(c_1)_{i_1}(c_2)_{i_2}(c_3)_{i_3}\,i_1!\,i_2!\,i_3!}\,x_1^{i_1}x_2^{i_2}x_3^{i_3}$$

for $|x_1|+|x_2|+|x_3|<1$ and

$$F_B^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3,c;x_1,x_2,x_3)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{(a_1)_{i_1}(a_2)_{i_2}(a_3)_{i_3}(b_1)_{i_1}(b_2)_{i_2}(b_3)_{i_3}}{(c)_{i_1+i_2+i_3}\,i_1!\,i_2!\,i_3!}\,x_1^{i_1}x_2^{i_2}x_3^{i_3}$$

for $|x_1|<1,|x_2|<1,|x_3|<1$ and

$$F_C^{(3)}(a,b,c_1,c_2,c_3;x_1,x_2,x_3)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{(a)_{i_1+i_2+i_3}(b)_{i_1+i_2+i_3}}{(c_1)_{i_1}(c_2)_{i_2}(c_3)_{i_3}\,i_1!\,i_2!\,i_3!}\,x_1^{i_1}x_2^{i_2}x_3^{i_3}$$

for $|x_1|\frac{1}{2}+|x_2|\frac{1}{2}+|x_3|\frac{1}{2}<1$ and

$$F_D^{(3)}(a,b_1,b_2,b_3,c;x_1,x_2,x_3)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{(a)_{i_1+i_2+i_3}(b_1)_{i_1}(b_2)_{i_2}(b_3)_{i_3}}{(c)_{i_1+i_2+i_3}\,i_1!\,i_2!\,i_3!}\,x_1^{i_1}x_2^{i_2}x_3^{i_3}.$$

The notation $(x)_n$ refers to the Pochhammer symbol

$$(x)_n=x(x+1)(x+2)\cdots(x+n-1)$$

or in gamma function

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)},\quad (x)_n=\frac{(-1)^n\Gamma(1-x)}{\Gamma(1-x-n)}.$$

For special case, we obtain

$$F_D^{(3)}(\alpha,c,c,c,c;-\widehat{\psi}x,-\widehat{\kappa}y,-\widehat{\phi}t^{\alpha})=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}(\alpha)_{i_1+i_2+i_3}\frac{(-\widehat{\psi}x)^{i_1}(-\widehat{\kappa}y)^{i_2}(-\widehat{\phi}t^{\alpha})^{i_3}}{i_1!\,i_2!\,i_3!},$$

where $\widehat{\psi}$, $\widehat{\kappa}$ and $\widehat{\phi}$ are the coefficients in Eq.(3.2).

Now we proceed to define a Lauricella hypergeometric functions in the positive semi-space in order to introduce the probability of heat distribution. For non negative variables X,Y,T, we may describe the following distribution:

$$\Omega(x,y,t):=P(X\le x,Y\le y,T\le t)=1-F_D^{(3)}(\alpha,c,c,c,c;-\widehat{\psi}x,-\widehat{\kappa}y,-\widehat{\phi}t^{1/\alpha}), \tag{3.5}$$

$$(t,x,y\ge 0,\ \alpha\in(0,1]).$$

Also we define

$$\begin{aligned}\Theta(x,y,t)&:=P(X>x,Y>y,T>t)=1-\Omega(x,y,t)\\&:=\overline{F}_D^{(3)}(\alpha,c,c,c,c;-\widehat{\psi}x,-\widehat{\kappa}y,-\widehat{\phi}t^{1/\alpha}).\end{aligned} \tag{3.6}$$

Physically, the above equations correspond to the probability of heat transmission. We impose the following result

Theorem 3.1 Assume Ω and Θ as in (3.5) and (3.6) respectively. Then Eq.(3.1) has a solution in terms of Lauricella hypergeometric functions.

Proof. The probability density function corresponding to (3.1) can be written by

$$\omega(x,y,t) = \frac{\partial^3}{\partial x \partial y \partial t}\Omega(x,y,t)$$

with

$$\int_D \omega(x,y,t)dxdydt = 1, \quad D \in \mathbb{R}^3.$$

The probability of the transmission, during time t in 2-dimensional space (x,y), of object can be related to (x',y',t') to be between (x,y,t) and $(x+dx, y+dy, t+dt)$. It is read by

$$\partial^3 P(x,y,t) = |\frac{\partial^3\Theta(x,y,t)}{\partial x\partial y\partial t}|(\partial x\partial y\partial t).$$

Thus we pose that

$$\begin{aligned}\frac{\partial^3\Theta(x,y,t)}{\partial x\partial y\partial t}(\partial x\partial y\partial t) &= \frac{\partial}{\partial x\partial y\partial t}\Big(\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}(\alpha)_{i_1+i_2+i_3}\\ &\times \frac{(-\widehat{\psi}x)^{i_1}(-\widehat{\kappa}y)^{i_2}(-\widehat{\phi}t^{1/\alpha})^{i_3}}{i_1!\, i_2!\, i_3!}\Big)\partial x\partial y\partial t\\ &= -\frac{\widehat{\psi}\widehat{\kappa}\widehat{\phi}\, t^{\frac{1}{\alpha}-1}(\alpha)_3}{\alpha}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}(\alpha)_{i_1+i_2+i_3}\\ &\times \frac{(-\widehat{\psi}x)^{i_1}(-\widehat{\kappa}y)^{i_2}(-\widehat{\phi}t^{1/\alpha})^{i_3}}{i_1!\, i_2!\, i_3!}\partial x\partial y\partial t\\ &= -\widehat{\psi}\widehat{\kappa}\widehat{\phi}\, t^{\frac{1}{\alpha}-1}(\alpha+1)_2\ F_D^{(3)}(\alpha,c,c,c,c;-\widehat{\psi}x,-\widehat{\kappa}y,-\widehat{\phi}t^{1/\alpha})\ \partial x\partial y\partial t\end{aligned} \tag{3.7}$$

Now for a function $f(x,y,z)$ has a series expansion of the form

$$f(x,y,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_1(i)\lambda_2(j)\lambda_3(k)\frac{(-1)^i x^i}{i!}\frac{(-1)^j y^j}{j!}\frac{(-1)^k z^k}{k!},$$

with

$$\lambda_1(0) \neq 0, \quad \lambda_2(0) \neq 0, \quad \lambda_3(0) \neq 0,$$

then

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty x^{\alpha_1-1}y^{\alpha_2-1}z^{\alpha_3-1}f(x,y,z)dxdydz =\\ &\qquad \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\lambda_1(-\alpha_1)\lambda_2(-\alpha_2)\lambda_3(-\alpha_3).\end{aligned} \tag{3.8}$$

The last assertion is called the generalized Ramanujan Master Theorem. By applying (3.8) in (3.7), where

$$\lambda_1 = (\alpha)_{i_1}, \quad \lambda_2 = (\alpha)_{i_2}, \quad \lambda_3 = (\alpha)_{i_3}$$

and that

$$\lambda_1(0) = 1, \quad \lambda_2(0) = 1, \quad \lambda_3(0) = 1,$$

we have a solution of (3.1) which is in terms of Lauricella hypergeometric functions. □

4 Conclusion

Distributions of barriers do not elaborate in some physical states, e.g. in media with locative interconnections between particles. In this work, we utilized the 3- D fractional derivative Poisson process which can be viewed as a utility tool to put into account long domain interconnections between particles in the medium. In addition, we applied the generalized Lauricella hypergeometric functions to give various methods of the probability of the heat transmission, depending on the renewal process. Our main result showed a solution of 2- D fractional heat equation in terms of the Lauricella hypergeometric functions based on the generalized Ramanujan Master Theorem.

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Received: November 3, 2013; *Accepted*: December 5, 2013

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(1)(2014) 49–60

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Functional equation originating from sum of higher powers of arithmetic progression using difference operator is stable in Banach space: direct and fixed point methods

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Abstract

In this paper, the authors has proved the solution of a new type of functional equation

$$f\left(\sum_{j=1}^{k} j^p\, x_j\right) = \sum_{j=1}^{k} \left(j^p\, f(x_j)\right), \qquad k, p \geq 1$$

which is originating from sum of higher powers of an arithmetic progression. Its generalized Ulam - Hyers stability in Banach space using direct and fixed point methods are investigated. An application of this functional equation is also studied.

Keywords: Additive functional equations, stirling numbers, polynomial factorial, difference operator, generalized Ulam - Hyers stability, fixed point.

2010 MSC: 39B52, 39B72, 39B82.

1 Introduction

During the last seven decades, the perturbation problems of several functional equations have been extensively investigated by a number of authors [1, 2, 12, 13, 20, 21, 26, 28] . The terminology generalized Ulam - Hyers stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [5, 8, 9, 10, 14, 15, 16, 22, 23, 24, 27].

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function.

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The solution and stability of the following various additive functional equations

$$f(2x-y)+f(x-2y)=3f(x)-3f(y), \tag{1.2}$$
$$f(x+y-2z)+f(2x+2y-z)=3f(x)+3f(y)-3f(z), \tag{1.3}$$
$$f(m(x+y)-2mz)+f(2m(x+y)-mz)=3m[f(x)+f(y)-f(z)]\ m\geq 1, \tag{1.4}$$
$$f\left(a\sum_{i=1}^{n-1}x_i-2ax_n\right)+f\left(2a\sum_{i=1}^{n-1}x_i-ax_n\right)=3a\left(\sum_{i=1}^{n-1}f(x_i)-f(x_n)\right)\ n\geq 3, \tag{1.5}$$
$$f(2x\pm y\pm z)=f(x\pm y)+f(x\pm z) \tag{1.6}$$
$$f(qx\pm y\pm z)=f(x\pm y)+f(x\pm z)+(q-2)f(x),\ \ q\geq 2 \tag{1.7}$$

were discussed by D.O. Lee [11], K. Ravi, M. Arunkumar [25], M. Arunkumar [3, 4].

Also M. Arunkumar et. al., [7] investigated the generalized Ulam-Hyers stability of a functional equation

$$f\,(y)=\frac{f\,(y+z)+f\,(y-z)}{2}$$

which is originating from arithmetic mean of consecutive terms of an arithmetic progression using direct and fixed point methods. Infact M. Arunkumar et. al.,[6] has proved the solution and generalized Ulam - Hyers - Rassias stability of a n dimensional additive functional equation

$$f(x)=\sum_{\ell=1}^{n}\left(\frac{f\,(x+\ell y_\ell)+f\,(x-\ell y_\ell)}{2\,\ell}\right)$$

where n is a positive integer, which is originating from arithmetic mean of n consecutive terms of an arithmetic progression.

In this paper, the authors established the solution and the generalized Ulam - Hyers stability of a new type of additive functional equation

$$f\left(\sum_{j=1}^{k}j^p\,x_j\right)=\sum_{j=1}^{k}\left(j^p\,f(x_j)\right),\qquad k,p\geq 1 \tag{1.8}$$

which is originating from sum of higher powers of an arithmetic progression. An application of this functional equation is also studied.

In Section 2, some basic preliminaries about difference operator is discussed. In Section 3, the general solution of the functional equation (1.8) is given. In Section 4 and 5, the generalized Ulam - Hyers stability of the additive functional equation (1.8) using direct and fixed point methods are respectively proved. An application of the additive functional equation (1.8) is discussed in Section 6.

2 Basic preliminaries on difference operator

Definition 2.1. *[18] If $\{y_k\}$ is a sequence of numbers, then we define the difference operator Δ as*

$$\Delta(y_k)\ =\ y_{k+1}-y_k. \tag{2.1}$$

Lemma 2.1. *[18] From (2.1) and the shift relation, $E(y_k)\ =\ y_{k+1}$, we obtain*

$$E=\Delta+1. \tag{2.2}$$

Definition 2.2. *[18] If n is positive integer, then the positive polynomial factorial is defined as*

$$k^{(n)}=k(k-1)(k-2)...(k-(n-1)). \tag{2.3}$$

Lemma 2.2. *[18] If S_r^n's are the Stirling numbers of second kind, then*

$$k^n=\sum_{r=1}^{n}S_r^n k^{(r)}. \tag{2.4}$$

Definition 2.3. *[18] For the positive integer n, the inverse operators are defined as if*

$$\Delta^n(z_k) = y_k, \text{ then } z_k = \Delta^{-n}(y_k). \tag{2.5}$$

Lemma 2.3. *[18] If m, k are positive integers and $k > m$, then*

$$\Delta^{-1}k^{(m)} = \frac{k^{(m+1)}}{(m+1)} + c, \qquad \text{where } c \text{ is constant.} \tag{2.6}$$

Theorem 2.1. *[18] If k is positive integer, then*

$$\Delta^{-1}(y_k) = \sum_{r=1}^{k} y_{(k-r)} + c, \qquad \text{where } c \text{ is constant.} \tag{2.7}$$

Theorem 2.2. *If k and p are positive integers then*

$$\sum_{r=1}^{p}(k+1-r)^p = \sum_{r=1}^{n} S_r^n k^{(p)} \frac{[k+1]^{(r+1)}}{[r+1]}. \tag{2.8}$$

Proof. The proof follows by Lemmas 2.2, 2.3 and Theorem 2.1. □

3 General solution of the functional equation(1.8)

In this section, the general solution of the functional equation (1.8) is given.

Theorem 3.3. *Let X and Y be real vector spaces. The mapping $f : X \to Y$ satisfies the functional equation (1.1) for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation (1.8) for all $x_1, x_2, \cdots, x_k \in X$.*

Proof. The proof follows by the additive property. □

Hereafter though out this paper, let us consider X and Y to be a normed space and a Banach space, respectively.

4 Stability results: Direct method

In this section, the generalized Hyers - Ulam - Rassias stability of the additive functional equation (1.8) is provided.

Theorem 4.4. *Let $i \in \{-1, 1\}$ and $\alpha : X^k \to [0, \infty)$ be a function such that*

$$\sum_{t=0}^{\infty} \frac{\alpha\left[\wp^{ti}x_1, \wp^{ti}x_2, \cdots, \wp^{ti}x_k\right]}{\wp^{ti}} \quad \text{converges in } \mathbb{R} \tag{4.1}$$

for all $x_1, x_2, x_3 \cdots, x_k \in X$. Let $f : X \to Y$ be a function satisfying the inequality

$$\left\| f\left[\sum_{j=1}^{k} j^p\, x_j\right] - \sum_{j=1}^{k}\left[j^p\, f[x_j]\right] \right\| \le \alpha\left[x_1, x_2, x_3 \cdots, x_k\right] \tag{4.2}$$

for all $x_1, x_2, x_3 \cdots, x_k \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equation (1.8) and

$$\|f[x] - A[x]\| \le \frac{1}{\wp} \sum_{s=\frac{1-i}{2}}^{\infty} \frac{\alpha[\wp^{si}x, \wp^{si}x, \cdots, \wp^{si}x]}{\wp^{si}} \tag{4.3}$$

where

$$\wp = \sum_{r=1}^{p} S_r^p \frac{[k+1]^{(r+1)}}{[r+1]} \tag{4.4}$$

for all $x \in X$. The mapping $A[x]$ is defined by

$$A[x] = \lim_{t\to\infty} \frac{f[\wp^{ti}x]}{\wp^{ti}} \tag{4.5}$$

for all $x \in X$.

Proof. Assume $i = 1$. Replacing $[x_1, x_2, \cdots, x_k]$ by $[x, x, \cdots, x]$ in (4.2), we get

$$\|f[[1^p + 2^p + \cdots + k^p]x] - [1^p + 2^p + \cdots + k^p] f[x]\| \leq \alpha[x, x, \cdots, x] \tag{4.6}$$

for all $x \in X$. The above equation can be rewritten as

$$\left\| f\left[\left\{\sum_{r=1}^{k}[k+1-r]^p\right\} x\right] - \left[\sum_{r=1}^{k}[k+1-r]^p\right] f[x]\right\| \leq \alpha[x, x, \cdots, x] \tag{4.7}$$

for all $x \in X$. Using Theorem 2.2, we have

$$\left\| f\left[\left\{\sum_{r=1}^{p} S_r^p \ \frac{[k+1]^{(r+1)}}{[r+1]}\right\} x\right] - \left[\sum_{r=1}^{p} S_r^p \ \frac{[k+1]^{(r+1)}}{[r+1]}\right] f[x]\right\| \leq \alpha[x, x, \cdots, x] \tag{4.8}$$

for all $x \in X$. Define

$$\wp = \sum_{r=1}^{p} S_r^p \ \frac{[k+1]^{(r+1)}}{[r+1]}$$

in the above equation and re modifying, we arrive

$$\left\|\frac{f[\wp\, x]}{\wp} - f[x]\right\| \leq \frac{\alpha[x, x, \cdots, x]}{\wp} \tag{4.9}$$

for all $x \in X$. Now replacing x by $\wp x$ and dividing by $\wp$ in (4.9), we get

$$\left\|\frac{f[\wp x]}{\wp} - \frac{f[\wp^2 x]}{\wp^2}\right\| \leq \frac{\alpha[\wp x, \wp x, \cdots, \wp x]}{\wp^2} \tag{4.10}$$

for all $x \in X$. From (4.8) and (4.10), we obtain

$$\begin{aligned}\left\|f[x] - \frac{f[\wp^2 y]}{\wp^2}\right\| &\leq \left\|f[x] - \frac{f[\wp x]}{\wp}\right\| + \left\|\frac{f[\wp x]}{\wp} - \frac{f[\wp^2 x]}{\wp^2}\right\| \\ &\leq \frac{1}{\wp}\left\{\alpha[x, x, \cdots, x] + \frac{\alpha[\wp x, \wp x, \cdots, \wp x]}{\wp}\right\}\end{aligned} \tag{4.11}$$

for all $x \in X$. In general for any positive integer t , we get

$$\begin{aligned}\left\|f[x] - \frac{f[\wp^t x]}{\wp^t}\right\| &\leq \frac{1}{\wp}\sum_{s=0}^{t-1}\frac{\alpha[\wp^s x, \wp^s x, \cdots, \wp^s x]}{\wp^s} \\ &\leq \frac{1}{\wp}\sum_{s=0}^{\infty}\frac{\alpha[\wp^s x, \wp^s x, \cdots, \wp^s x]}{\wp^s}\end{aligned} \tag{4.12}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{\frac{f[\wp^t x]}{\wp^t}\right\},$$

replace x by $\wp^l x$ and dividing by $\wp^l$ in (4.12), for any $t, l > 0$, we deduce

$$\begin{aligned}\left\|\frac{f[\wp^l x]}{\wp^l} - \frac{f[\wp^{l+t} x]}{\wp^{[l+t]}}\right\| &= \frac{1}{\wp^l}\left\|f[\wp^l x] - \frac{f[\wp^t \cdot \wp^l x]}{\wp^t}\right\| \\ &\leq \frac{1}{\wp}\sum_{s=0}^{t-1}\frac{\alpha\left[\wp^{s+l} x, \wp^{s+l} x, \cdots, \wp^{s+l} x\right]}{\wp^{s+l}} \\ &\leq \frac{1}{\wp}\sum_{s=0}^{\infty}\frac{\alpha\left[\wp^{s+l} x, \wp^{s+l} x, \cdots, \wp^{s+l} x\right]}{\wp^{s+l}} \\ &\to 0 \quad as \ \ l \to \infty\end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{f[\wp^t x]}{\wp^t} \right\}$, is a Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$A[x] = \lim_{t\to\infty} \frac{f[\wp^t x]}{\wp^t} \quad \forall \ x \in X.$$

Letting $t \to \infty$ in (4.12) we see that (4.3) holds for all $x \in X$.

To show that A satisfies (1.8), replacing $[x_1, x_2, x_3 \cdots, x_k]$ by $[\wp^t x_1, \wp^t x_2, \cdots, \wp^t x_k]$ and dividing by $\wp^t$ in (4.2) and using the definition of $A(x)$, and then letting $t \to \infty$, we see that A satisfies (1.8) for all $x_1, x_2, \cdots, x_k \in X$. To prove that A is unique, let $B[x]$ be another additive mapping satisfying (1.8) and (4.3), then

$$\begin{aligned}
\|A[x] - B[x]\| &= \frac{1}{\wp^t} \left\|A[\wp^t x] - B[\wp^t x]\right\| \\
&\leq \frac{1}{\wp^t} \left\{ \left\|A[\wp^t x] - f[\wp^t x]\right\| + \left\|f[\wp^t x] - B[\wp^t x]\right\| \right\} \\
&\leq \sum_{s=0}^{\infty} \frac{2\,\alpha[\wp^{t+s}x, \wp^{t+s}x, \cdots, \wp^{t+s}x]}{\wp \cdot \wp^{t+s}} \\
&\to 0 \quad as \ \ s \to \infty
\end{aligned}$$

for all $x \in X$. Hence A is unique.

For $i = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.4 concerning the Ulam-Hyers [13], Ulam-Hyers-Rassias [21], Ulam-Gavruta-Rassias [20] and Ulam-JRassias [26] stabilities of (1.8).

Corollary 4.1. *Let λ and q be nonnegative real numbers. Let a function $f : X \to Y$ satisfies the inequality*

$$\left\| f\left[\sum_{j=1}^{k} j^p \, x_j\right] - \sum_{j=1}^{k} \left[j^p \, f[x_j]\right] \right\|
\leq \begin{cases} \lambda, & \\ \lambda \left\{ \sum_{j=1}^{k} ||x_j||^q \right\}, & q < 1 \ \ or \ \ q > 1; \\ \lambda \prod_{j=1}^{k} ||x_j||^q, & q < \frac{1}{k} \ \ or \ \ q > \frac{1}{k}; \\ \lambda \left\{ \prod_{j=1}^{k} ||x_j||^q + \left\{ \sum_{j=1}^{k} ||x_j||^{kq} \right\} \right\}, & q < \frac{1}{k} \ \ or \ \ q > \frac{1}{k}; \end{cases} \tag{4.13}$$

for all $x_1, x_2, \cdots, x_k \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f[x] - A[x]\| \leq \begin{cases} \frac{\lambda}{|\wp - 1|}, \\ \frac{k\lambda||x||^q}{|\wp - \wp^q|}, \\ \frac{\lambda||x||^{kq}}{|\wp - \wp^{kq}|} \\ \frac{(k+1)\lambda||x||^{kq}}{|\wp - \wp^{kq}|} \end{cases} \tag{4.14}$$

for all $x \in X$.

5 Stability results: Fixed point method

In this section, we apply a fixed point method for achieving stability of the additive functional equation (1.8).

Now, we present the following theorem due to B. Margolis and J.B. Diaz [17] for fixed point theory.

Theorem 5.5. *[17] Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \quad \forall \quad n \geq 0,$$

or there exists a natural number n_0 such that
(FP1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T
(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using the above theorem, we now obtain the generalized Ulam - Hyers stability of (1.8).

Theorem 5.6. *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^k \to [0, \infty)$ with the condition*

$$\sum_{t=0}^{\infty} \frac{\alpha\left[\mu_i^t x_1, \mu_i^t x_2, \cdots, \mu_i^t x_k\right]}{\mu_i^t} \quad \text{converges in } \mathbb{R} \tag{5.1}$$

where $\mu_i = \wp$ if $i = 0$ and $\mu_1 = \frac{1}{\wp}$ if $i = 1$ such that the functional inequality

$$\left\| f\left[\sum_{j=1}^{k} j^p\, x_j\right] - \sum_{j=1}^{k} \left[j^p\, f[x_j]\right] \right\| \leq \alpha\left[x_1, x_2, x_3 \cdots, x_k\right] \tag{5.2}$$

for all $x_1, x_2, x_3 \cdots, x_k \in X$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma[x] = \frac{1}{\wp}\, \alpha\left[\frac{x}{\wp}, \frac{x}{\wp}, \cdots, \frac{x}{\wp}\right],$$

has the property

$$\gamma[x] = L\, \mu_i\, \gamma\left[\mu_i x\right]. \tag{5.3}$$

Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equation (1.8) and

$$\| f[x] - A[x] \| \leq \frac{L^{1-i}}{1-L} \gamma[x] \tag{5.4}$$

for all $x \in X$.

Proof. Consider the set

$$\Omega = \{p / p : X \to Y,\ p[0] = 0\}$$

and introduce the generalized metric on Ω,

$$d(p, q) = \inf\{K \in (0, \infty) : \| p[x] - q[x] \| \leq K\gamma[x], x \in X\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by

$$Tp[x] = \frac{1}{\mu_i} p[\mu_i x],$$

for all $x \in E$. Now $p, q \in \Omega$,

$$\begin{aligned}
d(p, q) \leq K &\Rightarrow \| p[x] - q[x] \| \leq K\gamma[x], x \in X. \\
&\Rightarrow \left\| \frac{1}{\mu_i} p[\mu_i x] - \frac{1}{\mu_i} q[\mu_i x] \right\| \leq \frac{1}{\mu_i} K\gamma[\mu_i x], x \in X, \\
&\Rightarrow \left\| \frac{1}{\mu_i} p[\mu_i x] - \frac{1}{\mu_i} q[\mu_i x] \right\| \leq LK\gamma[x], x \in X, \\
&\Rightarrow \| Tp[x] - Tq[x] \| \leq LK\gamma[x], x \in X, \\
&\Rightarrow d(p, q) \leq LK.
\end{aligned}$$

This implies $d(Tp, Tq) \leq Ld(p, q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.
From (4.9), we arrive

$$\left\| \frac{f[\wp\, x]}{\wp} - f[x] \right\| \leq \frac{\gamma[x]}{\wp} \tag{5.5}$$

for all $x \in X$. Using (5.3) for the case $i = 0$ it reduces to

$$\left\| \frac{f[\wp x]}{\wp} - f[x] \right\| \leq L\gamma[x]$$

for all $x \in X$,

$$\text{i.e.,} \quad d(f, Tf) \leq L \Rightarrow d(f, Tf) \leq L = L^1 < \infty.$$

Again replacing $x = \dfrac{x}{\wp}$ in (5.5), we get,

$$\left\| f[x] - \wp f\left[\frac{x}{\wp}\right] \right\| \leq \gamma\left[\frac{x}{\wp}\right] \tag{5.6}$$

for all $x \in X$. Using (5.3) for the case $i = 1$ it reduces to

$$\left\| f[x] - \wp f\left(\frac{x}{\wp}\right) \right\| \leq \gamma[x]$$

for all $x \in X$,

$$\text{i.e.,} \quad d(f, Tf) \leq 1 \Rightarrow d(f, Tf) \leq 1 = L^0 < \infty.$$

In above cases, we arrive

$$d(f, Tf) \leq L^{1-i}.$$

Therefore (FP1) holds.

By (FP2), it follows that there exists a fixed point A of T in Ω such that

$$A[x] = \lim_{t\to\infty} \frac{f[\mu_i^t x]}{\mu_i^t} \qquad \forall\ x \in X. \tag{5.7}$$

To order to prove $A : X \to Y$ is additive, replacing $[x_1, \cdots, x_k]$ by $\left[\mu_i^t x_1, \cdots, \mu_i^t x_k\right]$ and dividing by μ_i^t in (5.2) and using the definition of $A(x)$, and then letting $t \to \infty$, we see that A satisfies (1.8) for all $x_1, \cdots, x_k \in X$.

By (FP3), A is the unique fixed point of T in the set $\Delta = \{A \in \Omega : d(f, A) < \infty\}$, A is the unique function such that

$$\|f[x] - A[x]\| \leq K\gamma[x]$$

for all $x \in X$ and $K > 0$. Finally by (FP4), we obtain

$$d(f, A) \leq \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f, A) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\|f[x] - A[x]\| \leq \frac{L^{1-i}}{1-L}\gamma[x]$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 5.6 concerning the Ulam-Hyers [13], Ulam-Hyers-Rassias [21], Ulam-Gavruta-Rassias [20] and Ulam-JRassias [26] stabilities of (1.8).

Corollary 5.2. *Let $f : X \to Y$ be a mapping and there exits real numbers λ and q such that*

$$\left\| f\left[\sum_{j=1}^{k} j^p\, x_j\right] - \sum_{j=1}^{k} \left[j^p\, f[x_j]\right] \right\| \leq \begin{cases} (i) & \lambda, \\ (ii) & \lambda \left\{ \sum_{j=1}^{k} ||x_j||^q \right\}, & q < 1 \quad or \quad q > 1; \\ (iii) & \lambda \prod_{j=1}^{k} ||x_j||^q, & q < \frac{1}{k} \quad or \quad q > \frac{1}{k}; \\ (iv) & \lambda \left\{ \prod_{j=1}^{k} ||x_j||^q + \left\{ \sum_{j=1}^{k} ||x_j||^{kq} \right\} \right\}, & q < \frac{1}{k} \quad or \quad q > \frac{1}{k}; \end{cases} \tag{5.8}$$

for all $x_1, x_2, \cdots, x_k \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f[x] - A[x]\| \leq \begin{cases} (i) & \dfrac{\lambda}{|\wp - 1|}, \\ (ii) & \dfrac{k\lambda ||x||^q}{|\wp - \wp^q|}, \\ (iii) & \dfrac{\lambda ||x||^{kq}}{|\wp - \wp^{kq}|} \\ (iv) & \dfrac{(k+1)\lambda ||x||^{kq}}{|\wp - \wp^{kq}|} \end{cases} \tag{5.9}$$

for all $x \in X$.

Proof. Setting

$$\alpha[x_1, x_2, \cdots, x_k] = \begin{cases} \lambda, \\ \lambda \left\{ \sum_{j=1}^{k} ||x_j||^q \right\}, \\ \lambda \prod_{j=1}^{k} ||x_j||^q, \\ \lambda \left\{ \prod_{j=1}^{k} ||x_j||^q + \left\{ \sum_{j=1}^{k} ||x_j||^{kq} \right\} \right\}, \end{cases}$$

for all$x_1, x_2, \cdots, x_k \in X$.. Now,

$$\frac{1}{\mu_i^t} \alpha[\mu_i^t x_1, \mu_i^t x_2, \cdots, \mu_i^t x_k] = \begin{cases} \frac{\lambda}{\mu_i^t}, \\ \frac{\lambda}{\mu_i^t} \left\{ \sum_{j=1}^{k} ||\mu_i^t x_j||^q \right\}, \\ \frac{\lambda}{\mu_i^t} \prod_{j=1}^{k} ||\mu_i^t x_j||^q, \\ \frac{\lambda}{\mu_i^t} \left\{ \prod_{j=1}^{k} ||\mu_i^t x_j||^q + \left\{ \sum_{j=1}^{k} ||\mu_i^t x_j||^{kq} \right\} \right\}, \end{cases} = \begin{cases} \to 0 \text{ as } t \to \infty, \\ \to 0 \text{ as } t \to \infty, \\ \to 0 \text{ as } t \to \infty, \\ \to 0 \text{ as } t \to \infty. \end{cases}$$

Thus, (5.1) is holds.

But we have $\gamma[x] = \frac{1}{\wp}\, \gamma\,[x]$ has the property $\gamma[x] = L \cdot \mu_i\, \gamma\,[\mu_i x]$ for all $x \in X$. Hence

$$\gamma[x] = \frac{1}{\wp} \alpha[x, x, \cdots, x] = \begin{cases} \dfrac{\lambda}{\wp} \\ \dfrac{k\lambda}{\wp} ||x||^q, \\ \dfrac{\lambda}{\wp} ||x||^{kq}, \\ \dfrac{(k+1)\lambda}{\wp} ||x||^{kq}. \end{cases}$$

Now,

$$\frac{1}{\mu_i}\gamma[\mu_i x] = \begin{cases} \dfrac{\lambda}{\wp\mu_i} \\ \dfrac{k\lambda}{\wp\mu_i}||\mu_i x||^q, \\ \dfrac{\lambda}{\wp\mu_i}||\mu_i x||^{kqs}, \\ \dfrac{(k+1)\lambda}{\wp\mu_i}||\mu_i x||^{kq}. \end{cases}$$

$$= \begin{cases} \mu_i^{-1}\dfrac{\lambda}{\wp}, \\ \mu_i^{q-1}\dfrac{k\lambda}{\wp}||x||^q, \\ \mu_i^{kq-1}\dfrac{\lambda}{\wp}||x||^{kq}, \\ \mu_i^{kq-1}\dfrac{(k+1)\lambda}{\wp}||x||^{kq}. \end{cases}$$

$$= \begin{cases} \mu_i^{-1}\gamma[x], \\ \mu_i^{q-1}\gamma[x], \\ \mu_i^{kq-1}\gamma[x], \\ \mu_i^{kq-1}\gamma[x]. \end{cases}$$

Hence the inequality (5.3) holds either, $L = \wp^{-1}$ for $q = 0$ if $i = 0$ and $L = \frac{1}{\wp^{-1}}$ for $q = 0$ if $i = 1$. Now from (5.4), we prove the following cases for condition (ii).

Case:1 $L = \wp^{-1}$ for $q = 0$ if $i = 0$

$$\|f[x] - A[x]\| \leq \frac{\left(\wp^{(-1)(0-1)}\right)^{1-0}}{1-\wp^{(-1)(0-1)}}\frac{\lambda}{\wp} = \frac{\wp}{1-\wp}\frac{\lambda}{\wp} = \frac{\lambda}{1-\wp}$$

Case:2 $L = \frac{1}{\wp^{-1}}$ for $q = 0$ if $i = 1$

$$\|f[x] - A[x]\| \leq \frac{\left(\frac{1}{\wp^{(-1)(0-1)}}\right)^{1-1}}{1-\frac{1}{\wp^{(-1)(0-1)}}}\frac{\lambda}{\wp} = \frac{\wp}{\wp-1}\frac{\lambda}{\wp} = \frac{\lambda}{\wp-1}$$

Also the inequality (5.3) holds either, $L = \wp^{q-1}$ for $q < 1$ if $i = 0$ and $L = \frac{1}{\wp^{q-1}}$ for $q > 1$ if $i = 1$. Now from (5.4), we prove the following cases for condition (ii).

Case:1 $L = \wp^{q-1}$ for $q < 1$ if $i = 0$

$$\|f[x] - A[x]\| \leq \frac{\left(\wp^{(q-1)}\right)^{1-0}}{1-\wp^{(q-1)}}\frac{k\lambda}{\wp}||x||^q = \frac{\wp^{(q-1)}\cdot\wp}{\wp-\wp^s}\frac{k\lambda}{\wp}||x||^q = \frac{\wp^{(q-1)}k\lambda||x||^q}{\wp-\wp^q}$$

Case:2 $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$

$$\|f[x] - A[x]\| \leq \frac{\left(\frac{1}{\wp^{(q-1)}}\right)^{1-1}}{1-\frac{1}{\wp^{(q-1)}}}\frac{k\lambda}{\wp}||x||^q = \frac{\wp^{(q-1)}\cdot\wp}{\wp^s-\wp}\frac{k\lambda}{\wp}||x||^q = \frac{\wp^{(q-1)}k\lambda||x||^q}{\wp^s-\wp}$$

Again, the inequality (5.3) holds either, $L = \wp^{kq-1}$ for $q < \frac{1}{k}$ if $i = 0$ and $L = \frac{1}{\wp^{kq-1}}$ for $q > \frac{1}{k}$ if $i = 1$. Now from (5.4), we prove the following cases for condition (iii).

Case:1 $L = \wp^{kq-1}$ for $q < \frac{1}{k}$ if $i = 0$

$$\|f[x] - A[x]\| \leq \frac{\left(\wp^{(kq-1)}\right)^{1-0}}{1-\wp^{(kq-1)}}\frac{\lambda}{\wp}||x||^{kq} = \frac{\wp^{(kq-1)}\cdot\wp}{\wp-\wp^{kq}}\frac{\lambda}{\wp}||x||^{kq} = \frac{\wp^{(kq-1)}\lambda||x||^{kq}}{\wp-\wp^{kq}}$$

Case:2 $L = \frac{1}{\wp^{kq-1}}$ for $q > \frac{1}{k}$ if $i = 1$

$$\|f[x] - A[x]\| \leq \frac{\left(\frac{1}{\wp^{(kq-1)}}\right)^{1-1}}{1 - \frac{1}{\wp^{(kq-1)}}} \frac{\lambda}{\wp}||x||^{kq} = \frac{\wp^{(kq-1)} \cdot \wp}{\wp^{kq} - \wp} \frac{\lambda}{\wp}||x||^{kq} = \frac{\wp^{(kq-1)}\lambda||x||^{kq}}{\wp^{kq} - \wp}$$

Finally the inequality (5.3) holds either, $L = \wp^{kq-1}$ for $q < \frac{1}{k}$ if $i = 0$ and $L = \frac{1}{\wp^{kq-1}}$ for $q > \frac{1}{k}$ if $i = 1$. The proof of condition (iv) is similar lines to that of condition (iii). Hence the proof is complete. □

6 Application of the functional equation (1.8)

We know that the following sums of powers arithmetic progression

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k-1)}{2}$$

In general, using Stirling numbers of second kind, one can arrive

$$1^p + 2^p + 3^p + \cdots + k^p = \sum_{r=1}^{p} S_r^p \frac{[k+1]^{(r+1)}}{[r+1]}.$$

With the help of the above discussion, the authors transform the sum of p^{th} power of first k natural numbers as a functional equation

$$f\left(\sum_{j=1}^{k} j^p\, x_j\right) = \sum_{j=1}^{k} \left(j^p\, f(x_j)\right), \qquad k, p \geq 1$$

having additive solution.

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Received: April 4, 2013; *Accepted*: November 25, 2013

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(1)(2014) 61–71

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Some inequalities for the q, k-Gamma and Beta functions

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Abstract

Using q-integral inequalities we establish some new inequalities for the q-k Gamma, Beta and Psi functions.

Keywords: q, k-Gamma, q, k-Beta, q-integral inequalities.

2010 MSC: 33D05, 33D60, 41A17.

1 Introduction

The q-analogue Γ_q of the well known Gamma function was initially introduced by Thomae [11] and later deeply studied by Jackson [6]. The reader will find in the research literature more about this feature.
In [1], R. Diaz and C. Truel introduced a q, k-generalized Gamma and Beta functions and they proved integral representations for $\Gamma_{q,k}$ and $B_{q,k}$ functions.

This work is devoted to establish some inequalities for the generalized q, k-Gamma and Beta functions and this has been possible thanks to the inequalities that verify the q-Jackson's integral.

The paper is organized as follows: In section 2, we present some preliminaries and notations that will be useful in the sequel. In section 3, we recall the q-Čebyšev's integral inequality for q-synchronous (q-asynchronous) functions and in direct consequence, we deduce some inequalities involving q, k-Beta and q, k-Gamma functions. In section 4, we establish some inequalities for these functions owing to the q-Holder's inequality. Finally section 5 is devoted to some applications of q-Grüss integral inequality.

2 Notations and preliminaries

To make this paper self containing, we provide in this section a summary of the mathematical notations and definitions useful. All of these results can be found in [4], [8] or [9].
Throughout this paper, we will fix $q \in]0,1[$, $\quad k > 0$ a real number.
For $a \in \mathbb{C}$, we write

$$[a]_q = \frac{1-q^a}{1-q}, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \;\; n = 1, 2...\infty,$$

$$[n]_q! = [1]_q[2]_q...[n]_q, \qquad n \in \mathbb{N}.$$

The q-derivative D_q of a function f is given by

$$(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}, \;\text{ if }\; x \neq 0, \tag{2.1}$$

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and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.
The q-Jackson integrals from 0 to b and from 0 to ∞ are defined by (see [7])

$$\int_0^b f(x) d_q x = (1-q) b \sum_{n=0}^{\infty} f(bq^n) q^n \tag{2.2}$$

and

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n, \tag{2.3}$$

provided the sums converge absolutely.
The q-Jackson integral in a generic interval $[a,b]$ is given by (see [7])

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.4}$$

We denote by I one of the following sets:

$$\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}, \tag{2.5}$$

$$[0,b]_q = \{bq^n : n \in \mathbb{N}\}, \quad b > 0, \tag{2.6}$$

$$[a,b]_q = \{bq^r : \ 0 \leq r \leq n\}, \quad b > 0, \ a = bq^n, n \in \mathbb{N} \tag{2.7}$$

and we note $\int_I f(x) d_q x$ the q-integral of f on the correspondent I.

Definition 2.1. *let $x, y, s, t \in \mathbb{R}$ and $n \in \mathbb{N}$, we note by*

1. $(x+y)_{q,k}^n := \prod_{j=0}^{n-1} (x + q^{jk} y)$
2. $(1+x)_{q,k}^{\infty} := \prod_{j=0}^{\infty} (1 + q^{jk} x)$
3. $(1+x)_{q,k}^t := \frac{(1+x)_{q,k}^{\infty}}{(1+q^{kt}x)_{q,k}^{\infty}}$.

We have $(1+x)_{q,k}^{s+t} = (1+x)_{q,k}^s (1+q^{ks}x)_{q,k}^t$.

We recall the two q,k-analogues of the exponential function (see [1]) given by

$$E_{q,k}^x = \sum_{n=0}^{\infty} q^{\frac{kn(n-1)}{2}} \frac{x^n}{[n]_{q^k}!} = (1 + (1-q^k)x)_{q,k}^{\infty} \tag{2.8}$$

and

$$e_{q,k}^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q^k}!} = \frac{1}{(1-(1-q^k)x)_{q,k}^{\infty}}. \tag{2.9}$$

These q,k-exponential functions satisfy the following relations:
$D_{q^k} e_{q,k}^x = e_{q,k}^x, \qquad D_{q^k} E_{q,k}^x = E_{q,k}^{q^k x} \qquad$ *and* $\qquad E_{q,k}^{-x} e_{q,k}^x = e_{q,k}^x E_{q,k}^{-x} = 1.$

The q,k-Gamma function is defined by [1]

$$\Gamma_{q,k}(x) = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^x)_{q,k}^{\infty} (1-q)^{\frac{x}{k}-1}} \qquad x > 0. \tag{2.10}$$

When $k = 1$ it reduces to the known q-Gamma function Γ_q.

It satisfies the following functional equation:

$$\Gamma_{q,k}(x+k) = [x]_q \Gamma_{q,k}(x), \quad \Gamma_{q,k}(k) = 1 \tag{2.11}$$

and having the following integral representation (see [1])

$$\Gamma_{q,k}(x) = \int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} t^{x-1} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} d_q t, \;\; x > 0. \tag{2.12}$$

The previous integral representation, give that $\Gamma_{q,k}$ is an infinitely differentiable function on $]0, \;\; +\infty[$ and

$$\Gamma_{q,k}^{(i)}(x) = \int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} t^{x-1} (lnt)^i E_{q,k}^{-\frac{q^k t^k}{[k]_q}} d_q t, \quad x > 0, \;\; i \in \mathbb{N}. \tag{2.13}$$

The q,k-Beta function is defined by (see [1])

$$B_{q,k}(t,s) = [k]_q^{-\frac{t}{k}} \int_0^{[k]_q^{\frac{1}{k}}} x^{t-1} (1 - q^k \frac{x^k}{[k]_q})_{q,k}^{\frac{s}{k}-1} d_q x, \qquad s > 0, t > 0. \tag{2.14}$$

By using the following change of variable $u = \frac{x}{[k]_q^{\frac{1}{k}}}$, the last equation becomes

$$B_{q,k}(t,s) = \int_0^1 u^{t-1} (1 - q^k u^k)_{q,k}^{\frac{s}{k}-1} d_q u, \qquad s > 0, t > 0. \tag{2.15}$$

It satisfies

$$B_{q,k}(t,s) = \frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{\Gamma_{q,k}(t+s)}, \qquad s > 0, t > 0. \tag{2.16}$$

3 q-Čebyšev's integral inequality and applications

We begin this section by recalling the q-Čebyšev's integral inequality for q-synchronous (q-asynchronous) mappings [3] and as applications we give some inequalities for the q,k-Beta and the q,k-Gamma functions.

Definition 3.2. *Let f and g be two functions defined on I. The functions f and g are said q-synchronous (q-asynchronous) on I if*

$$(f(x) - f(y))(g(x) - g(y)) \geq (\leq) 0 \quad \forall x, y \in I. \tag{3.17}$$

Note that if f and g are both q-increasing or q-decreasing on I then they are q-synchronous on I.

Proposition 3.1. *Let f, g and h be three functions defined on I such that:*

1. $h(x) \geq 0, \quad x \in I,$
2. *f and g are q-synchronous (q-asynchronous) on I.*

Then

$$\int_I h(x) d_q x \int_I h(x) f(x) g(x) d_q x \geq (\leq) \int_I h(x) f(x) d_q x \int_I h(x) g(x) d_q x. \tag{3.18}$$

Proof. We have

$$\int_I h(x) d_q x \int_I h(x) f(x) g(x) d_q x - \int_I h(x) f(x) d_q x \int_I h(x) g(x) d_q x =$$
$$1/2 \int_I \int_I h(x) h(y) \left[f(x) - f(y)\right] \left[g(x) - g(y)\right] d_q x d_q y.$$

So, the result follows from the conditions (1) and (2). □

The following theorem is a direct consequence of the previous proposition.

Theorem 3.1. *Let m, n, p and p' be some positive reals such that*

$$(p-m)(p'-n) \le (\ge) 0.$$

Then

$$B_{q,k}(p,p')B_{q,k}(m,n) \ge (\le) B_{q,k}(p,n)B_{q,k}(m,p') \tag{3.19}$$

and

$$\Gamma_{q,k}(p+n)\Gamma_{q,k}(p'+m) \ge (\le) \Gamma_{q,k}(p+p')\Gamma_{q,k}(m+n). \tag{3.20}$$

Proof. Fix m, n, p and p' in $]0, \ +\infty[$, satisfying the condition of the theorem and the functions f, g and h defined on $[0,1]_q$ by

$$f(u) = u^{p-m}, \quad g(u) = (1-q^n u^k)_{q,k}^{\frac{p'-n}{k}} \quad \textit{and} \quad h(u) = u^{m-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1}.$$

From the relations

$$D_q f(u) = [p-m]_q u^{p-m-1} \tag{3.21}$$

and

$$D_q g(u) = [n-p']_q \, q^{p'} \, u^{k-1}(1-q^{n+k}u^k)_{q,k}^{\frac{p'-n}{k}-1}, \tag{3.22}$$

one can see that f and g are q-synchronous (q-asynchronous) on $I = [0,1]_q$.
So, by using the relation (2.15) and Proposition 3.1,
we obtain

$$\int_0^1 u^{m-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1} d_q u \int_0^1 u^{p-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1}(1-q^n u^k)_{q,k}^{\frac{p'-n}{k}} d_q u \ge$$
$$(\le) \int_0^1 u^{p-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1} d_q u \int_0^1 u^{m-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1}(1-q^n u^k)_{q,k}^{\frac{p'-n}{k}} d_q u,$$

which implies that

$$B_{q,k}(m,n)B_{q,k}(p,p') \ge (\le) B_{q,k}(p,n)B_{q,k}(m,p'). \tag{3.23}$$

Now, according to the relations (2.16) and (3.19), we obtain

$$\frac{\Gamma_{q,k}(m)\Gamma_{q,k}(n)}{\Gamma_{q,k}(m+n)} \frac{\Gamma_{q,k}(p)\Gamma_{q,k}(p')}{\Gamma_{q,k}(p+p')} \ge (\le) \frac{\Gamma_{q,k}(p)\Gamma_{q,k}(n)}{\Gamma_{q,k}(p+n)} \frac{\Gamma_{q,k}(m)\Gamma_{q,k}(p')}{\Gamma_{q,k}(m+p')}. \tag{3.24}$$

Therefore

$$\Gamma_{q,k}(p+n)\Gamma_{q,k}(p'+m) \ge (\le) \Gamma_{q,k}(p+p')\Gamma_{q,k}(m+n). \tag{3.25}$$

□

Corollary 3.1. *For all $p, m > 0$, we have*

$$B_{q,k}(p,m) \ge \Big[B_{q,k}(p,p)B_{q,k}(m,m)\Big]^{1/2} \tag{3.26}$$

and

$$\Gamma_{q,k}(p+m) \le \Big[\Gamma_{q,k}(2p)\Gamma_{q,k}(2m)\Big]^{1/2}. \tag{3.27}$$

Proof. A direct application of Theorem 3.1, with $p' = p$ and $n = m$, gives the results. □

Corollary 3.2. *For all $u, v > 0$, we have*

$$\Gamma_{q,k}\left(\frac{u+v}{2}\right) \le \sqrt{\Gamma_{q,k}(u)\Gamma_{q,k}(v)}. \tag{3.28}$$

Proof. The inequality follows from (3.27), by taking $p = \frac{u}{2}$ and $m = \frac{v}{2}$. □

Theorem 3.2. *Let m, p and r be real numbers satisfying $m, p > 0$ and $p > r > -m$ and let n be a nonnegative integer.*
If

$$r(p-m-r) \geq (\leq) 0 \tag{3.29}$$

then

$$\Gamma_{q,k}^{(2n)}(p)\Gamma_{q,k}^{(2n)}(m) \geq (\leq) \Gamma_{q,k}^{(2n)}(p-r)\Gamma_{q,k}^{(2n)}(m+r). \tag{3.30}$$

Proof. Let f, g and h be the functions defined on $I = [0, (\frac{[k]_q}{(1-q^k)})^{\frac{1}{k}}]_q$ by

$$f(x) = x^{p-m-r}, \quad g(x) = x^r \quad and \quad h(x) = x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n}.$$

We have

$$D_q f(x) = [p-m-r]_q x^{p-m-r-1} \quad and \quad D_q g(x) = [r]_q x^{r-1}.$$

If the condition (3.29) holds, one can show that the functions f and g are q-synchronous (q-asynchronous) on I and Proposition 3.1 gives

$$\begin{aligned} &\int_I x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \int_I x^{p-m-r} x^r x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \\ &\geq (\leq) \int_I x^{p-m-r} x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \int_I x^r x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int_I x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \int_I x^{p-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \\ &\geq (\leq) \int_I x^{p-r-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \int_I x^{r+m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x. \end{aligned}$$

Hence, the relation

$$\Gamma_{q,k}^{(i)}(x) = \int_I t^{x-1} (lnt)^i E_{q,k}^{-q^k \frac{t^k}{[k]_q}} d_q t, \quad x > 0, \ i \in \mathbb{N},$$

gives

$$\Gamma_{q,k}^{(2n)}(m)\Gamma_{q,k}^{(2n)}(p) \geq (\leq) \Gamma_{q,k}^{(2n)}(p-r)\Gamma_{q,k}^{(2n)}(m+r). \tag{3.31}$$

□

Taking $n = 0$ in the previous theorem, we obtain the following result.

Corollary 3.3. *Let m, p and r be some real numbers under the conditions of Theorem 3.2, we have*

$$\Gamma_{q,k}(p)\Gamma_{q,k}(m) \geq (\leq) \Gamma_{q,k}(p-r)\Gamma_{q,k}(m+r) \tag{3.32}$$

and

$$B_{q,k}(p,m) \geq (\leq) B_{q,k}(p-r, m+r). \tag{3.33}$$

Corollary 3.4. *Let n be a nonnegative integer, $p > 0$ and $p' \in \mathbb{R}$ such that $| p' | < p$. Then*

$$\left[\Gamma_{q,k}^{(2n)}(p)\right]^2 \leq \Gamma_{q,k}^{(2n)}(p-p')\Gamma_{q,k}^{(2n)}(p+p'). \tag{3.34}$$

Proof. By choosing $m = p$ and $r = p'$, we obtain

$$r(p-m-r) = -(p')^2 \leq 0$$

and the result turns out from Theorem 3.2. □

Taking in the previous result $p = \frac{u+v}{2}$ and $p' = \frac{u-v}{2}$, we obtain the following result:

Corollary 3.5. *Let u, v be two positive real numbers and n be a nonnegative integer. Then*

$$\Gamma_{q,k}^{(2n)}(\frac{u+v}{2}) \leq \sqrt{\Gamma_{q,k}^{(2n)}(u)\Gamma_{q,k}^{(2n)}(v)}. \tag{3.35}$$

Corollary 3.6. *Let $p > 0$ and $p' \in \mathbb{R}$ such that $\mid p' \mid < p$.*
Then

$$\Gamma_{q,k}^{2}(p) \leq \Gamma_{q,k}(p-p')\Gamma_{q,k}(p+p') \tag{3.36}$$

and

$$B_{q,k}(p,p) \leq B_{q,k}(p-p', p+p'). \tag{3.37}$$

Proof. For $n = 0$, the inequality (3.34) becomes

$$\Gamma_{q,k}^{2}(p) \leq \Gamma_{q,k}(p-p')\Gamma_{q,k}(p+p').$$

The inequality (3.37) follows from (2.16). □

Theorem 3.3. *Let a and b be two positive real numbers such*

$$(a-k)(b-k) \geq (\leq) 0$$

and n a nonnegative integer. Then

$$\Gamma_{q,k}^{(2n)}(2k)\Gamma_{q,k}^{(2n)}(a+b) \geq (\leq)\Gamma_{q,k}^{(2n)}(a+k)\Gamma_{q,k}^{(2n)}(b+k). \tag{3.38}$$

Proof. In Theorem 3.2, set $m = 2k$, $p = a+b$ and $r = b-k$. The condition (3.29) becomes

$$r(p-m-r) = (a-k)(b-k) \geq (\leq) 0. \tag{3.39}$$

So,

$$\Gamma_{q,k}^{(2n)}(2k)\Gamma_{q,k}^{(2n)}(a+b) \geq (\leq)\Gamma_{q,k}^{(2n)}(a+k)\Gamma_{q,k}^{(2n)}(b+k). \tag{3.40}$$

□

Corollary 3.7. *If $a, b > 0$ such $(a-k)(b-k) \geq (\leq) 0$. Then*

$$\Gamma_{q,k}(a+b) \geq (\leq) \frac{[a]_q[b]_q}{[k]_q}\Gamma_{q,k}(a)\Gamma_{q,k}(b) \tag{3.41}$$

and

$$B_{q,k}(a,b) \leq (\geq) \frac{[k]_q}{[a]_q[b]_q}. \tag{3.42}$$

Proof. The inequality (3.41) follows from the previous theorem by taking $n = 0$ and using the facts that $\Gamma_{q,k}(2k) = [k]_q$, $\Gamma_{q,k}(a+k) = [a]_q\Gamma_{q,k}(a)$ and
$\Gamma_{q,k}(b+k) = [b]_q\Gamma_{q,k}(b)$. (2.16) together with (3.41) give (3.42). □

Corollary 3.8. *The function $\ln\Gamma_{q,k}$ is superadditive for $x \geq k$ and $k \geq 1$, in the sense that*

$$\ln\Gamma_{q,k}(a+b) \quad \geq \quad \ln\Gamma_{q,k}(a) + \ln\Gamma_{q,k}(b).$$

Proof. For all $a, b \geq k$, we have

$$\begin{aligned} \ln\Gamma_{q,k}(a+b) &\geq \ln\frac{[a]_q[b]_q}{[k]_q} + \ln\Gamma_{q,k}(a) + \ln\Gamma_{q,k}(b) \\ &\geq \ln\Gamma_{q,k}(a) + \ln\Gamma_{q,k}(b), \end{aligned}$$

which completes the proof. □

Corollary 3.9. *For $a \geq k$ and $n = 1, 2, \ldots,$ we have*

$$\Gamma_{q,k}(na) \geq \frac{[n-1]_{q^a}![a]_q^{2(n-1)}}{[k]_q^{n-1}}[\Gamma_{q,k}(a)]^n. \tag{3.43}$$

Proof. We proceed by induction on n.
It is clear that the inequality is true for $n = 1$.
Suppose that (3.43) holds for an integer $n \geq 1$ and let us prove it for $n + 1$.
By (3.41), we have

$$\Gamma_{q,k}((n+1)a) = \Gamma_{q,k}(na+a) \geq \frac{[na]_q[a]_q}{[k]_q}\Gamma_{q,k}(na)\Gamma_{q,k}(a) \tag{3.44}$$

and by hypothesis, we have

$$\Gamma_{q,k}(na) \geq \frac{[n-1]_{q^a}![a]_q^{2(n-1)}}{[k]_q^{n-1}}[\Gamma_{q,k}(a)]^n. \tag{3.45}$$

The use of the fact that $[na]_q = [n]_{q^a}[a]_q$, gives

$$\begin{aligned}\Gamma_{q,k}((n+1)a) &\geq \frac{[na]_q[a]_q[n-1]_{q^a}![a]_q^{2(n-1)}}{[k]_q^n}[\Gamma_{q,k}(a)]^n\Gamma_{q,k}(a)\\ &\geq \frac{[n]_{q^a}![a]_q^{2n}}{[k]_q^n}\left[\Gamma_{q,k}(a)\right]^{n+1}.\end{aligned}$$

The inequality (3.43) is then true for $n + 1$. □

For a given real $m > 0$ and a nonnegative integer n, consider the mapping

$$\Gamma_{q,k,m,n}(x) = \frac{\Gamma_{q,k}^{(2n)}(x+m)}{\Gamma_{q,k}^{(2n)}(m)}.$$

We have the following result.

Corollary 3.10. *The mapping $\Gamma_{q,k,m,n}(.)$ is suppermultiplicative on $[0, \infty)$, in the sense*

$$\Gamma_{q,k,m,n}(x+y) \geq \Gamma_{q,k,m,n}(x)\Gamma_{q,k,m,n}(y).$$

Proof. Fix x, y in $[0, \infty)$ and put $p = x + y + m$ and $r = y$. We have

$$y(x+y+m-m-y) = xy \geq 0.$$

So, the theorem 3.2 leads to

$$\Gamma_{q,k}^{(2n)}(m)\Gamma_{q,k}^{(2n)}(x+y+m) \geq \Gamma_{q,k}^{(2n)}(x+m)\Gamma_{q,k}^{(2n)}(y+m), \tag{3.46}$$

which is equivalent to

$$\Gamma_{q,k,m,n}(x+y) \geq \Gamma_{q,k,m,n}(x)\Gamma_{q,k,m,n}(y). \tag{3.47}$$

This achieves the proof. □

4 Inequalities via the q- Hölder's one

We begin this section by recalling the q-analogue of the Hölder's integral inequality [3].

Lemma 4.1. *Let p and p' be two positive reals satisfying $\frac{1}{p} + \frac{1}{p'} = 1$, f and g be two functions defined on I. Then*

$$\left|\int_I f(x)g(x)d_qx\right| \leq \left(\int_I |f(x)|^p\, d_qx\right)^{\frac{1}{p}}\left(\int_I |g(x)|^{p'}\, d_qx\right)^{\frac{1}{p'}}. \tag{4.48}$$

Owing this lemma, one can establish some new inequalities involving the q,k-Gamma and q,k-Beta functions.

Theorem 4.4. *Let n be a nonnegative integer, x,y be two positive real numbers and a,b be two nonnegative real numbers such that $a+b=1$. Then*

$$\Gamma_{q,k}^{(2n)}(ax+by) \leq \left[\Gamma_{q,k}^{(2n)}(x)\right]^a \left[\Gamma_{q,k}^{(2n)}(y)\right]^b, \tag{4.49}$$

that is, the mapping $\Gamma_{q,k}^{(2n)}$ is logarithmically convex on $(0,\infty)$.

Proof. Consider the following functions defined on $I=[0,(\frac{[k]_q}{(1-q^k)})^{\frac{1}{k}}]_q$,

$$f(t)=t^{a(x-1)}\left(E_{q,k}^{-q^k\frac{t^k}{[k]_q}}(lnt)^{2n}\right)^a \quad \text{and} \quad g(t)=t^{b(y-1)}\left(E_{q,k}^{-q^k\frac{t^k}{[k]_q}}(lnt)^{2n}\right)^b.$$

By application of the q-Hölder's integral inequality, with $p=\dfrac{1}{a}$, we get

$$\begin{aligned}\int_I t^{a(x-1)}t^{b(y-1)}E_{q,k}^{-q^k\frac{t^k}{[k]_q}}(lnt)^{2n}d_qt \quad &\leq \quad \left[\int_I t^{a(x-1).(1/a)}E_{q,k}^{-q^k\frac{t^k}{[k]_q}}(lnt)^{2n}d_qt\right]^a \times \\ &\left[\int_I t^{b(y-1).(1/b)}E_{q,k}^{-q^k\frac{t^k}{[k]_q}}(lnt)^{2n}d_qt\right]^b,\end{aligned}$$

which is equivalent to

$$\int_I t^{ax+by-1}E_{q,k}^{-q^k\frac{t^k}{[k]_q}}(lnt)^{2n}d_qt \leq \left[\int_I t^{x-1}E_{q,k}^{-q^k\frac{t^k}{[k]_q}}(lnt)^{2n}d_qt\right]^a \left[\int_I t^{y-1}E_{q,k}^{-q^k\frac{t^k}{[k]_q}}(lnt)^{2n}d_qt\right]^b.$$

Then, (4.49) is a direct consequence of (2.13). □

Corollary 4.11. *Let $(p,p'),(m,m')\in(0,\infty)^2$ such that $p+p'=m+m'$ and $a,b\geq 0$ with $a+b=1$. Then, we have*

$$B_{q,k}(a(p,p')+b(m,m')) \leq \left[B_{q,k}(p,p')\right]^a \left[B_{q,k}(m,m')\right]^b. \tag{4.50}$$

Proof. On the one hand, we have

$$\begin{aligned}B_{q_k}(a(p,p')+b(m,m')) &= B_{q,k}(ap+bm,ap'+bm') = \frac{\Gamma_{q,k}(ap+bm)\Gamma_{q,k}(ap'+bm')}{\Gamma_{q,k}(ap+bm+ap'+bm')} \\ &= \frac{\Gamma_{q,k}(ap+bm)\Gamma_{q,k}(ap'+bm')}{\Gamma_{q,k}(a(p+p')+b(m+m'))}.\end{aligned}$$

Since $p+p'=m+m'$ and $a+b=1$, we have

$$\Gamma_{q,k}(a(p+p')+b(m+m'))=\Gamma_{q,k}(p+p')=\Gamma_{q,k}(m+m'). \tag{4.51}$$

On the other hand, from Theorem 4.4, with $n=0$, we obtain

$$\Gamma_{q,k}(ap+bm) \leq \left[\Gamma_{q,k}(p)\right]^a \left[\Gamma_{q,k}(m)\right]^b \tag{4.52}$$

and

$$\Gamma_{q,k}(ap'+bm') \leq \left[\Gamma_{q,k}(p')\right]^a \left[\Gamma_{q,k}(m')\right]^b. \tag{4.53}$$

Thus

$$\Gamma_{q,k}(ap+bm)\Gamma_{q,k}(ap'+bm') \leq \left[\Gamma_{q,k}(p)\Gamma_{q,k}(p')\right]^a \left[\Gamma_{q,k}(m)\Gamma_{q,k}(m')\right]^b. \tag{4.54}$$

From (4.51), we deduce that

$$\frac{\Gamma_{q,k}(ap+bm)\Gamma_{q,k}(ap'+bm')}{\Gamma_{q,k}(a(p+p')+b(m+m'))} \leq \left[\frac{\Gamma_{q,k}(p)\Gamma_{q,k}(p')}{\Gamma_{q,k}(p+p')}\right]^a \left[\frac{\Gamma_{q,k}(m)\Gamma_{q,k}(m')}{\Gamma_{q,k}(m+m')}\right]^b, \tag{4.55}$$

which completes the proof. □

Now, we recall that the logarithmic derivative of the q,k-Gamma function is defined on $(0,\infty)$, by

$$\Psi_{q,k}(x) = \frac{\Gamma'_{q,k}(x)}{\Gamma_{q,k}(x)}.$$

The following result gives some properties of the function $\Psi_{q,k}$.

Theorem 4.5. *$\Psi_{q,k}$ is monotonic non-decreasing and concave on $(0,\infty)$.*

Proof. By taking $n=0$ in Theorem 4.4, we obtain

$$\Gamma_{q,k}(ax+by) \leq \left[\Gamma_{q,k}(x)\right]^a \left[\Gamma_{q,k}(y)\right]^b,$$

for $x,\ y>0$ and $a,\ b\geq 0$ such that $a+b=1$.
So the function $ln\Gamma_{q,k}$ is convex. Then the monotonicity of $\Psi_{q,k}$ follows from the relation

$$\frac{d}{dx}[ln\Gamma_{q,k}(x)] = \frac{\Gamma'_{q,k}(x)}{\Gamma_{q,k}(x)} = \Psi_{q,k}(x), \quad x>0.$$

On the other hand, since

$$\Gamma_{q,k}(x) = \frac{(1-q^k)^{\infty}_{q,k}}{(1-q^x)^{\infty}_{q,k}(1-q)^{\frac{x}{k}-1}}, \tag{4.56}$$

we obtain, for $x>0$,

$$\begin{aligned}\Psi_{q,k}(x) &= \frac{d}{dx}[ln\Gamma_{q,k}(x)] = -\frac{1}{k}\,ln(1-q) + lnq\sum_{j=0}^{\infty}\frac{q^{x+jk}}{1-q^{x+jk}}\\ &= -\frac{1}{k}\,ln(1-q) + lnq\sum_{j=0}^{\infty}q^{x+jk}\sum_{n=0}^{\infty}q^{(x+jk)n} = -\frac{1}{k}\,ln(1-q) + lnq\sum_{n=0}^{\infty}\frac{q^{(n+1)x}}{1-q^{(n+1)k}}\\ &= -\frac{1}{k}\,ln(1-q) + \frac{lnq}{(1-q)}\int_0^q \frac{t^{x-1}}{1-t^k}d_qt.\end{aligned}$$

Now, let $x,y>0$ and $a,b\geq 0$ such that $a+b=1$. Then

$$\Psi_{q,k}(ax+by) + \frac{1}{k}\,ln(1-q) = \frac{lnq}{(1-q)}\int_0^q \frac{t^{ax+by-1}}{1-t^k}d_qt = \frac{lnq}{(1-q)}\int_0^q \frac{t^{a(x-1)+b(y-1)}}{1-t^k}d_qt. \tag{4.57}$$

Since the mapping $x\mapsto t^x$ is convex on $\mathbb{R}$ for $t\in(0,1)$, we have

$$t^{a(x-1)+b(y-1)} \leq at^{x-1} + bt^{y-1}, \quad for\ \ t\in[0,q]_q, \quad x,y>0.$$

Thus,

$$\frac{lnq}{(1-q)}\int_0^q \frac{t^{ax+by-1}}{1-t^k}d_qt \geq a\left(\frac{lnq}{(1-q)}\int_0^q \frac{t^{x-1}}{1-t^k}d_qt\right) + b\left(\frac{lnq}{(1-q)}\int_0^q \frac{t^{y-1}}{1-t^k}d_qt\right). \tag{4.58}$$

According to the relations (4.57) and (4.58), we have

$$\begin{aligned}\Psi_{q,k}(ax+by) + \frac{1}{k}\,ln(1-q) &\geq a(\Psi_{q,k}(x) + \frac{1}{k}\,ln(1-q)) + b(\Psi_{q,k}(y) + \frac{1}{k}\,ln(1-q))\\ &\geq a\Psi_{q,k}(x) + b\Psi_{q,k}(y) + \frac{1}{k}\,ln(1-q).\end{aligned}$$

This proves the concavity of the function $\Psi_{q,k}$. □

5 Inequalities via the q-Grüss's one

In [5] H. Gauchman gave a q-analogue of the Grüss' integral inequality namely.

Lemma 5.2. *Assume that* $m \leq f(x) \leq M$, $\varphi \leq g(x) \leq \Phi$, *for each* $x \in [a,b]$, *where* m, M, φ, Φ *are given real constants. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)d_qx - \frac{1}{(b-a)^2} \int_a^b f(x)d_qx \int_a^b g(x)d_qx \right| \leq \frac{1}{4}(M-m)(\Phi-\varphi). \tag{5.59}$$

As application of the previous inequality we state the following result

Theorem 5.6. *Let* $m, n > 0$, *we have*

$$\left| B_{q,k}(m+k, n+k) - \frac{1}{[m+1]_q [n+k]_q} \right| \leq \frac{1}{4}. \tag{5.60}$$

Remark that from the relations (2.16) and (2.11), the inequality (5.60) is equivalent to

$$| \Gamma_{q,k}(m+n+2k) - \Gamma_{q,k}(n+2k)\Gamma_{q,k}(m+k)[m+1]_q | \leq \frac{1}{4}[m+1]_q[n+k]_q\Gamma_{q,k}(m+n+2k). \tag{5.61}$$

Proof. Consider the functions

$$f(u) = u^m, \quad g(u) = u^{k-1}(1-q^ku^k)_{q,k}^{\frac{n}{k}}, \quad u \in [0,1]_q, \quad m,n > 0.$$

We have

$$0 \leq f(u) \leq 1 \quad and \quad 0 \leq g(u) \leq 1 \qquad \forall\, u \in [0,1]_q.$$

Then, using the q-Grüss' integral inequality, we obtain

$$\left| \int_0^1 u^{m+k-1}(1-q^ku^k)_{q,k}^{\frac{n}{k}}\, d_qu - \int_0^1 u^m\, d_qu \int_0^1 u^{k-1}(1-q^ku^k)_{q,k}^{\frac{n}{k}}\, d_qu \right| \leq \frac{1}{4}. \tag{5.62}$$

The inequality (5.60) follows from the definition of the q,k-Beta function (2.15) and the following facts: $\displaystyle\int_0^1 u^m d_qu = \frac{1}{[m+1]_q}$ and
$\displaystyle\int_0^1 u^{k-1}(1-q^ku^k)_{q,k}^{\frac{n}{k}}\, d_qu = B_{q,k}(k, n+k) = \frac{1}{[n+k]_q}$. □

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Received: April 3, 2013; *Accepted*: November 25, 2013

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Website: http://www.malayajournal.org/

Malaya J. Mat. 1(1)(2014) 72–81

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

A New Legendre Wavelets Decomposition Method for Solving PDEs

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Abstract

In this paper, we present a novel technique based on the Legendre wavelets decomposition. The properties of Legendre wavelets are used to reduces the PDEs problem into the solution of ODEs system. To illustrate our results, two examples are studied using a special software package which implements the proposed algorithms.

Keywords: Legendre Wavelets, Legendre Polynomials, Operational Matrix of integration, Telegraph Equation.

2000 MSC: 49M37, 30E25.

1 Introduction

Since the introduction of Legendre wavelets method (LWM), for the resolution of variational problems, by Rezzaghi and Yousefi in 2000 and 2001 [5, 7], several works applying this method were born. To mention a few, we give : the resolution of differential equations [3, 11, 12], the study of optimal control problem with constraints [6], the resolution of linear integro-differential equations [8], the numerical resolution of Abel equation [10], the resolution of fractional differential equations [2, 4].

The LWM transforms a boundary value problem (BVP) into a system of algebraic equations [5]. The unknown parameter of this system is the vector whose components are the decomposition coefficients of the BVP solution into Legendre wavelets basis.

In this paper, we apply the Legendre wavelets method to solve a partial differential equation, whose unknown function depends on spatial and temporal variables. But the decomposition of this unknown function into Legendre wavelets basis will be done only on the spatial variable. Obviously, the coefficients of this decomposition will depend on the temporal variable. Hence, via this technique, the solution of a partial differential equation is reduced to the solution of a time-dependent differential equation. As an application of this procedure, we present a numerical simulation of the telegraph equation. This equation modelled several phenomenas in electronics and electricity. It appears in particular during the description of the propagation of the electric signals along a transmission line.

This paper is organized as follows: In section 2, we give a detailed description of the Legendre wavelets decomposition of a function dependent on the temporal variable t and the space variable x. Section 3 is devoted to the operational matrix of integration. In section 4, we apply our technique on the telegraph equation. In section 5, we present formulas of errors calculation. For the last section, the performance of the new method is illustrated with two numerical examples.

2 Decomposition in Legendre wavelets basis

We start this section by recalling that the Legendre wavelets [5] are defined on the interval $[0,1]$ as follows : for all $j \in \mathbb{N}^*$, $n = 1,2,3,...,2^{j-1}$ (the number of levels), $m = 0,1,...,nc-1$ (the order of the Legendre

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polynomials) and nc is the number of collocation points ;

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{j/2}L_m\left(2^jx-2n+1\right) & \frac{n-1}{2^{j-1}} \leq x < \frac{n}{2^{j-1}} \\ 0, \quad otherwise, \end{cases} \tag{2.1}$$

where $L_m(x)$ are the Legendre polynomials of order m defined on the interval $[-1,1]$ and satisfy the following recursive formula:

$$L_{k+1}(x) = \left(\frac{2k+1}{k+1}\right)xL_k(x) - \left(\frac{k}{k+1}\right)L_{k-1}(x) \tag{2.2}$$

with $L_0(x) = 1$, $L_1(x) = x$ and $k = 1,2,3,....nc-2$.

It is established [3, 5–7] that the family $\{\psi_{n,m} : n \geq 1 \;\;,\;\; m \geq 0\}$ forms an orthonormal basis of the Hilbert space $L^2([0,1])$, i.e. any element $h \in L^2([0,1])$ may be expanded as

$$h(x) = \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} C_{n,m}\psi_{n,m}(x), \tag{2.3}$$

where the approximation coefficients are entirely determined by $C_{n,m} = \langle h, \psi_{n,m}\rangle$, in which $\langle .,.\rangle$ denotes the inner product of $L^2([0,1])$.

Since the series (2.3) converges on $[0,1]$, the function h can be approximated as

$$h(x) \simeq \sum_{n=1}^{2^{j-1}}\sum_{m=0}^{nc-1} C_{n,m}\psi_{n,m}(x) = C^T\Psi(x), \tag{2.4}$$

where C and $\Psi(x)$ are $2^{j-1}nc$ vectors given by

$$C = \left[C_{1,0}, C_{1,1}, ..., C_{1,nc-1}, C_{2,0}, ..., C_{2,nc-1}, ..., C_{2^{j-1},0}, ..., C_{2^{j-1},nc-1}\right]^T, \tag{2.5}$$

and

$$\Psi(x) = \left[\psi_{1,0}(x), ..., \psi_{1,nc-1}(x), ..., \psi_{2^{j-1},0}(x), ..., \psi_{2^{j-1},nc-1}(x)\right]^T. \tag{2.6}$$

Decomposition in an other L^2 -space

Let us consider the space

$$H = L^2\left([0,T]\,; L^2([0,1])\right), \tag{2.7}$$

see [1]. We want to give a Legendre wavelets decomposition of an element h in H:

$$\begin{array}{rccc} h: & [0,T] & \rightarrow & L^2([0,1]) \\ & t & \rightarrow & h(t,.) \,. \end{array} \tag{2.8}$$

As the function $h(t,.)$ belongs to $L^2([0,1])$, then by (2.3) we have

$$h(t,x) = \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} C_{n,m}(t)\,\psi_{n,m}(x), \tag{2.9}$$

where the coeficients $C_{n,m}(t)$ depending on the variable t are defined by

$$C_{n,m}(t) = \langle h(t,.), \psi_{n,m}\rangle = \int_0^1 h(t,x)\psi_{n,m}(x)\,dx. \tag{2.10}$$

The functions $h(t,.)$ and $\psi_{n,m}$, being both in $L^2([0,1])$, their product is in $L^1([0,1])$ (according to Cauchy-Schwartz inequality), which allows us to conclude that the coefficients $C_{n,m}(t)$ are well defined for all $t \in [0,T]$. Consequently, the relations (2.9) and (2.10) are justified.

For the sequel, we need the following lemmas.

Lemma 2.1. *If $h \in C\left([0,T], L^2([0,1])\right)$ then the function coefficients $C_{n,m}(t)$ are continuous in $[0,T]$.*

Proof. It arises from the fact that the inner product is a continuous function of its both arguments. □

Lemma 2.2. *If $h \in C^1\left(]0,T[\,, L^2\left([0,1]\right)\right)$, then the function coefficients $C_{n,m}(t)$ belong to $C^1(]0,T[)$. Forthermore, if $\frac{\partial h}{\partial t} \in L^2\left([0,T]\,, L^2\left([0,1]\right)\right)$, then*

$$\frac{dC_{n,m}(t)}{dt} = \int_0^1 \frac{\partial h(t,x)}{\partial t}\psi_{n,m}(x)\,dx. \tag{2.11}$$

Proof. Lemma 2.2 is based on

$$\frac{C_{n,m}(t+\Delta t) - C_{n,m}(t)}{\Delta t} = \int_0^1 \frac{h(t+\Delta t,x) - h(t,x)}{\Delta t}\psi_{n,m}(x)\,dx,$$

and

$$\frac{h(t+\Delta t,x) - h(t,x)}{\Delta t} = \frac{\partial h}{\partial t}(t,x) + \varepsilon(t,\Delta t,x),$$

with

$$\lim_{\Delta t \to 0} \varepsilon(t,\Delta t,x) = 0.$$

□

Lemma 2.3. *(Generalization) If $h \in C^k\left(]0,T[\,, L^2\left([0,1]\right)\right)$, then the function coefficients $C_{n,m}(t)$ belong to $C^k(]0,T[)$ $(k \geq 2)$.*

For all $t \in [0,T]$, the series (2.9) is convergent, we can thus approach any function h in $L^2\left([0,T]\,; L^2\left([0,1]\right)\right)$ as

$$h(t,x) \simeq \sum_{n=1}^{2^{j-1}} \sum_{m=0}^{nc-1} C_{n,m}(t)\,\psi_{n,m}(x) = C^T(t)\,\Psi(x), \tag{2.12}$$

where $C(t)$ and $\Psi(x)$ are $\left(2^{j-1}nc\right)$ functions vectors given by

$$C(t) = \left[C_{1,0}(t), C_{1,1}(t), ..., C_{1,nc-1}(t), ..., C_{2^{j-1},0}(t), ..., C_{2^{j-1},nc-1}(t)\right]^T, \tag{2.13}$$

and

$$\Psi(x) = \left[\psi_{1,0}(x), ..., \psi_{1,nc-1}(x), ..., \psi_{2^{j-1},0}(x), ..., \psi_{2^{j-1},nc-1}(x)\right]^T. \tag{2.14}$$

3 Operational matrix of integration

In this section, the operational matrix of integration [7] will be obtained. The integration into $[0,x]$, where $x \in]0,1]$ of the vector $\Psi(x)$ defined in Eq. (2.14) can be written as

$$\int_0^x \Psi(t)dt = P\Psi(x), \tag{3.1}$$

where

$$P = \frac{1}{2^j}\begin{bmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & & & & F \\ 0 & 0 & \cdots & 0 & L \end{bmatrix}, \tag{3.2}$$

is the $(2^{j-1}nc) \times (2^{j-1}nc)$ operational matrix of integration, F and L are $nc \times nc$ matrices given by

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & & 0 \end{bmatrix} \tag{3.3}$$

and

$$L = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \ddots & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & -\frac{\sqrt{2nc-3}}{(2nc-3)\sqrt{2nc-5}} & 0 & \frac{\sqrt{2nc-3}}{(2nc-3)\sqrt{2nc-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\sqrt{2nc-1}}{(2nc-1)\sqrt{2nc-3}} & 0 \end{bmatrix}. \tag{3.4}$$

4 Application to telegraph equation

The standard form of the telegraph equation is given by

$$\frac{\partial^2 u(x,t)}{\partial x^2} = a\frac{\partial^2 u(x,t)}{\partial t^2} + b\frac{\partial u(x,t)}{\partial t} + cu(x,t), \tag{4.1}$$

with boundary conditions

$$u(0,t) = \alpha(t) \quad \text{and} \quad \frac{\partial u(0,t)}{\partial x} = \beta(t) \tag{4.2}$$

and initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u(x,0)}{\partial t} = g(x), \tag{4.3}$$

where

a, b, c are constants related respectively to resistance, induction, capacity and conductibility of the cable.
$\alpha(t), \beta(t)$ are continuous functions in $[0,T]$ and $f(x), g(x)$ are continuous in $[0,1]$
We are interested in the evolution of the tension $u(x,t)$ in a coaxial cable.
Let

$$\frac{\partial^2 u(x,t)}{\partial x^2} = C^T(t)\,\Psi(x). \tag{4.4}$$

Integrating (4.4) with respect to second variable over $[0,x]$, we get

$$\frac{\partial u(x,t)}{\partial x} = C^T(t)\,P\Psi(x) + \beta(t). \tag{4.5}$$

By a second integration with respect to x into $[0,x]$, we have

$$u(x,t) = C^T(t)\,P^2\Psi(x) + \beta(t)\,x + \alpha(t). \tag{4.6}$$

As well as

$$\frac{\partial u(x,t)}{\partial t} = \frac{dC^T(t)}{dt}P^2\Psi(x) + \frac{d\beta(t)}{dt}x + \frac{d\alpha(t)}{dt} \tag{4.7}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{d^2C^T(t)}{dt^2}P^2\Psi(x) + \frac{d^2\beta(t)}{dt^2}x + \frac{d^2\alpha(t)}{dt^2} \tag{4.8}$$

We have also

$$\begin{cases} 1 = d^T\Psi(x) \\ x = e^T\Psi(x). \end{cases} \tag{4.9}$$

Substituting (4.4) to (4.9) in (4.1), we obtain

$$\begin{aligned} C^T(t)\Psi(x) &= a\left(\frac{d^2C^T(t)}{dt^2}P^2\Psi(x) + \frac{d^2\beta(t)}{dt^2}e^T\Psi(x) + \frac{d^2\alpha(t)}{dt^2}d^T\Psi(x)\right) \\ &+ b\left(\frac{dC^T(t)}{dt}P^2\Psi(x) + \frac{d\beta(t)}{dt}e^T\Psi(x) + \frac{d\alpha(t)}{dt}d^T\Psi(x)\right) \\ &+ c\left(C^T(t)\,P^2\Psi(x) + \beta(t)\,e^T\Psi(x) + \alpha(t)d^T\Psi(x)\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} & -a\left(P^2\right)^T C''(t) - b\left(P^2\right)^T C'(t) + \left(I - c\left(P^2\right)^T\right) C(t) \\ = & \left(a\beta''(t) + b\beta'(t) + c\beta(t)\right) e + \left(a\alpha''(t) + b\alpha'(t) + c\alpha(t)\right) d \end{aligned}$$

This system can be solved for unknown coefficients of the vector $C(t)$. Consequently, the solution $u(t,x)$ given in (4.6) can be calculated.

5 Error calculation

A reasonable scalar index for the closeness of two functions is the Euclidian norm.

The absolute error for each solution is produced by cumulates of truncation, Legendre Wavelets method and Finite Difference errors. This error is estimated when we know the exact solution by

$$E_A = \|u - u_e\|_2, \tag{5.1}$$

where u_e is the analytic solution and u is the approximate solution. Also, we consider the pointwise error :

$$E_{A,i} = |u_i - u_e(x_i)|. \tag{5.2}$$

6 Numerical results

In this section, we consider two examples to show the efficiency and the accuracy of our method.

6.1 Example 1

Consider in $[0,1]$ the following boundary values problem

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}, \tag{6.1}$$

with

$$u(0,t) = \cos(t) \quad \textit{and} \quad \frac{\partial u(0,t)}{\partial x} = 0, \tag{6.2}$$

and

$$u(x,0) = \cos(x) \quad \textit{and} \quad \frac{\partial u(x,0)}{\partial t} = 0. \tag{6.3}$$

The exact solution of the problem (6.1)-(6.2) and (6.3) is given by

$$u_{ex}(t,x) = \cos(x)\cos(t). \tag{6.4}$$

Suppose that the derivatives on x can be expressed as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = C^T(t)\,\Psi(x), \tag{6.5}$$

$$\frac{\partial u(x,t)}{\partial x} = C^T(t)\,P\Psi(x), \tag{6.6}$$

$$u(x,t) = \left(C^T(t)\,P^2 + \cos(t)d^T\right)\Psi(x), \tag{6.7}$$

then their derivates on t are given by

$$\frac{\partial u(x,t)}{\partial t} = \left(\frac{\partial C^T(t)}{\partial t}P^2 - \sin(t)d^T\right)\Psi(x), \tag{6.8}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \left(\frac{\partial^2 C^T(t)}{\partial t^2}P^2 - \cos(t)d^T\right)\Psi(x). \tag{6.9}$$

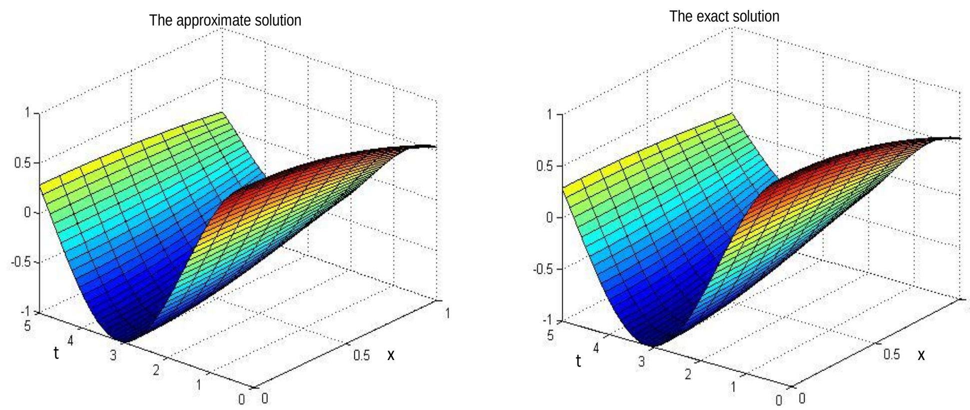

Figure 1: The analytical and approximate solutions.

Substituting (6.5) to (6.9) in (6.1), we obtain

$$C^T(t)\Psi(x) = \left(\frac{\partial^2 C^T(t)}{\partial t^2} P^2 - \cos(t) d^T\right)\Psi(x), \tag{6.10}$$

and

$$\frac{\partial^2 C(t)}{\partial t^2} - ((P^2)^T)^{-1} C(t) = ((P^2)^T)^{-1}\cos(t) d, \tag{6.11}$$

with

$$C^T(0) = \left(e^T - d^T\right)\left(P^2\right)^{-1} \quad and \quad \frac{\partial C(0)}{\partial t} = \overrightarrow{0}, \tag{6.12}$$

where

$$\begin{cases} 1 = d^T\Psi(x) \\ \cos(x) = e^T\Psi(x). \end{cases} \tag{6.13}$$

For the resolution of this problem, we can use a second time the wavelets Legendre method or a finite difference schemes.

We observe a good agreement between the analytical and approximate solutions (see Figure 1). However, the obtained result shows that this technique can provide good performance even when the mesh discretization has low resolution.

We also calculate the absolute error by using formula (5.1), for $j = 2$ and $nc = 5$

$$E_A = 1.29e - 2. \tag{6.14}$$

6.2 Example 2

We consider the telegraph equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} + 4\frac{\partial u(x,t)}{\partial t} + 4u(x,t), \tag{6.15}$$

with the conditions

$$u(0,t) = \left(1 + e^{-2t}\right) \quad and \quad \frac{\partial u(0,t)}{\partial x} = 2, \tag{6.16}$$

and

$$u(x,0) = 1 + e^{2x} \quad and \quad \frac{\partial u(x,0)}{\partial t} = -2. \tag{6.17}$$

Suppose that the second derivative of $u(x,t)$ can be expressed approximately as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = C^T(t)\Psi(x). \tag{6.18}$$

Using boundary condition (6.16), we get

$$\frac{\partial u(x,t)}{\partial x} = C^T(t)\,P\Psi(x) + 2d^T\Psi(x), \tag{6.19}$$

$$u(x,t) = \left(C^T(t)\,P^2 + 2l^T + \left(1+e^{-2t}\right)d^T\right)\Psi(x). \tag{6.20}$$

We can also express the functions of the right-hand sides of (6.19) and (6.20) as

$$\begin{cases} 1 = d^T\Psi(x) \\ x = l^T\Psi(x). \end{cases} \tag{6.21}$$

The derivates on t are given by

$$\frac{\partial u(x,t)}{\partial t} = \left(\frac{\partial C^T(t)}{\partial t}P^2 - 2e^{-2t}d^T\right)\Psi(x), \tag{6.22}$$

and

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \left(\frac{\partial^2 C^T(t)}{\partial t^2}P^2 + 4e^{-2t}d^T\right)\Psi(x). \tag{6.23}$$

Now by inserting Eqs. (6.18) to (6.23) into equation (6.15), we get

$$\begin{aligned} C^T(t)\,\Psi(x) \;=\; & \left(\frac{\partial^2 C^T(t)}{\partial t^2}P^2 + 4e^{-2t}d^T\right)\Psi(x) \\ & +4\left(\frac{\partial C^T(t)}{\partial t}P^2 - 2e^{-2t}d^T\right)\Psi(x) + \\ & 4\left(C^T(t)\,P^2 + 2l^T + \left(1+e^{-2t}\right)d^T\right)\Psi(x). \end{aligned}$$

Thus

$$\frac{\partial^2 C(t)}{\partial t^2} + 4\frac{\partial C(t)}{\partial t} + (4I - ((P^2)^T)^{-1})C(t) = -((P^2)^T)^{-1}(8l+4d), \tag{6.24}$$

where

$$C(0) = ((P^2)^T)^{-1}(ex - 2l - 2d) \quad and \quad \frac{\partial C(0)}{\partial t} = 0, \tag{6.25}$$

and

$$1 + e^{2x} = ex^T\Psi(x) \tag{6.26}$$

The exact solution of the problem (6.15)-(6.16) and (6.17) is

$$u_{ex}(t,x) = e^{2x} + e^{-2t}, \tag{6.27}$$

Applying the same technique as the preceding example, we observe a good agreement between the analytical and approximate solutions with an absolute errors of order $1.0e-013$ (see figure 2). However, the results obtained show that this technique can provide good performance even when the mesh discretization has low resolution.

We also calculate the absolute error by using formula (5.1), for $j=3$ and $nc=10$

$$E_A = 1.7705e-013. \tag{6.28}$$

We calculated the absolute error for different values of j and nc (see figure 3), we observe that :

The effect of the levels number on the solution is shown in Table 1. For $j = 1,2,...,6$ (2^{j-1} levels), the absolute errors with respect to analytic solutions are presented. As j increase from 1 to 6 for $nc=3$, the errors decrease to 10^{-4}.

The considerable effect of collocation points on the solution is shown in Table 2. For $nc = 2,3,...,10$, the absolute errors with respect to analytic solutions are presented. As nc changes from 2 to 4 for $j=3$, then from 6 to 10 for $j=3$, the errors decrease by a factor of 10^2 and 10^6 respectively.

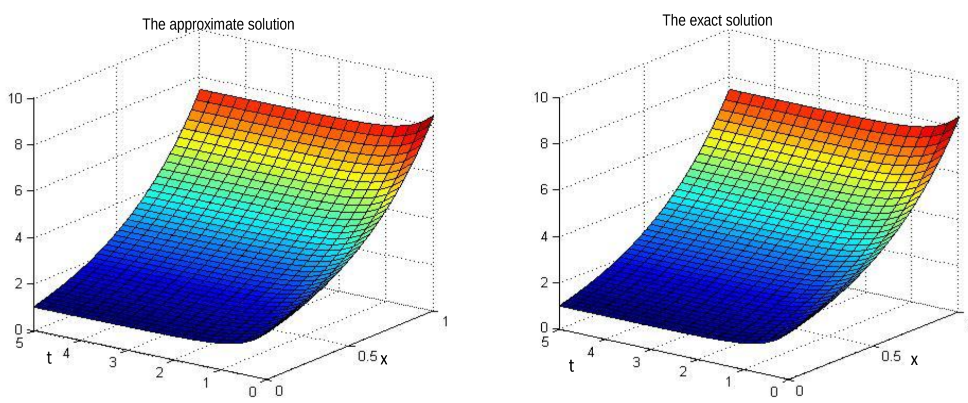

Figure 2: The analytical and approximate solutions.

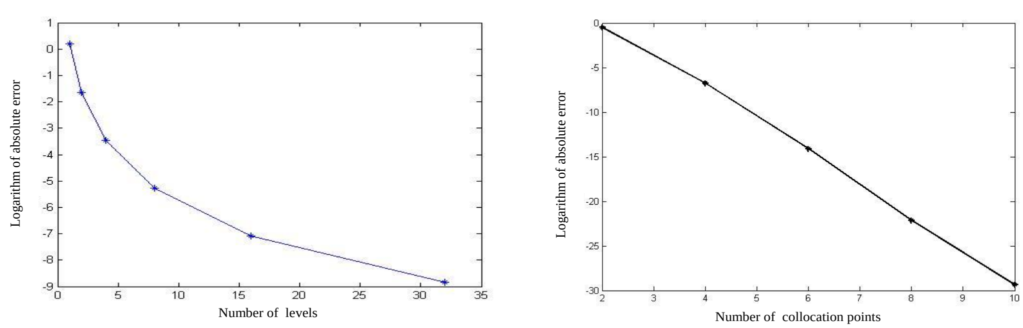

Figure 3: Evolution of absolute errors.

Table 1: Evolution of absolute errors for $nc = 3$

	j = 1	j = 2	j = 3	j = 4	j = 5	j = 6
Number of levels 2^{j-1}	1	2	4	8	16	32
Absolute errors	1.2048	1.910E-001	3.1442E-002	5.0610E-003	8.4109E-004	1.43699E-004

Table 2: Evolution of absolute errors for $j = 3$.

nc	2	4	6	8	10
Absolute errors	6.1105e-001	1.1544e-003	7.3398e-007	2.3729e-010	1.7700e-013

7 Conclusion

This work shows that the Legendre wavelets method is a very effective technique for reducing partial differential equations into a set of ordinary differential equations. This numerical method has been tested under different examples. Satisfactory results have been obtained even for a small number of collocation points. The convergence, stability of the solution and accuracy of the results prove the high quality of this method.

8 Acknowledgments

The authors wish to express their sincere thanks to the editor and the reviewer for valuable suggestions that improved the final manuscript.

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Received: October 11, 2013; *Accepted*: December 21, 2013

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(1)(2014) 82–90

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Common fixed points for a class of multi-valued mappings and application to functional equations arising in dynamic programming

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Abstract

In this paper, we give an existence theorem for hybrid generalized multi-valued α-contractive type mappings which extends, improves and unifies the corresponding main result of Sintunavart and Kumam [W. Sintunavart, P. Kumam, Common fixed point theorem for hybrid generalized multi-valued contraction mappings, Appl. Math. Lett. 25 (2012), 52-57] and some main results in the literature. As an application, we give some existence and uniqueness results for solutions of a certain class of functional equations arising in dynamic programming to illustrate the efficiency and usefulness of our main result.

Keywords: Common fixed point, Multi-valued contraction, Dynamic programming, Functional equations.

2010 MSC: 47H10, 54C60, 90C39.

1 Introduction and preliminaries

Fixed point theory is one of the classical and most powerful tools which plays an essential role in nonlinear analysis. Under the mathematical point of view, this is utilized in so many nonlinear problems arising from the most applicable areas of sciences such as engineering, economics, dynamic system and physics (more precisely in the theory of Phase Transitions), etc.
The idea of combining the multi-valued mappings, Lipschitz mapping and fixed point theorems was initiated by Nadler [14]. He was the first one that gave a famous generalization of the Banach contraction principle for multi-valued mappings from metric space X into $\mathcal{CB}(X)$ using the Hausdorff metric which is defined by

$$H(A,B) := \max\left\{\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\right\}, \quad A,B\in\mathcal{CB}(X),$$

where $d(x,A) = \inf_{y\in A} d(x,y)$ and $\mathcal{CB}(X)$ is a collection of nonempty closed bounded subsets of X.

Definition 1.1 ([9]). *Let H be the Hausdorff metric on $\mathcal{CB}(X)$, $f : X \to X$ be a single-valued mapping and $T : X \to \mathcal{CB}(X)$ be multi-valued mapping. We say an element $x \in X$ is a fixed point of f (resp. T) if $fx = x$ (resp. $x \in Tx$). We denote the set of all fixed points of f (resp. T) by $Fix(f)$ (resp. $Fix(T)$). A point $x \in X$ is a coincidence point of f and T if $fx \in Tx$ and is a common fixed point of f and T if $x = fx \in Tx$.*

Let (X,d) be a metric space, $f : X \to X$ be a single-valued mapping and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. T is said to be a f-weakly Picard mapping if and only if for each $x \in X$ and $fy \in Tx$ $(y \in X)$, there exists a sequence $\{x_n\}$ in X such that

(i) $x_0 = x, x_1 = y$;
(ii) $fx_{n+1} \in Tx_n$ for all $n = 0,1,2,...$;
(iii) the sequence $\{fx_n\}$ converges to fu, where u is the coincidence point of f and T.

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Theorem 1.1 (*Nadler*[14]). *Let (X,d) be a complete metric space and let $T: X \to \mathcal{CB}(X)$ be a multi-valued mapping such that for a fixed constant $h \in (0,1)$ and for each $x, y \in X$,*

$$H(Tx, Ty) \leq hd(x,y).$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

Afterward, several authors have focused on generalization of Nadler's fixed point theorem in various types. For example, in 2007, M. Berinde and V. Berinde [2] presented a nice fixed point result based on so-called $\mathcal{MT}$-functions which recently have been characterized by Du [6].

Definition 1.2 ([4,5,7]). *A function $\phi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-function if it satisfies Mizoguchi-Takahashi's condition, i.e.* $\limsup_{s\to t^+} \phi(s) < 1$ *for all $t \in [0,\infty)$.*

Theorem 1.2 ([2]). *Let (X,d) be a complete metric space and $T: X \to \mathcal{CB}(X)$ be a multi-valued mapping. If there exist an $\mathcal{MT}$-function $\phi : [0,\infty) \to [0,1)$ and $L \geq 0$ such that*

$$H(Tx, Ty) \leq \phi(d(x,y))d(x,y) + Ld(y, Tx) \ \text{ for all } \ x, y \in X,$$

then $\mathrm{Fix}(T) \neq \emptyset$.

We remark that letting $L = 0$ in Theorem 1.2 we can easily obtain the following result as Mizoguchi-Takahashi's fixed point theorem [13] which is considered as a partial answer of Problem 9 in Reich [17]. Also, Yu-Qing [21] gave an affirmative answer to a fixed point problem of Reich which was raised in [16].

Theorem 1.3 ([13]). *Let (X,d) be a complete metric space, $T: X \to \mathcal{CB}(X)$ be a multi-valued mapping and $\phi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Assume that*

$$H(Tx, Ty) \leq \phi(d(x,y))d(x,y) \ \text{ for all } \ x, y \in X.$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

Definition 1.3 ([9]). *Let (X,d) be a metric space, $f : X \to X$ be a single-valued mapping and $T: X \to \mathcal{CB}(X)$ be multi-valued mapping. If the sequence $\{x_n\}$ in X satisfying conditions (i) and (ii) in Definition 1.1, then the sequence $O_f(x_0) = \{fx_n : n = 1, 2, ...\}$ is said to be an f-orbit of T at x_0.*

Recently, Sintunavart and Kumam [20] presented a generalization of main result of Kamran [8, 9] and then unified and complemented the result by introducing the notion of hybrid generalized multi-valued contraction mapping in [19] as follows.

Definition 1.4 ([19]). *Let (X,d) be a metric space, $f : X \to X$ be a single-valued mapping and $T: X \to \mathcal{CB}(X)$ be a multi-valued mapping. T is said to be a hybrid generalized multi-valued contraction mapping if and only if there exist two functions $\phi : [0,\infty) \to [0,1)$ satisfying* $\limsup_{r\to t^+} \phi(r) < 1$ *for every $t \in [0,\infty)$ and $\varphi : [0,\infty) \to [0,\infty)$, such that*

$$H(Tx, Ty) \leq \phi(M(x,y))M(x,y) + \varphi(N(x,y))N(x,y),$$

for all $x, y \in X$, where

$$M(x,y) := \max\{d(fx, fy), d(fy, Tx)\}$$

and

$$N(x,y) := \min\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}.$$

Theorem 1.4 ([19]). *Let (X,d) be a metric space, $f : X \to X$ be a single-valued mapping and $T: X \to \mathcal{CB}(X)$ be a multi-valued mapping. Suppose that fX is a complete subspace of X and $TX \subset fX$. Then f and T have a coincidence point $z \in X$. Moreover, if $ffz = fz$, then f and T have a common fixed point.*

The following lemmas are crucial for the proof of our main result.

Lemma 1.1 ([9]). *Let (X,d) be a metric space, $\{A_k\}$ be a sequence in $\mathcal{CB}(X)$ and $\{x_k\}$ be a sequence in X such that $\{x_k\} \in A_{k-1}$. Let $h : [0,\infty) \to [0,1)$ be a function satisfying $\limsup_{r\to t^+} h(r) < 1$ for every $t \in [0,\infty)$. Suppose $d(x_{k-1}, x_k)$ to be a nonincreasing sequence such that*

$$\begin{aligned} H(A_{k-1}, A_k) &\leq h(d(x_{k-1}, x_k))d(x_{k-1}, x_k), \\ d(x_k, x_{k+1}) &\leq H(A_{k-1}, A_k) + h^{n_k}(d(x_{k-1}, x_k)), \end{aligned} \tag{1.1}$$

where $n_1 < n_2 < \cdots$ which $k, n_k \in \mathbb{N}$. Then $\{x_k\}$ is a Cauchy sequence in X.

Lemma 1.2 ([14]). *If $A, B \in \mathcal{CB}(X)$ and $a \in A$, then for each $\epsilon > 0$, there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \epsilon$.*

Throughout this work, we introduce the notions of a hybrid generalized multi-valued α-contractive type mapping based on Definition 1.4 and an α-admissible multi-valued mapping to obtain some fixed point theorems which either generalize or improve the corresponding recent fixed point results of Sintunavart and Kumam [19] (Theorem 1.4 of the present paper) and some ones in the literature. As an application, to show the applicability of our results, we give an existence theorem for certain class of functional equations arising in dynamic programming.

2 Main results

Denote with Ψ the family of functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy

(i) ψ is non-decreasing,
(ii) $\lim_{n\to\infty} \psi(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 0$ for $t_n \in [0,\infty)$,
(iii) ψ is subadditive, i.e., $\psi(t+s) \leq \psi(t) + \psi(s)$ for all $t, s \in [0,\infty)$,
(iv) for any $t > 0$ there exists an $s > 0$ such that $\psi(s) \leq t$,
(v) if $t < s$ then $t < \psi(s)$.

For example, function $\psi(t) = at$, where $a \geq 1$ is in Ψ.

In order to proceed with developing our work and obtain our results we need the following definition inspired by Definition 1.4 as follows.

Definition 2.5. *Let (X,d) be a metric space, $f : X \to X$ be a single-valued mapping and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. The mapping T is said to be a hybrid generalized multi-valued α-contractive type mapping if and only if there exist an $\mathcal{MT}$-function $\phi : [0,\infty) \to [0,1)$ and function $\varphi : [0,\infty) \to [0,\infty)$, such that*

$$\alpha(x,y)\psi(H(Tx,Ty)) \leq \phi(M(x,y))M(x,y) + \varphi(N(x,y))N(x,y),$$

for all $x, y \in X$, where $\psi \in \Psi$, $\alpha : X \times X \to [0,\infty)$ is a given function and

$$M(x,y) := \max\{d(fx,fy), d(fy,Tx)\}$$

and

$$N(x,y) := \min\{d(fx,fy), d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}.$$

Definition 2.6. *Let $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping, $f : X \to X$ be a single-valued mapping and $\alpha : X \times X \to [0,\infty)$. We say T is α-admissible if*

$$x, y \in X, \quad \alpha(x,y) \geq 1 \Longrightarrow \Lambda_f(Tx,Ty) \geq 1$$

where $\Lambda_f : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0,\infty)$ is given by $\Lambda_f(A,B) = \inf_{a,b\in X}\{\alpha(a,b) | fa \in A, fb \in B\}$.

Example 2.1. *Let $\alpha : X \times X \to [r,\infty)$ where $r \geq 1$. Then any arbitrary multi-valued mapping $T : X \to \mathcal{CB}(X)$ is an α-admissible.*

We first present the following simple lemma which is significant to prove our main result.

Lemma 2.3. *Let (X,d) be a metric space, $f : X \to X$ be a single-valued mapping and $T : X \to \mathcal{CB}(X)$ be an α-admissible multi-valued mapping. Suppose that $\Lambda_f(\{fx_0\}, Tx_0) \geq 1$ for some $x_0 \in X$ and $\{fx_k\}$ is an f-orbit of T at x_0. Then for all $n \in \mathbb{N}$, $\alpha(x_n, x_{n+1}) \geq 1$.*

Proof. According to the hypotheses let $x_0 \in X$ such that $\Lambda_f(\{fx_0\}, Tx_0) \geq 1$ and $\{fx_k\}$ be an f-orbit of T at x_0 so that $fx_n \in Tx_{n-1}$. Since T is α-admissible, we get the following implication

$$\alpha(x_0, x_1) \geq \Lambda_f(\{fx_0\}, Tx_0) \geq 1 \Longrightarrow \alpha(x_1, x_2) \geq \Lambda_f(Tx_0, Tx_1) \geq 1.$$

By the induction and using the α-admissibility of T, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}$$

which completes the proof. □

Now we give an immediate consequence of Lemma 1.2 as follows.

Lemma 2.4. *Let $A, B \in \mathcal{CB}(X)$, $a \in A$ and $\psi \in \Psi$. Then for each $\epsilon > 0$, there exists $b \in B$ such that $\psi(d(a,b)) \leq \psi(H(A,B)) + \epsilon$.*

Proof. Suppose, to the contrary, that there exist $a \in A$, $\epsilon > 0$ and $\psi \in \Psi$ such that

$$\psi(d(a,b)) > \psi(H(A,B)) + \epsilon \quad \text{for all } b \in B.$$

Following property (iv) of ψ we can choose an $s > 0$ such that $\psi(s) \leq \epsilon$. So for all $b \in B$ we get

$$\begin{aligned} \psi(d(a,b)) &> \psi(H(A,B)) + \epsilon \\ &\geq \psi(H(A,B)) + \psi(s) \\ &\geq \psi(H(A,B) + s), \end{aligned} \tag{2.2}$$

which implies

$$d(a,b) > H(A,B) + s \quad \text{for all } b \in B.$$

On the other hand, it follows from Lemma 1.2 that there is an element $b' \in B$ such that

$$d(a,b') \leq H(A,B) + s,$$

which shows an obvious contradiction and completes the proof. □

By the virtue of the proof of Lemma 1.1 and using properties (ii) and (iii) of function $\psi \in \Psi$ we can prove the lemma given below.

Lemma 2.5. *Let (X,d) be a metric space and $h : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Suppose $d(x_{k-1}, x_k)$ to be a nonincreasing sequence such that*

$$\psi(d(x_k, x_{k+1})) \leq h(d(x_{k-1}, x_k))\psi(d(x_{k-1}, x_k)) + h^{n_k}(d(x_{k-1}, x_k)),$$

where $\psi \in \Psi$ and $n_1 < n_2 < \cdots$ which $k, n_k \in \mathbb{N}$. Then $\{x_k\}$ is a Cauchy sequence in X.

The following is our main result and can be considered as a generalization of Theorem 1.4 and a series of results in the literature.

Theorem 2.5. *Let (X,d) be a metric space, $f : X \to X$ be a single-valued mapping and $T : X \to \mathcal{CB}(X)$ be a hybrid generalized multi-valued α-contractive type mapping such that fX is a complete subspace of X and $TX \subset fX$. Suppose that all the following conditions hold:*

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\Lambda_f(\{fx_0\}, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists an n_0 such that $\alpha(x_n, x) \geq 1$ for all $n \geq n_0$.

Then f and T have a coincidence point $u \in X$. Moreover, if $fu \in \mathrm{Fix}(f)$ and $\alpha(u, fu) \geq 1$, then fu is a common fixed point of f and T.

Proof. Let x_0 be an arbitrary element in X and $y_0 = fx_0$. We define the sequence $\{x_n\}$ and $\{y_n\}$ recursively as follows. Since $TX \subset fX$, we can choose an $x_1 \in X$ such that $y_1 = fx_1 \in Tx_0$. Since $0 \leq \phi(t) < 1$ for all $t \in [0, \infty)$ we can choose a positive integer n_1 such that

$$\phi^{n_1}(d(y_0, y_1)) \leq [1 - \phi(M(x_0, x_1))]M(x_0, x_1). \tag{2.3}$$

Following Lemma 2.4, we may select $y_2 = fx_2 \in Tx_1$ such that

$$\psi(d(y_1, y_2)) \leq \psi(H(Tx_0, Tx_1)) + \phi^{n_1}(d(y_0, y_1)).$$

Using (2.3), (ii) and the notion of a hybrid generalized multi-valued α-contractive type mapping, we get

$$\begin{aligned}
\psi(d(y_1, y_2)) &\leq \psi(H(Tx_0, Tx_1)) + \phi^{n_1}(d(y_0, y_1)) \\
&\leq \alpha(x_0, x_1)\psi(H(Tx_0, Tx_1)) + [1 - \phi(M(x_0, x_1))]M(x_0, x_1) \\
&\leq \phi(M(x_0, x_1))M(x_0, x_1) + \varphi(N(x_0, x_1))N(x_0, x_1) + [1 - \phi(M(x_0, x_1))]M(x_0, x_1) \\
&= M(x_0, x_1) \\
&= \max\{d(fx_0, fx_1), d(fx_1, Tx_0)\} \\
&= d(y_0, y_1).
\end{aligned}$$

Now, we take a positive integer n_2 which is greater than n_1 such that

$$\phi^{n_2}(d(y_1, y_2)) \leq [1 - \phi(M(x_1, x_2))]M(x_1, x_2).$$

Again following Lemma 2.4 since $TX \subset fX$, we may select $y_3 = fx_3 \in Tx_2$ such that

$$\psi(d(y_2, y_3)) \leq \psi(H(Tx_1, Tx_2)) + \phi^{n_2}(d(y_1, y_2)).$$

Similar to the previous case and using the α-admissibility of T and Lemma 2.3 we obtain the following

$$\begin{aligned}
\psi(d(y_2, y_3)) &\leq \psi(H(Tx_1, Tx_2)) + \phi^{n_2}(d(y_1, y_2)) \\
&\leq \alpha(x_1, x_2)\psi(H(Tx_1, Tx_2)) + [1 - \phi(M(x_1, x_2))]M(x_1, x_2) \\
&\leq \phi(M(x_1, x_2))M(x_1, x_2) + \varphi(N(x_1, x_2))N(x_1, x_2) + [1 - \phi(M(x_1, x_2))]M(x_1, x_2) \\
&= M(x_1, x_2) \\
&= \max\{d(fx_1, fx_2), d(fx_2, Tx_1)\} \\
&= d(y_1, y_2).
\end{aligned}$$

Continuing the procedure as above and applying Lemma 2.3, for all $k \in \mathbb{N}$, we can find a positive integer n_k such that

$$\phi^{n_k}(d(y_{k-1}, y_k)) \leq [1 - \phi(M(x_{k-1}, x_k))]M(x_{k-1}, x_k).$$

By applying Lemma 2.4 we may choose $y_{k+1} = fx_{k+1} \in Tx_k$ such that

$$\psi(d(y_k, y_{k+1})) \leq \psi(H(Tx_{k-1}, Tx_k)) + \phi^{n_k}(d(y_{k-1}, y_k)) \tag{2.4}$$

for each $k = 1, 2, \ldots$. The recent inequalities together with the notion of a hybrid generalized multi-valued α-contractive type mapping imply

$$\begin{aligned}
\psi(d(y_k, y_{k+1})) &\leq \psi(H(Tx_{k-1}, Tx_k)) + \phi^{n_k}(d(y_{k-1}, y_k)) \\
&\leq \alpha(x_{k-1}, x_k)\psi(H(Tx_{k-1}, Tx_k)) + [1 - \phi(M(x_{k-1}, x_k))]M(x_{k-1}, x_k) \\
&\leq \phi(M(x_{k-1}, x_k))M(x_{k-1}, x_k) + \varphi(N(x_{k-1}, x_k))N(x_{k-1}, x_k) \\
&\quad + [1 - \phi(M(x_{k-1}, x_k))]M(x_{k-1}, x_k) \qquad (2.5) \\
&= M(x_{k-1}, x_k) \\
&= \max\{d(fx_{k-1}, fx_k), d(fx_k, Tx_{k-1})\} \\
&= d(y_{k-1}, y_k)
\end{aligned}$$

for all $k \in \mathbb{N}$. If for some $k \in \mathbb{N}$, $y_{k-1} = y_k$ then there is nothing to prove. Otherwise, we claim that $d(y_{k-1}, y_k)$ is a nonincreasing sequence of nonnegative numbers. Suppose the opposite is true. It follows from (2.5) and property (v) of ψ that $d(y_{k-1}, y_k)$ is a nonincreasing sequence of nonnegative numbers. On the other hand

$$\begin{aligned}\psi(H(Tx_{k-1}, Tx_k)) &\leq \alpha(x_{k-1}, x_k)\psi(H(Tx_{k-1}, Tx_k))\\ &\leq \phi(M(x_{k-1}, x_k))M(x_{k-1}, x_k) + \varphi(N(x_{k-1}, x_k))N(x_{k-1}, x_k)\\ &\leq \phi(M(x_{k-1}, x_k))M(x_{k-1}, x_k)\\ &= \phi(d(y_{k-1}, y_k))d(y_{k-1}, y_k).\end{aligned}$$

This together with (2.4) imply

$$\psi(d(y_k, y_{k+1})) \leq \phi(d(y_{k-1}, y_k))d(y_{k-1}, y_k) + \phi^{n_k}(d(y_{k-1}, y_k)),$$

which shows that $\{y_k\} = \{fx_k\}$ is a Cauchy sequence in fX followed by Lemma 2.5. Hence, the sequence $\{fx_k\}$ is convergent to fu for some $u \in X$. Now following condition (iii) of assumptions and Lemma 2.3, since $\alpha(x_k, x_{k+1}) \geq 1$ we easily conclude that there exists a positive integer k_0 such that

$$\begin{aligned}\psi(d(fu, Tu)) &\leq \psi(d(fu, fx_k)) + \psi(d(fx_k, Tu))\\ &\leq \psi(d(fu, fx_k)) + \psi(H(Tx_{k-1}, Tu))\\ &\leq \psi(d(fu, fx_k)) + \alpha(x_{k-1}, u)\psi(H(Tx_{k-1}, Tu))\\ &\leq \psi(d(fu, fx_k)) + \phi(M(x_{k-1}, u))M(x_{k-1}, u) + \varphi(N(x_{k-1}, u))N(x_{k-1}, u)\end{aligned} \tag{2.6}$$

for all $k \geq k_0$. On the other hand, since $fx_k \to fu$ as $k \to \infty$ and $fx_k \in Tx_{k-1}$ so $d(fu, Tx_{k-1}) \to 0$ as $k \to \infty$. Therefore, the following equalities

$$\begin{aligned}M(x_{k-1}, u) &= \max\{d(fx_{k-1}, fu), d(fu, Tx_{k-1})\},\\ N(x_{k-1}, u) &= \min\{d(fx_{k-1}, fu), d(fx_{k-1}, Tx_{k-1}), d(fu, Tu), d(fx_{k-1}, Tu), d(fu, Tx_{k-1})\}\end{aligned}$$

together with (2.6) and property (ii) of ψ imply $d(fu, Tu) = 0$, that is, $fu \in Tu$ since $Tu \in \mathcal{CB}(X)$. This shows that u is a coincidence point of f and T. To prove that fu is a common fixed point of f and T, using the fact that $fu \in \mathrm{Fix}(f)$ and taking $v := fu \in Tu$ we have $fv = ffu = fu = v$ and hence

$$\begin{aligned}\psi(H(Tu, Tv)) &\leq \alpha(u, v)\psi(H(Tu, Tv))\\ &\leq \phi(M(u, v))M(u, v) + \varphi(N(u, v))N(u, v)\\ &= \phi(\max\{d(fu, fv), d(fv, Tu)\}) \max\{d(fu, fv), d(fv, Tu)\}\\ &\quad + \varphi(\min\{d(fu, fv), d(fu, Tu), d(fv, Tu), d(fu, Tv), d(fv, Tu)\})\\ &\quad \times \min\{d(fu, fv), d(fu, Tu), d(fv, Tv), d(fu, Tv), d(fv, Tu)\}\\ &= 0.\end{aligned}$$

Now it follows from $\psi(d(fv, Tv)) = \psi(d(fu, Tv)) \leq \psi(H(Tu, Tv)) = 0$ that $v = fv \in Tv$. Therefore, $v = fu$ is a common fixed point of f and T. □

Now we have the following immediate consequence.

Corollary 2.1. *Let (X, d) be a metric space and $f, T : X \to X$ be single-valued mappings. Suppose that T satisfies*

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \phi(M(x, y))M(x, y) + \varphi(N(x, y))N(x, y),$$

where $M(x, y), N(x, y), \phi, \varphi$ and ψ are given as in Definition 2.5, fX is a complete subspace of X and $TX \subset fX$. Suppose also that

(i) T is α-admissible, that is,

$$\text{if } x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \Lambda_f(Tx, Ty) \geq 1$$

where $\Lambda_f : X \times X \to [0, \infty)$ is given by $\Lambda_f(x, y) = \inf_{a,b \in X}\{\alpha(a, b) | fa = x, fb = y\}$;
(ii) there exists $x_0 \in X$ such that $\Lambda_f(fx_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists an n_0 such that $\alpha(x_n, x) \geq 1$ for all $n \geq n_0$.

Then f and T have a coincidence point $u \in X$. Moreover, if $fu \in \mathrm{Fix}(f)$ and $\alpha(u, fu) \geq 1$, then fu is a common fixed point of f and T.

To prove this, considering T as a single-valued mapping in Theorem 2.5 one can easily utilize the proof of Theorem 2.5, step by step, to conclude the result.

3 Functional equations in dynamic programming

Throughout this section, to verify the practicability of our main result we prove an existence theorem for a certain class of functional equations arising in dynamic programming which has been studied by utilizing various fixed point theorems in several papers (see [3,10-12,15,18] and the references therein).

From now on, we suppose that U and V are Banach spaces, $W \subset U$ and $D \subset V$. Also let $B(W)$ denote the set of all bounded real valued functions on W which is a metric space under the usual metric

$$d_B(h,k) = \sup_{x \in W} |h(x) - k(x)|, \quad h,k \in B(W).$$

Bellman and Lee [1] pointed out that the basic form of the functional equation of dynamic programming is as follows:

$$p(x) = \sup_{y \in D} H(x, y, p(\tau(x,y))), \;\; x \in W$$

where x and y stand for the state and decision vectors, respectively, $\tau : W \times D \to W$ represents the transformation of the process and $p(x)$ denotes the optimal return function with initial state x. Viewing W and D as the state and decision spaces, respectively, the problem of dynamic programming reduces to the problem of solving the functional equation

$$p(x) = \sup_{y \in D}\{g(x,y) + G(x,y,p(\tau(x,y)))\}, \;\; x \in W \tag{3.7}$$

where $g : W \times D \to \mathbb{R}$ and $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions. We investigate the existence and uniqueness of solutions for functional equation (3.7). Suppose that the functions ϕ, φ and ψ are given as in Definition 2.5 and the mapping F is defined by

$$F(h(x)) = \sup_{y \in D}\{g(x,y) + G(x,y,h(\tau(x,y)))\}, \;\; h \in B(W), \;\; x \in W. \tag{3.8}$$

Suppose that following conditions hold:

(i) that there exists a mapping $\Omega : B(W) \times B(W) \to \mathbb{R}$ such that

$$\Omega(h,k) \geq 0 \Longrightarrow \Omega(\sup_{y \in D}\{g(x,y) + G(x,y,h(\tau(x,y)))\}, \sup_{y \in D}\{g(x,y) + G(x,y,k(\tau(x,y)))\}) \geq 0, \;\; x \in W$$

where $h, k \in B(W)$;
(ii) there is an $h_0 \in B(W)$ such that

$$\Omega(h_0(x), \sup_{y \in D}\{g(x,y) + G(x,y,h_0(\tau(x,y)))\}) \geq 0 \;\text{ for }\; x \in W;$$

(iii) if $\{h_n\}$ is a sequence of functions in $B(W)$ such that $\Omega(h_n, h_{n+1}) \geq 0$ for all n and $\{h_n\}$ is pointwise convergent to $h \in B(W)$, then there exists an n_0 such that $\Omega(h_n, h) \geq 0$ for all $n \geq n_0$.

Then we have the following result.

Theorem 3.6. *Suppose that all the conditions (i)-(iii) are satisfied. In addition, for all $h, k \in B(W)$ with $\Omega(h,k) \geq 0$ we have*

$$\psi(d_B(F(h), F(k))) \leq \phi(M(h,k))M(h,k) + \varphi(N(h,k))N(h,k), \tag{3.9}$$

where

$$M(h,k) := \max\{d_B(h,k), d_B(k, F(h))\}$$

and

$$N(h,k) := \min\{d_B(h,k), d_B(h, F(h)), d_B(k, F(k)), d_B(h, F(k)), d_B(k, F(h))\}.$$

Then functional equation (3.7) possesses a solution in $B(W)$.

Proof. Bearing in mind that $(B(W), d_B)$ is a complete metric space, where d_B is the metric induced by the usual metric on $B(W)$. It is easy to see that $B(W)$ is F-invariant, that is, for any $h \in B(W)$ we get $F(h) \in B(W)$. Clearly, F is both hybrid generalized multi-valued α-contractive type mapping and α-admissible with

$$\alpha(h,k) = \begin{cases} 1, & \Omega(h,k) \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

Now, replacing f by the identity mapping and taking $h_{n+1} = F(h_n)$ in the proof of Theorem 2.5 we can find a positive integer n_k such that

$$\phi^{n_k}(d_B(h_{k-1}, h_k)) \leq [1 - \phi(M(h_{k-1}, h_k))]M(h_{k-1}, h_k)$$

and

$$\psi(d_B(h_k, h_{k+1})) \leq \psi(d_B(F(h_{k-1}), F(h_k))) + \phi^{n_k}(d_B(h_{k-1}, h_k)) \tag{3.10}$$

for each $k = 1, 2, \ldots$. These inequalities together with the fact that F enjoys condition (3.9) implies that $d_B(h_{k-1}, h_k)$ is a nondecreasing sequence, hence h_k converges to some h in $B(W)$. Now by subadditivity of ψ and using (3.9) we easily conclude that $d_B(h, F(h)) = 0$ which completes the proof. □

Theorem 3.7. *Suppose in addition to conditions of Theorem 3.6, that*

$$\Omega(h,k) \geq 0 \Longrightarrow \left(\frac{1}{d_B} - \left(\frac{\phi}{\psi}\right) \circ d_B\right)(h,k) > 0 \tag{3.11}$$

for $h, k \in B(W)$ such that $h \neq k$. Then functional equation (3.7) has a unique solution in $B(W)$.

Proof. Let $u, v \in B(W)$ be two distinct solutions of functional equation (3.7), that is, $Tu = u$ and $Tv = v$. Then using (3.9) we get

$$\psi(d_B(u,v)) \leq \phi(d_B(u,v))d_B(u,v)$$

which contradicts (3.11) and the conclusion follows. □

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Received: May 5, 2013; *Accepted*: January 8, 2014

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