ISSN 2319-3786
VOLUME 2, ISSUE 3, JULY 2014

M
J
M

Malaya Journal of Matematik

an international journal of mathematical sciences

UNIVERSITY PRESS
5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),
Coimbatore- 641114, Tamil Nadu, India.
www.malayajournal.org

The Malaya Journal of Matematik is published quarterly in single volume annually and four issues constitute one volume appearing in the months of January, April, July and October.

Subscription

The subscription fee is as follows:

USD 350.00 For USA and Canada

Euro 190.00 For rest of the world

Rs. 4000.00 In India. (For Indian Institutions in India only)

Prices are inclusive of handling and postage; and issues will be delivered by Registered Air-Mail for subscribers outside India.

Subscription Order

Subscription orders should be sent along with payment by Cheque/ D.D. favoring "Malaya Journal of Matematik" payable at COIMBATORE at the following address:

MKD Publishing House

5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),

Coimbatore- 641114, Tamil Nadu, India.

Contact No. : +91-9585408402

E-mail : info@mkdpress.com; editorinchief@malayajournal.org; publishingeditor@malayajournal.org

Website : https://mkdpress.com/index.php/index/index

Vol. 2 No. 03 (2014): Malaya Journal of Matematik (MJM)

Malaya J. Mat. 2(3)(2014) 180-187

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On k-step Hamiltonian Bipartite and Tripartite Graphs

Gee-Choon Lau[a,*], Sin-Min Lee[b], Karl Schaffer[c] and Siu-Ming Tong[d]

[a] *Faculty of Comp. & Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Malaysia .*

[b] *34803, Hollyhock St., Union City, CA 94587, USA.*

[c] *Dept. of Mathematics, De Anza College, Cupertino, CA 95014, USA.*

[d] *Dept. of Computer Science, Northwestern Polytechnic University, Fremont, CA 94539, USA.*

Abstract

For integer $k \geq 1$, a (p,q)-graph $G = (V, E)$ is said to admit an $AL(k)$-traversal if there exist a sequence of vertices $(v_1, v_2, \ldots, v_p)$ such that for each $i = 1, 2, \ldots, p-1$, the distance between v_i and v_{i+1} is k. We call a graph k-step Hamiltonian (or admits a k-step Hamiltonian tour) if it admits an $AL(k)$-traversal and $d(v_1, v_p) = k$. In this paper we consider k-step Hamiltonicity of bipartite and tripartite graphs. As an application, we found that a 2-step Hamiltonian tour of a graph could sometimes induce a super-edge-magic labeling of the graph.

Keywords: Hamiltonian tour, 2-step Hamiltonian tour, bipartite & tripartite graphs, NP-complete problem, super-edge-magic labeling.

2010 MSC: 05C78, 05C25.

1 Introduction

In 1856, Kirkman wrote a paper [13] in which he considered graphs with a cycle which passes through every vertex exactly once. The dodecahedron (see Figure 1) is a graph with such property that Hamilton played cycle games. Hence, such a graph is said to be Hamiltonian. The Hamiltonicity of a graph is the problem of determining for a given graph whether it contains a path/cycle that visits every vertex exactly once.

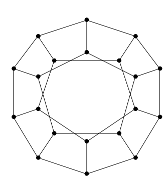


Figure 1: Dodecahedron.

There is no simple characterization on Hamiltonian graphs though they are related to the traveling salesman problem. So there are potential practical applications. In general we know very little about Hamiltonian graphs though their properties have been widely studied. A good reference for recent developments and open problems is [9].

In this paper we consider simple graphs with no loops. For integer $k \geq 1$, a (p,q)-graph $G = (V, E)$ is said to admit an $AL(k)$-traversal if there exist a sequence $(v_1, v_2, \ldots, v_p)$ such that for each $i = 1, 2, \ldots, p-1$, the

*Corresponding author.
E-mail address: geeclau@yahoo.com (G.C. Lau)

distance between v_i and v_{i+1} is k. We call a graph k-step Hamiltonian (or admits a k-step Hamiltonian tour) if it admits an $AL(k)$-traversal and $d(v_1, v_p) = k$.

For example, the cubic graph in Figure 2 is 2-step Hamiltonian and two others admit an $AL(2)$-traversal but are not 2-step Hamiltonian.

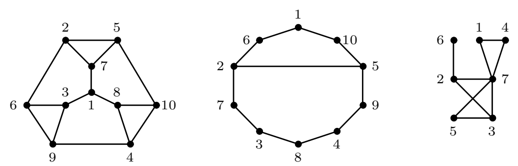


Figure 2: Example on 2-step Hamiltonicity.

There has been much research on Hamiltonicity of bipartite graphs [1, 2, 6, 10, 11, 14]. Clearly, 1-step Hamiltonian is Hamiltonian. In this paper we consider bipartite and tripartite graphs. As an application, we found that a 2-step Hamiltonian tour of a graph could sometimes induce a super-edge-magic labeling of the graph. For terms used but not defined, we refer to [3].

Definition 1.1. *For a graph G, let $D_k(G)$ denote the graph generated from G such that $V(D_k(G)) = V(G)$ and $E(D_k(G)) = \{uv | d(u,v) = k \text{ in } G.\}$.*

Lemma 1.1. *A graph G is k-step Hamiltonian or admits an $AL(k)$-traversal if and only if $D_k(G)$ is Hamiltonian or has a Hamiltonian path, respectively.*

Proof. It follows directly from Definition 1.1.

2 Main Results

We first give a sufficient condition for a graph to admit no k-step Hamiltonian tour.

Theorem 2.1. *Suppose G has a clique subgraph K_p. If $|V(G \backslash K_p)| < p$, then G is not k-step Hamiltonian for all $k \geq 2$.*

Proof. Observe that for any 2 vertices u, v in the clique subgraph K_p of G, $d(u,v) = 1$. Hence, K_p induces an empty graph in $D_k(G)$. If $D_k(G)$ is Hamiltonian, then these p vertices must be not adjacent in a Hamiltonian tour of $D_k(G)$. This implies that we need at least p more vertices to form such a Hamiltonian tour. Since $|V(G \backslash K_p)| < p$, it follows that no Hamiltonian tour exists in $D_k(G)$. By Lemma 1.1, the theorem follows. □

Theorem 2.2. *The vertex gluing of a graph G and an end-vertex of a path of length $n \geq k$ is not k-step Hamiltonian.*

Proof. Let $G(P_n)$ denote the graph such obtained. Observe that $D_k(G(P_n))$ has a cut-vertex and is not Hamiltonian. □

Theorem 2.3. *If graphs G and H are both k-step Hamiltonian, then so is $G \times H$.*

Proof. By Lemma 1.1, G is k-step Hamiltonian if and only if $D_k(G)$ is Hamiltonian. We show that $D_k(G) \times D_k(H)$ is a subgraph of $D_k(G \times H)$. Then any Hamiltonian cycle in $D_k(G) \times D_k(H)$ will also exist in $D_k(G \times H)$ and implies that $G \times H$ is also k-step Hamiltonian. Suppose that edge $e = (u,v)(u,w)$ is an edge in $D_k(G) \times D_k(H)$. Then (v,w) must be an edge in $D_k(H)$, so the distance between v and w in H is k. Let $v = v_0, v_1, v_2, \ldots, v_k = w$ be a length k path from v to w in H. Then (u,v), (u,v_1), (u,v_2),..., (u,w) is a length k path from (u,v) to (u,w) in $G \times H$, so the distance from (u,v) to (u,w) within $G \times H$ is no more than k. Suppose, however, that the distance from (u,v) to (u,w) is less than k in $G \times H$, and let $e_1, e_2, \ldots, e_m$, be a sequence of edges from (u,v) to (u,w) with $m < k$. All edges in this sequence will either be of the form $(z,x)(z,y)$ where xy is an edge in H, or $(x,z)(y,z)$ where xy is an edge in G. Consider the subsequence of edges which are of the first type, $(z,x)(z,y)$. This subsequence must be of the form $(z_1, x_0)(z_1, x_1), (z_2, x_1)(z_2, x_2), \ldots, (z_n, x_{n?})(z_n, x_n)$, where $x_0 = v, x_n = w$, and $n = m < k$. Furthermore, $x_0x_1, x_1x_2, \ldots, x_{n?}x_n$ must be a sequence of edges in H from $v = x_0$ to $w = x_n$, which has length n, where $n < k$. This contradicts the fact that the distance from v to w in H is actually k. Therefore the distance from

(u,v) to (u,w) in $G \times H$ is also k, and so $e = (u,v)(u,w)$ is also an edge of $D_k(G \times H)$; the argument for edges of the form $e = (u,v)(w,v)$ is identical. Since all edges and vertices of $D_k(G) \times D_k(H)$ are also in $D_k(G \times H)$, $D_k(G) \times D_k(H)$ is a subgraph of $D_k(G \times H)$. Since G and H are k-step Hamiltonian, $D_k(G) \times D_k(H)$ is Hamiltonian, and so is $D_k(G \times H)$, implying that $G \times H$ is k-step Hamiltonian. □

A Hamiltonian graph need not be 2-step Hamiltonian. The simplest example is the complete bipartite graph $K(2,2)$ that does not admit an $AL(2)$-Hamiltonian traversal, and hence cannot be 2-step Hamiltonian.

Theorem 2.4. *All bipartite graphs are not k-step Hamiltonian for even $k \geq 2$.*

Proof. Suppose $G = (V,E)$ is bipartite graph with bipartition (X,Y). If $k \geq 2$ is even, the vertex in X cannot connect with vertex in Y, vice versa, in $D_k(G)$. Thus $D_k(G)$ is a disconnected graph with two components X and Y. Hence $D_k(G)$ cannot have a Hamiltonian path. By Lemma 1.1, G is not k-step Hamiltonian. □

We now give a necessary and sufficient condition for cycles to admit a k-step Hamiltonian tour.

Theorem 2.5. *For integers $n \geq 3$ and $k \geq 2$, the cycle C_n is k-step Hamiltonian if and only if $n \geq 2k+1$ and $gcd(n,k) = 1$.*

Proof. If $n \leq 2k$, we have either $\text{diam}(C_n) < k$ or $D_k(C_n)$ is disconnected. Hence, C_n is not k-step Hamiltonian. We may now assume that $n \geq 2k+1$.

Without loss of generality, we may assume that a k-step Hamiltonian tour of C_n is given by the sequence $u_1, u_{k+1}, u_{2k+1}, \ldots, u_{(n-1)k+1}$. Note that $\{1, k+1, 2k+1, 3k+1, \ldots, (n-1)k+1\} \pmod{n}$ is a set of distinct integers if and only if $ik+1 \not\equiv jk+1 \pmod{n}$ for $0 \leq i < j \leq n-1$ if and only if $(j-i)k \not\equiv 0 \pmod{n}$ if and only if $k/n \neq r/(j-i)$ for some integer r if and only if $gcd(n,k) = 1$. Hence, the theorem holds and the k-step Hamiltonian tour of C_n is obtained. □

Theorem 2.6. *The cylinder graph $C_n \times P_m$ is 2-step Hamiltonian for odd $n \geq 3$ and all $m \geq 3$.*

Proof. Case 1. $n = 3$. This case is handled separately since the three vertices in any 3-cycle within $C_3 \times P_m$ are distance 1 from each other. Figure 3 shows 2-step Hamiltonian tours for $C_3 \times P_2$ and $C_3 \times P_3$. Figure 4 shows 2-step Hamiltonian tours for $C_3 \times P_{4k}$. It is based on the 2-step Hamiltonian tour for $C_3 \times P_2$. Here vertices are labeled (a,b) and we may denote edges by listing their vertices: $(a,b)(c,d)$. Then to modify the 2-step Hamiltonian tour for $C_3 \times P_{4k}$ to one for $C_3 \times P_{4k-2}$, replace edge $(1,4k-2)(2,4k-1)$ by $(1,4k-3)(2,4k-2)$, shown as a dotted line, and also remove edge $(2,4k-2)(2,4k)$. The cases in which $n = 3$ and m is odd are handled with similar constructions, based instead on the 2-step Hamiltonian tour for $C_3 \times P_3$ also as shown below. The diagram shows the 2-step Hamiltonian tour for $C_3 \times P_{4k+3}$ which may be modified for the cases $C_3 \times P_{4k-1}$ by adding edge $(1,4k)(2,4k-1)$, again shown shown as a dotted line.

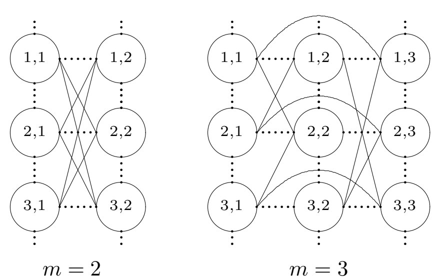


Figure 3: A 2-step Hamiltonian tour in $C_3 \times P_m, m = 2,3$.

Case 2. $n = 2k+1 \geq 5$. In this case, we consider two subcases.

Subcase 2.1. $m = 2j+1 \geq 3$. Figure 5 gives a 2-step Hamiltonian cycle of this subcase. Note that we have partitioned the vertices in a checkerboard pattern so those whose coordinates have even sum are shown by circular vertices, and those whose coordinates have odd sum are shown by square vertices. The circular vertices compose one cycle and the square vertices compose another, except that the two cycles cross over and connect through the edges $(2j+1, 2k+1)(2j,1)$ and $(2j+1,2k)(2j+1,1)$.

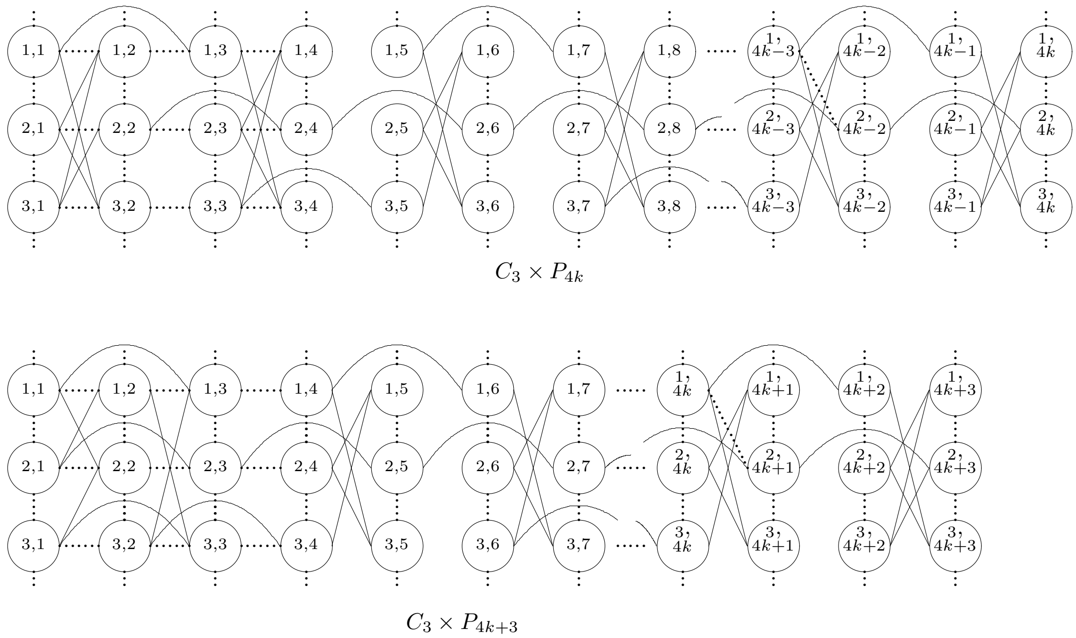


Figure 4: A 2-Hamiltnoian tour in $C_3 \times P_m$.

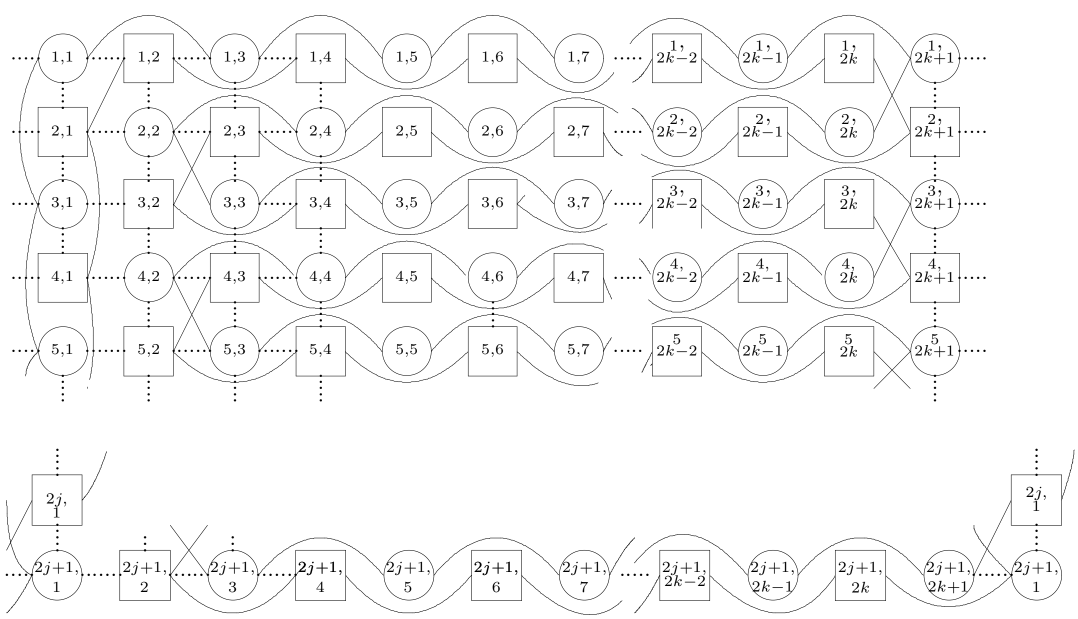


Figure 5: A 2-step Hamiltonian tour in $C_{2k+1} \times P_{2j+1}$.

Subcase 2.2. $m = 2j \geq 2$. Figure 6 gives a 2-step Hamiltonian cycle of this subcase. We have again partitioned the vertices in a checkerboard pattern so those whose coordinates have even sum are shown by circular vertices, and those whose coordinates have odd sum are shown by square vertices. The circular vertices compose one cycle and the square vertices compose another, except in this case the two cycles cross over and connect through the edges $(2j,2k)(2j,1)$ and $(2j-1,2k+1)(2j-2,1)$. Note that the vertices shown in the rightmost column are identical to those at the bottom of the first column.

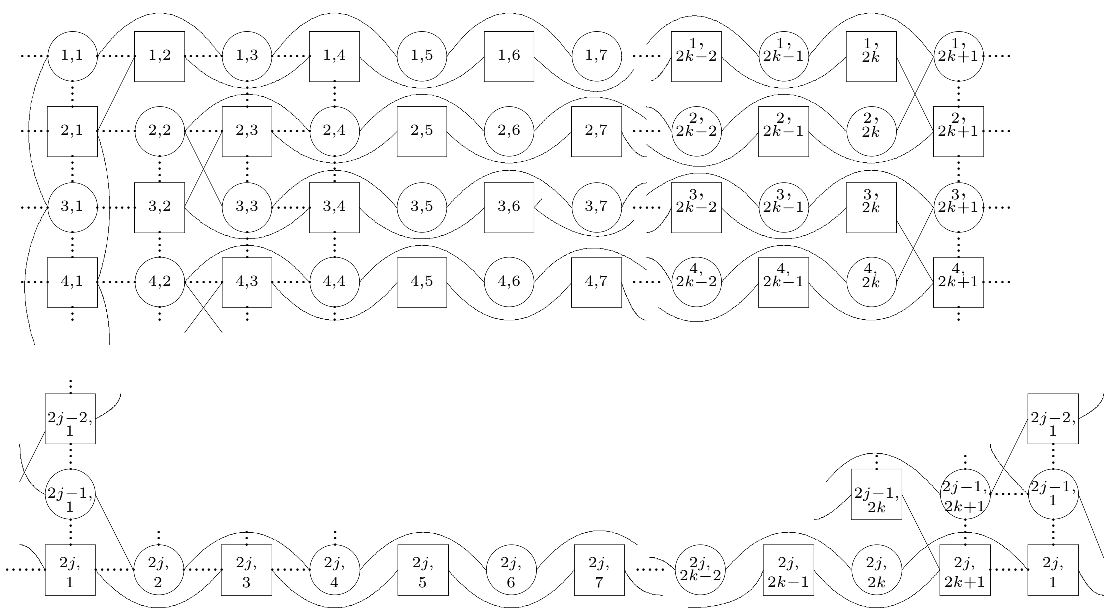


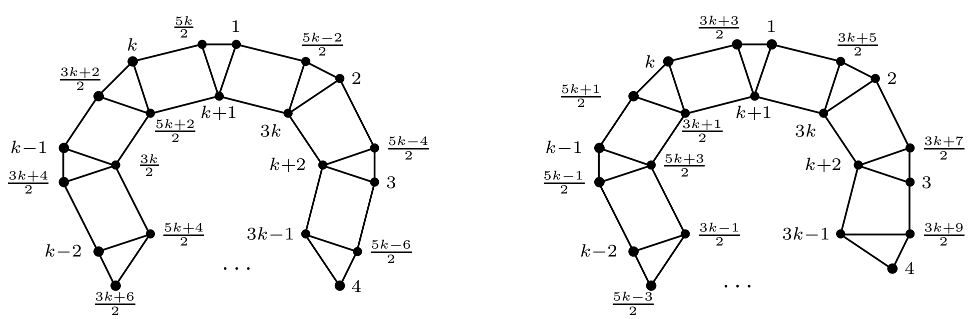
Figure 6: A 2-step Hamiltonian tour in $C_{2k+1} \times P_{2j}$.

□

Since $C_n \times P_m$ is a subgraph of $C_n \times C_m$, the same 2-step Hamiltonian cycles work for $C_n \times C_m$, when n is odd, and we have

Corollary 2.1. . *The graph $C_n \times C_m$ is 2-step Hamiltonian for odd n and all m.*

Let $D(n)$ denote the tripartite donut graph shown in Figure 7 with a given vertex labeling.



(a) $n = 3k \cong 0 \bmod (6)$, $k \geq 4$ even (b) $n = 3k \cong 4 \bmod (6)$, $k \geq 5$ odd

Figure 7: A 2-step Hamiltonian tour in $D(n)$.

Theorem 2.7. *The vertex labeling in graph $D(n)$ gives a 2-step Hamiltonian tour for all $n = 3k, k \geq 4$.*

A ring-worm is a unicyclic graph $U_n(a_1, a_2, \ldots, a_n)$ obtained from a cycle C_n with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ by identifying vertex v_i to the center, c_i of a star S_i having $a_i + 1 \geq 1$ vertices, $\{c_i, u_{i,1}, u_{i,2}, \ldots, u_{i,a_i}\}$. The ring-worm has $n + a_1 + a_2 + \ldots + a_n$ vertices and edges, respectively. We can arrange the vertices of the ring-worm as in Figure 8.

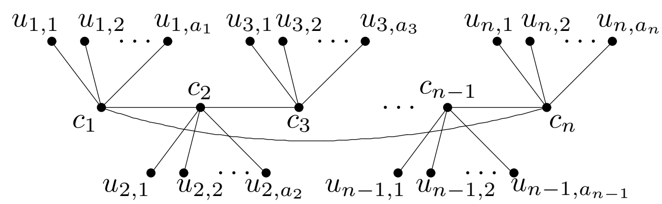


Figure 8: Ring worm graph.

Theorem 2.8. *If $n = 3$ and $a_i > 0$ $(i = 1, 2, 3)$, or $n \geq 5$ is odd and $a_i \geq 0$ $(1 \leq i \leq n)$, the ring worm $U_n(a_1, a_2, \ldots, a_n)$ is 2-step Hamiltonian.*

Proof. The case $n = 3$ is obvious. Without loss of generality, we assume that $n \geq 5$ and not all $a_i = 0$. Suppose $n = 2s + 1, s \geq 2$. We can label the vertices by consecutive integers as described below to get a 2-step Hamiltonian tour for the graph:
$f(c_{2i+1}) = a_2 + a_4 + \ldots + a_{2i} + i + 1$, for $i = 0, 1, 2, \ldots, s$,
$f(c_{2i+2}) = f(c_{2s+1}) + a_1 + a_3 + \ldots + a_{2i+1} + i + 1$ for $i = 0, 1, 2, \ldots, s - 1$,
$f(u_{2i+2,j}) = f(c_{2i+1}) + j$ for $i = 0, 1, 2, \ldots, s - 1, j = 1, 2, \ldots, a_{2i+2}$,
$f(u_{1,j}) = f(c_{2s+1}) + j, j = 1, 2, \ldots, a_1$,
$f(u_{2i+1,j}) = f(c_{2i}) + j$ for $i = 1, 2, \ldots, s - 1, j = 1, 2, \ldots, a_{2i+1}$. □

We next define two families of cubic graphs. Let n be a positive integer. The Möbius ladder (also known as the Möbius wheel) is the cycle C_{2n}, with n additional edges joining diagonally opposite vertices. We will denote this graph by M_{2n}, and its vertices by $v_1, v_2, \ldots, v_{2n}$. Then the edges are $v_1v_2, v_2v_3, \ldots, v_{2n}v_1$ of the cycle, and the n diagonals are $v_1v_{n+1}, v_2v_{n+2}, \ldots, v_nv_{2n}$. Figure 9 shows the Möbius ladder M_{2n} for $n = 3, 4$, drawn in both the circulant form and the ladder form.

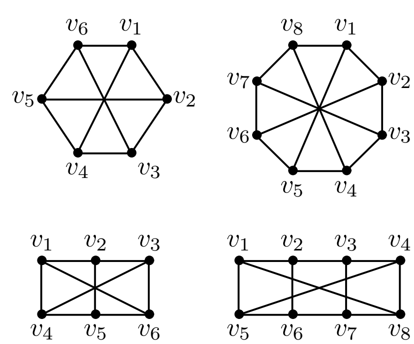


Figure 9: Möbius ladder for $n = 3, 4$.

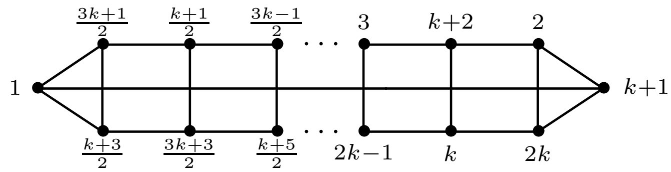
Observe that for odd n, M_{2n} is not 2-step Hamiltonian since it is bipartite. For even n, M_{2n} is tripartite.

Theorem 2.9. *For $m \geq 1$, M_{4m} is 2-step Hamiltonian.*

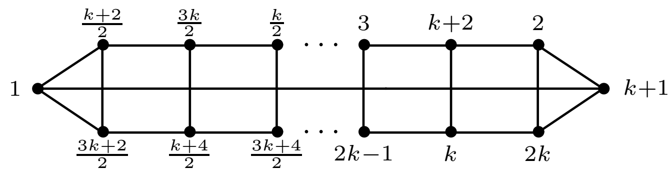
Proof. A 2-step Hamiltonian tour is given by the sequence $v_1, v_3, v_5, \ldots, v_{4m-1}, v_{2m}, v_{2m-2}, \ldots, v_2, v_{4m}, v_{4m-2}, \ldots, v_{2m+2}, v_1$. □

We now consider the cubic turtle shell graph, $TS(n)$, n even, with a given vertex labeling as shown in Figure 10.

Theorem 2.10. *The vertex labeling of the graph $TS(n)$ is a 2-step Hamiltonian tour for all $n = 2k, k \geq 3$.*



(a) $n = 2k \cong 0 \pmod 4$, $k \geq 4$ even



(b) $n = 2k \cong 2 \pmod 4$, $k \geq 3$ odd

Figure 10: Graph $TS(n), n$ even.

3 An Application

For a (p,q)-graph G, a labeling of the vertices and edges of G given by bijections $f : V(G) \to \{1,2,\ldots,p\}$ and $f^+ : E(G) \to \{p+1, p+2, \ldots, p+q\}$ is called a super-edge-magic (SEM) labeling if $f(u) + f(v) + f^+(uv)$ is a constant for every edge uv in $E(G)$. Such a graph is called SEM.

Theorem 3.11. *([4, 7]) A graph G is SEM if and only if it admits a bijection $f(V(G))$ such that $\{f(u) + f(v)\}$ consists of q consecutive integers.*

Observe that a 2-step Hamiltonian labeling for each odd cycle and the ring worm with odd cycle correspond to a vertex labeling that induces an edge labeling f^+ such that the edge labels form a sequence of consecutive integers. However, we are not able to find another 2-step Hamiltonian labeling that corresponds to a SEM labeling.

Problem 1. *Does there exist infinitely many families of 2-step Hamiltonian graphs whose labeling corresponds to a SEM labeling?*

The problem of determining whether a graph is Hamiltonian is NP-complete even for planar graphs. In 1972, Karp [12] proved that finding such a path in a directed or undirected graph is NP-complete. Later, Garey and Johnson [8] proved that the directed version restricted to planar graphs is also NP-complete, and the undirected version remains NP-complete even for cubic planar graphs. In 1980, Akiyama, Nishizeki, and Saito [1] showed that the problem is NP-complete even when restricted to bipartite graphs. We end this paper with the following conjecture and problem.

Conjecture 1. *The 2-step Hamiltonian problem for tripartite graphs is NP-complete.*

Problem 2. *Study the k-step Hamiltonicity of complete multipartite graph with certain edges deleted.*

References

[1] T.S.N. Akiyama, T. Nishizeki, NP-completeness of the Hamiltonian Cycle Problem for Bipartite Graphs, *Journal of Information Processing*, 3(1980), 73–76.

[2] J.-C. Bermond, Hamiltonian Graphs. In *Selected Topics in Graph Theory*. Edited by L. W. Beineke and R. J. Wilson. Academic, London(1978), 127–167.

[3] G. Chartrand and P. Zhang, Introduction to Graph Theory, Walter Rudin Student Series in Advanced Mathematics, McGraw-Hill, 2004.

[4] Z. Chen, On super edge-magic graphs, *J. of Comb. Math. Comb. Comput.,* 38 (2001), 53–64.

[5] V.V. Dimakopoulos, L. Palios and A.S. Poulakidas, On the Hamiltonicity of the Cartesian Product, available online: http://paragroup.cs.uoi.gr/Publications/120ipl2005.pdf (accessed October 2012).

[6] M.N. Ellingham, J.D. Horton, Non-Hamiltonian 3-connected Cubic Bipartite Graphs, *J. of Comb. Theory B*, 34(3)(1983), 350–353.

[7] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.,* 231(2001), 153–168

[8] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, New York, 1979.

[9] R. Gould, Advances on the Hamiltonian Problem, A Survey, *Graphs and Combinatorics*, 19(2003), 7–52.

[10] D. Holton and R.E.L. Aldred, Planar Graphs, Regular Graphs, Bipartite Graphs and Hamiltonicity, *Australasian J. of Combinatorics,* 20(1999), 111–131.

[11] A. Itai, C.H. Papadimitriou and J.L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing*, 11(4)(1982), 676–686

[12] R.M. Karp, Reducibility among combinatorial problems, *Complexity of Computer Computations*, (1972) 85–103.

[13] T.P. Kirkman, On the representation of polyhedra, *Phil. Trans. Royal Soc.*, 146(1856), 413–418.

[14] J. Moon and L. Moser, On Hamiltonian bipartite graphs, *Israel J. of Math.*, 1(3)(1963), 163–165.

Received: October 29, 2013; *Accepted*: February 26, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 188–196

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Product cordial labeling for alternate snake graphs

S. K. Vaidya[a,*] and N B Vyas[b]

[a]Department of Mathematics, Saurashtra University, Rajkot-360 005, Gujarat, India.

[b]Department of Mathematics, Atmiya Institute of Technology and Science, Rajkot-360 005, Gujarat, India.

Abstract

The product cordial labeling is a variant of cordial labeling. Here we investigate product cordial labelings for alternate triangular snake and alternate quadrilateral snake graphs.

Keywords: Cordial labeling, Product cordial labeling, Snake graph.

2010 MSC: 05C78.

1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. For standard terminology and notations we follow West[1].

If the vertices are assigned values subject to certain condition(s) then it is known as *graph labeling*. A mapping $f : V(G) \rightarrow \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the *label* of vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let $v_f(0)$ and $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$and $e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^*.

A binary vertex labeling of graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called *cordial* if admits a cordial labeling.

The concept of cordial labeling was introduced by Cahit[2] and in the same paper he investigated several results on this newly introduced concept. A latest survey on various graph labeling problems can be found in Gallian [3].

Motivated through the concept of cordial labeling, Sundaram *et al.* [4] have introduced a labeling which has the flavour of cordial lableing but absolute difference of vertex labels is replaced by product of vertex labels.

A binary vertex labeling of graph G with induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ defined by $f^*(e = uv) = f(u)f(v)$ is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *product cordial* if it admits a product cordial labeling.

Many researchers have explored this concept, Sundaram *et al.* [4] have proved that trees, unicyclic graphs of odd order, triangular snakes, dragons, helms and union of two path graphs are product cordial. They

*Corresponding author.
E-mail address: samirkvaidya@yahoo.co.in (S. K. Vaidya), niravbvyas@gmail.com(N B Vyas).

have also proved that a graph with p vertices and q edges with $p \geq 4$ is product cordial then $q < \frac{p^2-1}{4} + 1$. Vaidya and Dani[5] have proved that the graphs obtained by joining apex vertices of k copies of stars, shells and wheels to a new vertex are product cordial while Vaidya and Kanani[6] have reported the product cordial labeling for some cycle related graphs and investigated product cordial labeling for the shadow graph of cycle C_n. The same authors have investigated some new product cordial graphs in [7]. Vaidya and Vyas [8] have investigated product cordial labeling in the context of tensor product of some standard graphs. The product cordial labelings for closed helm, web graph, flower graph, double triangular snake and gear graph are investigated by Vaidya and Barasara [9].

An alternate triangular snake $A(T_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternately) to a new vertex v_i. That is every alternate edge of a path is replaced by C_3. *An alternate quadrilateral snake* $A(QS_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i, u_{i+1} (alternately) to new vertices v_i, w_i respectively and then joining v_i and w_i. That is every alternate edge of a path is replaced by a cycle C_4. A *double alternate triangular snake* $DA(T_n)$ consists of two alternate triangular snakes that have a common path. That is, double alternate triangular snake is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternately) to two new vertices v_i and w_i. A *double alternate quadrilateral snake* $DA(QS_n)$ consists of two alternate quadrilateral snakes that have a common path. That is, it is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternately) to new vertices v_i, x_i and w_i, y_i respectively and adding the edges $v_i w_i$ and $x_i y_i$.

2 Main results

Theorem 2.1. *$A(T_n)$ is product cordial where $n \not\equiv 3(mod\,4)$.*

Proof. Let $A(T_n)$ be alternate triangular snake obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternately) to new vertex v_i where $1 \leq i \leq n-1$ for even n and $1 \leq i \leq n-2$ for odd n. Therefore $V(A(T_n)) = \{u_i, v_j / 1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}$. We note that

$$|V(A(T_n))| = \begin{cases} \frac{3n}{2} & , n \equiv 0(\text{mod } 2); \\ \frac{3n-1}{2} & , n \equiv 1(\text{mod } 2). \end{cases},$$

$$|E(A(T_n))| = \begin{cases} 2n-1 & , n \equiv 0(\text{mod } 2); \\ 2n-2 & , n \equiv 1(\text{mod } 2). \end{cases}$$

We define $f : V(A(T_n)) \rightarrow \{0, 1\}$ as follows.

Case 1: $n \equiv 0(mod\,4)$

For $1 \leq i \leq n$:

$$f(u_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n}{2}; \\ 1 & , otherwise. \end{cases}$$

For $1 \leq i \leq \frac{n}{2}$:

$$f(v_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n}{4}; \\ 1 & , otherwise. \end{cases}$$

In view of above defined labeling patterns we have

$$v_f(0) = v_f(1) = \frac{3n}{4}\ , e_f(0) = e_f(1) + 1 = n$$

Case 2: $n \equiv 1(mod\,4)$

For $1 \leq i \leq n$:

$$f(u_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n-1}{2}; \\ 1 & , otherwise. \end{cases}$$

For $1 \leq i \leq \dfrac{n-1}{2}$:

$$f(v_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-1}{4}; \\ 1 & ,otherwise. \end{cases}$$

In view of above defined labeling patterns we have

$$v_f(0) + 1 = v_f(1) = \frac{3n+1}{4}, e_f(0) = e_f(1) = n-1$$

Case 3: $n \equiv 2(mod\, 4)$

For $1 \leq i \leq n$:

$$f(u_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-2}{2}; \\ 1 & ,otherwise. \end{cases}$$

For $1 \leq i \leq \dfrac{n-2}{2}$:

$$f(v_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-2}{4}; \\ 1 & ,otherwise. \end{cases}$$

For $i = \dfrac{n}{2}$

$$f(v_i) = 0$$

In view of above defined labeling patterns we have

$$v_f(0) + 1 = v_f(1) = \frac{3n+2}{4}, e_f(0) = e_f(1) + 1 = n$$

Thus, in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, $A(T_n)$ is a product cordial graph where $n \not\equiv 3(mod\, 4)$. □

Remark 2.1. *$A(T_n)$ is not product cordial graph for $n \equiv 3(mod\, 4)$. Because to satisfy the vertex condition for product cordial labeling it is essential to assign label 0 to $\frac{3n-1}{4}$ vertices out of $\frac{3n-1}{2}$ vertices. The vertices with label 0 will give rise to at least $n+2$ edges with label 0 and n edges with label 1. Consequently $|e_f(0) - e_f(1)| \geq 2$.*

Example 2.1. *$A(T_{10})$ and its product cordial labeling is shown in below Figure 1.*

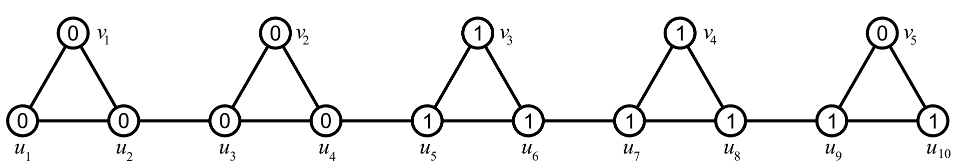


Figure 1

Theorem 2.2. *$A(QS_n)$ is product cordial where $n \not\equiv 2(mod\, 4)$.*

Proof. Let $A(QS_n)$ be an alternate quadrilateral snake obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i, u_{i+1} (alternately) to new vertices v_i, w_i respectively and then joining v_i and w_i where $1 \leq i \leq n-1$ for even n and $1 \leq i \leq n-2$ for odd n. Therefore $V(A(T_n)) = \{u_i, v_j, w_j / 1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}$. We note that

$$|V(A(QS_n))| = \begin{cases} 2n & ,n \equiv 0(\text{mod } 2); \\ 2n-1 & ,n \equiv 1(\text{mod } 2). \end{cases},$$

$$|E(A(QS_n))| = \begin{cases} \frac{5n-2}{2} & ,n \equiv 0(\text{mod } 2); \\ \frac{5n-5}{2} & ,n \equiv 1(\text{mod } 2). \end{cases}$$

We define $f : V(A(QS_n)) \rightarrow \{0, 1\}$ as follows.

Case 1: $n \equiv 0(mod\, 4)$

For $1 \leq i \leq n$:

$$f(u_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n}{2}; \\ 1 & ,otherwise. \end{cases}$$

For $1 \leq i \leq \dfrac{n}{2}$:

$$f(v_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n}{4}; \\ 1 & ,otherwise. \end{cases}$$

$$f(w_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n}{4}; \\ 1 & ,otherwise. \end{cases}$$

In view of above defined labeling patterns we have

$$v_f(0) = v_f(1) = n \, , e_f(0) = e_f(1) + 1 = \frac{5n}{4}$$

Case 2: $n \equiv 1(mod\, 4)$

For $1 \leq i \leq n$:

$$f(u_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-1}{2}; \\ 1 & ,otherwise. \end{cases}$$

For $1 \leq i \leq \dfrac{n-1}{2}$:

$$f(v_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-1}{4}; \\ 1 & ,otherwise. \end{cases}$$

$$f(w_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-1}{4}; \\ 1 & ,otherwise. \end{cases}$$

In view of above defined labeling patterns we have

$$v_f(0) + 1 = v_f(1) = n \, , e_f(0) = e_f(1) = \frac{5n-5}{4}$$

Case 3: $n \equiv 3(mod\, 4)$

For $1 \leq i \leq n$:

$$f(u_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-3}{2}; \\ 1 & ,otherwise. \end{cases}$$

For $1 \leq i \leq \dfrac{n-3}{2}$:

$$f(v_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-3}{4}; \\ 1 & ,otherwise. \end{cases}$$

$$f(w_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-3}{4}; \\ 1 & ,otherwise. \end{cases}$$

For $i = \dfrac{n-1}{2}$

$$f(v_i) = 0, f(w_i) = 0$$

In view of above defined labeling patterns we have

$$v_f(0) + 1 = v_f(1) = n \, , e_f(0) = e_f(1) + 1 = \frac{5n-3}{2}$$

Thus in each case we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

Hence $A(QS_n)$ is a product cordial graph where $n \not\equiv 2(mod\,4)$. □

Remark 2.2. *$A(QS_n)$ is not product cordial graph for $n \equiv 2(mod\,4)$. Because to satisfy the vertex condition for product cordial labeling it is essential to assign label 0 to n vertices out of $2n$ vertices. The vertices with label 0 will give rise to at least $\frac{5n+2}{4}$ edges with label 0 and $\frac{5n-6}{4}$ edges with label 1. Consequently $|e_f(0) - e_f(1)| \ge 2$.*

Example 2.2. *$A(QS_{11})$ and its product cordial labeling. Figure 2.*

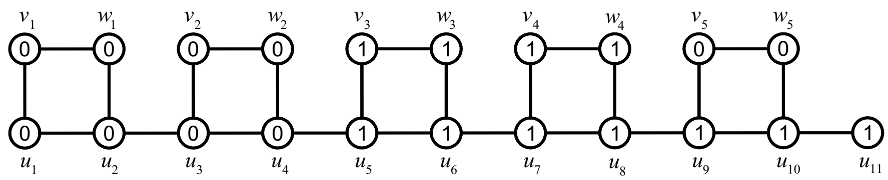


Figure 2

Theorem 2.3. *$DA(T_n)$ ia a product cordial graph where $n \not\equiv 2(mod\,4)$.*

Proof. Let G be a double alternate triangular snake $DA(T_n)$ then $V(G) = \{u_i, v_j, w_j / 1 \le i \le n, 1 \le j \le \lfloor \frac{n}{2} \rfloor\}$. We note that

$$|V(G)| = \begin{cases} 2n & , n \equiv 0(\text{mod } 2); \\ 2n-1 & , n \equiv 1(\text{mod } 2). \end{cases},$$

$$|E(G)| = \begin{cases} 3n-1 & , n \equiv 0(\text{mod } 2); \\ 3n-3 & , n \equiv 1(\text{mod } 2). \end{cases}$$

We define $f : V(A(QS_n)) \to \{0, 1\}$ as follows.
Case 1: $n \equiv 0(mod\,4)$

For $1 \le i \le n$:

$$f(u_i) = \begin{cases} 0 & , 1 \le i \le \frac{n}{2}; \\ 1 & , otherwise. \end{cases}$$

For $1 \le i \le \frac{n}{2}$:

$$f(v_i) = \begin{cases} 0 & , 1 \le i \le \frac{n}{4}; \\ 1 & , otherwise. \end{cases}$$

$$f(w_i) = \begin{cases} 0 & , 1 \le i \le \frac{n}{4}; \\ 1 & , otherwise. \end{cases}$$

In view of above defined labeling patterns we have

$$v_f(0) = v_f(1) = n \, , e_f(0) = e_f(1) + 1 = \frac{3n}{2}$$

Case 2: $n \equiv 1(mod\,4)$

For $1 \le i \le n$:

$$f(u_i) = \begin{cases} 0 & , 1 \le i \le \frac{n-1}{2}; \\ 1 & , otherwise. \end{cases}$$

For $1 \le i \le \frac{n-1}{2}$:

$$f(v_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-1}{4}; \\ 1 & ,otherwise. \end{cases}$$

$$f(w_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-1}{4}; \\ 1 & ,otherwise. \end{cases}$$

In view of above defined labeling patterns we have

$$v_f(0) + 1 = v_f(1) = n \,, e_f(0) = e_f(1) = \frac{3n-3}{2}$$

Case 3: $n \equiv 3(mod\, 4)$

For $1 \leq i \leq n-1$:

$$f(u_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-3}{2}; \\ 1 & ,otherwise. \end{cases}$$

$$f(u_n) = 0$$

For $1 \leq i \leq \frac{n-3}{2}$:

$$f(v_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-3}{4}; \\ 1 & ,otherwise. \end{cases}$$

$$f(w_i) = \begin{cases} 0 & ,1 \leq i \leq \frac{n-3}{4}; \\ 1 & ,otherwise. \end{cases}$$

For $i = \frac{n-1}{2}$

$$f(v_i) = 1, f(w_i) = 0$$

In view of above defined labeling patterns we have

$$v_f(0) + 1 = v_f(1) = n \,, e_f(0) = e_f(1) = \frac{3n-3}{2}$$

Thus, in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, $DA(T_n)$ is a product cordial graph where $n \not\equiv 2(mod\, 4)$. □

Remark 2.3. *$DA(T_n)$ is not product cordial graph for $n \equiv 2(mod\, 4)$. Because to satisfy the vertex condition for product cordial labeling it is essential to assign label 0 to n vertices out of $2n$ vertices. The vertices with label 0 will give rise to at least $\frac{3n+2}{2}$ edges with label 0 and $\frac{3n-4}{2}$ edges with label 1. Consequently $|e_f(0) - e_f(1)| \geq 2$.*

Example 2.3. *$DA(T_{11})$ and its product cordial labeling is shown in Figure 3.*

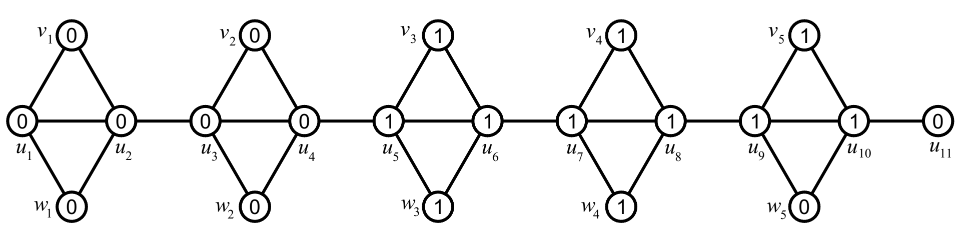


Figure 3

Theorem 2.4. *$DA(QS_n)$ is a product cordial graph where $n \not\equiv 2(mod\, 4)$.*

Proof. Let G be a double alternate quadrilateral snake $DA(T_n)$ then

$V(G) = \{u_i, v_j, w_j, x_j, y_j / 1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}$. We note that

$$|V(G)| = \begin{cases} 3n & , n \equiv 0(\text{mod } 2); \\ 3n-2 & , n \equiv 1(\text{mod } 2). \end{cases},$$

$$|E(G)| = \begin{cases} 4n-1 & , n \equiv 0(\text{mod } 2); \\ 4n-4 & , n \equiv 1(\text{mod } 2). \end{cases}$$

We define $f : V(G) \rightarrow \{0,1\}$ as follows.
Case 1: $n \equiv 0(mod\, 4)$

For $1 \leq i \leq n$:

$$f(u_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n}{2}; \\ 1 & , otherwise. \end{cases}$$

For $1 \leq i \leq \dfrac{n}{2}$:

$$f(v_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n}{4}; \\ 1 & , otherwise. \end{cases}$$

$$f(w_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n}{4}; \\ 1 & , otherwise. \end{cases}$$

$$f(x_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n}{4}; \\ 1 & , otherwise. \end{cases}$$

$$f(y_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n}{4}; \\ 1 & , otherwise. \end{cases}$$

In view of above defined labeling patterns we have

$$v_f(0) = v_f(1) = \frac{3n}{2}, e_f(0) = e_f(1) + 1 = 2n$$

Case 2: $n \equiv 1(mod\, 4)$

For $1 \leq i \leq n$:

$$f(u_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n-1}{2}; \\ 1 & , otherwise. \end{cases}$$

For $1 \leq i \leq \dfrac{n-1}{2}$:

$$f(v_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n-1}{4}; \\ 1 & , otherwise. \end{cases}$$

$$f(w_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n-1}{4}; \\ 1 & , otherwise. \end{cases}$$

$$f(x_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n-1}{4}; \\ 1 & , otherwise. \end{cases}$$

$$f(y_i) = \begin{cases} 0 & , 1 \leq i \leq \frac{n-1}{4}; \\ 1 & , otherwise. \end{cases}$$

In view of above defined labeling patterns we have

$$v_f(0) + 1 = v_f(1) = \frac{3n-1}{2}, e_f(0) = e_f(1) = 2n - 2$$

Case 3: $n \equiv 3(mod\, 4)$

For $1 \leq i \leq n-1$:

$$f(u_i) = \begin{cases} 0 & ,1 \le i \le \frac{n-3}{2}; \\ 1 & ,otherwise. \end{cases}$$

$f(u_n) = 0$

For $1 \le i \le \dfrac{n-3}{2}$:

$$f(v_i) = \begin{cases} 0 & ,1 \le i \le \frac{n-3}{4}; \\ 1 & ,otherwise. \end{cases}$$

$$f(w_i) = \begin{cases} 0 & ,1 \le i \le \frac{n-3}{4}; \\ 1 & ,otherwise. \end{cases}$$

$$f(x_i) = \begin{cases} 0 & ,1 \le i \le \frac{n-3}{4}; \\ 1 & ,otherwise. \end{cases}$$

$$f(y_i) = \begin{cases} 0 & ,1 \le i \le \frac{n-3}{4}; \\ 1 & ,otherwise. \end{cases}$$

For $i = \dfrac{n-1}{2}$

$f(v_i) = 1, f(w_i) = 1$

$f(x_i) = 0, f(y_i) = 0$

In view of above defined labeling patterns we have

$$v_f(0) + 1 = v_f(1) = \frac{3n-1}{2}, e_f(0) = e_f(1) = 2n - 2$$

Thus, in each case we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

Hence, $DA(QS_n)$ is a product cordial graph where $n \not\equiv 2(mod\,4)$. □

Remark 2.4. *$DA(QS_n)$ is not product cordial graph for $n \equiv 2(mod\,4)$. Because to satisfy the vertex condition for product cordial labeling it is essential to assign label 0 to $\frac{3n}{2}$ vertices out of $3n$ vertices. The vertices with label 0 will give rise to at least $2n+1$ edges with label 0 and $2n-2$ edges with label 1. Consequently $|e_f(0) - e_f(1)| \ge 2$.*

Example 2.4. *$DA(QS_8)$ and its product cordial labeling is shown in Figure 4.*

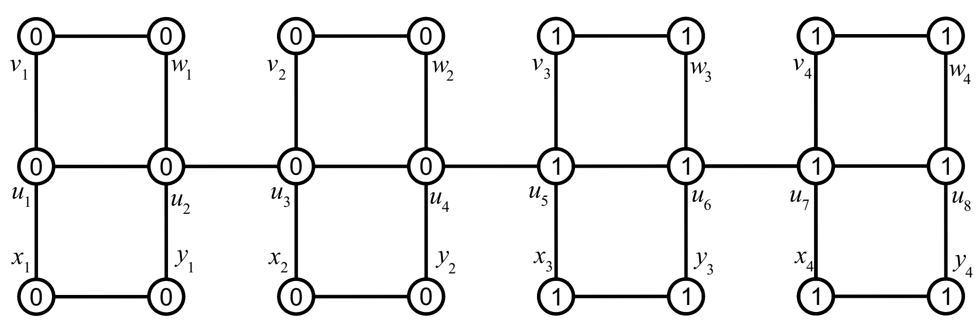


Figure 4

3 Concluding remarks

The labeling of discrete structures is one of the potential areas of research. Here we investigate product cordial labeling for some alternate snake graphs. To derive similar results for other graph families is an open area of research.

4 Acknowledgement

Our thanks are due to the anonymous referees for their constructive comments on the first draft of this paper.

References

[1] D. B. West, *Introduction to Graph Theory*, Printice-Hall of India, 2001.

[2] I. Cahit, Cordial Graphs: A weaker version of graceful and harmonious Graphs, *Ars Combinatoria*, 23(1987), 201-207.

[3] J. A. Gallian, A dynamic survey of graph labeling, *The Electronics Journal of Combinatorics*, 16(#DS6), 2013.

[4] M. Sundaram, R. Ponraj and S. Somsundaram, Product cordial labeling of graphs, *Bull. Pure and Applied Sciences(Mathematics and Statistics)*, 23E(2004), 155-163.

[5] S. K. Vaidya and N. A. Dani, Some new product cordial graphs, *Journal of App. Comp. Sci. Math.*, 8(4)(2010), 63-66.

[6] S. K. Vaidya and K. K. Kanani, Some cycle related product cordial graphs, *International Journal of Algorithms*, 3(1)(2010), 109-116.

[7] S. K. Vaidya and K. K. Kanani, Some new product cordial graphs, *Mathematics Today*, 27 (2011), 64-70.

[8] S. K. Vaidya and N. B. Vyas, Product Cordial Labeling in the Context of Tensor Product of Graphs, *Journal of Mathematics Research*, 3(3)(2011), 83-88.

[9] S. K. Vaidya and C. M. Barasara, Further results on product cordial labeling, *International Journal of Math. Combin.*, 3(2012), 64-71.

Received: December 6, 2013; *Accepted*: January 11, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 197–202

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On fuzzy soft metric spaces

Thangaraj Beaula[a,*] and Christinal Gunaseeli[b]

[a,b]Department of Mathematics, TMBL College, Porayar-609307, Tamil Nadu.

Abstract

In this paper an idea of fuzzy soft point is introduced and using it fuzzy soft metric space is established .The concepts like fuzzy soft open balls and fuzzy soft closed balls are introduced. Some properties of fuzzy soft metric spaces are developed.

Keywords: Fuzzy set, fuzzy metric space.

2010 MSC: 03E72.

1 Introduction

In daily life, the problems in many fields deal with uncertain data and are not successfully modelled in classical mathematics. There are two types of mathematical tools to deal with uncertainities namely fuzzy set theory introduced by Zadeh [12] and the theory of soft sets initiated by Molodstov [8] which helps to solve problems in all areas. Maji et al. [6] introduced several operations in soft sets and has also coined fuzzy soft sets. Chang [4] has introduced the theory of fuzzy topological spaces and Sanjay Roy et al.[10] has defined open and closed sets on fuzzy topological spaces.

In this paper we have defined fuzzy soft metric space in terms of fuzzy soft points. Fuzzy soft open ball and fuzzy soft closed ball are introduced in fuzzy soft metric space. Fuzzy soft Hausdorff metric is also defined and further some equivalent conditions in a fuzzy soft metric space is developed. Some other properties of fuzzy soft metric spaces are also established.

2 Preliminaries

Definition 2.1. *A fuzzy soft point F_e over (U, t) is a special fuzzy soft set defined by $F_e(a) = \mu F_e$, where $\mu F_e \neq \sigma$ $= \sigma$ if $a \neq e$*

Definition 2.2. *Let F_A be a fuzzy soft set over (U, E) and G_e be a fuzzy soft point over (U, E) then $G_e \in F_A$ if and only if $\mu_{G_e} \subseteq \mu_{F_{A^e}} = F_A(e)$ that is $\mu_{G_e}(x) \leq \mu_{F_{A^e}} \forall x \in U$*

Definition 2.3. *Two fuzzy soft points F_{e1}, F_{e2} are said to be equal if $\mu_{F_{e1}}(a) = \mu_{F_{e2}}(a)$. Thus $F_{e1} \neq F_{e2}$ if and only $\mu_{F_{e1}}(a) \neq \mu_{F_{e2}}(a)$.*

Proposition 2.1. *The union of any collection of fuzzy soft points can be considered as a fuzzy soft set and every fuzzy soft set can be expressed as the union of all fuzzy soft points.*

$$F = \left[\bigcup_{F_e \in F_A} F_e \right]$$

*Corresponding author.

E-mail address: edwinbeaula@yahoo.co.in (Thangaraj Beaula).

Proposition 2.2. *Let F_A, G_B be two fuzzy soft sets then $F_A \subseteq G_B$ if and only if $F_e \in F_A$ implies $F_e \in G_B$ and hence $F_A = G_B$ if and only if $F_e \in F_A$f and only if $F_e \in G_B$.*

Note 2.1. *Let $\mathcal{F}$ be a collection of fuzzy soft points then the fuzzy soft set generated by $\mathcal{F}$ be denoted by FSG($\mathcal{F}$) and the collection of all fuzzy soft points of a fuzzy soft set F_A be denoted by FSC($\mathcal{F}$).*

3 Fuzzy soft metric and fuzzy soft metric space

Let $A \subseteq E$ and let F_E be the absolute fuzzy soft set that is $F_E(e) = \tau$ for all $e \in E$. Let (A)* denote the set of all non negative fuzzy soft real numbers. The fuzzy soft metric using fuzzy soft points is defined as follows:

Definition 3.1. *A mapping $d : FSC(F_E) \times FSC(F_E) \to \mathbb{R}(A)^*$ is said to be a fuzzy soft metric on FE if d satisfies the following conditions.*
$(FSM_1) d(F_{e^1}, F_{e^2}) \geq 0$ *for all* $F_{e^1}, F_{e^2} \in FSM(F_E)$
$(FSM_2) d(F_{e^1}, F_{e^2}) = 0$ *if and only if* $F_{e^1} = F_{e^2}$
$(FSM_3) d(F_{e^1}, F_{e^2}) = d(F_{e^2}, F_{e^1})$ *for all* $F_{e^1}, F_{e^2} \in FSM(F_E)$
$(FSM_4) d(F_{e^1}, F_{e^3}) = d(F_{e^1}, F_{e^2}) + d(F_{e^2}, F_{e^3})$ *for all* $F_{e^1}, F_{e^2}, F_{e^3} \in FSM(F_E)$
The fuzzy soft set F_E with the fuzzy soft metric d is called the fuzzy soft metric space and is denoted by (F_E, d).

Definition 3.2. *Let $A \subseteq R$ and $E \subseteq R$, let F_E be the absolute fuzzy soft set that is $F_E(e) = \tau$ for all $e \in E$.*
Define $d : FSC(F_E) \times FSC(F_E) \to (\mathbb{R}(A))$ by $d(F_{e^1}, F_{e^2}) = \inf\{ |\mu_{F_{e1}}(a) - \mu F_{e^2}(a)| / a \in A\}$ for all $F_{e^1}, F_{e^2} \in FSM(F_E)$ then d is a fuzzy soft metric over F_E, let us verify $(FSM_1) - (FSM_5)$

(i) $d(F_{e^1}, F_{e^2}) \geq 0$ *for all* $F_{e^1}, F_{e^2} \in FSM(F_E)$

(ii) $d(F_{e^1}, F_{e^2}) = \inf\{ |\mu_{F_{e1}}(a) - \mu_{F_{e2}}(a)| / a \in A\}$
$=\inf\{ |\mu_{F_{e2}}(a) - \mu F_{e^1}(a)|\}$
$= d(F_{e^2}, F_{e^1})$

(iii) $d(F_{e^1}, F_{e^3}) = \inf\{ |\mu_{F_{e1}}(a) - \mu_{F_{e3}}(a)| / a \in A\}$
$= \inf\{ |\mu_{F_{e1}}(a) - \mu_{F_{e2}}(a) + \mu_{F_{e2}}(a) - \mu_{F_{e3}}| / a \in A\}$
$\leq \inf\{ |\mu_{F_{e1}}(a) - \mu_{F_{e2}}(a) + \mu_{F_{e2}}(a) - \mu_{F_{e3}}|\}$
$= d(F_{e^1}, F_{e^2}) + d(F_{e^2}, F_{e^3})$ *Thus d is a fuzzy soft metric on $FSC(F_E)$.*

Definition 3.3. *Let (F_E, d) be a fuzzy soft metric space and G_E, be a fuzzy soft subspace of F_E then distance between a fuzzy soft point F_e and G_E is defined by*
$d(F_e, G_E) = \sup\{ d(F_e, G_{e'}) /$ *for every fuzzy soft point $G_{e'}$ in G_E*$\}$

Definition 3.4. *A fuzzy soft subspace G_E is said to be a bounded if there exists a positive number M such that $d(G_{e^1}, G_{e^2}) \leq M$ for all $G_{e^1}, G_{e^2} \in G_E$.*
The diameter of the subspace G_E is defined as
diam $G_E = \sup\{ d(G_{e^1}, G_{e^2}) / G_{e^1}, G_{e^2} \in G_E\}$

Definition 3.5. *(F_E, d) be a fuzzy soft metric space and $\tilde{\tilde{t}}$ be a fuzzy soft real number. An open ball centered at fuzzy soft point $F_e \in F_E$ and radius t is a collection of all fuzzy soft points G_e of F_E such that $d(G_e, F_e) < t$.*
It is denoted by $\tilde{\tilde{B}}(F_e, \tilde{\tilde{t}})$. i.e. $B(F_e, t) = \{ G_e \in F_E / d(G_e, F_e) < t\}$
The fuzzy soft closed ball denoted by $B[F_e, t] = \{ G_e \in F_E / d(G_e, F_e) \leq t\}$

Definition 3.6. *Let (F_E, d) be a fuzzy soft metric space having atleast two fuzzy soft points (F_E, d) is said to be Hausdorff if for any points F_{e^1}, F_{e^2} in F_E such that $d(F_{e^1}, F_{e^2}) > 0$, then there exists two fuzzy soft open ball $B(F_{e^1}, t)$ and $B(F_{e^2}, t)$ with centre F_{e^1}, F_{e^2} and radius $\tilde{\tilde{t}}$ such that $\tilde{\tilde{B}}(F_{e^1}, t) \cap \tilde{\tilde{B}}(F_{e^2}, t) = \phi$.*

Theorem 3.1. *Every fuzzy soft metric space is Hausdorff.*

Proof. Let (F_E, d) be a fuzzy soft metric space having atleast two points. Let F_{e^1}, F_{e^2} be two fuzzy soft points in F_E such that $d(F_{e^1}, F_{e^2}) > 0$. Choose any fuzzy soft real number $\tilde{\tilde{t}}$ such that $0 < t < \frac{1}{2} d(F_{e^1}, F_{e^2})$. Consider two fuzzy soft open balls $\tilde{\tilde{B}}(F_{e_1}, \tilde{\tilde{t}}) = \{ F_e' : d(F_e', F_e) < \tilde{\tilde{t}}\}$ and $\tilde{\tilde{B}}(F_{e_2}, \tilde{\tilde{t}}) = \{ F_e'' : d(F_e, F_e'') < \tilde{\tilde{t}}\}$

Suppose $F_{e_3} \in \tilde{\tilde{B}}(F_{e^1}, \tilde{\tilde{t}}) \cap \tilde{\tilde{B}}(F_{e^2}, \tilde{\tilde{t}})$ then

$F_{e_3} \in \tilde{\tilde{B}}(F_{e^1}, \tilde{\tilde{t}}) \implies d(F_{e_1}, F_{e_3}) < \tilde{\tilde{t}}$
$F_{e_3} \in \tilde{\tilde{B}}(F_{e^2}, \tilde{\tilde{t}}) \implies d(F_{e_2}, F_{e_3}) < \tilde{\tilde{t}}$
By (FSM_4)
$d(F_{e_1}, F_{e_2}) \leq d(F_{e_1}, F_{e_3}) + d(F_{e_3}, F_{e_2})$
$< \tilde{\tilde{t}} + \tilde{\tilde{t}} = 2\tilde{\tilde{t}}$
therefore $\tilde{\tilde{t}} > \frac{1}{2}d(F_{e^1}, F_{e^2})$ which contradicts the hypothesis. So clearly, $\tilde{\tilde{B}}(F_{e^1}, t) \cap \tilde{\tilde{B}}(F_{e^2}, t) = \phi$ and hence (F_E, d) is Hausdorff. □

Definition 3.7. *Let $\{ F_{(e,n)1} \}_n$ be a sequence of fuzzy soft points in a fuzzy soft metric space (F_E, d). The sequence $\{ F_{(e,n)1} \}_n$ is said to converge in (F_E, d) if there is a fuzzy soft point $F^{\sigma}_{e'} \in F_E$ such that . $d(F_{(e,n)}, F^{\sigma}_{e'}) \to \tilde{\tilde{0}}$ as $n \to \infty$*
That is for every $\tilde{\tilde{\varepsilon}} > \tilde{\tilde{0}}$ there exist a positive integer $N = N(\tilde{\tilde{\varepsilon}})$ such that whenever $d(F_{(e,n)}, F^{\sigma}_{e'}) \geq \tilde{\tilde{\varepsilon}}$.
It is denoted as $\lim_{n\to\infty} F_{(e,n)1} = F^{\sigma}_{e'}$

Definition 3.8. *A sequence $\{ F_{(e,n)1} \}_n$ of fuzzy soft points in (F_E, d) is a Cauchy sequence if to every $\tilde{\tilde{\varepsilon}} > \tilde{\tilde{0}}$, there exists N a positive integer such that $d(F_{(e,i)1}, F_{(e,j)1} < \tilde{\tilde{\varepsilon}}$ for all $i, j \geq N$ i.e. $d(F_{(e,i)1}, F_{(e,j)1}) < \tilde{\tilde{\varepsilon}} \to 0$ as $i, j \to \infty$*

Definition 3.9. *A fuzzy soft metric space (F_E, d) is said to complete if every Cauchy sequence in F_E converges to some fuzzy soft point of F_E.*

Definition 3.10. *Let F_{CB} be the soft set of non-empty closed and bounded subspace of the soft metric space (F_E, d). Define a function on $F_{CB} \times F_{CB}$ as*
$H_d(R_A, R_B) = \max\{ \sup_{R^a_e \in R_A} d(R^a_e, R_B), \sup_{R^b_e \in R_B} d(R_A, R^b_e)\}$

Theorem 3.2. *For a fuzzy soft metric space (F_E, d) the following are equivalent.*
a) For each sequence of fuzzy soft real numbers $\{ \tilde{\tilde{t_n}} : n \in N\}$, there a sequence of $\{ F^{a_n}_{e_n}\}$ finite fuzzy soft points of F_E such that each finite fuzzy soft set $G_E \subset F_E$ is contained in $\{ F^{a_n}_e, \tilde{\tilde{t_n}}\}$ for some n.
b) For each sequence of fuzzy soft real numbers $\{ \tilde{\tilde{t_n}} : n \in N\}$ there is a sequence of $\{ F^{a_n}_{e_n}\}$ of finite fuzzy points of F_E such that for each finite fuzzy soft set $G_E \subset F_E$ contained in $\bigcup_{n_k \leq n \leq n_{k+1}} B(F^{a_n}_e, \tilde{\tilde{t_n}})$ for some k.

Proof. Let us prove $(b) \implies (a)$
Let $\{ \tilde{\tilde{t_n}} : n \in N\}$ be a sequence of fuzzy soft real numbers. For each $n \in N$
Let $\tilde{\tilde{S_n}} = \min\{ \tilde{\tilde{t_i}} = \mu_{\tilde{\tilde{t_i}}}$ for $i \leq n\}$
Applying (b) to $\{ \tilde{\tilde{S_n}} : n \in N\}$. Then there is an increasing sequence $n_1 < n_2 < ...$ in N. Such that each finite fuzzy point set $G_E \subset F_E$ is contained in $\bigcup_{n_k \leq n \leq n_{k+1}} B(F^{a_i}_e, \tilde{\tilde{S_i}})$ for some k.
Let $H^A_E = \bigcup_{i<n} F^{a_i}_e$ for $n < n_1$
$H^n_E = \bigcup_{n_k \leq i \leq n_{k+1}} F^{a_i}_e$, for each n and $n_k \leq n \leq n_{k+1}$
Let us prove that the sequence $\{ H^n_E : n \in N\}$ satisfies (a). Let S_E be a finite subset of F_E choose $\mathcal{K}$ such that $S_E \subset \bigcup_{n_k \leq i \leq n_{k+1}} B(F_{e_i}, \{ \tilde{\tilde{S_n}})$
Let $H^n_E = \bigcup_{n_k \leq i \leq n_{k+1}} F_{e_i}$, where $n_k \leq n \leq n_{k+1}$
Then for each $F_e \in S_E$ there is j, $n_k \leq j \leq n_{k+1}$ and F_{e_i} with $F_e \in B(F_{e_i}, \tilde{\tilde{S_j}})$ we also have $B(F_{e_i}, \tilde{\tilde{S_j}})$ and since $F_{e_j} \in H^n_E$. We have $F_e \in B(H_{E_j}, \tilde{\tilde{t_j}})$
Proof for (a) implies (b) is trivial. □

Theorem 3.3. *Cartesian product of two fuzzy soft hausdorff metric spaces is hausdroff.*

Proof. Let (F_E, d) and (G_E, d) be two fuzzy soft hausdorff metric spaces. Let $(F_{e_1}, G^1_{e_1})$ and $(F_{e_2}, G^1_{e_2})$ be points in $F_E \times G_E$, in such a way that $d((F_{e_1}, F_{e_2}) > \tilde{\tilde{0}}$ and $d((G^1_{e_1}, G^1_{e_2}) > \tilde{\tilde{0}}$. So either $F_{e_1} \neq G^1_{e_1}$ or $F_{e_2} \neq G^1_{e_2}$
Suppose $F_{e_1} \neq F_{e_2}$, since (F_E, d) is a fuzzy soft hausdorff metric space, there exists two fuzzy soft open balls $\tilde{\tilde{B}}(F_{e_1}, \tilde{\tilde{t_1}})$ and $\tilde{\tilde{B}}(F_{e_2}, \tilde{\tilde{t_2}})$ where $\tilde{\tilde{t_1}}$ and $\tilde{\tilde{t_2}}$ are fuzzy soft real numbers such that $\tilde{\tilde{0}} < \tilde{\tilde{t_1}} < \frac{1}{2}d(F_{e_1}, F_{e_2})$ and $\tilde{\tilde{0}} < \tilde{\tilde{t_2}} < \frac{1}{2}d(F_{e_1}, F_{e_2})$ and $\tilde{\tilde{B}}(F_{e_1}, \tilde{\tilde{t_1}}) \cap \tilde{\tilde{B}}(F_{e_2}, \tilde{\tilde{t_2}})$ is empty. Since every metric space is metrizeable, each F_E and G_E are open and so.
$\tilde{\tilde{B}}(F_{e_1}, \tilde{\tilde{t_1}}) \times F_E$ and $\tilde{\tilde{B}}(F_{e_2}, \tilde{\tilde{t_2}}) \times G_E$ are the fuzzy soft open sets on $F_E \times G_E$.
Hence $(\tilde{\tilde{B}}(F_{e_1}, \tilde{\tilde{t_1}}) \times F_E) \cap (\tilde{\tilde{B}}(F_{e_2}, \tilde{\tilde{t_2}}) \times G_E) = \phi$ □

Theorem 3.4. *Every convergent sequence in a fuzzy soft hausdorff metric space (F_E, d) has a unique limit.*

Proof. Assume $\{ F_{(e,n)m}\}$ a sequence of fuzzy soft points in the fuzzy soft metric space converges to $F_{e'_{\sigma_1}}$ and $F_{e'_{\sigma_2}}$.

Since (F_E, d) is hausdorff there exist $\tilde{\tilde{t_1}}$ and $\tilde{\tilde{t_2}}$ fuzzy soft real numbers such that $\tilde{\tilde{B}}(F_{e_{\sigma_1}}, \tilde{\tilde{t_1}})$ and $\tilde{\tilde{B}}(F_{e_{\sigma_1}}, \tilde{\tilde{t_2}})$ are disjoint.

Since $\{ F_{(e,n)m}\}$ converges to $F_{e'_{\sigma_1}}$ there exists a positive integer N_1 such that $d(F_{(e,n)m}, F_{e_{\sigma_1}}) < \tilde{\tilde{\varepsilon_1}}$ where $\tilde{\tilde{\varepsilon_1}} < \tilde{\tilde{t_1}}$ for all $n \leq N_1$, Again since $\{ F_{(e,n)m}\}$ converges to $F_{e_{\sigma_2}}$ there exist a positive integer N_2 such that $d(F_{(e,n)m}, F_{e_{\sigma_2}}) < \tilde{\tilde{\varepsilon_2}}$ where $\tilde{\tilde{\varepsilon_2}} < \tilde{\tilde{t_2}}$ for all $n \leq N_2$,

Let $N = max\{ N_1, N_2\}$ then for all $n \geq N, F_{(e,n)m} \in \tilde{\tilde{B}}(F_{e_{\sigma_1}}, \tilde{\tilde{\varepsilon_1}})$ and $F_{(e,n)m} \in \tilde{\tilde{B}}(F_{e_{\sigma_1}}, \tilde{\tilde{\varepsilon_2}})$ which is a contradiction to the fact that $\tilde{\tilde{B}}(F_{e_{\sigma_1}}, \tilde{\tilde{\varepsilon_1}})$ and $\tilde{\tilde{B}}(F_{e_{\sigma_1}}, \tilde{\tilde{\varepsilon_2}})$ are disjoint.

Suppose $\tilde{\tilde{\varepsilon_1}} > \tilde{\tilde{t_1}}$ and $\tilde{\tilde{\varepsilon_2}} > \tilde{\tilde{t_2}}$, then $F_{(e,n)m} \in \tilde{\tilde{B}}(F_{e_{\sigma_1}}, \tilde{\tilde{\varepsilon_1}})$ for all $n \geq N_1$ and for all $F_{(e,n)m} \in \tilde{\tilde{B}}(F_{e_{\sigma_1}}, \tilde{\tilde{\varepsilon_2}})$ for all $n \geq N_2$ cannot happen and so again we arrive at a contradiction. □

Definition 3.11. *F_A is called fuzzy soft open if and only if for every $G_e \in F_A$ there exists $r > 0$. Such that $B(G_e, r) \subseteq F_A$ where $B(G_e, r) = \{ H_e : d(H_e, G_e) < r\}$ and $B(G_e, r)$ is called a sphere with center G_e and radius r.*

Definition 3.12. *Let $\delta = \{ F_A : F_A$is fuzzy soft open$\}$ which satisfies the axioms of A fuzzy soft set F_A is called a neighbourhood of a fuzzy soft point G_e if and only if there exists $H_B \in \delta$ such that $G_e \in H_B \subseteq F_A$.*

Theorem 3.5. *Let $\gamma_{G_e} = \{ F_A : F_A$ is a neighbourhoood of the fuzzy soft point $G_e\}$ then the family γ_{G_e} at any point G_e over (U, E) satisfies the following properties.*
(i) if $F_A \in \gamma_{G_e}$ then $G_e \in F_A$
(ii) if $F_A \in \gamma_{G_e}$ and $F_A \subseteq$ then $H_B \in \gamma_{G_e}$
(iii) if $F_A, H_B \in \gamma_{G_e}$ then $F_A \cap H_B \in \gamma_{G_e}$

Proof. (i) if $F_A \in \gamma_{G_e}$ then F_A is a neighbourhood of the fuzzy soft point G_e. so, there exists a fuzzy soft open set H_B containing G_e and $H_B \subset F_A$. Thus there exists a $r > 0$ such that $B(G_e, r) \subset H_B$.

Hence $B(G_e, r) \subset H_B \subset F_A$ and so $G_e \in F_A$

(ii) Given $F_A \in \gamma_{G_e}$ and $F_A \subseteq H_B$ there exists fuzzy soft open set V_c containing G_e and $V_c \subset F_A$ also there exists $r > 0$, such that $B(G_e, r) \subset V_c \subset F_A$ by the given condition, $B(G_e, r) \subset V_c \subset F_A \subseteq H_B$ and so $B(G_e, r) \subset V_c \subseteq H_B$ implies that $H_B \in \gamma_{G_e}$

(iii) if $F_A, H_B \in \gamma_{G_e}$, then there exists fuzzy soft open sets V_c and W_D such that $V_c \subset F_A$ and $W_D \subset H_B$.

Thus there exists $r_1 > 0$ such that $B(G_e, r_1) \subset V_c \subset F_A$ and there exists $r_2 > 0$ such that $B(G_e, r_2) \subset W_D \subset H_B$ choose $r = min\{ r_1, r_2\}$

then $B(G_e, r_1) \subset V_c \cap W_D \subset F_A \cap H_B$

□

Definition 3.13. *A dual fuzzy soft point is a fuzzy soft point F_{e^d} of F_e over (U, E) where $F_{e^d}(a) = 1 - \mu_{F_e}$ if $a = e$, where $\mu_{F_e}\bar{0}$*
= 1 if $a \neq e$

Definition 3.14. *A fuzzy soft matric is a mapping $d : FSC(F_A) \times FSC(F_A) \rightarrow \mathbb{R}(A)^*$ on F_A which is continuous for membership grade and satisfies for all $F_{e_1}, F_{e_2}, F_{e_3} \in FSC(F_A)$ the following axioms*
(i) if $F_{e_2} \subset F_{e_1}$ then $d(F_{e_1}, F_{e_2}) = 0$
(ii) $d(F_{e_1}, F_{e_3}) \leq d(F_{e_1}, F_{e_2}) + d(F_{e_2}, F_{e_3})$
(iii) $d(F_{e_1}, F_{e_2}) = d(F_{d_{e_2}}, F_{d_{e_3}})$
(iv) if $F_{e_2} \subseteq F_{e_1}$, then $\bar{d}(F_{e_1}, F_{e_2}) > 0$
if in the definition mentioned above, if (iv) is omitted then d is called a soft fuzzy pseudo metric. if (iii) and (iv) are omitted then d is called a fuzzy soft quasi metric.

Definition 3.15. *Let d be a fuzzy soft quasi metric on the fuzzy soft set F_A then for any $F_e \in FSC(F_A)$ and $\varepsilon > 0$ then $B_\varepsilon(F_e) = \cup\{ F_{e'} : d(F_e, F_{e'}) < \varepsilon\}$ is a fuzzy soft set which is called an ε - open ball of F_e and $B_\varepsilon(F_e) = \cup\{ F_{e'} : d(F_e, F_{e'}) < \varepsilon\}$ called a fuzzy soft closed ball of F_e.*

Definition 3.16. *The family of all fuzzy soft open balls is known as the base of a fuzzy soft topology τ_F for F_A corresponding to fuzzy soft (quasi, pseudo). This is called as a fuzzy soft metric topology and (F_A, A, τ_F) is a fuzzy soft (quasi, pseudo) metric space.*

Theorem 3.6. *Let (F_A, A, τ_F) be a fuzzy soft quasi metric space then for any fuzzy soft point $F_e \in F_A$ and $\varepsilon > 0$, then the fuzzy soft ε - open ball. $B_\varepsilon(F_E)$ is a fuzzy soft open neighbourhood of F_e.*

Proof. We have to show that $F_e \in B_\varepsilon(F_E)$ for a particular $a \in A$, $F_e(a) = \mu F_e(a)$ if $a = e$, $\mu F_e \neq 0 = 0$ if $a \neq e$ in this case $d(F_{e_1}, F_e) < \varepsilon$ and so $F_e \in B_\varepsilon(F_E)$

For different elements in A, let us show that result is true.

If $a, b \in A$, such that $a < b$ or $b < a$. Then $\mu F_e(a) < \mu F_e(b)$ or $\mu F_e(b) < \mu F_e(a)$

In either cases we conclude from above that $d(F_{e_1}, F_e) < \varepsilon$ and so $F_e \in B_\varepsilon(F_E)$

□

Theorem 3.7. *Let (F_A, A, τ_F) be a fuzzy soft pseudo metric space, if $F_e = \cup_{a \in U} \mu_{F^e}(a)$*

Definition 3.17. *A fuzzy soft point G_e is said to be quasi - coincident with F_A denoted by $G_e q F_A$ if and only if $\mu_{G_e}(x) + \mu^e_{G_e}(x)$ for some $x \in U$*

Definition 3.18. *A fuzzy soft set F_A is said to be a Q - neighbourhood of G_e iff there exists $H_B \in \tau$ such that $G_e q H_B$ and $H_B \subseteq F_A$.*

Theorem 3.8. *A fuzzy soft point $G_e \in F_A$ if and only if each Q - neighbourhood of G_e is quasi - coincident with F_A.*

Definition 3.19. *A family of fuzzy soft sets in (F_A, A, τ_F) is said to be fuzzy soft locally finite if and only if every fuzzy soft point $F_e \in FSC(F_A)$ has a neighbourhood H_A which is quasi - coincident with atmost finite number of $\mathcal{S}$.*

Theorem 3.9. *If $\mathcal{S}$ is a fuzzy soft locally finite family in (F_A, A, τ_F) then*

$$\overline{\bigcup_{G_A \in \mathcal{S}} G_A} = \bigcup_{G_A \in \mathcal{S}} \overline{G_A}$$

Proof. Given any point $F_e \in \overline{\bigcup_{G_A \in \mathcal{S}} G_A}$ then each Q - neighbourhood of F_e is a quasi - coincident with $\bigcup_{G_A \in \mathcal{S}} G_A$. By fuzzy soft locally finite property, there exists a Q - neighbourhood H_A of F_e. which is quasi - coincident with atmost finite number of $\mathcal{S}$. But H_A is quasi - coincident with $\bigcup_{G_A \in \mathcal{S}} G_A$ and hence H_A is quasi - coincident with $\bigcup_{i=1}^n G^i_A$ for all $G^i_A \in \mathcal{S}$

Let us prove that every Q - neighbourhood of F_e is quasi - coincident with $\bigcup_{i=1}^n G^i_A$ for all $G^i_A \in \mathcal{S}$

If for every Q - neighbourhood K_A of F_e which is contained in G_A such that $\mu^e_{K_A}(x) \leq \mu^e_{G_A}(x)$ then we have to show that K_A is quasi - coincident with $\bigcup_{i=1}^n G^i_A$ for all $G^i_A \in \mathcal{S}$. If K_A is contained in G_A then K_A is quasi - coincident with $G^i_A, i = 1, 2, ..., n$.

But K_A and $\bigcup_{G_A \in \mathcal{S}} G^i_A$ are quasi - coincident and thus we have proved that every Q - neighbourhood of F_e is quasi - coincident with $\bigcup_{i=1}^n G^i_A$ and so we have, $F_e \subseteq \overline{\bigcup_{i=1}^n G^i_A} = \bigcup_{i=1}^n \overline{G^i_A} \subset \bigcup_{G_A \in \mathcal{S}} G^i_A$ □

References

[1] U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, *Comput. Math. Appl.*, 59(2010), 3458-3463.

[2] H. Aktas and N. Cagman, Soft sets and soft groups, *Inform. Sci.*, 177(2007), 2726-2735.

[3] A. Ayglu and H. Aygn, Introduction to fuzzy soft groups, *Comput. Math. Appl.*, 58 (2009), 1279-1286.

[4] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24(1968), 191-201.

[5] J. Ghosh, B. Dinda and T. K. Samanta, Fuzzy soft rings and fuzzy soft ideals, *Int. J. Pure Appl. Sci. Technol.*, 2(2)(2011), 66-74.

[6] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.*, 44(8-9)(2002), 1077-1083.

[7] W. K. Min, A note on soft topological spaces, *Comput. Math. Appl.*, 62(2011), 3524-3528.

[8] D. Molodtsov, Soft set theory First results, *Comput. Math. Appl.*, 37(4/5) (1999), 19-31.

[9] Bekir Tanay and M. Burc Kandemir, Topological structure of fuzzy soft sets, *Comput. Math. Appl.*, 61(2011), 2952-2957.

[10] S. Roy and T. K. Samanta, A note on fuzzy soft topological spaces, *Ann. Fuzzy Math. Inform.*, 3(2)(2012), 305-311.

[11] C. K. Wong, Covering properties of fuzzy topological spaces, *J. Math. Anal. Appl.*, 43 (1973), 697-704.

[12] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8(1965), 338-353.

Received: January 4, 2014; *Accepted*: March 15, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 203–214

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Existence results for an impulsive neutral integro-differential equation with infinite delay via fractional operators

Alka Chadha[*] and Dwijendra N Pandey

Indian Institute of Technology Roorkee-247667, Uttarakhand, India.

Abstract

In this present work, we consider an impulsive neutral integro-differential equation with infinite delay in an arbitrary Banach space X. The existence of mild solution is established by using resolvent operator and Hausdorff measure of noncompactness.

Keywords: Resolvent operator, Impulsive differential equation, Neutral integro-differential equation, Measure of noncompactness.

2010 MSC: 34K37, 34K30, 35R11, 47N20.

1 Introduction

In recent years, impulsive differential equations have become an active area of research due to their demonstrated applications in widespread fields of science and engineering such as biology, physics, control theory, population dynamics, economics, chemical technology, medicine and many others. Neutral differential equations arise in many areas of applied mathematics. The system of rigid heat conduction with finite wave spaces can be modeled in the form of the integro-differential equation of neutral type with delay. In addition, the development of the theory of the functional differential equation with infinite delay depends on a suitable choice of phase space. There are various phases spaces which have been studied in a book by Hale and Kato [9] and they introduced a common phase space $\mathfrak{B}$. For more detail on phase space, we refer to book by Hale and Kato [9] and Y. Hino et al. [20].

On the other hand, many real world processes and phenomena which are subjected during their development to short-term external influences can be modeled as impulsive differential equation with fractional order. Their duration is negligible compared to the total duration of the entire process or phenomena. Such process is investigated in various fields such as biology, physics, control theory, population dynamics, medicine and so on. For the general theory of such differential equations, we refer to the monographs [12], [18], and papers [5], [6], [14], [17], [19], [21]-[22], and references given therein.

The purpose of this paper is to study the following integro-differential equation with infinite delay in a Banach space $(X, \|\cdot\|)$,

$$\frac{d}{dt}[u(t) - F(t, u_t)] = A[u(t) + \int_0^t f(t-s)u(s)ds] + G(t, u_t, \int_0^t E(t,s,u_s)ds),$$
$$t \in J = [0, T_0],\ t \neq t_k,\ k = 1, 2, \cdots, m, \tag{1.1}$$
$$u_0 = \phi \in \mathfrak{B}, \tag{1.2}$$
$$\Delta u(t_i) = I_i(u_{t_i}),\ i = 1, 2, \cdots, m, \tag{1.3}$$

[*]Corresponding author.
E-mail address: alkachadda23@gmail.com(Alka Chadha), dwij.iitk@gmail.com(Dwijendra N Pandey).

where $0 < T_0 < \infty$, A is a closed linear operator defined on a Banach space $(X; \| \cdot \|)$ with dense domain $D(A) \subset X$; $f(t), t \in [0, T_0]$ is a bounded linear operator. The functions $F : [0, T_0] \times \mathfrak{B} \to X$, $G : [0, T_0] \times \mathfrak{B} \times X \to X$, $E : [0, T_0] \times [0, T_0] \times \mathfrak{B} \to X$, $I_i : X \to X$, $i = 1, \cdots, m$ are appropriate functions to be specified later, where $\mathfrak{B}$ is the phase space defined axiomatically later in section 2 and $0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T_0$ are pre-fixed numbers. The symbol $\Delta u(t) = u(t^+) - u(t^-)$ denotes the jump of the function u at t i.e., $u(t^-)$ and $u(t^+)$ denotes the end limits of the $u(t)$ at t. The history $u_t : (-\infty, 0] \to X$ is a continuous function defined as $u_t(s) = u(t+s), s \leq 0$ belongs to the abstract phase space $\mathfrak{B}$.
Hernandez et al, [4] has discussed the existence of solution for the neutral integro-differential problem

$$\frac{d}{dt}[u(t) + f(t, u_t)] = Au(t) + g(t, u_t), \quad t \in [0, T_0], \tag{1.4}$$

$$u_0 = \phi, \quad \phi \in \mathfrak{B}, \tag{1.5}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup and $f,\ g : [0, T_0] \times \mathfrak{B} \to X$ are appropriate functions. The existence of the mild solution for impulsive neutral integro-differential inclusions with nonlocal conditions

$$\frac{d}{dt}[u(t) - F(t, u(h_1(t)))] = A[u(t) + \int_0^t f(t-s)u(s)ds] + G(t, u(h_2(t))), \quad t \in [0, T_0],\ t \neq t_k, \tag{1.6}$$

$$\Delta u(t_k) = I_k(u(t_k^-)),\ k = 1, \cdots, m, \tag{1.7}$$

$$u(0) + g(u) = u_0, \tag{1.8}$$

has been established by Chang and Nieto in [22]. Where A is the infinitesimal generator of a compact, analytic resolvent operator $R(t), t > 0$ on a Banach space X and F, G, g, I_k are appropriated functions.

In this work, our work is spurred by the works [4]-[7], [14], [17], [21]-[22] to establish some existence results for the system (1.1)-(1.3) by using measure of noncompactness and resolvent operator. The tool of measure of noncompactness has been used in linear operator theory, theory of differential and integral equations, the fixed point theory and many others. For an initial study of the theory of the measure of noncompactness, we refer to book of Józef Banas [10], Akhmerov et. al.[16] and references given therein.

The organization of the article is as follows: In section 2, we provide some basic definitions, lemmas and theorems as preliminaries as these are useful for proving our results. In section 3, we prove the existence of mild solution to (1.1)-(1.3). An example is also considered at the end of the article.

2 Preliminaries

In this segment, we provide some fundamental definition, Lemmas and Theorems which will be utilized all around this paper.
Let X be a Banach space. The symbol $C([a, b]; X)$, $(a, b \in \mathbb{R})$ stands for the Banach space of all the continuous functions from $[a, b]$ into X equipped with the norm $\| z(t) \|_C = \sup_{t \in [a,b]} \| z(t) \|_X$ and $L^p((a, b); X)$ stands for Banach space of all Bochner-measurable functions from (a, b) to X with the norm

$$\| z \|_{L^p} = (\int_{(a,b)} \| z(s) \|_X^p ds)^{1/p}.$$

Let $0 \in \rho(A)$ i.e. A is invertible. Then it can be conceivable to characterize the fractional power A^α for $0 < \alpha \leq 1$ as a closed linear operator with domain $D(A^\alpha) \subset X$. It is easy to see that $D(A^\alpha)$ which is dense in X is a Banach space endowed with the norm $\| z \| = \| A^\alpha z \|$, for $z \in D(A^\alpha)$. Henceforth, we use X_α as notation of $D(A^\alpha)$. Also, we have that $X_\kappa \hookrightarrow X_\alpha$ for $0 < \alpha < \kappa$ and therefore, the embedding is continuous. Then, we define $X_{-\alpha} = (X_\alpha)^*$, for each $\alpha > 0$. The space $X_{-\alpha}$ stands for the dual space of X_α, is a Banach space with the norm $\| z \|_{-\alpha} = \| A^{-\alpha} z \|$. For additional parts on the fractional powers of closed linear operators, we allude to book by Pazy [1].

For the differential equation with infinite delay, Kato and Hale [9] was proposed the phase space $\mathfrak{B}$ satisfying certain fundamental axioms.

Definition 2.1. *The linear space of all functions from $(-\infty, 0]$ into Banach space X with a seminorm $\| \cdot \|_{\mathfrak{B}}$ is known as phase space $\mathfrak{B}$. The fundamental axioms assumed on $\mathfrak{B}$ are the followings:*

(A) *If $u : (-\infty, d + T_0] \to X$, $T_0 > 0$ is a continuous function on $[d, d + T_0]$ such that $u_d \in \mathfrak{B}$ and $u|_{[d,d+T_0]} \in \mathfrak{B} \in \mathcal{PC}([d, d + T_0]; X)$, then for every $t \in [d, d + T_0)$, the following conditions are hold:*

(i) $u_t \in \mathfrak{B}$,

(ii) $H\| u_t\|_{\mathfrak{B}} \geq \| u(t)\|$,

(iii)$\| u_t\|_{\mathfrak{B}} \leq N(t + d)\| u_d\|_{\mathfrak{B}} + K(t - d) \sup\{\| u(s)\| : d \leq s \leq t\}$,

where H is a positive constant; $N,\ K : [0, \infty) \to [1, \infty)$, N is a locally bounded, K is continuous and $K,\ H,\ N$ are independent of $u(\cdot)$.

(A1) *For the function u in $(A1)$, u_t is a $\mathfrak{B}$-valued continuous function for $t \in [d, d + T_0]$.*

(B) *The space $\mathfrak{B}$ is complete.*

To set the structure for our primary existence results, we have to introduce the following definitions.

Definition 2.2. *A family $\{R(t)\}_{t\in J}$ of bounded linear operators is said to be a resolvent operator (Fractional operators) for following equation*

$$x'(t) = A[x(t) + \int_0^t f(t - s)x(s)ds], \tag{2.9}$$

if the following conditions are satisfied

(i) $R(0) = I$, *where I is the identity operator on X.*

(ii) $R(t)$ *is strongly continuous for $t \in [0, T_0]$.*

(iii) $R(t) \in B(Z),\ t \in [0, T_0]$. *For $z \in Z$ and $R(\cdot)z \in C([0, T_0]; Z) \cap C^1([0, T_0]; Z)$, we have*

$$\frac{d}{dt}R(t)z = A[R(t)z + \int_0^t f(t - s)R(s)zds], \tag{2.10}$$

$$= R(t)Az + \int_0^t R(t - s)Af(s)zds, \quad t \in [0, T_0]. \tag{2.11}$$

Where $B(Z)$ denotes the space of bounded linear operators defined on Z and Z is a Banach space formed from $D(A)$ with the graph norm.

We assume that A generates a resolvent operator $\{R(t)\}_{t\geq 0}$ on a Banach space X and there exists a positive constant M_1 such that $\| R(t)\| \leq M_1$. For any $0 \leq \alpha \leq 1$, there exists a positive constant M_α such that

$$\| A^\alpha R(t)\| \leq \frac{M_\alpha}{t^\alpha},\ \ t \in [0, T_0]. \tag{2.12}$$

To consider the mild solution for the impulsive problem, we propose the set $\mathcal{PC}([0, T_0]; X) = \{u : [0, T_0] \to X : u$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and $u(t_i^+)$ exists, for all $i = 1, \cdots, m\}$. Clearly, $\mathcal{PC}([0, T_0]; X)$ is a Banach space endowed the norm $\| u\|_{\mathcal{PC}} = \sup_{t\in[0,T_0]} \| u(s)\|$. For a function $u \in \mathcal{PC}([0, T_0]; X)$ and $i \in \{0, 1, \cdots, m\}$, we define the function $\widetilde{u_i} \in C([t_i, t_{i+1}], X)$ such that

$$\widetilde{u_i}(t) = \begin{cases} u(t), & \text{for } t \in (t_i, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases} \tag{2.13}$$

For $W \subset \mathcal{PC}([0, T_0]; X)$ and $i \in \{0, 1, \cdots, m\}$, we have $\widetilde{W_i} = \{\widetilde{u_i} : u \in W\}$ and following Accoli-Arzel*à* type criteria. Now, we discuss some basic definition of measure of noncompactness (MNC).

Lemma 2.1. *[3]. A set $W \subset \mathcal{PC}([0, T_0]; X)$ is relatively compact if and only if each set $\widetilde{W_i} \subset C([t_i, t_{i+1}], X)$ $(i = 0, 1 \cdots, m)$ is relatively compact.*

Definition 2.3. *The Hausdorff's measure of noncompactness (H'MNC) χ_Y is defined as*

$$\chi_Y(U) = \inf\{\varepsilon > 0 : \ U \text{ can be covered by finite number of balls with radius } \varepsilon\}, \tag{2.14}$$

for the bounded set $U \subset Y$, where Y is a Banach space.

Lemma 2.2. *For any bounded set $U,\ V \subset Y$, where Y is a Banach space. Then, the following properties are fulfilled:*

(i) $\chi_Y(U) = 0$ *if and only if U is pre-compact;*

(ii) $\chi_Y(U) = \chi_Y(conv\ U) = \chi_Y(\overline{U})$, *where conv U and $\overline{U}$ denotes the convex hull and closure of U respectively;*

(iii) $\chi_Y(U) \subset \chi_Y(V)$, *when $U \subset V$;*

(iv) $\chi_Y(U+V) \leq \chi_Y(U) + \chi_Y(V)$, *where $U+V = \{u+v : u \in U,\ v \in V\}$;*

(v) $\chi_Y(U \cup V) \leq \max\{\chi_Y(U), \chi_Y(V)\}$;

(vi) $\chi_Y(\lambda U) = \lambda \cdot \chi_Y(U)$, *for any $\lambda \in \mathbb{R}$;*

(vii) If the map $P : D(P) \subset Y \to \mathcal{Z}$ is continuous and satisfy the Lipschitsz condition with constant κ. Then, we have that $\chi_{\mathcal{Z}}(PU) \leq \kappa\chi_Y(U)$ for any bounded subset $U \subset D(P)$, where Y and $\mathcal{Z}$ are Banach space;

Definition 2.4. *[10] A bounded and continuous map $P : D \subset Z \to Z$ is a χ_Z-contraction if there exists a constant $0 < \kappa < 1$ such that $\chi_Z(P(U)) \leq \kappa\chi_Z(U)$, for any bounded closed subset $U \subset D$, where Z is a Banach space.*

Lemma 2.3. *[15] Let $D \subset Z$ be a closed, convex with $0 \in D$ and the continuous map $P : D \to D$ be a χ_Z-contraction. If the set $\{u \in D : u = \lambda Pu,\ for\ \ 0 < \lambda < 1\}$ is bounded, then the map P has a fixed point in D.*

Lemma 2.4. *(Darbo-Sadovskii)[10]. Let $D \subset Z$ be bounded, closed and convex. If the continuous map $P : D \to D$ is a χ_Z-contraction, then the map P has a fixed point in D.*

In this paper, we consider that χ denotes the Hausdorff's measure of noncompactness (H'MNC)in X, χ_C denotes the Hausdorff's measure of noncompactness in $C([0, T_0]; X)$ and $\chi_{\mathcal{PC}}$ denotes the Hausdorff's measure of noncompactness in $\mathcal{PC}([0, T_0]; X)$.

Lemma 2.5. *([10]. If U is bounded subset of $C([0, T_0]; X)$. Then, we have that $\chi(U(t)) \leq \chi_C(U)$, $\forall\ \ t \in [0, T_0]$, where $U(t) = \{u(t); u \in U\} \subseteq X$. Furthermore, if U is equicontinuous on $[0, T_0]$, then $\chi(U(t))$ is continuous on the interval $[0, T_0]$ and*

$$\chi_C(U) = \sup_{t\in[0,T_0]} \{\chi(U(t))\}. \tag{2.15}$$

Lemma 2.6. *[10] If $U \subset C([0, T_0]; X)$ is bounded and equicontinuous, then $\chi(U(t))$ is continuous and*

$$\chi(\int_0^t U(s)ds) \leq \int_0^t \chi(U(s))ds,\ \forall\ \ t \in [0, T_0], \tag{2.16}$$

where $\int_0^t U(s)ds = \{\int_0^t u(s)ds, u \in U\}$.

Lemma 2.7. *[14]*

(1) If $U \subset \mathcal{PC}([0, T_0]; X)$ is bounded, then $\chi(U(t)) \leq \chi_{\mathcal{PC}}(U)$, $\forall\ \ t \in [0, T_0]$, where $U(t) = \{u(t) : u \in U\} \subset X$;

(2) If U is piecewise equicontinuous on $[0, T_0]$, then $\chi(U(t))$ is piecewise continuous for $t \in [0, T_0]$ and

$$\chi_{\mathcal{PC}}(U) = \sup\{\chi(U(t)) : t \in [0, T_0]\}; \tag{2.17}$$

(3) If $U \subset \mathcal{PC}([0, T_0]; X)$ is bounded and equicontinuous, then $\chi(U(t))$ is piecewise continuous for $t \in [0, T_0]$ and

$$\chi(\int_0^t U(s)ds) \leq \int_0^t \chi(U(s))ds, \forall\ \ t \in [0, T_0], \tag{2.18}$$

where $\int_0^t U(s)ds = \{\int_0^t u(s)ds : u \in U\}$.

3 Main Results

In this segment, the existence of the mild solution for the equation (1.1)-(1.3) is studied. Now we introduce following conditions:

(HR) Since $R(t)$ is a resolvent operator and f is bounded operator. Without loss of generality we assume that there exist positive constants $N_1,\ N_2$ such that $\| R(t)\| \leq N_1,\ \| f(t)\| \leq N_2,\ t \in [0, T_0]$. We assume that $R(t), t \geq 0$ satisfies the following property;

(R_1) The map $t \mapsto R(t)$ is continuous from $(0, T_0]$ to $\mathcal{L}(X)$ with the uniform operator norm $\| \cdot \|_{\mathcal{L}(X)}$.

(HF) The function $F : [0, T_0] \times \mathfrak{B} \to X$ is Lipschitz continuous and there exist constants $L_F > 0$ and $0 < \beta \leq 1$ such that

$$\| A^{\beta} F(t, x_1) - A^{\beta} F(s, x_2)\| \leq L_F[\| x_1 - x_2\|_{\mathfrak{B}}], \tag{3.19}$$

and

$$\| A^{\beta} F(t, x)\| \leq C_1\| x\|_{\mathfrak{B}} + C_2, \tag{3.20}$$

for all $x, x_1, x_2 \in \mathfrak{B}$ and $t \in [0, T_0]$, where C_1, C_2 are positive constants.

(HG) $G : [0, T_0] \times \mathfrak{B} \times X \to X$ is a nonlinear function such that
(1) For each $u : (-\infty, T_0] \to X$, $u_0 = \phi \in \mathfrak{B}$, $G(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0, T_0]$ and function $t \mapsto G(t, u_t, \int_0^t E(t, s, u_s)ds)$ is strongly measurable for $u \in \mathcal{PC}([0, T_0]; X)$.
(2) There is an integrable function $\alpha : J \to [0, \infty)$ and a monotone increasing continuous function $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\| G(\tau, x, y)\| \leq \alpha(\tau)\Omega(\| x\|_{\mathfrak{B}} + \| y\|),\ \tau \in [0, T_0],\ (x, y) \in \mathfrak{B} \times X. \tag{3.21}$$

(3) There is an integrable function $\eta : J \to [0, \infty)$ such that for any bounded subset $E_1 \subset \mathcal{PC}((-\infty, 0]; X)$, $E_2 \subset X$, we have that

$$\chi(R(\tau)G(\tau, E_1, E_2)) \leq \xi(\tau)\{ \sup_{-\infty \leq \theta \leq 0} \chi(E_1(\theta)) + \chi(E_2)\}, \tag{3.22}$$

for a.e. $t \in [0, T_0]$. Where $E_1(\theta) = \{u(\theta) : u \in E_1\}$.

(HE) (1) There is a constant $E_1 > 0$ such that

$$\| \int_0^{\tau} [E(\tau, s, u) - E(\tau, s, v)]ds\| \leq E_1\| u - v\|_{\mathfrak{B}},\ \tau, s \in [0, T_0], u, v \in \mathfrak{B}. \tag{3.23}$$

(2)The map $E(t, s, \cdot) : \mathfrak{B} \to X$ is continuous for each $(t, s) \in [0, T_0] \times [0, T_0]$ and $E(\cdot, \cdot, u) : [0, T_0] \times [0, T_0] \to X$ is a strongly measurable function for each $u \in \mathfrak{B}$. There exist a constant $\zeta > 0$ and integrable function $m_E : J \to [0, \infty)$ such that

$$\| E(\tau, s, x)\| \leq \zeta m_E(s)\varphi(\| x\|),\ \tau, s \in [0, T_0], \tag{3.24}$$

where $\varphi \in C([0, \infty); [0, \infty))$ is a increasing function and $\int_0^{\infty} \zeta m_E(s)ds \leq L_0$.

(HI) (1) The functions $I_i : \mathfrak{B} \to X,\ i = 1, 2, \cdots, m$ are continuous and there are constant $L_i > 0$ $(i = 1, 2, \cdots, m)$ such that

$$\| I_i(x) - I_i(y)\| \leq L_i\| x - y\|_{\mathfrak{B}},\ \forall\ x, y \in \mathfrak{B}. \tag{3.25}$$

(2) There exist positive constant K_i^1 and K_i^2,$(i = 1, \cdots, m)$ such that

$$\| I_i(x)\| = K_i^1\| x\|_{\mathfrak{B}} + K_i^2,\ x \in \mathfrak{B}. \tag{3.26}$$

(H')

$$\mu_1 = [(K_{T_0}N_1H + M_{T_0}) + K_{T_0}N_1\| A^{-\beta}\|C_1]\| \phi\|_{\mathfrak{B}} + K_{T_0}[\| A^{-\beta}\|C_2 + \frac{M_{1-\beta}T_0^{\beta}}{\beta}C_2 + N_2\frac{M_{1-\beta}T_0^{\beta+1}}{\beta}C_2 + N_1\sum_{0<t_i<t} K_i^1], \tag{3.27}$$

$$\mu_2 = [\| A^{-\beta}\|C_1 + \frac{M_{1-\beta}T_0^{\beta}}{\beta}C_1 + N_2\frac{M_{1-\beta}T_0^{\beta+1}}{\beta}C_1 + N_1\sum_{0<t_i<t} K_i^1] < 1 \tag{3.28}$$

and

$$\int_0^{T_0} \widehat{m_E}(s)ds \leq \int_b^{\infty} \frac{ds}{\Omega(s)+\varphi(s)} \quad , \tag{3.29}$$

where $b = \frac{\mu_1}{1-\mu_2}$.

Definition 3.5. *A piecewise continuous function $u : (-\infty, T_0] \to X$ is said to be a solution for the system (1.1)-(1.3) if $u_0 = \phi$, $u(\cdot)|_{[0,T_0]} \in \mathcal{PC}$ and following impulsive integral equation*

$$\begin{aligned} u(t) &= R(t)[\phi(0) - F(0,\phi)] + F(t,u_t) + \int_0^t AR(t-s)F(s,u_s)ds \\ &+ \int_0^t AR(t-s)\int_0^s f(s-\tau)F(\tau,u_\tau)d\tau ds \\ &+ \int_0^t R(t-s)G(s,u_s,\int_0^s E(s,\tau,u_\tau)d\tau)ds \\ &+ \sum_{0<t_i<t} R(t-t_i)I_i(u_{t_i}), \qquad t \in [0,T_0], \end{aligned} \tag{3.30}$$

is verified.

Let $z : (-\infty, T_0] \to X$ be a function defined by $z_0 = \phi$ and $z(t) = R(t)\phi(0)$ on $[0, T_0]$. It is clear that $\| z_t\| \leq (K_{T_0}N_1H + M_{T_0})\| \phi\|_{\mathfrak{B}}$, where $K_{T_0} = \sup_{t\in[0,T_0]} K(t)$, $M_{T_0} = \sup_{t\in[0,T_0]} M(t)$.

Theorem 3.1. *Suppose $(HR), (HF), (HG), (HE), (HI), (H')$ holds and*

$$K_{T_0}[L_F + \frac{M_{1-\beta}T_0^{\beta}}{\beta}L_F + \frac{N_2L_FM_{1-\beta}T_0^{\beta+1}}{\beta} + N_1\sum_{i=1}^m L_i] + (1+L_0\Omega_1)\int_0^t \xi(s)ds \leq 1. \tag{3.31}$$

Then, the impulsive system (1.1)-(1.3) has a mild solution.

Proof. Let $S(T_0) = \{u : (-\infty, T_0] \to X,\ u_0 = 0,\ u|_{[0,T_0]} \in \mathcal{PC}\}$ endowed with the supremum norm $\| \cdot \|$ be the space. Define operator $P : S(T_0) \to S(T_0)$ as

$$Pu(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ -R(t)F(0,\phi) + F(t,u_t+z_t) + \int_0^t AR(t-s)F(s,u_s+z_s)ds \\ + \int_0^t AR(t-s)\int_0^s f(s-\tau)F(\tau,u_\tau+z_\tau)d\tau ds \\ + \int_0^t R(t-s)G(s,u_s+z_s,\int_0^s E(s,\tau,u_\tau+z_\tau)d\tau)ds \\ + \sum_{0<t_i<t} R(t-t_i)I_i(u_{t_i}+z_{t_i}), & t \in [0,T_0]. \end{cases} \tag{3.32}$$

Also we have $\| u_t + z_t\|_{\mathfrak{B}} \leq (K_{T_0}N_1H + M_{T_0})\| \phi\|_{\mathfrak{B}} + K_{T_0}\| u\|_t$, where $\| u\|_t = \sup_{s\in[0,t]}\| u(s)\|$. From the axioms A, our assumptions and the strongly continuity of $R(t)$, we can see that $Pu \in \mathcal{PC}$. For $u \in S(T_0)$, we get

$$\begin{aligned} \| AR(t-s)F(s,u_s+z_s)\| &= \| A^{1-\beta}R(t-s)A^{\beta}F(s,u_s+z_s)\|, \\ &\leq \frac{M_{1-\beta}}{(t-s)^{1-\beta}}[C_1\| u_s+z_s\|_{\mathfrak{B}} + C_2], \end{aligned} \tag{3.33}$$

thus, from the Bocher theorem it takes after that $AR(t-s)F(s,u_s+z_s)$ is integrable. So, we obtain that P is well defined on $S(T_0)$. We give the demonstration of Theorem 3.1 in the numerous steps.

Step 1. The set $\{x \in \mathcal{PC}([0,T_0];X) : u(t) = \lambda Pu(t),\ \text{for } 0<\lambda<1\}$ is bounded.
For $1>\lambda>0$, let u_λ be a solution for $u=\lambda Pu$. We have that

$$\| u_{\lambda t}+z_t\| \le (K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}} + K_{T_0}\| u_\lambda\|_t. \tag{3.34}$$

Let $\nu_\lambda(t) = (K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}} + K_{T_0}\| u_\lambda\|_t$, for each $t\in[0,T_0]$. Then, we have

$$\begin{aligned}
\| u_\lambda(t)\| &= \| \lambda Pu_\lambda(t)\| \le \| Pu_\lambda(t)\|,\\
&\le \| R(t)F(0,\phi)\| + \| F(t,u_{\lambda t}+z_t)\|\\
&\quad + \int_0^t \| A^{1-\beta}R(t-s)\|\| A^\beta F(t,u_{\lambda s}+z_s)\|ds\\
&\quad + \int_0^t \| A^{1-\beta}R(t-s)\| \int_0^s f(s-\tau)\|A^\beta F(\tau,u_\tau+z_\tau)\|d\tau ds\\
&\quad + \int_0^t \| R(t-s)G(s,u_s+z_s,\int_0^s E(s,\tau,u_\tau+z_\tau)d\tau)\|ds\\
&\quad + \sum_{0<t_i<t}\| R(t-t_i)I_i(u_{t_i}+z_{t_i})\|,\\
&\le N_1\| A^{-\beta}\|[C_1\| \phi\|_{\mathfrak{B}}+C_2] + \| A^{-\beta}\|[C_1\nu_\lambda(t)+C_2]\\
&\quad + \frac{M_{1-\beta}{T_0}^\beta}{\beta}(C_1\nu_\lambda(s)+C_2) + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}(C_1\nu_\lambda(s)+C_2)\\
&\quad + N_1\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds\\
&\quad + N_1\sum_{0<t_i<t}(K_i^1\nu_\lambda(t)+K_i^2),\\
&\le N_1\| A^{-\beta}\|[C_1\| \phi\|_{\mathfrak{B}}+C_2] + \| A^{-\beta}\|C_2 + \frac{M_{1-\beta}b^\beta}{\beta}C_2 + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}C_2\\
&\quad + N_1\sum_{0<t_i<t}K_i^2 + [\| A^{-\beta}\|C_1 + \frac{M_{1-\beta}{T_0}^\beta}{\beta}C_1 + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}C_1\\
&\quad + N_1\sum_{0<t_i<t}K_i^1]\nu_\lambda(t) + N_1\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta\ m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds,
\end{aligned}$$

which gives that

$$\begin{aligned}
\nu_\lambda(t) &\le [(K_{T_0}N_1H+M_{T_0}) + K_{T_0}N_1\| A^{-\beta}\|C_1]\| \phi\|_{\mathfrak{B}} + K_{T_0}[\| A^{-\beta}\|C_2\\
&\quad + \frac{M_{1-\beta}{T_0}^\beta}{\beta}C_2 + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}C_2 + N_1\sum_{0<t_i<t}K_i^1] + [\| A^{-\beta}\|C_1\\
&\quad + \frac{M_{1-\beta}{T_0}^\beta}{\beta}C_1 + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}C_1 + N_1\sum_{0<t_i<t}K_i^1]\nu_\lambda(t)\\
&\quad + N_1\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta\ m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds,\\
\nu_\lambda(t) &\le \frac{\mu_1}{1-\mu_2} + \frac{N_1K_{T_0}}{1-\mu_2}\int_0^t \alpha(s)\Omega(v_\lambda(s)+\int_0^s \zeta\ m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds,
\end{aligned}$$

Take $b=\frac{\mu_1}{1-\mu_2}$, therefore we get

$$\nu_\lambda(t) \le b + \frac{N_1K_{T_0}}{1-\mu_2}\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta\ m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds, \tag{3.35}$$

Let $\beta_\lambda(t) = b + \frac{N_1K_{T_0}}{1-\mu_2}\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta\cdot m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds$, then we have $\beta_\lambda(0)=b$ and

$$\nu_\lambda(t) \le \beta_\lambda(t), \quad 0\le t\le T_0. \tag{3.36}$$

Also, we get

$$\beta_\lambda'(t) \leq \frac{N_1 K_{T_0}}{1-\mu_2}\alpha(t)\Omega(\nu_\lambda(t) + \int_0^t \zeta\, m_E(s)\varphi(\nu_\lambda(s))ds). \tag{3.37}$$

Since we have that Ω is nondecreasing. Therefore we get

$$\beta_\lambda'(t) \leq \frac{N_1 K_{T_0}}{1-\mu_2}\alpha(t)\Omega(\beta_\lambda(t) + \int_0^t \zeta\, m_E(s)\varphi(\beta_\lambda(s))ds). \tag{3.38}$$

Now we take $B_\lambda(t) = \beta_\lambda(t) + \int_0^t \zeta\, m_E(s)\varphi(\beta_\lambda(s))ds$ and we have $B_\lambda(0) = \beta_\lambda(0)$ and $B_\lambda(t) \leq \beta_\lambda(t)$.

$$\begin{aligned} B_\lambda'(t) &= \beta_\lambda'(t) + \zeta\, m_E(t)\varphi(\beta_\lambda(t)), \\ &\leq \frac{N_1 K_{T_0}}{1-\mu_2}\alpha(t)\Omega(B_\lambda(t)) + \zeta\, m_E(t)\varphi(B_\lambda(t)), \\ &\leq \widehat{m_E}(t)(\Omega(B_\lambda(t)) + \varphi(B_\lambda(t))), \end{aligned} \tag{3.39}$$

which gives that

$$\int_{B_\lambda(0)}^{B_\lambda(t)} \frac{1}{\Omega(s)+\varphi(s)}ds \leq \int_0^{T_0} \widehat{m_E}(s)ds \leq \int_b^\infty \frac{1}{\Omega(s)+\varphi(s)}ds. \tag{3.40}$$

It implies that functions $\beta_\lambda(t)$ are bounded on $[0, T_0]$. Therefore, the function $\nu_\lambda(t)$ are bounded on $[0, T_0]$ and $u_\lambda(\cdot)$ are bounded on $[0, T_0]$.

Step 2. P is χ-contraction.

We introduce the decomposition of $P = P_1 + P_2$ such that

$$\begin{aligned} P_1 u(t) &= R(t)[-F(0,\phi)] + F(t, u_t + z_t) + \int_0^t AR(t-s)F(s, u_s + z_s)ds \\ &\quad + \int_0^t AR(t-s)\int_0^s f(s-\tau)F(\tau, u_\tau + z_\tau)d\tau ds \\ &\quad + \sum_{0<t_i<t} R(t-t_i)I_i(u_{t_i} + z_{t_i}), \end{aligned} \tag{3.41}$$

$$P_2 u(t) = \int_0^t R(t-s)G(s, u_s + z_s, \int_0^s E(s,\tau,u_\tau + z_\tau)d\tau)ds. \tag{3.42}$$

To prove the result, firstly we show that P_1 satisfies the Lipschitz condition. For $u_1,\ u_2 \in S(T_0)$, we have $\| P_1 u_1(t) - P_1 u_2(t)\|$

$$\begin{aligned} &\leq \| A^\beta F(t, u_{1t} + z_t) - A^\beta F(t, u_{2t} + z_t)\| \\ &\quad + \int_0^t \|A^{1-\beta}R(t-s)\|\|A^\beta F(s, u_{1s} + z_s) - F(s, u_{2s} + z_s)\|ds \\ &\quad + \int_0^t \|A^{1-\beta}R(t-s)\| \int_0^s \|f(s-\tau)\|\|A^\beta F(\tau, u_{1\tau} + z_\tau) - F(\tau, u_{2\tau} + z_\tau)\|d\tau ds \\ &\quad + \sum_{0<t_i<t} \|R(t-t_i)\|\| I_i(u_{1t_i} + z_{t_i}) - I_i(u_{2t_i} + z_{t_i})\|, \\ &\leq L_F\| u_{1t} - u_{2t}\|_{\mathfrak{B}} + \frac{M_{1-\beta}{T_0}^\beta}{\beta}L_F\| u_{1t} - u_{2t}\|_{\mathfrak{B}} \\ &\quad + \frac{N_2 L_F M_{1-\beta}{T_0}^{\beta+1}}{\beta}\| u_{1t} - u_{2t}\|_{\mathfrak{B}} + N_1\sum_{i=1}^m L_i\| u_{1t} - u_{2t}\|_{\mathfrak{B}}, \\ &\leq K_{T_0}[L_F + \frac{M_{1-\beta}{T_0}^\beta}{\beta}L_F + \frac{N_2 L_F M_{1-\beta}{T_0}^{\beta+1}}{\beta} + N_1\sum_{i=1}^m L_i]\| u_1 - u_2\|_{T_0}, \end{aligned} \tag{3.43}$$

it gives that P_1 is Lipschitz continuous with Lipschitz constant $L = K_{T_0}[L_F + \frac{M_{1-\beta}{T_0}^\beta}{\beta}L_F + \frac{N_2 L_F M_{1-\beta}{T_0}^{\beta+1}}{\beta} + N_1\sum_{i=1}^m L_i]$.

Let B be an arbitrary subset of $S(T_0)$. Since $R(t)$ is equicontinuous resolvent. Therefore, from the assumption (HG) and the strongly continuity of $R(t)$, we have that $R(t-s)G(s, x_s + y_s, \int_0^s E(s,\tau, x_\tau + y_\tau)d\tau)$ is piecewise equicontinuous. Then, by the Lemma 2.6 we have
$\chi(P_2(B(t)))$

$$\begin{aligned}
&\leq \chi(\int_0^t R(t-s)G(s, B_s + z_s, \int_0^s E(s,\tau, B_\tau + z_\tau)d\tau)ds), \\
&\leq \int_0^t \xi(s)\cdot(\sup_{-\infty<\theta\leq 0} \chi(B(s+\theta)+z(s+\theta)) + \chi(\int_0^s E(s,\tau,B_\tau+z_\tau)d\tau))ds, \\
&\leq \int_0^t \xi(s) \sup_{-\infty<\theta\leq 0} [\chi(B(s+\theta)+z(s+\theta)) + L_0\chi(\Omega(B(s+\theta)+z(s+\theta)))]ds, \\
&\leq \int_0^t \xi(s) \sup_{0\leq\tau\leq s} (\chi(B(\tau)) + L_0\chi(\Omega(B(\tau))))ds, \\
&\leq \chi_{\mathcal{PC}}(B)[1+\Omega_1 L_0] \int_0^t \xi(s)ds,\ [\chi(\Omega(B(\tau))) \leq \Omega_1\chi(B(\tau))],
\end{aligned} \tag{3.44}$$

for every bounded set $B \subset \mathcal{PC}$. Where Ω_1 is a constant.
Now we can see that for any bounded subset $B \in \mathcal{PC}$

$$\begin{aligned}
\chi_{\mathcal{PC}}(P(B)) &= \chi_{\mathcal{PC}}(P_1B + P_2B), \\
&\leq \chi_{\mathcal{PC}}(P_1B) + \chi_{\mathcal{PC}}(P_2B), \\
&\leq (L + (1+L_0\Omega_1)\int_0^t \xi(s)ds)\chi_{\mathcal{PC}}(B),
\end{aligned} \tag{3.45}$$

from the above inequality we obtain that P is χ-contraction. Hence P has at least one fixed point in B by Darbo fixed point theorem. Let u be the fixed point of the map Q on $S(T_0)$. Thus $y = u + z$ is a mild solution for the problem (1.1)-(1.3). Therefore this completes the proof of the theorem. □

Theorem 3.2. *Suppose that $(HR), (HF), (HG), (HE), (HI)$ and (H') are satisfied and*

$$\begin{aligned}
&K_{T_0}[\| A^{-\beta}\|C_1 + \frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_1 + \frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_1 \\
&+N_1\sum_{i=1}^m K_i^1] + N_1K_{T_0}\int_0^{T_0}\alpha(s)ds \lim_{\tau\to\infty}\sup\frac{\tau+L_0\varphi(\tau)}{\tau} < 1.
\end{aligned} \tag{3.46}$$

Then, the impulsive system (1.1)-(1.3) has a mild solution.

Proof. Thus proof of the above theorem is like that of Theorem 3.1, We characterize the operator P as (3.32). Now, we show that there exist a $r > 0$ such that $Q(B_r) \subset B_r$, where B_r is a closed and convex ball with center at the origin and radius r i.e., $B_r = \{u \in S(T_0) : \| u\|_{T_0} \leq r\}$. To prove it, we assume that for any $r > 0$, there exists $u_r \in B_r$ and $t_r \in [0, T_0]$ such that $r < \| Qu_r(t_r)\|$. For $u_r \in B_r$ and $t_r \in [0, T_0]$, we have

$$\begin{aligned}
r &< \| Qu_r(t_r)\| \leq N_1\| F(0,\phi)\| + \| A^{-\beta}\|[C_1\| u_{rt_r} + z_{t_r}\|_{\mathfrak{B}} + C_2] \\
&+ \int_0^{t_r} \|A^{1-\beta}R(t_r-s)\|\|A^{\beta}F(s,u_{rs}+z_s)\|ds \\
&+ \int_0^{t_r} \|A^{1-\beta}R(t_r-s)\| \int_0^s \|f(s-\tau)\|\ \|A^{\beta}F(\tau,u_{r\tau}+z_s)\|d\tau ds \\
&+N_1\int_0^{t_r} \| G(s,u_{rs}+z_s, \int_0^s E(s,\tau,u_{r\tau}+z_s)\tau)\|ds \\
&+N_1\sum_{i=1}^m (K_i^1\| u_{rt}+z_t\|_{\mathfrak{B}} + K_i^2),
\end{aligned}$$

$$
\begin{aligned}
&\le N_1\| A^{-\beta}\|(C_1\|\phi\|_{\mathfrak{B}}+C_2)+\| A^{-\beta}\|[C_1\| u_{rt_r}+z_{t_r}\|_{\mathfrak{B}}+C_2] \\
&+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}(C_1\| u_{rt_r}+z_{t_r}\|_{\mathfrak{B}}+C_2)+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}(C_1\| u_{rt_r}+z_{t_r}\|_{\mathfrak{B}}+C_2) \\
&+\int_0^{t_r}\alpha(s)\Omega(\| u_{rt_r}+z_{t_r}\|_{\mathfrak{B}}+\| \int_0^s E(s,\tau,u_{r\tau}+z_\tau)d\tau\|)ds \\
&+N_1\sum_{i=1}^m(K_i^1\| u_{rt}+z_t\|_{\mathfrak{B}}+K_i^2), \\
&\le N_1\| A^{-\beta}\|(C_1\|\phi\|_{\mathfrak{B}}+C_2)+\| A^{-\beta}\|C_2+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_2+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_2 \\
&+N_1\sum_{i=1}^m K_i^2+[\| A^{-\beta}\|C_1+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_1+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_1+N_1\sum_{i=1}^m K_i^1] \\
&\times[(K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}}+K_{T_0}r]+\int_0^{t_r}\alpha(s)\Omega((K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}} \\
&+K_{T_0}r+L_0\varphi((K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}}+K_{T_0}r))ds,
\end{aligned}
\tag{3.47}
$$

it gives that

$$
\begin{aligned}
1 &< K_{T_0}[\| A^{-\beta}\|C_1+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_1+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_1+N_1\sum_{i=1}^m K_i^1] \\
&+N_1\int_0^{T_0}\alpha(s)ds \\
&\times\limsup_{r\to\infty}\frac{\Omega((K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}}+K_{T_0}r+L_0\varphi((K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}}+K_{T_0}r))}{r}, \\
&\le K_{T_0}[\| A^{-\beta}\|C_1+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_1+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_1+N_1\sum_{i=1}^m K_i^1] \\
&+N_1K_{T_0}\int_0^{T_0}\alpha(s)ds\limsup_{\tau\to\infty}\frac{\tau+L_0\varphi(\tau)}{\tau},
\end{aligned}
\tag{3.48}
$$

which is the contradiction of the inequality (3.46). Hence we conclude that $QB_r \subset B_r$.
As the proof of the Theorem 3.1, we obtain that there exists at least a mild solution for the problem (1.1)-(1.3). □

4 Example

In this section, we consider an example to illustrate the application of the theory. Here we take the space $C_0 \times L^2(h,X)$ as phase space $\mathfrak{B}$(see, [5]).
We consider the following first order neutral integro-differential equation with unbounded delay

$$
\begin{aligned}
&\frac{d}{dt}[x(t,u)-\int_{-\infty}^t\int_0^\pi B(t-s,\xi,u)x(s,\xi)d\xi ds]=\frac{\partial^2}{\partial u^2}[x(t,u)+\int_0^t f(t-s,u)x(s,u)ds] \\
&\quad+\int_0^t a(t,u,s-t)G(x(s,u),\int_0^s E(s,\tau,x_\tau)d\tau)ds,\quad t\in[0,T_0],\quad u\in[0,\pi],
\end{aligned}
\tag{4.49}
$$

$$x(t,0)=x(t,\pi)=0,\quad t\in[0,T_0], \tag{4.50}$$

$$x(\tau,u)=\phi(\tau,u),\quad \tau\le 0,\ 0\le u\le\pi, \tag{4.51}$$

$$\Delta x(t_i)(u)=\int_{-\infty}^t c_i(t_i-s)x(s,u)ds, \tag{4.52}$$

where $\phi \in C_0 \times L^2(h,X)$ and $0 < t_1 < t_2 < \cdots < t_m < b$ are fixed numbers.
The function B, f, a, G, E, c_i are satisfied the following conditions:

(A1) The function $B(s,\xi,u)$, $\frac{\partial}{\partial u}B$ are measurable and $B(s,\xi,0) = B(s,\xi,\pi) = 0$. Also

$$L_B = \max\{(\int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{h(s)}(\frac{\partial^i B(s,\xi,u)}{\partial u^i})d\xi ds du)^{1/2} : i = 0,1\} < \infty; \tag{4.53}$$

(A2) The operator $f(t), t \geq 0$ is bounded and $\| f(t,u)\| \leq N_2$;

(A3) $a(t,u,\tau)$ is continuous function on $[0,T_0] \times [0,\pi] \times (-\infty, 0]$ with $\int_{-\infty}^0 a(t,u,\tau)d\tau = n(t,u) < \infty$;

(A4) G is a continuous function, satisfying $G(x_1,x_2) \leq \Omega'(\| x_1\| + \| x_2\|)$, where $\Omega'(\cdot)$ is continuous, increasing and positive on $[0,\infty)$;

(A5) The function $E(\cdot)$ is a continuous function, satisfying $0 \leq E(t,s,u) \leq \omega(\| u\|)$, where ω is a positive increasing continuous function on $[0,\infty)$;

(A6) The functions $c_i \in C([0,\infty);\mathbb{R})$ and $K_i^3 = (\int_{-\infty}^0 \frac{(c_i(s))^2}{h(s)}ds)^{1/2} < 0, \forall\, i = 1,\cdots,m$;

Let $Ax = x''$, $A : D(A) \subset X \to X$ is a linear operator with domain

$$D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}. \tag{4.54}$$

It is known that A is the infinitesimal generator of an analytic resolvent operator $R(t), t \geq 0$. We assume that the $(A1) - (A6)$ are established.
Now, the system (4.49)-(4.52) can be reformulated as the abstract impulsive Cauchy problem (1.1)-(1.3) giving by

$$F(t,y)(u) = \int_{-\infty}^0 \int_0^\pi B(s,z,u)y(s,z)dzds, \tag{4.55}$$

$$G_1(t,w,y)(u) = \int_{-\infty}^0 a(t,u,\tau)G(w(\tau,u), \int_0^\tau y(\tau,\theta,x_\theta)d\theta)d\tau, \tag{4.56}$$

$$I_i(y)(u) = \int_{-\infty}^0 c_i(s)y(s,u)ds. \tag{4.57}$$

It is easy to see that $F(t,\cdot), G_1(t,\cdot,\cdot), I_i(i = 1,\cdots,m)$ are bounded linear operators. Applying the Theorem 3.1, we conclude that the problem (4.49)-(4.52) has at least one mild solution.

Acknowledgment

The authors would like to thank the referee for valuable comments and suggestions. The work of the first author is supported by the University Grants Commission (UGC), Government of India, New Delhi and Indian Institute of Technology, Roorkee.

References

[1] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer, New York, (1983).

[2] B. D. Andrade and J. P. Carvalho Dos Santos, Existence of solutions for a fractional neutral integro-differential equation with unbounded delay, *Elect. J. Diff. Equ.*, 90 (2012), 1-13.

[3] E. Hernández and D. O' Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, 141 (2012) 1641-1649.

[4] E. Hernández and H. R. Henríquez, Existence results for partial neutral functional differential equations with bounded delay, *J. Math. Anal. and Appl.*, 221 (1998), 452-475.

[5] E. Hernández, M. Pierri and G. Goncalves, Existence results for an impulsive abstract partial differential equation with state-dependent delay, *Comput. Math. Appl.*, 52 (2006), 411-420.

[6] E. Hernández, R. Sakthivel and S. Tanaka Aki, Existence results for impulsive evolution differential equations with state-dependent delay, *Elect. J. Differ. Equ.*, 28 (2008), 1-11.

[7] H. R. Henríquez and J. P. C. Dos Santos, Existence results for abstract partial neutral integro-differential equation with unbounded delay, *Elect. J. Qual. The. Diff. Equ.*, 29 (2009), 1-23.

[8] H. P. Heinz, On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions, *Nonlinear Analysis: TMA*, 7 (1983), 1351-1371.

[9] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.*, 21 (1978), 11-41.

[10] J. Banas and K. Goebel, Measure of noncompactness in Banach spaces, *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, USA, 1980.

[11] J. Prüss, Evolutionary Integral Equations and Applications, in Monographs Math., Vol. 87, Birkhauser-Verlag, 1993.

[12] M. Benchohra, J. Henderson and S. K. Ntouyas, Impulsive differential equations and inclusions, *Contemporary Mathematics and Its Applications, Vol.2*, Hindawi Publishing Corporation, New York, 2006.

[13] R. P. Agarwal, M. Benchohra and D. Seba, On the application of measure of noncompactness to the existence of solutions for fractional differential equations, *Results Math.*, 55 (2009), 221-230.

[14] Runping Ye, Existence of solutions for impulsive partial neutral functional differential equations with infinite delay, *Nonlinear Analysis: TMA*, 73 (2010), 155-162.

[15] R. Agarwal, M. Meehan and D. O'regan, Fixed point theory and applications, in: Cambridge Tracts in Mathematics, Cambridge University Press, New York, 2001, pp-178-179.

[16] R. R. Akhmerov, M. I. Kamenskiǐ, A. S. Potapov, A. E. Rodkina and B. N. Sadovskiǐ, Measures of noncompactness and Condensing operators, *Birkhäuser, Boston-Basel*, Berlin, Germany, 1992.

[17] T. Gunasekar, F. P. Samuel and M. M. Arjunan, Existence results for impulsive neutral functional integro-differential equation with infinite delay, *J. Nonlinear Sci. Appl.*, 6 (2013), 234-243.

[18] V. Lakshmikantham, D. Baǐnov and Pavel S. Simeonov, Theory of impulsive differential equations, Series in Modern Applied Mathematics, *World Scientific Publishing Co., Inc.*, Teaneck, NJ, 1989.

[19] X. Zhang, X. Huang and Z. Liu, The Existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay, *Noninear Analysis: Hybrid Systems*, 4 (2010), 775-781.

[20] Y. Hino, S. Murakami and T. Naito, Functional Differential Equations with Infinite Delay, in Lecture Notes in Math., vol. 1473, Springer-Verlag, Berlin, 1991.

[21] Y. K. Chang and W. S. Li, Solvability for impulsive neutral integro-differential equations with State-dependent delay via Fractional Operator, *J. Optimi. The. Appl.*, 2010 (144), 445-459.

[22] Y. K. Chang and Juan J. Nieto, Existence of solutions for impulsive neutral integro-differential inclusions with nonlocal initial conditions via fractional operators, *Nume. Funct. Anal. Optimi.*, 30 (2009), 227-244.

Received: January 17, 2014; *Accepted*: April 25, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 215–221

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Conversion of number systems and factorization

Janusz Wlodarczyk,[a] Djilali Behloul,[b] and Sui Sun Cheng[c]*

[a]*Space Research Centre, Polish Academy of Sciences, Bartycka 18A, Warsaw, 00-716, Poland.*

[b]*Department of Computer Science, USTHB, BP32 El Alia, Bab Ezzouar, 16111 Algiers, Algeria.*

[c]*Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, R. O. China.*

Abstract

In this paper one can see a new method for conversion of number systems. As an application we give an algorithm of factorization of an integer n with arithmetic complexity $O(\sqrt{n}\ln^2 n)$.

Keywords: conversion, number systems, factorization.

2010 MSC: 11Y05 11Y16.

1 Introduction

Let us first start with a whole number n described in the number system with base $\mathbf{p}$:

$$n = a_{m,\mathbf{p}} \cdots_{1,\mathbf{p}} a_{0,\mathbf{p}} = a_{m,\mathbf{p}}\mathbf{p}^m + \cdots + a_{1,\mathbf{p}}\mathbf{p} + a_{0,\mathbf{p}} \tag{1.1}$$

where $a_{i,\mathbf{p}}$ is the digit at position i. If the least significant number $a_{0,\mathbf{p}} = 0$, then $\mathbf{p}$ is a divisor of n.

In this note, we are interested in converting a number in the number system with base $\mathbf{p}$ to that with base $\mathbf{p}+2$. As a consequence, we are able to design an algorithm for factorizing a number n with arithmetic complexity $O(\sqrt{n}\ln^2 n)$. Here we use arithmetic complexity models, where cost is measured by the number of machine instructions performed on a single processor with addition and substraction of m-bit integers that costs $O(m)$ (see [1]).

2 Conversion

The conversion of a number n in the $\mathbf{p}$ base number system to the $\mathbf{p}+2$ base number system uses Horner's scheme 'illustrated' as follows :

$$\begin{aligned} n &= \cdots + (a\mathbf{p} + b)\mathbf{p} + c \\ &= \cdots + (a(\mathbf{p}+2) + (-2a + b))\mathbf{p} + c \\ &= \cdots + (a(\mathbf{p}+2)\mathbf{p} + (-2a + b)\mathbf{p}) + c \\ &= \cdots + (a(\mathbf{p}+2)(\mathbf{p}+2) - 2a(\mathbf{p}+2) + (-2a + b)(\mathbf{p}+2) + (-2a + b)(-2)) + c \\ &= \cdots + (a(\mathbf{p}+2) - 2a + (-2a + b))(\mathbf{p}+2) + (-2a + b)(-2) + c. \end{aligned}$$

Note that the conversion only employs additions and/or substractions. This idea of conversion is first announced by Walter Soden (see [2, p. 320]), but expressed in special cases. Knuth [2] also mentions this idea for numbers, not for digits.

*Corresponding author.
E-mail addresses: u_mnicha@inetia.pl (Janusz Wlodarczyk), dbehloul@yahoo.fr (Djilali Behloul), suisuncheng@gmail.com (Sui Sun Cheng).

Lemma 2.1. *Let* $n = a_{m,\mathbf{p}}\mathbf{p}^m + \cdots + a_{1,\mathbf{p}}\mathbf{p} + a_{0,\mathbf{p}}$ *be an integer written in base* $\mathbf{p}$. *If* $a_{m,\mathbf{p}} \neq 0$, *then* $m = \left[\log_{\mathbf{p}} n\right]$.

Proof. All the digits $a_{i,\mathbf{p}}$, except $a_{m,\mathbf{p}}$, are between 0 and $\mathbf{p}-1$. Hence

$$\mathbf{p}^m \leq n \leq (\mathbf{p}-1)\cdot\mathbf{p}^m + \cdots + (\mathbf{p}-1)\cdot\mathbf{p} + (\mathbf{p}-1).$$

But

$$(\mathbf{p}-1)\cdot\mathbf{p}^m + \cdots + (\mathbf{p}-1)\cdot\mathbf{p} + (\mathbf{p}-1) = (\mathbf{p}^{m+1}-1).$$

Hence

$$\mathbf{p}^m \leq n \leq \mathbf{p}^{m+1} - 1 < \mathbf{p}^{m+1}.$$

Taking $\log_{\mathbf{p}}$ on both sides, we see that

$$m \leq \log_{\mathbf{p}} n < m+1,$$

which implies $m = \left[\log_{\mathbf{p}} n\right]$. □

Lemma 2.2. *Let* $n = a\mathbf{p} + b$ *be an integer written in base* $\mathbf{p}$. *Then* n *can be written in base* $\mathbf{p}+2$ *as* $n = a'(\mathbf{p}+2) + b'$, *where*

1) if $b - 2a \geq 0$ *then* $a' = a$ *and* $b' = b - 2a$,

2) if $b - 2a < 0$ *and* $b - 2a + (\mathbf{p}+2) \geq 0$ *then* $a' = a - 1$ *and* $b' = b - 2a + \mathbf{p} + 2$, *and*

3) if $b - 2a + \mathbf{p} + 2 < 0$, *then* $a' = a - 2$ *and* $b' = b - 2a + \mathbf{p} + 2 + \mathbf{p} + 2$.

The arithmetic complexity is at most $O(\log_2 \mathbf{p})$.

Proof. We have,

$$\begin{aligned} n &= a(\mathbf{p}+2-2) + b \\ &= a(\mathbf{p}+2) + (b-2a). \end{aligned}$$

It is easy to see that $b - 2a \leq b < \mathbf{p} + 2$.

First case: if $b - 2a \geq 0$ then $a' = a$ and $b' = b - 2a$.

Second case: if $b - 2a < 0$ and $b - 2a + (\mathbf{p}+2) \geq 0$ then we substract 1 from the digit a and we add $(\mathbf{p}+2)$ to the number $(b-2a)$,

$$n = (a-1)(\mathbf{p}+2) + (b - 2a + \mathbf{p} + 2).$$

Then $a' = a - 1$ and $b' = b - 2a + \mathbf{p} + 2$

Third case: if $b - 2a + \mathbf{p} + 2 < 0$ then we substract 1 from the digit $a - 1$ and we add $(\mathbf{p}+2)$ to the number $(b - 2a + \mathbf{p} + 2)$, we obtain

$$n = (a-2)(\mathbf{p}+2) + (b - 2a + \mathbf{p} + 2 + \mathbf{p} + 2).$$

It is easy to see that $(b - 2a + \mathbf{p} + 2 + \mathbf{p} + 2) \geq 0$. Then $a' = a - 2$ and $b' = b - 2a + \mathbf{p} + 2 + \mathbf{p} + 2$.

It is easy to see that the number of additions or substractions manipulating the numbers $1, 2, a, b$ and $\mathbf{p}$ is 9. We have 5 additions, 4 substractions where we consider $b - 2a$ as $b - a - a$. The lengths of a, b and $\mathbf{p}$ are $\leq \log_2 \mathbf{p}$, the complexity (i.e., the number of binary arithmetic operations) is then $9\log_2 \mathbf{p} \in O(\log_2 \mathbf{p})$. □

3 Transformation I

Let $n = (a(\mathbf{p}+2) + b)\mathbf{p} + c$, where $0 \leq a, b < \mathbf{p} + 2$ and $0 \leq c < \mathbf{p}$. Then n can be written in base $\mathbf{p}+2$ as

$$n = a'(\mathbf{p}+2)^2 + b'(\mathbf{p}+2) + c'.$$

The transformation will be achieved in two steps: transform step and correction step.

Transform step: Write

$$\begin{aligned} n &= (a(\mathbf{p}+2) + b)\mathbf{p} + c \\ &= (a(\mathbf{p}+2) + b)(\mathbf{p}+2-2) + c \\ &= a(\mathbf{p}+2)^2 + (b-2a)(\mathbf{p}+2) + c - 2b \\ &= A(\mathbf{p}+2)^2 + B(\mathbf{p}+2) + C \end{aligned}$$

where $A = a, B = b - 2a$ and $C = c - 2b$.

Correction step:

1) If $C \geq 0$ then $c' = C$.

2) If $C < 0$ and $C + \mathbf{p} + 2 \geq 0$ then we substract 1 from B and we add $\mathbf{p} + 2$ to C. Then $c' = C + \mathbf{p} + 2$.

3) If $C + \mathbf{p} + 2 < 0$ then we substract 1 from $B - 1$ and we add $\mathbf{p} + 2$ to $C + \mathbf{p} + 2$. Then $c' = C + \mathbf{p} + 2 + \mathbf{p} + 2$.

Now we assume that C is corrected. Then

$$n = A(\mathbf{p} + 2)^2 + \tilde{B}(\mathbf{p} + 2) + c'$$

where $\tilde{B} = B$ or $B - 1$ or $B - 2$.

1) If $\tilde{B} \geq 0$ then $b' = \tilde{B}$ and $a' = A$.

2) If $\tilde{B} < 0$ and $\tilde{B} + \mathbf{p} + 2 \geq 0$, we substract 1 from A and we add $\mathbf{p} + 2$ to $\tilde{B}$, then $b' = \tilde{B} + \mathbf{p} + 2$ and $a' = A - 1$.

3) If $\tilde{B} + \mathbf{p} + 2 < 0$, we substract 1 from $A - 1$ and we add $\mathbf{p} + 2$ to $\tilde{B} + \mathbf{p} + 2$, then $b' = \tilde{B} + \mathbf{p} + 2 + \mathbf{p} + 2$ and $a' = A - 2$.

It is easy to see that the number of additions or substractions involving $1, 2, \tilde{a}, \tilde{b}, \mathbf{p}$ and c is 16. We have 8 additions and 8 substractions. The lengths of a, b, c and $\mathbf{p}$ are $\leq \log_2(\mathbf{p} + 1)$. Evaluation of a', b', c' involves $16 \log_2(\mathbf{p} + 1)$ binary arithmetic operations.

4 Transformation II

Let

$$\Delta_k = (...((a_k(\mathbf{p} + 2) + a_{k-1})(\mathbf{p} + 2) + a_{k-2})(\mathbf{p} + 2) + \cdots + a_1)\mathbf{p} + a_0$$

where $0 \leq a_i < \mathbf{p} + 2$ for $i = 1, 2..., k$ and $0 \leq a_0 < \mathbf{p}$. Then Δ_k can be written in base $\mathbf{p}$+2 in the form

$$\Delta_k = a'_k(\mathbf{p}{+}2)^k + a'_{k-1}(\mathbf{p} + 2)^{k-1} + \cdots + a'_1(\mathbf{p} + 2) + a'_0.$$

Again, this can be done in two steps : transform step and correction step.

Transform step: Write

$$\begin{aligned}
\Delta_k &= (a_k(\mathbf{p}{+}2)^{k-1} + a_{k-1}(\mathbf{p} + 2)^{k-2} + \cdots + a_2(\mathbf{p} + 2) + a_1)\mathbf{p} + a_0 \\
&= (a_k(\mathbf{p}{+}2)^{k-1} + a_{k-1}(\mathbf{p} + 2)^{k-2} + \cdots + a_2(\mathbf{p} + 2) + a_1)(\mathbf{p} + 2 - 2) + a_0 \\
&= a_k(\mathbf{p}{+}2)^k + (a_{k-1} - 2a_k)(\mathbf{p} + 2)^{k-1} + \cdots + (a_1 - 2a_2)(\mathbf{p} + 2) + a_0 - 2a_1.
\end{aligned}$$

Put $A_k = a_k$ and $A_{i-1} = a_{i-1} - 2a_i$ for $i = 1, ..., k$, then

$$\Delta_k = A_k(\mathbf{p}{+}2)^k + A_{k-1}(\mathbf{p} + 2)^{k-1} + \cdots + A_1(\mathbf{p} + 2) + A_0$$

Correction step:

1) If $A_0 \geq 0$ then $a'_0 = C$.

2) If $A_0 < 0$ and $A_0 + \mathbf{p} + 2 \geq 0$, we substract 1 to A_1 and we add $\mathbf{p} + 2$ to A_0 then $a'_0 = A_0 + \mathbf{p} + 2$

3) If $A_0 + \mathbf{p} + 2 < 0$, we substract 1 to $A_1 - 1$ and we add $\mathbf{p} + 2$ to $A_0 + \mathbf{p} + 2$ then $a'_0 = A_0 + \mathbf{p} + 2 + \mathbf{p} + 2$.

Now we assume that A_i is corrected, inductively, we will correct A_{i+1}:

$$\Delta_k = A_k(\mathbf{p}{+}2)^k + \cdots + A_{i+2}(\mathbf{p}{+}2)^{i+2} + \tilde{A}_{i+1}(\mathbf{p} + 2)^{i+1} + a'_i(\mathbf{p} + 2)^i + \cdots + a'_0$$

where $\tilde{A}_{i+1} = A_{i+1}$ or $A_{i+1} - 1$ or $A_{i+1} - 2$.

1) If $\tilde{A}_{i+1} \geq 0$ then $a'_{i+1} = \tilde{A}_{i+1}$.

2) If $\tilde{A}_{i+1} < 0$ and $\tilde{A}_{i+1} + \mathbf{p} + 2 \geq 0$, we substract 1 from A_{i+2} and we add $\mathbf{p} + 2$ to $\tilde{A}_{i+1}$, then $a'_{i+1} = \tilde{A}_{i+1} + \mathbf{p} + 2$.

3) If $\tilde{A}_{i+1} + \mathbf{p} + 2 < 0$, we substract 1 from $A_{i+2} - 1$ and we add $\mathbf{p} + 2$ to $\tilde{A}_{i+1} + \mathbf{p} + 2$, then $a'_{i+1} = \tilde{A}_{i+1} + \mathbf{p} + 2 + \mathbf{p} + 2$.

Number of operations :

1) The transform step needs $2k$ substractions

2) Correction step needs at most 4 additions, 2 substractions to correct A_0; 4 additions, 2 substractions to correct $\tilde{A}_1$; 4 additions and 2 substractions to correct $\tilde{A}_{k-1}$.

In total we need $2k + 6k = 8k$ operations.

The lengths of a_i and $\mathbf{p}$ are $\leq \log_2(\mathbf{p}+1)$. Evaluation of a_i', $i = 0,..,k$, involves at most $8k\log_2(\mathbf{p}+1)$ binary arithmetic operations.

Theorem 4.1. *Let $n = a_{m,\mathbf{p}}\mathbf{p}^m + \cdots + a_{1,\mathbf{p}}\mathbf{p} + a_{0,\mathbf{p}}$ be an integer written in base $\mathbf{p}$. Then we can write n in the base $\mathbf{p}+2$ in a systematic manner, as $n = a'_{m',\mathbf{p}+2}(\mathbf{p}+2)^{m'} + \cdots + a'_{1,\mathbf{p}+2}(\mathbf{p}+2) + a'_{0,\mathbf{p}+2}$ where $m' = \left[\log_{\mathbf{p}+2} n\right]$. Furthermore, the arithmetic complexity is at most $O\left(\frac{\log_2^2 n}{\log_2 \mathbf{p}}\right)$.*

Proof. The numbers $a'_{i,\mathbf{p}+2}$ are determined by the Lemma 2.2, Transforms I and II described above (and is implemented in the conversion algorithm below). The total number $T(n,\mathbf{p})$ of operations is given by:

$$\begin{aligned} T(n,\mathbf{p}) &= 9\log_2 \mathbf{p} + 16\log_2(\mathbf{p}+1) + ... + 8k\log_2(\mathbf{p}+1) + ... + 8m\log_2(\mathbf{p}+1) \\ &= 9\log_2 \mathbf{p} + 8(2+3+...+m)\log_2(\mathbf{p}+1) \\ &= 9\log_2 \mathbf{p} + 8\log_2(\mathbf{p}+1)\left(\frac{m(m+1)}{2} - 1\right) \\ &= O(m^2\log_2 \mathbf{p}). \end{aligned}$$

But $m = \left[\log_{\mathbf{p}} n\right]$ and $\log^2_{\mathbf{p}} n \log_2 \mathbf{p} = \dfrac{\log_2^2 n}{\log_2 \mathbf{p}}$, hence the complexity is $O(\log_2^2 n / \log_2 \mathbf{p})$. □

We may now summarize our previous discussions by means of the following

Conversion Algorithm:

INPUT: number $n = a_{m,\mathbf{p}} \ldots a_{1,\mathbf{p}} a_{0,\mathbf{p}}$ in the number system with base $\mathbf{p}$.

OUTPUT: number n in the number system with base $\mathbf{p}+2$ expressed in the form $n = a_{m,\mathbf{p}+2} \ldots a_{1,\mathbf{p}+2} a_{0,\mathbf{p}+2}$.

1. for $i = 0$ to m step 1 $\{a_i \leftarrow a_{i,\mathbf{p}}\}$ end for;
2. for $k = m-1$ to 0 step -1
3.   borrow index $b \leftarrow 0$
4.   for $i = k$ to m step 1 $\{a_{m+1} \leftarrow 0; a_i \leftarrow a_i - 2a_{i+1} - b; b \leftarrow 0;\}$
5.   if ($a_i < 0$) { $b \leftarrow b+1; a_i \leftarrow a_i + \mathbf{p} + 2;$ }
6.   if ($a_i < 0$) { $b \leftarrow b+1; a_i \leftarrow a_i + \mathbf{p} + 2;$}
7.   end for
8. end for
9. $m \leftarrow \left[\log_{\mathbf{p}+2} n\right]$ which is the actual number of digits n;
10. for $i = 0$ to m step 1 $\{a_{i,\mathbf{p}+2} \leftarrow a_i\}$ end for;

We remark that at the end of our algorithm, we correct the length of our number and the output number often has less digits. Thanks to lemma 2.1, the number of digits is related to the roots of the number n : when the current base $\mathbf{p}$ is greater than $\sqrt[k]{n}$, then $\ln \mathbf{p} > \ln \sqrt[k]{n}$ which implies $\log_{\mathbf{p}} n < k$, we have only k digits at most.

There are now numerous conversion algorithms, the present one has one interesting consequence.

5 Factorization

Factorization Algorithm:
INPUT: positive number $n = a_{m,2} \cdots a_{1,2}a_{0,2}$.
OUTPUT: table of divisors of the number n.

1. while ($a_{0,2} = 0$) { delete the least significant digit; table.insert(2) } end while;
2. $n \leftarrow$ the actual number n in which the least significant digit > 0;
3. conversion n into tertiary number $n = a_{m,3} \dots a_{1,3}a_{0,3}$ the base of system $\mathbf{p} \leftarrow 3$;
4. while ($a_{0,3} = 0$) { delete the least significant digit; table.insert(3) } end while;
5. $n \leftarrow$ the actual number n in which the least significant digit > 0;
6. **while** number of digits of number $n > 1$
7. convert the number $n = a_{m,\mathbf{p}} \dots a_{1,\mathbf{p}}a_{0,\mathbf{p}}$ into a number in the number system with base $\mathbf{p}+2$ with the form $n = a_{m,\mathbf{p}+2} \dots a_{1,\mathbf{p}+2}a_{0,\mathbf{p}+2}$ by means of the algorithm given by Theorem 4.1;
8. while ($a_{0,\mathbf{p}+2} = 0$),{delete the least significant digit;table.insert($\mathbf{p}+2$)} end while;
9. $n \leftarrow$ the actual number n in which the least significant digit > 0;
10. if $(\mathbf{p}+2)^2 > n$ then {table.insert(n); exit;}
11. $\mathbf{p} \leftarrow \mathbf{p}+2$;
12. end **while;**

Theorem 5.2. *The complexity of the factorization algorithm is* $O(\sqrt{n}\ln^2 n)$

Proof. In the above algorithm, we look for divisors by checking the least significant number. We delete zeros if necessary and call the conversion algorithm repeatedly. This algorithm starts with the base $\mathbf{p} = 3$ and terminates when the base $\mathbf{p}$ is greater than $\sqrt{n}$.

The total number $T(n)$ of operations is given by

$$\begin{aligned} T(n) &= O\left(\frac{\log_2^2 n}{\log_2 3}\right) + O\left(\frac{\log_2^2 n}{\log_2 5}\right) + \cdots + O\left(\frac{\log_2^2 n}{\log_2 \sqrt{n}}\right) \\ &\leq \frac{\sqrt{n}}{2} O\left(\frac{\log_2^2 n}{\log_2 3}\right) \\ &= O(\sqrt{n}\ln^2 n). \end{aligned}$$

□

As an example, we factorize the number 2525. It is even and in the number system with base 3, it has the form $1011\,0112_3$. The least significant number 3 is not equal to 0. Let us start converting it as a number in the number system with base 5 :

base 3:	1	0	1	1	0	1	1	2
	1	0						
	1	-2						
correct	0	3	1					
		3	-5					
correct		2	0	1				
		2	-4	1				
correct		1	1	1	0			
		1	-1	-1	-2			
correct		0	3	3	3	1		
			3	-3	-3	-5		
correct			2	1	1	0	1	
			2	-3	-1	-2	1	
correct			1	1	3	3	1	2
			1	-1	1	-3	-5	0
base 5:				4	0	1	0	0

The number in the number system with base 5 has the form 40100_5. The least significant digit is 0, so there exists a divisor which is equal to the base 5. After removing the least significant digit, the resulting number also has the digit 0 as its least significant digit. Hence, we have 2 divisors 5 and 5^2. Then the divisor 5 can be placed on a stack twice. Removing the least significant digit again, we have 401_5. Repeating the conversion procedure, we have

base 5:	4	0	1
	4	0	
	4	-8	
correct	2	6	1
	2	2	-11
base 7:	2	0	3

Continuing, we have

base 7:	2	0	3
	2	0	
	2	-4	
correct	1	5	3
	2	3	-7
base 9:	1	2	2

We continue with the number 122_9:

base 9:	1	2	2
	1	2	
	1	0	2
	1	-2	2
base 11:		9	2

We can stop the algorithm now, because $92_{11} < 100_{11}$. The number 2525 does not have any more divisors. The last is $9 \cdot 11 + 2 = 101$ in decimal system. After placing the divisor 101 on our stack, we see that all divisors can be obtained from our stack which contains $5, 5, 101$.

References

[1] R. P. Brent and P. Zimmerann, *Modern Computer Arithmetic*, arXiv 1004.4710.

[2] D. Knuth, *The Art of Computer Programming*, Vol. 2, Addison-Wesley 1997, 1998

Received: November 30, 2013; *Accepted*: April 2, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 222–227

Malaya Journal of Matematik

MJM

an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Lattice for covering rough approximations

Dipankar Rana[a,*] and Sankar Kumar Roy[b]

[a,b] *Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore-721102, West Bengal, India.*

Abstract

Covering is a common type of data structure and covering-based rough set theory is an efficient tool to process this type of data. Lattice is an important algebraic structure and used extensively in investigating some types of generalized rough sets. This paper presents the lattice based on covering rough approximations and lattice for covering numbers. An important result is investigated to illustrate the paper.

Keywords: Covering, Rough Set, Lattice, Covering approximation.

2010 MSC: 03G10, 14E20, 18B35.

1 Introduction

Theory of rough sets was introduced by Z. Pawlak [7], assumed that sets are chosen from a universe U, but that elements of U can be specified only upto an indiscernibility equivalence relation E on U. If a subset $X \subseteq U$ contains an element indiscernible from some elements not in X, then X is rough. Also a rough set X is described by two approximations. Basically, in rough set theory, it is assume that our knowledge is restricted by an indiscernibility relation. An indiscernibility relation is an equivalence relation E such that two elements of an universe of discourse U are E-equivalent if we can not distinguish these two elements by their properties known by us. By the means of an indiscernibility relation E, we can partition the elements of U into three disjoint classes respect to any set $X \subseteq U$, defined as follows:

- The elements which are certainly in X. These are elements $x \in U$ whose E-class x/E is included in X.
- The elements which certainly are not in X. These are elements $x \in U$ such that their E-class x/E is included in X^{co}, which is the complement of X
- The elements which are possibly belongs to X. These are elements whose E-class intersects with both X and X^{co}. In other words, x/E is not included in X nor in X^{co}.

From this observation, Pawlak [7] defined lower approximation set $X\downarrow$ of X to be the set of those elements $x \in U$ whose E-class is included in X, i.e, $X\downarrow = \{x \in U : x/E \subseteq X\}$ and for the upper approximation set $X\uparrow$ of X consists of elements $x \in U$ whose E-class intersect with X, i.e, $X\uparrow = \{x \in U : x/E \cap X \neq \emptyset\}$. The difference between $X\downarrow$ and $X\uparrow$ is treated as the actual area of uncertainty.
Covering-based rough set theory ([17], [19]) is a generalization of rough set theory. The structure of covering-based rough sets ([18],[19],[20]) have been a interested field of study. The classical rough set theory is based on equivalence relations. An equivalence relation corresponds to a partition, while a covering is an extension of a partition. This paper focuses on establishing algebraic structure of covering-based rough sets through down-sets and up-sets. Firstly, we connect posets with covering-based rough sets, then covering-based rough sets can be investigated in posets. Down-sets and up-sets are defined in the poset environment. In order to

*Corresponding author.
E-mail address: sankroy2006@gmail.com (Sankar Kumar Roy).

achieve this goal, many theories and methods have been proposed, for example, fuzzy set theory ([4], [16]), computing with words ([9], [15]), rough set theory ([5], [14]) and granular computing ([1], [3], [11], [13]). From the structures of these theories, two structures are mainly used, that is, algebraic structure ([2], [10],[12]) and topological structure [17]. This paper focuses on establish the algebraic structures of covering-based rough lattice through down sets, up sets and lattice for covering numbers.

2 Preliminaries

In this section, we present some definition and fundamental concept on covering lattice.

Definition 2.1. *Let U be a domain of discourse, and C be a family of subsets of U. If none of subsets in C is empty and $\cup C = U$, then C is called a covering of U. We call (U, C) the covering approximation space and the covering C is called the family of approximation sets. It is clear that a partition of U is certainly a covering of U, so the concept of a covering is an extension of a partition.*
Let (U, C) be an approximation space and x be any element of U. then the family. $Mind(x) = \{ K \in C : x \in K \wedge (\forall S \in C \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\}$ is called the minimal description of the object x. In order to describe an object we need only the essential characteristics related to this object. This is the purpose of the minimal description concept.

Definition 2.2. *A relation R on a set P is called a partial order if R is reflexive, antisymmetric, and transitive. If R is a partial order on P, then (P, R) is called a poset.*

Definition 2.3. *An upper semi-lattice is a poset (P, R) in which every subset $\{a, b\}$ has a least upper bound $a \vee b$. A lower semi-lattice is a poset (P, R) in which every subset $\{a, b\}$ has a greatest lower bound $a \wedge b$. The upper semi-lattice and the lower semi-lattice are also called semi-lattices.*

Definition 2.4. *[8] The lattice as a poset will be denoted by $(L, \leq)$, and the lattice as an algebra by $(L, \wedge, \vee)$.We write simply L to denote the lattice in both senses. A poset $(L, \leq)$ is a lattice if $sup\{a, b\}$ and $inf\{a, b\}$ exist for all $a, b \in L$.*

Definition 2.5. *Let C be a covering of domain U and $K \in C$. If K is a union of some sets in $C - \{K\}$, we say K is reducible in C, otherwise K is irreducible.*

Definition 2.6. *(Down-set and Up-set) Let $(P, \prec)$ be a poset. For all $A \subseteq P$, one can define $\downarrow A = \{x \in P : \exists a \in A, x \prec a\}$,*
$\uparrow A = \{x \in P : \exists a \in A, a \prec x\}$. $\downarrow A$ is called a down-set of A on the poset $(P, \prec)$; $\uparrow A$ is called an up-set A on the poset $(P, \prec)$. When there is no confusion, we say $\downarrow A$ is a down-set of A, and $\uparrow A$ an up-set of A.

Let (U, C) be a covering approximation space and $N(x) = \{K \in C : x \in K\}$ neighborhood of point x for each $x \in U$. There are six types of covering approximation operations that are defined as follows: for $X \subseteq U$,

- $X \downarrow_{C_1} = \cup\{K : K \in C \wedge K \subseteq X\}; X \uparrow_{C_1} = \cup\{K : K \in C \wedge K \cap X \neq \emptyset\}$;
- $X \downarrow_{C_2} = \cup\{K : K \in C \wedge K \subseteq X\}; X \uparrow_{C_2} = U - (U - X) \uparrow_{C_2}$;
- $X \downarrow_{C_3} = \{x \in U : N(x) \subseteq X\}; X \uparrow_{C_3} = \{x \in U : N(x) \cap X \neq \emptyset\}$;
- $X \downarrow_{C_4} = \{x \in U : \exists a(a \in N(x) \wedge N(a) \subseteq X)\}; X \uparrow_{C_4} = \{x \in U : \forall a(a \in N(x) \rightarrow N(a) \cap X \neq \emptyset)\}$;
- $X \downarrow_{C_5} = \{x \in U : \forall a(a \in N(a) \rightarrow N(a) \subseteq X)\}; X \uparrow_{C_5} = \cup\{N(x) : x \in U \wedge N(x) \cap X \neq \emptyset\}$;
- $X \downarrow_{C_6} = \{x \in U : \forall a(a \in N(a) \rightarrow a \in X)\}; X \uparrow_{C_6} = \cup\{N(x) : x \in X\}$. We call $X \downarrow_{C_n}$ the covering lower approximation operation and $X \uparrow_{C_n}$ the covering upper approximation operation $(n = 1, 2, 3, 4, 5, 6)$.

3 Rough set approximations based on covering

Let X^{co} be the complement of X in U, $X^{co} = U - X$. Let (U, C) be a covering approximation space. For any subset, $X \subseteq U$, the covering lower approximation of X be defined by $X \downarrow = \bigcup C \downarrow (X)$ and the covering upper approximation of X be defined by $X \uparrow = \cap\{K : K \subseteq X^{co}$ and $K \in C\}$. The set X is called new type covering based rough when $X \downarrow \neq X \uparrow$, otherwise X is called an exact set. The boundary of X denoted by $BN_C(X) = X \uparrow - X \downarrow$ is called as the boundary region of X of the new type covering C. With this concept, we construct the following proposition as:

Proposition 3.1. *$X\downarrow = X$ if and only if X is the union of some elements of C and also $X\uparrow = X$ if and only if X^{co} is the union of some elements of C.*

Proposition 3.2. *Let C be a covering of a universe U. If K is a reducible element of $C, C-\{K\}$ is still a covering of U.*

Proposition 3.3. *Let C be a covering of a universe $U, K \in C, K$ is a reducible element of C, and $K_1 \in C-\{K\}$, then K_1 is a reducible element of C if and only if it is a reducible element of $C-\{K\}$.*

3.1 Lattice based on covering rough approximation

Definition 3.7. *Let C be a covering of U. We define $L_C = \{X \subseteq U : C_6\downarrow(X) = X\}$. L_C is called the fixed point set of neighborhoods induced by C. We omit the subscript C when there is no confusion.*

Theorem 3.1. *$(L, \subseteq)$ is a lattice, where $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$ for any $X, Y \in L$.*

Proof. For any $X, Y \in L$, if $X \cup Y \notin L$, then there exists $x \in X \cup Y$ such that $N(x) \nsubseteq X \cup Y$. Since $x \in X \cup Y, \Rightarrow x \in X$ or $x \in Y$. Hence $N(x) \nsubseteq X$ or $N(x) \nsubseteq Y$, which is contradictory with $X, Y \in L$. Therefore, $X \cup Y \in L$. For any $X, Y \in L$, if $X \cap Y \ /\!\in L$, then there exists $y \in X \cap Y$ such that $N(x) \nsubseteq X \cap Y$. Since $x \in X \cap Y, x \in X$ and $x \in Y$. Hence there exist three cases as follows:
(1) $N(y) \nsubseteq X$ and $N(y) \nsubseteq Y$,
(2) $N(y) \nsubseteq X$ and $N(y) \subseteq Y$,
(3) $N(y) \subseteq X$ and $N(y) \nsubseteq Y$. But these three cases are all contradictory with $X, Y \in L$. Therefore, $X \cap Y \in L$. Thus $(L, \subseteq)$ is a lattice. $\emptyset$ and U are the least and greatest elements of $(L, \subseteq)$. In fact, $(L, \cap, \cup)$ is defined from the viewpoint of algebra and $(L, \subseteq)$ is defined from the viewpoint of partially ordered set. Both of them are lattices. Therefore, we no longer differentiate $(L, \cap, \cup)$ and $(L, \subseteq)$, and both of them are called lattice L. □

Proposition 3.4. *Let C be a covering of U. For all $a \in U, N(a) \in L$.*

Proof. For any $b \in N(a), N(b) \subseteq N(a)$, which implies $b \in c : N(c) \subseteq N(a) = C_6\downarrow(N(a))$. Hence $N(a) \subseteq C_6\downarrow(N(a))$. According to last definition of approximation $C_6\downarrow(N(a)) \subseteq N(a)$. Thus $C_6\downarrow(N(a)) = N(a)$, i.e., $N(a) \in L$. □

Theorem 3.2. *Let C be a covering of U, then L is a complete distributive lattice.*

Proof. For any $D \subseteq L$, we need to prove that $\cap D \in L$ and $\cup D \in L$. If $\cap D \notin L$, then there exists $y \in \cap D$ such that $N(y) \nsubseteq \cap D$, i.e., there are two index sets $I, J \subseteq \{1, 2, ..., |D|\}$ with $I \cap J = \emptyset$ and $|I \cup J| = |D|$ such that $N(y) \nsubseteq X_i$ and $N(y) \subseteq X_j$ for any $i \in I, j \in J$, where $X_i, X_j \in D$. This is contradictory with $X_i(i \in I), X_j(j \in J) \in L$. Hence $\cap D \in L$. If $\cup D \notin L$, then there exists $x \in \cup D$ such that $N(x) \nsubseteq \cup D$, i.e., there exists $X \in D$ such that $x \in X$and $N(x) \nsubseteq X$, which is contradictory with $X \in L$. Hence $\cup D \in L$. Again for any $X, Y, Z \in L, X, Y, Z \subseteq U$. It is straightforward that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$, $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$. Hence L is a distributive lattice. □

Theorem 3.3. *If $\{N(x) : x \in U\}$ is a partition of U, then L is a Boolean lattice.*

Proof. According to Theorem 3.6, L is a distributive lattice. Moreover, L is a bounded lattice. Therefore, we need to prove only that L is a complemented lattice. In other words, we need to prove that $X^{co} \in L$ for any $X \in L$. If $X^{co} \notin L$, i.e., $\cup_{x \in X^{co}} N(x) \neq X^{co}$, then there exists $y \in \cup_{x \in X^{co}} N(x)$ such that $y \notin X^{co}$. Since $y \in \cup_{x \in X^{co}} N(x)$, then there exists $z \in X^{co}$ such that $y \in N(z)$. Since $\{N(x) : x \in U\}$ is a partition of $U, z \in N(y)$. Therefore, $N(y) \nsubseteq X, i.e., \cup_{x \in X} N(x) \neq X$, which is contradictory with $X \in L$. Hence, $X^{co} \in L$ for any $X \in L$, i.e., L is a complemented lattice. Consequently, L is a Boolean lattice. □

4 Covering numbers

Various techniques have been proposed to characterize rough sets ([2], [3], [5]). Similarly, we establish some measurements to describe covering- based rough sets quantitatively.

4.1 Definitions and Properties of Covering Numbers

The upper covering number of a subset of a domain is the minimal number of some elements in a covering which can cover the subset. The lower covering number of a subset is the maximal number of some elements in a covering which can be included in the subset.

Definition 4.1. *Let C be a covering of U. For all $X \subseteq U$, we define*

- $N \uparrow_C (X) = min\{|B| : (X \subseteq \cup B) \wedge (B \subseteq C)\}$.
- $N \downarrow_C (X) = |\{K \in C | K \subseteq X\}|$.

$N \uparrow_C (X)$ and $N \downarrow_C (X)$ are called the upper and lower covering numbers of X with respect to C. When there is no confusion, $N \uparrow_C (X)$ is denoted simply by $N \uparrow (X)$, and $N \downarrow_C (X)$ by $N \downarrow (X)$.

Example-1: Let $U = \{a, b, c, d\}, D_1 = \{a, b\}, D_2 = \{a, c\}, D_3 = \{b, c\}, D_4 = \{d\}, C = \{D_1, D_2, D_3, D_4\}, X = \{a, d\}, Y = \{a, b, c\}$. Then $B_1 = \{D_1, D_4\}, B_2 = \{D_2, D_4\}, B_3 = \{D_1, D_2, D_4\}, B_4 = \{D_1, D_3, D_4\}, B_5 = \{D_2, D_3, D_4\}$, and $B_6 = \{D_1, D_2, D_3, D_4\}$ are also coverings of X; in other words, $X \subseteq \cup B_i$ for $i \in \{1, 2, 3, 4, 5, 6\}$. So $N \uparrow (X) = min\{|B_i| : 1 \leq i \leq 6\} = 2$. $N \downarrow (X) = |\{K \in C | K \subseteq X\}| = |\{K_4\}| = 1$. Similarly, $N \uparrow (Y) = 2$ and $N \downarrow (Y) = 3$. In particular, $N \uparrow \emptyset = 0$ since, $\{\emptyset\} \subseteq \cup\{\emptyset\}$ and $\{\emptyset\} \subseteq C$. The result makes the concept of the covering numbers more reasonable.

Lemma 4.1. *Let C be a covering of U. For all $K \in C, N \uparrow (K) = 1$.*

Lemma 4.2. *Let C be a covering of U. For all $x \in U, N \uparrow (\{x\}) = 1$.*

5 Lattice for covering numbers

Lattices are important algebraical structures, and have a variety of applications in the real world. This subsection establishes a lattice structure and two semi- lattices in covering-based rough sets with covering numbers.

Definition 5.1. *Let C be a covering of U. For all $X, Y \subseteq U$, if $X \subseteq Y$ and $N \uparrow (X) = N \uparrow (Y)$, we call Y an upper-set of X, and X a lower-set of Y . The family of all upper-sets and the family of all lower-sets are semi-lattices.*

Proposition 5.1. *Let C be a covering of U. For all $X \subseteq U$, we call D_X, D'_X the family of all upper-sets, lower-sets of X, respectively, i.e., $D_X = \{Y \subseteq U : (X \subseteq Y) \wedge (N \uparrow (X) = N \uparrow (Y))\}, D'_X = \{Y \subseteq U : (Y \subseteq X) \wedge (N \uparrow (X) = N \uparrow (Y))\}$. Then $(D_X, \cap)$, and $(D'_X, \cup)$ are semi-lattices.*

Proof. In fact, we only need to prove $Y_1 \cap Y_2 \in D_X$ for all $Y_1, Y_2 \in D$, and $Y_1 \cup Y_2 \in D'_X$ for all $Y_1, Y_2 \in D'_X$. For all $Y_1, Y_2 \in D, N \uparrow (Y_1) = N \uparrow (X), X \subseteq Y_1$ and $N \uparrow (Y2) = N \uparrow (X), X \subseteq Y_2$. So $X \subseteq Y_1 \cap Y_2 \subseteq Y_1$. Thus $N \uparrow (X) \leq N \uparrow (Y_1 \cap Y_2) \leq N \uparrow (Y_1) = N \uparrow (X)$, that is, $N \uparrow (Y_1 \cap Y_2) = N \uparrow (X)$. Therefore, $Y_1 \cap Y_2 \in D$. Similarly, we can prove $Y_1 \cup Y_2 \in D'_X$ for all $Y_1, Y_2 \in D'_X$. □

Definition 5.2. *Let C be a covering of U and $|C| = n$. For $X \subseteq U$, if $N \downarrow (X) + N \downarrow (X^{co}) = n$, we call X a detached-set of U with respect to C. With the detached-set, a covering is divided into two smaller coverings of two smaller domains. Moreover, the concept of the detached-set leads to a lattice structure.*

Proposition 5.2. *Let C be a covering of U and $|C| = n$. $\mathbf{D}$ is denoted as the family of all detached-sets of U, i.e., $\mathbf{D} = \{X \subseteq U : N \downarrow (X) + N \downarrow (X^{co}) = n\}$. Then $(\mathbf{D}, \bigcup, \bigcap)$ is a lattice.*

Proof. For all $X, Y \in \mathbf{D}, N \downarrow (X) + N \downarrow (X^{co}) = n, N \downarrow (Y) + N \downarrow (Y^{co}) = n$. $2n = (N \downarrow (X) + N \downarrow (X^{co})) + (N \downarrow (Y) + N \downarrow (Y^{co})) \leq (N \downarrow (X \bigcup Y) + N \downarrow (X \bigcap Y)) + (N \downarrow (X^{co} \bigcup Yc)) + N \downarrow (X^{co} \bigcap Y^{co}) = [N \downarrow (X \bigcup Y) + N \downarrow ((X \bigcup Y)^{co})] + [N \downarrow (X \bigcap Y) + N \downarrow ((X \bigcap Y)^{co})]$. $N \downarrow (X \bigcup Y) + N \downarrow ((X \bigcup Y)^{co}) = n$ and $N \downarrow (X \bigcap Y) + N \downarrow ((X \bigcap Y)^{co})$ since $N \downarrow (X) + N \downarrow (X^{co}) \leq n$ for all $X \subseteq U$. Thus $X \bigcup Y \in \mathbf{D}$ and $X \bigcap Y \in \mathbf{D}$. □

Proposition 5.3. *The covering lower and upper approximations have the following properties:*
(1) $X \downarrow_C \subseteq X \uparrow_C$
(2) $\emptyset \downarrow_C = \emptyset \uparrow_C = \emptyset$ *and* $U \downarrow_C = U \uparrow_C = U$
(3) $(X \cap Y) \downarrow_C = X \downarrow_C \cap Y \downarrow_C$ *and* $(X \cup Y) \uparrow_C = X \uparrow_C \cup Y \uparrow_C$
(4) $(X \downarrow_C) \downarrow_C = X \downarrow_C$ *and* $(X \uparrow_C) \uparrow_C$
(5) If $X \subseteq Y$ *then* $X \downarrow_C \subseteq Y \downarrow_C$ *and* $X \uparrow_C \subseteq Y \uparrow_C$
(6) $X \uparrow_C = \sim (\sim X) \downarrow_C$.

6 Conclusion

In this paper, we investigated some fundamental issues of approximation in the context of rough set theory based on covering based rough set approximation. Lattice based on covering rough approximation and lattice for covering numbers are also introduced. Our discussion is based on the notion of lattice that represents the relationships between elements of a universe with neighborhood system. Furthermore one can find the lattice for successive rough approximation and stratified rough approximation based on covering system.

References

[1] Bargiela, A., Pedrycz, W .: Granular Computing: An Introduction, *Kluwer Academic Publishers, Boston,* 180, 2002.

[2] Comer, S .: An algebraic approach to the approximation of information, *Fundamenta Informaticae,* 14(1991), 492-502.

[3] Lin, T.Y .: Granular computing - structures, representations, and applications, *In: LNAI*, 2639(2003), 16 - 24.

[4] Liu, G., Sai, Y .: Invertible approximation operators of generalized rough sets and fuzzy rough sets, *Information Sciences,* 180(2010), 2221-2229.

[5] Pawlak, Z.: Rough sets: A new approach to vagueness. In: L.A. Zadeh and J. Kacprzyk, Eds. *Fuzzy Logic for the Management of Uncertainty*, New York, John Wiley and Sons, 105-118, 1992.

[6] Pawlak, Z .: Rough sets : Theoretical aspects of reasoning about data, *Kluwer Academic Publishers, Boston,* 1991.

[7] Pawlak, Z., Some issues on rough sets, *Transactions on Rough Set, I, Journal Subline, Lecture Notes in Computer Science,* 3100(2004), 1-58.

[8] Rana, D., Roy, S. K .: Rough Set Approach on Lattice, *Journal of Uncertain Systems,* 5(1)(2011), 72-80.

[9] Wang, F.Y .: Outline of a computational theory for linguistic dynamical systems: Toward computing with words, *International Journal of Intelligent Control and Systems,* 2(1998), 211-224.

[10] Yao, Y .: Algebraic approach to rough sets, *Bull. Polish Acad. Sci. Math,* 35(1987), 673 - 683.

[11] Yao, Y .: A partition model of granular computing, *LNCS,* 3100(2004), 232 - 253.

[12] Yao, Y .: Constructive and algebraic methods of theory of rough sets, *Information Sciences,* 109, 21-47, 1998.

[13] Yao, Y .: Granular computing: basic issues and possible solutions, *In: Proceedings of the 5th Joint Conference on Information Sciences,* 1(2000), 186 - 189.

[14] Yao, Y .: Three-way decisions with probabilistic rough sets, *Information Science*, 180(2010), 341-353.

[15] Zadeh, L. A .: Fuzzy Logic = computing with words, *IEEE Transactions on Fuzzy Systems,* 4(1996), 103-111.

[16] Zadeh, L. A .: Fuzzy Sets, *Information and Control*, 8(1965), 338-353.

[17] Zhu, W.: Topological Approaches to Covering Rough Sets, *Information Sciences*, 177(2007), 1499-1508.

[18] Zhu, W.: Relationship among Basic Concepts in Covering-based Rough Sets, *Information Sciences*, 179(14)(2007), 2478-2486.

[19] Zhu, W.: Basic Concepts in Covering-Based Rough Sets, in ICNC'07, *Haikou, China, 24-27 August, 2007,* 5, 283-286, 2007.

[20] Zhu, W., Wang, F.Y.: On Three Types of Covering Rough Sets, *IEEE Transactions On Knowledge and Data Engineering,* 19(8)(2007), 1131-1144.

Received: October 19, 2012; *Accepted*: April 14, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 228–235

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

An improved proxy blind signature scheme based on ECDLP

Manoj Kumar Chande*

Shri Shankaracharya Institute Of Professional Management & Technology, Raipur, 492015, Chhattisgarh, India.

Abstract

In a proxy blind signature scheme, there is an integration of the properties as well as advantages of both signature schemes namely proxy signature and blind signature. The concept of this signature scheme with a salient feature that, it allows a designated person say proxy signer to sign on behalf of original signer, in such a way that he/she neither has any idea about the content of the message, nor he/she can make a linkage between the signature and the identity of the requester. Therefore, it is very suitable and easily adoptable for electronic commerce, e-cash applications. Recently, Pradhan and Mohapatra et al.'s claims that their proposed signature scheme satisfies all the properties mandatory for a proxy blind signature scheme. Unfortunately, their scheme fails to fulfil the unlinkability property. To overcome with this weakness, an improved proxy blind signature scheme is presented with the same intractable problem ECDLP. The analysis shows that the new scheme resolves the problem in the former scheme and meets all the aspects of security features needed by proxy blind signature scheme. The analytic results prove that the new scheme is more secure and practicable.

Keywords: Digital Signature, Discrete Logarithm Problem, Forward Security, Proxy Blind Signature.

2010 MSC: 94A60.

1 Introduction

David Chaum [1], presented the concept of blind signature in 1983, which allows the signature requester to have a given message signed by the signer without revealing any information about the message or its signature. Firstly, in the year 1996, Mambo [2, 3], introduced the concept of proxy signatures and proposed several constructions. It allows an original signer to delegate his signing power to a designated person, called the proxy signer, who has the power to act on behalf of the original signer. Proxy blind signature is an important extension of basic proxy signature; it can be widely used in many practical applications.

The first proxy blind signature scheme was introduced by Lin and Jan [4]. Later, there are two new schemes have been proposed, one is Tan's scheme [5], using schnorr's blind signature scheme based on discrete logarithm problem (DLP) and elliptic curve discrete logarithm problem (ECDLP) respectively. The other one is Lal et al.'s scheme [6], which is based on Mambo [2, 3], proxy signature scheme. Afterwards, Wang and Wang [7], proposed a proxy blind signature scheme based on ECDLP in 2005. However, Yang and Yu [8], proved that Wang and Wang's scheme did not meet the security properties and proposed an improved proxy blind signature scheme in 2008, but their scheme does not satisfy the unforgeabilty property. The proxy blind signature scheme focuses on both privacy and authentication, it should meet the following security properties -

Distinguishability: The normal signature made by the original signer, and the proxy blind signature made by the proxy signer both are distinguishable.

Identifiability: Anybody can confirm the identities of the original signer and the proxy signer.

**E-mail addresses*: manojkumarchande@gmail.com (Manoj Kumar Chande) .

Prevention of misuse: The proxy key pair should be used only for creating a proxy signature, which conforms to delegation information.

Nonrepudiation: The original signer and the proxy signer both cannot later falsely claim that they have not performed the signing procedures.

Unforgeability: No one, other than the proxy signer, can produce a valid proxy blind signature.

Unlinkability: The proxy signer or the original signer unable to link the relevance between the blinded message he signed and the revealed signature.

Verifiability: Any arbitrary verifier can be able to verify the proxy blind signature correctly.

Recently, Pradhan and Mohapatra [9], also proposed a new proxy blind signature scheme based on ECDLP. They claim that their scheme is secure and satisfy all the required properties. Unfortunately, their scheme cannot hold the unlinkability property. In this paper the scheme of Pradhan and Mohapatra [9], is improved in such a way, that the presented signature scheme fulfill the unlinkability property.

2 Preliminaries

2.1 Elliptic curve cryptography

The modern-day elliptic curve cryptography (ECC) begins with Koblitz [10] and Miller [11], they provide attractive alternative cryptosystem independently, because its security is based on ECDLP, and it is more efficient as compared with the traditional exponential cryptosystem like RSA [12] and ElGamal [13]. ECC operates over a group of points on an elliptic curve and offers a level of security comparable to classical cryptosystems that uses much larger key's. ECC offers the same security level with a shorter key's [14]. Therefore, the applications that use ECC for such devices will require fewer processor loops, less memory size, smaller key lengths, and less power consumption when compared with the applications using other public key cryptosystem algorithms. With growing potential in e-commerce, ECC systems will be considered to be an important alternative solution to ensure robust security.

2.2 Elliptic curve over finite galois field F_q

Let $q \geq 3$ be any prime number and $a, b \in F_q$, such that $4a^3 + 27b^2 \neq 0$ in F_q, this condition ensures that the defined elliptic curve has no multiple roots of unity. An elliptic curve $E(F_q)$, defined by the parameters a and b is the set of all solutions $(x, y) \in F_q$, to the equation $y^2 = x^3 + ax + b$. These points (x, y) together with an extra point at infinity, form an abelian group.

2.3 Addition law for points on elliptic curve

1. Point of identity - The point O is said to be the point of identity if,

$$P + O = O + P = P, \forall P \in E(F_q).$$

2. Negation of a point - Let a point $P(x, y) \in E(F_q)$, then any point with coordinate values $(x, -y)$ is said to be negation of P. The negation of point P is denoted by $-P$. This is because, their sum gives identity element, particularly $(x, y) + (x, -y) = O$.

3. Addition of points - Let $P(x_1, y_1)$, $Q(x_2, y_2) \in E(F_q)$, then $P + Q = R \in E(F_q)$ and coordinate (x_3, y_3) of R is given by

$$x_3 = \lambda^2 - x_1 - x_2$$
$$y_3 = \lambda(x_1 - x_3) - y_1,$$
$$where\ \lambda = \frac{y_2 - y_1}{x_2 - x_1}$$

4. Doubling of point - Let us take a point $P(x_1, y_1) \in E(F_q)$, where $P \neq -P$, then $P + P = 2P = (x_3, y_3)$, and coordinate values (x_3, y_3) are obtained as follows

$$x_3 = (\frac{3x_1^2 + a}{2y_1})^2 - 2x_1$$
$$y_3 = (\frac{3x_1^2 + a}{2y_1})(x_1 - x_3) - y_1 \ .$$

2.4 Elliptic curve discrete logarithm problem (ECDLP)

The security of elliptic curve cryptosystem relies on the intractability of ECDLP. Let us consider an elliptic curve $E(F_q)$ over a finite field and a point P of order n. For an element Q ($Q \neq P$), the problem is to find an integer d such that $Q = dP$, where $1 \leq d \leq n-1$. The number $d = log_P Q$, is called the discrete logarithm of Q to the base P.

3 Review of Pradhan and Mohaptra's proxy blind signature scheme

In this section, Pradhan and Mohapatra's [9], efficient proxy blind signature scheme based on ECDLP given in detail. The proposed scheme is divided into five phases: (1) System parameters, (2) Proxy delegation, (3) Blind signing, (4) Signature extraction and (5) Signature verification.

(1) **System Parameters and Notations**

U_o	–	Original Signer
U_p	–	Proxy Signer
U_r	–	Signature Requester
B	–	Base Point
$h(.)$	–	Hash Function
x_o	–	Private key of Original Signer
y_o	–	Public key of Original Signer, $y_o = x_o B$
x_p	–	Private key of proxy signer
y_p	–	Public key of proxy signer, $y_p = x_p B$
m_w	–	Warrant, contains the identity's information of the original signer and the proxy signer, validation periods of delegation, limits of authority.

(2) **Proxy Delegation**

The proxy signing key pair (S_{pr}, y_{pr}) is generated as follows:

- Original Signer U_o, randomly chooses k_o, where $(1 < k_o < n)$ and computes

$$R_o = k_o B = (x_{R_o}, y_{R_o})$$
$$r_o = x_{R_o} \mod n$$
$$s_o = x_o + k_o h(m_w \| r_o) \mod n$$

- Now U_o, sends (R_o, s_o, m_w) to the proxy signer U_p, through a secure channel.

- Then U_p checks,

$$s_o B = R_o h(m_w \| r_o) + y_o$$

If it is correct, U_p accepts it, and computes the proxy signer's secret key

$$S_{pr} = x_p + s_o$$

and the corresponding proxy public key is

$$y_{pr} = y_o + y_p + R_o h(m_w \| r_o) = BS_{pr}$$

(3) **Blind Signing**

- Proxy signer U_p, select a number k_p randomly, such that $1 < k_p < n$ and compute

$$R_p = k_p B = (x_{R_p}, y_{R_p})$$
$$r_p = (R_p)_x$$

and then send (R_o, R_p, m_w), to signature requester U_r.

- Signature requester U_r randomly select three numbers a, b, c and compute

$$r = R_p + bB - y_{pr}(a + c) \mod n$$

provided $r \neq 0$, otherwise select a, b, c again. Now signature requester U_r computes

$$e^* = h(r \| m)$$
$$e = e^* - c - a \tag{3.1}$$

and sends e to the proxy signer U_p.

- After receiving e, U_p computes

$$S' = eS_{pr} + k_p$$

and send S' to receiver.

(4) **Signature Extraction**

After receiving S', the receiver U_r computes

$$S = S' + b \tag{3.2}$$

Finally, the proxy blind signature of the message m is (m_w, r_o, m, e^*, S).

(5) **Signature Verification**

The recipient of proxy blind signature verifies (m_w, r_o, m, e^*, S), by checking

$$e^* = h((SB - e^* y_{pr}) \| m) \tag{3.3}$$

if it is true, then the proxy blind signature is valid one else reject it.

4 Absence of unlinkability in Pradhan and Mohapatra's scheme

In Pradhan and Mohapatra's scheme, the signature requester U_r, uses three blinding factor a, b and c. The signature requester U_r, verify the proxy blind signature (m_w, r_o, m, e^*, S), and after this the signature is made open by the requester. The proxy signer uses his signing data (S'_i, e_i, R_{p_i}), which he stores purposely, to find link between proxy blind signatures and his signed messages. Using stored records, he can find one of the blinding factor b from the equation (3.2), as $b = S - S'_i$. It is difficult to find the rest of the blind factors a and c separately, so he find sum of the blinding factors a and c from the equation (3.1). Let the sum of the blinding factors a and c is, $a + c = e^* - e = \alpha$, so with this sum α and previously calculated blinding factor b, proxy signer compute

$$\bar{R} = SB - e^* y_{pr} \tag{4.1}$$

Finally, the proxy signer can check the equation

$$\bar{R} = R_{p_i} + bB - y_{pr}\,\alpha \tag{4.2}$$

if the values from equations (4.1) and (4.2) are same then, the proxy signer is able to find linkage between the proxy blind signature and his signed blind message. This shows that Pradhan and Mohapatra's scheme is insecure, because there is absence of unlinkability.

5 Improved proxy blind signature based on ECDLP

In this section the proposed improved proxy blind signature is given. The system parameters and notations are same as used in Pradhan and Mohapatra [9].

(1) **Proxy Delegation**

- The original signer U_o , randomly chooses $1 < k_o < n$, and computes

$$R_o = k_o B = (x_1, y_1)$$
$$r_o = x_1 \mod n$$
$$s_o = x_o + k_o h(m_w \| r_o)$$

- The original signer U_o , sends (R_o, s_o, m_w), to the proxy signer U_p , through secure manner.

- When the proxy signer U_p , receives (R_o, s_o, m_w), from the original signer, he checks the following equation

$$s_o B = R_o h(m_w \| r_o) + y_o$$

If this equation holds, the proxy signer accepts the proxy delegation, and computes the proxy secret key as

$$s_{pr} = x_p + s_o \mod n$$

and the corresponding proxy public key is

$$y_{pr} = y_o + y_p + R_o h(m_w \| r_o) \mod n$$

(2) **Blind Signing**

- The proxy signer U_p randomly chooses $1 < k_p < n$ and computes $R_p = k_p = (x_2, y_2)$ and $r_p = x_2 \mod n$ and sends (R_o, R_p, m_w) to the requester.

- The requester U_r, randomly chooses three blinding factors a, b and c, then he computes

$$\bar{R} = aR_p + bB + cy_{pr}$$

If $\bar{R} = O$, then the requester must attempt other combinations of (a, b, c) until $\bar{R} \neq O$. The requester then computes

$$e^* = h(\bar{R} \| m) \mod n$$

and

$$e = a^{-1}(e^* + c) \mod n$$

and sends the blind message e to the proxy signer.

- After receiving e, the proxy signer computes

$$S'' = es_{pr} + k_p \mod n$$

and sends S'' back to the requester.

- The requester computes

$$S = S''a + b \mod n$$

Finally, the proxy blind signature of the message is (m_w, r_o, m, e^*, S).

(3) **Verification**

The verifier verifies the validity of the proxy blind signature by checking the following equation

$$e^* = h((SB - e^* y_{pr}) \| m) \quad (5.1)$$

If this equation holds then only the signature is valid otherwise invalid.

6 Security analysis of the proposed scheme

In this section, it is shown that the presented improved proxy blind signature scheme satisfies the security requirements according to the definitions in [9].

(a) **Distinguishbility**

The warrant m_w, is one of the component of the presented proxy blind signature (m_w, r_o, m, e^*, S), so anyone can distinguish the proxy blind signature from the normal signature.

(b) **Identifiability**

Using the verification equation (5.1), and the content of the warrant m_w, the verifier or other users can determine the identity of the corresponding proxy signer U_p, from the proxy signature.

(c) **Nonrepudiation**

In the presented scheme, since only the proxy signer U_p , know the proxy secret key s_{pr}, so no one can else produce S''. Therefore, the proxy signer U_p, cannot deny having signed the message on behalf of original signer.

(d) **Prevention of Misuse**

The message warrant m_w, is very vital part of proposed proxy blind signature scheme. This m_w, includes information regarding the identity of the original signer U_o, the proxy signer U_p, message type to be signed by the proxy signer, and delegation period, etc. Using the proxy key, the proxy signer U_p cannot sign messages that have not been authorized by the original signer. In this way the misuse of key's of original signer and proxy signer is prevented.

(e) **Unforgeability**

If an adversary wants to forge a valid proxy blind signature $(m_w, r_o, \bar{m}, \bar{e}, \bar{S})$, such that it can pass the verification equation $\bar{e} = h((\bar{S}B - \bar{e}y_{pr}) \| \bar{m})$, the adversary has to solve $\bar{S}$. It is difficult to do that because he has to solve the elliptic curve discrete logarithm problem (ECDLP) which is assumed to be infeasible.

(f) **Unlinkability**

Suppose that the proxy signer records all messages he signed (S_i'', e_i, R_{p_i}). After the proxy, blind signature (m_w, r_o, m, e^*, S), is revealed in the public by the requester, the proxy signer still unable to find the blinding factor a, b and c by computing the following equation:

$$e_i = a^{-1}(e^* + c) \mod n$$
$$S = S_i''a + b \mod n$$

Thus, he cannot check if the equation $\tilde{R} = aR_{p_i} + bB + cy_{pr}$, holds, meaning the proxy signer unable to trace the proxy blind signature with the corresponding signature transcript.

(g) **Verifiability**

The verifier U_r, can verify the proxy blind signature by checking the equation (5.1). The correctness of the proxy blind signature is obtained as follows

$$\begin{aligned} SB - e^*y_{pr} &= (S''a + b)B - e^*y_{pr} \\ &= (es_{pr} + k_p)aB + bB - e^*y_{pr} \\ &= es_{pr}aB + k_paB + bB - e^*y_{pr} \\ &= a^{-1}(e^* + c)ay_{pr} + aR_p + bB - e^*y_{pr} \\ &= e^*y_{pr} + cy_{pr} + aR_p + bB - e^*y_{pr} \\ &= aR_p + bB + cy_{pr} \\ &= \tilde{R} \end{aligned}$$

In summary, it is shown that the construction based on ECDLP is secure because, it can achieve the unlinkability property. Hence the proposed scheme satisfies all the security requirements of the proxy blind signature.

7 Conclusion

In this article, a linkability attack mounted on Pradhan and Mohapatra's proxy blind signature scheme, and it is demonstrated that how their scheme is insecure due to the absence of unlinkability property. This proposed proxy blind signature scheme holds all the security properties of both proxy and blind signature scheme. The security of the proposed schemes is based on the difficulty of the elliptic curve discrete logarithm problem (ECDLP).

References

[1] David Chaum, Blind signature for untraceable payments, *Advances in Cryptology, proceeding of CRYPTO'82, Springer-Verlag, New York*, 199–203, 1983.

[2] M. Mambo, K. Usuda, E. Okamoto, Proxy signatures: delegation of the power to sign message, *IEICE Transactions on Fundamentals*, E79-A(9)(1996), 1338–1354.

[3] M. Mambo, K. Usuda, and E. Okamoto, Proxy signatures for delegating signing operation, *In: Proceeding of 3rd ACM Conference on Computer and Communications Security (CCS'96)*, 48–57, ACM Press, (1996).

[4] W. D. Lin and J.K. Jan, A security personal learning tools using a proxy blind signature scheme, *Procceedings of International Conference on Chinese Language Computing*, Illinois, USA, July 2000, 273–277.

[5] Z. Tan, Z. Liu, C. Tang, Digital proxy blind signature schemes based on DLP and ECDLP, *MM Research Preprints, MMRC, AMSS, Academia, Sinica, Beijing*, 21, 212–217, (2002).

[6] A.K. Awasthi and S. Lal, Proxy blind signature scheme, *Journal of Information Science and Engineering, 2003, Cryptology ePrint Archive*, Report 2003/072, Available at $< http://eprint.iacr.org >$.

[7] H. Y. Wang and R. C. Wang, A proxy blind signature scheme based on ECDLP, *Chinese Journal of Electronics*, 14(2)(2005), 281–284.

[8] X. Yang and Z. Yu, Security Analysis of a proxy blind signature scheme based on ECDLP, *in Proceeding of* 4^{th} *International Conference on Wireless Communications, Networking and Mobile Computing (WiCOM'08)*, Oct. 2008, 1–4.

[9] S. Pradhan and R. K. Mohapatra, Proxy blind signature scheme based on ECDLP, *International Journal of Engineering Science & Technology*, 3(3)(2011), 2244–2248.

[10] N. Koblitz, Elliptic Curve Cryptosystems, *Math. Comp.*, 48(1987), 203–209.

[11] V. S. Miller, Use of elliptic curves in cryptography. *In Advances in Cryptology-CRYPTO'85, Santa Barbara, CA, 1985, Lecture Notes in Computer Science, 218, Springer-Verlag, Berlin*, 417–426, 1986.

[12] R.L. Rivest, A. Shamir and L.M. Adleman, A method for obtaining digital signatures and public-key cryptosystems, *Communications of the ACM* (2)(21)(1978), 120–126.

[13] T. ElGamal, A public-key cryptosystem and a signature scheme based on discrete logarithms, *IEEE Transactions on Information Theory*, 31(1985), 469–472.

[14] A. Lenstra, E. Verhuel, Selecting cryptographic key sizes, *Journal of Cryptography*, 14(2001), 255–293.

Received: January 19, 2013; *Accepted*: February 16, 2013

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 236–242

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Generating relations involving 2-variable Hermite matrix polynomials

Ahmed Ali Al-Gonah[a,*]

[a]*Department of Mathematics, Aden University, Aden, Yemen.*

Abstract

In the present paper, some generating relations involving the 2-variable Hermite matrix polynomials are derived by using operational techniques. Further, some new and known generating relations for the scalar Hermite polynomials are obtained as applications of the main results.

Keywords: Hermite matrix polynomials, Generating relations, Operational techniques.

2010 MSC: 15A60, 33C05, 33C25, 33C45, 33C50.

1 Introduction

An important generalization of special functions is special matrix functions. The study of special matrix polynomials is important due to their applications in certain areas of statistics, physics and engineering. The Hermite matrix polynomials are introduced by Jódar and Company in [12]. Some properties of the Hermite matrix polynomials are given in [9, 10, 12, 13, 14]. The extensions and generalizations of Hermite matrix polynomials have been introduced and studied in [2, 3, 15, 16, 19] for matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues are all situated in the right open half-plane.

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and if A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A)\subset\Omega$, then the matrix functional calculus [11] yields that

$$f(A)g(A)=g(A)f(A).$$

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of z, then $z^{\frac{1}{2}}$ represents $\exp(\frac{1}{2}\log(z))$. If A is a matrix with $\sigma(A)\subset D_0$, then $A^{\frac{1}{2}}=\sqrt{A}=\exp(\frac{1}{2}\log(A))$ denotes the image by $z^{\frac{1}{2}}=\sqrt{z}=\exp(\frac{1}{2}\log(z))$ of the matrix functional calculus acting on the matrix A. We say that A is a positive stable matrix [10] if

$$Re(z)>0,\quad \text{for all } z\in\sigma(A). \tag{1.1}$$

We recall that the 2-variable Hermite matrix polynomials (2VHMaP) $H_n(x,y,A)$ are defined by the series [2; p.84]

$$H_n(x,y,A)=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k y^k(x\sqrt{2A})^{n-2k}}{(n-2k)!k!}\quad (n\geq 0) \tag{1.2}$$

and specified by the generating function

$$\exp(xt\sqrt{2A}-yt^2I)=\sum_{n=0}^{\infty}H_n(x,y,A)\frac{t^n}{n!}. \tag{1.3}$$

*Corresponding author.

E-mail address: gonah1977@yahoo.com (Ahmed Ali Al-Gonah).

It is worth to mention that these matrix polynomials are linked to 2-variable Hermite-Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [1] by the following relation:

$$H_n(x,y,A) = H_n(x\sqrt{2A}, -y), \tag{1.4a}$$

or, equivalently

$$H_n\left((\sqrt{2A})^{-1}x, -y, A\right) = H_n(x,y), \tag{1.4b}$$

where $H_n(x,y)$ are defined by the series [1]

$$H_n(x,y) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{y^k x^{n-2k}}{k!(n-2k)!} \tag{1.5}$$

and specified by the generating function

$$\exp(xt+yt^2) = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.6}$$

Also, for $A = \frac{1}{2} \in \mathbb{C}^{1\times 1}$ in equation (1.3) and in view of generating function (1.6), we have

$$H_n\left(x, -y, \frac{1}{2}\right) = H_n(x,y). \tag{1.7}$$

In particular, we note that

$$H_n(x,y,A) = y^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{y}}, A\right), \tag{1.8}$$

$$H_n(x,1,A) = H_n(x,A), \tag{1.9}$$

where $H_n(x,A)$ denotes the Hermite matrix polynomials (HMaP) defined by [12]

$$\exp(xt\sqrt{2A} - t^2 I) = \sum_{n=0}^{\infty} H_n(x,A)\frac{t^n}{n!} \tag{1.10}$$

and linked to the classical Hermite polynomials $H_n(x)$ [18] by the following relation:

$$H_n(x,A) = H_n\left(x\sqrt{\frac{A}{2}}\right), \tag{1.11}$$

where $H_n(x)$ are defined by [18]

$$H_n(x) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k(2x)^{n-2k}}{k!(n-2k)!}. \tag{1.12}$$

The 2VHMaP $H_n(x,y,A)$ are also defined by the following operational rule [2; p.90]:

$$H_n(x,y,A) = \exp\left(-y(2A)^{-1}\frac{\partial^2}{\partial x^2}\right)\left\{(x\sqrt{2A})^n\right\} \tag{1.13}$$

and have the following representation [3; p.99]:

$$H_n(x,y,A) = \left(x\sqrt{2A} - 2y(\sqrt{2A})^{-1}\frac{\partial}{\partial x}\right)^n \{I\}. \tag{1.14}$$

Recently, Dattoli and his co-workers have shown that operational methods can be used to simplify the derivations of many properties of ordinary and generalized special functions and also provide a unique tool to treat various polynomials from a general and unified point of view, see for example [4-8]. In this paper, we derive some generating relations involving the 2VHMaP $H_n(x,y,A)$ which further prove the usefulness of the methods of operational nature.

2 Generating relations

We prove the following results by using operational techniques:

Theorem 2.1. *For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition (1.1), the following generating relation involving the 2VHMaP $H_n(x,y,A)$ holds true:*

$$\sum_{n=0}^{\infty} H_{2n}(x,y,A)\frac{t^n}{n!} = \frac{1}{\sqrt{1+4yt}}\exp\left(\frac{2Ax^2t}{1+4yt}\right). \tag{2.1}$$

Proof. By making use of equation (1.13) in the l.h.s. of equation (2.1), we find

$$\sum_{n=0}^{\infty} H_{2n}(x,y,A)\frac{t^n}{n!} = \exp\left(-y(2A)^{-1}\frac{\partial^2}{\partial x^2}\right)\sum_{n=0}^{\infty}(x\sqrt{2A})^{2n}\frac{t^n}{n!}, \tag{2.2}$$

which on using the exponential function becomes

$$\sum_{n=0}^{\infty} H_{2n}(x,y,A)\frac{t^n}{n!} = \exp\left(-y(2A)^{-1}\frac{\partial^2}{\partial x^2}\right)\ \exp(2Ax^2t). \tag{2.3}$$

Using the generalized Glaisher identity [6]

$$\exp\left(\lambda\frac{d^2}{dx^2}\right)\left\{\exp(-ax^2+bx)\right\} = \frac{1}{\sqrt{1+4a\lambda}}\exp\left(-\frac{ax^2-bx-b^2\lambda}{1+4a\lambda}\right), \tag{2.4}$$

with $b=0$ in the r.h.s. of equation (2.3), we get assertion (2.1) of Theorem 2.1. □

Remark 2.1. *Taking $y=1$ and replacing t by $-\left(\frac{t}{2}\right)^2$ in assertion (2.1) of Theorem 2.1 and using equation (1.9), we get the result [9; p.122]*

$$\sum_{n=0}^{\infty}(-1)^n\ H_{2n}(x,A)\frac{t^n}{n!\,2^{2n}} = (1-t^2)^{-\frac{1}{2}}\exp\left(\frac{A}{2}\ \frac{-x^2t^2}{(1-t^2)}\right). \tag{2.5}$$

Theorem 2.2. *For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition (1.1), the following generating relation involving the 2VHMaP $H_n(x,y,A)$ holds true:*

$$\sum_{n=0}^{\infty} H_{n+k}(x,y,A)\frac{t^n}{n!} = \exp(xt\sqrt{2A}-yt^2I)\ H_k\left(xI-yt\left(\sqrt{\frac{A}{2}}\right)^{-1},y,A\right). \tag{2.6}$$

Proof. By making use of equation (1.14) in the l.h.s. of equation (2.6), we find

$$\sum_{n=0}^{\infty} H_{n+k}(x,y,A)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(x\sqrt{2A}-2y(\sqrt{2A})^{-1}\frac{\partial}{\partial x}\right)^{n+k}\frac{t^n}{n!}, \tag{2.7}$$

which on simplifying the r.h.s. and again using equation (1.14) becomes

$$\sum_{n=0}^{\infty} H_{n+k}(x,y,A)\frac{t^n}{n!} = \exp\left(xt\sqrt{2A}-2yt(\sqrt{2A})^{-1}\frac{\partial}{\partial x}\right)\ H_k(x,y,A). \tag{2.8}$$

Now, decoupling the exponential operator in the r.h.s. of the above equation by using the Weyl identity [7]

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-k/2} \qquad \left([\hat{A},\hat{B}]=k,\ k\in\mathbb{C}\right), \tag{2.9}$$

we get

$$\sum_{n=0}^{\infty} H_{n+k}(x,y,A)\frac{t^n}{n!} = \exp(xt\sqrt{2A}-yt^2I)\exp\left(-2yt(\sqrt{2A})^{-1}\frac{\partial}{\partial x}\right)H_k(x,y,A). \tag{2.10}$$

Using the shift operator [7]

$$\exp\left(\lambda\frac{\partial}{\partial x}\right)f(x) = f(x+\lambda), \tag{2.11}$$

in the r.h.s. of equation (2.10), we get assertion (2.6) of Theorem 2.2. □

Remark 2.2. *Taking $y = 1$ in assertion (2.6) of Theorem 2.2 and using equation (1.9), we get the result [15, p. 170] with $b = 1$*

$$\sum_{n=0}^{\infty} H_{n+k}(x, A)\frac{t^n}{n!} = \exp(xt\sqrt{2A} - t^2 I)\; H_k\left(xI - t\left(\sqrt{\frac{A}{2}}\right)^{-1}, A\right). \tag{2.12}$$

Theorem 2.3. *For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition (1.1), the following bilinear generating relation of the 2VHMaP $H_n(x, y, A)$ holds true:*

$$\sum_{n=0}^{\infty} H_n(x, y, A)H_n(z, w, A)\frac{t^n}{n!} = \frac{1}{\sqrt{1-4ywt^2}}\;\exp\left(\frac{2A(xzt-(x^2w+z^2y)t^2)}{1-4ywt^2}\right). \tag{2.13}$$

Proof. By making use of equation (1.13) in the l.h.s. of equation (2.13), we find

$$\sum_{n=0}^{\infty} H_n(x, y, A)H_n(z, w, A)\frac{t^n}{n!} = \exp\left(-y(2A)^{-1}\frac{\partial^2}{\partial x^2}\right)\sum_{n=0}^{\infty} H_n(z, w, A)\;\frac{(xt\sqrt{2A})^n}{n!}, \tag{2.14}$$

which on using generating function (1.3) becomes

$$\sum_{n=0}^{\infty} H_n(x, y, A)H_n(z, w, A)\frac{t^n}{n!} = \exp\left(-y(2A)^{-1}\frac{\partial^2}{\partial x^2}\right)\;\exp(2Axzt - 2Aw(xt)^2). \tag{2.15}$$

Using the generalized Glaisher identity (2.4) in the r.h.s. of equation (2.15), we get assertion (2.13) of Theorem 2.3. □

Remark 2.3. *Taking $w = 1$ in assertion (2.13) of Theorem 2.3 and using equation (1.9), we deduce the following consequence of Theorem 2.3.*

Corollary 2.1. *For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition (1.1), the following generating relation involving the 2VHMaP $H_n(x, y, A)$ and HMaP $H_n(z, A)$ holds true:*

$$\sum_{n=0}^{\infty} H_n(x, y, A)H_n(z, A)\frac{t^n}{n!} = \frac{1}{\sqrt{1-4yt^2}}\;\exp\left(\frac{2A(xzt-(x^2+z^2y)t^2)}{1-4yt^2}\right). \tag{2.16}$$

Remark 2.4. *Taking $y = w = 1$ and replacing t by $\frac{t}{2}$ in assertion (2.13) of Theorem 2.3 and using equation (1.9), we get the result [13] (see [9])*

$$\sum_{n=0}^{\infty} H_n(x, A)H_n(z, A)\frac{t^n}{n!\,2^n} = (1-t^2)^{-\frac{1}{2}}\;\exp\left(\frac{A}{2}\;\frac{2xzt-(x^2+z^2)t^2}{(1-t^2)}\right). \tag{2.17}$$

It is worthy to mention that all the above main results can be proved alternately by using the series rearrangement techniques.

3 Special cases

In this section, we derive some new generating relations for Hermite polynomials in terms of matrix argument as applications of the results derived in Section 2.

I. Replacing y by $-y$ in equation (2.1) and making use of equation (1.4a) in the resultant equation, we get

$$\sum_{n=0}^{\infty} H_{2n}(x\sqrt{2A}, y)\frac{t^n}{n!} = \frac{1}{\sqrt{1-4yt}}\exp\left(\frac{2Ax^2t}{1-4yt}\right), \tag{3.1}$$

which is new generating relation for the 2VHKdFP $H_n(x, y)$ in terms of matrix argument and is a generalization of the generating relation [8, p. 412]

$$\sum_{n=0}^{\infty} H_{2n}(x, y)\frac{t^n}{n!} = \frac{1}{\sqrt{1-4yt}}\exp\left(\frac{x^2t}{1-4yt}\right). \tag{3.2}$$

Again, making use of equation (1.11)in equation (2.5), we get

$$\sum_{n=0}^{\infty} H_{2n}\left(x\sqrt{\frac{A}{2}}\right)\frac{t^n}{n!\,2^{2n}} = (1-t^2)^{-\frac{1}{2}}\exp\left(\frac{A}{2}\,\frac{-x^2t^2}{(1-t^2)}\right), \tag{3.3}$$

which is new generating relation for the classical Hermite polynomials $H_n(x)$ in terms of matrix argument.

II. Replacing y by $-y$ in equation (2.6) and making use of equation (1.4a) in the resultant equation, we get

$$\sum_{n=0}^{\infty} H_{n+k}(x\sqrt{2A},y)\frac{t^n}{n!} = \exp(xt\sqrt{2A}+yt^2I)\;H_k(x\sqrt{2A}+2yt,y), \tag{3.4}$$

which is new generating relation for the 2VHKdFP $H_n(x,y)$ in terms of matrix argument and is a generalization of the generating relation [17, p. 452]

$$\sum_{n=0}^{\infty} H_{n+k}(x,y)\frac{t^n}{n!} = \exp(xt+yt^2)\;H_k(x+2yt,y). \tag{3.5}$$

Again, making use of equation (1.11) in equation (2.12), we get

$$\sum_{n=0}^{\infty} H_{n+k}\left(x\sqrt{\frac{A}{2}}\right)\frac{t^n}{n!} = \exp(xt\sqrt{2A}-t^2I)\;H_k\left(x\sqrt{\frac{A}{2}}-tI\right), \tag{3.6}$$

which is new generating relation for the classical Hermite polynomials $H_n(x)$ in terms of matrix argument and is a generalization of the generating relation [18, p. 197]

$$\sum_{n=0}^{\infty} H_{n+k}(x)\frac{t^n}{n!} = \exp(2xt-t^2)\;H_k(x-t). \tag{3.7}$$

Next, replacing t by $t\sqrt{2A}$ in equation (2.12), we obtain the generating relation [20, p. 191]

$$\sum_{n=0}^{\infty} H_{n+k}(x,A)\frac{(t\sqrt{2A})^n}{n!} = \exp(2xtA-2t^2A)\;H_k\left(x-2t,A\right). \tag{3.8}$$

III. Replacing y by $-y$ and w by $-w$ in equation (2.13) and making use of equation (1.4a) in the resultant equation, we get

$$\sum_{n=0}^{\infty} H_n(x\sqrt{2A},y)H_n(z\sqrt{2A},w)\frac{t^n}{n!} = \frac{1}{\sqrt{1-4ywt^2}}\;\exp\left(\frac{2A(xzt+(x^2w+z^2y)t^2)}{1-4ywt^2}\right), \tag{3.9}$$

which is new bilinear generating relation for the 2VHKdFP $H_n(x,y)$ in terms of matrix argument and is a generalization of the generating relation [5, p. 116] (see also [17, p. 453])

$$\sum_{n=0}^{\infty} H_n(x,y)H_n(z,w)\frac{t^n}{n!} = \frac{1}{\sqrt{1-4ywt^2}}\;\exp\left(\frac{xzt+(x^2w+z^2y)t^2}{1-4ywt^2}\right). \tag{3.10}$$

Again, replacing y by $-y$ in equation (2.16) and making use of equations (1.4a) and (1.11) in the resultant equation, we get

$$\sum_{n=0}^{\infty} H_n(x\sqrt{2A},y)H_n\left(z\sqrt{\frac{A}{2}}\right)\frac{t^n}{n!} = \frac{1}{\sqrt{1+4yt^2}}\;\exp\left(\frac{2A(xzt-(x^2-z^2y)t^2)}{1+4yt^2}\right), \tag{3.11}$$

which is new generating relation for the 2VHKdFP $H_n(x,y)$ and the classical Hermite polynomials $H_n(x)$ in terms of matrix argument.

Further, making use of equation (1.11) in equation (2.17), we get

$$\sum_{n=0}^{\infty} H_n\left(x\sqrt{\frac{A}{2}}\right)H_n\left(z\sqrt{\frac{A}{2}}\right)\frac{t^n}{n!\,2^n} = (1-t^2)^{-\frac{1}{2}}\;\exp\left(\frac{A}{2}\,\frac{2xzt-(x^2+z^2)t^2}{(1-t^2)}\right), \tag{3.12}$$

which is new bilinear generating relation for the classical Hermite polynomials $H_n(x)$ in terms of matrix argument and is a generalization of the generating relation [18, p. 198]

$$\sum_{n=0}^{\infty} H_n(x)H_n(z)\frac{t^n}{n!} = \frac{1}{\sqrt{1-4t^2}}\;\exp\left(\frac{4(xzt-(x^2+z^2)t^2)}{1-4t^2}\right). \tag{3.13}$$

4 Concluding remarks

Recently, Subuhi Khan and Raza [15] introduced the 2-variable Hermite matrix polynomials of the second form $\mathcal{H}_n(x,y;A)$, defined by the series [15, p. 162]

$$\mathcal{H}_n(x,y;A) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{y^k \left(x\sqrt{\frac{A}{2}}\right)^{n-2k}}{k!(n-2k)!} \tag{4.1}$$

and specified by the generating function

$$\exp\left(xt\sqrt{\frac{A}{2}} + yt^2 I\right) = \sum_{n=0}^{\infty} \mathcal{H}_n(x,y;A)\frac{t^n}{n!}. \tag{4.2}$$

From generating functions (1.3) and (4.2), we note that the 2VHMaP of the second form $\mathcal{H}_n(x,y;A)$ are linked to the 2VHMaP $H_n(x,y;A)$ by the following relation:

$$\mathcal{H}_n(x,y;A) = H_n\left(\frac{x}{2}, -y, A\right). \tag{4.3}$$

In view of equation (4.3), we conclude that all the properties of the 2VHMaP of the second form $\mathcal{H}_n(x,y;A)$ can be deduced from the corresponding ones for 2VHMaP $H_n(x,y;A)$. For example, replacing x by $\frac{x}{2}$ and y by $-y$ in the main results (2.1), (2.6) and (2.13), we get the following generating relations involving the 2VHMaP of the second form $\mathcal{H}_n(x,y;A)$:

$$\sum_{n=0}^{\infty} \mathcal{H}_{2n}(x,y,A)\frac{t^n}{n!} = \frac{1}{\sqrt{1-4yt}} \exp\left(\frac{Ax^2t}{2(1-4yt)}\right), \tag{4.4}$$

$$\sum_{n=0}^{\infty} \mathcal{H}_{n+k}(x,y,A)\frac{t^n}{n!} = \exp\left(xt\sqrt{\frac{A}{2}} + yt^2 I\right) \mathcal{H}_k\left(xI + 2yt\left(\sqrt{\frac{A}{2}}\right)^{-1}, y, A\right) \tag{4.5}$$

and

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x,y,A)\, \mathcal{H}_n(z,w,A)\frac{t^n}{n!} = \frac{1}{\sqrt{1+4ywt^2}} \exp\left(\frac{A(4xzt-(x^2w-4z^2y)t^2)}{1+4ywt^2}\right), \tag{4.6}$$

respectively. It is therefore clear that by making use of relation (4.3) in some other generating functions obtained in Section 2, we may get a number of interesting results for the 2VHMaP of the second form $\mathcal{H}_n(x,y;A)$.

In this article, generating relations involving the Hermite matrix polynomials are introduced by making use of operational identities for decoupling of exponential operators. The approach presented here can be explored further to derive the results for some other suitable families of special matrix functions.

5 Acknowledgment

The author is thankful to the referee for valuable comments for improving the presentation of the paper.

References

[1] P. Appell, Kampé de Fériet J., 1926, *Fonctions hypergéométriques et hypersph-ériques: Polynômes d' Hermite*, Gauthier-Villars, Paris.

[2] R.S. Batahan, A new extension of Hermite matrix polynomials and its applications, *Linear Algebra Appl.* **419** (2006), 82-92.

[3] R.S. Batahan, Volterra integral equation of Hermite matrix polynomials, *Anal. Theory Appl.* **29** (2) (2013), 97–103.

[4] G. Dattoli, Hermite-Bessel and Laguerre-Bessel functions: a by product of the monomiality principle. *Advanced special functions and applications* (*Melfi*, 1999), 147-164, Proc. Melfi Sch. Adv. Top. Math. Phys., **1**, *Aracne, Rome*, (2000).

[5] G. Dattoli, Generalized polynomials, operational identities and their applications, *J. Comput. Appl. Math.* **118** (2000), 111-123.

[6] G. Dattoli, Subuhi Khan, P. E. Ricci, On Crofton-Glaisher type relations and derivation of generating functions for Hermite polynomials including the multi-index case. *Integral Transforms Spec. Funct.* **19** (1) (2008), 1–9.

[7] G. Dattoli, P.L. Ottaviani , A. Torre, L. V*á*zquez, Evolution operator equations: integration with algebraic and finite-difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory, *Riv. Nuovo Cimento Soc. Ital. Fis.*(4) **20** (1997), 1-133.

[8] G. Dattoli, A. Torre, S. Lorenzutta, Operational identities and properties of ordinary and generalized special functions, *J. Math. Anal. Appl.* **236** (1999), 399–414.

[9] E. Defez, A. Herv*á*s, L. J*ó*dar, Bounding Hermite matrix polynomials, *Math. Comput. Modelling* **40** (2004) 117-125.

[10] E. Defez, L. J*ó*dar, Some applications of the Hermite matrix polynomials series expansions, *J. Comput. Appl. Math.* **99** (1998), 105-117.

[11] N. Dunford, J. Schwartz, *Linear operators*, Part I, Interscience, New York, 1957.

[12] L. J*ó*dar, R. Company, Hermite matrix polynomials and second order matrix differential equations, *Approx. Theory Appl. (N.S.)* **12** (1996), 20-30.

[13] L. J*ó*dar, E. Defez, A matrix formula for the generating function of the product of Hermite matrix polynomials, *In International Workshop on Orthogonal Polynomials in Mathematical Physics*, (1996) 93–101.

[14] L. J*ó*dar, E. Defez, On Hermite matrix polynomials and Hermite matrix functions, *Approx. Theory Appl. (N.S.)* **14** (1998), 36-48.

[15] Subuhi Khan, N. Raza, 2-variable generalized Hermite matrix polynomials and Lie algebra representation, *Rep. Math. Phys.* **66** (2010), 159-174.

[16] M. S. Metwally, M. T. Mohamed, A. Shehata, Generalizations of two-index two-variable Hermite matrix polynomials, *Demonstratio Math.* **42** (2009), 687-701.

[17] M. I. Qureshi, Yasmeen, M. A. Pathan, Linear and bilinear generating functions involving Gould-Hopper polynomials, *Math. Sci. Res. J.* **6** (9) (2002), 449–456.

[18] E.D. Rainville, *Special functions*, Reprint of 1960 first edition, Chelsea Publishing Co., Bronx, New York, 1971.

[19] K.A.M. Sayyed, M.S. Metwally, R.S. Batahan, On generalized Hermite matrix polynomials, *Electron. J. Linear Algebra* **10** (2003), 272-279.

[20] M.J.S. Shahwan, M.A. Pathan, Generating relations of Hermite matrix polynomials by Lie algebraic method, *Ital. J. Pure Appl. Math.* **25** (2009), 187–192.

Received: January 31, 2014; *Accepted*: March 23, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 243–248

Malaya Journal of Matematik

www.malayajournal.org

MJM an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

A stochastic model for a single grade system with backup resource of manpower

T. Roja Mary,[a] and K.P. Uma[b,*]

[a] Department of Mathematics, Sree Sakthi Engineering College, Coimbatore, Tamil Nadu, India.

[b,] Department of Mathematics, Hindusthan College of Engineering and Technology, Coimbatore, Tamil Nadu, India.

Abstract

In this paper, an organization with single grade subjected to exodus of personnel due to policy decisions taken by it, is considered. In order to avoid the crisis of the organization reaching a breakdown point, a suitable univariate policy recruitment based on shock model approach and cumulative damage process is suggested. A mathematical model is constructed and a performance measure namely the mean time to recruitment is obtained. The analytical results are numerically illustrated and the influences of nodal parameters on the performance measures are studied and relevant conclusions are presented.

Keywords: Single grade system, Univariate policy of recruitment, shock model.

2010 MSC: 90B70, 91B40, 91D35.

1 Introduction

Frequent wastage or exit of personnel is common in many administrative and production oriented organization. Whenever the organization announces revised policies regarding sales target, revision of wages, incentives and perquisites, the exodus is possible. Reduction in the total strength of marketing personnel adversely affects the sales turnover in the organization. As the recruitment involves several costs, it is usual that the organization has the natural reluctance to go in for frequent recruitments. Once the total amount of wastage crosses a certain threshold level, the organization reaches an uneconomic status which otherwise be called the breakdown point and recruitment is done at this point of time. The time to attain the breakdown point is an important characteristic for the management of the organization. Many models could be seen in, Barthlomew [1] and Barthlomew and Forbes [2].Many researchers [3] [4] and[6] have considered the problem of time to recruitment in a marketing organization under different conditions. Srinivasan and Saavithri [5] have considered a single grade system under univariate policy of recruitment with the assumption that survival times follow geometric process and the threshold level as a non-negative constant. They have obtained mean time to recruitment and the long run average cost. Uma.et.al [7] have studied the work of Srinivasan and Saavithri [5] by considering the threshold level of the organization as continuous random variable following exponential distribution and having SCBZ property. Recently, Vijaysankar et.al [8] have constructed a stochastic model by assuming the threshold with two components namely the level of wastage which can be allowed and the manpower which is available from what is known as backup resource. The threshold can be treated now as the total of the maximum allowable attrition and the maximum available backup resource. The backup resource is similar to the manpower inventory on hand which can be utilized whenever it becomes necessary. The present paper studies the problem of time to recruitment for a single grade system with survival times follow geometric process and the threshold level has two components.

*Corresponding author.

E-mail addresses: trojamary@gmail.com (T. Roja Mary), umamaths95@gmail.com (K.P. Uma).

2 Model Description

Consider a single grade organization with univariate policy of recruitment which takes decisions at random epoch. At every decision making epoch a random number of persons quit the organization. There is an associated loss of manhours to the organization if a person quits. The loss of manhours at any decision forms a sequence of independent and identically distributed random variables. The survival time process is a geometric process and it is independent of process of loss of manhour. There is a threshold level for the level of wastage and also a resource backup available. If the total loss of manhours crosses the sum of the threshold and the resource backup available the break down occurs. The process that generates the loss of manhours and the threshold put together with the backup are linearly independent. Recruitment takes place only at decision points and time of recruitment is negligible. . The recruitment is made whenever the cumulative loss of manhours exceeds its threshold.

3 Notations

S_n : survival time after $(n-1)^{th}$ decision.

X_n : the loss of manhours at then^{th} decision.

T_n : the cumulative loss of manhours in the first $n$decisions.

$K(\cdot)$: distribution function of S_n with mean $\dfrac{\lambda}{a^{n-1}}, \quad a > 1.$

$G(\cdot)$: distribution function of $X_n,\ n = 1, 2, 3, \ldots.$

$G_n(\cdot)$: distribution function ofT_n.

T : The threshold of manpower depletion and$T = Y_1 + Y_2$.

$(i) Y_1 =$ the maximum allowable attrition.

$(ii) Y_2 =$ the maximum available backup resource.

$F(\cdot)$: distribution function of$T$.

W : time to recruitment under the given recruitment policy.

4 Results

In this section the expected time to recruitment is derived.

By assumption the recruitment is made whenever the cumulative loss of manhours exceeds the threshold T. Accordingly the time to recruitment $W = S_1$, if $T_1 > T$. If $T_1 \leq T$ then no recruitment is made till the next decision. If T_2 exceeds T then recruitment is made and $W = S_1 + S_2$, otherwise no recruitment. In general, if $T_k > T$ then recruitment is made and $W = S_1 + S_2 + ... + S_k$ and if $T_k \leq T$ no recruitment is made till the next decision.

Consequently,

$$W = \sum_{i=0}^{\infty} \sum_{j=0}^{i} S_{j+1} \chi(T_i \leq T \leq T_{i+1}), \tag{1.1}$$

where

$$\chi(e) = \begin{cases} 1, & \text{if the event } e \text{ happens} \\ 0, & \text{if the event } e \text{ does not happen} \end{cases} \tag{1.2}$$

The expected time to recruitment $E(W)$ is given by

$$E(W) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} E(S_{j+1})P(T_i \leq T \leq T_{i+1}) \tag{1.3}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i} \left(\frac{\lambda}{a^j}\right) P(0 \leq T - T_i < T_{i+1} - T_i)$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i} \left(\frac{\lambda}{a^j}\right) \int_0^{\infty}\int_0^{\infty}\int_t^{t+s} dF(u)dG(s)dG_i(t). \tag{1.4}$$

Assume that the loss of manhours at the i^{th} decision X_i, follows exponential distribution with parameter θ_1. Then the cumulative loss of manhours T_i follows gamma distribution with parameter θ_1 and i. Hence

$$dG_i(t) = \theta_1^i t^{i-1}\frac{e^{-\theta_1 t}}{(i-1)!}dt, \quad i = 1,2,3,\ldots,$$

Let $Y_1 \sim exp(\theta_2)$ and $Y_2 \sim exp(\theta_3)$. Since the p.d.f of T is the convolution of $T_1 + T_2$ and it is given by

$$f(u) = \frac{\theta_2\theta_3}{\theta_2 - \theta_3}\left[e^{-\theta_3 u} - e^{-\theta_2 u}\right].$$

Now the time to recruitment in equation (1.4) becomes

$$E(W) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} \left(\frac{\lambda}{a^j}\right) \int_0^{\infty}\int_0^{\infty}\int_t^{t+s} \frac{\theta_2\theta_3}{\theta_2 - \theta_3}\left[e^{-\theta_3 u} - e^{-\theta_2 u}\right] dudG(s)dG_i(t)$$

$$= \sum_{i=0}^{\infty}\left(\frac{a\lambda}{a-1}\right)\left(1 - \frac{1}{a^{i+1}}\right) \int_0^{\infty}\int_0^{\infty}\int_t^{t+s} \frac{\theta_2\theta_3}{\theta_2 - \theta_3}\left[e^{-\theta_3 u} - e^{-\theta_2 u}\right] dudG(s)dG_i(t)$$

$$= \sum_{i=0}^{\infty}\left(\frac{a\lambda}{a-1}\right)\left(1 - \frac{1}{a^{i+1}}\right)\left(\frac{\theta_2\theta_3}{\theta_2 - \theta_3}\right) \int_0^{\infty}\int_0^{\infty}\int_t^{t+s} \left[e^{-\theta_3 u} - e^{-\theta_2 u}\right] dudG(s)dG_i(t)$$

On simplification, the time to recruitment is

$$E(W) = \frac{a\lambda\theta_2\theta_3}{(a-1)(\theta_2 - \theta_3)}\left\{\frac{a(\theta_3 + \theta_1) - \theta_1 - \theta_3}{a\theta_3(\theta_1 + \theta_3) - \theta_1\theta_3} - \frac{a(\theta_2 + \theta_1) - \theta_2 - \theta_1}{a\theta_2(\theta_1 + \theta_2) - \theta_1\theta_2}\right\}.$$

5 Numerical Illustration

The value of $E(W)$ can be determined numerically using the above expression when the values of the various parameters are given. The changes in $E(W)$ consequent to the changes in each of these parameters when other parameters are kept fixed are also possible.

Effect of loss of manhours on performance measure:

θ_1	$E(W)$
0.1	14.7429
0.2	15.2800
0.3	15.6848
0.4	16.0000
0.5	16.2517
0.6	16.4571
0.7	16.6277
0.8	16.7714
0.9	16.8941
1.0	17.0000

Table 1.1
($\lambda = 2, a = 2, \theta_2 = 0.3, \theta_3 = 0.4$)

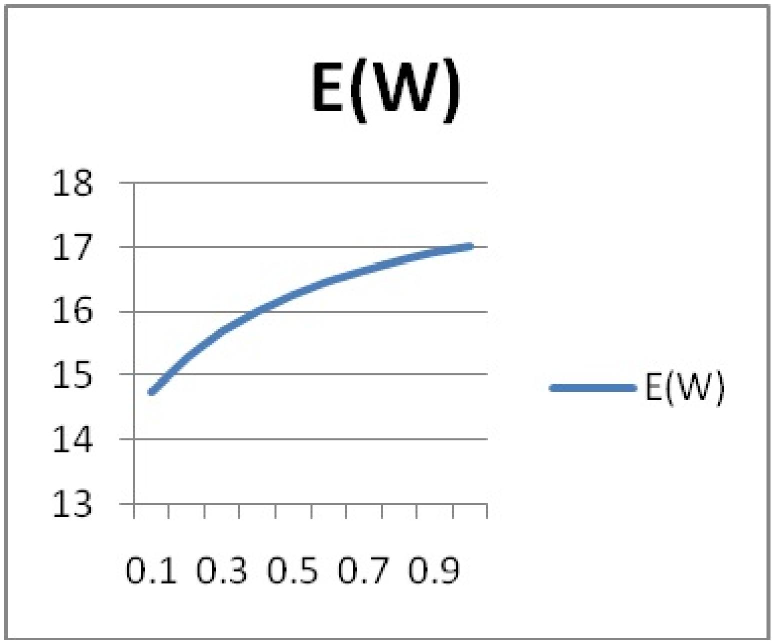


Figure 1: Figure 1.11

Effect of λ on performance measure

λ	$E(W)$
1.05	7.7400
1.10	8.1086
1.15	8.4771
1.20	8.8457
1.25	9.2143
1.30	9.5829
1.35	9.9514
1.40	10.3200
1.45	10.6886
1.50	11.0571

Table 1.2
$(\theta_1 = 0.1, a = 2, \theta_2 = 0.3, \theta_3 = 0.4)$

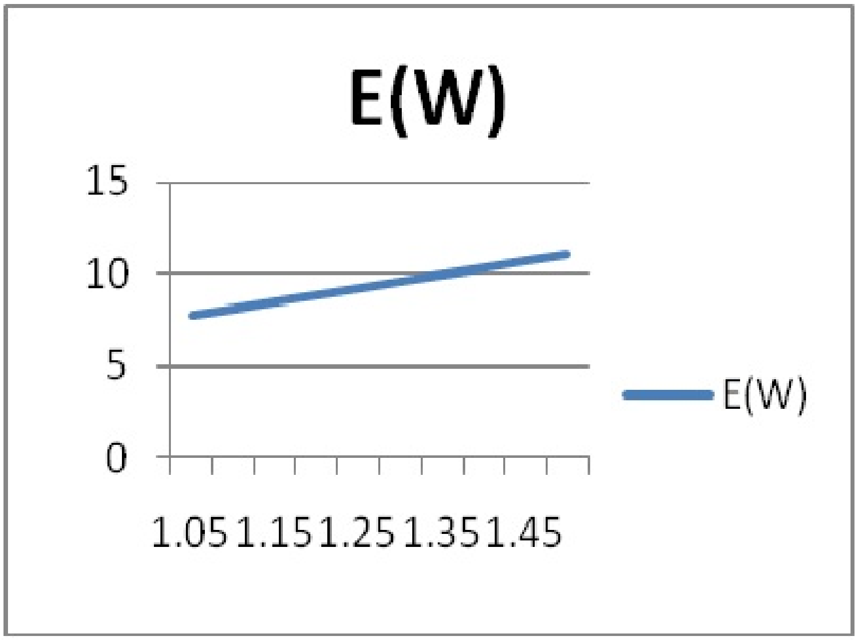


Figure 2: Figure 1.21

Effect of a on performance measure

a	$E(W)$
2	7.3714
3	5.5851
4	4.9825
5	4.6793
6	4.4967
7	4.3746
8	4.2873
9	4.2217
10	4.1706

Table 1.3
$(\theta_1 = 0.1, a = 2, \theta_2 = 0.3, \lambda = 2)$

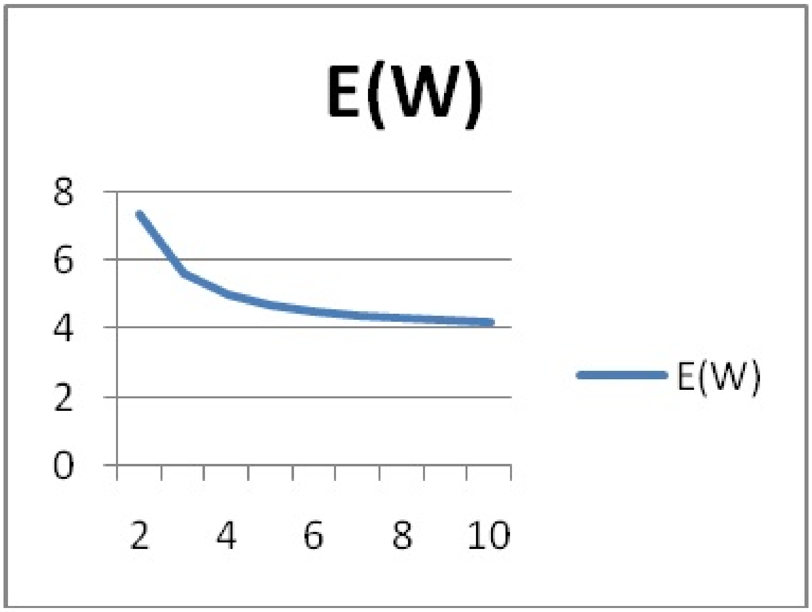


Figure 3: Figure 1.31

6 Conclusion

From the above tables we observe the following :

Case(i): If the value of the parameter θ_1 increases, the mean loss of manhours decreases and hence the expected time to recruitment increases as shown in Table 1.1 and Figure 1.11.

Case(ii): If the value of λ increases ,the expected time to recruitment $E(W)$ also increase as shown in Table 1.2 and Figure 1.21.

Case(iii): As a increases the mean survival time $\frac{\lambda}{a^{i-1}}$ decreases and hence the expected time to recruitment decreases as in Table 1.3 and Figure1.31.

References

[1] D.J. Bartholomew, *Stochastic Model for Social Process*, 2nd Edition, John Wiley and Sons, New York, 1973.

[2] D.J. Bartholomew and A.F.Forbes, *Statistical Techniques for Manpower Planning*, John Wiley, Chichester, 1979.

[3] R. Sathyamoorthi and R. Elangovan, A shock model approach to determine the expected time to recruitment , *Journal of Decision and Mathematical Sciences*, 2(1-3)(1998), 67-68.

[4] R. Sathyamoorthi and S. Parthasarathy, On the expected time to recruitment in a two grade marketing organization, *IAPQR Transactions*, 27(1)(2002), 77-80.

[5] A. Srinivasan and V. Saavithri, Cost analysis on univariate policies of recruitment, *International Journal of Management and Systems*, 18(3)(2002), 249-264.

[6] A. Srinivasan and K. Kasturri, Expected ttime for recruitment with correlated inter-decision times of exits when threshold distribution has SCBZ property, *Acta Ciencia Indica*, XXXI M.(1)(2005), 277-283.

[7] A. Srinivasan, K.P. Uma, K. Udayachandrika and V. Saavithri, *Expected time to recruitment for an univariate policy when threshold distribution has SCBZ Property*, Proceedings of the third national Conference On Mathematical and Computational Models, Narosa Publishing Book House, New Delhi, 2005, 242-246.

[8] N. Vijayasankar, R. Elangovan and R. Sathiyamoorthi, Determination of expected time to recruitment when backup resource of manpower exists, *Ultra Scientist*, 25(1)B(2013), 61-68.

Received: January 30, 2014; *Accepted*: March 5, 2014.

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 249–253

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

The b-chromatic number of some degree splitting graphs

S. K. Vaidya[a,*] and Rakhimol V. Isaac[b]

[a]Department of Mathematics, Saurashtra University, Rajkot - 360005, Gujarat, India.

[b]Department of Mathematics, Christ College, Rajkot - 360005, Gujarat, India.

Abstract

A b-coloring of a graph G is a variant of proper coloring in which each color class contains a vertex that has a neighbor in all the other color classes. We investigate some results on b-coloring in the context of degree splitting graph of P_n, $B_{n,n}$, S_n and G_n.

Keywords: graph coloring, b-coloring, b-vertex, degree splitting graph.

2010 MSC: 05C15; 05C76,.

1 Introduction

In this paper we deal with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. The notations and terminology here are used in the sense of Clark and Holton [1]. A *proper k-coloring* of a graph G is a function $c : V(G) \rightarrow \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The color class c_i is the subset of vertices of G that is assigned to color i. The chromatic number $\chi(G)$ is the minimum number k for which G admits proper k-coloring.

A proper k-coloring c of a graph G is a *b-coloring* if for every color class c_i, there is a vertex with color i which has at least one neighbor in every other color classes. Such vertex is called a *b-vertex*. The *b-chromatic number* of a graph G, denoted by $\varphi(G)$, is the largest integer k for which G admits a b-coloring.

The concept of b-coloring was introduced by Irving and Manlove [2]. If G has a b-coloring by k colors for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ then G is called *b-continuous*. The *b-spectrum* $S_b(G)$ of a graph G is the set of integers k such that G has a b-coloring by k colors.

The concept of b-coloring is explored by many researchers. The bounds for the b-chromatic number of a graph is investigated by Kouider and Mahéo [3] while b-chromatic number for Peterson graph and power of a cycle is discussed by Chandrakumar and Nicholas [6]. The b-continuity of chordal graphs is discussed by Faik [7].

Definition 1.1. *([2], [4])The m-degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m - 1$.*

Proposition 1.2. *([1]) For any graph G, $\chi(G) \geq 3$ if and only if G has an odd cycle.*

Proposition 1.3. *([2]) If G admits a b-coloring with m colors, then G must have at least m vertices with degree at least $m - 1$.*

Proposition 1.4. *([3]) $\chi(G) \leq \varphi(G) \leq m(G)$.*

It is obvious that if $\chi(G) = k$, then every coloring of a graph G by k colors is a b-coloring of G.

*Corresponding author.
E-mail address: samirkvaidya@yahoo.co.in (S. K. Vaidya), rakhiisaac@yahoo.co.in (Rakhimol V. Isaac).

Proposition 1.5. *([5]) If P_n, C_n, K_n, $K_{m,n}$ and $W_n : C_n + K_1$ are respectively path, cycle, complete graph, complete bipartite graph and wheel graph, then*

1. $\chi(C_{2n}) = 2$, $\chi(C_{2n+1}) = 3$.
2. $\chi(W_n) = 3$, *if n is odd and* $\chi(W_n) = 4$, *if n is even.*
3. $\chi(K_{m,n}) = 2$.
4. $\varphi(P_n) = 2$, *if* $1 < n < 5$ *and* $\varphi(P_n) = 3$, *if* $n \geq 5$.
5. $\varphi(C_n) = 2$, *if* $n = 4$ *and* $\varphi(C_n) = 3$, *if* $n \neq 4$.
6. $\varphi(W_n) = 3$, *if* $n = 4$ *and* $\varphi(W_n) = 4$, *if* $n \neq 4$.
7. $\chi(K_n) = \varphi(K_n) = n$.

2 Main Results

Definition 2.1. *Let $G = (V(G), E(G))$ be a graph with $V(G) = S_1 \cup S_2 \cup .... \cup S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) \setminus \bigcup_{i=1}^{t} S_i$. The degree splitting graph of G, denoted by $DS(G)$, is obtained from G by adding vertices $w_1, w_2, ..., w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$.*

Lemma 2.2. $\chi(DS(P_n)) = \begin{cases} 2, & n = 3 \\ 3, & n \neq 3. \end{cases}$

Proof. The path P_n has two pendant vertices and the remaining $n-1$ vertices are of degree two. Thus $V(P_n) = \{v_i\,; 1 \leq i \leq n\} = S_1 \cup S_2$ where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i\,; 2 \leq i \leq n-1\}$. For obtaining $DS(P_n)$ from P_n, add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus $V(DS(P_n)) = V(P_n) \cup \{w_1, w_2\}$ and $E(DS(P_n)) = E(P_n) \cup \{w_1 v_i$ where $v_i \in S_1; i = 1, n\} \cup \{w_2 v_j$ where $v_j \in S_2; 2 \leq j \leq n-1\}$. $|V(DS(P_n))| = n+2$ and $|E(DS(P_n))| = 2n-1$.

When $n = 3$, the graph $DS(P_3)$ is isomorphic to C_4. Then by Proposition 1.5, $\chi(DS(P_3)) = 2$. But when $n \neq 3$, $DS(P_n)$ contains a cycle C_3. Then by Proposition 1.2, $\chi(DS(P_n)) \geq 3$. If we assign the proper coloring as $c(w_1) = c(w_2) = 1$, $c(v_{2k+1}) = 2$, $c(v_{2k}) = 3$; $k \in \mathrm{N}$ then $\chi(DS(P_n)) = 3$. □

Theorem 2.3. $\varphi(DS(P_n)) = \begin{cases} 2, & n = 3 \\ 3, & n = 2, 4 \\ 4, & n \geq 5. \end{cases}$

Proof. The graphs $DS(P_2)$ and $DS(P_3)$ are isomorphic to C_3 and C_4 respectively. Then by Proposition 1.5, $\varphi(DS(P_2)) = 3$ and $\varphi(DS(P_3)) = 2$.

In the graph $DS(P_4)$ there are four vertices of degree 2. Then the m-degree, $m(DS(P_4))$ =3. Then by Proposition 1.4, $\varphi(DS(P_4)) \leq 3$. Moreover $DS(P_4)$ induces a path of length greater than four, $\varphi(DS(P_4)) \geq 3$. Hence $\varphi(DS(P_4)) = 3$.

For $n \geq 5$, the graph $DS(P_n)$ has at least four vertices of degree at least 3. Then the m-degree, $m(DS(P_n)) = 4$. Then by Proposition 1.4, $\varphi(DS(P_n)) \leq 4$. Moreover $DS(P_n)$ induces a path of length greater than four, $\varphi(DS(P_n)) \geq 3$. We suppose that $DS(P_n)$ has a b-coloring using four colors. By assigning the proper coloring as $c(w_1) = c(w_2) = 1$, $c(v_{3k-2}) = 2$, $c(v_{3k-1}) = 3$, $c(v_{3k}) = 4$; $k \in \mathrm{N}$ then the vertices w_2, v_4, v_2 and v_3 are the b-vertices for the color classes 1, 2, 3 and 4 respectively. Thus $\varphi(DS(P_n)) = 4$. Hence the result. □

Definition 2.4. *The bistar $B_{n,n}$ is a graph obtained by joining the center(apex) vertices of two copies of $K_{1,n}$ by an edge.*

Lemma 2.5. *For all n,* $\chi(DS(B_{n,n})) = 3$.

Proof. In $B_{n,n}$, $V(B_{n,n}) = \{u, v, u_i, v_i; 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uu_i, vv_i; 1 \leq i \leq n\} \cup \{uv\}$. The graph bistar $B_{n,n}$ contains two types of vertices - pendant vertices and vertices of degree $n+1$. Thus $V(B_{n,n}) = S_1 \cup S_2$ where $S_1 = \{u_i, v_i; 1 \leq i \leq n\}$ and $S_2 = \{u, v\}$. For obtaining $DS(B_{n,n})$ from $B_{n,n}$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus $V(DS(B_{n,n})) = V(B_{n,n}) \cup \{w_1, w_2\}$ and $E(DS(B_{n,n})) = E(B_{n,n}) \cup \{u_i w_1, v_i w_1, uw_2, vw_2\}$. Hence $|V(DS(B_{n,n}))| = 2n + 4$ and $|E(DS(B_{n,n}))| = 4n + 3$.

As the graph $DS(B_{n,n})$ contains a C_3, $\chi(DS(B_{n,n})) \geq 3$. If we assign the proper coloring as $c(w_2) = 1$, $c(u) = 2$, $c(v) = 3$, $c(w_1) = 2$, $c(u_i) = c(v_i) = 1$, for $i = 1, 2, .., n$, then $\chi(DS(B_{n,n})) = 3$ for all n. □

Theorem 2.6. *For all n, $\varphi(DS(B_{n,n})) = 3$.*

Proof. By Lemma 2.5, $\varphi(DS(B_{n,n})) \geq \chi(DS(B_{n,n})) = 3$. The graph $DS(B_{n,n})$ has at least three vertices of degree at least two. Then $m(DS(B_{n,n})) = 3$ and hence by Proposition 1.4, $\varphi(DS(B_{n,n})) \leq 3$. Thus $\varphi(DS(B_{n,n})) = 3$ for all n. □

Definition 2.7. *A shell S_n is the graph obtained by taking $n - 3$ concurrent chords in cycle C_n. That is, $S_n = P_{n-1} + K_1$.*

Lemma 2.8. $\chi(DS(S_n)) = \begin{cases} 4, & n = 3 \\ 3, & n \neq 3. \end{cases}$

Proof. In the shell graph S_n, $V(S_n) = \{u, v_1, v_2, ..., v_{n-1}\}$ where u is the apex vertex and $E(S_n) = \{uv_i\ ;\ 1 \leq i \leq n - 1\} \cup \{v_i v_{i+1}; 1 \leq i \leq n - 2\}$. Clearly $|V(S_n)| = n$ and $|E(S_n)| = 2n - 3$.
There are three types of vertices

(i) vertices of degree 2,

(ii) vertices of degree 3,

(iii) a vertex of degree $n - 1$.

Thus $V(S_n) = \{u, v_1, v_2, ..., v_{n-1}\} = S_1 \cup S_2 \cup T$ where $S_1 = \{v_1, v_{n-1}\}$, $S_2 = \{v_i\ ;\ 2 \leq i \leq n - 2\}$ and $T = \{u\} = V(S_n) \setminus \bigcup_{i=1}^{2} S_i$. For obtaining $DS(S_n)$ from S_n, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus $V(DS(S_n)) = V(S_n) \cup \{w_1, w_2\}$ and $E(DS(S_n)) = E(S_n) \cup \{v_1 w_1, v_{n-1} w_1\} \cup \{v_i w_2; 2 \leq i \leq n - 2\}$.

When $n = 3$, the graph $DS(S_3)$ is isomorphic to K_4. Then by Proposition 1.5, $\chi(DS(S_3)) = 4$. But when $n \neq 3$, $DS(S_n)$ contains a C_3, then by Proposition 1.2, $\chi(DS(S_n)) \geq 3$. If we assign the colors as $c(w_1) = c(w_2) = c(u) = 1$, $c(v_{2k+1}) = 2$, $c(v_{2k}) = 3$; $k \in \mathrm{N}$, then $\chi(DS(S_n)) = 3$. □

Theorem 2.9. $\varphi(DS(S_n)) = \begin{cases} 3, & n = 4 \\ 4, & n \neq 4. \end{cases}$

Proof. When $n = 3$, the graph $DS(S_3)$ is isomorphic to K_4, by Proposition 1.5, $\varphi(DS(S_3)) = 4$.

When $n = 4$, the graph $DS(S_4)$ has four vertices of degree at least three. Then $m(DS(S_4)) = 4$. Then by Proposition 1.4, $\varphi(DS(S_4)) \leq 4$. Suppose that $DS(S_4)$ does have a b-chromatic 4-coloring. By assigning the proper coloring as $c(u) = 1$, $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$ which in turn forces to assign $c(w_1)$ is either by the color 1 or 3 and $c(w_2)$ is either by the color 2 or 4. This proper coloring gives the b-vertices for the color classes 1 and 3 but not for 2 and 4. Similarly all other proper coloring using 4 colors will generate b-vertices at most for two color classes only. Hence $\varphi(DS(S_4)) \neq 4$. Thus $\varphi(DS(S_4)) \leq 3$. Also by Lemma 2.8, $\varphi(DS(S_4)) \geq 3$. Hence $\varphi(DS(S_4)) = 3$.

When $n = 5$ and 6, the graph $DS(S_n)$ has the m-degree four. Thus $\varphi(DS(S_n)) \leq 4$. Suppose that $DS(S_n)$ does have a b-chromatic 4-coloring. By assigning the proper coloring as $c(u) = 1$, $c(v_1) = c(v_4) = 2$, $c(v_2) = 3$, $c(v_3) = c(w_1) = 4$ which gives the b-vertices u, v_1, v_2, v_3 for the color classes 1, 2, 3, and 4 respectively. Thus $\varphi(DS(S_n)) = 4$.

When $n \geq 7$, the graph $DS(S_n)$ has the m-degree five. Thus $\varphi(DS(S_n)) \leq 5$. Suppose that $DS(S_n)$ does have a b-chromatic 5-coloring. By assigning the proper coloring as $c(v_2) = 1$, $c(v_1) = 2$, $c(u) = 4$, $c(v_3) = 5$, $c(w_2) = 3$, $c(v_4) = 2$ which in turn forces to assign $c(v_5) = 1$. This proper coloring gives the b-vertices for the color classes 1, 2 and 5 but not for 3 and 4. Similarly all other proper coloring with 5 colors will generate b-vertices at most for three color classes only. Hence $\varphi(DS(S_n)) \neq 5$. Thus $\varphi(DS(S_n)) \leq 4$. If we assign the colors as $c(w_1) = c(w_2) = 1$, $c(v_{3k-2}) = 2$, $c(v_{3k-1}) = 3$, $c(v_{3k}) = 4$; $k \in \mathrm{N}$ gives the b-vertices u, v_2, v_3, v_4 for the color classes 1, 3, 4 and 2 respectively. Thus $\varphi(DS(S_n)) = 4$. □

Definition 2.10. *The gear Graph, G_n, is obtained from the wheel by subdividing each of its rim edge.*

That is, let $W_n = C_n + K_1$ be the wheel graph with apex vertex v and the rim vertices $v_1, v_2, ..., v_n$. To obtain the gear graph G_n, subdivide each rim edge of wheel W_n by the vertices $u_1, u_2, ..., u_n$ where each u_i subdivides the edge v_iv_{i+1} for $i = 1, 2, .., n-1$ and u_n subdivides the edge v_1v_n. Then $|V(G_n| = 2n+1$ and $|E(G_n| = 3n$.

Lemma 2.11. $\chi(DS(G_n)) = \begin{cases} 3, & n = 3 \\ 2, & n \neq 3. \end{cases}$

Proof. The gear graph G_n has three types of vertices

(i) vertices of degree 2,

(ii) vertices of degree 3

(iii) a vertex of degree n.

Thus $V(G_n) = \{v_i, u_i, v\} = S_1 \cup S_2 \cup T$ where $S_1 = \{v_i\}$, $S_2 = \{u_i\}$, $T = \{v\} = V(G_n) \setminus \bigcup_{i=1}^{2} S_i$. For obtaining $DS(G_n)$ from G_n, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus $V(DS(G_n)) = V(G_n) \cup \{w_1, w_2\}$ and $E(DS(G_n)) = E(G_n) \cup \{v_iw_1, u_iw_2\}$.

When $n = 3$, $DS(G_3)$ contains a K_3 (formed by the vertices v, w_1 and v_2), $\chi(DS(G_3)) \geq 3$. If we assign the colors as $c(v) = 1$, $c(w_1) = 2$, $c(w_2) = 3$, $c(u_i) = 2$, $c(v_i) = 3$ for $i = 1, 2, ..., n$ gives the proper coloring using 3 colors. Thus $\chi(DS(G_3)) = 3$. But when $n \neq 3$, $DS(G_n)$ contains no odd cycles and it is a bipartite graph. Hence by Proposition 1.5, $\chi(DS(G_n)) = 2$. □

Theorem 2.12. $\varphi(DS(G_n)) = \begin{cases} 5, & n = 3 \\ 4, & n \neq 3. \end{cases}$

Proof. When $n = 3$, the graph $DS(G_3)$ contains five vertices of degree 4. Consequently $m(DS(G_3) = 5$. Then by Proposition 1.4, $\varphi(DS(G_3)) \leq 5$. Suppose that $DS(G_3)$ does have a b-chromatic 5-coloring. By assigning the proper coloring as $c(u_1) = 1$, $c(u_2) = 3$, $c(u_3) = 2$, $c(v_1) = 3$, $c(v_2) = 2$, $c(v_3) = 1$, $c(v) = 4$, $c(w_2) = 4$, $c(w_1) = 5$ then the vertices v_3, v_2, v_1, v, and w_1 are the b-vertices for the color classes 1, 2, 3, 4 and 5 respectively. Thus $\varphi(DS(G_3)) = 5$.

When $n \neq 3$, the graph $DS(G_n)$ contains at least five vertices of degree 4. Then $m(DS(G_n) = 5$. Then by Proposition 1.4, $\varphi(DS(G_n)) \leq 5$. Suppose that $DS(G_n)$ does have a b-chromatic 5-coloring. By assigning the proper coloring as $c(v) = 1$, $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$, $c(v_4) = 5$ gives the b- vertex v for the color class 1. Again assume that $c(u_1) = 4$ and $c(u_n) = 3$ which in turn forces to assign $c(w_1) = 5$ which is not possible as the adjacent vertices w_1 and v_4 will receive the same color. Thus v_1 is not a b- vertex for the color class 2. Similarly we can prove that no v_i's are b-vertices when five colors are used for b-coloring. Hence $\varphi(DS(G_n)) \neq 5$. But if we assign the colors as $c(v) = 1$, $c(v_{3k-2}) = 2$, $c(v_{3k-1}) = 3$, $c(v_{3k}) = 4$; $k \in \mathrm{N}$ which gives the b- vertices v, v_1, v_2 and v_3 for the color classes 1, 2, 3 and 4 respectively. Thus $\varphi(DS(G_n)) = 4$. Hence the result.

We have the following obvious result stating the b-spectrum of $DS(G_n)$ as any proper coloring with $\chi(G)$ colors is a b-coloring. □

Corollary 2.13. $S_b(DS(G_n)) = \begin{cases} \{3,4,5\}, & n = 3 \\ \{2,3,4\}, & n \neq 3 \end{cases}$ *and $DS(G_n)$ is b-continuous.*

Proof. When $n = 3$, by assigning the colors as $c(v) = 1$, $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$, $c(w_1) = c(w_2) = 4$ and $c(u_i) = 1$ for $i = 1, 2$ and 3, the graph $DS(G_3)$ has the b-chromatic 4-coloring. But when $n \neq 3$, by assigning the colors as $c(v) = c(w_1) = c(w_2) = 1$, $c(v_i) = 2$, $c(u_i) = 3$ for $i = 1, 2, .., n$, $DS(G_n)$ has the b-chromatic 3-coloring. Thus by Lemma 2.11 and Theorem 2.12, $DS(G_n)$ is b-continuous and the b-spectrum $S_b(DS(G_n)) = \begin{cases} \{3,4,5\}, & n = 3 \\ \{2,3,4\}, & n \neq 3. \end{cases}$ □

3 Concluding Remarks

The study of b-coloring is important due to its applications in many real life problems like scheduling problem, channel assignment problem, routing networks etc. Here we investigate b-chromatic number and related parameters for the degree splitting graph of some graphs. We show that the degree splitting graph of G_n is b-continuous. The degree splitting graph of P_n, $B_{n,n}$ and S_n are obviously b-continuous as any proper coloring with $\chi(G)$ colors is a b-coloring.

References

[1] J. Clark and D. A. Holton, *A First Look at Graph Theory*, World Scientific, (1969).

[2] R. W. Irving and D. F. Manlove, The b-chromatic number of a graph, *Discrete Applied Mathematics*, 91, (1999), 127-141.

[3] M. Kouider and M. Mahéo, Some bounds for the b-chromatic number of a graph, *Discrete Mathematics*, 256, (2002), 267-277.

[4] F. Havet, C. L. Sales and L. Sampaio, b-coloring of tight graphs, *Discrete Applied Mathematics*, 160, (2012), 2709-2715.

[5] M. Alkhateeb, On b-colorings and b-continuity of graphs, Ph.D Thesis, Technische Universität Bergakademie, Freiberg, Germany, (2012).

[6] S. Chandrakumar and T. Nicholas, b-continuity in Peterson graph and power of a cycle, *International Journal of Modern Engineering Research*, 2, (2012), 2493-2496.

[7] T. Faik, About the b-continuity of graphs, *Electronic Notes in Discrete Math.*, 17, (2004), 151-156.

Received: March 14, 2014; *Accepted*: April 6, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 254–262

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

A relational reformulation of the Phelps–Cardwell lemma

Árpád Száz[a,*]

[a]Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary.

Abstract

By using some results on translation and superadditive relations, we give some relational reformulations of the Phelps–Cardwell lemma in terms of open and closed surroundings.

These reformulations have mainly been suggested by a unifying scheme for continuities of relations in relator spaces and a projective generation of translation relators by superadditive relations.

Keywords: Translation and superadditive relations, open and closed surroundings, Phelps–Cardwell lemma.

2010 MSC: 46B20, 47A07,08A02, 54E15.

1 Introduction

In 1960, by using the classical Hahn-Banach extension theorem, Phelps [9] proved the following lemma which has had some applications in [2] and [5, Proposition 8], and also in [4] which has not been available to the author.

Lemma 1.1. *Suppose that E is a real normed linear space and that $\varepsilon > 0$. Let U and S^* denote the closed unit ball of E and the unit sphere of the dual space E^*, respectively. If $f, g \in S^*$ are such that*

$$f^{-1}(0) \cap U \subset g^{-1}[-\varepsilon/2, \varepsilon/2],$$

then either $\| f - g \| \leq \varepsilon$ or $\| f + g \| \leq \varepsilon$.

Remark 1.1. The above inclusion in a detailed form means only that if $x \in E$ such that $f(x) = 0$ and $\| x \| \leq 1$, then $| g(x) | \leq \varepsilon/2$.

In 2006, by using a quite elementary, but rather tricky computation, Cardwell [3] proved the following partial generalization of Lemma 1.1.

Lemma 1.2. *Let X be a complex Banach space and let ε be such that $0 < \varepsilon < 1/2$. Let $\varphi, \psi \in X^*$ be such that $\| \varphi \| = \| \psi \| = 1$. Suppose that for all $x \in X$ with $\| x \| \leq 1$ and $\varphi(x) = 0$, it holds that $\| \psi(x) \| \leq \varepsilon$. Then there is some complex number α such that $| \alpha | = 1$ and $\| \varphi - \alpha \psi \| \leq 5 \varepsilon$.*

Remark 1.2. If in particular φ and ψ are real-valued, then by slightly modifying the proof of Lemma 1.2 one can choose α to be either 1 or -1. Thus, Lemma 1.1, with bound ε replaced by $(5/2)\varepsilon$, can also be proved in an elementary way.

In 2007, by modifying the original proof of Phelps, Aron et al. [1] proved the following improvement of Lemma 1.2.

*Corresponding author.
E-mail address: szaz@science.unideb.hu (Árpád Száz).

Lemma 1.3. *Let X be a complex Banach space and S_X its unit sphere. If $f, g : X \to \mathbb{C}$ are linear forms of norm one and $\varepsilon > 0$ such that*

$$S_X \cap \{ f(x) = 0 \} \subset S_X \cap \{ |g(x)| \leq \varepsilon \},$$

then $\| g - \alpha f \| \leq 2\varepsilon$ for some $|\alpha| = 1$.

Remark 1.3. Note that if $x \in X$ such that $0 \neq \| x \| \leq 1$ and $f(x) = 0$, then by taking $u = \| x \|^{-1} x$, we have $\| u \| = 1$ and $f(u) = \| x \|^{-1} f(x) = 0$. Therefore, if the condition of Lemma 1.3 holds, then $|g(u)| \leq \varepsilon$, and thus $|g(x)| = \big| g\big(\| x \| u \big) \big| = \| x \| \, |g(u)| \leq \varepsilon$ also holds. Hence, since $|g(0)| = 0 \leq \varepsilon$, we can note that the condition of Lemma 1.2 also holds.

Now, by using the closed surroundings $\bar{B}_r = \big\{ (x, y) : \quad d(x, y) \leq r \big\}$, we shall prove the following relational reformulation of Lemma 1.3.

Lemma 1.4. *Let X be a normed space over $\mathbb{C}$, and assume that φ and ψ are linear functions of X to $\mathbb{C}$ such that $\| \varphi \| = 1$ and $\| \psi \| = 1$. Moreover, assume that $r > 0$ and $s > 0$ such that*

$$\Big(\bar{B}_r \cap \varphi^{-1} \Big)(0) \subset \Big(\psi^{-1} \circ \bar{B}_{rs} \Big)(0). \tag{1.1}$$

Then, there exists $\alpha \in \mathbb{C}$, with $|\alpha| = 1$, such that

$$(\varphi - \alpha\psi) \circ \bar{B}_r \subset \bar{B}_{2rs} \circ (\varphi - \alpha\psi). \tag{1.2}$$

Remark 1.4. We shall show that (1.1) is equivalent to the inclusions

$$\Big(B_r \cap \varphi^{-1} \Big)(0) \subset \Big(\psi^{-1} \circ \bar{B}_{rs} \Big)(0)$$

and

$$\Big((\bar{B}_r \setminus B_r) \cap \varphi^{-1} \Big)(0) \subset \Big(\psi^{-1} \circ \bar{B}_{rs} \Big)(0).$$

Moreover, we shall also show that (1.1) and (1.2) are equivalent to the inclusions

$$\Big(\psi \circ \big(\bar{B}_r \cap \varphi^{-1} \big) \Big)(0) \subset \bar{B}_{rs}(0)$$

and

$$\bar{B}_r \subset (\varphi - \alpha\psi)^{-1} \circ \bar{B}_{2rs} \circ (\varphi - \alpha\psi).$$

The relational reformulations of Lemma 1.3 have been mainly suggested by a unifying scheme for continuities of relations in relator spaces [15, Definition 4.1] and a basic theorem on translation and superadditive relations [13, Theorem 4.8] which allows of a projective generation of translation relators by superadditive relations.

2 A few basic facts on relations

A subset F of a product set $X \times Y$ is called a *relation on* X *to* Y. If in particular $F \subset X^2$, then we may simply say that F is a *relation on* X. Thus, in particular $\Delta_X = \{ (x, x) : \ x \in X \}$ is a relation on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{ y \in Y : \ (x, y) \in F \}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images of* x *and* A *under* F, respectively.

Moreover, the sets $D_F = \{ x \in X : \ F(x) \neq \emptyset \}$ and $R_F = F[D_F]$ are called the *domain and range of* F, respectively. If in particular $D_F = X$, then we say that F is a *relation of* X *to* Y, or that F is a *total relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the *inverse relation* F^{-1} of F can be naturally defined such that $F^{-1}(y) = \{ x \in X : \ y \in F(x) \}$ for all $y \in Y$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ of G and F can be naturally defined such that $(G \circ F)(x) = F\big[\,G(x)\big]$ for all $x \in X$. Thus, we also have $(G \circ F)(A) = G\big[\,F(A)\big]$ for all $A \subset X$.

If d is a nonnegative function of X^2, then for each $r > 0$ we may naturally define two relations B_r^d and $\bar{B}_r^d$ on X such that

$$B_r^d(x) = \big\{\, y \in X : \ d(x, y) < r \,\big\} \qquad \text{and} \qquad \bar{B}_r^d(x) = \big\{\, y \in X : \ d(x, y) \leq r \big\}$$

for all $x \in X$.

In the distance space $X(d) = (X, d)$, the *r–sized open and closed surroundings B_r^d and $\bar{B}_r^d$* are usually more convenient means, than the *open and closed subsets of $X(d)$*, or even the *distance function d* itself.

For instance, a function f of one *distance space $X(d)$* to another $Y(\rho)$ can be easily seen to be *uniformly continuous* if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f \circ B_\delta^d \subset B_\varepsilon^\rho \circ f, \qquad \text{or equivalently} \qquad B_\delta^d \subset f^{-1} \circ B_\varepsilon^\rho \circ f.$$

To more nicely express this notion and some other more complicated ones, instead of the relator $\mathcal{R}_d = \{\, B_r^d : \ r > 0\}$, it is necessary to work with the various refinements and modifications of $\mathcal{R}_d$ considered in [8].

For instance, if $\mathcal{R}$ is a *relator (relational system) on X to Y*, then the relator

$$\mathcal{R}^\wedge = \big\{\, U \subset X \times Y : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subset U(x) \,\big\}$$

may be naturally called the *topological refinement or closure of $\mathcal{R}$*.

Thus, a pair (F, G) of relations on one *relator space* $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$ may be naturally called *topologically upper semicontinuous*, resp. *topologically mildly continuous* if

$$\mathcal{S}^\wedge \circ F \subset \big(G \circ \mathcal{R}^\wedge\big)^\wedge, \qquad \text{resp.} \qquad G^{-1} \circ \mathcal{S}^\wedge \circ F \subset \mathcal{R}^\wedge.$$

3 A few basic facts on translation relations

Definition 3.1. A relation R on a groupoid X is called a *translation relation* if for any $x, y \in X$ we have

$$x + R(y) \subset R(x + y).$$

Remark 3.5. By using the notation $u\,R\,v$ instead of $v \in R(u)$, the above inclusion can be expressed by saying that $y R z$ implies $(x+y) R (x+z)$ for all $x \in X$. Thus, in particular, the usual inequality relations $<$ and $\leq$ on $\mathbb{R}$ are translation relations.

Remark 3.6. However, it is now more important to note that if p is a nonnegative function of a group X and

$$d(x, y) = p(-x + y)$$

for all $x, y \in X$, then the surroundings $B_r^p = B_r^{d_p}$ and $\bar{B}_r^p = \bar{B}_r^{d_p}$ are translation relations on X.

To check the translation property of B_r^p, note that if $x, y \in X$ and $z \in B_r^p(y)$, then $p(-y + z) = d(y, z) < r$, and thus

$$d(x + y, x + z) = p\big(-(x + y) + x + z\big) = p(-y - x + x + z) = p(-y + z) < r.$$

Therefore, $x + z \in B_r^p(x + y)$. Thus, $x + B_r^p(y) \subset B_r^p(x + y)$ also holds.

The above facts and the following theorem has been first established in [13].

Theorem 3.1. *For a relation R on a group X, the following assertions are equivalent :*

(1) R is a translation relation ;

(2) $R(x) = x + R(0)$ for all $x \in X$;

(3) $R(x + y) = x + R(y)$ for all $x, y \in X$;

(4) $R(x + y) \subset x + R(y)$ for all $x, y \in X$.

Proof. For instance, if (4) holds, then

$$R(x) = R(x+0) \subset x + R(0) = x + R(-x+x) \subset x - x + R(x) = R(x)$$

for all $x \in X$. Therefore, (2) also holds. □

Remark 3.7. Now, in addition to Remark 3.6, we can also state that

$$B_r^p(x+y) = x + B_r^p(y) \qquad \text{and} \qquad \bar{B}_r^p(x+y) = x + \bar{B}_r^p(y)$$

for all $x, y \in X$.

Some further basic properties of the above surrounding can also be derived from the following theorems of [13] .

Theorem 3.2. *If R is a translation relation on a groupoid X, then for any $A, B \subset X$ we have*

$$A + R[B] \subset R[A+B].$$

Moreover, if in particular X is a group, then the corresponding equality is also true.

Theorem 3.3. *If R is a translation relation on a groupoid X, then R^{-1} is also a translation relation on X. Moreover, if in particular X is a commutative group, then for any $A \subset X$ we have*

$$R^{-1}[A] = -R[-A].$$

Theorem 3.4. *If R and S are translation relation on a groupoid X, then $S \circ R$ is also a translation relation on X. Moreover, if in particular X is a commutative group, then for any $A, B \subset X$ we have*

$$(S \circ R)[A+B] = R[A] + S[B].$$

Remark 3.8. In this respect, it is also worth mentioning that the family of all translation relations on a groupoid is also closed under complementation, and arbitrary unions and intersections.

4 A few basic facts on superadditive relations

Definition 4.2. A relation F on one groupoid X to another Y is called *superadditive* if for any $x, y \in X$ we have

$$F(x) + F(y) \subset F(x+y).$$

Remark 4.9. By using the notation $u F v$ instead of $v \in F(u)$, the above inclusion can be expressed by saying that xFz and yFw implies $(x+y)F(z+w)$. Thus, in particular, the usual inequality relations $<$ and $\leq$ on $\mathbb{R}$ are superadditive relations.

Remark 4.10. It is clear that a reflexive and superadditive relation R on a groupoid X is a translation relation. Moreover, by [16, Theorem 3.14] , a translation relation R on a commutative group X is superadditive if and only if it is transitive.

Definition 4.3. A relation F on a group X to a groupoid Y with zero is called *quasi-odd* if for any $x \in D_F$ we have

$$0 \in F(x) + F(-x).$$

Remark 4.11. Thus, a reflexive relation R on a group X is quasi-odd. Moreover, if F is an *odd relation* on one group X to another Y in the sense that $F(-x) = -F(x)$ for all $x \in X$, then F is in particular quasi-odd.

Now, as certain counterparts of Theorem 3.1, we can also prove the following two theorems.

Theorem 4.5. *If F is a quasi-odd and superadditive relation on a group X to a monoid Y, then*

$$F(x+y) = F(x) + F(y)$$

for all $x, y \in X$ with either $x \in D_F$ or $y \in D_F$.

Proof. If $x \in D_F$, then $0 \in F(x) + F(-x) \subset F(0)$. Moreover,

$$F(x+y) \subset F(x) + F(-x) + F(x+y) \subset F(x) + F(y)$$

for all $y \in X$. The case $x \in X$ and $y \in D_F$ can be treated quite similarly. □

Theorem 4.6. *If F is a quasi-odd and superadditive relation on one group X to another Y then there exists a function f on X to Y such that for all $x \in X$ we have*

$$F(x) = f(x) + F(0) \qquad \text{and} \qquad F(x) = F(0) + f(x).$$

Proof. Now, for any $x \in D_F$, we have $0 \in F(x) + F(-x)$. Therefore, there exist $y \in F(x)$ and $z \in F(-x)$ such that $0 = y + z$. Hence, we can already infer that $y = -z \in -F(-x)$. Therefore, $y \in F(x) \cap \big(-F(-x)\big)$, and thus $F(x) \cap \big(-F(-x)\big) \neq \emptyset$. Hence, by the Axiom of Choice, it is clear that there exists a function f of D_F to Y such that $f(x) \in F(x) \cap (-F(-x)$, and thus $f(x) \in F(x)$ and $f(x) \in -F(-x)$ for all $x \in D_F$.

Now, if $x \in D_F$, then we can see that $f(x) + F(0) \subset F(x) + F(0) \subset F(x)$. Moreover, since $-f(x) \in F(-x)$, we can also see that

$$F(x) \subset f(x) - f(x) + F(x) \subset f(x) + F(-x) + F(x) \subset f(x) + F(0).$$

Therefore, $F(x) = f(x) + F(0)$. Hence, since $F(x) = \emptyset$ and $f(x) = \emptyset$ if $x \in X \setminus D_F$, it is clear that the first part of the required assertion is true. The second part can be proved quite similarly. □

Remark 4.12. Note that if F is a relation on one group X to another Y and f is a selection function of F such that either F or f is odd, then we also have $-f(x) \in F(-x)$ for all $x \in D_F$. Therefore, if in particular F is superadditive, then by the above argument we also have $F(x) = f(x) + F(0)$ for all $x \in X$.

Various conditions in order that a relation F could have an additive selection function f have been given by several authors dealing with relational generalizations of the Hahn-Banach extension theorems and the Hyers-Ulam stability theorems. (For a rapid overview on the subjects, see [18] and the reference therein.)

The close relationship between translation and supperadditive relations can also be clarified by the following generalization of [13, Theorem 4.8].

Theorem 4.7. *If F and G are superadditive relations of one groupoid X to another Y such that $G \subset F$, and S is a translation relation on Y, then $R = G^{-1} \circ S \circ F$ is a translation relation on X.*

Proof. If $x, y \in X$ and $z \in R(y)$, then by the corresponding definitions we also have

$$z \in \big(G^{-1} \circ S \circ F\big)(y) = G^{-1}\big[S\big[F(y)\big]\big].$$

Thus, there exists $w \in S\big[F(y)\big]$ such that $z \in G^{-1}(w)$, and hence $w \in G(z)$. Consequently, we also have $G(z) \cap S\big[F(y)\big] \neq \emptyset$. Hence, since $G(x) \neq \emptyset$, it follows that

$$\big(G(x) + G(z)\big) \cap \big(G(x) + S\big[F(y)\big]\big) \neq \emptyset.$$

Now, by using that $G(x) + G(z) \subset G(x+z)$ and

$$G(x) + S\big[F(y)\big] \subset F(x) + S\big[F(y)\big] \subset S\big[F(x) + F(y)\big] \subset S\big[F(x+y)\big],$$

we can see that

$$G(x+z) \cap S\big[F(x+y)\big] \neq \emptyset.$$

Thus, there exists $\omega \in S\big[F(x+y)\big]$ such that $\omega \in G(x+z)$, and hence $x + z \in G^{-1}(\omega)$. Consequently, we also have

$$x + z \in G^{-1}\big[S\big[F(x+y)\big]\big] = \big(G^{-1} \circ S \circ F\big)(x+y) = R(x+y).$$

Therefore, the inclusion $x + R(y) \subset R(x+y)$ is also true. □

Finally, we note that, analogously to the corresponding results of Section 3, the following theorems can also be proved.

Theorem 4.8. *If F is a superadditive relation on one groupoid to another Y, then for any A, $B \subset X$ we have*

$$F[A] + F[B] \subset F[A+B].$$

Remark 4.13. If in particular X and Y are groups, and F is in addition quasi-odd, then the corresponding equality is also true with either $A \subset D_F$ or $B \subset D_F$.

Theorem 4.9. *If F is a superadditive relation on one groupoid to another Y, then F^{-1} is a superadditive relation on Y to X.*

Theorem 4.10. *If F is a superadditive relation on one groupoid to another Y and G is a superadditive relation on Y to another groupoid Z, then $G \circ F$ is a superadditive relation on X to Z.*

Remark 4.14. In this respect, it is also worth noticing that if F and G are superadditive relations on a groupoid X to a commutative semigroup Y, then their pointwise sum $F+G$ is also a superadditive relation on X to Y.

Thus, in particular if f is an additive function on X to Y and Z is a subsemigroup of Y, then the relation $f+Z$, defined such that $(f+Z)(x) = f(x) + Z$ for all $x \in X$, is an additive relation on X to Y. Note that, by Theorems 3.1 and 4.6, some translation and superadditive relations are of the latter form.

5 A relational reformulation of Lemma 1.3

Now, by using our former results on translation and superadditive relations, we can prove the following

Lemma 5.5. *Let X be a normed space over $\mathbb{C}$, and assume that φ and ψ are linear functions of X to $\mathbb{C}$ such that*

$$\|\varphi\| = 1 \qquad \text{and} \qquad \|\psi\| = 1.$$

Moreover, assume that $r > 0$ and $s > 0$ such that

$$\left(\bar{B}_r \cap \varphi^{-1}\right)(0) \subset \left(\psi^{-1} \circ \bar{B}_{rs}\right)(0). \tag{5.3}$$

Then, there exists $\alpha \in \mathbb{C}$, with $|\alpha| = 1$, such that

$$(\varphi - \alpha\psi) \circ \bar{B}_r \subset \bar{B}_{2rs} \circ (\varphi - \alpha\psi). \tag{5.4}$$

Proof. If $x \in X$ such that

$$\|x\| = 1 \qquad \text{and} \qquad \varphi(x) = 0,$$

then we also have

$$\|rx\| = r\|x\| = r \qquad \text{and} \qquad \varphi(rx) = r\varphi(x) = 0.$$

Hence, we can already infer that

$$rx \in \bar{B}_r(0) \qquad \text{and} \qquad rx \in \varphi^{-1}(0),$$

and thus

$$rx \in \bar{B}_r(0) \cap \varphi^{-1}(0) = \left(\bar{B}_r \cap \varphi^{-1}\right)(0) \subset \left(\psi^{-1} \circ \bar{B}_{rs}\right)(0) = \psi^{-1}\left[\bar{B}_{rs}(0)\right].$$

This implies that $\psi(rx) \in \bar{B}_{rs}(0)$. Therefore,

$$r|\psi(x)| = |r\psi(x)| = |\psi(rx)| \le rs, \qquad \text{and thus} \qquad |\psi(x)| \le s.$$

Now, by Lemma 1.3, we can state that here exists $\alpha \in \mathbb{C}$, with $|\alpha| = 1$, such that under the notation

$$f = \varphi - \alpha\psi$$

we have $\|f\| \le 2s$. This implies that

$$|f(x)| \le \|f\|\|x\| \le 2s\|x\|$$

for all $x \in X$. Hence, if in particular

$$x \in \bar{B}_r(0), \qquad \text{and thus} \qquad \| x \| \leq r,$$

we can see that

$$| f(x) | \leq 2rs, \qquad \text{and thus} \qquad f(x) \in \bar{B}_{rs}(0).$$

Therefore, we also have

$$x \in f^{-1}[\bar{B}_{2rs}(0)] = f^{-1}[\bar{B}_{2rs}(f(0))] = f^{-1}[(\bar{B}_{2rs} \circ f)(0)] = \Big(f^{-1} \circ \bar{B}_{2rs} \circ f\Big)(0).$$

This proves that

$$\bar{B}_r(0) \subset (f^{-1} \circ \bar{B}_{2rs} \circ f)(0).$$

Hence, by using Remark 3.6 and Theorems 3.1 and 4.7, we can already infer that

$$\bar{B}_r(x) = x + \bar{B}_r(0) \subset x + \Big(f^{-1} \circ \bar{B}_{2rs} \circ f\Big)(0) = \Big(f^{-1} \circ \bar{B}_{2rs} \circ f\Big)(x) = f^{-1}[(\bar{B}_{2rs} \circ f)(x)],$$

and thus

$$(f \circ \bar{B}_r)(x) = f[\bar{B}_r(x)] \subset f\Big[f^{-1}[(\bar{B}_{2rs} \circ f)(x)]\Big] \subset (\bar{B}_{2rs} \circ f)(x)$$

for all $x \in X$. Therefore, $f \circ \bar{B}_r \subset \bar{B}_{2rs} \circ f$, and thus the required inclusion is also true. □

Remark 5.15. The above proof shows that condition (5.3) can be weakened by requiring only that

$$\Big((\bar{B}_r \setminus B_r) \cap \varphi^{-1}\Big)(0) \subset \Big(\psi^{-1} \circ \bar{B}_{rs}\Big)(0)$$

The forthcoming Proposition 6.1 will shows that the latter inclusion is actually equivalent to condition (5.3).

6 Equivalent reformulations of condition (5.3)

Condition (5.3) can also be naturally weakened with the help of the following

Proposition 6.1. *If φ and ψ are continuous and homogeneous functions of one normed space X to another Y, then for any $r > 0$ and $s > 0$ the following inclusions are equivalent:*

(a) $(B_r \cap \varphi^{-1})(0) \subset (\psi^{-1} \circ \bar{B}_s)(0)$;

(b) $(\bar{B}_r \cap \varphi^{-1})(0) \subset (\psi^{-1} \circ \bar{B}_s)(0)$;

(c) $((\bar{B}_r \setminus B_r) \cap \varphi^{-1})(0) \subset (\psi^{-1} \circ \bar{B}_s)(0)$.

Proof. Since $\bar{B}_r = B_r \cup (\bar{B}_r \setminus B_r)$, it is clear that (b) implies both (a) and (c) even if φ and ψ are arbitrary relations. Therefore, we need only show that both (a) and (c) imply (b).

If $x \in (\bar{B}_r \cap \varphi^{-1})(0)$, then $x \in \bar{B}_r(0) \cap \varphi^{-1}(0)$, and thus

$$\| x \| \leq r \qquad \text{and} \qquad \varphi(x) = 0.$$

Hence, if $x \neq 0$, then by taking

$$u = r \, \| x \|^{-1} x,$$

we can infer that

$$\| u \| = r \, \| x \|^{-1} \| x \| = r \qquad \text{and} \qquad \varphi(u) = r \, \| x \|^{-1} \varphi(x) = 0.$$

This implies that

$$u \in (\bar{B}_r(0) \setminus B_r(0)) \cap \varphi^{-1}(0) = (\bar{B}_r \setminus B_r)(0) \cap \varphi^{-1}(0) = \Big((\bar{B}_r \setminus B_r) \cap \varphi^{-1}\Big)(0).$$

Therefore, if (c) holds, then we also have

$$u \in \Big(\psi^{-1} \circ \bar{B}_s\Big)(0) = \psi^{-1}[\bar{B}_s(0)].$$

This implies that $\psi(u)\in\bar{B}_s(0)$, and thus $\|\psi(u)\|\le s$. Hence, we can infer that

$$\|\psi(x)\|=\left\|\psi\left(r^{-1}\|x\|u\right)\right\|=r^{-1}\|x\|\,\|\psi(u)\|\le s,$$

and thus $\psi(x)\in\bar{B}_s(0)$. Moreover, we can note that $\psi(0)=0\in\bar{B}_s(0)$ also holds. Therefore,

$$x\in\psi^{-1}\left[\bar{B}_s(0)\right]=\left(\psi^{-1}\circ\bar{B}_s\right)(0)$$

even if $x=0$. This shows that (c) implies (b) even if φ and ψ are only assumed to be homogeneous.

On the other hand, if $x\in\left(\bar{B}_r\cap\varphi^{-1}\right)(0)$, and thus $\|x\|\le r$ and $\varphi(x)=0$, then by taking

$$x_n=n\,(n+1)^{-1}x$$

for each $n\in\mathbb{N}$, we can see that

$$\|x_n\|=n\,(n+1)^{-1}\|x\|<r\qquad\text{and}\qquad\varphi(x_n)=n\,(n+1)^{-1}\varphi(x)=0.$$

This implies that

$$x_n=B_r(0)\cap\varphi^{-1}(0)=\left(B_r\cap\varphi^{-1}\right)(0).$$

Therefore, if (a) holds, then we also have

$$x_n\in\left(\psi^{-1}\circ\bar{B}_{rs}\right)(0)=\psi^{-1}\left[\bar{B}_s(0)\right],$$

and thus $\psi(x_n)\in\bar{B}_s(0)$. This implies that $\|\psi(x_n\|\le s$. Hence, by using that

$$\lim_{n\to\infty}x_n=x,\qquad\text{and thus}\qquad\lim_{n\to\infty}\psi(x_n)=\psi(x),$$

we can infer already that $\|\psi(x)\|\le s$. Therefore, $\psi(x)\in\bar{B}_s(0)$, and thus $x\in\left(\psi^{-1}\circ\bar{B}_s\right)(0)$ also holds. This shows that (a) implies (b) even if φ and ψ are only assumed to be homogeneous and continuous, respectively. □

In this respect, it is also worth mentioning that we also have the following

Proposition 6.2. *If φ is a continuous homogeneous functions of one normed space X to another Y, then for any $r>0$ we have*

$$\left(\bar{B}_r\cap\varphi^{-1}\right)(0)=\overline{\left(B_r\cap\varphi^{-1}\right)(0)}.$$

Proof. From the proof of the implication (a) $\Longrightarrow$ (b) we can see that

$$\left(\bar{B}_r\cap\varphi^{-1}\right)(0)\subset\overline{\left(B_r\cap\varphi^{-1}\right)(0)}$$

even if φ is only assumed to be homogeneous.

Moreover, we can note that $\varphi(0)=\varphi^{-1}[\{0\}]$ is a closed subset of X even if φ is only assumed to be continuous. Thus,

$$\left(\bar{B}_r\cap\varphi^{-1}\right)(0)=\bar{B}_r(0)\cap\varphi^{-1}(0)$$

is also a closed subset of X. Hence, it is clear that

$$\overline{\left(B_r\cap\varphi^{-1}\right)(0)}\subset\overline{\left(\bar{B}_r\cap\varphi^{-1}\right)(0)}=\left(\bar{B}_r\cap\varphi^{-1}\right)(0),$$

and thus the required equality is also true. □

Remark 6.16. The latter proposition allows of a shorter proof of the implication (a) $\Longrightarrow$ (b) in Proposition 6.1.

Namely, if (a) holds, then by noticing that

$$\left(\psi^{-1}\circ\bar{B}_s\right)(0)=\psi^{-1}\left[\bar{B}_s(0)\right]$$

is also a closed subset of X, we can at once see that

$$\left(\bar{B}_r\cap\varphi^{-1}\right)(0)=\overline{\left(B_r\cap\varphi^{-1}\right)(0)}\subset\overline{\left(\psi^{-1}\circ\bar{B}_s\right)(0)}=\left(\psi^{-1}\circ\bar{B}_s\right)(0),$$

and thus (b) also holds.

7 Equivalent reformulations of inclusions (5.3) and (5.4)

Inclusions (5.3) and (5.4) can also be reformulated by using the following basic proposition whose proof is included here only for the reader's convenience.

Proposition 7.3. *If Ψ is a relation on one set X to another Y, then*

(a) $\Delta_X \subset \Psi^{-1} \circ \Psi$ *if* Ψ *is total;*

(b) $\Psi \circ \Psi^{-1} \subset \Delta_Y$ *if* Ψ *is a function.*

Proof. If Ψ is total, then for each $x \in X$ there exists $y \in Y$ such that $y \in \Psi(x)$. Hence, it is clear that $x \in \Psi^{-1}(y)$, and thus

$$x \in \Psi\left[\Psi^{-1}(x)\right] = \left(\Psi \circ \Psi^{-1}\right)(x).$$

Therefore, $(x, x) \in \Psi \circ \Psi^{-1}$. This shows that (a) is true.

On the other hand, if $(y, z) \in \Psi \circ \Psi^{-1}$, then we can note that

$$z \in \left(\Psi \circ \Psi^{-1}\right)(y) = \Psi\left[\Psi^{-1}(y)\right].$$

Therefore, there exists $x \in \Psi^{-1}(y)$ such that $z \in \Psi(x)$. Hence, if Ψ is a function, we can already infer that $y = \Psi(x) = z$. Therefore, (b) is also true. □

By this proposition, it is clear that in particular we also have the following

Proposition 7.4. *If Ψ is a relation on one set X to another Y, then for any $A \subset X$ and $B \subset Y$*

(a) $\Psi[A] \subset B$ *implies* $A \subset \Psi^{-1}[B]$ *if* Ψ *is total;*

(b) $A \subset \Psi^{-1}[B]$ *implies* $\Psi[A] \subset B$ *if* Ψ *is a function.*

Proof. If for instance $\Psi[A] \subset B$ and Ψ is total, then Proposition 7.3 we have

$$A = \Delta_X[A] \subset \left(\Psi^{-1} \circ \Psi\right)[A] = \Psi^{-1}\left[\Psi[A]\right] \subset \Psi^{-1}[B].$$

□

A simple application of this proposition gives the following

Proposition 7.5. *If Ψ is a relation on one set X to another Y, and R and S are relations on X and Y, respectively, then for any $A \subset X$*

(1) $(\Psi \circ R)[A] \subset S[A]$ *implies* $R[A] \subset \left(\Psi^{-1} \circ S\right)[A]$ *if* Ψ *is total;*

(2) $R[A] \subset \left(\Psi^{-1} \circ S\right)[A]$ *implies* $(\Psi \circ R)[A] \subset S[A]$ *if* Ψ *is a function.*

Proof. If for instance $(\Psi \circ R)[A] \subset S[A]$ holds, then we also have $\Psi\left[R[A]\right] \subset S[A]$. Hence, if Ψ is total, then by using Proposition 7.4 we can infer that

$$R[A] \subset \Psi^{-1}\left[S[A]\right] = \left(\Psi^{-1} \circ S\right)[A].$$

□

Remark 7.17. By this proposition, it is clear that condition (5.3) of Lemma 5.5 is equivalent to the inclusion

$$\Big(\psi \circ \left(\bar{B}_r \cap \varphi^{-1}\right)\Big)(0) \subset \bar{B}_{rs}(0).$$

Moreover, by using Proposition 7.4, we can also easily establish the following

Proposition 7.6. *If F is a relation on one set X to another Y, and R and S are relations on X and Y, respectively, then for any $A \subset X$*

(1) $(F \circ R)[A] \subset (S \circ F)[A]$ *implies* $R[A] \subset \left(F^{-1} \circ S \circ R\right)[A]$ *if* F *is total;*

(2) $R[A] \subset \left(F^{-1} \circ S \circ R\right)[A]$ *implies* $(F \circ R)[A] \subset (S \circ F)[A]$ *if* F *is a function.*

Proof. If for instance $(F \circ R)[A] \subset (S \circ F)[A]$ holds, then we also have $F\big[R[A]\big] \subset (S \circ F)[A]$. Hence, if F is total, then by using Proposition 7.5 we can infer that

$$R[A] \subset F^{-1}\Big[\big(F^{-1} \circ S \circ R\big)[A]\Big] = \big(F^{-1} \circ S \circ R\big)[A].$$

□

Remark 7.18. By this proposition, it is clear that conclusion (5.4) of Lemma 5.5 is equivalent to the inclusion

$$\bar{B}_r \subset (\varphi - \alpha\,\psi)^{-1} \circ \bar{B}_{2rs} \circ (\varphi - \alpha\,\psi).$$

8 Acknowledgment

The work of the author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK-81402.

References

[1] R. Aron, A. Cardwell, D. García, and I. Zalduendo, *A multilinear Phelps' lemma*, Proc. Amer. Math. Soc. 135 (2007), 2549–2554.

[2] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. 67 (1961), 97–98.

[3] A. E. Cardwell, *A new proof of a lemma by Phelps*, Int. J. Math. Math. Sci. 2006, Art. ID 28063, 3 pp.

[4] M. Fabian, P. Habala, P. Hájek, J. Pelant, V. Montesinos and V. Zizler, *Functional Analysis and Infinite Dimensional Geometry*, Canad. Math. Soc. Books in Math., Springer-Verlag, New York, 2001.

[5] M. Fabian, V. Montesinos and V. Zizler, *A characterization of subspaces of weakly compactly generated Banach spaces*, J. London Math. Soc. 69 (2004), 457-464.

[6] T. Glavosits and Á. Száz, *On the existence of odd selections*, Adv. Stud. Contemp. Math. (Kyungshang) 8 (2004), 155–164.

[7] T. Glavosits and Á. Száz, *Constructions and extensions of free and controlled additive relations*, Tech. Rep., Inst. Math., Univ. Debrecen 2010/1, 49 pp.

[8] G. Pataki and Á. Száz, *A unified treatment of well-chainedness and connectedness properties*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 19 (2003), 101–165.

[9] R. R. Phelps, *A representation theorem for bounded convex sets*, Proc. Amer. Math. Soc. 11 (1960), 976–983.

[10] Á. Száz, *Projective generation of preseminormed spaces by linear relations*, Studia Sci. Math. Hungar. 23 (1988), 297–313.

[11] Á. Száz, *Projective and inductive generations of relator spaces*, Acta Math. Hungar. 53 (1989), 407–430.

[12] Á. Száz, *The intersection convolution of relations and the Hahn–Banach type theorems*, Ann. Polon. Math. 69 (1998), 235–249.

[13] Á. Száz, *Translation relations, the building blocks of compatible relators*, Math. Montisnigri 12 (2000), 135–156.

[14] Á. Száz, *Preseminorm generating relations and their Minkowski functionals*, Univ. Beograd., Publ. Elektotechn. Fak., Ser. Mat. 12 (2001), 16–34.

[15] Á. Száz, *Somewhat continuity in a unified framework for continuities of relations*, Tatra Mt. Math. Publ. 24 (2002), 41–56.

[16] Á. Száz, *Relationships between translation and additive relations*, Acta Acad. Paedagog. Agriensis, Sect. Math. (N.S.) 30 (2003), 179–190.

[17] Á. Száz, *Foundations of the theory of vector relators*, Adv. Stud. Contemp. Math. (Kyungshang) 20 (2010), 139–195.

[18] Á. Száz, *The Hyers–Ulam and Hahn–Banach theorems and some elementary operations on relations motivated their set-valued generalizations* In: P. M. Pardalos, P. G. Georgiev and H. M. Srivastava (eds.), Stability, Approximations, and Inequalities, Dedicated to Professor Th. M. Rassias on the occasion of his 60th birthday, Springer Optimization and Its Applications 68 (2012), 631–705.

[19] Á. Száz, *Lower semicontinuity properties of relations in relator spaces*, Adv. Stud. Contemp. Math. (Kyungshang) 23 (2013), 107–158.

[20] Á. Száz, *An extension of an additive selection theorem of Z. Gajda and R. Ger to vector relator spaces*, Scientia, Ser A, Math. Sci. 24 (2013), 33–54.

Received: October 12, 2013; *Accepted*: April 20, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 263–272

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Square-mean asymptotically almost automorphic mild solutions to non-autonomous stochastic differential equations

Zhi-Hong Li[a,*] and Zhi-Han Zhao[b]

[a]Department of Mathematics, Lanzhou Jiaotong University, Lanzhou - 730070, P. R. China.

[b]Department of Information Engineering, Sanming University, Sanming - 365004, P. R. China.

Abstract

This paper is mainly concerned with square-mean asymptotically almost automorphic mild solutions to a class of non-autonomous stochastic differential equations in a real separable Hilbert space. Some existence results of square-mean asymptotically almost automorphic mild solutions have been established by properties and composition theroems of square-mean asymptotically almost automorphic functions and fixed point theorems.

Keywords: Non-autonomous differential equation, Square-mean asymptotically almost automorphic.

2010 MSC: 34K14, 60H10, 35B15, 34F05.

1 Introduction

In this paper, we study the existence of square-mean asymptotically almost automorphic solutions for the following non-autonomous stochastic differential equations in the form

$$\begin{cases} dx(t) = A(t)x(t)dt + f(t, B_1x(t))dt + g(t, B_2x(t))dW(t), \ t \geq 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $A(t) : D(A(t)) \subset L^2(,) \to L^2(,)$ is a family of densely defined closed linear operators satisfying the so called "Acquistapace-Terreni" conditions, B_i, $i = 1, 2$ are bounded linear operators, and $W(t)$ is a two sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, , \mathcal{F}_t)$ where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. x_0 is an $\mathcal{F}_0$-adapted, -valued random variable independent of the Wiener process W, and $f, g : [0, +\infty) \times L^2(,) \to L^2(,)$ are appropriate functions to be specified later.

The asymptotically almost automorphic functions were firstly introduced by G. M. N'Gu'er'ekata in [14]. Since then these functions have become of great interest to several mathematicians and generated lots of developments and applications, we refer the reader to [3, 11, 12] and the references therein.

Recently, the existence of almost periodic, almost automorphic and pseudo almost automorphic solutions to some stochastic differential equations have been considered in many publications such as [4, 5, 7, 8, 10, 18] and references therein. In a very recent paper [8], the authors introduced a new concept of S^2-almost automorphy for stochastic processes including a composition theorem. In paper [16], the authors introduced the notion of square-mean asymptotically almost automorphic stochastic process and established some basic results not only on the completeness of the space that consists of the square-mean asymptotically almost automorphic processes but also on the composition of such processes. They apply this new concept to investigate the existence of square-mean asymptotically almost automorphic mild solutions to the following abstract stochastic

*Corresponding author.
E-mail address: 1950290025@qq.com (Zhi-Hong Li), zhaozhihan841110@126.com (Zhi-Han Zhao).

integro-differential equations

$$\begin{cases} dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s)ds\right] dt + f(t, x(t))dW(t),\ t \geq 0, \\ x(0) = x_0, \end{cases}$$

where A and $B(t)$, $t \geq 0$ are densely defined and closed linear operators in a Hilbert space $L^2(,)$, and $W(t)$ is a two sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, , \mathcal{F}_t)$ where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. x_0 is an $\mathcal{F}_0$-adapted, -valued random variable independent of the Wiener process W.

Motivated by the works [8, 11, 16, 17], the main purpose of this paper is to investigate the existence of square-mean asymptotically almost automorphic mild solutions to the problems (1.1). The obtained results can be seen as a contribution to this emerging field.

The present paper is organized as follows. In section 2, we introduce the notion of square-mean asymptotically almost automorphic processes and study some of their basic properties. In section 3, we prove the existence of existence of square-mean asymptotically almost automorphic mild solutions to the problem (1.1).

2 Preliminary

In this section, we introduce some basic definitions, notations, lemmas and technical results which will be used in the sequel. For more details on this section, we refer the reader to [7, 13].

Throughout the paper, we assume that $(, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(, \|\cdot\|, \langle\cdot,\cdot\rangle)$ are two real separable Hilbert spaces. Let $(\Omega, \mathcal{F},)$ be a complete probability space. The notation $L^2(,)$ stands for the space of all -valued random variable x such that

$$E\|x\|^2 = \int_\Omega \|x\|^2 d < \infty.$$

For $x \in L^2(,)$, let

$$\|x\|_2 = \left(\int_\Omega \|x\|^2 d\right)^{\frac{1}{2}}.$$

Then it is routine to check that $L^2(,)$ is a Hilbert space equipped with the norm $\|\cdot\|_2$. We let $L(,)$ denote the space of all linear bounded operators from into , equipped with the usual operator norm $\|\cdot\|_{L(,)}$; in particular, this is simply denoted by $L()$ when $=$. The notation $C_0(\mathbb{R}^+; L^2(,))$ stands for the collection of all bounded continuous stochastic processes φ from $\mathbb{R}^+$ into $L^2(,)$ such that $\lim_{t\to+\infty} E\|\varphi(t)\|^2 = 0$. Similarly, $C_0(\mathbb{R}^+ \times L^2(,); L^2(,))$ stands for the space of the continuous stochastic processes $f : \mathbb{R}^+ \times L^2(,) \to L^2(,)$ such that

$$\lim_{t\to+\infty} E\|f(t,x)\|^2 = 0$$

uniformly for $x \in K$, where $K \subset L^2(,)$ is any bounded subset. In addition, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, , \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$.

Definition 2.1. *[13] A stochastic process $x : \mathbb{R} \to L^2(,)$ is said to be stochastically continuous if*

$$\lim_{t\to s} E\|x(t) - x(s)\|^2 = 0.$$

Definition 2.2. *[9] A stochastically continuous stochastic process $x : \mathbb{R} \to L^2(,)$, $(t,x) \to f(t,x)$ is said to be square-mean almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a stochastic process $y : \mathbb{R} \to L^2(,)$ such that*

$$\lim_{n\to\infty} E\|x(t+s_n) - y(t)\|^2 = 0 \ \ and \ \ \lim_{n\to\infty} E\|y(t-s_n) - x(t)\|^2 = 0$$

hold for each $t \in \mathbb{R}$. The collection of all square-mean almost automorphic stochastic processes $x : \mathbb{R} \to L^2(,)$ is denoted by $AA(\mathbb{R}; L^2(,))$.

Definition 2.3. [9] *A function $f : \mathbb{R} \times L^2(,) \to L^2(,)$, $(t, x) \to f(t, x)$, which is jointly continuous, is said to be square-mean almost automorphic if $f(t, x)$ is square-mean almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$ is any bounded subset of $L^2(,)$. That is to say, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $\widetilde{f} : \mathbb{R} \times L^2(,) \to L^2(,)$ such that*

$$\lim_{n\to\infty} E\|f(t+s_n, x) - \widetilde{f}(t,x)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|\widetilde{f}(t-s_n, x) - f(t,x)\|^2 = 0$$

for each $t \in \mathbb{R}$ and each $x \in K$. Denote by $AA(\mathbb{R} \times L^2(,); L^2(,))$ the set of all such functions.

Lemma 2.1. [13] *$(AA(\mathbb{R}; L^2(,)), \|\cdot\|_\infty)$ is a Banach space when it is equipped with the norm*

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_2 = \sup_{t\in\mathbb{R}} (E\|x(t)\|^2)^{\frac{1}{2}},$$

for $x \in AA(\mathbb{R}; L^2(,))$.

Lemma 2.2. [9] *Let $f : \mathbb{R} \times L^2(,) \to L^2(,)$, $(t, x) \to f(t, x)$ be square-mean almost automorphic, and assume that $f(t, \cdot)$ is uniformly continuous on each bounded subset $K \subset L^2(,)$ uniformly for $t \in \mathbb{R}$, that is for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in K$ and $E\|x - y\|^2 < \delta$ imply that $E\|f(t, x) - f(t, y)\|^2 < \varepsilon$ for all $t \in \mathbb{R}$. Then for any square-mean almost automorphic process $x : \mathbb{R} \to L^2(,)$, the stochastic process $F : \mathbb{R} \to L^2(,)$ given by $F(\cdot) := f(\cdot, x(\cdot))$ is square-mean almost automorphic.*

Definition 2.4. [16] *A stochastically continuous process $f : \mathbb{R}^+ \to L^2(,)$ is said to be square-mean asymptotically almost automorphic if it can be decomposed as $f = g + h$, where $g \in AA(\mathbb{R}; L^2(,))$ and $h \in C_0(\mathbb{R}^+; L^2(,))$. Denote by $AAA(\mathbb{R}^+; L^2(,))$ the collection of all the square-mean asymptotically almost automorphic processes $f : \mathbb{R}^+ \to L^2(,)$.*

Definition 2.5. [16] *A function $f : \mathbb{R}^+ \times L^2(,) \to L^2(,)$, $(t, x) \to f(t, x)$, which is jointly continuous, is said to be square-mean asymptotically almost automorphic if it can be decomposed as $f = g + h$, where $g \in AA(\mathbb{R} \times L^2(,); L^2(,))$ and $h \in C_0(\mathbb{R}^+ \times L^2(,); L^2(,))$. Denote by $AAA(\mathbb{R}^+ \times L^2(,); L^2(,))$ the set of all such functions.*

Lemma 2.3. [16] *If f, f_1 and f_2 are all square-mean asymptotically almost automorphic stochastic processes, then the following hold true:*
(I) $f_1 + f_2$ is square-mean asymptotically almost automorphic ;
(II) λf is square-mean asymptotically almost automorphic for any scalar λ;
(III) There exists a constant $M > 0$ such that $\sup_{t\in\mathbb{R}^+} E\|f(t)\|^2 \leq M$.

Lemma 2.4. [16] *Suppose that $f \in AAA(\mathbb{R}^+; L^2(,))$ admits a decomposition $f = g + h$, where $g \in AA(\mathbb{R}; L^2(,))$ and $h \in C_0(\mathbb{R}^+; L^2(,))$. Then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}^+\}}$.*

Corollary 2.1. [16] *The decomposition of a square-mean asymptotically almost automorphic process is unique.*

Lemma 2.5. [16] *$AAA(\mathbb{R}^+; L^2(,))$ is a Banach space when it is equipped with the norm:*

$$\|f\|_{AAA(\mathbb{R}^+;L^2(,))} := \sup_{t\in\mathbb{R}} \|g(t)\|_2 + \sup_{t\in\mathbb{R}^+} \|h(t)\|_2,$$

where $f = g + h \in AAA(\mathbb{R}^+; L^2(,))$ with $g \in AA(\mathbb{R}; L^2(,))$, $h \in C_0(\mathbb{R}^+; L^2(,))$.

Lemma 2.6. [16] *$AAA(\mathbb{R}^+; L^2(,))$ is a Banach space with the norm:*

$$\|f\|_\infty := \sup_{t\in\mathbb{R}^+} \|f(t)\|_2 = \sup_{t\in\mathbb{R}^+} (E\|f(t)\|^2)^{\frac{1}{2}}.$$

Remark 2.1. [16] *In view of the previous Lemmas it is clear that the two norms are equivalent in $AAA(\mathbb{R}^+; L^2(,))$.*

Lemma 2.7. [16] *Let $f \in AA(\mathbb{R} \times L^2(,); L^2(,))$ and let $f(t, x)$ be uniformly continuous in any bounded subset $K \subset L^2(,)$ uniformly for $t \in \mathbb{R}^+$. Then $f(t, x)$ is uniformly continuous in any bounded subset $K \subset L^2(,)$ uniformly for $t \in \mathbb{R}$.*

Lemma 2.8. [16] *Let $f \in AAA(\mathbb{R}^+ \times L^2(,); L^2(,))$ and suppose that $f(t, x)$ be uniformly continuous in any bounded subset $K \subset L^2(,)$ uniformly for $t \in \mathbb{R}^+$. If $u(t) \in AAA(\mathbb{R}^+; L^2(,))$, then $f(\cdot, u(\cdot)) \in AAA(\mathbb{R}^+; L^2(,))$.*

Lemma 2.9. *Let $\mathcal{L} \in L(H)$ and assume that $f \in AAA(\mathbb{R}^+; L^2(,))$. Then $\mathcal{L}f \in AAA(\mathbb{R}^+; L^2(,))$.*

Proof. Since $f \in AAA\left(\mathbb{R}^+; L^2(,)\right)$, we have by definition that $f = g + h$, where $g \in AA\left(\mathbb{R}; L^2(,)\right)$ and $h \in C_0\left(\mathbb{R}^+; L^2(,)\right)$. Then, by [6, Lemma 2.4], we see that $\mathcal{L}g \in AA(\mathbb{R}; L^2(,))$. On the other hand, since $\mathcal{L} \in L(H)$, then we have

$$E\|\mathcal{L}h(t)\|^2 \le \|\mathcal{L}\|^2_{L(H)} E\|h(t)\|^2$$

which shows that $\lim_{t\to+\infty} E\|\mathcal{L}h(t)\|^2 = 0$, since $h \in C_0\left(\mathbb{R}^+; L^2(,)\right)$. Thus, $\mathcal{L}f \in AAA(\mathbb{R}^+; L^2(,))$. This ends the proof. □

The following Lemma hold by [1, Theorem 2.3] and [2].

Lemma 2.10. *If the Acquistapace-Terreni conditions (ATCs) are satisfied, that is, there exists a constant $K_0 > 0$ and a set of real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k, \beta_1, \cdots, \beta_k$ with $0 \le \beta_i < \alpha_i \le 2$, $i = 1, 2, \cdots, k$, such that*

$$\|A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\| \le K_0 \sum_{i=1}^{k} (t-s)^{\alpha_i} |\lambda|^{\beta_i - 1},$$

for $t, s \in \mathbb{R}$, $\lambda \in S_{\theta_0} \backslash \{0\}$, where

$$\rho(A(t)) \supset S_{\theta_0} = \{\lambda \in \mathbb{C} : |arg\lambda| \le \theta_0\} \cup \{0\}, \theta_0 \in (\frac{\pi}{2}, \pi)$$

and there exists a constant $K_1 \ge 0$ such that

$$\|(\lambda - A(t))^{-1}\| \le \frac{K_1}{1+|\lambda|}, \lambda \in S_{\theta_0}.$$

Then there exists a unique evolution family $\{U(t,s), t \ge s > -\infty\}$ on $L^2(,)$.

Throughout the rest of the paper we assume that (ATCs) are satisfied.

Definition 2.6. *An $\mathcal{F}_t$-adapted stochastic process $x : [0, \infty) \to L^2(,)$ is called a mild solution of problem (1.1) if $x(0) = x_0$ is $\mathcal{F}_0$-measurable and $x(t)$ satisfies the corresponding stochastic integral equation*

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s) f(s, B_1 x(s)) ds + \int_0^t U(t,s) g(s, B_2 x(s)) dW(s).$$

for all $t \ge 0$ and $0 \le s \le t$.

3 Extension Principle

In this section, we establish the existence of square-mean asymptotically automorphic mild solutions to (1.1). For that, we give the following assumptions:
(H1) The evolution family $U(t,s)$ generated by $A(t)$ is exponentially stable, that is, there exist $M \ge 1$ and $\delta > 0$ such that $\|U(t,s)\| \le Me^{-\delta(t-s)}$ for all $t \ge s$.
(H2) $U(t,s)$, $t \ge s$, satisfies the condition that, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\|U(t+s_n, s+s_n) - U(t,s)\| \le \varepsilon e^{-\delta(t-s)},$$

for all $n > N$ and all $t \ge s$, moreover

$$\|U(t-s_n, s-s_n) - U(t,s)\| \le \varepsilon e^{-\delta(t-s)},$$

for all $n > N$ and all $t \ge s$.
(H3) The operators $B_i : L^2(,) \to L^2(,)$ for $i = 1, 2$, are bounded linear operators and we let $\overline{\omega} := \max_{i=1,2} \left\{\|B_i\|_{\mathcal{L}(L^2(,))}\right\}$.
(H4) The functions $f, g \in AAA(\mathbb{R}^+ \times L^2(,); L^2(,))$ and there are positive numbers L_f, L_g such that

$$E\|f(t,\varphi) - f(t,\psi)\|^2 \le L_f E\|\varphi - \psi\|^2,$$

and

$$E\|g(t,\varphi) - g(t,\psi)\|^2 \le L_g E\|\varphi - \psi\|^2,$$

for all $t \in \mathbb{R}^+$ and $\varphi, \psi \in L^2(,)$.

Theorem 3.1. *Assume that the conditions (H1)-(H4) are satisfied, then the problem (1.1) has a unique square-mean asymptotically almost automorphic mild solution $x(\cdot) \in AAA(\mathbb{R}^+; L^2(,))$ provided that*

$$L_0 = M^2\overline{\omega}^2 \left[\frac{2}{\delta^2} L_f + \frac{1}{\delta} L_g\right] < 1.$$

Proof. Let $\Gamma : AAA(\mathbb{R}^+; L^2(,)) \to AAA(\mathbb{R}^+; L^2(,))$ be the operator defined by

$$\Gamma x(t) := U(t,0)x_0 + \int_0^t U(t,s)f(s, B_1x(s))ds + \int_0^t U(t,s)g(s, B_2x(s))dW(s), \quad t \geq 0.$$

Let us prove that Γx is well defined, for this, let $x \in AAA(\mathbb{R}^+; L^2(,))$. We need to prove that $\Gamma x(t) \in AAA(\mathbb{R}^+; L^2(,))$. Let us consider the nonlinear operators Γ_0, Γ_1 and Γ_2 acting on the Banach space $AAA(\mathbb{R}^+; L^2(,))$ defined by $\Gamma_0 x(t) = U(t,0)x_0$,

$$\Gamma_1 x(t) = \int_0^t U(t,s)f(s, B_1x(s))ds \text{ and } \Gamma_2 x(t) = \int_0^t U(t,s)g(s, B_2x(s))dW(s)$$

respectively.

First, we will show that $\Gamma_1 x(t) \in AAA(\mathbb{R}^+; L^2(,))$. Indeed, let $x \in AAA\left(\mathbb{R}^+; L^2(,)\right)$, then $s \to B_i x(s)$ is in $AAA\left(\mathbb{R}^+; L^2(,)\right)$ as $B_i \in L\left(L^2(,)\right)$, $i = 1, 2$. And hence, by Lemma 2.8, the functions $s \to f(s, B_1x(s))$ belongs to $AAA(\mathbb{R}^+; L^2(,))$. Then we let $F(t) = f(t, B_1x(t)) \in AAA(\mathbb{R}; L^2(,))$. Now we can write $F(t) = f_1(t) + f_2(t)$, where $f_1(t) \in AA(\mathbb{R}; L^2(,))$ and $f_2(t) \in C_0(\mathbb{R}^+; L^2(,))$. Observe

$$\begin{aligned}
\Gamma_1 x(t) &= \int_0^t U(t,s)f_1(s)ds + \int_0^t U(t,s)f_2(s)ds \\
&= \int_{-\infty}^t U(t,s)f_1(s)ds - \int_{-\infty}^0 U(t,s)f_1(s)ds + \int_0^t U(t,s)f_2(s)ds \\
&= \gamma_1(t) + \gamma_2(t),
\end{aligned}$$

where $\gamma_1(t) = \int_{-\infty}^t U(t,s)f_1(s)ds$ and $\gamma_2(t) = \int_0^t U(t,s)f_2(s)ds - \int_{-\infty}^0 U(t,s)f_1(s)ds$.

First we prove that $\gamma_1(t) \in AA(\mathbb{R}; L^2(,))$. Let $\{s'_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $f_1 \in AA(\mathbb{R}; L^2(,))$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ such that for a certain stochastic process $\widetilde{f_1}$

$$\lim_{n\to\infty} E\|f_1(t+s_n) - \widetilde{f_1}(t)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|\widetilde{f_1}(t-s_n) - f_1(t)\|^2 = 0 \tag{3.1}$$

hold for each $t \in \mathbb{R}$. By condition (H2), for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, it follows that $\|U(t+s_n, s+s_n)\| \leq \varepsilon e^{-\delta(t-s)}$ for all $t \geq s \in \mathbb{R}$. Moreover, if we let $\widetilde{\gamma_1}(t) = \int_{-\infty}^t U(t,s)\widetilde{f_1}(s)ds$, then by using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& E\|\gamma_1(t+s_n) - \widetilde{\gamma_1}(t)\|^2 \\
= & E\left\|\int_{-\infty}^{t+s_n} U(t+s_n, s)f_1(s)ds - \int_{-\infty}^t U(t,s)\widetilde{f_1}(s)ds\right\|^2 \\
= & E\left\|\int_{-\infty}^{t} U(t+s_n, s+s_n)f_1(s+s_n)ds - \int_{-\infty}^t U(t,s)\widetilde{f_1}(s)ds\right\|^2 \\
\leq & 2E\left\|\int_{-\infty}^{t} [U(t+s_n, s+s_n) - U(t,s)]f_1(s+s_n)ds\right\|^2 \\
& +2E\left\|\int_{-\infty}^{t} U(t,s)[f_1(s+s_n) - \widetilde{f_1}(s)]ds\right\|^2 \\
\leq & 2\varepsilon^2 E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\|f_1(s+s_n)\|ds\right)^2 \\
& +2M^2 E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\|f_1(s+s_n) - \widetilde{f_1}(s)\|ds\right)^2
\end{aligned}$$

$$\begin{aligned}
&\leq 2\varepsilon^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}E\|f_1(s+s_n)\|^2ds\right)\\
&\quad +2M^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}E\|f_1(s+s_n)-\widetilde{f_1}(s)\|^2ds\right)\\
&\leq 2\varepsilon^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)^2\sup_{t\in\mathbb{R}}E\|f_1(t+s_n)\|^2\\
&\quad +2M^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)\sup_{t\in\mathbb{R}}E\|f_1(t+s_n)-\widetilde{f_1}(t)\|^2\\
&\leq \frac{2}{\delta^2}\varepsilon^2\sup_{t\in\mathbb{R}}E\|f_1(t)\|^2+\frac{2}{\delta^2}M^2\sup_{t\in\mathbb{R}}E\|f_1(t+s_n)-\widetilde{f_1}(t)\|^2
\end{aligned}$$

for all $t \geq s \in \mathbb{R}$ and all $n > N$. Since $f_1(\cdot)$ is bounded and satisfies (3.1), then we immediately obtain that

$$\lim_{n\to\infty} E\|\gamma_1(t+s_n)-\widetilde{\gamma_1}(t)\|^2 = 0 \ \text{ for each } \ t\in\mathbb{R}.$$

A similar reasoning establishes that

$$\lim_{n\to\infty} E\|\widetilde{\gamma_1}(t-s_n)-\gamma_1(t)\|^2 = 0 \ \text{ for each } \ t\in\mathbb{R}.$$

Thus we conclude that $\gamma_1(\cdot) \in AA(\mathbb{R}; L^2(,))$.

Next, let us show that $\gamma_2 \in C_0(\mathbb{R}^+; L^2(,))$. Since $f_2 \in C_0(\mathbb{R}^+; L^2(,))$, for any sufficiently small $\varepsilon_0 > 0$, there exists a constant $T > 0$ such that $E\|f_2(s)\|^2 \leq \varepsilon_0$ for all $s \geq T$. Then, for all $t \geq 2T$, we obtain

$$\begin{aligned}
E\|\gamma_2(t)\|^2 &= E\left\|\int_0^{\frac{t}{2}} U(t,s)f_2(s)ds+\int_{\frac{t}{2}}^{t} U(t,s)f_2(s)ds-\int_{-\infty}^{0} U(t,s)f_1(s)ds\right\|^2\\
&\leq 3E\left\|\int_0^{\frac{t}{2}} U(t,s)f_2(s)ds\right\|^2+3E\left\|\int_{\frac{t}{2}}^{t} U(t,s)f_2(s)ds\right\|^2+3E\left\|\int_{-\infty}^{0} U(t,s)f_1(s)ds\right\|^2\\
&\leq 3EM^2\left(\int_0^{\frac{t}{2}} e^{-\delta(t-s)}\|f_2(s)\|ds\right)^2+3EM^2\left(\int_{\frac{t}{2}}^{t} e^{-\delta(t-s)}\|f_2(s)\|ds\right)^2\\
&\quad +3EM^2\left(\int_{-\infty}^{0} e^{-\delta(t-s)}\|f_1(s)\|ds\right)^2\\
&\leq 3M^2\left(\int_0^{\frac{t}{2}} e^{-\delta(t-s)}ds\right)\left(\int_0^{\frac{t}{2}} e^{-\delta(t-s)}E\|f_2(s)\|^2ds\right)\\
&\quad +3M^2\left(\int_{\frac{t}{2}}^{t} e^{-\delta(t-s)}ds\right)\left(\int_{\frac{t}{2}}^{t} e^{-\delta(t-s)}E\|f_2(s)\|^2ds\right)\\
&\quad +3M^2\left(\int_{-\infty}^{0} e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{0} e^{-\delta(t-s)}E\|f_1(s)\|^2ds\right)\\
&\leq 3M^2\left(\int_0^{\frac{t}{2}} e^{-\delta(t-s)}ds\right)^2\sup_{t\in\mathbb{R}^+}E\|f_2(t)\|^2+3M^2\varepsilon_0\left(\int_{\frac{t}{2}}^{t} e^{-\delta(t-s)}ds\right)^2\\
&\quad +3M^2\left(\int_{-\infty}^{0} e^{-\delta(t-s)}ds\right)^2\sup_{t\in\mathbb{R}}E\|f_1(t)\|^2\\
&\leq \frac{3M^2}{\delta^2}\left[e^{-\frac{\delta t}{2}}-e^{-\delta t}\right]\sup_{t\in\mathbb{R}^+}E\|f_2(t)\|^2+\frac{3M^2\varepsilon_0}{\delta^2}\left[1-e^{-\frac{\delta t}{2}}\right]+\frac{3M^2e^{-\delta t}}{\delta^2}\sup_{t\in\mathbb{R}^+}E\|f_1(t)\|^2\\
&\leq \frac{3M^2}{\delta^2}e^{-\frac{\delta t}{2}}M_f+\frac{3M^2\varepsilon_0}{\delta^2}+\frac{3M^2}{\delta^2}M_g e^{-\delta t}.
\end{aligned}$$

where $M_f = \sup_{t\in\mathbb{R}^+} E\|f_2(t)\|^2$ and $M_g = \sup_{t\in\mathbb{R}} E\|f_1(t)\|^2$. Therefore, the last estimation converges to zero as $t \to +\infty$, since ε_0 is arbitrary. Thus, it leads to $\lim_{t\to+\infty} E\|\gamma_2(t)\|^2 = 0$. Recalling that $\Gamma_1 x(t) = \gamma_1(t) + \gamma_2(t)$ for all $t \geq 0$, we get $\Gamma_1 x(t) \in AAA(\mathbb{R}^+; L^2(,))$.

Now we prove that $\Gamma_2 x(t) \in AAA(\mathbb{R}^+; L^2(,))$. Similarly, by using Lemma 2.8 one can easily see that $s \to g(s, B_2 x(s))$ is in $AAA(\mathbb{R}^+; L^2(,))$ whenever $B_2 x \in AAA(\mathbb{R}^+; L^2(,))$. Then we let $G(t) = g(t, B_2 x(t)) = g_1(t) + g_2(t) \in AAA(\mathbb{R}^+; L^2(,))$ where $g_1 \in AA(\mathbb{R}; L^2(,))$ and $g_2 \in C_0(\mathbb{R}^+; L^2(,))$, then

$$\begin{aligned}
\Gamma_2 x(t) &= \int_0^t U(t,s) g_1(s) dW(s) + \int_0^t U(t,s) g_2(s) dW(s) \\
&= \int_{-\infty}^t U(t,s) g_1(s) dW(s) - \int_{-\infty}^0 U(t,s) g_1(s) dW(s) + \int_0^t U(t,s) g_2(s) dW(s) \\
&= M_1(t) + N_1(t),
\end{aligned}$$

where $M_1(t) = \int_{-\infty}^t U(t,s) g_1(s) dW(s)$ and $N_1(t) = \int_0^t U(t,s) g_2(s) dW(s) - \int_{-\infty}^0 U(t,s) g_1(s) dW(s)$.

The next step we prove that $M_1(t) \in AA(\mathbb{R}; L^2(,))$. Let $\{s_n'\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $g_1 \in AA(\mathbb{R}; L^2(,))$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s_n'\}_{n\in\mathbb{N}}$ such that for a certain stochastic process $\widetilde{g}_1$

$$\lim_{n\to\infty} E\|g_1(t+s_n) - \widetilde{g}_1(t)\|^2 = 0 \ \text{ and } \ \lim_{n\to\infty} E\|\widetilde{g}_1(t-s_n) - g_1(t)\|^2 = 0 \tag{3.2}$$

hold for each $t \in \mathbb{R}$ and by condition (H2), for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, it follows that $\|U(t+s_n, s+s_n) - U(t,s)\| \le \varepsilon e^{-\delta(t-s)}$. Now, let $\widetilde{W}(\sigma) := W(\sigma+s_n) - W(s_n)$ for each $\sigma \in \mathbb{R}$. Note that $\widetilde{W}$ is also a Brownian motion and has the same distribution as W. Moreover, if we let $\widetilde{M}_1(t) = \int_{-\infty}^t U(t,s)\widetilde{g}_1(s) dW(s)$, then by making a change of variable $\sigma = s - s_n$ to get

$$\begin{aligned}
& E\|M_1(t+s_n) - \widetilde{M}_1(t)\|^2 \\
= \ & E\left\|\int_{-\infty}^{t+s_n} U(t+s_n, s) g_1(s) dW(s) - \int_{-\infty}^t U(t,s)\widetilde{g}_1(s) dW(s)\right\|^2 \\
= \ & E\left\|\int_{-\infty}^{t} U(t+s_n, \sigma+s_n) g_1(\sigma+s_n) d\widetilde{W}(\sigma) - \int_{-\infty}^t U(t,\sigma)\widetilde{g}_1(\sigma) d\widetilde{W}(\sigma)\right\|^2 \\
\le \ & 2E\left\|\int_{-\infty}^{t} [U(t+s_n, \sigma+s_n) - U(t,\sigma)] g_1(\sigma+s_n) d\widetilde{W}(\sigma)\right\|^2 \\
& +2E\left\|\int_{-\infty}^{t} U(t,\sigma)[g_1(\sigma+s_n) - \widetilde{g}_1(\sigma)] d\widetilde{W}(\sigma)\right\|^2.
\end{aligned}$$

Thus using an estimate on the Ito integral established in [15], we obtain that

$$\begin{aligned}
& E\|M_1(t+s_n) - \widetilde{M}_1(t)\|^2 \\
\le \ & 2\int_{-\infty}^{t} \|U(t+s_n, \sigma+s_n) - U(t,\sigma)\|^2 E\|g_1(\sigma+s_n)\|^2 d\sigma \\
& +2\int_{-\infty}^{t} \|U(t,\sigma)\|^2 E\|g_1(\sigma+s_n) - \widetilde{g}_1(\sigma)\|^2 d\sigma \\
\le \ & 2\varepsilon^2 \int_{-\infty}^{t} e^{-2\delta(t-\sigma)} E\|g_1(\sigma+s_n)\|^2 d\sigma \\
& +2M^2 \int_{-\infty}^{t} e^{-2\delta(t-\sigma)} E\|g_1(\sigma+s_n) - \widetilde{g}_1(\sigma)\|^2 d\sigma \\
\le \ & \frac{1}{\delta}\varepsilon^2 \sup_{t\in\mathbb{R}} E\|g_1(t)\|^2 + \frac{1}{\delta} M^2 \sup_{t\in\mathbb{R}} E\|g_1(\sigma+s_n) - \widetilde{g}_1(\sigma)\|^2,
\end{aligned}$$

for all $t \ge s$ and all $n > N$. Since $g_1(\cdot)$ is bounded and satisfies (3.2), then we immediately obtain that

$$\lim_{n\to\infty} E\|M_1(t+s_n) - \widetilde{M}_1(t)\|^2 = 0 \ \text{ for all } \ t \in \mathbb{R}.$$

Arguing in a similar way, we infer that $\lim_{n\to\infty} E\|\widetilde{M}_1(t-s_n) - M_1(t)\|^2 = 0$, for all $t \in \mathbb{R}$. This implies that $M_1(t) \in AA(\mathbb{R}; L^2(,))$.

The next step consists of showing that $N_1(t) \in C_0(\mathbb{R}^+; L^2(,))$, since $g_2 \in C_0(\mathbb{R}^+; L^2(,))$, for any sufficient small $\varepsilon_0 > 0$, there exists a constant $T > 0$ such that $E\|g_2(s)\|^2 \le \varepsilon_0$ for all $s \ge T$. Then, for all $t \ge 2T$, we obtain

$$
\begin{aligned}
& E\|N_1(t)\|^2 \\
= \ & E\left\|\int_0^{\frac{t}{2}} U(t,s)g_2(s)dW(s) + \int_{\frac{t}{2}}^t U(t,s)g_2(s)dW(s) - \int_{-\infty}^0 U(t,s)g_1(s)dW(s)\right\|^2 \\
\leq \ & 3E\left(\int_0^{\frac{t}{2}} \|U(t,s)g_2(s)\|^2 ds\right) + 3E\left(\int_{\frac{t}{2}}^t \|U(t,s)g_2(s)\|^2 ds\right) \\
& +3E\left(\int_{-\infty}^0 \|U(t,s)g_1(s)\|^2 ds\right) \\
\leq \ & 3M^2 \int_0^{\frac{t}{2}} e^{-2\delta(t-s)} E\|g_2(s)\|^2 ds + 3M^2 \int_{\frac{t}{2}}^t e^{-2\delta(t-s)} E\|g_2(s)\|^2 ds \\
& +3M^2 \int_{-\infty}^0 e^{-2\delta(t-s)} E\|g_1(s)\|^2 ds \\
\leq \ & \frac{3M^2}{2\delta}[e^{-\delta t} - e^{-2\delta t}] \sup_{t\in\mathbb{R}^+} E\|g_2(s)\|^2 ds + \frac{3M^2}{2\delta}[1 - e^{-\delta t}]\varepsilon_0 \\
& +\frac{3M^2}{2\delta} e^{-2\delta t} \sup_{t\in\mathbb{R}} E\|g_1(s)\|^2 ds \\
\leq \ & \frac{3M^2}{2\delta} e^{-\delta t} M_u + \frac{3M^2}{2\delta}\varepsilon_0 + \frac{3M^2}{2\delta} M_v e^{-2\delta t}
\end{aligned}
$$

where $M_u = \sup_{t\in\mathbb{R}^+} E\|g_2(t)\|^2$ and $M_v = \sup_{t\in\mathbb{R}} E\|g_1(t)\|^2$. Therefore, the last estimation converges to zero as $t \to +\infty$ since ε_0 is arbitrary. Thus, it leads to $\lim_{t\to+\infty} E\|N_1(t)\|^2 = 0$. Recalling that $\Gamma_2 x(t) = M_1(t) + N_1(t)$ for all $t \geq 0$, we get $\Gamma_2 x(t) \in AAA(\mathbb{R}^+; L^2(,))$.

On the other hand, since the evolution family $U(t,s)$ is exponentially stable, it follows that $\lim_{t\to+\infty} E\|\Gamma_0 x(t)\|^2 = 0$. Thus, $\Gamma x(\cdot) \in AAA(\mathbb{R}^+; L^2(,))$. Hence, in view of the above, it is clear that Γ maps $AAA(\mathbb{R}^+; L^2(,))$ into itself.

Now to complete the proof, we have to prove that Γ is a contraction mapping on $AAA(\mathbb{R}^+; L^2(,))$. Indeed, for each $x(t), y(t) \in AAA(\mathbb{R}^+; L^2(,))$, we see that

$$
\begin{aligned}
& E\|(\Gamma x)(t) - (\Gamma y)(t)\|^2 \\
= \ & E\left\|\int_0^t U(t,s)[f(s,B_1x(s)) - f(s,B_1y(s))]ds + \int_0^t U(t,s)[g(s,B_2x(s)) - g(s,B_2y(s))]dW(s)\right\|^2 \\
\leq \ & 2E\left\|\int_0^t U(t,s)[f(s,B_1x(s)) - f(s,B_1y(s))]ds\right\|^2 \\
& +2E\left\|\int_0^t U(t,s)[g(s,B_2x(s)) - g(s,B_2y(s))]dW(s)\right\|^2 \\
\leq \ & 2M^2 E\left(\int_0^t e^{-\delta(t-s)}\|f(s,B_1x(s)) - f(s,B_1y(s))\|\right)^2 \\
& +2E\left(\int_0^t \|U(t,s)[g(s,B_2x(s)) - g(s,B_2y(s))]\|^2 ds\right) \\
\leq \ & 2M^2 E\left[\left(\int_0^t e^{-\delta(t-s)} ds\right)\left(\int_0^t e^{-\delta(t-s)}\|f(s,B_1x(s)) - f(s,B_1y(s))\|^2 ds\right)\right] \\
& +2M^2 \int_0^t e^{-2\delta(t-s)} E\|g(s,B_2x(s)) - g(s,B_2y(s))\|^2 ds \\
\leq \ & 2M^2 L_f \left(\int_0^t e^{-\delta(t-s)} ds\right)\left(\int_0^t e^{-\delta(t-s)} E\|B_1x(s) - B_1y(s)\|^2 ds\right) \\
& +2M^2 L_g \int_0^t e^{-2\delta(t-s)} E\|B_2x(s) - B_2y(s)\|^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2M^2 L_f \overline{w}^2 \left(\int_0^t e^{-\delta(t-s)} ds \right)^2 \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2 \\
&\quad +2M^2 L_g \overline{w}^2 \left(\int_0^t e^{-2\delta(t-s)} ds \right) \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2 \\
&\leq \frac{2M^2}{\delta^2} L_f \overline{w}^2 \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2 + \frac{M^2}{\delta} L_g \overline{w}^2 \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2 \\
&\leq M^2 \overline{w}^2 \left[\frac{2}{\delta^2} L_f + \frac{1}{\delta} L_g \right] \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2,
\end{aligned}
$$

that is,

$$
\|(\Gamma x)(t) - (\Gamma y)(t)\|_2^2 \leq L_0 \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2. \tag{3.3}
$$

Note that

$$
\sup_{t\in\mathbb{R}^+} \|x(t)-y(t)\|_2^2 \leq \left(\sup_{t\in\mathbb{R}^+} \|x(t)-y(t)\|_2 \right)^2, \tag{3.4}
$$

and (3.3) together with (3.4) gives, for each $t \in \mathbb{R}$.

$$
\|(\Gamma x)(t) - (\Gamma y)(t)\|_2 \leq \sqrt{L_0}\|x-y\|_\infty.
$$

Hence, we obtain

$$
\|\Gamma x - \Gamma y\|_\infty = \sup_{t\in\mathbb{R}^+} \|(\Gamma x)(t) - (\Gamma y)(t)\|_2 \leq \sqrt{L_0}\|x-y\|_\infty.
$$

which implies that Γ is a contraction by (3.1). So by the Banach contraction principle, we conclude that there exists a unique fixed point $x(\cdot)$ for Γ in $AAA(\mathbb{R}^+; L^2(,))$ such that $\Gamma x = x$, that is

$$
x(t) = U(t,0)x_0 + \int_0^t U(t,s) f(s, B_1 x(s)) ds + \int_0^t U(t,s) g(s, B_2 x(s)) dW(s)
$$

for all $t \in \mathbb{R}^+$. It is clear that x is a square-mean asymptotically almost automorphic mild solution of Eq. (1.1). The proof is now completed.

□

4 Acknowledgements

The second author's work was supported by Research Fund for Young Teachers of Sanming University (B201107/Q) and Science Foundation of the Education Department of Fujian Province (Grant No.JB12227).

References

[1] P. Acquistapace, Evolution operators and strong solution of abstract linear parabolic equations, *Differential Integral Equations*, 1(1988), 433-457.

[2] P. Acquistapace, B. Terreni, A unified approach to abstract linear nonautonomous parabolic equations, *Rend. Sem. Mat. Univ. Padova*, 78(1987), 47-107.

[3] D. Bugajewski and G. M. N'Guérékata, On the topological structure of almost automorphic and asymptotically almost automorphic solutions of differential and integral equations in abstract spaces, *Nonlinear Anal.*, 59(2004), 1333-1345.

[4] P. Bezandry and T. Diagana, Existence of almost periodic solutions to some stochastic differential equations, *Appl. Anal.*, 86(2007), 819-827.

[5] P. Bezandry and T. Diagana, Square-mean almost periodic solutions to nonautonomous stochastic differential equations, *Electron. J. Differertial Equations*, 2007, 1-10.

[6] Y. K. Chang, Z. H. Zhao, G. M. N'Guérékata, Square-mean almost automorphic mild solutions to non-autonomous stochastic differential equations in Hilbert spaces, *Comput. Math. Appl.,* 61(2011), 384-391.

[7] Y. K. Chang, Z. H. Zhao, G. M. N'Gu'er'ekata, Square-mean almost automorphic mild solutions to some stochastic differential equations in Hilbert space, *Advances in Difference Equations*, 2011, 2011:9.

[8] Y. K. Chang, Z. H. Zhao, G. M. N'Guérékata and R. Ma, Stepanov-like almost automorphic for stochastic processes and applications to stochastic differential equations, *Nonlinear Anal. RWA*, 12(2011), 1130-1139.

[9] Y. K. Chang, Z. H. Zhao and G. M. N'Guérékata, A new composition theorem for square-mean almost automorphic functions and applications to stochastic differential equations, *Nonlinear Anal.*, 74(2011), 2210-2219.

[10] Z. Chen, W. Lin, Square-mean pseudo almost automorphic process and its application to stochastic evolution equations, *J. Funct. Anal.,* 261(2011), 69-89.

[11] H. S. Ding, T. J. Xiao and J. Liang, Asymptotically almost automorphic solutions for some integral-differential equations with nonlocal initial conditions, *J. Math. Anal. Appl.*, 338(2008), 141-151.

[12] T. Diagana, E. Hernández and J. P. C. dos Santos, Existence of asymptotically almost automorphic solutions to some abstract partial neutral integral-differential equations, *Nonlinear Anal.*, 71(2009), 248-257.

[13] M. M. Fu and Z. X. Liu, Square-mean almost automorphic solutions for some stchastic differential equations, *Proc. Amer. Math. Soc.*, 138(2010), 3689-3701.

[14] G. M. N'Guérékata, Sur les solutions presqu'automorphes d''equations diff'erentielles abstraites, *Ann. Sci. Math. Qu'ebec*, 5(1981), 69-79.

[15] A. Ichikawa, Stability of semilinear stochastic evolution equations, *J. Math. Anal. Appl.,* 90(1982), 12-14.

[16] Z. H. Zhao, Y. K. Chang and J. J. Nieto, square-mean asymptotically almost automorphic process and its application to stochastic integro-differential equations, *Dynam. Syst. Appl.,* 22(2013), 269-284.

[17] Z. H. Zhao, Y. K. Chang and J. J. Nieto, Asymptotic behavior of solutions to abstract stochastic fractional partial integrodifferential equations, *Abstr. Appl. Anal.,* Vol. 2013, Article ID 138068, 8 pages, 2013.

[18] Z. H. Zhao, Y. K. Chang and J. J. Nieto, Almost automorphic solutions to some stochastic functional differential equations with delay, *Afr. Diaspora J. Math.,* 15(2013), 7-25.

Received: January 4, 2014; *Accepted*: March 3, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 273–276

Malaya Journal of Matematik

www.malayajournal.org

MJM an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Distributive Lattice: A Rough Set Approach

Sankar Kumar Roy[a,*] and Susanta Bera[b]

[a,b]Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore-721102, West Bengal, India.

Abstract

This paper studies the distributive lattice under the rough set environment and thereby forms a concept of rough distributive lattice. We have discussed the properties of lattice theory in the approximation space and defined rough lattice, rough sublattice and complete rough lattice. We have defined approximation space by using an equivalence relation and then present rough set as a pair of two ordinary sets namely lower and upper approximation sets in the approximation space. The objective of this paper is to study the lattice theory based on rough set by using indiscernibility relation. Some important result are proved. Finally we cite some examples to illustrate the definitions and theories.

Keywords: Distributive Lattice, Rough Set, Approximation Space, Rough Lattice.

2010 MSC: 09D99, 06B75, 06D75.

1 Introduction

The concept of rough set was introduced by Pawlak [1] which is a mathematical tool for systematic study of incomplete knowledge. The basic notions of rough set include rough set approximations and information systems. In Pawlak's rough set the approximations are constructed by equivalence classes of an equivalence relation. For every rough set we associate two crisp sets, called lower and upper approximation sets and viewed as the set of elements which certainly and possibly belong to the set. J. Pomykala and J. A. Pomykala [7] showed that set of rough sets form a stone algebra. Davey and Priestly [4] introduced the concept of lattice theory and order. Iwinski [2] defined rough lattice and order without using any concept of indiscernibility of rough set. Rana and Roy [8] introduced rough set approach on lattice. Most of these are motivated to form lattice by inducing some order relation on rough sets. That is rough set is used as an example of lattice. But rough set is the generalization of ordinary set and therefore one of the main directions of the paper is to study the algebraic structure (lattice) based on rough set.

2 Basic Concept and Definitions

In this section, we give some basic properties and definitions related to lattice under the light of rough set environment which will be needed in the following sections. Let ρ be an equivalence relation defined over a set U.

Definition 2.1. *An equivalence class of $x(\in U)$ is denoted by $[x]_\rho$ and defined as follows: $[x]_\rho = \{y \in U : (x, y) \in \rho\}$.*

Definition 2.2. *The sets $A_\star(X) = \{x \in U : [x]_\rho \subseteq X\}$ and $A^\star(X) = \{x \in U : [x]_\rho \cap X \neq \phi\}$ are respectively called lower and upper approximations of $X \subseteq U$. The pair $S = (U, \rho)$ is called an approximation space and the pair $(A_\star(X), A^\star(X))$ is called the rough set of X in S and is denoted by $A(X)$. The difference $B(X) = A^\star(X) - A_\star(X)$ is called boundary region of X and treated as the area of uncertainty.*

*Corresponding author.

E-mail address: sankroy2006@gmail.com (Sankar Kumar Roy).

Definition 2.3. *The cartesian product of two rough sets* $A(X) = (A_\star(X), A^\star(X))$ *and* $A(Y) = (A_\star(Y), A^\star(Y)$ *is denoted by* $A(X) \times A(Y)$ *and defined as follows:*
$A(X) \times A(Y) = \{(x, y) : x \in A^\star(X)$ *and* $y \in A^\star(Y)\}$

Definition 2.4. *A rough set* $A(Y)$ *is said to be rough subset of a rough set* $A(X)$ *if* $A_\star(Y) \subseteq A_\star(X)$ *and* $A^\star(Y) \subseteq A^\star(X)$ *and it is denoted by* $A(Y) \subseteq A(X)$.

Definition 2.5. *[3] A poset* $(L, \leq)$ *is called a meet-semilattice if for all* $a, b \in L$, $Inf\{a, b\}$ *exists. The definition of join-semilattice is dual one.*

Definition 2.6. *[3] A non-empty set L together with two binary operations '$\vee$' and '$\wedge$' is said to form a lattice if* $\forall a, b, c \in L$, *the following conditions hold:*
1. $a \wedge a = a, \;\; a \vee a = a$ *(Idempotency)*
2. $a \wedge b = b \wedge a, \;\; a \vee b = b \vee a$ *(Commutativity)*
3. $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \;\; a \vee (b \vee c) = (a \vee b) \vee c$ *(Associativity)*
4. $a \wedge (a \vee b) = a, \;\; a \vee (a \wedge b) = a$ *(Absorption).*

Definition 2.7. *[3] A lattice L is said to be complete lattice if for every non-empty subset X of L has a least upper bound and greatest lower bound in X.*

Definition 2.8. *[3] A lattice L is said to be modular lattice if* $\forall x, y, z \in L$ *with* $x \geq y$ *such that* $x \wedge (y \vee z) = y \vee (x \wedge z)$.

Definition 2.9. *[3] A lattice L is said to be distributive lattice if* $\forall x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Now we state two well known lemmas without proof.

Lemma 2.1. *A sublattice of a distributive lattice is distributive.*

Lemma 2.2. *A distributive lattice is always modular.*

2.1 Rough Lattice

In this section we present rough lattice and some properties of them based on Pawlak's notion of roughness. Let $< L, \vee, \wedge >$ be a lattice with two binary operations '$\vee$' and '$\wedge$' defined over L and also let $S = (L, \rho)$ be an approximation space. Let $X \subseteq U$ and $A(X) = (A_\star(X), A^\star(X))$ be the rough set of X in S.

Definition 2.10. $A(X)$ *is said to be rough join semi-lattice if* $x \vee y \in A^\star(X), \forall\, x, y \in X$.
$A(X)$ *is said to be rough meet semi-lattice if* $x \wedge y \in A^\star(X), \forall x, y \in X$.

Definition 2.11. $A(X)$ *is said to be rough lattice in* $S[= (L, \rho)]$ *with respect to the operations '$\vee$' and '$\wedge$' if* $\forall\, x, y \in X$
(i) $x \vee y \in A^\star(X)$
(ii) $x \wedge y \in A^\star(X)$

A rough lattice $< A(X), \vee, \wedge >$ satisfy the following properties $\forall x, y, z \in X$:
(i) $x \vee x = x$ and $x \wedge x = x$ (Idempotency)
(ii) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (Commutativity)
(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (Associativity)
(iv) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ (Absorption)
(v) $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$ (Consistency)

Proposition 2.1. *If* $A(X) = (A_\star(X), A^\star(X))$ *is a rough lattice in an approximation space* $S[= (L, \rho)]$ *such that* $A^\star(X) = X$, *then* $A^\star(X)$ *is a sublattice of L.*

Proof. Since $A(X) = (A_\star(X), A^\star(X))$ is a rough lattice in the approximation space $S[= (L, \rho)]$, therefore clearly, $\forall x, y \in A^\star(X)$, $x \vee y \in A^\star(X)$ and $x \wedge y \in A^\star(X)$. Hence the proved. ☐

Proposition 2.2. *If L is a distributive lattice and* $A(X)$ *is a rough lattice in* $S[= (L, \rho)]$ *such that* $A^\star(X) = X$ *then* $A^\star(X)$ *is a distributive lattice.*

Proof. Since $A^{\star}(X) = X$ and $A(X)$ is a rough lattice, therefore by **Proposition** 2.1, $A^{\star}(X)$ is a sublattice of L and hence by **Lemma 2.1**, $A^{\star}(X)$ is a distributive lattice. □

Definition 2.12. *A rough subset $A(Y)$ of a rough lattice $A(X)$ in an approximation space $S[= (L,\rho)]$ is said to be rough sublattice if $A(Y)$ itself form a rough lattice with respect to the same operations.*

Definition 2.13. *A rough lattice $A(X)$ under an approximation space $S = (L,\rho)$ is said to be a complete rough lattice if every non-empty subset of X has least upper bound and greatest lower bound in $A^{\star}(X)$.*

Proposition 2.3. *A rough lattice $A(X)$ under an approximation space $S[= (L,\rho)]$ is complete rough lattice if $A^{\star}(X)$ is a complete sublattice of L.*

Proof. Let $A^{\star}(X)$ is a complete sublattice of L. Then every non-empty subset of $A^{\star}(X)$ has a least upper bound and greatest lower bound in $A^{\star}(X)$. Since X is a non-empty subset of $A^{\star}(X)$, therefore X has a least upper bound and greatest lower bound in $A^{\star}(X)$. Hence $A(X)$ is a complete rough lattice. □

Definition 2.14. *Let $< A(X), \vee, \wedge >$ is a rough lattice under an approximation space $S[= (L,\rho)]$, then $< A(X), \vee, \wedge >$ is said to be rough modular lattice (RML) if $\forall\, x, y, z \in A^{\star}(X)$ with $x \geq y$, $x \wedge (y \vee z) = y \vee (x \wedge z)$.*

2.2 Rough Distributive Lattice

Definition 2.15. *Let $< A(X), \vee, \wedge >$ is a rough lattice under an approximation space $S[= (L,\rho)]$, then $< A(X), \vee, \wedge >$ is said to be rough distributive lattice (RDL) if $\forall\, x, y, z \in A^{\star}(X)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*

Example 2.1. *The set $L = \{1, 2, 4, 5, 10, 20\}$ of factors of 20 form a lattice where the operators '$\vee$' and '$\wedge$' are defined as $a \vee b$ =least common multiple of $\{a, b\}$ and $a \wedge b$ =greatest common divisor of $\{a, b\}$ and the order relation is divisibility. Let us consider an equivalence relation ρ on L by $x\rho y$ iff "x is congruent to y modulo 2" $\forall x, y \in L$. Let $X = \{2, 4\}$. Then $A_{\star}(X) = \phi$ and $A^{\star}(X) = \{2, 4, 10, 20\}$. Clearly $A(X)$ is rough lattice. Also $\forall x, y, z \in A^{\star}(x)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Therefore $A(X)$ is a RDL.*

Proposition 2.4. *Rough sublattice of a RDL is RDL.*

Proof. Let $A(X)$ is a RDL and $A(Y)$ is a rough sublattice of $A(X)$. Therefore $A^{\star}(Y) \subseteq A^{\star}(X)$ and hence if $x, y, z \in A^{\star}(Y)$ then $x, y, z \in A^{\star}(X)$ and since $A(X)$ is RDL, therefore,$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Therefore if $x, y, z \in A^{\star}(Y)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. □

Proposition 2.5. *If L is a distributive lattice and $A(X)$ is a rough lattice then $A(X)$ is a RDL.*

Proof. Since $A(X)$ is a rough lattice, $A^{\star}(X) \subseteq L$. Therefore $\forall x, y, z \in A^{\star}(X)$ imply, $x, y, z \in L$ and since L is distributive, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Therefore $\forall x, y, z \in A^{\star}(X)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ i.e, $A(X)$ is a RDL. □

Proposition 2.6. *If $A(X)$ is a RDL in $S = (L,\rho)$ and if$A^{\star}(X) = X$ then X is distributive sublattice of L and vice-versa.*

Proof. Since $A(X)$ is RDL and $A^{\star}(X) = X$, therefore $\forall x, y, z \in A^{\star}(X) = X$,
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Also by **proposition 2.1** $A^{\star}(X)$ is sublattice of L and since sublattice of a distributive lattice is distributive, therefore $A^{\star}(X) = X$ is distributive sublattice of L.
Conversely, let $A^{\star}(X) = X$ is distributive sublattice of L. Therefore $A^{\star}(X)$ is rough lattice with $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z \in A^{\star}(X)$. So $A(X)$ is RDL. □

Proposition 2.7. *Two rough lattices $A(X)$ and $A(Y)$ are RDL iff $A(X) \times A(Y)$ is RDL.*

Proof. Let $A(X)$ and $A(Y)$ be RDL and also let $A(X) = (A_{\star}(X), A^{\star}(X))$ and $A(Y) = (A_{\star}(Y), A^{\star}(Y))$. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^{\star}(X) \times A^{\star}(Y)$. Then $x_1, x_2, x_3 \in A^{\star}(X)$ and $y_1, y_2, y_3 \in A^{\star}(Y)$. Therefore

$$\begin{aligned}(x_1, y_1) \wedge \{(x_2, y_2) \vee (x_3, y_3)\} &= (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) \\ &= ((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2) \vee (y_1 \wedge y_3)) \\ &= (x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3) \\ &= \{(x_1, y_1) \wedge (x_2, y_2)\} \vee \{(x_1, y_1) \wedge (x_3, y_3).\end{aligned}$$

Hence $A(X) \times A(Y)$ is RDL.
Conversely, let $A(X) \times A(Y)$ be RDL. Let $x_1, x_2, x_3 \in A^\star(X)$ and $y_1, y_2, y_3 \in A^\star(Y)$. Then $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^\star(X) \times A^\star(Y)$. As $A(X) \times A(Y)$ is RDL, therefore

$$\begin{aligned}
& (x_1, y_1) \wedge \{(x_2, y_2) \vee (x_3, y_3)\} && = \{(x_1, y_1) \wedge (x_2, y_2)\} \vee \{(x_1, y_1) \wedge (x_3, y_3)\} \\
i.e \quad & (x_1, y_1) \wedge (x_2 \vee x_3, y_2 \vee y_3) && = (x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3) \\
or, \quad & (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) && = ((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2)g \vee (y_1 \wedge y_3)).
\end{aligned}$$

Which gives $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$ and $y_1 \wedge (y_2 \vee y_3) = y_1 \wedge y_2) \vee (y_1 \wedge y_3)$. This imply $A(X)$ and $A(Y)$ are RDL. □

Proposition 2.8. *Every RDL is RML but converse is not true.*

Proof. Let $A(X)$ is a RDL. Therefore $x, y, z \in A^\star(Y)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. If $x \geq y$ and $x, y, z \in A^\star(Y)$, then $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = y \vee (x \wedge z)$. Hence $A(X)$ is a RDL.
The converse is not true which is illustrated by the following example: □

Example 2.2. *Let $K_4 = \{e, a, b, c\}$ be the Klein's four group. Let L be the set of all subgroups of K_4. Then $L = \{\{e\}, \{e, a\}, \{e, b\}, \{e, c\}, K_4\}$. L forms a lattice under set inclusion and the operations '$\vee$' and '$\wedge$' are defined by $A \vee B = A \cup B$ and $A \wedge B = A \cap B$, $\forall A, B \in L$. Let us consider an equivalence relation ρ on L defined by $A \rho B$ iff $O(A) = O(B)$ $\forall A, B \in L$. Let $X = \{\{e\}, \{e, a\}, \{K_4\}\}$. Then $A_\star(X) = \{\{e\}, \{K_4\}\}$ and $A^\star(X)) = \{\{e\}, \{e, a\}, \{e, b\}, \{e, c\}, \{K_4\}\}$. Clearly, $A(X)$ is a rough modular lattice but $A(X)$ is not rough distributive lattice.*

3 Conclusion

In this paper the concept of rough distributive lattice is introduced based on Pawlak's indiscernibility relation. At first we construct rough lattice in an approximation space and then we study various properties of them compare to ordinary lattice. Our rough lattice(as we defined) is a rough set with two binary operations and it behaves in a lattice like manner within the rough boundary. As we defined RDL it is seen that the distributivity property of lattice is extended to the area of uncertainty. So we may use lattice structure when the elements of the set are not crisp.

References

[1] Pawlak, Z., Rough Sets, *International Journal of Computer and Information Sciences*, 11(5)(1982), 341-356.

[2] Iwinski,T.B., Algebraic approach to rough sets, *Bull. Polish. Acad. Sci. (Math)*, 35(9-10)(1987), 673-683.

[3] Grätzer, G.,*General Lattice Theory*, 2nd Edition, Birkhauser Verlag, Basel-Boston, Berlin, 2003.

[4] Davey, B.A. and H.A. Priestley, *Introduction to Lattices and Order*, Cambridge, UK: Cambridge University Press, 2001.

[5] Pawlak, Z., Some issues on rough sets, *Transactions on Rough Set, I, Journal Subline, Lecture Notes in Computer Science*, 3100(2004), 1-58.

[6] Järvinen, J., Lattice theory for rough sets, *Transaction on Rough sets VI, Lecture notes in Computer Science*, 4374(2007), 400-498.

[7] J. Pomykala, J. A. Pomykala The stone algebra of rough sets,*Bull. Polish Acad. sci. (Math)*, 36(1998), 495-508.

[8] Rana, D. and Roy, S. K. Rough Set Approach on Lattice, *Journal of Uncertain Systems*, 5(1)(2011), 72-80.

Received:September 12, 2013; *Accepted*: March 13, 2014

Malaya J. Mat. 2(3)(2014) 277–286

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Asymptotic behavior of solutions to a nonautonomous semilinear evolution equation

Xiao-Xia Luo[a,*] and Yong-Long Wang[b]

[a,b]Department of Mathematics, Lanzhou Jiaotong University,Lanzhou - 730070, P. R. China.

Abstract

In this paper, we shall deal with μ-pseudo almost automorphic solutions to the nonautonomous semilinear evolution equations: $u'(t) = A(t)u(t) + f(t, u(t-h))$, $t \in \mathbb{R}$ in a Banach space $\mathbb{X}$, where $A(t)$, $t \in \mathbb{R}$ generates an exponentially stable evolution family $\{U(t,s)\}$ and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a μ-pseudo almost automorphic function satisfying some suitable conditions. We obtain our main results by properties of μ-pseudo almost automorphic functions combined with theories of exponentially stable evolution family.

Keywords: μ-pseudo almost automorphic function, nonautonomous semilinear evolution equations, fixed point.

2010 MSC: 34K14, 60H10, 35B15, 34F05.

1 Introduction

This paper is mainly concerned with the existence of μ-pseudo almost automorphic mild solutions to the following nonautonomous semilinear evolution equations such as

$$u'(t) = A(t)u(t) + f(t, u(t-h)),\ t \in \mathbb{R}, \tag{1.1}$$

where $h \geq 0$ is a fixed constant, and $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the *Acquistapace-Terreni* condition in [1].

The concept of almost automorphy was first introduced in the literature by Bochner [2, 3], it is an important generalization of the classical almost periodicity. For more details about this topic we refer the reader to [4–6]. Since then, there have been several interesting, natural and powerful generalizations of the classical almost automorphic functions. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata in [7]. Liang, Xiao and Zhang in [8, 9] presented the concept of pseudo almost automorphy. In [10], N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy and applied this concept to investigate the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation. Blot et al. introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space in [11], which generalizes that of pseudo-almost automorphic functions. Zhang et al. investigated some properties and new composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [12, 13]. Recently, Blot, Cieutat and Ezzinbi in [14] applied the measure theory to define an ergodic function and they investigate many powerful properties of μ-pseudo almost automorphic functions, and thus the classical theory of pseudo almost automorphy becomes a particular case of this approach.

In [15], the authors studied the existence and uniqueness of Stepanov-like almost solutions to Eq. (1.1). However, few results are available for μ-pseudo asymptotic behavior of solutions to the nonautonomous semilinear evolution equation (1.1). Inspired by the methods in [14, 15], the main aim of the present paper is

*Corresponding author.
E-mail address: lzluoxx0931@163.com (Xiao-Xia Luo), wangyl0931@126.com (Yong-Long Wang).

to investigate μ-pseudo behavior of solutions to the problem (1.1). Some sufficient conditions are established via composition theorems of μ-pseudo almost automorphic functions combined with theories of exponentially stable evolution family.

The rest of this paper is organized as follows. In section 2, we introduce some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In section 3, we prove the existence of μ-pseudo almost automorphic mild solutions to the nonautonomous semilinear evolution equation (1.1).

2 Preliminary

In this section, we fix some basic definitions, notations, lemmas and preliminary facts which will be used in the sequel. Throughout the paper, the notation $(\mathbb{X}, \|\cdot\|)$ is a complex Banach space and $BC(\mathbb{R}, \mathbb{X})$ denotes the Banach space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$.

Throughout this work, we denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all $a, b \in \mathbb{R}(a < b)$.

Definition 2.1. *[3] A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that*

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{R}, \mathbb{X})$.

Definition 2.2. *[16] A continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ is said to be bi-almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that*

$$g(t,s) := \lim_{n\to\infty} f(t + s_n, s + s_n)$$

is well defined for each $t, s \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n, s - s_n) = f(t,s)$$

for each $t, s \in \mathbb{R}$. The collection of all such functions will be denoted by $bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$.

Define

$$PAA_0(\mathbb{R}, \mathbb{X}) := \left\{ \phi \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma)\| d\sigma = 0 \right\}.$$

In the same way, we define by $PAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ as the collection of jointly continuous functions $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ which belong to $BC(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and satisfy

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma, x)\| d\sigma = 0$$

uniformly in compact subset of $\mathbb{X}$.

Definition 2.3. *[16, 17] A continuous function $f : \mathbb{R} \to \mathbb{X}$ (respectively $\mathbb{R} \times \mathbb{X} \to \mathbb{X}$) is called pseudo-almost automorphic if it can be decomposed as $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$(respectively $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$) and $\phi \in PAA_0(\mathbb{R}, \mathbb{X})$(respectively $PAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X})$). Denote by $PAA(\mathbb{R}, \mathbb{X})$ (respectively $PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$) the set of all such functions.*

Definition 2.4. *[14] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-ergodic if*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| d\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

Definition 2.5. *[14] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{X}, \mu)$.*

Obviously, we have $AA(\mathbb{R}, \mathbb{X}) \subset PAA(\mathbb{R}, \mathbb{X}, \mu) \subset BC(\mathbb{R}, \mathbb{X})$.

Lemma 2.1. *[14, Proposition 2.13] Let $\mu \in \mathcal{M}$. Then $(\varepsilon(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 2.2. *[14, Theorem 4.1] Let $\mu \in \mathcal{M}$ and $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ be such that $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. If $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$, (the closure of the range of f).*

Lemma 2.3. *[14, Theorem 2.14] Let $\mu \in \mathcal{M}$ and I be the bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent.*
(i) $f \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.
(ii) $\lim_{r\to+\infty} \frac{1}{\mu([-r,r]\setminus I)} \int_{[-r,r]\setminus I} \|f(t)\| d\mu(t) = 0$.
(iii)For any $\varepsilon > 0$, $\lim_{r\to+\infty} \frac{\mu(\{t\in[-r,r]\setminus I : \|f(t)\|>\varepsilon\})}{\mu([-r,r]\setminus I)} = 0$.

Lemma 2.4. *[14, Theorem 4.7] Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = g + \phi$ where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is unique.*

Lemma 2.5. *[14, Theorem 4.9] Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Theorem 2.1. *[18] Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$. Assume that*
(a1) $f(t, x)$ is uniformly continuous on any bounded subset $Q \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.
(a2) $g(t, x)$ is uniformly continuous on any bounded subset $Q \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.
Then the function defined by $F(\cdot) := f(\cdot, \phi(\cdot)) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ if $\phi \in PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Now, we recall a useful compactness criterion.

Let $h' : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h'(t) \geq 1$ for all $t \in \mathbb{R}$ and $h'(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_{h'}(\mathbb{X}) = \left\{ u \in C(\mathbb{R}, \mathbb{X}) : \lim_{|t|\to\infty} \frac{u(t)}{h'(t)} = 0 \right\}.$$

Endowed with the norm $\|u\|_{h'} = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h'(t)}$, it is a Banach space (see [19, 20]).

Lemma 2.6. *[19, 20] A subset $R \subseteq C_{h'}(\mathbb{X})$ is a relatively compact set if it verifies the following conditions:*
(c-1) The set $R(t) = \{u(t) : u \in R\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.
(c-2) The set R is equicontinuous.
(c-3) For each $\varepsilon > 0$ there exists $L > 0$ such that $\|u(t)\| \leq \varepsilon h'(t)$ for all $u \in R$ and all $|t| > L$.

Lemma 2.7. *[21] (Leray-Schauder Alternative Theorem) Let D be a closed convex subset of a Banach space $\mathbb{X}$ such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

Theorem 2.2. *[22] Assume that $A(t)$, $t \in \mathbb{R}$ is a bounded linear operator on a Banach space $\mathbb{X}$ and $t \to A(t)$ is continuous in the uniform operator topology, then for $-\infty < s \leq t < \infty$, $U(t, s)$ generated by $A(t)$, is a bounded linear operator satisfying the following:*
(i) $\|U(t, s)\| \leq \exp(\int_s^t \|A(\tau)\| d\tau)$.
(ii) $U(t, t) = I$, $U(t, s) = U(t, r)U(r, s)$, for $-\infty < s \leq r \leq t < \infty$.
(iii) $(t, s) \to U(t, s)$ is continuous in the uniform operator topology for $-\infty < s \leq t < \infty$.
(iv) $\partial U(t, s)/\partial t = A(t)U(t, s)$ for $-\infty < s \leq t < \infty$.
(v) $\partial U(t, s)/\partial s = -U(t, s)A(s)$ for $-\infty < s \leq t < \infty$.

3 Main results

In this paper we assume that $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the *Acquistapace-Terreni* conditions introduced in [1, 23], that is,

(H1) There exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $\mathcal{L}, \mathcal{K} \geq 0$ and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \ \ \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{\mathcal{K}}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq \mathcal{L}|t - s|^\alpha |\lambda|^{-\beta}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| \leq \theta\}$.

Remark 3.1. *[1, 24] If the condition (H1) holds, then there exists a unique evolution family $\{U(t,s)\}_{-\infty<s\leq t<\infty}$ on $\mathbb{X}$, which satisfies the homogeneous equation $u'(t) = A(t)u(t)$, $t \in \mathbb{R}$.*

We further suppose that

(H2) The evolution family $U(t,s)$ generated by $A(t)$ is exponentially stable, that is, there are constants K, $\omega > 0$ such that $\|U(t,s)\| \leq Ke^{-\omega(t-s)}$ for all $t \geq s$, $U(t,s)x \in bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$ uniformly for all x in any bounded subset of $\mathbb{X}$.

Consider the following abstract differential equation in the Banach space $(\mathbb{X}, \|\cdot\|)$:

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R} \tag{3.1}$$

where $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the condition (H1) and $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Lemma 3.8. *Let $\mu \in \mathcal{M}$. Assume that (H1) and (H2) hold. Then the Eq. (3.1) has a unique μ-pseudo almost automorphic mild solution given by*

$$u(t) = \int_{-\infty}^{t} U(t,\sigma)f(\sigma)d\sigma \tag{3.2}$$

Proof. First, it is conducted similarly as in the proof of [15, Theorem 3.2], we can prove the uniqueness of the μ-pseudo almost automorphic solution.

Now let us investigate the existence of the μ-pseudo almost automorphic solution. Since $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, there exist $g \in AA(\mathbb{R}, \mathbb{X})$ and $h \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ such that $f = g + h$. So

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} U(t,\sigma)f(\sigma)d\sigma \\ &= \int_{-\infty}^{t} U(t,\sigma)g(\sigma)d\sigma + \int_{-\infty}^{t} U(t,\sigma)h(\sigma)d\sigma \\ &= \Phi(t) + \Psi(t), \end{aligned}$$

where $\Phi(t) = \int_{-\infty}^{t} U(t,\sigma)g(\sigma)d\sigma$, and $\Psi(t) = \int_{-\infty}^{t} U(t,\sigma)h(\sigma)d\sigma$. We just need to verify $\Phi(t) \in AA(\mathbb{R}, \mathbb{X})$ and $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. In view of [15, Theorem 3.2], we see that $\Phi(t) \in AA(\mathbb{R}, \mathbb{X})$.

Next, we prove that $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. It is obvious that $\Psi(t) \in BC(\mathbb{R}, \mathbb{X})$, the left task is to show that

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\Psi(t)\| d\mu(t) = 0.$$

For $r > 0$, we notice that

$$\begin{aligned} & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\Psi(t)\| d\mu(t) \\ \leq\ & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left\| \int_{-\infty}^{t} U(t,\sigma)h(\sigma)d\sigma \right\| d\mu(t) \\ \leq\ & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left\| \int_{0}^{\infty} U(t,t-\sigma)h(t-\sigma)d\sigma \right\| d\mu(t) \\ \leq\ & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_{0}^{\infty} \|U(t,t-\sigma)\| \|h(t-\sigma)\| d\sigma d\mu(t) \\ \leq\ & K \int_{0}^{\infty} e^{-\omega\sigma} \left(\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h(t-\sigma)\| d\mu(t) \right) d\sigma \\ =\ & K \int_{0}^{\infty} e^{-\omega\sigma} \Omega_r(\sigma) d\sigma, \end{aligned}$$

where $\Omega_r(\sigma) = \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|h(t-\sigma)\| d\mu(t)$.

By the fact that the space $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, it follows that $t \to h(t-\sigma)$ belongs to $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ for each $\sigma \in \mathbb{R}$ and hence $\Omega_r(\sigma) \to 0$ as $r \to +\infty$. Since Ω_r is bounded ($\|\Omega_r\| \leq \|h\|_\infty$) and $e^{-\omega\sigma}$ is integrable in $[0,\infty)$, using the Lebesgue dominated convergence theorem it follows that $\lim_{r\to+\infty}\int_0^\infty e^{-\omega\sigma}\Omega_r(\sigma)d\sigma = 0$. We deduce that $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Therefore $u(t) = \Phi(t) + \Psi(t)$ is μ-pseudo almost automorphic.

Finally, let us prove that $u(t)$ is a mild solution of the Eq. (3.1). Indeed, if we let

$$u(s) = \int_{-\infty}^{s} U(s,\sigma)f(\sigma)d\sigma, \tag{3.3}$$

and multiply both sides of (3.3) by $U(t,s)$, then

$$U(t,s)u(s) = \int_{-\infty}^{s} U(t,\sigma)f(\sigma)d\sigma.$$

If $t \geq s$, then

$$\begin{aligned}\int_s^t U(t,\sigma)f(\sigma)d\sigma &= \int_{-\infty}^{t} U(t,\sigma)f(\sigma)d\sigma - \int_{-\infty}^{s} U(t,\sigma)f(\sigma)d\sigma \\ &= u(t) - U(t,s)u(s).\end{aligned}$$

It follows that

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\sigma)f(\sigma)d\sigma.$$

This completes the proof of the theorem. □

Since the space $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, we can easily obtain the following lemma.

Lemma 3.9. *If $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ and $h \geq 0$, then $u(\cdot - h) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$.*

Let us list the following basic assumptions:

(H3) $f \in PAA(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu)$ and there exists a constant $L_f > 0$, such that

$$\|f(t,x) - f(t,y)\| \leq L_f\|x-y\|$$

for all $t \in \mathbb{R}$ and each $x, y \in \mathbb{X}$.

(H4) The function $f = g + \varphi \in PAA(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu)$, where $g \in AA(\mathbb{R}\times\mathbb{X}, \mathbb{X})$ is uniformly continuous in any bounded subset $M \subset \mathbb{X}$ uniformly on $t \in \mathbb{R}$ and $\varphi \in \varepsilon(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu)$.

The following theorems are the main results of this section.

Theorem 3.3. *Let $\mu \in \mathcal{M}$ and suppose that the conditions (H1)-(H3) are satisfied. Then Eq. (1.1) has a unique μ-pseudo almost automorphic mild solution on $\mathbb{R}$ and provided that $\frac{KL_f}{\omega} < 1$.*

Proof. We define the nonlinear operator $\Gamma : PAA(\mathbb{R}, \mathbb{X}, \mu) \to PAA(\mathbb{R}, \mathbb{X}, \mu)$ by

$$(\Gamma u)(t) := \int_{-\infty}^{t} U(t,s)f(s,u(s-h))ds, \quad t \in \mathbb{R}.$$

First, let us prove that $\Gamma(PAA(\mathbb{R}, \mathbb{X}, \mu)) \subset PAA(\mathbb{R}, \mathbb{X}, \mu)$. For each $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, by using the fact that the range of an almost automorphic function is relatively compact combined with the above Theorem 2.1 and Lemma 3.9, one can easily see that $f(\cdot, u(\cdot - h)) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. Hence, from the proof of Lemma 3.8, we know that $(\Gamma u)(\cdot) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. That is, Γ maps $PAA(\mathbb{R}, \mathbb{X}, \mu)$ into $PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Now it suffices to show that the operator Γ has a unique fixed point in $PAA(\mathbb{R}, \mathbb{X}, \mu)$. For this, let u and v be in $PAA(\mathbb{R}, \mathbb{X}, \mu)$, we have

$$\begin{aligned}\|(\Gamma u)(t) - (\Gamma v)(t)\|_\infty &= \sup_{t\in\mathbb{R}}\left\|\int_{-\infty}^{t} U(t,s)[f(s,u(s-h)) - f(s,v(s-h))]ds\right\| \\ &\leq K\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\omega(t-s)}\|f(s,u(s-h)) - f(s,v(s-h))\|ds \\ &\leq KL_f\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\omega(t-s)}\|u(s-h) - v(s-h)\|ds \\ &\leq KL_f\int_{-\infty}^{t} e^{-\omega(t-s)}ds\|u-v\|_\infty \\ &\leq \frac{KL_f}{\omega}\|u-v\|_\infty.\end{aligned}$$

So $\|\Gamma u-\Gamma v\|_\infty \leq \frac{KL_f}{\omega}\|u-v\|_\infty$. By the Banach contraction principle with $\frac{KL_f}{\omega}<1$, Γ has a unique fixed point u in $PAA(\mathbb{R},\mathbb{X},\mu)$, which is the μ-pseudo almost automorphic solution to Eq. (1.1). The proof is complete. $\square$

We next study the existence of μ-pseudo almost automorphic mild solutions of Eq. (1.1) when the perturbation f is not necessarily Lipschitz continuous. For that, we require the following assumptions:
(H5) $f\in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ and $f(t,x)$ is uniformly continuous in any bounded subset $M\subset\mathbb{X}$ uniformly for $t\in\mathbb{R}$ and for every bounded subset $M\subset\mathbb{X}$, $\{f(\cdot,x):x\in M\}$ is bounded in $PAA(\mathbb{R},\mathbb{X},\mu)$.
(H6) There exists a continuous nondecreasing function $W:[0,\infty)\to(0,\infty)$ such that

$$\|f(t,x)\|\leq W(\|x\|)\quad \text{for all}\quad t\in\mathbb{R}\ \text{ and }\ x\in\mathbb{X}.$$

Remark 3.2. *For condition (H6), an interesting results (see Corollary 3.1) is given for the perturbation f satisfying the Hölder type condition.*

Theorem 3.4. *Let $\mu\in\mathcal{M}$ and conditions (H1) and (H2) hold. Let $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ be a function satisfying conditions (H4)-(H6) and the following additional conditions:*
(i) For each $z\geq 0$, the function $t\to\int_{-\infty}^t e^{-\omega(t-s)}W(zh'(s-h))\,ds$ belongs to $BC(\mathbb{R})$. We set

$$\beta(z)=K\left\|\int_{-\infty}^t e^{-\omega(t-s)}W(zh'(s-h))ds\right\|_{h'},$$

where K is the constant given in (H2).
(ii) For each $\varepsilon>0$ there is $\delta>0$ such that for every u, $v\in C_{h'}(\mathbb{X})$, $\|u-v\|_{h'}\leq\delta$ implies that

$$\int_{-\infty}^t e^{-\omega(t-s)}\|f(s,u(s-h))-f(s,v(s-h))\|ds\leq\varepsilon,$$

for all $t\in\mathbb{R}$.
(iii) $\liminf_{\xi\to\infty}\frac{\xi}{\beta(\xi)}>1$.
(iv) For all $a,b\in\mathbb{R}$, $a<b$, and $z>0$, the set $\{f(s,h'(s-h)x):a\leq s-h\leq b, x\in C_{h'}(\mathbb{X}),\|x\|_{h'}\leq z\}$ is relatively compact in $\mathbb{X}$.
Then Eq. (1.1) has at least one μ-pseudo almost automorphic mild solution.

Proof. We define the nonlinear operator $\Gamma:C_{h'}(\mathbb{X})\to C_{h'}(\mathbb{X})$ by

$$(\Gamma u)(t):=\int_{-\infty}^t U(t,s)f(s,u(s-h))ds,\ t\in\mathbb{R}.$$

We will show that Γ has a fixed point in $PAA(\mathbb{R},\mathbb{X},\mu)$. For the sake of convenience, we divide the proof into several steps.

(I) For $u\in C_{h'}(\mathbb{X})$, we have that

$$\begin{aligned}\|(\Gamma u)(t)\| &\leq K\int_{-\infty}^t e^{-\omega(t-s)}W(\|u(s-h)\|)ds\\ &\leq K\int_{-\infty}^t e^{-\omega(t-s)}W(\|u\|_{h'}h'(s-h))ds.\end{aligned}$$

It follows from condition (i) that Γ is well defined.

(II) The operator Γ is continuous. In fact, for any $\varepsilon>0$, we take $\delta>0$ involved in condition (ii). If u, $v\in C_{h'}(\mathbb{X})$ and $\|u-v\|_{h'}\leq\delta$, then

$$\|(\Gamma u)(t)-(\Gamma v)(t)\|\leq K\int_{-\infty}^t e^{-\omega(t-s)}\|f(s,u(s-h))-f(s,v(s-h))\|ds\leq\varepsilon,$$

which shows the assertion.

(III) We will show that Γ is completely continuous. We set $B_z(\mathbb{X})$ for the closed ball with center at 0 and radius z in the space $\mathbb{X}$. Let $V=\Gamma(B_z(C_{h'}(\mathbb{X})))$ and $v=\Gamma(u)$ for $u\in B_z(C_{h'}(\mathbb{X}))$. First, we will prove that $V(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t\in\mathbb{R}$. It follows from condition (i) that the function $s \to Ke^{-\omega s}W(zh'(t-s-h))$ is integrable on $[0,\infty)$. Hence, for $\varepsilon > 0$, we can choose $a \geq 0$ such that $K\int_a^\infty e^{-\omega s}W(zh'(t-s-h))ds \leq \varepsilon$. Since

$$v(t) = \int_0^a U(t,t-s)f(t-s,u(t-s-h))ds + \int_a^\infty U(t,t-s)f(t-s,u(t-s-h))ds$$

and

$$\left\| \int_a^\infty U(t,t-s)f(t-s,u(t-s-h))ds \right\| \leq K\int_a^\infty e^{-\omega s}W(zh'(t-s-h))ds \leq \varepsilon,$$

we get $v(t) \in \overline{ac_0(N)} + B_\varepsilon(\mathbb{X})$, where $c_0(N)$ denotes the convex hull of N and $N = \{U(t,t-s)f(\xi, h'(\xi-h)x) : 0 \leq s \leq a, t-a \leq \xi - h \leq t, \|x\|_{h'} \leq z\}$. Using the strong continuous of $U(t,s)$ and property (iv) of f, we infer that N is a relatively compact set, and $V(t) \subseteq \overline{ac_0(N)} + B_\varepsilon(\mathbb{X})$, which establishes our assertion.

Second, we show that the set V is equicontinuous. In fact, we can decompose

$$\begin{aligned} v(t+s) - v(s) &= \int_0^s U(t,t-\sigma)f(t+s-\sigma, u(t+s-h-\sigma))d\sigma \\ &\quad + \int_0^a [U(t,t-\sigma-s) - U(t,t-\sigma)]f(t-\sigma,u(t-h-\sigma))d\sigma \\ &\quad + \int_a^\infty [U(t,t-\sigma-s) - U(t,t-\sigma)]f(t-\sigma,u(t-h-\sigma))d\sigma. \end{aligned}$$

For each $\varepsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned} &\left\| \int_0^s U(t,t-\sigma)f(t+s-\sigma,u(t+s-h-\sigma))d\sigma \right. \\ &\left. + \int_a^\infty [U(t,t-\sigma-s) - U(t,t-\sigma)]f(t-\sigma,u(t-h-\sigma))d\sigma \right\| \\ \leq\ & K\int_0^s e^{-\omega\sigma}W(zh'(t+s-h-\sigma))d\sigma + K\int_a^\infty (e^{-\omega(\sigma+s)} + e^{-\omega\sigma})W(zh'(t-h-\sigma))d\sigma \\ \leq\ & \frac{\varepsilon}{2} \end{aligned}$$

for $s \leq \delta_1$. Moreover, since $\{f(t-\sigma,u(t-h-\sigma)) : 0 \leq \sigma - h \leq a, x \in B_z(C_{h'}(\mathbb{X}))\}$ is a relatively compact set and $U(t,s)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[U(t,t-\sigma-s) - U(t,t-\sigma)]f(t-\sigma,u(t-h-\sigma))\| \leq \frac{\varepsilon}{2a}$ for $s \leq \delta_2$. Combining these estimates, we get $\|v(t+s) - v(t)\| \leq \varepsilon$ for s small enough and independent of $u \in B_z(C_{h'}(\mathbb{X}))$.

Finally, applying condition (i), it is easy to see that

$$\frac{\|v(t)\|}{h'(t)} \leq \frac{K}{h'(t)}\int_{-\infty}^t e^{-\omega(t-s)}W(zh'(s-h))ds \to 0, \quad |t| \to \infty,$$

and this convergence is independent of $x \in B_z(C_{h'}(\mathbb{X}))$. Hence, by Lemma 2.6, V is a relatively compact set in $(C_{h'}(\mathbb{X}))$.

(IV) Let us now assume that $u^\lambda(\cdot)$ is a solution of equation $u^\lambda = \lambda\Gamma(u^\lambda)$ for some $0 < \lambda < 1$. We can estimate

$$\begin{aligned} \left\| u^\lambda(t) \right\| &= \lambda \left\| \int_{-\infty}^t U(t,s)f(s,u^\lambda(s-h)) \right\| \\ &\leq K\int_{-\infty}^t e^{-\omega(t-s)}W(\|u^\lambda\|_{h'}h'(s-h))ds \\ &\leq \beta(\|u^\lambda\|_{h'})h'(t). \end{aligned}$$

Hence, we get

$$\frac{\|u^\lambda\|_{h'}}{\beta(\|u^\lambda\|_{h'})} \leq 1$$

and combining with condition (iii), we conclude that the set $\{u^\lambda : u^\lambda = \lambda\Gamma(u^\lambda), \lambda \in (0,1)\}$ is bounded.

(V) It follows from assumption (H5), Theorem 2.1 and Lemma 3.9 that the function $t \to f(t,u(t-h))$ belongs to $PAA(\mathbb{R},\mathbb{X},\mu)$, whenever $u \in PAA(\mathbb{R},\mathbb{X},\mu)$. Moreover, from Lemma 3.8 we infer that $\Gamma(PAA(\mathbb{R},\mathbb{X},\mu)) \subset PAA(\mathbb{R},\mathbb{X},\mu)$ and noting that $PAA(\mathbb{R},\mathbb{X},\mu)$ is a closed subspace of $C_{h'}(\mathbb{X})$, consequently, we can consider $\Gamma : PAA(\mathbb{R},\mathbb{X},\mu) \to PAA(\mathbb{R},\mathbb{X},\mu)$. Using properties (I)-(III), we deduce that this map is completely continuous. Applying Lemma 2.7 we infer that Γ has a fixed point $u \in PAA(\mathbb{R},\mathbb{X},\mu)$, which completes the proof. □

Corollary 3.1. *Let $\mu \in \mathcal{M}$. Assume that conditions (H1)-(H2) hold. Let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a function that satisfying assumption (H4)-(H5) and the Hölder type condition*

$$\|f(t,u) - f(t,v)\| \leq \gamma \|u - v\|^{\alpha}, \; 0 < \alpha < 1,$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$, where $\gamma > 0$ is a constant. Moreover, assume the following conditions are satisfied:
(a) $f(t,0) = q$.
(b) $\sup_{t\in\mathbb{R}} K \int_{-\infty}^{t} e^{-\omega(t-s)} h'(s-h)^{\alpha} ds = \gamma_2 < \infty$.
(c) For all $a, b \in \mathbb{R}$, $a < b$, and $z > 0$, the set $\{f(s, h'(s-h)x) : a \leq s - h \leq b, x \in C_{h'}(\mathbb{X}), \|x\|_{h'} \leq z\}$ is relatively compact in $\mathbb{X}$.
Then Eq.(1.1) has a μ-pseudo almost automorphic mild solution.

Proof. Let $\gamma_0 = \|q\|$, $\gamma_1 = \gamma$. We take $W(\xi) = \gamma_0 + \gamma_1 \xi^{\alpha}$. Then condition (H6) is satisfied. It follows from (b), we can see that function f satisfies (i) in Theorem 3.4. To verify condition (ii), note that for each $\varepsilon > 0$ there is $0 < \delta^{\alpha} < \frac{\varepsilon}{\gamma_1 \gamma_2}$ such that for every $u, v \in C_{h'}(\mathbb{X})$, $\|u - v\|_{h'} \leq \delta$ implies that $K \int_{-\infty}^{t} e^{-\omega(t-s)} \|f(s, u(s-h)) - f(s, v(s-h))\| ds \leq \varepsilon$ for all $t \in \mathbb{R}$. On the other hand, the hypothesis (iii) in the statement of Theorem 3.4 can be easily verified using the definition of W. This completes the proof. □

Example 3.1. *Consider the following problem:*

$$\begin{cases} \frac{\partial u}{\partial t} u(t,x) = \frac{\partial^2 u}{\partial^2 x} u(t,x) + u(t,x) \sin \frac{1}{2+\cos t + \cos \sqrt{2}t} + f(t, u(t-h, x)), \\ u(t,0) = u(t,1) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{3.4}$$

where $h > 0$, $\mathbb{X} = L^2(0,1)$, and

$D(B) := \{x \in C^1[0,1]; x'$ is absolutely continuous on $[0,1], x'' \in \mathbb{X}, x(0) = x(1) = 0\}$, $Bx(r) = x''(r)$, $r \in (0,1)$, $x \in D(B)$.

Then B generates a C_0-semigroup $T(t)$ on $\mathbb{X}$ given by

$$(T(t)x)(r) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < x, e_n >_{L^2} e_n(r),$$

where $e_n(r) = \sqrt{2} \sin n\pi r$, $n = 1, 2, \cdots$. Moreover, $\|T(t)\| \leq e^{-\pi^2 t}$, $t \geq 0$.

Define a family of linear operators $A_1(t)$ by

$$\begin{cases} D(A_1(t)) = D(B), \;\; t \in \mathbb{R} \\ A_1(t)x = \left(B + \sin \frac{1}{2+\cos t + \cos \sqrt{2}t}\right) x, \quad x \in D(A_1(t)). \end{cases}$$

Then, $\{A_1(t), t \in \mathbb{R}\}$ generates an evolution family $\{U_1(t,s)\}_{t\geq s}$ such that

$$U_1(t,s)x = T(t-s) e^{\int_s^t \sin \frac{1}{2+\cos t + \cos \sqrt{2}t} d\tau} x.$$

Hence

$$\|U_1(t,s)\| \leq e^{-(\pi^2-1)(t-s)}, \;\; t \geq s.$$

It is easy to see that $U_1(t,s)$ satisfies conditions (H1)-(H2) with $K = 1$, $\omega = \pi^2 - 1$.

Set

$$f(t,u) = u \sin \frac{1}{\cos^2 t + \cos^2 \pi t} + \max_{k\in\mathbb{Z}} \{e^{-(t\pm k^2)^2}\} \sin u, \;\; t \in \mathbb{R}.$$

According to [16, 17], f clearly satisfies conditions (H3) and (H4). From Theorem 3.3, the problem (3.4) has a unique μ-pseudo almost automorphic mild solution.

References

[1] P. Acquistapace, B. Terreni, A unified approach to abstract linear parabolic equations, *Rend. Sem. Mat. Univ. Padova*, 78(1987), 47-107.

[2] S. Bochner, Continuous mappings of almost automorphic and almost periodic functions, *Proc. Natl. Acad. Sci.*, 52(1964), 907-910.

[3] S. Bochner, A new approach to almost periodicity, *Proc. Natl. Acad. Sci.,* 48(1962), 2039-2043.

[4] G. M. N'Guérékata, Almost Automorphic and Almost Periodic Functions in Abstract Spaces, *Kluwer Academic, Plenum Publishers*, New York, 2001.

[5] G. M. N'Guérékata, Topics in Almost Automorphy, *Springer*, New York, 2005.

[6] H. S. Ding, J. Liang, T. J. Xiao, Almost automorphic solutions to nonautonomous semilinear evolution equations in Banach spaces, *Nonlinear Anal.,* 73(2010), 1426-1438.

[7] G. M. N'Guérékata, Sue les solutions presqu'automorphes d' équations différentielles abstraites, *Annales des Sciences Mathématqiues du Québec*, 51(1981), 69-79.

[8] T. J. Xiao, J. Liang, J. Zhang, Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces, *Semigroup Forum,* 76(2008), 518-524.

[9] J. Liang, J. Zhang, T. J.Xiao, Composition of pseudo almost automorphic and asymptotically almost automorphic functions, *J. Math. Anal. Appl.* 340(2008), 1493-1499.

[10] G. M. N'Guérékata, A. Pankov, Stepanov-like almost automorphic functions and monotone evolution equations, *Nonlinear Anal.,* 68(2008), 2658-2667.

[11] J. Blot, G. M. Mophou, G. M. N'Guérékata, D. Pennequin, Weighted pseudo almost automorphic functions and applications to abstract differential equations, *Nonlinear Anal.,* 71(2009), 903-909.

[12] R. Zhang, Y. K. Chang, G. M. N'Guérékata, Weighted pseudo almost automorphic mild solutions for non-autonomous neutral functional differential equations with infinite delay (in Chinese), *Sci. Sin. Math.,* 43(2013), 273-292.

[13] R. Zhang, Y. K. Chang, G. M. N'Guérékata, New composition theorems of Stepanov-like weighted pseudo almost automorphic functions and applications to nonautonomous evolution equations, *Nonlinear Anal. RWA.,* 13(2012), 2866-2879.

[14] J. Blot, P. Cieutat, K. Ezzinbi, Measure theory and pseudo almost automorphic functions : New developments and applications, *Nonlinear Anal.,* 75(2012), 2426-2447.

[15] H. Lee, H. Alkahby, Stepanov-like almost automorphic solutions of nonautonomous semilinear evolution equations with delay, *Nonlinear Anal.,* 69(2008), 2158-2166.

[16] T. J. Xiao, X. X. Zhu, J. Liang, Pseudo almost automorphic mild solutions to nonautonomous differential equations and applications, *Nonlinear Anal. TMA.,* 70(2009), 4079-4085.

[17] J. Liang, G. M. N'Guérékata, T. J. Xiao, et al., Some properties of pseudo almost automrphic functions and applications to abstract differential equations, *Nonlinear Anal. TMA.,* 70(2009), 2731-2735.

[18] Y. K. Chang, X. X. Luo, Existence of μ-pseudo almost automorphic solutions to a neutral differential equation by interpolation theory, *Filomat*, Accepted.

[19] H. R. Henríquez, C. Lizama, Compact almost automorphic solutions to integral equations with infinite delay, *Nonlinear Anal.,* 71(2009), 6029-6037.

[20] R. P. Agarwal, B. Andrade, C. Cuevas, Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations, *Nonlinear Anal. RWA.,* 11(2010), 3532-3554.

[21] A. Granas, J. Dugundji, Fixed Point Theory, *Springer-Verlag*, New York, 2003.

[22] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, in: Applied Mathematical Sciences, vol. 44, *Springer-Verlag*, New York, 1983.

[23] K. J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, *Springer*, New York, 2000.

[24] H. S. Ding, J. Liang, G. M. N'Guérékata, T. J. Xiao, Pseudo-almost periodicity of some nonautonomous evolution equations with delay, *Nonlinear Anal. TMA.*, 67(2007), 1412-1418.

[25] T. Diagana, G. M. N'Guérékata, Stepanov-like almost automorphic functions and applications to some semilinear equations, *Appl. Anal.*, 86(2007), 723-733.

Received: February 13, 2014; *Accepted*: April 25, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 287–292

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

New representation of a fuzzy set

Thangaraj Beaula[a,*] and J. Partheeban[b]

[a,b]*Department of Mathematics, TMBL College, Porayar-609 307, Tamil Nadu.*

Abstract

A new representation of a fuzzy set is introduced. Moreover, the decomposition theorem for the new representation is proved. Fuzzy number is defined using this definition and some properties are established.

Keywords: New representation of a fuzzy set, decomposition theorem, fuzzy number.

2010 MSC: 03E72

1 Introduction

Fuzzy sets have been introduced by LoftiA.Zadeh(1965)[5]. Fuzzy set theory permits the gradual assessment of the membership of elements in a set which is described in the interval[0,1]. It can be used in a wide range of domains where information is incomplete and imprecise.

The representation of fuzzy set in terms of their α-cuts was introduced by Zadeh(1971)[6] in the form of the decomposition theorem. The extension principal is an important tool by which classical mathematical theories can be fuzzified.

The concept of fuzzy numbers have been introduced by Chang and Zadeh(1972)[1].The thory of fuzzy numbers has been studied by Fuller, Majlender[3] and Fodor, Bede[2].

In this paper, some new representation of a fuzzy set is introduced based on α-cut and then some related theorems are proved.

2 Preliminaries

Definition 2.1. *Let X be a universe of discourse, then a fuzzy set is defined as $A = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, which is characterized by a membership function $\mu_A(x) : X \to [0,1]$, where $\mu_A(x)$ denotes the degree of membership of the element x to the set A.*

Definition 2.2. *The α-cut ${}^{\alpha}A$ of a set A is the crisp subset of A with membership grades of at least α. So, ${}^{\alpha}A = \{x|\mu_A(x) \geq \alpha\}$.*

Definition 2.3. *Define for each $x \in X$, a fuzzy set ${}_{\alpha}A(x) = \alpha.{}^{\alpha}A(x)$, where ${}^{\alpha}A$ is the α- cut of the fuzzy set A.*

Definition 2.4. *A fuzzy set $\tilde{A}$ is convex if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$, for $x_1, x_2 \in X, \lambda \in [0,1]$ Alternatively, a fuzzy set is convex if all α-level sets are convex.*

Definition 2.5. *A fuzzy subset A of a classical set X is called normal if there exists an $x \in X$ such that $A(x) = 1$. Otherwise A is subnormal.*

Definition 2.6. *The support of a set A is the crisp subset of A with nonzero membership grades. So,* $\sup(A) = \{x|\mu_A(x) > 0\}$.

*Corresponding author.
E-mail address: edwinbeaula@yahoo.co.in (Thangaraj Beaula).

Definition 2.7. *A fuzzy number A must possess the following three properties:*

1 . *A must be a normal fuzzy set,*

2 . *The alpha levels* ${}^{\alpha}A$ *must be closed for every* $\alpha \in (0,1]$.

3 . *The support of A,* ${}^{0+}A$, *must be bounded.*

3 New representation of a fuzzy set

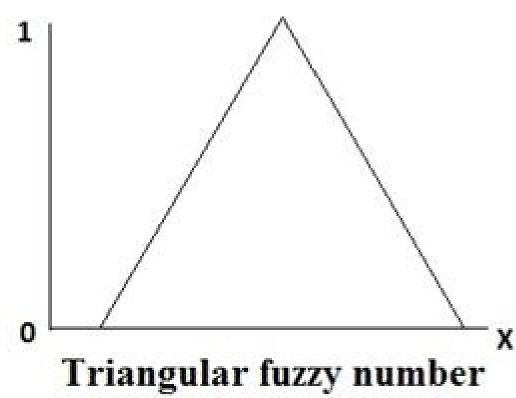
Definition 3.8. *(Power Level Fuzzy Set)*

For a fuzzy set A, $\alpha \in (0,1]$ *define a fuzzy set* $A^{(\alpha)}$, *for each* $x \in X$, *as follows.*
$A^{(\alpha)} = \{(x, \alpha_i) | \alpha_i \geq_{\alpha} A(x) \quad and \quad x \in^{\alpha} A \quad for \quad \alpha \in [0,1]\}$
where ${}_{\alpha}A(x) = \alpha.{}^{\alpha}A$ *and* ${}^{\alpha}A$ *is the* α*-cut of the fuzzy set A.*

Theorem 3.1. *(Decomposition theorem for the new representation)*

For every $A \in \mathcal{F}(X)$, $A = \bigcup_{\alpha \in [0,1]} A^{(\alpha)}$ *where* $A^{(\alpha)} = \sum \alpha_i / x$, *where* $\alpha_i \geq_{\alpha} A(x)$ *and* $x \in^{\alpha} A$.

Proof. If $A(X) = a$, then choose $\alpha = a$, for $x \in^{\alpha} A$ and for all $\alpha_i \geq A(x)$,
Clearly $a \in A^{(\alpha)}$, hence, $A \subseteq \bigcup_{\alpha \in [0,1]} A^{(\alpha)}$
Suppose $\gamma' \in \bigcup_{\alpha \in [0,1]} A^{(\alpha)}, \gamma' \in A(x_{\gamma}) = \gamma$
$\gamma' \in A^{(\alpha)}$ for some α where $\gamma' = (x_{\gamma}, \gamma)$ then $x_{\gamma} \in^{\alpha} A$ and $A(x_{\gamma}) \leq \alpha_i$
Therefore $A(x_{\gamma}) \geq \alpha$ and $\gamma \leq \alpha_i$ and so $A = \bigcup_{\alpha \in [0,1]} A^{(\alpha)}$ □



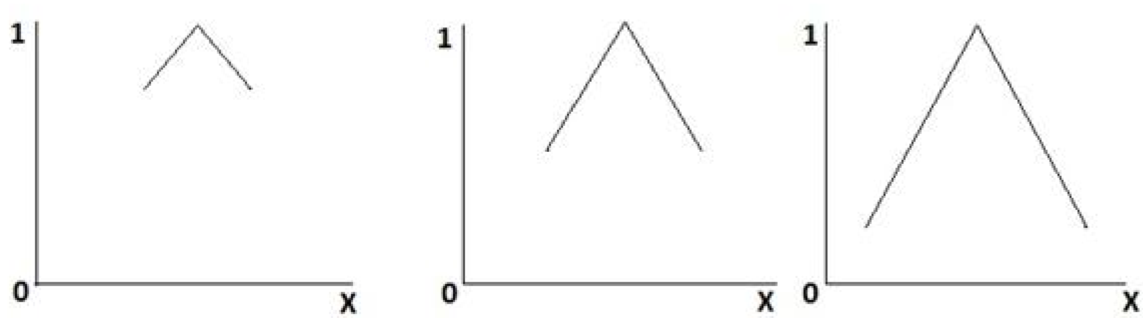


Figure 1: Various $A^{(\alpha)}$ for $\alpha \in (0,1]$

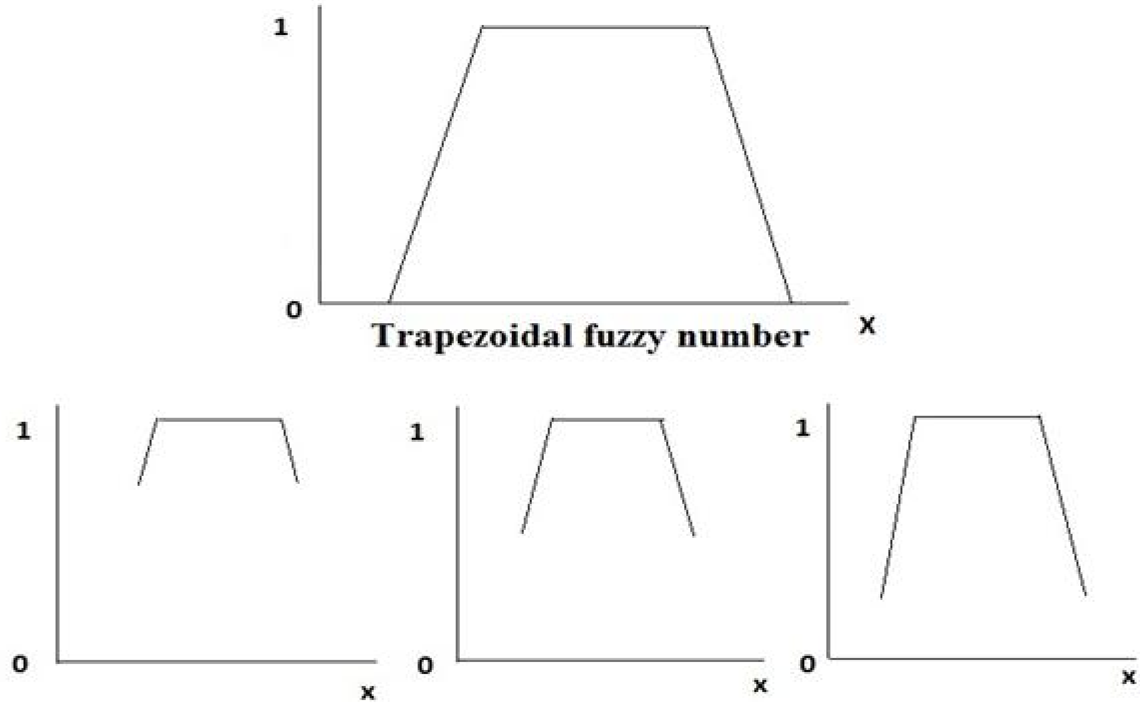


Figure 2: Various $A^{(\alpha)}$ for $\alpha \in (0,1]$

3.1 Extension Principle

Any given function $f : X \to Y$ induces two functions, $f : \mathcal{F}(X) \to \mathcal{F}(Y), f^{-1} : \mathcal{F}(Y) \to \mathcal{F}(X)$, which are defined by $[f(A)](Y) sup_{x|y=f(x)} A(x)$ for all $A \in \mathcal{F}(X)$ and $[f^{-1}(B)](x) = B(f(x))$ for all $B \in \mathcal{F}(X)$.

Theorem 3.2. *Let $f : X \to Y$ be an arbitrary crisp function, for any $A \in \mathcal{F}(X)$ and all $\alpha \in [0,1]$ the following property of f fuzzified by extension principle satisfies the equation $[f(A)]^{(\alpha+)} = f(A^{(\alpha+)})$.*

Proof. For all $y \in Y$,

$$\begin{aligned} y \in [f(A)]^{(\alpha+)} &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(y)] \quad \text{and} \quad y \in^\alpha [f(A)] \quad \text{for} \quad \alpha \in [0,1] \\ &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(Y)] \quad \text{and} \quad f(A)(y) \geq \alpha \\ &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(Y)] \quad \text{and} \quad \sup_{x|y=f(x)} A(x) \geq \alpha \\ &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(Y)] \quad \text{and there exist} \quad x_0 \in X \quad \text{with} \quad y = f(x_0) \\ &\qquad \text{and} \quad A(x_0) \geq \alpha \\ &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(Y)] \quad \text{and there exist} \quad x_0 \in X \quad \text{with} \quad y = f(x_0) \\ &\qquad \text{and} \quad x_0 \in^{\alpha+} A \end{aligned}$$

hence $[f(A)]^{(\alpha+)} = f(A^{(\alpha+)})$.

□

Theorem 3.3. *Let $f : X \to Y$ be an arbitrary crisp function. Then for any $A \in \mathcal{F}(X)$,f fuzzified by the extension principle satisfies the equation, $f(A) = \bigcup_{\alpha \in [0,1]} f(A^{(\alpha+)})$.*

Proof. Applying theorem(3.1) to f(A), which is a fuzzy set on Y,
we obtain $f(A) = \bigcup_{\alpha \in [0,1]} [f(A)]^{(\alpha+)}$

by theorem(3.2)

$$[f(A)]^{(\alpha+)} = f(A^{(\alpha+)}),$$
$$\text{Hence,} \quad f(A) = \bigcup_{\alpha \in [0,1]} f(A^{(\alpha+)}).$$

□

Definition 3.9. *Define $A^{c(\alpha)}$,for each $x \in X$ as $A^{c(\alpha)} = \{(x, \alpha_i) | \alpha_i <_\alpha A(x)$ and $x \in^\alpha A^c\}$.
where ${}^\alpha A^c = \{x | \mu_A(x) \geq 1 - \alpha\}$.*

Theorem 3.4. *For every $A \in \mathcal{F}(X)$,then $A^c = \bigcup_{\alpha \in [0,1]} A^{c(\alpha)}$*

Proof. If $A^{c(\alpha)} = a$, then choose $\alpha = a$, for $x \in^\alpha A^c$, then for all $\alpha_i <_a A(x)$,where $a \in A^{c(\alpha)}$ hence,

$$A^c \subseteq \bigcup_{\alpha \in [0,1]} A^{c(\alpha)} \tag{1}$$

Conversely, suppose $x \in \bigcup_{\alpha \in [0,1]} A^{c(\alpha)}$
then $x \in A^{c(\alpha)}$ for some $\alpha \in [0,1]$ this is true when $x \notin A^{(\alpha)}$ which means that $A^c(x) \leq \alpha$ and $\alpha_i <_\alpha A(x)$, by the definition, we get $x \in A^c$ and hence

$$A^c \supseteq \bigcup_{\alpha \in [0,1]} A^{c(\alpha)} \tag{2}$$

From (1) and (2) we have $A^c = \bigcup_{\alpha \in [0,1]} A^{c(\alpha)}$. □

Definition 3.10.

$$A^{(\alpha)} = \{(x, \alpha_i) | \alpha_i \geq_\alpha A(x) \quad and \quad x \in^\alpha A \quad for \quad \alpha \in (0,1]\}$$

$A^{(\alpha)}$ is convex if and only if ${}_\alpha A$ and ${}^\alpha A$ are convex for any $\alpha \in (0,1]$.

Theorem 3.5. *A fuzzy set $A^{(\alpha)}$ is convex if and only if
$A^{(\alpha)}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A^{(\alpha)}(x_1), A^{(\alpha)}(x_2)\}$ for all $x_1, x_2 \in \Re$ and all $\lambda \in [0,1]$.*

Proof. Assume that $A^{(\alpha)}$ is convex if and only if ${}_\alpha A$ and ${}^\alpha A$ are convex for any $\alpha \in (0,1]$.
By definition, $A^{(\alpha)} = \{(x, \alpha_i) | \alpha_i \geq_\alpha A(x)$ and $x \in^\alpha A$ for $\alpha \in (0,1]\}$.
Let $x_1^{(1)} = (x_1, \alpha_{1i})$ and $x_2^{(1)} = (x_2, \alpha_{2i}) \in A^{(\alpha)}$,
to show that $\lambda(x_1, \alpha_{1i}) + (1-\lambda)(x_2, \alpha_{2i})$ belongs to $A^{(\alpha)}$
We have to prove that $\lambda x_1 + (1-\lambda)x_2 \in A^{(\alpha)}$ and $\left(\frac{\alpha_1 i}{\lambda}\right) + \left(\frac{\alpha_2 i}{1-\lambda}\right) \geq_\alpha A$
as $(x_1, \alpha_{1i}), (x_2, \alpha_{2i})$, by definition $x_1, x_2 \in^\alpha A$ and $\alpha_{1i} \geq_\alpha A(x)$ and $\alpha_{2i} \geq_\alpha A(x)$
If $x_1, x_2 \in^\alpha A$, then $A(x_1) \geq \alpha$ and $A(x_2) \geq \alpha$ and for any $\lambda \in [0,1]$, by the given condition

$$A^{(\alpha)}(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A^{(\alpha)}(x_1^{(1)}, A^{(\alpha)}(x_2^{(1)})\} \tag{1}$$

Consider $\lambda(x_1, \alpha_{1i}) + (1-\lambda)(x_2, \alpha_{2i}) = \left(\lambda x_1 + (1-\lambda)x_1, \frac{\alpha_{1i}}{(\lambda)} + \frac{\alpha_{2i}}{(1-\lambda)}\right)$
From(1), we get $A(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A(x_1^{(1)}), A(x_2^{(1)})\} \geq \min\{\alpha, \alpha\} = \alpha$
Therefore, $\lambda x_1 + (1-\lambda)x_2 \in^\alpha A$, since $\alpha_{1i} \geq_\alpha A(x)$ and $\alpha_{2i} \geq_\alpha A(x)$

From(1),

$$A^{(\alpha)}\left(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}\right) \geq \min\{\alpha_{1i}, \alpha_{2i}\} \tag{2}$$

Suppose $\alpha_{1i} \leq \alpha_{2i}$, then(2) becomes
$\left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) \geq \alpha_{1i}$
and so, $\left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) \geq \alpha_{1i} \geq_\alpha A(x)$
hence $\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)} \in A^{(\alpha)}$
Let $\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)} \in A^{(\alpha)}$
then $\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)} \in^\alpha A$ and $\left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) \geq_\alpha A(x)$
to prove $A^{(\alpha)}(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A^{(\alpha)}(x_1^{(1)}), A^{(\alpha)}(x_2^{(1)})\}$
We already prove that $A(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A(x_1^{(1)}), A(x_2^{(1)})\} \geq \min\{\alpha, \alpha\} = \alpha$
Now let $\alpha_{1i} \leq \alpha_{2i}$, we get

$$\begin{aligned}
\left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) &\geq \left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{1i}}{1-\lambda}\right) \\
&\geq \left(\frac{(1-\lambda)\alpha_{1i} + \lambda\alpha_{1i}}{\lambda(1-\lambda)}\right) \\
&= \left(\frac{\alpha_{1i}}{\lambda(1-\lambda)}\right) \\
\left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) &\geq_\alpha A(x)
\end{aligned}$$

hence,
$A^{(\alpha)}(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A^{(\alpha)}(x_1^{(1)}), A^{(\alpha)}(x_2^{(1)})\}$. □

Definition 3.11. *A fuzzy set A on $\Re$ is a fuzzy number if,*

(i) $A^{(\alpha)}$ *is a normal fuzzy set,*

(ii) *α-cut of* $\{A^{(\alpha)}|\alpha \in (0,1]\}$ *must be nested sequence of closed intervals,*

(iii) *Support of* $A^{(\alpha)}$ *is bounded.*

Definition 3.12. *The fuzzy set* $A^{(\alpha)}$ *is normal if* $supA^{(\alpha)}(x) = 1$ *for* $x \in X$.

Definition 3.13. *By definition* $A^{(\alpha)} = \{(x, \alpha_i)|\alpha_i \geq_\alpha A(x)$ *and* $x \in^\alpha A$ *for* $\alpha \in [0,1]\}$,*then α-cut of* $A^{(\alpha)}$ *consists of all elements* (x, α_i) *such that* $A^{(\alpha)}(x, \alpha_i) \geq \alpha$.

Theorem 3.6. $A \in \mathcal{F}(\Re)$ *is a fuzzy number if and only if for each* $\alpha \in (0,1]$

$$A^{(\alpha)}(x) = \begin{cases} 1 \quad for \quad x \in [0,1] \\ l^\alpha(x) \quad for \quad x \in (-\infty, a) \\ r^\alpha(x) \quad for \quad x \in (b, +\infty) \end{cases}$$

l^α is a monotonically increasing function from $(-\infty, a)$ to $[0,1]$ such that $l^\alpha(x) = 0$ for $x \in (-\infty, a)$ and r^α is a monotonically decreasing function from $(b, +\infty)$ to $[0,1]$ such that $r^\alpha(x) = 0$ for $x \in (b, +\infty)$.

Proof. Necessity. Suppose A is a fuzzy number then $A = \bigcup_{\alpha \in [0,1]} A^{(\alpha)}$
and we obtain a sequence $A^{(\alpha)}$ of fuzzy set for each $\alpha \in [0,1]$.
So, $A^{(\alpha)}$ is a normal fuzzy set and sup $A^{(\alpha)}(x) = 1$ for $x \in X$
by definition, α-cut of $\{A^{(\alpha)}|\alpha \in (0,1]\}$ is a nested sequence of closed intervals
we know that, $A^{(\alpha)} = \{(x, \alpha_i)|\alpha_i \geq_\alpha A(x)$ and $x \in^\alpha A$ for $\alpha \in [0,1]\}$ and its α-cut, ${}^\alpha A(\alpha)$ is a closed interval, for $\alpha = 1$,

Consider ${}^1A^{(\alpha)} = \{x|A^{(\alpha)}(x) \geq 1\}$ and so, $\alpha_i \geq_1 A(x)$ with $x \in^1 A$ for $\alpha \in (0,1]$
for $x \in^1 A^{(\alpha)}, A(x) \geq 1$ with $\alpha_i \geq A(x)$, choose ${}^1A^{(\alpha)} = [a,b]$
define, $l^\alpha(x) = A^\alpha(x)$ for $(-\infty, a)$, then $0 \leq l^\alpha(x) < 1$
Corresponding to each $\alpha \in (0,1]$, there exists a sequence $\{l^{(\alpha)}\}$ of functions from $(-\infty, a)$ to $[0,1]$
Similarly we obtain a sequence $\{r^{(\alpha)}\}$ of functions from $(b, +\infty)$ to $[0,1]$ for each $\alpha \in (0,1]$.
Sufficiency. Every fuzzy set $A^{(\alpha)}$ defined by (1) is clearly normal and support of $A^{(\alpha)}$ is bounded. Finally it remains to prove that α-cut of $A^{(\alpha)}$, for $\alpha \in (0,1]$ is a closed interval.
By definition, $A^{(\alpha)} = \{(x,\alpha_i)|\alpha_i \geq_\alpha A(x) \text{ and } x \in^\alpha A \text{ for } \alpha \in [0,1]\}$

$$\begin{aligned} Let \quad l_{x_\alpha}^{(\alpha)} &= \inf\{(x,\alpha_i)|{}^\alpha l^\alpha(x) \geq \alpha, x < a \quad \text{and} \quad \alpha_i \geq_\alpha l^\alpha(x)\} \\ l_{y_\alpha}^{(\alpha)} &= \sup\{(x,\alpha_i)|{}^\alpha r^\alpha(x) \geq \alpha, y > b \quad \text{and} \quad \alpha_i \geq_\alpha r^\alpha(x)\} \end{aligned}$$

we have

$$\begin{aligned} l_{x\alpha}^{(\alpha)} &= \left(l_{x_\alpha}, \alpha_{l_{x\alpha}}\right) \\ l_{y\alpha}^{(\alpha)} &= \left(l_{y_\alpha}, \alpha_{l_{y\alpha}}\right) \end{aligned}$$

To prove that $[l_{x_\alpha}, l_{y_\alpha}]$ and $[\alpha_{l_{x\alpha}}, \alpha l_{y\alpha}]$ are closed intervals.
If$\left(x_0, \alpha_{x0,i}\right)$ belongs to α-cut of $A^{(\alpha)}$ and if $x_0 < a$
then $l^{(\alpha)}(x_0) = A^{(\alpha)}(x_0)$ and so $x_0 \in^\alpha l^{(\alpha)}$ and $\alpha_i \geq_\alpha l^{(\alpha)}(x)$
$\Rightarrow x_0 \in^\alpha A^{(\alpha)}$ and $\alpha_i \geq_\alpha A^{(\alpha)}$
$\Rightarrow x_0 \in l^{(\alpha)}(x_\alpha)$ and $\alpha_i \geq_\alpha l^{(\alpha)}(x)$
$\Rightarrow \left(x_0, \alpha_{x_{0,i}}\right) \geq l_{x_\alpha}^{(\alpha)}$
if $\left(x_0, \alpha_{x_{0,i}}\right)$ belongs to α-cut of $A^{(\alpha)}$ and if $x_0 > b$
then $r^{(\alpha)}(x_0) = A^{(\alpha)}(x_0)$ and so $x_0 \in^\alpha r^\alpha$ and $\alpha_i \geq_\alpha l^{(\alpha)}$
$\Rightarrow x_0 \in^\alpha A^{(\alpha)}$ and $\alpha_i \geq_\alpha A^{(\alpha)}$
$\Rightarrow x_0 \in l^{(\alpha)}(y_\alpha)$ and $\alpha_i \geq_\alpha l^{(\alpha)}(y)$
$\Rightarrow \left(x_0, \alpha_{x_{0,i}}\right) \leq l_{ya}^{(\alpha)}$
Therefore, $x_0 \in [l_{x_\alpha}^{(\alpha)}, l_{y_\alpha}^{(\alpha)}]$ and hence α-cut of $A^{(\alpha)}$ is a subset of $[l_{x_\alpha}^{(\alpha)}, l_{y_\alpha}^{(\alpha)}]$
by the definition of $l_{x_\alpha}^{(\alpha)}$ there must exist a sequence $\left(x_n, \alpha_{i,n}\right)$ in
$\{(x,\alpha_i)|\alpha_i \geq_\alpha A(x) \text{ and } x \in^\alpha A \text{ for } \alpha \in [0,1]\}$ such that $\lim\limits_{n\to\infty} \left(x_n, \alpha_{i,n}\right) = l_{x_\alpha}^{(\alpha)}$ where $x_n \geq x_\alpha$ for any α, since $l^{(\alpha)}$ is right continuous,$l(l_{x_\alpha}^{(\alpha)}) = l\left(\lim\limits_{n\to\infty}(x_n, \alpha_{i,n})\right) = \lim\limits_{n\to\infty} l(x_n, \alpha_{i,n})$
Then ${}^\alpha l^{(\alpha)}(x_n) \geq \alpha$ and $\alpha_i \geq_\alpha l^{(\alpha)}(x_n)$ and so $l_{x_\alpha}^{(\alpha)}$ lies in the α-cut of $A^{(\alpha)}$
Hence both the necessary and sufficient part of the theorem is proved. □

References

[1] Chang, S.S.L.,Zadeh,L.A., On fuzzy mappings and control, *IEEE Trans.Syst. Man Cyber.*, 2(1972), 30-34.

[2] Fodor,J.,Bede,B., Arithmetic with fuzzy number; a comparative over view, *SAMI 2006 Conference*, Herlany, Slovakia,[ISBN 963 7154 442],PP. 54-68.

[3] Fuller, R.,Majlender. P., On interactive fuzzy numbers, *Fuzzy Sets and Systems*, 143(2004), 355-369.

[4] George J. Klir and Bo Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall of India, 2005.

[5] Zadeh, L.A., Fuzzy sets, *Information and Control*, 8(1965), 338-353.

[6] Zadeh, L.A., Similarity relation and fuzzy orderings, *Information Sciences*, 3(2)(1971), 177-200.

[7] Zadeh, L.A., The concept of fuzzy linguistic variable and its applications reasoning I,II,III, *Information Sciences*, 8(1975), 199-251, 301-357, 43-80.

Received: March 9, 2014; *Accepted*: May 12, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 293–300

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

On Hermite-Hadamard type integral inequalities for functions whose second derivative are nonconvex

Mehmet Zeki SARIKAYA[a,*], Hakan Bozkurt and Mehmet Eyüp KİRİŞ[b]

[a]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY.

[b]Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyon-TURKEY.

Abstract

In this paper, we extend some estimates of the right hand side of a Hermite- Hadamard type inequality for nonconvex functions whose second derivatives absolute values are φ-convex, log-φ-convex, and quasi-φ-convex.

Keywords: Hermite-Hadamard's inequalities, φ-convex functions, log-φ-convex, quasi-φ-convex, Hölder's inequality.

2010 MSC: 26D07, 26D10, 26D99.

1 Introduction

It is well known that if f is a convex function on the interval $I = [a,b]$ and $a,b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which is known as the Hermite-Hadamard inequality for the convex functions. Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1]-[4], [10]-[18]).

The following lemma was proved for twice differentiable mappings in [3]:

Lemma 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^o, $a,b \in I$ with $a < b$ and f'' of integrable on $[a,b]$, the following equality holds:*

$$\frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)\,f(ta+(1-t)b)\,dt.$$

A simple proof of this equality can be also done by twice integrating by parts in the right hand side.

In [4], by using Lemma 1.1, Hussain et al. proved some inequalities related to Hermite-Hadamard's inequality for s-convex functions:

---

*Corresponding author.

E-mail address: sarikayamz@gmail.com (Mehmet Zeki SARIKAYA).

Theorem 1.1. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is $s-$convex on $[a,b]$ for some fixed $s \in [0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2 \times 6^{\frac{1}{p}}} \left[\frac{|f''(a)|^q + |f''(b)|^q}{(s+2)(s+3)} \right]^{\frac{1}{q}}, \tag{1.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1.1. *If we take $s = 1$ in (1.2), then we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{12} \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}}.$$

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(tx + (1-t)y) \leq \sup \{f(x), f(y)\}$$

for all $x, y \in [a,b]$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [10]).

Alomari, Darus and Dragomir in [1] introduced the following theorems for twice differentiable quasiconvex functions:

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o, $a, b \in I^o$ with $a < b$ and f'' is integrable on $[a,b]$. If $|f''|$ is quasiconvex on $[a,b]$, then the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{(b-a)^2}{12} \max\left\{ |f''(a)|, |f''(b)| \right\}.$$

Theorem 1.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o, $a, b \in I^o$ with $a < b$ and f'' is integrable on $[a,b]$. If $|f''|^{\frac{p}{p-1}}$ is a quasiconvex on $[a,b]$, for $p > 1$, then the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right|$$
$$\leq \frac{(b-a)^2}{8} \left(\frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)} \right)^{\frac{1}{p}} \left(\max\left\{ |f''(a)|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o, $a, b \in I^o$ with $a < b$ and f'' is integrable on $[a,b]$. If $|f''|^q$ is a quasiconvex on $[a,b]$, for $q \geq 1$, then the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{(b-a)^2}{12} \left(\max\left\{ |f''(a)|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}}.$$

2 Preliminaries

Let $f, \varphi : K \to \mathbb{R}$, where K is a nonempty closed set in $\mathbb{R}^n$, be continuous functions. First of all, we recall the following well known results and concepts, which are mainly due to Noor and Noor [5] and Noor [9] as follows:

Definition 2.1. *Let $u, v \in K$. Then the set K is said to be $\varphi -$ convex at u with respect to φ, if*

$$u + te^{i\varphi}(v-u) \in K, \ \forall u, v \in K, \ t \in [0,1].$$

Remark 2.2. *We would like to mention that Definition 2.1 of a φ-convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K. We do not require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points, $u, v \in K$, then $e^{i\varphi}(v-u) = v-u$ if and only if, $\varphi = 0$, and consequently φ-convexity reduces to convexity. Thus, it is true that every convex set is also an φ-convex set, but the converse is not necessarily true, see [5]-[9] and the references therein.*

Definition 2.2. *The function f on the φ-convex set K is said to be φ-convex with respect to φ, if*

$$f\left(u+te^{i\varphi}(v-u)\right) \le (1-t)f(u)+tf(v),\ \forall u,v\in K,\ t\in[0,1].$$

The function f is said to be φ-concave if and only if $-f$ is φ-convex. Note that every convex function is a φ-convex function, but the converse is not true.

Definition 2.3. *The function f on the φ-convex set K is said to be logarithmic φ-convex with respect to φ, such that*

$$f\left(u+te^{i\varphi}(v-u)\right) \le (f(u))^{1-t}(f(v))^{t},\ u,v\in K,\ t\in[0,1]$$

where $f(.) > 0$.

Now we define a new definition for quasi-φ-convex functions as follows:

Definition 2.4. *The function f on the quasi φ-convex set K is said to be quasi φ-convex with respect to φ, if*

$$f\left(u+te^{i\varphi}(v-u)\right) \le \max\{f(u), f(v)\}.$$

From the above definitions, we have

$$\begin{aligned} f\left(u+te^{i\varphi}(v-u)\right) &\le (f(u))^{1-t}(f(v))^{t} \\ &\le (1-t)f(u)+tf(v) \\ &\le \max\{f(u), f(v)\}. \end{aligned}$$

Clearly, any φ-convex function is a quasi φ-convex function. Furthermore, there exist quasi φ-convex functions which are neither φ-convex nor continuous. For example, for

$$\varphi(v,u) = \begin{cases} 2k\pi, & u.v \ge 0,\ k\in\mathbb{Z} \\ k\pi, & u.v < 0,\ k\in\mathbb{Z} \end{cases}$$

the floor function $f_{loor}(x) = \lfloor x \rfloor$, is the largest integer not greater than x, is an example of a monotonic increasing function which is quasi φ-convex but it is neither φ-convex nor continuous.

In [7], Noor proved the Hermite-Hadamard inequality for the $\varphi-$convex functions as follows:

Theorem 2.5. *Let $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ be a φ-convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a+e^{i\varphi}(b-a)$ and $0 \le \varphi \le \frac{\pi}{2}$. Then the following inequality holds:*

$$\begin{aligned} f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) &\le \frac{1}{e^{i\varphi}(b-a)}\int_{a}^{a+e^{i\varphi}(b-a)} f(x)\,dx \\ &\le \frac{f(a)+f\left(a+e^{i\varphi}(b-a)\right)}{2} \le \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.3}$$

This inequality can easily show that using the φ-convex function's definition and $f\left(a+e^{i\varphi}(b-a)\right) < f(b)$.

In [19] and [20], the authors proved some generalization inequalities connected with Hermite-Hadamard's inequality for diferentiable φ-convex functions.

In this article, using functions whose second derivatives absolute values are φ-convex, log-φ-convex and quasi-φ-convex, we obtained new inequalities related to the right side of Hermite-Hadamard inequality given with (2.3).

3 Hermite-Hadamard Type Inequalities

We will start the following theorem:

Theorem 3.6. *Let $K \subset \mathbb{R}$ be an open interval, $a, a+e^{i\varphi}(b-a) \in K$ with $a<b$ and $f: K=\left[a, a+e^{i\varphi}(b-a)\right] \rightarrow (0,\infty)$ a twice differentiable mapping such that f'' is integrable and $0 \leq \varphi \leq \frac{\pi}{2}$. If $|f''|$ is φ-convex function on $\left[a, a+e^{i\varphi}(b-a)\right]$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx-\frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ &\frac{e^{2i\varphi}(b-a)^2}{24}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].
\end{aligned}$$

Proof. If the partial integration method is applied twice, then it follows that

$$\begin{aligned}
&\frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)f''(a+te^{i\varphi}(b-a))dt \qquad (3.4)\\
=\ &\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx-\frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}.
\end{aligned}$$

Thus, by φ-convexity function of $|f''|$, we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx-\frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ &\frac{e^{2i\varphi}(b-a)^2}{2}\left|\int_0^1 (t-t^2)f''(a+te^{i\varphi}(b-a))dt\right| \\
\leq\ &\frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left[(1-t)\left|f''(a)\right|+t\left|f''(b)\right|\right]dt \\
\leq\ &\frac{e^{2i\varphi}(b-a)^2}{24}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]
\end{aligned}$$

which the proof is completed. □

Theorem 3.7. *Let $f: K=\left[a, a+e^{i\varphi}(b-a)\right] \rightarrow (0,\infty)$ be a twice differentiable mapping on K^0 and f'' be integrable on $\left[a, a+e^{i\varphi}(b-a)\right]$. Assume $p \in \mathbb{R}$ with $p>1$. If $|f''|^{p/p-1}$ is φ-convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a+e^{i\varphi}(b-a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx-\frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ &\frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^{\frac{p}{p-1}}+|f''(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Proof. By assumption, Hölder's inequality and (3.4), we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 \left|t-t^2\right|\left|f''(a+te^{i\varphi}(b-a))\right|dt \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\left(\int_0^1 (t-t^2)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(a+te^{i\varphi}(b-a))\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}} \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\int_0^1 \left[(1-t)\left|f''(a)\right|^{\frac{p}{p-1}}+t\left|f''(b)\right|^{\frac{p}{p-1}}\right]dt\right)^{\frac{p-1}{p}} \\
=\ & \frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^{\frac{p}{p-1}}+|f''(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}
\end{aligned}$$

where we use the fact that

$$\int_0^1 (t-t^2)^p dt = \frac{2^{-1-2p}\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}$$

which completes the proof. □

Let us denote by $A(a,b)$ the arithmetic mean of the nonnegative real numbers, and by $L(a,b)$ the logaritmic mean of the same numbers.

Theorem 3.8. *Let $K \subset \mathbb{R}$ be an open interval, $a, a+e^{i\varphi}(b-a) \in K$ with $a < b$ and $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ a twice differentiable mapping such that f'' is integrable and $0 \leq \varphi \leq \frac{\pi}{2}$. If $|f''|$ is log φ-convex function on $\left[a, a+e^{i\varphi}(b-a)\right]$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \left(\frac{e^{i\varphi}(b-a)}{\log|f''(b)|-\log|f''(a)|}\right)^2\left[A\left(|f''(b)|,|f''(a)|\right)-L\left(|f''(b)|,|f''(a)|\right)\right].
\end{aligned}$$

Proof. By using (3.4) and log φ-convexity of $|f''|$, we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left|f''(a+te^{i\varphi}(b-a))\right|dt \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left(\left|f''(a)\right|^{1-t}\left|f''(b)\right|^{t}\right)dt \\
=\ & \frac{e^{2i\varphi}(b-a)^2}{2}\left[\frac{|f''(b)|+|f''(a)|}{(\log|f''(b)|-\log|f''(a)|)^2}-\frac{2\left(|f''(b)|-|f''(a)|\right)}{(\log|f''(b)|-\log|f''(a)|)^3}\right] \\
=\ & \left(\frac{e^{i\varphi}(b-a)}{\log|f''(b)|-\log|f''(a)|}\right)^2\left[A\left(|f''(b)|,|f''(a)|\right)-L\left(|f''(b)|,|f''(a)|\right)\right].
\end{aligned}$$

The proof of Theorem 3.8 is completed. □

Theorem 3.9. *Let $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ be a twice differentiable mapping on K^o and f'' be integrable on $\left[a, a+e^{i\varphi}(b-a)\right]$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f''|^{p/p-1}$ is log φ-convex function on the interval of real numbers*

K^o (the interior of K) and $a, b \in K^o$ with $a < a + e^{i\varphi}(b-a)$ and $0 \le \varphi \le \frac{\pi}{2}$. Then, the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2} \right| \\ \le & \frac{e^{2i\varphi}(b-a)^2}{8} \left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}} \left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left(\frac{|f''(a)|^{\frac{p}{p-1}} - |f''(b)|^{\frac{p}{p-1}}}{\log|f''(b)| - \log|f''(a)|}\right)^{\frac{p-1}{p}}. \end{aligned}$$

Proof. By using (3.4) and the well known Hölder's integral inequality, we obtain

$$\begin{aligned} & \left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2} \right| \\ \le & \frac{e^{2i\varphi}(b-a)^2}{2} \int_0^1 (t-t^2) \left| f''(a+te^{i\varphi}(b-a)) \right| dt \\ \le & \frac{e^{2i\varphi}(b-a)^2}{2} \left(\int_0^1 (t-t^2)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f''(a+te^{i\varphi}(b-a)) \right|^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \\ \le & \frac{e^{2i\varphi}(b-a)^2}{2} \left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(\int_0^1 |f''(a)|^{\frac{p}{p-1}(1-t)} |f''(b)|^{\frac{p}{p-1}t} dt \right)^{\frac{p-1}{p}} \\ = & \frac{e^{2i\varphi}(b-a)^2}{8} \left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}} \left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left(\frac{|f''(a)|^{\frac{p}{p-1}} - |f''(b)|^{\frac{p}{p-1}}}{\log|f''(b)| - \log|f''(a)|}\right)^{\frac{p-1}{p}}. \end{aligned}$$

□

Theorem 3.10. *Let $f : K = [a, a+e^{i\varphi}(b-a)] \to (0,\infty)$ be a differentiable mapping on K^0 and f'' be integrable on $[a, a+e^{i\varphi}(b-a)]$. If $|f''|$ is a quasi φ-convex function on the interval of real numbers K^o (the interior of K) and $a, b \in K^o$ with $a < a + e^{i\varphi}(b-a)$ and $0 \le \varphi \le \frac{\pi}{2}$. Then, the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2} \right| \\ \le & \frac{e^{i\varphi}(b-a)}{4} \max\{|f'(a)|, |f'(b)|\}. \end{aligned}$$

Proof. By using (3.4) and the quasi φ-convexity of $|f''|$, we have

$$\begin{aligned} & \left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2} \right| \\ \le & \frac{e^{2i\varphi}(b-a)^2}{2} \int_0^1 (t-t^2) \left| f''(a+te^{i\varphi}(b-a)) \right| dt \\ \le & \frac{e^{2i\varphi}(b-a)^2}{2} \max\{|f''(a)|, |f''(b)|\} \int_0^1 (t-t^2)dt \\ \le & \frac{e^{2i\varphi}(b-a)^2}{24} \max\{|f''(a)|, |f''(b)|\}. \end{aligned}$$

□

Theorem 3.11. *Let $f : K = [a, a+e^{i\varphi}(b-a)] \to (0,\infty)$ be a differentiable mapping on $K^o$and f'' be integrable on $[a, a+e^{i\varphi}(b-a)]$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f''|^{p/p-1}$ is a quasi φ-convex function on the interval of real numbers K^o (the interior of K) and $a, b \in K^o$ with $a < a + e^{i\varphi}(b-a)$ and $0 \le \varphi \le \frac{\pi}{2}$. Then, the following inequality* *holds:*

$$\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right|$$
$$\leq \frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left[\max\{\left|f''(a)\right|^{\frac{p}{p-1}},\left|f''(b)\right|^{\frac{p}{p-1}}\}\right]^{\frac{p-1}{p}}.$$

Proof. By using (3.4) and the well known Hölder's integral inequality, we get

$$\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right|$$
$$\leq \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left|f''(a+te^{i\varphi}(b-a))\right| dt$$
$$\leq \frac{e^{i\varphi}(b-a)}{2}\left(\int_0^1 (t-t^2)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(a+te^{i\varphi}(b-a))\right|^{\frac{p-1}{p}} dt\right)^{\frac{p}{p-1}}$$
$$\leq \frac{e^{i\varphi}(b-a)}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\int_0^1 \max\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\}dt\right)^{\frac{p}{p-1}}$$
$$\leq \frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left[\max\{\left|f''(a)\right|^{\frac{p}{p-1}},\left|f''(b)\right|^{\frac{p}{p-1}}\}\right]^{\frac{p-1}{p}}.$$

□

Theorem 3.12. *Let $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ be a differentiable mapping on K^o and f'' be integrable on $\left[a, a+e^{i\varphi}(b-a)\right]$. Assume $q \in \mathbb{R}$ with $q \geq 1$. If $|f''|^q$ is a quasi φ-convex function on the interval of real numbers K^o (the interior of K) and $a, b \in K^o$ with $a < a+e^{i\varphi}(b-a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then, the following inequality holds:*

$$\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right|$$
$$\leq \frac{e^{2i\varphi}(b-a)^2}{12}\left[\max\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\}\right]^{\frac{1}{q}}.$$

Proof. By using (1.1) and the well known power mean integral inequality, we have

$$\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right|$$
$$\leq \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left|f''(a+te^{i\varphi}(b-a))\right| dt$$
$$\leq \frac{e^{i\varphi}(b-a)}{2}\left(\int_0^1 (t-t^2)dt\right)^{\frac{1}{p}}\left(\int_0^1 (t-t^2)\left|f'(a+te^{i\varphi}(b-a))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{e^{i\varphi}(b-a)}{2}\left(\frac{1}{6}\right)^{\frac{1}{p}}\left(\max\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\}\int_0^1 (t-t^2)dt\right)^{\frac{1}{q}}$$
$$\leq \frac{e^{2i\varphi}(b-a)^2}{12}\left[\max\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\}\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$. □

References

[1] M. Alomari, M. Darus and S. S. Dragomir, New inequalities of Hermite-Hadamard's type for functions whose second derivatives absolute values are quasiconvex, *Tamk. J. Math.*, 41(2010), 353-359.

[2] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, *Appl. Math. Lett.*, 11(5)(1998), 91–95.

[3] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[4] S. Hussain, M. I. Bhatti and M. Iqbal, Hadamard-type inequalities for s-convex functions I, *Punjab Univ. Jour. of Math.*, 41(2009), 51-60.

[5] M. A. Noor, Some new classes of nonconvex functions, *Nonl. Funct. Anal. Appl.*, 11(2006), 165-171.

[6] M. A. Noor, On Hadamard integral inequalities involving two log-preinvex functions, *J. Inequal. Pure Appl. Math.*, 8(3)(2007), 1-6.

[7] M. A. Noor, Hermite-Hadamard integral inequalities for log-φ-convex *functions*, *Nonlinear Analysis Forum*, 13(2)(2008), 119–124.

[8] M. A. Noor, On a class of general variotional inequalities, *J. Adv. Math. Studies*, 1(2008), 31-42.

[9] K. I. Noor and M. A. Noor, Relaxed strongly nonconvex functions, *Appl. Math. E-Notes*, 6(2006), 259-267.

[10] D.A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, *Annals of University of Craiova Math. Comp. Sci. Ser.*, 34(2007), 82-87.

[11] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, *Appl. Math. Lett.*, 13(2)(2000), 51–55.

[12] J. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial ordering and statistical applications*, Academic Press, New York, 1991.

[13] A. Saglam, M. Z. Sarikaya and H. Yildirim, Some new inequalities of Hermite-Hadamard's type, *Kyungpook Mathematical Journal*, 50(2010), 399-410.

[14] M. Z. Sarikaya, A. Saglam and H. Yıldırım, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, *International Journal of Open Problems in Computer Science and Mathematics (IJOPCM)*, 5(3)(2012).

[15] M. Z. Sarikaya, A. Saglam and H. Yıldırım, On some Hadamard-type inequalities for h-convex functions, *Journal of Mathematical Inequalities*, 2(3)(2008), 335-341.

[16] M. Z. Sarikaya, M. Avci and H. Kavurmaci, *On some inequalities of Hermite-Hadamard type for convex functions*, ICMS Iternational Conference on Mathematical Science. AIP Conference Proceedings 1309, 852 (2010).

[17] M. Z. Sarikaya and N. Aktan, On the generalization some integral inequalities and their applications *Mathematical and Computer Modelling*, 54(9-10)(2011), 2175-2182.

[18] M. Z. Sarikaya, E. Set and M. E. Ozdemir, On some new inequalities of Hadamard type involving h-convex functions, *Acta Mathematica Universitatis Comenianae*, Vol. LXXIX, 2(2010), 265-272.

[19] M. Z. Sarikaya, H. Bozkurt and N. Alp, On Hadamard Type Integral Inequalities for nonconvex Functions, *Mathematical Sciences And Applications E-Notes*, in press, arXiv:1203.2282v1.

[20] M. Z. Sarikaya, N. Alp and H. Bozkurt, On Hermite-Hadamard Type Integral Inequalities for preinvex and log-preinvex functions, *Contemporary Analysis and Applied Mathematics*, 1(2)(2013), 237-252.

Received: December 3, 2013; *Accepted*: April 15, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 301–307

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Modified new operations for triangular intuitionistic fuzzy numbers(TIFNS)

R. Irene Hepzibah[a,*] and R.Vidhya[b]

[a]Department of Mathematics, AVC College of Engineering, Mayiladuthurai, Tamil Nadu, India.

[b]Department of Mathematics, AS-SALAM College of Engineering and Technology, Aduthurai, Tamil Nadu, India.

Abstract

Intuitionistic fuzzy sets (IFS) are a generalization of the concept of fuzzy set. In standard intuitionistic fuzzy arithmetic operations, we have some grievances in subtraction and division operations. In this paper, modified new operations for subtraction and division on triangular intuitionistic fuzzy numbers (TIFNS) are defined. Finally an illustrative example for solving Intuitionistic fuzzy multi-objective linear programming problem (IFMOLPP) using these modified operators is provided.

Keywords: Intuitionistic fuzzy arithmetic, Triangular intuitionistic fuzzy number (TIFN), Intuitionistic fuzzy multi-objective linear programming problem (IFMOLPP).

2010 MSC: 65K05, 90C90, 90C70, 90C29.

1 Introduction

In real world, we frequently deal with vague or imprecise information. Information available is sometimes vague, sometimes inexact or sometimes insufficient. The concept of Fuzzy sets was introduced by Zadeh in 1965. The usual arithmetic operation on real numbers can be extended to the ones defined on fuzzy numbers by means of Zadeh's extension Principle [22-24].Then some of the noteworthy contributions on Fuzzy numbers and its applications have been made by Dubois and Prade [6, 7], Kaufmann [9], Kaufmann and Gupta [10], Mizumoto and Tanaka [13], Nahmias [18] and Nguyen [19]. Interval Arithmetic was first suggested by Dwyer [8] in 1951. The same was developed by Moore [14, 15].Various operations on fuzzy numbers were also available in the literature [4, 17] which includes a new operation on Triangular fuzzy number for solving linear programming problem. But these operations are not adequate explicitly. Out of several higher order fuzzy sets, intuitionistic fuzzy sets (IFS) [1,2] have been found to be highly useful to deal with vagueness. There are situations where due to insufficiency in the information available, the evaluation of membership values is not also always possible and consequently there remains a part indeterministic on which hesitation survives. Certainly fuzzy set theory is not appropriate to deal with such problems; rather intuitionistic fuzzy set (IFS) theory is more suitable. The Intuitionistic fuzzy set was introduced by Atanassov.K.T [1] in 1986. For the fuzzy multiple criteria decision making problems, the degree of satisfiability and non- satisfiability of each alternative with respect to a set of criteria is often represented by an intuitionistic fuzzy number (IFN), which is an element of IFS [11, 21].This Intuitionistic fuzzy mathematics is very little studied subject and the extension of fuzzy arithmetic operations to Intuitionistic fuzzy set is needed. Modified new arithmetic operations on intuitionistic triangular fuzzy numbers (TIFNS) are developed in this paper. According to intuitionistic fuzzy arithmetic operation using function principle [4,16], we have $\tilde{A}^I - \tilde{A}^I = \{(0,0,0);(0,0,0)\}$and

*Corresponding author.

E-mail address: ireneraj74@gmail.com (R. Irene Hepzibah).

$\frac{\tilde{A}^I}{\tilde{A}^I} \neq \tilde{1}^I = \{(1,1,1);(1,1,1)\}$. However, in optimization and many engineering applications, it can be desirable to have crisp values for $\tilde{A}^I - \tilde{A}^I$and $\frac{\tilde{A}^I}{\tilde{A}^I}$. i.e., the crisp values 0 and 1 respectively. To overcome the above, the standard intuitionistic fuzzy arithmetic operations on intuitionistic triangular fuzzy number are modified for subtraction and division operations with some necessary conditions and its application is also provided here to enhance the robustness of the new operations developed by us.

The paper is organized as follows. Section 2 deals with some preliminary definitions and the modified operations on triangular fuzzy number using function principle. In section 3 and 4, the new intuitionistic fuzzy arithmetic operations on intuitionistic triangular fuzzy number and its properties are discussed. In section 5, the definition of Intuitionistic fuzzy multi-objective linear programming problem with accuracy function is given. An application of this new operation is discussed with intuitionistic fuzzy multi-objective linear programming problem (IFMOLPP) in section 6 and some concluding remarks are given in section 7.

2 Preliminaries

Definition 2.1. *[1] Given a fixed set $X = \{x_1, x_2, ..., x_n\}$, an intuitionistic fuzzy set (IFS)is defined as $\tilde{A}^I = (\langle x_i, \mu_{\tilde{A}^I(x_i)}, v_{\tilde{A}^I}(x_i)\rangle x_i \in X)$ which assigns to each element x_i, a membership degree $\mu_(x_i)$ and a non-membership degree $v_A(x_i)$ under the condition $0 \leq \mu_A(x_i) + v_A(x_i) \leq 1$, for all $x_i \in X$.*

Definition 2.2. *[10] A triangular intuitionistic fuzzy number (TIFN)$\tilde{A}^I$ is an intuitionistic fuzzy set in R with the following membership function $\mu_{\tilde{A}^I}(x)$ and non-membership function $v_{\tilde{A}^I}(x)$*

$$\mu_{\tilde{A}^I}(x)=\begin{cases} \frac{x-a_1}{a_2-a_1}, a_1 \leq x \leq a_2 \\ \frac{x-a_3}{a_2-a_3}, a_2 \leq x \leq a_3 \\ 0, otherwise \end{cases} \quad and \; v_{\tilde{A}^I}(x)=\begin{cases} \frac{a_2-x}{a_2-a_1^{'}}, a_1^{'} \leq x \leq a_2 \\ \frac{x-a_2}{a_3^{'}-a_2}, a_2 \leq x \leq a_3^{'} \\ 1, otherwise \end{cases},$$

where $a_1^{'} \leq a_1 \leq a_2 \leq a_3 \leq a_3^{'}$ and $\mu_{\tilde{A}^I}(x) + v_{\tilde{A}^I}(x) \leq 1$ or $\mu_{\tilde{A}^I}(x) = v_{\tilde{A}^I}(x)$ for all $x \in R$
This TIFN is denoted by $\tilde{A}^I = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'}) = \{(a_1, a_2, a_3); (a_1^{'}, a_2, a_3^{'})\}$

2.1 Positive triangular intuitionistic fuzzy number

A positive triangular intuitionistic fuzzy number is denoted as $\{(a_1, a_2, a_3); (a_1^{'}, a_2, a_3^{'})\}$ where all $a_i^{\prime}s$ and $a_i^{''}s > 0$ for all $i = 1, 2, 3$.

2.2 Negative triangular intuitionistic fuzzy number

A negative triangular intuitionistic fuzzy number is denoted as $\{(a_1, a_2, a_3); (a_1^{'}, a_2, a_3^{'})\}$ where all $a_i^{\prime}s$ and $a_i^{''}s < 0$ for all $i = 1, 2, 3$.

2.3 Modified operations of triangular intuitionistic fuzzy numbers using function principle

The following are the modified operations that can be performed on triangular intuitionistic fuzzy numbers: Let $\tilde{A}^I = \{(a_1, a_2, a_3); (a_1^{'}, a_2, a_3^{'})\}$ and $\tilde{B}^I = \{(b_1, b_2, b_3); (b_1^{'}, b_2, b_3^{'})\}$

Then

1. Addition:$\tilde{A}^I + \tilde{B}^I = \{(a_1 + b_1, a_2 + b_2, a_3 + b_3); (a_1^{'} + b_1^{'}, a_2 + b_2, a_3^{'} + b_3^{'})\}$.

2. Subtraction: $\tilde{A}^I - \tilde{B}^I = \{(a_1 - b_3, a_2 - b_2, a_3 - b_1); (a_1^{'} - b_3^{'}, a_2 - b_2, a_3^{'} - b_1^{'})\}$.

3. Multiplication:$\tilde{A}^I \times \tilde{B}^I = \{(min(a_1b_1, a_1b_3, a_3b_1, a_3b_3), a_2b_2, max(a_1b_1, a_1b_3, a_3b_1, a_3b_3));$
$(min(a_1^{'}b_1^{'}, a_1^{'}b_3^{'}, a_3^{'}b_1^{'}, a_3^{'}b_3^{'}), a_2b_2, max(a_1^{'}b_1^{'}, a_1^{'}b_3^{'}, a_3^{'}b_1^{'}, a_3^{'}b_3^{'}))\}$.

4. Division: $\frac{\tilde{A}^I}{\tilde{B}^I} = \{(min(\frac{a_1}{b_1}, \frac{a_1}{b_3}, \frac{a_3}{b_1}, \frac{a_3}{b_3}), \frac{a_2}{b_2}, max(\frac{a_1}{b_1}, \frac{a_1}{b_3}, \frac{a_3}{b_1}, \frac{a_3}{b_3}))$
$(min(\frac{a_1^{'}}{b_1^{'}}, \frac{a_1^{'}}{b_3^{'}}, \frac{a_3^{'}}{b_1^{'}}, \frac{a_3^{'}}{b_3^{'}}), \frac{a_2}{b_2}, max(\frac{a_1^{'}}{b_1^{'}}, \frac{a_1^{'}}{b_3^{'}}, \frac{a_3^{'}}{b_1^{'}}, \frac{a_3^{'}}{b_3^{'}}))\}$.

Example 2.1. *Let* $\tilde{A}^I = \{(2,4,6);(1,4,7)\}$ *and* $\tilde{B}^I = \{(1,2,3);(0.5,2,3.5)\}$
Then

1. $\tilde{A}^I + \tilde{B}^I = \{(3,6,9);(1.5,6,10.5)\}$
2. $\tilde{A}^I - \tilde{B}^I = \{(-1,2,5);(-2.5,2,6.5)\}$
3. $\tilde{A}^I \times \tilde{B}^I = \{(2,8,18);(0.5,8,24.5)\}$
4. $\frac{\tilde{A}^I}{\tilde{B}^I} = \{(0.6,2,6);(0.286,2,14)\}$
5. $\frac{\tilde{A}^I}{\tilde{A}^I} = \{(0.333,1,3);(0.143,1,7)\}$

Remark 2.1. *As mentioned earlier that* $\tilde{A}^I - \tilde{A}^I \neq \tilde{0}^I = \{(0,0,0);(0,0,0)\}$;
$\frac{\tilde{A}^I}{\tilde{A}^I} \neq \tilde{1}^I = \{(1,1,1);(1,1,1)\}$.
It follows that $\tilde{C}^I$ *is the solution of the intuitionistic fuzzy linear equation* $\tilde{A}^I + \tilde{B}^I = \tilde{C}^I$. *Then we would expect* $\tilde{B}^I = \tilde{C}^I - \tilde{A}^I$.
For example, $\tilde{A}^I + \tilde{B}^I = \{(2,4,6);(1,4,7)\} + \{(1,2,3);(0.5,2,3.5)\} = \{(3,6,9);(1.5,6,10.5)\}$.
But $\{(1,2,3);(0.5,2,3.5)\} = \{(1,2,3);(0.5,2,3.5)\} - \{(2,4,6);(1,4,7)\}$
$\{(1,2,3);(0.5,2,3.5)\} = \{(-3,2,7);(-5.5,2,9.5)\} \neq \tilde{B}^I$.
The same thing appears when solving the intuitionistic fuzzy equation $\tilde{A}^I \times \tilde{B}^I = \tilde{C}^I$ *whose solution is not given by* $\tilde{B}^I = \frac{\tilde{C}^I}{\tilde{A}^I} = \frac{\{(2,8,18);(0.5,8,24.5)\}}{\{(2,4,6);(1,4,7)\}}$
$\tilde{B}^I = \{(\frac{2}{6}, \frac{8}{4}, \frac{18}{2});(\frac{0.5}{7}, \frac{8}{4}, \frac{24.5}{1})\} = \{(0.333,2,9);(0.071,2,24.5)\} \neq \tilde{B}^I$
Therefore, the addition and subtraction (respectively multiplication and division) of intuitionistic triangular fuzzy numbers are not reciprocal operations. According to this statement, it is not possible to solve inverse problems exactly using the standard fuzzy arithmetic operators. To overcome this in function principle operation of triangular intuitionistic fuzzy numbers, a new operation is proposed that allows exact inversion.

3 A New Operation for Subtraction on intuitionistic Triangular fuzzy Number:

In this section our objective is to develop a new subtraction operator on triangular intuitionistic fuzzy number, which is the exact inverse of the addition $'+'$.

3.1 Condition on Subtraction Operator

Let $\tilde{A}^I = \{(a_1,a_2,a_3);(a_1^{'},a_2,a_3^{'})\}$ and $\tilde{B}^I = \{(b_1,b_2,b_3);(b_1^{'},b_2,b_3^{'})\}$
Then $\tilde{A}^I - \tilde{B}^I = \{(a_1-b_1,a_2-b_2,a_3-b_3);(a_1^{'}-b_1^{'},a_2-b_2,a_3^{'}-b_3^{'})\}$.
The new subtraction operation exists only if the following conditions are satisfied $D(\tilde{A}^I) \geq D(\tilde{B}^I)$ and $D(\tilde{A}^{I'}) \geq D(\tilde{B}^{I'})$, where $D(\tilde{A}^I) = \frac{a_3-a_1}{2}, D(\tilde{B}^I) = \frac{b_3-b_1}{2}, D(\tilde{A}^{I'}) = \frac{a_3^{'}}{2}$ and $D(\tilde{B}^I) = \frac{b_3^{'}-b_2^{'}}{2}$. Here D denotes difference point of a intuitionistic triangular fuzzy number.

3.2 Properties of Subtraction Operator

1. Inverse operator of $+$: $\tilde{B}^I + (\tilde{A}^I - \tilde{B}^I) = (\tilde{A}^I - \tilde{B}^I) + \tilde{B}^I$
2. Multiplication by a scalar: $\lambda(\tilde{A}^I - \tilde{B}^I) = \lambda\tilde{A}^I - \lambda\tilde{B}^I$
3. Neutral element: $\tilde{A}^I - \tilde{0}^I = \tilde{A}^I$
4. Associativity: $\tilde{A}^I - (\tilde{B}^I - \tilde{C}^I) = (\tilde{A}^I - \tilde{B}^I) - \tilde{C}^I$

5. Inverse element: Any intuitionistic triangular fuzzy number is its own inverse under the modified subtraction i.e., $\tilde{A}^I - \tilde{A}^I = \tilde{0}^I$

6. Regularity: $\tilde{A}^I - \tilde{B}^I = \tilde{A}^I - \tilde{C}^I \Rightarrow \tilde{B}^I = \tilde{C}^I$

7. Pseudo - distributivity with respect to $+$: $(\tilde{A}^I + \tilde{B}^I) - (\tilde{C}^I + \tilde{D}^I) = (\tilde{A}^I - \tilde{C}^I) + (\tilde{B}^I - \tilde{D}^I)$

3.3 Mid point of a intuitionistic triangular fuzzy number

Let $\tilde{A}^I = \{(a_1, a_2, a_3); (a_1^{'}, a_2, a_3^{'})\}$ and $\tilde{B}^I = \{(b_1, b_2, b_3); (b_1^{'}, b_2, b_3^{'})\}$
Then $M(\tilde{A}^I) = \frac{a_3+a_1}{2}$,$M(\tilde{B}^I) = \frac{b_3+b_1}{2}$,$M(\tilde{A}^{I'}) = \frac{a_3^{'}+a_1^{'}}{2}$, $M(\tilde{B}^{I'}) = \frac{b_3^{'}+b_1^{'}}{2}$.Here M denotes midpoint of a intuitionistic triangular fuzzy number.

3.4 Necessary Existence Condition for Subtraction

Proposition 3.1. *The new subtraction operations exists only if the following conditions are satisfied* $D(\tilde{A}^I) \geq D(\tilde{B}^I)$ *and* $D(\tilde{A}^{I'}) \geq D(\tilde{B}^{I'})$.

Proof 1. *We have derived the necessary existence condition for* $\tilde{A}^I - \tilde{B}^I$ *which is equal to* $\{(C_1, C_2, C_3); (C_1^{'}, C_2, C_3^{'})\}$.
Let as take $C_1 \leq C_2 \leq C_3 \Rightarrow C_1 \leq C_3 \Rightarrow a_1 - b_1 \leq a_3 - b_3$
$\Rightarrow [M(\tilde{A}^I) - D(\tilde{A}^I)] - [M(\tilde{B}^I) - D(\tilde{B}^I)] \leq [M(\tilde{A}^I) + D(\tilde{A}^I)] - [M(\tilde{B}^I) + D(\tilde{B}^I)]$
$\Rightarrow [M(\tilde{B}^I) + D(\tilde{B}^I)] - [M(\tilde{B}^I) - D(\tilde{B}^I)] \leq [M(\tilde{A}^I) + D(\tilde{A}^I)] - [M(\tilde{A}^I) - D(\tilde{A}^I)]$
$\Rightarrow 2D(\tilde{B}^I) \leq 2D(\tilde{A}^I)$
$\Rightarrow D(\tilde{B}^I) \leq D(\tilde{A}^I)$
$\Rightarrow D(\tilde{A}^I) \geq D(\tilde{B}^I)$
Similarly we can prove $D(\tilde{A}^{I'}) \geq D(\tilde{B}^{I'})$.
These are the necessary conditions for new subtraction operator.

4 Condition on Division Operator:

Let $\tilde{A}^I = \{(a_1, a_2, a_3); (a_1^{'}, a_2, a_3^{'})\}$ and $\tilde{B}^I = \{(b_1, b_2, b_3); (b_1^{'}, b_2, b_3^{'})\}$
Then $\frac{\tilde{A}^I}{\tilde{B}^I} = \{(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}); (\frac{a_1^{'}}{b_1^{'}}, \frac{a_2}{b_2}, \frac{a_3^{'}}{b_3^{'}})\}$.

The new division operator exists only if the following conditions are satisfied $| \frac{D(\tilde{A}^I)}{M(\tilde{A}^I)} | \geq | \frac{D(\tilde{B}^I)}{M(\tilde{B}^I)} |$;$| \frac{D(\tilde{A}^{I'})}{M(\tilde{A}^{I'})} | \geq | \frac{D(\tilde{B}^{I'})}{M(\tilde{B}^{I'})} |$ and the negative triangular intuitionistic fuzzy number should be changed into negative multiplication of positive triangular intuitionistic fuzzy number.

4.1 Properties of Division Operator

1. Inverse operator of X: $\tilde{B}^I \times (\frac{\tilde{A}^I}{\tilde{B}^I}) = (\frac{\tilde{A}^I}{\tilde{B}^I}) \times \tilde{B}^I$

2. Neutral element: The singleton $\tilde{1}^I = \{(1,1,1); (1,1,1)\}$ defined by constant profile equal to $\tilde{1}^I$ is a right neutral element of division $\frac{\tilde{A}^I}{\tilde{1}^I} = \{(\frac{a_1}{1}, \frac{a_2}{1}, \frac{a_3}{1}); (\frac{a_1^{'}}{1}, \frac{a_2}{1}, \frac{a_3^{'}}{1})\} = \{(a_1, a_2, a_3); (a_1^{'}, a_2, a_3^{'})\} = \tilde{A}^I$

3. Inverse element: Any triangular intuitionistic fuzzy number is its own inverse under modified division operator $\frac{\tilde{A}^I}{\tilde{A}^I} = \{(\frac{a_1}{a_1}, \frac{a_2}{a_2}, \frac{a_3}{a_3}); (\frac{a_1^{'}}{a_1^{'}}, \frac{a_2}{a_2}, \frac{a_3^{'}}{a_3^{'}})\} = \{(1,1,1); (1,1,1)\} = \tilde{1}^I$

4. Regularity: $\frac{\tilde{A}^I}{\tilde{B}^I} = \frac{\tilde{A}^I}{\tilde{C}^I} \Rightarrow \tilde{B}^I = \tilde{C}^I$

5. Distributivity with regard to $+$: $\frac{\tilde{A}^I + \tilde{B}^I}{\tilde{C}^I} = \frac{\tilde{A}^I}{\tilde{C}^I} + \frac{\tilde{B}^I}{\tilde{C}^I}$

4.2 Necessary Existence Condition for Division

Proposition:
We have derived the necessary existence condition for $\frac{\tilde{A}^I}{\tilde{B}^I}$ which is equal to $\{(C_1, C_2, C_3); (C_1^{'}, C_2, C_3^{'})\}$.
Let as take $C_1 \leq C_2 \leq C_3 \Rightarrow C_1 \leq C_3 \Rightarrow \frac{a_1}{b_1} \leq \frac{a_3}{b_3}$
$\Rightarrow \frac{[M(\tilde{A}^I)-D(\tilde{A}^I)]}{[M(\tilde{B}^I)-D(\tilde{B}^I)]} \leq \frac{[M(\tilde{A}^I)+D(\tilde{A}^I)]}{[M(\tilde{B}^I)+D(\tilde{B}^I)]}$
$\Rightarrow \{M(\tilde{A}^I)M(\tilde{B}^I)+M(\tilde{A}^I)D(\tilde{B}^I)-D(\tilde{A}^I)M(\tilde{B}^I)-D(\tilde{A}^I)D(\tilde{B}^I)\} \leq \{M(\tilde{A}^I)M(\tilde{B}^I)-M(\tilde{A}^I)D(\tilde{B}^I)+D(\tilde{A}^I)M(\tilde{B}^I)-D(\tilde{A}^I)D(\tilde{B}^I)\}$
$\Rightarrow 2M(\tilde{A}^I)D(\tilde{B}^I) \leq 2D(\tilde{A}^I)M(\tilde{B}^I)$
$\Rightarrow \frac{D(\tilde{B}^I)}{M(\tilde{B}^I)} \leq \frac{D(\tilde{A}^I)}{M(\tilde{A}^I)}$
In this $\tilde{B}^I$ may be positive or negative. So we take the condition as $\mid \frac{D(\tilde{A}^I)}{M(\tilde{A}^I)} \mid \geq \mid \frac{D(\tilde{B}^I)}{M(\tilde{B}^I)} \mid$. Similarly we can prove $\mid \frac{D(\tilde{A}^{I'})}{M(\tilde{A}^{I'})} \mid \geq \mid \frac{D(\tilde{B}^{I'})}{M(\tilde{B}^{I'})} \mid$.
These are the necessary conditions for new subtraction operator.

5 Intuitionistic Fuzzy Multi-Objective Linear Programming Problem (IF-MOLPP):

Multi objective linear Programming with Triangular Intuitionistic Fuzzy Variables is defined as Minimize: $[\tilde{C}_1^I\tilde{x}^I, \tilde{C}_2^I\tilde{x}^I, ..., \tilde{C}_n^I\tilde{x}^I]$
Subject to $\sum_{j=1}^{n} \tilde{a}_{ij}^I\tilde{x}_j^I \leq \tilde{b}_i^I, \tilde{x}_j^I \geq 0$
where $i = 1, 2, ..., m; j = 1, 2, ..., n$ where $\tilde{A}^I = (\tilde{a_{ij}^I}), \tilde{C}^I, \tilde{b}^I, \tilde{x}^I$ are $(m \times n), (1 \times n), (m \times 1), (n \times 1)$ intuitionistic fuzzy matrices consisting of triangular intuitionistic fuzzy numbers (TIFN).

5.1 Accuracy function [16]:

Let $\tilde{A}^I = \{(a_1, a_2, a_3); (a_1^{'}, a_2, a_3^{'})\}$ be a TIFN.
Then we define $(\tilde{A}^I) = \frac{\{(a_1+2a_2+a_3)+(a_1^{'}+2a_2+a_3^{'})\}}{8}$, an accuracy function of $\tilde{A}^I$, to defuzzify the given number.

Example 5.2. *Here we are going to solve fully intuitionistic fuzzy multi-objective linear programming problem using simplex algorithm and using new operators.*
Maximize $\{\tilde{Z}_1^I = \tilde{4}^I\tilde{x}_1^I + \tilde{10}^I\tilde{x}_2^I, \tilde{Z}_2^I = \tilde{2}^I\tilde{x}_1^I + \tilde{5}^I\tilde{x}_2^I\}$
Subject to the constraints

$$\tilde{2}^I\tilde{x}_1^I + \tilde{1}^I\tilde{x}_2^I = \tilde{5}^I$$
$$\tilde{2}^I\tilde{x}_1^I + \tilde{5}^I\tilde{x}_2^I = \tilde{10}^I$$
$$\tilde{2}^I\tilde{x}_1^I + \tilde{3}^I\tilde{x}_2^I = \tilde{9}^I$$

We consider the first objective with the given constraints and it can be written as
Maximize $\{\tilde{Z}_1^I = \tilde{4}^I\tilde{x}_1^I + \tilde{10}^I\tilde{x}_2^I + \tilde{0}^I\tilde{s}_1^I + \tilde{0}^I\tilde{s}_2^I + \tilde{0}^I\tilde{s}_3^I\}$
Subject to the constraints

$$\tilde{2}^I\tilde{x}_1^I + \tilde{1}^I\tilde{x}_2^I + \tilde{1}^I\tilde{s}_1^I = \tilde{5}^I$$
$$\tilde{2}^I\tilde{x}_1^I + \tilde{5}^I\tilde{x}_2^I + \tilde{1}^I\tilde{s}_2^I = \tilde{10}^I$$
$$\tilde{2}^I\tilde{x}_1^I + \tilde{3}^I\tilde{x}_2^I + \tilde{1}^I\tilde{s}_3^I = \tilde{9}^I$$

Using Simplex Algorithm (by5.1),the current solution is
$\tilde{x}_1^I = \{(0,0,0); (0,0,0)\}$ *and* $\tilde{x}_2^I = \{(1.5, 2, 2.75); (1.385, 2, 2.875)\}$
Hence, Maximize $\tilde{Z}_1^I = \{(13.5, 20, 30.250); (12.465, 20, 33.063)\} = \tilde{20}^I$

Using Simplex Algorithm (by5.1, then the current solution is
Here, $\tilde{x}_1^I = \{(0,0,0); (0,0,0)\}$ *and* $\tilde{x}_2^I = \{(1.5, 2, 2.75); (1.385, 2, 2.875)\}$

Hence, Maximize $\tilde{Z}_2^I = \{(6, 10, 16.5); (5.540, 10, 18.688)\} = \tilde{10}^I$

6 Conclusion

The main aim of this paper is to introduce a new operation for subtraction and division on intuitionistic triangular fuzzy number which will be the inverse operations of addition and multiplication. These operations may help us to reduce the computational complexities exist in solving many optimization problems.

References

[1] Atanassov.K.T., More on intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 33(1986), 37-46.

[2] Atanassov.K.T., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20(1986), 87-96.

[3] Alefeld, G., and Herzberger, J., *Introduction to Interval Computation*, Academic Press, New York, 1983.

[4] Chen, S.H ., Operations on fuzzy numbers with function principle, *Tamkang Journal of Management*, 6(1)(1985), 13 - 26.

[5] Compas.L and Verdegay.J.L ., Linear Programming Problems and ranking of fuzzy numbers, *Fuzzy Sets and Systems*, 32(1989), 1- 11.

[6] Dubois,D.,and Prade,H., perations of Fuzzy Number's, *Internat.J. Systems Sci.,* 9(6)(1978), 613-626.

[7] Dubois, D., and Prade, H., *Fuzzy Sets and Systems, Theory and Applications* Academic Press, New York, 1980.

[8] Dwyer,P.S., *Linear Computation*, New York, 1951.

[9] Kaufmann,A., *Introduction to theory of Fuzzy Subsets*, Academic Press, New York, 1975.

[10] Kaufmann,A.,andGupta,M.M., *Introduction to Fuzzy Arithmetic*, Van Nostrand Reinhold, New York, 1985.

[11] Liu H.W. Synthetic decision based on intuitionistic fuzzy relations, *Journal of Shandong University of Technology*, 33(5)(2003), 579-581.

[12] G.S.Mahapatra and T.K.Roy, *Reliability Evaluation using Triangular Intuitionistic Fuzzy numbers Arithmetic operations*., World Academy of science, Engineering and Technology 50(2009), 574-581.

[13] Mizumoto,M.,andTanaka,K.,*The four Operations of Arithmetic on Fuzzy Numbers*., Systems Comput. Controls 7(5) (1977)73-81.

[14] Moore,R.E., *Interval Analysis*.,Printice Hall,Inc. Englewood & Cliffs, N.J., 1966.

[15] Moore,R.E., *Methods andApplications of Interval Analysis*., SIAM, Philadelphia, 1979.

[16] A.Nagoorgani and K.Ponnalagu , *Solving Linear Programming Problem in an Intuitionistic Environment*., Proceedings of the Heber international conference on Applications of Mathematics and Statistics, HICAMS 5-7(2012).

[17] A. Nagoorgani and S.N. Mohamed Assarudeen, A new operation on Triangular fuzzy number for solving linear programming problem, *Applied Mathematical Sciences*,6(11)(2012), 525- 532.

[18] Nahmias, S., *Fuzzy variables, Fuzzy sets and systems*', 1(2) (1977)97-110.

[19] Neumaier, A., *Interval Methods for Systems of Equations*., Cambridge University Press, Cambridge, (1990).

[20] Nguyen,H.T., A Note on extension principle for fuzzysets, *J. Math. Anal. Appl.*, 64(1978), 369-380.

[21] Rardin, R.L.,*Optimization in Operations Research*, Pearson Education, New Delhi, 2003.

[22] Xu Z.S., Intuitionistic preference relations and their application in group decision making, *Information Sciences*, (177)(2007), 2363-2379.

[23] Zadeh,L.A., The concept of a Linguistic variable and its applications to approximate Reasoning-parts I, II and III"., *Information Sciences*, 8(1975), 199-249; 8, 1975301-357; 9(1976) 43-80.

[24] Zadeh, L.A., Fuzzy sets, *Information and Control*, 8(1965), 339-353.

[25] Zadeh,L.A., Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1(1978), 3-28.

Received: February 24, 2014; *Accepted*: April 25, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 308–314

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Comments on Jensen's Inequalities

Zlatko Pavić [a,*]

[a]Mechanical Engineering Faculty in Slavonski Brod, University of Osijek, Trg Ivane Brlić Mažuranić 2,35000 Slavonski Brod, Croatia.

Abstract

The paper gives generalizations of some Jensen type inequalities for convex functions of one variable. The work is based on the methods which use convex combinations in deriving inequalities. The main inequality is applied to the quasi-arithmetic means.

Keywords: Affine combination, convex combination, convex function, Jensen's inequality.

2010 MSC: 26A51, 52A40.

1 Introduction

1.1 Affine and Convex Combinations

The concept of affine and convex combinations refers to the sets of vectors. Through the paper we will only use the combinations

$$c = \sum_{i=1}^{n} p_i x_i \tag{1.1}$$

of the points $x_i \in \mathbb{R}$ and the coefficients $p_i \in \mathbb{R}$. A combination in (1.1) is affine if $\sum_{i=1}^{n} p_i = 1$. A combination in (1.1) is convex if all $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. The point c itself is called the combination center. If $\mathcal{I} \subseteq \mathbb{R}$ is an interval, then any convex combination of the points $x_i \in \mathcal{I}$ belongs to the interval $\mathcal{I}$.

1.2 Affine and Convex Functions

A function $f : \mathbb{R} \to \mathbb{R}$ which is represented by the equation $f(x) = kx + l$ where k and l are real constants is affine, and it verifies the equality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) = \sum_{i=1}^{n} p_i f(x_i) \tag{1.2}$$

for all affine combinations $\sum_{i=1}^{n} p_i x_i$ from $\mathbb{R}$. A function $f : \mathcal{I} \to \mathbb{R}$ which satisfies the inequality $f(px + qy) \leq pf(x) + qf(y)$ for all binomial convex combinations $px + qy$ from $\mathcal{I}$ is convex, and it verifies the equality or inequality in (1.2) for all convex combinations $\sum_{i=1}^{n} p_i x_i$ from $\mathcal{I}$.

1.3 Recent Results

Theorem 1.1. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $\sum_{i=1}^{n} p_i x_i$ be a convex combination from $[a, b]$.*

*Corresponding author.

E-mail address: Zlatko.Pavic@sfsb.hr (Zlatko Pavić).

Then every convex function $f : [a,b] \to \mathbb{R}$ verifies the inequality

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) - \sum_{i=1}^{n} p_i f(x_i) &\leq f\left(a+b-\sum_{i=1}^{n} p_i x_i\right) \\ &\leq f(a)+f(b)-\sum_{i=1}^{n} p_i f(x_i). \end{aligned} \tag{1.3}$$

Theorem 1.2. *Let $[a,b] \subset \mathbb{R}$ and $[c,d] \subset \mathbb{R}$ be bounded closed intervals where $a < b$ and $c < d$. Let $p : [c,d] \to \mathbb{R}$ be a non-negative continuous function with $\int_c^d p(x)\,dx > 0$, and $g : [c,d] \to [a,b]$ be a continuous function.*

Then every convex function $f : [a,b] \to \mathbb{R}$ verifies the inequality

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) - \frac{\int_c^d p(x) f(g(x))\,dx}{\int_c^d p(x)\,dx} &\leq f\left(a+b-\frac{\int_c^d p(x) g(x)\,dx}{\int_c^d p(x)\,dx}\right) \\ &\leq f(a)+f(b)-\frac{\int_c^d p(x) f(g(x))\,dx}{\int_c^d p(x)\,dx}. \end{aligned} \tag{1.4}$$

The right-hand side of the inequality in (1.3) was obtained in [3]. The left-hand side of the inequality in (1.3), and the inequality in (1.4) were obtained in [2]. Some new Jensen type inequalities have been recently derived in [4].

2 Three Methods of Deriving Convex Function Inequalities

2.1 Basic Method Using Affinity

If $a, b \in \mathbb{R}$ are different numbers, say $a < b$, then every number $x \in \mathbb{R}$ can be uniquely presented as the affine combination

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b. \tag{2.1}$$

The above binomial combination is convex if, and only if, the number x belongs to the interval $[a,b]$. Given the function $f : \mathbb{R} \to \mathbb{R}$, let $f_{\{a,b\}}^{\text{line}} : \mathbb{R} \to \mathbb{R}$ be the function of the line passing through the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of f. Applying the affinity of $f_{\{a,b\}}^{\text{line}}$ to the combination in (2.1), we get

$$f_{\{a,b\}}^{\text{line}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b). \tag{2.2}$$

Assume that the function f is convex. Applying its convexity to the combination in (2.1) and connecting it with the equation in (2.2), we get the basic inequalities of convex functions:

Lemma 2.1. *Let $[a,b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$.*

Then every convex function $f : \mathbb{R} \to \mathbb{R}$ verifies the inequality

$$f(x) \leq f_{\{a,b\}}^{line}(x) \;\; \text{if } x \in [a,b], \tag{2.3}$$

and the reverse inequality

$$f(x) \geq f_{\{a,b\}}^{line}(x) \;\; \text{if } x \notin (a,b). \tag{2.4}$$

If f is concave, then the reverse inequalities are valid in (2.3) and (2.4).

2.2 Discrete Method Using Common Center

The following lemma deals with two convex combinations of the same center (one convex combination with two "sub-combinations" has been studied in [5, Proposition 2]). Applying a convex function on such convex combinations, we obtain the Jensen type inequality:

Lemma 2.2. *Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, and $a, b \in \mathcal{I}$ be points such that $a \leq b$. Let $\sum_{i=1}^{n} p_i x_i$ be the convex combination with points $x_i \in [a, b]$. Let $\sum_{j=1}^{m} q_j y_j$ be the convex combination with points $y_j \in \mathcal{I} \setminus (a, b)$.*

If the convex combination center equality

$$\sum_{i=1}^{n} p_i x_i = \sum_{j=1}^{m} q_j y_j \tag{2.5}$$

is satisfied, then every convex function $f : \mathcal{I} \to \mathbb{R}$ verifies the inequality

$$\sum_{i=1}^{n} p_i f(x_i) \leq \sum_{j=1}^{m} q_j f(y_j). \tag{2.6}$$

If f is concave, then the reverse inequality is valid in (2.6).

Proof. Prove the convexity case. If $a < b$, relying on the convexity of f and the affinity of $f_{\{a,b\}}^{\text{line}}$, we get the series of inequalities

$$\begin{aligned} \sum_{i=1}^{n} p_i f(x_i) &\leq \sum_{i=1}^{n} p_i f_{\{a,b\}}^{\text{line}}(x_i) = f_{\{a,b\}}^{\text{line}}\left(\sum_{i=1}^{n} p_i x_i\right) \\ &= f_{\{a,b\}}^{\text{line}}\left(\sum_{j=1}^{m} q_j y_j\right) = \sum_{j=1}^{m} q_j f_{\{a,b\}}^{\text{line}}(y_j) \\ &\leq \sum_{j=1}^{m} q_j f(y_j) \end{aligned}$$

derived applying the inequality in (2.3) to x_i, and the inequality in (2.4) to y_j. If $a = b$, we use any support line $f_{\{a\}}^{\text{line}}$ instead of the chord line $f_{\{a,b\}}^{\text{line}}$. □

Remark 2.1. *Lemma 2.2 is the generalization of Jensen's inequality. Applying the lemma to the convex combination center equality*

$$1c = \sum_{i=1}^{n} p_i x_i, \tag{2.7}$$

with the assumption $a = b = c$, we come to the Jensen inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) = 1f(c) \leq \sum_{i=1}^{n} p_i f(x_i). \tag{2.8}$$

Respecting the Jensen inequality and our purposes in the main section, we give the following consequence of Lemma 2.2:

Corollary 2.1. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $\sum_{i=1}^{n} p_i x_i$ be a convex combination from $[a, b]$.*

If the convex combination center equality

$$\sum_{i=1}^{n} p_i x_i = \alpha a + \beta b \tag{2.9}$$

is satisfied, then every convex function $f : [a, b] \to \mathbb{R}$ verifies the inequality

$$f(\alpha a + \beta b) \leq \sum_{i=1}^{n} p_i f(x_i) \leq \alpha f(a) + \beta f(b). \tag{2.10}$$

If f is concave, then the reverse inequality is valid in (2.10).

Let us show the immediate application of the above corollary. Rewrite the inequality in (1.3) of Theorem 1.1 in the form

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} f\left(a + b - \sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} \frac{p_i}{2} f(x_i) \leq \frac{f(a) + f(b)}{2}, \tag{2.11}$$

and observe the the convex combination center equality

$$\frac{1}{2}\left(a+b-\sum_{i=1}^{n}p_ix_i\right)+\sum_{i=1}^{n}\frac{p_i}{2}x_i=\frac{1}{2}a+\frac{1}{2}b. \tag{2.12}$$

The middle member in (2.12) is the $(n+1)$-membered convex combination of the points $\bar{x}_1 = a+b-\sum_{i=1}^{n} p_ix_i$ and $\bar{x}_{i+1} = x_i$ from $[a,b]$ with the coefficients $\bar{p}_1 = 1/2$ and $\bar{p}_{i+1} = p_i/2$, including all $i = 1,\dots,n$. The right member in (2.12) is the two-membered convex combination, in fact the arithmetic center, of the endpoints a and b. So, we can apply the inequality in (2.10) of Corollary 2.1 to the equality in (2.12) to obtain the inequality in (2.11).

2.3 Integral Method Using Convex Combinations

Let $[a,b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $f : [a,b] \to \mathbb{R}$ be the Riemann integrable function. Given the positive integer n, let

$$[a,b]=\bigcup_{i=1}^{n}[a_{ni},b_{ni}] \tag{2.13}$$

where $a = a_{n1}$, $a_{ni} < b_{ni} = a_{n\,i+1}$ for $i = 1,\dots,n-1$ and $a_{nn} < b_{nn} = b$. It is assumed that every interval of the above union contracts to the point as n approaches infinity. Take one point $x_{ni} \in [a_{ni}, b_{ni}]$ for every index $i = 1,\dots,n$. Then the limit of the sequence $(c_n)_n$ of the convex combination centers

$$c_n=\sum_{i=1}^{n}\frac{b_{ni}-a_{ni}}{b-a}f(x_{ni}), \tag{2.14}$$

as n approaches infinity, is the point

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

As an application of the above procedure, insert the points $x_i = x_{ni}$ and the convex combination coefficients $p_i = (b_{ni} - a_{ni})/(b-a)$ in the inequality in (2.11). Letting n to infinity, we have

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{2}f\left(a+b-\frac{1}{b-a}\int_a^b x\,dx\right)+\frac{1}{2(b-a)}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}$$

which after arranging and using $(a+b)/2 = a+b-(a+b)/2$, gives the inequality

$$f\left(a+b-\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le f(a)+f(b)-f\left(\frac{a+b}{2}\right). \tag{2.15}$$

The integral method with convex combinations in deriving some variants of the known inequalities has been applied in [6].

3 Main Results

Lemma 3.3. *Let $[a,b] \subset \mathbb{R}$ be a bounded closed interval where $a \le b$, and $x_i \in [a,b]$ be points. Let $\alpha, \beta, p_i \in [0,1]$ be coefficients such that $\alpha + \beta - \sum_{i=1}^{n} p_i = 1$.*

Then the affine combination

$$\alpha a+\beta b-\sum_{i=1}^{n}p_ix_i \tag{3.1}$$

belongs to the interval $[a,b]$.

Proof. Take $\gamma = \sum_{i=1}^{n} p_i$, so $\alpha + \beta - \gamma = 1$ by the assumption. Note that $\gamma \le \alpha$ and $\gamma \le \beta$. In the case $\gamma = 0$, the combination in (3.1) is reduced to the convex combination $\alpha a + \beta b \in [a, b]$.

If $\gamma > 0$, then the convex combination $\sum_{i=1}^{n} (p_i/\gamma) x_i \in [a, b]$, so it is consequently equal to the binomial convex combination $\alpha_1 a + \beta_1 b$. In this case, we have

$$\begin{aligned} \alpha a + \beta b - \sum_{i=1}^{n} p_i x_i &= \alpha a + \beta b - \gamma(\alpha_1 a + \beta_1 b) \\ &= (\alpha - \gamma\alpha_1) a + (\beta - \gamma\beta_1) b \\ &= \alpha_2 a + \beta_2 b, \end{aligned}$$

where the coefficients $\alpha_2 = \alpha - \gamma\alpha_1 \ge \alpha - \gamma \ge 0$ and $\beta_2 = \beta - \gamma\beta_1 \ge \beta - \gamma \ge 0$, and their sum $\alpha_2 + \beta_2 = \alpha + \beta - \gamma(\alpha_1 + \beta_1) = 1$. □

Assigning the convex function to the affine combinations of the above lemma, our main result reads as follows:

Theorem 3.3. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $x_i \in [a, b]$ be points. Let $\alpha, \beta, p_i \in [0, 1]$ be coefficients such that $\alpha + \beta - \sum_{i=1}^{n} p_i = 1$.*

Then every convex function $f : [a, b] \to \mathbb{R}$ verifies the inequality

$$\begin{aligned} f\left(\frac{\alpha a + \beta b}{\alpha + \beta}\right) &\le \frac{1}{\alpha + \beta}\left[f\left(\alpha a + \beta b - \sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} p_i f(x_i)\right] \\ &\le \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta}. \end{aligned} \tag{3.2}$$

Proof. Briefly, since the convex combination center equality

$$\frac{1}{\alpha + \beta}\left(\alpha a + \beta b - \sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} \frac{p_i}{\alpha + \beta} x_i = \frac{\alpha a + \beta b}{\alpha + \beta} \tag{3.3}$$

is satisfied, we can apply the inequality in (2.10) of Corollary 2.1 to obtain the inequality in (3.2). Namely, the middle member in (3.3) should be taken as the $(n + 1)$-membered convex combination from $[a, b]$, and similarly the right member as the two-membered convex combination of the endpoints. □

The inequality in (3.2) with $\alpha = \beta = 1$ reduces to the inequality in (1.3). By application the integral method with convex combinations the inequality in (3.2) can be transferred to integrals.

Corollary 3.2. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$. Let $\alpha, \beta \in [0, 1]$ be coefficients such that $\alpha + \beta > 1$.*

Then every convex function $f : [a, b] \to \mathbb{R}$ verifies the inequality

$$\frac{\alpha + \beta}{\gamma} f\left(\frac{\alpha a + \beta b}{\alpha + \beta}\right) - \frac{1}{\gamma} f(c) \le \frac{1}{b - a}\int_a^b f(x)\,dx \le \frac{\alpha}{\gamma} f(a) + \frac{\beta}{\gamma} f(b) - \frac{1}{\gamma} f(c) \tag{3.4}$$

where $\gamma = \alpha + \beta - 1$ and

$$c = \frac{\alpha - \beta + 1}{2} a + \frac{\beta - \alpha + 1}{2} b.$$

Proof. Using the inequality in (3.2) with $x_i = x_{ni}$ and $p_i = \gamma(b_{ni} - a_{ni})/(b - a)$ in which case $\sum_{i=1}^{n} p_i x_i$ approaches

$$\frac{\gamma}{b - a}\int_a^b x\,dx = \frac{\gamma}{2}(a + b),$$

and $\sum_{i=1}^{n} p_i f(x_i)$ approaches

$$\frac{\gamma}{b - a}\int_a^b f(x)\,dx$$

as n approaches infinity, we get the inequality in (3.4). □

Corollary 3.3. *Let $[a,b] \subset \mathbb{R}$ and $[c,d] \subset \mathbb{R}$ be bounded closed intervals where $a < b$ and $c < d$. Let $p : [c,d] \to \mathbb{R}$ be a non-negative continuous function with $\int_c^d p(x)\,dx > 0$, and $g : [c,d] \to [a,b]$ be a continuous function. Let $\alpha, \beta \in [0,1]$ be coefficients such that $\alpha + \beta > 1$.*

Then every convex function $f : [a,b] \to \mathbb{R}$ verifies the inequality

$$\begin{aligned} f\left(\frac{\alpha a+\beta b}{\alpha+\beta}\right) &\leq \frac{1}{\alpha+\beta}\left[f\left(\alpha a+\beta b-\gamma\frac{\int_c^d p(x)g(x)\,dx}{\int_c^d p(x)\,dx}\right)+\gamma\frac{\int_c^d p(x)f(g(x))\,dx}{\int_c^d p(x)\,dx}\right] \\ &\leq \frac{\alpha f(a)+\beta f(b)}{\alpha+\beta} \end{aligned} \tag{3.5}$$

where $\gamma = \alpha + \beta - 1$.

Proof. The inequality in (3.5) follows from the inequality in (3.2) with the points $x_i = g(x_{ni})$ and the coefficients $p_i = \gamma(d_{ni} - c_{ni})p(x_{ni}) / \sum_{i=1}^n (d_{ni} - c_{ni})p(x_{ni})$. For that matter, the combination

$$\sum_{i=1}^n p_i x_i = \gamma \sum_{i=1}^n \frac{(d_{ni} - c_{ni})p(x_{ni})}{\sum_{i=1}^n (d_{ni} - c_{ni})p(x_{ni})} g(x_{ni}) = \gamma \frac{\sum_{i=1}^n (d_{ni} - c_{ni})p(x_{ni})g(x_{ni})}{\sum_{i=1}^n (d_{ni} - c_{ni})p(x_{ni})}$$

passes to the integral quotient

$$\gamma \frac{\int_c^d p(x)g(x)\,dx}{\int_c^d p(x)\,dx}$$

as n approaches infinity. The same goes for the combination $\sum_{i=1}^n p_i f(x_i)$. □

The inequality in (3.5) with $\gamma = 1$, and consequently $\alpha = \beta = 1$, reduces to the inequality in (1.4).

4 Applications

We want to apply the combination in (3.1), and the right-hand side of the inequality in (3.2),

$$f\left(\alpha a + \beta b - \sum_{i=1}^n p_i x_i\right) \leq \alpha f(a) + \beta f(b) - \sum_{i=1}^n p_i f(x_i), \tag{4.1}$$

to discrete quasi-arithmetic means. The excellent book on means and their inequalities in [1] can always be recommended.

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval. In the applications of convexity, we often use strictly monotone continuous functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ such that ψ is convex with respect to φ (ψ is φ-convex), that is, $f = \psi \circ \varphi^{-1}$ is convex on $\varphi(\mathcal{I})$. A similar notation is used for the concavity.

Let $\sum_{i=1}^n p_i x_i$ be a convex combination from $\mathcal{I}$. The discrete φ-quasi-arithmetic mean of the points x_i with the coefficients p_i is the point

$$M_\varphi(x_i; p_i) = \varphi^{-1}\left(\sum_{i=1}^n p_i \varphi(x_i)\right) \tag{4.2}$$

which belongs to $\mathcal{I}$. The point $M_\varphi(x_i; p_i)$ can also be called the φ-quasi-center of the convex combination center $c = \sum_{i=1}^n p_i x_i$. The idea of the formula in (4.2) may be applied for a quasi-arithmetic mean of the affine combination $\alpha a + \beta b - \sum_{i=1}^n p_i x_i$ that belongs to $[a,b]$, in this way:

$$M_\varphi(a, b, x_i; \alpha, \beta, p_i) = \varphi^{-1}\left(\alpha\varphi(a) + \beta\varphi(b) - \sum_{i=1}^n p_i \varphi(x_i)\right). \tag{4.3}$$

The mean defined in (4.3) belongs to $[a,b]$ because $\alpha\varphi(a) + \beta\varphi(b) - \sum_{i=1}^n p_i \varphi(x_i)$ belongs to $\varphi([a,b])$.

We have the following application of the formula in (4.1) to the quasi-arithmetic means in (4.3):

Corollary 4.4. *Let $[a,b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $\varphi, \psi : [a,b] \to \mathbb{R}$ be strictly monotone continuous functions. Let $x_i \in [a,b]$ be points, and $\alpha, \beta, p_i, \in [0,1]$ be coefficients such that $\alpha + \beta - \sum_{i=1}^{n} p_i = 1$.*

If ψ is either φ-convex and increasing or φ-concave and decreasing, then the inequality

$$M_\varphi(a,b,x_i;\alpha,\beta,p_i) \le M_\psi(a,b,x_i;\alpha,\beta,p_i) \tag{4.4}$$

holds.

If ψ is either φ-convex and decreasing or φ-concave and increasing, then the reverse inequality is valid in (4.4).

Proof. Prove the case that ψ is φ-convex and increasing. Since φ is monotone, the endpoints of the interval $[c,d] = \varphi([a,b])$ are $\varphi(a)$ and $\varphi(b)$. Using the inequality in (4.1) with the convex function $f = \psi \circ \varphi^{-1} : [c,d] \to \mathbb{R}$, we get

$$\psi \circ \varphi^{-1}\left(\alpha\varphi(a) + \beta\varphi(b) - \sum_{i=1}^{n} p_i\varphi(x_i)\right) \le \alpha\psi(a) + \beta\psi(b) - \sum_{i=1}^{n} p_i\psi(x_i),$$

and assigning the increasing function ψ^{-1} to the above inequality, we attain the mean inequality in (4.4). □

Using the pairs of functions $\varphi(x) = x^{-1}$, $\psi(x) = \ln x$ and $\varphi(x) = \ln x$, $\psi(x) = x$ in the inequality in (4.4) with $a, b > 0$, we get the harmonic-geometric-arithmetic inequality for the means defined in (4.3):

$$\left(\frac{\alpha}{a} + \frac{\beta}{b} - \sum_{i=1}^{n} \frac{p_i}{x_i}\right)^{-1} \le a^\alpha b^\beta \prod_{i=1}^{n} x_i^{-p_i} \le \alpha a + \beta b - \sum_{i=1}^{n} p_i x_i. \tag{4.5}$$

A further application of the inequality in (4.1) could be related to the definition of the variant of Jensen's functional by the formula

$$J_f(a,b,x_i;\alpha,\beta,p_i) = \alpha f(a) + \beta f(b) - \sum_{i=1}^{n} p_i f(x_i) - f\left(\alpha a + \beta b - \sum_{i=1}^{n} p_i x_i\right). \tag{4.6}$$

Some new results relating to the bounds of Jensen's functional have been latterly achieved in [7].

References

[1] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, Reidel, Dordrecht, NL, 1988.

[2] A. E. Farissi, B. Belaidi, and Z. Latreuch, A variant of Jensen's inequalities, *Malaya Journal of Matematik*, 4(2013), 54-60.

[3] A. McD. Mercer, A variant of Jensen's inequality, *Journal of Inequalities in Pure and Applied Mathematics*, 4(2003), Article 73, 2 pages.

[4] C. P. Niculescu, and C. I. Spiridon, New Jensen-type inequalities, arXiv:1207.6877 [math.CA], 7 pages, 2012.

[5] Z. Pavić, J. Pečarić, and I. Perić, Integral, discrete and functional variants of Jensen's inequality, *Journal of Mathematical Inequalities*, 5(2011), 253-264.

[6] Z. Pavić, Convex combinations, barycenters and convex functions, *Journal of Inequalities and Applications*, 2013(2013), Article 61, 13 pages.

[7] J. Sándor, On global bounds for generalized Jensen's inequality", *Annals of Functional Analysis*, 4(2013), 18-24.

Received: August 20, 2014; *Accepted*: May 2, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 315–321

Malaya Journal of Matematik

www.malayajournal.org

MJM an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

On pre-$\mathcal{I}_s$-open sets and pre-$\mathcal{I}_s$-continuous functions

R. Santhi[a,*] and M. Rameshkumar[b]

[a,]Department of Mathematics, NGM College, Pollachi-642001, Tamil Nadu, India.

[b]Department of Mathematics, P. A College of Engineering and Technology, Pollachi-642002, Tamil Nadu, India.

Abstract

We study the notion of pre-$\mathcal{I}_s$-open and pre-$\mathcal{I}_s$-continuous and obtain some properties. Then, we introduce and investigate pre-$\mathcal{I}_s$-open functions and pre-$\mathcal{I}_s$-closed functions. Also we obtain a decomposition of continuity via idealization.

Keywords: pre-$\mathcal{I}_s$-open set, pre-$\mathcal{I}_s$-continuous function.

2010 MSC: 54C05, 54C10.

1 Introduction

Ideal in topological space have been considered since 1930 by Kuratowski[9] and Vaidyanathaswamy[15]. After several decades, in 1990, Jankovic and Hamlett[6] investigated the topological ideals which is the generalization of general topology. Where as in 2010, Khan and Noiri[7] introduced and studied the concept of semi-local functions. The notion of pre-open sets and pre-continuity was first introduced and investigated by Mashhour et. al. [11] in 1982. Finally in 1996, Dontchev [3] introduced the notion of pre-$\mathcal{I}$-open sets and pre-$\mathcal{I}$-continuity in ideal topological spaces. Recently we introduced pre-$\mathcal{I}_s$-open sets and pre-$\mathcal{I}_s$-continuity to obtain decomposition of continuity.

In this paper we study the notion of pre-$\mathcal{I}_s$-open and pre-$\mathcal{I}_s$-continuous and obtain some properties. We introduce and investigate pre-$\mathcal{I}_s$-open functions and pre-$\mathcal{I}_s$-closed functions. Also we obtain a decomposition of continuity via idealization.

2 Preliminaries

Let (X,τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X,τ), $cl(A)$ and $int(A)$ denote the closure and interior of A in (X,τ) respectively.

An ideal $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies: (1) $A\in\mathcal{I}$ and $B\subseteq A$ implies $B\in\mathcal{I}$ (2)$A\in\mathcal{I}$ and $B\in\mathcal{I}$ implies $A\cup B\in\mathcal{I}$.

If $(X,\tau)$is a topological space and $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal topological space or an ideal space.

Let $P(X)$ be the power set of X. Then the operator $()^*: P(X)\to P(X)$ called a local function [9] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A\subseteq X$, $A^*(\mathcal{I},\tau)=\{x\in X : U\cap A\notin\mathcal{I}$ for every open set U containing $x\}$. We simply write A^* instead of $A^*(\mathcal{I},\tau)$ in case there is no confusion. For every ideal topological space $(X,\tau,\mathcal{I})$ there exists topology τ^* finer than τ, generated by $\beta(\mathcal{I},\tau)=\{U\setminus J : U\in\tau$ and $J\in\mathcal{I}\}$ but in general $\beta(\mathcal{I},\tau)$ is not always a topology. Additionally $cl^*(A)=A\cup A^*$ defines Kuratowski closure operator for a topology τ^* finer than τ. Throughout this paper X denotes the ideal topological space $(X,\tau,\mathcal{I})$ and also $int^*(A)$ denotes the interior of A with respect to τ^*.

*Corresponding author.

E-mail address: santhifuzzy@yahoo.co.in (R. Santhi), rameshngm@gmail.com (M. Rameshkumar).

Definition 2.1. *Let (X,τ) be a topological space. A subset A of X is said to be semi-open [10] if there exists an open set U in X such that $U \subseteq A \subseteq cl(U)$. The complement of a semi-open set is said to be semi-closed. The collection of all semi-open (resp. semi-closed) sets in X is denoted by SO(X) (resp. SC(X)). The semi-closure of A in (X,τ) is denoted by the intersection of all semi-closed sets containing A and is denoted by $scl(A)$.*

Definition 2.2. *For $A \subseteq X$, $A_*(\mathcal{I},\tau) = \{\, x \in X / U \cap A \notin \mathcal{I}$ for every $U \in SO(X)\}$ is called the semi-local function [7] of A with respect to $\mathcal{I}$ and τ, where $SO(X,x) = \{U \in SO(X) : x \in U\}$. We simply write A_* instead of $A_*(\mathcal{I},\tau)$ in this case there is no ambiguity.*

It is given in [2] that $\tau^{*s}(\mathcal{I})$ is a topology on X, generated by the sub basis $\{U - E : U \in SO(X)$ and $E \in I\}$ or equivalently $\tau^{*s}\mathcal{I} = \{U \subseteq X : cl^{*s}(X-U) = X-U\}$. The closure operator cl^{*s} for a topology $\tau^{*s}(\mathcal{I})$ is defined as follows: for A $\subseteq X$, $cl^{*s}(A) = A \cup A_*$ and int^{*s} denotes the interior of the set A in $(X,\tau^{*s},\mathcal{I})$. It is known that $\tau \subseteq \tau^*(\mathcal{I}) \subseteq \tau^{*s}(\mathcal{I})$. A subset A of $(X,\tau,\mathcal{I})$ is called semi-$*$-perfect [8] if $A = A_*$. $A \subseteq (X,\tau,\mathcal{I})$ is called $*$-semi dense in-itself [8](resp. semi-$*$-closed [8]) if $A \subset A_*$ (resp. $A_* \subseteq A$).

Lemma 2.1. *[7] Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A, B be subsets of X. Then for the semi-local function the following properties hold:*

1. *If $A \subseteq B$ then $A_* \subseteq B_*$.*
2. *If $U \in \tau$ then $U \cap A_* \subseteq (U \cap A)_*$*
3. *$A_* = scl(A_*) \subseteq scl(A)$ and A_* is semi-closed in X.*
4. *$(A_*)_* \subseteq A_*$.*
5. *$(A \cup B)_* = A_* \cup B_*$.*
6. *If $\mathcal{I} = \{\phi\}$, then $A_* = scl(A)$.*

Definition 2.3. *A subset A of a topological space X is said to be*

1. *α-open [12]if $A \subseteq int(cl(int(A)))$,*
2. *pre-open [11] if $A \subseteq int(cl(A))$,*
3. *β-open [1] if $A \subseteq cl(int(cl(A)))$.*

Definition 2.4. *A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be*

1. *α-$\mathcal{I}$-open [4]if $A \subseteq int(cl^*(int(A)))$,*
2. *pre-$\mathcal{I}$-open [3] if $A \subseteq int(cl^*(A))$,*
3. *semi-$\mathcal{I}$-open [4] if $A \subseteq cl^*(int(A))$.*

Definition 2.5. *A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be*

1. *α-$\mathcal{I}_s$-open [13]if $A \subseteq int(cl^{*s}(int(A)))$,*
2. *pre-$\mathcal{I}_s$-open [13] if $A \subseteq int(cl^{*s}(A))$,*
3. *semi-$\mathcal{I}_s$-open [13] if $A \subseteq cl^{*s}(int(A))$.*

By $PISO(X,\tau)$, we denote the family of all pre-$\mathcal{I}_s$-open sets of a space $(X,\tau,\mathcal{I})$.

Remark 2.1. *In [13], the authors obtained the following diagram:*

$$\begin{array}{ccccccc}
open & \longrightarrow & \alpha\text{-}\mathcal{I}_s\text{-open} & \longrightarrow & \text{semi-}\mathcal{I}_s\text{-open} & \longrightarrow & \text{semi-open} \\
 & & \downarrow & & & & \downarrow \\
 & & \text{pre-}\mathcal{I}_s\text{-open} & \longrightarrow & \text{pre-open} & \longrightarrow & \beta\text{-open}
\end{array}$$

3 Pre-$\mathcal{I}_s$-open sets

Theorem 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space.*

1. *If $\{A_\alpha : \alpha \in \Delta\} \subseteq PISO(X)$, then $\cup\{A_\alpha : \alpha \in \Delta\} \in PISO(X)$*
2. *If $A \in PISO(X)$ and $U \in \tau$, then $A \cap U \in PISO(X)$.*
3. *If $A \in PISO(X)$ and $B \in \tau^\alpha$, then $A \cap B \in PO(X)$*

Proof. (1) Since $\{A_\alpha : \alpha \in \Delta\} \subseteq PISO(X)$, then $A_\alpha \subseteq int(cl^{*s}(A_\alpha))$ for every $\alpha \in \Delta$. Thus

$$\bigcup_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} int(cl^{*s}(A_\alpha)) \subseteq int(\bigcup_{\alpha\in\Delta} cl^{*s}(A_\alpha)) = int(\bigcup_{\alpha\in\Delta}((A_\alpha)_* \cup A_\alpha))$$

$$= int(\bigcup_{\alpha\in\Delta}(A_\alpha)_* \cup \bigcup_{\alpha\in\Delta} A_\alpha) \subseteq int((\bigcup_{\alpha\in\Delta} A_\alpha)_* \cup \bigcup_{\alpha\in\Delta} A_\alpha) = int(cl^{*s}(\bigcup_{\alpha\in\Delta} A_\alpha)).$$

(2) By assumption $A \subseteq int(cl^{*s}(A))$ and $U \subseteq int(U)$. By Lemma 2.1, $A \cap U \subseteq int(cl^{*s}(A)) \cap int(U) \subseteq int(cl^{*s}(A) \cap U) = int((A_* \cup A) \cap U) = int((A_* \cap U) \cup (A \cap U)) \subseteq int((A \cap U)_* \cup (A \cap U)) = int(cl^{*s}(A \cap U))$.

(3) Every pre-$\mathcal{I}_s$-open set is pre-open and the intersection of pre-open set and α-set is always pre-open set. □

Remark 3.1. *Intersection of even two pre-$\mathcal{I}_s$-open sets need not be pre-$\mathcal{I}_s$-open set as shown in the following example.*

Example 3.1. *Let X = $\{a, b, c, d\}$, $\tau = \{\phi, X, \{b\}, \{c, d\}, \{b, c, d\}\}$ and $\mathcal{I} = \{\phi\}$. Then we put A = $\{a, b, c\}$ and B = $\{a, b, d\}$ are pre-$\mathcal{I}_s$-open but $A \cap B = \{a, b\}$ is not pre-$\mathcal{I}_s$-open.*

Definition 3.1. *A subset F of a space $(X, \tau, \mathcal{I})$ is said to be pre-$\mathcal{I}_s$-closed if its complement is pre-$\mathcal{I}_s$-open.*

Remark 3.2. *For a subset A of a space $(X, \tau, \mathcal{I})$, we have $X - cl^{*s}(int(A)) \neq int(cl^{*s}(X - A))$ as shown from the following example.*

Example 3.2. *Let X = $\{a, b, c, d\}$, $\tau = \{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $\mathcal{I} = \{\phi, X, \{c\}, \{d\}, \{c, d\}\}$. Then we put A = $\{b, d\}$, we have $int(cl^{*s}(X - A)) = int(cl^{*s}(\{a, c\}) = int(\{a, c\}) = \{a, c\}$ but $X - cl^{*s}(int(A)) = X - cl^{*s}(\{d\}) = X - \{d\} = \{a, b, c\}$.*

Theorem 3.2. *If a subset A of a space $(X, \tau, \mathcal{I})$ is pre-$\mathcal{I}_s$-closed, then $cl^{*s}(int(A)) \subseteq A$.*

Proof. Since A is pre-$\mathcal{I}_s$-closed, $X - A \in PISO(X, \tau)$. Since $\tau^{*s}(\mathcal{I})$ is finer than τ, we have $X - A \subseteq int(cl^{*s}(X - A)) \subseteq int(cl(X - A)) = X - cl(int(A)) \subseteq X - cl^{*s}(int(A))$. Therefore we obtain $cl^{*s}(int(A)) \subseteq A$. □

Corollary 3.1. *Let A be a subset of a space $(X, \tau, \mathcal{I})$ such that $X - cl^{*s}(int(A)) = int(cl^{*s}(X - A))$. Then A is pre-$\mathcal{I}_s$-closed if and only if $cl^{*s}(int(A)) \subseteq A$*

Proof. This is an immediate consequence of Theorem 3.2. □

Theorem 3.3. *[8] Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq Y \subseteq X$, where Y is α-open in X. Then $A_*(\mathcal{I}_Y, \tau|_Y) = A_*(\mathcal{I}, \tau) \cap Y$.*

Theorem 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $Y \in \tau$ and $W \in PISO(X)$, then $Y \cap W \in PISO(Y, \tau|_Y, \mathcal{I}_Y)$.*

Proof. Since Y is open, we have $int_Y(A) = int(A)$ for any subset A of Y. Now $Y \cap W \subseteq Y \cap int(cl^{*s}(W)) = Y \cap (int(W_* \cup W)) = Y \cap (int(W_*) \cup int(W)) = (Y \cap int(W_*)) \cup (Y \cap int(W)) = int_Y(Y \cap W_*) \cup int_Y(Y \cap W) = int_Y[(Y \cap W_*) \cup (Y \cap W)] = int_Y[((Y \cap W_*) \cup (Y \cap W)) \cap Y] = int_Y[Y \cap (Y \cap W_*) \cup Y \cap W] \subseteq int_Y[Y \cap (Y \cap W)_* \cup Y \cap W] = int_Y[(Y \cap W)_*(I_Y, \tau|_Y) \cup (Y \cap W)] = int_Y[cl^{*s}_Y(Y \cap W)]$. This shows that $Y \cap W \in PISO(Y, \tau|_Y, \mathcal{I}_Y)$. □

Theorem 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq U \in \tau$. Then, A is pre-$\mathcal{I}_s$-open in $(X, \tau, \mathcal{I})$ if and only if A is pre-$\mathcal{I}_s$-open in $(U, \tau|_U, \mathcal{I}_U)$.*

Proof. Let A be pre-$\mathcal{I}_s$-open in $(X, \tau, \mathcal{I})$. Then we have $A = U \cap A \subseteq U \cap int(cl^{*s}(A)) \subseteq int_U(U \cap cl^{*s}(A)) \subseteq int_U(cl^{*s}_U(A))$. This shows that A is pre-$\mathcal{I}_s$-open in $(U, \tau|_U, \mathcal{I}_U)$.

Sufficiency. Let A be pre-$\mathcal{I}_s$-open in $(U, \tau|_U, \mathcal{I}_U)$. Then we have $A \subseteq int_U(cl^{*s}_U(A)) = int(cl^{*s}(A) \cap U) \subseteq int(cl^{*s}(A))$. This shows that A is pre-$\mathcal{I}_s$-open in $(X, \tau, \mathcal{I})$. □

4 Pre-$\mathcal{I}_s$-continuous functions

Definition 4.1. *A function $f : (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$ is said to be pre-$\mathcal{I}_s$-continuous [13] (resp. pre-$\mathcal{I}$-continuous [3], pre-continuous [11]) if $f^{-1}(V)$ is pre-$\mathcal{I}_s$-open (resp. pre-$\mathcal{I}$-open, pre-open) in $(X,\tau,\mathcal{I})$ for each open set V of (Y,σ).*

Theorem 4.1. *Let $f : (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$ be a function. Then the following holds:*

a) *Every continuous function is pre-$\mathcal{I}_s$-continuous.*

b) *Every pre-$\mathcal{I}_s$-continuous is pre-continuous.*

c) *Every pre-$\mathcal{I}_s$-continuous is pre-$\mathcal{I}$-continuous.*

Proof. The proof is obvious. □

Remark 4.1. *Converse of the Theorem 4.1 need not be true as seen from the following examples.*

Example 4.1. *Let $X = \{a, b, c, d\}$, $Y = \{a, b, c\}$ $\tau = \{\phi, X, \{c\}\{a,b\}, \{a,b,c\}\}, \sigma = \{\phi, X, \{c\}, \{d\}, \{c,d\}\}$ and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b,c\}\}$. Define a function $f : (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$ as follows $f(a) = f(b) = c, f(c) = b, f(d) = a$. Then f is pre-$\mathcal{I}_s$-continuous but not continuous.*

Example 4.2. *Let $X = \{a,b,c,d\}$, $\tau = \{\phi, X, \{a\}, \{b\}\{a,b\}\}, \sigma = \{\phi, X, \{a\}, \{a,b\}, \{a,b,c\}\}$ and $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a,b\}\}$. The identity function $f : (X,\tau,\mathcal{I}) \rightarrow (X,\sigma)$ is pre-continuous and pre-$\mathcal{I}$-continuous, but it is not pre-$\mathcal{I}_s$-continuous.*

Theorem 4.2. *For a function $f : (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$, the following are equivalent:*

1. *f is pre-$\mathcal{I}_s$-continuous,*
2. *for each $x \in X$ and each $V \in \sigma$ containing $f(x)$, then there exists $W \in PISO(X,\tau)$ containing x such that $f(W) \subseteq V$,*
3. *for each $x \in X$ and each $V \in \sigma$ containing $f(x), cl^{*s}(f^{-1}(V))$ is neighborhood of X,*
4. *the inverse image of each closed set in (Y,σ) is pre-$\mathcal{I}_s$-closed.*

Proof. (1) ⇒ (2). Let $x \in X$ and V be any open set of Y containing $f(x)$. Set $W = f^{-1}(V)$, then by(1), W is pre-$\mathcal{I}_s$-open and clearly $x \in W$ and $f(W) \subseteq V$.

(2) ⇒ (3). Since $V \in \sigma$ and $f(x) \in V$. Then by(2) there exists $W \in PISO(X)$ containing x such that $f(W) \subseteq V$. Thus, $x \in W \subseteq int(cl^{*s}(W)) \subseteq int(cl^{*s}(f^{-1}(V))) \subseteq cl^{*s}(f^{-1}(V))$. Hence $cl^{*s}(f^{-1}(V))$ is a neighborhood of X.

(3) ⇒ (4) and (1) ⇔ (4) are obvious. □

Theorem 4.3. *Let $f : (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$ be pre-$\mathcal{I}_s$-continuous and $U \in \tau$. Then the restriction $f|_U : (U, \tau|_U, \mathcal{I}_U) \rightarrow (Y,\sigma)$ is pre-$\mathcal{I}_s$-continuous.*

Proof. Let V be any open set of (Y, σ). Since f is pre-$\mathcal{I}_s$-continuous, $f^{-1}(V) \in PISO(X,\tau)$ and by Theorem 3.5, $(f|_U)^{-1}(V) = f^{-1}(V) \cap U \in PISO(U,\tau|_U)$. This shows that $f|_U : (U,\tau|_U,\mathcal{I}_U) \rightarrow (Y,\sigma)$ is pre-$\mathcal{I}_s$-continuous. □

Theorem 4.4. *Let $f : (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$ be a function and $\{U_\alpha / \alpha \in \Delta\}$ be an open cover of X. Then f is pre-$\mathcal{I}_s$-continuous if and only if the restriction $f|_{U_\alpha} : (U_\alpha, \tau|_{U_\alpha}, \mathcal{I}_{U_\alpha}) \rightarrow (Y,\sigma)$ is pre-$\mathcal{I}_s$-continuous for each $\alpha \in \Delta$.*

Proof. This follows from Theorem 4.3.
Sufficiency Let V be any open set in (Y,σ). Since $f|_{U_\alpha}$ is pre-$\mathcal{I}_s$-continuous for each $\alpha \in \Delta$. $(f|_{U_\alpha})^{-1}(V)$ is pre-$\mathcal{I}_s$-open set of $(U_\alpha, \tau|_{U_\alpha}, \mathcal{I}_{U_\alpha})$ and hence by Theorem 3.5, $(f|_{U_\alpha})^{-1}(V)$ is pre-$\mathcal{I}_s$-open set in $(X,\tau,\mathcal{I})$ for each $\alpha \in \Delta$. Moreover, we have

$$f^{-1}(V) = \Big(\bigcup_{\alpha\in\Delta} U_\alpha\Big) \cap f^{-1}(V) = \bigcup_{\alpha\in\Delta}\Big(U_\alpha \cap f^{-1}(V)\Big) = \bigcup_{\alpha\in\Delta}\Big(f|_{U_\alpha}\Big)^{-1}(V).$$

□

Theorem 4.5. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta)$, be two functions where $\mathcal{I}$ and $\mathcal{J}$ are ideals of X and Y respectively. Then:*

1. *$g \circ f$ is pre-$\mathcal{I}_s$-continuous if f is pre-$\mathcal{I}_s$-continuous and g is continuous.*
2. *$g \circ f$ is pre-continuous if f is pre-$\mathcal{I}_s$-continuous and g is continuous.*

Proof. It is obvious. □

Theorem 4.6. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is pre-$\mathcal{I}_s$-continuous if and only if the graph function $g : X \to X \times Y$, defined by g(x) = (x, f(X)) for each $x \in X$, is pre-$\mathcal{I}_s$-continuous.*

Proof. Suppose that f is pre-$\mathcal{I}_s$-continuous. Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. Then there exists a basic open set $U \times V$ such that $g(x) = (x, f(x)) \in U \times V \subseteq W$. Since f is pre-$\mathcal{I}_s$-continuous, then there exists a pre-$\mathcal{I}_s$-open set $U_\circ$ of X containing x such that $f(U_\circ) \subseteq V$. By Theorem 3.1 $U_\circ \cap U \in PISO(X, \tau)$ and $g(U_\circ \cap U) \subseteq U \times V \subseteq W$. This shows that g is pre-$\mathcal{I}_s$-continuous.
Sufficiency: Suppose that g is pre-$\mathcal{I}_s$-continuous. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then $X \times V$ is open in $X \times Y$ and by pre-$\mathcal{I}_s$-continuity of g, there exists $U \in PISO(X, \tau)$ containing x such that $g(U) \subseteq X \times V$. Therefore we obtain $f(U) \subseteq V$. This shows that f is pre-$\mathcal{I}_s$-continuous. □

5 Pre-$\mathcal{I}_s$-open and pre-$\mathcal{I}_s$-closed functions

Definition 5.1. *[14] A function $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ is called pre-$\mathcal{I}_s$-open (resp. pre-$\mathcal{I}_s$-closed) if for each $U \in \tau$ (resp. U is closed)$f(U) \in PISO(Y, \sigma, \mathcal{J})$(resp. $f(U)$ is pre-$\mathcal{I}_s$-closed set).*

Definition 5.2. *[3] A function $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ is called pre-$\mathcal{I}$-open (resp. pre-$\mathcal{I}$-closed) if for each $U \in \tau$ (resp. U is closed) $f(U)$ is pre-$\mathcal{I}$-open (resp. $f(U)$ is pre-$\mathcal{I}$-closed) set in $(Y, \sigma, \mathcal{J})$.*

Remark 5.1.
1. *Every pre-$\mathcal{I}_s$-open (resp. pre-$\mathcal{I}_s$-closed) function is pre-open (resp. pre-closed) and the converses are false in general.*
2. *Every pre-$\mathcal{I}_s$-open (resp. pre-$\mathcal{I}_s$-closed) function is pre-$\mathcal{I}$-open (resp. pre-$\mathcal{I}$-closed) and the converses are false in general.*
3. *Every open function is pre-$\mathcal{I}_s$-open but the converse is not true in general.*

Example 5.1. *Let $X = \{a, b, c, d\}, \tau = \{\phi, X, \{a, b\}\}, \sigma = \{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $\mathcal{J} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. Define a function $f : (X, \tau) \to (X, \sigma, \mathcal{J})$ as follows $f(a) = c, f(b) = d, f(c) = b, f(d) = a$. Then f is pre-open, but it is not pre-$\mathcal{I}_s$-open.*

Example 5.2. *Let $X = \{a, b, c, d\}, \tau = \{\phi, X, \{b\}, \{b, c, d\}\}, \sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{J} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (X, \sigma, \mathcal{J})$ as follows $f(a) = f(d) = a, f(b) = b, f(c) = c$. Then f is pre-$\mathcal{I}$-open, but it is not pre-$\mathcal{I}_s$-open.*

Example 5.3. *Let $X = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}, \{a, b, c\}\}, \sigma = \{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $\mathcal{J} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. Define a function $f : (X, \tau) \to (X, \sigma, \mathcal{J})$ as follows $f(a) = a, f(b) = b, f(c) = d, f(d) = b$. Then f is pre-$\mathcal{I}_s$open, but it is not open.*

Theorem 5.1. *A function $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ is called pre-$\mathcal{I}_s$-open if and only if for each subset $W \subseteq Y$ and each closed set F of X containing $f^{-1}(W)$, there exists a pre-$\mathcal{I}_s$-closed set $H \subseteq Y$ containing W such that $f^{-1}(H) \subseteq F$.*

Proof. Let $H = Y - f(X - F)$. Since $f^{-1}(W) \subseteq F$, we have $f(X - F) \subseteq Y - W$. Since f is pre-$\mathcal{I}_s$-open, then H is pre-$\mathcal{I}_s$-closed and $f^{-1}(H) = X - f^{-1}(f(X - F)) \subseteq X - (X - F) = F$.
Sufficiency. Let U be any open set of X and $W = Y - f(U)$. Then $f^{-1}(W) = X - f^{-1}(f(U)) \subseteq X - U$ and $X - U$ is closed. By the hypothesis, there exists an pre-$\mathcal{I}_s$-closed set H of Y containing W such that $f^{-1}(H) \subseteq X - U$. Then we have $f^{-1}(H) \cap U = \phi$ and $H \cap f(U) = \phi$. Therefore we obtain $Y - f(U) \supseteq H \supseteq W = Y - f(U)$ and $f(U)$ is pre-$\mathcal{I}_s$-open in Y. This shows that f is pre-$\mathcal{I}_s$-open. □

Theorem 5.2. *For any bijective function $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$, the following are equivalent:*

1. $f^{-1} : (X, \sigma, \mathcal{J}) \to (X, \tau)$ *is pre-*$\mathcal{I}_s$*-continuous,*
2. *f is pre-*$\mathcal{I}_s$*-open,*
3. *f is pre-*$\mathcal{I}_s$*-closed,*

Proof. Obvious. □

6 Decomposition of Continuity

Definition 6.1. *A subset A of an ideal topological space* $(x, \tau, \mathcal{I})$ *is called* $\mathcal{I}_s$*-locally closed if* $A = U \cap V$, *where* $U \in \tau$ *and V is semi-∗-perfect.*

Proposition 6.1. *Let* $(x, \tau, \mathcal{I})$ *be an ideal topological space and A a subset of X. Then the following are equivalent:*

1. *A is open,*
2. *A is pre-*$\mathcal{I}_s$*-open and* $\mathcal{I}_s$*-locally closed.*

Proof. (1) ⇒ (2) Let A is open. Then A is pre-$\mathcal{I}_s$-open. On the other hand $A = A \cap X$, where $A \in \tau$ and X is semi-∗-perfect.

(2) ⇒ (1) By assumption $A \subseteq int(cl^{*s}(A)) = int(cl^{*s}(U \cap V))$, where $U \in \tau$ and V is semi-∗-perfect. Hence $A = U \cap A \subseteq U \cap int(cl^{*s}(U)) \cap int(cl^{*s}(V)) = U \cap int(V \cup V_*) = int(U) \cap int(V) = int(U \cap V) = int(A)$. Hence A is open. □

Definition 6.2. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, *is called* $\mathcal{I}_s$*-LC-continuous if for every* $V \in \sigma$, $f^{-1}(V)$ *is* $\mathcal{I}_s$*-locally closed.*

Proposition 6.2. *Let* $(x, \tau, \mathcal{I})$ *be an ideal topological spaces. Then, every continuous function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, *is* $\mathcal{I}_s$*-LC-continuous.*

Proof. Obvious. □

Remark 6.1. *Converse of the Theorem 6.2 need not be true as seen from the following example.*

Example 6.1. *Let* $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}, \sigma = \{\phi, X, \{d\}\}$ *and* $\mathcal{J} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. *The identity function* $f : (X, \tau) \to (X, \sigma, \mathcal{J})$ *is* $\mathcal{I}_s$*-LC-continuous but it is not continuous.*

Theorem 6.1. *Let* $(x, \tau, \mathcal{I})$ *be an ideal topological spaces. For a function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, *the following conditions are equivalent:*

1. *f is continuous,*
2. *f is pre-*$\mathcal{I}_s$*-continuous and* $\mathcal{I}_s$*-LC-continuous.*

Proof. This follows from Proposition 6.1. □

References

[1] M. E. Abd El-Monsef, S. N. El-Deep and R. A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assiut Univ.*, 12(1983), 77-90.

[2] M.E. Abd El-Monsef, E.F. Lashien and A.A. Nasef, Some topological operators via ideals, *Kyungpook Math. J.*, 32(2)(1992), 273-284.

[3] J. Dontchev, *Idealization of Ganster-Reilly decomposition theorems*, http://arxiv.org/abs/ Math. GN/9901017, 5 Jan. 1999(Internet).

[4] E. Hatir and T.Noiri, On decompositions of continuity via idealization, *Acta. Math. Hungar.*, 96(4)(2002), 341-349.

[5] E. Hatir and T. Noiri, On semi-I-open sets and semi-I-continuous functions, *Acta. Math. Hungar.*, 107(4)(2005), 345-353.

[6] D. Jankovic and T. R. Hamlett, New topologies from old via ideals, *Amer. Math. Monthly*, 97(4)(1990), 295-310.

[7] M. Khan and T. Noiri, Semi-local functions in ideal topological spaces, *J. Adv. Res. Pure Math.*, 2(1)(2010), 36-42.

[8] M. Khan and T. Noiri, On gI-closed sets in ideal topological spaces, *J. Adv. Stud. in Top.*, 1(2010), 29-33.

[9] K. Kuratowski, *Topology*, Vol. I, Academic press, New York, 1966.

[10] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.

[11] A. S. Mashhour, M. E. Abd. El-Monsef and S. N. El-deeb, On pre-continuous and weak pre-continuous mappings, *Proc. Math. Phys. Soc. Egypt*, 53(1982), 47-53.

[12] O. Njastad, On some classes of nearly open sets, *Pacific J. Math.*, 15(1965), 961-970.

[13] R. Santhi and M. Rameshkumar, *A decomposition of continuity in ideal by using semi-local functions*, (Submitted).

[14] R. Santhi and M. Rameshkumar, *On α-$\mathcal{I}_s$-open sets and α-$\mathcal{I}_s$-continuous functions*,(submitted).

[15] R. Vaidyanathaswamy, *Set Topology*, Chelsea Publishing Company, 1960.

Received: February 28, 2014; *Accepted*: May 2, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(3)(2014) 322–329

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Ostrowski inequality for generalized fractional integral and related inequalities

Hüseyin YILDIRIM[a,*] and Zeynep KIRTAY[b]

[a,b]Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, 46000, Kahramanmaraş, Turkey.

Abstract

In this article we obtain new generalizations for ostrowski inequality by using generalized Riemann-Liouville fractional integral.

Keywords: Fractional Integral,Ostrowski İnequality, Korkine identity, Riemann-Liouville Fractional Integral.

2010 MSC: 26A33, 26D10, 26D15,41A55.

1 Introduction

Let $f : [a,b] \to R$ be continuous on $[a,b]$ and differentiable in (a,b) and assume $|f'(x)| \leq M$ for all $x \in (a,b)$. Then the following holds [1]:

$$|f(x) - M(f;a,b)| \leq \frac{M}{b-a} \frac{(b-x)^2 + (x-a)^2}{2} \tag{1.1}$$

for all $x \in [a,b]$. Where $M(f;a,b) = \frac{1}{b-a} \int\limits_a^b f(x)dx$.

(1.1) inequality is well known in the literature as Ostrowski Inequality. Many researchers try to generalize this inequality. There are numerous generalizations, variants and extensions in the literature, see [4-17] and the references cited therein. Hu Yue makes the following generalizations by using Riemann-Liouville fractional integrals [4].

Definition 1.1. *([24])Let $f \in L^1[a,b]$. The Riemann-Liouville fractional integral $J_{a^+}^{\alpha} f(x)$ and $J_{b^-}^{\alpha} f(x)$ of order $\alpha \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \quad x > a \tag{1.2}$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt \quad x < b. \tag{1.3}$$

respectively. Where $\Gamma(\alpha)$ is Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

Many researches have studied various integral inequality types for Riemann-Liouville integral which are given in Definition 1 ($[18-21], [23-31]$).

Grüss proved the following inequality [2]:

$$|M(fg;a,b) - M(f;a,b)M(g;a,b)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2) \tag{1.4}$$

*Corresponding author.
E-mail address: hyildir@aku.edu.tr (Hüseyin YILDIRIM).

provided that f and g are two integral function on $[a,b]$ satisfying the condition $m_1 \leq f \leq M_1$ and $m_2 \leq g \leq M_2$ for all $x \in [a,b]$, where $m_1, m_2, M_1, M_2 \in R$. The constant $\frac{1}{4}$ is the best possible. So we call (1.4) the Grüss inequality.

Korkine's identity [3] states that if f and g are two integral function on $[a,b]$, then

$$M(fg;a,b) - M(f;a,b)M(g;a,b) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(t) - f(s))(g(t) - g(s))\, dsdt. \tag{1.5}$$

Hu Yue obtains new generalizations (the following theorems) for (1.1) by using (1.4) and (1.5).

Theorem 1.1. *([4])Let f be differentiable function on $[a,b]$ and $|f'(x)| \leq M$ for any $x \in [a.b]$. Then the following fractional inequality holds:*

$$\left| \frac{(x-a)^\alpha + (b-x)^\alpha}{\Gamma(\alpha+1)} f(x) - J_{x^+}^\alpha f(a) - J_{x^+}^\alpha f(b) \right| \leq M \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\Gamma(\alpha+2)} \tag{1.6}$$

for any $x \in [a,b]$ and $\alpha \geq 0$.

Theorem 1.2. *([4])Let $f : [a,b] \to R$ be a differentiable mapping and $f' \in L^2[a,b]$. If f' bounded on $[a,b]$ with $m \leq f'(x) \leq M$, then we have*

$$\begin{aligned} &\left| \frac{\alpha f(x) + f(a)}{\Gamma(\alpha)(\alpha+1)} (x-a)^{\alpha-1} - \frac{\alpha}{x-a} J_{x^-}^\alpha f(a) + \frac{\alpha f(x) + f(b)}{\Gamma(\alpha)(\alpha+1)} (b-x)^{\alpha-1} - \frac{\alpha}{b-x} J_{x^+}^\alpha f(b) \right| \\ &\leq \sqrt{\tfrac{1}{2\alpha+1} - \tfrac{1}{(\alpha+1)^2}} \, \tfrac{(x-a)^\alpha K_1 + (b-x)^\alpha K_2}{\Gamma(\alpha)} \\ &\leq \sqrt{\tfrac{1}{2\alpha+1} - \tfrac{1}{(\alpha+1)^2}} \, \tfrac{(x-a)^\alpha + (b-x)^\alpha}{2\Gamma(\alpha)} (M-m) \end{aligned} \tag{1.7}$$

for all $x \in [a,b]$ and $\alpha \geq 0$. Where

$$\begin{aligned} K_1^2 &= M(f'^2; a,x) - M^2(f'; a,x) \\ K_2^2 &= M(f'^2; x,b) - M^2(f'; x,b). \end{aligned}$$

Now we will give some definitions for fractional integrals which are called generalized fractional integrals.

Definition 1.2. *([22])A real valued function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a complex number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty]$.*

Definition 1.3. *([22])A function $f(t) \in C_\mu$, $t > 0$ is said to be in the $L_{p,k}(a,b)$ space if*

$$L_{p,k}(a,b) = \left\{ f : \|f\|_{L_{p,k(a,b)}} = \left(\int_a^b |f(t)|^p t^k dt \right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty,\ k \geq 0 \right\}.$$

Definition 1.4. *([22],[27])Consider the space $X_c^p(a,b)$ $(c \in R, 1 \leq p < \infty)$ of those real-valued lebesque measurable functions f on $[a,b]$ for which*

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty,\ (1 \leq p < \infty,\ c \in R)$$

and for the case $p = \infty$

$$\|f\|_{X_c^\infty} = ess \sup_{a \leq t \leq b} [t^c f(t)],\ c \in R.$$

In particular, when $c = \frac{k+1}{p}$ $(1 \leq p < \infty, k \geq 0)$ the space $X_c^p(a,b)$ coincides with the $L_{p,k}(a,b)$-space and also if we take $c = \frac{1}{p}$ $(1 \leq p < \infty)$ the space $X_c^p(a,b)$ coincides with the classical $L^p(a,b)$-space.

Definition 1.5. *([22],[27])Let $f \in L_{1,k}[a,b]$. The Generalized Riemann-Liouville fractional integral $J_{a^+}^{\alpha,k} f(x)$ and $J_{b^-}^{\alpha,k} f(x)$ of order $\alpha \geq 0$ and $k \geq 0$ are defined by*

$$J_{a^+}^{\alpha,k} f(x) = \tfrac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \left(x^{k+1} - t^{k+1} \right)^{\alpha-1} t^k f(t)\, dt \quad x > a \tag{1.8}$$

and

$$J_{b^-}^{\alpha,k} f\left(x\right) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \left(t^{k+1} - x^{k+1}\right)^{\alpha-1} t^k f\left(t\right) dt \quad b > x. \tag{1.9}$$

Where $\Gamma\left(\alpha\right)$ *is Gamma function and* $J_{a^+}^{0,k} f\left(x\right) = J_{b^-}^{0,k} f\left(x\right) = f(x)$.

(1.8) and (1.9) integral formulas are called right Generalized Riemann Liouville Integral and left Generalized Riemann Liouville Integral respectively.

Remark 1.1. *Letting* $k = 0$ *for (1.8) and (1.9) formulas we obtain the equalities in Definition 1.*

In this paper we will generalize (1.1), (1.5), (1.6) and (1.7) expressions by using Generalized Riemann-Liouville Fractional Integrals.

2 MAIN RESULTS

Theorem 2.3. *If* $f, g \in L_{1,k}[a,b]$, $k \geq 0$ *then*

$$\begin{aligned} J_{a^+}^{\alpha,k}[f\left(b\right)g(b)] - \frac{\Gamma(\alpha+1)(k+1)^\alpha}{\left(b^{k+1}-a^{k+1}\right)^\alpha} J_{a^+}^{\alpha,k}[f\left(b\right)] J_{a^+}^{\alpha,k}[g(b)] = \frac{\alpha(k+1)^{2-\alpha}}{2\left(b^{k+1}-a^{k+1}\right)^\alpha\Gamma(\alpha)} \int_a^b \int_a^b \left(f(t) - f(s)\right) \\ \times\left(g(t) - g(s)\right) \left(b^{k+1} - s^{k+1}\right)^{\alpha-1} \left(b^{k+1} - t^{k+1}\right)^{\alpha-1} t^k s^k ds dt. \end{aligned} \tag{2.10}$$

Proof. We have the following equality by $\left(f(t) - f(s)\right)\left(g(t) - g(s)\right)$;

$$\begin{aligned} &\int_a^b \int_a^b \left(f(t) - f(s)\right)\left(g(t) - g(s)\right) \left(b^{k+1} - s^{k+1}\right)^{\alpha-1} \left(b^{k+1} - t^{k+1}\right)^{\alpha-1} t^k s^k ds dt \\ &= \int_a^b \int_a^b f(t)g(t) \left(b^{k+1} - s^{k+1}\right)^{\alpha-1} \left(b^{k+1} - t^{k+1}\right)^{\alpha-1} t^k s^k ds dt \\ &- \int_a^b \int_a^b f(t)g(s) \left(b^{k+1} - s^{k+1}\right)^{\alpha-1} \left(b^{k+1} - t^{k+1}\right)^{\alpha-1} t^k s^k ds dt \\ &- \int_a^b \int_a^b f(s)g(t) \left(b^{k+1} - s^{k+1}\right)^{\alpha-1} \left(b^{k+1} - t^{k+1}\right)^{\alpha-1} t^k s^k ds dt \\ &+ \int_a^b \int_a^b f(s)g(s) \left(b^{k+1} - s^{k+1}\right)^{\alpha-1} \left(b^{k+1} - t^{k+1}\right)^{\alpha-1} t^k s^k ds dt. \\ &= 2\left[\int_a^b \left(b^{k+1} - s^{k+1}\right)^{\alpha-1} s^k dt \int_a^b f(t)g(t) \left(b^{k+1} - t^{k+1}\right)^{\alpha-1} t^k dt\right] \\ &-2\left[\int_a^b g(s) \left(b^{k+1} - s^{k+1}\right)^{\alpha-1} s^k dt \int_a^b f(t) \left(b^{k+1} - t^{k+1}\right)^{\alpha-1} t^k dt\right] \\ &= \frac{2\left(b^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \frac{\Gamma(\alpha)}{(k+1)^{1-\alpha}} J_{a^+}^{\alpha,k}[f\left(b\right)g(b)] - \frac{2\Gamma^2(\alpha)}{(k+1)^{2-2\alpha}} J_{a^+}^{\alpha,k}[f\left(b\right)] J_{a^+}^{\alpha,k}[g(b)]. \end{aligned} \tag{2.11}$$

So this proves theorem. □

Remark 2.2. *If we take* $\alpha = 1$ *in (2.1) we obtain the following identity:*

$$M_k(fg;a,b) - M_k(f;a,b)M_k(g;a,b) = \frac{(k+1)^2}{2\left(b^{k+1} - a^{k+1}\right)^2} \int_a^b \int_a^b (f(t) - f(s))(g(t) - g(s)) t^k s^k ds dt \tag{2.12}$$

where $M_k(f;a,b) = \frac{k+1}{b^{k+1}-a^{k+1}} \int_a^b f(t) t^k dt, k \geq 0.$

Remark 2.3. *For* $\alpha = 1$ *and* $k = 0$ *in (2.1), we obtain the Korkine's identity (1.5).*

Theorem 2.4. *Let* f *be differentiable function on* $[a,b]$ *and* $|f'(x)| \leq M$ *for any* $x \in [a,b]$. *Then the following generalized fractional inequality holds for* $\alpha \geq 0$ *and* $k \geq 0$

$$\begin{aligned} &\left| \frac{(k+1)^{-\alpha}}{\Gamma\left(\alpha+1\right)} [\left(x^{k+1} - a^{k+1}\right)^\alpha + \left(b^{k+1} - x^{k+1}\right)^\alpha] x^k f(x) \right. \\ &\left. - J_{x^-}^{\alpha,k}[a^k f\left(a\right)] - J_{x^+}^{\alpha,k}[b^k f\left(b\right)] - k\left[J_{x^-}^{\alpha+1,k}[\frac{f(a)}{a}] + J_{x^+}^{\alpha+1,k}[\frac{f(b)}{b}]\right] \right| \\ &\leq \frac{(k+1)^{-\alpha-1}}{\Gamma\left(\alpha+2\right)} M[\left(x^{k+1} - a^{k+1}\right)^{\alpha+1} + \left(b^{k+1} - x^{k+1}\right)^{\alpha+1}]. \end{aligned} \tag{2.13}$$

Proof. If we use integration by parts for fractional integrals in Definition 5, we have

$$J_{x^-}^{\alpha+1,k}f'(a)=\frac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)}x^k\left(x^{k+1}-a^{k+1}\right)^{\alpha}f(x)-J_{x^-}^{\alpha,k}[a^kf(a)]-kJ_{x^-}^{\alpha+1,k}[\frac{f(a)}{a}] \tag{2.14}$$

and

$$J_{x^+}^{\alpha+1,k}f'(b)=\frac{-(k+1)^{-\alpha}}{\Gamma(\alpha+1)}x^k\left(b^{k+1}-x^{k+1}\right)^{\alpha}f(x)+J_{x^+}^{\alpha,k}[b^kf(b)]+kJ_{x^+}^{\alpha+1,k}[\frac{f(b)}{b}]. \tag{2.15}$$

By (2.5) and (2.6) we obtain

$$\begin{aligned}&J_{x^-}^{\alpha+1,k}f'(a)-J_{x^+}^{\alpha+1,k}f'(b)\\&=\tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)}x^kf(x)[\left(x^{k+1}-a^{k+1}\right)^{\alpha}+\left(b^{k+1}-x^{k+1}\right)^{\alpha}]\\&-J_{x^-}^{\alpha,k}[a^kf(a)]-J_{x^+}^{\alpha,k}[b^kf(b)]-k[J_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}]+J_{x-}^{\alpha+1,k}[\tfrac{f(b)}{b}]].\end{aligned} \tag{2.16}$$

Using $|f'(x)|\le M$, $x\in[a,b]$ for the left part of the (2.7) formula we have

$$\begin{aligned}&\left|\tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)}\int_a^x\left(t^{k+1}-a^{k+1}\right)^{\alpha}t^kf'(t)\,dt-\tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)}\int_x^b\left(b^{k+1}-t^{k+1}\right)^{\alpha}t^kf'(t)\,dt\right|\\&\le\tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)}M[\int_a^x\left(t^{k+1}-a^{k+1}\right)^{\alpha}t^kdt+\int_x^b\left(b^{k+1}-t^{k+1}\right)^{\alpha}t^kdt]\\&\le\tfrac{(k+1)^{-\alpha-1}}{\Gamma(\alpha+2)}M[\left(x^{k+1}-a^{k+1}\right)^{\alpha+1}+\left(b^{k+1}-x^{k+1}\right)^{\alpha+1}].\end{aligned} \tag{2.17}$$

So the proof is completed. □

Remark 2.4. *If we take $k=0$ in inequality (2.4) we obtain the inequality (1.6) in Theorem 1.*

Remark 2.5. *Also letting $k=0$ and $\alpha=1$, formula (2.4) reduces Ostrowski Inequality:*

$$|f(x)-M(f;a,b)|\le\frac{M}{b-a}\frac{(b-x)^2+(x-a)^2}{2}.$$

Theorem 2.5. *Let $f:[a,b]\to R$ be a differentiable mapping and $f'\in L_{2,k}[a,b]$. If f' bounded on $[a,b]$ with $m\le f'(x)\le M$, then the following inequality holds :*

$$\begin{aligned}&\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(x^{k+1}-a^{k+1}\right)^{\alpha-1}[x^kf(x)-\tfrac{1}{(\alpha+1)}\int_a^xf'(t)\,t^kdt]\\&-\frac{\alpha(k+1)}{x^{k+1}-a^{k+1}}\left(J_{x^-}^{\alpha,k}[a^kf(a)]+kJ_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}]\right)\\&+\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(b^{k+1}-x^{k+1}\right)^{\alpha-1}[x^kf(x)+\tfrac{1}{(\alpha+1)}\int_x^bf'(t)\,t^kdt]\\&-\frac{\alpha(k+1)}{b^{k+1}-x^{k+1}}\left(J_{x^+}^{\alpha,k}[b^kf(b)]+kJ_{x^-}^{\alpha+1,k}[\tfrac{f(b)}{b}]\right)\\&\le\sqrt{\tfrac{1}{2\alpha+1}-\tfrac{1}{(\alpha+1)^2}}\tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha)}\left(x^{k+1}-a^{k+1}\right)^{\alpha}K_1+\left(b^{k+1}-x^{k+1}\right)^{\alpha}K_2\\&\le\sqrt{\tfrac{1}{2\alpha+1}-\tfrac{1}{(\alpha+1)^2}}\tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha)}\tfrac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}+\left(b^{k+1}-x^{k+1}\right)^{\alpha}}{2}(M-m)\end{aligned} \tag{2.18}$$

for all $x\in[a,b]$ and $\alpha\ge0$. Where

$$\begin{aligned}K_1^2&=M_k(f'^2;a,x)-M_k^2(f';a,x)\\K_2^2&=M_k(f'^2;x,b)-M_k^2(f';x,b).\end{aligned}$$

Proof. From (1.8) and (1.9) we have

$$\begin{aligned}\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)\left(x^{k+1}-a^{k+1}\right)}\int_a^x\left(t^{k+1}-a^{k+1}\right)^{\alpha}t^kf'(t)\,dt&=\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}x^k\left(x^{k+1}-a^{k+1}\right)^{\alpha-1}f(x)\\&-\frac{\alpha(k+1)}{x^{k+1}-a^{k+1}}[J_{x^-}^{\alpha,k}[a^kf(a)]+kJ_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}]]\end{aligned} \tag{2.19}$$

$$\begin{aligned}\frac{-(k+1)^{-\alpha+1}}{\Gamma(\alpha)\left(b^{k+1}-x^{k+1}\right)}\int_x^b\left(b^{k+1}-t^{k+1}\right)^{\alpha}t^kf'(t)\,dt&=\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}x^k\left(b^{k+1}-x^{k+1}\right)^{\alpha-1}f(x)\\&-\frac{\alpha(k+1)}{b^{k+1}-x^{k+1}}[J_{x^+}^{\alpha,k}[b^kf(b)]+kJ_{x^-}^{\alpha+1,k}[\tfrac{f(b)}{b}]].\end{aligned} \tag{2.20}$$

Then

$$\begin{aligned}
&\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(x^{k+1}-a^{k+1}\right)^{\alpha-1}[x^k f(x)-\tfrac{1}{(\alpha+1)}\textstyle\int_a^x f'(t)\,t^k dt] \\
&\quad -\frac{\alpha(k+1)}{x^{k+1}-a^{k+1}}[J_{x^-}^{\alpha,k}[a^k f(a)]+kJ_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}]] \\
&\quad +\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(b^{k+1}-x^{k+1}\right)^{\alpha-1}[x^k f(x)+\tfrac{1}{(\alpha+1)}\textstyle\int_x^b f'(t)\,t^k dt] \\
&\quad +\frac{\alpha(k+1)}{b^{k+1}-x^{k+1}}[J_{x^+}^{\alpha,k}[b^k f(b)]+kJ_{x^-}^{\alpha+1,k}[\tfrac{f(b)}{b}]] \\
&=\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)\left(x^{k+1}-a^{k+1}\right)}\int_a^x\left(t^{k+1}-a^{k+1}\right)^{\alpha}t^k f'(t)\,dt \\
&\quad -\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)(\alpha+1)}\left(x^{k+1}-a^{k+1}\right)^{\alpha-1}\int_a^x f'(t)\,t^k dt \\
&\quad -\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)\left(b^{k+1}-x^{k+1}\right)}\int_x^b\left(b^{k+1}-t^{k+1}\right)^{\alpha}t^k f'(t)\,dt \\
&\quad +\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)(\alpha+1)}\left(b^{k+1}-x^{k+1}\right)^{\alpha-1}\int_x^b f'(t)\,t^k dt.
\end{aligned}\tag{2.21}$$

If we use the Korkine's identity (2.3) for Generalized Riemann Liouville integral for (2.12), we obtain

$$\begin{aligned}
&\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(x^{k+1}-a^{k+1}\right)^{\alpha-1}[x^k f(x)-\tfrac{1}{(\alpha+1)}\textstyle\int_a^x f'(t)\,t^k dt] \\
&\quad -\frac{\alpha(k+1)}{x^{k+1}-a^{k+1}}[J_{x^-}^{\alpha,k}[a^k f(a)]+kJ_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}]] \\
&+\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(b^{k+1}-x^{k+1}\right)^{\alpha-1}[x^k f(x)+\tfrac{1}{(\alpha+1)}\textstyle\int_x^b f'(t)\,t^k dt] \\
&\quad +\frac{\alpha(k+1)}{b^{k+1}-x^{k+1}}[J_{x^+}^{\alpha,k}[b^k f(b)]+kJ_{x^-}^{\alpha+1,k}[\tfrac{f(b)}{b}]] \\
&=\frac{(k+1)^2(k+1)^{-\alpha}}{2\Gamma(\alpha)\left(x^{k+1}-a^{k+1}\right)^2}\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^k t^k ds dt \\
&+\frac{(k+1)^2(k+1)^{-\alpha}}{2\Gamma(\alpha)\left(b^{k+1}-x^{k+1}\right)^2}\int_x^b\int_x^b[\left(b^{k+1}-s^{k+1}\right)^{\alpha}-\left(b^{k+1}-t^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^k t^k ds dt.
\end{aligned}\tag{2.22}$$

Using the Cauchy-Schwarz inequality for double integrals in (2.13), we obtain

$$\begin{aligned}
&\left|\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^{\frac{k}{2}}t^{\frac{k}{2}}s^{\frac{k}{2}}t^{\frac{k}{2}}ds dt\right| \\
&\le\left(\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}]^2 s^k t^k ds dt\right)^{\frac{1}{2}}\left(\int_a^x\int_a^x[f'(t)-f'(s)]^2 s^k t^k ds dt\right)^{\frac{1}{2}}.
\end{aligned}\tag{2.23}$$

However

$$\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}]^2 s^k t^k ds dt=\frac{2\left(x^{k+1}-a^{k+1}\right)^{2\alpha+2}}{(k+1)^2}\left(\frac{1}{2\alpha+1}-\frac{1}{(\alpha+1)^2}\right)\tag{2.24}$$

and

$$\int_a^x\int_a^x[f'(t)-f'(s)]^2 s^k t^k ds dt=\frac{2\left(x^{k+1}-a^{k+1}\right)^2}{(k+1)^2}\left[M_k(f'^2;a,x)-M_k^2(f';a,x)\right].\tag{2.25}$$

By (2.14)-(2.16), we have

$$\begin{aligned}
&\left|\frac{(k+1)^{-\alpha+2}}{2\Gamma(\alpha)\left(x^{k+1}-a^{k+1}\right)}\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^k t^k ds dt\right| \\
&\qquad\le\frac{(k+1)^{-\alpha}\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\Gamma(\alpha)}\sqrt{\frac{1}{2\alpha+1}-\frac{1}{(\alpha+1)^2}}\left[M_k(f'^2;a,x)-M_k^2(f';a,x)\right]^{\frac{1}{2}}.
\end{aligned}\tag{2.26}$$

Similarly we have

$$\begin{aligned}
&\left|\frac{(k+1)^{-\alpha+2}}{2\Gamma(\alpha)\left(b^{k+1}-x^{k+1}\right)}\int_x^b\int_x^b[\left(b^{k+1}-s^{k+1}\right)^{\alpha}-\left(b^{k+1}-t^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^k t^k ds dt\right| \\
&\qquad\le\frac{(k+1)^{-\alpha}\left(b^{k+1}-x^{k+1}\right)^{\alpha}}{\Gamma(\alpha)}\sqrt{\frac{1}{2\alpha+1}-\frac{1}{(\alpha+1)^2}}\left[M_k(f'^2;x,b)-M_k^2(f';x,b)\right]^{\frac{1}{2}}.
\end{aligned}\tag{2.27}$$

Using (2.13),(2.17) and (2.18) we obtain (2.9) inequality. Moreover if $m \leq f'(x) \leq M$ on $[a,b]$, then by Grüss inequality, we have

$$0 \leq \frac{k+1}{x^{k+1}-a^{k+1}} \left\|f^{'}\right\|^2_{L_{2,k}(a,x)} - (M_k(f^{'};a,x))^2 \leq \frac{1}{2}(M-m)^2 \tag{2.28}$$

$$0 \leq \frac{k+1}{b^{k+1}-x^{k+1}} \left\|f^{'}\right\|^2_{L_{2,k(x,b)}} - (M_k(f^{'};x,b))^2 \leq \frac{1}{2}(M-m)^2 \tag{2.29}$$

which proves the last inequality of (2.9) □

Remark 2.6. *Letting $k=0$ in (2.9) we obtain the inequality (1.7) in Theorem 2.*

Corollary 2.1. *Under the assumptions of Theorem 5 with $\alpha = 1$, then the following inequality holds*

$$\begin{aligned} 2x^k f(x) + \tfrac{1}{2}\left[\int_x^b f'(t)t^k dt - \int_a^x f'(t)t^k dt\right] - \tfrac{k+1}{x^{k+1}-a^{k+1}}\left[J^{1,k}_{x^-}[a^k f(a)] + kJ^{2,k}_{x^+}[\tfrac{f(a)}{a}]\right] \\ - \tfrac{k+1}{b^{k+1}-x^{k+1}}\left[J^{1,k}_{x^+}[b^k f(b)] + kJ^{2,k}_{x^-}[\tfrac{f(b)}{b}]\right] \\ \leq \tfrac{1}{4\sqrt{3}}\tfrac{1}{k+1}(b^{k+1}-a^{k+1})(M-m). \end{aligned} \tag{2.30}$$

Remark 2.7. *Letting $k=0$ in (2.21) we obtain the inequality*

$$\begin{aligned} \left|f(x) + \tfrac{f(a)+f(b)}{2} - \tfrac{1}{x-a}\int_a^x f(t)dt - \tfrac{1}{b-x}\int_x^b f(t)dt\right| \\ \leq \tfrac{1}{4\sqrt{3}}(b-a)(M-m). \end{aligned} \tag{2.31}$$

Remark 2.8. *Letting $x = \frac{a+b}{2}$ in (2.22) we obtain*

$$\begin{aligned} \left|\tfrac{1}{2}\left(f(\tfrac{a+b}{2}) + \tfrac{f(a)+f(b)}{2}\right) - \tfrac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq \tfrac{1}{8\sqrt{3}}(b-a)(M-m). \end{aligned} \tag{2.32}$$

References

[1] Ostrowski, A.M., Über die Absolutabweichung einer differentiebaren Function von ihrem integralmittelwert Commentarii Mathematici Helvetici, 10(1938), 226-227.

[2] Grüss, G., Uber das maximum des absoluten Betrages von

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$$

Math. Z, 39(1935), 215-226.

[3] Korkine A.N., Sur une therome de M. Tchebychef, *C.R. Acad. Sci. Paris*, 96(1883), 316-327.

[4] Hu Yue, Ostrowski Inequality for Fractional Integrals and Related Fractional Inequalities, *TJMM5*, (2013), 85-89.

[5] Belarbi, S. and Dahmani, Z., On some new fractional integral inequalities, *J. Ineq. Pure Appl. Math.*, 10(3)(2009), Art. 86.

[6] Dahmani, Z., New inequalities in fractional integrals, *Int. J. Nonlinear Sci.*, 9(4)(2010), 493-497.

[7] Dahmani, Z., Tabharit, L. and Taf, S., Some fractional integral inequalities, *Nonlinear.Sci. Lett. A*, 1(2)(2010), 155-160.

[8] Dragomir, S.S., On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.*, 1(2(1998).

[9] Dragomir, S.S., The Ostrowski integral inequality for Lipschitzian mappings and applications, *Comput. Math. Appl.*, 38(1999), 33-37.

[10] Liu, Z., Some companions of an Ostrowski type inequality and application, *J. Inequal. Pure Appl. Math.*, 10(2)(2009), Art. 52.

[11] Pachpatte, B.G., On an inequality of Ostrowski type in three independent variables, *J. Math. Anal. Appl.*, 249(2000), 583-591.

[12] Pachpatte, B.G., On a new Ostrowski type inequality in two independent variables, *Tamkang J. Math.*, 32(1)(2001), 45-49.

[13] Sarikaya, M.Z., On the Ostrowski type integral inequality, *Acta Math. Univ. Comenianae,* LXXIX(1)(2010), 129-134.

[14] Ujevic, N., Sharp inequalities of Simpson type and Ostrowski type, *Comput. Math. Appl.,* 48(2004), 145-151.

[15] Zhongxue, L., On sharp inequalities of Simpson type and Ostrowski type in two independent variables, *Comput. Math. Appl.,* 56(2008), 2043-2047.

[16] Alomari, M. and Darus, M., Some Ostrowski type inequalities for convex functions with applications, *RGMIA*, 13(1)(2010), article No. 3, Preprint.

[17] Alomari, M., Darus, M., Dragomir, S.S. and Cerone, P., Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, *Appl. Math. Lett.,* 23(2010), 1071-1076.

[18] Gorenflo, R. and Mainardi, F., *Fractional Calculus: Integral and Differential Equations of Fractional Order*, Springer Verlag, Wien, 1997, 223-276.

[19] P. L. Butzer, A. A. Kilbas and J.J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *Journal of Mathematical Analysis and Applications*, 269(2002), 387-400.

[20] P. L. Butzer, A. A. Kilbas and J.J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *Journal of Mathematical Analysis and Applications*, 269(2002), 1-27.

[21] P. L. Butzer, A. A. Kilbas and J.J. Trujillo, Melling transform analysis and integration by parts for Hadamard-type fractional integrals, *Journal of Mathematical Analysis and Applications*, 270(2002), 1-15.

[22] U.N. Katugampola, New Approach to a generalized fractional integral, *Appl. Math. Comput.,* 218(3)(2011), 860-865.

[23] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Acamedic Press, New York,1974.

[24] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon et alibi, 1993.

[25] M.Z. Sarıkaya and H. Ogunmez, On new inequalities via Riemann-Liouville Fractional Integration, *arXiv:1005.1167v1 [math.CA]*, 7 May 2010.

[26] V.Kiryakov, *Generalized Fractional Calculus and Applications*, John Wiley and Sons Inc., New York,1994.

[27] A. A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Diferential Equations*, Elsevier B.V., Amsterdam, Netherlands, 2006

[28] M. Z. Sarikaya, Ostrowski type inequalities involving the right Caputo fractional derivatives belong to L_{p}, *Facta Universitatis, Series Mathematics and Informatics*, 27(2)(2012), 191-197.

[29] M. Z. Sarikaya and H. Yaldiz, On weighted Montogomery identities for Riemann-Liouville fractional integrals, *Konuralp Journal of Mathematics*, 1(1)(2013), 48-53.

[30] M. Z. Sarikaya and H. Yaldiz, New generalization fractional inequalities of Ostrowski-Grüss type, *Lobachevskii Journal of Math.*, 34(4)(2013), 326-331.

[31] M. Z. Sarikaya and H. Filiz, Note on the Ostrowski type inequalities for fractional integrals, *Vietnam Journal of Mathematics*, January 2014, DOI 10.1007/s10013-014-0056-4.

Received: December 18, 2013; *Accepted*: April 25, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/