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Meromorphic parabolic starlike functions with a fixed point involving Srivastava-Attiya operator

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Abstract

In the present investigation, we introduce a new class of meromorphic parabolic starlike functions with a fixed point defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ by making use of the Srivastava-Attiya Operator $\mathcal{J}_b^s$. We obtained Coefficient inequalities, growth and distortion inequalities, as well as closure results for functions $f \in \mathcal{M}_b^s(\lambda, \beta, \gamma)$. We further established some results concerning convolution and the partial sums.

Keywords: Meromorphic functions, starlike function, convolution, positive coefficients, coefficient inequalities, integral operator.

2010 MSC: 30C50.

1 Introduction

Let ξ be a fixed point in the unit disc $:= \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{H}()$ the class of functions which are regular and

$$\mathcal{A}(\xi) = \{f \in H() : f(\xi) = f'(\xi) - 1 = 0\}.$$

Also denote by

$$\mathcal{S}_\xi = \{f \in \mathcal{A}(\xi) : f \text{ is univalent in } \},$$

the subclass of $\mathcal{A}(\xi)$ consist of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n (z - \xi)^n, \tag{1.1}$$

that are analytic in the open unit disc . Note that $\mathcal{S}_0 = \mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in . By $\mathcal{S}_\xi^*(\gamma)$ and $\mathcal{K}_\xi(\gamma)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions

$$\Re\left(\frac{(z-\xi)f'(z)}{f(z)}\right) > \gamma, \Re\left(1 + \frac{(z-\xi)f''(z)}{f'(z)}\right) > \gamma$$

and $z \in$ for $0 \leq \gamma < 1$, introduced and studied by Kanas and Ronning [11]. The class $\mathcal{S}_\xi^*(0)$ is defined by geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathcal{K}_\xi(0)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [8] and [9] for uniformly starlike and convex functions, except that in this case the point ξ is fixed. In particular, $\mathcal{K}_0 = \mathcal{K}(0)$ and $\mathcal{S}_0^* = \mathcal{S}^*(0)$ respectively, are the well-known standard class of convex and starlike functions(see [21]).

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Let Σ denote the class of meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.2}$$

defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Denote by Σ_ξ be the subclass of $\mathcal{A}(\xi)$ consist of the functions of the form

$$f(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} a_n (z-\xi)^n, a_n \geq 0; z \neq \xi. \tag{1.3}$$

A function f of the form (1.3) is in the class of *meromorphic starlike of order* γ $(0 \leq \gamma < 1)$ denoted by $\Sigma_\xi^*(\gamma)$, if

$$-\Re\left(\frac{(z-\xi)f'(z)}{f(z)}\right) > \gamma, \quad z-\xi \in \Delta := \Delta^* \cup \{0\} \tag{1.4}$$

and is in the class of *meromorphic convex of order* γ $(0 \leq \gamma < 1)$ denoted by $\Sigma_\xi^K(\gamma)$, if

$$-\Re\left(1 + \frac{(z-\xi)f''(z)}{f'(z)}\right) > \gamma, \quad z-\xi \in \Delta := \Delta^* \cup \{0\}.$$

For functions $f(z)$ given by (1.3) and $g(z) = \frac{1}{(z-\xi)} + \sum_{n=1}^{\infty} b_n (z-\xi)^n$, $(b_n \geq 0)$ we define the Hadamard product or convolution of f and g by

$$(f * g)(z) := \frac{1}{z-\xi} + \sum_{n=1}^{\infty} a_n\, b_n (z-\xi)^n.$$

The study of operators plays a vital role in the geometric function theory and its associated fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical investigation and also helps to understand the geometric properties of such operators better.

We recall a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (see [24])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.5}$$

$$(a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}, \Re(s) > 1 \text{ and } |z| = 1)$$

where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$ $(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}; \mathbb{N} := \{1,2,3,...\})$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [5], Lin and Srivastava [12], Lin et al. [13], and see the references stated therein.

For the class of analytic functions denote by $\mathcal{A}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in)$$

Srivastava and Attiya [23] introduced and investigated the linear operator:

$$\mathcal{J}_{s,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product (or convolution) by

$$\mathcal{J}_{s,b} f(z) = G_{b,s} * f(z) \tag{1.6}$$

where, for convenience,

$$G_{b,s}(z) := (1+b)^s[\Phi(z,s,b) - b^{-s}] \tag{1.7}$$

$(z \in ; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}; f \in \mathcal{A})$. For $f \in \mathcal{A}$ it is easy to observe from (1.6) and (1.7) that

$$\mathcal{J}_{s,b} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^s a_n z^n, \quad (z \in .) \tag{1.8}$$

It is well known that the Srivastava-Attiya operator $\mathcal{J}_{s,b}$ contains, among its special cases, the integral operators introduced and investigated earlier by (for example) Alexander [1], Libera [14], Bernardi [4], and Jung et al. [10].

Motivated essentially by the above mentioned Srivastava-Attiya operator, in this paper we define a new linear operator

$$\mathcal{J}_b^s : \Sigma_\xi \to \Sigma_\xi$$

in terms of Hadamard product given by

$$\mathcal{J}_b^s f(z) = \mathcal{G}_{b,p}^s * f(z) \tag{1.9}$$

$$(z-\xi \in \Delta := \Delta^* \cup \{0\}; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}; f \in \Sigma_\xi),$$

where, for convenience

$$\mathcal{G}_{b,p}^s(z) := (1+b)^s[\Phi_p(z,s,b) - b^{-s}] \tag{1.10}$$

and

$$\Phi_p(z,s,b) = \frac{1}{b^s} + \frac{(z-\xi)^{-1}}{(1+b)^s} + \frac{(z-\xi)}{(2+b)^s} + \dots .$$

For $f \in \Sigma_\xi$, it is easy to observe from the above equations (1.9) and (1.10) that

$$\mathcal{J}_b^s f(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} C_b^s(n) a_n (z-\xi)^n, \quad (z-\xi \in \Delta := \Delta^* \cup \{0\}) \tag{1.11}$$

where

$$C_b^s(n) = \left| \left(\frac{1+b}{n+1+b} \right)^s \right| \tag{1.12}$$

and (throughout this paper unless otherwise mentioned) the parameters s, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}$.

Motivated by earlier works on meromorphic functions by function theorists(see [2, 3, 7, 15, 16, 17, 18, 19, 20, 25]), we define the following new subclass of functions in Σ_ξ by making use of the generalized operator $\mathcal{J}_b^s$.

For $0 \leq \gamma < 1$ and $0 \leq \lambda < 1/2$, we let $\mathcal{M}_b^s(\lambda, \beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (1.3) satisfying the condition that

$$\begin{aligned} & - \Re\left(\frac{(z-\xi)(\mathcal{J}_b^s f(z))' + \lambda (z-\xi)^2 (\mathcal{J}_b^s f(z))''}{(1-\lambda)\mathcal{J}_b^s f(z) + \lambda(z-\xi)(\mathcal{J}_b^s f(z))'} \right) \\ & > \beta \left| \frac{(z-\xi)(\mathcal{J}_b^s f(z))' + \lambda (z-\xi)^2 (\mathcal{J}_b^s f(z))''}{(1-\lambda)\mathcal{J}_b^s f(z) + \lambda(z-\xi)(\mathcal{J}_b^s f(z))'} + 1 \right| + \gamma \end{aligned} \tag{1.13}$$

where $\mathcal{J}_b^s f$ is given by (1.11).

Further shortly we can state this condition by

$$-\Re\left(\frac{(z-\xi)G'(z)}{G(z)} \right) > \beta \left| \frac{(z-\xi)G'(z)}{G(z)} + 1 \right| + \gamma, \tag{1.14}$$

where

$$G(z) = (1-\lambda)\mathcal{J}_b^s f(z) + \lambda(z-\xi)(\mathcal{J}_b^s f(z))' = \frac{1-2\lambda}{z-\xi} + \sum_{n=1}^{\infty} (n\lambda - \lambda + 1) C_b^s(n) a_n (z-\xi)^n, \quad a_n \geq 0. \tag{1.15}$$

It is of interest to note that, on specializing the parameters λ, β and s, b we can define various new subclasses of Σ_ξ. We illustrate two important subclasses in the following examples.

Example 1.1. *For $\lambda = 0$, we let $\mathcal{M}_b^s(0, \beta, \gamma) = \mathcal{M}_b^s(\beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (1.3) satisfying the condition that*

$$- \Re\left(\frac{(z-\xi)(\mathcal{J}_b^s f(z))'}{\mathcal{J}_b^s f(z)} \right) > \beta \left| \frac{(z-\xi)(\mathcal{J}_b^s f(z))'}{\mathcal{J}_b^s f(z)} + 1 \right| + \gamma \tag{1.16}$$

where $\mathcal{J}_b^s f(z)$ is given by (1.11).

Example 1.2. *For $\lambda = 0, \beta = 0$ we let $\mathcal{M}_b^s(0,0,\gamma) = \mathcal{M}_b^s(\gamma)$ denote a subclass of Σ_ξ consisting functions of the form (1.3) satisfying the condition that*

$$- \Re\left(\frac{(z-\xi)(\mathcal{J}_b^s f(z))'}{\mathcal{J}_b^s f(z)}\right) > \gamma \tag{1.17}$$

where $\mathcal{J}_b^s f(z)$ is given by (1.11).

In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, as well as closure results for the function class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. Properties of certain integral operator and convolution properties of the new class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$ are also discussed.

2 Coefficients Inequalities

In order to obtain the necessary and sufficient condition for a function $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$, we recall the following lemmas.

Lemma 2.1. *If γ is a real number and w is a complex number, then $\Re(w) \geq \gamma \Leftrightarrow |w + (1-\gamma)| - |w - (1+\gamma)| \geq 0$.*

Lemma 2.2. *If w is a complex number and γ, k are real numbers, then*

$$\Re(w) \geq k|w-1| + \gamma \Leftrightarrow \Re\{w(1+ke^{i\theta}) - ke^{i\theta}\} \geq \gamma, \quad -\pi \leq \theta \leq \pi.$$

Analogous to the lemma proved by Dziok et.al [7], we state the following lemma without proof.

Lemma 2.3. *Suppose that $\gamma \in [0,1)$, $r \in (0,1]$ and the function $H \in \Sigma_\xi(\gamma)$ is of the form $H(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} b_n (z-\xi)^n$, $\quad 0 < |z-\xi| < r$, with $b_n \geq 0$ then*

$$\sum_{n=1}^{\infty} (n+\gamma) b_n r^{n+1} \leq 1 - \gamma. \tag{2.1}$$

Theorem 2.1. *Let $f \in \Sigma_\xi$ be given by (1.3). Then $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty} [n(1+\beta) + (\gamma+\beta)](n\lambda - \lambda + 1)\ C_b^s(n) a_n \leq (1-2\lambda)(1-\gamma). \tag{2.2}$$

Proof. If $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$, then by (1.14) we have,

$$-\Re\left(\frac{(z-\xi)G'(z)}{G(z)}\right) > \beta\left|\frac{(z-\xi)G'(z)}{G(z)} + 1\right| + \gamma.$$

Making use of Lemma 2.2

$$-\Re\left(\frac{(z-\xi)(1+\beta e^{i\theta})G'(z) + \beta e^{i\theta} G(z)}{G(z)}\right) > \gamma,$$

where $G(z)$ is given by (1.15). Substituting for $G(z), G'(z)$ and letting $|z-\xi| < r \to 1^-$, we have

$$\left\{\frac{(1-2\lambda)(1-\gamma) - \sum_{n=1}^{\infty}[n(1+\beta)+(\gamma+\beta)](n\lambda-\lambda+1)C_b^s(n)a_n}{(1-2\lambda) - \sum_{n=1}^{\infty}(n\lambda-\lambda+1)C_b^s(n)a_n}\right\} > 0.$$

This shows that (2.2) holds.

Conversely, assume that (2.2) holds. Since $-\Re(w) > \gamma$, if and only if $|w+1| < |w-(1-2\gamma)|$, it is sufficient to show that

$$\left|\frac{w+1}{w-(1-2\gamma)}\right| < 1 \text{ and } |w-(1-2\gamma)| \neq 0 \text{ for } |z-\xi| < r \leq 1, \quad (z-\xi) \in \Delta.$$

Using (2.2) and taking $w(z) = \frac{(z-\xi)(1+\beta e^{i\theta})G'(z)+\beta e^{i\theta}G(z)}{G(z)}$ we get

$$\left|\frac{w+1}{w-(1-2\gamma)}\right| \leq \frac{\sum_{n=1}^{\infty}(n\lambda-\lambda+1)[(n+1)(1+\beta)]C_b^s(n)a_n}{2(1-\gamma)(1-2\lambda) - \sum_{n=1}^{\infty}(n\lambda-\lambda+1)[n(1+\beta)+(\beta+2\gamma-1)]C_b^s(n)a_n} \leq 1.$$

Thus we have $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$. □

For the sake of brevity throughout this paper we let

$$d_n(\lambda,\beta,\gamma) := [n(1+\beta)+(\gamma+\beta)](n\lambda-\lambda+1) \tag{2.3}$$
$$d_1(\lambda,\beta,\gamma) = (1+\gamma+2\beta)$$

unless otherwise stated.

Our next result gives the coefficient estimates for functions in $\mathcal{M}_b^s(\lambda,\beta,\gamma)$.

Theorem 2.2. *If $f\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$, then*

$$a_n \le \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}, \qquad n=1,2,3,\ldots.$$

The result is sharp for the functions $f_n(z)$ given by

$$f_n(z) = \frac{1}{z-\xi} + \frac{1-\gamma}{d_n(\lambda,\beta,\gamma)C_b^s(n)}(z-\xi)^n, \qquad n=1,2,3,\ldots.$$

Proof. If $f\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$, then we have, for each n,

$$d_n(\lambda,\beta,\gamma)C_b^s(n)a_n \le \sum_{n=1}^{\infty} d_n(\lambda,\beta,\gamma)C_b^s(n)a_n \le (1-\gamma)(1-2\lambda).$$

Therefore we have

$$a_n \le \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}.$$

Since

$$f_n(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}(z-\xi)^n$$

satisfies the conditions of Theorem 2.1, $f_n(z)\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$ and the equality is attained for this function. □

Theorem 2.3. *Suppose that there exists a positive number ν*

$$\nu = \inf_{n\in\mathbb{N}} \left\{d_n(\lambda,\beta,\gamma)C_b^s(n)\right\}. \tag{2.4}$$

If $f\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$, then

$$\left|\frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)}{\nu}r\right| \le |f(z)| \le \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r, \quad (|z-\xi|=r)$$

and

$$\left|\frac{1}{r^2} - \frac{(1-\gamma)(1-2\lambda)}{\nu}\right| \le |f'(z)| \le \frac{1}{r^2} + \frac{(1-\gamma)(1-2\lambda)}{\nu} \quad (|z-\xi|=r).$$

If $\nu = d_1(\lambda,\beta,\gamma)C_b^s(1) = (1+\gamma+2\beta)C_b^s(1)$, then the result is sharp for

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{(1+\gamma+2\beta)C_b^s(1)}(z-\xi). \tag{2.5}$$

Proof. Let the function f given by (1.3) we have

$$|f(z)| \le \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \le \frac{1}{r} + r\sum_{n=1}^{\infty} a_n.$$

Since,

$$\sum_{n=1}^{\infty} a_n \le \frac{(1-\gamma)(1-2\lambda)}{\nu}.$$

Using this, we have

$$|f(z)| \le \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r.$$

Similarly

$$|f(z)| \ge \left|\frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)}{\nu}r\right|.$$

The result is sharp for function (2.5) with $\nu = d_1(\lambda,\beta,\gamma)C_b^s(1) = (1+\gamma+2\beta)C_b^s(1)$.
Similarly we can prove the other inequality $|f'(z)|$. □

3 Radius of starlikeness

In the following theorem we obtain the radius of starlikeness for the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. We say that f given by (1.3) is meromorphically starlike of order ρ, $(0 \le \rho < 1)$, in $|z-\xi| < r$ when it satisfies the condition (1.4) in $|z-\xi| < r$.

Theorem 3.1. *Let the function f given by (1.3) be in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. Then, if there exists*

$$r_1(\lambda,\gamma,\rho) = \inf_{n\ge 1}\left[\frac{(1-\rho)d_n(\lambda,\beta,\gamma)C_b^s(n)}{(n+\rho)(1-\gamma)(1-2\lambda)}\right]^{\frac{1}{n+1}} \tag{3.1}$$

and it is positive, then f is meromorphically starlike of order ρ in $|z-\xi| < r \le r_1(\lambda,\gamma,\rho)$.

Proof. Let the function $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ be of the form (1.3). If $0 < r \le r_1(\lambda,\gamma,\rho)$, then by (3.1)

$$r^{n+1} \le \frac{(1-\rho)d_n(\lambda,\beta,\gamma)C_b^s(n)}{(n+\rho)(1-\gamma)(1-2\lambda)} \tag{3.2}$$

for all $n \in \mathbb{N}$. From (3.2) we get

$$\frac{n+\rho}{1-\rho}r^{n+1} \le \frac{d_n(\lambda,\beta,\gamma)C_b^s(n)}{(1-\gamma)(1-2\lambda)}$$

for all $n \in \mathbb{N}$, thus

$$\sum_{n=1}^{\infty}\frac{n+\rho}{1-\rho}a_n r^{n+1} \le \sum_{n=1}^{\infty}\frac{d_n(\lambda,\beta,\gamma)C_b^s(n)}{(1-\gamma)(1-2\lambda)}a_n \le 1 \tag{3.3}$$

because of (2.2). Hence, from (3.3) and (2.1), f is meromorphically starlike of order ρ in $|z-\xi| < r \le r_1(\lambda,\gamma,\rho)$. □

Suppose that there exists a number $\widetilde{r}, \widetilde{r} > r_1(\lambda,\gamma,\rho)$ such that each $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ is meromorphically starlike of order ρ in $|z-\xi| < \widetilde{r} \le 1$. The function

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}(z-\xi)^n$$

is in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$, thus it should satisfy (2.1) with $\widetilde{r}$:

$$\sum_{n=1}^{\infty}(n+\rho)a_n\widetilde{r}^{n+1} \le 1-\rho, \tag{3.4}$$

while the left–hand side of (3.4) becomes

$$(n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}\widetilde{r}^{n+1} > (n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}\frac{(1-\rho)d_n(\lambda,\beta,\gamma)C_b^s(n)}{(n+\rho)(1-\gamma)(1-2\lambda)} = 1-\rho$$

which contradicts with (3.4). Therefore the number $r_1(\lambda,\gamma,\rho)$ in Theorem 3.1, cannot be replaced with a greater number. This means that $r_1(\lambda,\gamma,\rho)$ is called radius of meromorphically starlikness of order ρ for the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$.

4 Results Involving Modified Hadamard Products

For functions

$$f_j(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty}a_{n,j}(z-\xi)^n, a_{n,j} \ge 0 \tag{4.5}$$

we define the Hadamard product or convolution of f_1 and f_2 by

$$(f_1 * f_2)(z) := \frac{1}{z-\xi} + \sum_{n=1}^{\infty}a_{n,1}a_{n,2}(z-\xi)^n.$$

Let

$$\Psi(n,\lambda) = \frac{(n\lambda-\lambda+1)}{(1-2\lambda)}C_b^s(n). \tag{4.6}$$

Theorem 4.2. *For functions $f_j (j = 1, 2)$ defined by (4.5), let $f_1 \in \mathcal{M}_b^s(\lambda, \beta, \gamma)$ and $f_2 \in \mathcal{M}_b^s(\lambda, \beta, \delta)$. Then $f_1 * f_2 \in \mathcal{M}_b^s(\lambda, \beta, \eta)$ where*

$$\eta = 1 - \frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)\Psi(1,\lambda) - 2(1-\gamma)(1-\delta)} \tag{4.7}$$

and $\Psi(1, \lambda) = \frac{C_b^s(1)}{1-2\lambda}$. The results is the best possible for

$$\begin{aligned} f_1(z) &= \frac{1}{z-\xi} + \frac{1-\gamma}{(1+\gamma+2\beta)\Psi(1,\lambda)}(z-\xi), \\ f_2(z) &= \frac{1}{z-\xi} + \frac{1-\delta}{(1+\delta+2\beta)\Psi(1,\lambda)}(z-\xi) \end{aligned}$$

where $\Psi(1, \lambda) = \frac{C_b^s(1)}{1-2\lambda}$.

Proof. In the view of Theorem 2.1, it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{[n(1+\beta) + (\eta+\beta)]}{(1-\eta)} \Psi(n,\lambda) a_{n,1} a_{n,2} \leq 1$$

where η is defined by (4.7) under the hypothesis. It follows from (2.2) and the Cauchy's-Schwarz inequality that

$$\sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}} \Psi(n,\lambda)\sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{4.8}$$

Thus we need to find largest η such that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{[n(1+\beta) + (\eta+\beta)]}{(1-\eta)} \Psi(n,\lambda) a_{n,1} a_{n,2} \\ \leq & \sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}} \Psi(n,\lambda)\sqrt{a_{n,1}a_{n,2}} \\ \leq & 1. \end{aligned}$$

By virtue of (4.8) it is sufficient to find the largest η, such that

$$\begin{aligned} & \frac{\sqrt{(1-\gamma)(1-\delta)}}{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}\Psi(n,\lambda)} \\ \leq & \frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}} \frac{1-\eta}{[n(1+\beta)+(\eta+\beta)]}, \end{aligned}$$

which yields

$$\eta \leq 1 - \frac{(1-\gamma)(1-\delta)(2n+1+\beta)}{[n(1+\beta)+(\gamma+\beta)][n(1+\beta)+(\delta+\beta)]\Psi(n,\lambda) - (1-\gamma)(1-\delta)(n+1)}$$

for $n \geq 1$ where $\Psi(n, \lambda)$ is given by (4.6) and since $\Psi(n, \lambda)$ is a decreasing function of n $(n \geq 1)$, we have

$$\eta = 1 - \frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)\Psi(1,\lambda) - 2(1-\gamma)(1-\delta)}$$

and $\Psi(1, \lambda) = \frac{C_b^s(1)}{1-2\lambda}$, which completes the proof. □

Theorem 4.3. *Let the functions $f_j, (j = 1, 2)$ defined by (4.5) be in the class $\mathcal{M}_b^s(\lambda, \beta, \gamma)$. Then $(f_1 * f_2)(z) \in \mathcal{M}_b^s(\lambda, \beta, \eta)$ where*

$$\eta = 1 - \frac{(1-\gamma)^2(3+\beta)}{(1+\gamma+2\beta)^2\Psi(1,\lambda) - 2(1-\gamma)^2}$$

with $\Psi(1, \lambda) = \frac{C_b^s(1)}{1-2\lambda}$.

Proof. By taking $\delta = \gamma$ in the above theorem, the results follows. □

For functions in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$ we can prove the following inclusion property.

Theorem 4.4. *Let the functions $f_j(j=1,2)$ defined by (4.5) be in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. Then the function h defined by*

$$h(z)=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}(a_{n,1}^2+a_{n,2}^2)(z-\xi)^n$$

is in the class $\mathcal{M}_b^s(\lambda,\beta,\delta)$ where

$$\delta\le 1-\frac{4(1-\gamma)^2(1+\beta)}{[1+\gamma+2\beta]^2\Psi(1,\lambda)+2(1-\gamma)^2}, \tag{4.9}$$

and $\Psi(1,\lambda)=\frac{C_b^s(1)}{1-2\lambda}$.

Proof. In view of Theorem 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\Psi(n,\lambda)\frac{[n(1+\beta)+(\delta+\beta)]}{(1-\delta)}(a_{n,1}^2+a_{n,2}^2)\le 1 \tag{4.10}$$

where $f_j\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$ $(j=1,2)$, we find from (4.5) and Theorem 2.1, that

$$\sum_{n=1}^{\infty}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2 a_{n,j}^2\le\sum_{n=1}^{\infty}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}a_{n,j}\right]^2\le 1, \tag{4.11}$$

which would yields

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2(a_{n,1}^2+a_{n,2}^2)\le 1. \tag{4.12}$$

On comparing (4.10) and (4.12) it can be seen that inequality (4.9) will be satisfied if

$$\Psi(n,\lambda)\frac{[n(1+\beta)+(\delta+\beta)]}{1-\delta}(a_{n,1}^2+a_{n,2}^2)\le\frac{1}{2}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2(a_{n,1}^2+a_{n,2}^2).$$

That is, if

$$\delta\le 1-\frac{2(1-\gamma)^2[(n+1)(1+\beta)]}{[n(1+\beta)+(\gamma+\beta)]^2\Psi(n,\lambda)+2(1-\gamma)^2} \tag{4.13}$$

where $\Psi(n,\lambda)$ is given by (4.6) and $\Psi(n,\lambda)$ is a decreasing function of n $(n\ge 1)$, we get (4.9), which completes the proof. □

5 Closure Theorems

We state the following closure theorems for $f\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$ without proof (see [7, 16, 18]).

Theorem 5.5. *Let the function $f_k(z)=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}a_{n,k}(z-\xi)^n$ be in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$ for every $k=1,2,...,m$. Then the function f defined by*

$$f(z)=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}a_{n,k}(z-\xi)^n,\ (a_{n,k}\ge 0)$$

belongs to the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$, where $a_{n,k}=\frac{1}{m}\sum_{k=1}^{m}a_{n,k},\ (n=1,2,..)$.

Theorem 5.6. *Let $f_0(z)=\frac{1}{z-\xi}$ and $f_n(z)=\frac{1}{z-\xi}+\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}(z-\xi)^n$ for $n=1,2,\ldots$. Then $f\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$ if and only if f can be expressed in the form $f(z)=\sum_{n=0}^{\infty}\eta_nf_n(z)$ where $\eta_n\ge 0$ and $\sum_{n=0}^{\infty}\eta_n=1$.*

Theorem 5.7. *The class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$ is closed under convex linear combination.*

Now, we prove that the class is$\mathcal{M}_b^s(\lambda,\beta,\gamma)$ closed under integral transforms .

Theorem 5.8. *Let the function $f(z)$ given by (1.3) be in $\mathcal{M}_b^s(\lambda, \beta, \gamma)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)du \qquad (0 < u \leq 1, 0 < c < \infty)$$

is in $\mathcal{M}_b^s(\lambda, \beta, \delta)$, where

$$\delta \leq \frac{n^2(1+\beta) + n[(\gamma+\beta) + (1+\beta)(1+c\gamma)] + (c+1)(\gamma+\beta) + c\beta(1-\gamma)}{n^2(1+\beta) + n[(\gamma+\beta) + (1+c)(1+\beta)] + (1+c)(\gamma+\beta) + c(1-\gamma)}.$$

The result is sharp for the function $f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{(1+\gamma+2\beta)C_b^s(1)}(z-\xi)$.

Proof. Let $f(z) \in \mathcal{M}_b^s(\lambda, \beta, \gamma)$. Then

$$F(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z-w} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n (z-\xi)^n.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c\, d_n(\lambda, \beta, \delta) C_b^s(n)}{(c+n+1)(1-\delta)} a_n \leq 1. \tag{5.14}$$

Since $f \in \mathcal{M}_b^s(\lambda, \beta, \gamma)$, we have

$$\sum_{n=1}^{\infty} \frac{d_n(\lambda, \beta, \gamma) C_b^s(n)}{(1-\gamma)(1-2\lambda)} a_n \leq 1.$$

Note that (5.14) is satisfied if

$$\frac{c\, d_n(\lambda, \beta, \delta) C_b^s(n)}{(c+n+1)(1-\delta)} \leq \frac{d_n(\lambda, \beta, \gamma) C_b^s(n)}{(1-\gamma)(1-2\lambda)}.$$

Solving for δ, we have

$$\delta \leq \frac{n^2(1+\beta) + n[(\gamma+\beta) + (1+\beta)(1+c\gamma)] + (c+1)(\gamma+\beta) + c\beta(1-\gamma)}{n^2(1+\beta) + n[(\gamma+\beta) + (1+c)(1+\beta)] + (1+c)(\gamma+\beta) + c(1-\gamma)} = \Phi(n).$$

A simple computation will show that $\Phi(n)$ is increasing and $\Phi(n) \geq \Phi(1)$. Using this, the results follows. □

6 Partial Sums

Silverman [22] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested to search results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [22] and Cho and Owa [6] we will investigate the ratio of a function of the form (1.3) to its sequence of partial sums

$$f_k(z) = \frac{1}{z-\xi} + \sum_{n=1}^{k} a_n (z-\xi)^n \tag{6.15}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} d_n(\lambda, \beta, \gamma) C_b^s(n)\, a_n \leq (1-\gamma)(1-2\lambda).$$

More precisely we will determine sharp lower bounds for $\Re\left(\frac{f(z)}{f_k(z}\right)$ and $\Re\left(\frac{f_k(z)}{f(z}\right)$. In this connection we make use of the well known results that $\Re\left(\frac{1+w(z)}{1-w(z)}\right) > 0, \quad (z-\xi \in \Delta)$ if and only if $w(z) = \sum_{n=1}^{\infty} c_n (z-\xi)^n$ satisfies the inequality $|w(z)| \leq |z-\xi|$.

Unless otherwise stated, we will assume that f is of the form (1.3) and its sequence of partial sums is denoted by (6.15).

Theorem 6.9. *Let $f(z) \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ be given by (1.3) satisfies condition, (2.2) and suppose that all of its partial sums (6.15) don't vanish in Δ. Moreover, suppose that*

$$2-2\sum_{n=1}^{k}|a_n| - \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\sum_{n=k+1}^{\infty}|a_n| > 0, for\ \ all\ \ k \in \mathbb{N}. \tag{6.16}$$

Then,

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \geq 1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)} \qquad (z-\xi \in \Delta) \tag{6.17}$$

where

$$d_n(\lambda,\beta,\gamma) \geq \begin{cases} (1-\gamma)(1-2\lambda), & if \quad n = 1,2,3,\ldots,k \\ d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1), & if \quad n = k+1,k+2,\ldots. \end{cases} \tag{6.18}$$

The result (6.17) is sharp with the function given by

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}(z-\xi)^{k+1}. \tag{6.19}$$

Proof. Define the function $w(z)$ by

$$w(z) = \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\left[\frac{f(z)}{f_k(z)} - \left(1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}\right)\right]$$

$$= 1 + \frac{\frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\sum\limits_{n=k+1}^{\infty} a_n(z-\xi)^{n+1}}{1+\sum\limits_{n=1}^{k} a_n(z-\xi)^{n+1}}. \tag{6.20}$$

It suffices to show that $\Re(w(z)) > 0$, hence we find that

$$\left|\frac{1+w(z)}{1-w(z)}\right| \leq \frac{\frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{k}|a_n| - \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\sum\limits_{n=k+1}^{\infty}|a_n|} \leq 1$$

From the condition (2.2),it readily yields the assertion (6.17) of Theorem 6.9.

To see that the function given by (6.19) gives the sharp result, we observe that for $z = re^{i\pi/(k+2)}$

$$\frac{f(z)}{f_k(z)} = 1 + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}(z-\xi)^n \to 1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}$$

when $r \to 1^-$ which shows the bound (6.17) is the best possible for each $k \in \mathbb{N}$. □

We next determine bounds for $f_k(z)/f(z)$.

Theorem 6.10. *Under the assumptions of Theorem 6.9, we have*

$$\Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)+(1-\gamma)(1-2\lambda)} \qquad (z-w \in \Delta), \tag{6.21}$$

The result (6.21) is sharp with the function given by (6.19).

Proof. By setting

$$w(z) = \left(1 + \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\right)\left[\frac{f_k(z)}{f(z)} - \frac{\frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}}{1+\frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}}\right]$$

proceeding as in Theorem 6.9, we get the desired result and so we omit the details. □

Remark 6.1. *We observe that, if we specialize the parameters λ and β as mentioned in Examples 1 and 2 , we obtain the analogous results for the classes $\mathcal{M}_b^s(\beta,\gamma)$ and $\mathcal{M}_b^s(\gamma)$. Further specializing the parameters s,b various other interesting results (as in Theorems 2.1 to 6.10) can be derived easily for the function class based on interesting integral operators. Further by taking $|\xi| = d$ and $|z-\xi| = r+d < 1$, one can easily prove analogous results as in Theorems 2.1 to 6.10. The details involved may be left as an exercise for the interested reader.*

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Invariant solutions of Barlett and Whitaker's equations

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Abstract

Lie symmetry group method is applied to study the Barlett and Whitaker's equations. The symmetry group and its optimal system are given,and group invariant solutions associated to the symmetries are obtained. Finally the structure of the Lie algebra symmetries is determined.

Keywords: Lie group analysis, Symmetry group, Optimal system, Invariant solution.

2010 MSC: 58G35, 35C05, 22E70.

1 Introduction

Enzymes electrodes are powerful tools for understanding the mechanism and kinetics of fast reactions. Owing to their specicity and sensitivity, enzyme elec-trodes including various amplication, schemes have been developed for many applications such as electrochemical immunoassays, [1, 2] water pollutant detection, [3, 4, 5, 6, 7] and monitoring of biological metabolities [8, 9, 10, 11]. The sensitivity of enzyme electrodes is very often increased by incorporation of a substrate-recycling scheme and several strategies including chemical, enzymatic, or electrochemical recycling have been developed. In the view of numerous application of such bio-sensor with amplied response,we are interested in investigating the concentration s and p in order to improve the metrological characteristics further.

In addition, this theoretical approach is of practical interest since this kind of bio-sensor can be used for the determination of phenolic compounds and catecholamine neurotransmitters in the field of environmental control and clinical analysis [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. Such a theoretical and kinetic analysis is a powerful approach to rationalize functions of biosensors. Desprez and Labbe [23] obtained the analytical expression concentration and current for the limiting cases only. The purpose of this communication is to derive a simple accurate polynomial expressions of concentrations generated at a enzyme electrode using Lie Symmetries.

2 Lie Symmetry of the System

We consider the BWEs (Barlett and Whitaker's equations) [24], Desprez and Labbe [23], describing the concentrations of s and p at steady state as follows (with one independent and two dependent):

$$\text{BWEs} \quad : \quad \frac{d^2s}{dx^2} - \frac{\gamma s}{\alpha s + 1} = 0, \qquad \frac{d^2p}{dx^2} + \frac{\gamma s}{\alpha s + 1} = 0, \tag{2.1}$$

where

$$\gamma = \frac{1}{\Lambda^2}, \qquad \alpha = \frac{1}{K_s}, \qquad \Lambda = \sqrt{\frac{mK_s}{K_c E_t}}, \tag{2.2}$$

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x is varible, s and p are functions, and $\gamma, \Lambda, \alpha, K_s, m, K_c$, and E_t are constants. Let

$$\mathbf{v} = \xi(x,s,p)\partial_x + \tau(x,s,p)\partial_s + \varphi(x,s,p)\partial_p, \tag{2.3}$$

be a general vector field on the space of independent and dependent variables. we need the second prolongation:

$$\mathrm{Pr}^{(2)}\mathbf{v} = \mathbf{v} + \tau^x\partial_{s_x} + \varphi^x\partial_{p_x} + \tau^{xx}\partial_{s_{xx}} + \varphi^{xx}\partial_{p_{xx}}, \tag{2.4}$$

of $\mathbf{v}$, with the coefficients

$$\begin{aligned}
\tau^x &= \tau_x + \tau_p p_x + \tau_s s_x - s_x\xi_x - s_x\xi_p p_x - \xi_s s_x^2,\\
\varphi^x &= \varphi_x + \varphi_p p_x + \varphi_s s_x - p_x\xi_x - \xi_p p_x^2 - p_x\xi_s s_x,\\
\tau^{xx} &= 2\tau_{xp}p_x + 2\tau_{xs}s_x - s_x\xi_{xx} - 2\xi_{xs}s_x^2 + \tau_{pp}p_x^2 + p_{xx}\tau_p + \tau_{ss}s_x^2 - \xi_{ss}s_x^3 + s_{xx}\tau_s\\
&\quad -2s_{xx}\xi_x - 2s_x\xi_{xp}p_x + 2p_x\tau_{sp}s_x - s_x\xi_{pp}p_x^2 - 2p_x\xi_{sp}s_x^2 - p_{xx}\xi_p s_x\\
&\quad -3s_{xx}\xi_s s_x - 2s_{xx}\xi_p p_x + \tau_{xx},\\
\varphi^{xx} &= 2\varphi_{xp}p_x + 2\varphi_{xs}s_x - p_x\xi_{xx} - 2\xi_{xp}p_x^2 + \varphi_{pp}p_x^2 - \xi_{pp}p_x^3 + p_{xx}\varphi_p - 2p_{xx}\xi_x + \varphi_{ss}s_x^2\\
&\quad +s_{xx}\varphi_s - 2p_x\xi_{xs}s_x + 2p_x\varphi_{sp}s_x - 2s_x\xi_{sp}p_x^2 - 3p_{xx}\xi_p p_x - 2p_{xx}\xi_s s_x\\
&\quad -p_x\xi_{ss}s_x^2 - s_{xx}\xi_s p_x + \varphi_{xx}.
\end{aligned} \tag{2.5}$$

Applying $\mathrm{Pr}^{(2)}\mathbf{v}$ to equations (2.1), we find the infinitesimal criterion system. determining equations yields:

$$\begin{aligned}
&\varphi_{ss} = \tau_{p,p} = \xi_{ss} = \xi_{p,p} = \xi_{sp} = 0,\\
&\tau_{sp} - \xi_{xp} = \tau_{ss} - 2\xi_{xs} = 2\xi_{xp} - \varphi_{p,p} = \xi_{xs} - \varphi_{sp} = 0,\\
&-2sK_cE_t\xi_p + 2\tau_{xp}mK_s + 2\tau_{xp}ms = 2sK_cE_t\xi_s + 2\varphi_{xs}mK_s + 2\varphi_{xs}ms = 0,\\
&2\tau_{xs}mK_s + 2\tau_{xs}ms - 3sK_cE_t\xi_s - \xi_{xx}mK_s - \xi_{xx}ms + KcE_ts\xi_p = 0,\\
&3sK_cE_t\xi_p - sK_cE_t\xi_s + 2\varphi_{xp}mK_s + 2\varphi_{xp}ms - \xi_{xx}mK_s - \xi_{xx}ms = 0,\\
&-\tau K_cE_tK_s - 2K_cE_ts\xi_xK_s - 2K_cE_ts^2\xi_x - K_cE_ts\tau_pK_s\\
&\qquad -K_cE_ts^2\tau_p + \tau_{xx}mK_s^2 + 2\tau_{xx}mK_ss + \tau_{xx}ms^2 + K_cE_ts\tau_sK_s + K_cE_ts^2\tau_s = 0,\\
&\tau K_cE_tK_s - K_cE_ts\varphi_pK_s - K_cE_ts^2\varphi_p + 2K_cE_ts\xi_xK_s + 2K_cE_ts^2\xi_x\\
&\qquad +K_cE_ts\varphi_sK_s + K_cE_ts^2\varphi_s + \varphi_{xx}mK_s^2 + 2\varphi_{xx}mK_ss + \varphi_{xx}ms^2 = 0.
\end{aligned} \tag{2.6}$$

The solution of the above system gives the following coefficients of the vector field $\mathbf{v}$:

$$\varphi = C_2\,x + C_4\,(s+p) + C_3, \qquad \tau = 0, \qquad \xi = C_1, \tag{2.7}$$

where $C_1, \cdots, C_4$ are arbitrary constants; Thus the Lie algebra $\mathbf{G}$ of the electoenzymatic processes involved in a PPO-rotating-disk-bioelectrode equation is spanned by the four vector fields

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = x\partial_p, \qquad \mathbf{v}_3 = \partial_p, \qquad \mathbf{v}_4 = (s+p)\partial_p. \tag{2.8}$$

The commutator table of $\mathbf{G}$ is

Table 1. Commutation relations satisfied by infinitesimal generators

$[\,,\,]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	$\mathbf{v}_3$	0	0
$\mathbf{v}_2$	$-\mathbf{v}_3$	0	0	$\mathbf{v}_2$
$\mathbf{v}_3$	0	0	0	$\mathbf{v}_3$
$\mathbf{v}_4$	0	$-\mathbf{v}_2$	$-\mathbf{v}_3$	0

Thus, $\mathbf{G}$ is a solvabel algebra with derived series $\mathbf{G} \geq \mathbf{G}^{(1)} \geq \{0\}$, where $\mathbf{G}^{(1)} = \mathrm{Span}\{\mathbf{v}_2, \mathbf{v}_3\} \cong R^2$, and $\mathbf{G}/\mathbf{G}^{(1)} \cong R^2$ are abelian, thus $\mathbf{G}$ is semidirect product of R^2 by itself.

The one-parameter groups G_i generated by the base of $\mathbf{G}$ are given in the following table.

$$\begin{aligned} G_1 &: (x,s,p) \longmapsto (x+\varepsilon, s, p), \\ G_2 &: (x,s,p) \longmapsto (x, s, x\varepsilon + p), \\ G_3 &: (x,s,p) \longmapsto (x, s, p+\varepsilon), \\ G_4 &: (x,s,p) \longmapsto (x, s, -s + e^{\varepsilon}(s+p)). \end{aligned} \tag{2.9}$$

Since each group G_i is a symmetry group and if $s = S(x), p = P(x)$ are solutions of the equations (2.1), so are the functions

$$\begin{array}{lll} 1) & s = S(x-\varepsilon), & p = P(x-\varepsilon), \\ 2) & s = S(x), & p = P(x) + x\varepsilon, \\ 3) & s = S(x), & p = P(x) + \varepsilon, \\ 4) & s = S(x), & p = e^{\varepsilon}(S(x)+P(x)) - S(x), \end{array} \tag{2.10}$$

where ε is a real number.

3 Optimal system of (2.1)

As is well known, the theoretical Lie group method plays an important role in finding exact solutions and performing symmetry reductions of differential equations. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the differential equation. So, a mean of determining which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. One of the applications of the adjoint representation is classifying group-invariant solutions.

The adjoint action is given by the Lie series

$$\mathbf{Ad}(\exp(\varepsilon \mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \varepsilon[\mathbf{v}_i, \mathbf{v}_j] + \frac{\varepsilon^2}{2}[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots \tag{3.1}$$

where $[\mathbf{v}_i, \mathbf{v}_j]$ is a commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 4$. The adjoint table

Table 2. Adjoint relations satisfied by infinitesimal generators

$[\,,\,]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2 - \epsilon\mathbf{v}_3$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_2$	$\mathbf{v}_1 + \epsilon\mathbf{v}_3$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 - \epsilon\mathbf{v}_2$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 - \epsilon\mathbf{v}_3$
$\mathbf{v}_4$	$\mathbf{v}_1$	$e^{\epsilon}\mathbf{v}_2$	$e^{\epsilon}\mathbf{v}_3$	$\mathbf{v}_4$

with (i,j)-th entry indicating $\mathbf{Ad}(\exp(\epsilon\mathbf{v}_i)\mathbf{v}_j)$ and ε is a real number. Here we can find the general group of the symmetries by considering a general linear combination $c_1\mathbf{v}_1 + \cdots + c_4\mathbf{v}_4$ of the given vector fields. In particular if g is the action of the symmetry group near the identity, it can be represented in the form $g = \exp(c_1\mathbf{v}_1) \circ \cdots \circ \exp(c_4\mathbf{v}_4)$.

Let $F_i^{\varepsilon} : \mathbf{G} \longrightarrow \mathbf{G}$ defined by $\mathbf{v} \longrightarrow \mathbf{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v})$ is a linear map, for $i = 1, ..., 4$. The matrices M_i^{ε} of F_i^{ε}, $i = 1, \cdots, 4$, with respect to basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_4\}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\varepsilon & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \varepsilon & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\varepsilon & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\varepsilon & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\varepsilon} & 0 & 0 \\ 0 & 0 & e^{\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.2}$$

respectively, by acting these matrices on a vector field $\mathbf{v}$ alternatively we can show that a one-dimensional optimal system of $\mathbf{G}$ is given by

$$1)\ \mathbf{v}_1, \qquad 2)\ \mathbf{v}_3, \qquad 3)\ \mathbf{v}_1+\mathbf{v}_2, \qquad 4)\ \mathbf{v}_1-\mathbf{v}_2, \qquad 5)\ \mathbf{v}_1+a\mathbf{v}_2,\ a\in R. \tag{3.3}$$

4 Conclusion

In this article group classification of (2.1) and the algebraic structure of the symmetry group is considered. Classification of one-dimensional subalgebra is determined by constructing one-dimensional optimal system. The structure of Lie algebra symmetries is analyzed.

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Ulam - Hyers stability of a 2- variable AC - mixed type functional equation in quasi - beta normed spaces: direct and fixed point methods

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Abstract

In this paper, we obtain the generalized Ulam - Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x+y,2z+w)-f(2x-y,2z-w)=4[f(x+y,z+w)-f(x-y,z-w)]-6f(y,w)$$

in Quasi - Beta normed space using direct and fixed point methods.

Keywords: Additive functional equations, cubic functional equations, Mixed type AC functional equations, generalized Ulam - Hyers stability, fixed point.

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1 Introduction

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

The first stability problem was raised by S.M. Ulam [24] during his talk at the University of Wisconsin in 1940. In 1941, D.H. Hyers [8] gave an first affirmative answer to Ulam problem for Banach spaces. It was further generalized and excellent results were obtained by a number of authors.

Over the last seven decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations including mixed type additive and cubic functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 4, 6, 7, 9, 10, 11, 16, 17, 19, 21, 23, 25].

Very recently, M. Arunkumar et.al., [3] first time introduced and investigated the solution and generalized Ulam-Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x+y,2z+w)-f(2x-y,2z-w)=4[f(x+y,z+w)-f(x-y,z-w)]-6f(y,w) \tag{1.1}$$

having solutions

$$f(x,y)=ax+by \tag{1.2}$$

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and

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \tag{1.3}$$

in Banach space using direct and fixed point approach.

The solution of the AC functional equation (1.1) is given in the following lemmas.

Lemma 1.1. *[3] If $f : U^2 \to V$ be a mapping satisfying (1.1) and let $g : U^2 \to V$ be a mapping given by*

$$g(x, x) = f(2x, 2x) - 8f(x, x) \tag{1.4}$$

for all $x \in U$ then

$$g(2x, 2x) = 2g(x, x) \tag{1.5}$$

for all $x \in U$ such that g is additive.

Lemma 1.2. *[3] If $f : U^2 \to V$ be a mapping satisfying (1.1) and let $h : U^2 \to V$ be a mapping given by*

$$h(x, x) = f(2x, 2x) - 2f(x, x) \tag{1.6}$$

for all $x \in U$ then

$$h(2x, 2x) = 8h(x, x) \tag{1.7}$$

for all $x \in U$ such that h is cubic.

Remark 1.1. *[3] If $f : U^2 \to V$ be a mapping satisfying (1.1) and let $g, h : U^2 \to V$ be a mapping defined in (1.4) and (1.6) then*

$$f(x, x) = \frac{1}{6}(h(x, x) - g(x, x)) \tag{1.8}$$

for all $x \in U$.

In this paper, the authors established the generalized Ulam-Hyers stability of the 2-variable AC functional equation (1.1) in Quasi-Beta Normed spaces using direct and fixed point methods are discussed in Section 3 and Section 4, respectively.

2 Preliminary results on quasi-beta normed spaces

In this section, we present some preliminary results concerning to quasi-β-normed spaces.

We fix a real number β with $0 < \beta \leq 1$ and let $\mathcal{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. *Let X be a linear space over $\mathcal{K}$. A quasi-β-norm $\| \cdot \|$ is a real-valued function on X satisfying the following:*

- (i) $\| x \| \geq 0$ *for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$.*
- (ii) $\| \lambda x \| = | \lambda |^{\beta} . \| x \|$ *for all $\lambda \in \mathcal{K}$ and all $x \in X$.*
- (iii) *There is a constant $K \geq 1$ such that $\| x + y \| \leq K(\| x \| + \| y \|)$ for all $x, y \in X$.*

The pair $(X, \| \cdot \|)$ is called quasi-β-normed space if $\| \cdot \|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\| \cdot \|$.

Definition 2.2. *A quasi-β-Banach space is a complete quasi-β-normed space.*

Definition 2.3. *A quasi-β-norm $\| \cdot \|$ is called a (β, p)-norm $(0 < p \leq 1)$ if*

$$\| x + y \|^p \leq \| x \|^p + \| y \|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

More details, one can refer [7, 25] for the concepts of quasi-normed spaces and p-Banach space.

3 Stability results: Direct method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.1) using direct method.

Through out this section, let us take U is a linear space over $\mathcal{K}$ and V is a (β, p) Banach space with $p-$norm $\|.\|_V$. Let K be the modulus of concavity of $\|.\|_V$. Define a mapping $F : U^2 \to V$ by

$$F(x,y,z,w) = f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) \\ + 4f(x-y,z-w) + 6f(y,w)$$

for all $x,y,z,w \in U$.

Theorem 3.1. *Let $j = \pm 1$. Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0,\infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{2^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}w) = 0 \tag{3.1}$$

such that the functional inequality

$$\|F(x,y,z,w)\|_V \leq \alpha(x,y,z,w) \tag{3.2}$$

for all $x,y,z,w \in U$. Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ satisfying the functional equation (1.1) and

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\|_V^p \leq \frac{K^{np}}{2^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{2^{kjp}} \tag{3.3}$$

where $\delta(2^{kj}x)$ and $A(x,x)$ are defined by

$$\delta(2^{kj}x) = 4^{\beta}\ \alpha(2^{kj}x, 2^{kj}x, 2^{kj}x, 2^{kj}x) + \alpha(2^{kj}x, 2^{(k+1)j}x, 2^{kj}x, 2^{(k+1)j}x) \tag{3.4}$$

$$A(x,x) = \lim_{n\to\infty} \frac{1}{2^{nj}} (f(2^{(n+1)j}x, 2^{(n+1)j}x) - 8f(2^{nj}x, 2^{nj}x)) \tag{3.5}$$

for all $x \in U$.

Proof. Assume $j = 1$. Letting (x,y,z,w) by (x,x,x,x) in (3.2), we obtain

$$\|f(3x,3x) - 4f(2x,2x) + 5f(x,x)\|_V \leq \alpha(x,x,x,x) \tag{3.6}$$

for all $x \in U$. Replacing (x,y,z,w) by $(x,2x,x,2x)$ in (3.2), we get

$$\|f(4x,4x) - 4f(3x,3x) + 6f(2x,2x) - 4f(x,x)\|_V \leq \alpha(x,2x,x,2x) \tag{3.7}$$

for all $x \in U$. Now, from (3.6) and (3.7), we have

$$\begin{aligned} &\|f(4x,4x) - 10f(2x,2x) + 16f(x,x)\|_V \\ &\leq K\left(4^{\beta}\ \|f(3x,3x) - 4f(2x,2x) + 5f(x,x)\|_V\ + \|f(4x,4x) - 4f(3x,3x) + 6f(2x,2x) - 4f(x,x)\|_V\right) \\ &\leq K(4^{\beta}\alpha(x,x,x,x) + \alpha(x,2x,x,2x)) \end{aligned} \tag{3.8}$$

for all $x \in U$. From (3.8), we arrive

$$\|f(4x,4x) - 10f(2x,2x) + 16f(x,x)\|_V \leq K\delta(x) \tag{3.9}$$

where

$$\delta(x) = 4^{\beta}\alpha(x,x,x,x) + \alpha(x,2x,x,2x) \tag{3.10}$$

for all $x \in U$. It is easy to see from (3.9) that

$$\|f(4x,4x) - 8f(2x,2x) - 2(f(2x,2x) - 8f(x,x))\|_V \leq K\delta(x) \tag{3.11}$$

for all $x \in U$. Using (1.4) in (3.11), we obtain

$$\|g(2x,2x) - 2g(x,x)\|_V \leq K\delta(x) \tag{3.12}$$

for all $x \in U$. From (3.12), we arrive

$$\left\| \frac{g(2x,2x)}{2} - g(x,x) \right\|_V \leq K \frac{\delta(x)}{2^\beta} \tag{3.13}$$

for all $x \in U$. Now replacing x by $2x$ and dividing by 2 in (3.13), we get

$$\left\| \frac{g(2^2x,2^2x)}{2^2} - \frac{g(2x,2x)}{2} \right\|_V \leq K \frac{\delta(2x)}{2^{\beta+1}} \tag{3.14}$$

for all $x \in U$. From (3.13) and (3.14), we obtain

$$\begin{aligned} \left\| \frac{g(2^2x,2^2x)}{2^2} - g(x,x) \right\|_V &\leq K \left(\left\| \frac{g(2x,2x)}{2} - g(x,x) \right\|_V + \left\| \frac{g(2^2x,2^2x)}{2^2} - \frac{g(2x,2x)}{2} \right\|_V \right) \\ &\leq \frac{K^2}{2^\beta} \left[\delta(x) + \frac{\delta(2x)}{2} \right] \end{aligned} \tag{3.15}$$

for all $x \in U$. Proceeding further and using induction on a positive integer n , we get

$$\left\| \frac{g(2^nx,2^nx)}{2^n} - g(x,x) \right\|_V \leq \frac{K^n}{2^\beta} \sum_{k=0}^{n-1} \frac{\delta(2^kx)}{2^k} \leq \frac{K^n}{2^\beta} \sum_{k=0}^{\infty} \frac{\delta(2^kx)}{2^k} \tag{3.16}$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{ \frac{g(2^nx,2^nx)}{2^n} \right\}$, replacing x by 2^mx and dividing by 2^m in (3.16), for any $m,n > 0$, we deduce

$$\begin{aligned} \left\| \frac{g(2^{n+m}x,2^{n+m}x)}{2^{(n+m)}} - \frac{g(2^mx,2^mx)}{2^m} \right\|_V &= \frac{1}{2^{m\beta}} \left\| \frac{g(2^n \cdot 2^mx,2^n \cdot 2^mx)}{2^n} - g(2^mx,2^mx) \right\|_V \\ &\leq \frac{K^n}{2^\beta} \sum_{k=0}^{\infty} \frac{\delta(2^{k+m}x)}{2^{k+m\beta}} \to 0 \quad as \quad m \to \infty \end{aligned}$$

for all $x \in U$. This shows that the sequence $\left\{ \frac{g(2^nx,2^nx)}{2^n} \right\}$ is a Cauchy sequence. Since V is complete, there exists a mapping $A(x,x) : U^2 \to V$ such that

$$A(x,x) = \lim_{n\to\infty} \frac{g(2^nx,2^nx)}{2^n}, \quad \forall \ x \in U.$$

Letting $n \to \infty$ in (3.16) and using (1.4), we see that (3.3) holds for all $x \in U$. To show that A satisfies (1.1), replacing (x,y,z,w) by $(2^nx,2^ny,2^nz,2^nw)$ and dividing by 2^n in (3.2), we obtain

$$\frac{1}{2^n} \left\| F(2^nx,2^ny,2^nz,2^nw) \right\|_V \leq \frac{1}{2^n} \alpha(2^nx,2^ny,2^nz,2^nw)$$

for all $x,y,z,w \in U$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x,x)$, we see that A satisfies (1.1) for all $x,y,z,w \in U$. To prove A is a unique 2-variable additive function satisfying (1.1), we let $B(x,x)$ be another 2-variable additive mapping satisfying (1.1) and (3.3), then

$$\begin{aligned} \|A(x,x) - B(x,x)\|_V \leq \frac{K}{2^{n\beta}} \Big\{ &\left\| A(2^nx,2^nx) - f(2^{n+1}x,2^{n+1}x) + 8f(2^nx,2^nx) \right\|_V \\ &+ \left\| f(2^{n+1}x,2^{n+1}x) - 8f(2^nx,2^nx) - B(2^nx,2^nx) \right\|_V \Big\} \\ \leq \frac{2K^{n+1}}{2^\beta} \sum_{k=0}^{\infty} \frac{\delta(2^{k+n}x)}{2^{(k+n\beta)}} &\to 0 \quad as \quad n \to \infty \end{aligned}$$

for all $x \in U$. Hence A is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.1).

Corollary 3.1. *Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x,y,z,w))\|_V \leq \begin{cases} \lambda, & \\ \lambda\left\{||x||^s+||y||^s+||z||^s+||w||^s\right\}, & s<1 \quad or \quad s>1; \\ \lambda\, ||x||^s||y||^s||z||^s||w||^s, & s<\frac{1}{4} \quad or \quad s>\frac{1}{4}; \\ \lambda\left\{||x||^s||y||^s||z||^s||w||^s+\left\{||x||^{4s}+||y||^{4s}+||w||^{4s}+||z||^{4s}\right\}\right\}, & s<\frac{1}{4} \quad or \quad s>\frac{1}{4}; \end{cases} \tag{3.17}$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable additive function $A : U^2 \to V$ such that

$$\|f(2x,2x)-8f(x,x)-A(x,x)\|_V^p \leq \begin{cases} \left(\dfrac{K^n 2\lambda(4^\beta+1)}{2^\beta}\right)^p, \\ \left(\dfrac{K^n 2\lambda(4^{\beta+1}+2^{\beta s+1}+2)\lambda||x||^s}{2^\beta|2-2^{\beta s}|}\right)^p, \\ \left(\dfrac{K^n 2\lambda(4^\beta+2^{2\beta s})\lambda||x||^{4s}}{2^\beta|2-2^{\beta 4s}|}\right)^p \\ \left(\dfrac{K^n 2\lambda(5\cdot 4^\beta+2^{2\beta s}+2^{4\beta s+1}+2)\lambda||x||^{4s}}{2^\beta|2-2^{\beta 4s}|}\right)^p \end{cases} \tag{3.18}$$

for all $x \in U$.

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $s = 1$ in condition (ii) of Corollary 3.1.

Example 3.1. *Let $\alpha : \mathcal{K} \to \mathcal{K}$ be a function defined by*

$$\alpha(x) = \begin{cases} \mu x, & if\ |x| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathcal{K}^2 \to \mathcal{K}$ by

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \qquad for\ all \quad x \in \mathcal{K}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)|_V \leq 32\,\mu\,(|x|+|y|+|z|+|w|) \tag{3.19}$$

for all $x, y, z, w \in \mathcal{K}$. Then there do not exist a additive mapping $A : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\rho > 0$ such that

$$|f(2x,2x)-8f(x,x)-A(x,x)|_V \leq \rho|x| \qquad for\ all \quad x \in \mathcal{K}. \tag{3.20}$$

Proof. Now

$$|f(x,x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\,\mu.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.19).

If $x = y = z = w = 0$ then (3.19) is trivial. If $|x|+|y|+|z|+|w| \geq \frac{1}{2}$ then the left hand side of (3.19) is less than 32μ. Now suppose that $0 < |x|+|y|+|z|+|w| < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^k} \leq |x|+|y|+|z|+|w| < \frac{1}{2^{k-1}}, \tag{3.21}$$

so that $2^{k-1}x < \frac{1}{2}$, $2^{k-1}y < \frac{1}{2}$, $2^{k-1}z < \frac{1}{2}$, $2^{k-1}w < \frac{1}{2}$ and consequently

$$2^{k-1}(y,w), 2^{k-1}(x+y, z+w), 2^{k-1}(x-y, z-w),$$
$$2^{k-1}(2x+y, 2z+w), 2^{k-1}(2x-y, 2z-w), \in (-1,1).$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$2^n(y, w), 2^n(x+y, z+w), 2^n(x-y, z-w), \\ 2^n(2x+y, 2z+w), 2^n(2x-y, 2z-w), \in (-1, 1)$$

and

$$\alpha(2^n(2x+y, 2z+w)) - \alpha(2^n(2x-y, 2z-w)) - 4\alpha(2^n(x+y, z+w)) \\ + 4\alpha(2^n(x-y, z-w)) + 6\alpha(2^n(y, w)) = 0$$

for $n = 0, 1, \ldots, k-1$. From the definition of f and (3.21), we obtain that

$$\begin{aligned} &\left| f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4f(x+y, z+w) + 4f(x-y, z-w) + 6f(y, w) \right|_V \\ &\le \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \alpha(2^n(2x+y, 2z+w)) - \alpha(2^n(2x-y, 2z-w)) - 4\alpha(2^n(x+y, z+w)) \\ &\qquad + 4\alpha(2^n(x-y, z-w)) + 6\alpha(2^n(y, w)) \Big|_V \\ &\le \sum_{n=k}^{\infty} \frac{1}{2^n} \Big| \alpha(2^n(2x+y, 2z+w)) - \alpha(2^n(2x-y, 2z-w)) - 4\alpha(2^n(x+y, z+w)) \\ &\qquad + 4\alpha(2^n(x-y, z-w)) + 6\alpha(2^n(y, w)) \Big|_V \\ &\le \sum_{n=k}^{\infty} \frac{1}{2^n} 16\mu = 16\,\mu \times \frac{2}{2^k} = 32\,\mu\,(|x| + |y| + |z| + |w|)\,. \end{aligned}$$

Thus f satisfies (3.19) for all $x, y, z, w \in \mathcal{K}$ with $0 < |x| + |y| + |z| + |w| < \dfrac{1}{2}$.

We claim that the additive functional equation (1.1) is not stable for $s = 1$ in condition (ii) Corollary 3.1. Suppose on the contrary that there exist a additive mapping $A : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\rho > 0$ satisfying (3.20). Since f is bounded and continuous for all $x \in \mathcal{K}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x, x) = cx$ for any x in $\mathcal{K}$. Thus we obtain that

$$|f(2x, 2x) - 8f(x, x)|_V \le (\rho + |c|)\,|x|. \tag{3.22}$$

But we can choose a positive integer m with $m\mu > \rho + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(2x, 2x) - 8f(x, x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \ge \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\rho + |c|)\,x$$

which contradicts (3.22). Therefore the additive functional equation (1.1) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality condition (ii) of (3.17). □

A counter example to illustrate the non stability in condition (iii) of Corollary 3.1 is given in the following example.

Example 3.2. *Let s be such that $0 < s < \dfrac{1}{4}$. Then there is a function $F : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\lambda > 0$ satisfying*

$$|F(x, y, z, w)|_V \le \lambda |x|^{\frac{s}{4}}\,|y|^{\frac{s}{4}}\,|z|^{\frac{s}{4}}\,|w|^{\frac{1-3s}{4}} \tag{3.23}$$

for all $x, y, z, w \in \mathcal{K}$ and

$$\sup_{x \neq 0} \frac{|f(2x, 2x) - 8f(x, x) - A(x, x)|_V}{|x|} = +\infty \tag{3.24}$$

for every additive mapping $A(x, x) : \mathcal{K}^2 \to \mathcal{K}$.

Proof. If we take

$$f(x,x) = \begin{cases} (x,x)\ln|x,x| & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (3.24), it follows that

$$\begin{aligned} \sup_{x\neq 0} \frac{|f(2x,2x) - 8f(x,x) - A(x,x)|_V}{|x|} &\geq \sup_{\substack{n\in\mathbb{N} \\ n\neq 0}} \frac{|f(2n,2n) - 8f(n,n) - A(n,n)|_V}{|n|} \\ &= \sup_{\substack{n\in\mathbb{N} \\ n\neq 0}} \frac{|n(2,2)\ln|2n,2n| - 8n(1,1)\ln|n,n| - n\,A(1,1)|_V}{|n|} \\ &= \sup_{\substack{n\in\mathbb{N} \\ n\neq 0}} |(2,2)\ln|2n,2n| - 8(1,1)\ln|n,n| - A(1,1)|_V = \infty. \end{aligned}$$

We have to prove (3.23) is true.
Case (i): If $x, y, z, w > 0$ in (3.23) then,

$$\begin{aligned} &|f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4f(x+y, z+w) + 4f(x-y, z-w) + 6f(y,w)|_V \\ &= |(2x+y, 2z+w)\ln|2x+y, 2z+w| - (2x-y, 2z-w)\ln|2x-y, 2z-w| \\ &\quad -4(x+y, z+w)\ln|x+y, z+w| + 4(x-y, z-w)\ln|x-y, z-w| + 6(y,w)\ln|y,w||_V. \end{aligned}$$

Set $x = v_1, y = v_2, z = v_3, w = v_4$ it follows that

$$\begin{aligned} &|f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4f(x+y, z+w) + 4f(x-y, z-w) + 6f(y,w)|_V \\ &= |(2v_1+v_2, 2v_3+v_4)\ln|2v_1+v_2, 2v_3+v_4| - |2v_1-v_2, 2v_3-v_4|\ln|2v_1-v_2, 2v_3-v_4| \\ &\quad -4(v_1+v_2, v_3+v_4)\ln|v_1+v_2, v_3+v_4| + 4|v_1-v_2, v_3-v_4|\ln|v_1-v_2, v_3-v_4| \\ &\quad +6(v_2,v_4)\ln|v_2,v_4||_V. \\ &= |f(2v_1+v_2, 2v_3+v_4) - f(2v_1-v_2, 2v_3-v_4) - 4f(v_1+v_2, v_3+v_4) \\ &\quad +4f(v_1-v_2, v_3-v_4) + 6f(v_2,v_4)|_V \\ &\leq \lambda|v_1|^{\frac{s}{4}}\ |v_2|^{\frac{s}{4}}\ |v_3|^{\frac{s}{4}}\ |v_4|^{\frac{1-3s}{4}} \\ &= \lambda|x|^{\frac{s}{4}}\ |y|^{\frac{s}{4}}\ |z|^{\frac{s}{4}}\ |w|^{\frac{1-3s}{4}}. \end{aligned}$$

Case (ii): If $x, y, z, w < 0$ in (3.23) then,

$$\begin{aligned} &|f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4f(x+y, z+w) + 4f(x-y, z-w) + 6f(y,w)|_V \\ &= |(2x+y, 2z+w)\ln|2x+y, 2z+w| - (2x-y, 2z-w)\ln|2x-y, 2z-w| \\ &\quad -4(x+y, z+w)\ln|x+y, z+w| + 4(x-y, z-w)\ln|x-y, z-w| + 6(y,w)\ln|y,w||_V. \end{aligned}$$

Set $-x = v_1, -y = v_2, -z = v_3, -w = v_4$ it follows that

$$\begin{aligned} &|f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4f(x+y, z+w) + 4f(x-y, z-w) + 6f(y,w)|_V \\ &= |(-2v_1-v_2, -2v_3-v_4)\ln|-2v_1-v_2, -2v_3-v_4| \\ &\quad -(-2v_1+v_2, -2v_3+v_4)\ln|-2v_1+v_2, -2v_3+v_4| \\ &\quad -4(-v_1-v_2, -v_3-v_4)\ln|-v_1-v_2, -v_3-v_4| \\ &\quad +4(-v_1+v_2, -v_3+v_4)\ln|-v_1+v_2, -v_3+v_4| \\ &\quad +6(-v_2,-v_4)\ln|-v_2,-v_4||_V. \\ &= |f(-2v_1-v_2, -2v_3-v_4) - f(-2v_1+v_2, -2v_3+v_4) - 4f(-v_1-v_2, -v_3-v_4) \\ &\quad +4f(-v_1+v_2, -v_3+v_4) + 6f(-v_2,-v_4)|_V \\ &\leq \lambda|-v_1|^{\frac{s}{4}}\ |-v_2|^{\frac{s}{4}}\ |-v_3|^{\frac{s}{4}}\ |-v_4|^{\frac{1-3s}{4}} \\ &= \lambda|x|^{\frac{s}{4}}\ |y|^{\frac{s}{4}}\ |z|^{\frac{s}{4}}\ |w|^{\frac{1-3s}{4}}. \end{aligned}$$

Case (iii): If $x, z > 0, y, w < 0$ then $2x + y, 2z + w, x + y, z + w > 0$, $2x - y, 2z - w, x - y, z - w < 0$ in (3.23) then,

$$
\begin{aligned}
&|f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) + 4f(x-y,z-w) + 6f(y,w)|_V \\
&= |(2x+y,2z+w)\ln|2x+y,2z+w| - (2x-y,2z-w)\ln|2x-y,2z-w| \\
&\quad -4(x+y,z+w)\ln|x+y,z+w| + 4(x-y,z-w)\ln|x-y,z-w| + 6(y,w)\ln|y,w||_V .
\end{aligned}
$$

Set $x = v_1, -y = v_2, z = v_3, -w = v_4$ it follows that

$$
\begin{aligned}
&|f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) + 4f(x-y,z-w) + 6f(y,w)|_V \\
&= |(2v_1 - v_2, 2v_3 - v_4)\ln(2v_1 - v_2, 2v_3 - v_4) \\
&\qquad -(2v_1 + v_2, 2v_3 + v_4)\ln|-(2v_1 + v_2, 2v_3 + v_4)| \\
&\qquad -4(v_1 - v_2, v_3 - v_4)\ln|v_1 - v_2, v_3 - v_4| \\
&\qquad +4(v_1 + v_2, v_3 + v_4)\ln|-(v_1 + v_2, v_3 + v_4)| \\
&\qquad +6(-v_2, -v_4)\ln(-v_2, -v_4)|_V . \\
&= |f(2v_1 - v_2, 2v_3 - v_4) - f(2v_1 + v_2, 2v_3 + v_4) - 4f(v_1 - v_2, v_3 - v_4) \\
&\qquad +4f(v_1 + v_2, v_3 + v_4) + 6f(-v_2, -v_4)|_V \\
&\le \lambda |v_1|^{\frac{s}{4}}\, |-v_2|^{\frac{s}{4}}\, |v_3|^{\frac{s}{4}}\, |-v_4|^{\frac{1-3s}{4}} \\
&= \lambda |x|^{\frac{s}{4}}\, |y|^{\frac{s}{4}}\, |z|^{\frac{s}{4}}\, |w|^{\frac{1-3s}{4}} .
\end{aligned}
$$

Case (iv): If $x, z > 0, y, w < 0$ then $2x + y, 2z + w, x + y, z + w < 0$, $2x - y, 2z - w, x - y, z - w > 0$ in (3.23) then,

$$
\begin{aligned}
&|f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) + 4f(x-y,z-w) + 6f(y,w)|_V \\
&= |(2x+y,2z+w)\ln|2x+y,2z+w| - (2x-y,2z-w)\ln|2x-y,2z-w| \\
&\quad -4(x+y,z+w)\ln|x+y,z+w| + 4(x-y,z-w)\ln|x-y,z-w| + 6(y,w)\ln|y,w||_V .
\end{aligned}
$$

Set $x = v_1, -y = v_2, z = v_3, -w = v_4$ it follows that

$$
\begin{aligned}
&|f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) + 4f(x-y,z-w) + 6f(y,w)|_V \\
&= |(2v_1 - v_2, 2v_3 - v_4)\ln|-(2v_1 - v_2, 2v_3 - v_4)| \\
&\qquad -(2v_1 + v_2, 2v_3 + v_4)\ln|2v_1 + v_2, 2v_3 + v_4| \\
&\qquad -4(v_1 - v_2, v_3 - v_4)\ln|-(v_1 - v_2, v_3 - v_4)| \\
&\qquad +4(v_1 + v_2, v_3 + v_4)\ln|v_1 + v_2, v_3 + v_4| \\
&\qquad +6(-v_2, -v_4)\ln(-v_2, -v_4)|_V . \\
&= |f(2v_1 - v_2, 2v_3 - v_4) - f(2v_1 + v_2, 2v_3 + v_4) - 4f(v_1 - v_2, v_3 - v_4) \\
&\qquad +4f(v_1 + v_2, v_3 + v_4) + 6f(-v_2, -v_4)|_V \\
&\le \lambda |v_1|^{\frac{s}{4}}\, |-v_2|^{\frac{s}{4}}\, |v_3|^{\frac{s}{4}}\, |-v_4|^{\frac{1-3s}{4}} \\
&= \lambda |x|^{\frac{s}{4}}\, |y|^{\frac{s}{4}}\, |z|^{\frac{s}{4}}\, |w|^{\frac{1-3s}{4}} .
\end{aligned}
$$

Case (v): If $x = y = z = w = 0$ in (3.23) then it is trivial. □

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $s = \dfrac{1}{4}$ in condition (iv) of Corollary 3.1.

Example 3.3. *Let $\alpha : \mathcal{K} \to \mathcal{K}$ be a function defined by*

$$
\alpha(x) = \begin{cases} \mu x, & \text{if } |x| < \dfrac{1}{4} \\ \dfrac{\mu}{4}, & \text{otherwise} \end{cases}
$$

where $\mu > 0$ *is a constant, and define a function* $f : \mathcal{K}^2 \to \mathcal{K}$ *by*

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \qquad for\ all \quad x \in \mathcal{K}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)|_V \leq 8\,\mu \left(|x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} + \{|x| + |y| + |w| + |z|\} \right) \tag{3.25}$$

for all $x, y, z, w \in \mathcal{K}$. *Then there do not exist a additive mapping* $A : \mathcal{K}^2 \to \mathcal{K}$ *and a constant* $\rho > 0$ *such that*

$$|f(2x,2x) - 8f(x,x) - A(x,x)|_V \leq \rho |x| \qquad for\ all \quad x \in \mathcal{K}. \tag{3.26}$$

Proof. Now

$$|f(x,x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{1}{2^n} \times \frac{\mu}{4} = \frac{\mu}{2}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.25).

If $x = y = z = w = 0$ then (3.25) is trivial.

If $|x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} + \{|x| + |y| + |w| + |z|\} \geq \frac{1}{2}$ then the left hand side of (3.25) is less than 8μ. Now suppose that $0 < |x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} + \{|x| + |y| + |w| + |z|\} < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^k} \leq |x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} + \{|x| + |y| + |w| + |z|\} < \frac{1}{2^{k-1}}, \tag{3.27}$$

so that $2^{k-1}|x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} < \frac{1}{2}$, $2^{k-1}|x| < \frac{1}{2}$, $2^{k-1}|y| < \frac{1}{2}$, $2^{k-1}|z| < \frac{1}{2}$, $2^{k-1}|w| < \frac{1}{2}$ and consequently

$$\begin{aligned} &2^{k-1}(y,w), 2^{k-1}(x+y, z+w), 2^{k-1}(x-y, z-w), \\ &\qquad 2^{k-1}(2x+y, 2z+w), 2^{k-1}(2x-y, 2z-w), \in \left(-\frac{1}{4}, \frac{1}{4}\right). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} &2^n(y,w), 2^n(x+y, z+w), 2^n(x-y, z-w), \\ &\qquad 2^n(2x+y, 2z+w), 2^n(2x-y, 2z-w), \in \left(-\frac{1}{4}, \frac{1}{4}\right) \end{aligned}$$

and

$$\begin{aligned} &\alpha(2^n(2x+y, 2z+w)) - \alpha(2^n(2x-y, 2z-w)) - 4\alpha(2^n(x+y, z+w)) \\ &\qquad + 4\alpha(2^n(x-y, z-w)) + 6\alpha(2^n(y,w)) = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f and (3.27), we obtain that

$$\begin{aligned} &\left| f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4f(x+y, z+w) + 4f(x-y, z-w) + 6f(y,w) \right|_V \\ &\leq \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \alpha(2^n(2x+y, 2z+w)) - \alpha(2^n(2x-y, 2z-w)) - 4\alpha(2^n(x+y, z+w)) \\ &\qquad + 4\alpha(2^n(x-y, z-w)) + 6\alpha(2^n(y,w)) \Big|_V \\ &\leq \sum_{n=k}^{\infty} \frac{1}{2^n} \Big| \alpha(2^n(2x+y, 2z+w)) - \alpha(2^n(2x-y, 2z-w)) - 4\alpha(2^n(x+y, z+w)) \\ &\qquad + 4\alpha(2^n(x-y, z-w)) + 6\alpha(2^n(y,w)) \Big|_V \end{aligned}$$

$$\leq \sum_{n=k}^{\infty} \frac{16\mu}{4} \times \frac{1}{2^n} = \frac{16\mu}{4} \times \frac{2}{2^k} = 8\,\mu\left(|x|^{\frac{1}{4}}\,|y|^{\frac{1}{4}}\,|z|^{\frac{1}{4}}\,|w|^{\frac{1}{4}} + \{|x|+|y|+|w|+|z|\}\right).$$

Thus f satisfies (3.25) for all $x, y, z, w \in \mathcal{K}$ with
$0 < |x|^{\frac{1}{4}}\,|y|^{\frac{1}{4}}\,|z|^{\frac{1}{4}}\,|w|^{\frac{1}{4}} + \{|x|+|y|+|w|+|z|\} < \frac{1}{2}$.

We claim that the additive functional equation (1.1) is not stable for $s = \frac{1}{4}$ in condition (iv) Corollary 3.1. Suppose on the contrary that there exist a additive mapping $A : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\rho > 0$ satisfying (3.26). Since f is bounded and continuous for all $x \in \mathcal{K}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x,x) = cx$ for any x in $\mathcal{K}$. Thus we obtain that

$$|f(2x,2x) - 8f(x,x)|_V \leq (\rho + |c|)\,|x|. \tag{3.28}$$

But we can choose a positive integer m with $m\mu > \rho + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(2x,2x) - 8f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\rho + |c|)\,x$$

which contradicts (3.28). Therefore the additive functional equation (1.1) is not stable in sense of Ulam, Hyers and Rassias if $s = \frac{1}{4}$, assumed in the inequality condition (iv) of (3.17).

□

Theorem 3.2. *Let $j = \pm 1$. Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{8^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}w) = 0 \tag{3.29}$$

such that the functional inequality

$$\|F(x,y,z,w)\|_V \leq \alpha(x,y,z,w) \tag{3.30}$$

for all $x, y, z, w \in U$. Then there exists a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1.1) and

$$\|f(2x,2x) - 2f(x,x) - C(x,x)\|_V^p \leq \frac{K^{np}}{8^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{8^{kjp}} \tag{3.31}$$

where $\delta(2^{kj}x)$ and $C(x,x)$ are defined by

$$\delta(2^{kj}x) = 4^{\beta}\alpha(2^{kj}x, 2^{kj}x, 2^{kj}x, 2^{kj}x) + \alpha(2^{kj}x, 2^{(k+1)j}x, 2^{kj}x, 2^{(k+1)j}x) \tag{3.32}$$

$$C(x,x) = \lim_{n\to\infty} \frac{1}{8^{nj}} (f(2^{(n+1)j}x, 2^{(n+1)j}x) - 2f(2^{nj}x, 2^{nj}x)) \tag{3.33}$$

for all $x \in U$.

Proof. It is easy to see from (3.9) that

$$\|f(4x,4x) - 2f(2x,2x) - 8(f(2x,2x) - 2f(x,x))\|_V \leq K\delta(x) \tag{3.34}$$

for all $x \in U$. Using (1.6) in (3.34), we obtain

$$\|h(2x,2x) - 8h(x,x)\|_V \leq K\delta(x) \tag{3.35}$$

for all $x \in U$. From (3.35), we arrive

$$\left\|\frac{h(2x,2x)}{8} - h(x,x)\right\|_V \leq K\frac{\delta(x)}{8^{\beta}} \tag{3.36}$$

for all $x \in U$. The rest of the proof is similar to that of Theorem 3.1 □

The following Corollary is an immediate consequence of Theorem 3.2 concerning the stability of (1.1).

Corollary 3.2. *Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x,y,z,w))\|_V \leq \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s < 3 \quad or \quad s > 3; \\ \lambda\; ||x||^s ||y||^s ||z||^s ||w||^s, & s < \frac{3}{4} \quad or \quad s > \frac{3}{4}; \\ \lambda\left\{||x||^s ||y||^s ||z||^s ||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s < \frac{3}{4} \quad or \quad s > \frac{3}{4}; \end{cases} \tag{3.37}$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable cubic function $C : U^2 \to V$ such that

$$\|f(2x,2x) - 2f(x,x) - C(x,x)\|_V^p \leq \begin{cases} \left(\dfrac{K^n 8\lambda(4^\beta + 1)}{7\cdot 8^\beta}\right)^p, \\ \left(\dfrac{K^n 8\lambda(4^{\beta+1} + 2^{\beta s+1} + 2)\lambda||x||^s}{8^\beta|8 - 2^{\beta s}|}\right)^p, \\ \left(\dfrac{K^n 8\lambda(4^\beta + 2^{2\beta s})\lambda||x||^{4s}}{8^\beta|8 - 2^{\beta 4s}|}\right)^p \\ \left(\dfrac{K^n 8\lambda(5\cdot 4^\beta + 2^{2\beta s} + 2^{4\beta s+1} + 2)\lambda||x||^{4s}}{8^\beta|8 - 2^{\beta 4s}|}\right)^p \end{cases} \tag{3.38}$$

for all $x \in U$.

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $s = 3$ in condition (ii) of Corollary 3.2.

Example 3.4. *Let $\alpha : \mathcal{K} \to \mathcal{K}$ be a function defined by*

$$\alpha(x) = \begin{cases} \mu x^3, & if\ |x| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathcal{K}^2 \to \mathcal{K}$ by

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{8^n} \qquad for\ all \quad x \in \mathcal{K}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)|_V \leq \frac{16\ \mu \times 8^3}{7}\left(|x|^3 + |y|^3 + |z|^3 + |w|^3\right) \tag{3.39}$$

for all $x, y, z, w \in \mathcal{K}$. Then there do not exist a cubic mapping $C : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\beta > 0$ such that

$$|f(2x,2x) - 2f(x,x) - C(x,x)|_V \leq \beta|x|^3 \qquad for\ all \quad x \in \mathcal{K}. \tag{3.40}$$

Proof. Now

$$|f(x,x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|8^n|} = \sum_{n=0}^{\infty} \frac{\mu}{8^n} = \frac{8\ \mu}{7}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.39).

If $x = y = z = w = 0$ then (3.39) is trivial. If $|x|^3 + |y|^3 + |z|^3 + |w|^3 \geq \frac{1}{8}$ then the left hand side of (3.39) is less than $\frac{16 \times 8\mu}{7}$. Now suppose that $0 < |x|^3 + |y|^3 + |z|^3 + |w|^3 < \frac{1}{8}$. Then there exists a positive integer k such that

$$\frac{1}{8^{k+2}} \leq |x|^3 + |y|^3 + |z|^3 + |w|^3 < \frac{1}{8^{k+1}}, \tag{3.41}$$

the rest of the proof is similar to that of Example 3.1. □

A counter example to illustrate the non stability in condition (iii) of Corollary 3.2 is given in the following example.

Example 3.5. *Let s be such that* $0 < s < \frac{3}{4}$. *Then there is a function* $F : \mathcal{K}^2 \to \mathcal{K}$ *and a constant* $\lambda > 0$ *satisfying*

$$|F(x,y,z,w)|_V \leq \lambda |x|^{\frac{s}{4}} \, |y|^{\frac{s}{4}} \, |z|^{\frac{s}{4}} \, |w|^{\frac{3-3s}{4}} \tag{3.42}$$

for all $x, y, z, w \in \mathcal{K}$ *and*

$$\sup_{x \neq 0} \frac{|f(2x,2x) - 2f(x,x) - C(x,x)|_V}{|x|^3} = +\infty \tag{3.43}$$

for every cubic mapping $C : \mathcal{K}^2 \to \mathcal{K}$.

Proof. If we take

$$f(x,x) = \begin{cases} (x,x)^3 \ln |x,x| & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (3.43), it follows that

$$\begin{aligned} \sup_{x \neq 0} & \frac{|f(2x,2x) - 2f(x,x) - C(x,x)|_V}{|x|^3} \\ & \geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|f(2n,2n) - 2f(n,n) - C(n,n)|_V}{|n|^3} \\ & = \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|n^3 (2,2)^3 \ln |n,n| - 2n^3 (1,1)^3 \ln |n,n| - n^3 \, C\,(1,1)\right|_V}{|n|^3} \\ & = \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \left|(2,2)^3 \ln |n,n| - 2(1,1)^3 \ln |n,n| - C\,(1,1)\right|_V = \infty. \end{aligned}$$

Rest of the proof is similar to that of Example 3.2. □

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $s = \frac{3}{4}$ in condition (iv) of Corollary 3.2.

Example 3.6. *Let* $\alpha : \mathcal{K} \to \mathcal{K}$ *be a function defined by*

$$\alpha(x) = \begin{cases} \mu x^3, & if \, |x| < \dfrac{3}{4} \\ \dfrac{3\mu}{4}, & otherwise \end{cases}$$

where $\mu > 0$ *is a constant, and define a function* $f : \mathcal{K}^2 \to \mathcal{K}$ *by*

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{8^n} \qquad for \; all \quad x \in \mathcal{K}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)|_V \leq \frac{96\mu \times 8^2}{7} \left(|x|^{\frac{3}{4}} \, |y|^{\frac{3}{4}} \, |z|^{\frac{3}{4}} \, |w|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |w|^3 + |z|^3 \right\} \right) \tag{3.44}$$

for all $x, y, z, w \in \mathcal{K}$. *Then there do not exist a cubic mapping* $C : \mathcal{K}^2 \to \mathcal{K}$ *and a constant* $\rho > 0$ *such that*

$$|f(2x,2x) - 2f(x,x) - C(x,x)|_V \leq \rho |x| \qquad for \; all \quad x \in \mathcal{K}. \tag{3.45}$$

Proof. Now

$$|f(x,x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|8^n|} = \sum_{n=0}^{\infty} \frac{1}{8^n} \times \frac{3\mu}{4} = \frac{6\,\mu}{7}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.44).

If $x = y = z = w = 0$ then (3.44) is trivial. If $|x|^{\frac{3}{4}}\ |y|^{\frac{3}{4}}\ |z|^{\frac{3}{4}}\ |w|^{\frac{3}{4}}$ $+ \{|x|^3 + |y|^3 + |w|^3 + |z|^3\} \geq \frac{1}{8}$ then the left hand side of (3.44) is less than $\frac{96\ \mu}{7}$. Now suppose that $0 < |x|^{\frac{3}{4}}\ |y|^{\frac{3}{4}}\ |z|^{\frac{3}{4}}\ |w|^{\frac{3}{4}} + \left\{|x|^3 + |y|^3 + |w|^3 + |z|^3\right\} < \frac{1}{8}$. Then there exists a positive integer k such that

$$\frac{1}{8^{k+2}} \leq |x|^{\frac{3}{4}}\ |y|^{\frac{3}{4}}\ |z|^{\frac{3}{4}}\ |w|^{\frac{3}{4}} + \left\{|x|^3 + |y|^3 + |w|^3 + |z|^3\right\} < \frac{1}{8^{k+1}}, \tag{3.46}$$

the rest of the proof is similar to that of Example 3.3. □

Now, we are ready to prove our main stability results.

Theorem 3.3. *Let $j = \pm 1$. Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the conditions given in (3.1) and (3.29) respectively, such that the functional inequality*

$$\|F(x,y,z,w)\|_V \leq \alpha(x,y,z,w) \tag{3.47}$$

for all $x, y, z, w \in U$. Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1.1) and

$$\|f(x,x) - A(x,x) - C(x,x)\|_V^p \leq \frac{K^p}{6^p}\left\{\frac{K^{np}}{2^{\beta p}}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\delta(2^{kj}x)^p}{2^{kjp}} + \frac{K^{np}}{8^{\beta p}}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\delta(2^{kj}x)^p}{8^{kjp}}\right\} \tag{3.48}$$

where $\delta(2^{kj}x)$, $A(x,x)$ and $C(x,x)$ are respectively defined in (3.4), (3.5) and (3.33) for all $x \in U$.

Proof. By Theorems 3.1 and 3.2, there exists a unique 2-variable additive function $A_1 : U^2 \to V$ and a unique 2-variable cubic function $C_1 : U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A_1(x,x)\|_V^p \leq \frac{K^{np}}{2^{\beta p}}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\delta(2^{kj}x)^p}{2^{kjp}} \tag{3.49}$$

$$\|f(2x,2x) - 2f(x,x) - C_1(x,x)\|_V^p \leq \frac{K^{np}}{8^{\beta p}}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\delta(2^{kj}x)^p}{8^{kjp}} \tag{3.50}$$

for all $x \in U$. Now from (3.49) and (3.50), one can see that

$$\begin{aligned}
&\left\|f(x,x) + \frac{1}{6}A_1(x,x) - \frac{1}{6}C_1(x,x)\right\|_V^p \\
&= \left\|\left\{-\frac{f(2x,2x)}{6} + \frac{8f(x,x)}{6} + \frac{A_1(x,x)}{6}\right\} + \left\{\frac{f(2x,2x)}{6} - \frac{2f(x,x)}{6} - \frac{C_1(x,x)}{6}\right\}\right\|_V^p \\
&\leq \frac{K^p}{6^p}\left\{\|f(2x,2x) - 8f(x,x) - A_1(x,x)\|_V^p + \|f(2x,2x) - 2f(x,x) - C_1(x,x)\|_V^p\right\} \\
&\leq \frac{K^p}{6^p}\left\{\frac{K^{np}}{2^{\beta p}}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\delta(2^{kj}x)^p}{2^{kjp}} + \frac{K^{np}}{8^{\beta p}}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\delta(2^{kj}x)^p}{8^{kjp}}\right\}
\end{aligned}$$

for all $x \in U$. Thus we obtain (3.50) by defining $A(x,x) = \frac{-1}{6}A_1(x,x)$ and $C(x,x) = \frac{1}{6}C_1(x,x)$, $\delta(2^{kj}x)$, $A(x,x)$ and $C(x,x)$ are respectively defined in (3.4), (3.5) and (3.33) for all $x \in U$. □

The following corollary is the immediate consequence of Theorem 3.3, using Corollaries 3.1 and 3.2 concerning the stability of (1.1).

Corollary 3.3. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$\|F(x,y,z,w))\|_V \leq \begin{cases} \lambda, \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s \neq 1, 3; \\ \lambda\ ||x||^s||y||^s||z||^s||w||^s, & s \neq \frac{1}{4}, \frac{3}{4}; \\ \lambda\left\{||x||^s||y||^s||z||^s||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s \neq \frac{1}{4}, \frac{3}{4}; \end{cases} \tag{3.51}$$

for all $x, y, z, w \in U$, then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ such that

$$\|f(x,x) - A(x,x) - C(x,x)\|_V^p \leq \begin{cases} \left(\frac{K^{n+1}\lambda(4^\beta + 1)}{6} \left[\frac{2}{2^\beta} + \frac{8}{7 \cdot 8^\beta} \right] \right)^p, \\ \left(\frac{K^{n+1}\lambda(4^{\beta+1} + 2^{\beta s+1} + 2)||x||^s}{6} \left[\frac{2}{2^\beta|2 - 2^{\beta s}|} + \frac{8}{8^\beta|8 - 2^{\beta s}|} \right] \right)^p, \\ \left(\frac{K^{n+1}\lambda(4^\beta + 2^{2\beta s})||x||^{4s}}{6} \left[\frac{2}{2^\beta|2 - 2^{\beta 4s}|} + \frac{8}{8^\beta|8 - 2^{\beta 4s}|} \right] \right)^p, \\ \left(\frac{K^{n+1}8\lambda(5 \cdot 4^\beta + 2^{2\beta s} + 2^{4\beta s+1} + 2)\lambda||x||^{4s}}{6} \left[\frac{2}{2^\beta|2 - 2^{\beta 4s}|} + \frac{8}{8^\beta|8 - 2^{\beta 4s}|} \right] \right)^p, \end{cases} \tag{3.52}$$

for all $x \in U$.

4 Stability results: Fixed point method

In this section, we apply a fixed point method for achieving stability of the 2-variable AC functional equation (1.1).

Now, we present the following theorem due to B. Margolis and J.B. Diaz [12] for fixed point Theory.

Theorem 4.1. *[12] Suppose that for a complete generalized metric space (Ω, β) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \quad \forall \quad n \geq 0,$$

or there exists a natural number n_0 such that
(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
(ii) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T
(iii) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using the above theorem, we now obtain the generalized Ulam - Hyers stability of (1.1).

Through out this section let U be a normed space and V is a (β, p) Banach space with $p-$norm $\|.\|_V$. Define a mapping $F : U^2 \to V$ by

$$F(x,y,z,w) = f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4f(x+y, z+w) \\ + 4f(x-y, z-w) + 6f(y,w)$$

for all $x, y, z, w \in U$.

Theorem 4.2. *Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the condition*

$$\lim_{n \to \infty} \frac{1}{\mu_i^n} \alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w) = 0 \tag{4.1}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality

$$\|F(x,y,z,w)\|_V \leq \alpha(x,y,z,w) \tag{4.2}$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = K\delta\left(\frac{x}{2}\right),$$

has the property

$$\gamma(x) \leq L\,\mu_i\,\gamma(\mu_i x). \tag{4.3}$$

for all $x \in U$. Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ satisfying the functional equation (1.1) and

$$\| f(2x,2x) - 8f(x,x) - A(x,x) \|_V^p \leq \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \tag{4.4}$$

for all $x \in U$.

Proof. Consider the set

$$\Omega = \{q_1 / q_1 : U^2 \to V,\ q_1(0,0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(q_1,q_2) = d_\gamma(q_1,q_2) = \inf\{M \in (0,\infty) : \| q_1(x,x) - q_2(x,x) \| \leq M\gamma(x), x \in U\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega^2 \to \Omega$ by

$$Tq_1(x,x) = \frac{1}{\mu_i} q_1(\mu_i x, \mu_i x),$$

for all $x \in U$. Now $q_1, q_2 \in \Omega$,

$$\begin{aligned} d(q_1,q_2) \leq M &\Rightarrow \| q_1(x,x) - q_2(x,x) \| \leq M\gamma(x), x \in U. \\ &\Rightarrow \left\| \frac{1}{\mu_i} q_1(\mu_i x, \mu_i x) - \frac{1}{\mu_i} q_2(\mu_i x, \mu_i x) \right\| \leq \frac{1}{\mu_i} M\gamma(\mu_i x), x \in U, \\ &\Rightarrow \left\| \frac{1}{\mu_i} q_1(\mu_i x, \mu_i x) - \frac{1}{\mu_i} q_2(\mu_i x, \mu_i x) \right\| \leq LM\gamma(x), x \in U, \\ &\Rightarrow \| Tq_1(x,x) - Tq_2(x,x) \| \leq LM\gamma(x), x \in U, \\ &\Rightarrow d_\gamma(q_1,q_2) \leq LM. \end{aligned}$$

This implies $d(Tq_1, Tq_2) \leq Ld(q_1,q_2)$, for all $q_1, q_2 \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.
From (3.12), we arrive

$$\left\| \frac{g(2x,2x)}{2} - g(x,x) \right\|_V \leq K \frac{\delta(x)}{2^\beta} \tag{4.5}$$

for all $x \in U$. Using (4.3) for the case $i = 0$ it reduces to

$$\left\| \frac{g(2x,2x)}{2} - g(x,x) \right\| \leq L\gamma(x)$$

for all $x \in U$,

$$\text{i.e.,} \quad d(g,Tg) \leq L = \frac{1}{2^\beta} \Rightarrow d(g,Tg) \leq L = L^1 < \infty. \tag{4.6}$$

Again replacing $x = \frac{x}{2}$ in (4.5), we get,

$$\left\| g(x,x) - 2g\left(\frac{x}{2}, \frac{x}{2}\right) \right\|_V \leq K\delta\left(\frac{x}{2}\right) \tag{4.7}$$

Using (4.3) for the case $i = 1$ it reduces to

$$\left\| g(x,x) - 2g\left(\frac{x}{2}, \frac{x}{2}\right) \right\|_V \leq \gamma(x)$$

for all $x \in U$,

$$\text{i.e.,} \quad d(g,Tg) \leq 1 \Rightarrow d(g,Tg) \leq 1 = L^0 < \infty. \tag{4.8}$$

From (4.6) and (4.8), we have

$$d(Tg,g) \leq L^{1-i}. \tag{4.9}$$

Now from from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x,x) = \lim_{n\to\infty} \frac{1}{\mu_i^n} (f(\mu_i^{(n+1)} x, \mu_i^{(n+1)} x) - 8f(\mu_i^n x, \mu_i^n x)) \tag{4.10}$$

for all $x \in U$.

To prove $A : U^2 \to V$ is additive. Replacing (x,y,z,w) by $(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)$ in (4.2) and dividing by μ_i^n, it follows from (4.1) that

$$\|A(x,y,z,w)\|_V = \lim_{n\to\infty} \frac{\|F(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)\|_V}{\mu_i^n} \le \lim_{n\to\infty} \frac{\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)}{\mu_i^n} = 0$$

for all $x,y,z,w \in U$ i.e., A satisfies the functional equation (1.1).

According to the fixed point alternative, since A is the unique fixed point of T in the set $\Delta = \{A \in \Omega : d(f,A) < \infty\}$, A is the unique function such that

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\|_V \le M\gamma(x)$$

for all $x \in U$ and $K > 0$. Again using the fixed point alternative, we obtain

$$d(f,A) \le \frac{1}{1-L} d(f,Tf)$$

this implies

$$d(f,A) \le \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(2x,2x) - 8f(x,x) - A(x,x) \|_V^p \le \left(\frac{L^{1-i}}{1-L}\right)^p \gamma(x)^p$$

this completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 4.2 concerning the stability of (1.1).

Corollary 4.4. *Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x,y,z,w))\|_V \le \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s<1 \ \text{or} \ s>1; \\ \lambda\, ||x||^s ||y||^s ||z||^s ||w||^s, & s<\frac{1}{4} \ \text{or} \ s>\frac{1}{4}; \\ \lambda\left\{||x||^s ||y||^s ||z||^s ||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s<\frac{1}{4} \ \text{or} \ s>\frac{1}{4}; \end{cases} \tag{4.11}$$

for all $x,y,z,w \in U$, then there exists a unique 2- variable additive function $A : U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\|_V^p \le \begin{cases} (K\lambda(4^\beta+1))^p, \\ \left(\frac{(2+2^{s+1}+4^{\beta+1})K\lambda||x||^s}{|2-2^{\beta s}|}\right)^p, \\ \left(\frac{(4^\beta+2^{2s})K\lambda||x||^{4s}}{|2-2^{\beta 4s}|}\right)^p \\ \left(\frac{(5\cdot 4^\beta+2^{2s}+2^{4s+1}+2)K\lambda||x||^{4s}}{|2-2^{\beta 4s}|}\right)^p \end{cases} \tag{4.12}$$

for all $x \in U$.

Proof. Setting

$$\alpha(x,y,z,w) = \begin{cases} \lambda, \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, \\ \lambda\ ||x||^s\ ||y||^s\ ||z||^s\ ||w||^s \\ \lambda\left\{||x||^s ||y||^s ||z||^s ||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\} \end{cases}$$

for all $x, y, z, w \in U$. Now

$$\frac{\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)}{\mu_i^n} = \begin{cases} \frac{\lambda}{\mu_i^n}, \\ \frac{\lambda}{\mu_i^n}\{||\mu_i^n x||^s + ||\mu_i^n y||^s + ||\mu_i^n z||^s + ||\mu_i^n w||^s\}, \\ \frac{\lambda}{\mu_i^n}\ ||\mu_i^n x||^s\ ||\mu_i^n y||^s\ ||\mu_i^n z||^s\ ||\mu_i^n w||^s \\ \frac{\lambda}{\mu_i^n}\ \left\{||\mu_i^n x||^s\ ||\mu_i^n y||^s\ ||\mu_i^n z||^s\ ||\mu_i^n w||^s \right. \\ \left\{||\mu_i^n x||^{4s} + ||\mu_i^n y||^{4s} + ||\mu_i^n z||^{4s} + ||\mu_i^n w||^{4s}\right\}\Big\} \end{cases} = \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (4.1) is holds.

But we have $\gamma(x) = K\delta\left(\frac{x}{2}\right)$ has the property $\gamma(x) \le L \cdot \mu_i\, \gamma(\mu_i x)$ for all $x \in U$. Hence

$$\gamma(x) = K\delta\left(\frac{x}{2}\right) = K\left(4\alpha\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + \alpha\left(\frac{x}{2}, x, \frac{x}{2}, x\right)\right) = \begin{cases} K\lambda\left(4^\beta + 1\right), \\ \frac{K\lambda}{2^s}\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s, \\ \frac{K\lambda}{2^{4s}}\left(2^{2s} + 4^\beta\right)||x||^{4s}, \\ \frac{K\lambda}{2^{4s}}\left(2^{2s} + 2^{s+1} + 2^{4s+1} + 5 \cdot 4^\beta\right)||x||^{4s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i}\gamma(\mu_i x) = \begin{cases} \mu_i^{-1} K\lambda\left(4^\beta + 1\right), \\ \mu_i^{s-1} K\frac{\lambda}{2^s}\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s, \\ \mu_i^{4s-1} K\frac{\lambda}{2^{4s}}\left(2^{2s} + 4^\beta\right)||x||^{4s}, \\ \mu_i^{4s-1} K\frac{\lambda}{2^{4s}}\left(2^{2s} + 2^{s+1} + 2^{4s+1} + 5 \cdot 4^\beta\right)||x||^{4s} \end{cases} = \begin{cases} \mu_i^{-1}\gamma(x), \\ \mu_i^{s-1}\gamma(x), \\ \mu_i^{4s-1}\gamma(x), \\ \mu_i^{4s-1}\gamma(x). \end{cases}$$

Hence the inequality (4.3) holds either, $L = 2^{s-1}$ for $s < 2$ if $i = 0$ and $L = \frac{1}{2^{s-1}}$ for $s > 2$ if $i = 1$. Now from (4.4), we prove the following cases for condition (ii).

Case:1 $L = 2^{s-1}$ for $s < 1$ if $i = 0$

$$\begin{aligned} \|f(2x, 2x) - 8f(x, x) - A(x, x)\| &\le \frac{\left(2^{(s-1)}\right)^{1-0}}{1 - 2^{(s-1)}} \frac{K\lambda}{2^s}\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s \\ &= \frac{K\lambda\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s}{2 - 2^s} \end{aligned}$$

Case:2 $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$

$$\begin{aligned} \|f(2x, 2x) - 8f(x, x) - A(x, x)\| &\le \frac{\left(\frac{1}{2^{(s-1)}}\right)^{1-1}}{1 - \frac{1}{2^{(s-1)}}} \frac{K\lambda}{2^s}\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s \\ &= \frac{K\lambda\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s}{2^s - 2} \end{aligned}$$

Similarly, the inequality (4.3) holds either, $L = 2^{-1}$ for $s = 0$ if $i = 0$ and $L = \frac{1}{2^{-1}}$ for $s = 0$ if $i = 1$ for condition (i), the inequality (4.3) holds either, $L = 2^{4s-1}$ for $s < 2$ if $i = 0$ and $L = \frac{1}{2^{4s-1}}$ for $s > 2$ if $i = 1$ for condition (iii) and the inequality (4.3) holds either, $L = 2^{4s-1}$ for $s < 2$ if $i = 0$ and $L = \frac{1}{2^{4s-1}}$ for $s > 2$ if $i = 1$ for condition (iv).

Hence the proof is complete □

The proof of the following Theorem and Corollary is similar to that of Theorem 4.2 and Corollary 4.4. Hence the details of the proof is omitted.

Theorem 4.3. *Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^{3n}} \alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w) = 0 \tag{4.13}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality

$$\|F(x, y, z, w)\|_V \le \alpha(x, y, z, w) \tag{4.14}$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = K\delta\left(\frac{x}{2}\right),$$

has the property

$$\gamma(x) \le L\, \mu_i^3\, \gamma(\mu_i x). \tag{4.15}$$

Then there exists a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1.1) and

$$\| f(2x, 2x) - 2f(x, x) - C(x, x) \|_V^p \le \left(\frac{L^{1-i}}{1-L}\right)^p \gamma(x)^p \tag{4.16}$$

for all $x \in U$.

Corollary 4.5. *Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\begin{aligned} &\|F(x, y, z, w))\|_V \\ &\le \begin{cases} \lambda, \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s < 1 \quad or \quad s > 1; \\ \lambda\, ||x||^s ||y||^s ||z||^s ||w||^s, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \\ \lambda\left\{||x||^s ||y||^s ||z||^s ||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s < \frac{3}{4} \quad or \quad s > \frac{3}{4}; \end{cases} \end{aligned} \tag{4.17}$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable cubic function $C : U^2 \to V$ such that

$$\|f(2x, 2x) - 2f(x, x) - C(x, x)\|_V^p \le \begin{cases} \left(\dfrac{K\lambda(4^\beta + 1)}{7}\right)^p, \\ \left(\dfrac{(2 + 2^{s+1} + 4^{\beta+1})K\lambda||x||^s}{|8 - 2^{\beta s}|}\right)^p, \\ \left(\dfrac{(4^\beta + 2^{2s})K\lambda||x||^{4s}}{|8 - 2^{\beta 4s}|}\right)^p \\ \left(\dfrac{(5 \cdot 4^\beta + 2^{2s} + 2^{4s+1} + 2)K\lambda||x||^{4s}}{|8 - 2^{\beta 4s}|}\right)^p \end{cases} \tag{4.18}$$

for all $x \in U$.

Now, we are ready to prove the main fixed point stability results.

Theorem 4.4. *Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the conditions (4.1) and (4.13) where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality*

$$\|F(x, y, z, w)\|_V \le \alpha(x, y, z, w) \tag{4.19}$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = K\delta\left(\frac{x}{2}\right),$$

has the properties (4.3) *and* (4.15) *Then there exists a unique 2-variable additive mapping* $A : U^2 \to V$ *and a unique 2-variable cubic mapping* $C : U^2 \to V$ *satisfying the functional equation* (1.1) *and*

$$\| f(x,x) - A(x,x) - C(x,x) \|_V^p \leq \frac{2\,K^p}{6^{\beta\,p}} \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \tag{4.20}$$

for all $x \in U$.

Proof. By Theorems 4.2 and 4.3, there exists a unique 2-variable additive function $A_1 : U^2 \to V$ and a unique 2-variable cubic function $C_1 : U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A_1(x,x)\|_V^p \leq \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \tag{4.21}$$

and

$$\|f(2x,2x) - 2f(x,x) - C_1(x,x)\|_V^p \leq \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \tag{4.22}$$

for all $x \in U$. Now from (4.21) and (4.22), one can see that

$$\begin{aligned}
&\left\| f(x,x) + \frac{1}{6}A_1(x,x) - \frac{1}{6}C_1(x,x) \right\|_V^p \\
&= \left\| \left\{ -\frac{f(2x,2x)}{6} + \frac{8f(x,x)}{6} + \frac{A_1(x,x)}{6} \right\} + \left\{ \frac{f(2x,2x)}{6} - \frac{2f(x,x)}{6} - \frac{C_1(x,x)}{6} \right\} \right\|_V^p \\
&\leq \frac{K^p}{6^{\beta p}} \left\{ \|f(2x,2x) - 8f(x,x) - A_1(x,x)\|_V^p + \|f(2x,2x) - 2f(x,x) - C_1(x,x)\|_V^p \right\} \\
&\leq \frac{K^p}{6^{\beta p}} \left\{ \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p + \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \right\}
\end{aligned}$$

for all $x \in U$. Thus we obtain (4.20) by defining $A(x,x) = \frac{-1}{6}A_1(x,x)$ and $C(x,x) = \frac{1}{6}C_1(x,x)$, for all $x \in U$. □

The following Corollary is an immediate consequence of Theorem 4.4, using Corollaries 4.4 and 4.5 concerning the stability of (1.1).

Corollary 4.6. *Let* $F : U^2 \to V$ *be a mapping and there exists real numbers* λ *and* s *such that*

$$\|F(x,y,z,w))\|_V \leq \begin{cases} \lambda, & \\ \lambda \left\{ ||x||^s + ||y||^s + ||z||^s + ||w||^s \right\}, & s \neq 1,3; \\ \lambda\, ||x||^s ||y||^s ||z||^s ||w||^s, & s \neq \frac{1}{4}, \frac{3}{4}; \\ \lambda \left\{ ||x||^s ||y||^s ||z||^s ||w||^s + \left\{ ||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s} \right\} \right\}, & s \neq \frac{1}{4}, \frac{3}{4}; \end{cases} \tag{4.23}$$

for all $x, y, z, w \in U$*, then there exists a unique 2-variable additive mapping* $A : U^2 \to V$ *and a unique 2-variable cubic mapping* $C : U^2 \to V$ *such that*

$$\|f(x,x) - A(x,x) - C(x,x)\|_V \leq \begin{cases} \left(\dfrac{K^2\lambda(4^\beta + 1)}{6} \left[1 + \dfrac{1}{7} \right] \right)^p, \\ \left(\dfrac{(2 + 2^{s+1} + 4^{\beta+1})K^2\lambda||x||^s}{6} \left[\dfrac{1}{|2 - 2^{\beta s}|} + \dfrac{1}{|8 - 2^{\beta s}|} \right] \right)^p, \\ \left(\dfrac{(4^\beta + 2^{2s})K^2\lambda||x||^{4s}}{6} \left[\dfrac{1}{|2 - 2^{\beta 4s}|} + \dfrac{1}{|8 - 2^{\beta 4s}|} \right] \right)^p \\ \left(\dfrac{(5 \cdot 4^\beta + 2^{2s} + 2^{4s+1} + 2)K^2\lambda||x||^{4s}}{6} \left[\dfrac{1}{|2 - 2^{\beta 4s}|} + \dfrac{1}{|8 - 2^{\beta 4s}|} \right] \right)^p \end{cases} \tag{4.24}$$

for all $x \in U$.

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On 3-Dissection Property

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Abstract

The purpose of this paper is to derive 3- dissection for $(q^2;q^2)_\infty^{-1}(q^4;q^4)_\infty^{-1}$, $(q^3;q^3)_\infty^{-1}(q^6;q^6)_\infty^{-1}$ and $(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty^{-1}(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty^{-1}$.

Keywords: Partition functions, Generating functions.

2010 MSC: 05A17, 05A15, 11P83.

1 Introduction

$x \sim y$

In 2010, Chan [1] has studied on Ramanujan's cubic continued fraction and defined a function a(n), as

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty\,(q^2;q^2)_\infty} \tag{1.1}$$

In 2011, Zhao and Zhong [2] have studied and investigated the arithmetic properties of a function b(n), as

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2\,(q^2;q^2)_\infty^2} \tag{1.2}$$

Through this paper, we assume

$$(a;q)_\infty = \prod_{n=1}^{\Pi}(1 - aq^{(n-1)}) \quad ; \quad |q| < 1 \tag{1.3}$$

Many properties of $a(n)$ and $b(n)$ are similar with the standard partition function $p(n)$, the function $p(n)$ is defined to be the number of ways of writing n as a sum of positive integers in non-increasing order. Mathematically it is defined as $\sum_{n\geq 0} p(n)q^n = \prod_{n=1}(1-q)^{-1}$. It is convention that, one sets $p(0) = 0$ and $p(n) = 0$ for $n < 0$. Chan[1] obtained the generating function of $a(3n+2)$, as

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3\,(q^6;q^6)_\infty^3}{(q;q)_\infty^4\,(q^2;q^2)_\infty^4} \tag{1.4}$$

This identity was prove by Cao[3] by using the 3-dissection for $(q;q)_\infty\,(q^2;q^2)_\infty$. The outline of this paper is as follows. In sections 2, we have recorded some well known results, those are useful to the rest of the paper. In section 3, we state and prove three new theorems, which are not available in the literature.

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2 Preliminaries

Let us recall the definition of cubic theta function $A(q)$,$B(q)$ and $C(q)$due to Borwein et al.[4], as

$$A(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{2.1}$$

$$B(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}; \ \ \omega = exp\left(\frac{2\Pi\iota}{3}\right) \tag{2.2}$$

$$C(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} \tag{2.3}$$

Borwein et al.[4] established the following relations

$$A(q) = A(q^3) + 2qC(q^3) \tag{2.4}$$

$$B(q) = A(q^3) - qC(q^3) \tag{2.5}$$

$$C(q) = \frac{3(q^3;q^3)_\infty^3}{(q;q)_\infty} \tag{2.6}$$

$$A(q)A(q^2) = B(q)B(q^2) + qC(q)C(q^2) \tag{2.7}$$

3 Main results

Now we derive following results by applying 3-dissection

Theorem-I:

$$\frac{1}{(q^2;q^2)_\infty(q^4;q^4)_\infty} = \frac{A(q^{12})C(q^6)}{3(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty^3} + \frac{q^2A(q^6)C(q^{12})}{3(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty^3} + \frac{q^4C(q^6)C(q^{12})}{3(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty^3} \tag{3.1}$$

Theorem-II:

$$\frac{1}{(q^3;q^3)_\infty(q^6;q^6)_\infty} = \frac{q^3A(q^9)C(q^{18})}{3(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3} + \frac{A(q^{18})C(q^9)}{3(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3} + \frac{q^6C(q^9)C(q^{18})}{3(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3} \tag{3.2}$$

Theorem-III:

$$\frac{1}{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty} = \frac{qA(q^2)C(q)}{3(q;q)_\infty^3(q^2;q^2)_\infty^3} + \frac{q^{\frac{1}{3}}A(q)C(q^2)}{3(q;q)_\infty^3(q^2;q^2)_\infty^3} + \frac{q^{\frac{2}{3}}C(q)C(q^2)}{3(q;q)_\infty^3(q^2;q^2)_\infty^3} \tag{3.3}$$

Proof of Theorem-I: In equation (2.6), by substituting $q = q^2$ and $q = q^4$,we get the values of $C(q^2)$ and $C(q^4)$ respectively. Now by multiplying $C(q^2)$ and $C(q^4)$, and after making suitable arrangement, we get

$$\frac{1}{(q^2;q^2)_\infty(q^4;q^4)_\infty} = \frac{C(q^2)C(q^4)}{9(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty^3} \tag{3.4}$$

In equation (2.4), by substituting $q = q^2$ and $q = q^4$,we get the values of $A(q^2)$ and $A(q^4)$ respectively. Now by multiplying $A(q^2)$ and $A(q^4)$,we get

$$A(q^2)A(q^4) = A(q^6)A(q^{12}) + 2q^2A(q^{12})C(q^6) + 2q^4A(q^6)C(q^{12}) + 4q^6C(q^6)C(q^{12}) \tag{3.5}$$

In equation (2.5), by substituting $q = q^2$ and $q = q^4$,we get the values of $B(q^2)$ and $B(q^4)$ respectively. Now by multiplying $B(q^2)$ and $B(q^4)$,we get

$$B(q^2)B(q^4) = A(q^6)A(q^{12}) - q^2A(q^{12})C(q^6) - q^4A(q^6)C(q^{12}) + q^8C(q^6)C(q^{12}) \tag{3.6}$$

In equation (2.7), by substituting $q = q^2$,we get the values of $C(q^2)C(q^4)$, as

$$q^2C(q^2)C(q^4) = A(q^2)A(q^4) - B(q^2)B(q^4) \tag{3.7}$$

By the equations (3.5),(3.6)and(3.7), we get

$$C(q^2)C(q^4) = 3A(q^{12})C(q^6) + 3q^2A(q^6)C(q^{12}) + 3q^4C(q^6)C(q^{12}) \tag{3.8}$$

By substituting the value of $C(q^2)C(q^4)$ in equation (3.4), from equation (3.8), after simplification, we get the required result as per equation (3.1), and we complete the proof of Theorem-I.

Proof of Theorem-II: In equation (2.6), by substituting $q = q^3$ and $q = q^6$,we get the values of $C(q^3)$ and $C(q^6)$ respectively. Now by multiplying $C(q^3)$ and $C(q^6)$, and after making suitable arrangement, we get

$$\frac{1}{(q^3;q^3)_\infty(q^6;q^6)_\infty} = \frac{C(q^3)C(q^6)}{9(q^9;q^9)^3_\infty(q^{18};q^{18})^3_\infty} \tag{3.9}$$

In equation (2.4), by substituting $q = q^3$ and $q = q^6$,we get the values of $A(q^3)$ and $A(q^6)$ respectively. Now by multiplying $A(q^3)$ and $A(q^6)$,we get

$$A(q^3)A(q^6) = A(q^9)A(q^{18}) + 2q^6A(q^9)C(q^{18}) + 2q^3A(q^{18})C(q^9) + 4q^9C(q^9)C(q^{18}) \tag{3.10}$$

In equation (2.5), by substituting $q = q^3$ and $q = q^6$,we get the values of $B(q^3)$ and $B(q^6)$ respectively. Now by multiplying $B(q^3)$ and $B(q^6)$,we get

$$B(q^3)B(q^6) = A(q^9)A(q^{18}) - q^6A(q^9)C(q^{18}) - q^3A(q^{18})C(q^9) + q^9C(q^9)C(q^{18}) \tag{3.11}$$

In equation (2.7), by substituting $q = q^3$,we get the values of $C(q^3)C(q^6)$, as

$$q^3C(q^3)C(q^6) = A(q^3)A(q^6) - B(q^3)B(q^6) \tag{3.12}$$

By the equations (3.10),(3.11)and(3.12), we get

$$C(q^3)C(q^6) = 3q^3A(q^9)C(q^{18}) + 3A(q^{18})C(q^9) + 3q^6C(q^9)C(q^{18}) \tag{3.13}$$

By substituting the value of $C(q^3)C(q^6)$ in equation (3.9), from equation (3.13), after simplification, we get the required result as per equation (3.2), and we complete the proof of Theorem-II.

Proof of Theorem-III: In equation (2.6), by substituting $q = q^{\frac{1}{3}}$ and $q = q^{\frac{2}{3}}$,we get the values of $C(q^{\frac{1}{3}})$ and $C(q^{\frac{2}{3}})$ respectively. Now by multiplying $C(q^{\frac{1}{3}})$ and $C(q^{\frac{2}{3}})$, and after making suitable arrangement, we get

$$\frac{1}{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty} = \frac{C(q^{\frac{1}{3}})C(q^{\frac{2}{3}})}{9(q;q)^3_\infty(q^2;q^2)^3_\infty} \tag{3.14}$$

In equation (2.4), by substituting $q = q^{\frac{1}{3}}$ and $q = q^{\frac{2}{3}}$,we get the values of $A(q^{\frac{1}{3}})$ and $A(q^{\frac{2}{3}})$ respectively. Now by multiplying $A(q^{\frac{1}{3}})$ and $A(q^{\frac{2}{3}})$,we get

$$A(q^{\frac{1}{3}})A(q^{\frac{2}{3}}) = A(q)A(q^2) + 2q^{\frac{1}{3}}A(q^2)C(q) + 2q^{\frac{2}{3}}A(q)C(q^2) + 4qC(q)C(q^2) \tag{3.15}$$

In equation (2.5), by substituting $q = q^{\frac{1}{3}}$ and $q = q^{\frac{2}{3}}$,we get the values of $B(q^{\frac{1}{3}})$ and $B(q^{\frac{2}{3}})$ respectively. Now by multiplying $B(q^{\frac{1}{3}})$ and $B(q^{\frac{2}{3}})$,we get

$$B(q^{\frac{1}{3}})B(q^{\frac{2}{3}}) = A(q)A(q^2) - q^{\frac{1}{3}}A(q^2)C(q) - q^{\frac{2}{3}}A(q)C(q^2) + qC(q)C(q^2) \tag{3.16}$$

In equation (2.7), by substituting $q = q^{\frac{1}{3}}$,we get the values of $C(q^{\frac{1}{3}})C(q^{\frac{2}{3}})$, as

$$q^{\frac{1}{3}}C(q^{\frac{1}{3}})C(q^{\frac{2}{3}}) = A(q^{\frac{1}{3}})A(q^{\frac{2}{3}}) - B(q^{\frac{1}{3}})B(q^{\frac{2}{3}}) \tag{3.17}$$

By the equations (3.15),(3.16)and(3.17), we get

$$C(q^{\frac{1}{3}})C(q^{\frac{2}{3}}) = 3A(q^2)C(q) + 3(q^{\frac{1}{3}})A(q)C(q^2) + 3q^{\frac{2}{3}}C(q)C(q^2) \tag{3.18}$$

By substituting the value of $C(q^{\frac{1}{3}})C(q^{\frac{2}{3}})$ in equation (3.14), from equation (3.18), after simplification, we get the required result as per equation (3.3), and we complete the proof of Theorem-III.

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The extended tanh method for certain system of nonlinear ordinary differential equations

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Abstract

We propose a method to obtain Tanh-solution based on leading order analysis of Painlevè test. The crucial aspect is that this point of view gives "exactly truncation of the series expansion applicable to Tanh-method". This approach gives all possible leading orders of solutions. Each branches can be treated separately and obtained closed form solutions.

Keywords: Ordinary differential equations, Tanh-method, Singularity analysis.

2010 MSC: 02; 02.30.Ik; 02.30.Hq.

1 Introduction

For many years, nonlinearity is playing an important role in various fields of mathematics, physics and biology. Finding the exact solutions of the nonlinear ordinary differential equations and partial differential equations are quite difficult. So far, many methods have been proposed by many authors for finding exact solutions of nonlinear differential equations. We mentioned some of them here: tanh−expansion method [1] − [7], the simplest equation method [11], the Jacobi elliptic−function method [12], the modified simplest equation method [13], the exp−function method [14] − [16], the G'/G-expansion method [18] and application of the Hirota method for non integrable nonlinear differential equation [17]. Recently, Willy Malfliet et al. and Abdul−Majid WazWaz [7] have successfully refined the tanh method for solving a lot of systems of autonomous partial differential equations and obtained solutions of them successfully. For the first time, best of our knowledge, we employ this method directly to ordinary differential equations. Here, we implement the leading order analysis or ARS method to determine all leading orders in the expansion of all solutions of differential equations. We remind the readers that we are not going to test the Painlevé property here. Thus, the approach is equally applicable for both integrable and non-integrable differential equations. We truncate the expression looking at the leading term. That is, if the leading term starts with $\tau^{-p},\ p > 0$ then the expression terminates at τ^{p}. To find the full expression of this expansion, we determine the each coefficients of the expansion by comparing the various powers of ξ and obtain an over-determined system of algebraic equation for the unknowns. Solving them consistently, we can obtain the values of the coefficients uniquely. Thus, tanh solution is determined uniquely for a given equation. If there are more than one leading orders then each order will give the appropriate series solutions separately. Interestingly the present approach gives a concrete way of finding all leading terms. That is if a given equation admits more than one branch of solutions then it could be determined uniquely.

In this paper, we explain the extended tanh-method with all possible leading orders and apply to certain physically important problems.

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2 Review of leading order analysis of Painlevé test [9]

Let us consider the system of ordinary differential equations

$$f_1(x, y, z, \dot{x}, \dot{y}, \dot{z}, ...) = 0, \tag{2.1}$$

$$f_2(x, y, z, \dot{x}, \dot{y}, \dot{z}, ...) = 0, \tag{2.2}$$

$$f_3(x, y, z, \dot{x}, \dot{y}, \dot{z}, ...) = 0, \tag{2.3}$$

where ‘$\cdot$’ denotes derivative with respect to t. Assume that the leading order of the solutions are in the form

$$x \sim \tau^p, \tag{2.4}$$

$$y \sim \tau^q, \tag{2.5}$$

$$z \sim \tau^r, \tag{2.6}$$

where p, q and r are the integers to be determined and $\tau = t - t_0$. Substituting Eqs.(2.4)-(2.6) into Eqs.(2.1)-(2.3) then equating the all dominant terms then we can get the all possible choices of p, q and r. Some times we may get two or more choices of p, q and r. We demonstrate these concepts with the following example

Example

Consider the third-order ordinary differential equation [9]

$$\dddot{x} + x\ddot{x} - 2x^3 + \lambda x^2 + \alpha x + \beta = 0. \tag{2.7}$$

Substituting Eq.(2.4) in Eq.(2.7) then we get

$$p(p-1)(p-2)\tau^{p-3} + p(p-1)\tau^{2p-2} - 3\tau^{3p} \approx 0. \tag{2.8}$$

Equating the various powers of τ and find p as follows

1. $p - 3 = 2p - 2$ this implies $p = -1$
2. $2p - 2 = 3p$ this implies $p = -2$.

Hence, there are two set of dominant terms $(\dddot{x}, x\ddot{x})$ and $(x\ddot{x}, x^3)$ which are balancing each other in Eq.(2.7) [9].

3 Review of extended Tanh-method $[1] - [7]$

Now we use the extended tanh-method $[1] - [7]$ for finding the exact solutions of system of nonlinear autonomous ordinary differential equations. we introduce a new independent variable

$$\xi = \tanh(\mu t), \tag{3.9}$$

then (3.10)

$$\frac{d}{dt} = \mu(1-\xi^2)\frac{d}{d\xi}, \tag{3.11}$$

$$\frac{d^2}{dt^2} = -2\mu^2\xi(1-\xi^2)\frac{d}{d\xi} + \mu^2(1-\xi^2)^2\frac{d^2}{d\xi^2}, \tag{3.12}$$

$$\frac{d^3}{dt^3} = 2\mu^3(1-\xi^2)(3\xi^2-1)\frac{d}{d\xi} - 6\mu^3\xi(1-\xi^2)^2\frac{d^2}{d\xi^2} + \mu^3(1-\xi^2)^3\frac{d^3}{d\xi^3}, \tag{3.13}$$

$$\begin{aligned}\frac{d^4}{dt^4} &= -8\mu^4\xi(1-\xi^2)(3\xi^2-2)\frac{d}{d\xi} + 4\mu^4(1-\xi^2)^2(9\xi^2-2)\frac{d^2}{d\xi^2} \\ &\quad -12\mu^4\xi(1-\xi^2)^3\frac{d^3}{d\xi^3} + \mu^4(1-\xi^2)^4\frac{d^4}{d\xi^4}.\end{aligned} \tag{3.14}$$

holds. Now consider the series expansion

$$\begin{aligned} x[t] &= X[\xi] = \sum_{i=-p}^{p} a_i \xi^i \\ y[t] &= Y[\xi] = \sum_{i=-q}^{q} b_i \xi^i \\ z[t] &= Z[\xi] = \sum_{i=-r}^{r} c_i \xi^i \end{aligned}$$

where p, q and r which were identified from leading order analysis.

4 Applications

4.1 Example

Consider the system of ODE [9]

$$\dddot{x} + x\ddot{x} - 2x^3 + \lambda x^2 + \alpha x + \beta = 0. \tag{4.15}$$

First, one has to change the given Eq.(4.15) in terms of new independent variable ξ by using Eqs.(3.11), (3.12) and (3.13). Thus, we obtain

$$\begin{aligned} &\mu^3\left(1-\xi^2\right)^3 x''' - 6\mu^3\xi\left(1-\xi^2\right)^2 x'' + x\left(-2\mu^2\xi\left(1-\xi^2\right)x' + \mu^2\left(1-\xi^2\right)^2 x''\right) \\ &+2\mu^3\left(1-\xi^2\right)\left(-1+3\xi^2\right)x' - 2x^3 + \lambda x^2 + \alpha x + \beta = 0, \end{aligned} \tag{4.16}$$

where $'$ denote the derivatives with respect to new independent variable ξ.

Since, we have obtained two possible leading orders $p = -1$ and $p = -2$, it is evident that there are two branches of solutions exist for Eq.(4.15). we treat each case separately.

Case (a) p=-1:
We assume that the solution of the form

$$x[t] = X[\xi] = a_{-1}\xi^{-1} + a_0 + a_1\xi. \tag{4.17}$$

On substitution Eq.(4.17) into Eq.(4.16) and collecting the coefficients of various powers of ξ than we obtain a system of over-determined equations for a_i, where $i = -1,\ 0$ and 1.

$$\begin{aligned} -6\mu^3 a_{-1} + 2\mu^2 a_{-1}^2 &= 0, \\ -2a_{-1}^3 + 2\mu^2 a_{-1}a_0 &= 0, \\ 8\mu^3 a_{-1} + \lambda a_{-1}^2 - 2\mu^2 a_{-1}^2 - 6a_{-1}^2 a_0 + 2\mu^2 a_{-1}a_1 &= 0, \\ \alpha a_{-1} + 2\lambda a_{-1}a_0 - 2\mu^2 a_{-1}a_0 - 6a_{-1}a_0^2 - 6a_{-1}^2 a_1 &= 0, \\ \beta - 2\mu^3 a_{-1} + \alpha a_0 + \lambda a_0^2 - 2a_0^3 - 2\mu^3 a_1 + 2\lambda a_{-1}a_1 & \\ -4\mu^2 a_{-1}a_1 - 12a_{-1}a_0a_1 &= 0, \\ \alpha a_1 + 2\lambda a_0 a_1 - 2\mu^2 a_0 a_1 - 6a_0^2 a_1 - 6a_{-1}a_1^2 &= 0, \\ 8\mu^3 a_1 + 2\mu^2 a_{-1}a_1 + \lambda a_1^2 - 2\mu^2 a_1^2 - 6a_0 a_1^2 &= 0, \\ 2\mu^2 a_0 a_1 - 2a_1^3 &= 0, \\ -6\mu^3 a_1 + 2\mu^2 a_1^2 &= 0. \end{aligned} \tag{4.18}$$

Solving them consistently, we arrive at solutions of a_i where $i = -1,\ 0$ and 1. We tabulate the results in table(1).

Table 1: Case (a): p=-1

Cases	Values	Conditions	Solutions
i	$a_{-1} = 0, a_0 = 9,$ $a_1 = \pm\sqrt{\frac{3(\alpha+486)}{10}},$	$\beta = \frac{(-69984 - 108\alpha + \alpha^2)}{150},$ $\lambda = \frac{1944-\alpha}{45},$ $\mu = \pm\sqrt{\frac{\alpha+486}{30}}$	$x[t] = 9 + \sqrt{\frac{3(\alpha+486)}{10}} \tan\left[\sqrt{\frac{\alpha+486}{30}}t\right],$
ii	$a_{-1} = \pm\frac{1}{2}\sqrt{\frac{3(\alpha+486)}{10}},$ $a_0 = 9,$ $a_1 = \pm\frac{1}{2}\sqrt{\frac{3(\alpha+486)}{10}}$	$\beta = \frac{(-69984 - 108\alpha + \alpha^2)}{150},$ $\lambda = \frac{1944-\alpha}{45},$ $\mu = \pm\frac{1}{2}\sqrt{\frac{\alpha+486}{30}}$	$x[t] = 9 + \frac{1}{2}\sqrt{\frac{3(\alpha+486)}{10}} \cot\left[\frac{1}{2}\sqrt{\frac{\alpha+486}{30}}t\right]$ $+\frac{1}{2}\sqrt{\frac{3(\alpha+486)}{10}} \tan\left[\frac{1}{2}\sqrt{\frac{\alpha+486}{30}}t\right],$
iii	$a_{-1} = \pm\sqrt{\frac{3(\alpha+486)}{10}},$ $a_0 = 9,$ $a_1 = 0$	$\beta = \frac{(-69984 - 108\alpha + \alpha^2)}{150},$ $\lambda = \frac{1944-\alpha}{45},$ $\mu = \pm\sqrt{\frac{\alpha+486}{30}}$	$x[t] = 9 + \sqrt{\frac{3(\alpha+486)}{10}} \cot\left[\sqrt{\frac{\alpha+486}{30}}t\right],$

Case (b): p=-2
Assume the solution in the form

$$X[\xi] = a_{-2}\xi^{-2} + a_{-1}\xi^{-1} + a_0 + a_1\xi + a_2\xi^2, \tag{4.19}$$

On substitution Eq.(4.19) into Eq.(4.16) and collecting the coefficients of various powers of ξ than we obtain a system of over-determined equations for a_i where $i = -2, -1, 0, 1$ and 2. The solutions are given in the table(2).

Table 2: Case (b): p=-2

Cases	Values	Conditions	Solutions
i	$a_0 = \frac{88}{25}, a_{-1} = a_{-2} = 0,$ $a_2 = \frac{12}{25}, a_1 = \pm\frac{24}{25}$	$\beta = \frac{75392}{625}, \alpha = -\frac{58848}{625},$ $\lambda = 24, \mu = \mp\frac{2}{5}$	$x[t] = \frac{88}{25} - \frac{24}{25}\tan\left[\frac{2t}{5}\right] + \frac{12}{25}\tan^2\left[\frac{2t}{5}\right]$
ii	$a_0 = \frac{88}{25}, a_{-2} = \frac{12}{25},$ $, a_1 = a_2 = 0, a_{-1} = \pm\frac{24}{25}$	$\beta = \frac{75392}{625}, \alpha = -\frac{58848}{625},$ $\lambda = 24, \mu = \mp\frac{2}{5}$	$x[t] = \frac{88}{25} - \frac{24}{25}\cot\left[\frac{2t}{5}\right] + \frac{12}{25}\cot^2\left[\frac{2t}{5}\right]$
iii	$a_0 = \frac{94}{25}, a_{-2} = a_2 = \frac{3}{25},$ $a_{-1} = a_1 = \pm\frac{12}{25}$	$\beta = \frac{75392}{625}, \alpha = -\frac{58848}{625},$ $\lambda = 24, \mu = \mp\frac{1}{5}$	$x[t] = \frac{94}{25} - \frac{12}{25}\left(\cot\left[\frac{t}{5}\right] + \tan\left[\frac{t}{5}\right]\right)$ $+\frac{3}{25}\left(\cot^2\left[\frac{t}{5}\right] + \tan^2\left[\frac{t}{5}\right]\right)$
iv	$a_0 = \frac{468}{25}, a_{-2} = a_2 = \frac{-162}{25},$ $a_{-1} = a_1 = \pm\frac{36i\sqrt{6}}{25}$	$\beta = \frac{2239488}{625}, \alpha = -\frac{82944}{125}$ $\lambda = \frac{1656}{25}, \mu = \mp\frac{3i\sqrt{6}}{5}$	$x[t] = -\frac{36}{25}\sqrt{6}\left(\coth\left[\frac{3\sqrt{6}t}{5}\right] + \tanh\left[\frac{3\sqrt{6}t}{5}\right]\right)$ $+\frac{468}{25} + \frac{162}{25}\left(\coth^2\left[\frac{3\sqrt{6}t}{5}\right] + \tanh^2\left[\frac{3\sqrt{6}t}{5}\right]\right)$
v	$a_0 = \frac{792}{25}, a_{-2} = a_{-1} = 0,$ $a_2 = -\frac{648}{25}, a_1 = \pm\frac{72i\sqrt{6}}{25}$	$\beta = \frac{2239488}{625}, \alpha = -\frac{82944}{125}$ $\lambda = \frac{1656}{25}, \mu = \mp\frac{6i\sqrt{6}}{5}$	$x[t] = \frac{792}{25} + \frac{72}{25}\sqrt{6}\tanh\left[\frac{6\sqrt{6}t}{5}\right]$ $+\frac{648}{25}\tanh^2\left[\frac{6\sqrt{6}t}{5}\right]$
vi	$a_0 = \frac{792}{25}, a_{-1} = \frac{\pm 72i\sqrt{6}}{25}$ $a_{-2} = -\frac{648}{25}, a_1 = a_2 = 0$	$\beta = \frac{2239488}{625}, \alpha = -\frac{82944}{125}$ $\lambda = \frac{1656}{25}, \mu = \mp\frac{6i\sqrt{6}}{5}$	$x[t] = \frac{792}{25} - \frac{72}{25}\sqrt{6}\coth\left[\frac{6\sqrt{6}t}{5}\right]$ $+\frac{648}{25}\coth^2\left[\frac{6\sqrt{6}t}{5}\right]$

4.2 Fourth order equation

Consider the fourth order ODE [19]

$$x^{(4)}+x(\ddot{x}+\beta)-\frac{3}{4}\dot{x}^2-3(\alpha+1)=0,. \tag{4.20}$$

In [19] expensive studies have been made from geometrical and numerical point of view. However, no exact analytical solutions been presented for Eq.(4.20). In this paper, we present a class of new exact closed form solutions for Eq.(4.20). Due to the importance of this equation from geometric point of view, we believe that the solutions presented here are significant in many ways. Painlené leading order analysis gives $p=-2$ for Eq.(4.20). On substitution this value into $X[\xi]$ and follow the tanh procedure then we tabulate the results below

Cases	Values	Conditions	Solutions
i	$a_0=5\sqrt{\frac{\beta}{21}}, a_{-1}=a_1=0,$ $a_{-2}=a_2=-\frac{5}{2}\sqrt{\frac{3\beta}{7}}$	$\alpha=\frac{\left(-63-10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu=\pm\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t]=-\frac{5}{2}\sqrt{\frac{3}{7}}\sqrt{\beta}\coth^2\left[\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right]$ $+\frac{5\sqrt{\beta}}{\sqrt{21}}-\frac{5}{2}\sqrt{\frac{3}{7}}\sqrt{\beta}\tanh^2\left[\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right]$
ii	$a_{-1}=a_1=0, a_0=20\sqrt{\frac{\beta}{21}},$ $a_2=-10\sqrt{\frac{3\beta}{7}}, a_{-2}=0$	$\alpha=\frac{\left(-63-10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu=\pm\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t]=\frac{20\sqrt{\beta}}{\sqrt{21}}$ $-10\sqrt{\frac{3}{7}}\sqrt{\beta}\tanh^2\left[\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right].$
iii	$a_{-1}=a_1=0, a_0=20\sqrt{\frac{\beta}{21}},$ $a_{-2}=-10\sqrt{\frac{3\beta}{7}}, a_2=0$	$\alpha=\frac{\left(-63-10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu=\pm\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t]=\frac{20\sqrt{\beta}}{\sqrt{21}}$ $-10\sqrt{\frac{3}{7}}\sqrt{\beta}\coth^2\left[\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right].$
iv	$a_0=-20\sqrt{\frac{\beta}{21}}, a_{-1}=0,$ $a_2=10\sqrt{\frac{3\beta}{7}}, a_{-2}=a_1=0,$	$\alpha=\frac{\left(-63+10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu=\pm\frac{1}{2}i\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t]=-\frac{20\sqrt{\beta}}{\sqrt{21}}$ $-10\sqrt{\frac{3}{7}}\sqrt{\beta}\tan^2\left[\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right].$
v	$a_0=-20\sqrt{\frac{\beta}{21}}, a_1=a_2=0$ $a_{-2}=10\sqrt{\frac{3\beta}{7}}, a_{-1}=0,$	$\alpha=\frac{\left(-63+10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu=\pm\frac{1}{2}i\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t]=-\frac{20\sqrt{\beta}}{\sqrt{21}}$ $-10\sqrt{\frac{3}{7}}\sqrt{\beta}\cot^2\left[\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right].$
vi	$a_0=-5\sqrt{\frac{\beta}{21}}, a_{-1}=a_1=0,$ $a_{-2}=a_2=\frac{5}{2}\sqrt{\frac{3\beta}{7}}$	$\alpha=\frac{\left(-63-10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu=\pm\frac{1}{4}i\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t]=-\frac{5}{2}\sqrt{\frac{3}{7}}\sqrt{\beta}\cot^2\left[\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right]$ $+\frac{5\sqrt{\beta}}{\sqrt{21}}-\frac{5}{2}\sqrt{\frac{3}{7}}\sqrt{\beta}\tan^2\left[\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right]$

5 Conclusions

In this paper, we have successfully employed extended tanh-method by using leading order analysis of Painlevé test. Thus we could able to find all possible branches of solutions for the given differential equations. Also the choice of the leading term and truncation is indeed not arbitrary uniquely determined by the leading order analysis. Our method is successful to find large class of solutions of certain well-known systems. Finally, we remark that this approach can equally applied to nonintegrable systems as well including systems from Biology [20].

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Peristaltic pumping of a Jeffrey fluid in an asymmetric channel with permeable walls

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Abstract

This paper deals with the peristaltic pumping of a Jeffrey fluid in an asymmetric channel with permeable walls under long wave length and low Reynolds number assumptions. The channel asymmetry is produced by choosing the peristaltic wave trains with phase difference on the walls of the channel. The flow is investigated in a wave frame of reference with the velocity of the wave. The effect of various parameters on the flow characteristics are discussed through graphs.

Keywords: Peristalsis, Jeffrey fluid, Asymmetric channel, Permeable walls.

2010 MSC: 92C10,92C30,76S05.

1 Introduction

Peristaltic transport is a form of material transport induced by a progressive wave of area contraction or expansion along the length of a distensible tube. Peristaltic pumping has been the object of scientific and engineering research during the past few decades. The pumping of fluids through muscular tubes by means of peristaltic waves is an important biological mechanism. Study of the mechanism of peristalsis from both the mechanical and physiological viewpoints has been the object of scientific research. The waves can be short, local reflexes or long, continuous contractions along the length of the organ. In the esophagus, peristaltic waves push food into the stomach. In the stomach, they help mix stomach contents and propel food to the small intestine, where they expose food to the intestinal wall for absorption and move it forward. Peristalsis in the large intestine pushes waste towards the anal canal and is important in removing gas and dislodging potential bacterial colonies.

Peristalsis plays an indispensable role in transporting many physiological fluids in the body such as urine transport from kidney to bladder, the movement of chyme in the gastrointestinal tract, the transport of spermatozoa in the ductus efferentes of the male reproductive tract, the movement of ovum in the fallopian tubes, the swallowing of food through oesophagus and the vasomotion of small blood vessels. Many modern mechanical devices have been designed on the principle of peristaltic pumping for transporting fluids without internal moving parts. The problem of mechanism of peristaltic transport has attracted the attention of many researchers since the experimental investigation of Latham [4]. Subsequently a number of analytical, numerical and experimental studies of peristaltic flow of different fluids have been reported under different conditions with reference to physiological and mechanical situations.

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The importance of the study of peristaltic transport in an asymmetric channel has been brought out by Eytan and Elad [1] with an application to uterine fluid flow in a non - pregnant uterus. Mishra and Rao [6] studied the peristaltic transport of a Newtonian fluid in an asymmetric channel. Srinivas [10] investigated the nonlinear peristaltic transport in an inclined asymmetric channel. Peristaltic motion of a power-law fluid in an asymmetric channel was investigated by Sreenadh et al. [11]. Ravi Kumar et al. [9] studied the unsteady peristaltic pumping in a finite length tube with permeable wall.

Kothandapani and Srinivas [3] have analyzed the MHD peristaltic flow of a viscous fluid in an asymmetrical channel with heat transfer. Wang et al. [14] have studied the MHD peristaltic motion of a Sisko fluid in an asymmetric channel. Peristaltic motion of a Carreau fluid in an asymmetric channel is studied by Ali and Hayat [7]. They used perturbation method to find the solution.

The study of peristaltic transport through and past porous media has become the important area of research because of its vast applications in the study of biofluids. Misra and Ghosh [5] proposed a mathematical model to study the blood flow taking the channel bounded by permeable walls. Gopalan [2] modeled the tissue region in the blood vessels as porous medium. Ravi Kumar et al. [8] studied the peristaltic transport of a power - law fluid in an asymmetric channel bounded by permeable walls. Ravi Kumar et al. [14] studied the unsteady peristaltic pumping in a finite length tube with permeable wall. Many of the physiological fluids are known to be non - Newtonian. Peristaltic transport of blood in small vessels is investigated by considering various non - Newtonian fluids such as power - law, Casson, Herschel - Bulkley, Micropolar. Krishna Kumari et al. [12, 13] considered Jeffrey fluid in their study. Jeffrey model is a relatively simpler linear model using time derivatives instead of convected derivatives.

The present paper deals with peristaltic pumping of Jeffrey fluid, in an asymmetric channel with permeable walls. The channel asymmetry is produced by choosing the peristaltic wave trains with phase difference on the walls. The governing equations are solved subject to relevant boundary conditions. The results are numerically evaluated and discussed through graphs.

2 Mathematical Formulation of the Problem

We consider the motion of an incompressible viscous fluid in a two dimensional channel induced by sinusoidal wave trains propagating with constant speed c along the channel walls. The wall deformations are given by

$$\overline{h_1}(\overline{X},\bar{t}) = d_1 + a_1 \cos\frac{2\pi}{\lambda}(\overline{X} - c\bar{t}) \quad \text{(Upper wall)}$$

$$\overline{h_2}(\overline{X},\bar{t}) = d_2 + a_2 \cos\frac{2\pi}{\lambda}[(\overline{X} - c\bar{t}) + \theta] \quad \text{(Lower wall)} \tag{2.1}$$

where a_1, a_2 are the amplitudes of waves, λ is the wave length, $d_1 + d_2$ is the width of the channel. The phase difference θ varies in the range $0 \leq \theta \leq \pi, \theta = 0$ corresponds to symmetric channel with waves out of phase and for $\phi = \pi$ the waves are in phase and further a_1, a_2, d_1, d_2 and θ satisfy the condition

$$a_1^2 + a_2^2 + 2a_1a_2\cos\theta \leq (d_1 + d_2)^2. \tag{2.2}$$

Equations of motion

The constitutive equations for an incompressible Jeffrey fluid are

$$\begin{aligned}\overline{T} &= -\overline{p}\overline{I} + \overline{S}\\ \overline{S} &= \frac{\mu}{1+\lambda_1}\left(\frac{\partial\overline{\gamma}}{\partial t} + \lambda_2\frac{\partial^2\overline{\gamma}}{\partial t^2}\right)\end{aligned} \tag{2.3}$$

where $\overline{T}$ and $\overline{S}$ are Cauchy stress tensor and extra stress tensor, $\overline{P}$ is the pressure, $\overline{I}$ is the identity tensor, λ_1 is the ratio of the relaxation to retardation times, λ_2 is the retardation time and γ is the shear rate.

In laboratory frame, the equations governing two dimensional motion of an incompressible Jeffrey fluid are

$$\rho\left[\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial\overline{X}} + \overline{V}\frac{\partial}{\partial\overline{Y}}\right]\overline{U} = -\frac{\partial\overline{p}}{\partial\overline{X}} + \frac{\partial}{\partial\overline{X}}(\overline{S}_{\overline{XX}}) + \frac{\partial}{\partial\overline{Y}}(\overline{S}_{\overline{XY}}) \tag{2.4}$$

$$\rho\left[\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial\overline{X}}+\overline{V}\frac{\partial}{\partial\overline{Y}}\right]\overline{V}=-\frac{\partial\overline{p}}{\partial\overline{Y}}+\frac{\partial}{\partial\overline{X}}(\overline{S}_{\overline{XY}})+\frac{\partial}{\partial\overline{Y}}(\overline{S}_{\overline{YY}}) \tag{2.5}$$

and the equation of continuity is

$$\frac{\partial\overline{U}}{\partial\overline{X}}+\frac{\partial\overline{V}}{\partial\overline{Y}}=0 \tag{2.6}$$

where S_{xx}, S_{yy} and S_{xy} are the stress components in laboratory frame.

We introduce a wave frame of reference $(\overline{x},\overline{y})$ moving with velocity c in which the motion becomes independent of time when the channel length is an integral multiple of wavelength and the pressure difference at the ends of the channel is a constant. The transformation from the fixed frame of reference $(\overline{X},\overline{Y})$ to wave frame of reference $(\overline{x},\overline{y})$ is given by

$$\overline{x}=\overline{X}-ct,\ \overline{y}=\overline{Y},\ u=\overline{U}-c,\ \overline{v}=\overline{V},\ \overline{p}(x)=\overline{P}(X,t) \tag{2.7}$$

where $\overline{u},\overline{v}$ are the velocity components in the wave frame $(\overline{x},\overline{y}), \overline{p}, \overline{P}$ are pressures in wave and fixed frame of references respectively.

Non - dimensionalisation of the flow quantities

Now introducing the non-dimensional quantities,

$$x=\frac{2\pi\overline{X}}{\lambda},\ y=\frac{\overline{Y}}{d},\ u=\frac{\overline{U}}{c},\ v=\frac{\overline{V}}{c\delta},\ \delta=\frac{2\pi d}{\lambda},\ p=\frac{2\pi d^2\overline{p}}{\mu c\lambda},$$

$$h_1=\frac{\overline{h_1}}{d},\ h_2=\frac{\overline{h_2}}{d},\ S=\frac{\overline{S}d}{\mu c},\ \phi_1=\frac{a_1}{d_1},\ \phi_2=\frac{a_2}{d_2}. \tag{2.8}$$

Using conditions (2.3) in (2.4) and (2.5), the equations of motion reduces to

$$\rho\left[u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]u=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}(S_{xx})+\frac{\partial}{\partial y}(S_{xy}) \tag{2.9}$$

$$\rho\left[u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]v=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}(S_{xy})+\frac{\partial}{\partial y}(S_{yy}) \tag{2.10}$$

and the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2.11}$$

Eliminating pressure from equations (2.9) and (2.10), we get

$$\delta Re\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi\right]=\left[\left(\frac{\partial^2}{\partial y^2}-\delta^2\frac{\partial^2}{\partial x^2}\right)S_{xy}\right]+\delta\left[\frac{\partial^2}{\partial x\partial y}(S_{xx}-S_{xy})\right] \tag{2.12}$$

in which

$$S_{xx}=\frac{2\delta}{1+\lambda_1}\left[1+\frac{\delta\lambda_2 c}{d}\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\right]\frac{\partial^2\psi}{\partial x\partial y} \tag{2.13}$$

$$S_{xy}=\frac{1}{1+\lambda_1}\left[1+\frac{\delta\lambda_2 c}{d}\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\right]\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right) \tag{2.14}$$

$$S_{yy}=\frac{2\delta}{1+\lambda_1}\left[1+\frac{\delta\lambda_2 c}{d}\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\right]\frac{\partial^2\psi}{\partial x\partial y} \tag{2.15}$$

$$\nabla^2=\delta^2\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2},\ \ \delta=\frac{2\pi d}{\lambda},\ Re=\frac{\rho c d}{\mu}. \tag{2.16}$$

Using the long wave length approximation and neglecting the wave number δ, we get

$$\frac{\partial^2}{\partial y^2}S_{xy}=0 \tag{2.17}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{1+\lambda_1}\frac{\partial^2\psi}{\partial y^2}\right)=\frac{dp}{dx} \tag{2.18}$$

or

$$\frac{\partial}{\partial y}\left(\frac{1}{1+\lambda_1}\frac{\partial u}{\partial y}\right)=\frac{dp}{dx}. \tag{2.19}$$

The corresponding boundary conditions are (Saffman slip conditions)

$$\frac{\partial u}{\partial y}=\frac{-\alpha}{\sqrt{k}}u \quad \text{at } y=h_1 \text{ (upper wall)} \tag{2.20}$$

$$\frac{\partial u}{\partial y}=\frac{\alpha}{\sqrt{k}}u \quad \text{at } y=h_2 \text{ (lower wall)} \tag{2.21}$$

After non dimensionalisation the governing equations and the boundary conditions become

$$\frac{\partial}{\partial y}\left(\frac{1}{1+\lambda_1}\frac{\partial u}{\partial y}\right)=\frac{dp}{dx}. \tag{2.22}$$

The corresponding boundary conditions are

$$\frac{\partial u}{\partial y}=-\alpha\sigma\cdot u \quad \text{at } y=h_1 \text{ (Upper wall)} \tag{2.23}$$

$$\frac{\partial u}{\partial y}=\alpha\sigma\cdot u \quad \text{at } y=h_2 \text{ (Lpper wall)} \tag{2.24}$$

3 Solution of the Problem

Solving the equation (2.22) together with the boundary conditions (2.23) and (2.24), we get the velocity as

$$u=\frac{P(1+\lambda_1)}{2}y^2+c_1y+c_2,\ P=\frac{dp}{dx} \tag{3.1}$$

where

$$c_1=\frac{-P(1+\lambda_1((h_1+h_2)}{2}\quad c_2=\frac{P((1+\lambda_1)(h_2-h_1)+\alpha P\sigma((1+\lambda_1)h_1h_2}{2\alpha\sigma}.$$

The volume flow rate 'q' in the wave frame of reference is given by

$$q=\int_{h_2}^{h_1}udy=P(1+\lambda_1)(h_2-h_1)\left[\frac{h_1^2+h_1h_2+h_2^2}{6}+\frac{(h_1+h_2)^2}{4}+\frac{(h_2-h_1)+\alpha\sigma\cdot h_2h_1}{2\alpha\sigma}\right]. \tag{3.2}$$

From (3.2), we get

$$\frac{dp}{dx}=\frac{q}{(1+\lambda_1)(h_2-h_1)D} \tag{3.3}$$

where

$$D=\left[\frac{h_1^2+h_1h_2+h_2^2}{6}+\frac{(h_1+h_2)^2}{4}+\frac{(h_2-h_1)+\alpha\sigma\cdot h_2h_1}{2\alpha\sigma}\right].$$

The instantaneous flux at any axial station is

$$Q(x,t)=\int_{h_2}^{h_1}(u+1)dy=q+h_1-h_2. \tag{3.4}$$

The average volume flow rate over one period $\left(T=\frac{\lambda}{c}\right)$ of the peristaltic wave is defined as

$$\overline{Q}=\frac{1}{T}\int_0^T Qdt=\frac{1}{T}\int_0^T(q+h_1-h_2)dt=q+1+d. \tag{3.5}$$

The dimensionless frictional forces at $y=h_1$ and $y=h_2$ are given by

$$F_1=\int_0^1 h_1^2\left(\frac{-dp}{dx}\right)dx$$

$$F_2=\int_0^1 h_2^2\left(\frac{-dp}{dx}\right)dx \tag{3.6}$$

4 Discussion of the Results

From the Eq. (3.6), we have calculated the pressure difference P as a function of time average flow rate $\overline{Q}$ to study the effects of various parameters on pumping characteristics.

Figs. (2) - (4) are drawn to study the effect of Jeffrey parameter on pumping characteristics for the values of $\theta = 0, \pi/4, \pi/6$. It is observed that the pumping rate decreases with the increase in the Jeffrey parameter λ_1 for pumping ($\Delta P > 0$) and as well as for free pumping ($\Delta P = 0$). Further, observed that the pumping is more for a Jeffrey fluid when compared with a Newtonian fluid. Fig. 2 corresponds to symmetric channel. From Figs. (2) - (4) it is also observed that the pumping rate decreases as the symmetry of the channel increases.

The variation of pressure rise with time averaged flow rate ($\overline{Q}$) is calculated from equation (30) for different values of α (slip parameter) and is shown in Figs. (5),(6) and (7). We observe that the lesser the slip parameter, the greater the pressure rise against which the pump works. For a given ΔP, the flux $\overline{Q}$ decreases with increasing α. For a given flux $\overline{Q}$, the pressure difference ΔP increases with increasing α.

The variation of $\overline{Q}$ with ΔP for different values of phase difference θ is shown in Fig. 8. It is observed that the pumping decreases as the phase difference θ increases. For a fixed $\Delta P, \overline{Q}$ decreases as θ increases, this is due to the asymmetry of the channel. Fig. 9 is drawn for the variation of the axial velocity u with y for varying Jeffrey parameter λ_1. It is observed that maximum velocity decreases as λ_1 decreases. It is observed that the velocity increases as slip parameter decreases from Fig. 10.

5 Conclusions

In this paper, peristaltic pumping of a Jeffrey fluid in an asymmetric channel with permeable walls has been studied. The effect of various parameters on the pumping characteristics is discussed. The following conclusions have been found and summarized as follows.

(1) Pumping rate decreases with the increasing Jeffrey parameter.

(2) The pressure rise decreases as the slip parameter increases.

(3) The axial velocity decreases as Jeffrey parameter increases.

(4) The axial velocity increases as slip parameter decreases.

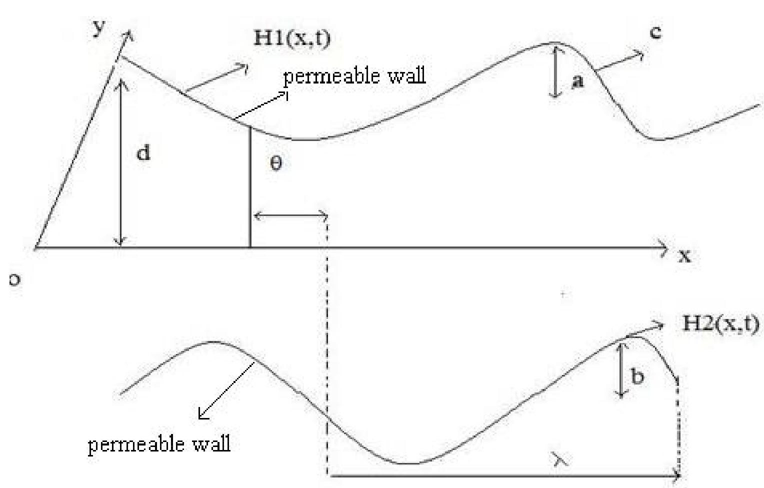

Fig. 1 : Physical Model

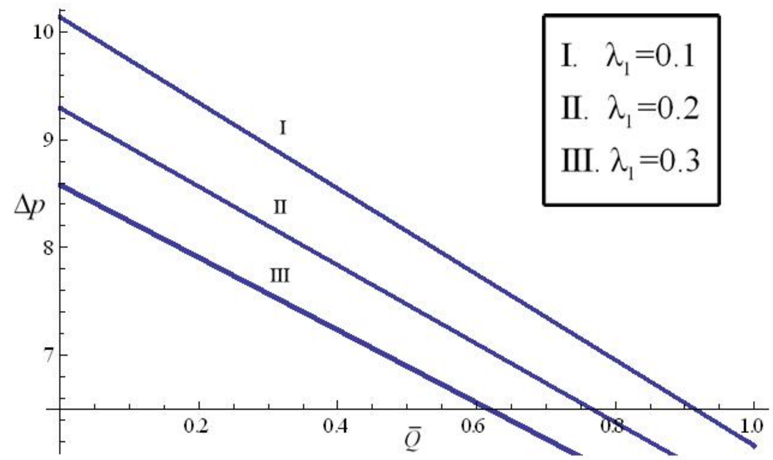

Fig. 2 : The variation of Δp with $\overline{Q}$ for different values of λ_1 with $\theta = 0$.

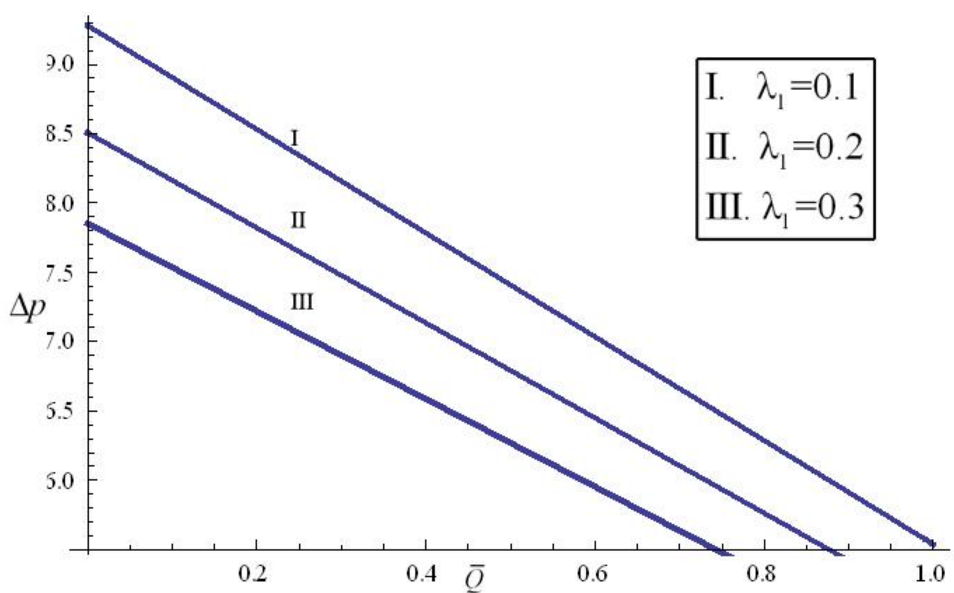

Fig. 3 : The variation of Δp with $\overline{Q}$ for different values of λ_1 with $\theta = \pi/4$.

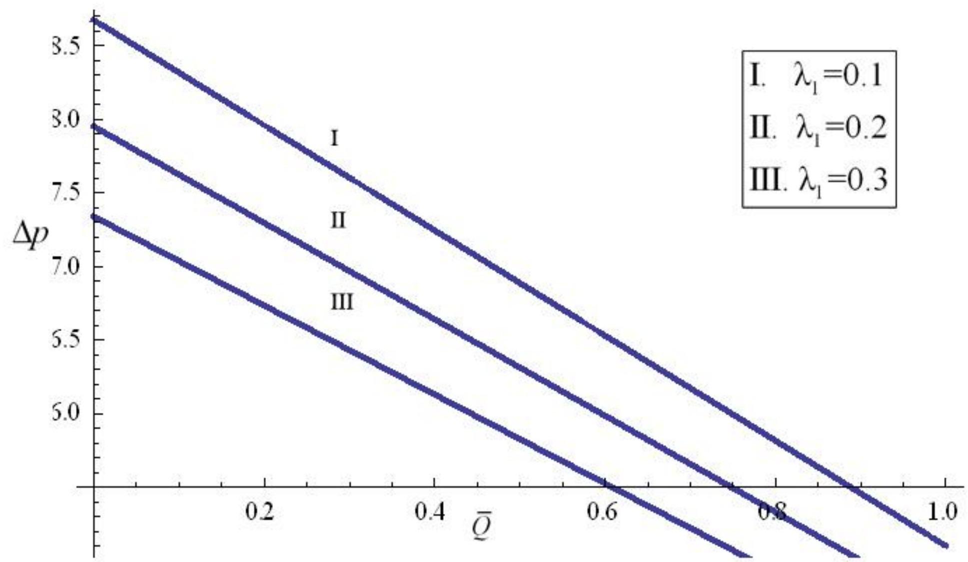

Fig. 4 : The variation of Δp with $\overline{Q}$ for different values of λ_1 with $\theta = \pi/3$.

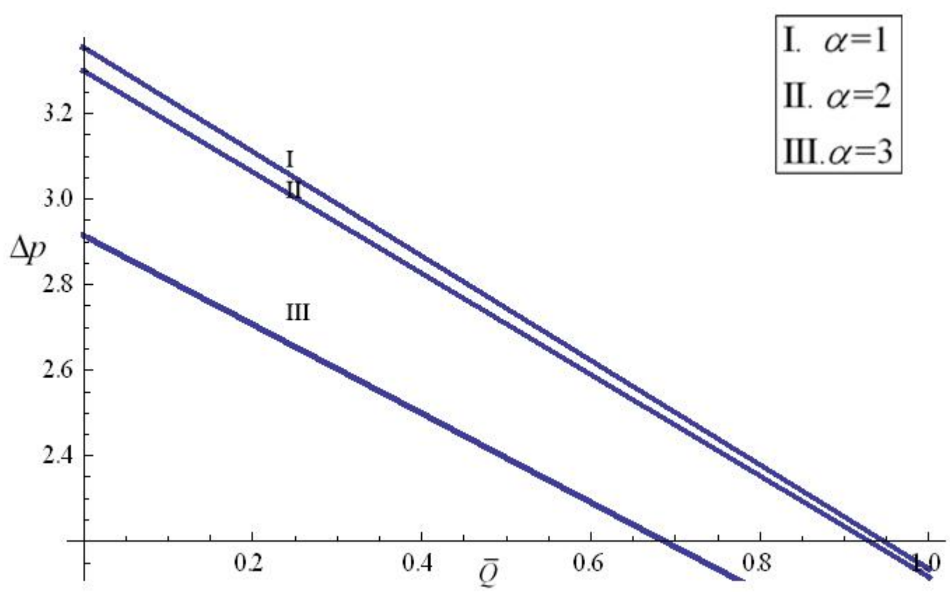

Fig. 5 : The variation of Δp with $\overline{Q}$ for different values of α with $\theta = 0$.

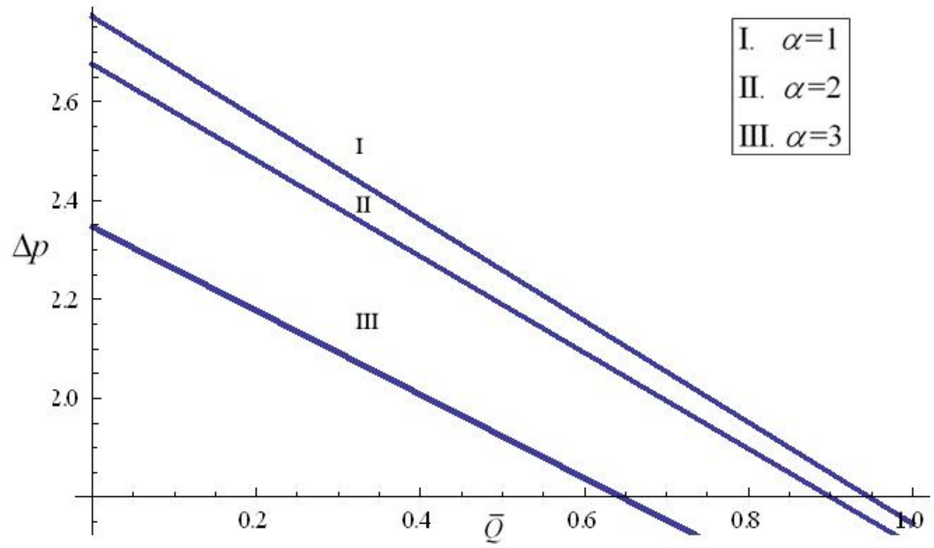

Fig. 6 : The variation of Δp with $\overline{Q}$ for different values of α with $\theta = \pi/4$.

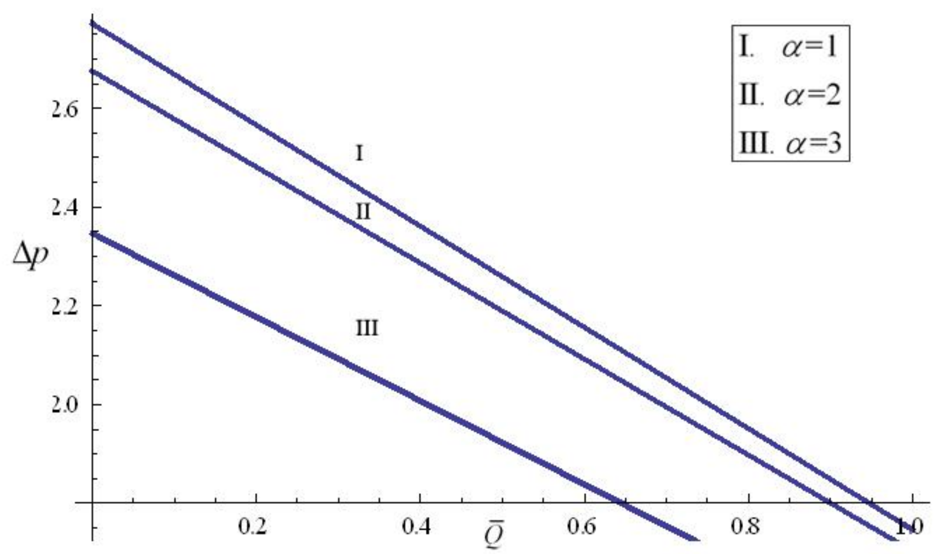

Fig. 7 : The variation of Δp with $\overline{Q}$ for different values of α with $\theta = \pi/3$.

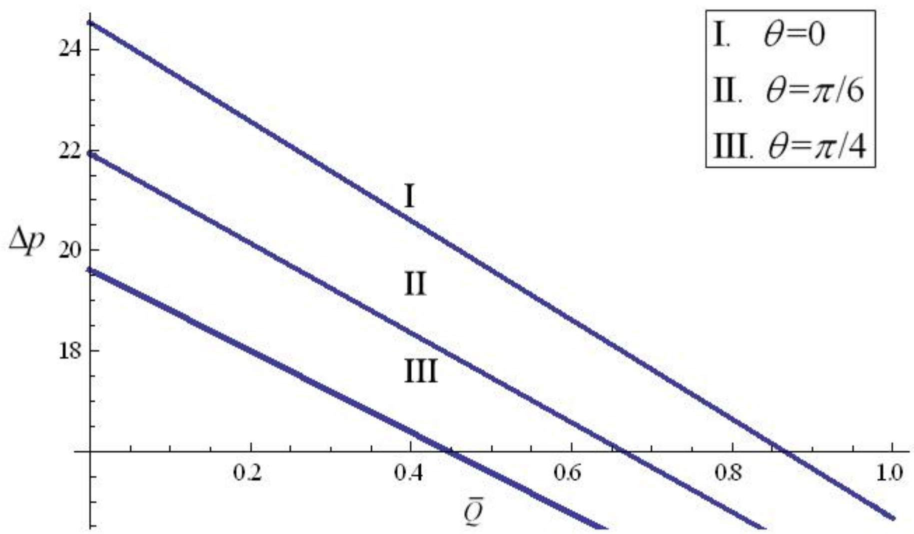

Fig. 8 : The variation of Δp with $\overline{Q}$ for different values of τ.

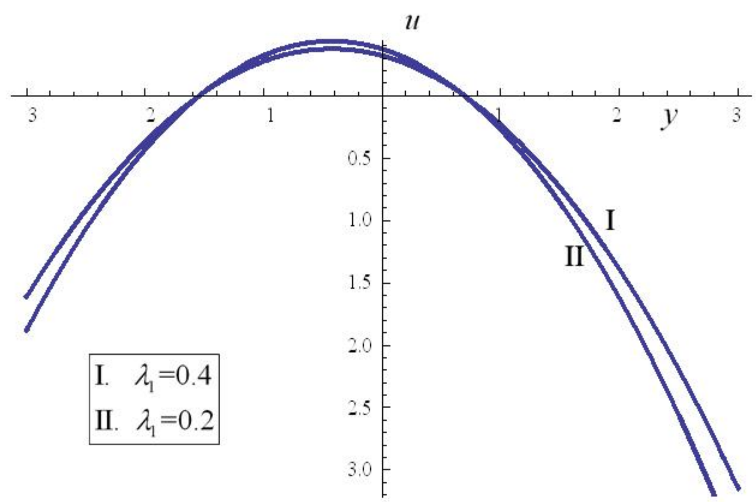

Fig. 9 : The variation of the velocity u with y for different values of λ_1 with $\theta = \pi/4$.

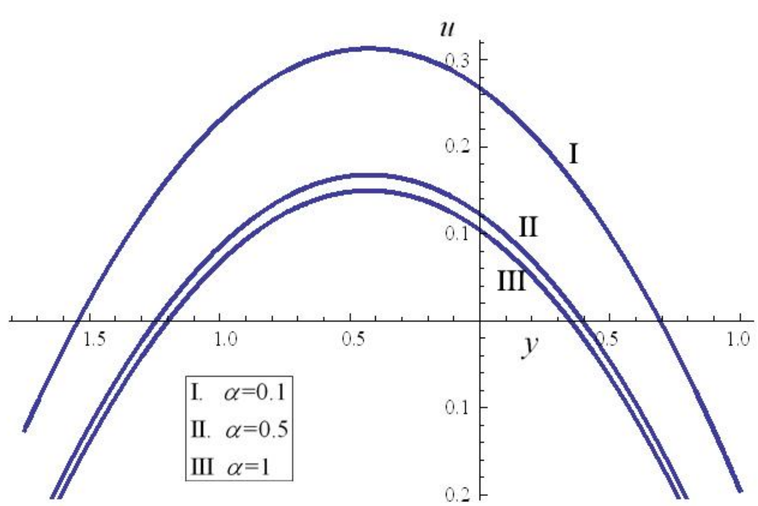

Fig. 10 : The variation of the velocity u with y for different values of al with $\theta = \pi/4$.

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An existence and uniqueness theorem for fuzzy H-integral equations of fractional order

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Abstract

We present an existence and uniqueness theorem for H- integral equations of fractional order involving fuzzy set valued mappings of a real variable whose values are normal, convex, upper semi continuous and compactly supported fuzzy sets in $\mathbb{R}^n$. The method of successive approximation is the main tool in our analysis.

Keywords: Fuzzy mapping, fractional orders, Riemann-Liouville H-differentiability, Fuzzy H-integral equation, Hausdorff metric, successive approximation.

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1 Introduction

Dubois and Prade [10] introduced the concept of integration of fuzzy functions. Alternative approaches were later suggested by Goetschel and Voxman [13], Kaleva [15], Nanda [21] and others. While Goetschel and Voxman preferred a Riemann integral type approach, Kaleva chose to define the integral of fuzzy function, using the Lebesgue-type concept of integration. For more information about integration of fuzzy functions and fuzzy integral equations, for instance, see [2, 8, 10, 13, 15, 21, 22, 24, 25] and references therein. On the other hand, the first serious attempt to give a logical definition of a fractional derivative is due to Liouville, see [14] and references therein. Now, the fractional calculus topic is enjoying growing interest among scientists and engineers, see [1, 8, 14, 16, 18, 23, 26].

By means of the fuzzy integral due to Kaleva [15], we investigate the fractional fuzzy integral equation, for the fuzzy set-valued mappings of a real variable whose values are normal, convex, upper semi-continuous and compactly supported fuzzy sets in $\mathbb{R}^n$. We consider the fuzzy integral equation of Riemann-Liouville fractional order generalized H-differentiability this equation takes the form

$$y(t) = f(t) + \frac{1}{\Gamma(1-q)} \int_0^t \frac{g(s, y(s))}{(t-s)^q}\, ds, \tag{1.1}$$

where $f : [0, T] \to E^n$ and $g : [0, T] \times E^n \to E^n$, and $q \in (0, 1)$. The definition of E^n is given in Section 2.
The paper is organized as follows: in Section 2 auxiliary facts and results are given which will be used later. In Section 3, the Riemann-Liouville H-differentiability is proposed for fuzzy-valued function and the some of important results of it are provided. In Section 4 the main theorem on the existence and uniqueness of solutions of equation (1.1) is given.

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2 Auxiliary facts and results

This section is devoted to collect some definitions and results which will be needed further on.

Definition 2.1. Let X be a nonempty set. A *fuzzy set* A in X is characterized by its membership function $A : X \to [0,1]$ and $A(x)$, called the membership function of fuzzy set A, is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

The value zero is used to represent complete non-membership, the value one is used to represent complete membership and values between them are used to represent intermediate degrees of membership. Let $P_k(\mathbb{R}^n)$ denote the collection of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $P_k(\mathbb{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}$$

where $d(b,A) = \inf\{d(b,a) : a \in A\}$. It is clear that $(P_k(\mathbb{R}^n), d)$ is a complete metric space [17].

A fuzzy set $u \in E^n$ is a function $u : \mathbb{R}^n \to [0,1]$ for which

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex, i.e., for $x,\ y \in \mathbb{R}^n$ and $\beta \in [0,1]$,

$$u(\beta x + (1-\beta)y) \geq \min(u(x), u(y))$$

(iii) u is upper semi-continuous, and

(iv) the closure of $\{x \in \mathbb{R}^n \ :\ u(x) > 0\}$, denoted by $[u]^0$, is compact.

For $0 < \gamma \leq 1$, the $\alpha-$level set $[u]^\gamma$ is define by $[u]^\gamma = \{x \in \mathbb{R}^n \ :\ u(x) \geq \gamma\}$. Then from $(i)-(iv)$, it follows that $[u]^\gamma \in P_k(\mathbb{R}^n)$ for all $0 \leq \gamma \leq 1$.

We define the supremum metric D on E^n by

$$D(u,\overline{u}) = \sup_{0<\gamma\leq 1} H_d([u]^\gamma, [\overline{u}]^\gamma)$$

for all $u, \overline{u} \in E^n$. (E^n, D) is a complete metric space.

3 Riemann-Liouville Fractional H-differentiability

Now, we define fuzzy Riemann-Liouville fractional derivatives of order $0 \leq r \leq 1$ for fuzzy-valued function f which is a direct extension of strongly generalized H-differentiability in the fractional literature [9].

Definition 3.2. *Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, it is denoted by $z = x \ominus y$.*
The sign $\ominus$ always stands for H-difference, also not that $x \ominus y \neq x + (-1)y$.
Also,we define some notations which are used throughout the paper.

- $L_p^F(a,b), 1 \leq p < \infty$ *is the set of all fuzzy-valued measurable and p-integrable functionsf on* $[a,b]$ *where*

$$\|f\|_p = \left(\int_0^1 (d(f(t),0))^p dt\right)^{\frac{1}{p}}.$$

- $C^F[a,b]$ *is a space of fuzzy-valued functions which are continuous on* $[a,b]$.

- $AC^F[a,b]$ *denotes the set of all fuzzy-valued functions which are absolutely continuous.*

Definition 3.3. *Let* $f : [a,b] \to E$, $x_0 \in (a,b)$ *and* $\Phi(x) = \frac{1}{\Gamma(1-q)} \int_a^x \frac{f(t)}{(x-t)^q} dt$. *We say that* $f(x)$ *is fuzzy Riemann-Liouville fractional H-differentiable about order* $0 \le q \le 1$ *at* x_0, *if there exists an element* $({}^{RL}D^q_{a^+} f)(x_0) \in C^F, 0 \le q \le 1$ *such that for all* $0 \le r \le 1, h > 0$

(*i*)

$$({}^{RL}D^q_{a^+} f)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0 + h) \ominus \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 - h)}{h}$$

or

(*ii*)

$$({}^{RL}D^q_{a^+} f)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{\Phi(x_0 - h) \ominus \Phi(x_0)}{-h}.$$

For sake of simplicity, we say that a fuzzy-valued function f is ${}^{RL}(1,q)$-differentiable if it is differentiable as in the definition 3.3 case (i), and is ${}^{RL}(2,q)$-differentiable if it is differentiable as in Definition 3.3 case (ii).

Definition 3.4. Let $f \in L^1(a,b)$, $0 \le a < b < \infty$, and let $0 < q < 1$ be a real number. The fractional integral of order q of Riemann-Liouville type is defined by (see; [16, 23]).

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds.$$

Let us consider the $r-$cut representation of fuzzy valued function f as $f(x;r) = [\underline{f}(x;r), \overline{f}(x;r)]$ for $0 \le r \le 1$, then we can indicate the Riemann-Liouville integral of fuzzy-valued function f based on its lower and upper functions as follows:

Theorem 3.1. *Let* $f : [a,b] \to E$ *be a fuzzy-valued function. The fuzzy Riemann-Liouville integral of* f *can be expressed as follows:*

$$(I^q f)(x;r) = [(I^q \underline{f})(x;r), (I^q \overline{f})(x;r)], 0 \le r \le 1$$

where

$$(I^q \underline{f})(x;r) = \frac{1}{\Gamma(q)} \int_a^x \frac{\underline{f}(t;r)}{(x-t)^{1-q}} dt$$

$$(I^q \overline{f})(x;r) = \frac{1}{\Gamma(q)} \int_a^x \frac{\overline{f}(t;r)}{(x-t)^{1-q}} dt.$$

Now, we define fuzzy Riemann-Liouville fractional derivatives of order $0 \le r \le 1$ for fuzzy-valued function f which is a direct extension of strongly generalized H-differentiability [9] in the fractional literature. Also, we denote by C^F the space of all fuzzy-valued functions which are continuous on $[a,b]$ and we assume that all fuzzy-valued functions in this work are placed in C^F. We define the fuzzy Riemann-Liouville H-integrals of fuzzy-valued function as follows:

Theorem 3.2. *Let* $f : [0,T] \to E^n$, $x_0 \in [0,T]$ *and* $0 \le q \le 1$ *such that for all* $0 \le r \le 1$.

(1) *if* $f(x)$ *be a* ${}^{RL}(1,q)$ *differentiable fuzzy-valued function, then*

$$({}^{RL}D^q_0 f)(x_0;r) = [{}^{RL}D^q_0 \underline{f}(x_0;r), {}^{RL}D^q_0 \overline{f}(x_0;r)],$$

(2) *if* $f(x)$ *be a* ${}^{RL}(2,q)$ *differentiable fuzzy-valued function, then*

$$({}^{RL}D^q_0 f)(x_0;r) = [{}^{RL}D^q_0 \overline{f}(x_0;r), {}^{RL}D^q_0 \underline{f}(x_0;r)].$$

Where

$$({}^{RL}D^q_0 \underline{f})(x_0) = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^q} dt \,|_{x=x_0},$$

and

$$({}^{RL}D^q_0 \overline{f})(x_0) = \ominus \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^q} dt \,|_{x=x_0} .$$

Rewrite Eq.(1.1) in the form

$$y(t) = f(t) +^{RL} I_0^q g(t, y(t)), \quad t \geq 0, \tag{3.2}$$

where ${}^{RL}I_0^q$ is the standard Riemann-Liouville fractional H-integral operator. Notice that, since f is assumed to be integrable and $(x-t)^{q-1}$ is a crisp function, we deduce that $\frac{f(t)}{(x-t)^{1-q}}$ is integrable and then, the existence of integral (1.1) is proved.

Theorem 3.3. *Let $f : [a, b] \to E$, $x_0 \in (a, b)$ and $0 \leq q \leq 1$ for all $0 \leq r \leq 1$, we have*

(1) *if f is ${}^{RL}(1, q)$ H-integrable then*

$$^{RL}I_0^q(f)(x_0; r) = [^{RL}I_0^q \underline{f}(x_0; r),^{RL} I_0^q \overline{f}(x_0; r)]$$

(2) *if f is ${}^{RL}(2, q)$ H-integrable then*

$$^{RL}I_0^q f(x_0; r) = [^{RL}I_0^q \overline{f}(x_0; r),^{RL} I_0^q \underline{f}(x_0; r)]$$

In this paper, we prove an existence and uniqueness theorem of a solution to the fuzzy integral equation (1.1). The method of successive approximation is the main tool in our analysis.

4 Main Theorem

In this section, we will study Eq(1.1) assuming that the following assumptions are satisfied, Let L and T be positive numbers:

(a_1) $f : [0, T] \to E^n$ is continuous and bounded.

(a_2) $g : [0, T] \times E^n \to E^n$ is continuous and satisfies the Lipschitz condition, i.e.,

$$D\left(g(t, y_2(t)), g(t, y_1(t))\right) \leq L\, D\left(y_2(t), y_1(t)\right), \; t \in [0, T],$$

where $y_i : [0, T] \to E^n$, $i = 1,\ 2$.

(a_3) $g(t, \hat{0})$ is bounded on $[0, T]$.

Now, we are in a position to state and prove our main result in paper

Theorem 4.4. Let the assumptions $(a_1) - (a_3)$ be satisfied. If

$$T < \left(\frac{\Gamma(2-q)}{L}\right)^{\frac{1}{1-q}},$$

then Eq(1.1) has a unique solution y on $[0, T]$ defined as the following:

(1) In the case ${}^{RL}(1; q)$ differentiability, the successive iterations

$$\begin{aligned} y_0(t) &= f(t) \\ y_{n+1}(t) &= f(t) + {}^{RL}{I_0}^q g(t, y_n(t)), \quad n = 0, 1, 2, \ldots \end{aligned} \tag{4.3}$$

(2) In the case ${}^{RL}(2; q)$ differentiability, the successive iterations

$$\begin{aligned} \hat{y}_0(t) &= f(t) \\ \hat{y}_{n+1}(t) &= f(t) \ominus {}^{RL}{I_0}^q g(t, \hat{y}_n(t)), \quad n = 0, 1, 2, \ldots \end{aligned} \tag{4.4}$$

are uniformly convergent to y on $[0, T]$.

Proof. (1) Case (1): If f is ${}^{RL}(1;q)$ differentiable
First we prove that y_n are bounded on $[0,T]$. We have $y_0 = f(t)$ is bounded, thanks (a_1). Assume that y_{n-1} is bounded. From (4.3) we have

$$\begin{aligned}
D(y_n(t),\hat{0}) &= D\left(f(t) + {}^{RL}I_0^q g(t,y_{n-1}(t)),\, \hat{0}\right)\\
&\leq D\left(f(t),\, \hat{0}\right) + D\left({}^{RL}I_0^q g(t,y_{n-1}(t)),\, \hat{0}\right)\\
&\leq D\left(f(t),\, \hat{0}\right) + \frac{1}{\Gamma(1-q)}\int_0^t D\left(\frac{g(s,y_{n-1}(s))}{(t-s)^q}\,,\, \hat{0}\right)\, ds\\
&\leq D\left(f(t),\, \hat{0}\right) + \frac{1}{\Gamma(1-q)}\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\, \hat{0})\int_0^t \frac{ds}{(t-s)^q}.
\end{aligned}$$

But

$$\begin{aligned}
D(g(t,y_{n-1}(t)),\, \hat{0}) &\leq D(g(t,y_{n-1}(t)),\, g(t,\, \hat{0})) + D(g(t,\, \hat{0}),\, \hat{0})\\
&\leq L\, D(y_{n-1}(t),\, \hat{0}) + D(g(t,\, \hat{0}),\, \hat{0}).
\end{aligned}$$

So

$$\begin{aligned}
D(y_n(t),\hat{0}) &\leq D\left(f(t),\, \hat{0}\right) + \frac{T^{1-q}}{\Gamma(2-q)}\sup_{0\leq t\leq T}[L\, D(y_{n-1}(t),\, \hat{0}) + D(g(t,\, \hat{0}),\, \hat{0})]\\
&\leq D\left(f(t),\, \hat{0}\right) + \sup_{0\leq t\leq T} D(y_{n-1}(t),\, \hat{0}) + \frac{T^{1-q}}{\Gamma(2-q)}\sup_{0\leq t\leq T} D(g(t,\, \hat{0}),\, \hat{0}).
\end{aligned}$$

This proves that y_n is bounded. Therefore, $\{y_n\}$ is a sequence of bounded functions on $[0,T]$. Second we prove that y_n are continuous on $[0,T]$. For $0\leq t\leq \tau\leq T$, we have

$$\begin{aligned}
D(y_n(t),y_n(\tau)) &\leq D(f(t),f(\tau)) + \frac{1}{\Gamma(1-q)}D\left(\int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^q}\,ds,\int_0^\tau \frac{g(s,y_{n-1}(s))}{(\tau-s)^q}\,ds\right)\\
&\leq D\left(f(t),f(\tau)\right) + \frac{1}{\Gamma(1-q)}D\left(\int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^q}\,ds,\int_0^t \frac{g(s,y_{n-1}(s))}{(\tau-s)^q}\,ds\right)\\
&\quad+\frac{1}{\Gamma(1-q)}D\left(\int_t^\tau \frac{g(s,y_{n-1}(s))}{(\tau-s)^q}\,ds,\, \hat{0}\right)\\
&\leq D\left(f(t),f(\tau)\right) + \frac{1}{\Gamma(1-q)}\int_0^t D\left(\frac{g(s,y_{n-1}(s))}{(t-s)^q},\frac{g(s,y_{n-1}(s))}{(\tau-s)^q}\right)\,ds\\
&\quad+\frac{1}{\Gamma(1-q)}\int_t^\tau D\left(\frac{g(s,y_{n-1}(s))}{(\tau-s)^q},\, \hat{0}\right)\,ds\\
&\leq D\left(f(t),f(\tau)\right) + \frac{1}{\Gamma(1-q)}\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\, \hat{0})\\
&\quad\int_0^t |(t-s)^{-q}-(\tau-s)^{-q}|\,ds\\
&\quad+\frac{1}{\Gamma(1-q)}\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\, \hat{0})\int_t^\tau \frac{ds}{(\tau-s)^q}\,ds\\
&\leq D\left(f(t),f(\tau)\right) + \frac{1}{\Gamma(1-q)}[|t-\tau|^{(1-q)} - |t^{(1-q)}-\tau^{(1-q)}|]\\
&\quad\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\, \hat{0})\\
&\quad+\frac{1}{\Gamma(2-q)}\,|t-\tau|^\alpha\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\, \hat{0})
\end{aligned}$$

$$
\begin{aligned}
&\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(2-q)}[2\,|t-\tau|^{(1-q)} - |t^{(1-q)} - \tau^{(1-q)}|] \\
&\qquad \sup_{0\le t\le T} D(g(t, y_{n-1}(t)),\ \hat{0}) \\
&\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(2-q)}[2\,|t-\tau|^{(1-q)} - |t^{(1-q)} - \tau^{(1-q)}|] \\
&\qquad \sup_{0\le t\le T} [L\, D(g(y_{n-1}(t)),\ \hat{0}) + D(g(t,\hat{0}),\ \hat{0})].
\end{aligned}
$$

The last inequality, by symmetry, is valid for all $t,\ \tau \in [0,T]$ regardless whether or not $t \le \tau$. Thus, $D(y_n(t), y_n(\tau)) \to 0$ as $t \to \tau$. Therefore, the sequence $\{y_n\}$ is continuous on $[0,T]$. For $n \ge 1$, we have

$$
\begin{aligned}
D(y_{n+1}(t), y_n(t)) &= \frac{1}{\Gamma(1-q)}\, D\left(\int_0^t \frac{g(s,y_n(s))}{(t-s)^q}\,ds, \int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^q}\,ds\right) \\
&\le \frac{1}{\Gamma(1-q)} \int_0^t D\left(\frac{g(s,y_n(s))}{(t-s)^q}, \int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^q}\right) ds \\
&\le \frac{1}{\Gamma(1-q)} \int_0^t D\left(g(s,y_n(s)), g(s,y_{n-1}(s))\right) \frac{ds}{(t-s)^q} \\
&\le \frac{1}{\Gamma(1-q)} \sup_{0\le t\le T} D(g(t,y_n(t)), g(t,y_{n-1}(t))) \int_0^t \frac{ds}{(t-s)^q} \\
&\le \frac{L\,T^{(1-q)}}{\Gamma(2-q)} \sup_{0\le t\le T} D(y_n(t), y_{n-1}(t)) \\
&\le \left(\frac{L\,T^{(1-q)}}{\Gamma(2-q)}\right)^2 \sup_{0\le t\le T} D(y_{n-1}(t), y_{n-2}(t)) \\
&\ \ \vdots \\
&\le \left(\frac{L\,T^{(1-q)}}{\Gamma(2-q)}\right)^n \sup_{0\le t\le T} D(y_1(t), y_0(t)).
\end{aligned} \tag{4.5}
$$

But

$$
\begin{aligned}
D(y_1(t), y_0(t)) &= \frac{1}{\Gamma((1-q))}\, D\left(\int_0^t \frac{g(s,f(s))}{(t-s)^q}\,ds, \hat{0}\right) \\
&\le \frac{1}{\Gamma((1-q))} \int_0^t D\left(\frac{g(s,f(s))}{(t-s)^q}, \hat{0}\right) ds \\
&\le \frac{1}{\Gamma((1-q))} \sup_{0\le t\le T} D(g(t,f(t)), \hat{0}) \int_0^t \frac{ds}{(t-s)^q}.
\end{aligned}
$$

Thus

$$
\sup_{0\le t\le T} D(y_1(t), y_0(t)) \le \frac{T^{(1-q)}}{\Gamma(2-q)}[LM+N] := R,
$$

where

$$
M = \sup_{0\le t\le T} D(f(t), \hat{0}) \ \text{ and } \ N = \sup_{0\le t\le T} D(g(t,\hat{0}), \hat{0}).
$$

Therefore (4.5) takes the form

$$
D(y_{n+1}(t), y_n(t)) \le R\, \left(\frac{L\,T^{(1-q)}}{\Gamma(2-q)}\right)^n. \tag{4.6}
$$

Next, we show that for each $t \in [0,T]$ the sequence $\{y_n(t)\}$ is a Cauchy sequence in E^n. Let $m_1,\ m_2$ be such that $m_2 > m_1$ and $t \in [0,T]$. Then, by using (4.6), we have

$$\begin{aligned} D(y_{m_1}(t), y_{m_2}(t)) &\leq D(y_{m_2}(t), y_{m_2-1}(t)) + D(y_{m_2-1}(t), y_{m_2-2}(t)) \\ &\quad + \ldots + D(y_{m_1+1}(t), y_{m_1}(t)) \\ &\leq R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-1} + R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-2} \\ &\quad + \ldots + R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_1} \\ &= R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-1}\left[1 + \frac{\Gamma(2-q)}{L\,T^{1-q}} + \left(\frac{\Gamma(2-q)}{L\,T^{1-q}}\right)^2 \right. \\ &\quad \left. + \ldots + \left(\frac{\Gamma(2-q)}{L\,T^{1-q}}\right)^{m_2-m_1-1}\right] \\ &= R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-1}\left[\frac{1-\left(\frac{\Gamma(2-q)}{L\,T^{1-q}}\right)^{m_2-m_1}}{1-\frac{\Gamma(2-q)}{L\,T^{1-q}}}\right]. \end{aligned}$$

The right hand side of the last inequality tends to zero as $m_1,\ m_2 \to \infty$. This implies that $\{y_n(t)\}$ is a Cauchy sequence. Consequently, the sequence $\{y_n(t)\}$ is convergent, thanks to the completeness of the metric space (E^n, D). If we denote $y(t) = \lim_{n\to\infty} y_n(t)$, then $y(t)$ satisfies (1.1). It is continuous and bounded on $[0, T]$. To prove the uniqueness, let $x(t)$ be a continuous solution of (1.1) on $[0, T]$. Then

$$x(t) = f(t) + {}^{RL}I^q g(t, x(t)), \quad t \geq 0.$$

Now, for $n \geq 1$, we have

$$\begin{aligned} D(x(t), y_n(t)) &= D\left({}^{RL}I^{1-q} g(t, x(t)), {}^{RL}I^{1-q} g(t, y_n(t))\right) \\ &\leq \frac{1}{\Gamma(1-q)} \int_0^t D\left(\frac{g(s, x(s))}{(t-s)^q}, \int_0^t \frac{g(s, y_n(s))}{(t-s)^q}\right) ds \\ &\leq \frac{1}{\Gamma(1-q)} \int_0^t D\left(g(s, x(s)), g(s, y_n(s))\right) \frac{ds}{(t-s)^q} \\ &\leq \frac{1}{\Gamma(1-q)} \sup_{0\leq t\leq T} D(g(t, x(t)), g(t, y_n(t))) \int_0^t \frac{ds}{(t-s)^q} \\ &\leq \frac{L\,T^{1-q}}{\Gamma(2-q)} \sup_{0\leq t\leq T} D(x(t), y_n(t)) \\ &\ \ \vdots \\ &\leq \left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^n \sup_{0\leq t\leq T} D(x(t), y_0(t)). \end{aligned}$$

Since $\frac{L\,T^{1-q}}{\Gamma(2-q)} < 1$

$$\lim_{n\to\infty} y_n(t) = x(t) = y(t), \quad t \in [0, T].$$

This completes the proof.

(2) Case (2): If f is ${}^{RL}(2;q)$ differentiable, with the same argument as above, we can prove that the solution is (4.4) with

$$\lim_{n\to\infty} \hat{y}_n(t) = \hat{x}(t) = \hat{y}(t), \quad t \in [0, T].$$

□

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Exact solution of the(2+1)-dimensional hyperbolic nonlinear Schrödinger equation by Adomian decomposition method

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Abstract

This paper studies the exact solution of the the(2+1)-dimensional hyperbolic nonlinear Schrödinger equation by the aid of Adomian decomposition method.

Keywords: Exact solution, Hyperbolic Schrödinger equation, Adomian decomposition method.

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1 Introduction

Nonlinear equations describe fundamental physical phenomena in nature ranging from chaotic behaviour in biological systems, plasma containment in tokamaks and stellarators for energy generation, to solitonic fibre optical communication devices. The construction of the exact solutions of nonlinear partial differential equations (PDEs) is one of the most important and essential tasks which help us for better understanding of nonlinear complex physical phenomena. In the past couple of decades, there are various mathematical techniques have been developed to carry out the integration of these equations. Some of these commonly studied techniques are Inverse Scattering Transform [5], bilinear transformation[4], the tanh-sech method[6, 7], adomian decomposition method [3], the tanh-coth method[8], homogeneous balance method[9], Exp-function method [10], and many others.
The Adomian decomposition method was introduced and developed by George Adomian in [11, 12] and is well addressed in the literature. A reliable modification of the Adomian decomposition method developed by Wazwaz and presented in [3]. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear equations for detail see [13, 14, 15, 16, 17, 18] and the references therein.

In this paper the Adomian decomposition method will determine exact solution to (2+1)-dimensional hyperbolic nonlinear Schrödinger equation. In Section 2, we described this method for finding exact solutions for nonlinear PDEs. In Section 3, we illustrated this method in detail with the hyperbolic Schrödinger equation. In Section 4, we gave some conclusions.

2 Adomian decomposition method for nonlinear PDEs

We first consider the nonlinear partial differential equation given in an operator form

$$L_x u(x,y) + L_y u(x,y) + R(u(x,y)) + F(u(x,y)) = g(x,y), \tag{2.1}$$

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where L_x is the highest order differential in x, L_y is the highest order differential in y, R contains the remaining linear terms of lower derivatives, $F(u(x,y))$ is an analytic nonlinear term, and $g(x,y)$ is an inhomogeneous or forcing term. the decision as to which operator L_x or L_y should be used to solve the problem depends mainly on two bases: (i) The operator of lowest order should be selected to minimize the size of computational work. (ii) The selected operator of lowest order should be of best known conditions to accelerate the evaluation of the components of the solution.For more detail see[3]. Assume that L_y meet these two conditions, therefore we set

$$L_y u(x,y) = g(x,y) - L_x u(x,y) - R(u(x,y)) - F(u(x,y)). \tag{2.2}$$

Applying L_y^{-1} to both sides of (2.2) gives

$$u(x,y) = \Phi_0 - L_y^{-1} g(x,y) - L_y^{-1} L_x u(x,y) - L_y^{-1} R(u(x,y)) - L_y^{-1} F(u(x,y)), \tag{2.3}$$

where

$$\Phi_0 = \begin{cases} u(x,0) & L = \frac{\partial}{\partial y}, \\ u(x,0) + y u_y(x,0) & L = \frac{\partial^2}{\partial y^2}, \\ u(x,0) + y u_y(x,0) + \frac{1}{2!} y^2 u_{yy}(x,0) & L = \frac{\partial^3}{\partial y^3}, \\ u(x,0) + y u_y(x,0) + \frac{1}{2!} y^2 u_{yy}(x,0) + \frac{1}{3!} y^3 u_{yyy}(x,0) & L = \frac{\partial^4}{\partial y^4}, \end{cases}$$

Take the solution $u(x,y)$ in a series form

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \tag{2.4}$$

and the nonlinear term $F(u(x,y))$ by

$$F(u(x,y)) = \sum_{n=0}^{\infty} A_n, \tag{2.5}$$

where A_n are Adomian polynomials that can be generated for all forms of nonlinearity and can be evaluated by using the following expression

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, n = 0,1,2 \tag{2.6}$$

Based on these assumptions, Eq. (2.3) become

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,y) = \Phi_0 - L_y^{-1} g(x,y) - L_y^{-1} L_x \left(\sum_{n=0}^{\infty} u_n(x,y) \right) \\ - L_y^{-1} R \left(\sum_{n=0}^{\infty} u_n(x,y) \right) - L_y^{-1} \left(\sum_{n=0}^{\infty} A_n \right). \end{aligned} \tag{2.7}$$

The components $u_n(x,y)$, $n \geq 0$ of the solution $u(x,y)$ can be recursively determined by using the relation

$$\begin{aligned} u_0(x,y) &= \Phi_0 - L_y^{-1} g(x,y), \\ u_{k+1}(x,y) &= -L_y^{-1} L_x u_k - L_y^{-1} R(u_k) - L_y^{-1}(A_k), \quad k \geq 0. \end{aligned} \tag{2.8}$$

Next find the components of $\sum_{n=0}^{\infty} u_n(x,y)$ by

$$\begin{aligned} u_0(x,y) &= \Phi_0 - L_y^{-1} g(x,y), \\ u_1(x,y) &= -L_y^{-1} L_x u_0(x,y) - L_y^{-1} R\left(u_0(x,y)\right) - L_y^{-1} A_0, \\ u_2(x,y) &= -L_y^{-1} L_x u_1(x,y) - L_y^{-1} R\left(u_1(x,y)\right) - L_y^{-1} A_1, \\ u_3(x,y) &= -L_y^{-1} L_x u_2(x,y) - L_y^{-1} R\left(u_2(x,y)\right) - L_y^{-1} A_2, \\ u_4(x,y) &= -L_y^{-1} L_x u_3(x,y) - L_y^{-1} R\left(u_3(x,y)\right) - L_y^{-1} A_3, \\ &. \\ &. \\ &. \end{aligned}$$

where each component can be determined by using the preceding component. Having the calculated the components $u_n(x,y)$, $n \geq 0$, the solution in a series form is readily obtained.

3 Exact solutions for(2+1)-dimensional hyperbolic Schrödinger equation

In this section we obtain exact solution of (2+1)-dimensional hyperbolic nonlinear Schrödinger equation by using the decomposition method. The hyperbolic nonlinear Schrödinger equation given by[1] is

$$iu_t + \frac{1}{2}u_{xx} - \frac{1}{2}u_{yy} + |u|^2 u = 0 \tag{3.1}$$

where u is a complex valued function, while x, y and t are the independent variables. In order to seek exact solution, we assume that $u(x,y,0) = e^{i(mx+ny)}$ Multiplying Eq.(3.1) by i, we may express this equation in an operator form as follows

$$L_t u(x,y,t) = \frac{i}{2}L_{xx}u(x,y,t) - \frac{i}{2}L_{yy}u(x,y,t) + i|u(x,y,t)|^2 u(x,y,t) \tag{3.2}$$

where L_t is defined by $L_t = \frac{\partial}{\partial t}$ and the inverse operator L_t^{-1} is identified by

$$L_t^{-1}(\cdot) = \int_0^t (\cdot)dt$$

Applying L_t^{-1} to both sides of (3.2) and using the initial condition we obtain

$$u(x,y,t) = e^{i(mx+ny)} + \frac{i}{2}L_t^{-1}(u(x,y,t))_{xx} - \frac{i}{2}L_t^{-1}(u(x,y,t))_{yy} + iL_t^{-1}|u(x,y,t)|^2 u(x,y,t), \tag{3.3}$$

where $|u(x,y,t)|^2 u(x,y,t)$is nonlinear term.
Substituting

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) \tag{3.4}$$

and nonlinear term

$$|u(x,y,t)|^2 u(x,y,t) = \sum_{n=0}^{\infty} A_n \tag{3.5}$$

into (3.3) gives

$$\sum_{n=0}^{\infty} u_n(x,y,t) = e^{i(mx+ny)} + \frac{i}{2}L_t^{-1}\left(\sum_{n=0}^{\infty} u_n(x,y,t)\right)_{xx} - \frac{i}{2}L_t^{-1}\left(\sum_{n=0}^{\infty} u_n(x,y,t)\right)_{yy} + iL_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{3.6}$$

Adomian's analysis introduces the recursive relation

$$\begin{aligned} &u_0(x,y,t) = e^{i(mx+ny)}, \\ &u_{k+1}(x,y,t) = \tfrac{i}{2}L_t^{-1}(u_k)_{xx} - \tfrac{i}{2}L_t^{-1}(u_k)_{yy} + iL_t^{-1}(A_k), k \geq 0. \end{aligned} \tag{3.7}$$

since u is a complex function so we can write

$$|u|^2 = u\bar{u} \tag{3.8}$$

where $\bar{u}$ is the conjugate of u. this means that (3.5) can be written as

$$u^2\bar{u} = \sum_{n=0}^{\infty} A_n \tag{3.9}$$

By using formal technique to find adomian polynomial used in [3] we find that (3.9)has the following polynomial representation

$$\begin{aligned} A_0 &= u_0^2\bar{u}_0, \\ A_1 &= 2u_0u_1\bar{u}_0 + u_0^2\bar{u}_1, \\ A_2 &= 2u_0u_2\bar{u}_0 + u_1^2\bar{u}_0 + 2u_0u_1\bar{u}_1 + u_0^2\bar{u}_2, \\ A_3 &= 2u_0u_3\bar{u}_0 + 2u_1u_2\bar{u}_0 + 2u_0u_2\bar{u}_1 + u_1^2\bar{u}_1 + 2u_0u_1\bar{u}_2 + u_0^2\bar{u}_3 \end{aligned} \tag{3.10}$$

that in turn gives the first few components by

$$\begin{aligned}
u_0(x,y,t) &= e^{i(mx+ny)},\\
u_1(x,y,t) &= \tfrac{i}{2}L_t^{-1}\left(u_{0_{xx}}\right) - \tfrac{i}{2}L_t^{-1}\left(u_{0_{yy}}\right) + iL_t^{-1}(A_0),\\
u_2(x,y,t) &= \tfrac{i}{2}L_t^{-1}\left(u_{1_{xx}}\right) - \tfrac{i}{2}L_t^{-1}\left(u_{1_{yy}}\right) + iL_t^{-1}(A_1),\\
u_3(x,y,t) &= \tfrac{i}{2}L_t^{-1}\left(u_{2_{xx}}\right) - \tfrac{i}{2}L_t^{-1}\left(u_{2_{yy}}\right) + iL_t^{-1}(A_2),
\end{aligned} \tag{3.11}$$

we obtain

$$\begin{aligned}
&u_0(x,y,t) = e^{i(mx+ny)}, A_0 = u_0^2\bar{u}_0 = e^{i(mx+ny)},\\
&u_1(x,y,t) = \tfrac{i}{2}L_t^{-1}\left(-m^2e^{i(mx+ny)}\right) - \tfrac{i}{2}L_t^{-1}\left(-n^2e^{i(mx+ny)}\right) + iL_t^{-1}(e^{i(mx+ny)}) = it(\tfrac{n^2}{2} - \tfrac{m^2}{2} + 1)e^{i(mx+ny)},\\
&u_2(x,y,t) = \tfrac{i}{2}L_t^{-1}\left(u_{1_{xx}}\right) - \tfrac{i}{2}L_t^{-1}\left(u_{1_{yy}}\right) + iL_t^{-1}(A_1) = \tfrac{(it)^2}{2!}\left(\tfrac{n^2}{2} - \tfrac{m^2}{2} + 1\right)^2 e^{i(mx+ny)}\\
&u_3(x,y,t) = \tfrac{i}{2}L_t^{-1}\left(u_{2_{xx}}\right) - \tfrac{i}{2}L_t^{-1}\left(u_{2_{yy}}\right) + iL_t^{-1}(A_2) = \tfrac{(it)^3}{3!}\left(\tfrac{n^2}{2} - \tfrac{m^2}{2} + 1\right)^3 e^{i(mx+ny)}
\end{aligned} \tag{3.12}$$

Accordingly, the series solution is given by

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) = u_1 + u_2 + u_3 + ...$$

$$u(x,y,t) = e^{i(mx+ny)}\left[1 + \frac{it}{1!}\left(\frac{n^2}{2} - \frac{m^2}{2} + 1\right) + \frac{(it)^2}{2!}\left(\frac{n^2}{2} - \frac{m^2}{2} + 1\right)^2 + ...\right] \tag{3.13}$$

that gives exact solution of (3.1) in closed form

$$u(x,y,t) = e^{i\left(mx+ny+\left(\frac{n^2}{2}-\frac{m^2}{2}+1\right)t\right)} \tag{3.14}$$

.

4 Conclusion

The Adomian decomposition method is successfully used to establish new exact solution. The performance of this method is found to be reliable and effective and can give more solutions, which may be important for the explanation of some nonlinear complex physical phenomena.

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Recurrence relations of multiparameter K-Mittag-Leffler function

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Abstract

In this paper we evaluate the functional relation between Multiparameter K-Mittag-Leffler function defined by [2] and K-Series defined by [3]. Also we evaluate the recurrence relations and integral representation of Multiparameter K-Mittag-Leffler function. Some particular cases have been discussed.

Keywords: Multiparameter K-Mittag-Leffler function, K-Series, K-Pochhammer symbol, K- Gamma function.

2010 MSC: 26A33, 33B15, 33E12.

1 Introduction

In [8] the author introduce the generalized K-Gamma Function $\Gamma_k(x)$ as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, k > 0, x \in C\backslash kZ^-, \tag{1.1}$$

where $(x)_{n,k}$ is the k-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k).....(x+(n-1)k), x \in C, k \in R, n \in N^+. \tag{1.2}$$

K-Gamma function is given by,

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, x \in C, k \in R, Re(x) > 0, \tag{1.3}$$

and it follows easily that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}). \tag{1.4}$$

$$\Gamma_k(x+k) = x\Gamma_k(x). \tag{1.5}$$

$$(x)_{n,k} = k^n(\frac{x}{k})_n. \tag{1.6}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}. \tag{1.7}$$

$$nk(x)_{n-1,k} = (x)_{n,k} - (x-k)_{n,k}. \tag{1.8}$$

$$(x)_{n+j,k} = (x)_{j,k}(x+jk)_{n,k} \tag{1.9}$$

The Multiparameter K-Mittag-Leffler function defined by [2], as

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Definition 1.1. *Let $k \in R_+ = (0, \infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R$ $(j = 1, 2, .., p; r = 1, 2, .., q; i = 1, 2, .., m)$. Then the Multiparameter K-Mittag-Leffler function defined as,*

$$ {}_pK_{q,k}^{(\beta,\eta)_m}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m; z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)}, \tag{1.10} $$

where $\Gamma_k(x)$ is the K-Gamma function given by (1.1) and $(\gamma)_{n,k}$ is the K-Pochhammer symbol given by (1.2).

The series (1.10) is defined when none of the parameter $b_r (r = 1, 2, .., q)$ is negative integer or zero. If any parameter $a_j (j = 1, 2, .., p)$ in (1.10) is zero or negative, the series terminates into polynomial in z.
Convergent conditions for the series (1.10) are given by Ratio test,
(i) If $p < q + \sum_{i=1}^m (\frac{\eta_i}{k})$, then the power series on the right of (1.10) is absolutely convergent for all $z \in C$.
(ii) If $p = q + \sum_{i=1}^m (\frac{\eta_i}{k})$, then the power series on the right of (1.10) is absolutely convergent for all $|k^{p-q-\sum_{i=1}^m (\frac{\eta_i}{k})} z| < \prod_{i=1}^m (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$ and $|k^{p-q-\sum_{i=1}^m (\frac{\eta_i}{k})} z| = \prod_{i=1}^m (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$, $Re(\sum_{r=1}^q (\frac{b_r}{k}) + \sum_{i=1}^m (\frac{\beta_i}{k}) - \sum_{j=1}^p (\frac{a_j}{k})) > \frac{2+q+m-p}{2}$.

2 Main Results

In this section we evaluate the functional relation between Multiparameter K-Mittag-Leffler Function and K-Series. Also we evaluate the recurrence relations and integral representation of Multiparameter K-Mittag-Leffler Function. Nine particular cases have been evaluated for different values of parameters.

Theorem 2.1. *The functional relation between Multiparameter K-Mittag-Leffler function and K-Series is given by,*

$$ {}_pK_{q,k}^{(\beta,\eta)_m}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m; z] $$

$$ = k^{\sum_{i=1}^m (1-\frac{\beta_i}{k})}\, {}_pK_q^{(\beta,\eta)_m}[(\frac{a_j}{k})_{j=1}^p; (\frac{b_r}{k})_{r=1}^q, (\frac{\beta_i}{k}, \frac{\eta_i}{k})_{i=1}^m; zk^{p-q-\sum_{i=1}^m \frac{\eta_i}{k}}]. \tag{2.1} $$

And its counter part is given by

$$ {}_pK_q^{(\beta,\eta)_m}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m; z] $$

$$ = k^{\sum_{i=1}^m (\beta_i - 1)}\, {}_pK_{q,k}^{(\beta,\eta)_m}[(ka_j)_{j=1}^p; (kb_r)_{r=1}^q, (k\beta_i, k\eta_i)_{i=1}^m; zk^{\sum_{i=1}^m \eta_i + q - p}]. \tag{2.2} $$

Proof. From equation (1.10), we have

$$ A \equiv {}_pK_{q,k}^{(\beta,\eta)_m}[z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)}, $$

using equations (1.4) and (1.6), we obtain

$$ A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p k^{pn} (\frac{a_j}{k})_n\, z^n}{\prod_{r=1}^q k^{qn} (\frac{b_r}{k})_n \prod_{i=1}^m k^{\frac{\eta_i n + \beta_i}{k} - 1} \Gamma(\frac{\eta_i}{k} n + \frac{\beta_i}{k})}, $$

$$ A \equiv k^{\sum_{i=1}^m (1-\frac{\beta_i}{k})} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\frac{a_j}{k})_n\, (zk^{p-q-\sum_{i=1}^m \frac{\eta_i}{k}})^n}{\prod_{r=1}^q (\frac{b_r}{k})_n \prod_{i=1}^m \Gamma(\frac{\eta_i}{k} n + \frac{\beta_i}{k})}, $$

$$ A \equiv k^{\sum_{i=1}^m (1-\frac{\beta_i}{k})}\, {}_pK_q^{(\beta,\eta)_m}[(\frac{a_j}{k})_{j=1}^p; (\frac{b_r}{k})_{r=1}^q, (\frac{\beta_i}{k}, \frac{\eta_i}{k})_{i=1}^m; zk^{p-q-\sum_{i=1}^m \frac{\eta_i}{k}}]. $$

□

Theorem 2.2. *Let $b \in C, \beta \in R$ and the convergent conditions of Multiparameter K-Mittag-Leffler function are satisfies, then*

$$ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m, (b, \beta); z] $$

$$ = b\, {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m, (b+k, \beta); z] $$

$$ +\beta z \frac{d}{dz}\, {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m, (b+k, \beta); z]. \tag{2.3} $$

Proof. Consider the right hand side of equation (2.3) and using equation (1.10), we have

$$A \equiv b\; {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(b+k,\beta);z]$$

$$+\beta z\frac{d}{dz}\; {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(b+k,\beta);z],$$

$$A \equiv b\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\, z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\beta n+b+k)}$$

$$+\beta z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\, z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\beta n+b+k)},$$

$$A \equiv \sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}(\beta n+b)\, z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\beta n+b+k)},$$

using equation (1.5), we obtain

$$A \equiv\; {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(b,\beta);z].$$

□

Theorem 2.3. *Let $a \in C$ and the convergent conditions of Multiparameter K-Mittag-Leffler Function are satisfies, then*

$$\begin{aligned}&{}_{p+1}K_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p,a+k;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(k,1);z]\\ &-\,{}_{p+1}K_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p,a;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(k,1);z]\\ &=\frac{kz\prod_{j=1}^p(a_j)}{\prod_{r=1}^q(b_r)}\;{}_pK_{q,k}^{(\beta+\eta,\eta)_{m+1}}[(a_j+k)_{j=1}^p;(b_r+k)_{r=1}^q,(\beta_i+\eta_i,\eta_i)_{i=1}^m,(1,1);z].\end{aligned} \tag{2.4}$$

Proof. Consider the left hand side of equation (2.4) and using equation (1.10), we have

$$A \equiv\; {}_{p+1}K_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p,a+k;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(k,1);z]$$

$$-\,{}_{p+1}K_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p,a;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(k,1);z],$$

$$A \equiv \sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\, z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(n+k)}[(a+k)_{n,k}-(a)_{n,k}],$$

using equation (1.8), we obtain

$$A \equiv \sum_{n=1}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\, z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(n+k)}[nk(a+k)_{n-1,k}],$$

replacing n by $n+1$, we obtain

$$A \equiv \sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n+1,k}\, z^{n+1}}{\prod_{r=1}^q(b_r)_{n+1,k}\prod_{i=1}^m \Gamma_k(\eta_i(n+1)+\beta_i)\Gamma_k(n+1+k)}[(n+1)k(a+k)_{n,k}],$$

using equations (1.5) and (1.9), we obtain

$$A \equiv \sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{1,k}(a_j+k)_{n,k}\, z^{n+1}[(n+1)k(a+k)_{n,k}]}{\prod_{r=1}^q(b_r)_{1,k}(b_r+k)_{n,k}\prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i+\eta_i)(n+1)\Gamma_k(n+1)},$$

$$A \equiv \frac{kz\prod_{j=1}^p(a_j)}{\prod_{r=1}^q(b_r)}\;{}_pK_{q,k}^{(\beta+\eta,\eta)_{m+1}}[(a_j+k)_{j=1}^p;(b_r+k)_{r=1}^q,(\beta_i+\eta_i,\eta_i)_{i=1}^m,(1,1);z].$$

□

Theorem 2.4. *Let $\beta \in C, Re(\beta) > 0, \alpha \in R$ and the convergent conditions of Multiparameter K-Mittag-Leffler Function are satisfies, then*

$$
\begin{aligned}
&{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);z] \\
&-k\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);z] \\
&= z^2\alpha^2\,{}_p\ddot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z] \\
&+z\{\alpha^2+2\alpha(\beta+k)\}\,{}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z] \\
&+\beta(\beta+2k)\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]. \qquad (2.5)
\end{aligned}
$$

Proof. From equations (1.10) and (1.5), we have

$$
{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);z]
$$

$$
=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_i n+\beta_i)(\alpha n+\beta)\Gamma_k(\alpha n+\beta)}. \qquad (2.6)
$$

Again,

$$
{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);z]
$$

$$
=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_i n+\beta_i)(\alpha n+\beta+k)(\alpha n+\beta)\Gamma_k(\alpha n+\beta)}. \qquad (2.7)
$$

$$
=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)}\frac{1}{k}\left[\frac{1}{(\alpha n+\beta)}-\frac{1}{(\alpha n+\beta+k)}\right]
$$

using equation (2.6), we obtain

$$
\begin{aligned}
S =&\ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);z] \\
&-k\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);z]. \qquad (2.8)
\end{aligned}
$$

Where

$$
S=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)(\alpha n+\beta+k)}. \qquad (2.9)
$$

Applying a simple identity $\frac{1}{u}=\frac{k}{u(u+k)}+\frac{1}{(u+k)}$, for $u=\alpha n+\beta+k$ to equation (2.9), we obtain

$$
S=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)}
\times\left[\frac{k}{(\alpha n+\beta+k)(\alpha n+\beta+2k)}+\frac{1}{(\alpha n+\beta+2k)}\right],
$$

$$
S=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)}
$$

$$
\times\left[\frac{k(\alpha n+\beta)}{(\alpha n+\beta)(\alpha n+\beta+k)(\alpha n+\beta+2k)}+\frac{(\alpha n+\beta)(\alpha n+\beta+k)}{(\alpha n+\beta)(\alpha n+\beta+k)(\alpha n+\beta+2k)}\right],
$$

using equation (1.5), we have

$$
S=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}\,z^n[n^2\alpha^2+2n\alpha(\beta+k)+\beta(\beta+2k)]}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta+3k)}. \qquad (2.10)
$$

We express each summation in right side of (2.5) as follows;

$$\frac{d}{dz}\{z\ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]\}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}(n+1)\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_in+\beta_i)\Gamma_k(\alpha n+\beta+3k)},$$

$$z\ {}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$+\ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}(n+1)\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_in+\beta_i)\Gamma_k(\alpha n+\beta+3k)},$$

$$z\ {}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$=\sum_{n=0}^{\infty}\frac{n\prod_{j=1}^p(a_j)_{n,k}\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_in+\beta_i)\Gamma_k(\alpha n+\beta+3k)}. \tag{2.11}$$

Again

$$\frac{d^2}{dz^2}\{z^2\ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]\}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_{n,k}(n+2)(n+1)\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_in+\beta_i)\Gamma_k(\alpha n+\beta+3k)},$$

$$z^2\ {}_p\ddot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$+4z\ {}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$+2\ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$=\sum_{n=0}^{\infty}\frac{(n^2+3n+1)\prod_{j=1}^p(a_j)_{n,k}\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_in+\beta_i)\Gamma_k(\alpha n+\beta+3k)},$$

using equation (2.11)

$$z^2\ {}_p\ddot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$+z\ {}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$=\sum_{n=0}^{\infty}\frac{n^2\prod_{j=1}^p(a_j)_{n,k}\,z^n}{\prod_{r=1}^q(b_r)_{n,k}\prod_{i=1}^m\Gamma_k(\eta_in+\beta_i)\Gamma_k(\alpha n+\beta+3k)}. \tag{2.12}$$

using equations (2.11), (2.12) in equation (2.10), we obtain

$$S=z^2\alpha^2\ {}_p\ddot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$+z\{\alpha^2+2\alpha(\beta+k)\}\ {}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$+\beta(\beta+2k)\ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z].$$

□

Theorem 2.5. *Let $\beta \in C, Re(\beta) > 0, \alpha \in R$ and the convergent conditions of Multiparameter K-Mittag-Leffler function are satisfies, then*

$$\int_0^1 t^{\beta+k-1} {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta,\alpha);t^\alpha]dt$$

$$= {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);1]$$

$$-k\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);1]. \tag{2.13}$$

Proof. Put $z = 1$ in equations (2.8) and (2.9), we have

$$S = {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);1]$$

$$-k\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);1]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)(\alpha n+\beta+k)}. \tag{2.14}$$

Consider the left hand side integral,

$$A \equiv \int_0^1 t^{\beta+k-1} {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta,\alpha);t^\alpha]dt,$$

using equation (1.10), we have

$$A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)} \int_0^1 t^{\alpha n+\beta+k-1} dt,$$

$$A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)(\alpha n+\beta+k)},$$

from equation (2.14), we obtain

$$A \equiv {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);1]$$

$$-k\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);1].$$

□

2.1 Particular Cases

The particular cases of this paper are given by particularzing the values of parameters, we obtain the result for different known Mittag-Leffler Functions, given as:

(a) If we set $k = 1$, then we obtain the results for K-Series definded by [3].

(b) If we set $k = 1, p = q = m$ and $b_1 = b_2 = ... = b_m = 1$, we obtain the results for the 3M-Parameter Multi-Index Mittag-Leffler function definded by [4].

(c) If we set $k = 1, p = q = 1, a_1 = \rho, b_1 = 1$, then we obtain the results for the Generalized Mittag-Leffler function studied by [5].

(d) If we set $k = 1, p = q = 1, a_1 = b_1 = 1$ and $\eta_i = \frac{1}{\alpha_i}$, then we obtain the results for the Multi-Index Mittag-Leffler function studied by [10].

(e) If we set $k = 1, m = 1$, then we obtain the results for Generalized M-Series definded by [9].

(f) If we set $p = q = m = 1, a_1 = \delta, b_1 = k$, then we obtain the results for the K- Mittag-Leffler function studied by [1].

(g) If we set $k = 1, p = q = m = 1, a_1 = \delta, b_1 = 1$, then we obtain the results for the Generalized Mittag-Leffler function studied by [7].

(h) If we set $k = 1, p = q = m = 1, a_1 = b_1 = 1$, then we obtain the results for the Mittag-Leffler function studied by [11].

(i) If we set $k = 1, p = q = m = 1, a_1 = b_1 = 1$ and $\beta = 1$, then we obtain the results for the Mittag-Leffler function studied by [6].

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Fractional integral inequalities for continuous random variables

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Abstract

By introducing new concepts on the probability theory, new integral inequalities are established for the fractional expectation and the fractional variance for continuous random variables. These inequalities generalize some interested results in [N.S. Barnett, P. Cerone, S.S. Dragomir and J. Roumeliotis: *Some inequalities for the dispersion of a random variable whose p.d.f. is defined on a finite interval*, J. Inequal. Pure Appl. Math., Vol. 2 Iss. 1 Art. 1 (2001), 1-18].

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1 Introduction

It is well known that the integral inequalities play a fundamental role in the theory of differential equations and applied sciences. Significant development in this theory has been achieved for the last two decades. For details, we refer to [4, 7, 11, 16, 19, 20, 21, 23] and the references therein. Moreover, the study of fractional type inequalities is also of great importance. We refer the reader to [2, 3, 6, 8, 10] for further information and applications. Let us introduce now the results that have inspired our work. The first one is given in [5]; in their paper, using Korkine identity and Holder inequality for double integrals, N.S. Barnett et al. established several integral inequalities for the expectation $E(X)$ and the variance $\sigma^2(X)$ of a random variable X having a probability density function (p.d.f.) $f : [a,b] \to \mathbb{R}^+$. In [13, 14], P. Kumar presented new inequalities for the moments and for the higher order central moments of a continuous random variable. In [15], Y. Miao and G. Yang gave new upper bounds for the standard deviation $\sigma(X)$, for the quantity $\sigma^2(X) + (t - E(X))^2, t \in [a,b]$ and for the L^p absolute deviation of a random variable X. Recently, G.A. Anastassiou et al. [2] proposed a generalization of the weighted Montgomery identity for fractional integrals with weighted fractional Peano kernel. More recently, M. Niezgoda [18] proposed new generalizations of the results of P. Kumar [14], by applying some Ostrowski-Gruss type inequalities. Other paper deal with these probability inequalities can be found in [1, 17, 22].

In this paper, we introduce new concepts on "fractional random variables". Then, we obtain new integral inequalities for the fractional dispersion and the fractional variance functions of a continuous random variable X having the probability density function $(p.d.f.) f : [a,b] \to \mathbb{R}^+$. We also present new results for the "fractional expectation and the fractional variance". For our results, some classical integral inequalities of Barnet et al. [5] can be deduced as some special cases.

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2 Preliminaries

Definition 2.1. *[12] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function h on $[a,b]$ is defined as*

$$\begin{array}{c} J^{\alpha}[h(t)] = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}h(\tau)d\tau; \quad \alpha > 0, a < t \leq b, \\ J^0[h(t)] = h(t), \end{array} \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du$.

We give the following properties:

$$J^{\alpha}J^{\beta}[h(t)] = J^{\alpha+\beta}[h(t)], \alpha \geq 0, \beta \geq 0, \tag{2.2}$$

and

$$J^{\alpha}J^{\beta}[h(t)] = J^{\beta}J^{\alpha}[h(t)], \alpha \geq 0, \beta \geq 0. \tag{2.3}$$

We introduce also the following new concepts and definitions:

Definition 2.2. *The fractional expectation function of order $\alpha \geq 0$, for a random variable X with a positive $p.d.f.$ f defined on $[a,b]$ is defined as*

$$E_{X,\alpha}(t) := J^{\alpha}[tf(t)] = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}\tau f(\tau)d\tau; \quad \alpha \geq 0, a < t \leq b. \tag{2.4}$$

In the same way, we define the fractional expectation function of $X - E(X)$ by:

Definition 2.3. *The fractional expectation function of order $\alpha \geq 0$, for a random variable $X - E(X)$ is defined as*

$$E_{X-E(X),\alpha}(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}(\tau - E(X))f(\tau)d\tau; \quad \alpha \geq 0, a < t \leq b, \tag{2.5}$$

where $f : [a,b] \to \mathbb{R}^+$ is the p.d.f. of X.

For $t = b$, we introduce the following concept:

Definition 2.4. *The fractional expectation of order $\alpha \geq 0$, for a random variable X with a positive p.d.f. f defined on $[a,b]$ is defined as*

$$E_{X,\alpha} = E_{X,\alpha} = \frac{1}{\Gamma(\alpha)}\int_a^b (b-\tau)^{\alpha-1}\tau f(\tau)d\tau; \quad \alpha \geq 0. \tag{2.6}$$

For the fractional variance of X, we introduce the two definitions:

Definition 2.5. *The fractional variance function of order $\alpha \geq 0$ for a random variable X having a p.d.f. $f : [a,b] \to \mathbb{R}^+$ is defined as*

$$\begin{array}{c} \sigma^2_{X,\alpha}(t) := J^{\alpha}[(t-E(X))^2 f(t)] = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}(\tau - E(X))^2 f(\tau)d\tau; \\ \alpha \geq 0, a < t \leq b. \end{array} \tag{2.7}$$

where $E(X) := \int_a^b \tau f(\tau)d\tau$ is the classical expectation of X.

Definition 2.6. *The fractional variance of order $\alpha \geq 0$, for a random variable X with a p.d.f. $f : [a,b] \to \mathbb{R}^+$ is defined as*

$$\sigma^2_{X,\alpha} = \frac{1}{\Gamma(\alpha)}\int_a^b (b-\tau)^{\alpha-1}(\tau - E(X))^2 f(\tau)d\tau; \alpha >\geq 0. \tag{2.8}$$

We give the following important properties:

$(P1*)$: If we take $\alpha = 1$ in Definition 2.4, we obtain the classical expectation: $E_{X,1} = E(X)$.
$(P2*)$: If we take $\alpha = 1$ in Definition 2.6, we obtain the classical variance: $\sigma^2_{X,1} = \sigma^2(X) = \int_a^b (\tau - E(X))^2 f(\tau)d\tau$.
$(P3*)$: For $\alpha > 0$, the $p.d.f.$ f satisfies $J^{\alpha}[f(b)] = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}$.
$(P4*)$: For $\alpha = 1$, we have the well known property $J^{\alpha}[f(b)] = 1$.

3 Main Results

In this section, we present new results for fractional continuous random variables. The first main result is the following theorem:

Theorem 3.1. *Let X be a continuous random variable having a p.d.f.* $f : [a,b] \to \mathbb{R}^+$. *Then we have:*
(a) : *For all* $a < t \leq b, \alpha \geq 0$,

$$J^\alpha[f(t)]\sigma^2_{X,\alpha}(t) - (E_{X-E(X),\alpha}(t))^2 \leq ||f||^2_\infty \Big[\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J^\alpha[t^2] - (J^\alpha[t])^2\Big], \tag{3.9}$$

provided that $f \in L_\infty[a,b]$.
(b) : *The inequality*

$$J^\alpha[f(t)]\sigma^2_{X,\alpha}(t) - (E_{X-E(X),\alpha}(t))^2 \leq \frac{1}{2}(t-a)^2(J^\alpha[f(t)])^2 \tag{3.10}$$

is also valid for all $a < t \leq b, \alpha \geq 0$.

Proof. Let us define the quantity

$$H(\tau,\rho) := (g(\tau) - g(\rho))(h(\tau) - h(\rho));\ \tau,\rho \in (a,t), a < t \leq b. \tag{3.11}$$

Taking a function $p : [a,b] \to \mathbb{R}^+$, multiplying (3.11) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} p(\tau);\ \tau \in (a,t)$, then integrating the resulting identity with respect to τ from a to t, we can state that

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} p(\tau) H(\tau,\rho) d\tau \\ &= J^\alpha[pgh(t)] - g(\rho) J^\alpha[ph(t)] - h(\rho) J^\alpha[pg(t)] + g(\rho)h(\rho) J^\alpha[p(t)]. \end{aligned} \tag{3.12}$$

Now, multiplying (3.12) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} p(\rho);\ \rho \in (a,t)$ and integrating the resulting identity with respect to ρ over (a,t), we can write

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} p(\tau)p(\rho) H(\tau,\rho) d\tau d\rho \\ &= 2J^\alpha[p(t)]J^\alpha[pgh(t)] - 2J^\alpha[pg(t)]J^\alpha[ph(t)]. \end{aligned} \tag{3.13}$$

In (3.13), taking $p(t) = f(t), g(t) = h(t) = t - E(X), t \in (a,b)$, we have

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho \\ &= 2J^\alpha[f(t)]J^\alpha[f(t)(t-E(X))^2] - 2\Big(J^\alpha[f(t)(t-E(X))]\Big)^2. \end{aligned} \tag{3.14}$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho \\ &\leq ||f||^2_\infty \tfrac{1}{\Gamma^2(\alpha)} \textstyle\int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}(\tau-\rho)^2 d\tau d\rho \\ &\leq ||f||^2_\infty \Big[2\tfrac{(t-a)^\alpha}{\Gamma(\alpha+1)} J^\alpha[t^2] - 2(J^\alpha[t])^2\Big]. \end{aligned} \tag{3.15}$$

Thanks to (3.14), (3.15), we obtain the part (a) of Theorem 3.1.
For the part (b), we have

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho \\ &\leq \sup_{\tau,\rho\in[a,t]} |(\tau-\rho)|^2 (J^\alpha[f(t)])^2 = (t-a)^2(J^\alpha[f(t)])^2. \end{aligned} \tag{3.16}$$

Then, by (3.14) and (3.16), we get the desired inequality (3.10). □

We give also the following corollary:

Corollary 3.1. *Let X be a continuous random variable with a p.d.f. f defined on* $[a,b]$. *Then:*
(i) : *If* $f \in L_\infty[a,b]$, *then for any* $\alpha \geq 0$, *we have*

$$\frac{(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\sigma^2_{X,\alpha} - E^2_{X,\alpha} \leq ||f||^2_\infty\Big[\frac{2(b-a)^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)} - \Big(\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\Big)^2\Big]. \tag{3.17}$$

(ii) : *The inequality*

$$\frac{(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\sigma^2_{X,\alpha} - E^2_{X,\alpha} \leq \frac{1}{2}\frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha)} \tag{3.18}$$

is also valid for any $\alpha \geq 0$.

Remark 3.1. $(r1)$: *Taking* $\alpha = 1$ *in* (i) *of Corollary 3.1, we obtain the first part of Theorem 1 in [5].*
$(r2)$: *Taking* $\alpha = 1$ *in* (ii) *of Corollary 3.1, we obtain the last part of Theorem 1 in [5].*

We shall further generalize Theorem 3.1 by considering two fractional positive parameters:

Theorem 3.2. *Let X be a continuous random variable having a p.d.f.* $f : [a,b] \rightarrow \mathbb{R}^+$. *Then we have:*
$(a*)$: *For all* $a < t \leq b, \alpha \geq 0, \beta \geq 0$,

$$\begin{aligned} &J^\alpha[f(t)]\sigma^2_{X,\beta}(t) + J^\beta[f(t)]\sigma^2_{X,\alpha}(t) - 2(E_{X-E(X),\alpha}(t))(E_{X-E(X),\beta}(t)) \\ &\leq ||f||^2_\infty\Big[\tfrac{(t-a)^\alpha}{\Gamma(\alpha+1)}J^\beta[t^2] + \tfrac{(t-a)^\beta}{\Gamma(\beta+1)}J^\alpha[t^2] - 2(J^\alpha[t])(J^\beta[t])\Big],\end{aligned} \tag{3.19}$$

where $f \in L_\infty[a,b]$.
$(b*)$: *The inequality*

$$\begin{aligned} &J^\alpha[f(t)]\sigma^2_{X,\beta}(t) + J^\beta[f(t)]\sigma^2_{X,\alpha}(t) - 2(E_{X-E(X),\alpha}(t))(E_{X-E(X),\beta}(t)) \\ &\leq (t-a)^2 J^\alpha[f(t)]J^\beta[f(t)]\end{aligned} \tag{3.20}$$

is also valid for any $a < t \leq b, \alpha \geq 0, \beta \geq 0$.

Proof. Using (3.11), we can write

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)p(\rho)H(\tau,\rho)d\tau d\rho \\ &= J^\alpha[p(t)]J^\beta[pgh(t)] + J^\beta[p(t)]J^\alpha[pgh(t)] \\ &-J^\alpha[ph(t)]J^\beta[pg(t)] - J^\beta[ph(t)]J^\alpha[pg(t)].\end{aligned} \tag{3.21}$$

Taking $p(t) = f(t), g(t) = h(t) = t - E(X), t \in (a,b)$ in the above identity, yields

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho \\ &= J^\alpha[f(t)]J^\beta[f(t)(t-E(X))^2] + J^\beta[f(t)]J^\alpha[f(t)(t-E(X))^2] \\ &-2J^\alpha[f(t)(t-E(X))]J^\beta[f(t)(t-E(X))].\end{aligned} \tag{3.22}$$

We have also

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho \\ &\leq ||f||^2_\infty \tfrac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho)^2 d\tau d\rho \\ &\leq ||f||^2_\infty\Big[\tfrac{(t-a)^\alpha}{\Gamma(\alpha+1)}J^\beta[t^2] + \tfrac{(t-a)^\beta}{\Gamma(\beta+1)}J^\alpha[t^2] - 2J^\alpha[t]J^\beta[t]\Big].\end{aligned} \tag{3.23}$$

Thanks to (3.22) and (3.23), we obtain $(a*)$.
To prove $(b*)$, we use the fact that $\sup_{\tau,\rho\in[a,t]}|(\tau-\rho)|^2=(t-a)^2$. We obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}f(\tau)f(\rho)(\tau-\rho)^2d\tau d\rho\\&\leq(t-a)^2J^\alpha[f(t)]J^\beta[f(t)].\end{aligned}\tag{3.24}$$

And, by (3.22) and (3.24), we get (3.20). □

Remark 3.2. $(r1)$: *Applying Theorem 3.2 for $\alpha=\beta$, we obtain Theorem 3.1.*
$(r2)$: *Taking $\alpha=\beta=1$ in $(a*)$ of Theorem 3.4, we obtain the first inequality of Theorem 1 in [5].*
$(r3)$: *Taking $\alpha=\beta=1$ in $(b*)$ of Theorem 3.2, we obtain the last part of Theorem 1 in [5].*

We give also the following fractional integral result:

Theorem 3.3. *Let f be the p.d.f. of X on $[a,b]$. Then for all $a<t\leq b,\alpha\geq0$, we have:*

$$J^\alpha[f(t)]\sigma^2_{X,\alpha}(t)-(E_{X-E(X),\alpha}(t))^2\leq\frac{1}{4}(b-a)^2(J^\alpha[f(t)])^2.\tag{3.25}$$

Proof. Using Theorem 3.1 of [9], we can write

$$\begin{aligned}&\left|J^\alpha[p(t)]J^\alpha[pg^2(t)]-(J^\alpha[pg(t)])^2\right|\\&\leq\tfrac{1}{4}\Big(J^\alpha[p(t)]\Big)^2(M-m)^2.\end{aligned}\tag{3.26}$$

Taking $p(t)=f(t),g(t)=t-E(X),t\in[a,b]$, then $M=b-E(X),m=a-E(X)$. Hence, (3.25) allows us to obtain

$$\begin{aligned}&0\leq J^\alpha[f(t)]J^\alpha[f(t)(t-E(X))^2]-\Big(J^\alpha[f(t)(t-E(X))]\Big)^2\\&\leq\tfrac{1}{4}(J^\alpha[f(t)])^2(b-a)^2.\end{aligned}\tag{3.27}$$

This implies that

$$J^\alpha[f(t)]\sigma^2_{X,\alpha}(t)-(E_{X-E(X),\alpha}(t))^2\leq\frac{1}{4}(J^\alpha[f(t)])^2(b-a)^2.\tag{3.28}$$

Theorem 3.3 is thus proved. □

For $t=b$, we propose the following interesting inequality:

Corollary 3.2. *Let f be the p.d.f. of X on $[a,b]$. Then for any $\alpha\geq0$, we have:*

$$\frac{(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\sigma^2_{X,\alpha}-(E_{X-E(X),\alpha})^2\leq\frac{1}{4\Gamma^2(\alpha)}(b-a)^{2\alpha}.\tag{3.29}$$

Remark 3.3. *Taking $\alpha=1$ in Corollary 3.2, we obtain Theorem 2 of [5].*

We also present the following result for the fractional variance function with two parameters:

Theorem 3.4. *Let f be the p.d.f. of the random variable X on $[a,b]$. Then for all $a<t\leq b,\alpha\geq0,\beta\geq0$, we have:*

$$\begin{aligned}&J^\alpha[f(t)]\sigma^2_{X,\beta}(t)+J^\beta[f(t)]\sigma^2_{X,\alpha}(t)\\&+2(a-E(X))(b-E(X))J^\alpha[f(t)]J^\beta[f(t)]\\&\leq(a+b-2E(X))\Big(J^\alpha[f(t)](E_{X-E(X),\beta}(t))+J^\beta[f(t)](E_{X-E(X),\alpha}(t))\Big).\end{aligned}\tag{3.30}$$

Proof. Thanks to Theorem 3.4 of [9], we can state that:

$$\begin{aligned}&\Big[J^{\alpha}[p(t)]J^{\beta}[pg^{2}(t)]+J^{\beta}[p(t)]J^{\alpha}(pg^{2}(t)]-2J^{\alpha}[pg(t)]J^{\beta}[pg(t)]\Big]^{2}\\&\leq\Big[\Big(MJ^{\alpha}[p(t)]-J^{\alpha}[pg(t)]\Big)\Big(J^{\beta}[pg(t)]-mJ^{\beta}[p(t)]\Big)+\\&\Big(J^{\alpha}[pg(t)]-mJ^{\alpha}[p(t)]\Big)\Big(MJ^{\beta}[p(t)]-J^{\beta}[pg(t)]\Big)\Big]^{2}.\end{aligned}\tag{3.31}$$

In (3.31), we take $p(t)=f(t), g(t)=t-E(X), t\in[a,b]$. We obtain

$$\begin{aligned}&\Big[J^{\alpha}[f(t)]J^{\beta}[f(t)(t-E(X))^{2}]+J^{\beta}[f(t)]J^{\alpha}[f(t)(t-E(X))^{2}]\\&-2J^{\alpha}[f(t)(t-E(X))]J^{\beta}[f(t)(t-E(X))]\Big]^{2}\\&\leq\Big[\Big(MJ^{\alpha}[f(t)]-J^{\alpha}[f(t)(t-E(X))]\Big)\Big(J^{\beta}[f(t)(t-E(X))]-mJ^{\beta}[f(t)]\Big)+\\&\Big(J^{\alpha}[f(t)(t-E(X))]-mJ^{\alpha}[f(t)]\Big)\Big(MJ^{\beta}[f(t)]-J^{\beta}[f(t)(t-E(X))]\Big)\Big]^{2}.\end{aligned}\tag{3.32}$$

Combining (3.22) and (3.32) and taking into account the fact that the left hand side of (3.22) is positive, we get:

$$\begin{aligned}&J^{\alpha}[f(t)]J^{\beta}[f(t)(t-E(X))^{2}]+J^{\beta}[f(t)]J^{\alpha}[f(t)(t-E(X))^{2}]\\&-2J^{\alpha}[f(t)(t-E(X))]J^{\beta}[f(t)(t-E(X))]\\&\leq\Big(MJ^{\alpha}[f(t)]-J^{\alpha}[f(t)(t-(EX))]\Big)\Big(J^{\beta}[f(t)(t-E(X))]-mJ^{\beta}[f(t)]\Big)+\\&\Big(J^{\alpha}[f(t)(t-E(X))]-mJ^{\alpha}[f(t)]\Big)\Big(MJ^{\beta}[f(t)]-J^{\beta}[f(t)(t-E(X))]\Big).\end{aligned}\tag{3.33}$$

Therefore,

$$\begin{aligned}&J^{\alpha}[f(t)]J^{\beta}[f(t)(t-E(X))^{2}]+J^{\beta}[f(t)]J^{\alpha}[f(t)(t-E(X))^{2}]\\&\leq M\Big(J^{\alpha}[f(t)](E_{X-E(X),\beta}(t))+J^{\beta}[f(t)](E_{X-E(X),\alpha}(t))\Big)\\&+m\Big(J^{\alpha}[f(t)](E_{X-E(X),\beta}(t))+J^{\beta}[f(t)](E_{X-E(X),\alpha}(t))\Big)\\&-2mMJ^{\alpha}[f(t)]J^{\beta}[f(t)].\end{aligned}\tag{3.34}$$

Substituting the values of m and M in (3.28) , then a simple calculation allows us to obtain (3.30). Theorem 3.4 is thus proved. □

To finish, we present to the reader the following corollary:

Corollary 3.3. *Let f be the p.d.f. of X on $[a,b]$. Then for all $a<t\leq b, \alpha\geq 0$, the inequality*

$$\begin{aligned}&\sigma^{2}_{X,\alpha}(t)+(a-E(X))(b-E(X))J^{\alpha}[f(t)]\\&\leq(a+b-2E(X))E_{X-E(X),\alpha}(t)\end{aligned}\tag{3.35}$$

is valid.

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