ISSN 2319-3786
VOLUME 2, ISSUE 4, OCTOBER 2014

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Malaya Journal of Matematik

an international journal of mathematical sciences

UNIVERSITY PRESS
5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),
Coimbatore- 641114, Tamil Nadu, India.
www.malayajournal.org

The Malaya Journal of Matematik is published quarterly in single volume annually and four issues constitute one volume appearing in the months of January, April, July and October.

Subscription

The subscription fee is as follows:

USD 350.00 For USA and Canada

Euro 190.00 For rest of the world

Rs. 4000.00 In India. (For Indian Institutions in India only)

Prices are inclusive of handling and postage; and issues will be delivered by Registered Air-Mail for subscribers outside India.

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Coimbatore- 641114, Tamil Nadu, India.

Contact No. : +91-9585408402

E-mail : info@mkdpress.com; editorinchief@malayajournal.org; publishingeditor@malayajournal.org

Website : https://mkdpress.com/index.php/index/index

Vol. 2 No. 04 (2014): Malaya Journal of Matematik (MJM)

Malaya J. Mat. 2(4)(2014) 330–344

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Higher order binaries with time dependent coefficients and two factors - model for defaultable bond with discrete default information

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Abstract

In this article, we consider a 2 factors-model for pricing defaultable bonds with discrete default intensity and barrier where the 2 factors are a stochastic risk free short rate process and firm value process. We assume that the default event occurs in an expected manner when the firm value reaches a given default barrier at predetermined discrete announcing dates or in an unexpected manner at the first jump time of a Poisson process with given default intensity given by a step function of time variable. Then our pricing model is given by a solving problem of several linear PDEs with variable coefficients and terminal value of binary type in every subinterval between the two adjacent announcing dates. Our main approach is to use higher order binaries. We first provide the pricing formulae of higher order binaries with time dependent coefficients and consider their integrals on the last expiry date variable. Then using the pricing formulae of higher binary options and their integrals, we give the pricing formulae of defaultable bonds in both cases of exogenous and endogenous default recoveries and perform credit spread analysis.

Keywords: higher order binary options, time dependent coefficients, defaultable bond, default intensity, default barrier, exogenous, endogenous, credit spread.

2010 MSC: 35C15, 35Q91, 91G20, 91G40, 91G50, 91G80.

1 Introduction

The study on defaultable corporate bonds and credit risk is now one of the most promising areas of cutting edge in financial mathematics [1]. As well known, there are two main approaches to pricing defaultable corporate bonds; one is the structural approach and the other one is the reduced form approach. In the structural method, we think that the default event occurs when the firm value is not enough to repay debt, that is, the firm value reaches a certain lower threshold (default barrier) from above. Such a default can be expected and thus we call it expected default. In the reduced-form approach, the default is treated as an unpredictable event governed by a default intensity process. In this case, the default event can occur without any correlation with the firm value and such a default is called *unexpected default*. In the reduced-form approach, if the default probability in time interval $[t, t+\Delta t]$ is $\lambda \Delta t$, then λ is called *default intensity* or *hazard rate*.

As for the history of the two approaches and their advantages and shortcomings, readers can refer to the introductions of [5, 8, 9, 13, 19]. To take the advantages and overcome the shortcomings of structural and reduced-form approaches, many authors used unified models of the two approaches. (See [5, 6, 8, 9, 13, 14, 17, 18, 19, 20].) As noted in [14, 17, 18], many researchers of unified model including [5, 6, 8, 9, 13, 19] tried to express the price of the bond in terms of the firm value or the related signal variable to the firm value and the value of default intensity together with default barrier at any time in the whole lifetime of the bond.

On the other hand, Duffie et al. [10] observe that it is typically difficult for investors in the secondary market for corporate bonds to observe a firm's assets directly, because of noisy or delayed accounting reports,

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or barriers to monitoring by other means. In [14, 17] the authors noted that it is difficult for investors outside the firm to know the firm's financial data except for some discrete dates (for example, once in a month or once in a three month etc.) to announce management data and studied the pricing problem for defaultable corporate bond under the assumption that we only know the firm value and the default barrier at 2 fixed discrete announcing dates, we don't know about any information of the firm value in another time and the default intensity between the adjoined two announcing dates is a constant determined by its announced firm value at the former announcing date. The computational error in [17] is corrected in [14]. The approach of [14, 17] is a kind of study of defaultable bond under insufficient information about the firm. It is interesting to note that E. Agliardi et al. [2, 3] studied bond pricing problem under imprecise information with the technique of fuzzy mathematics. The approach of [14, 17] can also be seen as a unified model of structural model and reduced form model. Agliardi [1] studied a structural model for defaultable bond with several (discrete) coupon dates where the default can occur only when the firm value is not large enough to pay its debt and coupon in those discrete coupon dates.

In [18], the authors studied a one-factor model for defaultable bond with discrete default intensity and discrete default barrier using higher order binary options and their integrals, where the 1 factor is the firm value process. In their credit risk model, the default event occurs in an expected manner when the firm value reaches a certain lower threshold - the default barrier at predetermined discrete announcing dates or in an unexpected manner at the first jump time of a Poisson process with given default intensity given by a step function of time variable, respectively. They considered both endogenous and exogenous default recovery and the pricing model is a solving problem of inhomogeneous or homogeneous Black-Scholes PDEs with different coefficients and terminal value of binary type in every subinterval between the two adjacent announcing dates. In order to deal with the inhomogenous term related to endogenous recovery, they introduced a special binary option called integral of i-th binary or nothing and using it obtained the pricing formulae of defaultable corporate bond. The approach of [18] to model credit risk seems similar with the one of [14] but the essential difference is that in [14] they assumed that they know the firm value only in the discrete announcing dates and the default intensity between two adjacent announcing dates is determined by the firm value in the former announcing date. Another different point is that [18] considered arbitrary number of announcing dates while [14] considered only 2 announcing dates.

As a continued study of [18] we here consider a two factors - model for pricing defaultable bond with discrete default intensity and barrier where the 2 factors are stochastic risk free short rate process and firm value process. Our pricing model is given by a solving problem of several PDEs with variable coefficients and terminal value of binary type in every subinterval between the two adjacent announcing dates. Through the change of numeraire, they are transformed into several homogeneous or inhomogeneous Black-Scholes PDEs with different time dependent coefficients and terminal value of binary type. The coefficients time dependency is the different point from [18]. Here we encounter the problems of higher order binaries with time dependent coefficients even if the drifts and volatilities of short rate and firm value processes are all constants. Therefore we first provide the pricing formulae of higher order binaries with time dependent coefficients and consider their integrals on the last expiry date variable. Then using the pricing formulae of higher binary options and their integrals, we give the pricing formulae of defaultable bonds in both cases of exogenous and endogenous default recoveries and credit spread analysis.

Finally we note that it is interesting to see that the Geske's compound option approach used in [1] for pricing of defaultable bond with discrete coupon payments in structural approach is the same technique as higher binary used here.

The remainder of the article is organized as follows. In section 2 we consider higher order binaries with time dependent coefficients and their properties. In section 3 we set the problem for defaultable bonds and provide the pricing formulae and credit spread analysis. In section 4 we provide the sketch of the proof of pricing formulae for defaultable bonds.

2 Higher order binaries with time dependent coefficients

First, we explain higher order bond and asset binaries with risk free rate $r(t)$, dividend rate $q(t)$ and volatility $\sigma(t)$.

$$\frac{\partial V}{\partial t}+\frac{\sigma^2(t)}{2}x^2\frac{\partial^2 V}{\partial x^2}+(r(t)-q(t))x\frac{\partial V}{\partial x}-r(t)V=0,\ 0\le t<T,\ 0<x<\infty, \tag{2.1}$$

$$V(x,T)=x\cdot 1(sx>s\xi), \tag{2.2}$$

$$V(x,T)=1(sx>s\xi). \tag{2.3}$$

Here $s=\pm 1$ and the signs refer to the call/put attribute of the option.

The solution to the problem (2.1) and (2.3) is called the *asset-or-nothing binaries* (or *asset* binaries) and denoted by $A_\xi^s(x,t;T)$. The solution to the problem (2.1) and (2.3) is called the *cash-or-nothing binaries* (or *bond* binaries) and denoted by $B_\xi^s(x,t;T)$. Asset binary and bond binary are called the *first order binary* options. If necessary, we will denote by $A_\xi^s(x,t;T;r(\cdot),q(\cdot),\sigma(\cdot))$ or $B_\xi^s(x,t;T;r(\cdot),q(\cdot),\sigma(\cdot))$, where the coefficients $r(t),q(t)$ and $\sigma(t)$ of Black-Scholes equation (2.1) are explicitly included in the notation.

Let assume that $0<T_0<T_1<\cdots<T_{n-1}$ and the $(n-1)$th order (asset or bond) binary options $A_{\xi_1\cdots\xi_{n-1}}^{s_1\cdots s_{n-1}}(x,t;T_1,\cdots,T_{n-1})$ and $B_{\xi_1\cdots\xi_{n-1}}^{s_1\cdots s_{n-1}}(x,t;T_1,\cdots,T_{n-1})$ are already defined. Let

$$V(x,T_0)=A_{\xi_1\cdots\xi_{n-1}}^{s_1\cdots s_{n-1}}(x,T_0;T_1,\cdots,T_{n-1})\cdot 1(s_0x>s_0\xi_0), \tag{2.4}$$

$$V(x,T_0)=B_{\xi_1\cdots\xi_{n-1}}^{s_1\cdots s_{n-1}}(x,T_0;T_1,\cdots,T_{n-1})\cdot 1(s_0x>s_0\xi_0), \tag{2.5}$$

The solution to the problem (2.1) and (2.4) is called the *n-th order asset binaries* and denoted by $A_{\xi_0\xi_1\cdots\xi_{n-1}}^{s_0s_1\cdots s_{n-1}}(x,t;T_0,T_1,\cdots,T_{n-1})$. The solution to the problem (2.1) and (2.5) is called the *n-th order bond binaries* and denoted by $B_{\xi_0\xi_1\cdots\xi_{n-1}}^{s_0s_1\cdots s_{n-1}}(x,t;T_0,T_1,\cdots,T_{n-1})$.

Next, we provide the pricing formulae of asset and bond binaries with time dependent coefficients. In the sequel the terminal value of the option price is a given function $f(x)$, that is

$$V(x,T)=f(x). \tag{2.6}$$

Lemma 2.1. *Assume that there exist nonnegative constants M and α such that $|f(x)|\le Mx^{\alpha\ln x}, x>0$. Then the solution of (2.1) and (2.6) is provided as follows:*

$$\begin{aligned}V(x,t;T)&=e^{-\bar r(t,T)}\int_0^\infty\frac{1}{\sqrt{2\pi\overline{\sigma^2}(t,T)}}\frac{1}{z}e^{-\frac{\left(\ln\frac{x}{z}+\bar r(t,T)-\bar q(t,T)-\frac12\overline{\sigma^2}(t,T)\right)^2}{2\overline{\sigma^2}(t,T)}}f(z)dz\\&=xe^{-\bar q(t,T)}\int_0^\infty\frac{1}{\sqrt{2\pi\overline{\sigma^2}(t,T)}}\frac{1}{z^2}e^{-\frac{\left(\ln\frac{x}{z}+\bar r(t,T)-\bar q(t,T)+\frac12\overline{\sigma^2}(t,T)\right)^2}{2\overline{\sigma^2}(t,T)}}f(z)dz.\end{aligned} \tag{2.7}$$

Here

$$\bar r(t,T)=\int_t^T r(s)ds,\ \bar q(t,T)=\int_t^T q(s)ds,\ \overline{\sigma^2}(t,T)=\int_t^T\sigma^2(s)ds. \tag{2.8}$$

Proof. It is well known that the solution to Black-Scholes equation with time dependent coefficients $r(t),q(t)$ and $\sigma(t)$ can be obtained by replacing $r(T-t),q(T-t)$ and $\sigma^2(T-t)$ in the solution representation of Black-Scholes equation with constant coefficients r,q and σ into $\bar r(t,T),\bar q(t,T)$ and $\overline{\sigma^2}(t,T)$. Using this fact and the proposition 1 at page 249 in [15], we soon have (2.7). A way of direct proof is as follows. As in [12], in (2.1) we use the changes of variable and unknown function . Then (2.1) is changed to

$$\begin{cases}\frac{\partial U}{\partial t}+\frac12\sigma^2(t)y^2\frac{\partial^2 U}{\partial y^2}=0, & 0<t<T,\ y>0,\\ U(y,T)=f(y), & y>0.\end{cases}$$

If we change time variable into $\tau=\int_0^t\sigma^2(s)ds,\hat T=\int_0^T\sigma^2(s)ds$, then we have

$$\begin{cases}\frac{\partial U}{\partial \tau}+\frac12y^2\frac{\partial^2 U}{\partial y^2}=0, & 0<\tau<\hat T,\ y>0,\\ U(y,\hat T)=f(y), & y>0.\end{cases}$$

This is the Black-Scholes equation with constant coefficients 0, 0 and 1 and thus we apply the proposition 1 at page 249 in [15] to get the representation of $U(y,\tau)$. Returning to original variables and unknown function, we get (2.7). □

Theorem 2.1. (The Pricing Formulae of Higher Order Binary Options with Time Dependent Coefficients) *The prices of higher order bond and asset binaries with risk free short rate $r(t)$, dividend rate $q(t)$ and volatility $\sigma(t)$ are as follows.*

$$\begin{aligned} &A_K^s(x,t;T;r(\cdot),q(\cdot),\sigma(\cdot)) = xe^{-\overline{q}(t,T)}N(sd),\\ &B_K^s(x,t;T;r(\cdot),q(\cdot),\sigma(\cdot)) = e^{-\overline{r}(t,T)}N(sd'),\ s=+\ or\ -, \end{aligned} \tag{2.9}$$

$$N(x) = \left(\sqrt{2\pi}\right)^{-1}\int_{-\infty}^{x}\exp[-y^2/2]dy,$$

$$d = \left(\sqrt{\overline{\sigma^2}(t,T)}\right)^{-1}\left[\ln(x/K)+\overline{r}(t,T)-\overline{q}(t,T)+\overline{\sigma^2}(t,T)/2\right],\ d' = d-\sqrt{\overline{\sigma^2}(t,T)},$$

$$\begin{aligned} &A_{K_1K_2}^{s_1s_2}(x,t;T_1,T_2) = xe^{-\overline{q}(t,T_2)}N_2(s_1d_1,s_2d_2;s_1s_2\rho),\\ &B_{K_1K_2}^{s_1s_2}(x,t;T_1,T_2) = e^{-\overline{r}(t,T_2)}N_2(s_1d_1',s_2d_2';s_1s_2\rho),\ s_1,s_2=+\ or\ -, \end{aligned} \tag{2.10}$$

$$N_2(a,b;\rho) = \int_{-\infty}^{a}\int_{-\infty}^{b}\left(2\pi\sqrt{1-\rho^2}\right)^{-1}e^{-\frac{y^2-2\rho yz+z^2}{2(1-\rho^2)}}dydz,\ \rho=\sqrt{\overline{\sigma^2}(t,T_1)/\overline{\sigma^2}(t,T_2)},$$

$$\begin{aligned} &A_{\xi_1\cdots\xi_m}^{s_1\cdots s_m}(x,t;T_1,\cdots,T_m) = xe^{-\overline{q}(t,T_m)}N_m(s_1d_1,\cdots,s_md_m;A_{s_1\cdots s_m}),\ s_i=+\ or\ -, m\ge 3,\\ &B_{\xi_1\cdots\xi_m}^{s_1\cdots s_m}(x,t;T_1,\cdots,T_m) = e^{-r(t,T_m)}N_m(s_1d_1',\cdots,s_md_m';A_{s_1\cdots s_m}),\ i=1,\cdots,m, \end{aligned} \tag{2.11}$$

$$N_m(a_1,\cdots,a_m;A) = \int_{-\infty}^{a_1}\cdots\int_{-\infty}^{a_m}\frac{1}{\left(\sqrt{2\pi}\right)^m}\sqrt{\det A}\exp\left(-\frac{1}{2}y^{\perp}Ay\right)dy,$$

$$d_i = \left(\sqrt{\overline{\sigma^2}(t,T_i)}\right)^{-1}\left[\ln(x/K_i)+\overline{r}(t,T_i)-\overline{q}(t,T_i)+\overline{\sigma^2}(t,T_i)/2\right],$$

$$d_i' = d_i-\sqrt{\overline{\sigma^2}(t,T_i)},\ i=1,\cdots,m,$$

$$A_{s_1\cdots s_m} = (s_is_ja_{ij})_{i,j=1}^m.$$

Here $y^{\perp}=(y_1,\cdots,y_m)$ and the matrix $(a_{ij})_{i,j=1}^m$ is given as follows:

$$\begin{aligned} &a_{11} = \overline{\sigma^2}(t,T_2)/\overline{\sigma^2}(T_1,T_2),\ \ a_{mm} = \overline{\sigma^2}(t,T_m)/\overline{\sigma^2}(T_{m-1},T_m),\\ &a_{ii} = \overline{\sigma^2}(t,T_i)/\overline{\sigma^2}(T_{i-1},T_i)+\overline{\sigma^2}(t,T_i)/\overline{\sigma^2}(T_i,T_{i+1}),\ 2\le i\le m-1,\\ &a_{i,i+1} = a_{i+1,i} = -\sqrt{\overline{\sigma^2}(t,T_i)\cdot\overline{\sigma^2}(t,T_{i+1})}/\overline{\sigma^2}(T_i,T_{i+1}),\ 1\le i\le m-1,\\ &a_{ij} = 0 \text{ for another } i,j=1,\cdots,m. \end{aligned}$$

Proof. $A_K^s(x,t;T)$ is just the solution to the problems (2.1) and (2.6) when $f(x)=x\cdot 1(sx>sK)$. If we substitute $f(x)=x\cdot 1(sx>sK)$ into the second formula of (2.7), we soon get the formula for $A_K^s(x,t;T)$ of (2.9). Similarly, if we substitute $f(x)=1(sx>sK)$ into the first formula of (2.7), we soon get the formula for $B_K^s(x,t;T)$ of (2.9).

$A_{K_1K_2}^{s_1s_2}(x,t;T_1,T_2)$ is just the solution to the problems (2.1) and (2.6) when $T=T_1$ and $f(x)=A_{K_2}^{s_2}(x,T_1;T_2)\cdot 1(s_1x>s_1K_1)$. If we substitute $f(x)=A_{K_2}^{s_2}(x,T_1;T_2)\cdot 1(s_1x>s_1K_1)$ and the formula for $A_{K_2}^{s_2}(x,T_1;T_2)$ of (2.9) into the second formula of (2.7) and represent the integral with the cumulative distribution function of bivariate normal distribution, we get the formula for $A_{K_1K_2}^{s_1s_2}(x,t;T_1,T_2)$ of (2.10). Similarly, if we substitute $f(x)=B_{K_2}^{s_2}(x,T_1;T_2)\cdot 1(s_1x>s_1K_1)$ and the formula for $B_{K_2}^{s_2}(x,T_1;T_2)$ of (2.9) into the first formula of (2.7) and represent the integral with the cumulative distribution function of bivariate normal distribution, we get the formula for $B_{K_1K_2}^{s_1s_2}(x,t;T_1,T_2)$ of (2.10).

In the case of $m>2$, we use induction to prove (2.11). Assume that (2.11) holds for $m=n-1$. Then from (2.4) $A_{\xi_1\xi_2\cdots\xi_n}^{s_1s_2\cdots s_n}(x,t;T_1;T_2,\cdots,T_n)$ satisfies (2.1) when $T=T_1$ and

$$V(x,T_1) = A_{\xi_2\cdots\xi_n}^{s_2\cdots s_n}(x,T_1;T_2,\cdots,T_n)\cdot 1(s_1x>s_1\xi_1).$$

Then from the second formula of (2.7), $A^{s_1 s_2 \cdots s_n}_{\xi_1 \xi_2 \cdots \xi_n}(x,t;T_1,T_2,\cdots,T_n)$ is provided as follows:

$$A^{s_1 s_2 \cdots s_n}_{\xi_1 \xi_2 \cdots \xi_n}(x,t;T_1,T_2,\cdots,T_n)$$
$$= xe^{-\overline{q}(t,T_1)} \int_0^\infty \frac{1}{\sqrt{2\pi\overline{\sigma^2}(t,T_1)}} \frac{1}{z^2} e^{-\frac{\left(\ln\frac{x}{z}+\overline{r}(t,T_1)-\overline{q}(t,T_1)+\frac{1}{2}\overline{\sigma^2}(t,T_1)\right)^2}{2\overline{\sigma^2}(t,T_1)}} A^{s_2 \cdots s_n}_{\xi_2 \cdots \xi_n}(z,T_1;T_2,\cdots,T_n)\cdot 1(s_1 z > s_1 \xi_1)dz.$$

Here $A^{s_2 \cdots s_n}_{\xi_2 \cdots \xi_n}(z,T_1;T_2,\cdots,T_n)$ is the price of the underlying $(n-1)$-th order asset binary option and thus by induction-assumption and (2.11) we have

$$A^{s_2 \cdots s_n}_{\xi_2 \cdots \xi_n}(z,T_1;T_2,\cdots,T_n) = ze^{-\overline{q}(T_1,T_n)} N_{n-1}(s_2 d_2,\cdots,s_n d_n; A_{s_2 \cdots s_n}).$$

If we substitute this equality into the above integral representation and represent the integral with the cumulative distribution function of n-variate normal distribution, we can get the first formula of (2.11) for $m=n$. The result for bond binaries (the second formula of (2.11)) is similarly proved. □

Remark 2.1. *Recently (higher order) binary options were priced in the literature, in particular, in [11] the Black-Scholes framework was adopted and in [4] the more general Lévy framework was studied. Theorem 2.1 is a generalization of the corresponding results of [7, 15]. In Theorem 2.1, $N_2(a,b;\rho)$ is the cumulative distribution function of bivariate normal distribution with a mean vector [0, 0] and a covariance matrix $[1,\rho;\rho,1]$ (symbols in the software "Matlab"), and $N_m(a_1,\cdots,a_m;A)$ is the cumulative distribution function of m-variate normal distribution with zero mean vector and a covariance matrix $A^{-1}=(r_{ij})_{i,j=1}^m$ where $r_{ij}=\sqrt{\overline{\sigma^2}(t,T_i)/\overline{\sigma^2}(t,T_j)}$, $r_{ji}=r_{ij}$, $i\le j$. Such special functions can easily be calculated by standard functions supplied in standard software for mathematical calculation (for example, Matlab).*

Next, we consider a relation between prices of higher order binaries with constant difference between risk free rates and dividend rates. From the formulae (2.9), (2.10) and (2.11), when b is a constant, we have:

$$\begin{aligned} A^{s_1\cdots s_m}_{K_1\cdots K_m}(x,t;T_1,\cdots,T_m;r_1(\cdot),r_1(\cdot)+b,\sigma(\cdot)) &= e^{-\overline{(r_1-r_2)}(t,T_m)} A^{s_1\cdots s_m}_{K_1\cdots K_m}(x,t;T_1,\cdots,T_m;r_2(\cdot),r_2(\cdot)+b,\sigma(\cdot)),\\ B^{s_1\cdots s_m}_{K_1\cdots K_m}(x,t;T_1,\cdots,T_m;r_1(\cdot),r_1(\cdot)+b,\sigma(\cdot)) &= e^{-\overline{(r_1-r_2)}(t,T_m)} B^{s_1\cdots s_m}_{K_1\cdots K_m}(x,t;T_1,\cdots,T_m;r_2(\cdot),r_2(\cdot)+b,\sigma(\cdot)). \end{aligned} \tag{2.12}$$

Next, as in [18], we can prove the following lemma. The proof is easy and omitted.

Lemma 2.2. (Integral of binary or nothing) *Assume that $g(\tau)$ is a continuous function of $\tau\in[T_{i-1},T]$. Let*

$$V(x,T_0) = 1(s_0 x > s_0\xi_0)\int_{T_{i-1}}^T g(\tau)\cdot F^{s_1\cdots s_{i-1}s_i}_{\xi_1\cdots\xi_{i-1}\xi_i}(x,T_0;T_1,\cdots,T_{i-1},\tau)d\tau. \tag{2.13}$$

Then the solution of (2.1) and (2.13) is given as follows:

$$V(x,t) = \int_{T_{i-1}}^T g(\tau)\cdot F^{s_0 s_1\cdots s_{i-1}s_i}_{\xi_0\xi_1\cdots\xi_{i-1}\xi_i}(x,t;T_0,T_1,\cdots,T_{i-1},\tau)d\tau. \tag{2.14}$$

Here $F=A$ or $F=B$.

3 The problem of defaultable bonds with discrete default information, the pricing formulae and credit spread analysis

3.1 The problem

Assumptions: 1) Short rate follows the law

$$dr_t = a_r(r,t)dt + s_r(t)dW_1(t),\ a_r(r,t) = a_1(t) - a_2(t)r \tag{3.1}$$

under the risk neutral martingale measure and a standard Wiener process W_1.

2) $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, $t_1, \cdots, t_N$ are announcing dates and T is the maturity of our corporate bond with face value 1 (unit of currency). For every $i = 0, \cdots, N-1$ the unexpected default probability in $[t, t+dt] \bigcap [t_i, t_{i+1}]$ is $\lambda_i dt$. Here the default intensity λ_i is a constant.

3) The firm value $V(t)$ follows a geometric Brown motion $dV(t) = (r_t - b)V(t)dt + s_V(t)V(t)dW_2(t)$ under the risk neutral martingale measure and a standard Wiener process W_1 and $E(dW_1, dW_2) = \rho dt$. The firm continuously pays out dividend in rate b (constant) for a unit of firm value.

4) The expected default barrier is only given at time t_i and the expected default event occurs when

$$V(t_i) \le K_i Z(r, t_i; T) \ (i = 1, \cdots, N).$$

Here K_i is a constant that reflects the quantity of debt and $Z(r, t; T)$ is default free zero coupon bond price.

5) The expected default recovery R_{ed} is given by $R_e \cdot Z(r, t; T)$, the unexpected default recovery R_{ud} as $R_u \cdot Z(r, t; T)$ and the recovery rates $0 \le R_e, R_u \le 1$ are constants. (*Exogenous* recovery.)

5)' The expected default recovery is given by $R_{ed} = R_e \cdot \alpha \cdot V$, the unexpected default recovery by $R_{ud} = \min\{Z(r, t),\ R_u \cdot \alpha \cdot V\}$ and the recovery rates $0 \le R_e, R_u \le 1$ are constants and $0 < \alpha < 1$ is a constant that reflects the quantity of debt (*Endogenous* recovery). Here the reason why the expected default recovery and unexpected recovery are given in different forms is to avoid the possibility of paying more than the current price of risk free bond as a default recovery when the unexpected default event occurs.

6) In the subinterval (t_i, t_{i+1}), the price of our corporate bond is given by a sufficiently smooth function $C_i(V, r, t)(i = 0, \cdots, N-1)$.

Problem. Find the representation of the price function $C_i(V, r, t)(i = 0, \cdots, N-1)$ under the above assumptions.

3.2 The Pricing Model

Under the assumption 1), the price $Z(r, t; T)$ of default free bond is the solution to the following problem

$$\begin{cases} \frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2(t) \frac{\partial^2 Z}{\partial r^2} + a_r(r, t) \frac{\partial Z}{\partial r} - rZ = 0, \\ Z(r, T) = 1. \end{cases} \tag{3.2}$$

The solution is given by

$$Z(r, t; T) = e^{A(t,T) - B(t,T)r}. \tag{3.3}$$

Here $A(t, T)$ and $B(t, T)$ are differently given dependent on the specific model of short rate [20]. For example, if the short rate follows the Vasicek model, that is, if the coefficients $a_1(t), a_2(t), s_r(t)$ in (3.1) are all constants (that is, $a_1(t) \equiv a_1,\ a_2(t) \equiv a_2,\ s_r(t) \equiv s_r$), then $B(t, T)$ and $A(t, T)$ are respectively given as follows:

$$B(t, T) = \frac{1 - e^{-a_2(T-t)}}{a_2}, \ \ A(t, T) = -\int_t^T \left[a_2 B(u, T) - \frac{1}{2} s_r^2 B^2(u, T) \right] du. \tag{3.4}$$

See [20] for $B(t, T)$ and $A(t, T)$ in Ho-Lee model, Hull-White model and CIR model.

According to [20], under the above assumptions the price of defaultable bond with a constant default intensity λ and unexpected default recovery R_{ud} satisfies the following PDE:

$$\begin{aligned} \frac{\partial C}{\partial t} &+ \frac{1}{2} \left[s_V^2(t) V^2 \frac{\partial^2 C}{\partial V^2} + 2\rho s_V(t) s_r(t) V \frac{\partial^2 C}{\partial V \partial r} + s_r^2(t) \frac{\partial^2 C}{\partial r^2} \right] \\ &+ (r - b)V \frac{\partial C}{\partial V} + a_r(r, t) \frac{\partial C}{\partial r} - (r + \lambda)C + \lambda R_{ud} = 0. \end{aligned}$$

Therefore if we let $C_N(V, r, t) \equiv 1$, then the price model of our bond is given as follows:

$$\begin{cases} \frac{\partial C_i}{\partial t} + \frac{1}{2} \left[s_V^2(t) V^2 \frac{\partial^2 C_i}{\partial V^2} + 2\rho s_V(t) s_r(t) V \frac{\partial^2 C_i}{\partial V \partial r} + s_r^2(t) \frac{\partial^2 C_i}{\partial r^2} \right] + (r - b)V \frac{\partial C_i}{\partial V} \\ \quad + a_r(r, t) \frac{\partial C_i}{\partial r} - (r + \lambda_i)C_i + \lambda_i R_{ud} = 0, \ t_i \le t < t_{i+1}, \\ C_i(t_{i+1}) = C_{i+1}(t_{i+1}) \cdot 1\{V > K_{i+1}Z\} + R_{ed} \cdot 1\{V \le K_{i+1}Z\}, \ i = 0, \cdots, N-1. \end{cases} \tag{3.5}$$

Here the default recoveries R_{ed}, R_{ud} are differently given whether we choose to take the assumption 5) or 5)′. Under the assumption 5) the model is as follows:

$$\begin{cases} \frac{\partial C_i}{\partial t} + \frac{1}{2}\left[s_V^2(t)V^2\frac{\partial^2 C_i}{\partial V^2} + 2\rho s_V(t)s_r(t)V\frac{\partial^2 C_i}{\partial V\partial r} + s_r^2(t)\frac{\partial^2 C_i}{\partial r^2}\right] + (r-b)V\frac{\partial C_i}{\partial V} \\ \quad + a_r(r,t)\frac{\partial C_i}{\partial r} - (r+\lambda_i)C_i + \lambda_i R_u \cdot Z(r,t;T) = 0, \ \ t_i \le t < t_{i+1}, \\ C_i(t_{i+1}) = C_{i+1}(t_{i+1}) \cdot 1\{V > K_{i+1}Z\} + R_e \cdot Z(r,t;T) \cdot 1\{V \le K_{i+1}Z\}, \ \ i = 0, \cdots, N-1. \end{cases} \tag{3.6}$$

Under the assumption 5)′ the model is as follows:

$$\begin{cases} \frac{\partial C_i}{\partial t} + \frac{1}{2}\left[s_V^2(t)V^2\frac{\partial^2 C_i}{\partial V^2} + 2\rho s_V(t)s_r(t)V\frac{\partial^2 C_i}{\partial V\partial r} + s_r^2(t)\frac{\partial^2 C_i}{\partial r^2}\right] + (r-b)V\frac{\partial C_i}{\partial V} \\ \quad + a_r(r,t)\frac{\partial C_i}{\partial r} - (r+\lambda_i)C_i + \lambda_i \min\{Z(r,t), R_u \cdot \alpha \cdot V\} = 0, \ \ t_i \le t < t_{i+1}, \\ C_i(t_{i+1}) = C_{i+1}(t_{i+1}) \cdot 1\{V > K_{i+1}Z\} + R_e \cdot \alpha \cdot V \cdot 1\{V \le K_{i+1}Z\}, \ \ i = 0, \cdots, N-1. \end{cases} \tag{3.7}$$

3.3 The pricing formulae

Theorem 3.2. (Exogenous recovery) *Under the assumptions* 1) − 6)*, the price of our bond is represented as follows:*

$$C_i(V,r,t) = W_i(V/Z,t) \cdot Z + [1 - W_i(V/Z,t)] \cdot R_u \cdot Z, \ t_i \le \forall t < t_{i+1}, \ i = 0, \cdots, N-1. \tag{3.8}$$

Here

$$\begin{aligned} W_i(x,t) =& e^{-\lambda_i(t_{i+1}-t)} \Bigg\{ e^{-\sum_{k=i+1}^{N-1}\lambda_k(t_{k+1}-t_k)} B^{+\ \cdots\ +}_{K_{i+1}\cdots K_N}(x,t;t_{i+1},\cdots,t_N;0,b,S_X(\cdot)) \\ &+ \frac{R_e - R_u}{1-R_u}\sum_{m=i}^{N-1} e^{-\sum_{k=i+1}^{m}\lambda_k(t_{k+1}-t_k)} B^{+\ \cdots\ +\ -}_{K_{i+1}\cdots K_m K_{m+1}}(x,t;t_{i+1},\cdots,t_m,t_{m+1};0,b,S_X(\cdot)) \Bigg\} \end{aligned} \tag{3.9}$$

$$t_{N-2} \le t < t_{N-1}, \ x > 0, \ i = 0, \cdots, N-1,$$

$$S_X^2(t) = s_V^2(t) + 2\rho \cdot s_V(t) \cdot s_r(t) \cdot B(t,T) + [s_r(t) \cdot B(t,T)]^2 \ge 0. \tag{3.10}$$

and $B(t,T)$ is given in (3.4)*; $B^{+\ \cdots\ +}_{K_1\cdots K_m}(x,t;t_1,\cdots,t_m;0,b,s_X(\cdot))$ is the price of m-th order bond binary with 0-risk free rate, b-dividend rate and $S_X(t)$-volatility.*

(See Theorem 2.1.)

Remark 3.2. *Theorem 3.2 is a generalization of the theorem 2 of [18] to the case of stochastic risk free rate. That is, if we let r is a constant and $R_e = R_u$, we have the theorem 2 of [18]. The financial meaning of the pricing formulae* (3.8) *is very clear when $R = R_u = R_e$ and just the same as the one of the theorem 2 in [18]. $W_i(V/Z,t)$ is the survival probability after the time $t \in [t_i, t_{i+1})$, that is, the probability with which no default event occurs in the interval $[t,T]$ and $1 - W_i(V/Z,t)$ is the ruin probability after the time $t \in [t_i, t_{i+1})$, that is, the probability with which default event occurs in the interval $[t,T]$ when $t_i \le t < t_{i+1}$. The formulae* (3.8) *can be written as follows:*

$$C_i(V,r,t) = R \cdot Z + (1-R)W_i(V/Z,t) \cdot Z, \ t_i \le \forall t < t_{i+1}, \ i = 0, \cdots, N-1. \tag{3.11}$$

The financial meaning of (3.11) *is that the first term of* (3.11) *is the current price of the part to be given to bond holder regardless of default occurs or not, and the second term is the allowance dependent on the survival probability after time t.*

Theorem 3.3. (Endogenous recovery) *Under the assumptions* 1) − 5)′ *and* 6)*, the price of our bond is provided as follows:*

$$C_i(V,r,t) = Z(r,t) \cdot u_i(V/Z(r,t),t), \ t_i \le t < t_{i+1}, \ i = 0, \cdots, N-1. \tag{3.12}$$

Here

$$
\begin{aligned}
u_i(x,t) =& e^{-\lambda_i(t_{i+1}-t)} \Big\{ e^{-\sum_{k=i+1}^{N-1}\lambda_k(t_{k+1}-t_k)} B^{+\ \cdots\ +}_{K_{i+1}\cdots K_N}(x,t;t_{i+1},\cdots,t_N;0,b,S_X(\cdot)) \\
&+ R_e\alpha \sum_{m=i}^{N-1} e^{-\sum_{k=i+1}^{m}\lambda_k(t_{k+1}-t_k)} A^{+\ \cdots\ +\ -}_{K_{i+1}\cdots K_m K_{m+1}}(x,t;t_{i+1},\cdots,t_m,t_{m+1};0,b,S_X(\cdot)) \\
&+ \sum_{m=i+1}^{N-1} \lambda_m e^{-\sum_{k=i+1}^{m-1}\lambda_k(t_{k+1}-t_k)} \int_{t_m}^{t_{m+1}} e^{-\lambda_m(\tau-t_m)} \Big[B^{+\ \cdots\ +\ +}_{K_{i+1}\cdots K_m \frac{1}{R_u\alpha}}(x,t;t_{i+1},\cdots,t_m,\tau;0,b,S_X(\cdot)) \\
&+ R_u\cdot\alpha\cdot A^{+\ \cdots\ +\ -}_{K_{i+1}\cdots K_m \frac{1}{R_u\alpha}}(x,t;t_{i+1},\cdots,t_m,\tau;0,b,S_X(\cdot)) \Big] d\tau \Big\} \\
&+ \lambda_i \int_t^{t_{i+1}} e^{-\lambda_i(\tau-t)} \Big[B^{+}_{\frac{1}{R_u\alpha}}(x,t;\tau;0,b,S_X(\cdot)) + R_u\cdot\alpha\cdot A^{-}_{\frac{1}{R_u\alpha}}(x,t;\tau;0,b,S_X(\cdot)) \Big] d\tau.
\end{aligned} \tag{3.13}
$$

$S_X^2()t$ and $B(t,T)$ are given in (3.10) and (3.4); $B^{+\ \cdots\ +}_{K_1\cdots K_m}(x,t;t_1,\cdots,t_m;0,b,S_X(\cdot))$ and $A^{+\ \cdots\ +\ -}_{K_1\cdots K_{m-1}K_m}(x,t;t_1,\cdots,t_{m-1},t_m;0,b,S_X(\cdot))$ are the prices of m-th order bond and asset binaries with 0-risk free rate, b-dividend rate and $S_X(t)$-volatility.

(See Theorem 2.1.)

Remark 3.3. *Theorem 3.3 is a generalization of the theorem 1 of [18] into the case of stochastic risk free rate. That is, if we let r is a constant, $\alpha = 1/n$ and $R_e = R_u$, then we have the theorem 1, i) of [18]. The financial meaning of $u_i(x,t)$ is that its the relative price of our bond in a subinterval with respect to the risk free zero coupon bond.*

3.4 Credit spread analysis

In this subsection, we will illustrate the effect of several parameters including recovery rate, volatility of firm value, the relative price of the firm value and etc. on credit spreads. The *credit spread* is defined as the difference between the yields of defaultable bond C and default-free bond Z and is given by the following expression:

$$CS = -\frac{\ln C - \ln Z}{T-t}.$$

In the case of exogenous recovery (considered in Theorem 3.2), the credit spread feature is similar with that of [18]. Here we consider the case of endogenous recovery (considered in Theorem 3.3). In this case, the credit spread is differently given in every subinterval.

$$CS_i = -\frac{\ln(C_i(V,r,t)/Z(r,t))}{T-t} = -\frac{\ln(u_i(V/Z(r,t),t)}{T-t},\ t_i \le \forall t < t_{i+1},\ i=0,\cdots,N-1. \tag{3.14}$$

Let $N=2,\ t_1=3,\ t_2=T=6$ (annum)

Basic data for calculation of CS is as follows: short rate model parameters: $a_1(t) \equiv 0.379 * 0.098$, $a_2(t) \equiv 0.379$, $s_r(t) \equiv 0.077$ (Vasicek model); firm value process parameters: dividend rate $b = 0.05$, volatility $s_V = 1.0$; $x = V/Z = 200$; correlation of short rate and firm value: $\rho = 0.5$; $\lambda_0 = 0.1, \lambda_1 = 0.3$ are respectively default intensity in the intervals $[0,t_1],[t_1,t_2]$; $K_1 = K_2 = 100$ is default barrier at time t_1, t_2; recovery rate: $R_e = R_u = 0.5; \alpha = 1/150$.

We will analyze $(t : CS)$ plot changing one of $R, s_V, \rho, x = V/Z, \lambda$ and K under keeping the remainder of data as above.

In what follows, Figure 1 shows that increase of recovery rate results in decrease of credit spread. Figure 2 shows that increase of volatility of firm value results in increase of credit spread. The reason is that when s_V increases, the firm value fluctuates more seriously and there are more risks of default, which results in increase of credit spread. Figure 3 shows that increase of correlation between firm value and short rate results in increase of credit spread. Figure 4 shows that increase of firm value results in decrease of credit spread. Figures 5, 6 and 7 show that in the time interval close to the maturity increase of the default intensity results in increase of credit spread but in other time region the circumstance is not so simple. Figures 8, 9 and 10 show that the effect of default barrier on credit spread is different in the time intervals $[0,t_1]$ and $[t_1,T]$. The reason of such a complexity of the effect of default intensity and default barrier is in the formula (3.13).

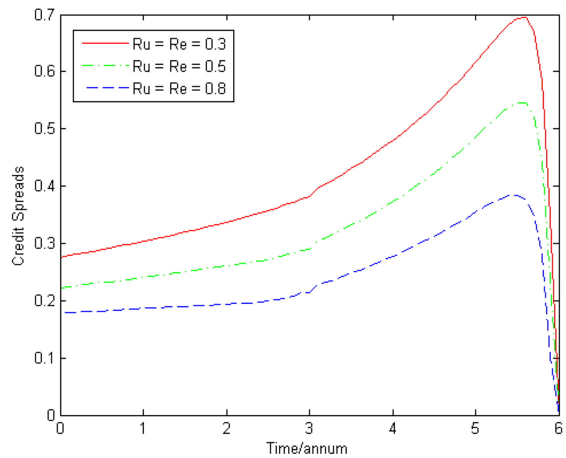

Figure 1: Plot $(t : CS)$ when $R_e = R_u = 0.3, 0.5, 0.8$

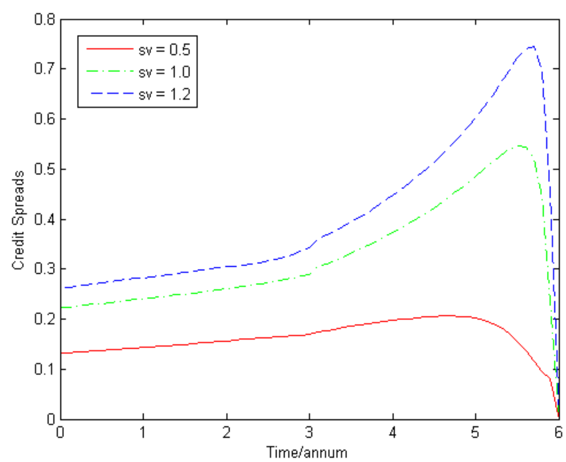

Figure 2: Plot $(t : CS)$ when $s_V = 0.5, 1.0, 1.2$

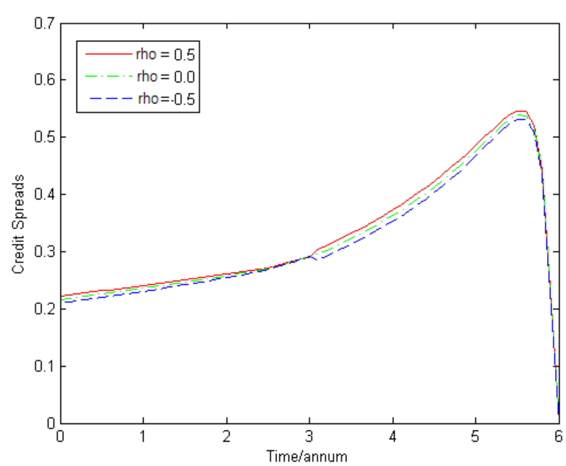

Figure 3: Plot $(t : CS)$ when $\rho = 0.5, 0, -0.5$

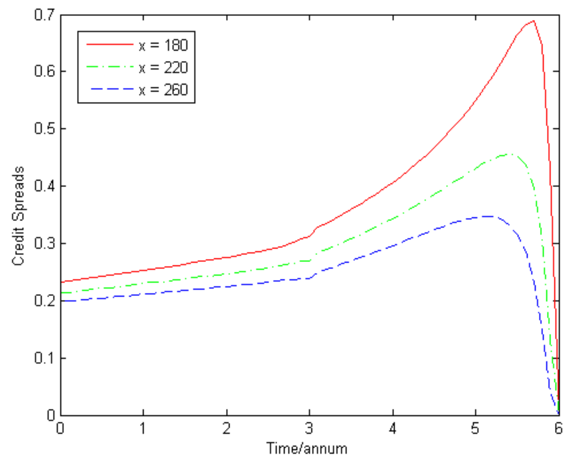

Figure 4: Plot $(t : CS)$ when $x = V/Z = 180, 220, 260$

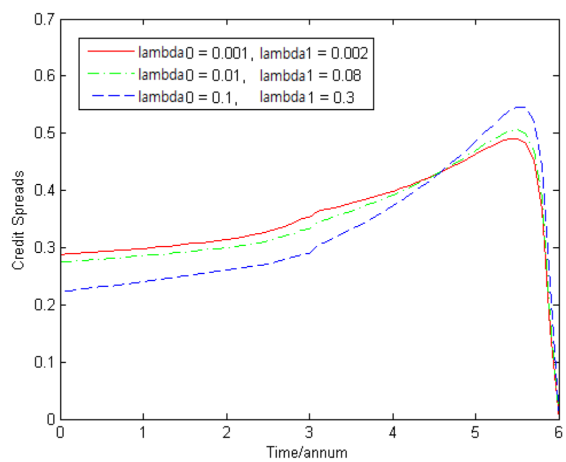

Figure 5: Plot $(t : CS)$ when $(\lambda_0, \lambda_1) = (0.001, 0.002), (0.01, 0.008), (0.1, 0.3)$

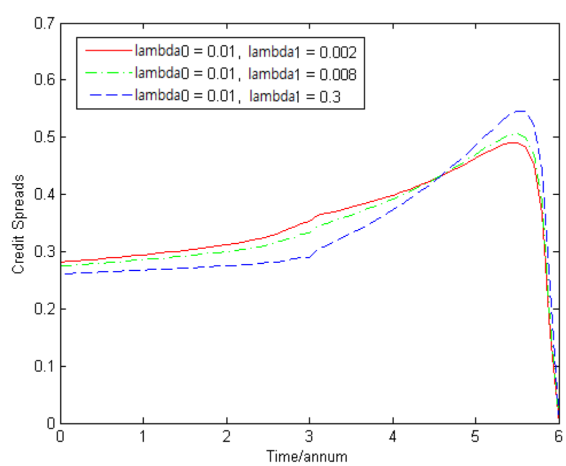

Figure 6: Plot $(t : CS)$ when $(\lambda_0 = 0.01; \lambda_1 = 0.002, 0.008, 0.3$

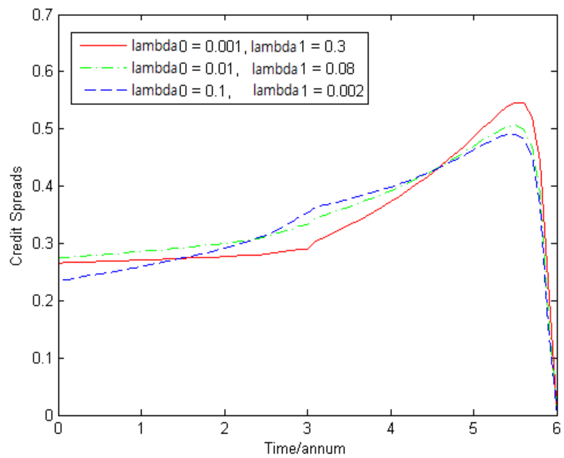

Figure 7: Plot $(t : CS)$ when $(\lambda_0, \lambda_1) = (0.001, 0.3), (0.01, 0.08), (0.1, 0.002)$

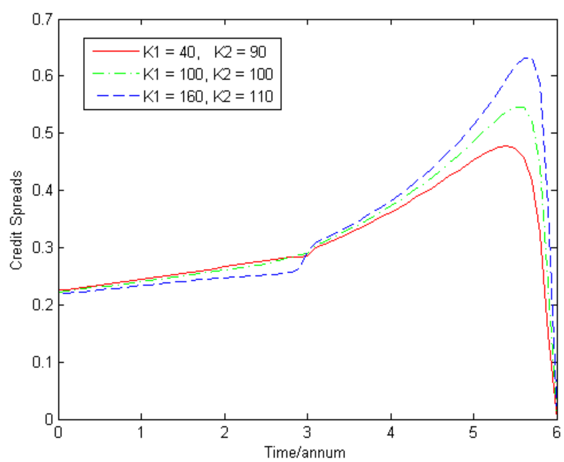

Figure 8: Plot $(t : CS)$ when $(K_1, K_2) = (40, 90), (100, 100), (160, 110)$

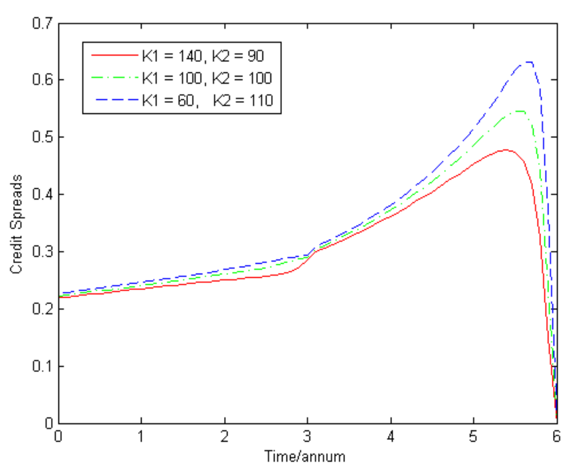

Figure 9: Plot $(t : CS)$ when $(K_1, K_2) = (140, 90), (100, 100), (60, 110)$

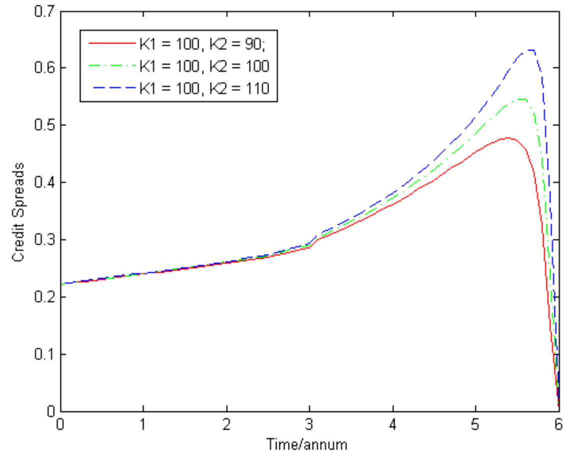

Figure 10: Plot $(t : CS)$ when $K_1 = 100; K_2 = 90, 100, 110$

4 The proofs of the pricing formulae

The proof of Theorem 3.2. Under the assumptions 1) − 6), the price model of our bond is given by (3.6). In (3.6), we use change of numeraire

$$x = V/Z(r,t),\ u_i(x,t) = C_i(V,r,t)/Z(r,t),\ t_i \le t < t_{i+1},\ i = 0, \cdots, N-1. \tag{4.1}$$

Here $Z(r,t)$ is the price of default free zero coupon bond given by (3.3). If we substitute (4.1) into the first equation of (3.6), note that

$$\begin{aligned} &C_t = u_t Z - x u_x Z_t + u Z_t,\ VC_V = x u_x Z,\ C_r = Z_r(u - x u_x),\ \ V^2 C_{VV} = x^2 u_{xx} Z, \\ &VC_{Vr} = -x^2 u_{xx} Z_r, \qquad C_{rr} = Z_{rr}(u - x u_x) + x^2 u_{xx} Z_r^2/Z,\ Z_r/Z = -B(t,T) \end{aligned}$$

and consider the equation (3.2) on $Z(r,t)$, then we have

$$u_t Z + \frac{1}{2} x^2 u_{xx} Z \left[s_V^2(t) + 2\rho s_V(t) s_r(t) B(t) + (s_r(t)B(t))^2 \right] - b x u_x Z - \lambda_i u Z + \lambda_i R_u Z(r,t) = 0.$$

Divide the two hands by Z and let $S_X^2(t) = s_V^2(t) + 2\rho s_V(t) \cdot s_r(t) \cdot B(t) + (s_r(t)B(t))^2$, then the problem (3.6) is changed to the following one dimensional problem:

$$\begin{cases} \frac{\partial u_i}{\partial t} + \frac{1}{2} S_X^2(t) x^2 \frac{\partial^2 u_i}{\partial x^2} - b x \frac{\partial u_i}{\partial x} - \lambda_i(u_i - R_u) = 0, & t_i < t < t_{i+1},\ x > 0, \\ u_i(x, t_{i+1}) = u_{i+1}(x, t_{i+1}) \cdot 1(x > K_{i+1}) + R_e \cdot 1(x \le K_{i+1}), & x > 0,\ i = 0, \cdots, N-1, \end{cases} \tag{4.2}$$

Here $u_N(x,t) \equiv 1$. We use the change of unknown function

$$u_i = (1 - R_u) W_i + R_u,\ (i = 0, 1, \cdots, N-1) \tag{4.3}$$

to have

$$\begin{cases} \frac{\partial W_i}{\partial t} + \frac{1}{2} S_X^2(t) x^2 \frac{\partial^2 W_i}{\partial x^2} - b x \frac{\partial W_i}{\partial x} - \lambda_i W_i = 0, & t_i \le t < t_{i+1},\ x > 0, \\ W_i(x, t_{i+1}) = W_{i+1}(x, t_{i+1}) \cdot 1(x > K_{i+1}) + \frac{R_e - R_u}{1 - R_u} \cdot 1(x < K_{i+1}), & x > 0,\ i = 0, \cdots, N-1. \end{cases} \tag{4.4}$$

Here $W_N(x,t) \equiv 1$. Then using this W_i, our bond price is provided by (3.8). The equation (4.4) is called the *equation for the survival probability* after the time $t \in [t_i, t_{i+1})$.

(4.4) is a set of Black-Scholes equations just like (4.22) in [18], but here the coefficient $S_X^2(t)$ is not a constant. So we use Theorem 2.1 instead of the theorems of [7, 15, 16].

Now we solve the problem (4.4). When $i = N-1$, (4.4) is

$$\begin{cases} \frac{\partial W_{N-1}}{\partial t} + \frac{1}{2}S_X^2(t)x^2\frac{\partial^2 W_{N-1}}{\partial x^2} - bx\frac{\partial W_{N-1}}{\partial x} - \lambda_{N-1}W_{N-1} = 0, & t_{N-1} \le t < t_N,\ x > 0, \\ W_{N-1}(x, t_N) = 1(x > K_N) + \frac{R_e - R_u}{1 - R_u}\cdot 1(x < K_N), & x > 0. \end{cases} \tag{4.5}$$

This is a pricing problem of binary options with coefficients $\lambda_{N-1}, \lambda_{N-1}+b, S_X(t)$ whose expiry payoff is a combination of bond and asset binaries. By the definition of bond binary, we have

$$W_{N-1}(x,t) = B^{+}_{K_N}(x,t;t_N;\lambda_{N-1},\lambda_{N-1}+b,S_X(\cdot)) + \frac{R_e - R_u}{1 - R_u}\cdot B^{-}_{K_N}(x,t;t_N;\lambda_{N-1},\lambda_{N-1}+b,S_X(\cdot)), \tag{4.6}$$
$$t_{N-1} \le t < t_N,\ x > 0.$$

Here $B^{s}_{K}(x,t;T;r(\cdot),q(\cdot),\sigma(\cdot))$ is given by the formula (2.9) of Theorem 2.1.

For our further purpose, using the relations (2.12) we rewrite (4.6) by the prices of bond and asset binaries with the coefficients $r = 0,\ q = b,\ \sigma(t) = S_X(t)$:

$$W_{N-1}(x,t) = e^{-\lambda_{N-1}(t_N - t)}B^{+}_{K_N}(x,t;t_N;0,b,S_X(\cdot)) + \frac{R_e - R_u}{1 - R_u}\cdot e^{-\lambda_{N-1}(t_N - t)}B^{-}_{K_N}(x,t;t_N;0,b,S_X(\cdot)), \tag{4.7}$$
$$t_{N-1} \le t < t_N,\ x > 0.$$

In order to solve (4.4) when $i = N-2$, we need to rewrite (4.6) by the prices of bond and asset binaries with the coefficients $r = \lambda_{N-2},\ q = \lambda_{N-2}+b,\ \sigma(t) = S_X(t)$ just as noted in the remark 3 in [18].

$$\begin{aligned} W_{N-1}(x,t) =& e^{-(\lambda_{N-1}-\lambda_{N-2})(t_N - t)}\Big[B^{+}_{K_N}(x,t;t_N;\lambda_{N-2},\lambda_{N-2}+b,S_X(\cdot)) \\ &+ \frac{R_e - R_u}{1 - R_u}\cdot B^{-}_{K_N}(x,t;t_N;\lambda_{N-2},\lambda_{N-2}+b,S_X(\cdot))\Big],\ t_{N-1} \le t < t_N,\ x > 0. \end{aligned} \tag{4.8}$$

When $i = N-2$ using (4.8), (4.4) is written as follows:

$$\begin{cases} \frac{\partial W_{N-2}}{\partial t} + \frac{1}{2}S_X^2(t)x^2\frac{\partial^2 W_{N-2}}{\partial x^2} - bx\frac{\partial W_{N-2}}{\partial x} - \lambda_{N-2}W_{N-2} = 0,\ t_{N-2} \le t < t_{N-1},\ x > 0, \\ W_{N-2}(x,t_{N-1}) = e^{-(\lambda_{N-1}-\lambda_{N-2})(t_N - t_{N-1})}\Big[B^{+}_{K_N}(x,t_{N-1};t_N;\lambda_{N-2},\lambda_{N-2}+b,S_X(\cdot))\cdot 1(x > K_{N-1}) \\ \quad + \frac{R_e - R_u}{1 - R_u}\cdot B^{-}_{K_N}(x,t;t_N;\lambda_{N-2},\lambda_{N-2}+b,S_X(\cdot))\cdot 1(x > K_{N-1})\Big] + \frac{R_e - R_u}{1 - R_u}\cdot 1(x < K_{N-1}),\ x > 0. \end{cases} \tag{4.9}$$

This is a pricing problem of binary options with coefficients $\lambda_{N-2},\ \lambda_{N-2}+b,\ S_X(t)$ whose expiry payoff is a combination of bond and asset binaries. By the definition of second order binary, we have

$$\begin{aligned} W_{N-2}(x,t) =& e^{-(\lambda_{N-1}-\lambda_{N-2})(t_N - t_{N-1})}\Big[B^{+\,+}_{K_{N-1}K_N}(x,t;t_{N-1};t_N;\lambda_{N-2},\lambda_{N-2}+b,S_X(\cdot)) \\ &+ \frac{R_e - R_u}{1 - R_u}\cdot B^{+\,-}_{K_{N-1}K_N}(x,t;t_{N-1};t_N;\lambda_{N-2},\lambda_{N-2}+b,S_X(\cdot))\Big] \\ &+ \frac{R_e - R_u}{1 - R_u}\cdot B^{-}_{K_{N-1}}(x,t;t_{N-1};\lambda_{N-2},\lambda_{N-2}+b,S_X(\cdot)),\ t_{N-2} \le t < t_{N-1},\ x > 0. \end{aligned}$$

Here $B^{s_1 s_2}_{K_1 K_2}(x,t;T_1,T_2;r(\cdot),q(\cdot),\sigma(\cdot))$ is given by the formula (2.10) of Theorem 2.1.

For our further purpose, using the relations (2.12) we rewrite $W_{N-2}(x,t)$ by the prices of bond and asset binaries with the coefficients $r = 0, q = b, \sigma(t) = S_X(t)$:

$$\begin{aligned} W_{N-2}(x,t) =& e^{-\lambda_{N-2}(t_{N-1}-t)-\lambda_{N-1}(t_N - t_{N-1})}B^{+\,+}_{K_{N-1}K_N}(x,t;t_{N-1};t_N;0,b,S_X(\cdot)) \\ &+ \frac{R_e - R_u}{1 - R_u}\cdot\Big[e^{-\lambda_{N-2}(t_{N-1}-t)-\lambda_{N-1}(t_N - t_{N-1})}B^{+\,-}_{K_{N-1}K_N}(x,t;t_{N-1};t_N;0,b,S_X(\cdot)) \\ &+ e^{-\lambda_{N-2}(t_{N-1}-t)}B^{-}_{K_{N-1}}(x,t;t_{N-1};0,b,S_X(\cdot))\Big],\ t_{N-2} \le t < t_{N-1},\ x > 0. \end{aligned} \tag{4.10}$$

By induction we have (3.9). Returning to original variables through (4.1) and (4.3), then we have the formula (3.8). □

The proof of Theorem 3.3. Under the assumptions 1) − 5)′ and 6), the price model of our bond is given by (3.7). In (3.7), we use change of numeraire (4.1), then we have

$$\begin{cases} \frac{\partial u_i}{\partial t} + \frac{1}{2}S_X^2(t)x^2\frac{\partial^2 u_i}{\partial x^2} - bx\frac{\partial u_i}{\partial x} - \lambda_i(u_i) + \lambda_i \min\{1, R_u\alpha \cdot x\} = 0, \quad t_i < t < t_{i+1},\ x > 0, \\ u_i(x, t_{i+1}) = u_{i+1}(x, t_{i+1}) \cdot 1\{x > K_{i+1}\} + R_e\alpha \cdot x \cdot 1\{x \le K_{i+1}\}, \quad x > 0,\ i = 0, \cdots, N-1, \end{cases} \tag{4.11}$$

(4.11) is a similar problem with the problem (4.5) in [18]. The only difference is that the (4.11) is a set of terminal value problems for inhomogenous Black-Scholes equations with time dependent coefficients but the (4.5) in [18] is a set of terminal value problems for inhomogenous Black-Scholes equations with constant coefficients. If we follow the way of solving (4.5) in [18] using Theorem 2.1, Lemma 2.2 and the relations (2.12), then we can get the formula (3.13). Then returning to the original variable V and the unknown function C using (4.1) we can soon obtain the formula (3.12). The detail is omitted. □

5 Conclusions

1) We proved the pricing formula of higher order binary option with time dependent coefficients (Theorem 2.1). This is a generalization of the corresponding results of [7, 15]. Moreover, we generalized the integral formula of higher order binary option on the last expiry date variable into the case with time dependent coefficients (Lemma 2.2).

2) We obtained the pricing formulae of Two factor-model for defaultable bonds with discrete default intensity and discrete default barrier in both cases of exogenous and endogenous recoveries (Theorem 3.2 and Theorem 3.3) using the pricing formulae of higher order binary options with time dependent coefficients.

3) In further study the method can seemingly be applied to generalization of the study of [1] into the pricing of defaultable bonds by combining the structural approach and the reduced form approach.

Acknowledgments The authors wish to thank anonymous referees for their comments and suggestions.

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Received: April 25, 2014; *Accepted*: August 4, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 345–351

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Existence of solution of a Coupled system of differential equation with nonlocal conditions

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Abstract

In this paper, we study the existence of at least one solution of the coupled system of differential equations with nonlocal conditions. Also, a coupled system of differential equations with the nonlocal integral conditions will be considered.

Keywords: Coupled systems, nonlocal conditions, at least one solution, integral conditions.

2010 MSC: 34B18, 34B10.

1 Introduction

Problems with nonlocal conditions have been extensively studied by several authors in the last decades.The reader is referred to([2]-[20]) and references therein.
In [13] the authors studied nonlocal cauchy problem

$$\dot{x} = f(t, x(t)),\ t \in [0, T]$$

$$\sum_{j=1}^{m} b_j x(\eta_j) = x_1,\ \eta_j \in (0, a) \subset [0, T].$$

Also, in [7] the authors studied the local and global existence of solutions of the nonlocal problem

$$\frac{dx}{dt} = f_1(t, y(t)),\ \ t \in (0, T] \tag{1.1}$$

$$\frac{dy}{dt} = f_2(t, x(t)),\ \ t \in (0, T] \tag{1.2}$$

with the nonlocal conditions

$$x(0) + \sum_{k=1}^{n} a_k x(\tau_k) = x_0,\ \ a_k > 0,\ \tau_k \in (0,\ T) \tag{1.3}$$

$$y(0) + \sum_{j=1}^{m} b_j y(\eta_j) = y_0,\ \ b_j > 0,\ \eta_j \in (0,\ T) \tag{1.4}$$

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Here we are studied the existence of at least one solution of the nonlocal problem (1.1)-(1.4), the problem with nonlocal integral conditions

$$x(0) + \int_0^T x(s)ds = x_0, \quad (1.5)$$

$$y(0) + \int_0^T y(s)ds = y_0. \quad (1.6)$$

are studied.

2 Preliminaries

we need the following definitions.

Definition 2.1. *[19] Let $F = \{f_i : X \to Y , i \in I\}$ be a family of functions with Y being a set of real (or complex) numbers, then we call F uniformly bounded if there exists a real number c such that $| f_i(x) | \leq c \ \forall i \in I , x \in X$.*

Definition 2.2. *[19] Let $F = \{f(x)\}$ is the class of functions defined on A where $A = [a, b] \subset R$, the class of functions $F = \{f(x)\}$ is equicontinuous if $\forall \epsilon > 0$, $\exists\delta(\epsilon)$ such that $| x - y |< \delta$, implies that $| f(x) - f(y) |< \epsilon \ \forall f \in F , \ x, y \in A$.*

Theorem 2.1. *[1] The function $f(x) = (f_1(x) , f_2(x) , ..., f_n(x))$ is uniformly continuous in $I = [a, b]$ if and only if each f_i is uniformly continuous in $[a, b]$.*

Theorem 2.2. *[19](Lebesgue Dominated Convergence Theorem)*
let f_n be a sequence of functions converging to a limit f of A, and suppose that
$| f_n(t) | \leq \phi(t) , t \in A , n = 1 , 2 , 3 , ...$ where ϕ is integrable on A . Then

1. *f is integrable on A*
2. *$\lim_{n\to\infty} \int_A f_n(t)d\mu = \int_A f(t)d\mu$.*

Theorem 2.3. *[18](Schauder)*
Let Q be a convex subset of a Banach space X, $T : Q \to Q$ be a compact and continuous map, then T has at least one fixed point in Q.

3 Integral Representation

Let X be the class of all columns vectors $\begin{pmatrix} x \\ y \end{pmatrix}$, $x, y \in C(0, T]$ with the norm

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_X = ||x|| + ||y|| = \sup_{t\in[0,T]} | x(t) | + \sup_{t\in[0,T]} | y(t) | .$$

Throughout the paper we assume that the following assumptions hold:

i. $f_i : [0, T] \times R \to R$ satisfies Caratheodory conditions, that is f_i is
   1. measurable in $t \in (0, T]$, for any $x \in R$.
   2. continuous in $x \in R$, for almost all $t \in (0, T]$.

ii. There exist two integrable functions $m_i \in L_1[0, T]$, $i = 1, 2$ such that
   $|f_i (t, x)| \leq m_i(t)$,
   $\int_0^t m_i(s)\, ds < k_i , i = 1, 2 \ \ \forall t \in [0, T]$.

Lemma 3.1. *The solution of the nonlocal problem (1.1)-(1.4) can be expressed by the system of the integral equations*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a\, x_0 + \int_0^t f_1(s, y(s))\, ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, y(s)) ds \\ \\ b\, y_0 + \int_0^t f_2(s, x(s))\, ds - b \sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s, x(s)) ds \end{pmatrix},$$

where $\left(1 + \sum_{k=1}^{n} a_k\right)^{-1} = a, \quad \left(1 + \sum_{j=1}^{m} b_j\right)^{-1} = b.$

3.1 Existence of solution

Here, we study the existence of at least one solution of the nonlocal problem (1.1)-(1.4). Define the superposition operator F by

$$F\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ax_0 + \int_0^t f_1(s,y(s))ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y(s))ds \\ \\ by_0 + \int_0^t f_2(s,x(s))ds - b \sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s,x(s))ds \end{pmatrix} = \begin{pmatrix} F_1 y \\ F_2 x \end{pmatrix}.$$

Now we have the following theorem.

Theorem 3.4. *Consider the assumptions (i)-(ii) are satisfied, then there exists at least one solution of the nonlocal problem (1.1)-(1.4).*

Proof. Define the operator $F(x,y) = (F_1 x, F_2 y)$, where

$$F_1 y = a\, x_0 + \int_0^t f_1(s,y(s))\, ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y(s))\, ds,$$

$$F_2 x = b\, y_0 + \int_0^t f_2(s,x(s))\, ds - a \sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s,x(s))\, ds.$$

Now

$$\begin{aligned} | F_1 y | &= \left| a\, x_0 + \int_0^t f_1(s,y(s))\, ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y(s))\, ds \right| \\ &\le | ax_0 | + \int_0^t | f_1(s,y(s)) | ds + | a | \sum_{k=1}^{n} | a_k | \int_0^{\tau_k} | f_1(s,y(s)) | ds \\ &\le a | x_0 | + \int_0^t m_1(s)\, ds + a \sum_{k=1}^{n} | a_k | \int_0^{\tau_k} m_1(s)\, ds \\ &\le a | x_0 | + K_1 + a \sum_{k=1}^{n} a_k K_1 \le a | x_0 | + K_1(1 + a \sum_{k=1}^{n} a_k) \\ &\le a | x_0 | + K_1 \left(1 + \frac{\sum_{k=1}^{n} a_k}{1 + \sum_{k=1}^{n} a_k} \right) \le a | x_0 | + 2K_1 = M_1, \end{aligned}$$

then F_1 is uniformly bounded.
Similarly

$$| F_2x | \leq b | y_0 | + 2K_2 = M_2,$$

then F_2 is uniformly bounded.
Hence $\| F(x,y) \|_X = \| F_1y \| + \| F_2x \| \leq M_1 + M_2 = M,$
and then F is uniformly bounded.
For $t_1, t_2 \in (0,T]$, $t_1 < t_2$, let $| t_2 - t_1 | < \delta$, then

$$\begin{aligned} | F x(t_2) - F x(t_1) | &= | F_1y(t_2) - F_1y(t_1) | \\ &= \left| \int_0^{t_2} f_1(s,y(s))\, ds - \int_0^{t_1} f_1(s,y(s))\, ds \right| \\ &= \left| \int_{t_1}^{t_2} f_1(s,y(s))\, ds \right| \\ &\leq \int_{t_1}^{t_2} | f_1(s,y(s)) |\, ds \\ &\leq \int_{t_1}^{t_2} m_1(s)\, ds \leq \epsilon, \end{aligned}$$

then $\{F_1y\}$ is a class of equicontinuous functions.
Similarly

$$| F y(t_2) - F y(t_1) | = | F_2x(t_2) - F_2x(t_1) | \leq \int_{t_1}^{t_2} m_2(s)\, ds \leq \epsilon,$$

then $\{F_2x\}$ is a class of equicontinuous functions.
Therefore the operator F is equicontinuous and uniformly bounded.
Let
$\{y_N(t)\} \in C[0,T]$, $y_N(t) \to y(t)$, $\{x_N(t)\} \in C[0,T]$, $x_N(t) \to x(t)$,
So,

$$\lim_{N\to\infty} F_1(y_N) = \lim_{N\to\infty} \left(a\, x_0 + \int_0^t f_1(s,y_N(s))\, ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y_N(s))\, ds \right),$$

but $| f_i\,(s,\, y_N\,(s)) | \leq m_i$, and $f_i\,(s,\, y_N\,(s)) \to f_i\,(s,\, y(s))$
applying Lebesgue dominated convergence theorem [19], then we deduce that

$$\lim_{N\to\infty} \int_0^t f_1(s,y_N(s))ds = \int_0^t \lim_{N\to\infty} f_1(s,y_N(s))ds = \int_0^t f_1(s, \lim_{N\to\infty} y_N(s))ds = \int_0^t f_1(s,y(s))\, ds,$$

and

$$\lim_{N\to\infty} a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y_N(s))ds = a\sum_{k=1}^{n} a_k \lim_{N\to\infty}\int_0^{\tau_k} f_1(s,y_N(s))ds,$$

$$= a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \lim_{N\to\infty} f_1(s,y_N(s))\, ds,$$

$$= a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, \lim_{N\to\infty} y_N(s))\, ds,$$

$$= a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y(s))\, ds,$$

then

$$\lim_{N\to\infty} F_1(y_N) = a\, x_0 + \int_0^{t} f_1(s,y_N(s))\, ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y_N(s))\, ds = F_1 y.$$

This proves that $F_1 y$ is continuous operator,
Similarly, we can prove that

$$\lim_{N\to\infty} F_2(x_N) = a\, y_0 + \int_0^{t} f_2(s,x_N(s))\, ds - b\sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s,x_N(s))\, ds = F_2 x,$$

then $F_2 x$ is continuous operator.
Then $F : X \to X$ is continuous and compact.
Now we show that X is convex,
let $(x_1\,,\, y_1)\,,\, (x_2\,,\, y_2) \in X$

$$\|\ (x_i\,,\, y_i)\ \|_X = \|\ x_i\ \| + \|\ y_i\ \| < M,\ \ i = 1,2.$$

For $\lambda \in [0\,,\, T]$

$$\begin{aligned}
&\|\ \lambda\,(x_1\,,\, y_1) + (1-\lambda)\,(x_2\,,\, y_2)\ \|_X \\
&\quad = \|\ (\lambda\, x_1\,,\, \lambda\, y_1) + ((1-\lambda)\, x_2\,,\, (1-\lambda)\, y_2)\ \| \\
&\quad = \|\ (\lambda\, x_1 + (1-\lambda)\, x_2\,,\, \lambda\, y_1 + (1-\lambda)\, y_2)\ \| \\
&\quad \le \|\ \lambda\, x_1 + ((1-\lambda)\, x_2\ \| + \|\ \lambda\, y_1 + (1-\lambda)\, y_2)\ \| \\
&\quad \le \lambda\ \|\ x_1\ \| + (1-\lambda)\ \|\ x_2\ \| + \lambda\ \|\ y_1\ \| + (1-\lambda)\ \|\ y_2\ \| \\
&\quad = \lambda\,[\ \|\ x_1 + \|\ y_1\ \|\ ] + (1-\lambda)\,[\ \|\ x_2\ \| + \|\ y_2\ \|\ ] \\
&\quad \le \lambda\, M + (1-\lambda)\, M = M,
\end{aligned}$$

this means that X is convex.
Then F has a fixed point $(x\,,\, y) \in X$ which proves that there exists at least one solution of the nonlocal problem (1.1)-(1.4). □

4 Nonlocal Integral Condition

Let $a_k = (t_k - t_{k-1})$, $\tau_k \in (t_{k-1}, t_k)$, and $b_j = (t_j - t_{j-1})$, $\eta_j \in (t_{j-1}, t_j)$,
where $0 < t_1 < t_2 < t_3 < .... < 1$.
Then, the nonlocal conditions (1.3)-(1.4) will be in the form

$$x(0) + \sum_{k=1}^{n} (t_k - t_{k-1})\, x(\tau_k) = x_0, \qquad y(0) + \sum_{j=1}^{m} (t_j - t_{j-1})\, x(\eta_j) = y_0.$$

From the continuity of the solution of the nonlocal problem (1.1)-(1.4), we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n} (t_k - t_{k-1})\, x(\tau_k) = \int_0^T x(s)ds, \qquad \lim_{m\to\infty}\sum_{j=1}^{m} (t_j - t_{j-1})\, y(\eta_j) = \int_0^T y(s)ds,$$

that is, the nonlocal conditions (1.3)-(1.4) is transformed to the integral condition

$$x(0) + \int_0^T x(s)ds = x_0, \qquad y(0) + \int_0^T y(s)ds = y_0.$$

Now, we have the following theorem.

Theorem 4.5. *Let the assumption (i)-(ii) be satisfied, then the coupled system of differential equations (1.1) and (1.4) with the nonlocal integral condition (1.5)and(1.6) has at least one solution represented in the form*

$$U = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a^\star x_0 + \int_0^t f_1(\theta, y(\theta))\, d\theta - a^\star \int_0^T\int_0^s f_1(\theta, y(\theta)) d\theta ds \\ \\ a^\star y_0 + \int_0^t f_2(\theta, x(\theta))\, d\theta - a^\star \int_0^T\int_0^s f_2(\theta, x(\theta)) d\theta ds \end{pmatrix},$$

where $a^\star = (1+T)^{-1}$.

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Received: ?, 2014; *Accepted*: ?, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 363–372

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Mathematical Modelling for nutrient uptake by plant root which is considered as cylindrical

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Abstract

In this article, we drive mathematical model for nutrient uptake by the plant root which is considered as cylindrical, i.e, we obtain concentration of nutrient entering into the root surface by advection diffusion equation. The equation is written in the radial form and solved using Michal Menten boundary condition, which is nonlinear boundary condition. It is found that generally advection diffusion is solved taking Peclet number as zero, then equation reduces to the diffusion equation and solved by Laplace method[9]. But we solve the advection diffusion equation without taking Plect number as zero and solved by re-scaling and using separation of variable which reduces it into Bessel's equation. For particular solution, we use extreme parameters.

Keywords: Solution of advection diffusion equation, Re-scaling variable.

2010 MSC: 35XX-35K xx , 76XX-76Rxx, 76XX-7605, 76XX-76Zxx-76Z05, 92XX-92Bxx.

1 Introduction

The primary physiological function of root is uptaking the water as well as nutrients and transport to leaves for photosynthesis. Investigations and observation of the uptake of water and nutrient in plant root and stem can be traced back to many years ago, it possesses importance in point of view of agricultural production and economical development[3,7-9]. In traditional farming like planting and agricultural the mechanism of water and nutrients is invaluable for utilizing water and fertilizer for increasing production. Now a new trend of planting inedible plant, emerge on industrial basis. The view of planting inedible plant are prevent the salinization, desertification of soil, to clean pollution of heavy metals, radioelement and plant's mining. To collect the valuable metals, like gold, from soil by planting some plants whose roots possess a special capability of absorbing the valuable metals. The plants of genus Bauhina have many species out of which Bauhinia Variegata plant extract is analyzed and found it contain micro-particles of gold. Since ancient times Bauhinia Racemosa Lam. family: Caesalpinaceae has been an integral part of life in India. Leaves of Bauhinia Racemosa are traditionally used on occasion of Dashera festival as symbol of gold in India. Recently proved that Bauhina Racemosa extract also contain micro particles of gold. In recent years, a number of researchers from various fields, such as physics, applied mathematics and plant physiology, paid more attention to develop mathematical model for water and nutrient uptake. The outstanding work in this field is done by T.Roose and proposed a mathematical model for uptake of water and nutrient. Roose work is the development of Nye, Tinker and Barber model for water and nutrient uptake assuming that the root is an infinitely long cylinder. To develop Mathematical model, we first derive advection diffusion equation of nutrient transport in the groundwater and then try to solve the advection diffusion equation by transforming it

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into non-dimensional form and using Michal Menten boundary condition as boundary condition. We re-scale the equation and reduce into the Bessel's equation, so we write solution in terms of Bessel's function.

2 Uptake in saturated zone

The root surrounded by soil is mainly divided in three parts namely solid, liquid and gas. We indicate ϕ_l volume fraction of soil occupied by the liquid, ϕ_s volume fraction of soil occupied by the solid and ϕ_g volume fraction of soil occupied by gas. Other phases like microbes, mucigel etc are neglected. The conservation of soil volume equation is written as:

$$\phi_s + \phi_g + \phi_l = 1. \tag{2.1}$$

The porosity ϕ of the soil is defined as $\phi_g + \phi_l = \phi$ or $\phi = 1 - \phi_s$. Soil is described as fully saturated if the pore space is full of water, i.e. $\phi = \phi_l$. Nutrients in solid phase can be exchanged with the liquid phase and diffuse in the solid phase. The diffusion of ions in this phase is negligible, so we neglect it. Thus the equation for the ion the solid phase becomes

$$\frac{\partial c_s}{\partial t} = d_s. \tag{2.2}$$

Where c_s indicate the amount of ions in the solid form and d_s indicate the rate of liquid-solid inter-facial ion transport.

Nutrient comes in contact with surface of the root by flow of pore water in which diffusion of nutrient takes place. Then the equation for ions in the liquid phase is written as

$$\frac{\partial}{\partial t}(\phi_l c_l) + \nabla.(c_l \mathbf{u}) = \nabla.(\phi_l D \nabla c_l) + d_l, \tag{2.3}$$

where $\mathbf{u}$ is the Darcy flux of water in the soil, c_l is the nutrient concentration in the liquid phase of the soil D is the diffusion coefficient in the liquid phase of the soil and d_l is the rate of solid-liquid inter-facial ion transport. Addition of equation (2.2) and (2.3), we get

$$\frac{\partial}{\partial t}(\phi_l c_l + c_s) + \nabla.(c_l \mathbf{u}) = \nabla.(\phi_l D \nabla c_l) + d_s + d_l, \tag{2.4}$$

assuming mass conservation during the inter-facial transport of ions

$$d_s + d_l = 0. \tag{2.5}$$

Hence, the equation (2.4) in terms of c_l becomes,

$$(b + \phi_l)\frac{\partial c_l}{\partial t} + \nabla.(c_l \mathbf{u}) = \nabla.(\phi_l D \nabla c_l). \tag{2.6}$$

Noting $c_l = c$ and writing equation (2.6) in radial polar coordinates we get

$$(b + \phi_l)\frac{\partial c}{\partial t} - \frac{aV}{r}\frac{\partial c}{\partial r} = D\phi_l \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}), \tag{2.7}$$

where a is the radius of the root. The water flux is given by $\mathbf{u} = -\frac{aV}{r}$, which derives from the law of mass conservation for water, i.e, $\nabla.\mathbf{u} = 0$. The quantity V is the Darcy flux of water into the root.

3 Boundary condition

Root surface accept the nutrient up to a certain level even if the nutrient concentration in liquid increases indefinitely. It is also verified that the root surface accept nutrient up to a critical level(low) of nutrient in liquid phase near the root surface below which first it stop the uptake of nutrient and then start bleeding in the soil. The experimentally measured, heuristic Michaelis-Menten type nutrient uptake boundary condition is therefore given by, see [5]

$$\phi_l D\frac{\partial c}{\partial r} + Vc = \frac{F_m c}{K_m + c} - E, \tag{3.1}$$

at $r = a$. Where c indicate the concentration of nutrient in the liquid phase of the soil, K_m indicate the Michaelis-Menten constant that is equal to the root surface nutrient concentration when the flux of nutrient into the root is half of the maximum possible, F_m indicate the maximum flux of nutrient into the root, $E = \frac{F_m c_{min}}{K_m + c_{min}}$ where c_{min} indicate the minimum concentration when the roots stop the uptake of nutrients, and a is the radius of the root.

4 Initial Condition and boundary condition

Initial condition can be write as for $t = 0$

$$c = c_0 \; at \; t = 0 \; for \; a < r < \infty, \tag{4.1}$$

for later time

$$c \to c_0 \; as \; r \to \infty \; for \; t > 0. \tag{4.2}$$

5 Non-dimensionalisation of Nutrient Transport equation

Choosing time, space, and concentration-scale as follows and substitute in (2.7)

$$t = \frac{a^2(\phi_l + b)}{D\phi_l} t^*, \; r = ar^*, \; c = K_m c^*. \tag{5.1}$$

Where c^*, t^* *and* $r^*$are dimensionless nutrient concentration, time, and radial variables,respectively, we obtain (after dropping $^*s$) the following dimensionless model

$$\frac{\partial c}{\partial t} - P_e \frac{1}{r}\frac{\partial c}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}), \tag{5.2}$$

with boundary conditions

$$\frac{\partial c}{\partial r} + P_e c = \lambda \frac{c}{1+c} - \epsilon \; at \; r = 1. \tag{5.3}$$

$$c \to c_\infty \; as \; r \to \infty \; for \; t > 0, \tag{5.4}$$

the dimensionless initial condition is given by

$$c = c_\infty \; at \; t = 0 \; for \; 1 < r < \infty. \tag{5.5}$$

the dimensionless parameters in above equations are defined as

$$P_e = \frac{aV}{D\phi_l}, \; \lambda = \frac{F_m a}{DK_m\phi_l}, \; \epsilon = \frac{Ea}{DK_m\phi_l}, \; c_\infty = \frac{c_0}{K_m}. \tag{5.6}$$

equation (5.2) write as

$$\frac{\partial c}{\partial t} - (\frac{P_e + 1}{r})\frac{\partial c}{\partial r} = \frac{\partial^2 c}{\partial r^2}, \tag{5.7}$$

implies

$$\frac{\partial c}{\partial t} = (\frac{P_e + 1}{r})\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}, \tag{5.8}$$

re-scaling with $r = (1 + P_e)R$, then $\partial r = (1 + P_e)\partial R$. Then equation (5.8) become

$$(1 + P_e)\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial R^2} + \frac{1}{R}\frac{\partial c}{\partial R}, \tag{5.9}$$

Corresponding boundary condition changes

$$\frac{\partial c}{\partial R} + (1 + P_e)P_e c = \lambda(1 + P_e)[\frac{c}{1+c} - \epsilon], \; at \; R = \frac{1}{1 + P_e}, \tag{5.10}$$

for $\lambda = \frac{F_m a}{DK_m\phi_l}$ value of λ with large value of ϕ and small radius R we have

$$\lambda \equiv 0. \tag{5.11}$$

Then the boundary condition becomes

$$\frac{\partial c}{\partial R} + (1 + P_e)P_e c = 0, \tag{5.12}$$

Consider $c(R,t) = U(R)T(t)$ substituting in (5.9) and (5.12) then it becomes

$$\frac{1}{T}(1+P_e)\frac{\partial T}{\partial t} = \frac{1}{U}[\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}], \tag{5.13}$$

corresponding boundary condition becomes

$$\frac{\partial U}{\partial R} + (1+P_e)P_e U = 0. \tag{5.14}$$

From the equation (5.9) we can write

$$\frac{1}{T}(1+P_e)\frac{\partial T}{\partial t} = \frac{1}{U}[\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}] = -\beta^2. \tag{5.15}$$

We have the Bessel equation with boundary condition

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} + \beta^2 U = 0. \tag{5.16}$$

$$\frac{\partial U}{\partial R} + (1+P_e)P_e U = 0, \; at \; R = \frac{1}{1+P_e} \tag{5.17}$$

and

$$\frac{\partial T}{\partial t} = -\frac{\beta^2}{1+P_e}T. \tag{5.18}$$

$$c = c_\infty \; at \; t = 0 \; as \; 1 < R < \frac{1}{1+P_e} \tag{5.19}$$

Solution of Bessels equation is given by,

$$\begin{aligned} U(\beta,R) = {} & J_0(\beta R)[\beta Y_1(\beta\frac{1}{(1+P_e)}) + P_e(-1-P_e)Y_0(\beta\frac{1}{(1+P_e)})] \\ & -Y_0(\beta R)[\beta J_1(\beta\frac{1}{(1+P_e)}) + P_e(-1-P_e)J_0(\beta\frac{1}{1+P_e})], \end{aligned} \tag{5.20}$$

also

$$\begin{aligned} N(\beta) = {} & [\beta J_1(\beta\frac{1}{(1+P_e)}) + (-1-P_e)J_0(\beta\frac{1}{(1+P_e)})]^2 \\ & +[\beta Y_1(\beta\frac{1}{(1+P_e)}) + (-1-P_e)Y_0(\beta\frac{1}{(1+P_e)})]^2. \end{aligned} \tag{5.21}$$

Replacing R by $R = \frac{r}{(1+P_e)}$ in equation (5.20)
Above solution of Bessels equation become

$$\begin{aligned} U(\beta,r) = {} & J_0(\beta\frac{r}{(1+P_e)})[\beta Y_1(\beta\frac{1}{(1+P_e)}) + P_e(-1-P_e)Y_0(\beta\frac{1}{(1+P_e)})] \\ & -Y_0(\beta\frac{r}{(1+P_e)})[\beta J_1(\beta\frac{1}{(1+P_e)}) + P_e(-1-P_e)J_0(\beta\frac{1}{1+P_e})]. \end{aligned} \tag{5.22}$$

Then the complete solution is given by, see [4]

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\frac{1}{(1+P_e)}\beta^2 t} U(\beta,r) d\beta \int_{r'=1}^{\infty} r' U(\beta,r') c_\infty dr'. \tag{5.23}$$

Amount of nutrient absorb by root is given as, [1-2]

$$M = 2\pi r t \frac{\partial c}{\partial t}. \tag{5.24}$$

6 Steady state uptake of nutrient

Consider equation (5.7) with boundary condition (5.3) and (5.5) in steady state it takes the form

$$\frac{\partial^2 c}{\partial r^2} + \frac{(1+P_e)}{r}\frac{\partial c}{\partial r} = 0, \tag{6.25}$$

with the substitution $r = (1+P_e)R$ equation (6.1) changes to the form

$$\frac{\partial^2 c}{\partial R^2} + \frac{1}{R}(\frac{\partial c}{\partial R}) = 0. \tag{6.26}$$

With the assumption of section (5.1), λ approaches to zero and ϵ is of order zero then boundary condition for (5.12) is the equation changes to the form,

$$\frac{\partial c}{\partial R} + (1+P_e)P_e c = 0. \tag{6.27}$$

And initial condition changes to $c \rightarrow c_\infty \; as \; R \rightarrow \infty for t > 0$

$$c = c_\infty, \; at \; t = o \; for \; \frac{1}{1+P_e} < R < \infty, \tag{6.28}$$

we may take for large R as L Solution of equation (6.2) is given by, see [1-2],

$$c = A + BlogR. \tag{6.29}$$

We can find the arbitrary constant A and B by applying initial and boundary condition as follwes $\frac{B}{R} + (1+P_e)P_e(A + BlogR) = 0$ at $R = \frac{1}{1+P_e}$,

$$B = -\frac{(1+P_e)P_e c_\infty}{[(1+P_e) + (1+P_e)P_e \; log\frac{1}{L(1+P_e)}]}. \tag{6.30}$$

$$A = c_\infty + \frac{(1+P_e)P_e c_\infty}{[(1+P_e) + (1+P_e)P_e \; log\frac{1}{L(1+P_e)}]}logL. \tag{6.31}$$

Then the general solution for equation is given by
$c = c_\infty + \frac{(1+P_e)P_e c_\infty}{[(1+P_e)+(1+P_e)P_e \; log\frac{1}{L(1+P_e)}]}logL - \frac{(1+P_e)P_e c_\infty}{[(1+P_e)+(1+P_e)P_e \; log\frac{1}{L(1+P_e)}]}logR.$

$$c = c_\infty + \frac{P_e c_\infty}{[1 + P_e \; log\frac{1}{L(1+P_e)}]}logL - \frac{P_e c_\infty}{[1 + P_e \; log\frac{1}{L(1+P_e)}]}logR, \tag{6.32}$$

solution modified as

$$c(R) = c_\infty[1 + \frac{P_e log\frac{L}{R}}{[1 + P_e \; log\frac{1}{L(1+P_e)}]}], \tag{6.33}$$

replacing value of R is

$$c(r) = c_\infty[1 + \frac{P_e log\frac{L(1+P_e)}{r}}{[1 + P_e \; log\frac{1}{L(1+P_e)}]}]. \tag{6.34}$$

Solution of steady state advection diffusion equation is written as

$$c(r) = c_\infty[\frac{1 - P_e logr}{1 - P_e logL(1+P_e)}], \tag{6.35}$$

total nutrient uptake per unit length is given by

$$Q = -2\pi D c_\infty \frac{r - P_e}{1 - P_e logL(1+P_e)}. \tag{6.36}$$

7 Nutrient transport equation with $c_\infty << 1$ and $\epsilon < P_e << 1$

In this section we consider P_e, ϵ and c_∞ are negligible. If Michaelis-Menten coefficient K_∞ much larger than the far field concentration c_0 ,i.e., $c_\infty << 1$, the equation (5.2) reduces to the form

$$\frac{\partial c}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}). \tag{7.37}$$

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}. \tag{7.38}$$

Corresponding boundary condition reduces to the form

$$\frac{\partial c}{\partial r} = \lambda\frac{c}{1+c}, \tag{7.39}$$

re-scaling $c = c_\infty C$ then the model in scaled concentration is written as

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r}, \tag{7.40}$$

scaled boundary condition are as follows

$$\frac{\partial C}{\partial r} = \lambda\frac{C}{1+c_\infty C},\ r = 1\ and\ C \to 1\ as\ r \to \infty. \tag{7.41}$$

for $c_\infty << 1$ we can approximate the root surface boundary condition,using the binomial expansion,at the leading order given by

$$\frac{\partial C}{\partial r} \approx \lambda C\ at\ r = 1. \tag{7.42}$$

Initial condition scaled in following manner

$$C = 1\ at\ t = 0\ for\ 1 < r < \infty. \tag{7.43}$$

We solve the above boundary value problem by separation of the variables .substituting the substitution $C(r,t) = T(t)U(r)$ the value in equation(7.4) we have

$$\frac{1}{U}[\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}] = \frac{1}{T}[\frac{\partial T}{\partial t}] = -\beta^2. \tag{7.44}$$

Now consider the boundary value problem

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \beta^2 U = 0. \tag{7.45}$$

With the boundary condition

$$\frac{dU}{dr} - \lambda U = 0. \tag{7.46}$$

The complete solution is given by, see [4],

$$C(r,t) = \int_{\beta=0}^{\infty}\frac{\beta}{N(\beta)}e^{-\beta^2 t}U(\beta,r)d\beta\int_{r=1}^{\infty}r'U(\beta,r')dr', \tag{7.47}$$

where $U(\beta_m, r)$ is eigenvalue function.

$$U(\beta,r) = J_0(\beta r)[\beta Y_1(\beta) + \lambda Y_0(\beta)] - Y_0(\beta r)[\beta J_1(\beta) + \lambda J_0(\beta)]. \tag{7.48}$$

$$N(\beta) = [\beta J_1(\beta) + \lambda J_0(\beta)]^2 + [\beta Y_1(\beta) + \lambda Y_0(\beta)]^2. \tag{7.49}$$

So the general solution of equation is given by

$$c(r,t) = \int_{\beta=0}^{\infty}\frac{\beta}{N(\beta)}e^{-\beta^2 t}R(\beta,r)d\beta\int_{r=1}^{\infty}r'R(\beta,r')dr'. \tag{7.50}$$

8 Advection diffusion equation with Case $c_\infty << 1$and $\epsilon << 1$

With the very very small space concentration ϵ value is negligible for the advection diffusion equation (5.2) with boundary condition (5.3) can be reduced in the diffusion equation by re-scaling $r = (1 + P_e)R$ and $c = c_\infty C$

$$`(1+P_e)\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial R} + \frac{1}{R}\frac{\partial C}{\partial R}. \tag{8.51}$$

$$\frac{\partial C}{\partial R} + (1+P_e)P_e C = \lambda(1+P_e)[\frac{C}{1+c_\infty C}] \; at \; R = \frac{1}{1+P_e}, \tag{8.52}$$

for $c_\infty << 1$ we can approximate the root surface boundary condition,using the binomial expansion,at the leading order given by

$$\frac{\partial C}{\partial R} + (1+P_e)P_e C = \lambda(1+P_e)C \; at \; R = \frac{1}{1+P_e}. \tag{8.53}$$

$$\frac{\partial C}{\partial R} + [(1+P_e)P_e - \lambda(1+P_e)]C = 0 \; at \; R = \frac{1}{1+P_e}. \tag{8.54}$$

$$\frac{\partial C}{\partial R} + (1+P_e)(P_e - \lambda)C = 0 \; at \; R = \frac{1}{1+P_e}. \tag{8.55}$$

The complete solution is given by separation of variable as similar to equation (7.8) with the substitution $C(R,t) = U(R)T(t)$

$$C(R,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\beta^2 t} U(\beta, R) d\beta \int_{R=\frac{1}{1+P_e}}^{\infty} R' U(\beta, R') dR' \tag{8.56}$$

where $U(\beta_m, R)$ is solution of Bessel equation.

$$\begin{aligned} U(\beta,R) = {} & J_0(\beta R)[\beta Y_1(\beta\frac{1}{1+P_e}) + (1+P_e)(P_e-\lambda)Y_0(\beta R)] \\ & - Y_0(\beta r)[\beta J_1(\beta\frac{1}{1+P_e}) + (1+P_e)(P_e-\lambda)J_0(\beta\frac{1}{1+P_e})]. \end{aligned} \tag{8.57}$$

$$\begin{aligned} N(\beta) = {} & [\beta J_1(\beta\frac{1}{1+P_e}) + (1+P_e)(P_e-\lambda)J_0(\beta\frac{1}{1+P_e})]^2 \\ & + [\beta\frac{1}{1+P_e} Y_1(\beta) + (1+P_e)(P_e-\lambda)Y_0(\beta\frac{1}{1+P_e})]^2. \end{aligned} \tag{8.58}$$

Re-substituting value of $R = \frac{r}{1+P_e}$

$$\begin{aligned} U(\beta,r) = {} & J_0(\beta\frac{r}{1+P_e})[\beta Y_1(\beta\frac{1}{1+P_e}) + (1+P_e)(P_e-\lambda)Y_0(\beta R)] \\ & - Y_0(\beta r)[\beta J_1(\beta\frac{1}{1+P_e}) + (1+P_e)(P_e-\lambda)J_0(\beta\frac{1}{1+P_e})], \end{aligned} \tag{8.59}$$

so the general solution of equation is given by

$$c(r,t) = c_\infty \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\beta^2 t} U(\beta, r) d\beta \int_{r=1}^{\infty} r' U(\beta, r') dr'. \tag{8.60}$$

9 High Nutrient uptake for$\lambda >> 1$

If the gradient of nutrient concentration near root surface is high, i.e., $\frac{\partial c}{\partial r}|_{r=1} = \lambda >> 1$ for $c \sim O(1)$. Then re-scaling the independent variables r and t to stretched variables R and T i.e. $r = 1 + \frac{R}{\lambda}$ and $t = \frac{T}{\lambda^2}$, the problem reduces to

$$\frac{\partial c}{\partial T} = \frac{\partial^2 c}{\partial R^2} + \frac{1}{R+\lambda}\frac{\partial c}{\partial R}. \tag{9.61}$$

Which at the leading order simplifies to

$$\frac{\partial c}{\partial T} = \frac{\partial^2 c}{\partial R^2}, \tag{9.62}$$

since $\frac{1}{\lambda+R} << 1$ for $\lambda >> 1$. The re-scaled boundary conditions are

$$\frac{\partial c}{\partial R} = c \; at \; R = 0 \; and \; c \to 1 \; as \; R \to \infty, \tag{9.63}$$

and the initial condition is $c = 1$ at $T = 0$ for $0 < R < \infty$. Then the general solution to this leading order problem is given by

$$c(R,T) = erf(\frac{R}{2\sqrt{T}}) + e^{R+T} erfc(\frac{R}{2\sqrt{T}} + \sqrt{T}), \tag{9.64}$$

with the flux $F(T) = \frac{\partial c}{\partial R}\frac{\partial R}{\partial r}|_{R=0}$, of nutrient into the root given by

$$F(T) = \lambda e^T erfc(\sqrt{T}). \tag{9.65}$$

As $T \to \infty$, the concentration of nutrient at the surface $c \to 0$ and $F \to 0$, since $e^T erfc(\sqrt{T}) \to 0$ as $T \to \infty$.

10 Zero-sink Model

For $t > t_c \sim \frac{1}{\lambda^2}$ the root surface nutrient concentration has dropped to a very low level then we take the boundary condition at the root surface at the leading order to be $c = 0$ at $r = 1$, i.e, the problem to be solved is, see [6],

$$\frac{\partial c}{\partial t} + \frac{(-P_e)}{r}\frac{\partial c}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}) \tag{10.66}$$

$$c = 0 \; at \; r = 1 \; and \; c \to 1 \; as \; r \to \infty, \tag{10.67}$$

Let $q = P_e + 1$ the equation (10.1) becomes

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial r^2} + \frac{q}{r}\frac{\partial c}{\partial r}. \tag{10.68}$$

Using variable separation technique where λ is the separation constant yield

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{U}[\frac{\partial^2 U}{\partial r^2} + \frac{q}{r}\frac{\partial U}{\partial r}] = -\lambda. \tag{10.69}$$

Then above equation reduces to the equations

$$\frac{\partial T}{\partial t} + \lambda T = 0. \tag{10.70}$$

$$r\frac{\partial^2 U}{\partial r^2} + q\frac{\partial U}{\partial r} + r\lambda U = 0, \tag{10.71}$$

$$R(1) = 0 \tag{10.72}$$

the time function $T(t)$ is the exponential solution of equation (10.5) is

$$T_i(t) = e^{-\lambda_i t}. \tag{10.73}$$

The solution of spatial function $R(r)$ is obtained by power series method used for bessel equation

$$R_i(r) = \sum_{n=0}^{\infty} \frac{(-1)^n (r\sqrt{\lambda_i})^{2n}}{2^{2n-\gamma} n!.\Gamma(\nu - \gamma + 1).\lambda_i^{\frac{\gamma}{2}}} \; with \; \gamma = \frac{1-q}{2} = -\frac{P_e}{2}, \tag{10.74}$$

given solution can be represented using a negative γ -order Bessel function $J_{-\gamma}$ of the first kind. The separation constant λ_i of a specific problem is a scaled version of the general Bessel function roots to accommodate the boundary condition at r=1

$$R_i(r) = r^{\gamma}.J_{-\gamma}(r\sqrt{\lambda_i})_{r=1} = 0, \; \sqrt{\lambda_i} = s_i, \tag{10.75}$$

combining the spatial and time function solution we get desired solution as an infinite sum of eigenfunctions as

$$C(r,t) = \Sigma_{t=0}^{\infty}[A_i r^{\gamma}.J_{-\gamma}(r\sqrt{\lambda_i})e^{-\lambda_i t}]. \tag{10.76}$$

According to the Sturm-Liouville theory orthogonal base functions correspond to the weights r^q. The coefficient A_i can be adjusted using a Fourier-Bessel decomposition

$$A_i = \frac{\int_0^1 J(s_i r).r^{\gamma+q}dr}{\int_0^1 [J(s_i r)]^2.r^{2\gamma+q}dr}. \tag{10.77}$$

11 Zero-sink Model with $P_e << 1$

The equation (10.1) is reduced to the form as,

$$\frac{\partial c}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}). \tag{11.78}$$

$$c = 0 \; at \; r = 1 \; and \; c \to 1 \; as \; r \to \infty, \tag{11.79}$$

$c = 1 \; at \; t = 0 \; as \; 1 < r < \infty$.
Separating the variables solution for time-variable function is given by $e^{-\beta^2 t}$ and space variable function $U(\beta, r)$ is the solution of the following problem

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + \beta^2 U = 0 \; for \; 1 < r < \infty, \tag{11.80}$$

$$c = 0 \; at \; r = 1. \tag{11.81}$$

Then the complete solution for c(r, t) is constructed as

$$c(r,t) = \int_{\beta=0}^{\infty} C(\beta)e^{-\beta^2 t}R(\beta,r)d\beta, \tag{11.82}$$

with the application of initial condition we get

$$1 = \int_{\beta=0}^{\infty} c(\beta)U(\beta,r)d\beta \; in \; 1 < r < \infty, \tag{11.83}$$

using the orthogonality of eigenvalue functions we have

$$C(\beta) \equiv \frac{1}{N(\beta)}\beta\int_{r'=1}^{\infty} r'R(\beta,r')dr'. \tag{11.84}$$

Substituting equation (10.7) into equation (10.5) gives

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)}e^{-\beta^2 t}U(\beta,r)d\beta\int_{r'=1}^{\infty} r'R(\beta,r')dr'. \tag{11.85}$$

Where

$$U(\beta,r) = J_0(\beta r)Y_0(\beta) - Y_0(\beta r)J_0(\beta), \tag{11.86}$$

and

$$N(\beta) = [J_0^2(\beta) + Y_0^2(\beta)]. \tag{11.87}$$

Then complete integral is given by

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{J_0^2(\beta)+Y_0^2(\beta)}e^{-\beta^2 t}[Y_0(\beta r)J_0(\beta)]d\beta - J_0(\beta r)Y_0(\beta) \\ .\int_{r'=1}^{\infty} r'[J_0(\beta r')Y_o(\beta) - Y_0(\beta r')J_0(\beta)]dr'. \tag{11.88}$$

12 Conclusion

We solved radial advection diffusion by re-scaling and reduced it by separation of variables into Bessel's equation rather than Laplace method used in [9], in which whenever Laplace method is used for solving advection diffusion, we have to choose always $P_e << 1$. The method used in this article is one of the best alternative to Laplace method used in [9] and not always necessary to choose $P_e << 1$ due to which it reducing the advection diffusion equation into diffusion form.

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Received: February 05, 2014; *Accepted*: May 20, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 373–391

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Analytical solutions of incomplete elliptic integrals motivated by the work of Ramanujan

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Abstract

In this paper, we obtain exact solutions of some unsolved incomplete elliptic integrals of first, second and third kinds, given in Entry 7 of Chapter XVII of second notebook of Srinivasa Ramanujan. Furthermore, we generalize these elliptic integrals in the forms of multiple series identities involving bounded multiple sequences.

Keywords: Gaussian Hypergeometric Function; Incomplete Elliptic Integrals; Multiple Series Identities; Srivastava-Daoust double Hypergeometric Function ; Kampé de Fériet double Hypergeometric Function; Srivastava's Triple Hypergeometric Function.

2010 MSC: 33B10, 33C05,, 33C65.

1 Introduction and Preliminaries

Some Interesting Series Identities

We recall the following identities which are potentially useful in the series rearrangement techniques.

$$\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\Theta(m,\ r)=\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ r) \tag{1.1}$$

$$\sum_{m=0}^{\infty}\sum_{r=0}^{2m-1}\Theta(m,\ r)=\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ r)+\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ 2r+m+1) \tag{1.2}$$

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$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n-1}\Psi(m,\ n,\ r)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(0,\ n+r+1,\ r)+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+r+1,\ n,\ r)+$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+1,\ n+r+1,\ r+m+1) \tag{1.3}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n}\Psi(m,\ n,\ r)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+r,\ n,\ r)+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m,\ n+r+1,\ r+m+1) \tag{1.4}$$

where $\{\Theta(m,r)\}_{m,r=0}^{\infty}$ and $\{\Psi(m,n,r)\}_{m,n,r=0}^{\infty}$ are suitably bounded double and triple sequences of essentially arbitrary(real or complex) parameters respectively.

Legendre's Normal Forms of Incomplete Elliptic Integrals

Following integrals arise in the solutions of certain classes of vibration problems and problems involving computations of the radiation field off axis from a uniform circular disc radiating according to an arbitrary angular distribution law.

Following elliptic integrals (R.H.S.) have been represented in different notations (L.H.S.) by researchers

First Kind :
$$F(x,\ \phi)=\int_0^{\phi}\frac{d\theta}{\sqrt{(1-x^2\sin^2\theta)}} \tag{1.5}$$

Second Kind :
$$E(x,\ \phi)=\int_0^{\phi}\sqrt{(1-x^2\sin^2\theta)}\,d\theta \tag{1.6}$$

Third Kind :
$$\prod(a,\ x,\ \phi)=\int_0^{\phi}\frac{d\theta}{(1-a\sin^2\theta)\sqrt{(1-x^2\sin^2\theta)}} \tag{1.7}$$

where $\quad 0\le\ x\le\ 1,\ 0\le\ \phi\le\ \frac{\pi}{2},\ -\infty<a<\infty,\ a\neq 1$

The integrands of elliptic integrals are periodic functions with a period π. Here x, ϕ and a are called modulus, amplitude and characteristic parameter respectively. In case $x=\sin\delta$, δ is called modular angle.

Some Useful Indefinite Integrals

When $m=0,1,2,3,\cdots,$ then

$$\int\sin^{2m}(c\theta)d\theta=\left\{\frac{-(\frac{1}{2})_m\sin(c\theta)\cos(c\theta)}{(1)_m\ c}\sum_{r=0}^{m-1}\frac{(1)_r\sin^{2r}(c\theta)}{(\frac{3}{2})_r}\right\}+\left\{\frac{\theta\,(\frac{1}{2})_m}{(1)_m}\right\}+Constant \tag{1.8}$$

$$\int\cos^{2m}(c\theta)d\theta=\left\{\frac{(\frac{1}{2})_m\sin(c\theta)\cos(c\theta)}{(1)_m\ c}\sum_{r=0}^{m-1}\frac{(1)_r\cos^{2r}(c\theta)}{(\frac{3}{2})_r}\right\}+\left\{\frac{\theta\,(\frac{1}{2})_m}{(1)_m}\right\}+Constant \tag{1.9}$$

$$\int\sin^{2m+1}(c\theta)d\theta=\frac{-(1)_m\cos(c\theta)}{(\frac{3}{2})_m\ c}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r\sin^{2r}(c\theta)}{(1)_r}+Constant \tag{1.10}$$

$$\int\cos^{2m+1}(c\theta)d\theta=\frac{(1)_m\sin(c\theta)}{(\frac{3}{2})_m\ c}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r\cos^{2r}(c\theta)}{(1)_r}+Constant \tag{1.11}$$

Above formulas (1.8)-(1.11) can be verified for $m=0,1,2,3,\cdots$ and it is the convention that the empty sum $\sum_{r=0}^{-1}F(r)$ is treated as zero.

2 Seventh entry of Chapter Seventeenth of Second Notebook of Ramanujan[9,pp.104-107,pp.112-113]

Ramanujan's notebooks have been divided into several chapters and contains large numbers of important and useful results on elliptic integrals. Many results on elliptic integrals have been proved by B. C. Berndt [8, pp.104-113] and R. Y. Denis *et al.*[15].

We have also verified the following entries of Ramanujan by using the methods of B. C. Berndt and R. Y. Denis *et al.*

Entry 7(i): If $\sin\alpha = \sqrt{x}\,\sin\beta$ or $\frac{\tan\alpha}{\tan\beta} = \frac{\sqrt{x}\,\cos\beta}{\sqrt{(1-x\sin^2\beta)}}$, then

$$\int_0^\alpha \frac{d\theta}{\sqrt{(x-\sin^2\theta)}} = \int_0^\beta \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \tag{2.1}$$

In a paper of Denis *et al.* [15, p.113(1)], a misprint condition is written in Entry 7(i).

Entry 7(ii): If $\tan\alpha = \sqrt{(1-x)}\,\tan\beta$ or $\sin\alpha = \frac{\sqrt{(1-x)}\,\sin\beta}{\sqrt{(1-x\,\sin^2\beta)}}$, then

$$\int_0^\alpha \frac{d\theta}{\sqrt{(1-x\cos^2\theta)}} = \int_0^\beta \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \tag{2.2}$$

Entry 7(iii): If $\tan\alpha = \sqrt{(1-b)}\,\tan\beta$ or $\sin\beta = \frac{\sin\alpha}{\sqrt{(1-b\cos^2\alpha)}}$, then

$$\int_0^\alpha \frac{d\theta}{\sqrt{\{1-(\frac{a-b}{1-b})\sin^2\theta\}}} = \sqrt{(1-b)}\int_0^\beta \frac{d\theta}{\sqrt{(1-a\sin^2\theta)\,(1-b\sin^2\theta)}} \tag{2.3}$$

Entry 7(iv): If $\tan\alpha = \sqrt{(1+x)}\,\tan\beta$, then

$$\int_0^\alpha \frac{d\theta}{\sqrt{(1+x\cos 2\theta)}} = \int_0^\beta \frac{d\theta}{\sqrt{(1-x^2\sin^4\theta)}} \tag{2.4}$$

which is the correct form of a misprint result of Denis *et al.* [15, p.115(4)].

Entry 7(v): Degenerate form of addition theorem

If $\cot\alpha\,\cot\beta = \sqrt{(1-x)}$, then

$$\int_0^\alpha \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} + \int_0^\beta \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \frac{\pi}{2}\,{}_2F_1\left[\begin{array}{lll}\frac{1}{2},\frac{1}{2} & ; & \\ & & x\\ 1 & ; & \end{array}\right] \tag{2.5}$$

Entry 7(vi): Classical duplication formula (Special Case of the converse of Entry 7(viii)a)

If $\cot\alpha\,\tan(\frac{\beta}{2}) = \sqrt{(1-x\sin^2\alpha)}$, then

$$2\int_0^\alpha \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \int_0^\beta \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \tag{2.6}$$

Entry 7(vii): Jacobi's imaginary transformation (For incomplete elliptic integral of first kind having imaginary amplitude)

If $\alpha = \ln\{\tan(\frac{\pi}{4}+\frac{\beta}{2})\}$ or $e^{\alpha} = \frac{\cos\beta}{1-\sin\beta} = \frac{1+\sin\beta}{\cos\beta}$ or $\sinh(\alpha) = \tan\beta$, then

$$\int_0^{i\alpha} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = i\int_0^{\beta} \frac{d\theta}{\sqrt{(1-(1-x)\sin^2\theta)}} \tag{2.7}$$

which is the correct form of a misprint result of Denis *et al.* [15, p.116(7)].

Entry 7(viii): Addition theorem

If

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} + \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \int_0^{\gamma} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \tag{2.8}$$

then four different sets of hypothesis (implications) are given by

$$(a)\ \ \tan\left(\frac{\gamma}{2}\right) = \frac{\sin\alpha\sqrt{(1-x\sin^2\beta)}+\sin\beta\sqrt{(1-x\sin^2\alpha)}}{\cos\alpha+\cos\beta}$$

which is the correct form of a misprint condition of Denis *et al.* [15, p.117(i)].

$$(b)\ \ \gamma = \tan^{-1}\{\tan\alpha\sqrt{(1-x\sin^2\beta)}\} + \tan^{-1}\{\tan\beta\sqrt{(1-x\sin^2\alpha)}\}$$

(c) Legendre's canonical form of the addition theorem

$$\cot\alpha\ \cot\beta = \frac{\cos\gamma}{\sin\alpha\sin\beta} + \sqrt{(1-x\sin^2\gamma)}$$

$$(d)\ \ \frac{\sqrt{x}}{2} = \frac{\sqrt{\{\sin(s)\ \sin(s-\alpha)\ \sin(s-\beta)\ \sin(s-\gamma)\}}}{\sin\alpha\ \sin\beta\ \sin\gamma}\ \ \text{;where}\ \ s = \frac{\alpha+\beta+\gamma}{2}$$

The four different sets of hypothesis (implications) $(a)-(d)$ in Entry 7(viii) are both necessary and sufficient.

Entry 7(xii): Gauss' transformation

If $(1+x\sin^2\alpha)\sin\beta = (1+x)\ \sin\alpha$, then

$$(1+x)\int_0^{\alpha} \frac{d\theta}{\sqrt{(1-x^2\sin^2\theta)}} = \int_0^{\beta} \frac{d\theta}{\sqrt{\{1-\frac{4x}{(1+x)^2}\sin^2\theta\}}} \tag{2.9}$$

which is the correct form of misprint result of Denis *et al.* [15, p.118(9)].

Entry 7(xiii): Landen's transformation (i.e. The first geometric representation)

If $x\sin\alpha = \sin(2\beta-\alpha)$, then

$$(1+x)\int_0^{\alpha}\frac{\mathrm{d}\theta}{\sqrt{(1-x^2\sin^2\theta)}}=2\int_0^{\beta}\frac{\mathrm{d}\theta}{\sqrt{\{1-\frac{4x}{(1+x)^2}\sin^2\theta\}}} \tag{2.10}$$

Entries 7(xii) and 7(xiii) are very similar in appearance.

3 A General Family of Multiple-Series Identities

Theorem 3.1. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^{2m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$$
$$=-\frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\,y^m(y\sin^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m y^m}{(m!)^2} \tag{3.1}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.2. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cos^{2m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$$
$$=\frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\,y^m(y\cos^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m y^m}{(m!)^2} \tag{3.2}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.3. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^{4m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}=\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{4})_m(\frac{3}{4})_m y^m}{(\frac{1}{2})_m(m!)^2}-$$
$$-\frac{3y\sin(c\gamma)\cos(c\gamma)}{8c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(1)_r\,y^m(y\sin^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_r}-$$
$$-\frac{y\sin^3(c\gamma)\cos(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(2)_{m+2r}\,(y\sin^2(c\gamma))^m(y\sin^4(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{5}{2})_{m+2r}} \tag{3.3}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.4. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cos^{4m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}=\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{4})_m(\frac{3}{4})_m y^m}{(\frac{1}{2})_m(m!)^2}+$$
$$+\frac{3y\sin(c\gamma)\cos(c\gamma)}{8c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(1)_r\,y^m(y\cos^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_r}+$$
$$+\frac{y\sin(c\gamma)\cos^3(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(2)_{m+2r}\,(y\cos^2(c\gamma))^m(y\cos^4(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{5}{2})_{m+2r}} \tag{3.4}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.5. *Let* $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ *be a suitably bounded double sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\left(\int_0^{\gamma}\sin^{2m+2n}(c\theta)\mathrm{d}\theta\right)\frac{y^m z^n}{m!\; n!}$$

$$= \gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(\frac{1}{2})_{m+n}y^m z^n}{(m+n)!\; m!\; n!} - \frac{z\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{0,n+r+1}\frac{(\frac{3}{2})_{n+r}(1)_r\, z^n(z\sin^2(c\gamma))^r}{(2)_{n+r}(2)_{n+r}(\frac{3}{2})_r} -$$
$$- \frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r+1,n}\frac{(\frac{3}{2})_{m+n+r}(1)_r\, y^m z^n(y\sin^2(c\gamma))^r}{(2)_{m+n+r}(2)_{m+r}(\frac{3}{2})_r(1)_n} -$$
$$- \frac{yz\sin^3(c\gamma)\cos(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+1,n+r+1}\frac{(\frac{5}{2})_{m+n+r}(2)_{m+r}\,(y\sin^2(c\gamma))^m z^n(z\sin^2(c\gamma))^r}{(3)_{m+n+r}(\frac{5}{2})_{m+r}(2)_{n+r}(2)_m} \tag{3.5}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.6. *Let* $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ *be a suitably bounded double sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\left(\int_0^{\gamma}\cos^{2m+2n}(c\theta)\mathrm{d}\theta\right)\frac{y^m z^n}{m!\; n!}$$

$$= \gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(\frac{1}{2})_{m+n}y^m z^n}{(m+n)!\; m!\; n!} + \frac{z\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{0,n+r+1}\frac{(\frac{3}{2})_{n+r}(1)_r\, z^n(z\cos^2(c\gamma))^r}{(2)_{n+r}(2)_{n+r}(\frac{3}{2})_r} +$$
$$+ \frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r+1,n}\frac{(\frac{3}{2})_{m+n+r}(1)_r\, y^m z^n(y\cos^2(c\gamma))^r}{(2)_{m+n+r}(2)_{m+r}(\frac{3}{2})_r(1)_n} +$$
$$+ \frac{yz\sin(c\gamma)\cos^3(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+1,n+r+1}\frac{(\frac{5}{2})_{m+n+r}(2)_{m+r}\,(y\cos^2(c\gamma))^m z^n(z\cos^2(c\gamma))^r}{(3)_{m+n+r}(\frac{5}{2})_{m+r}(2)_{n+r}(2)_m} \tag{3.6}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.7. *Let* $\{\Omega_m\}_{m=0}^{\infty}$ *be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^m(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$$

$$= \gamma\sum_{m=0}^{\infty}\Omega_{2m}\frac{y^{2m}}{4^m\,(m!)^2} - \frac{y^2\sin(c\gamma)\cos(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+2}\frac{(1)_r\, y^{2m}(y\sin(c\gamma))^{2r}}{4^{m+r}(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r} +$$
$$+\frac{y}{c}\sum_{m=0}^{\infty}\Omega_{2m+1}\frac{y^{2m}}{4^m(\frac{3}{2})_m(\frac{3}{2})_m} - \frac{y\cos(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+1}\frac{(\frac{1}{2})_r\, y^{2m}(y\sin(c\gamma))^{2r}}{4^{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_{m+r}(1)_r} \tag{3.7}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.8. *Let* $\{\Omega_m\}_{m=0}^{\infty}$ *be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cos^m(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$$

$$= \gamma\sum_{m=0}^{\infty}\Omega_{2m}\frac{y^{2m}}{4^m\,(m!)^2} + \frac{y^2\sin(c\gamma)\cos(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+2}\frac{(1)_r\, y^{2m}(y\cos(c\gamma))^{2r}}{4^{m+r}(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r} +$$
$$+\frac{y\sin(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+1}\frac{(\frac{1}{2})_r\, y^{2m}(y\cos(c\gamma))^{2r}}{4^{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_{m+r}(1)_r} \tag{3.8}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.9. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^{2m+1}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}=\frac{1}{c}\sum_{m=0}^{\infty}\Omega_m\frac{y^m}{(\frac{3}{2})_m}-\frac{\cos(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r}\frac{(\frac{1}{2})_r\,y^m(y\sin^2(c\gamma))^r}{(\frac{3}{2})_{m+r}(1)_r}\tag{3.9}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.10. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cos^{2m+1}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}=\frac{\sin(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r}\frac{(\frac{1}{2})_r\,y^m(y\cos^2(c\gamma))^r}{(\frac{3}{2})_{m+r}(1)_r}\tag{3.10}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.11. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers. Then*

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\left(\int_0^{\gamma}\sin^{2m+2n+1}(c\theta)\mathrm{d}\theta\right)\frac{y^mz^n}{m!\,n!}\\&=-\frac{\cos(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r,n}\frac{(1)_{m+n+r}(\frac{1}{2})_r\,y^m\,z^n(y\sin^2(c\gamma))^r}{(\frac{3}{2})_{m+n+r}(1)_{m+r}(1)_n\,(1)_r}-\\&-\frac{z\cos(c\gamma)\sin^2(c\gamma)}{3c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m,n+r+1}\frac{(2)_{m+n+r}(\frac{3}{2})_{m+r}\,(y\sin^2(c\gamma))^m\,z^n\,(z\sin^2(c\gamma))^r}{(\frac{5}{2})_{m+n+r}(2)_{n+r}(2)_{m+r}\,m!}+\\&+\frac{1}{c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(1)_{m+n}\,y^m\,z^n}{(\frac{3}{2})_{m+n}\,m!\,n!}\end{aligned}\tag{3.11}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.12. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers. Then*

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\left(\int_0^{\gamma}\cos^{2m+2n+1}(c\theta)\mathrm{d}\theta\right)\frac{y^mz^n}{m!\,n!}\\&=\frac{\sin(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r,n}\frac{(1)_{m+n+r}(\frac{1}{2})_r\,y^m\,z^n(y\cos^2(c\gamma))^r}{(\frac{3}{2})_{m+n+r}(1)_{m+r}(1)_n\,(1)_r}+\\&+\frac{z\sin(c\gamma)\cos^2(c\gamma)}{3c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m,n+r+1}\frac{(2)_{m+n+r}(\frac{3}{2})_{m+r}(y\cos^2(c\gamma))^m\,z^n\,(z\cos^2(c\gamma))^r}{(\frac{5}{2})_{m+n+r}(2)_{m+r}(2)_{n+r}\,m!}\end{aligned}\tag{3.12}$$

provided that each of the series involved is absolutely convergent.

Proof of (3.1) : $\displaystyle\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^{2m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$

$$\begin{aligned}&=-\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\Omega_m\frac{\sin(c\gamma)\cos(c\gamma)(\frac{1}{2})_m(1)_r\,y^m(\sin^2(c\gamma))^r}{c(1)_m(1)_m(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_my^m}{(m!)^2}\\&=-\sum_{m=0}^{\infty}\sum_{r=0}^{m}\Omega_{m+1}\frac{\sin(c\gamma)\cos(c\gamma)(\frac{1}{2})_{m+1}(1)_r\,y^{m+1}(\sin^2(c\gamma))^r}{c(1)_{m+1}(1)_{m+1}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_my^m}{(m!)^2}\\&=-\frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\,y^m(y\sin^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_my^m}{(m!)^2}\end{aligned}$$

Similarly we can derive (3.2) to (3.12) by means of series identities (1.1) to (1.4).

4 Hypergeometric Generalizations of Incomplete Elliptic Integrals of Ramanujan and their solutions

Putting $c = 1$ in theorems (3.1) to (3.6), (3.9) to (3.12) and setting

$$\Omega_m = \frac{(a_1)_m(a_2)_m(a_3)_m\cdots(a_A)_m}{(b_1)_m(b_2)_m(b_3)_m\cdots(b_B)_m} = \frac{((a_A))_m}{((b_B))_m},\ \Lambda_{m,n} = \frac{((a_A))_{m+n}((d_D))_m((g_G))_n}{((b_B))_{m+n}((e_E))_m((h_H))_n},$$

in theorems (3.1) to (3.12), using some algebraic properties of Pochhammer symbol and interpreting the multiple power series in hypergeometric forms of Gauss [34, p. 42(1)], Kampé de Fériet [34, p.63(16); see also 33], Srivastava [34, p.69(39,40)] and Srivastava-Daoust [33, p.37(21, 22, 23); 34, pp.64-65(18, 19, 20)], we get the analytical solutions of generalized incomplete elliptic integrals.

$$\int_0^\gamma {}_AF_B\left[\begin{array}{ll}(a_j)_{j=1}^A ; & \\ & y\sin^2\theta \\ (b_j)_{j=1}^B ; & \end{array}\right]\mathrm{d}\theta = \gamma\ {}_{A+1}F_{B+1}\left[\begin{array}{ll}\frac{1}{2},(a_j)_{j=1}^A ; & \\ & y \\ 1,(b_j)_{j=1}^B ; & \end{array}\right] - $$
$$- \frac{y\sin\gamma\ \cos\gamma\prod_{i=1}^A(a_i)}{2\prod_{i=1}^B(b_i)} F^{A+1:1;2}_{B+2:0;1}\left[\begin{array}{ll}\frac{3}{2},(1+a_j)_{j=1}^A : 1 ; 1,1 ; & \\ & y, y\sin^2\gamma \\ 2,2,(1+b_j)_{j=1}^B : - ; \frac{3}{2} ; & \end{array}\right] \tag{4.1}$$

$$\int_0^\gamma {}_AF_B\left[\begin{array}{ll}(a_j)_{j=1}^A ; & \\ & y\cos^2\theta \\ (b_j)_{j=1}^B ; & \end{array}\right]\mathrm{d}\theta = \gamma\ {}_{A+1}F_{B+1}\left[\begin{array}{ll}\frac{1}{2},(a_j)_{j=1}^A ; & \\ & y \\ 1,(b_j)_{j=1}^B ; & \end{array}\right] + $$
$$+ \frac{y\sin\gamma\ \cos\gamma\prod_{i=1}^A(a_i)}{2\prod_{i=1}^B(b_i)} F^{A+1:1;2}_{B+2:0;1}\left[\begin{array}{ll}\frac{3}{2},(1+a_j)_{j=1}^A : 1 ; 1 , 1 ; & \\ & y, y\cos^2\gamma \\ 2,2,(1+b_j)_{j=1}^B : - ; \frac{3}{2} ; & \end{array}\right] \tag{4.2}$$

$$\int_0^\gamma {}_AF_B\left[\begin{array}{ll}(a_j)_{j=1}^A ; & \\ & y\sin^4\theta \\ (b_j)_{j=1}^B ; & \end{array}\right]\mathrm{d}\theta = \gamma\ {}_{A+2}F_{B+2}\left[\begin{array}{ll}\frac{1}{4},\frac{3}{4},(a_j)_{j=1}^A ; & \\ & y \\ 1,\frac{1}{2},(b_j)_{j=1}^B ; & \end{array}\right] - $$
$$- \frac{3y\ \sin\gamma\cos\gamma\prod_{i=1}^A(a_i)}{8\prod_{i=1}^B(b_i)} F^{A+2:1;2}_{B+3:0;1}\left[\begin{array}{ll}\frac{5}{4},\frac{7}{4},(1+a_j)_{j=1}^A : 1; 1,1 ; & \\ & y,\ y\sin^2\gamma \\ 2,2,\frac{3}{2},(1+b_j)_{j=1}^B : - ; \frac{3}{2} ; & \end{array}\right] - $$
$$- \frac{y\sin^3\gamma\ \cos\gamma\prod_{i=1}^A(a_i)}{4\prod_{i=1}^B(b_i)} F^{A+3:1;1}_{B+4:0;0}\left(\begin{array}{l}[(1+a_j):1,1]_{j=1}^A,\ [\frac{5}{4}:1,1],\ [\frac{7}{4}:1,1],\ [2:1,2] \quad : \\ \\ [(1+b_j):1,1]_{j=1}^B,\ [2:1,1],\ [2:1,1],\ [\frac{3}{2}:1,1],\ [\frac{5}{2}:1,2] \ :\end{array}\right.$$

$$\left.\begin{matrix} [1{:}1]\,;[1{:}1] & ; & \\ & & y\sin^2\gamma,\, y\sin^4\gamma \\ \text{—}\,;\text{—} & ; & \end{matrix}\right) \tag{4.3}$$

$$\int_0^\gamma {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^A & ; & \\ & & y\cos^4\theta \\ (b_j)_{j=1}^B & ; & \end{matrix}\right]\mathrm{d}\theta=\gamma\ {}_{A+2}F_{B+2}\left[\begin{matrix}\frac14,\frac34,(a_j)_{j=1}^A & ; & \\ & & y \\ 1,\frac12,(b_j)_{j=1}^B & ; & \end{matrix}\right]+$$

$$+\frac{3y\ \sin\gamma\cos\gamma\prod_{i=1}^A(a_i)}{8\prod_{i=1}^B(b_i)}F^{A+2:1;2}_{B+3:0;1}\left[\begin{matrix}\frac54,\frac74,(1+a_j)_{j=1}^A\ :\ 1;1,1 & ; & \\ & & y,\,y\cos^2\gamma \\ 2,2,\frac32,(1+b_j)_{j=1}^B:\text{—};\frac32 & ; & \end{matrix}\right]+$$

$$+\frac{y\sin\gamma\ \cos^3\gamma\prod_{i=1}^A(a_i)}{4\prod_{i=1}^B(b_i)}F^{A+3:1;1}_{B+4:0;0}\left(\begin{matrix}[(1+a_j){:}1,1]_{j=1}^A,\ [\frac54{:}1,1],\ [\frac74{:}1,1],\ [2{:}1,2] & : \\ [(1+b_j)\ {:}1,1]_{j=1}^B,\ [2{:}1,1],\ [2{:}1,1],\ [\frac32{:}1,1],\ [\frac52{:}1,2] & :\end{matrix}\right.$$

$$\left.\begin{matrix} [1{:}1]\,;[1{:}1] & ; & \\ & & y\cos^2\gamma,\, y\cos^4\gamma \\ \text{—}\,;\text{—} & ; & \end{matrix}\right) \tag{4.4}$$

$$\int_0^\gamma F^{A:D;G}_{B:E;H}\left[\begin{matrix}(a_j)_{j=1}^A:(d_j)_{j=1}^D;(g_j)_{j=1}^G & ; & \\ & & y\sin^2\theta,\,z\sin^2\theta \\ (b_j)_{j=1}^B:(e_j)_{j=1}^E;(h_j)_{j=1}^H & ; & \end{matrix}\right]\mathrm{d}\theta$$

$$=\gamma\ F^{A+1:D;G}_{B+1:E;H}\left[\begin{matrix}(a_j)_{j=1}^A,\ \frac12:(d_j)_{j=1}^D;(g_j)_{j=1}^G & ; & \\ & & y,\,z \\ (b_j)_{j=1}^B,\ 1:(e_j)_{j=1}^E;(h_j)_{j=1}^H & ; & \end{matrix}\right]-\frac{z\ \sin\gamma\ \cos\gamma\prod_{i=1}^A(a_i)\ \prod_{i=1}^G(g_i)}{2\prod_{i=1}^B(b_i)\ \prod_{i=1}^H(h_i)}\times$$

$$\times\ F^{A+G+1:1;2}_{B+H+2:0;1}\left[\begin{matrix}\frac32,\ (1+a_j)_{j=1}^A,\ (1+g_j)_{j=1}^G\ \ :\ 1\,;1,1 & ; & \\ & & z,\,z\ \sin^2\gamma \\ 2,\,2,\ (1+b_j)_{j=1}^B,\ (1+h_j)_{j=1}^H:\text{—};\ \frac32 & ; & \end{matrix}\right]-$$

$$-\frac{y\ \sin\gamma\ \cos\gamma\prod_{i=1}^A(a_i)\ \prod_{i=1}^D(d_i)}{2\prod_{i=1}^B(b_i)\ \prod_{i=1}^E(e_i)}\times$$

$$\times\ F^{(3)}\left[\begin{matrix}\frac32,(1+a_j)_{j=1}^A::\text{—};\text{—};(1+d_j)_{j=1}^D\ \ :1\,;(g_j)_{j=1}^G;1,1 & ; & \\ & & y,\,z,\,y\sin^2\gamma \\ 2,\ (1+b_j)_{j=1}^B::\text{—};\text{—};(1+e_j)_{j=1}^E,2{:}\text{—};(h_j)_{j=1}^H;\frac32 & ; & \end{matrix}\right]-$$

$$-\frac{y\,z\,\sin^3\gamma\,\cos\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{A}(1+a_i)\,\prod_{i=1}^{D}(d_i)\,\prod_{i=1}^{G}(g_i)}{4\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{B}(1+b_i)\,\prod_{i=1}^{E}(e_i)\,\prod_{i=1}^{H}(h_i)}\times$$

$$\times F^{(3)}\left[\begin{matrix}\frac{5}{2},(2+a_j)_{j=1}^{A}::—;(1+g_j)_{j=1}^{G}\;\;;2:1,(1+d_j)_{j=1}^{D};1;1\;\;; \\ \\ 3,\,(2+b_j)_{j=1}^{B}::—;(1+h_j)_{j=1}^{H},2;\frac{5}{2}:2,(1+e_j)_{j=1}^{E};—;—\;\;;\end{matrix}\;y\sin^2\gamma,\,z,\,z\sin^2\gamma\right] \tag{4.5}$$

$$\int_0^{\gamma} F_{B:E;H}^{A:D;G}\left[\begin{matrix}(a_j)_{j=1}^{A}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G}\;\;; \\ \\ (b_j)_{j=1}^{B}:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H}\;\;;\end{matrix}\;y\cos^2\theta,\,z\cos^2\theta\right]\mathrm{d}\theta$$

$$=\gamma\,F_{B+1:E;H}^{A+1:D;G}\left[\begin{matrix}(a_j)_{j=1}^{A},\,\frac{1}{2}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G}\;\;; \\ \\ (b_j)_{j=1}^{B},\,1:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H}\;\;;\end{matrix}\;y,\,z\right]+\frac{z\,\sin\gamma\,\cos\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{G}(g_i)}{2\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{H}(h_i)}\times$$

$$\times\,F_{B+H+2:0;1}^{A+G+1:1;2}\left[\begin{matrix}\frac{3}{2},\,(1+a_j)_{j=1}^{A},\,(1+g_j)_{j=1}^{G}\;:1;1,1\;\;; \\ \\ 2,\,2,\,(1+b_j)_{j=1}^{B},\,(1+h_j)_{j=1}^{H}:—;\,\frac{3}{2}\;\;;\end{matrix}\;z,\,z\,\cos^2\gamma\right]+$$

$$+\frac{y\,\sin\gamma\,\cos\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{D}(d_i)}{2\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{E}(e_i)}\times$$

$$\times\,F^{(3)}\left[\begin{matrix}\frac{3}{2},(1+a_j)_{j=1}^{A}::—;—;(1+d_j)_{j=1}^{D}\;\;:1;(g_j)_{j=1}^{G};\,1,1\;\;; \\ \\ 2,\,(1+b_j)_{j=1}^{B}::—;—;(1+e_j)_{j=1}^{E},2:—;(h_j)_{j=1}^{H};\;\frac{3}{2}\;\;;\end{matrix}\;y,\,z,\,y\cos^2\gamma\right]+$$

$$+\frac{y\,z\,\sin\gamma\,\cos^3\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{A}(1+a_i)\,\prod_{i=1}^{D}(d_i)\,\prod_{i=1}^{G}(g_i)}{4\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{B}(1+b_i)\,\prod_{i=1}^{E}(e_i)\,\prod_{i=1}^{H}(h_i)}F^{(3)}\left[\begin{matrix}\frac{5}{2},(2+a_j)_{j=1}^{A}::—;(1+g_j)_{j=1}^{G}\;\;;2: \\ \\ 3,\,(2+b_j)_{j=1}^{B}::—;(1+h_j)_{j=1}^{H},2\;;\frac{5}{2}:\end{matrix}\right.$$

$$\left.\begin{matrix}1,(1+d_j)_{j=1}^{D};\,1\;;\;1\;\;; \\ \\ 2,(1+e_j)_{j=1}^{E};—;—\;\;;\end{matrix}\;y\cos^2\gamma,\,z,\,z\cos^2\gamma\right] \tag{4.6}$$

$$\int_0^{\gamma}{}_AF_B\left[\begin{matrix}(a_j)_{j=1}^{A}\;\;; \\ \\ (b_j)_{j=1}^{B}\;\;;\end{matrix}\;y\sin(c\theta)\right]\mathrm{d}\theta$$

$$= \gamma\ {}_{2A}F_{2B+1}\left[\begin{array}{lc}(\frac{a_j}{2})_{j=1}^{A},\ (\frac{1+a_j}{2})_{j=1}^{A}\ ; & \\ & \frac{y^2}{4^{(1+B-A)}} \\ 1,\ (\frac{b_j}{2})_{j=1}^{B},\ (\frac{1+b_j}{2})_{j=1}^{B}\ ; & \end{array}\right] - \frac{y\cos(c\gamma)\prod_{i=1}^{A}(a_i)}{c\prod_{i=1}^{B}(b_i)}\times$$

$$\times F^{2A\ :1;1}_{2B+2:0;0}\left[\begin{array}{lc}(\frac{1+a_j}{2})_{j=1}^{A},\ (\frac{2+a_j}{2})_{j=1}^{A}\ :1\ ;\ \frac{1}{2}\ ; & \\ & \frac{y^2}{4^{(1+B-A)}},\ \frac{y^2\sin^2(c\gamma)}{4^{(1+B-A)}} \\ \frac{3}{2},\ \frac{3}{2},\ (\frac{1+b_j}{2})_{j=1}^{B},\ (\frac{2+b_j}{2})_{j=1}^{B}:-;-\ ; & \end{array}\right] -$$

$$- \frac{y^2\sin(c\gamma)\cos(c\gamma)\prod_{i=1}^{A}(a_i)\prod_{i=1}^{A}(1+a_i)}{4c\prod_{i=1}^{B}(b_i)\prod_{i=1}^{B}(1+b_i)}\times$$

$$\times F^{2A\ :1;2}_{2B+2:0;1}\left[\begin{array}{lc}(\frac{2+a_j}{2})_{j=1}^{A},\ (\frac{3+a_j}{2})_{j=1}^{A}\ :1\ ;\ 1,1\ ; & \\ & \frac{y^2}{4^{(1+B-A)}},\ \frac{y^2\sin^2(c\gamma)}{4^{(1+B-A)}} \\ 2,\ 2,\ (\frac{2+b_j}{2})_{j=1}^{B},\ (\frac{3+b_j}{2})_{j=1}^{B}:-;\ \frac{3}{2}\ ; & \end{array}\right] +$$

$$+ \frac{y\prod_{i=1}^{A}(a_i)}{c\prod_{i=1}^{B}(b_i)}\ {}_{2A+1}F_{2B+2}\left[\begin{array}{lc}1,(\frac{1+a_j}{2})_{j=1}^{A},\ (\frac{2+a_j}{2})_{j=1}^{A}\ ; & \\ & \frac{y^2}{4^{(1+B-A)}} \\ \frac{3}{2},\ \frac{3}{2},\ (\frac{1+b_j}{2})_{j=1}^{B},\ (\frac{2+b_j}{2})_{j=1}^{B}\ ; & \end{array}\right] \tag{4.7}$$

$$\int_0^{\gamma} {}_AF_B\left[\begin{array}{lc}(a_j)_{j=1}^{A}\ ; & \\ & y\cos(c\theta) \\ (b_j)_{j=1}^{B}\ ; & \end{array}\right]\mathrm{d}\theta$$

$$= \gamma\ {}_{2A}F_{2B+1}\left[\begin{array}{lc}(\frac{a_j}{2})_{j=1}^{A},\ (\frac{1+a_j}{2})_{j=1}^{A}\ ; & \\ & \frac{y^2}{4^{(1+B-A)}} \\ 1,\ (\frac{b_j}{2})_{j=1}^{B},\ (\frac{1+b_j}{2})_{j=1}^{B}\ ; & \end{array}\right] + \frac{y\sin(c\gamma)\prod_{i=1}^{A}(a_i)}{c\prod_{i=1}^{B}(b_i)}\times$$

$$\times F^{2A\ :1;1}_{2B+2:0;0}\left[\begin{array}{lc}(\frac{1+a_j}{2})_{j=1}^{A},\ (\frac{2+a_j}{2})_{j=1}^{A}\ :1\ ;\ \frac{1}{2}\ ; & \\ & \frac{y^2}{4^{(1+B-A)}},\ \frac{y^2\cos^2(c\gamma)}{4^{(1+B-A)}} \\ \frac{3}{2},\ \frac{3}{2},\ (\frac{1+b_j}{2})_{j=1}^{B},\ (\frac{2+b_j}{2})_{j=1}^{B}:-;-\ ; & \end{array}\right] +$$

$$+ \frac{y^2\sin(c\gamma)\cos(c\gamma)\prod_{i=1}^{A}(a_i)\prod_{i=1}^{A}(1+a_i)}{4c\prod_{i=1}^{B}(b_i)\prod_{i=1}^{B}(1+b_i)}\times$$

$$\times F^{2A\ :1;2}_{2B+2:0;1}\left[\begin{array}{lc}(\frac{2+a_j}{2})_{j=1}^{A},\ (\frac{3+a_j}{2})_{j=1}^{A}\ :1\ ;1,1\ ; & \\ & \frac{y^2}{4^{(1+B-A)}},\ \frac{y^2\cos^2(c\gamma)}{4^{(1+B-A)}} \\ 2,\ 2,\ (\frac{2+b_j}{2})_{j=1}^{B},\ (\frac{3+b_j}{2})_{j=1}^{B}:-;\frac{3}{2}\ ; & \end{array}\right] \tag{4.8}$$

$$\int_0^\gamma \sin\theta\, {}_AF_B\left[\begin{array}{ll}(a_j)_{j=1}^A & ; \\ & y\sin^2\theta \\ (b_j)_{j=1}^B & ;\end{array}\right] d\theta = {}_{A+1}F_{B+1}\left[\begin{array}{ll}1,(a_j)_{j=1}^A & ; \\ & y \\ \frac{3}{2},(b_j)_{j=1}^B & ;\end{array}\right] -$$

$$- \cos\gamma\, F^{A\ :1;1}_{B+1:0;0}\left[\begin{array}{ll}(a_j)_{j=1}^A\ :1\ ;\frac{1}{2} & ; \\ & y,\ y\sin^2\gamma \\ \frac{3}{2},\ (b_j)_{j=1}^B:-;- & ;\end{array}\right] \quad (4.9)$$

$$\int_0^\gamma \cos\theta\, {}_AF_B\left[\begin{array}{ll}(a_j)_{j=1}^A & ; \\ & y\cos^2\theta \\ (b_j)_{j=1}^B & ;\end{array}\right] d\theta = \sin\gamma\, F^{A\ :1;1}_{B+1:0;0}\left[\begin{array}{ll}(a_j)_{j=1}^A\ :1\ ;\ \frac{1}{2} & ; \\ & y,\ y\cos^2\gamma \\ \frac{3}{2},\ (b_j)_{j=1}^B:-;- & ;\end{array}\right] \quad (4.10)$$

$$\int_0^\gamma \sin\theta\, F^{A:D;G}_{B:E;H}\left[\begin{array}{ll}(a_j)_{j=1}^A:(d_j)_{j=1}^D;(g_j)_{j=1}^G & ; \\ & y\sin^2\theta,\ z\sin^2\theta \\ (b_j)_{j=1}^B:(e_j)_{j=1}^E;(h_j)_{j=1}^H & ;\end{array}\right] d\theta$$

$$= F^{A+1:D;G}_{B+1:E;H}\left[\begin{array}{ll}1,\ (a_j)_{j=1}^A:(d_j)_{j=1}^D;(g_j)_{j=1}^G & ; \\ & y,\ z \\ \frac{3}{2},\ (b_j)_{j=1}^B:(e_j)_{j=1}^E;(h_j)_{j=1}^H & ;\end{array}\right] -$$

$$- \cos\gamma\, F^{(3)}\left[\begin{array}{ll}1,\ (a_j)_{j=1}^A::-;-;(d_j)_{j=1}^D\ :1\ ;(g_j)_{j=1}^G;\ \frac{1}{2} & ; \\ & y,\ z,\ y\sin^2\gamma \\ \frac{3}{2},\ (b_j)_{j=1}^B::-;-;1,\ (e_j)_{j=1}^E:-;(h_j)_{j=1}^H;- & ;\end{array}\right] -$$

$$- \frac{z\sin^2\gamma\cos\gamma \prod_{i=1}^A (a_i) \prod_{i=1}^G (g_i)}{3\prod_{i=1}^B (b_i) \prod_{i=1}^H (h_i)} \times$$

$$\times F^{(3)}\left[\begin{array}{ll}2,\ (1+a_j)_{j=1}^A::-;(1+g_j)_{j=1}^G\ ;\frac{3}{2}:(d_j)_{j=1}^D;\ 1\ ;\ 1 & ; \\ & y\sin^2\gamma,\ z,\ z\sin^2\gamma \\ \frac{5}{2},\ (1+b_j)_{j=1}^B::-;2,\ (1+h_j)_{j=1}^H;2:(e_j)_{j=1}^E;-;- & ;\end{array}\right] \quad (4.11)$$

$$\int_0^\gamma \cos\theta\, F^{A:D;G}_{B:E;H}\left[\begin{array}{ll}(a_j)_{j=1}^A:(d_j)_{j=1}^D;(g_j)_{j=1}^G & ; \\ & y\cos^2\theta,\ z\cos^2\theta \\ (b_j)_{j=1}^B:(e_j)_{j=1}^E;(h_j)_{j=1}^H & ;\end{array}\right] d\theta$$

$$= \sin\gamma\, F^{(3)}\left[\begin{array}{ll}1,\ (a_j)_{j=1}^A::-;-;(d_j)_{j=1}^D\ :1\ ;(g_j)_{j=1}^G;\ \frac{1}{2} & ; \\ & y,\ z,\ y\cos^2\gamma \\ \frac{3}{2},\ (b_j)_{j=1}^B::-;-;1,\ (e_j)_{j=1}^E:-;(h_j)_{j=1}^H;- & ;\end{array}\right] +$$

$$+ \frac{z\sin\gamma\cos^2\gamma \prod_{i=1}^A (a_i) \prod_{i=1}^G (g_i)}{3\prod_{i=1}^B (b_i) \prod_{i=1}^H (h_i)} \times$$

$$\times F^{(3)}\left[\begin{array}{l} 2,\,(1+a_j)_{j=1}^{A} :: -\,;\;\; (1+g_j)_{j=1}^{G}\;\; ;\, \frac{3}{2} : (d_j)_{j=1}^{D}\,;\;\; 1\,;\; 1 \quad ; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y\cos^2\gamma,\, z,\, z\cos^2\gamma \\ \frac{5}{2},\,(1+b_j)_{j=1}^{B} :: -\,;\, 2,\,(1+h_j)_{j=1}^{H}\,;\, 2 : (e_j)_{j=1}^{E}\;\, ;\, -\,;\, - \quad ; \end{array}\right] \tag{4.12}$$

provided that each of the series as well as associated integrals involved are convergent.

5 Solutions of Ramanujan's incomplete elliptic integrals

Setting $A = 1, B = 0$ and $a_1 = \frac{1}{2}$ in (4.1) and (4.2) respectively, we get

$$F(\sqrt{y},\,\gamma) = \int_0^{\gamma} \frac{d\theta}{\sqrt{(1-y\sin^2\theta)}} = \gamma\,{}_2F_1\left[\begin{array}{l} \frac{1}{2}, \frac{1}{2}\;\; ; \\ \qquad\qquad y \\ 1 \qquad ; \end{array}\right] - \frac{y\sin\gamma\,\cos\gamma}{4} \times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{l} \frac{3}{2}, \frac{3}{2} : 1\,;\, 1, 1 \quad ; \\ \qquad\qquad\qquad y,\, y\sin^2\gamma \\ 2, 2 : -\,;\; \frac{3}{2} \qquad ; \end{array}\right] \quad ; |y| < 1 \tag{5.1}$$

which is the exact solution of incomplete elliptic integral of first kind.

$$\int_0^{\gamma} \frac{d\theta}{\sqrt{(1-y\cos^2\theta)}} = \gamma\,{}_2F_1\left[\begin{array}{l} \frac{1}{2}, \frac{1}{2}\;\; ; \\ \qquad\qquad y \\ 1 \qquad ; \end{array}\right] + \frac{y\sin\gamma\,\cos\gamma}{4} \times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{l} \frac{3}{2}, \frac{3}{2} : 1;\, 1, 1 \quad ; \\ \qquad\qquad\qquad y,\, y\cos^2\gamma \\ 2, 2 : -\,;\; \frac{3}{2} \qquad ; \end{array}\right] \quad ; |y| < 1 \tag{5.2}$$

Putting $\gamma = \frac{\pi}{2}$ in (5.1) and (5.2), we get

$$\mathbf{K}(\sqrt{y}) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-y\sin^2\theta)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-y\cos^2\theta)}} = \frac{\pi}{2}\,{}_2F_1\left[\begin{array}{l} \frac{1}{2}, \frac{1}{2}\;\; ; \\ \qquad\qquad y \\ 1 \qquad ; \end{array}\right] \quad ; |y| < 1 \tag{5.3}$$

which is well-known complete elliptic integral of first kind.

Putting $A = 1, B = 0$ and $a_1 = -\frac{1}{2}$ in (4.1)and (4.2)respectively, we get

$$E(\sqrt{y},\,\gamma) = \int_0^{\gamma} \sqrt{(1-y\sin^2\theta)}\,d\theta = \gamma_2F_1\left[\begin{array}{l} \frac{1}{2}, -\frac{1}{2}\;\; ; \\ \qquad\qquad y \\ 1 \qquad ; \end{array}\right] + \frac{y\sin\gamma\cos\gamma}{4} \times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{l} \frac{1}{2},\frac{3}{2}:1;1,1\ ; \\ \qquad\qquad\qquad y, y\sin^2\gamma \\ 2,2:—;\ \frac{3}{2}\ ; \end{array}\right] \quad ;|y|<1 \tag{5.4}$$

which is the exact solution of incomplete elliptic integral of second kind.

$$\int_0^{\gamma}\sqrt{(1-y\cos^2\theta)}\,\mathrm{d}\theta = \gamma\,{}_2F_1\left[\begin{array}{l} \frac{1}{2},-\frac{1}{2}\ ; \\ \qquad\qquad y \\ 1\ ; \end{array}\right] - \frac{y\sin\gamma\cos\gamma}{4}\times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{l} \frac{1}{2},\frac{3}{2}:1;1,1\ ; \\ \qquad\qquad\qquad y, y\cos^2\gamma \\ 2,2:—;\ \frac{3}{2}\ ; \end{array}\right] \quad ;|y|<1 \tag{5.5}$$

Setting $\gamma = \frac{\pi}{2}$ in (5.4) and (5.5), we get

$$\mathbf{E}(\sqrt{y}) = \int_0^{\frac{\pi}{2}}\sqrt{(1-y\sin^2\theta)}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sqrt{(1-y\cos^2\theta)}\,\mathrm{d}\theta = \frac{\pi}{2}\,{}_2F_1\left[\begin{array}{l} -\frac{1}{2},\frac{1}{2}\ ; \\ \qquad\qquad y \\ 1\ ; \end{array}\right] \quad ;|y|<1 \tag{5.6}$$

which is well-known complete elliptic integral of second kind.

Putting $A=1, B=0,\ a_1=\frac{1}{2},\ \gamma=\beta$ and $y=x^2$ in (4.3), we get

$$\int_0^{\beta}\frac{\mathrm{d}\theta}{\sqrt{(1-x^2\sin^4\theta)}} = \beta\,{}_2F_1\left[\begin{array}{l} \frac{1}{4},\frac{3}{4}\ ; \\ \qquad\qquad x^2 \\ 1\ ; \end{array}\right] - \frac{3x^2\sin\beta\cos\beta}{16}\times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{l} \frac{5}{4},\frac{7}{4}:1;1,1\ ; \\ \qquad\qquad\qquad x^2, x^2\sin^2\beta \\ 2,2:—;\ \frac{3}{2}\ ; \end{array}\right] - \frac{x^2\sin^3\beta\cos\beta}{8}\times$$

$$\times F^{3:1;1}_{3:0;0}\left(\begin{array}{l} [\frac{5}{4}:1,1],\ [\frac{7}{4}:1,1],\ [2:1,2]:\ [1:1];\ [1:1]\ ; \\ \qquad\qquad\qquad\qquad\qquad x^2\sin^2\beta,\ x^2\sin^4\beta \\ [2:1,1],\ [2:1,1],\ [\frac{5}{2}:1,2]\ :\ —;\ —\ ; \end{array}\right) \quad ;|x|<1 \tag{5.7}$$

which is the exact solution of unsolved-incomplete elliptic integral given in Ramanujan's Entry 7(iv).

Setting $A=B=E=H=0,\ D=G=1,\ d_1=g_1=\frac{1}{2},\ \gamma=\beta,\ y=a,\ z=b$ in (4.5), we get

$$\int_0^{\beta}\frac{\mathrm{d}\theta}{\sqrt{(1-a\sin^2\theta)(1-b\sin^2\theta)}} = \beta\,\mathrm{F}^{1:1;1}_{1:0;0}\left[\begin{array}{l} \frac{1}{2}:\ \frac{1}{2};\ \frac{1}{2}\ ; \\ \qquad\qquad a,\ b \\ 1:—;—\ ; \end{array}\right] -$$

$$-\frac{b\sin\beta\cos\beta}{4}F^{2:1;2}_{2:0;1}\left[\begin{array}{l} \frac{3}{2},\frac{3}{2}:1;1,1\ ; \\ \qquad\qquad\qquad b,\ b\sin^2\beta \\ 2,2:—;\ \frac{3}{2}\ ; \end{array}\right] -$$

$$-\frac{a\,\sin\beta\,\cos\beta}{4}\,F^{(3)}\left[\begin{array}{ll}\frac{3}{2}::-;-;\frac{3}{2}:1\,;\,\frac{1}{2};1,1\;\;; & \\ & a,\,b,\,a\sin^2\beta \\ 2::-;-;2:-;-;\;\frac{3}{2}\;\;; & \end{array}\right]-$$

$$-\frac{a\,b\,\sin^3\beta\,\cos\beta}{16}\,F^{(3)}\left[\begin{array}{ll}\frac{5}{2}::-;\frac{3}{2}\,;2:1,\frac{3}{2};\,1\,;\,1\;\;; & \\ & a\sin^2\beta,\,b,\,b\sin^2\beta \\ 3::-;2\;\;;\frac{5}{2}:\,2\,;-;-\;\;; & \end{array}\right]\quad;\max\{|a|,|b|\}<1 \tag{5.8}$$

which is the exact solution of unsolved-incomplete elliptic integral given in Ramanujan's Entry 7(iii). Here $F^{1:1;1}_{1:0;0}[\,\cdot\,]$ is Appell's function of first kind, in the notation of Srivastava and Panda.

Setting $A=B=E=H=0,\ D=G=1,\ d_1=1,\ g_1=\frac{1}{2},\ \gamma=\phi,\ y=a,\ z=b^2$ in (4.5), we get

$$\prod(a,\,b,\,\phi)=\int_0^{\phi}\frac{\mathrm{d}\theta}{(1-a\sin^2\theta)\sqrt{(1-b^2\sin^2\theta)}}=\phi\,F^{1:1;1}_{1:0;0}\left[\begin{array}{ll}\frac{1}{2}:1\,;\,\frac{1}{2}\;\;; & \\ & a,\,b^2 \\ 1:-;-\;\;; & \end{array}\right]-$$

$$-\frac{b^2\,\sin\phi\,\cos\phi}{4}\,F^{2:1;2}_{2:0;1}\left[\begin{array}{ll}\frac{3}{2},\,\frac{3}{2}\,:1\,;\,1,1\;\;; & \\ & b^2,\,b^2\,\sin^2\phi \\ 2,\,2\,:-;\frac{3}{2}\;\;; & \end{array}\right]-$$

$$-\frac{a\,\sin\phi\,\cos\phi}{2}\,F^{(3)}\left[\begin{array}{ll}\frac{3}{2}::-;-;-:1\;\;;\,\frac{1}{2}\;\;;1,1\;\;; & \\ & a,\,b^2,\,a\sin^2\phi \\ 2::-;-;-:-;-\;\;;\frac{3}{2}\;\;; & \end{array}\right]-$$

$$-\frac{a\,b^2\,\sin^3\phi\,\cos\phi}{8}\,F^{(3)}\left[\begin{array}{ll}\frac{5}{2}::-;\frac{3}{2}\,;2:1\;\;;1\;\;;1\;\;; & \\ & a\sin^2\phi,\,b^2,\,b^2\sin^2\phi \\ 3::-;2\,;\frac{5}{2}\;:-;-;-\;\;; & \end{array}\right] \tag{5.9}$$

$$\text{where}\qquad \max\{|a|,|b|\}<1$$

which is the exact solution of incomplete elliptic integral of third kind.

In (5.9), putting $\phi=\frac{\pi}{2}$ we get the exact solution of complete elliptic integral of third kind.

$$\prod\left(a,\,b,\,\frac{\pi}{2}\right)=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{(1-a\sin^2\theta)\sqrt{(1-b^2\sin^2\theta)}}=\frac{\pi}{2}\,F_1\left[\frac{1}{2};1,\,\frac{1}{2};1;a,b^2\right]\quad;\max\{|a|,|b|\}<1 \tag{5.10}$$

where $F_1\left[\frac{1}{2};1,\frac{1}{2};1;a,b^2\right]$ is Appell's function of first kind[34,p.53(1.6.4)].

These solutions are not found in Ramanujan's notebooks[29-31], Five notebooks of B. C. Berndt [6-10], Three volumes of R. P. Agarwal [2-4] and other literature [5; 17; 18; 20; 22(pp.100-106); 24; 25; 26; 28; 32; 35] on special functions.

Setting $A = 1,\ B = 0,\ a_1 = \frac{1}{2},\ \gamma = \alpha,\ y = -x,\ c = 2$ in (4.8), we get

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{(1+x\cos\ 2\theta)}} = \alpha\, {}_2F_1\left[\begin{array}{ll} \frac{1}{4}, \frac{3}{4} & ; \\ & x^2 \\ 1, & ; \end{array}\right] - \frac{x\sin(2\alpha)}{4} \times$$

$$\times\, F^{2:1;1}_{2:0;0}\left[\begin{array}{ll} \frac{3}{4}, \frac{5}{4} : 1 ; \frac{1}{2} & ; \\ & x^2,\ x^2\ \cos^2(2\alpha) \\ \frac{3}{2}, \frac{3}{2} : - ; - & ; \end{array}\right] +$$

$$+ \frac{3x^2\sin(4\alpha)}{64}\, F^{2:1;2}_{2:0;1}\left[\begin{array}{ll} \frac{5}{4}, \frac{7}{4} : 1 ; 1, 1 & ; \\ & x^2,\ x^2\cos^2(2\alpha) \\ 2, 2 : - ; \frac{3}{2} & ; \end{array}\right] ; |x| < 1 \tag{5.11}$$

Further on using (5.2) in above integral, we get an elegant formula in the following form:

$$\sqrt{(1-x)}\int_0^{\alpha} \frac{d\theta}{\sqrt{(1+x\cos\ 2\theta)}} = \int_0^{\alpha} \frac{d\theta}{\sqrt{\{1-(\frac{2x}{x-1})\cos^2\theta)\}}}$$

$$= \alpha\, {}_2F_1\left[\begin{array}{ll} \frac{1}{2}, \frac{1}{2} & ; \\ & \frac{2x}{x-1} \\ 1 & ; \end{array}\right] + \frac{x\sin 2\alpha}{4(x-1)}\, F^{2:1;2}_{2:0;1}\left[\begin{array}{ll} \frac{3}{2}, \frac{3}{2} : 1 ; 1, 1 & ; \\ & \frac{2x}{x-1},\ \frac{2x\cos^2\alpha}{x-1} \\ 2, 2 : - ; \frac{3}{2} & ; \end{array}\right] ; \left|\frac{2x}{x-1}\right| < 1 \tag{5.12}$$

which is the exact solution of Ramanujan's unsolved-incomplete elliptic integral given in Entry 7(iv).

6 Special cases

In (4.1), set $A = 2, B = 1, a_1 = b, a_2 = 1-b, b_1 = \frac{1}{2}$ and $\gamma = \frac{\pi}{2}$, we obtain

$$\int_0^{\frac{\pi}{2}} \frac{\cos\{(2b-1)\sin^{-1}(\sqrt{y}\sin\theta)\}}{\sqrt{(1-y\sin^2\theta)}}\, d\theta = \frac{\pi}{2}\, {}_2F_1\left[\begin{array}{ll} b, 1-b & ; \\ & y \\ 1 & ; \end{array}\right] \tag{6.1}$$

which is the known result of Ramanujan [8, p.88(Entry 1)].

In (4.1), put $A = 2, B = 1, a_1 = \frac{1}{3}, a_2 = \frac{2}{3}, b_1 = \frac{1}{2}$ and $\gamma = \frac{\pi}{2}$, we obtain

$$\int_0^{\frac{\pi}{2}} {}_2F_1\left[\begin{array}{ll} \frac{1}{3}, \frac{2}{3} & ; \\ & y\sin^2\theta \\ \frac{1}{2} & ; \end{array}\right] d\theta = \frac{\pi}{2}\, {}_2F_1\left[\begin{array}{ll} \frac{1}{3}, \frac{2}{3} & ; \\ & y \\ 1 & ; \end{array}\right] \tag{6.2}$$

which is a known result of B.C.Berndt[10, p.133].

In (5.7), set $\beta = \frac{\pi}{2}$ and $x^2 = y$, we have

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{(1 - y\,\sin^4\theta)}} = \frac{\pi}{2}\,{}_2F_1\left[\begin{array}{lll} \frac{1}{4}, \frac{3}{4} & ; & \\ & & y \\ 1 & ; & \end{array}\right] \tag{6.3}$$

which is another known result of B.C.Berndt[8, p.110].

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Received: March 14, 2014; *Accepted*: August 23, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 392–410

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Nonparametric estimation of some characteristics of the conditional distribution in single functional index model

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Abstract

The aim of this paper is to establish a nonparametric estimation of some characteristics of the conditional distribution. Kernel type estimators for the conditional cumulative distribution function and the successive derivatives of the conditional density are introduced of a scalar response variable Y given a Hilbertian random variable X when the observations are linked with a single-index structure. We establish the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of the kernel estimate of this model. Asymptotic properties are stated for each of these estimates, and they are applied to the estimations of the conditional mode and conditional quantiles.

Keywords: Conditional single-index, Conditional cumulative distribution, Derivatives of conditional density, Nonparametric estimation, Conditional mode, Conditional quantile, Kernel estimator, semi-metric choice.

2010 MSC: 62G07, 62G5, 62G20.

1 Introduction

The single-index models are becoming increasingly popular because of their importance in several areas of science such as econometrics, biostatistics, medicine, financial econometric and so on. The single-index model, a special case of projection pursuit regression, has proven to be a very efficient way of coping with the high dimensional problem in nonparametric regression. Härdle *et al.* [16], Hristache *et al.* [18]. Delecroix et al. [6] have studied the estimation of the single-index approach of regression function and established some asymptotic properties. The recent literature in this domain shows a great potential of these functional statistical methods. The most popular case of functional random variable corresponds to the situation when we observe random curve on different statistical units. The first work in the fixed functional single-model was given by Ferraty *et al.* [10], where authors have obtained almost complete convergence (with the rate) of the regression function in the i.i.d. case. Their results have been extended to dependent case by Aït Saidi *et al.* [1]. Aït Saidi *et al.* [2] studied the case where the functional single-index is unknown. The authors have proposed for this parameter an estimator, based on the the cross-validation procedure.

In the present work we study a single- index modeling in the case of the functional explanatory variable. More precisely, we consider the problem of estimating some characteristics of the conditional distribution of a real variable Y given a functional variable X when the explanation of Y given X is done through its projection on one functional direction. The conditional distribution plays an important role in prediction problems, such as the conditional mode the conditional median or the conditional quantiles. Nonparametric estimation of the conditional density has been widely studied, when the data are real. The first related result in nonparametric functional statistic was obtained by Ferraty *et al.* [12], the authors have established the almost complete convergence (with rate) in the independent and identically distributed (i.i.d.) random variables. The

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asymptotic normality of this kernel estimator has been studied in the dependent data by Ezzahrioui and Ould Saïd [9].

The goal of this paper is to establish a nonparametric estimation of some characteristics of the conditional distribution where Kernel type estimators for the conditional cumulative distribution function and the successive derivatives of the conditional density in the single functional index model are introduced. We establish the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of the kernel estimate of this model. Asymptotic properties are stated for each of these estimates, and they are applied to the estimations of the conditional mode and conditional quantiles.

Now, let us outline the paper. At first, in section 2, we present general notations and some conditions necessary for our study, Then, in sections 3 we propose the estimator of the conditional cumulative distribution function and that of the conditional density derivatives, and we give their pointwise almost complete convergence (with rate). Then, in section 4, we study the uniform almost complete convergence of the conditional cumulative distribution function (resp. the conditional density derivatives) estimator given in section 3. Section 5 is devoted to some applications, in this part, we first consider the problem of the estimation of the conditional mode in functional single-index model, then we investigate the asymptotic properties of the conditional quantile function of a scalar response and functional covariate when the observations are in single functional index model and data are independent and identically distributed (i.i.d.), after that the cross-validation method is given, which is so important in guarding against testing hypotheses suggested by the data, especially where further samples are hazardous, costly or impossible to collect.

In the end, we finish our paper by giving technical proofs of lemmas and corollary (Appendix).

2 General notations and conditions

All along the paper, when no confusion will be possible, we will denote by C, C' or/and $C_{\theta,x}$ some generic constant in $\mathbb{R}_+^*$, and in the following, any real function with an integer in brackets as exponent denotes its derivative with the corresponding order.

Let X be a functional random variable, *frv* its abbreviation. Let (X_i, Y_i) be a sample of independent pairs, each having the same distribution as (X, Y), our aim is to build nonparametric estimates of several functions related with the conditional probability distribution (*cond-cdf*) of Y given $< X, \theta > = < x, \theta >$.

Let

$$\forall y \in \mathbb{R},\ F(\theta, y, x) = (Y \leq y| < X, \theta > = < x, \theta >).$$

be the *cond-cdf* of Y given $< X, \theta > = < x, \theta >$, for $x \in \mathcal{H}$, which also shows the relationship between X and Y but is often unknown.

If this distribution is absolutely continuous with respect to the Lebesgues measure on $\mathbb{R}$, then we will denote by $f(\theta, \cdot, x)$. (*resp.* $f^{(j)}(\theta, \cdot, x)$) the conditional density (*resp.* its j^{th} order derivative) of Y given $< X, \theta > = < x, \theta >$. In Sections 3 and 4, we will give almost complete convergence[1] results (with rates of convergence[2]) for nonparametric estimates of both functions $F(\theta, \cdot, x)$ and $f^{(j)}(\theta, \cdot, x)$.

In the following, for any $x \in \mathcal{H}$ and $y \in \mathbb{R}$, let $\mathcal{N}_x$ be a fixed neighborhood of x in $\mathcal{H}$, $\mathcal{S}_\mathbb{R}$ will be a fixed compact subset of $\mathbb{R}$, and we will use the notation $B_\theta(x, h) = \{X \in \mathcal{H}/0 < | < x - X, \theta > | < h\}$. Our nonparametric models will be quite general in the sense that we will just need the following simple assumption for the marginal distribution of $< \theta, X >$:

(H1) $(X \in B_\theta(x, h)) = \phi_{\theta,x}(h) > 0,$

together with some usual smoothness conditions on the function to be estimated. According to the type of estimation problem to be considered, we will assume either

(H2) $\forall (y_1, y_2) \in \mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R}$, $\forall (x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x$, $|F(\theta, y_1, x_1) - F(\theta, y_2, x_2)| \leq C_{\theta,x} \left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2} \right),$

[1]Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to converge almost completely to some variable T, if for any $\epsilon > 0$, we have $\sum_n (|T_n - T| > \epsilon) < \infty$. This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre, 1987).

[2]Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to be of order of complete convergence u_n, if there exists some $\epsilon > 0$ for which $\sum_n (|T_n| > \epsilon u_n) < \infty$. This is denoted by $T_n = O(u_n)$, *a.co.* (or equivalently by $T_n = O_{a.co.}(u_n)$).

$b_1 > 0, b_2 > 0,$

(H3) $\forall (y_1, y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall (x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x, |f^{(j)}(\theta, y_1, x_1) - f^{(j)}(\theta, y_2, x_2)| = C_{\theta,x}\left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}\right),$
$b_1 > 0, b_2 > 0.$

3 Pointwise almost complete estimation

In this section we give the pointwise almost complete estimation (with rate) of the conditional cumulative distribution as of the successive derivatives of the conditional density.

3.1 Conditional cumulative distribution estimation

The purpose of this section is to estimate the *cond-cdf* $F^x(\theta, \cdot, x)$. We introduce a kernel type estimator $\widehat{F}^x(\theta, \cdot, x)$ of $F^x(\theta, \cdot, x)$ as follows:

$$\widehat{F}(\theta, y, x) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(y - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, \tag{3.1}$$

where K is a kernel, H is a cumulative distribution function (*cdf*) and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers which goes to zero as n tends to infinity, and with the convention $0/0 = 0$. Note that a similar estimate was already introduced in the case where X is a valued in some semi-metric space which can be of infinite dimension by Ferraty *et al.* [11]. In our single functional index context, we need the following conditions for our estimate:

(H4) H is such that, for all $(y_1, y_2) \in \mathbb{R}^2$, $|H(y_1) - H(y_2)| \leq C|y_1 - y_2|$
$\int |t|^{b_2} H^{(1)}(t) dt < \infty,$

(H5) K is a positive bounded function with support $[-1, 1]$,

(H6) $\lim_{n\to\infty} h_K = 0$ with $\lim_{n\to\infty} \frac{\log n}{n\phi_{\theta,x}(h_K)} = 0,$

(H7) $\lim_{n\to\infty} h_H = 0$ with $\lim_{n\to\infty} n^{\alpha} h_H = \infty$ for some $\alpha > 0$.

• **Comments on the assumptions**

Our assumptions are very standard for this kind of model. Assumptions (H1) and (H5) are the same as those given in Ferraty et al. [10]. Assumptions (H2) and (H3) is a regularity conditions which characterize the functional space of our model and is needed to evaluate the bias term of our asymptotic results. Assumptions (H4) and (H6)-(H7) are technical conditions and are also similar to those done in Ferraty *et al.* [12].

Theorem 3.1. *Under the hypotheses (H1), (H2) and (H4)-(H7), and for any fixed y, we have*

$$|\widehat{F}(\theta, y, x) - F(\theta, y, x)| \ = \ O(h_K^{b_1}) + O(h_H^{b_2}) + O\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right), \qquad a.co. \tag{3.2}$$

Proof. For $i = 1, \ldots, n$, we consider the quantities $K_i(\theta, x) := K(h_K^{-1}(< x - X_i, \theta >))$ and, for all $y \in \mathbb{R}$ $\quad H_i(y) = H\left(h_H^{-1}(y - Y_i)\right)$ and let $\widehat{F}_N(\theta, y, x)$ (resp. $\widehat{F}_D(\theta, x)$) be defined as

$$\widehat{F}_N(\theta, y, x) \ = \ \frac{1}{n\left(K_1(\theta, x)\right)} \sum_{i=1}^{n} K_i(\theta, x) H_i(y) \qquad \text{(resp. } \widehat{F}_D(\theta, x) \ = \ \frac{1}{n\left(K_1(\theta, x)\right)} \sum_{i=1}^{n} K_i(\theta, x)).$$

This proof is based on the following decomposition

$$\begin{aligned}\widehat{F}(\theta,y,x)-F(\theta,y,x) &= \frac{1}{\widehat{F}_D(\theta,x)}\left\{\left(\widehat{F}_N(\theta,y,x)-\widehat{F}_N(\theta,y,x)\right)-\left(F(\theta,y,x)-\widehat{F}_N(\theta,y,x)\right)\right\}\\ &\quad+\frac{F(\theta,y,x)}{\widehat{F}_D(\theta,x)}\left\{1-\widehat{F}_D(\theta,x)\right\}\end{aligned} \tag{3.3}$$

and on the following intermediate results.

Lemma 3.1. *([1]) Under the hypotheses (H1) and (H5)-H6), we have*

$$|\widehat{F}_D(\theta,x)-1| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right), \tag{3.4}$$

Corollary 3.1. *Under the hypotheses of Lemma 3.1, we have*

$$\sum_{n=1}^{\infty}\left(|\widehat{F}_D(\theta,x)|\leq 1/2\right) < \infty. \tag{3.5}$$

Lemma 3.2. *Under the hypotheses (H1), (H2) and (H4)-(H.6), we have*

$$|F(\theta,y,x)-\widehat{F}_N(\theta,y,x)| = O\left(h_K^{b_1}\right)+O\left(h_H^{b_2}\right), \tag{3.6}$$

Lemma 3.3. *Under the hypotheses (H1), (H2) and (H4)-(H7), we have*

$$|\widehat{F}_N(\theta,y,x)-\widehat{F}_N(\theta,y,x)| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right), \tag{3.7}$$

□

3.2 Estimating successive derivatives of the conditional density

The main objective of this part is the estimation of successive derivatives of the conditional density of Y given $< X,\theta > = < x,\theta >$, denoted by $f(\theta,\cdot,x)$. It is well known that, in nonparametric statistics, this latter provides an alternative approach to study the links between Y and X and it can be also used, in single index modelling, to estimate the functional index θ if it is unknown.

So, at first, we propose to define the estimator $\widehat{f}^{(j)}(\theta,y,x)$ of $f^{(j)}(\theta,y,x)$ as follows:

$$\widehat{f}^{(j)}(\theta,y,x) = \frac{h_H^{-1-j}\sum_{i=1}^{n}K(h_K^{-1}(< x-X_i,\theta >))H^{(j+1)}(h_H^{-1}(y-Y_i))}{\sum_{i=1}^{n}K(h_K^{-1}(< x-X_i,\theta >))},\ y\in\mathbb{R} \tag{3.8}$$

Similar estimate was already introduced in the case where X is a valued in some semi-metric space which can be of infinite dimension; Ferraty *et al.* [11], then widely studied (see for instance by Attaoui *et al.* [3], for several asymptotic results and references). In addition to the conditions introduced along the previous section, we need the following ones, which are technical conditions and are also similar to those given in Ferraty *et al.* [12]:

(H8) $\begin{cases}\forall(y_1,y_2)\in\mathbb{R}^2,\ |H^{(j+1)}(y_1)-H^{(j+1)}(y_2)|\leq C_{\theta,x}|y_1-y_2|\\ \exists\nu>0,\ \forall j'\leq j+1,\ \lim_{y\to\infty}|y|^{1+\nu}\ \left|H^{(j'+1)}(y)\right| = 0.\end{cases}$

(H9) $\lim_{n\to\infty}h_K = 0$ with $\lim_{n\to\infty}\dfrac{\log n}{nh_H^{2j+1}\phi_{\theta,x}(h_K)} = 0.$

The next result concerns the asymptotic behavior of the kernel functional estimator $\widehat{f}^{(j)}(\theta,\cdot,x)$ of the j^{th} order derivative of the conditional density function.

Theorem 3.2. *Under Assumptions (H1), (H3)-(H5), and (H7)-(H9), and for any fixed y, we have, as n goes to infinity*

$$|\widehat{f}^{(j)}(\theta,y,x) - f^{(j)}(\theta,y,x)| = O\left(h_K^{b_1}\right) + O\left(h_H^{b_2}\right) + O\left(\sqrt{\frac{\log n}{n h_H^{2j+1}\phi_{\theta,x}(h_K)}}\right) \quad a.c.o \tag{3.9}$$

Proof. This result is based on the same kind of decomposition as (3.3). Indeed, we can write:

$$\begin{aligned}\widehat{f}^{(j)}(\theta,y,x) - f^{(j)}(\theta,y,x) &= \frac{1}{\widehat{F}_D(\theta,x)}\left(\widehat{f}_N^{(j)}(\theta,y,x) - (\widehat{f}_N^{(j)}(\theta,y,x)\right) - \frac{1}{\widehat{F}_D(\theta,x)} \\ &\left(f^{(j)}(\theta,y,x) - \widehat{f}_N^{(j)}(\theta,y,x)\right) \\ &+\frac{f^{(j)}(\theta,y,x)}{\widehat{F}_D(\theta,x)}\left(1-\widehat{F}_D(\theta,x)\right)\end{aligned} \tag{3.10}$$

where

$$\widehat{f}_N^{(j)}(\theta,y,x) = \frac{1}{n\, h_H^{j+1}\,(K_1(\theta,x))}\sum_{i=1}^{n} K_i(\theta,x)H_i^{(j+1)}(y).$$

Then, Theorem 3.2 can be deduced from both following lemmas, together with Lemma 3.1 and Corollary 3.1.

Lemma 3.4. *Under the hypotheses (H1), (H2), (H3), (H5) and (H6) we have*

$$|f^{(j)}(\theta,y,x) - (\widehat{f}_N^{(j)}(\theta,y,x)| = O\left(h_K^{b_1}\right) + O\left(h_H^{b_2}\right),$$

Lemma 3.5. *Under the hypotheses (H1)-(H7), we have*

$$|\widehat{f}_N^{(j)}(\theta,y,x) - (\widehat{f}_N^{(j)}(\theta,y,x)| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\, h_H^{2j+1}\,\phi_{\theta,x}(h_K)}}\right),$$

The proofs of the the above lemmas and corollary are given in the same manner as it was done in [12], since they are a special case of the Lemmas 2.3.2, 2.3.3, 2.3.4 and 2.3.5. It suffices to replace $\widehat{f}^{(j)}(y,x)$ (resp. $f^{(j)}(y,x)$) by $\widehat{f}^{(j)}(\theta,y,x)$ (resp. $f^{(j)}(\theta,y,x)$), and $\widehat{F}_D(x)$, (resp. $F_D(x)$) by $\widehat{F}_D(\theta,x)$ (resp. $F_D(\theta,x)$) with $d(x_1,x_2) =< x_1 - x_2, \theta >$ □

4 Uniform almost complete convergence

In this section we derive the uniform version of Theorem 3.1 and Theorem 3.2. The study of the uniform consistency is an indispensable tool for studying the asymptotic properties of all estimates of the functional index if is unknown. In the multivariate case, the uniform consistency is a standard extension of the pointwise one, however, in the functional case, it requires some additional tools and topological conditions (see Ferraty *et al.*, 2009). Thus, in addition to the conditions introduced previously, we need the following ones. Firstly, Consider

$$\mathcal{S}_{\mathcal{H}} \subset \bigcup_{k=1}^{d_n^{\mathcal{S}_{\mathcal{H}}}} B(x_k, r_n) \text{ and } \Theta_{\mathcal{H}} \subset \bigcup_{j=1}^{d_n^{\Theta_{\mathcal{H}}}} B(t_j, r_n) \tag{4.11}$$

with x_k (resp. t_j) $\in \mathcal{H}$ and $r_n, d_n^{\mathcal{S}_{\mathcal{H}}}, d_n^{\Theta_{\mathcal{H}}}$ are sequences of positive real numbers which tend to infinity as n goes to infinity.

4.1 Conditional cumulative distribution estimation

In this section we propose to study the uniform almost complete convergence of our estimator defined above (3.1) for this, we need the following assumptions:

(A1) There exists a differentiable function $\phi(\cdot)$ such that $\forall x \in \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$0 < C\phi(h) \le \phi_{\theta,x}(h) \le C'\phi(h) < \infty \text{ and } \exists \eta_0 > 0,\ \forall \eta < \eta_0,\ \phi'(\eta) < C,$$

(A2) $\forall (y_1, y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall (x_1, x_2) \in \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$|F(\theta, y_1, x_1) - F(\theta, y_2, x_2)| \leq C_{(x,\theta)} \left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2} \right),$$

(A3) The kernel K satisfy (H3) and Lipschitz's condition holds

$$|K(x) - K(y)| \leq C\|x - y\|,$$

(A4) For $r_n = O\left(\frac{\log n}{n}\right)$ the sequences $d_n^{\mathcal{S}_{\mathcal{H}}}$ and $d_n^{\Theta_{\mathcal{H}}}$ satisfy:

$$\frac{(\log n)^2}{n\phi(h_K)} < \log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \frac{n\phi(h_K)}{\log n},$$

and $\sum_{n=1}^{\infty} n^{1/2b_2} (d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})^{1-\beta} < \infty$ for some $\beta > 1$

Remark 4.1. *Note that Assumptions (A1) and (A2) are, respectively, the uniform version of (H1) and (H2). Assumptions (A1) and (A4) are linked with the the topological structure of the functional variable, see Ferraty* et al. *[13].*

Theorem 4.3. *Under Assumptions (A1)-(A4) and (H4), as n goes to infinity, we have*

$$\sup_{\theta \in \Theta_{\mathcal{H}}} \sup_{x \in \mathcal{S}_{\mathcal{H}}} \sup_{y \in \mathcal{S}_{\mathbb{R}}} |\widehat{F}(\theta, y, x) - F(\theta, y, x)| = O(h_K^{b_1}) + O(h_H^{b_2}) + O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \tag{4.12}$$

In the particular case, where the functional single-index is fixed we get the following result.

Corollary 4.2. *Under Assumptions (A1)-(A4) and (H4), as n goes to infinity, we have*

$$\sup_{x \in \mathcal{S}_{\mathcal{H}}} \sup_{y \in \mathcal{S}_{\mathbb{R}}} |\widehat{F}(\theta, y, x) - F(\theta, y, x)| = O(h_K^{b_1}) + O(h_H^{b_2}) + O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{n\phi(h_K)}}\right) \tag{4.13}$$

Clearly The proofs of these two results namely the Theorem 4.3 and Corollary 4.2 can be deduced from the following intermediate results which are only uniform version of Lemmas 3.1-3.3 and Corollary 3.1.

Lemma 4.6. *Under Assumptions (A1), (A3) and (A4), we have as $n \to \infty$*

$$\sup_{\theta \in \Theta_{\mathcal{H}}} \sup_{x \in \mathcal{S}_{\mathcal{H}}} |\widehat{F}_D(\theta, x) - 1| = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

Corollary 4.3. *Under the assumptions of Lemma 4.6, we have,*

$$\sum_{n=1}^{\infty} \left(\inf_{\theta \in \Theta_{\mathcal{H}}} \inf_{x \in \mathcal{S}_{\mathcal{H}}} \widehat{F}_D(\theta, x) < \frac{1}{2} \right) < \infty$$

Lemma 4.7. *Under Assumptions (A1), (A2) and (H4), we have, as n goes to infinity*

$$\sup_{\theta \in \Theta_{\mathcal{H}}} \sup_{x \in \mathcal{S}_{\mathcal{H}}} \sup_{y \in \mathcal{S}_{\mathbb{R}}} |F(\theta, y, x) - (\widehat{F}_N(\theta, y, x))| = O(h_K^{b_1}) + O(h_H^{b_2}) \tag{4.14}$$

Lemma 4.8. *Under the assumptions of Theorem 4.3, we have, as n goes to infinity*

$$\sup_{\theta \in \Theta_{\mathcal{H}}} \sup_{x \in \mathcal{S}_{\mathcal{H}}} \sup_{y \in \mathcal{S}_{\mathbb{R}}} |\widehat{F}_N(\theta, y, x) - \left[\widehat{F}_N(\theta, y, x)\right]| = O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + log\ d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

4.2 Estimating successive derivatives of the conditional density

In this part we focus on the study of uniform almost complete convergence of our estimator defined above (3.8). Thus, in addition to the conditions introduced in the section 4, we need the following ones.

(A5) $\forall(y_1, y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall(x_1, x_2) \in \mathcal{S}_{\mathcal{F}} \times \mathcal{S}_{\mathcal{F}}$ and $\forall\theta \in \Theta_{\mathcal{F}}$,

$$|f^{(j)}(\theta, y_1, x_1) - f^{(j)}(\theta, y_2, x_2)| \leq C \left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}\right),$$

(A6) For some $\gamma \in (0, 1)$, $\lim_{n\to\infty} n^{\gamma} h_H = \infty$, and for $r_n = O\left(\frac{\log n}{n}\right)$ the sequences $d_n^{\mathcal{S}_{\mathcal{F}}}$ and $d_n^{\Theta_{\mathcal{F}}}$ satisfy:

$$\frac{(log\ n)^2}{n h_H^{2j+1} \phi(h_K)} < log\ d_n^{\mathcal{S}_{\mathcal{F}}} + \log d_n^{\Theta_{\mathcal{F}}} < \frac{n h_H^{2j+1} \phi(h_K)}{\log n},$$

and $\sum_{n=1}^{\infty} n^{(3\gamma+1)/2} (d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}})^{1-\beta} < \infty$, for some $\beta > 1$

Theorem 4.4. *Under Hypotheses (A1), (A3) ,(A5)-(A6) and (H8), as n goes to infinity, we have*

$$\sup_{\theta\in\Theta_{\mathcal{F}}} \sup_{x\in\mathcal{S}_{\mathcal{F}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} |\widehat{f}^{(j)}(\theta, y, x) - f^{(j)}(\theta, y, x)| = O(h_K^{b_1}) + O(h_H^{b_2}) + O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}} + \log d_n^{\Theta_{\mathcal{F}}}}{n h_H^{2j+1} \phi(h_K)}}\right) \tag{4.15}$$

Proof. This result is based on the same kind of decomposition (3.10), therefore, Theorem 4.4 can be deduced from both following lemmas, together with Lemma 4.6 and Corollary 4.3.

Lemma 4.9. *Under Assumptions (A1), (A5) and (H8), we have, as n goes to infinity*

$$\sup_{\theta\in\Theta_{\mathcal{F}}} \sup_{x\in\mathcal{S}_{\mathcal{F}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} |f^{(j)}(\theta, y, x) - (\widehat{f}_N^{(j)}(\theta, y, x))| = O(h_K^{b_1}) + O(h_H^{b_2})$$

Lemma 4.10. *Under the assumptions of Theorem 4.4, we have, as n goes to infinity*

$$\sup_{\theta\in\Theta_{\mathcal{F}}} \sup_{x\in\mathcal{S}_{\mathcal{F}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} \left|\widehat{f}_N^{(j)}(\theta, y, x)] - \left[\widehat{f}_N^{(j)}(\theta, y, x)\right]\right| = O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}} + \log d_n^{\Theta_{\mathcal{F}}}}{n h_H^{2j+1} \phi_{\theta,x}(h_K)}}\right)$$

□

5 Applications

5.1 The conditional mode in functional single-index model

In this section we will consider the problem of the estimation of the conditional mode in the functional single-index model. The main objective, here, is to establish the almost complete convergence of the kernel estimator of the conditional mode of Y given $< X, \theta > = < x, \theta >$ denoted by $M_\theta(x)$, uniformly on fixed subset $\mathcal{S}_{\mathcal{H}}$ of $\mathcal{H}$. To this end, we suppose that $M_\theta(x)$ satisfies on $\mathcal{S}_{\mathcal{H}}$ the following uniform uniqueness property (see, Ould-saïd and Cai [23], for the multivariate case).

(A6) $\forall\varepsilon_0 > 0, \exists\eta > 0, \forall\varphi : \mathcal{S}_{\mathcal{H}} \longrightarrow \mathcal{S}_{\mathbb{R}}$,

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}} |M_\theta(x) - \varphi(x)| \geq \varepsilon_0 \Longrightarrow \sup_{x\in\mathcal{S}_{\mathcal{H}}} |f(\theta, \varphi(x), x) - f(\theta, M_\theta(x), x)| \geq \eta.$$

We estimate the conditional mode $\widehat{M_\theta}(x)$ with a random variable M_θ such as

$$\widehat{M_\theta}(x) = \arg\sup_{y\in\mathcal{S}_{\mathbb{R}}} \widehat{f}(\theta, y, x). \tag{5.16}$$

Note that the estimate $\widehat{M_\theta}$ is not necessarily unique, and if this is the case all the remaining of our paper will concern any value $\widehat{M_\theta}$ satisfying (5.16). The difficulty of the problem is naturally linked with the flatness of the function $f(\theta, y, x)$ around the mode M_θ. This flatness can be controlled by the number of vanishing derivatives at point M_θ, and this parameter will also have a great influence on the asymptotic rates of our estimates. More precisely, we introduce the following additional smoothness condition.

$$(A7)\begin{cases} f^{(l)}(\theta, M_\theta(x), x) = 0, & \text{if;} 1 \leq l < j \\ \text{and } f^{(j)}(\theta, \cdot, x), & \text{is uniformly continuous on } \mathcal{S}_{\mathbb{R}} \\ \text{such that,} & |f^{(j)}(\theta, \cdot, x)| > C > 0 \end{cases}$$

Theorem 5.5. *Under the assumptions of Theorem 4.4 hold together with (A6)-(A7) we have*

$$\sup_{x \in \mathcal{S}_{\mathcal{H}}} |\widehat{M}_\theta(x) - M_\theta(x)| = O(h_K^{\frac{b_1}{j}}) + O(h_H^{\frac{b_2}{j}}) + O_{a.co.}\left(\left(\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{n^{1-\gamma}\phi(h_K)}\right)^{\frac{1}{2j}}\right)$$

Let us now define the application framework of our results to prediction problem by applying the result in the above Theorem, we obtain the following result.

Corollary 5.4. *Under the assumptions of Theorem 5.5, we have as n goes to infinity*

$$\widehat{M}_\theta(x) - M_\theta(x) \longrightarrow 0 \quad a.co.$$

5.2 Conditional quantile in functional single-index model

In this part of paper we investigate the asymptotic properties of the conditional quantile function of a scalar response and functional covariate when the observations are from a single functional index model and data are independent and identically distributed (i.i.d.)

We will consider the problem of the estimation of the conditional quantiles. Saying that, we are implicitly assuming the existence of a regular version for the conditional distribution of Y given $< X, \theta >$. Now, let $t_\theta(\alpha)$ be the α-order quantile of the distribution of Y given $< X, \theta >=< x, \theta >$. From the *cond-cdf* $F(\theta, \cdot, x)$, it is easy to give the general definition of the α-order quantile:

$$t_\theta(\alpha) = \inf\{t \in \mathbb{R} : \; F(\theta, t, x) \geq \alpha\}, \qquad \forall \alpha \in (0, 1).$$

In order to simplify our framework and to focus on the main interest of our part (the functional feature of $< X, \theta >$), we assume that $F(\theta, \cdot, x)$ is strictly increasing and continuous in a neighborhood of $t_\theta(\alpha)$. This is insuring unicity of the conditional quantile $t_\theta(\alpha)$ which is defined by:

$$t_\theta(\alpha) \; = \; F^{-1}(\theta, \alpha, x). \tag{5.17}$$

In what remains, we wish to stay in a free distribution framework. This will lead to assume only smoothness restrictions for the *cond-cdf* $F(\theta, \cdot, x)$ through nonparametric modelling (see Section 2).
As by-product of (5.17) and (3.1), it is easy to derive an estimator $\widehat{t}_\theta(\alpha)$ of $t_\theta(\alpha)$:

$$\widehat{t}_\theta(\alpha) \; = \; \widehat{F}^{-1}(\theta, \alpha, x). \tag{5.18}$$

As we will see later on, such an estimator is unique as soon as H is an increasing continuous function.
Naturally, we will estimate this quantile by mean of the conditional distribution estimator studied in previous sections. Here also, as far as we know, the literature on (conditional and/or unconditional) quantile estimation is quite important when the explanatory variable X is real (see for instance Samanta, 1989, for previous results and Berlinet *et al.*, 2001, for recent advances and references). In the functional case, the conditional quantiles for scalar response and a scalar/multivariate covariate have received considerable interest in the statistical literature. For completely observed data, several nonparametric approaches have been proposed, for instance, Gannoun et al., (2003) introduced a smoothed estimator based on double kernel and local constant kernel methods and Berlinet et al., (2001) established its asymptotic normality. Under random censoring, Gannoun et al., (2005) introduced a local linear (LL) regression (see Koenker and Bassett (1978) for the definition) and El Ghouch and Van Keilegom (2009) studied the same LL estimator. Ould-Saïd (2006) constructed a kernel estimator of the conditional quantile under independent and identically distributed (i.i.d.) censorship model and established its strong uniform convergence rate. Liang and De Uña-Àlvarez (2011) established the strong uniform convergence (with rate) of the conditional quantile function under α-mixing assumption.

Recently, many authors are interested in the estimation of conditional quantiles for a scalar response and functional covariate. Ferraty *et al.*, (2005) introduced a nonparametric estimator of conditional quantile defined as the inverse of the conditional cumulative distribution function when the sample is considered as an α-mixing sequence. They stated its rate of almost complete consistency and used it to forecast the well-known El Niño time series and to build confidence prediction bands. Ezzahrioui and Ould-Saïd (2008) established the asymptotic normality of the kernel conditional quantile estimator under α-mixing assumption. Recently, and within the same framework, Dabo-Niang and Laksaci (2012) provided the consistency in L^p norm of the conditional quantile estimator for functional dependent data.

So, in this work we propose to estimate $t_\theta(\alpha)$ by the estimate $\widehat{t}_\theta(\alpha)$ defined as (5.18) or as

$$\widehat{F}(\theta, \widehat{t}_\theta(\alpha), x) = \alpha. \tag{5.19}$$

To insure existence and unicity of this quantile, we will assume that

(A8) $F(\theta, \cdot, x)$ is strictly increasing,

Note that, because H is a *cdf* satisfying (H4), such a value $\widehat{t}_\theta(\alpha)$ is always existing. It could be the case that it is not unique, but if this happens all the remaining of the paper will concern any among all the values $\widehat{t}_\theta(\alpha)$ satisfying (5.19).

In order to insure unicity of $\widehat{t}_\theta(\alpha)$ we will make the following, quite unrestrictive, assumption:

(A9) H is is strictly increasing,

As for the mode estimation problem discussed before, the difficulty occur in estimating the conditional quantile $t_\theta(\alpha)$ is linked with the flatness of the curve of the conditional distribution $F(\theta, \cdot, x)$ around $t_\theta(\alpha)$. More precisely, we will suppose that there exists some integer $j > 0$ such that:

$$(A10) \begin{cases} F^{(l)}(\theta, t_\theta(\alpha), x) = 0, & \text{if; } 1 \le l < j \\ \text{and } F^{(j)}(\theta, \cdot, x), & \text{is uniformly continuous on; } \mathcal{S}_\mathbb{R} \\ \text{such that,} & |F^{(j)}(\theta, t_\theta(\alpha), x)| > C > 0 \end{cases}$$

Theorem 5.6. *If the conditions of Theorem 4.4 hold together with (A8)-(A10), we have*

$$\sup_{x \in \mathcal{S}_\mathcal{H}} |\widehat{t}_\theta(\alpha) - t_\theta(\alpha)| = O\left(h_K^{\frac{b_1}{j}} + h_H^{\frac{b_2}{j}}\right) + O\left(\left(\frac{\log d_n^{\mathcal{S}_\mathcal{H}}}{n\ \phi_x(h_K)}\right)^{\frac{1}{2j}}\right), \qquad a.co. \tag{5.20}$$

Proof. Let us write the following Taylor expansion of the function $\widehat{F}(\theta, \cdot, x)$:

$$\begin{aligned} \widehat{F}(\theta, t_\theta(\alpha), x) - \widehat{F}(\theta, \widehat{t}_\theta(\alpha), x) &= \sum_{l=1}^{j-1} \frac{(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^l}{l!} \widehat{F}^{(l)}(\theta, t_\theta(\alpha), x) \\ &+ \frac{(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^j}{j!} \widehat{F}^{(j)}(\theta, t^*, x), \end{aligned}$$

where t^* is some point between $t_\theta(\alpha)$ and $\widehat{t}_\theta(\alpha)$. It suffices now to use the first part of condition (A10) to be able to rewrite this expression as:

$$\begin{aligned} \widehat{F}(\theta, t_\theta(\alpha), x) - \widehat{F}(\theta, \widehat{t}_\theta(\alpha), x) &= \sum_{l=1}^{j-1} \frac{(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^l}{l!} \left(\widehat{f}^{(l-1)}(\theta, t_\theta(\alpha), x) - f^{(l-1)}(\theta, t_\theta(\alpha), x)\right) \\ &+ \frac{(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^j}{j!} \widehat{f}^{(j-1)}(\theta, t^*, x), \end{aligned}$$

As long as we could be able to check that

$$\exists \tau > 0, \sum_{n=1}^{n=\infty} \left(f^{(j-1)}(\theta, t^*, x) < \tau \right) < \infty, \tag{5.21}$$

we would have

$$\begin{aligned} (t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^j &= O\left(\widehat{F}(\theta, t_\theta(\alpha), x) - F(\theta, t_\theta(\alpha), x)\right) \\ &+ O\left(\sum_{l=1}^{j-1} (t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^l (\widehat{f}^{(l-1)}(\theta, t_\theta(\alpha), x) - f^{(l-1)}(\theta, t_\theta(\alpha), x)\right), \ a.co. \end{aligned} \tag{5.22}$$

By comparing the rates of convergence given in Theorems 4.3 and 4.4, we see that the leading term in right hand side of equation (5.22) is the first one. So we have

$$(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^j = O_{a.co.}\left(\widehat{F}(\theta, t_\theta(\alpha), x) - F(\theta, t_\theta(\alpha), x)\right),$$

Because of Theorem 4.4, this is enough to get the claimed result, and so (5.21) is the only result that remains to check. This will be done directly by using the uniform continuity of the function $f^{(j-1)}(\theta, \cdot, x)$ given by second part of (A10) together with the third part of (A7) and with the following lemma.

Lemma 5.11. *If the conditions of Theorem 4.3 hold together with (A8) and (A9), then we have:*

$$\widehat{t}_\theta(\alpha) - t_\theta(\alpha) \rightarrow 0, \ a.co. \tag{5.23}$$

□

The next part is devoted to another type of application called the cross-validation method, this application has been already given in [2].

5.3 The cross-validation method

This method is widely applied, it can be used to compare the performances of different predictive modeling procedures. For instance, in optical character recognition; a mechanical or electronic conversion of scanned or photographed images of typewritten or printed text into machine-encoded/computer-readable text, this later is widely used as a form of data entry from some sort of original paper data source, whether passport documents, invoices, bank statement, receipts, business card, mail, or any number of printed records. It can also be used in variable selection; the process of selecting a subset of relevant features for use in model construction.

After this short introduction let's give an application of the method:

1. The regression operator $\widehat{r}_\theta(x)$ depends on the functional parameter θ, So, a crucial question arises: how to choose the functional index θ? The answer is nontrivial and a firstway consists in extending the standard cross-validation procedure to our functional context. For this, one considers various quadratic distances, namely the averaged squared error

$$\mathrm{ASE}(\theta) = n^{-1} \sum_{j=1}^{n} \left(r_{\theta_0}(X_j) - \widehat{r}_\theta(X_j) \right)^2, \tag{5.24}$$

the integrated squared error

$$\mathrm{ISE}(\theta) = \left[(r\theta_0(X_0) - \widehat{r}_\theta(X_0))^2 \, | Z_1, \ldots, Z_n \right], \tag{5.25}$$

and the mean integrated squared error

$$\mathrm{MISE}(\theta) = [ISE(\theta)]. \tag{5.26}$$

The main goal consists in finding a θ which minimizes (in some sense) over Θ_n these quantities. However, because all these quadratic distances depend on the unknown regression operator r_{θ_0}, the criterion used in practice for choosing θ is

$$CV(\theta) = n^{-1} \sum_{j=1}^{n} \left(Y_j - \widehat{r}_\theta^{-j}(X_j) \right)^2 \tag{5.27}$$

where $\widehat{r}_\theta^{-j}$ is the leave-one-out estimate of $r_\theta(x)$, given by

$$\widehat{r}_\theta^{-j}(x) = \frac{(n-1)^{-1} \sum_{\substack{i=1 \\ i\neq j}}^{n} Y_i K \left(h_K^{-1}(< x - X_i, \theta >) \right)}{(n-1)^{-1} \sum_{\substack{i=1 \\ i\neq j}}^{n} K \left(h_K^{-1}(< x - X_i, \theta >) \right)}. \tag{5.28}$$

So, the selection rule will be to choose θ_{CV} which minimizes the so-called cross-validation criterion $CV(\theta)$. Clearly, for a given θ, $CV(\theta)$ is a computable quantity. It measures a quadratic distance between $(Y_1, \ldots, Y_n)$ and its prediction $\widehat{r}_\theta^{-j}(X_1), \ldots, \widehat{r}_\theta^{-j}(X_n)$ when, for each i, $\widehat{r}_\theta^{-i}(.)$ is built without the ith data (X_i, Y_i). So, the method of cross-validation consists in choosing among several candidates θ, the one who is the most adapted to our data set (X_i, Y_i) in terms of prediction. This method is inspired by the cross-validation ideas which have been proposed in various standard nonparametric estimation problems (see [17] for the regression problem, [22] for the density and [26] for the hazard function).

From a practical point of view, some questions arise in order to implement this single-functional index model. What about the identifiability of the model given a sample of observed curves $(x_1, \ldots, x_n)$? How to build the set of functional indexes $\Theta_{\mathcal{F}}$? What about the choice of the bandwidth h?

Emphasizes the good behaviour of this simple cross-validated procedure, even in pathological situations. To see that, one focuses on a favourable case (i.e. $\theta_0 \in \Theta_{\mathcal{F}}$).

First of all, one builds a sample of n curves curves as follows:

$$x_i(t_j) = a_i cos(2\pi t_j) + b_i sin(4\pi t_j) + 2c_i(t_j - 0.25)(t_j - 0.5),$$

where $0 = t_1 < t_2 < \ldots < t_{n-1} < t_n = 1$ are equispaced points, the a_i's, b_i's and c_i's being independent observations uniformly distributed on $[0,1]$. Once the curves are defined, one simulates a single-functional index model as follows:

- Choose one $\theta_0(\cdot)$.
- Choose one link function $r(\cdot)$.
- Compute the inner products $< \theta_0, x_1 >, \ldots, < \theta_0, x_n >$.
- Generate independently $\varepsilon_1, \ldots, \varepsilon_n$, from a centred Gaussian of variance equal to 0.05 times the empirical variance of
  $r(< \theta_0, x_1 >), \ldots, r(< \theta_0, x_n >)$ (i.e. signal-to-noise ratio $= 0.05$).
- Simulate the corresponding responses: $Y_i = r(< \theta_0, x_i >) + \varepsilon_i$.

Finally, the observations $(x_k, Y_k)_{k=1,\ldots,m}$ are used for the learning step and the others (i.e. $(x_l, Y_l)_{l=m+1,\ldots,n}$ allow the computation of the mean square error of prediction:

$$\text{MSEP} = \frac{1}{n-m} \sum_{j=n-m}^{n} \left(Y_j \widehat{r}\left(< \theta_{CV}, x_j >\right) \right)^2.$$

In order to highlight the specificity of such a single-functional index model, the obtained predictions are compared with those coming from a pure nonparametric functional data analysis (NPFDA) method (see [12] for details and references therein). Actually, the NPFDA regression method uses the following kernel estimator:

$$\forall x \in \mathcal{H}, \quad \widehat{r}(x) = \frac{\sum_{i=1}^{n} Y_i K\left(h^{-1}(d(X_i, x)\right)}{\sum_{i=1}^{n} K\left(h^{-1}(d(X_i, x)\right)} \tag{5.29}$$

for estimating the regression operator m in the nonparametric model $Y_i = r(X_i) + \varepsilon_i$, for all $i = 1, \ldots, n$, where $d(.,.)$ is a fixed semi-metric.

If one looks at the NPFDA kernel estimator (5.29), it suffices to replace the fixed semi-metric $d(\cdot,\cdot)$ with $d_{\theta_{CV}}(\cdot,\cdot)$. What does this mean? It means that the functional index model can be seen as one way of building an nonparametric functional data analysis (NPFDA) kernel estimator with a data-driven semi-metric. In particular, in pure nonparametric functional models when one has no idea of the semi-metric, the functional index model appears to be a method for performing an adaptative one. The functional index model makes the NPFDA method more flexible. In this sense, the functional index model is not a competitive statistical technique with respect to the NPFDA method, but rather a complementary one.

2. If we wish to predict a real characteristic denoted Y of X_n knowing the curve X_{n-1}, we have to consider the observations (X_i, y_i) where y_i is the characteristic we want to provide at the instant i. For example:

   - If we want to predict the value of the process at time t_j knowing the curve X_{n-1}, we set $Y_i = X_{i+1}(t_j)$.
   - For the sup, we pose $Y_i = \sup_t X_{i+1}(t)$.
   - If we look for the time where the process reaches maximum, we set $Y_i = \arg\sup_t X_{i+1}(t)$.

   By using the conditional mode as a prediction tool, we can predict Y by $\widehat{M_\theta(X_{n-1})}$.

6 Appendix

Proof of Lemma 4.6 For all $x \in \mathcal{S}_\mathcal{H}$ and $\theta \in \Theta_\mathcal{H}$, we set

$$k(x) = \arg\min_{k\in\{1\ldots r_n\}} \|x - x_k\| \text{ and } j(\theta) =) = \arg\min_{j\in\{1\ldots l_n\}} \|\theta - t_j\|.$$

Let us consider the following decomposition

$$\begin{aligned}
\sup_{x\in\mathcal{S}_\mathcal{H}} \sup_{\theta\in\Theta_\mathcal{H}} \left|\widehat{F}_D(\theta,x) - \left(\widehat{F}_D(\theta,x)\right)\right| \leq & \underbrace{\sup_{x\in\mathcal{S}_\mathcal{H}} \sup_{\theta\in\Theta_\mathcal{H}} \left|\widehat{F}_D(\theta,x) - (\widehat{F}_D(\theta,x_{k(x)})\right|}_{\Pi_1} \\
& + \underbrace{\sup_{x\in\mathcal{S}_\mathcal{H}} \sup_{\theta\in\Theta_\mathcal{H}} \left|\widehat{F}_D(\theta,x_{k(x)}) - \widehat{F}_D(t_{j(\theta)},x_{k(x)})\right|}_{\Pi_2} \\
& + \underbrace{\sup_{x\in\mathcal{S}_\mathcal{H}} \sup_{\theta\in\Theta_\mathcal{H}} \left|\widehat{F}_D(t_{j(\theta)},x_{k(x)}) - \left(\widehat{F}_D(t_{j(\theta)},x_{k(x)})\right)\right|}_{\Pi_3} \\
& + \underbrace{\sup_{x\in\mathcal{S}_\mathcal{H}} \sup_{\theta\in\Theta_\mathcal{H}} \left|\left(\widehat{F}_D(t_{j(\theta)},x_{k(x)})\right) - \left(\widehat{F}_D(\theta,x_{k(x)})\right)\right|}_{\Pi_4} \\
& + \underbrace{\sup_{x\in\mathcal{S}_\mathcal{H}} \sup_{\theta\in\Theta_\mathcal{H}} \left|\left(\widehat{F}_D(\theta,x_{k(x)})\right) - \left(\widehat{F}_D(\theta,x)\right)\right|}_{\Pi_5}
\end{aligned}$$

For Π_1 and Π_2, we employe the Hölder continuity condition on K, Cauchy Schwartz's and the Bernstein's inequalities, we get

$$\Pi_1 = O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right), \ \Pi_2 = O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \tag{6.30}$$

Then, by using the fact that $\Pi_4 \leq \Pi_1$ and $\Pi_5 \leq \Pi_2$, we get for n tending to infinity

$$\Pi_4 = O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right), \ \Pi_5 = O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \tag{6.31}$$

Now, we deal with Π_3, for all $\eta > 0$, we have

$$\left(\Pi_3 > \eta\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)\right) \leq$$

$$d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}} \max_{k\in\{1\ldots d_n^{S_{\mathcal{H}}}\}} \max_{j\in\{1\ldots d_n^{\Theta_{\mathcal{H}}}\}} \left(\left|\widehat{F}_D(t_{j(\theta)}, x_{k(x)}) - \left(\widehat{F}_D(t_{j(\theta)}, x_{k(x)})\right)\right| > \eta\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)\right).$$

Applying Bernstein's exponential inequality to

$$\frac{1}{\phi(h_K)}\left(K_i\left(t_{j(\theta)}, x_{k(x)}\right) - \left(K_i\left(t_{j(\theta)}, x_{k(x)}\right)\right)\right),$$

then under (A7), we get

$$\Pi_3 = O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

Lastly the result will be easily deduced from the latter together with (6.30) and (6.31).

Proof Corollary 4.3 It is easy to see that,

$\inf_{\theta\in\Theta_{\mathcal{H}}} \inf_{x\in\mathcal{S}_{\mathcal{H}}} |\widehat{F}_D(\theta, x)| \leq 1/2 \implies \exists x \in \mathcal{S}_{\mathcal{H}}, \ \exists \theta \in \Theta_{\mathcal{H}}$, such that

$$1 - \widehat{F}_D(\theta, x) \geq 1/2 \implies \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} |1 - \widehat{F}_D(\theta, x)| \geq 1/2.$$

We deduce from Lemma 4.6 the following inequality

$$\left(\inf_{\theta\in\Theta_{\mathcal{H}}} \inf_{x\in\mathcal{S}_{\mathcal{H}}} |\widehat{F}_D(\theta, x)| \leq 1/2\right) \leq \left(\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} |1 - \widehat{F}_D(\theta, x)| \leq 1/2\right).$$

Consequently,

$$\sum_{n=1}^{\infty}\left(\inf_{\theta\in\Theta_{\mathcal{H}}} \inf_{x\in\mathcal{S}_{\mathcal{H}}} \widehat{F}_D(\theta, x) < \frac{1}{2}\right) < \infty$$

□

Proof of Lemma 4.7 One has

$$\begin{aligned}
\widehat{F}_N(\theta, y, x) - F(\theta, y, x) &= \frac{1}{K_1(x,\theta)}\left[\sum_{i=1}^{n} K_i(x,\theta) H_i(y)\right] - F(\theta, y, x) \\
&= \frac{1}{K_1(x,\theta)}\left(K_1(x,\theta)\left[E\left(H_1(y)| < X_1, \theta >\right) - F(\theta, y, x)\right]\right).
\end{aligned} \tag{6.32}$$

Moreover, we have

$$\left(H_1(y)| < X_1, \theta >\right) = \int_{\mathbb{R}} H\left(h_H^{-1}(y - z)\right) f(\theta, z, X_1) dz,$$

now, integrating by parts and using the fact that H is a *cdf*, we obtain

$$(H_1(y)| < X_1, \theta >) = \int_{\mathbb{R}} H^{(1)}(t) F(\theta, y - h_H t, X_1) dt.$$

Thus, we have

$$|(H_1(y)| < X_1, \theta >) - F(\theta, y, x)| \leq \int_{\mathbb{R}} H^{(1)}(t) \, |F(\theta, y - h_H t, X_1) - F(\theta, y, x)| \, dt.$$

Finally, the use of (A2) implies that

$$|(H_1(y)|X_1) - F^x(y)| \leq C_{\theta,x} \int_{\mathbb{R}} H^{(1)}(t) \left(h_K^{b_1} + |t|^{b_2} h_H^{b_2} \right) dt. \tag{6.33}$$

Because this inequality is uniform on $(\theta, y, x) \in \Theta_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathbb{R}}$ and because of (H4), (4.14) is a direct consequence of (6.32), (6.33) and of Corollary 4.3.

□

Proof of Lemma 4.8 We keep the notation of the Lemma 4.6 and we use the compact of $\mathcal{S}_{\mathbb{R}}$, we can write that, for some, $t_1, \ldots, t_{z_n} \in \mathcal{S}_{\mathbb{R}}$, $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{m=1}^{z_n} (y_m - l_n, y_m + l_n)$ with $l_n = n^{-1/2b_2}$ and $z_n \leq C n^{-1/2b_2}$. Taking $m(y) = \arg \min_{\{1,2,\ldots,z_n\}} |y - t_m|$.

Thus, we have the following decomposition:

$$\begin{aligned}
\left| \widehat{F}_N(\theta, y, x) - \left(\widehat{F}_N(\theta, y, x) \right) \right| = & \underbrace{\left| \widehat{F}_N(\theta, y, x) - \widehat{F}_N(\theta, y, x_{k(x)}) \right|}_{\Gamma_1} \\
& + \underbrace{\left| \widehat{F}_N(\theta, y, x_{k(x)}) - \left(\widehat{F}_N(\theta, y, x_{k(x)}) \right) \right|}_{\Gamma_2} \\
& + 2 \underbrace{\left| \widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)}) \right|}_{\Gamma_3} \\
& + 2 \underbrace{\left| \left(\widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) \right) - \left(\widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)}) \right) \right|}_{\Gamma_4} \\
& + \underbrace{\left| \left(\widehat{F}_N(\theta, y, x_{k(x)}) \right) - \left(\widehat{F}_N(\theta, y, x) \right) \right|}_{\Gamma_5}
\end{aligned}$$

↪ Concerning Γ_1 we have

$$\left| \widehat{F}_N(\theta, y, x) - \widehat{F}_N(\theta, y, x_{k(x)}) \right| \leq \frac{1}{n} \sum_{i=1}^{n} \left| \frac{1}{K_1(\theta, x)} K_i(\theta, x) H_i(y) - \frac{1}{K_1(\theta, x_{k(x)})} K_i(\theta, x_{k(x)}) H_i(y) \right|.$$

We use the Hölder continuity condition on K, the Cauchy-Schwartz inequality, the Bernstein's inequality and the boundness of H (assumption (H4)). This allows us to get:

$$\begin{aligned}
\left| \widehat{F}_N(\theta, y, x) - \widehat{F}_N(\theta, y, x_{k(x)}) \right| & \leq \frac{C}{\phi(h_K)} \frac{1}{n} \sum_{i=1}^{n} \left| K_i(\theta, x) H_i(y) - K_i(\theta, x_{k(x)}) H_i(y) \right| \\
& \leq \frac{C}{\phi(h_K)} \frac{1}{n} \sum_{i=1}^{n} |H_i(y)| \left| K_i(\theta, x) - K_i(\theta, x_{k(x)}) \right| \\
& \leq \frac{C' r_n}{\phi(h_K)}
\end{aligned}$$

↪ Concerning Γ_2, the monotony of the functions $\widehat{F}_N(\theta, \cdot, x)$ and $\widehat{F}_N(\theta, \cdot, x)$ permits to write, $\forall m \leq z_n$, $\forall x \in \mathcal{S}_{\mathcal{H}}$, $\forall \theta \in \Theta_{\mathcal{H}}$

$$\begin{aligned}
& \widehat{F}_N(\theta, y_{m(y)} - l_n, x_{k(x)}) \leq \sup_{y \in (y_{m(y)} - l_n, y_{m(y)} + l_n)} \widehat{F}_N(\theta, y, x) \leq \widehat{F}_N(\theta, y_{m(y)} + l_n, x_{k(x)}) \\
& \widehat{F}_N(\theta, y_{m(y)} - l_n, x_{k(x)}) \leq \sup_{y \in (y_{m(y)} - l_n, y_{m(y)} + l_n)} \widehat{F}_N(\theta, y, x) \leq \widehat{F}_N(\theta, y_{m(y)} + l_n, x_{k(x)}).
\end{aligned} \tag{6.34}$$

Next, we use the Hölder's condition on $F(\theta, y, x)$ and we show that, for any $y_1,\ y_2 \in \mathcal{S}_{\mathbb{R}}$ and for all $x \in \mathcal{S}_{\mathcal{H}},\ \theta \in \Theta_{\mathcal{H}}$

$$\begin{aligned}\left|\widehat{F}_N(\theta, y_1, x) - \widehat{F}_N(\theta, y_2, x)\right| &= \frac{1}{K_1(x,\theta)} \left|\left(K_1(x,\theta)F(\theta, y_1, X_1)\right) - \left(K_1(x,\theta)F(\theta, y_2, X_1)\right)\right| \\ &\leq C|y_1 - y_2|^{b_2}. \end{aligned} \tag{6.35}$$

Now, we have, for all $\eta > 0$

$$\left(\left|\widehat{F}_N(\theta, y, x_{k(x)}) - \widehat{F}_N(\theta, y, x_{k(x)})\right| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

$$=$$

$$\left(\max_{j\in\{1\ldots d_n^{\Theta_{\mathcal{H}}}\}} \max_{k\in\{1\ldots d_n^{\mathcal{S}_{\mathcal{H}}}\}} \max_{1\leq m\leq z_n} \left|\widehat{F}_N(\theta, y, x_{k(x)}) - \widehat{F}_N(\theta, y, x_{k(x)})\right| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

$$\leq$$

$$z_n d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}} \max_{j\in\{1\ldots d_n^{\Theta_{\mathcal{H}}}\}} \max_{k\in\{1\ldots d_n^{\mathcal{S}_{\mathcal{H}}}\}} \max_{1\leq m\leq z_n} \left(\left|\widehat{F}_N(\theta, y, x_{k(x)}) - \widehat{F}_N(\theta, y, x_{k(x)})\right| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}}}{n\phi(h_K)}}\right)$$

$$\leq$$

$$2z_n d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}} \exp\left(-C\eta^2 \log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\right)$$

choising $z_n = O\left(l_n^{-1}\right) = O\left(n^{\frac{1}{2b_2}}\right)$, we get

$$\left(\left|\widehat{F}_N(\theta, y, x_{k(x)}) - \widehat{F}_N(\theta, y, x_{k(x)})\right| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \leq C' z_n \left(d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\right)^{1-C\eta^2}$$

putting $C\eta^2 = \beta$ and using (A4), we get

$$\Gamma_2 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

$\hookrightarrow$ Concerning the terms Γ_3 and Γ_4, using Lipschitz's condition on the kernel H, one can write

$$\begin{aligned}\left|\widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)})\right| &\leq C\frac{1}{n\phi(h_K)} \sum_{i=1}^{n} K_i(t_{j(\theta)}, x_{k(x)}) \left|H_i(y) - H_i(y_{m(y)})\right| \\ &\leq \frac{Cl_n}{nh_H\phi(h_K)} \sum_{i=1}^{n} K_i(t_{j(\theta)}, x_{k(x)}).\end{aligned}$$

Once again a standard exponential inequality for a sum of bounded variables allows us to write

$$\widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)}) = O\left(\frac{l_n}{h_H}\right) + O_{a.co}\left(\frac{l_n}{h_H}\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right).$$

Now, the fact that $\lim_{n\to\infty} n^{\gamma} h_H = \infty$ and $l_n = n^{-1/2b_2}$ imply that:

$$\frac{l_n}{h_H\phi(h_K)} = o\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right),$$

then

$$\Gamma_3 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

Hence, for n large enough, we have

$$\Gamma_3 \leq \Gamma_4 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

$\hookrightarrow$ Concerning Γ_5, we have

$$\left(\widehat{F}_N(\theta, y, x_{k(x)})\right) - \left(\widehat{F}_N(\theta, y, x)\right) \leq \sup_{x \in \mathcal{S}_\mathcal{H}} \left|\widehat{F}_N(\theta, y, x) - \widehat{F}_N(\theta, y, x_{k(x)})\right|,$$

then following similar proof used in the study of Γ_1 and using the same idea as for $\left(\widehat{F}_D(\theta, x_{k(x)})\right) - \left(\widehat{F}_D(\theta, x)\right)$ we get, for n tending to infinity,

$$\Gamma_5 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

□

Proof of Lemma 4.9. Let $H_i^{(j+1)}(y) = H^{(j+1)}\left(h_H^{-1}(y - Y_i)\right)$, note that

$$\widehat{f}_N^{(j)}(\theta, y, x) - f^{(j)}(\theta, y, x) = \frac{1}{h_H^{j+1} K_1(x, \theta)}\left(K_1(x, \theta)\left[\left(H_1^{(j+1)}(y) | < X, \theta >\right) - h_H^{j+1} f^{(j)}(\theta, y, x)\right]\right). \quad (6.36)$$

Moreover,

$$\begin{aligned}\left(H_1^{(j+1)}(y) | < X, \theta >\right) &= \int_{\mathbb{R}} H^{(j+1)}\left(h_H^{-1}(y - z)\right) f(\theta, z, X) dz, \\ &= -\sum_{l=1}^{j} h_H^l \left[H^{(j-l+1)}\left(h_H^{-1}(y - z)\right) f^{(l-1)}(\theta, z, X)\right]_{-\infty}^{+\infty} \\ &\quad + h_H^j \int_{\mathbb{R}} H^{(1)}\left(h_H^{-1}(y - z)\right) f^{(j)}(\theta, z, X) dz. \end{aligned} \quad (6.37)$$

Condition (H8) allows us to cancel the first term in the right side of (6.37) and we can write:

$$\left|\left(H_1^{(j+1)}(y) | < X, \theta >\right) - h_H^{j+1} f^{(j)}(\theta, y, x)\right| \leq h_H^{j+1} \int_{\mathbb{R}} H^{(1)}(t)\left|f^{(j)}(\theta, y - h_H t, X) - f^{(j)}(\theta, y, x)\right| dt.$$

Finally, (A5) allows to write

$$\left|\left(H_1^{(j+1)}(y) | < X, \theta >\right) - h_H^{j+1} f^{(j)}(\theta, y, x)\right| \leq C_{\theta, x} h_H^{j+1} \int_{\mathbb{R}} H^{(1)}(t)\left(h_K^{b_1} + |t|^{b_2} h_H^{b_2}\right) dt. \quad (6.38)$$

This inequality is uniform on $(\theta, y, x) \in \Theta_\mathcal{F} \times \mathcal{S}_\mathcal{F} \times \mathcal{S}_\mathbb{R}$, now to finish the proof it is sufficient to use (H4).

□

Proof of Lemma 4.10. Let $l_n = n^{-\frac{3}{2}\gamma - \frac{1}{2}}$ and $z_n \leq C n^{-\frac{3}{2}\gamma - \frac{1}{2}}$.

Consider the following decomposition

$$\begin{aligned}\left|\widehat{f}_N^{(j)}(\theta, y, x) - \left(\widehat{f}_N^{(j)}(\theta, y, x)\right)\right| = & \underbrace{\left|\widehat{f}_N^{(j)}(\theta, y, x) - \widehat{f}_N^{(j)}(\theta, y, x_{k(x)})\right|}_{\Delta_1} \\ & + \underbrace{\left|\widehat{f}_N^{(j)}(\theta, y, x_{k(x)}) - \left(\widehat{f}_N^{(j)}(\theta, y, x_{k(x)})\right)\right|}_{\Delta_2} \\ & + 2\underbrace{\left|\widehat{f}_N^{(j)}(t_{j(\theta)}, y, x_{k(x)}) - \widehat{f}_N^{(j)}(t_{j(\theta)}, y_{m(y)}, x_{k(x)})\right|}_{\Delta_3} \\ & + 2\underbrace{\left|\left(\widehat{f}_N^{(j)}(t_{j(\theta)}, y, x_{k(x)})\right) - \left(\widehat{f}_N^{(j)}(t_{j(\theta)}, y_{m(y)}, x_{k(x)})\right)\right|}_{\Delta_4} \\ & + \underbrace{\left|\left(\widehat{f}_N^{(j)}(\theta, y, x_{k(x)})\right) - \left(\widehat{f}_N^{(j)}(\theta, y, x)\right)\right|}_{\Delta_5}\end{aligned}$$

$\rightsquigarrow$ Concerning Δ_1, we use the Hölder continuity condition on K, the Cauchy-Schwartz's inequality and the Bernstein's inequality. With theses arguments we get

$$\Delta_1 = O\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n h_H^{2j+1}\phi(h_K)}}\right).$$

Then using the fact that $\Delta_5 \le \Delta_1$, we obtain

$$\Delta_5 \le \Delta_1 = O\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n h_H^{2j+1}\phi(h_K)}}\right). \tag{6.39}$$

$\rightsquigarrow$ For Δ_2, we follow the same idea given for Γ_2, we get

$$\Delta_2 = O\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n h_H^{2j+1}\phi(h_K)}}\right)$$

$\rightsquigarrow$ Concerning Δ_3 and Δ_4, Using Lipschitz's condition on the kernel H,

$$\left|\widehat{f}_N^{(j)}(t_{j(\theta)}, y, x_{k(x)}) - \widehat{f}_N^{(j)}(t_{j(\theta)}, y_{m(y)}, x_{k(x)})\right| \le \frac{l_n}{h_H^{j+2}\phi(h_k)},$$

using the fact that $\lim_{n\to\infty} n^{\gamma} h_H = \infty$ and choosing $l_n = n^{-\frac{3}{2}\gamma - \frac{1}{2}}$ implies

$$\frac{l_n}{h_H^{j+2}\phi(h_k)} = o\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n h_H^{2j+1}\phi(h_K)}}\right)$$

So, for n large enough, we have

$$\Delta_3 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n h_H^{2j+1}\phi(h_K)}}\right).$$

And as $\Delta_4 \le \Delta_3$, we obtain

$$\Delta_4 \le \Delta_3 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n h_H^{2j+1}\phi(h_K)}}\right). \tag{6.40}$$

Finally, the lemma can be easily deduced from (6.39) and (6.40)

□

Proof of Lemma 5.11. Because of (H4) and (A9) the function $\widehat{F}(\theta,\cdot,x)$ is uniformly continuous and strictly increasing. So, we have:

$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0, \forall y, |\widehat{F}(\theta, y, x) - \widehat{F}(\theta, t_\theta(\alpha), x)| \le \delta(\epsilon) \Rightarrow |y - t_\theta(\alpha)| \le \epsilon.$$

This leads directly to

$$\begin{aligned}\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0,\ \ \left(|\widehat{t}_\theta(\alpha) - t_\theta(\alpha)| > \epsilon\right) &\le \left(|\widehat{F}(\theta, \widehat{t}_\theta(\alpha), x) - \widehat{F}(\theta, t_\theta(\alpha), x)| \ge \delta(\epsilon)\right) \\ &= \left(|F(\theta, t_\theta(\alpha), x) - \widehat{F}(\theta, t_\theta(\alpha), x)| \ge \delta(\epsilon)\right).\end{aligned}$$

Finally, It suffices to use the result of Theorem 4.3 to get the claimed result.

□

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Received: March 8, 2014; *Accepted*: May 21, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 411–418

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On solutions for classes of fractional differential equations

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Abstract

We provide a new solution of diffusion fractional differential equation using fractal index method. Also we shall impose a new solution for Riccati equation of arbitrary order. The fractional operators are taken in sense of the Riemann-Liouville operators.

Keywords: Fractional calculus; fractional differential equations; fractal index.

2010 MSC: 34A12.

1 Introduction

Fractional differential equations are viewed as alternative models to nonlinear differential equations. Varieties of them play important roles and tools not only in mathematics but also in physics, dynamical systems, control systems and engineering to create the mathematical modeling of many physical phenomena. Furthermore, they employed in social science such as food supplement, climate and economics. Fractional differential equations concerning the Riemann-Liouville fractional operators or Caputo derivative have been recommended by many authors (see [1-5]).

Transform is a significant technique to solve mathematical problems. Many useful transforms for solving various problems were appeared in open literature such as wave transformation, Laplace transform, the Fourier transform, the Bücklund transformation, the integral transform, the local fractional integral transforms and the fractional complex transform and Mellin transform (see [6-10]).

One of the most tools in the theory of fractional calculus is viewed by the RiemannLiouville operators. It imposes advantages of fast convergence, higher stability and higher accuracy to derive different types of numerical algorithms. In this note, we shall deal with scalar linear time-space fractional differential equations. The time and the space are taken in sense of the Riemann-Liouville fractional operators. Also, This type of differential equation arises in many interesting applications [11-17].

Several researchers have studied fractional dynamic equations generalizing the diffusion or wave equations in terms of R-L or Caputo time fractional derivatives, and their fundamental solutions have been represented in terms of the Mittag-Leffler (M-L) functions and their generalizations. In this work we shall provide a new solution of diffusion fractional differential equation using fractal index method. Also we shall impose a new solution for Riccati equation of arbitrary order. The fractional operators are taken in sense of the Riemann-Liouville operators.

2 Preliminaries

The idea of the fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) was planted over 300 years ago. Abel in 1823 investigated the generalized tautochrone prob-

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lem and for the first time applied fractional calculus techniques in a physical problem. Later Liouville applied fractional calculus to problems in potential theory. Since that time the fractional calculus has haggard the attention of many researchers in all area of sciences (see [1-5]).

Definition 2.1. *The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by*

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denoted the convolution product, $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\phi_\alpha(t) = 0,\ t \leq 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2. *The fractional (arbitrary) order derivative of the function f of order $0 < \alpha \leq 1$ is defined by*

$$D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

In the sequel, we use the notation $\frac{\partial^\alpha}{\partial t^\alpha}$.

Remark 2.1. *From Definition 2.1 and Definition 2.2, $a = 0$, we have*

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \ \ \mu > -1;\ 0 < \alpha < 1$$

and

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \ \ \mu > -1;\ \alpha > 0.$$

The Leibniz rule is

$$D_a^\alpha [f(t)g(t)] = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} D_a^{\alpha-k} f(t) D_a^k g(t) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} D_a^{\alpha-k} g(t) D_a^k f(t),$$

where

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)}.$$

Definition 2.3. *The Caputo fractional derivative of order $\mu > 0$ is defined, for a smooth function $f(t)$ by*

$${}^cD^\mu f(t) := \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\mu-n+1}} d\zeta,$$

where $n = [\mu] + 1$, (the notation $[\mu]$ stands for the largest integer not greater than μ).

Note that there is a relationship between Riemann-Liouville differential operator and the Caputo operator

$$D_a^\mu f(t) = \frac{1}{\Gamma(1-\mu)} \frac{f(a)}{(t-a)^\mu} + {}^cD_a^\mu f(t);$$

and they are equivalent in a physical problem (i.e., a problem which specifies the initial conditions) [16].

3 The fractal index

We consider the equation

$$t^\alpha D_t^\alpha u(t,x) + \Big(a(x^\beta - t) + b(ux^\beta - t^2) + c(u^2 x^\beta - t^3)\Big) + eD_x^{3\beta} u(t,x) = 0, \tag{3.1}$$

$$\Big(t \in J := [0,T],\ x \in \mathbb{R}\Big)$$

where $u(0,x) = 0$, $e \neq 0$, $a, b, c \in \mathbb{R}$ and $\alpha, \beta \in (0,1]$.

Eq.(1) involves well known time-space fractional diffusion equations. Several researchers have studied fractional dynamic equations generalizing the diffusion or wave equations in terms of R-L or Caputo time fractional derivatives, and their fundamental solutions have been represented in terms of the MittagLeffler (M-L) functions and their generalizations. The mathematical study of fractional diffusion equations began with the work of Kochubei [18,19]. Later this study followed by the work of Metzler and Klafter [20] and Zaslavsky [21]. Recently, Mainardi et all obtained the time fractional diffusion equation from the standard diffusion equation [22,23].

Let $X = x^\alpha$ and $f = X^n$, $n \neq 0$ then we obtain

$$\begin{aligned}\frac{\partial^\alpha f}{\partial x^\alpha} &= \frac{\partial f}{\partial X}\frac{\partial^\alpha X}{\partial x^\alpha}\\ &= \frac{\Gamma(1+n\alpha)x^{n\alpha-\alpha}}{\Gamma(1+n\alpha-\alpha)} := \frac{\partial f}{\partial X}\theta_\alpha\\ &= n\theta_\alpha x^{n\alpha-\alpha}\end{aligned}$$

we receive

$$\theta_\alpha = \frac{\Gamma(1+n\alpha)}{n\Gamma(1+n\alpha-\alpha)}.$$

Consequently

$$\begin{aligned}\frac{\partial^{2\alpha} f}{\partial x^{2\alpha}} &= \frac{\partial^2 f}{\partial X^2}\theta_{\alpha\alpha}\\ &:= n(n-1)\theta_{\alpha\alpha}x^{n\alpha-2\alpha}\end{aligned}$$

where

$$\theta_{\alpha\alpha} = \frac{\Gamma(1+n\alpha)}{n\Gamma(1+n\alpha-\alpha)}\frac{\Gamma(1+n\alpha-\alpha)}{(n-1)\Gamma(1+n\alpha-2\alpha)}.$$

And

$$\begin{aligned}\frac{\partial^{3\alpha} f}{\partial x^{3\alpha}} &= \frac{\partial^3 f}{\partial X^3}\theta_{\alpha\alpha\alpha}\\ &:= n(n-1)(n-2)\theta_{\alpha\alpha\alpha}x^{n\alpha-3\alpha}\end{aligned}$$

where

$$\begin{aligned}\theta_{\alpha\alpha\alpha} &= \frac{\Gamma(1+n\alpha)}{n\Gamma(1+n\alpha-\alpha)}\frac{\Gamma(1+n\alpha-\alpha)}{(n-1)\Gamma(1+n\alpha-2\alpha)}\frac{\Gamma(1+n\alpha-2\alpha)}{(n-2)\Gamma(1+n\alpha-3\alpha)}\\ &= \frac{\Gamma(1+n\alpha)}{n(n-1)(n-2)\Gamma(1+n\alpha-3\alpha)}.\end{aligned}$$

Now we proceed to impose a solution for the Eq. (1) using the fractal method.

Let the solution takes the form

$$u(t,x) = \sum_{n=1}^{\infty}\mu_n(x)t^n, \tag{3.2}$$

where u is analytic in J. By balancing the first two terms of the equation (1) (w.r.t t), we have $n = 2$. Therefore,

$$u(t,x) = \mu_1(x)t + \mu_2(x)t^2, \quad \mu_i(0) = 1,\ i = 1,2. \tag{3.3}$$

By using some properties of the fractional calculus, we obtain

$$\begin{aligned}D_t^\alpha u(t,x) &= \mu_1(x)D_t^\alpha t + \mu_2(x)D_t^\alpha t^2\\ &= \frac{\mu_1(x)}{\Gamma(2-\alpha)}t^{1-\alpha} + \frac{\mu_2(x)}{\Gamma(3-\alpha)}t^{2-\alpha}.\end{aligned}$$

This implies

$$t^\alpha D_t^\alpha u(t,x) = \frac{\mu_1(x)}{\Gamma(2-\alpha)}t + \frac{\mu_2(x)}{\Gamma(3-\alpha)}t^2. \tag{3.4}$$

Moreover,

$$eD_x^{3\beta}u(t,x) = etD_x^{3\beta}\mu_1(x) + et^2D_x^{3\beta}\mu_2(x) \tag{3.5}$$

and

$$\begin{aligned}(a+bu+cu^2)x^\beta - \left(at+bt^2+ct^3\right) = &\Big(a + b[\mu_1(x)t + \mu_2(x)t^2 + \mu_3(x)t^3] \\ &+ c[\mu_1^2t^2 + 2\mu_1\mu_2t^3 + ...]\Big)x^\beta \\ &- \left(at+bt^2+ct^3\right)\end{aligned} \tag{3.6}$$

Next we shall calculate the functions $\mu_i(x)$, $i = 1,2$. Comparing the coefficients of Eq. (4-6) with respect to t, t^2, yields

$$D_x^{3\beta}\mu_1(x) + (\phi_1 + \psi_1)\mu_1(x) - \frac{a}{e} = 0, \tag{3.7}$$

where

$$\phi_1 := \frac{1}{e\Gamma(2-\alpha)}, \quad \psi_1(x^\beta) := \frac{b}{e}x^\beta,\ e \neq 0.$$

To calculate the fractal index for the equation (7), we assume the transform $X = x^\beta$ and the solution can be expressed in a series of the form

$$\mu_1(X) = \sum_{m=0}^{\infty}\gamma_m X^m, \quad \mu_1(0) = 1 \tag{3.8}$$

where γ_m are constants. Substitute (8) in (7) and by using the fractal index we receive

$$\begin{aligned}&\frac{\partial^3}{\partial X^3}\sum_{m=0}^{\infty}\theta_{\beta\beta\beta m}\gamma_m X^m + \phi_1\sum_{m=0}^{\infty}\gamma_m X^m + \frac{b}{e}\sum_{m=0}^{\infty}\gamma_m X^{m+1} = 0 \\ &\sum_{m=3}^{\infty}\frac{\Gamma(1+m\beta)}{\Gamma(1+m\beta-3\beta)}\gamma_m X^{m-3} + \phi_1\sum_{m=0}^{\infty}\gamma_m X^m + \frac{b}{e}\sum_{m=0}^{\infty}\gamma_m X^{m+1} - \frac{a}{e} = 0\end{aligned} \tag{3.9}$$

where

$$\theta_{\beta\beta\beta m} = \frac{\Gamma(1+m\beta)}{m(m-1)(m-2)\Gamma(1+m\beta-3\beta)}.$$

Comparing the coefficients of X^0, we have

$$\Gamma(1+3\beta)\gamma_3 + \phi_1\gamma_0 + \frac{b}{e}\gamma_{-1} = \frac{a}{e};$$

but $\gamma_0 = 1$ and $\gamma_{-1} = 0$, in general we obtain

$$\gamma_m = \frac{(\frac{a}{e} - \phi_1)^m}{\Gamma(1+m\beta)}, \quad m \geq 3.$$

Hence

$$\mu_1(x) = E_\beta((\frac{a}{e} - \phi_1)x^\beta),$$

where E_β is a Mittag-Leffler function.

Similarly for μ_2; if we let

$$\mu_2(X) = \sum_{m=0}^{\infty}\delta_m X^m, \quad \mu_2(0) = 1 \tag{3.10}$$

we can receive

$$\mu_2(x) = E_\beta((\frac{b}{e} - \phi_2)x^\beta), \quad \phi_2 := \frac{1}{e\Gamma(3-\alpha)}.$$

Thus we have the following solution of the Eq.(1):

$$u(t,x) = tE_\beta((\frac{a}{e} - \phi_1)x^\beta) + t^2E_\beta((\frac{b}{e} - \phi_2)x^\beta). \tag{3.11}$$

4 Fractional Riccati equation

$$D_x^\alpha \psi(x) = \sigma + \psi^2(x), \tag{4.12}$$

where $\sigma \in \mathbb{R}$. To calculate the fractal index for the equation (12), we assume the transform $X = x^\alpha$ and the solution can be expressed in a series of the form

$$\psi(X) = \sum_{m=0}^{\infty} \psi_m X^m, \quad \psi(0) = 1 \tag{4.13}$$

where ψ_m are constants. Substitute (13) in (12) and by applying the fractal index we impose

$$\begin{aligned} \frac{\partial}{\partial X} \sum_{m=0}^{\infty} \theta_{\alpha m} \psi_m X^m &= \sigma + \Big(\sum_{m=0}^{\infty} \psi_m X^m \Big) \Big(\sum_{m=0}^{\infty} \psi_m X^m \Big), \\ \sum_{m=1}^{\infty} \frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha-\alpha)} \psi_m X^{m-1} &= \sigma + \Big(\sum_{m=0}^{\infty} \psi_m X^m \Big) \Big(\sum_{m=0}^{\infty} \psi_m X^m \Big), \end{aligned} \tag{4.14}$$

where

$$\theta_{\alpha m} = \frac{\Gamma(1+m\alpha)}{m\Gamma(1+m\alpha-\alpha)}.$$

Comparing the coefficients of X^0, we have

$$\Gamma(1+\alpha)\psi_1 = \sigma + \psi_0^2;$$

but $\psi_0 = 1$ so in general we obtain

$$\psi_m = \frac{(\sigma+1)^m}{\Gamma(1+m\alpha)}, \quad m \geq 1.$$

Hence

$$\psi(x) = E_\alpha((\sigma+1)x^\alpha). \tag{4.15}$$

Note that Zhang and Zhang [17] derived some exact solutions to Eq.(15) take the form

$$\psi(x) = \begin{cases} -\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}x) & \text{for } \sigma < 0 \\ -\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}x) & \text{for } \sigma < 0 \\ \sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}x) & \text{for } \sigma > 0 \\ -\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}x) & \text{for } \sigma > 0 \\ -\frac{\Gamma(1+\alpha)}{x^\alpha+v} & \text{for } \sigma = 0, \end{cases} \tag{4.16}$$

where v is a constant. In the next section, we shall use (15) to locate exact solution of fractional differential equation using Bäcklund transformation of fractional Riccati equation.

5 Bäcklund transformation method

In this section, based on the Bäcklund transformation method and the known seed solutions, we will impose a technique for solving fractional partial differential equations. It will be shown that the use of the Bäcklund transformation permits us to get new exact solutions from the known seed solutions. The Bäcklund transformation for the fractional Riccati equation is determined by

$$\phi(\eta) = \frac{-\sigma B + D\psi(\eta)}{D + B\psi(\eta)}, \tag{5.17}$$

where $\phi(\eta)$ satisfies the fractional Riccati equation (12) and $B \neq 0, D$ are arbitrary parameters, and ψ are the known solutions of Eq. (12).

Our method can be summarized as follows:

Step 1: Using the wave transform

$$u(t,x_1,...,x_j)=u(\eta),$$
$$\eta=\eta_0+\lambda t+\lambda_1 x_1+...+\lambda_j x_j,$$

where $\lambda,\lambda_i(i=1,...,j)$ are constants. Hence the equation

$$F\Big(u,u_t,u_{x_1},...,u_{x_j},u_{x_1x_1},...,D^{\alpha}_{x_1}u,...,D^{\alpha}_{x_j}u,...\Big)=0, \tag{5.18}$$

becomes

$$\Phi\Big(u(\eta),u'(\eta),u''(\eta),...,D^{\alpha}_{\eta}u\Big)=0, \tag{5.19}$$

where $(')=\frac{d}{d\eta}$.

Step 2: Assuming a solution of the form

$$u(\eta)=\sum_{m=0}^{n}a_m\phi^m(\eta), \tag{5.20}$$

where a_m $(m=0,...,n)$ are constants to be calculated and ϕ computes from the Bäcklund transform.

Step 3: Substituting (20) in (19) and setting the coefficients of the powers of ϕ to be zero, we impose a nonlinear algebraic system in a_m and λ.

Step 4: Solving the system to obtain these values and substituting them into Eq.(20) we receive the exact solutions of (18).

6 Applications

In this section we shall illustrate two examples to examine our method.

6.1 Example

Water as a liquid moves through the vadose region in response to gravity and gradients of pressure. Recall that the vadose region has hole spaces filled with both air and liquid water. The water pressure depends on the water saturation and related capillary forces. Because the soil is only partially saturated the pressure is negative due to capillarity. If the soil is uniform in its properties such as composition, capillary pressures are most negative where the soil is dry, and most positive where it is wet. As a FDE it can be represented as

$$D^{\alpha}_{t}u-\kappa uD^{\alpha}_{x}u-\delta D^{2\alpha}_{x}u=0, \tag{6.21}$$

where x is the position in this model and u is the so-called volumetric water content. It denotes the proportion of the space filled by water. δ is the so-called soil moisture diffusivity and κ is the saturation dependent hydraulic conductivity. Equation (21) describes the infiltration in the vadose region. The advection is due the gravity and the diffusion is due to capillary wicking.

Using the wave transform

$$u(t,z)=u(\eta),\quad \eta=\lambda t+x,$$

we receive

$$\lambda^{\alpha}D^{\alpha}_{\eta}u-\kappa uD^{\alpha}_{\eta}u-\delta D^{2\alpha}_{\eta}u=0. \tag{6.22}$$

By applying the above method yields

$$u(\eta)=a_0+a_1\frac{-\sigma B+D\psi(\eta)}{D+B\psi(\eta)},$$

where ψ defined in (15) and

$$a_0=\frac{\lambda^{\alpha}}{\kappa},\quad a_1=\frac{-2\delta}{\kappa},\ \kappa\neq 0.$$

Thus we have exact solutions of (21) as follows:

$$u(\eta)=\frac{\lambda^{\alpha}}{\kappa}-\frac{2\delta}{\kappa}\Big(\frac{-\sigma B+DE_{\alpha}((\sigma+1)\eta^{\alpha})}{D+BE_{\alpha}((\sigma+1)\eta^{\alpha})}\Big).$$

6.2 Example

In 1973, Fischer Black and Myron Scholes [24] suggested the famous theoretical valuation formula for options. The main fictional idea of Black and Scholes excites in the texture of a riskless portfolio taking positions in bonds (cash), option and the underlying stock. Such an process strengthens the use of the no-arbitrage principle as well. The Black-Scholes model for the value of an option can be described by the fractional equation

$$D_t^\alpha u + \rho(t,x) D_x^{2\alpha} u + u D_x^\alpha u - r(t,x)u = 0, \quad t \in (0,T) \tag{6.23}$$

where $u(t,x)$ is the European call option price at asset price x(positive real number) and at time t; $r(t,x)$ is the risk free interest rate, and $\rho(t,x)$ represents the volatility function of underlying asset. By employing the

wave transform

$$u(t,x) = u(\eta), \quad \eta = \lambda t + \lambda_1 x,$$

we extradite

$$\lambda^\alpha D_\eta^\alpha u + \lambda_1^{2\alpha} \rho(\eta) D_\eta^{2\alpha} u + \lambda_1^\alpha u D_\eta^\alpha u - r(\eta)u = 0. \tag{6.24}$$

Now in virtue of the above method, we have

$$a_0 = \frac{-\lambda^\alpha \sigma}{\lambda_1^\alpha \sigma - r}, \quad a_1 = \frac{-2\lambda_1^{2\alpha} \rho\sigma}{\lambda_1^\alpha \sigma - r},$$

where $\lambda_1^\alpha \sigma \neq r$. Hence some exact solutions of (23) can be expressed as follows:

$$u(\eta) = \frac{-\lambda^\alpha \sigma}{\lambda_1^\alpha \sigma - r} - \frac{2\lambda_1^{2\alpha} \rho\sigma}{\lambda_1^\alpha \sigma - r} \Big(\frac{-\sigma B + D E_\alpha((\sigma+1)\eta^\alpha)}{D + B E_\alpha((\sigma+1)\eta^\alpha)} \Big).$$

7 Conclusion

From above we conclude that the transform method of fractional differential equation affected on the exact solutions of fractional differential equations. This method has more advantages: it is direct and concise. Thus, we realize that the proposed method can be extended to solve many systems of nonlinear fractional partial differential equations. Moreover, these solutions are analytic in their domains. The applications are taken for liquid move equation (Eq.(21)) and the Black-Scholes model (Eq.(23)).

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Received: May 18, 2014; *Accepted*: May 28, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 419–425

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Fractional differintegral operators of the generalized Mittag-Leffler type function

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Abstract

In the present paper we study a new function called as R-function [6], which is an extension of the generalized Mittag-Leffler functions. We derive the relations that exist between the R-function and Saigo-Maeda fractional calculus operators. Some results derived by Kumar and Kumar [6], Kilbas [4], Kilbas and Saigo [5]; and Sharma and Jain [23] are special cases of the main results derived in this paper.

Keywords: Fractional calculus, fractional differintegral operators, generalized Mittag-Leffler function, R-function.

2010 MSC: 26A33, 33C45, 33E12, 33E20.

1 Introduction and preliminaries

The Mittag-Leffler function has gained importance and popularity during the last one and a half decades due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differintegral equations.

In 1903, the Swedish mathematician Gosta Mittag-Leffler [9, 10] introduced studied the function $E_\alpha(z)$, defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \ (\alpha \in C, Re(\alpha) > 0). \tag{1.1}$$

A generalization of this series given by Wiman [27] who defined the function $E_{\alpha,\beta}(z)$ as follows

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ (\alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0). \tag{1.2}$$

The function $E_{\alpha,\beta}(z)$ is now known as Wiman function, which was later studied by Agarwal [1] and others. The generalization of (1.2) was introduced by Prabhakar [11] in terms of the series representation as given following:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!}, \ (\alpha, \beta, \gamma \in C, Re(\alpha) > 0, Re(\beta) > 0), \tag{1.3}$$

Shukla and Prajapati [24] defined and investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ as

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta) n!}, \ (\alpha, \beta, \gamma \in C, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0), \tag{1.4}$$

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where $q \in (0,1) \cup N$ and $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol which in particular reduces to

$$q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma + r - 1}{q} \right)_n, \quad q \in N.$$

Srivastava and Tomovski [26] introduced and investigated a further generalization of (1.3), which is defined in the following way:

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} z^n}{\Gamma(\alpha n + \beta) n!}, \; (z, \beta, \gamma \in C; Re(\alpha) > max\{0, Re(k) - 1\}; Re(k) > 0), \tag{1.5}$$

which, in the special case when $k = q \;\; (q \in (0,1) \cup N)$ and $min\{Re(\beta), Re(\gamma)\} > 0$, is given by (1.4). It is an entire function of order $\rho = [Re(\alpha)]^{-1}$. Some special cases of (1.3) are

$$E_\alpha(z) = E_{\alpha,1}^1(z), E_{\alpha,\beta}(z) = E_{\alpha,\beta}^1(z), \phi(\beta,\gamma;z) = {}_1F_1(\beta,\gamma;z) = \Gamma(\gamma) E_{1,\gamma}^{\beta}(z), \tag{1.6}$$

An interesting generalization of (1.2) is recently introduced by Kilbas and Saigo [5] in terms of a special entire function as given below

$$E_{\alpha,m,r}(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{1.7}$$

where $c_n = \prod_{i=0}^{n-1} \frac{\Gamma[\alpha(im+r)+1]}{\Gamma[\alpha(im+r+1)+1]}$ and an empty product is to be interpreted as unity.

In order to prove our main results we only provide here the basic definitions of left-sided fractional calculus operators. The readers can refer for detailed account of fractional calculus operators in several papers [15, 16, 17] and many more

Let $\alpha, \alpha', \beta, \beta', \gamma \in C, \; x > 0$, then the left-sided $(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma})$ generalized fractional integral operators of a function $f(x)$ for $Re(\gamma) > 0$ is defined by Saigo and Maeda [16], in the following form:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t) \mathrm{d}t, \tag{1.8}$$

This operator reduce to the left-sided Saigo fractional integral operator [15] due to the following relation:

$$I_{0+}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{0+}^{\gamma,\alpha-\gamma,-\beta} f(x) \quad (\gamma \in C), \tag{1.9}$$

Further, if we set $\beta = -\alpha$, then operator (1.9) reduces to left-sided Riemann-Liouville fractional integral operator

$$I_{0+}^{\alpha,-\alpha,\gamma} f(x) = I_{0+}^{\alpha} f(x), \tag{1.10}$$

Let $\alpha, \alpha', \beta, \beta', \gamma \in C$, and $x \in R_+$, then the left-sided generalized fractional differentiation operator [16] involving the Appell function F_3 as a kernel are defined by the following equation:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f \right)(x) \tag{1.11}$$

$$= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx} \right)^n \left(x^{\alpha'} \right) \int_0^x (x-t)^{n-\gamma-1} t^{\alpha}$$

$$\times F_3\left(-\alpha', -\alpha, n - \beta', -\beta, n - \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) \, dt, \tag{1.12}$$

The above operator reduce to the left-sided Saigo fractional derivative operator [15, 18] as

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f \right)(x) = \left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f \right)(x), \quad (Re(\gamma) > 0); \tag{1.13}$$

If we set $\beta = -\alpha$, then operator (1.13) reduces to left-sided Riemann-Liouville fractional derivative operator

$$D_{0+}^{\alpha,-\alpha,\gamma} f(x) = D_{0+}^{\alpha} f(x). \tag{1.14}$$

Under various fractional calculus operators, the computations of image formulas for special functions are very important from the point of view of the usefulness of these results in the evaluation of generalized integrals and the solution of differential and integral equations. Therefore, in the literature we found several papers on the subject, see for instance [12], [13], [19]-[21] and [2] and references cited therein.

2 The generalized Mittag-Leffler type function (R-function)

The R-function is introduced and studied by Kumar and Kumar [6] as follows:

$$ {}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(z) = {}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(a_1,...,a_p;b_1,...,b_q;z) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{(\gamma)_{kn} z^n}{n!\Gamma(\alpha n+\beta)}, \tag{2.1} $$

where $\alpha,\beta,\gamma \in C, Re(\alpha) > max\{0, Re(k)-1\}; Re(k) > 0; (a_j)_n$ and $(b_j)_n$ are the Pochhammer symbols. The series (2.1) is defined when none of the parameters b_j's, $j = \overline{1,q}$ is a negative integer or zero. If any parameter a_j is a negative integer or zero, then the series (2.1) terminates to a polynomial in z , and the series is convergent for all z if $p < q+1$. It can also converge in some cases if we have $p = q+1$ and $|z| = 1$. Let $\gamma = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$, it can be shown that if $Re(\gamma) > 0$ and $p = q+1$ the series is absolutely convergent for $|z| = 1$, in order convergent for $z = -1$ when $0 \le Re(\gamma) < 1$ and divergent for $|z| = 1$ when $1 \le Re(\gamma)$.

Special Cases of the R-function:

(i) If we set $a_j = b_j = 1$, we have

$$ {}_{0}^{k}R_{0}^{\alpha,\beta;\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} z^n}{n!\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}^{\gamma,k}(z), \tag{2.2} $$

where $E_{\alpha,\beta}^{\gamma,k}(z)$ is the generalized Mittag-Leffler function which introduced by Srivastava and Tomovski [26].

(ii) In the special case of (2.2), when $k = q \ \ (q \in (0,1) \cup N)$ and $min\{Re(\beta),\ Re(\gamma)\} > 0$, we have the following:

$$ {}_{0}^{q}R_{0}^{\alpha,\beta;\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{n!\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}^{\gamma,q}(z), \tag{2.3} $$

where $E_{\alpha,\beta}^{\gamma,q}(z)$ was considered earlier by Shukla and Prajapati [24].

(iii) If we set $a_j = b_j = k = 1$ in (2.1), we have

$$ {}_{0}^{1}R_{0}^{\alpha,\beta;\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n} z^n}{n!\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}^{\gamma}(z), \tag{2.4} $$

where $E_{\alpha,\beta}^{\gamma}(z)$ is generalization of the Mittag-Leffler function which introduced by Prabhakar [11], and studied by Haubold et al. [3] and others.

(iv) If we put $\gamma = 1$ in (2.4), we have

$$ {}_{0}^{1}R_{0}^{\alpha,\beta;1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}^{1}(z) = E_{\alpha,\beta}(z), \tag{2.5} $$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function [27] (also known as Wiman function), which was later studied by Agarwal [1] and others.

(v) If we take $\beta = \gamma = 1$ in (2.4), we have

$$ {}_{0}^{1}R_{0}^{\alpha,1;1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} = E_{\alpha,1}^{1}(z) = E_{\alpha}(z), \tag{2.6} $$

where $E_{\alpha}(z)$ is the Mittag-Leffler function [9, 10], compare (1.1).

(vi) If we take $\alpha = \beta = \gamma = 1$ in (2.4), we obtain

$$ {}_{0}^{1}R_{0}^{1,1;1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = E_{1,1}^{1}(z) = E_{1}(z) = e^{x}, \tag{2.7} $$

where e^x is the Exponential function [14].

(vii) If we set $\gamma = k = 1$ in (2.1), then the R-function can be represented in the Wright generalized hypergeometric function [28] ${}_p\psi_q(z)$ and the H-function [4, 8] as given below

$$
\begin{aligned}
{}_p^1R_q^{\alpha,\beta;1}(z) = {}_p^1R_q^{\alpha,\beta;1}(a_1,...,a_p;b_1,...,b_q;z) &= \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\, {}_{p+1}\psi_{q+1}\left[z\middle|\begin{matrix}(a_1,1),\cdots,(a_p,1),(1,1)\\ (b_1,1),\cdots,(b_q,1),(\beta,\alpha)\end{matrix}\right] \\
&= \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} H^{1,p+1}_{p+1,q+2}\left[-z\middle|\begin{matrix}(1-a_j,1)_{1,p},(0,1)\\ (0,1),(1-b_j,1)_{1,q},(1-\beta,\alpha)\end{matrix}\right], \qquad (2.8)
\end{aligned}
$$

where H-function is as defined in the monograph by Mathai et al. [8].

(viii) If we set $p = q = 0$, and $\gamma = k = 1$ in (2.1), then we obtain another special case of R-function in terms of the Wright generalized hypergeometric function as given below:

$$
{}_0^1R_0^{\alpha,\beta;1}(z) = {}_0^1R_0^{\alpha,\beta;1}(-;1;z) = \sum_{n=0}^{\infty}\frac{\Gamma(n+1)\,z^n}{\Gamma(\alpha n+\beta)\,n!} = \frac{(1)_n\,z^n}{\Gamma(\alpha n+\beta)\,n!} = {}_1\psi_1\left[z\middle|\begin{matrix}(1,1)\\ (\beta,\alpha)\end{matrix}\right], \qquad (2.9)
$$

(ix) If we set $\alpha = \beta = \gamma = k = 1$ in (2.1), then the R-function reduces to the generalized hypergeometric function ${}_pF_q$ (see for detail [7, 14, 17]) as given

$$
{}_p^1R_q^{1,1;1}(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!} = {}_pF_q\left((a_j)_{1,p};(b_j)_{1,q};z\right). \qquad (2.10)
$$

3 Main results

This section deals with results, which established well defined ralations for generalized fractional differintegrals (fractional integral and differential operators) and generalized Mittag-Leffler type function (R-function), defined by (2.1).

Theorem 3.1. *Let $\vartheta,\vartheta',\eta,\eta',\delta,\alpha,\beta,\gamma \in C$, $Re(\delta) > 0$, $Re(\alpha) > 0$ and $(a)_n = \Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$
\begin{aligned}
I_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right) &= x^{-\vartheta-\vartheta'+\delta}\frac{\Gamma(1+\delta-\vartheta-\vartheta'-\eta)\,\Gamma(1+\eta'-\vartheta')}{\Gamma(1+\delta-\vartheta-\vartheta')\,\Gamma(1+\delta-\vartheta'-\eta)\,\Gamma(1+\eta')} \\
&\times {}_{p+3}^{\ \ k}R_{q+3}^{\alpha,\beta;\gamma}(a_1,\ldots,a_p,1,1+\delta-\vartheta-\vartheta'-\eta,1+\eta'-\vartheta'; \\
&\qquad b_1,\ldots,b_q,1+\delta-\vartheta-\vartheta',1+\delta-\vartheta'-\eta,1+\eta';x). \qquad (3.1)
\end{aligned}
$$

Proof. Following the definition of Saigo-Maeda fractional integral [16] as given in (1.8), we have the following relation:

$$
I_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right) = \frac{x^{-\vartheta}}{\Gamma(\delta)}\int_0^x (x-t)^{\delta-1}\,t^{-\vartheta'}F_3\left(\vartheta,\vartheta',\eta,\eta',\delta;1-\frac{t}{x},1-\frac{x}{t}\right)\,{}_p^kR_q^{\alpha,\beta;\gamma}(t)\,dt
$$

by virtue of (2.1), we obtain

$$
\begin{aligned}
I_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right) &= \frac{x^{-\vartheta}}{\Gamma(\delta)}\int_0^x (x-t)^{\delta-1}\,t^{-\vartheta'}F_3\left(\vartheta,\vartheta',\eta,\eta',\delta;1-\frac{t}{x},1-\frac{x}{t}\right) \\
&\times \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{(\gamma)_{kn}\,t^n}{n!\Gamma(\alpha n+\beta)}\,dt. \qquad (3.2)
\end{aligned}
$$

Interchanging the order of integration and evaluating the inner integral with the help of Beta function, we arrive at

$$
\begin{aligned}
I_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right) &= x^{-\vartheta-\vartheta'+\delta}\frac{\Gamma(1+\delta-\vartheta-\vartheta'-\eta)\,\Gamma(1+\eta'-\vartheta')}{\Gamma(1+\delta-\vartheta-\vartheta')\,\Gamma(1+\delta-\vartheta'-\eta)\,\Gamma(1+\eta')} \\
&\times \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,(1)_n\,(1+\delta-\vartheta-\vartheta'-\eta)_n\,(1+\eta'-\vartheta')_n}{\prod_{j=1}^{q}(b_j)_n\,(1+\delta-\vartheta-\vartheta')_n\,(1+\delta-\vartheta'-\eta)_n\,(1+\eta')_n}\frac{(\gamma)_{kn}\,x^n}{n!\Gamma(\alpha n+\beta)}
\end{aligned}
$$

$$= x^{-\vartheta-\vartheta'+\delta} \frac{\Gamma(1+\delta-\vartheta-\vartheta'-\eta)\,\Gamma(1+\eta'-\vartheta')}{\Gamma(1+\delta-\vartheta-\vartheta')\,\Gamma(1+\delta-\vartheta'-\eta)\,\Gamma(1+\eta')}$$

$$\times\ {}_{p+3}^{\ \ k}R_{q+3}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\delta-\vartheta-\vartheta'-\eta,1+\eta'-\vartheta'\,;\right.$$

$$\left. b_1,\ldots,b_q,1+\delta-\vartheta-\vartheta',1+\delta-\vartheta'-\eta,1+\eta';x\right).$$

The interchange of the order of summation is permissible under the conditions stated along with the theorem. This shows that a Saigo-Maeda fractional integral of the R-function is again the R-function with increased order $(p+3,q+3)$.

This completes the proof of the Theorem 1. □

In view of the relation (1.9), we obtain the result given by Kumar and Kumar [6] concerning Saigo fractional integral operator asserted by the following corollary.

Corollary 3.1. *Let $\vartheta,\eta,\delta,\alpha,\beta,\gamma\in C$, $Re(\vartheta)>0$, $Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$I_{0+}^{\vartheta,\eta,\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right)=\frac{x^{-\eta}\Gamma(1+\delta-\eta)}{\Gamma(1+\vartheta+\delta)\Gamma(1-\eta)}$$

$$\times\ {}_{p+2}^{\ \ k}R_{q+2}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\delta-\eta;b_1,\ldots,b_q,1+\vartheta+\delta,1-\eta;x\right). \tag{3.3}$$

Further, if we put $\eta=-\vartheta$ in (3.3) then we obtain following Corollary concerning Riemann-Liouville fractional integral operator [17]:

Corollary 3.2. *Let $\vartheta,\alpha,\beta,\gamma\in C, Re(\vartheta)>0, Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$I_{0+}^{\vartheta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right)=\frac{x^{\vartheta}}{\Gamma(1+\vartheta)}\ {}_{p+1}^{\ \ k}R_{q+1}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1;b_1,\ldots,b_q,1+\vartheta;x\right). \tag{3.4}$$

Theorem 3.2. *Let $\vartheta,\vartheta',\eta,\eta',\delta,\alpha,\beta,\gamma\in C$, $Re(\delta)>0$, $Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$D_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right)=x^{\vartheta+\vartheta'-\delta}\frac{\Gamma(1+\vartheta+\vartheta'+\eta'-\delta)\,\Gamma(1+\vartheta-\eta)}{\Gamma(1+\vartheta+\vartheta'-\delta)\,\Gamma(1+\vartheta+\eta'-\delta)\,\Gamma(1-\eta)}$$

$$\times\ {}_{p+3}^{\ \ k}R_{q+3}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\vartheta+\vartheta'+\eta'-\delta,1+\vartheta-\eta\,;\right.$$

$$\left. b_1,\ldots,b_q,1+\vartheta+\vartheta'-\delta,1+\vartheta+\eta'-\delta,1-\eta;x\right). \tag{3.5}$$

Proof. Following the definition of Saigo-Maeda fractional derivative [16] as given in (1.12), we have the following relation:

$$D_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right)=\frac{x^{\vartheta'}}{\Gamma(r-\delta)}\left(\frac{d}{dx}\right)^r\int_0^x(x-t)^{r-\delta-1}\,t^{\vartheta}$$

$$\times F_3\left(-\vartheta',-\vartheta,r-\eta',-\eta,r-\delta;1-\frac{t}{x},1-\frac{x}{t}\right)\ {}_p^kR_q^{\alpha,\beta;\gamma}(t)\,dt$$

where $r=[-Re(\delta)]+1$ by virtue of (2.1), we obtain

$$D_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right)=\frac{x^{\vartheta'}}{\Gamma(r-\delta)}\left(\frac{d}{dx}\right)^r\int_0^x(x-t)^{r-\delta-1}\,t^{\vartheta}$$

$$\times F_3\left(-\vartheta',-\vartheta,r-\eta',-\eta,r-\delta;1-\frac{t}{x},1-\frac{x}{t}\right)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_n}{\prod_{j=1}^q(b_j)_n}\frac{(\gamma)_{kn}\,t^n}{n!\Gamma(\alpha n+\beta)}\,dt. \tag{3.6}$$

By using $\frac{d^r}{dx^r}x^m=\frac{\Gamma(m+1)}{\Gamma(m-r+1)}x^{m-r}$ $(m,r\in N_0;\ m\geq r)$ in (3.6) and interchanging the order of integration and evaluating the inner integral with the help of Beta function, we arrive at

$$D_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right)=x^{\vartheta+\vartheta'-\delta}\frac{\Gamma(1+\vartheta+\vartheta'+\eta'-\delta)\,\Gamma(1+\vartheta-\eta)}{\Gamma(1+\vartheta+\vartheta'-\delta)\,\Gamma(1+\vartheta+\eta'-\delta)\,\Gamma(1-\eta)}$$

$$\times\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p(a_j)_n\,(1)_n\,(1+\vartheta+\vartheta'+\eta'-\delta)_n\,(1+\vartheta-\eta)_n}{\prod_{j=1}^q(b_j)_n\,(1+\vartheta+\vartheta'-\delta)_n\,(1+\vartheta+\eta'-\delta)_n\,(1-\eta)_n}\frac{(\gamma)_{kn}\,x^n}{n!\Gamma(\alpha n+\beta)}$$

$$= x^{\vartheta+\vartheta'-\delta} \frac{\Gamma(1+\vartheta+\vartheta'+\eta'-\delta)\,\Gamma(1+\vartheta-\eta)}{\Gamma(1+\vartheta+\vartheta'-\delta)\,\Gamma(1+\vartheta+\eta'-\delta)\,\Gamma(1-\eta)}$$
$$\times\ {}_{p+3}^{\ k}R_{q+3}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\vartheta+\vartheta'+\eta'-\delta,1+\vartheta-\eta\,;\right.$$
$$\left. b_1,\ldots,b_q,1+\vartheta+\vartheta'-\delta,1+\vartheta+\eta'-\delta,1-\eta;x\right).$$

This shows that a Saigo-Maeda fractional derivative of the R-function is again the R-function with increased order $(p+3,q+3)$.

This completes the proof of the Theorem 2. □

Now, on making use the relation (1.13), we obtain the result concerning Saigo fractional derivative operator given by [6] asserted by the following corollary.

Corollary 3.3. *Let $\vartheta,\eta,\delta,\alpha,\beta,\gamma\in C$, $Re(\vartheta)>0$, $Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$D_{0+}^{\vartheta,\eta,\delta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right)=\frac{x^{\eta}\,\Gamma(1+\vartheta+\eta+\delta)}{\Gamma(1+\delta)\Gamma(1+\eta)}$$
$$\times\ {}_{p+2}^{\ k}R_{q+2}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\vartheta+\eta+\delta;b_1,\ldots,b_q,1+\delta,1+\eta;x\right). \tag{3.7}$$

Again, if we further put $\eta=-\vartheta$ in (3.7), then we obtain following corollary concerning Riemann-Liouville fractional derivative operator [17]:

Corollary 3.4. *Let $\vartheta,\alpha,\beta,\gamma\in C, Re(\vartheta)>0, Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$D_{0+}^{\vartheta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right)=\frac{x^{-\vartheta}}{\Gamma(1-\vartheta)}\,{}_{p+1}^{\ k}R_{q+1}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1;b_1,\ldots,b_q,1-\vartheta;x\right) \tag{3.8}$$

It is remarked in passing that a number of known and new results can be obtained as special cases of the Theorems 3.1 and 3.2.

4 Conclusion

In this paper we derive a new generalization of Mittag-Leffler function and obtain the relations between the R-function and Saigo-Maeda fractional calculus operators. The results are also extension of work done by Kumar and Kumar [6] and Sharma [22]. The provided results are new and have uniqueness identity in the literature. A number of known results can easily be found as special cases of our main results.

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Received: April 3, 2014; *Accepted*: July 10, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 426–427

Malaya Journal of Matematik
www.malayajournal.org

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ISSN : 2319-3786

On null sets in measure spaces

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Abstract

In this short work, first, we have a review on null sets in measure spaces. Next, we present an interesting example of a null set.

Keywords: Null set, Measure space, Sard's lemma.

2010 MSC: 28C20.

1 Introduction

In the section, we have a brief review on some properties of null sets.
In mathematics, a null set is a set that is negligible in some sense. In measure theory, any set of measure 0 is called a null set (or simply a measure-zero set). More generally, whenever an ideal is taken as understood, then a null set is any element of that ideal.
Null sets play a key role in the definition of the Lebesgue integral: if functions f and g are equal except on a null set, then f is integrable if and only if g is, and their integrals are equal. Indeed, via null sets we give a sufficient and necessary condition for integrability of a bounded real function:

Theorem 1.1. *If $f(x)$ is bounded in $[a,b]$, then a necessary and sufficient condition for the existence of $\int_a^b f(x)dx$ is that the set of discontinuities have measure zero [1].*

A measure in which all subsets of null sets are measurable is complete. Any non-complete measure can be completed to form a complete measure by asserting that subsets of null sets have measure zero. Lebesgue measure is an example of a complete measure; in some constructions, it's defined as the completion of a non-complete Borel measure.
A famous example for a null set is given by Sard's lemma.

Example 1.1 (Sard's lemma). *The set of critical values of a smooth function has measure zero [2].*

In the following, we present some another examples of null sets.

Example 1.2. *Any countable set has zero measure [1].*

Example 1.3. *All the subsets of $\mathbb{R}^n$ whose dimension is smaller than n have null Lebesgue measure in $\mathbb{R}^n$.*

Note that it may possible an uncountable set has zero measure; For instance, the standard construction of the Cantor set is an example of a null uncountable set in $\mathbb{R}$; however other constructions are possible which assign the Cantor set any measure whatsoever.
It is well-known and easy to show that a subset of a set of measure zero also has measure zero and a countable union of sets of measure zero also has measure zero.

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Remark 1.1. *Isomorphic sets may have different measures; In the other hand, a measure is not preserved by bijections. The most famous example would be the Cantor set* $\mathbf{C}$*. One can show that* $\mathbf{C}$ *has measure zero, yet there exists a bijection between* $\mathbf{C}$ *and* $[0,1]$*, which does not have measure zero.*

Let's end with an interesting example showing that a sum of two measure zero sets may has positive measure.

Example 1.4. *Let* $\mathbf{C}$ *be the Cantor set. Define*

$$\mathbf{C}+\mathbf{C}=\{a+b:\ a,b\in\mathbf{C}\}$$

It can be seen easily that $\mathbf{C}+\mathbf{C}=[0,2]$*. Hence we have a sum of two measure zero sets which has positive measure.*

Another properties of null sets and measurable spaces can be found in [3, 4].

2 An interesting Null Set

In the following theorem, we have presented a null set.

Theorem 2.2. *Let* X *be a nonempty set and* $\mu : 2^X \to [0,\infty)$ *an outer measure. Suppose that* (A_n) *be a sequence of subsets in* 2^X *such that* $\sum_n \mu(A_n) < \infty$*. Consider the set* $F = \{x \in X :\ x \text{ belong to infinitely many of } A_k's\}$*. Then* $\mu(F) = 0$*.*

Proof. By Example 1.2, it is enough to prove that F is countable. Evidently, for each $x \in F$, there is $n_x \in \mathbb{N}$ so that $x \in \bigcap_{k=n_x}^{\infty} A_k$. Define the relation $\sim$ on X as follow:

$$x \sim y \Leftrightarrow n_x = n_y$$

It is easy to verify that $\sim$ is an equivalence relation on F. Set $N_F := \{n_x :\ x \in F\}$. Clearly $N_F \subset \mathbb{N}$. Now, define the function $f : E\mathbf{C}(F) \to N_F$ by $f([x]) = n_x$, where $E\mathbf{C}(F)$ denotes the set of all equivalent classes of F. Since equivalence classes partite F, so f is well-defined. Obviously, f is onto. Let $n_x = n_y$. This implies that $x \sim y$, i.e., $x \in [y]$. Also, it follows that $y \in [x]$. Therefore, $x = y$. Thus f is an one to one corresponding. Hence $E\mathbf{C}(F)$ is a countable set. Finally, by defining the function $g : E\mathbf{C}(F) \to F$, $g([x]) = x$, we conclude that F is countable, as desired. □

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Received: March 12, 2014; *Accepted*: July 20, 2014

Malaya J. Mat. 2(3)(2014) 428–437

Malaya Journal of Matematik
www.malayajournal.org

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ISSN : 2319-3786

Mild solution for fractional functional integro-differential equation with not instantaneous impulse

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Abstract

In this investigation, we prove the existence uniqueness and continuous dependence results of mild solution for nonlocal fractional differential equations with state dependent delay subject to not instantaneous impulse. We illustrate the existence result by an example involving partial derivatives.

Keywords: Fractional order differential equation, Functional differential equations, Impulsive conditions, Fixed point theorem.

2000 MSC: 26A33,34K05,34A12,34A37,26A33.

1 Introduction

The generalization of the theory of ordinary differential equation is a differential equation with an arbitrary non integer order. Fractional differential equations are widely used in modeling of several fields such as Science, Physics, Engineering and Economy due to this reason differential equations with fractional order have received increasing attentions in recent years, see [1, 2, 3, 4, 5, 6, 7]. Fractional equations with delay properties arise in serval fields such as biological and physical with state dependent delay or non constant delay. Presently, existence results of mild solutions for such problems became very attractive and several researchers are working on it. Many number of papers have been written on the fractional order problems with state dependent delay [13, 14, 17, 18, 22, 25, 27] and references therein.

Impulsive differential equations with fractional order have gained much attention, since it is much richer in terms of its applications. Impulsive effect exist widely in many phenomena in which their states are changed abruptly at certain time of moments. Recently, the results of existence and uniqueness of impulsive evolution equations in infinite dimensional spaces have been investigated by several authors [8, 9, 10, 11, 12, 15, 19, 20, 21, 23, 24, 26, 29].

Araya et al. [6] study the following problem:

$$D_t^{\alpha} u(t) = Au(t) + t^n f(t, u(t), u'(t)),\ t \in \mathbb{R}, n \in Z_+, 1 \leq \alpha \leq 2,$$

and introduce the concept of α-resolvent families and then proved the existence and uniqueness results of almost automorphism mild solution. Mophou et al. [7] established the existence and uniqueness of mild solution of the following Cauchy problem

$$D_t^{\alpha} x(t) = Ax(t) + t^n f(t, x(t), Bx(t)),\ t \in [0, T], n \in Z_+,\ x(0) = x_0 + g(x).$$

Recently, Herndez et al. [9] have introduced a new class of abstract impulsive differential equations for which the impulses are not instantaneous

$$u'(t) = Au(t) + f(t, u(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N, \tag{1.1}$$

$$u(t) = g_i(t, u(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N,\ u(0) = x_0, \tag{1.2}$$

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and established the existence and uniqueness results of mild and classical solutions by using classical fixed point theorems. In the model equation (1.1)-(1.2), the impulses start abruptly at the points t_i and their action continue on a finite time interval $[t_i, s_i]$. As pointed in [9], there are many different motivations for the study of this type of problem. For example as in [9], we note the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorbtion for the body are gradual and continuous processes, we can interpret this situation as an impulsive action which starts abruptly and stays active on a finite time interval.

Further, Pierri et al. [10] have extended the results of [9] by using the theory of analytic semigroup and fractional power of closed operators and established the existence results of solutions for a class of semi-linear abstract impulsive differential equations with not instantaneous impulses. Further, Wang et al. [12] study the problem (1.1)-(1.2) for the cases if $\alpha \in (0,1]$ and $\alpha = 1$ with $A = 0$ and with periodic boundary condition $u(0) = u(T)$.

Kumar et al. [11] have studied the the following fractional order problem with not instantaneous impulse

$$^{C}D_t^{\beta} u(t) + Au(t) = f(t, u(t), g(u(t))),\ t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N, \tag{1.3}$$

$$u(t) = g_i(t, u(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N,\ u(0) = u_0 \in H, \tag{1.4}$$

by using the Banach fixed point theorem with condensing map established the existence and uniqueness results.

Motivated by the above mention works [6, 7, 8, 9], we consider the following fractional differential equation with not instantaneous impulses of the form

$$D_t^{\alpha} u(t) = Au(t) + t^n f(t, u_{\rho(t,u_t)}) + \int_0^t q(t-s) h(s, u_{\rho(s,u_s)}) ds,\ t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N, \tag{1.5}$$

$$u(t) + l(u) = \phi(t),\ t \in (-\infty, 0], \tag{1.6}$$

$$u(t) = g_i(t, y(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N, \tag{1.7}$$

where D_t^{α} is Caputo's fractional derivative of order $\alpha \in (0,1], n \in Z^+$ and $J = [0, T]$ is operational interval. The map $A : D(A) \subset X \to X$ is the a closed linear sectorial type operator defined on a Banach space $(X, \|\cdot\|)$. Here $f, h : J \times \mathfrak{B}_h \to X, q : J \to X$, and $\rho : J \times \mathfrak{B}_h \to (-\infty, T]$ are appropriate functions and satisfied some axioms. The functions $g_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, 2, \cdots, N$, is stand for impulsive conditions and $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_N \le s_N \le t_{N+1} = T$, are pre-fixed numbers. The nonlocal condition $l : X \to X$, defined as $l(u) = \sum_{k=1}^{r} c_k u(t_k)$, where $c_k, k = 1, \cdots, r$, are given constants and $0 < t_1 < t_2 < \cdots < t_r < T$. The history function $u_t : (-\infty, 0] \to X$ is element of $\mathfrak{B}_h$ and defined by $u_t(\theta) = u(t + \theta),\ \theta \in (-\infty, 0]$ respectively. The nonlocal condition [28], $u(0) + l(u)$ to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local condition $u(0) = u_0$. The Problem (1.5)-(1.7) appears in mathematical models of viscoelasticity and other fields of science which gives the better result using nonlocal condition.

Equation (1.5) is very important due to its appearance in mathematical modeling of viscoelasticity and other fields of science and engineering. This fact motivate us to study the existence results of the equation (1.5) with not instantaneous impulses and nonlocal condition. To the best of our knowledge the existence results for the considered problem (1.5)-(1.7) in the present paper are new. This paper has four sections, in which second section provides some basic definitions, theorems, notations and lemma. Third section is equipped with existence results of the mild solution of the considered problem and fourth section contained an example to verify the results.

2 Preliminaries and Definitions

Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the norm $\|u\|_X = \sup_{t \in J}\{|u(t)| : u \in X\}$ and $L(X)$ denotes the Banach space of bounded linear operators from X into X equipped with its natural topology. Due to infinite delay, we use abstract phase space $\mathfrak{B}_h$ as defined in [15] details are as follow:

Assume that $h : (-\infty, 0] \to (0, \infty)$ is a continuous functions with $l = \int_{-\infty}^{0} h(s) ds < \infty, t \in (-\infty, 0]$. For any $a > 0$, we define

$$\mathfrak{B} = \{\psi : [-a, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable}\},$$

equipped the space $\mathfrak{B}$ with the norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|_X, \forall\, \psi \in \mathfrak{B}$. Let us define

$$\mathfrak{B}_h = \{\psi : (-\infty, 0] \to X, \text{ s.t. for any } a \geq c > 0, \psi\,|_{[-c,0]} \in \mathfrak{B} \;\&\; \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds < \infty\}.$$

If $\mathfrak{B}_h$ is endowed with the norm $\|\psi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds,\ \forall\ \psi \in \mathfrak{B}_h$, then it is clear that $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a complete Banach space. We consider the space

$$\mathfrak{B}'_h := PC((-\infty, T]; X),\ T < \infty,$$

be a Banach space of all such functions $u : (-\infty, T] \to X$, which are continuous every where except for a finite number of points $t_i,\ i = 1, 2, \ldots, N$, at which $u(t_i^+)$ and $u(t_i^-)$ exists and endowed with the norm

$$\|u\|_{\mathfrak{B}'_h} = \sup\{\|u(s)\|_X : s \in [0, T]\} + \|\phi\|_{\mathfrak{B}_h}, u \in \mathfrak{B}'_h,$$

where $\|\cdot\|_{B'_h}$ to be a semi-norm in $\mathfrak{B}'_h$.

For a function $u \in \mathfrak{B}'_h$ and $i \in \{0, 1, ..., N\}$, we introduce the function $\bar{u}_i \in C([t_i, t_{i+1}]; X)$ given by

$$\bar{u}_i(t) = \begin{cases} u(t), \text{ for } \ t \in (t_i, t_{i+1}], \\ u(t_i^+), \text{ for } \ t = t_i. \end{cases}$$

If $u : (-\infty, T] \to X$ s.t. $u \in \mathfrak{B}'_h$ then for all $t \in J$, the following conditions hold:

(C_1) $u_t \in \mathfrak{B}_h$.

(C_2) $\|u(t)\|_X \leq H\|u_t\|_{\mathfrak{B}_h}$.

(C_3) $\|u_t\|_{\mathfrak{B}_h} \leq K(t)\sup\{\|u(s)\|_X : 0 \leq s \leq t\} + M(t)\|\phi\|_{\mathfrak{B}_h}$, where $H > 0$ is constant; $K, M : [0,\infty) \to [0,\infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and K, M are independent of $u(t)$.

(C_{4_ϕ}) The function t$\to$ ϕ_t is well defined and continuous from the set

$$\Re(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in [0, T] \times \mathfrak{B}_h\}$$

into $\mathfrak{B}_h$ and there exists a continuous and bounded function $J^\phi : \Re(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathfrak{B}_h} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}_h}$ for every $t \in \Re(\rho^-)$.

Lemma 2.1. *([14])Let $u : (-\infty, T] \to X$ be function such that $u_0 = \phi, u\,|_{J_k} \in C(J_k, X)$ and if (C_{4_ϕ}) hold, then*

$$\|u_s\|_{\mathfrak{B}_h} \leq (M_b + J^\phi)\|\phi\|_{\mathfrak{B}_h} + K_b \sup\{\|u(\theta)\|_X;\ \theta \in [0, \max\{0, s\}]\},\ s \in \Re(\rho^-) \cup J,$$

where $J^\phi = \sup_{t\in\Re(\rho^-)} J^\phi(t)$, $M_b = \sup_{s\in[0,T]} M(s)$ and $K_b = \sup_{s\in[0,T]} K(s)$.

Example 2.1. *[27] Let $g : (-\infty, 0) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that $g(\xi,\theta) \leq \gamma(\xi)g(\theta)$ for all $\xi \leq 0$ and $\theta \in (-\infty, 0) \setminus N_\xi$ where $N_\xi \subseteq (-\infty, 0)$ is a set with Lebesgue measure zero. The space $\mathfrak{B}_h = C_0 \times L(g; X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous at zero, Lebesgue measurable and $g\|\varphi\|$ is Lebesgue integrable on $(-\infty, 0)$. The seminorm in $C_0 \times L(g; X)$ is defined by*

$$\|\varphi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} g(\theta)\|\varphi(\theta)\|d\theta.$$

It is clear that $C_0 \times L(g; X)$ is complete Banach space.

Definition 2.1 ([5]). Caputo's derivative of order $\alpha > 0$ with lower limit a, for a function $f : [0,\infty) \to \mathbb{R}$ such that $f \in C^n(\mathbb{R}_+, X)$ is defined as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = {}_a J_t^{n-\alpha} f^{(n)}(t),$$

where $a \geq 0,\ n \in N$. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{{}_0D_t^\alpha f(t); \lambda\} = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^k(0); \quad n-1 < \alpha \leq n.$$

Definition 2.2 ([5]). The Riemann-Liouville fractional integral operator of order $\alpha > 0$ with lower limit a, for a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ is defined by

$$ {}_aJ_t^0 f(t) = f(t),\ {}_aJ_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,\ t > 0, $$

where $a \geq 0,\ n \in N$ and $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. A two parameter function of the Mittag-Lefler type is defined by the series expansion

$$ E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k+\beta)} = \frac{1}{2\pi\iota} \int_c \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^\alpha - y} d\mu,\ \alpha,\ \beta > 0,\ y \in C, $$

where c is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |y|^{\frac{1}{\alpha}}$ counter clockwise. The Laplace integral of this function given by

$$ \int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega},\ Re\lambda > \omega^{\frac{1}{\alpha}},\ \omega > 0. $$

For more details on the above definition one can see the monographs of I. Podlubny [5].

Definition 2.4. *([16])* A closed and linear operator A is said to be sectorial if there are constants $\omega \in R,\ \theta \in [\frac{\pi}{2}, \pi],\ M > 0$, such that the following two conditions are satisfied:

(1) $\sum_{(\theta,\omega)} = \{\lambda \in C : \lambda \neq \omega,\ |arg(\lambda - \omega)| < \theta\} \subset \rho(A)$,

(2) $\|R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda-\omega|},\ \lambda \in \sum_{(\theta,\omega)}$,

where X is the complex Banach space with norm denoted $\|.\|_{L(X)}$.

Definition 2.5. *([6])* Let A be a closed and linear operator with domain D(A) defined on a Banach space X. Let $\rho(A)$ be the resolvent set of A, we call A is the generator of an α-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $T_\alpha : R^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and

$$ (\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} T_\alpha(t) x dt,\ Re\lambda > \omega,\ x \in X. $$

In this case, $T_\alpha(t)$ is called α-resolvent family generated by A.

Definition 2.6. *([13])* Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and

$$ \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} u = \int_0^\infty e^{-\lambda t} S_\alpha(t) u dt,\ Re\lambda > \omega,\ u \in X. $$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

Lemma 2.2. Consider the following Cauchy problem of order $0 < \alpha \leq 1$

$$ {}_aD_t^\alpha u(t) \quad = \quad Au(t) + f(t),\ t \in J = [a, T],\ a \geq 0,\ u(a) = u_0, \tag{2.8} $$

then a function $u(t) \in C([a, T], \mathbb{R})$ is called the solution of the equation (2.8) if f satisfies the uniform Holder condition with exponent $\beta \in (0, 1]$ and A is a sectorial operator and also satisfy the following integral equation

$$ u(t) = S_\alpha(t-a)u_0 + \int_a^t T_\alpha(t-s) f(s) ds, \tag{2.9} $$

where $S_\alpha(t), T_\alpha(t)$ are analytic solution operator and α-resolvent family generated by A and defined as

$$ S_\alpha(t) \quad = \quad \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} d\lambda, $$
$$ T_\alpha(t) \quad = \quad \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda^\alpha I - A)^{-1} d\lambda, $$

where Γ is a suitable path lying on $\sum_{\theta,\omega}$.

Proof. Let $t = w + a$, then the problem (2.8) translated into the form

$$_0D_w^\alpha \tilde{u}(w) = A\tilde{u}(w) + \tilde{f}(w),\ \tilde{u}(0) = u_0.$$

Now, applying the Laplace transform, we have

$$\begin{aligned}\lambda^\alpha L\{\tilde{u}(w)\} - \lambda^{\alpha-1}\tilde{u}(0) &= AL\{\tilde{u}(w)\} + L\{\tilde{f}(w)\}\\ L\{\tilde{u}(w)\}[\lambda^\alpha - A] &= \lambda^{\alpha-1}\tilde{u}(0) + L\{\tilde{f}(w)\}.\end{aligned} \tag{2.10}$$

Since $(\lambda^\alpha I - A)^{-1}$ exists, that is $\lambda^\alpha \in \rho(A)$, from (2.10), we obtain

$$L\{\tilde{u}(w)\} = \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}\tilde{u}(0) + (\lambda^\alpha I - A)^{-1}L\{\tilde{f}(w)\}.$$

Therefore, by taking the inverse Laplace transformation, we have

$$\tilde{u}(w) = E_{\alpha,1}(Aw^\alpha)\tilde{u}(0) + \int_0^w E_{\alpha,\alpha}(A(w-\tau)^\alpha)\tilde{f}(\tau)d\tau. \tag{2.11}$$

Putting $w = t - a$, in equation (2.11) then we obtain

$$u(t) = E_{\alpha,1}(A(t-a)^\alpha)u_0 + \int_0^{t-a} (t-a-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-a-\tau)^\alpha)f(\tau)d\tau.$$

This is the same as

$$u(t) = E_{\alpha,1}(A(t-a)^\alpha)u_0 + \int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s)ds. \tag{2.12}$$

Let $S_\alpha(t) = E_{\alpha,1}(At^\alpha)$, and $T_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$, then equation (2.12) we have

$$u(t) = S_\alpha(t-a)u_0 + \int_a^t T_\alpha(t-s)f(s)ds.$$

This completes the proof of the lemma. □

Now, we state the definition of mild solution based on definition 2.1 in [9].

Definition 2.7. A function $u : (-\infty, T] \to X$ such that $u \in \mathfrak{B}_h'$ is called a mild solution of the problem (1.5)-(1.7) if $u(0) = \phi(0),\ u(t) = g_j(t, u(t))$ for $t \in (t_j, s_j]$ and each $j = 1, 2, \cdots, N$, satisfies the following integral equation

$$u(t) = \begin{cases} S_\alpha(t)(\phi(0) - l(u)) + \int_0^t T_\alpha(t-s)s^n f(s, u_{\rho(s,u_s)})ds \\ + \int_0^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi, u_{\rho(\xi,u_\xi)})d\xi ds, \quad \text{for all } \ t \in [0, t_1], \\ S_\alpha(t-s_i)g_i(s_i, u(s_i)) + \int_{s_i}^t T_\alpha(t-s)s^n f(s, u_{\rho(s,u_s)})ds \\ + \int_{s_i}^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi, u_{\rho(\xi,u_\xi)})d\xi ds, \quad \text{for all } \ t \in [s_i, t_{i+1}], \end{cases}$$

for every $i = 1, 2, \cdots, N$. It can be verified easily from the lemma (2.2).

3 Existence and Uniqueness Result

In this section, we prove the existence results of mild solutions for the impulsive system (1.5)-(1.7). If $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$, then $S_\alpha(t) \leq Me^{\omega t}$ and $T_\alpha(t) \leq Ce^{\omega t}(1 + t^{\alpha-1})$. Let $\widetilde{M_S} := \sup_{0\leq t\leq T} \|S_\alpha(t)\|_{L(X)}$, $\widetilde{M_T} := \sup_{0\leq t\leq T} Ce^{\omega t}(1 + t^{1-\alpha})$. So we have $\|S_\alpha(t)\|_{L(X)} \leq \widetilde{M_S}$, $\|T_\alpha(t)\|_{L(X)} \leq t^{\alpha-1}\widetilde{M_T}$.

To prove our results we shall assume the function $\rho : [0, T] \times \mathfrak{B}_h \to (-\infty, T]$ is continuous and $\phi \in \mathfrak{B}_h$. If $y \in \mathfrak{B}_h$ we defined $\bar{y} : (-\infty, T) \to X$ as the extension of y to $(-\infty, T]$ such that $y(\bar{t}) = \phi$. We defined $\widetilde{y} : (-\infty, T) \to X$ such that $\widetilde{y} = y + x$ where $x : (-\infty, T) \to X$ is the extension of $\phi \in \mathfrak{B}_h$ such that $x(t) = S_\alpha(t)\phi(0)$ for $t \in J$. In the sequel we introduce the following axioms:

(H_1) There exists positive constants L_f, L_h, L_{g_i}, L_l such that

$$\|f(t,\varphi)-f(t,\psi)\|_X \leq L_f\|\varphi-\psi\|_{\mathfrak{B}_h},\ \|h(t,\varphi)-h(t,\psi)\|_X \leq L_h\|\varphi-\psi\|_{\mathfrak{B}_h},$$
$$\|g_i(t,u)-g_i(t,v)\|_X \leq L_{g_i}\|u-v\|_X,\ \|l(u)-l(v)\|_X \leq L_l\|u-v\|_X,$$

$t \in J, u, v \in X,\ \varphi, \psi \in \mathfrak{B}_h$ and each $i = 1, 2, \cdots, N$.

Theorem 3.1. Let the assumption (H_1) hold and the constant

$$\Delta = c^* + \frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_fK_b + \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_hK_b < 1,$$

where $c^* = \max\{\widetilde{M_S}L_{g_i}, \widetilde{M_S}L_l\}$ and for $i = 1, 2, \cdots, N$. Then there exists a unique mild solution $u(t)$ on J for the system (1.5)-(1.7).

Proof. Let $\bar{\phi} : (-\infty, T) \to X$ be the extension of ϕ to $(-\infty, T]$ such that $\bar{\phi(t)} = \phi(0)$ on J. Consider the space $\mathfrak{B}_h'' = \{y \in \mathfrak{B}_h' : y(0) = \phi(0)\}$ and $y(t) = \phi(t)$, for $t \in (-\infty, 0]$ endowed with the uniform convergence topology. Let us consider a operator $P : \mathfrak{B}_h'' \to \mathfrak{B}_h''$ defined as $Pu(0) = \phi(0),\ Pu(t) = g_i(t, \bar{u}(t))$ for $t \in (t_i, s_i]$ and

$$Pu(t) = \begin{cases} S_\alpha(t)(\phi(0) - l(\bar{u})) + \int_0^t T_\alpha(t-s)s^n f(s, \bar{u}_{\rho(s,\bar{u}_s)})ds \\ + \int_0^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi, \bar{u}_{\rho(\xi,\bar{u}_\xi)})d\xi ds, \quad \text{for all } t \in [0, t_1], \\ S_\alpha(t-s_i)g_i(s_i, \bar{u}(s_i)) + \int_{s_i}^t T_\alpha(t-s)s^n f(s, \bar{u}_{\rho(s,\bar{u}_s)})ds \\ + \int_{s_i}^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi, \bar{u}_{\rho(\xi,\bar{u}_\xi)})d\xi ds, \quad \text{for all } t \in [s_i, t_{i+1}], \end{cases}$$

where $\bar{u} : (-\infty, T] \to X$ is such that $\bar{u}(0) = \phi$ and $\bar{u} = u$ on J. It is obvious that P is well defined. We will show that the operator $P : \mathfrak{B}_h'' \to \mathfrak{B}_h''$ has a fixed point. So let $u(t), u^*(t) \in \mathfrak{B}_h''$ and $t \in [0, t_1]$, we get

$$\begin{aligned} \|Pu(t) - Pu^*(t)\|_X \leq\ & \|S_\alpha(t)\|_{L(X)}\|l(\bar{u})) - l(\bar{u}^*))\|_X + \int_0^t \|T_\alpha(t-s)\|_{L(X)} \\ & \times s^n\|f(s, \bar{u}_{\rho(s,\bar{u}_s)}) - f(s, \bar{u}^*_{\rho(s,\bar{u}^*_\xi)})\|_X ds + \int_0^t \|T_\alpha(t-s)\|_{L(X)} \\ & \times \int_0^s q(s-\xi)\|h(\xi, \bar{u}_{\rho(\xi,\bar{u}_\xi)}) - h(\xi, \bar{u}^*_{\rho(\xi,\bar{u}^*_\xi)})\|_X d\xi ds \\ \leq\ & \widetilde{M_S}L_l\|u - u^*\|_{\mathfrak{B}_h''} + \frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_fK_b\|u - u^*\|_{\mathfrak{B}_h''} \\ & + \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_hK_b\|u - u^*\|_{\mathfrak{B}_h''}. \end{aligned}$$

For $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned} \|Pu(t) - u^*(t)\|_X \leq\ & \|S_\alpha(t-s_i)\|_{L(X}\|g_i(s_i, \bar{u}(s_i)) - g_i(s_i, \bar{u^*}(s_i))\|_X \\ & + \int_{s_i}^t \|T_\alpha(t-s)\|_{L(X)}s^n\|f(s, \bar{u}_{\rho(s,\bar{u}_s)}) - g_i(s_i, \bar{u^*}(s_i))\|_X ds \\ & + \int_{s_i}^t \|T_\alpha(t-s)\|_{L(X)}\int_0^s q(s-\xi)\|h(\xi, \bar{u}_{\rho(\xi,\bar{u}_\xi)}) - h(\xi, \bar{u^*}_{\rho(\xi,\bar{u^*}_\xi)})\|_X d\xi ds \\ \leq\ & \widetilde{M_S}L_{g_i}\|u - u^*\|_{\mathfrak{B}_h''} + \frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_fK_b\|u - u^*\|_{\mathfrak{B}_h''} \\ & + \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_hK_b\|u - u^*\|_{\mathfrak{B}_h''}. \end{aligned}$$

For $t \in (t_j, s_j]$, we get $\|Pu(t) - u^*(t)\|_X \leq L_{g_j}\|u - u^*\|_{\mathfrak{B}_h''},\ j = 1, 2, \cdots, N$, gathering above results, we obtain

$$\|Pu(t) - u^*(t)\|_X \leq \Delta\|u - u^*\|_{\mathfrak{B}_h''}.$$

Since $\Delta < 1$, which implies that P is a contraction map and there exists a unique fixed point which is the mild solution of problem (1.5)-(1.7). This completes the proof of the theorem. □

4 Continuous Dependence of Mild Solutions

Theorem 4.2. *Suppose that the assumptions (H_1) are satisfied and the following inequalities hold:*

$$\widetilde{M_S}L_{g_i} + C'K_b < 1.$$

Then for each ϕ, ϕ^*, let u, u^* be the corresponding mild solutions of the system (1.5)-(1.7), then the following inequalities hold:

$$\begin{aligned}\|u - u^*\|_X &\leq \frac{\widetilde{M_S} + C'(M_b + J^\phi)}{1 - [\widetilde{M_S}L_l + C'K_b]}\|\phi - \phi^*\|, t \in [0, t_1],\\ \|u - u^*\|_X &\leq \frac{C'(M_b + J^\phi)}{1 - [\widetilde{M_S}L_{g_i} + C'K_b]}\|\phi - \phi^*\|, t \in [s_i, t_{i+1}],\end{aligned}$$

for $i = 1, 2, \cdots, N$.

Proof. Estimating for $t \in [0, t_1]$, we have

$$\begin{aligned}\|u - u^*\|_X &\leq \|S_\alpha(t)\|_{L(X)}(\|\phi(0) - \phi^*(0)\|_{\mathfrak{B}_h} + \|l(\bar{u}) - l(\bar{u}^*)\|_X)\\ &+ \int_0^t \|T_\alpha(t-s)\|_{L(X)} s^n \|f(s, \bar{u}_{\rho(s,\bar{u}_s)}) - f(s, \bar{u}^*_{\rho(s,\bar{u}^*_s)})\|_X ds\\ &+ \int_0^t \|T_\alpha(t-s)\|_{L(X)} \int_0^s q(s-\xi)\|h(\xi, \bar{u}_{\rho(\xi,\bar{u}_\xi)}) - h(\xi, \bar{u}^*_{\rho(\xi,\bar{u}^*_\xi)})\|_X d\xi ds\\ &\leq \widetilde{M_S}(\|\phi - \phi^*\| + L_l\|u - u^*\|_X) + (\frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_f\\ &+ \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_h) \times ((M_b + J^\phi)\|\phi - \phi^*\|_{\mathfrak{B}_h} + K_b\|u - u^*\|),\\ \|u - u^*\|_X &\leq \frac{\widetilde{M_S} + C'(M_b + J^\phi)}{1 - [\widetilde{M_S}L_l + C'K_b]}\|\phi - \phi^*\|,\end{aligned}$$

where

$$C' = \frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_f + \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_h. \tag{4.13}$$

Similar way, when $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned}\|u - u^*\|_X &\leq \|S_\alpha(t - s_i)\|_{L(X}\|g_i(s_i, \bar{u}(s_i)) - g_i(s_i, \bar{u^*}(s_i))\|_X\\ &+ \int_{s_i}^t \|T_\alpha(t-s)\|_{L(X)} s^n \|f(s, \bar{u}_{\rho(s,\bar{u}_s)}) - g_i(s_i, \bar{u^*}(s_i))\|_X ds\\ &+ \int_{s_i}^t \|T_\alpha(t-s)\|_{L(X)} \int_0^s q(s-\xi)\|h(\xi, \bar{u}_{\rho(\xi,\bar{u}_\xi)}) - h(\xi, \bar{u^*}_{\rho(\xi,\bar{u^*}_\xi)})\|_X d\xi ds\\ &\leq \widetilde{M_S}L_{g_i}\|u - u^*\|_X + (\frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_f\\ &+ \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_h) \times ((M_b + J^\phi)\|\phi - \phi^*\|_{\mathfrak{B}_h} + K_b\|u - u^*\|),\\ \|u - u^*\|_X &\leq \frac{C'(M_b + J^\phi)}{1 - [\widetilde{M_S}L_{g_i} + C'K_b]}\|\phi - \phi^*\|,\end{aligned}$$

where C' is given in equation (4.13). This completes the proof of the theorem. □

5 Example

Consider the following nonlocal impulsive fractional partial differential equation:

$$\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) &= \frac{\partial^2}{\partial y^2}u(t,x)+\frac{t}{9}\int_{-\infty}^{t} e^{2(s-t)}\frac{u(s-\sigma_1(s)\sigma_2(\|u\|),x)}{16}ds \\
&\quad +\int_0^t \sin(t-s)\int_{-\infty}^{\xi} e^{2(\nu-\xi)}\frac{u(\nu-\sigma_1(\nu)\sigma_2(\|u\|),x)}{25}d\nu ds, \\
&\quad (t,y)\in\cup_{i=1}^{N}[s_i,t_{i+1}]\times[0,\pi], && (5.14)\\
u(t,0) &= u(t,\pi)=0,\ \ t\geq 0, && (5.15)\\
u(t,x) &+ \sum_{k=1}^{r} c_k u(s_k,x)=\phi(t,x), t\in(-\infty,0], x\in[0,\pi], && (5.16)\\
u(t,x) &= G_i(t,u(t,x)),\ x\in[0,\pi],\ t\in(t_i,s_i], && (5.17)
\end{aligned}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo's fractional derivative of order $\alpha\in(0,1]$, $0=t_0=s_0<t_1\leq s_1<\cdots<t_N\leq s_N<t_{N+1}=1$ are fixed real numbers, $\phi\in\mathfrak{B}_h$, and r is a positive integer, $0<t_0<t_1,\cdots,<t_r<1$. Let $X=L^2[0,\pi]$ and define the operator $A:D(A)\subset X\to X$ by $Aw=w''$ with the domain $D(A):=\{w\in X:\ w,w'$ are absolutely continuous, $w''\in X, w(0)=0=w(\pi)\}$. Then

$$Aw=\sum_{n=1}^{\infty} n^2(w,w_n)w_n,\ w\in D(A),$$

where $w_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx),\ n\in\mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in X and is given by

$$T(t)\omega=\sum_{n=1}^{\infty} e^{-n^2t}(\omega,\omega_n)\omega_n,\ \text{for all } \omega\in X,\ \text{and every } t>0.$$

The subordination principle of solution operator implies that A is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\geq 0}$, s.t. $\|S_\alpha(t)\|_{L(X)}\leq \widetilde{M}_S$ for $t\in[0,1]$.

Let $h(s)=e^{2s},\ s<0$ then $l=\int_{-\infty}^{0}h(s)ds=\frac{1}{2}<\infty$, for $t\in(-\infty,0]$ and define

$$\|\phi\|_{\mathfrak{B}_h}=\int_{-\infty}^{0}h(s)\sup_{\theta\in[s,0]}\|\phi(\theta)\|_{L^2}ds.$$

Hence for $(t,\phi)\in[0,1]\times\mathfrak{B}_h$, where $\phi(\theta)(x)=\phi(\theta,x),\ (\theta,x)\in(-\infty,0]\times[0,\pi]$.

Set $u(t)(x)=u(t,x)$, and $\rho(t,\phi)=\rho_1(t)\rho_2(\|\phi(0)\|)$, we have

$$f(t,\phi)(x)=\frac{1}{9}\int_{-\infty}^{0}e^{2(s)}\frac{\phi}{16}ds;\ g(t,\phi)(x)=\int_{-\infty}^{0}e^{2(s)}\frac{\phi}{25}ds,$$

$$g_i(t,u)(x)=G_i(t,u(t,x)),\ l(u)=\sum_{k=1}^{r}c_k u(s_k,x),$$

then with these settings the equations (5.14)-(5.17) can be written in the abstract form of equations (1.5)-(1.7). We assume that $\rho_i:[0,\infty)\to[0,\infty),\ i=1,2,$ are continuous functions. Now, we can see that for

$(t,\phi),(t,\psi) \in J \times \mathfrak{B}_h$, we have

$$\begin{aligned}
&\|f(t,\phi)-f(t,\psi)\|_{L^2}\\
&= \left[\int_0^\pi \left\{ \| \frac{1}{9}\int_{-\infty}^0 e^{2(s)}\frac{\phi}{16}ds - \frac{1}{9}\int_{-\infty}^0 e^{2(s)}\frac{\psi}{16}ds\| \right\}^2 dy\right]^{1/2}\\
&\le \left[\int_0^\pi \left\{ \frac{1}{9}\int_{-\infty}^0 e^{2(s)} \| \frac{\phi}{16} - \frac{\psi}{16}\| ds \right\}^2 dy\right]^{1/2}\\
&\le \left[\int_0^\pi \left\{ \frac{1}{144}\int_{-\infty}^0 e^{2(s)} \sup \|\phi-\psi\| ds \right\}^2 dy\right]^{1/2}\\
&\le \frac{\sqrt{\pi}}{144}\|\phi-\psi\|_{\mathfrak{B}_h}.
\end{aligned}$$

Similarly,
$$\|h(t,\phi)-h(t,\psi)\|_{L^2} \le \frac{\sqrt{\pi}}{25}\|\phi-\psi\|_{\mathfrak{B}_h},$$
$$\|l(u)-l(v)\|_{L^2} \le \sum_{k=1}^{r} c_k\|u-v\|_{L^2},$$
$$\|g_i(t,u)-g_i(t,v)\|_{L^2} \le L_{G_i}\|u-v\|_{L^2}.$$

Hence all the function $f, g_i,\ h$ and l satisfy assumptions of (H_1). We deduced that the system (5.14)-(5.17) has a unique mild solution on $[0,1]$.

Acknowledgement: The authors express their deep gratitude to the referees for their valuable suggestions and comments for improvement of the paper. The research of J. Dabas has been partially supported by Department of Science & Technology, project No.SR/FTP/MS-030/2011).

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Received: February 19, 2014; *Accepted*: June 04, 2014

Malaya J. Mat. 2(4)(2014) 438–444

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Third Hankel determinant for a subclass of analytic univalent functions

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Abstract

This paper focuses on attaining the upper bounds on $H_3(1)$ for a class C_α^β $(0 \leq \beta < 1, \alpha \geq 0)$ in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

Keywords: Bounded turning, coefficient bounds, convex functions, Hankel determinant, starlike functions.

2010 MSC: 30C45, 30C50.

1 Introduction

Let A be the class of functions

$$f(z) = z + a_2 z^2 + \ldots \tag{1.1}$$

which are analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in A$ is said to be of bounded turning, starlike and convex respectively if and only if for $z \in \Delta$, $Re\ f'(z) > 0$, $Re\ \frac{zf'(z)}{f(z)} > 0$ and $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. By usual notations, we denote these classes of functions respectively by R, S^* and C. Let $n \geq 0$ and $q \geq 1$. The q^{th} Hankel determinant is defined as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & & \cdots \\ \vdots & & \vdots & \\ a_{n+q-1} & \cdots & & a_{n+2(q-1)} \end{vmatrix}.$$

This determinant has been considered by several authors. For example, Noor in [11] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by authors of articles [5, 6, 7, 13, 14] for different classes of functions.

The class C_α^β is defined as follows.

Definition 1.1. *Let f be given by (1.1). Then $f \in C_\alpha^\beta$ if and only if*

$$Re\left\{\frac{(zf'(z) + \alpha z^2 f''(z))'}{f'(z)}\right\} > \beta, \quad z \in \Delta, 0 \leq \beta < 1, 0 \leq \alpha \leq 1.$$

The choice of $\alpha = 0$, $\beta = 0$ yields $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$, $z \in \Delta$, the class of convex functions C [12].
The choice of $\alpha = 0$, yields $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta$, $z \in \Delta$, the class of convex functions of order β denoted by $C(\beta)$ [12].

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In the present investigation, our focus is on the Hankel determinant, $H_3(1)$ for the class C_α^β in Δ. By definition, $H_3(1)$ is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

for $f \in A$, $a_1 = 1$, so that

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and by triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.2}$$

In this paper, we find the sharp upper bound for the functional $|a_2a_3 - a_4|$, $|a_2a_4 - a_3^2|$ and $|a_3 - a_2^2|$ respectively for the functions belonging to the class C_α^β. Our proofs are based on the techniques employed by [8, 9] which has been widely used by many authors (see for example [5, 6, 7, 14]).

2 Preliminary Results

Let P denote the class of functions

$$p(z) = 1 + c_1z + c_2z^2 + \cdots \tag{2.3}$$

which are regular in Δ and satisfy $Re\ p(z) > 0$, $z \in \Delta$. Throughout this paper, we assume that $p(z)$ is given by (2.3) and $f(z)$ is given by (1.1). To prove the main results we shall require the following lemmas.

Lemma 2.1. *[3] Let $p \in P$. Then $|c_k| \leq 2$, $k = 1, 2, \ldots$ and the inequality is sharp.*

Lemma 2.2. *[8, 9] Let $p \in P$. Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.4}$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \tag{2.5}$$

for some x, y such that $|x| \leq 1$ and $|y| \leq 1$.

Lemma 2.3. *[2] Let $p \in P$. Then*

$$\left| c_2 - \sigma \frac{c_1^2}{2} \right| = \begin{cases} 2(1-\sigma) & \text{if } \sigma \leq 0, \\ 2 & \text{if } 0 \leq \sigma \leq 2, \\ 2(\sigma - 1) & \text{if } \sigma \geq 2. \end{cases}$$

3 Main Results

Lemma 3.1. *Let $f \in C_\alpha^\beta$. Then, we have the best possible bound for*

$$|a_2a_3 - a_4| \leq \begin{cases} \frac{4}{9\sqrt{3}} & \alpha = 0, \beta = 0 \\ \frac{(1-\beta)}{MA_2}\sqrt{\frac{A_1}{A_2}}[B_1 + (4A_2 - A_1)(B_2 + B_3)] & 0 < \alpha \leq 1, 0 < \beta < 1, \end{cases}$$

where,
$A_1 = 4(4 + 23\alpha + 48\alpha^2 + 36\alpha^3 - \beta - 2\alpha\beta)$,
$A_2 = 3(4 + 20\alpha + 64\alpha^2 + 48\alpha^3 + 2\beta + 20\alpha\beta - 2\beta^2 - 12\alpha\beta^2)$,
$B_1 = -3\alpha + 3\beta + 22\alpha\beta - 2\beta^2 - 12\alpha\beta^2 + 16\alpha^2 + 12\alpha^3$,
$B_2 = 3 + 16\alpha + 32\alpha^2 + 24\alpha^3$,
$B_3 = 1 + 7\alpha + 16\alpha^2 + 12\alpha^3$,
$M = 48(1 + 2\alpha)^2(1 + 3\alpha)(1 + 4\alpha)$.

Proof. For $f \in C_\alpha^\beta$, there exists a $p \in P$ such that

$$f'(z) + zf''(z) + \alpha z^2 f'''(z) + 2\alpha z f''(z) = [(1-\beta)p(z) + \beta]f'(z).$$

Equating the coefficients,

$$a_2 = \frac{c_1(1-\beta)}{2(1+2\alpha)}, \quad a_3 = \frac{c_2(1-\beta)}{6(1+3\alpha)} + \frac{c_1^2(1-\beta)^2}{6(1+2\alpha)(1+3\alpha)},$$
$$a_4 = \frac{c_3(1-\beta)}{12(1+4\alpha)} + \frac{c_1c_2(3+8\alpha)(1-\beta)^2}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)},$$
$$a_5 = \frac{1}{20(1+5\alpha)}\left\{\frac{c_1c_3(1-\beta)^2(4+4\alpha)}{3(1+2\alpha)(1+4\alpha)} + \frac{c_1^4(1-\beta)^4}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_2^2(1-\beta)^2}{2(1+3\alpha)}\right.$$
$$\left. + \frac{c_1^2c_2(1-\beta)^3(6+20\alpha)}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + c_4(1-\beta)\right\}.$$

Thus, we have

$$|a_2a_3 - a_4| = \left|\frac{c_1c_2(1-\beta)^2(-1)}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3(1+6\alpha)}{24(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} - \frac{c_3(1-\beta)}{12(1+4\alpha)}\right| \tag{3.6}$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$,using the Lemma 2.1, we may assume without restriction $c \in [0,2]$. Substituting for c_2 and c_3, from Lemma 2.2 and applying the triangle inequality with $\rho = |x|$, we obtain

$$\begin{aligned}|a_2a_3 - a_4| &\leq \frac{c^3(1-\beta)[-3\alpha + 3\beta + 22\alpha\beta - 2\beta^2 - 12\alpha\beta^2 + 16\alpha^2 + 12\alpha^3]}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} \\ &+ \frac{\rho c(1-\beta)(4-c^2)(3+10\alpha+12\alpha^2-\beta)}{48(1+2\alpha)(1+3\alpha)(1+4\alpha)} \\ &+ \frac{\rho^2(4-c^2)(1-\beta)(c-2)}{48(1+4\alpha)} + \frac{2(1-\beta)(4-c^2)}{48(1+4\alpha)} \\ &= F(\rho).\end{aligned}$$

Differentiating $F(\rho)$, we get

$$F'(\rho) = \frac{c(1-\beta)(4-c^2)(3+10\alpha+12\alpha^2-\beta)}{48(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{\rho(4-c^2)(1-\beta)(c-2)}{24(1+4\alpha)}.$$

Note also that $F'(\rho) \geq F'(1) \geq 0$. Then there exist a $c^* \in [0,2]$ such that $F'(\rho) > 0$ for $c \in (c^*, 2]$ and $F'(\rho) \leq 0$ otherwise.
Then, for a $c \in (c^*, 2]$, $F(\rho) \leq F(1)$, that is:

$$\begin{aligned}|a_2a_3 - a_4| &\leq \frac{c^3(1-\beta)[-3\alpha + 3\beta + 22\alpha\beta - 2\beta^2 - 12\alpha\beta^2 + 16\alpha^2 + 12\alpha^3]}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} \\ &+ \frac{c(1-\beta)(4-c^2)(3+10\alpha+12\alpha^2-\beta)}{48(1+2\alpha)(1+3\alpha)(1+4\alpha)} \\ &+ \frac{(4-c^2)(1-\beta)(c-2)}{48(1+4\alpha)} + \frac{2(1-\beta)(4-c^2)}{48(1+4\alpha)} \\ &= G(c).\end{aligned}$$

If $\alpha = 0, \beta = 0$, we have $G(c) = \frac{c(4-c^2)}{12}$. By elementary calculus, we have $G'(c) = \frac{4-3c^2}{12}$, $G''(c) = -\frac{c}{2} < 0$. Since $c \in [0,2]$ by our assumption, it follows that $G(c)$ is maximum at $c = 2/\sqrt{3}$. Otherwise, again by elementary calculus $G(c)$ is maximum at $c = \sqrt{\frac{A_1}{A_2}}$ and is given by

$$G(c) \leq \frac{(1-\beta)}{MA_2}\sqrt{\frac{A_1}{A_2}}[B_1 + (4A_2 - A_1)(B_2 + B_3)]$$

Now suppose $c \in [0, c^*]$, then $F(\rho) \leq F(0)$, that is:

$$F(\rho) \leq \frac{c^3(1-\beta)[-3\alpha+3\beta+22\alpha\beta-2\beta^2-12\alpha\beta^2+16\alpha^2+12\alpha^3]}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} + \frac{2(1-\beta)(4-c^2)}{48(1+4\alpha)}$$
$$= G(c),$$

which implies that $G(c)$ turns at $c = 0$ and $c = \frac{4(1+2\alpha)^2(1+3\alpha)}{[-3\alpha+3\beta+22\alpha\beta-2\beta^2-12\alpha\beta^2+16\alpha^2+12\alpha^3]}$
with its maximum at $c = 0$. That is $G(c) \leq \frac{(1-\beta)}{6(1+4\alpha)}$.
Thus for all admissible $c \in [0, 2]$, the maximum of the functional $|a_2a_3 - a_4|$ are given by the inequalities of the theorem.
If $p(z) \in P$, with $c_1 = 2/\sqrt{3}$, $c_2 = -2/3$ and $c_3 = -10/3\sqrt{3}$, then we obtain $p(z) = 1 + \frac{2}{\sqrt{3}}z - \frac{2}{3}z^2 - \frac{10}{3\sqrt{3}}z^3 + \cdots \in P$ which shows that the result is sharp. □

Lemma 3.2. *Let $f \in C_\alpha^\beta$. Then ,we have the best possible bound for*

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{1}{8} & \alpha = 0, \beta = 0 \\ \frac{(1-\beta)^2}{N}[M_1V_1V_2 + (4V_2 - V_1)\{M_2V_1 + V_1P_1 + P_2\}] & 0 < \alpha \leq 1, 0 < \beta \leq 1, \end{cases}$$

where,
$M_1 = [22\alpha^3 + 31\alpha^2 + 11\alpha - 2\beta^2 - 5\beta - 3\alpha\beta - 8\alpha^2\beta]$,
$M_2 = 3 + 118\alpha^2 - 45\alpha + 44\alpha^3 - \beta - 3\alpha\beta - 8\alpha^2\beta$,
$P_1 = (1 + 27\alpha^2 - 10\alpha)(1 + 2\alpha)$,
$P_2 = (8 + 48\alpha + 64\alpha^2)(1 + 2\alpha)$,
$V_1 = 2M_1 + 8M_2 + 8P_1 - 2P_2$,
$V_2 = 4M_2 + 4P_1$,
$N = 288(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)$.

Proof. Let $f \in C_\alpha^\beta$. Then proceeding as in Lemma 3.1, we have

$$|a_2a_4 - a_3^2| = \left| \frac{c_1c_3(1-\beta)^2}{24(1+2\alpha)(1+4\alpha)} + \frac{c_1^2c_2(3+8\alpha)(1-\beta)^3}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} + \frac{c_1^4(1-\beta)^4}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} - \frac{c_2^2(1-\beta)^2}{36(1+3\alpha)^2} - \frac{c_1^4(1-\beta)^4}{36(1+2\alpha)^2(1+3\alpha)^2} - \frac{2c_1^2c_2(1-\beta)^3}{36(1+2\alpha)(1+3\alpha)^2} \right|. \tag{3.7}$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, Using Lemma 2.1, we may assume without restriction $c \in (0, 2]$. Substituting for c_2 and c_3, from Lemma 2.2 and applying triangle inequality with $\rho = |x|$, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{1}{144}\left\{ \frac{(1-\beta)^2c^4[22\alpha^3 + 23\alpha^2 + 8\alpha^2 + 11\alpha - 2\beta^2 - 5\beta - 3\alpha\beta - 8\alpha^2\beta}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \right.$$
$$+ \frac{\rho c^2(4-c^2)(1-\beta)^2[3 + 118\alpha^2 - 45\alpha + 44\alpha^3 - \beta - 3\alpha\beta - 8\alpha^2\beta]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}$$
$$+ \frac{\rho^2(4-c^2)(1-\beta)^2[8 + c^2 + 48\alpha + 64\alpha^2 + 27c^2\alpha^2 - 10c^2\alpha]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}$$
$$\left. + \frac{3(4-c^2)(1-\rho)(1-\beta)^2}{(1+2\alpha)(1+4\alpha)} \right\}$$
$$= F(\rho).$$

Differentiating $F(\rho)$, we get,

$$F'(\rho) = \frac{1}{144}\left\{ \frac{c^2(4-c^2)(1-\beta)^2[3 + 118\alpha^2 - 45\alpha + 44\alpha^2 - \beta - 3\alpha\beta - 8\alpha^2\beta]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \right.$$
$$+ \frac{2\rho(4-c^2)(1-\beta)^2[8 + c^2 + 48\alpha + 64\alpha^2 + 27c^2\alpha - 10c^2\alpha]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}$$
$$\left. + \frac{3\rho(4-c^2)(1-\beta)^2}{(1+2\alpha)(1+4\alpha)} \right\}.$$

Note also that $F'(\rho) \geq F'(1) \geq 0$. Then there exist a $c^* \in [0,2]$ such that $F'(\rho) > 0$ for $c \in (c^*, 2]$ and $F'(\rho) \leq 0$ otherwise.
Then for a $c \in (c^*, 2]$, $F(\rho) \leq F(1)$, that is:

$$\begin{aligned}|a_2a_4 - a_3^2| \leq \frac{1}{144}\Bigg\{&\frac{(1-\beta)^2[22\alpha^3 + 31\alpha^2 + 11\alpha - 2\beta^2 - 5\beta - 3\alpha\beta - 8\alpha^2\beta]c^4}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\ &+ \frac{(1-\beta)^2[3 + 118\alpha^2 - 45\alpha + 44\alpha^3 - \beta - 3\alpha\beta - 8\alpha^2\beta]c^2(4-c^2)}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\ &+ \frac{(1-\beta)^2(4-c^2)(8 + c^2 + 48\alpha + 64\alpha^2 + 27c^2\alpha^2 - 10c^2\alpha)}{2(1+2\alpha)(1+3\alpha)^2(1+4\alpha)}\Bigg\} \\ = G(c).&\end{aligned}$$

If $\alpha = 0$, $\beta = 0$, we have $G(c) = \frac{3c^2(4-c^2)}{2} + \frac{(4-c^2)(c^2+8)}{2}$. By elementary calculus we have, $G'(c) = 8c - 8c^3$, $G''(c) = 8 - 24c^2 < 0$. Since $c \in (0,2]$, by our assumption it follows that $G(c)$ is maximum at $c = 1$. Otherwise, again by elementary calculus $G(c)$ is maximum at $c = \sqrt{\frac{V_1}{V_2}}$ and is given by

$$G(c) \leq \frac{(1-\beta)^2}{N}[M_1V_1V_2 + (4V_2 - V_1)\{M_2V_1 + V_1P_1 + P_2\}].$$

Now suppose $c \in [0, c^*]$, then $F(\rho) \leq F(0)$, that is:

$$\begin{aligned}F(\rho) \leq \frac{1}{144}\Bigg\{&\frac{(1-\beta)^2[22\alpha^3 + 31\alpha^2 + 11\alpha - 2\beta^2 - 5\beta - 3\alpha\beta - 8\alpha^2\beta]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\ &+ \frac{3(4-c^2)(1-\beta)^2}{(1+2\alpha)(1+4\alpha)}\Bigg\} \\ = G(c),&\end{aligned}$$

which implies that $G(c)$ turns at $c = 0$ and $c = \sqrt{\frac{(22\alpha^3+31\alpha^2+11\alpha-2\beta^2-5\beta-3\alpha\beta-8\alpha^2\beta)}{3(1+2\alpha)(1+3\alpha)^2}}$, with its maximum at $c = 0$. That is, $G(c) \leq \frac{12(1-\beta)^2}{(1+2\alpha)(1+4\alpha)}$.
Thus for all admissible $c \in [0,2]$, the maximum of the functional $|a_2a_4 - a_3^2|$ are given by the inequalities of the theorem.
If $p(z) \in P$, with $c_1 = 1$, $c_2 = -1$, $c_3 = -2$, then $p(z) = \frac{1-z^2}{1-z+z^2} = 1 + z - z^2 - 2z^3 + \cdots \in P$ which shows that the result is sharp. □

Lemma 3.3. *Let $f \in C_\alpha^\beta$. Then we have the best possible bound for*

$$|a_3 - a_2^2| \leq \begin{cases} \frac{1}{3} & \alpha = 0, \beta = 0 \\ \frac{1-\beta}{3(1+3\alpha)} & 0 < \alpha \leq 1, 0 < \beta \leq 1. \end{cases}$$

Proof. Let $f \in C_\alpha^\beta$. Then proceeding as in Lemma 3.1, we have

$$|a_3 - a_2^2| = \left|\frac{c_2(1-\beta)}{6(1+3\alpha)} - \frac{c_1^2(1-\beta)^2(1+5\alpha)}{12(1+2\alpha)^2(1+3\alpha)}\right| \tag{3.8}$$

and

$$|a_3 - a_2^2| = \frac{(1-\beta)}{6(1+3\alpha)}\left|c_2 - \frac{c_1^2(1-\beta)(1+5\alpha)}{2(1+2\alpha)^2}\right|.$$

Setting $\sigma = \frac{(1-\beta)(1+5\alpha)}{(1+2\alpha)^2}$, using Lemma 2.3, we have $|a_3 - a_2^2| \leq \frac{(1-\beta)}{3(1+3\alpha)}$.
If $p(z) \in P$ with $c_1 = 0$, $c_2 = 2$, then $p(z) = \frac{1+z^2}{1-z^2} = 1 + 2z^2 + 2z^4 + \cdots \in P$, which shows that the result is sharp. □

Remark 3.1. *Let $f \in C_\alpha^\beta$. By Lemma 2.1, we have*

$$\begin{aligned}
|a_3| &= \left| \frac{c_2(1-\beta)}{6(1+3\alpha)} + \frac{c_1^2(1-\beta)^2}{6(1+2\alpha)(1+3\alpha)} \right|, \\
&\leq \frac{(1-\beta)(3+2\alpha-2\beta)}{3(1+2\alpha)(1+3\alpha)}, \\
|a_4| &= \left| \frac{c_3(1-\beta)}{12(1+4\alpha)} + \frac{c_1c_2(3+8\alpha)(1-\beta)^2}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} \right|, \\
&\leq \frac{(1-\beta)(6+6\alpha^2+2\beta^2+13\alpha-7\beta-8\alpha\beta)}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)}, \\
|a_5| &= \left| \frac{1}{20(1+5\alpha)} \left\{ \frac{c_1c_3(1-\beta)^2(4+4\alpha)}{3(1+2\alpha)(1+4\alpha)} + \frac{c_1^4(1-\beta)^4}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_2^2(1-\beta)^2}{2(1+3\alpha)} \right.\right. \\
&\quad \left.\left. + \frac{c_1^2c_2(1-\beta)^3(6+20\alpha)}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + c_4(1-\beta) \right\} \right| \\
&\leq \frac{(1-\beta)(120+408\alpha^2+500\alpha-576\alpha\beta-432\alpha^2\beta+160\alpha\beta^2+188\beta+96\beta^2)}{120(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)}.
\end{aligned}$$

This leads to the next theorem which gives a sharp result by using Lemmas 3.1,3.2 and 3.3 and Remark 3.1.

Theorem 3.1. *Let $f \in C_\alpha^\beta$. Then*

$$\begin{aligned}
|H_3(1)| \leq\ & \frac{(1-\beta)^2(3+2\alpha-2\beta)}{3(1+2\alpha)(1+3\alpha)} \\
& \left\{ \frac{(1-\beta)^2}{N} [M_1V_1V_2 + (4V_2 - V_1)\{M_2V_1 + V_1P_1 + P_2\}] \right\} \\
& + \frac{(1-\beta)(6+6\alpha^2+2\beta^3+13\alpha-7\beta-8\alpha\beta)}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} \\
& \left\{ \frac{(1-\beta)}{MA_2} \sqrt{\frac{A_1}{A_2}} [B_1 + (4A_2 - A_1)(B_2 + B_3)] \right\} \\
& + \frac{(1-\beta)(120+408\alpha^2+500\alpha-576\alpha\beta-432\alpha^2\beta+160\alpha\beta^2+188\beta+96\beta^2)}{120(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)} \\
& \times \frac{(1-\beta)}{3(1+3\alpha)}.
\end{aligned}$$

When $\alpha = 0$, $\beta = 0$, we have the following corollary due to [1].

Corollary 3.1. *If $\alpha = 0$, $\beta = 0$, then*

$$|H_3(1)| \leq \frac{32+33\sqrt{3}}{72\sqrt{3}} = 0.714933452973167.$$

4 Acknowledgement

The authors thank the referees for their valuable comments and suggestions to improve the presentation of the paper.

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Received: March 05, 2014; *Accepted*: July 05, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 445–459

Malaya Journal of Matematik
www.malayajournal.org

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an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Some new Ostrowski type inequalities for functions whose second derivative are h-convexe via Riemann-Liouville fractionnal

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Abstract

A new identity similar to an identity proved in Erhan Set. (2012) [16] for fractional integrals is established.By making use of the established identity, some new Ostrowski type inequalities for Riemann–Liouville fractional integral are obtained.

Keywords: Ostrowski type inequalities, Riemann-Liouville integrals, (s,m) −convex function.

2010 MSC: 26D15, 26D20, 39A12.

1 Introduction

In 1938, A.M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

Theorem 1.1. *[13] Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mappingin the interior $I^{\circ}$of I, and $a, b \in I^{\circ}$, with $a < b$.*

If $|f'| \leq M$ for all $x \in [a,b]$, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a,b] \tag{1.1}$$

This is well-know Ostrowski inequality.In recent years, a number of authors have written about generalizations,extensions and variants of such inequalities (see [1, 2, 3]).

Let us recall definitions of some kinds of convexity as follows.

Definition 1.1. *We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ $(I \neq \phi)$ is convex function if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.2}$$

holds for all $x, y \in I$, and $t \in [0,1]$.

Definition 1.2. *[7] We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ $(I \neq \phi)$ is P- function if f is non-negative and the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y) \tag{1.3}$$

holds for all $x, y \in I$, and $t \in [0,1]$.

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Definition 1.3. *[8] We say that $f : [0,\infty) \to \mathbb{R}$ is s-convex function in the second sense, if the inequality*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.4}$$

holds for all $x, y \in (0, b]$, $t \in [0,1]$ and for fixed $s \in (0,1]$

Definition 1.4. *[15] Let $h : J \subset \mathbb{R} \to \mathbb{R}$,be a positive function.We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ $(I \neq \phi)$ is h-convex function,*

if f is non-negative and

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{1.5}$$

holds for all $x, y \in I$,and $t \in [0,1]$.

Definition 1.5. *[17] We say that $f : [0,b] \to \mathbb{R}$ $(0 < b)$ is said to be $m-$convex, where $m \in (0,1]$ and $b > 0$, if for every $x, y \in [0,b]$ and $t \in [0,1]$, we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)\, f(y) \tag{1.6}$$

Definition 1.6. *[12] We say that $f : [0,b] \to \mathbb{R}$ $(0 < b)$ is said to be $(s,m)-$convex, where $(s,m) \in (0,1]^2$ and $b > 0$ if for every $x, y \in [0,b]$ and $t \in [0,1]$, we have*

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s)f(y) \tag{1.7}$$

Definition 1.7. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}f(x), J^{\alpha}_{b^-}f(x)$ of order $\alpha > 0$,with $a > 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1.8}$$

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b \tag{1.9}$$

and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$
where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt \tag{1.10}$$

noting also

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1.11}$$

Motivated by the recent results given in [1, 2, 6, 9], in the present paper, we provide some companions of Ostrowski type inequalities involving Riemann – Liouville fractional integrals for functions whose second derivatives absolute value are h-convex.

2 OSTROWSKI TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS

In order to prove our main results we need the following identity.

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < b$.If $f'' \in L_1[a,b]$,then the following equaliy for fractional integrals holds for any $x \in [a,b]$*

$$L_{\alpha}(x) = (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int_0^1 t^{\alpha+1}f''(tx+(1-t)a)\,dt + (x-b)\int_0^1 t^{\alpha+1}f''(tx+(1-t)b)\,dt\right] \tag{2.12}$$

where

$$L_{\alpha}(x) = (\alpha+1)(b-x)^{\alpha}(x-a)^{\alpha}(b-a)f(x) - \Gamma(\alpha+2)\left[(b-x)^{\alpha+1} J_{x-}^{\alpha} f(a) + (x-a)^{\alpha+1} J_{x+}^{\alpha} f(b)\right] \quad (2.13)$$

Proof. We have

$$\begin{aligned} J_{x-}^{\alpha} f(a) &= \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} f(t)dt \quad (2.14)\\ &= \frac{1}{\Gamma(\alpha)} \left[\frac{(x-a)^{\alpha}}{\alpha} f(x) - \int_a^x \frac{(t-a)^{\alpha}}{\alpha} f'(t)dt \right] \\ &= \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} f(x) - \left[\frac{(x-a)^{\alpha+1}}{\alpha+1} f'(x) - \int_a^x \frac{(t-a)^{\alpha+1}}{\alpha+1} f''(t)dt \right] \right] \\ &= \frac{1}{\Gamma(\alpha+2)} \left[(\alpha+1)(x-a)^{\alpha} f(x) - (x-a)^{\alpha+1} f'(x) + \int_a^x (t-a)^{\alpha+1} f''(t)dt \right], \end{aligned}$$

multiplying both side of (2.14) by $\Gamma(\alpha+2)(b-x)^{\alpha+1}$, we get

$$\Gamma(\alpha+2)(b-x)^{\alpha+1} J_{x-}^{\alpha} f(a) = \left[\begin{array}{c} (\alpha+1)(b-x)^{\alpha+1}(x-a)^{\alpha} f(x) - (b-x)^{\alpha+1}(x-a)^{\alpha+1} f'(x) + \\ (b-x)^{\alpha+1}(x-a)^{\alpha+2} \int_0^1 t^{\alpha+1} f''(tx+(1-t)a)\,dt \end{array} \right]. \quad (2.15)$$

And

$$\begin{aligned} J_{x+}^{\alpha} f(b) &= \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} f(t)dt \quad (2.16)\\ &= \frac{1}{\Gamma(\alpha)} \left[\frac{(b-x)^{\alpha}}{\alpha} f(x) + \int_x^b \frac{(b-t)^{\alpha}}{\alpha} f'(t)dt \right] \\ &= \frac{1}{\Gamma(\alpha+1)} \left[(b-x)^{\alpha} f(x) + \left[\frac{(b-x)^{\alpha+1}}{\alpha+1} f'(x) - \int_x^b \frac{(b-t)^{\alpha+1}}{\alpha+1} f''(t)dt \right] \right] \\ &= \frac{1}{\Gamma(\alpha+2)} \left[(\alpha+1)(b-x)^{\alpha} f(x) + (b-x)^{\alpha+1} f'(x) + \int_x^b (b-t)^{\alpha+1} f''(t)dt \right], \\ &= \frac{1}{\Gamma(\alpha+2)} \left[(\alpha+1)(b-x)^{\alpha} f(x) + (b-x)^{\alpha+1} f'(x) + (b-x)^{\alpha+2} \int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt \right] \end{aligned}$$

multiplying both side of (2.16) by $\Gamma(\alpha+2)(x-a)^{\alpha+1}$, we get

$$\Gamma(\alpha+2)(x-a)^{\alpha+1} J_{x+}^{\alpha} f(b) = \left[\begin{array}{c} (\alpha+1)(x-a)^{\alpha+1}(b-x)^{\alpha} f(x) + (x-a)^{\alpha+1}(b-x)^{\alpha+1} f'(x) + \\ (x-a)^{\alpha+1}(b-x)^{\alpha+2} \int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt \end{array} \right]. \quad (2.17)$$

Summing (2.15) and (2.17), we obtain

$$\Gamma(\alpha+2)(b-x)^{\alpha+1} J_{x-}^{\alpha} f(a) + \Gamma(\alpha+2)(x-a)^{\alpha+1} J_{x+}^{\alpha} f(b) =$$

$$(\alpha+1)(b-a)(x-a)^{\alpha}(b-x)^{\alpha} f(x) +$$

$$(x-a)^{\alpha+1}(b-x)^{\alpha+1} \left[(x-a) \int_0^1 t^{\alpha+1} f''(tx+(1-t)a)\,dt + (b-x) \int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt \right] \quad (2.18)$$

we can rewrite (2.18) as follows

$$(\alpha+1)(b-a)(x-a)^{\alpha}(b-x)^{\alpha} f(x)-\left[\Gamma(\alpha+2)(b-x)^{\alpha+1} J_{x^-}^{\alpha} f(a)+\Gamma(\alpha+2)(x-a)^{\alpha+1} J_{x^+}^{\alpha} f(b)\right]$$

$$=(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int_0^1 t^{\alpha+1} f''(tx+(1-t)a)\,dt+(x-b)\int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt\right] \tag{2.19}$$

thus (2.19) implies (2.12). □

Theorem 2.1. *Let $f: I \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a, b \in I$, with $a<b$.*

If $|f''|$ is convex function on $[a,b]$, and f'' is bounded,i.e., $\|f''\|_{\infty}=\sup_{x\in[a,b]}|f''|<\infty$,for any $x\in[a,b]$,then the following inequality holds

$$|L_{\alpha}(x)| \leq \frac{(x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)}{(\alpha+2)}\|f''\|_{\infty} \tag{2.20}$$

Proof. By lemma 2.1, and Under the given assumptions on f'' we have

$$|L_{\alpha}(x)|=\left|(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int_0^1 t^{\alpha+1} f''(tx+(1-t)a)\,dt+(x-b)\int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt\right]\right|$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}$$
$$\times\left[(x-a)\int_0^1 t^{\alpha+1}(t|f''(x)|+(1-t)|f''(a)|)dt+(b-x)\int_0^1 t^{\alpha+1}(t|f''(x)|+(1-t)|f''(b)|)dt\right]$$

$$\leq \|f''\|_{\infty}(x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-x+x-a)\int_0^1 t^{\alpha+1}dt$$

$$=\frac{(x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)}{(\alpha+2)}\|f''\|_{\infty}$$

□

Remark 2.1. *Under the same hypotheses of Theorem 2.1 at the exception of the convexity of $f''$the inequality (2.20) remains valid.*

Corollary 2.1. *With the assumptions in Theorem 2.1, in the case where $\alpha=1$,one has the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{24}\|f''\|_{\infty} \tag{2.21}$$

Proof. Choose $x=\frac{a+b}{2}$ and $\alpha=1$ in (2.9), we get

$$\frac{(b-a)^3}{2}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)\,5}{48}\|f''\|_{\infty} \tag{2.22}$$

dividing both side of (2.22) by $\frac{(b-a)^3}{2}$ we obtain (2.21). □

Remark 2.2. *The inequality (2.21) is obtained in [9], choose $x=\frac{a+b}{2}$ in theorem 2.2.*

Corollary 2.2. *With the assumptions in Theorem 2.1, in the case where $\alpha = 1$,one has the inequality.*

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2}{96}\ \|f''\|_\infty \tag{2.23}$$

Proof. Apply Theorem 1.1, a faith on the interval $\left[a,\frac{a+b}{2}\right]$,taking $\alpha = 1$ in (2.9) , and replace x by $\frac{3a+b}{4}$, we get

$$\frac{(b-a)^3}{16}\left[\left|f(\frac{3a+b}{4})-\frac{2}{b-a}\int_a^{\frac{b+a}{2}} f(t)dt\right|\right] \leq \frac{(b-a)^5}{1536}\ \|f''\|_\infty \tag{2.24}$$

(2.24) implies

$$\left|f(\frac{3a+b}{4})-\frac{2}{b-a}\int_a^{\frac{b+a}{2}} f(t)dt\right| \leq \frac{(b-a)^2}{96}\ \|f''\|_\infty \tag{2.25}$$

Apply Theorem 1.1 another faith on the interval $\left[\frac{a+b}{2},b\right]$, taking $\alpha = 1$ in (2.9) , and replace x by $\frac{a+3b}{4}$, we get

$$\left|f(\frac{a+3b}{4})-\frac{2}{b-a}\int_{\frac{b+a}{2}}^{b} f(t)dt\right| \leq \frac{(b-a)^2}{96}\ \|f''\|_\infty \tag{2.26}$$

summing (2.25) and (2.26), dividing the result by 2 we obtain (2.23). □

Remark 2.3. *The inequality (2.23) is obtained in [9] corollary 2.3*

Corollary 2.3. *Let $f : I \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a, b \in I$, with $a < b$.*

If $|f''|^q$ is convex function on $[a,b]$, $p, q \geq 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and and f'' is bounded,i.e., $\|f''\|_\infty = \sup_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$,

then the following inequality holds

$$|L_\alpha(x)| \leq\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)\left(\frac{1}{(\alpha+1)\,p+1}\right)^{\frac{1}{p}}\ \|f''\|_\infty \tag{2.27}$$

Proof. under the assumptions given on $f''$and using the well-known Hölder's inequality for lemma 2.1, we get

$$|L_\alpha(x)| = \left|(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int_0^1 t^{\alpha+1}f''(tx+(1-t)a)\,dt+(x-b)\int_0^1 t^{\alpha+1}f''(tx+(1-t)b)\,dt\right]\right|$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(x-a)\int_0^1 t^{\alpha+1}\left|f''(tx+(1-t)a)\right|dt+(b-x)\int_0^1 t^{\alpha+1}\left|f''(tx+(1-t)b)\right|dt\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}+\\(b-x)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c} (x-a)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int\limits_0^1 \left(t\left|f''(x)\right|^q+(1-t)\left|f''(a)\right|^q\right)dt\right)^{\frac{1}{q}}+ \\ (b-x)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int\limits_0^1 \left(t\left|f''(x)\right|^q+(1-t)\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left\|f''\right\|_\infty$$

□

Theorem 2.2. *Let $f: I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a,b \in I$, with $a<b$.*

If $|f''|$ is P-convex on $[a,b]$, and and f'' is bounded,i.e., $\|f''\|_\infty = \sup\limits_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$, then the following inequality holds

$$|L_\alpha(x)| \leq \frac{2\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)}{(\alpha+2)}\left\|f''\right\|_\infty \tag{2.28}$$

Proof. by lemma 2.1, and Under the given assumptions on f'', we have

$$|L_\alpha(x)| = \left|(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int\limits_0^1 t^{\alpha+1}f''(tx+(1-t)a)\,dt+(x-b)\int\limits_0^1 t^{\alpha+1}f''(tx+(1-t)b)\,dt\right]\right|$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(x-a)\int\limits_0^1 t^{\alpha+1}(\left|f''(x)\right|+\left|f''(a)\right|)dt+(b-x)\int\limits_0^1 t^{\alpha+1}(\left|f''(x)\right|+\left|f''(b)\right|)dt\right]$$

$$= 2\left\|f''\right\|_\infty (x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)\int\limits_0^1 t^{\alpha+1}dt = \frac{2\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)}{(\alpha+2)}\left\|f''\right\|_\infty$$

□

Corollary 2.4. *With the assumptions in Theorem 2.2, in the case where $\alpha=1$,one has the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int\limits_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{12}\left\|f''\right\|_\infty \tag{2.29}$$

Proof. just take in (2.28), $\alpha=1$, $x=\frac{a+b}{2}$ and dividing both side of the result by $\frac{(b-a)^3}{2}$ we obtain (2.29). □

Corollary 2.5. *With the assumptions in Theorem 2.2, in the case where $\alpha=1$,one has the inequality*

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int\limits_a^b f(t)dt\right| \leq \frac{(b-a)^2}{48}\left\|f''\right\|_\infty \tag{2.30}$$

Proof. The steps of the proof are similar to that of Corollary 2.2, we start by applying Theorem 2.2 a faith on the interval $\left[a,\frac{a+b}{2}\right]$,taking $\alpha=1$ and $x=\frac{3a+b}{4}$, and a second time on the interval $[\frac{a+b}{2},b]$ for $\alpha=1$ and $x=\frac{a+3b}{4}$, make the sum and dividing the results by 2, we obtain (2.30). □

Corollary 2.6. *Let $f: I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a,b \in I$, with $a<b$.*

If $|f''|^{q}$ is P-convex on $[a,b]$, $p,q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$ and and f'' is bounded,i.e., $\|f''\|_{\infty}=\sup\limits_{x\in[a,b]}|f''|<\infty$, for any $x\in[a,b]$,

then the following inequality holds

$$|L_{\alpha}(x)|\leq 2^{\frac{1}{q}}\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\|f''\|_{\infty} \tag{2.31}$$

Proof. by lemma 2.1, the assumptions given on $f''$and using the well-known Hölder's inequality, we have

$$|L_{\alpha}(x)|\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\begin{bmatrix}(x-a)\int\limits_0^1 t^{\alpha+1}|f''(tx+(1-t)a)|\,dt+\\(b-x)\int\limits_0^1 t^{\alpha+1}|f''(tx+(1-t)b)|\,dt\end{bmatrix}$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\begin{bmatrix}(x-a)\left(\int\limits_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int\limits_0^1|f''(tx+(1-t)a)|^{q}\ dt\right)^{\frac{1}{q}}+\\(b-x)\left(\int\limits_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int\limits_0^1|f''(tx+(1-t)a)|^{q}\ dt\right)^{\frac{1}{q}}\end{bmatrix}$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\begin{bmatrix}(x-a)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left(|f''(x)|^{q}+|f''(a)|^{q}\right)\ dt\right)^{\frac{1}{q}}+\\(b-x)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left(|f''(x)|^{q}+\ |f''(b)|^{q}\right)\ dt\right)^{\frac{1}{q}}\end{bmatrix}$$

$$\leq 2^{\frac{1}{q}}\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\|f''\|_{\infty}$$

□

Theorem 2.3. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_{1}[a,b]$, where $a,b\in I$, with $a<b$.*

If $|f''|$ is s-convex on $[a,b]$ with $s\in(0,1)$, and and f'' is bounded,i.e., $\|f''\|_{\infty}=\sup\limits_{x\in[a,b]}|f''|<\infty$, for any $x\in[a,b]$,

then the following inequality holds

$$|L_{\alpha}(x)|\leq\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\frac{1}{\alpha+s+2}+\beta(\alpha+2,s+1)\right]\|f''\|_{\infty} \tag{2.32}$$

Proof. by lemma 2.1, and since $|f''|$ is s-convex and $|f''|\leq M$, then we have

$$|L_{\alpha}(x)|\leq\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}\begin{bmatrix}(x-a)\int\limits_0^1 t^{\alpha+1}|f''(tx+(1-t)a)|\ dt+\\(b-x)\int\limits_0^1 t^{\alpha+1}|f''(tx+(1-t)b)|\,dt\end{bmatrix}$$

$$\leq\quad (x-a)^{\alpha+1}(b-x)^{\alpha+1}$$
$$\times\begin{bmatrix}(x-a)\int\limits_0^1 t^{\alpha+1}\left(t^{s}|f''(x)|\ +(1-t)^{s}|f''(a)|\right)\ dt+\\(b-x)\int\limits_0^1 t^{\alpha+1}\left(t^{s}|f''(x)|\ +(1-t)^{s}|f''(b)|\right)\ dt\end{bmatrix}$$

$$\leq\quad \|f''\|_{\infty}\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}$$
$$\times\left[(x-a)\left(\int\limits_0^1 t^{\alpha+s+1}dt+\int\limits_0^1 t^{\alpha+1}\ (1-t)^{s}\,dt\right)+(b-x)\left(\int\limits_0^1 t^{\alpha+s+1}dt+\int\limits_0^1 t^{\alpha+1}\ (1-t)^{s}\,dt\right)\right]$$

$$= (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\frac{1}{\alpha+s+2}+\beta(\alpha+2,s+1)\right]\|f''\|_{\infty}$$

□

Corollary 2.7. *With the assumptions in Theorem 2.3, in the case where $\alpha=1$,one has the inequality*

$$\left|f(\frac{a+b}{2})-\frac{1}{(b-a)}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{8}\left[\frac{s^2+3s+4}{(s+3)(s+2)(s+1)}\right]\|f''\|_{\infty} \tag{2.33}$$

Proof. The proof is similar to that of Corollary 2.1 □

Corollary 2.8. *With the assumptions in Theorem 2.3, in the case where $\alpha=1$,one has the inequality*

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2}{32}\left[\frac{s^2+3s+4}{(s+3)(s+2)(s+1)}\right]\|f''\|_{\infty} \tag{2.34}$$

Proof. The proof is similar to that of Corollary 2.2 □

Corollary 2.9. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$, with $a<b$.*

If $|f''|^q$ is s-convex on $[a,b]$ with $s\in(0,1)$, $p,q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$ and and f'' is bounded,i.e., $\|f''\|_{\infty}=\sup_{x\in[a,b]}|f''|<\infty$, for any $x\in[a,b]$,

then the following inequality holds

$$|L_{\alpha}(x)| \leq 2^{\frac{1}{q}}\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\|f''\|_{\infty} \tag{2.35}$$

Proof. by lemma 2.1, the assumptions given on $f''$and using the well-known Hölder's inequality, we have

$$|L_{\alpha}(x)| \leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)a)|\,dt+\\ (b-x)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)b)|\,dt\end{array}\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}+\\ (b-x)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(t^s|f''(x)|^q+(1-t)^s|f''(a)|^q\right)dt\right)^{\frac{1}{q}}+\\ (b-x)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(t^s|f''(x)|^q+(1-t)^s\ |f''(b)|^q\right)dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(t^s\ +(1-t)^s\ \right)dt\right)^{\frac{1}{q}}\|f''\|_{\infty}$$

$$2^{\frac{1}{q}}\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\|f''\|_{\infty}$$

□

Theorem 2.4. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a,b \in I$, with $a<b$.*

If $|f''|$ is h-convex on $[a,b]$, and f'' is bounded,i.e., $\|f''\|_\infty = \sup_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$, then the following inequality holds

$$|L_\alpha(x)| \le \|f''\|_\infty (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1} \int_0^1 (t^{\alpha+1} + (1-t)^{\alpha+1}) h(t)dt. \tag{2.36}$$

Proof. by lemma 2.1, and since $|f''|$ is h-convex and $|f''| \le M$, then we have

$$|L_\alpha(x)| \le (x-a)^{\alpha+1}(b-x)^{\alpha+1} \begin{bmatrix} (x-a)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)a)|\,dt+ \\ (b-x)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)b)|\,dt \end{bmatrix}$$

$$\le (x-a)^{\alpha+1}(b-x)^{\alpha+1} \times \begin{bmatrix} (x-a)\int_0^1 t^{\alpha+1}(h(t)|f''(x)| + h(1-t)|f''(a)|)dt+ \\ (b-x)\int_0^1 t^{\alpha+1}(h(t)|f''(x)| + h(1-t)|f''(b)|)dt \end{bmatrix}$$

$$\le \|f''\|_\infty (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1} \int_0^1 t^{\alpha+1}(h(t)+h(1-t))dt$$

$$= \|f''\|_\infty (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1} \int_0^1 (t^{\alpha+1}+(1-t)^{\alpha+1})h(t)dt.$$

□

Corollary 2.10. *With the assumptions in Theorem 2.4, in the case where $\alpha = 1$,one has the inequality*

$$\left| f(\frac{a+b}{2}) - \frac{1}{(b-a)} \int_a^b f(t)\,dt \right| \le \frac{\|f''\|_\infty (b-a)^2}{8} \int_0^1 (2t^2 - 2t + 1)h(t)dt. \tag{2.37}$$

Proof. The proof is similar to that of Corollary 2.1 □

Corollary 2.11. *With the assumptions in Theorem 2.4, in the case where $\alpha = 1$,one has the inequality*

$$\left| \frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \frac{\|f''\|_\infty (b-a)^2}{32} \int_0^1 (2t^2 - 2t + 1)h(t)dt. \tag{2.38}$$

Proof. The proof is similar to that of Corollary 2.2 □

Corollary 2.12. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a,b \in I$, with $a<b$.*

If $|f''|^q$ is h-convex on $[a,b]$, $p,q \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and and f'' is bounded,i.e., $\|f''\|_\infty = \sup_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$, then the following inequality holds

$$|L_\alpha(x)| \le 2^{\frac{1}{q}} (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1} \left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}} \left(\int_0^1 h(t)\,dt\right)^{\frac{1}{q}} \|f''\|_\infty \tag{2.39}$$

Proof. By lemma 2.1, the assumptions given on $f''$and using the well-known Hölder's inequality, we have

$$|L_\alpha(x)| \leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\begin{bmatrix} (x-a)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)a)|\,dt+ \\ (b-x)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)b)|\,dt \end{bmatrix}$$

$$\leq \frac{1}{(\alpha+1)(b-a)}$$
$$\times\left[(b-x)(x-a)^2\int_0^1 t^{\alpha+1}\left|f''(tx+(1-t)a)\right|\,dt+(a-x)(b-x)^2\int_0^1 t^{\alpha+1}\left|f''(tx+(1-t)b)\right|dt\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\begin{bmatrix} (x-a)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}+ \\ (b-x)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}} \end{bmatrix}$$

$$=(b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(h(t)\left|f''(x)\right|^q+h(1-t)\left|f''(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right]$$

$$\leq 2^{\frac{1}{q}}(b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1 h(t)\,dt\right)^{\frac{1}{q}}\|f''\|_\infty.$$

□

Now, using the above reasoning we can obtain some new Ostrowski Type inequalities involving Riemann-Liouville fractional integrals for functions whose derivatives are m-convex.

Theorem 2.5. *Let $f: I \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a,b \in I$, with $0<a<b$.*

If $|f''|$ is $m-$convex function on $[a,b]$, $m \in (0,1]$ and f'' is bounded,i.e., $\|f''\|_\infty = \sup_{x\in[a,b]}|f''| < \infty$,for any $x \in [a,b]$,then the following inequality holds

$$|L_\alpha(x)| \leq (1-m)\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}(\Upsilon_1+\Upsilon_2)\|f''\|_\infty \tag{2.40}$$

where $\Upsilon_1=(x-a)\left[\frac{1}{\alpha+3}\left(\frac{x-a}{x-ma}\right)+\frac{1}{\alpha+2}\left(\frac{(1-m)a}{x-ma}+\frac{m}{(1-m)}\right)\right]$,

and $\Upsilon_2=(b-x)\left[-\frac{1}{\alpha+3}\left(\frac{b-x}{b-mx}\right)+\frac{1}{\alpha+2}\left(\frac{1}{1-m}\right)\right]$.

Proof. By lemma 2.1, and Under the given assumptions on f'' we have

$$|L_\alpha(x)| = \left|(b-x)^{\alpha+1}\int_a^x (y-a)^{\alpha+1}f''(y)\,dy+(x-a)^{\alpha+1}\int_x^b (b-y)^{\alpha+1}f''(y)\,dy\right|$$

$$=\left|\begin{matrix}(b-x)^{\alpha+1}(x-ma)\int_{\frac{(1-m)a}{x-ma}}^{1}(tx+m(1-t)a-a)^{\alpha+1}f''(tx+m(1-t)a)\,dt+ \\ (x-a)^{\alpha+1}(b-mx)\int_{\frac{(1-m)x}{b-mx}}^{1}(b-(tb+m(1-t)x))^{\alpha+1}f''(tb+m(1-t)x)\,dt\end{matrix}\right|$$

$$\leq \left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2} \int\limits_{\frac{(1-m)a}{x-ma}}^{1} \left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1} |f''(tx+m(1-t)a)|\,dt + \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2} \int\limits_{\frac{(1-m)x}{b-mx}}^{1} (1-t)^{\alpha+1} |f''(tb+m(1-t)x)|\,dt \end{array} \right]$$

$$\leq (1-m)\left\|f''\right\|_{\infty} \left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2} \int\limits_{\frac{(1-m)a}{x-ma}}^{1} \left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1} \left(t+\frac{m}{1-m}\right)dt + \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2} \int\limits_{\frac{(1-m)x}{b-mx}}^{1} (1-t\ )^{\alpha+1} \left(t+\frac{m}{1-m}\right)dt \end{array} \right]$$

$$\leq \left\|f''\right\|_{\infty} \left[\begin{array}{c} (1-m)(b-x)^{\alpha+1}(x-ma)^{\alpha+2} \left[\begin{array}{c} \int\limits_{\frac{(1-m)a}{x-ma}}^{1} \left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+2} dt + \\ \left(\frac{(1-m)a}{x-ma}+\frac{m}{(1-m)}\right) \int\limits_{\frac{(1-m)a}{x-ma}}^{1} \left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1} dt \end{array} \right] + \\ (1-m)(x-a)^{\alpha+1}(b-mx)^{\alpha+2} \left[\begin{array}{c} \int\limits_{\frac{(1-m)x}{b-mx}}^{1} -(1-t\ )^{\alpha+2} dt + \\ \left(\frac{1}{1-m}\right) \int\limits_{\frac{(1-m)x}{b-mx}}^{1} (1-t\ )^{\alpha+1}\ dt \end{array} \right] \end{array} \right]$$

$$\leq (1-m)\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}(\Upsilon_1+\Upsilon_2)\left\|f''\right\|_{\infty}$$

□

Corollary 2.13. *With the assumptions in Theorem 2.5, in the case where $\alpha = 1$, one has the inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(t)\,dt \right| \leq \frac{(1-m)(b-a)^2}{16}\varphi \left\|f''\right\|_{\infty} \tag{2.41}$$

where $\varphi = \frac{(1-m)(b-a)^2}{4(b+(1-2m)a)(2b-m(b+a))} + \frac{1}{3}\left[\frac{1+m}{1-m} + \frac{2(1-m)a}{b+(1-2m)a}\right]$.

Proof. Choose $x = \frac{a+b}{2}$ and $\alpha = 1$ in (2.40), we get

$$\frac{(b-a)^3}{2} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(t)\,dt \right| \leq \varphi \frac{(1-m)(b-a)^5}{32} \left\|f''\right\|_{\infty} \tag{2.42}$$

dividing both side of (2.42) by $\frac{(b-a)^3}{2}$ we obtain (2.41). □

Corollary 2.14. *With the assumptions in Theorem 2.5, in the case where $\alpha = 1$,one has the inequality.*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq (\Psi_1+\Psi_2) \frac{(1-m\ )(b-a)^2}{128} \left\|f''\right\|_{\infty} \tag{2.43}$$

where $\Psi_1 = \frac{b-a}{4}\left[\frac{1}{b+(3-4m)a} - \frac{1}{(2-m)b+(2-3m)a}\right] + \frac{1}{3}\left[\frac{1+m}{1-m} + \frac{4(1-m)a}{b+(3-4m)a}\right]$,
and $\Psi_2 = \frac{b-a}{4}\left[\frac{1}{(3-2m)b+(1-2m)a} - \frac{1}{(4-3m)b-ma}\right] + \frac{1}{3}\left[\frac{1+m}{1-m} + \frac{2(1-m)(a+b)}{(3-2m)b+(1-2m)a}\right]$.

Proof. Apply Theorem 2.5, a faith on the interval $\left[a, \frac{a+b}{2}\right]$,taking $\alpha = 1$ in (2.40) , and replace x by $\frac{3a+b}{4}$, we get

$$\frac{(b-a)^3}{16}\left[\left|f(\frac{3a+b}{4})-\frac{2}{b-a}\int_a^{\frac{b+a}{2}} f(t)dt\right|\right] \leq \frac{(1-m)(b-a)^5}{1024}\left\|f''\right\|_\infty \Psi_1 \tag{2.44}$$

(2.44) implies

$$\left|f(\frac{3a+b}{4})-\frac{2}{b-a}\int_a^{\frac{b+a}{2}} f(t)dt\right| \leq \frac{(1-m)(b-a)^2}{64}\left\|f''\right\|_\infty \Psi_1 \tag{2.45}$$

Apply Theorem 2.5 another faith on the interval $\left[\frac{a+b}{2}, b\right]$, taking $\alpha = 1$ in (2.40) , and replace x by $\frac{a+3b}{4}$, we get

$$\left|f(\frac{a+3b}{4})-\frac{2}{b-a}\int_{\frac{b+a}{2}}^{b} f(t)dt\right| \leq \frac{(1-m)(b-a)^2}{64}\left\|f''\right\|_\infty \Psi_2 \tag{2.46}$$

summing (2.45) and (2.46), dividing the result by 2 we obtain (2.43). □

Theorem 2.6. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a, b \in I$, with $a < b$.*

If $|f''|$ is (s,m) $-$convex on $[a,b]$, where $(s,m) \in (0,1]^2$, and and f'' is bounded,i.e., $\|f''\|_\infty = \sup_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$, then the following inequality holds

$$|L_\alpha(x)| \leq (1-m)\ \left\|f''\right\|_\infty \left[\frac{m}{(1-m)(\alpha+2)}\ (b-x)^{\alpha+1}(x-\ a)^{\alpha+1}\ \ (b-a)+\chi_1+\chi_2\right] \tag{2.47}$$

where $\chi_1 = (b-x)^{\alpha+1}(x-ma)^{-s}((1-m)a)^{\alpha+s+2}\beta(\alpha+2, -s-\alpha-2)$,
and $\chi_2 = (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\beta(\alpha+2, s+1)$.

Proof. by lemma 2.1, and Under the given assumptions on f'', we have

$$|L_\alpha(x)| = \left|(b-x)^{\alpha+1}\int_a^x (y-a)^{\alpha+1} f''(y)\,dy + (x-a)^{\alpha+1}\int_x^b (b-y)^{\alpha+1} f''(y)\,dy\right|$$

$$= \left|\begin{array}{l}(b-x)^{\alpha+1}(x-ma)\displaystyle\int_{\frac{(1-m)a}{x-ma}}^{1}(tx+m(1-t)a-a)^{\alpha+1} f''(tx+m(1-t)a)\,dt+ \\ (x-a)^{\alpha+1}(b-mx)\displaystyle\int_{\frac{(1-m)x}{b-mx}}^{1}(b-(tb+m(1-t)x))^{\alpha+1} f''(tb+m(1-t)x)\,dt\end{array}\right|$$

$$\leq \left[\begin{array}{l}\left|(b-x)^{\alpha+1}(x-ma)^{\alpha+2}\displaystyle\int_{\frac{(1-m)a}{x-ma}}^{1}\left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1} f''(tx+m(1-t)a)\,dt\right| + \\ \left|(x-a)^{\alpha+1}(b-mx)^{\alpha+2}\displaystyle\int_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1} f''(tb+m(1-t)x)\,dt\right|\end{array}\right]$$

$$\leq \left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\int\limits_{\frac{(1-m)a}{x-ma}}^{1}\left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1}|f''(tx+m(1-t)a)|\,dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1}|f''(tb+m(1-t)x)|\,dt \end{array}\right]$$

$$\leq \left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\int\limits_{\frac{(1-m)a}{x-ma}}^{1}\left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1}\left(t^s|f''(x)|+m(1-t^s)|f''(a)|\right)dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1}\left(t^s|f''(b)|+m(1-t^s)|f''(x)|\right)dt \end{array}\right]$$

$$\leq (1-m)\|f''\|_{\infty}\left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\int\limits_{\frac{(1-m)a}{x-ma}}^{1}\left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1}\left(t^s+\frac{m}{1-m}\right)dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1}\left(t^s+\frac{m}{1-m}\right)dt \end{array}\right]$$

$$\leq (1-m)\|f''\|_{\infty}\left[\begin{array}{c} \frac{m}{(1-m)(\alpha+2)}(b-x)^{\alpha+1}(x-a)^{\alpha+2}+\frac{m}{(1-m)(\alpha+2)}(x-a)^{\alpha+1}(b-x)^{\alpha+2}+ \\ (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\left(\frac{(1-m)a}{x-ma}\right)^{\alpha+s+2}\int\limits_{\frac{(1-m)a}{x-ma}}^{1}(1-t)^{\alpha+1}t^{-(s+\alpha+3)}dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1}t^s dt \end{array}\right]$$

$$\leq (1-m)\|f''\|_{\infty}\left[\begin{array}{c} \frac{m}{(1-m)(\alpha+2)}(b-x)^{\alpha+1}(x-a)^{\alpha+2}+\frac{m}{(1-m)(\alpha+2)}(x-a)^{\alpha+1}(b-x)^{\alpha+2}+ \\ (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\left(\frac{(1-m)a}{x-ma}\right)^{\alpha+s+2}\int\limits_{0}^{1}(1-t)^{\alpha+1}t^{-(s+\alpha+3)}dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_{0}^{1}(1-t)^{\alpha+1}t^s dt \end{array}\right]$$

$$\leq (1-m)\|f''\|_{\infty}\left[\frac{m}{(1-m)(\alpha+2)}(b-x)^{\alpha+1}(x-a)^{\alpha+1}(b-a)+\chi_1+\chi_2\right]$$

□

Corollary 2.15. *With the assumptions in Theorem 2.6, in the case where $\alpha=1$,one has the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int\limits_a^b f(t)\,dt\right|\leq\Gamma\left(\frac{1-m}{b-a}\right)\|f''\|_{\infty} \tag{2.48}$$

where $\Gamma=\frac{m(b-a)^3}{24m}+\frac{(b+(1-2m)a)^{-s}((1-m)a)^{s+3}}{2^{1-s}}\beta(3,-s-3)+\frac{((2-m)b-ma)^3}{2^4}\beta(3,s+1)$.

Proof. just take in (2.47), $\alpha=1$, $x=\frac{a+b}{2}$ and dividing both side of the result by $\frac{(b-a)^3}{2}$ we obtain (2.48). □

Corollary 2.16. *With the assumptions in Theorem 2.6, in the case where $\alpha=1$,one has the inequality*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq(\zeta_1+\zeta_2)\frac{1-m}{2(b-a)}\|f''\|_{\infty}, \tag{2.49}$$

where $\zeta_1 = \frac{m(b-a)^3}{48(1-m)} + \left(\frac{b+(3-4m)a}{4}\right)^{-s}((1-m)\,a)^{s+3}\,\beta(3,-s-3) + \frac{((2-m)b-ma)^3}{64}\beta(3,\,s+1)$,
and $\zeta_2 = \frac{m(b-a)^3}{48(1-m)} + \left(\frac{(3-2m)b+(1-2m)a}{4}\right)^{-s}\left(\frac{(1-m)(b+a)}{2}\right)^{s+3}\beta(3,-s-3) + \frac{((4-3m)b+(2-3m)a)^3}{64}\beta(3,\,s+1)$.

Proof. The steps of the proof are similar to that of Corollary 2.2, we start by applying Theorem 2.6 a faith on the interval $\left[a, \frac{a+b}{2}\right]$,taking $\alpha = 1$ and $x = \frac{3a+b}{4}$, and a second time on the interval $[\frac{a+b}{2}, b]$ for $\alpha = 1$ and $x = \frac{a+3b}{4}$, make the sum and dividing the results by $\frac{(b-a)^3}{32}$, we obtain (2.49). □

3 Acknowledgements

The author would like to thank the anonymous referee for his/her valuable suggestions. This work has been supported by CNEPRU–MESRS–B01120120103 project grants.

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Received: June 13, 2014; *Accepted*: July 19, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 460–471

Malaya Journal of Matematik
www.malayajournal.org

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ISSN : 2319-3786

Oscillation theorems for second-order half-linear neutral difference equations

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Abstract

In this article, some new oscillation criteria are established for the second order neutral difference equation of the form

$$\Delta(a(n)\Delta(z(n))^{\alpha}) + q(n)x^{\alpha}(\sigma(n)) = 0, \; n \geq n_0,$$

where $z(n) = x(n) + p(n)x(\tau(n))$. Our results improve and extend some known results in the literature. Some examples are also provided to show the importance of these results.

Keywords: Second order, half-linear, neutral, oscillation, difference equations.

2010 MSC: 39A10.

1 Introduction

This article deals with the oscillation of all solutions of the second order neutral difference equation of the form

$$\Delta(a(n)\Delta(z(n))^{\alpha}) + q(n)x^{\alpha}(\sigma(n)) = 0, \; n \geq n_0, \tag{1.1}$$

where $z(n) = x(n) + p(n)x(\tau(n))$. Throughout this article, we assume the following hypotheses:

(H_1) α is a ratio of odd positive integers;

(H_2) $\{a(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive real numbers;

(H_3) $\{\sigma(n)\}$ and $\{\tau(n)\}$ are sequences of nonnegative integers with $\tau \circ \sigma = \sigma \circ \tau$.

By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined and satisfying the equation (1.1) for all $n \geq n_0$. A nontrivial solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

It is well-known that second order neutral difference equations find applications in so many problems in the field of population dynamics, economics, biology etc. Therefore, there has been much interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of different types of second order difference equations, see for example [1, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Here, we recall some of the previous works that motivate our study.

In[1, 4, 9], the authors discussed the oscillatory behavior of all solutions of equation

$$\Delta^2(x(n) + p(n)x(n-\tau)) + q(n)x(n-\sigma) = 0$$

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under different conditions on the sequence $\{p_n\}$ and $\{q_n\}$. In [8], the authors studied the oscillation of nonlinear difference equation

$$\Delta(a(n)\Delta(x(n)+p(n)x(n-\tau))+q(n)f(x(n-\sigma))=0,$$

under the assumptions

$$\frac{f(u)}{u}\leq M>0,\ \sum_{n=n_0}^{\infty}\frac{1}{a(n)}=\infty \text{ and } 0\leq p(n)<1.$$

In [10, 11], the authors established several oscillation results for the equation

$$\Delta(a(n)(\Delta(x(n)+p(n)x(n-\tau)))^{\gamma})+f(n,x(n-\tau))=0$$

under the assumptions

$$f(n,u)sgn(u)\geq q(n)u^{\gamma},\ \sum_{n=n_0}^{\infty}\frac{1}{(a(n))^{1/\alpha}}=\infty \text{ and}$$

In [7, 8, 10, 11], the authors studied the oscillatory properties of the nonlinear neutral difference equation of the form

$$\Delta(a(n)(\Delta(x(n)+p(n)x(\tau(n))))^{\alpha})+q(n)x^{\beta}(\sigma(n))=0$$

with the condition $0\leq p(n)\leq p<\infty$ and $\tau\circ\sigma=\sigma\circ\tau$. Following this trend, in this paper we establish some new oscillation criteria for the equation (1.1) with the following conditions:

(i)

$$\sum_{n=n_0}^{\infty}\frac{1}{[a(n)]^{1/\alpha}}=\infty \tag{1.2}$$

and

(ii)

$$\sum_{n=n_0}^{\infty}\frac{1}{[a(n)]^{1/\alpha}}<\infty. \tag{1.3}$$

In Sections 2 and 3, we use the following notations for our convenience:

$$Q(n)=\min\{q(n),q[\tau(n)]\} \text{ and } \delta(n)=\sum_{s=\eta(n)}^{\infty}\frac{1}{[a(s)]^{1/\alpha}}.$$

2 Oscillation Results

In this section, we present the following lemma, which will be useful in proving the main results.

Lemma 2.1. *Let $A\geq 0$, $B\geq 0$ and $\alpha\geq 1$. Then*

$$(A+B)^{\alpha}\leq 2^{\alpha-1}(A^{\alpha}+B^{\alpha}). \tag{2.4}$$

Proof. The proof can be found in [16, Lemma 2.1]. □

Theorem 2.1. *Suppose that condition (1.2) holds, $\Delta\sigma(n)>0$, $\sigma(n)\leq n$ and $\sigma(n)\leq\tau(n)$ for all $n\geq n_0$. If there exists a positive real sequence $\{\rho(n)\}$ such that*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\rho(s)\left[\frac{Q(s)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(\frac{\Delta\rho(s)}{\rho(s)}\right)^{\alpha+1}a[\sigma(s)]\left[1+\frac{1}{(p^{\alpha}[\sigma(s)])^2}\right]\right]=\infty, \tag{2.5}$$

then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $n_1 \geq n_0$ such that $x(n) > 0$, $x[\tau(n)] > 0$ and $x[\sigma(n)] > 0$ for all $n \geq n_1$. Then by the definition of $z(n)$, we have $z(n) > 0$. From equation (1.1), for all sufficiently large n, we have

$$\begin{aligned} \Delta(a(n)\Delta(z(n))^{\alpha}) + q(n)x^{\alpha}(\sigma(n)) + q(\tau(n))p^{\alpha}(\sigma(n))x^{\alpha}(\sigma(\tau(n))) \\ + p^{\alpha}(\sigma(n))\Delta(a(\tau(n)))(\Delta z(\tau(n)))^{\alpha} = 0, \end{aligned} \tag{2.6}$$

Using (2.4), $\tau \circ \sigma = \sigma \circ \tau$ and the definition of $\{z(n)\}$ in (2.6), we conclude that

$$\Delta(a(n)\Delta(z(n))^{\alpha}) + \frac{1}{2^{\alpha-1}}Q(n)z^{\alpha}(\sigma(n)) + p^{\alpha}(\sigma(n))\Delta(a(\tau(n)))(\Delta z(\tau(n)))^{\alpha} \leq 0. \tag{2.7}$$

From the equation (1.1), we have

$$\Delta(a(n)\Delta(z(n))^{\alpha}) = -q(n)x^{\alpha}(\sigma(n)) < 0, \; n \geq n_1. \tag{2.8}$$

Thus $\{a(n)(\Delta z(n))^{\alpha}\}$ is a decreasing sequence. Here, we have two possible cases for $\Delta z(n)$, namely, (i) $\Delta z(n) < 0$ eventually or (ii) $\Delta z(n) > 0$ eventually.

Case (i): Suppose that $\Delta z(n) < 0$ for all $n \geq n_2 \geq n_1 \geq n_0$. Then, from (2.8), we have

$$a(n)\Delta(z(n))^{\alpha} \leq a(n_2)\Delta(z(n_2))^{\alpha} < 0, \; n \geq n_2 \tag{2.9}$$

which implies that

$$z(n) \leq z(n_2) + a^{1/\alpha}(n_2)\Delta z(n_2)\sum_{s=n_2}^{n-1}\frac{1}{a^{1/\alpha(s)}}. \tag{2.10}$$

Letting $n \to \infty$, by (1.2) we see that $z(n) \to -\infty$, which is a contradiction for the positivity of $z(n)$.

Case (ii): Suppose that $\Delta z(n) > 0$ for all $n \geq n_2 \geq n_1 \geq n_0$. Define

$$w(n) = \rho(n)\frac{a(n)(\Delta z(n))^{\alpha}}{(z(\sigma(n))^{\alpha}}, \; n \geq n_2, \tag{2.11}$$

then $w(n) > 0$ for all $n \geq n_2$. By (2.8), we have

$$\Delta(z(\sigma(n)) \geq \Delta z(n)\left(\frac{a(n)}{a(\sigma(n))}\right)^{1/\alpha}. \tag{2.12}$$

From (2.11), we obtain

$$\begin{aligned} \Delta w(n) &= \Delta\rho(n)\frac{a(n+1)(\Delta z(n+1))^{\alpha}}{(z[\sigma(n+1)])^{\alpha}} + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}} \\ &\quad -\frac{\rho(n)a(n)(\Delta z(n))^{\alpha}}{(z[\sigma(n)])^{\alpha}(z[\sigma(n+1)])^{\alpha}}\Delta(z[\sigma(n)])^{\alpha} \\ &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) - \frac{\rho(n)a(n+1)(\Delta z(n+1))^{\alpha}}{(z[\sigma(n)])^{\alpha}(z[\sigma(n+1)])^{\alpha}}\Delta(z[\sigma(n)])^{\alpha} \\ &\quad +\rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}} \\ &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}} \\ &\quad -\frac{\rho(n)}{\rho(n+1)}w(n+1)\frac{\Delta(z[\sigma(n)])^{\alpha}}{(z[\sigma(n)])^{\alpha}}. \end{aligned} \tag{2.13}$$

By using Mean Value Theorem, we have

$$\begin{aligned} \Delta w(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}} \\ &\quad -\frac{\rho(n)}{\rho(n+1)}w(n+1)\alpha\frac{w^{1/\alpha}(n+1)}{\rho^{1/\alpha}(n+1)a^{1/\alpha}(\sigma(n))} \\ &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}} \\ &\quad -\alpha\frac{\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)}\frac{w^{\frac{\alpha+1}{\alpha}}(n+1)}{a^{1/\alpha}(\sigma(n))}. \end{aligned} \tag{2.14}$$

Define

$$v(n) = \rho(n)\frac{a[\tau(n)](\Delta z[\tau(n)])^{\alpha}}{(z[\sigma(n)])^{\alpha}},\ n \geq n_2, \tag{2.15}$$

then, we have $v(n) > 0$ and

$$\begin{aligned} \Delta v(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}v(n+1) + \frac{\rho(n)\Delta(a[\tau(n)](\Delta z[\tau(n)])^{\alpha})}{(z[\sigma(n+1)])^{\alpha}} \\ &\quad - \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)}\frac{v^{\frac{\alpha+1}{\alpha}}(n+1)}{a^{1/\alpha}[\sigma(n)]}. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we have

$$\begin{aligned} \Delta w(n) + p^{\alpha}[\sigma(n)]\Delta v(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \frac{\Delta\rho(n)}{\rho(n+1)}p^{\alpha}[\sigma(n)]v(n+1) \\ &+ \rho(n)\left[\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n)])^{\alpha}} + p^{\alpha}[\sigma(n)]\frac{\Delta(a[\tau(n)](\Delta z[\tau(n)]))^{\alpha}}{(z[\sigma(n)])^{\alpha}}\right] \\ &- \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]}\left[w^{\frac{\alpha+1}{\alpha}}(n+1) + p^{\alpha}[\sigma(n)]v^{\frac{\alpha+1}{\alpha}}(n+1)\right]. \end{aligned} \tag{2.17}$$

Using (2.7) in (2.17), we have

$$\begin{aligned} \Delta w(n) + p^{\alpha}[\sigma(n)]\Delta v(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}\left[w(n+1) + p^{\alpha}[\sigma(n)]v(n+1)\right] - \frac{\rho(n)Q(n)}{2^{\alpha-1}} \\ &- \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]}\left[w^{\frac{\alpha+1}{\alpha}}(n+1) + p^{\alpha}[\sigma(n)]v^{\frac{\alpha+1}{\alpha}}(n+1)\right]. \end{aligned} \tag{2.18}$$

Summing the last inequality from n_2 to $n-1$, we obtain

$$\begin{aligned} w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) &\leq -\frac{1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s) \\ &+ \sum_{s=n_2}^{n-1}\left[\frac{\Delta\rho(s)}{\rho(s+1)}w(s+1) - \frac{\alpha\rho(s)}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\sigma(s)]}w^{\frac{\alpha+1}{\alpha}}(s+1)\right] \\ &+ \sum_{s=n_2}^{n-1}\left[\frac{\Delta\rho(s)}{\rho(s+1)}v(s+1) - \frac{\alpha p(s)p^{\alpha}[\sigma(s)]}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\sigma(s)]}v^{\frac{\alpha+1}{\alpha}}(s+1)\right]. \end{aligned} \tag{2.19}$$

$$\begin{aligned} \text{Let } A(n) &= \left(\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]}\right)^{\frac{\alpha}{\alpha+1}} w(n+1) \text{ and} \\ B(n) &= \left(\frac{\alpha}{\alpha+1}\frac{\Delta\rho(n)}{\rho(n+1)}\left(\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]}\right)^{\frac{-\alpha}{\alpha+1}}\right)^{\alpha}. \end{aligned} \tag{2.20}$$

Now, using the inequality

$$\frac{\alpha+1}{\alpha}AB^{1/\alpha} - A^{\frac{\alpha+1}{\alpha}} \leq \frac{1}{\alpha}B^{\frac{\alpha+1}{\alpha}}, \tag{2.21}$$

by taking $A = A(n)$ and $B = B(n)$ in the second part of the right hand side of the inequality (2.19), we have

$$\begin{aligned} w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) &\leq \frac{-1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s) \\ &+ \sum_{s=n_2}^{n-1}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}}a[\sigma(s)] \\ &+ \sum_{s=n_2}^{n-1}\left[\frac{\Delta\rho(s)}{\rho(s+1)}v(s+1) - \frac{\alpha\rho(s)}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\sigma(s)]}v^{\frac{\alpha+1}{\alpha}}v(s+1)\right]. \end{aligned} \tag{2.22}$$

Now, let

$$C(n) = \left(\frac{\alpha \rho(n) p^{\alpha}[\sigma(n)]}{\rho^{\frac{\alpha+1}{\alpha}}(n+1) a^{1/\alpha}[\sigma(n)]} \right)^{\frac{\alpha}{\alpha+1}} v(n+1)$$

and

$$D(n) = \left[\frac{\alpha}{\alpha+1} \frac{\Delta\rho(n)}{\rho(n+1)} \left(\frac{\alpha \rho(n) p^{\alpha}[\sigma(n)]}{\rho^{\frac{\alpha+1}{\alpha}}(n+1) a^{1/\alpha}[\sigma(n)]} \right)^{\frac{-\alpha}{\alpha+1}} \right]^{\alpha}. \tag{2.23}$$

Now, using the inequality (2.21) by taking $A = C(n)$ and $B = D(n)$ in the third part of the right hand side of (2.32), we get

$$w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \leq \frac{-1}{2^{\alpha-1}} \sum_{s=n_2}^{n-1} \rho(s)Q(s)$$
$$+ \sum_{s=n_2}^{n-1} \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}} a[\sigma(s)] + \sum_{s=n_2}^{n-1} \frac{(\Delta\rho(s))^{\alpha+1}}{((p^{\alpha}\sigma(s)))^2}. \tag{2.24}$$

Now,

$$w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \leq - \sum_{s=n_2}^{n-1} \rho(s) \left[\frac{1}{2^{\alpha-1}} Q(s) \right.$$
$$\left. - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{(\Delta\rho(s))}{(\rho(s))} \right)^{\alpha+1} a[\sigma(s)] \left[1 + \frac{1}{(p^{\alpha}[\sigma(s)])^2} \right] \right]. \tag{2.25}$$

Letting $n \to \infty$ in the last inequality and using (2.5), we see that $w(n) + p^{\alpha}[\sigma(n)]v(n) \to -\infty$, which contradicts the positivity of $w(n) + p^{\alpha}[\sigma(n)]v(n)$. This completes the proof. □

Theorem 2.2. *Assume that condition (1.2) holds, $p(n) \leq p_0 < \infty$, $\Delta\sigma(n) > 0$, $\sigma(n) \leq n$ and $\sigma(n) \leq \tau(n)$ for all $n \geq n_0$. Further, suppose that there exists a sequence $\{\rho(n)\}$ of positive real numbers such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \rho(s) \left[\frac{Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{\Delta\rho(s)}{\rho(s)} \right)^{\alpha+1} a[\sigma(s)] \left(1 + \frac{1}{(p_0^{\alpha}[\sigma(s)])^2} \right) \right] = \infty. \tag{2.26}$$

Then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $n_1 \geq n_0$ such that $x(n) > 0$, $x[\tau(n)] > 0$ and $x[\sigma(n)] > 0$ for all $n \geq n_1$. Using the equation (1.1), for all sufficiently large n, we have

$$(a(n)(\Delta z(n))^{\alpha}) + q(n)x^{\alpha}[\sigma(n)] + p_0^{\alpha} q[\tau(n)] x^{\alpha}[\sigma(\tau(n))]$$
$$+ p_0^{\alpha}(a[\tau(n)])(\Delta z[\tau(n)]^{\alpha}) \leq 0. \tag{2.27}$$

By applying (2.4) and the definition of $z(n)$, we conclude that

$$\Delta(a(n)(\Delta z(n))^{\alpha}) + \frac{1}{2^{\alpha-1}} Q(n) z^{\alpha}[\sigma(n)] + p_0^{\alpha}(a[\tau(n)](\Delta z[\tau(n)])^{\alpha}) \leq 0. \tag{2.28}$$

The remainder of the proof is similar to that of Theorem 2.1 and hence it is omitted.

□

Theorem 2.3. *Assume that conditions (1.2) holds, $\tau(n) \leq n$ and $\sigma(n) \geq \tau(n)$ for all $n \geq n_0$. Furthermore assume that there exists a positive real sequence $\{\rho(n)\}$ such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \rho(s) \left[\frac{Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{\Delta\rho(s)}{\rho(s)} \right)^{\alpha+1} a[\tau(s)] \left(1 + \frac{1}{(p^{\alpha}[\sigma(s)])^2} \right) \right] = \infty. \tag{2.29}$$

Then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $n_1 \geq n_0$ such that $x(n) > 0$, $x[\tau(n)] > 0$ and $x[\sigma(n)] > 0$ for all $n \geq n_1$. Proceeding as in the proof of Theorem 2.1, we have (2.7). From (1.1), we have $\{a(n)(\Delta z(n))^\alpha\}$ is a decreasing sequence. Then, we have two possible cases for $\Delta z(n)$, namely, (i) $\Delta z(n) < 0$ eventually or (ii) $\Delta z(n) > 0$ eventually.

Case (i): If $\Delta z(n) < 0$ for all $n \geq n_2 \geq n_1$, then by the similar proof of case (i) of Theorem 2.1, we get a contradiction.

Case (ii): If $\Delta z(n) > 0$ for $n \geq n_2 \geq n_1$, then we define

$$w(n) = \rho(n)\frac{a(n)(\Delta z(n))^\alpha}{z[\tau(n)])^\alpha},\ n \geq n_2, \tag{2.30}$$

and $w(n) > 0$. for all $n \geq n_2$. By (2.8), we have

$$\Delta z[\tau(n)] \leq \left(\frac{a(n)}{a[\tau(n)]}\right)^{1/\alpha} \Delta z(n),\ n \geq n_2. \tag{2.31}$$

From (2.30), we have

$$\begin{aligned}
\Delta w(n) &= \Delta\rho(n)\frac{a(n+1)(\Delta z(n+1))^\alpha}{(z[\tau(n+1)])^\alpha} + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} \\
&\quad - \frac{\rho(n)a(n)(\Delta z(n))^\alpha}{(z[\tau(n)])^\alpha(z[\tau(n+1)])^\alpha}\Delta(z[\tau(n)])^\alpha \\
&\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) - \frac{\rho(n)a(n+1)(\Delta z(n+1))^\alpha}{(z[\tau(n)])^\alpha(z[\tau(n+1)])^\alpha}\Delta(z[\tau(n)])^\alpha \\
&\quad + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} \\
&\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} - \frac{\rho(n)w(n+1)}{\rho(n+1)}\frac{\Delta(z[\tau(n)])^\alpha}{(z[\tau(n)])^\alpha}.
\end{aligned} \tag{2.32}$$

By using Mean Value Theorem, we have

$$\begin{aligned}
\Delta w(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} \\
&\quad - \frac{\rho(n)w(n+1)}{\rho(n+1)}\frac{\alpha w^{1/\alpha}(n+1)}{\rho^{1/\alpha}(n+1)a^{1/\alpha}[\tau(n)]} \\
&\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} - \alpha\frac{\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)}\frac{w^{\frac{\alpha+1}{\alpha}}(n+1)}{a^{1/\alpha}[\tau(n)]}.
\end{aligned} \tag{2.33}$$

Define

$$v(n) = \rho(n)\frac{a[\tau(n)](\Delta z[\tau(n)])^\alpha}{(z[\tau(n)])^\alpha},\ n \geq n_2, \tag{2.34}$$

then we get $v(n) > 0$ and

$$\Delta v(n) \leq \frac{\Delta\rho(n)}{\rho(n+1)}v(n+1) + \frac{\rho(n)(a[\tau(n)](\Delta z[\tau(n)])^\alpha)}{(z[\tau(n)])^\alpha} - \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)}\frac{v^{\frac{\alpha+1}{\alpha}}(n+1)}{a^{1/\alpha}[\tau(n)]}. \tag{2.35}$$

From (2.33) and (2.35), we have

$$\begin{aligned}
\Delta w(n) + p^\alpha[\sigma(n)]\Delta v(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \frac{\Delta\rho(n)}{\rho(n+1)}p^\alpha[\sigma(n)]v(n+1) \\
&\quad + \rho(n)\left[\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} + \frac{p^\alpha[\sigma(n)]\Delta(a(\tau(n))(\Delta z(\tau(n)))^\alpha)}{(z[\tau(n+1)])^\alpha}\right] \\
&\quad - \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{\frac{1}{\alpha}}[\tau(n)]}\left[w^{\frac{\alpha+1}{\alpha}}(n+1) + p^\alpha[\sigma(n)]v^{\frac{\alpha+1}{\alpha}}(n+1)\right].
\end{aligned} \tag{2.36}$$

Using (2.7) in (2.36), we have

$$\Delta w(n) + p^{\alpha}[\sigma(n)]\Delta v(n) \le \frac{\Delta\rho(n)}{\rho(n+1)}\Big[w(n+1) + p^{\alpha}[\sigma(n)]v(n+1)\Big] - \frac{\rho(n)Q(n)}{2^{\alpha-1}} - \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{\frac{1}{\alpha}}[\tau(n)]}\Big[w^{\frac{\alpha+1}{\alpha}}(n+1) + p^{\alpha}[\sigma(n)]v^{\frac{\alpha+1}{\alpha}}(n+1)\Big]. \quad (2.37)$$

Summing the last inequality from n_2 to $n-1$, we have

$$w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \le -\frac{1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s) + \sum_{s=n_2}^{n-1}\left[\frac{\Delta\rho(s)}{\rho(s+1)}w(s+1) - \frac{\alpha\rho(s)}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{\frac{1}{\alpha}}[\tau(s)]}w^{\frac{\alpha+1}{\alpha}}(s+1)\right] + \sum_{s=n_2}^{n-1}\Big[\frac{p^{\alpha}(\sigma(s))\Delta\rho(s)}{\rho(s+1)}v(s+1) - \frac{\alpha\rho(s)p^{\alpha}[\sigma(s)]}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\tau(s)]}v^{\frac{\alpha+1}{\alpha}}(s+1)\Big]. \quad (2.38)$$

Let

$$A(n) = \left(\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\tau(n)]}\right)^{\frac{\alpha+1}{\alpha}} w(n+1)$$

and

$$B(n) = \left(\frac{\alpha}{\alpha+1}\frac{\Delta\rho(n)}{\rho(n+1)}\left(\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\tau(n)]}\right)^{-\frac{\alpha}{\alpha+1}}\right)^{\alpha}.$$

Now, using the inequality

$$\frac{\alpha+1}{\alpha}AB^{\frac{1}{\alpha}} - A^{\frac{\alpha+1}{\alpha}} \le \frac{1}{\alpha}B^{\frac{\alpha+1}{\alpha}} \quad (2.39)$$

by taking $A = A(n)$ and $B = B(n)$ on the second part of the right hand side of the inequality (2.38), we have

$$w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \le -\frac{1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s) + \sum_{s=n_2}^{n-1}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}}a[\tau(s)] + \sum_{s=n_2}^{n-1}\Big[\frac{\Delta\rho(s)}{\rho(s+1)}v(s+1) - \frac{\alpha\rho(s)p^{\alpha}[\sigma(s)]}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\tau(s)]}v^{\frac{\alpha+1}{\alpha}}(s+1)\Big]. \quad (2.40)$$

Now, let

$$C(n) = \left(\frac{\alpha\rho(n)p^{\alpha}[\sigma(n)]}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\tau(n)]}\right)^{\frac{\alpha}{\alpha+1}} v(n+1)$$

and

$$D(n) = \left[\frac{\alpha}{\alpha+1}\frac{\Delta\rho(n)}{\rho(n+1)}\left(\frac{\alpha\rho(n)p^{\alpha}[\sigma(n)]}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\tau(n)]}\right)^{-\frac{\alpha}{\alpha+1}}\right]^{\alpha}.$$

Now, using the inequality (2.39) by taking $A = C(n)$ and $B = D(n)$ in the third part of right hand side of (2.40), we have

$$w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \le -\frac{1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s) + \sum_{s=n_2}^{n-1}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}}a[\tau(s)] + \sum_{s=n_2}^{n-1}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}}\frac{a[\tau(s)]}{(p^{\alpha}[\sigma(s)])^2}. \quad (2.41)$$

Now,

$$w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \leq -\sum_{s=n_2}^{n-1} \rho(s) \Bigg[\frac{1}{2^{\alpha-1}} Q(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{\Delta\rho(s)}{\rho(s)} \right)^{\alpha+1} a[\tau(s)] \left[1 + \frac{1}{(p^{\alpha}[\sigma(s)])^2} \right] \Bigg]. \tag{2.42}$$

Letting $n \to \infty$ in the last inequality and using (2.29), we see that

$$w(n) + p^{\alpha}[\sigma(n)]v(n) \to -\infty,$$

which contradicts the positivity of $w(n) + p^{\alpha}[\sigma(n)]v(n)$. This completes the proof. □

Theorem 2.4. *Assume that condition (1.2) holds, $p(n) \leq p_0 < \infty$, $\tau(n) \leq n$ and $\sigma(n) \geq \tau(n)$ for all $n \geq n_0$. Furthermore, if there exists a positive real sequence $\{\rho(n)\}$ such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \rho(s) \left[\frac{Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{\Delta\rho(s)}{\rho(s)} \right)^{\alpha+1} a[\tau(s)] \left(1 + \frac{1}{(p_0^{\alpha}[\sigma(s)])^2} \right) \right] = \infty, \tag{2.43}$$

then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $n_1 \geq n_0$ such that $x(n) > 0, x[\tau(n)] > 0$ and $x[\sigma(n)] > 0$ for all $n \geq n_1$. Using equation (1.1) and the definition of $z(n)$, we get (2.28) for all sufficiently large n. The remainder of the proof is similar to that of Theorem 2.3 and hence it is omitted. □

Theorem 2.5. *Assume that condition (1.3) holds, $p(n) \leq p_0 < \infty$, $\Delta\tau(n) > 0$, $\Delta\sigma(n) > 0$, $\sigma(n) \leq n$ and $\sigma(n) \leq \tau(n)$ for all $n \geq n_0$. Further assume that there exists a positive real sequence $\{\rho(n)\}$ such that (2.26) holds. If there exists a sequence $\{\eta(n)\}$ of positive real numbers with $\eta(n) \geq n, \Delta\eta(n) > 0$ for all $n \geq n_0$ such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[\frac{Q(s)\delta^{\alpha}(s)}{2^{\alpha-1}} + (1+p_0^{\alpha}) \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{1}{\delta(s)a^{\frac{1}{\alpha}}[\eta(s)]} \right] = \infty, \tag{2.44}$$

then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be a positive solution of equation (1.1). Then there exists $n_1 \geq n_0$ such that $x(n) > 0, x[\tau(n)] > 0$ and $x[\sigma(n)] > 0$ for all $n \geq n_1$. Proceeding as in Theorem 2.1, we get

$$\Delta(a(n))(\Delta z(n))^{\alpha} + p_0^{\alpha}\Delta(a[\tau(n)])(\Delta z[\tau(n)])^{\alpha} + \frac{1}{2^{\alpha-1}} Q(n) z^{\alpha}[\sigma(n)] \leq 0 \tag{2.45}$$

for all $n \geq n_1$. Also from equation (1.1), we have $a(n)(\Delta z(n))^{\alpha}$ is decreasing. Then we have two cases for $\Delta z(n)$, namely, (i) $\Delta z(n) < 0$ or (ii) $\Delta z(n) > 0$ for all $n \geq n_2 \geq n_1$.

Case(i): Suppose that $\Delta z(n) > 0$ for all $n \geq n_2 \geq n_1 \geq n_0$. Then the proof is similar to that of Theorem 2.2.

Case(ii): Suppose that $\Delta z(n) < 0$ for all $n \geq n_2 \geq n_1 \geq n_0$. Now define

$$u(n) = \frac{-a(n)(-\Delta z(n))^{\alpha}}{z^{\alpha}[\eta(n)]} \text{ for all } n \geq n_2. \tag{2.46}$$

Then $u(n) < 0$ for all $n \geq n_2$. Since $a(n)(\Delta z(n))^{\alpha}$ is decreasing, we have $a(n)(-\Delta z(n))^{\alpha}$ is increasing and we get

$$a^{\frac{1}{\alpha}}(s)\Delta z(s) \leq a^{\frac{1}{\alpha}}(n)\Delta z(n) \text{ for all } s \geq n \geq n_2. \tag{2.47}$$

Dividing the last inequality by $a^{\frac{1}{\alpha}}(s)$ and then summing from $\eta(n)$ to $n-1$, we have

$$z(n) \leq z[\eta(n)] + a^{\frac{1}{\alpha}}(n)\Delta z(n) \sum_{s=\eta(n)}^{n-1} \frac{1}{a^{\frac{1}{\alpha}}(s)}. \tag{2.48}$$

Letting $n \to \infty$ in the last inequality, we see that

$$0 \leq z[\eta(n)] + a^{\frac{1}{\alpha}}(n)\Delta z(n)\delta(n), \tag{2.49}$$

that is,

$$-\frac{a^{\frac{1}{\alpha}}(n)\Delta z(n)\delta(n)}{z[\eta(n)]} \leq 1. \tag{2.50}$$

Hence by (2.46), we have

$$-u(n)\delta^{\alpha}(n) \leq 1. \tag{2.51}$$

Now, define

$$v(n) = \frac{-a[\tau(n)](-\Delta z[\tau(n)])^{\alpha}}{z^{\alpha}[\eta(n)]}, \; n \geq n_2. \tag{2.52}$$

Then, we have $v(n) < 0$. By using the monotonicity of $a(n)(-z(n))^{\alpha}$ and using $\tau(n) \leq n$, we get

$$a(n)(-\Delta z(n))^{\alpha} \geq a[\tau(n)](-\Delta z[\tau(n)])^{\alpha} \text{ for all } n \geq n_2. \tag{2.53}$$

Thus

$$0 < -v(n) \leq -u(n). \tag{2.54}$$

From (2.50) and (2.54), we have

$$-\delta^{\alpha}(n)v(n) \leq 1. \tag{2.55}$$

Now, from (2.46), we have

$$\begin{aligned}\Delta u(n) &= \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} + \frac{a(n+1)\Delta(-z(n+1))^{\alpha}}{z^{\alpha}[\eta(n)]z^{\alpha}[\eta(n+1)]}\Delta^{\alpha}z[\eta(n)] \\ &\leq \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} + \frac{(-u(n+1))}{z^{\alpha}[\eta(n)]}\Delta^{\alpha}z[\eta(n)].\end{aligned} \tag{2.56}$$

By Mean value Theorem, we have

$$\Delta z^{\alpha}[\eta(n)] \leq \alpha z^{\alpha-1}[\eta(n)]\Delta z[\eta(n)]. \tag{2.57}$$

Using (2.57) in (2.56), we get

$$\Delta u(n) \leq \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} - u(n+1)\alpha\Delta z[\eta(n)]. \tag{2.58}$$

Using monotonicity of $a(n)(\Delta z(n))^{\alpha}$, we have

$$\Delta z[\eta(n)] \leq \left(\frac{a(n)}{a[\eta(n)]}\right)^{1/\alpha}\Delta z(n). \tag{2.59}$$

Using (2.59) in (2.58), we get

$$\begin{aligned}\Delta u(n) &\leq \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} - (-u(n+1))\frac{a^{1/\alpha}(n)}{a^{1/\alpha}[\eta(n)]}\Delta z(n) \\ &\leq \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} - \frac{\alpha[-u(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(n)]}.\end{aligned} \tag{2.60}$$

Similarly, we have

$$\Delta v(n) \leq \frac{\Delta(-a(\tau(n))(-\Delta z(\tau(n)))^{\alpha})}{z^{\alpha}[\eta(\tau(n))]} - \frac{\alpha[-v(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(n))]}. \tag{2.61}$$

From (2.60) and (2.61)

$$\begin{aligned}\Delta u(n) + p_0^{\alpha}\Delta v(n) \leq& \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} + p_0^{\alpha}\frac{\Delta(-a(\tau(n))(-\Delta z(\tau(n)))^{\alpha})}{z^{\alpha}[\eta(\tau(n))]} \\ &- \frac{\alpha[-u(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(n)]} - \frac{\alpha p_0^{\alpha}[-v(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(n))]}.\end{aligned} \tag{2.62}$$

Using (2.7) in (2.62), we have

$$\Delta u(n)+p_0^{\alpha}\Delta v(n)\leq \frac{-1}{2^{\alpha-1}}Q(n)-\frac{\alpha[-u(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(n)]}-\frac{\alpha p_0^{\alpha}[-v(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(n))]}. \tag{2.63}$$

Multiplying (2.63) by $\delta^{\alpha}(n)$ and summing the resulting inequality from n_2 to $n-1$, we get

$$\sum_{s=n_2}^{n-1}\Delta u(s)\delta^{\alpha}(s)+\sum_{s=n_2}^{n-1}\Delta v(s)\delta^{\alpha}(s)p_0^{\alpha}+\sum_{s=n_2}^{n-1}\delta^{\alpha}(s)\left[\frac{1}{2^{\alpha-1}}Q(s)\right.$$
$$\left.+\frac{\alpha[-u(s+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(s)]}-\frac{\alpha p_0^{\alpha}[-v(s+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(s))]}\right]\leq 0. \tag{2.64}$$

By using summation by parts formula, we obtain

$$[u(s)\delta^{\alpha}(s)]_{n_2}^{n}-\sum_{s=n_2}^{n-1}u(s+1)\Delta\delta^{\alpha}(s)+[p_0^{\alpha}v(s)\delta^{\alpha}(s)]_{n_2}^{n}$$
$$-p_0^{\alpha}\sum_{s=n_2}^{n-1}v(s+1)\Delta\delta^{\alpha}(s)\sum_{s=n_2}^{n-1}\delta^{\alpha}(s)\left[\frac{1}{2^{\alpha-1}}Q(s)\right.$$
$$\left.+\frac{\alpha[-u(s+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(s)]}-\frac{\alpha p_0^{\alpha}[-v(s+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(s))]}\right]\leq 0.$$

Now

$$u(n)\delta^{\alpha}(n)-u(n_2)\delta^{\alpha}(n_2)+p_0^{\alpha}v(n)\delta^{\alpha}(n)-p_0^{\alpha}v(n_2)\delta^{\alpha}(n_2)$$
$$+\alpha\sum_{s=n_2}^{n-1}\left[\frac{\delta^{\alpha-1}(s)u(s+1)}{a^{\frac{1}{\alpha}}\eta(s)}-\frac{\delta^{\alpha}(s)u(s+1)^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}\eta(s)}\right]$$
$$+\alpha p_0^{\alpha}\sum_{s=n_2}^{n-1}\left[\frac{\delta^{\alpha-1}(s)v(s+1)}{a^{\frac{1}{\alpha}}\eta(s)}-\frac{\delta^{\alpha}(s)v(s+1)^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}\eta(s)}\right]+\sum_{s=n_2}^{n-1}\frac{\delta^{\alpha}(s)}{2^{\alpha-1}}Q(s)\leq 0. \tag{2.65}$$

By using the inequality

$$Bu-Au^{\alpha+1/\alpha}\leq\frac{\alpha^{\alpha}}{(\alpha+1)^{(\alpha+1)}}\frac{B^{\alpha+1}}{A^{\alpha}} \tag{2.66}$$

in fifth and sixth parts of the left hand side of the last inequality, we have

$$\sum_{s=n_2}^{n-1}\left[\frac{Q(s)\delta^{\alpha}(s)}{2^{\alpha-1}}+(1+p_0^{\alpha})\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{\delta(s)a^{1/\alpha}[\eta(s)]}\right)\right]$$
$$\leq u(n_2)\delta^{\alpha}(n_2)+p_0^{\alpha}v(n_2)\delta^{\alpha}(n_2)+1+p_0^{\alpha}. \tag{2.67}$$

Letting $n\to\infty$, we get a contradiction with (2.45). This completes the proof. □

3 Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. *Consider the neutral difference equation*

$$\Delta(n(\Delta(x(n)+\frac{1}{2}x(n-2)))^3)+n^2x^3(n-3)=0,\ n\geq 3. \tag{3.68}$$

Here $a(n)=n$, $p(n)=\dfrac{1}{2}$, $q(n)=n^2$, $\alpha=3$, $\tau(n)=n-2$ and $\sigma(n)=n-3$. By taking $\rho(n)=n$, it is easy to see that all conditions of Theorem 2.1 are satisfied and hence all solutions of equation (3.68) are oscillatory.

Example 3.2. *Consider the neutral difference equation*

$$\Delta(n^4(\Delta(x(n)+\frac{1}{3}x(n-2)))^3)+n^6x(n-4)=0,\ n\geq 4. \tag{3.69}$$

Here $a(n)=n^4$, $p(n)=\frac{1}{3}$, $q(n)=n^6$, $\alpha=3$, $\tau(n)=n-2$ *and* $\sigma(n)=n-4$. *By taking* $\rho(n)=1$ *and* $\eta(n)=n$, *it is easy to see that all conditions of Theorem 2.5 are satisfied and hence all solutions of equation (3.69) are oscillatory.*

We conclude this paper with the following remark.

Remark 3.1. *The method used in this paper can be applied to the following difference equation*

$$\Delta(a(n)\Delta(x(n)+p(n)x(\tau(n))))+q(n)|x(\delta(n))|^{\alpha-1}x(\delta(n))=0$$

where $\alpha\geq 1$, *to obtain oscillation results. Also it would be interesting to find oscillation criteria for the equation (1.1) when* $\tau\circ\sigma\neq\sigma\circ\tau$.

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Received: May 28, 2014; *Accepted*: July 19, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 472–481

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Drazin invertibility of sum and product of closed linear operators

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Abstract

The paper present a survey of results concerning the fundamental properties of the Drazin inverse for bounded operators and an interesting study of the Drazin inverse for a closed operator in a Banach space. Some necessary and sufficient conditions for A closed linear operator to possess a Drazin inverse A^D are given, we obtain also a useful caracterization and explicit formula for the Drazin inverse $(A+B)^D$ and $(AB)^D$ if A and B are closed operators.

Keywords: Drazin inverse, Closed linear operators, Gap metric.

2010 MSC: 47A05, 47B33.

1 Introduction

In recent years, the representation and characterization of the Drazin inverses of matrices or operators on a Banach space have been considered by many authors (see [2], [6], [7], [14], [15], [31],...). It is shown that the Drazin inverse has proved helpful in analyzing Markov chains, difference equation, differential equations, Cauchy problems and iterative procedures.

For bounded linear operators and elements of a Banach algebra the Drazin inverse was introduced and studied by Ben-Israel in [2], Caradus [5], Koliha in [15] and other authors.

In this paper, we give a survey of results concerning the fundamental properties of the Drazin inverse for bounded operators on a Banach space. We continue our investigation with a representation and characterization of Drazin inverse for a closed linear operator in a Banach space. We give some necessary and sufficient conditions for A closed to possess a Drazin inverse A^D, we obtain also a useful characterization and explicit formula for the Drazin inverse $(A+B)^D$ and $(AB)^D$ if A and B are closed linear operators satisfying certain topological conditions via the gap metric between their respective graphs.

Precisely, if A and B are two closed linear operators Drazin invertible, it is a question when $A+B$ and AB are closed and Drazin invertible. This question was partially studied by Messirdi and Mortad [21], Azzouz, Messirdi and Djellouli [1] and Koliha and Tran [18], it finds its applications in a number of areas such that differential and difference equations, linear and non linear analysis. Koliha and Tran consider in [18], the Drazin inverse of $A+B$ and AB where A is closed and B bounded, what assures the closedness. Their method can not be applied to arbitrary closed operators.

As main result, we give sufficient conditions for a sum and product of two Drazin invertible closed operators to be closed and Drazin invertible in a Hilbert space. Thus, using a different method via the gap metric, we generalize the result from [18].

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2 A concise introduction to Drazin inverse for bounded operators

Let E be a complex Banach space (resp. H be a separable Hilbert space). Let us denote by $B(E)$ the algebra of bounded linear operators on E and $C(E)$ the set of all densely defined closed linear operators on E. If $A \in C(E)$, the domain of A is denoted by $D(A)$ and $G(A) = \{(x, Ax) \,;\, x \in D(A)\}$ is its graph, in particular $G(A)$ is a closed subspace of $E \times E$. The null space and the range of A will be denoted by $N(A)$ and $R(A)$ respectively. A^* is the adjoint of A and I is the operator identity on E. The orthogonal complement of a subset M of H is denoted by $M^{\perp}$.

We write $\sigma(A), \rho(A)$ and $r(A)$ for the spectrum, the resolvent set and the spectral radius of A, respectively. For $\lambda \in \rho(A)$ we denote the resolvent $(\lambda I - A)^{-1}$ by $R(\lambda, A)$. If 0 is an isolated point of $\sigma(A)$, then the spectral projection of A associated with $\{0\}$ is defined by (see [13]) :

$$P_A = \frac{1}{2\pi i}\int_{\gamma} R(\lambda, A)d\lambda$$

where γ is a small circle surrounding 0 and separating 0 from $\sigma(A)\backslash\{0\}$. An element $A \in B(E)$ whose spectrum $\sigma(A)$ consists of the set $\{0\}$ is said to be quasinilpotent. It is clear that A is quasinilpotent if and only if the spectral radius $r(A) = 0$.

For bounded linear operators and elements of a Banach algebra the Drazin inverse was introduced and studied by Ben-Israel [2], Koliha in [15] and others.

Let $A \in B(E)$ if there exists an operator $X \in B(E)$ satisfied the following three operator equations

$$\begin{cases} AX = XA \\ XAX = X \\ A^{k+1}X = A^k \end{cases}$$

then X is called a Drazin inverse of A and denoted by A^D. The smallest non-negative integer k in the third equation is called the index of A, denoted by $i(A)$.

The above conditions are equivalent to $AA^D = A^D A$; $A^D A A^D = A^D$ and $A(I - AA^D)$ is nilpotent.

If A is Drazin invertible and $i(A) = r \geq 1$, then $R(\lambda, A)$ has a pole of order r at $\lambda = 0$ and it can be expressed in the region $0 < |\lambda| < (r(A^D))^{-1}$, by (see [5]) :

$$R(\lambda, A) = \sum_{n=1}^{r} \frac{A^{n-1}P_A}{\lambda^n} - \sum_{n=0}^{\infty} \lambda^n (A^D)^{n+1}$$

An operator $A \in B(E)$ has its Drazin inverse A^D if and only if it has finite ascent and descent, which is equivalent with that 0 is a finite order pole of the resolvent operator $R(\lambda, A)$, say of order p. In such case $i(A) = asc(A) = des(A) = p$. Recall that $asc(A)$ (resp. $des(A)$), the ascent (resp. descent) of $A \in B(E)$, is the smallest non-negative integer n such that $N(A^n) = N(A^{n+1})$ (resp. $R(A^n) = R(A^{n+1})$). If no such n exists, then $asc(A) = \infty$ (resp. $des(A) = \infty$). It is well known, $des(A) = asc(A)$ if $asc(A)$ and $des(A)$ are finite [12, 14]. Otherwise, we say $i(A) = +\infty$. When $i(A) = 0$, then the Drazin inverse is reduced into the regular inverse, i.e., $A^D = A^{-1}$. Moreover, we know that for $A \in B(E)$, A^D exists if and only if $0 \notin acc[\sigma(A)]$ ($acc[\sigma(A)]$ is the set of all accumulation points of the spectrum $\sigma(A)$ of A) and in that case A^D is unique [15]. We note that if A is nilpotent, then it is Drazin invertible, $A^D = 0$, and $i(A) = r$, where r is the power of nilpotency of A. Chen F. King showed in [14] that if $A \in B(E)$ has a Drazin inverse with index $i(A) = k$ then A can be written as $A = S + T$ where S has index 0 or 1, T is nilpotent of order k and $ST = TS = 0$.

The problems of the invertibility of sums and products of idempotent operators on a Hilbert space were studied by several researchers, Groß and Trenkler in [12]; Buckholtz in [3], Rakocevic in [25], Vidav in [29] and Wimmer in [30]. Rakocevic and Wei in [26] investigated this question in the case of C^*-algebra. The formulae for the Drazin inverse of sums, differences and products of idempotents are established by Deng and Wei in [10].

Generally, commutation relations between operators in a Banach space E play an important role in the representations of the Drazin inverse. Some properties of the Drazin inverse according to such relations have been extensively studied in the mathematical literature. Djordjevic and Wei in 2002 showed in [11] the following results, see also [18]:

Theorem 2.1. *Let $A, B \in B(E)$ be Drazin invertible.*
If $AB = BA = 0$ then $(A+B)^D = A^D + B^D$.
If $AB = 0$, then $A + B$ is Drazin invertible and

$$(A+B)^D = (I - BB^D)[\sum_{n=0}^{\infty} B^n (A^D)^n] A^D + B^D [\sum_{n=0}^{\infty} (B^D)^n A^n](I - AA^D)$$

This result was later refined by Castro-González, Dopazo and Martínez-Serrano in [6] :

Theorem 2.2. *Let $A, B \in B(E)$ be Drazin invertible. If $B^2A = BA^2 = 0$, and let BA de Drazin invertible, then $A + B$ is Drazin invertible and*

$$\begin{aligned}(A+B)^D &= UP_B + P_A V + X(I+YB)P_B + P_A(I+AX)Y + AUV + UVB \\ &+ \sum_{k=0}^{2r+t-2} (A^D)^{k+1}\Gamma_{k+2} B + \sum_{k=0}^{2r+s-2} A\Lambda_{k+2}(B^D)^{k+1}\end{aligned}$$

where $s = i(A)$, $t = i(B)$ and $r = i(BA)$. Γ_{k+2} is the coefficient at $(\frac{1}{\lambda})^{k+2}$ of $R(\lambda^2, BA)R(\lambda, B)$, Λ_{k+2} is the coefficient at $(\frac{1}{\lambda})^{k+2}$ of $R(\lambda, A)R(\lambda^2, BA)$.

$$\begin{aligned}X &= \sum_{j=1}^{[s/2]} A^{2j-1} P_A((BA)^D)^j \, , \; Y = \sum_{j=1}^{[t/2]} ((BA)^D)^j B^{2j-1} P_B \\ U &= \sum_{j=0}^{r-1} (A^D)^{2j+1} (BA)^j P_{BA} \, , \; V = \sum_{j=0}^{r-1} P_{BA}(BA)^j (B^D)^{2j+1}\end{aligned}$$

Moreover, $i(A+B) \leq 2r + s + t - 1$.
If B is nilpotent, then

$$(A+B)^D = U + X(I+YB) + P_A(I+AX)Y + \sum_{k=0}^{2r+t-2} (A^D)^{k+1}(T_k - YB^k)B$$

where $T_k = \sum_{j=0}^{k'} P_{BA}(BA)^j B^{k-1-2j} P_B$, $k \geq 1$ with $T_0 = 0$.
In the case $B^2 = 0$, then

$$(A+B)^D = U + X + P_A(I+AX)(BA)^D B + A^D UB$$

If A and B are nilpotent, then

$$(A+B)^D = X(I+YB) + (I+AX)Y$$

Patricio and Hartwig in 2009 investigated in [24] the existence of the Drazin inverse $(A+B)^D$ under the condition $A^2B + AB^2 = 0$.

Theorem 2.3. *Let $A, B \in B(E)$. Supose that $A^2 + AB$ and $AB + B^2$ are Drazin invertible, and that $A^2B + AB^2 = 0$. Then $A + B$ is Drazin invertible with :*

$$\begin{aligned}(A+B)^D &= (A^2+AB)^D A + B(AB+B^2)^D + BCA \\ C &= -(AB+B^2)^D(A+B)(A^2+AB)^D + \\ &\quad [I - (AB+B^2)(AB+B^2)^D] C_k [(A^2+AB)^D]^{k+1} \\ &\quad + [(AB+B^2)^D]^{k+1} C_k [I - (A^2+AB)(A^2+AB)^D]\end{aligned}$$

$$C_k = \sum_{r=0}^{k-1} (AB+B^2)^{k-r-1}(A+B)(AB+A^2)^r$$

and $\max(i(A^2+AB), i(B^2+AB)) \leq k \leq i(A^2+AB) + i(B^2+AB)$.

Xiaoji Liu, Liang Xu and Yaoming Yu in 2010 gave the explicit representations of $(A \pm B)$ when $n \times n$ matrices A, B satisfied $AB = B^3A$, $BA = A^3B$ (see [31]). Precisely, if a pair of bounded invertible operators A and B have dual power commutativity $AB = B^mA$ and $BA = A^nB$, $m, n \geq 1$, then $A + B$ is always Drazin invertible.

On the other hand, for a given pair of bounded operators (A, B) and an arbitrary operator X, expressions for the inverse and the Drazin inverse of the operator $A - XB$ are established :

Theorem 2.4. *(i) Let A and B be matrices of the same size. If A is invertible, then $A - XB$ is invertible iff the Schur complement $I - BA^{-1}X$ is invertible and*

$$(A - XB)^{-1} = A^{-1} + A^{-1}X(I - BA^{-1}X)^{-1}BA^{-1}$$

(ii) Let $A, B \in B(E)$ and A be Drazin invertible
- If $(I - AA^D)X = 0$, $B(I - AA^D) = 0$ and $X(I - AA^D)B = 0$, then

$$(A - XB)^D = A^D + A^DX(I - BA^DX)^DBA^D$$

- If there exists an idempotent operator P (ie $P^2 = P$) such that $AP = PA$ and $PX = 0$, then

$$\begin{aligned}(A - XB)^D \;=\;& R^D + PA^D + R^DXBPA^D \\ & - \sum_{n=0}^{\infty}(R^D)^{n+2}XBPA^n(I - AA^D) \\ & -(I - RR^D)\sum_{n=0}^{\infty}(A - XB)^nXBP(A^D)^{n+2}\end{aligned}$$

where $R = (A - XB)(I - P)$.

Let us recall that the original Sherman-Morrison-Woodbury formula has been used to consider the inverse of matrices. We will consider here directly the more generalized case for bounded linear operators.

Theorem 2.5. *Let $A, B, Y, Z \in B(E)$.*

(i) Suppose that A and B are both invertible.Then, $A + YBZ^*$ is invertible iff $B^{-1} + Z^*A^{-1}Y$ is invertible. In which case

$$(A + YBZ^*)^{-1} = A^{-1} - A^{-1}Y(B^{-1} + Z^*A^{-1}Y)^{-1}Z^*A^{-1}$$

(ii) Suppose that A and B are both Drazin invertible. Let $C = A + YBZ^*$ and $T = B^D + Z^*A^DY$. If

$$\begin{aligned} R(A^D) &\subset R(C^D)\,;\, N(A^D) \subset N(C^D) \\ N(B^D) &\subset N(Y)\,;\, N(T^D) \subset N(B) \end{aligned}$$

then,

$$(A + YBZ^*)^D = A^D - A^DY(B^D + Z^*A^DY)^DZ^*A^D$$

The question of the invertibility of $P - Q$ where P and Q are idempotent operators on a Hilbert space H, is of great interest in operator theory as it is connected with the question of when the space H is the direct sum $H = R(P) \oplus R(Q)$ of the ranges, and with the existence of an idempotent operator X satisfying the equations $PX = X$, $XP = P$, $Q(I - X) = I - X$ and $(I - X)Q = Q$.

These problems were studied by several researchers, Groß and Trenkler in [12] considered the case of general matrix projectors; Buckholtz [3], [4], Rakocevic [25], Vidav [29], Wimmer [30] discussed the invertibility in the setting of Hilbert spaces. Koliha, Rakocevic and Straskraba [17], Rakocevic and Wei [26] investigated this question in the setting of C* -algebra. Consequently, if P and Q are idempotents some equivalent conditions for the Drazin invertibility of $P + Q$, $P - Q$, PQ and $PQ \pm QP$ are listed as following.

Theorem 2.6. *Let P and Q be idempotents.*

(i) The following statements are equivalent :
$P - Q$ is Drazin invertible,
$P + Q$ is Drazin invertible,
$I - PQ$ is Drazin invertible.

(ii) PQ is Drazin invertible iff $I-P-Q$ is Drazin invertible.
(iii) $PQ-QP$ is Drazin invertible,
$PQ+QP$ is Drazin invertible,
PQ and $P-Q$ are Drazin invertible.

Let P and Q be idempotents and $P-Q$ is invertible, Koliha and Rakocevic in [16] first use the denotations $F=P(P-Q)^{-1}$ and $G=(P-Q)^{-1}P$ to give the representation of $(P-Q)^{-1}$. To give explicit formulae for $(P-Q)^D$ and $(P+Q)^D$ we define $F=P(P-Q)^D$ and $G=(P-Q)^DP$ and $K=(P-Q)^D(P-Q)$. The following results are showed by Deng and Wei in [10].

Theorem 2.7. *Let $P,Q\in B(E)$ be idempotents, then*
(i) $(P-Q)^D=F+G-K$.
(ii) $(P+Q)^D=(2G-K)(F+G-K)$.
(iii) $(P+Q)^D=(P-Q)^D(P+Q)(P-Q)^D$.
(iv) $(P-Q)^D=(P+Q)^D(P-Q)(P+Q)^D$.
(v) $(P-Q)^D=(I-PQ)^D(P-PQ)+(P+Q-PQ)^D(PQ-Q)$.
(vi) $(PQ-QP)^D=(PQP)^D(P-Q)^D-(P-Q)^D(PQP)^D$.
(vii) $(PQ+QP)^D=(P+Q)^D(P+Q-I)^D$.
(viii) $(I-PQP)^D=I-P+P[(P-Q)^D]^2$.
(ix) if PQ is Drazin invertible,

$$\begin{aligned}(PQP)^D &= [(I-P-Q)^D]^2P\\ (PQ)^D &= [(PQP)^D]^2Q=[(I-P-Q)^D]^4Q\end{aligned}$$

As is well known, that AB is invertible does not imply that BA is invertible for A and $B\in B(E)$ (let S be the unilateral shift operator on H, then $S^*S=I$ is invertible, but SS^* is not invertible). The following result insure the equivalence between Drazin invertibility of AB and BA, it was successively shown by Dajic and Koliha [8], Schmoeger [27], Deng [9] and Lu Jian Ming, Du Hong Ke and Wei Xiao Mei [20].

Theorem 2.8. *Let $A,B\in B(E)$ be Drazin invertible. AB is Drazin invertible if and only if BA is Drazin invertible, $i(AB)\le i(BA)+1$ and $(AB)^D=A[(BA)^D]^2B$.*
If A is idempotent, then $A^D=A$.
If $AB=BA$, then $(AB)^D=B^DA^D=A^DB^D$, $A^DB=BA^D$ and $AB^D=B^DA$.

In the following section we introduce the notion of Drazin inverse into the class of the closed operators on a Banach space E and we investigate some basic properties of A^D.

3 Drazin inverse of closed operators

The conventional Drazin inverse was extended to closed linear operators by Nashed and Zhao in [23]; it exists if and only if 0 is at most a pole of the resolvent $R(\lambda,A)$ of the operator A.

The purpose of this section is to introduce the Drazin inverse A^D of a closed linear operator A on a Banach space E which is defined if 0 is merely an isolated spectral point of A, and to investigate basic properties of A^D. We also study, the Drazin invertibility of sums and products of closed linear operators in the case where the respective domains are not trivial and where the sum and the product remain closed operators. The result of Theorem 4.11 is a generalization to densely defined closed linear operators of results obtained by several authors on bounded operators (see e.g. [7, 15, 18, 24, 27]).

We start with a definition of the Drazin inverse of a closed operator and we recall afterward the results on the stability of the closedness of sum and product of closed operators established by Azzouz, Messirdi and Djellouli in [1], and Messirdi, Mortad, Azzouz and Djellouli in [22].

Definition 3.1. *([18]) Let $A\in C(E)$. A is called Drazin invertible (or generalized Drazin invertible) if it can be expressed in the form $A=A_1\oplus A_2$ where A_1 is bounded and quasinilpotent and A_2 is closed and invertible on E. Thus, $A_2^{-1}\in B(E)$, the operators $A^D=0\oplus A_2^{-1}$ is the Drazin inverse of A.*

The Drazin index $i(A)$ is defined to be $i(A)=0$ if A is invertible, $i(A)=q$ if A is not invertible and A_1 is nilpotent of index q, and $i(A)=\infty$ otherwise.

Remark 3.1. *Drazin invertible operators include closed invertible and quasinilpotent operators when $A_1 = 0$ and $A_2 = 0$ respectively and projections.*

We deduct directly from the definition the following properties for a Drazin invertible densely defined closed linear operator.

Lemma 3.1. *Let $A \in C(E)$ be Drazin invertible with Drazin inverse $A^D \in B(E)$. Then,*
(i) A^D is unique and $R(A^D) \subset D(A)$.
(ii) $R(I - AA^D) \subset D(A)$.
(iii) $A^D AA^D = A^D$.
(iv) $AA^D = A^D A$.
(v) $(AA^D)^2 = AA^D$.
(vi) $P_A = I - AA^D$ is the spectral projection of A corresponding to 0.
(vii) $\sigma(A(I - AA^D)) = \{0\}$.

Proof. The properties (i) to (v) follow from Definition 3.1. (v) implies $P_A^2 = P_A$. We can deduce from [28] that 0 is not an accumulation point of the spectrum of A and P_A is the spectral projection of A at 0. □

Remark 3.2. *Koliha and Tran showed in [18] that the properties (i) to (iv) and (vi) in the Lemma 3.1 are necessary and sufficient conditions for $A \in C(X)$ to possess a Drazin inverse.*

Example 3.1. *Let $L^2([0,1]) = \left\{ f \,;\, f \;:\; [0,1] \longrightarrow \mathbb{C} \text{ such that } \int_0^1 |f(x)|^2 \, dx < +\infty \right\}$ be a Hilbert space with the natural inner product. Set $M : [0,1] \longrightarrow \mathbb{C}$ by*

$$M(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \end{cases}$$

and define the maximal operator of multiplication A by M on $L^2([0,1])$, that is,

$$Af = Mf, \text{for } f \in D(A) = \{f \in L^2([0,1]) : Mf \in L^2([0,1])\}$$

Then A is a densely defined closed linear operator on $L^2([0,1])$. Since $|M(x)| \geq 1$ for all $x \in [0,1]$, $R(A) = L^2([0,1])$ and A has a bounded inverse $A^{-1} : L^2([0,1]) \longrightarrow L^2([0,1])$ defined by $A^{-1}g = \frac{g}{M}$, for all $g \in L^2([0,1])$.
Therefore, A is a closed operator on $L^2([0,1])$ with a bounded inverse A^{-1}which is the Drazin inverse of A.

The following results, easily verifiable by a manipulation of direct operator sums, are often useful (see e.g. [18]).

Theorem 3.9. *Let $A \in C(E)$ be Drazin invertible. Then,*
(i) P_A and A commute on $D(A)$.
(ii) $A + P_A$ is invertible and $A^D = (A + P_A)^{-1}(I - P_A) \in B(E)$.
(iii) If $B \in C(E)$ be such that $R(B) \subset D(A)$, $R(A) \subset D(B)$ and $AB = BA$, then, $A^D B = BA^D$.
(iv) For each $n \geq 1$, A^n is Drazin invertible and $(A^n)^D = (A^D)^n$.
(v) A^* is Drazin invertible and $(A^D)^* = (A^*)^D$.
(vi) If $B \in B(E)$ is quasinilpotent such that $R(B) \subset D(A)$ and $AB = BA$, then $A + B \in C(E)$ is Drazin invertible and

$$(A + B)^D = (A + B + P_A)^{-1}(I - P_A)$$

(vii) If $B \in B(E)$ is Drazin invertible such that $R(B) \subset D(A)$, $R(A) \subset D(B)$ and $AB = BA = 0$, then $(A + B)^D = A^D + B^D$.
(viii) There exist $B \in C(E)$, $D(A) = D(B)$ and B is Drazin invertible with index $i(B) \leq 1$, and $C \in B(E)$ quasinilpotent with $R(C) \subset D(A)$ such that $A = B + C$ and $BC = CB = 0$. $B^D = A^D$ and a such decomposition is unique.

Example 3.2. *([18]) Let the operator A_1 on l^1 the space of all complex sequences $(x_k)_k$ such that $\sum_{k=0}^{\infty} |x_k| < +\infty$, defined by*

$$A_1 x = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...)$$

Then $A_1 \in B(l^1)$ and is quasinilpotent but not nilpotent. The right shift $Rx = (0, x_1, x_2, ...)$ is an injective bounded operator on l^1 with spectrum equal to the closed unit ball in $\mathbb{C}$. Its inverse A_2 is a closed linear operator with the domain $D(A_2) = \{x \in l^1\ ;\ x_1 = 0\}$ and $\sigma(A_2) = \{\lambda \in \mathbb{C}\ ;\ |\lambda| \geq 1\}$.

Define $A = A_1 \oplus A_2$ on $E = l^1 \oplus l^1$. Then, $\sigma(A) = \{0\} \cup \{\lambda \in \mathbb{C}\ ;\ |\lambda| \geq 1\}$, 0 is isolated in $\sigma(A)$,A is Drazin invertible with $A^D = 0 \oplus A_2^{-1} = 0 \oplus R$ and $i(A) = \infty$. A^D is not Drazin invertible since $\sigma(A^D) = \{\lambda \in \mathbb{C}\ ;\ |\lambda| \leq 1\}$ and 0 is an accumulation point of $\sigma(A)$.

4 Drazin inverse of sum and product of closed operators

Let us remind at first a perturbation result of Castro Gonzalez, Koliha and Yimin Wei for the Drazin inverse of closed linear operators (see [7]).

Theorem 4.10. *Let $A \in C(E)$ be a Drazin invertible operator. Let $B \in C(E)$ with $D(A) = D(B)$ such that $(A - B)A^D \in B(E)$.*

If $\|(A-B)A^D\| < 1$, $P_A B A^D = A^D B P_A = 0$ and $\sigma(BP_A) = \{0\}$, then B is Drazin invertible operator and

$$\begin{aligned} B^D &= A^D(I - (A-B)A^D)^{-1} \\ \frac{\|B^D - A^D\|}{\|A^D\|} &\leq \frac{\|(A-B)A^D\|}{1 - \|(A-B)A^D\|} \\ \frac{\|A^D\|}{1 + \|(A-B)A^D\|} &\leq \left\|B^D\right\| \leq \frac{\|A^D\|}{1 - \|(A-B)A^D\|} \end{aligned}$$

A general result of stability of the Drazin inverse is obtained via the gap metrics in the case of Hilbert spaces.

Let $\mathcal{P}_F$ be the orthogonal projection on a closed linear subspace F of H. If $M,\ N$ are two closed linear subspaces of H let us put :

$$g(M, N) = \|\mathcal{P}_M - \mathcal{P}_N\|_{B(H)}$$

We notice that g is a distance on the set of all closed linear subspaces of H and we can easily verify that g have the following properties (see [21], [19]).

Proposition 4.1. *Let M, N be closed linear subspaces of H, we have*

$g(M, N) < 1 \Rightarrow M \cap N^\perp = M^\perp \cap N = \{0\}$.

$g(M, N) < 1 \Longleftrightarrow M \oplus N^\perp = H$.

Now $C(H)$ equipped with the metrics g called "gap" metric becomes a metric space :

$$A, B \in C(H), \qquad g(A, B) = g(G(A), G(B)) = \left\|\mathcal{P}_{G(A)} - \mathcal{P}_{G(B)}\right\|_{B(H\times H)}$$

where $\mathcal{P}_{G(A)}$ and $\mathcal{P}_{G(B)}$ denote respectively the orthogonal projection in $H \times H$ on the graph $G(A)$ of the operator A and the graph $G(B)$ of the operator B. Nevertheless, $C(H)$ is not complete for the metric g, and the natural laws of addition and multiplication are partially defined without being stable in $C(H)$. In fact, the sum and product of two operators A, B of $C(H)$ can be trivial ($D(A + B) = \{0\}$ and $D(AB) = \{0\}$) or else an operator not closed on H.

Azzouz, Messirdi and Djellouli in [1] have established, by means of the metrics g and by using Proposition 4.1, sufficient conditions under different perturbations to ensure that the closed and selfadjoint character is preserved and that the adjoint of the sum is the sum of adjoints. They showed that $(A + B) \in C(H)$ and $(A + B)^* = \overline{A^* + B^*}$, if $A, B \in C(H)$ such that $D(A) \cap D(B)$, $D(A^*) \cap D(B^*)$ and $D((A^* + B^*)^*)$ are dense in H, $0 \notin \sigma(A + B)$ and $g\left(G(A), G(-B)^\perp\right) < 1$.

Messirdi, Mortad, Azzouz and Djellouli in [22] have already found a topological condition such that the product of two operators of $C(H)$ remains in $C(H)$. Indeed, they show that if $A,\ B \in C(H)$ are such that $g(A, B^*) < 1$ then $D(AB)$ and $D(BA)$ are dense in H and $AB,\ BA \in C(H)$.

We show here the main results of this paper. We establish a topological characterization on the Drazin inverse of the sum and product of two operators of $C(H)$.

Theorem 4.11. *Let $A, B \in C(H)$ be Drazin invertible operators such that $R(B) \subset D(A)$, $R(A) \subset D(B)$ and $AB = BA$ on $D(A)$.*

(i) If $AB = BA = 0$ on $D(A)$, $0 \notin \sigma(A+B) \cup \sigma(A^* + B^*)$ and $g\left(G(A), G(-B)^{\perp}\right) < 1$, then $(A+B), (A^* + B^*) \in C(H)$, $(A+B)^* = A^* + B^*$ and

$$\begin{aligned} (A+B)^{-1} &= A^D + B^D \\ (A^* + B^*)^{-1} &= (A^D)^* + (B^D)^* = (A^*)^D + (B^*)^D \end{aligned}$$

(ii) If $P_A = P_B$ on $D(A)$ and $g(A, B^*) < 1$, then $AB \in C(H)$ is Drazin invertible and

$$(AB)^D = A^D B^D$$

Proof. (i) Under the assumptions $0 \notin \sigma(A+B) \cup \sigma(A^* + B^*)$ and $g\left(G(A), G(-B)^{\perp}\right) < 1$, Theorem 2.8 in [1] implies that $(A+B), (A^* + B^*) \in C(H)$ and $(A+B)^* = A^* + B^*$, furthermore $(A+B)^{-1}, (A^* + B^*)^{-1} \in B(H)$.

Remark that A, B, A^D, B^D all commute. Then from Lemma 3.1 (iii) and (iv), we obtain

$$AB^D = ABB^D B^D = 0\,;\, A^D B = A^D A^D AB = 0$$

Hence

$$\begin{aligned} (A^D + B^D)(A+B)(A^D + B^D) &= (A+B)(A^D + B^D)^2 \\ &= A^D + B^D \end{aligned}$$

Furthermore, $(A+B)[I - (A+B)(A^D + B^D)]$ is well defined on H since

$$(A+B)[I - (A+B)(A^D + B^D)] = AP_A + BP_B$$

As A and B are Drazin invertible $\sigma(AP_A) = \sigma(BP_B) = \{0\}$. Thus, $(A+B)[I - (A+B)(A^D + B^D)]$ is quasinilpotent operator on H, which shows that $A+B$ is Drazin invertible in H and $(A+B)^{-1} = (A+B)^D = A^D + B^D$ by uniqueness of the inverse.

By the same process we obtain the Drazin inverse of $A^* + B^*$.

(ii) Under the assumption $g(A, B^*) < 1$, then $AB \in C(H)$ by Theorem 2 in [22]. The operators A, B, A^D, B^D all commute. Then from Lemma 3.1 (iii) and (iv), we have

$$\begin{aligned} A^D B^D ABA^D B^D &= A(A^D)^2 B(B^D)^2 \\ &= A^D B^D \end{aligned}$$

Furthermore, $AB[I - ABA^D B^D] \in B(H)$ since

$$AB[I - ABA^D B^D] = AP_A BP_B + AP_A(I - P_A)BP_B + BP_B(I - P_B)AP_A$$

and $AP_A BP_B, AP_A(I - P_A)BP_B, BP_B(I - P_B)AP_A \in B(H)$. Consequently, $AB[I - ABA^D B^D]$ is quasinilpotent since the operators AP_A and BP_B are too. □

Remark 4.3. *(1) The assuption $P_A = P_B$ become useless if the operators A and B are bounded.*

(2) Drazin invertible operators having the same spectral projection were studied by Castro Gonzalez, Koliha and Yimin Wei in the paper [7].

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Received: May 15, 2014; *Accepted*: July 12, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 482–488

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

b-Chromatic number of some wheel related graphs

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Abstract

A proper coloring f is a b-coloring of the vertices of graph G such that in each color class there exists a vertex that has neighbours in every other color classes. The b-chromatic number $\varphi(G)$ of a graph G is the largest integer k for which G admits a b-coloring with k colors. If $\chi(G)$ is the chromatic number of G and b-coloring exists for every integer k satisfying the inequality $\chi(G) \leq k \leq \varphi(G)$ then G is called b-continuous. The b-spectrum $S_b(G)$ of a graph G is the set of k integers(colors) for which G has a b-coloring. We investigate b-chromatic number for the graphs obtained from wheel W_n by means of duplication of vertices. We also discuss b-continuity and b-spectrum for such graphs.

Keywords: b-Coloring, b-Continuity, b-Spectrum.

2010 MSC: 05C15, 05C76

1 Introduction

A proper k-coloring of a graph $G = (V(G), E(G))$ is a mapping $f : V(G) \rightarrow \{1, 2, ..., k\}$ such that every two adjacent vertices receive different colors. The chromatic number of a graph G is denoted by $\chi(G)$, is the minimum number for which G has a proper k-coloring. The set of vertices with a specific color is called a color class. A b-coloring of a graph G is a variant of proper k-coloring such that every color class has a vertex which is adjacent to at least one vertex in every other color classes and such a vertex is called a color dominating vertex. If v is a color dominating vertex of color class c then we denote it as $cdv(c) = v$. The b-chromatic number $\varphi(G)$ is the largest integer k such that G admits a b-coloring with k colors. The concept of b-coloring was originated by Irving and Manlove [6] and they also observed that every coloring of a graph G with $\chi(G)$ colors is obviously a b-coloring. In the same paper they have introduced the concepts of b-continuity and b-spectrum. If the b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ then G is called b-continuous and the b-spectrum $S_b(G)$ of a graph G is the set of k integers(colors) for which G has a b-coloring. The concept of b-coloring has been extensively studied by Faik [4], Kratochvil *et al.*[7], Alkhateeb [1], Balakrishnan [2], Chandrakumar and Nicholas [3]. The b- chromatic number of some cycle realated graphs have investigated by Vaidya and Shukla [8] while b-chromatic number of some degree splitting graphs is studied by Vaidya and Rakhimol [9].
Throughout this work wheel W_n we mean $W_n = C_n + K_1$.

Proposition 1.1. *[2] For any graph G, $\varphi(G) \leq \Delta(G) + 1$.*

Definition 1.1. *[5] The m-degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m - 1$.*

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Proposition 1.2. *[6] If graph G admits a b-coloring with m-colors, then G must have at least m vertices with degree at least* $m-1$.

Proposition 1.3. $\chi\left(W_n\right)=\begin{cases}3, & n \text{ is even}\\ 4, & n \text{ is odd.}\end{cases}$

2 Some general Results

Definition 2.2. *Duplication of a vertex* v *of graph* G *produces a new graph* G' *by adding a new vertex* v' *such that* $N(v')=N(v)$.

Theorem 2.1. *Let* G_1 *be the graph obtained from graph* G *by duplication of vertices(vertex) then* $\chi\left(G\right)=\chi\left(G_1\right)$.

Proof. Let $v\in V(G)$ be an arbitrary vertex of G and $v'\in V(G_1)$ be its duplicated vertex. As $N(v)=N(v')$ in G_1 and v and v' are independent vertices we can assign the same color to v and v'. Thus no extra color is required for proper coloring of G_1.

As all the duplicated vertices are independent in G_1 this argument can be extended in the case when arbitrary number of vertices are duplicated . Hence $\chi\left(G\right)=\chi\left(G_1\right)$. □

Theorem 2.2. *Let* G *be the graph obtained by duplicating all the rim vertices in* W_n *then*

$$\varphi\left(G\right)=\begin{cases}4, & n=3\\ 3, & n=4\\ 5, & n=5,6,8\\ 6, & n=7\\ 6, & n\geq 9.\end{cases}$$

Proof. Let $v_1,v_2,...,v_n$ be the rim vertices and u be the apex vertex of W_n and G be the graph obtained by duplication of all the rim vertices of W_n. Let $v'_1,v'_2,...,v'_n$ be the duplicated vertices corresponding to $v_1,v_2,...,v_n$. Then $|V\left(G\right)|=2n+1$ and $|E\left(G\right)|=5n$. To define proper coloring we consider the following cases.
Case-1: $n=3$.
In this case we have $V(G)=\{v_1,v_2,v_3,v'_1,v'_2,v'_3,u\}$ and $|V\left(G\right)|=7$. More precisely G has three vertices of degree three, three vertices of degree five and one vertex of degree six. Then by Proposition 1.1, $\varphi\left(G\right)\leq 7$ as $\Delta\left(G\right)=6$.
If $\varphi\left(G\right)=7$, then according to Proposition 1.3, the graph G must have seven vertices of degree six which is not possible as there is only one vertex of degree six. Hence $\varphi\left(G\right)\neq 7$.
If $\varphi\left(G\right)=6$ then according to Proposition 1.3, the graph G must have six vertices of degree five which is not possible as there are only three vertices of degree five and the remaining vertices are of degree three. Hence $\varphi\left(G\right)\neq 6$.
We claim that $\varphi\left(G\right)\neq 5$ because to achieve $\varphi\left(G\right)=5$ we need minimum five vertices of degree four, which is not possible by Proposition 1.3 as there are only three vertices of degree five and the remaining one vertex is of degree six while three vertices are of degree three. Hence $\varphi\left(G\right)\neq 5$.
If $\varphi\left(G\right)=4$ then according to Proposition 1.3, the graph G must have four vertices of degree three, which is possible for G. For b-coloring consider the color class $c=\{1,2,3,4\}$ and to assign the proper coloring to the vertices define the color function $f:V\rightarrow\{1,2,3,4\}$ as $f(v_1)=f(v'_1)=1, f(v_2)=f(v'_2)=2, f(v_3)=f(v'_3)=3, f(u)=4$. This proper coloring gives $cdv(1)=v'_1,\ cdv(2)=v'_2,\ cdv(3)=v'_3,\ cdv(4)=u$. Hence $\varphi\left(G\right)=4$.
Case-2: $n=4$.
For graph G we have $V(G)=\{v_1,v_2,v_3,v_4,v'_1,v'_2v'_3,v'_4,u\}$ and $|V\left(G\right)|=9$. More precisely graph G has four vertices of degree three, four vertices of degree five and one vertex of degree eight. Then by Proposition 1.3 we have $\varphi\left(G\right)\leq 9$ as $\Delta\left(G\right)=8$. If $\varphi\left(G\right)=9,8,7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi\left(G\right)=6$ then according to Proposition 1.3, the graph G must have six vertices of degree five, which is not possible as there are only four vertices of degree five, four vertices of degree three and one vertex of degree

eight. Hence $\varphi(G) \neq 6$
If $\varphi(G) = 5$ then according to Proposition 1.3, the graph G must have five vertices of degree four that is not possible as there is no vertex of degree four. Hence $\varphi(G) \neq 5$
If $\varphi(G) = 4$ then by Proposition 1.3, the graph G must have four vertices of degree three which is possible. But due to nature of the graph G any proper coloring with four colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph G. Hence $\varphi(G) \neq 4$. Thus we can color the graph by three colors.
For b-coloring consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3\}$ as $f(v_1) = f(v'_1) = 1$, $f(v_2) = f(v'_2) = 2$, $f(v_3) = f(v'_3) = 1, f(v_4) = f(v'_4) = 2, f(u) = 3$. This proper coloring gives $cdv(1) = v'_1, cdv(2) = v'_2, cdv(3) = u$. Hence $\varphi(G) = 3$.
Case-3: When $n = 5, 6, 8$.
Subcase-1: For $n = 5$.
In this case we have $V(G) = \{v_1, v_2, v_3, v_4, v_5, v'_1, v'_2, v'_3, v'_4, v'_5, u\}$ and $|V(G)| = 11$. More precisely G has five vertices of degree three, five vertices of degree five and one vertex of degree ten. Then by Proposition 1.1, $\varphi(G) \leq 11$ as $\Delta(G) = 10$.
If $\varphi(G) = 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi(G) = 6$ then the graph G must have six vertices of degree at least five which is not possible as there are only five vertices of degree five and the remaining vertices are of degree three while one vertex is of degree ten. Hence $\varphi(G) \neq 6$
If $\varphi(G) = 5$ then the graph G must have five vertices of degree at least four which is possible for the graph G. Thus we can color the graph by five colors.
Now for b-coloring consider the set of colors $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 2, f(v'_1) = 3, f(v'_2) = 3, f(v'_3) = 4, f(v'_4) = 4, f(v'_5) = 1, f(u) = 5$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_1, cdv(5) = u$. Hence $\varphi(G) = 5$.
Subcase-2: For $n = 6$.
For graph G we have $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, u\}$ and $|V(G)| = 13$. More precisely G has six vertices of degree three, six vertices of degree five and the remaining one vertex is of degree twelve. Then by Proposition 1.1, $\varphi(G) \leq 13$ as $\Delta(G) = 12$.
If $\varphi(G) = 12, 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi(G) = 6$ then the graph G must have six vertices of degree at least five which is possible. But due to the nature of the graph G any proper coloring with six colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph G. Thus $\varphi(G) \neq 6$.
For b-coloring with five colors consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = f(v'_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 1, f(v'_5) = 1, f(v'_6) = 1, f(u) = 5$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = u$. Hence $\varphi(G) = 5$.
Subcase-3: For $n = 8$.
In this case we have $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, u\}$ and $|V(G)| = 17$. More precisely G has eight vertices of degree three, eight vertices of degree five and one vertex of degree sixteen. Then by Proposition 1.1, $\varphi(G) \leq 17$ as $\Delta(G) = 16$.
If $\varphi(G) = 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi(G) = 6$, then graph G must have six vertices of degree at least five which is possible. But due to nature of the graph G any proper coloring with six colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph G. Thus $\varphi(G) \neq 6$.
For b-coloring with five colors consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = f(v'_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_7) = 3, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 1, f(v'_5) = 1, f(v'_6) = 1, f(v'_7) = 3, f(v_8) = f(v'_8) = 1, f(u) = 5$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = u$. Hence $\varphi(G) = 5$.
Case-4: $n = 7$.

For graph G we have $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, u\}$ and $|V(G)| = 15$. More precisely G has seven vertices of degree three, seven vertices of degree five and one vertex is of degree fourteen. Then by Proposition 1.1, $\varphi(G) \leq 15$ as $\Delta(G) = 14$.
If $\varphi(G) = 15, 14, 13, 12, 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi(G) = 6$ then according to Proposition 1.3, we need minimum six vertices of degree at least five which is possible. For b-coloring consider the color class $c = \{1, 2, 3, 4, 5, 6\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5, 6\}$ as $f(v_1) = 5, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(v_6) = 4, f(v_7) = 2, f(v'_1) = 3, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 5, f(v'_5) = 5, f(v'_6) = 4, f(v'_7) = 3, f(u) = 6$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_6, cdv(5) = v_1, cdv(6) = u$. Which conforms that $\varphi(G) = 6$.
<u>Case-5</u>: $n \geq 9$.
For $n = 9$, $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v'_9, u\}$ and $|V(G)| = 19$. More precisely G has nine vertices of degree five, nine vertices of degree three and one vertex of degree eighteen. Then by Proposition 1.1, $\varphi(G) \leq 19$ as $\Delta(G) = 18$.
If $\varphi(G) = 19, 18, 17, 16, 15, 14, 13, 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
According to Proposition 1.3 if $\varphi(G) = 6$ then we need minimum six vertices of degree at least five which is possible. For b-coloring consider the color class $c = \{1, 2, 3, 4, 5, 6\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5, 6\}$ as $f(v_1) = 4, f(v_2) = 2, f(v_3) = 5, f(v_4) = 1, f(v_5) = 2, f(v_6) = 3, f(v_7) = 1, f(v_8) = 4, f(v_9) = 2, f(v'_1) = 4, f(v'_2) = 3, f(v'_3) = 3, f(v'_4) = 4, f(v'_5) = 4, f(v'_6) = 5, f(v'_7) = 5, f(v'_8) = 4, f(v'_9) = 3, f(u) = 6$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_6, cdv(5) = v_1, cdv(6) = u$. Hence $\varphi(G) = 6$.
When $n > 9$ we repeat the colors as in the above graph G for the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v'_9, u\}$ and for the remaining vertices assign the colors as follows $f(v_i) = 1,\ f(v'_i) = 2;\ when\ i\ is\ even$ and $f(v_i) = 2,\ f(v'_i) = 1;\ when\ i\ is\ odd$. Hence $\varphi(G) = 6,\ n \geq 9$. □

Theorem 2.3. *G is b-continuous.*

Proof. To prove this result we continue with the terminology and notations used in Theorem 2.3 and consider the following cases.
<u>Case-1</u>: $n = 3$.
In this case the graph G is b-continuous as $\chi(G) = \varphi(G) = 4$.
<u>Case-2</u>: $n = 4$.
In this case the graph G is b-continuous as $\chi(G) = \varphi(G) = 3$.
<u>Case-3</u>: $n = 5$.
In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_5) = 4$. Also by Theorem 2.3, $\varphi(G) = 5$. Thus b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$(Here $k = 4, 5$). Hence G is b-continuous.
<u>Case-4</u>: $n = 6$.
In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_6) = 3$. Also by Theorem 2.3, $\varphi(G) = 5$. It is obvious that b-coloring for the graph G is possible using the number of colors $k = 3, 5$.
Now for $k = 4$ the b-coloring for the graph G is as follows. Consider the color class $c = 1, 2, 3, 4$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4\}$ as $f(v_1) = 1 = f(v'_1), f(v_2) = 2 = f(v'_2), f(v_3) = 3 = f(v'_3), f(v_4) = 1 = f(v'_4), f(v_5) = 2 = f(v'_5), f(v_6) = 3 = f(v'_6), f(u) = 4$.
This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = u$. Thus G is four colorable. Hence b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ (Here $k = 3, 4, 5$).Hence G is b-continuous.
<u>Case-5</u>: $n = 7$.
In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_7) = 4$. Also by Theorem 2.3,$\varphi(G) = 6$. It is obvious that b-coloring for the graph G is possible using the number of colors $k = 4, 6$. Now for $k = 5$ the b-coloring for the graph G is as follows. Consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 1, f(v_5) = 4, f(v_6) = 2, f(v_7) = 4, f(v'_1) = 1, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 1, f(v'_5) = 4, f(v'_6) = 3, f(v'_7) = 3, f(u) = 5$. This proper coloring gives dominating vertices $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_5, cdv(5) = u$. So the graph G is five colorable. Hence b-coloring exists for every integer k satis-

fying $\chi(G) \leq k \leq \varphi(G)$ (Here $k = 4, 5, 6$). Thus G is b-continuous.
Case-6: $n = 8$.
In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_8) = 3$. Also by Theorem 2.3, $\varphi(G) = 5$. It is obvious that b-coloring for the graph G is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring for the graph G is as follows. Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4\}$ as $f(v_1) = 1 = f(v'_1), f(v_2) = 2 = f(v'_2), f(v_3) = 3 = f(v'_3), f(v_4) = 1 = f(v'_4), f(v_5) = 2 = f(v'_5), f(v_6) = 3 = f(v'_6), f(v_7) = 1 = f(v'_7), f(v_8) = 3 = f(v'_8), f(u) = 4$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = u$. Thus G is four colorable. Hence b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ (Here $k = 3, 4, 5$). Consequently G is b-continuous.
Case-7: $n = 9$.
In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_9) = 4$. Also by Theorem 2.3, $\varphi(G) = 6$. It is obvious that b-coloring for the graph G is possible using the number of colors $k = 4, 6$. Now for $k = 5$ the b-coloring for the graph G is as follows. Consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = 2, f(v_2) = 4, f(v_3) = 1, f(v_4) = 2, f(v_5) = 3, f(v_6) = 1, f(v_7) = 2, f(v_8) = 1, f(v_9) = 3, f(v'_1) = 2, f(v'_2) = 3, f(v'_3) = 3, f(v'_4) = 4, f(v'_5) = 4, f(v'_6) = 1, f(v'_7) = 2, f(v'_8) = 1, f(v'_9) = 3, f(u) = 5$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_9, cdv(5) = u$. Thus G is five colorable. Hence b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ (Here $k = 4, 5, 6$).
Case-8: $n > 9$.
When $n > 9$ we repeat the color assignment as in the case $n = 9$ discussed above for the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v'_9, u\}$ and for the remaining vertices give the colors as follows.
<u>When $k = 5$</u>

$$f(v_i) = f(v'_i) = \begin{cases} 1, & i \text{ even} \\ 3, & i \text{ odd} \end{cases}$$

Hence G is b-continuous. □

As any coloring with $\chi(G)$ colors is a b-coloring, we state the following obvious result.

Corollary 2.1.

$$S_b(G) = \begin{cases} \{4\} & n = 3 \\ \{3\} & n = 4 \\ \{4,5\} & n = 5 \\ \{3,4,5\} & n = 6,8 \\ \{4,5,6\} & n = 7 \\ \{4,5,6\} & n \geq 9 \end{cases}$$

Theorem 2.4. *Let G_1 be the graph obtained by duplicating the apex vertex in W_n then*

$$\varphi(G_1) = \begin{cases} 4, & n = 3 \\ 3, & n = 4 \\ 4, & n \geq 5 \end{cases}$$

Proof. For W_n, $v_1, v_2, ..., v_n$ be the vertices and u be the apex vertex of W_n. Let G_1 be the graph obtained by duplication of the vertex u of W_n. Let u' be the duplicated vertices corresponding to u. Then $|V(G_1)| = n + 2$ and $|E(G_1)| = 3n$. To define the proper coloring we consider the following two cases.
Case-1: $n = 3$.
In this case $V(G_1) = \{v_1, v_2, v_3, u, u'\}$ and $|V(G_1)| = 5$. More precisely G_1 has two vertices of degree three, three vertices of degree four. Then by Proposition 1.1, $\varphi(G_1) \leq 5$ as $\Delta(G_1) = 4$. If $\varphi(G_1) = 4$ then according to Proposition 1.3, the graph G_1 must have four vertices of degree at least three which is possible.
For b-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(u) = 4 = f(u')$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = u$. Hence $\varphi(G_1) = 4$.
Case-2: $n = 4$.

In this case $V(G_1) = \{v_1, v_2, v_3, v_4, u, u'\}$ and $|V(G_1)| = 6$. More precisely G_1 has six vertices of degree four. Then by Proposition 1.1, $\varphi(G_1) \leq 5$ as $\Delta(G_1) = 4$. If $\varphi(G_1) = 5$ then according to Proposition 1.3 the graph G_1 must have five vertices of degree at least four which is possible. But due to the nature of graph G_1 any proper coloring with five colors have at least one color class which does not have color dominating vertices. Hence the graph G_1 is not b-colorable using five colors. Hence $\varphi(G_1) \neq 5$.
If possible let $\varphi(G_1) = 4$ and $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(u) = 4$, which in turn forces us to assign $f(v_4) = 2, f(u') = 4$. This proper coloring gives the color dominating vertices for color classes 2 and 4 but not for 1 and 3 which is contradiction. Thus $\varphi(G_1) \neq 4$. Hence we can color the graph by three colors.
For b-coloring consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 1, f(v_4) = 2, f(u) = 3, f(u') = 3$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = u$. Hence $\varphi(G_1) = 3$.
Case-3: $n = 5$.
For graph G_1 we have $V(G_1) = \{v_1, v_2, v_3, v_4, v_5, u, u'\}$ and $|V(G_1)| = 7$. More precisely G_1 has five vertices of degree four and two vertices of degree two. Then by Proposition 1.1, $\varphi(G_1) \leq 6$ as $\Delta(G_1) = 5$. According to Proposition 1.3, if $\varphi(G_1) = 6$ then we need six vertices of degree at least five, which is not possible as there are only two vertices of degree five and the remaining vertices are of degree four. Hence $\varphi(G_1) \neq 6$.
If $\varphi(G_1) = 5$ then according to Proposition 1.3 the graph G_1 must have five vertices of degree at least four which is possible. But due to the nature of graph G_1 any proper coloring with five colors have at least one color class which does not have any color dominating vertex. Hence G_1 is not b-colorable with five colors. Hence $\varphi(G_1) \neq 5$. Thus we can color the graph by four colors.
For b-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(u) = 4, f(u') = 4$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = u$. Hence $\varphi(G_1) = 4$.
Case-3: $n > 5$.
When $n > 5$ we repeat the color assignment as in the case when $n = 5$ for the vertices $\{v_1, v_2, v_3, v_4, v_5, u, u'\}$ and for the remaining vertices assign the colors as follows.

$$f(v_i) = \begin{cases} 2, & i \text{ is even} \\ 1, & i \text{ is odd} \end{cases}$$

Hence $\varphi(G_1) = 4;\ n \geq 5$. □

3 Concluding Remarks

We explore the concept of b-coloring in the context of duplication of vertex in graph and prove that the chromatic number of graph G is same as the chromatic number of the graph obtained by duplication of vertices in G. We investigate the b-chromatic number for the larger graphs obtained from wheel W_n by means of duplication of a vertex. The graph obtained by duplication of the apex of W_n is obviously b-continuous while we have shown that the graph obtained by duplication of rim vertices altogether is a b-continuous. We also determine the b-spectrum for the same.

4 Acknowledgement

Our thanks are due to the anonymous referee for careful reading of the paper.

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Received: May 15, 2014; *Accepted*: July 31, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 489–496

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On the oscillation of third order quasilinear delay differential equations with Maxima

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Abstract

In this paper, we study the oscillation and asymptotic properties of third order quasilinear neutral delay differential equation

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))''\right)^{\alpha}\right)'+q(t)\max_{[\sigma(t),t]}x^{\alpha}(s)=0,\;\; t\geq t_0\geq 0 \tag{0.1}$$

where α is a ratio of odd positive integers and $\int_{t_0}^{\infty}\frac{1}{a^{1/\alpha}(t)}dt=\infty$. We establish a new condition which guarantees that every solution of (0.1) is either oscillatory or converges to zero. There results extend some known results in the literature without "maxima". Examples are given to illustrate the main results.

Keywords: Oscillation, quasilinear, neutral, delay, third order, differential equations with maxima.

2010 MSC: 34K15.

1 Introduction

We are concerned with the oscillation problem of third order quasilinear neutral delay differential equation with "maxima" of the form

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))''\right)^{\alpha}\right)'+q(t)\max_{[\sigma(t),t]}x^{\alpha}(s)=0,\;\; t\geq t_0\geq 0 \tag{1.1}$$

where $\alpha>0$ is the quotient of odd positive integers. Throughout this paper, we will assume that the following conditions hold:

(C_1) $\tau(t)\leq t$ and $\sigma(t)<t$ are continuous functions in $[t_0,\infty)$l;

(C_2) $p(t)\in C^3([t_0,\infty),\ R)$ with $0\leq p(t)\leq p<1$, and $q(t)\in C([t_0,\infty),\ R_+)$ with $q(t)$ is not identically zero on any ray of the form $[t_*,\infty)$ for any $t_*\geq t_0$;

(C_3) $a(t)\in C^1([t_0,\infty),\ a(t)>0$ and nondecreasing for all $t\geq t_0$ and $\int_{t_0}^{\infty}\frac{1}{a^{\frac{1}{\alpha}}(1)}dt=\infty$.

By a solution of equation (1.1) we mean a continuous function $x(t)\in C^2([T_x,\infty)),\ T_x\geq t_)$, which has the property $((x(t)+p(t)x(\tau(t)))'')^{\alpha}$ are continuously differentiable and $x(t)$ satisfies the equation (1.1) on $[T_x,\infty)$. We consider only those solution $x(t)$ of equation (1.1) which satisfy $\sup\{|x(t)|:t\geq T\}>0$ for all $t\geq T_x$. We assume that the equation (1.1) is called oscillatory if it has arbitrary large zeros on $[T_x,\infty)$, otherwise it is called nonoscillatory. A solution $x(t)$ of equation (1.1) is said to be almost oscillatory if $x(t)$ is either oscillatory or $|x(t)|\to 0$ monotonically as $t\to\infty$.

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In the last few years, the qualitative theory of differential equations with “maxima” received very little attention even though such equations often arise in the problem of automatic regulation of various real system, see for example [1, 10, 12]. The oscillatory behavior of solutions of differential equations with “maxima” are discussed in [1-6, 11, 13, 14], and the references cited therein.

The great attention has been devoted to the oscillation of third order differential equation without “maxima” see for example [15-24, 26, 27] and the references cited therein. Compared to second order differential equations with “maxima” less attention has received the third order differential equation with “maxiam”. Motivated by these observations, in this paper, we present some sufficient conditions for the oscillation of all solutions of equation (1.1). The result extend that of in [25] for equation (1.1) with $p(t) \equiv 0$ and without “maxima”.

In Section 2, we obtain criteria for the oscillation of all solution of equation (1.1) and is Section 3 we present some examples to illustrate the main results.

Remark 1.1. *All functional inequalities consider in this paper assumed to hold eventually, that is they are satisfied for all t large enough.*

Remark 1.2. *Without loss of generality we can deal only with the positive solution of equation (1.1).*

2 Oscillation Results

In this section, we obtain a oscillatory criterion for equation (1.1). For a solution $x(t)$ of (1.1) we define the corresponding function $z(t)$ by

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{2.2}$$

To obtain sufficient condition for the oscillation of solutions of equation (1.1), we need the following lemmas.

Lemma 2.1. *Let $x(t)$ be a positive solution of equation (1.1), then there are only the following two cases for $z(t)$ defined in (2.2) hold:*

(I) $z(t) > 0,\ z'(t) > 0$ and $z''(t) > 0$;

(II) $z(t) > 0,\ z'(t) < 0$ and $z''(t) > 0$ for $t \geq t_1 \geq t_0$;

where t_1 is sufficiently large.

Proof. Assume that $x(t)$ is a positive solution of (1.1) on $[t_0, \infty)$. We see that $z(t) > x(t) > 0$ and

$$\left(a(t)\left((x(t) + p(t)x(\tau(t)))''\right)^{\alpha}\right)' = -q(t) \max_{[\sigma(t),t]} x^{\alpha}(s) < 0. \tag{2.3}$$

Thus, $a(t)(z''(t))^{\alpha}$ is nonincreasing and of one sign. Therefore $z''(t)$ is also of one sign and so we have two possibilities

$$z''(t) < 0 \text{ or } z''(t) > 0 \text{ for } t \geq t_1.$$

If we admit that $z''(t) < 0$, then there exists a constant $M > 0$ such that

$$aa(t)(z''(t))^{\alpha} \leq -M < 0.$$

Integrating the last inequality from t_1 to t we obtain

$$z'(t) \leq z'(t_1) - M^{1/\alpha} \int_{t_1}^{t} a^{-1/\alpha}(s)ds.$$

Letting $t \to \infty$ and using (C_2) we get $z'(t) \to \infty$. Thus $z'(t) < 0$ eventually. But $z''(t) < 0$ and $z'(t) < 0$ eventually imply $z(t) < 0$ for $t \geq t_1$ a contradiction. This contradiction proves that $z''(t) > 0$ and we have only tow cases (I) and (II) for $z(t)$. The proof is now complete. □

Lemma 2.2. *Assume that $u(t) > 0,\ u'(t) \geq 0,\ u''(t) \leq 0$, on $[t_0, \infty)$. Then for each $\ell \in (0,1)$ there exists a $T_\ell \geq t_0$ such that*

$$\frac{u(\tau(t))}{u(t)} \geq \ell \frac{u(t)}{t} \text{ for } t \geq T_\ell.$$

Lemma 2.3. *Assume that* $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leq 0$, *on* $[T_\ell, \infty)$. *Then*

$$\frac{z(t)}{z'(t)} \geq \frac{t - T_\ell}{2} \quad for \quad t \geq T_\ell.$$

The proofs of Lemma 2.2 and Lemma 2.3 are found in [25].

Lemma 2.4. *The function* $x(t)$ *is a negative solutions of equation* (1.1) *if and only if* $-x(t)$ *is a positive solution of the equation*

$$\left(a(t)\left((x(t) + p(t)x(\tau(t)))''\right)^{\alpha}\right)' + q(t) \min_{[\sigma(t),t]} x^{\beta}(s) = 0. \tag{2.4}$$

Proof. The assertion of Lemma 2.4 can be verified easily. □

Lemma 2.5. *Let* $x(t)$ *be a positive solution of equation* (1.1) *and let the corresponding* $z(t)$ *satisfy Lemma 2.1 (II). If*

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{a(u)} \int_{u}^{\infty} q(s)ds\right)^{\frac{1}{\alpha}} du dv = \infty \tag{2.5}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Proof. The proof is similar to that of in [25] and hence the details are omitted. □

Lemma 2.6. *Assume that* $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leq 0$ *on* $[T_\ell, \infty)$. *Then*

$$(t - T_\ell)\frac{z''(t)}{z'(t)} \leq 1 \quad for \quad t \geq T_\ell.$$

Proof. The proof is similar to that of in [25] and hence the details are omitted. □

Now, we present the main results. For simplicity we introduce the following notations:

$$p_* = \lim_{t\to\infty} \frac{t^{\alpha}}{a(t)} \int_{t}^{\infty} P_\ell(s)ds,$$

$$q_* = \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds$$

where

$$P_\ell(s) = \ell^{\alpha} \max_{[\sigma(t),t]} (1 - p(s))^{\alpha} q(s) \left(\frac{\tau(s)}{s}\right)^{\alpha} \left(\frac{\tau(s) - T_\ell}{2}\right)^{\alpha} \tag{2.6}$$

with $\ell \in (0,1)$ arbitrarily chosen and T_ℓ large enough. Moreover for $z(t)$ satisfying case (I), we define

$$w(t) = a(t) \left(\frac{z''(t)}{z(t)}\right)^{\alpha} \tag{2.7}$$

$$r = \liminf_{t\to\infty} \frac{t^{\alpha}}{a(t)},$$

and

$$r = \limsup_{t\to\infty} \frac{t^{\alpha}}{a(t)}. \tag{2.8}$$

Theorem 2.1. *Assume that condition* (2.5) *holds and* $a'(t) \geq 0$ *for all* $t \geq t_0$. *If*

$$p_* = \liminf_{t\to\infty} \frac{t^{\alpha}}{a(t)} \int_{t}^{\infty} P_\ell(s)ds > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}. \tag{2.9}$$

Then the solution $x(t)$ *of equation* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ is a positive solution of equation (1.1) and the corresponding function $z(t)$ satisfies case(I) of Lemma 2.1. First note that

$$x(t) = z(t) - p(t)x(\tau(t)) \geq (1 - p(t))z(t) \tag{2.10}$$

or

$$\max_{[\tau(t),t]} x^{\alpha}(s) \geq z^{\alpha} \max_{[\sigma(t),t]} (1 - p(s))^{\alpha}.$$

Using the above inequality in (1.1) we obatin

$$(a(t)(z''(t))^{\alpha})' \leq 0 \tag{2.11}$$

The last inequality together with $a'(t) \geq 0$ gives that $z(t)$ satisfies $z(\tau(t)) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leq 0$ for $t \in [T, \infty]$. From the definition of $w(t)$ we see that $w(t) > 0$ and from (1.1) we have

$$\begin{aligned} w'(t) &= \frac{(z'(t))^{\alpha}(a(t)(z''(t))^{\alpha})' - (a(t)(z''(t))^{\alpha})\alpha(z'(t))^{\alpha-1}z''(t)}{(z'(t))^{2\alpha}} \\ &= \frac{-q(t)z^{\alpha}(t)\max_{[\sigma(t),t]}(1-p(s))^{\alpha}}{(z'(t))^{\alpha}} - \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t) \end{aligned} \tag{2.12}$$

From Lemma 2.2 with $u(t) = z'(t)$, we have for ℓ the same $P_{\ell}(t)$,

$$\frac{1}{z'(t)} \geq \ell \frac{\tau(t)}{t} \frac{1}{z'(tau(t))}, \quad t \geq T_{\ell}$$

which with (2.12) gives

$$w'(t) \leq -q(t)\ell^{\alpha} \left(\frac{\tau(t)}{t}\right)^{\alpha} \frac{z^{\alpha}(t)}{(z'(\tau(t)))^{\alpha}} \max_{[\sigma(s),s]} (1 - p(s))^{\alpha} - \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t).$$

Using the fact from Lemma 2.3 that $z(t) \geq \frac{(t-T_{\ell})}{2} z'(t)$, we have

$$w'(t) + P_{\ell}(t) + \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t) \leq 0. \tag{2.13}$$

Since $P_{\ell}(t) > 0$ and $w(t) > 0$ for $t \geq T_{\ell}$, we have from (2.13) that $w'(t) \leq 0$ and

$$-\left(\frac{w'(t)}{\alpha w^{\frac{\alpha+1}{\alpha}}(t)}\right) > \frac{1}{a^{1/\alpha}(t)}, \quad \text{for } t \geq T_{\ell}. \tag{2.14}$$

This implies that

$$\left(\frac{1}{w^{1/\alpha}(t)}\right)' > \frac{1}{a^{1/\alpha}(t)} \tag{2.15}$$

Integrating the last inequality from T_{ℓ} to t, we obtain

$$w(t) = \frac{1}{\left(\int_{T_{\ell}}^{t} \frac{ds}{a^{1/\alpha}(t)}\right)^{\alpha}} \tag{2.16}$$

which inview of (C_3) implies that $\lim_{t\to\infty} w(t) = 0$. On the otherhand, from the definition of $w(t)$, and Lemma 2.3, we see that

$$0 \leq r \leq R \leq 1. \tag{2.17}$$

Now, let $\varepsilon > 0$, then from the definitions of p_* and r we can pick $t_2 \in [T_{\ell}, \infty)$ sufficiently large that

$$\frac{t^{\alpha}}{a(t)} \int_{t}^{\infty} P_{\ell}(s)ds \geq p_* - \varepsilon,$$

and

$$\frac{t^{\alpha} w(t)}{a(t)} \geq t - \varepsilon, \quad \text{for } t \in [t_0, \infty).$$

Integrating (2.13) from t to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w(t) \geq \int_t^\infty P_\ell(s)ds + \alpha \int_t^\infty \frac{w^{1+1/\alpha}(s)}{a^{1/\alpha}(s)}ds, \text{ for } t \in [t_2, \infty). \tag{2.18}$$

Assume $p_* = \infty$, then from (2.18), we have

$$\frac{t^\alpha w(t)}{a(t)} \geq \frac{t^\alpha}{a(t)} \int_t^\infty P_\ell(s)ds.$$

Taking the limit infimum on both sides as $t \to \infty$, we get in view of (2.17) that $1 \geq r \geq \infty$. This is a contradiction. Next assume that $p_* < \infty$. Now from (2.18) and the fact $a'(t) \geq 0$, we have

$$\begin{aligned} \frac{t^\alpha}{a(t)} w(t) &\geq \frac{t^\alpha}{a(t)} \int_t^\alpha P_\ell(s)ds + \alpha \frac{t^\alpha}{a(t)} \int_t^\infty \frac{s^{\alpha+1} a(s) w^{\frac{1}{\alpha}+1}(s)}{s^{\alpha+1} a^{\frac{1}{\alpha}+1}(s)} ds \\ &\geq (p_* - \varepsilon) + \frac{t^\alpha (r-\varepsilon)^{1+\frac{1}{\alpha}}}{a(t)} \int_t^\infty \frac{\alpha a(s)}{s^{\alpha+1}} ds \\ &\geq (p_* - \varepsilon) + (r-\varepsilon)^{1+\frac{1}{\alpha}} t^\alpha \int_t^\infty \frac{\alpha}{s^{\alpha+1}} ds \end{aligned} \tag{2.19}$$

or

$$\frac{t^\alpha w(t)}{a(t)} \geq (p_* - \varepsilon) + (r-\varepsilon)^{1+\frac{1}{\alpha}}.$$

Taking the limit infimum on both sides as $t \to \infty$, we get

$$r \geq p_* - \varepsilon + (r-\varepsilon)^{1+\frac{1}{\alpha}}.$$

Since $\varepsilon > 0$ is arbitrary, we get the desired result

$$p_* \leq r - r^{1+\frac{1}{\alpha}}.$$

Using the inequality $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$. With $A = B = 1$, we get $p_* \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$, which contradicts (2.9). This completes the proof. □

Corollary 2.1. *Assume that (2.5) holds and $a'(t) \geq 0$. Let $x(t)$ be a solution of equation (1.1). If*

$$\liminf_{t\to\infty} \frac{t^\alpha}{a(t)} \int_t^\infty q(s) \max_{[\sigma(t),t]} (1-p(s))^\alpha \frac{\tau^{2\alpha}(s)}{s^\alpha} P_\ell(s)ds > \frac{(2\alpha)^\alpha}{(\alpha+1)^{\alpha+1}} \tag{2.20}$$

then $x(t)$ is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. We shall now show that (2.20) implies (2.19). First note that for any $\ell \in (0,1)$ there exists a t_1 such that $\tau(t) - T_\ell \geq \ell\tau(t)$, $t \geq t_1$. Therefore

$$P_\ell \geq \frac{\ell^{2\alpha} \max_{[\sigma(t),t]} (1-p(t))^\alpha}{2^\alpha} \frac{q(t)\tau^{2\alpha}(t)}{t^\alpha}, \quad t \geq t_1. \tag{2.21}$$

On the otherhand, (2.20) implies that for some $\ell \in (0,1)$

$$\liminf_{t\to\infty} \frac{t^\alpha}{a(t)} \int_t^\infty q(s) \max_{[\sigma(t),t]} (1-p(s))^\alpha \frac{\tau^{2\alpha}(s)}{s^\alpha} > \frac{1}{\ell^{2\alpha}} \frac{(2\alpha)^\alpha}{(\alpha+1)^(\alpha+1)} \tag{2.22}$$

Combining (2.21) with (2.22) we get (2.9). □

Theorem 2.2. *Assume that the condition (2.5) holds and $a'(t) \geq 0$ for all $t \geq t_0$. Let $x(t)$ be a solution of equation (1.1). If $p_* + q_* > 1$, then $x(t)$ is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. Assume that $x(t)$ be a positive solution of equation (1.1) and the corresponding function $z(t)$ satisfies case(I) of Lemma 2.1. Now multiply (2.13) by $\frac{t^{\alpha+1}}{a(t)}$, and integrating from t_2 to t $(t \geq t_2)$, we get

$$\int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} w'(s)ds \leq \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds - \alpha \int_{t_2}^{t} \left(\frac{s^\alpha w(t)}{a(s)} \right)^{\frac{s+1}{s}} ds \tag{2.23}$$

Using integration by parts, we obtain

$$\begin{aligned} \frac{t^{\alpha+1}}{a(t)} w(t) \quad &\leq \quad \frac{t_2^{\alpha+1} w(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds \\ &\quad - \alpha \int_{t_2}^{t} \left(\frac{s^\alpha w(t)}{a(s)} \right)^{\frac{s+1}{s}} ds + \int_{t_2}^{t} \left(\frac{s^{\alpha+1}}{a(s)} \right)' w(s)ds. \end{aligned}$$

Since $a'(t) \geq 0$, we have

$$\left(\frac{s^{\alpha+1}}{a(s)} \right)' = \frac{a(s)(\alpha+1)s^\alpha - a'(s)s^\alpha}{(a(s))^2} \leq \frac{(\alpha+1)s^\alpha}{a(s)}.$$

Hence,

$$\begin{aligned} \frac{t^{\alpha+1}}{a(t)} w(t) \quad &\leq \quad \frac{t_2^{\alpha+1} w(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds \\ &\quad + \int_{t_2}^{t} \left[\frac{(\alpha+1)s^\alpha w(s)}{a(s)} - \alpha \left(\frac{s^\alpha w(s)}{a(s)} \right)^{\alpha+1} \right] ds. \end{aligned}$$

Using the inequality $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$, with $u(s) = \frac{s^\alpha w(s)}{a(s)} > 0$, and positive constants. $A = \alpha$, $B = \alpha + 1$, we get

$$\frac{t^{\alpha+1}}{a(t)} w(t) \leq \frac{t_2^{\alpha+1}}{a(t_2)} w(t_2) - \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds + \frac{t - t_2}{t}. \tag{2.24}$$

Taking limit supreme on both sides as $t \to \infty$ we obtain $R \leq q_* + 1$. Combining this with the inequality (2.20) we get

$$p_* + q_* \leq 1. \tag{2.25}$$

This is a contradiction. If $z(t)$ satisfies condition (2.5) then by Lemma 2.1 of case(II) with $\lim_{t\to\infty} z(t) = 0$. This completes the proof. □

Corollary 2.2. *Assume that (2.5) holds and $a'(t) \geq 0$. Let $x(t)$ be a solution of equation (1.1). If*

$$q_* = \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds > 1 \tag{2.26}$$

then $x(t)$ is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

As a matter of fact we can slightly simplify the function $P_\ell(t)$ in (2.26).

Corollary 2.3. *Assume that (2.5) holds and $a'(t) \geq 0$. Let $x(t)$ be a solution of equation (1.1). If*

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s\tau^{2\alpha}(s) q(s) \max_{[\sigma(t),t]} (1 - p(s))^\alpha}{a(s)} ds > 2^\alpha$$

then $x(t)$ is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

3 Examples

In this section we present some examples to illustrate the main results.

Example 3.1. *Consider the differential equation*

$$\left(t^3\left((x(t)+\frac{1}{3}x(t/2))''\right)^3\right)'+\frac{750}{27t^4}\max_{[t/2,t]}x^3(s)=0,\ \ t\geq 0. \tag{3.1}$$

One can easily verify that all conditions of Theorem 2.1 are satisfied and hence every solution of equation (1.1) is almost oscillatory. Infact $x(t)=\frac{1}{t}$ is one such solution of equation (3.1).

Example 3.2. *Consider the differential equation*

$$\left(t^{1/3}\left((x(t)+\frac{1}{2}x(t/2))''\right)^{1/3}\right)'+\frac{1}{3}\left(\frac{2}{t}\right)^{4/3}\max_{[t/2,t]}x^{1/3}(s)=0,\ \ t\geq 1. \tag{3.2}$$

One can easily verify that all conditions of Theorem 2.2 are satisfied and hence every solution of equation (1.1) is almost oscillatory. In fact $x(t)=\frac{1}{t}$ is one such solution of equation (3.2).

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Received: March 27, 2014; *Accepted*: August 02, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 497–501

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Adapted linear approximation for singular integral equations

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Abstract

The aim of this work is to solve singular integral equations (S.I.E), of Cauchy type on a closed smooth curve. This method presented by the author is based on the adapted linear approximation of the singular integral of the dominant part, where we reduce a (S.I.E) to an algebraic linear system and we realize numerically this approach by examples.

Keywords: singular integral, interpolation, linear approximation, Holder space and Holder condition.

2010 MSC: 45D05, 45E05, 65R20.

1 Introduction

Many studies devoted to the numerical procedure are developed for solving singular integral equations (S.I.E) over a contour. Cauchy type singular integral equations are often encountered in electrostatics, fluid dynamics and in simulation of cracks. Computational efficient quadrature methods for the solution of (S.I.E) have recently been introduced and analyzed in the case of a closed curve [5]. A different quadrature method for a closed curve, involving subtraction of the singularity, was analyzed in [8]. Noting that, the solution of a large class of boundary-value problems in mathematical physics can be reduced to singular integral equations (S.I.E) of the form

$$a_0(t_0)\varphi(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt + \frac{1}{\pi i}\int_\Gamma k(t,t_0)\,\varphi(t)\,dt = f(t_0). \tag{1.1}$$

It will often be useful to write equation (1) in the form

$$(a_0(t_0)\,I + b_0(t_0)\,S_\Gamma + K_\Gamma)\,\varphi(t_0) = f(t_0), \tag{1.2}$$

where I is the identity operator and the operators S_Γ and K_Γ are defined by

$$S_\Gamma\varphi(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt; \qquad K_\Gamma\varphi(t_0) = \frac{1}{\pi i}\int_\Gamma k(t,t_0)\varphi(t)dt. \tag{1.3}$$

In this work we present a direct method for an approximative solution of a singular integral equation (S.I.E) on a piecewise smooth integration path Γ, where Γ is any piecewise smooth closed contour [2], t_0 and t are points on Γ, the known functions $a_0(t)$, $b_0(t)$ and $k(t,t_0)$ are defined on Γ and satisfying the Holder condition $H(\alpha)$, $0<\alpha\leq 1$ [2]. Further, anywhere on Γ we have

$$a_0^2(t) - b_0^2(t) \neq 0. \tag{1.4}$$

As it is known, the integral of the dominant part of the above equation (1) exists in the sense of a Cauchy principal value integral for all density φ satisfies the Holder condition $H(\alpha)$ and also exists for all function $\varphi \in L^2(\Gamma)$.

The present note is divided into two parts. In the first one, we present a formulation of the quadrature formula for the evaluation of Cauchy type integrals proposed by the author [5], this quadrature formula is based on the adapted quadratic approximation of the density $\varphi(x)$.

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In the second part, we present the numerical realization of this approximation; also the estimate of the error of the approximation integral was established. Besides, pointwise convergence of the approximate solutions to an exact solution is obtained [5,6].

A method to proceed is to solve the (S.I.E) by numerical means, like the reduction to a system of linear algebraic equations after the use of an appropriate quadrature rule.

We denote by t the parametric complex function $t(s)$ of the curve Γ defined by

$$t(s) \ = x(s) + iy(s), \ \ a \le s \le b,$$

where $x(s)$ and $y(s)$ are continuous functions on the finite interval of definition $[a, b]$ and have continuous first derivatives $x'(s)$ and $y'(s)$ never simultaneously null.

2 The Quadrature

Theorem 2.1. *Let N be an arbitrary natural number, generally we take it large enough and divide the interval $[a, b]$ into N equal subintervals $I_1, I_2, ..., I_N$ by the points*

$$s_\sigma = a + \sigma \frac{l}{N} \ , \ l = b - a \ , \ \sigma = 0, 1, 2, ..., N.$$

We introduce the notation

$$t_\sigma = t(s_\sigma); \ \ \sigma = 0, 1, 2, ..., N.$$

Assuming that, for the indices $\sigma, \nu = 0, 1, 2, ..., N-1$, the points t and t_0 belong respectively to the arcs $\overset{\frown}{t_\sigma t_{\sigma+1}}$ and $\overset{\frown}{t_\nu t_{\nu+1}}$ where $\overset{\frown}{t_\alpha t_{\alpha+1}}$ designates the smallest arc with ends t_α and $t_{\alpha+1}$ [3,5,6,7].

Following [6], we define the approximation $\psi_{\sigma\nu}(\varphi; t, t_0)$ for the density $\varphi(t)$ by the following expression

$$\begin{aligned} \psi_{\sigma\nu}(\varphi; t, t_0) &= \varphi(t_0) + \beta_{\sigma\nu}(\varphi; t, t_0) \\ &= \varphi(t_0) + U(\varphi; t, t_0, \sigma) - V(\varphi; t, t_0, \sigma, \nu), \end{aligned} \tag{2.1}$$

where the expression $\psi_{\sigma\nu}(\varphi; t_0, t)$, designates the approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ [5], destined for the first integral of the left hand side of the equation (1).

Indeed, for $t_\sigma \le t \le t_{\sigma+1}$ we put

$$U(\varphi; t, t_0, \sigma) = \frac{(t_{\sigma+1} - t)}{(t_{\sigma+1} - t_\sigma)} \varphi(t_\sigma) \frac{t - t_0}{t_\sigma - t_0} + \frac{(t - t_\sigma)}{(t_{\sigma+1} - t_\sigma)} \varphi(t_{\sigma+1}) \frac{t - t_0}{t_{\sigma+1} - t_0},$$

and the function $V(\varphi; t_0, \sigma, \nu)$ is given by

$$V(\varphi; t, t_0, \sigma, \nu) = \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t_{\sigma+1} - t)}{(t_{\sigma+1} - t_\sigma)} + \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t - t_\sigma)}{(t_{\sigma+1} - t_\sigma)},$$

with the function $S_1(\varphi; t_0, \nu)$ represents the piecewise linear interpolating polynomial of the function density $\varphi(t_0)$ given by

$$S_1(\varphi; t, \nu) = \frac{(t_{\nu+1} - t)}{(t_{\nu+1} - t_\nu)} \varphi(t_\nu) + \frac{(t - t_\nu)}{(t_{\nu+1} - t_\nu)} \varphi(t_{\nu+1}).$$

Let $A\varphi(t_0)$ denote the left side of the equation (1)

$$\begin{aligned} A\varphi(t_0) &= (a_0(t_0)I + b_0(t_0)S_\Gamma + K_\Gamma)\varphi(t_0) \\ &= a_0(t_0)\varphi(t_0) + \frac{b_0(t_0)}{\pi i} \int_\Gamma \frac{\varphi(t)}{t - t_0} dt + \frac{1}{\pi i} \int_\Gamma k(t, t_0)\varphi(t) dt \\ &= (a_0(t_0) + b_0(t_0))\varphi(t_0) + \frac{b_0(t_0)}{\pi i} \int_\Gamma \frac{\varphi(t) - \varphi(t_0)}{t - t_0} dt + \frac{1}{\pi i} \int_\Gamma k(t, t_0)\varphi(t) dt \\ &= ((a_0(t_0) + b_0(t_0))I + b_0(t_0)S^1_\Gamma + K_\Gamma)\varphi(t_0), \end{aligned}$$

where the operator S^1_Γ is defined by

$$S^1_\Gamma \varphi(t_0) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(t) - \varphi(t_0)}{t - t_0} dt, \tag{2.2}$$

and $\widetilde{A}\widetilde{\varphi}(t_0)$ be the adapted linear interpolation formula for the operator $A\varphi(t)$ given by

$$\begin{aligned}\widetilde{A}\widetilde{\varphi}(t_0) &= (a_0(t_0)I + b_0(t_0)\widetilde{S}_\Gamma + \widetilde{K}_\Gamma)\widetilde{\varphi}(t_0)\\ &= a_0(t_0)\widetilde{\varphi}(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \frac{\psi_{\sigma v}(\varphi;t,t_0)}{t-t_0}dt + \frac{1}{\pi i}\int_\Gamma \widetilde{k}(t,t_0)\widetilde{\varphi}(t)dt\\ &= (a_0(t_0)+b_0(t_0))\widetilde{\varphi}(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \frac{\beta_{\sigma v}(\varphi;t,t_0)}{t-t_0}dt + \frac{1}{\pi i}\int_\Gamma \widetilde{k}(t,t_0)\widetilde{\varphi}(t)dt\\ &= ((a_0(t_0)+b_0(t_0))I + b_0(t_0)\widetilde{S}^1_\Gamma + \widetilde{K}_\Gamma)\widetilde{\varphi}(t_0),\end{aligned}$$

where the function $\widetilde{\varphi}(t)$ represents the approximate solution of equation (1), obtained by the equality of the functions $\widetilde{A}\widetilde{\varphi}(t_0)$ and $f(t_0)$ at the points $t_\sigma, \sigma = 0,1,..,N-1$.

3 Main results

Theorem 3.1. *he singular integral equation of the form* (1) *with the condition* (2) *has a unique solution $\varphi(t)$ and an approximate solution $\widetilde{\varphi}(t)$ converges to the solution $\varphi(t)$ with the following estimation*

$$\|\varphi(t) - \widetilde{\varphi}(t)\| \leq \frac{C_1 \ln(N)}{N^\mu} + \frac{C_2}{N^2}; \quad N > 1,$$

where the constant C_1 and C_2 depend only on the curve Γ and the Holder constant of the function φ.

We can written the integral equation (1) as

$$A\varphi(t_0) = (I + S_\Gamma + K_\Gamma)\varphi(t_0) = f(t_0),$$

while as an approximating equation n the space $H(\alpha)$ we consider

$$\widetilde{A}\widetilde{\varphi}(t_0) = (I + \widetilde{S}_\Gamma + \widetilde{K}_\Gamma)\widetilde{\varphi}(t_0) = f.$$

It follows from [5] that, for all $\varphi(t)$ in $H^\alpha(\Gamma)$ we have

$$\|S_\Gamma\varphi(t_0) - \widetilde{S}_\Gamma\widetilde{\varphi}(t_0)\| \leq \frac{C\ln(N)}{N^\mu}, \quad C > 0,$$

and also it is known that

$$\|K_\Gamma\varphi(t_0) - \widetilde{K}_\Gamma\widetilde{\varphi}(t_0)\| \leq \frac{C'}{N^2}, \quad C' > 0,$$

for all K compact and $\varphi \in H(\alpha)$.

It is easily to see that

$$\begin{aligned}\|\varphi(t_0) - \widetilde{\varphi}(t_0)\| &= \|\frac{1}{a_0(t_0)+b_0(t_0)}\|\|b_0(t_0)(S_\Gamma\varphi(t_0) - \widetilde{S}_\Gamma\widetilde{\varphi}(t_0)) + (K_\Gamma\varphi(t_0) - \widetilde{K}_\Gamma\widetilde{\varphi}(t_0))\|\\ &\leq \|\frac{b_0(t)}{a_0(t_0)+b_0(t_0)}\|\|S_\Gamma\varphi(t_0) - \widetilde{S}_\Gamma\widetilde{\varphi}(t_0)\| + \|\frac{1}{a_0(t_0)+b_0(t_0)}\|\|K_\Gamma\varphi(t_0) - \widetilde{K}_\Gamma\widetilde{\varphi}(t_0)\|\\ &\leq \frac{C_1\ln(N)}{N^\mu} + \frac{C_2}{N^2}; \quad N > 1.\end{aligned}$$

4 Numerical Experiments

In this section we describe some of the numerical experiments performed in solving the singular integral equations (1). In all cases, the curve Γ designate the unit circle and we chose the right hand side $f(t)$ in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with our method is correct.

We apply the algorithms described in [1,3,5] to solve S.I.E and we present results concerning the accuracy of the calculations; in this numerical experiments it is easily to see that the matrix of the system of algebraic equation given by our approximation is invertible, confirmed in [3,6,7].

In each table, φ represents the exact solution given in the sense of the principal value of Cauchy and $\widetilde{\varphi}$ corresponds to the approximate solution produced by the approximation at points values interpolation [3,4,5].

Consider the singular integral equation

$$\varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt = 2t_0,$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = t$$

Table 1. The values of points, the exact solution of the singular integral equation, the approximate solution and the error for N=50.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
8.0902e-001 +5.8779e-001i	8.0902e-001 +5.8779e-001i	8.0902e-001 +5.8779e-001i	1.2311e-014
-3.0902e-001 +9.5106e-001i	-3.0902e-001 +9.5106e-001i	-3.0902e-001 +9.5106e-001i	1.2152e-014
-1.0000e+000 +1.2246e-016i	-1.0000e+000 +1.2246e-016i	-1.0000e+000 +2.2760e-015i	8.7082e-015
-3.0902e-001 -9.5106e-001i	-3.0902e-001 -9.5106e-001i	-3.0902e-001 -9.5106e-001i	2.2892e-014
8.0902e-001 -5.8779e-001i	8.0902e-001 -5.8779e-001i	8.0902e-001 -5.8779e-001i	2.5337e-014

Consider the singular integral equation

$$\frac{2}{t_0}\varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt = \frac{2-t_0}{t_0^2},$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t}$$

Table 2. The values of points, the exact solution of the singular integral equation, the approximate solution and the error for N=50.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
8.0902e-001 +5.8779e-001i	8.0902e-001 -5.8779e-001i	8.0676e-001 -5.8909e-001i	2.6098e-003
-3.0902e-001 +9.5106e-001i	-3.0902e-001 -9.5106e-001i	-3.1278e-001 -9.5190e-001i	3.8573e-003
-1.0000e+000 +1.2246e-016i	-1.0000e+000 -1.2246e-016i	-1.0055e+000 -5.0190e-003i	7.4789e-003
-3.0902e-001 -9.5106e-001i	-3.0902e-001 +9.5106e-001i	-3.1026e-001 +9.4740e-001i	3.8618e-003
8.0902e-001 -5.8779e-001i	8.0902e-001 +5.8779e-001i	8.0747e-001 +5.8568e-001i	2.6087e-003

Consider the singular integral equation

$$t_0(t_0+3)\varphi(t_0) + \frac{t_0^2+2}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt = \frac{t_0(t_0+3)-(t_0^2+2)}{2t_0+1},$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{2t+1}$$

Table 3. The values of points, the exact solution of the singular integral equation, the approximate solution and the error for N=50.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
8.0902e-001 +5.8779e-001i	3.1787e-001 -1.4273e-001i	3.1957e-001 -1.4146e-001i	2.1209e-003
-3.0902e-001 +9.5106e-001i	1.0148e-001 -5.0535e-001i	9.9874e-002 -5.0612e-001i	1.7823e-003
-1.0000e+000 +1.2246e-016i	-1.0000e+000 -2.4493e-016i	-1.0023e+000 -1.3759e-002i	1.3942e-002
-3.0902e-001 -9.5106e-001i	1.0148e-001 +5.0535e-001i	1.0204e-001 +5.0625e-001i	1.0605e-003
8.0902e-001 -5.8779e-001i	3.1787e-001 +1.4273e-001i	3.1644e-001 +1.4133e-001i	2.0082e-003

Consider the singular integral equation

$$-t_0(t_0-2)\varphi(t_0) - \frac{t_0(t_0+5)}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt + \frac{1}{\pi i}\int_\Gamma \frac{t_0(t+2)}{t}\varphi(t)dt = \frac{t_0}{t_0+2},$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t+2}$$

Table 4. The values of points, the exact solution of the singular integral equation, the approximate solution and the error for N=50.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
8.0902e-001 +5.8779e-001i	3.4106e-001 -7.1367e-002i	3.4502e-001 -7.0920e-002i	3.9853e-003
-3.0902e-001 +9.5106e-001i	4.4926e-001 -2.5268e-001i	4.5626e-001 -2.5387e-001i	7.1025e-003
-1.0000e+000 +1.2246e-016i	1.0000e+000 -1.2246e-016i	1.0139e+000 -1.1282e-002i	1.7870e-002
-3.0902e-001 -9.5106e-001i	4.4926e-001 +2.5268e-001i	4.5100e-001 +2.5844e-001i	6.0213e-003
8.0902e-001 -5.8779e-001i	3.4106e-001 +7.1367e-002i	3.4398e-001 +7.3563e-002i	3.6542e-003

5 Conclusion

We have considered the numerical solution of singular integral equations and have presented an efficient scheme to compute this singular integrals. The essential idea is to find a combination of functions of approximation for the function density where we can be used it to remove integrable singularities. The regular part where it is the remaining integrands are well behaved and pose no serious numerical problem. Typical examples taken from the literature with known closed form solutions, were used to illustrate the stability and convergence of the approach. The stability of the numerical solution was verified by comparing the analytical and numerical solutions which agree well.

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Received: June 26, 2014; *Accepted*: August 10, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 502–509

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Some curvature tensors on a generalized Sasakian space form

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Abstract

In the present paper, we have studied the geometry of generalized Sasakian space form with the condition satisfying $W^*(\xi, X)W^* = 0$, $W^*(\xi, X)S = 0$, $W^*(\xi, X)P = 0$ and $P(\xi, X)P = 0$.

Keywords: Generalized Sasakian space form, M-projective curvature tensor, Projective curvature tensor.

2010 MSC: 53D10, 53D15, 53C25.

1 Introduction

A Sasakian manifold (M, ϕ, ξ, η, g) is said to be a Sasakian space form [3], if all the ϕ-sectional curvatures $K(X \wedge \phi X)$ are equal to a constant C, where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to ξ and ϕX. In such a case, the Riemannian curvature tensor of M is given by,

$$\begin{aligned} R(X,Y)Z &= \frac{C+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{C-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ \frac{C-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{1.1}$$

As a natural generalization of these manifolds, Alegre P., Blair D. E. and Carriazo A. [1, 3] introduced the notion of generalized Sasakian space form.

Sasakian space form and generalized Sasakian space form have been studied by several authors, viz., [2], [3], [5], [9], [14], [15].

De U. C. and Sarkar A. [9] studied properties of projective curvature tensor to generalized Sasakian space form. Mehmet Atceken [10] studied generalized Sasakian space form satisfying certain conditions on the concircular curvature tensor.

The properties of the M-projective curvature tensor in Sasakian and Kaehler manifolds were studied by Ojha R. H. [11, 12]. He showed that it bridges the gap between the conformal curvature tensor, coharmonic curvature tensor and concircular curvature tensor. Chaubey S. K. and Ojha R. H. [7] studied the properties of the M-projective curvature tensor in Riemannian and Kenmotsu manifolds. Chaubey S. K. [8] also studied the properties of M-projective curvature tensor in LP-Sasakian manifold. Present authors [4] have studied some properties of M-projective curvature tensor in a generalized Sasakian space form. Motivated by these ideas, in the present paper we have extended the study of further properties of M-projective curvature tensor to generalized Sasakian space form. The present paper is organized as follows:

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In section 2, we review some preliminary results. From section 3 onwards we have obtained necessary and sufficient condition for a generalized Sasakian space form satisfying the derivation conditions $W^*(\xi, X)W^* = 0$, $W^*(\xi, X)S = 0$, $W^*(\xi, X)P = 0$ and $P(\xi, X)P = 0$. We have proved that these conditions are satisfied if and only if $f_3 = \frac{3f_2}{(1-2n)}$.

2 Preliminaries

An odd-dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists on M, a $(1, 1)$ tensor field ϕ, a vector field ξ and a 1-form η [6] such that,

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$\eta(\phi X) = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$\phi\xi = 0, \quad \eta(\xi) = 0, \quad g(X, \xi) = \eta(X), \tag{2.5}$$

for any vector fields X, Y on M.

If in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well known that a Contact metric manifold is a K-contact manifold if and only if $(\nabla_X \xi) = -\phi(X)$ for any vector field X on M.

Given an almost contact metric manifold (M, ϕ, ξ, η, g) we say that M is an generalized Sasakian space form [1], if there exists three functions f_1, f_2 and f_3 on M such that

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.6}$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M. This kind of manifold appears as a natural generalization of the well-known Sasakian space form $M(C)$, which can be obtained as particular cases of generalized Sasakian space form by taking $f_1 = \frac{C+3}{4}$ and $f_2 = f_3 = \frac{C-1}{4}$. Further in a $(2n+1)$-dimensional generalized Sasakian space form, we have [1]

$$(\nabla_X \phi)(Y) = (f_1 - f_3)(g(X,Y)\xi - \eta(Y)X), \tag{2.7}$$

$$(\nabla_X \xi) = -(f_1 - f_3)\phi(X), \tag{2.8}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.9}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.10}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.11}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.12}$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.13}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.14}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X). \tag{2.15}$$

In 1971, Pokhariyal G. P. and Mishra R. S. [13] defined M-projective curvature tensor W^* on a Riemannian manifold as

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{2.16}$$

and projective curvature tensor [16] is defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{2.17}$$

3 Generalized Sasakian space form satisfying $W^*(\xi, X)W^* = 0$

Let us consider a generalized Sasakian space form satisfying

$$W^*(\xi, X)W^* = 0. \tag{3.18}$$

The above equation can be written as

$$\begin{aligned} &W^*(\xi, X)W^*(Y, Z)U - W^*(W^*(\xi, X)Y, Z)U \\ &-W^*(Y, W^*(\xi, X)Z)U - W^*(Y, Z)W^*(\xi, X)U = 0, \end{aligned} \tag{3.19}$$

for any vector field X, Y, Z, U on M.
In view of (2.5), (2.9), (2.10) and (2.13), (2.16) becomes

$$W^*(\xi, X)Y = \frac{1}{4n}[(1-2n)f_3 - 3f_2](g(X, Y)\xi - \eta(Y)X) \tag{3.20}$$

and

$$\eta(W^*(X, Y)Z) = \frac{1}{4n}[(1-2n)f_3 - 3f_2][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{3.21}$$

From (2.16) and (3.20), we find

$$\begin{aligned} W^*(\xi, X)W^*(Y, Z)U &= \frac{1}{4n}[(1-2n)f_3 - 3f_2][g(X, W^*(Y, Z)U)\xi \\ &- \frac{1}{4n}[(1-2n)f_3 - 3f_2][g(Z, U)\eta(Y)X - g(Y, U)\eta(Z)X]] \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} W^*(W^*(\xi, X)Y, Z)U &= \frac{1}{4n}[(1-2n)f_3 - 3f_2][\frac{(1-2n)f_3 - 3f_2}{4n}[g(X, Y)g(Z, U)\xi \\ &- g(X, Y)\eta(U)Z] - \eta(Y)W^*(X, Z)U]. \end{aligned} \tag{3.23}$$

Substituting $Z = \xi$ in (2.16), we get

$$W^*(X, Y)\xi = \frac{1}{4n}[(1-2n)f_3 - 3f_2](\eta(Y)X - \eta(X)Y), \tag{3.24}$$

Substituting (3.20), (3.22), (3.23) in (3.19), we get

$$\begin{aligned} &\frac{(1-2n)f_3 - 3f_2}{4n}[g(W^*(Y, Z)U, X)\xi - \frac{(1-2n)f_3 - 3f_2}{4n}[g(Z, U)\eta(Y)X \\ -\ &g(Y, U)\eta(Z)X + g(X, Y)g(Z, U)\xi - g(X, Y)\eta(U)Z + g(X, Z)\eta(U)Y \\ -\ &g(X, Z)g(U, Y)\xi + g(X, U)\eta(Z)Y - g(X, U)\eta(Y)Z] \\ +\ &\eta(Y)W^*(X, Z)U + \eta(Z)W^*(Y, X)U + \eta(U)W^*(Y, Z)X] = 0. \end{aligned} \tag{3.25}$$

Taking inner product of (3.25) with respect to ξ and using (2.16) and (3.21), we get

$$\begin{aligned} &\frac{(1-2n)f_3 - 3f_2}{4n}[g(R(Y, Z)U, X) \\ -\ &\frac{4nf_1 + 3f_2 - (1+2n)f_3}{4n}[g(X, Y)g(Z, U) - g(X, Z)g(Y, U)] \\ -\ &\frac{3f_2 + (2n-1)f_3}{4n}[g(X, Z)\eta(U)\eta(Y) + g(Y, U)\eta(Z)\eta(X) \\ -\ &g(X, Y)\eta(Z)\eta(U) - g(Z, U)\eta(X)\eta(Y)]] = 0. \end{aligned} \tag{3.26}$$

This implies either

$$f_3 = \frac{3f_2}{(1-2n)} \tag{3.27}$$

or

$$\begin{aligned} g(R(Y,Z)U,X) &= \frac{4nf_1+3f_2-(1+2n)f_3}{4n}[g(X,Y)g(Z,U) \\ &- g(X,Z)g(Y,U)] + \frac{3f_2+(2n-1)f_3}{4n}[g(X,Z)\eta(U)\eta(Y) \\ &+ g(Y,U)\eta(Z)\eta(X) - g(X,Y)\eta(Z)\eta(U) - g(Z,U)\eta(X)\eta(Y)]. \end{aligned} \tag{3.28}$$

Let $\{e_i\}, i = 1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the space form. Then putting $X = Y = e_i$, in (3.28) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned} S(Z,U) &= \frac{1}{4n}[[2n(4nf_1+3f_2-(1+2n)f_3) \\ &- (3f_2+(2n-1)f_3)]g(Z,U) \\ &- (2n-1)(3f_2+(2n-1)f_3)\eta(U)\eta(Z)]. \end{aligned} \tag{3.29}$$

Contracting the above equation we get,

$$r = \frac{1}{2}[(2n+1)(4nf_1+3f_2-(1+2n)f_3) - 2(3f_2+(2n-1)f_3)], \tag{3.30}$$

using (2.11) we get

$$f_3 = \frac{3f_2}{(1-2n)}. \tag{3.31}$$

This leads us to state the following:

Theorem 3.1. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form satisfies the condition $W^*(\xi,X)W^* = 0$ if and only if $f_3 = \frac{3f_2}{(1-2n)}$.*

4 Generalized Sasakian space form satisfying $W^*(\xi,X)S = 0$

The condition $W^*(\xi,X)S = 0$ implies that

$$S(W^*(\xi,X)Y,Z) + S(Y,W^*(\xi,X)Z) = 0. \tag{4.32}$$

Substituting (3.20) in (4.32), we obtain

$$\begin{aligned} &\frac{(1-2n)f_3-3f_2}{4n}[g(X,Y)S(\xi,Z) - \eta(Y)S(X,Z) \\ &+ S(Y,\xi)g(X,Z) - \eta(Z)S(X,Y)] = 0. \end{aligned} \tag{4.33}$$

Again substituting $Z = \xi$ in (4.33), we get

$$\frac{(1-2n)f_3-3f_2}{4n}[S(X,Y) - 2n(f_1-f_3)g(X,Y)] = 0. \tag{4.34}$$

This implies either

$$f_3 = \frac{3f_2}{(1-2n)} \tag{4.35}$$

or

$$S(X,Y) = 2n(f_1-f_3)g(X,Y). \tag{4.36}$$

On contracting (4.36), we find

$$r = 2n(2n+1)(f_1-f_3) \quad and \;\; so \quad f_3 = \frac{3f_2}{(1-2n)}. \tag{4.37}$$

Thus, we state

Theorem 4.2. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form satisfies the condition $W^*(\xi,X)S = 0$ if and only if $f_3 = \frac{3f_2}{(1-2n)}$.*

5 Generalized Sasakian space form satisfying $W^*(\xi, X)P = 0$

We know that,

$$
\begin{aligned}
(W^*(\xi,X)P)(Y,Z)U &= W^*(\xi,X)P(Y,Z)U - P(W^*(\xi,X)Y,Z)U \\
&- P(Y,W^*(\xi,X)Z)U - P(Y,Z)W^*(\xi,X)U.
\end{aligned} \tag{5.38}
$$

But as we assume $W^*(\xi, X)P = 0$, (5.38) takes the form

$$
\begin{aligned}
& W^*(\xi,X)P(Y,Z)U - P(W^*(\xi,X)Y,Z)U \\
- & P(Y,W^*(\xi,X)Z)U - P(Y,Z)W^*(\xi,X)U = 0.
\end{aligned} \tag{5.39}
$$

In view of (2.14), we obtain from (2.17) that

$$
\eta(P(X,Y)Z) = \frac{1}{2n}[(1-2n)f_3 - 3f_2][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{5.40}
$$

From (2.17) and (3.20), we find

$$
\begin{aligned}
W^*(\xi,X)P(Y,Z)U &= \frac{1}{4n}[(1-2n)f_3 - 3f_2][g(X,R(Y,Z)U)\xi \\
&- \frac{1}{2n}[S(U,Z)g(X,Y) - S(Y,U)g(X,Z)]\xi \\
&- \frac{1}{2n}[(1-2n)f_3 - 3f_2][g(Z,U)\eta(Y)X - g(Y,U)\eta(Z)X]]
\end{aligned} \tag{5.41}
$$

and

$$
\begin{aligned}
P(W^*(\xi,X)Y,Z)U &= \frac{1}{4n}[(1-2n)f_3 - 3f_2][(f_1 - f_3)g(X,Y)g(Z,U)\xi \\
&- \frac{1}{2n}S(U,Z)g(X,Y)\xi - \eta(Y)P(X,Z)U].
\end{aligned} \tag{5.42}
$$

Also

$$
P(Y,Z)W^*(\xi,X)U = -\frac{1}{4n}[(1-2n)f_3 - 3f_2]\eta(U)P(Y,Z)X]. \tag{5.43}
$$

Substituting (5.41), (5.42) and (5.43) in (5.39), we get

$$
\begin{aligned}
& \frac{(1-2n)f_3 - 3f_2}{4n}[g(R(Y,Z)U,X)\xi \\
- & \frac{1}{2n}[S(U,Z)g(X,Y) - S(Y,U)g(X,Z)]\xi \\
- & \frac{1}{2n}[(1-2n)f_3 - 3f_2][g(Z,U)\eta(Y)X - g(Y,U)\eta(Z)X] \\
- & (f_1 - f_3)g(X,Y)g(Z,U)\xi + \frac{1}{2n}S(U,Z)g(X,Y)\xi \\
+ & (f_1 - f_3)g(X,Z)g(Y,U)\xi - \frac{1}{2n}S(Y,U)g(X,Z)\xi \\
+ & \eta(Y)P(X,Z)U + \eta(Z)P(Y,X)U + \eta(U)P(Y,Z)X] = 0
\end{aligned} \tag{5.44}
$$

Taking inner product of (5.44) with respect to the Riemannian metric g and then using (2.5) and (5.40), we have

$$
\begin{aligned}
& \frac{1}{4n}[(1-2n)f_3 - 3f_2][g(R(Y,Z)U,X) \\
- & (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\} \\
+ & \frac{1}{2n}[(1-2n)f_3 - 3f_2][g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(Z)\eta(U)]] = 0.
\end{aligned} \tag{5.45}
$$

This implies either

$$
f_3 = \frac{3f_2}{(1-2n)} \tag{5.46}
$$

or

$$g(R(Y,Z)U,X) = (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\} - \frac{1}{2n}[(1-2n)f_3 - 3f_2][g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(Z)\eta(U)]. \tag{5.47}$$

Let $\{e_i\}, i = 1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the space form. Then putting $X = Y = e_i$, in (5.47) and taking summation over i, $1 \le i \le 2n+1$, we get

$$S(Z,U) = 2n(f_1 - f_3)g(Z,U) + [(1-2n)f_3 - 3f_2]\eta(Z)\eta(U). \tag{5.48}$$

Contracting (5.48), we find

$$r = 2n(2n+1)(f_1 - f_3) + (1-2n)f_3 - 3f_2. \tag{5.49}$$

Using (2.11), the above equation gives

$$f_3 = \frac{3f_2}{(1-2n)} \tag{5.50}$$

Thus, we state

Theorem 5.3. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form satisfies the condition $W^*(\xi, X)P = 0$ if and only if $f_3 = \frac{3f_2}{(1-2n)}$.*

6 An generalized Sasakian space form satisfying $P(\xi, X)P = 0$

The condition $P(\xi, X)P = 0$ implies that

$$(P(\xi,X)P)(Y,Z)U = P(\xi,X)P(Y,Z)U - P(P(\xi,X)Y,Z)U - P(Y,P(\xi,X)Z)U - P(Y,Z)P(\xi,X)U = 0. \tag{6.51}$$

In view of (2.5), (2.10) and (2.13), (2.17) becomes

$$P(\xi,X)Y = (f_1 - f_3)g(X,Y)\xi - \frac{1}{2n}S(X,Y)\xi \tag{6.52}$$

Using (6.52)in (6.51), we get

$$\begin{aligned}
&(f_1 - f_3)g(P(Y,Z)U,X)\xi - \frac{1}{2n}S(P(Y,Z)U,X)\xi \\
-\ &[(f_1 - f_3)g(X,Y) - \frac{1}{2n}S(X,Y)][(f_1 - f_3)g(Z,U) - \frac{1}{2n}S(Z,U)]\xi \\
-\ &[(f_1 - f_3)g(X,Z) - \frac{1}{2n}S(X,Z)][\frac{1}{2n}S(Y,U) - (f_1 - f_3)g(Y,U)]\xi \\
-\ &[(f_1 - f_3)g(X,U) - \frac{1}{2n}S(X,U)]P(Y,Z)\xi = 0,
\end{aligned} \tag{6.53}$$

Taking inner product of (6.53) with respect to the Riemannian metric g and then using (2.10), (2.17) and (5.40), we have

$$\frac{1}{2n}[(1-2n)f_3 - 3f_2][g(R(Y,Z)U,X) - (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}] = 0. \tag{6.54}$$

$\Rightarrow$ $f_3 = \frac{3f_2}{(1-2n)}$ or

$$g(R(Y,Z)U,X) = (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}. \tag{6.55}$$

(6.55) implies

$$R(Y,Z)U = (f_1 - f_3)\{g(Z,U)Y - g(Y,U)Z\}. \tag{6.56}$$

Contracting (6.56) with respect to the vector field Y, we find

$$S(Z,U) = 2n(f_1 - f_3)g(Z,U). \tag{6.57}$$

On contracting the above equation, we get

$$r = 2n(2n+1)(f_1 - f_3) \quad and \ so \quad f_3 = \frac{3f_2}{(1-2n)}. \tag{6.58}$$

Thus, we state

Theorem 6.4. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form satisfies the condition $P(\xi, X)P = 0$ if and only if $f_3 = \frac{3f_2}{(1-2n)}$.*

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Received: July 09, 2014; *Accepted*: August 19, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 510–516

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Some new general integral inequalities for P-functions

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Abstract

In this paper, we derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are P-functions. Some applications to special means of real numbers are also given.

Keywords: Convex function, P-function, Simpson's inequality, Hermite-Hadamard's inequality.

2010 MSC: 26A33, 26D10, 26D15,41A55.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This double inequality is well known as Hermite-Hadamard integral inequality for convex functions in the literature .

In [2] Dragomir et al. defined the concept of P-function as the following:

Definition 1.1. *We say that $f : I \to \mathbb{R}$ is a P-function, or that f belongs to the class $P(I)$, if f is a non-negative function and for all $x, y \in I$, $\alpha \in [0,1]$, we have*

$$f(\alpha x + (1-\alpha)y) \leq f(x) + f(y).$$

$P(I)$ contain all nonnegative monotone convex and quasi convex functions.

In [2], Dragomir et al., proved following inequalities of Hadamard's type for P-function

Theorem 1.1. *Let $f \in P(I), a, b \in I$ with $a < b$ and $f \in L[a,b]$. Then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2\left[f(a)+f(b)\right]. \tag{1.2}$$

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The following inequality is well known in the literature as Simpson's inequality .

Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

In recent years many authors have studied error estimations for Simpson's inequality and Hermite-Hadamard inequalitiy; for refinements, counterparts, generalizations, see ([1]-[10]).

In [3], Iscan obtained a new generalization of some integral inequalities for differentiable convex mapping which are connected Simpson and Hadamard type inequalities by using the following lemma.

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$ and $\alpha,\lambda\in[0,1]$. Then the following equality holds:

$$\begin{aligned}&\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx\\=&(b-a)\left[\int_0^{1-\alpha}(t-\alpha\lambda)f'(tb+(1-t)a)\,dt\right.\\&\left.+\int_{1-\alpha}^1(t-1+\lambda(1-\alpha))f'(tb+(1-t)a)\,dt\right].\end{aligned}$$

The aim of this paper is to establish some new general integral inequalities for functions whose derivatives in absolute value at certain power are P-functions. Some applications of these results to special means is to give as well.

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, the interior of I. Throughout this section we will take

$$\begin{aligned}&I_f(\lambda,\alpha,a,b)\\=&\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx\end{aligned}$$

where $a,b\in I^\circ$ with $a<b$ and $\alpha,\lambda\in[0,1]$.

Theorem 1.2. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$ and $\alpha,\lambda\in[0,1]$. If $|f'|^q$ is P-function on $[a,b]$, $q\geq1$, then the following inequality holds:*

$$\begin{aligned}&\left|I_f(\lambda,\alpha,a,b)\right|\leq(b-a)\left(\left|f'(b)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}\\&\times\left\{\begin{array}{ll}\gamma_2(\alpha,\lambda)+\gamma_2(1-\alpha,\lambda)&\alpha\lambda\leq1-\alpha\leq1-\lambda(1-\alpha)\\\gamma_2(\alpha,\lambda)+\gamma_1(1-\alpha,\lambda)&\alpha\lambda\leq1-\lambda(1-\alpha)\leq1-\alpha\\\gamma_1(\alpha,\lambda)+\gamma_2(1-\alpha,\lambda)&1-\alpha\leq\alpha\lambda\leq1-\lambda(1-\alpha)\end{array}\right.,\end{aligned}\tag{1.3}$$

where

$$\begin{aligned}\gamma_1(\alpha,\lambda)&=(1-\alpha)\left[\alpha\lambda-\frac{(1-\alpha)}{2}\right],\\\gamma_2(\alpha,\lambda)&=(\alpha\lambda)^2-\gamma_1(\alpha,\lambda).\end{aligned}\tag{1.4}$$

Proof. Suppose that $q\geq1$.Since $|f'|^q$ is P-function on $[a,b]$, from Lemma 1 and using the well known power mean inequality, we have

$$\begin{aligned}&\left|I_f(\lambda,\alpha,a,b)\right|\\\leq&(b-a)\left[\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|dt+\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|dt\right]\end{aligned}$$

$$\leq (b-a)\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|\,dt\right)^{1-\frac{1}{q}}\left(\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq (b-a)\left(\left|f'(b)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}\left\{\int_0^{1-\alpha}|t-\alpha\lambda|\,dt+\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|\,dt\right\} \tag{1.5}$$

Additionally, by simple computation

$$\int_0^{1-\alpha}|t-\alpha\lambda|\,dt=\begin{cases}\gamma_2(\alpha,\lambda), & \alpha\lambda\leq 1-\alpha\\ \gamma_1(\alpha,\lambda), & \alpha\lambda\geq 1-\alpha\end{cases}, \tag{1.6}$$

$$\gamma_1(\alpha,\lambda)=(1-\alpha)\left[\alpha\lambda-\frac{(1-\alpha)}{2}\right],\ \gamma_2(\alpha,\lambda)=(\alpha\lambda)^2-\gamma_1(\alpha,\lambda),$$

$$\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|\,dt=\int_0^{\alpha}|t-(1-\alpha)\lambda|\,dt \tag{1.7}$$

$$=\begin{cases}\gamma_1(1-\alpha,\lambda), & 1-\lambda(1-\alpha)\leq 1-\alpha\\ \gamma_2(1-\alpha,\lambda), & 1-\lambda(1-\alpha)\geq 1-\alpha\end{cases},$$

Thus, using (1.6) and (1.7) in (1.5), we obtain the inequality (1.3). This completes the proof. □

Corollary 1.1. *Under the assumptions of Theorem 1.2 with $q=1$, we have*

$$\left|I_f(\lambda,\alpha,a,b)\right|\leq(b-a)\left(\left|f'(b)\right|+\left|f'(a)\right|\right)$$
$$\times\begin{cases}\gamma_2(\alpha,\lambda)+\gamma_2(1-\alpha,\lambda) & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ \gamma_2(\alpha,\lambda)+\gamma_1(1-\alpha,\lambda) & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha\\ \gamma_1(\alpha,\lambda)+\gamma_2(1-\alpha,\lambda) & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases},$$

Corollary 1.2. *In Theorem 1.2, if we take $\alpha=\frac{1}{2}$ and $\lambda=\frac{1}{3}$, then we have the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{5(b-a)}{36}\left(\left|f'(b)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}$$

Corollary 1.3. *In Theorem 1.2, if we take $\alpha=\frac{1}{2}$ and $\lambda=0$, then we have following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\left|f'(b)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}$$

Corollary 1.4. *In Theorem 1.2, if we take $\alpha=\frac{1}{2}$, and $\lambda=1$, then we get the following trapezoid inequality which is identical to the inequality in [1, Theorem 2.3].*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\left|f'(b)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}$$

Using Lemma 1 we shall give another result for convex functions as follows.

Theorem 1.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is P-function on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\left|I_f(\lambda,\alpha,a,b)\right| \le (b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{1.8}$$

$$\times \begin{cases} \left[\varepsilon_1^{1/p}(\alpha,\lambda,p)c_f^{1/q}(\alpha,q)+\varepsilon_1^{1/p}(1-\alpha,\lambda,p)k_f^{1/q}(\alpha,q)\right], & \alpha\lambda \le 1-\alpha \le 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{1/p}(\alpha,\lambda,p)c_f^{1/q}(\alpha,q)+\varepsilon_2^{1/p}(1-\alpha,\lambda,p)k_f^{1/q}(\alpha,q)\right], & \alpha\lambda \le 1-\lambda(1-\alpha) \le 1-\alpha \\ \left[\varepsilon_2^{1/p}(\alpha,\lambda,p)c_f^{1/q}(\alpha,q)+\varepsilon_1^{1/p}(1-\alpha,\lambda,p)k_f^{1/q}(\alpha,q)\right], & 1-\alpha \le \alpha\lambda \le 1-\lambda(1-\alpha) \end{cases},$$

where

$$\begin{aligned} c_f(\alpha,q) &= (1-\alpha)\left[\left|f'((1-\alpha)b+\alpha a)\right|^q+\left|f'(a)\right|^q\right], \\ k_f(\alpha,q) &= \alpha\left[\left|f'((1-\alpha)b+\alpha a)\right|^q+\left|f'(b)\right|^q\right], \end{aligned} \tag{1.9}$$

$$\varepsilon_1(\alpha,\lambda,p) = (\alpha\lambda)^{p+1}+(1-\alpha-\alpha\lambda)^{p+1},\ \varepsilon_2(\alpha,\lambda,p) = (\alpha\lambda)^{p+1}-(\alpha\lambda-1+\alpha)^{p+1},$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f'|^q$ is P-function on $[a,b]$, from Lemma 1 and by Hölder's integral inequality, we have

$$\begin{aligned} &\left|I_f(\lambda,\alpha,a,b)\right| \\ \le\ & (b-a)\left[\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|dt+\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|dt\right] \\ \le\ & (b-a)\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|^p dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{1.10}$$

By the inequality (1.2), we get

$$\begin{aligned} \int_0^{1-\alpha}\left|f'(tb+(1-t)a)\right|^q dt &= (1-\alpha)\left[\frac{1}{(1-\alpha)(b-a)}\int_a^{(1-\alpha)b+\alpha a}\left|f'(x)\right|^q dx\right] \\ &\le (1-\alpha)\left[\left|f'((1-\alpha)b+\alpha a)\right|^q+\left|f'(a)\right|^q\right]. \end{aligned} \tag{1.11}$$

The inequality (1.11) also holds for $\alpha = 1$. Similarly, for $\alpha \in (0,1]$ by the inequality (1.2), we have

$$\begin{aligned} \int_{1-\alpha}^1\left|f'(tb+(1-t)a)\right|^q dt &= \alpha\left[\frac{1}{\alpha(b-a)}\int_{(1-\alpha)b+\alpha a}^b\left|f'(x)\right|^q dx\right] \\ &\le \alpha\left[\left|f'((1-\alpha)b+\alpha a)\right|^q+\left|f'(b)\right|^q\right]. \end{aligned} \tag{1.12}$$

The inequality (1.12) also holds for $\alpha = 0$. By simple computation

$$\int_0^{1-\alpha}|t-\alpha\lambda|^p dt = \begin{cases} \frac{(\alpha\lambda)^{p+1}+(1-\alpha-\alpha\lambda)^{p+1}}{p+1}, & \alpha\lambda \le 1-\alpha \\ \frac{(\alpha\lambda)^{p+1}-(\alpha\lambda-1+\alpha)^{p+1}}{p+1}, & \alpha\lambda \ge 1-\alpha \end{cases}, \tag{1.13}$$

and

$$\int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)|^{p} dt = \begin{cases} \frac{[\lambda(1-\alpha)]^{p+1}+[\alpha-\lambda(1-\alpha)]^{p+1}}{p+1}, & 1-\alpha \leq 1-\lambda(1-\alpha) \\ \frac{[\lambda(1-\alpha)]^{p+1}-[\lambda(1-\alpha)-\alpha]^{p+1}}{p+1}, & 1-\alpha \geq 1-\lambda(1-\alpha) \end{cases}, \tag{1.14}$$

thus, using (1.11)-(1.14) in (1.10), we obtain the inequality (1.8). This completes the proof. □

Corollary 1.5. *In Theorem 1.3, if we take $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, then we have the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right\}.$$

Corollary 1.6. *In Theorem 1.3, if we take $\alpha = \frac{1}{2}$ and $\lambda = 0$, then we have the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right\}.$$

We note that by inequality

$$\left|f'\left(\frac{a+b}{2}\right)\right|^q \leq |f'(a)|^q+|f'(b)|^q$$

we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(|f'(b)|^q+2|f'(a)|^q\right)^{\frac{1}{q}}+\left(|f'(a)|^q+2|f'(b)|^q\right)^{\frac{1}{q}}\right\}.$$

Corollary 1.7. *In Theorem 1.3, if we take $\alpha = \frac{1}{2}$ and $\lambda = 1$, then we have the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right\}.$$

2 Some applications for special means

We now recall the following well-known concepts. For arbitrary real numbers a, b, $a \neq b$, we define

1. The unweighted arithmetic mean

$$A(a,b) := \frac{a+b}{2},\ a,b \in \mathbb{R}.$$

2. Then $n-$Logarithmic mean

$$L_n(a,b) := \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}},\ n \in \mathbb{N},\ n \geq 1,\ a,b \in \mathbb{R},\ a < b.$$

Now we give some applications of the new results derived in section 2 to special means of real numbers.

Proposition 2.1. *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $n \in \mathbb{N}, n \geq 2$. *Then*

$$\left|\frac{1}{3}A(a^n, b^n) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq \frac{5n(b-a)}{36}\left(|b|^{(n-1)q} + |a|^{(n-1)q}\right)^{\frac{1}{q}}$$

Proof. The assertion follows from Corollary 1.2 applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, because $|f'|^q$ is a P-function. □

Proposition 2.2. *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $n \in \mathbb{N}, n \geq 2$. *Then*

$$|A^n(a,b) - L_n^n(a,b)| \leq \frac{n(b-a)}{4}\left(|b|^{(n-1)q} + |a|^{(n-1)q}\right)^{\frac{1}{q}}$$

and

$$|A(a^n, b^n) - L_n^n(a,b)| \leq \frac{n(b-a)}{4}\left(|b|^{(n-1)q} + |a|^{(n-1)q}\right)^{\frac{1}{q}}$$

Proof. The assertion follows from Corollary 1.3 and Corollary 1.4 applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, because $|f'|^q$ is a P-function. □

Proposition 2.3. *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $n \in \mathbb{N}, n \geq 2$. *Then*

$$\left|\frac{1}{3}A(a^n, b^n) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq \frac{n(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(|A(a,b)|^{(n-1)q} + |a|^{(n-1)q}\right)^{\frac{1}{q}} + \left(|A(a,b)|^{(n-1)q} + |b|^{(n-1)q}\right)^{\frac{1}{q}}\right\}.$$

Proof. The assertion follows from Corollary 1.5 applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, because $|f'|^q$ is a P-function. □

Proposition 2.4. *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $n \in \mathbb{N}, n \geq 2$. *Then*

$$|A^n(a,b) - L_n^n(a,b)| \leq \frac{n(b-a)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(|A(a,b)|^{(n-1)q} + |a|^{(n-1)q}\right)^{\frac{1}{q}} + \left(|A(a,b)|^{(n-1)q} + |b|^{(n-1)q}\right)^{\frac{1}{q}}\right\}.$$

and

$$|A(a^n, b^n) - L_n^n(a,b)| \leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(|A(a,b)|^{(n-1)q} + |a|^{(n-1)q}\right)^{\frac{1}{q}} + \left(|A(a,b)|^{(n-1)q} + |b|^{(n-1)q}\right)^{\frac{1}{q}}\right\}.$$

Proof. The assertion follows from Corollary 1.6 and Corollary 1.7 applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, because $|f'|^q$ is a P-function. □

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Received: April 25, 2014; *Accepted*: Julu 31, 2014

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Website: http://www.malayajournal.org/

Malaya J. Mat. 2(4)(2014) 517-526

Malaya Journal of Matematik
www.malayajournal.org

MJM
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ISSN : 2319-3786

Comparison of four different obstacle models of fluid flow with a slip-like boundary condition

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Abstract

In this paper, we investigate a time-discretized 2-dimensional Navier-Stokes equation with a slip-like boundary condition, which arises in the melting ice problem with obstacle. We study the existence and uniqueness of a approximate solution. We also study the numerical solution of melting ice problem using Continuous Galerkin method.

Keywords: Navier-Stokes equation; obstacle modeling; slip-like boundary; Continuous Galerkin finite element method.

2010 MSC: 65L60; 34K28, 37D35.

1 Introduction

The incompressible Navier-Stokes system is one of the main equations studied in mathematical physics and fluid mechanics fields and there is a huge literature written on the subject. For example, we quote [1] for finite difference methods, [2, 3, 4] for finite element methods, and [5] for finite volume methods. Computational fluid dynamics models are in general based on the solution of the Navier-Stokes equations and its discretization scheme, for instance, finite element methods and finite volume methods. To accurately capture the physical properties of the fluid flow being simulated, we usually need highly refined meshes on the entire flow domain which can cause a large scale computation possibly beyond the capability of a single computer. Therefore, to utilize the computational power of modern high-performance computers, much effort is thrown into the development of efficient computing methods for the Navier-Stokes equations.

Let Ω be an open and bounded domain in R^2 with Lipschitz continuous boundary Γ. Throughout the paper we will use the standard notation for Sobolev spaces $W^{m,p}(\Omega)$ with norm $||.||_{m,p,\Omega}$ (see[6]). Specially $H^m(\Omega) = W^{m,2}(\Omega)$, where m is an integer greater than zero, will denotes the Sobolev space of real-valued functions with square integrable derivatives of order up to m equipped with the usual norm which we denote $||.||_m$. We will denote $H^0(\Omega)$ by $L^2(\Omega)$ and the standard L^2 inner product by $(.,.)$. Also $H^m(\Omega)$ will denote the space of vector-valued functions each of whose n components belong to $H^m(\Omega)$, and the dual space of $H^m(\Omega)$ will be denoted by $H^{-m}(\Omega)$ of particular interest to us will be the constrained space

$$L_0^2(\Omega) = \{\xi \in L^2(\Omega), \quad \int_\Omega \xi d\Omega = 0\}$$

and

$$H_0^1(\Omega) = \{v \in H^1(\Omega), \quad v|_\Gamma = 0\}.$$

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In [8], the authors studied only the existence and uniqueness of weak solution of Navier-Stokes equation with slip-like boundary condition. In this paper we study the existence and uniqueness of approximate solution and numerical solution of melting ice problem using Continuous Galerkin finite element method.

2 Problem Formulation

Consider an ice plate, placed upright, whose vertical face is exposed to the air and melting. So this face is covered by the layer of flowing water, and the shapes of the ice and the water-layer vary as time t goes on. Therefore, in the water region, this system can be described by Navier-Stokes equations with two free boundaries of the ice-water interface Γ_1 and the water-air interface Γ_2, whose movements would depend on the unknown functions. However, as a first step of analysis, we here consider the discretized Navier-Stokes equation in the time variable t with the discretization parameters $\tau > 0$ in the fixed domain Ω with given interfaces Γ_1 and Γ_2. Experiments for this kind of problems can be found in [14] and mathematical treatments for problems similar to ours are discussed by several authors see [11, 12].

Fix the x-axis vertically and downward, the y-axis in the direction of the thickness and outward, and the z-axis orthogonally to the x and y axes. The ice-water interface and the water-air interface are represented by $y = l(x, z)$ and $y = d(x, z)$ respectively. Further suppose that the size of ice plate in z-direction is so large that we can regard l and d as constant in z. So our problem can be formulated in the following 2-dimensional setting.

Define the domain Ω which is occupied by water by

$$\Omega = \{(x, y) : 0 < x < 1,\ \ l(x) < y < d(x)\},$$

where $l, d \in C^{0,1}([0, 1])$; that is, l and d are Lipschitsz continuous on $[0, 1]$ and

$$0 \le l(x) < d(x) \le 1 \quad \text{for all } \ 0 \le x \le 1.$$

Hence Ω is of class $C^{0,1}([0, 1])$. Define the ice-water interface Γ_1, the water-air interface Γ_2, the lower boundary Γ_3, and the upper boundary Γ_4 by

$$\begin{aligned}
\Gamma_1 &= \{(x, y) : 0 \le x \le 1,\ \ y = l(x)\},\\
\Gamma_2 &= \{(x, y) : 0 \le x \le 1,\ \ y = d(x)\},\\
\Gamma_3 &= \{(x, y) : x = 1,\ \ l(1) \le y \le d(1)\},\\
\Gamma_4 &= \{(x, y) : x = 0,\ \ l(0) \le y \le d(0)\},
\end{aligned}$$

by respectively, We consider the following two-dimensional Navier-Stokes equations with Slip-like boundary condition

$$\begin{gathered}
\frac{1}{\tau}(\mathbf{u} - \mathbf{u}_0) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p - \nu\Delta\mathbf{u} = \mathbf{g} \quad \text{in } \Omega,\\
\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega,
\end{gathered} \tag{1}$$

for the fixed discretization parameter $\tau > 0$ with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{2}$$

$$U_Y = V_Y = 0 \quad \text{on } \Gamma_2, \tag{3}$$

$$v = 0 \quad \text{on } \Gamma_3, \tag{4}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_4. \tag{5}$$

Here, the velocity vector $\mathbf{u} = (u, v)$ and the pressure p are unknown functions of (x, y). The initial velocity $\mathbf{u}_0$, the gravity force $\mathbf{g}$, the density ρ, and the kinematic viscosity ν are given data. The unit time τ is to be determined later. Put $U = \mathbf{u} \cdot \mathbf{t}$, $V = \mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ designates the outer unit normal vector of Γ_2 and $\mathbf{t}$ designates the downward unit tangential vector of Γ_2. Denote by (X, Y) the local coordinate with directions $\mathbf{t}$ and $\mathbf{n}$. The original slip boundary condition is stated as

$$U_Y + V_X = 0 \quad \text{on } \Gamma_2,$$

(see [13]) and condition (3) is its linearized version. In the original problem, both Γ_1 and Γ_2 move after the unit time τ. But in our setting, the interfaces stay invariant.

To approximate the solutions of the governing equations derived in section 2 we use the Continuous Galerkin finite element method, which is also know as Ritz-Galerkin method. In this method we formulate a weak formulation of the 2-dimensional Navier-Stokes equation with a slip-like boundary condition that we observe. Discretizing the equations offers the possibility to obtain the approximated solution numerically.

First we discretize the Navier-Stokes equation with a slip-like boundary condition by using the arbitrariness of the variational derivatives with respect to each variable. Note that they are elements of the test function-space $C_0^\infty(D_h)$ on the domain, which can be restricted to the test functions on each element K_i with the test function space $C_0^\infty(K+)$. Hereby then we formulate the finite element weak formulations.

3 The Variational Formulation

The variational formulation for problem (1) is written as

$$\begin{aligned}(\frac{1}{\tau}(\mathbf{u}-\mathbf{u_0})-\mathbf{g},\delta\mathbf{u})+\nu a(\mathbf{u},\delta\mathbf{u})+a_1(\mathbf{u},\mathbf{u},\delta\mathbf{u})-b(p,\delta\mathbf{u}) &= 0 \quad \forall\delta\mathbf{u}\in\mathbf{V} \\ b(q,\delta\mathbf{u}) &= 0 \qquad \forall q\in\mathbf{H}\end{aligned} \tag{6}$$

where

$$\mathbf{V}=\{\mathbf{u}\in\left(H^1(\Omega)\right)^2:\operatorname{div}\mathbf{u}=0,\ \mathbf{u}=\mathbf{0}\text{ on }\Gamma_1\text{ and }\Gamma_4, v=0\text{ on }\Gamma_3\},$$
$$\mathbf{H}=\{\mathbf{u}\in(L_2(\Omega))^2:\operatorname{div}\mathbf{u}=0\},$$
$$P_\sigma\text{is the orthogonal projection from }(L_2(\Omega))^2\text{ onto }\mathbf{H},$$
$$\mathbf{L}_4=\{\mathbf{u}\in(L_4(\Omega))^2:\operatorname{div}\mathbf{u}=0\}$$

and let $(\cdot,\cdot)$ and $|\cdot|$ denote the inner product and the norm of the space $\mathbf{H}$.

Define a bounded positive bilinear form $a(\cdot,\cdot)$ on $\mathbf{V}$ by

$$a(\mathbf{u},\delta\mathbf{u}):=\int_\Omega(\nabla u\cdot\nabla\,\delta u+\nabla\,v\cdot\nabla\,\delta v)\,d\mathbf{x}$$

for $\mathbf{u}=(u,v)$, $\delta\mathbf{u}=(\delta u,\delta v)\in\mathbf{V}$. Also define a trilinear form $b(\cdot,\cdot,\cdot)$ on $(\mathbf{L}_4)^2\times\mathbf{V}$ by

$$a_1(\mathbf{w},\mathbf{u},\delta\mathbf{u}):=\int_\Omega(w_1u_x\delta u+w_1v_x\delta v+w_2u_y\delta u+w_2v_y\delta v)\,d\mathbf{x}$$

for $\mathbf{w}=(w_1,w_2)\in\mathbf{L}_4$, $\mathbf{u}=(u,v)\in\mathbf{V}$, $\delta\mathbf{u}=(\delta u,\delta v)\in\mathbf{L}_4$, where $\mathbf{x}=(x,y)$.

The above trilinear form $a_1(.;.,.)$ satisfies the following properties [2, 3]

$$a_1(\mathbf{u};\delta\mathbf{u},\delta\mathbf{u})=0,\qquad a_1(\mathbf{u};\delta\mathbf{u},\mathbf{w})=-a_1(\mathbf{u};\mathbf{w},\delta\mathbf{u}),\forall\mathbf{u},\delta\mathbf{u},\mathbf{w}\in\mathbf{V}$$

$$|a_1(\mathbf{u};\delta\mathbf{u},\mathbf{w})|\le N||\nabla\mathbf{u}||_0||\nabla\delta\mathbf{u}||_0||\nabla\mathbf{w}||_0,\qquad\forall\mathbf{u},\delta\mathbf{u},\mathbf{w}\in\mathbf{V}$$

where

$$N=\sup_{\mathbf{u},\delta\mathbf{u},\mathbf{w}\in\mathbf{V}}\frac{|a_1(\mathbf{u};\delta\mathbf{u},\mathbf{w})|}{||\nabla\mathbf{u}||_0||\nabla\delta\mathbf{u}||_0||\nabla\mathbf{w}||_0}$$

is a positive constant depending only on the domain Ω.

We note that Hölder's inequality gives

$$|a_1(\mathbf{w},\mathbf{u},\delta\mathbf{u})|\le|\mathbf{w}|_4|\nabla\mathbf{u}||\delta\mathbf{u}|_4\quad\text{for }\tilde{\mathbf{u}}\in\mathbf{L}_4,\ \mathbf{u}\in\mathbf{V},\ \delta\mathbf{u}\in\mathbf{L}_4. \tag{7}$$

Here $|\cdot|_4$ denotes the norm of $\mathbf{L}_4$ and

$$|\nabla\mathbf{u}|=|(|\nabla u|,|\nabla v|)|,\quad|\nabla\mathbf{u}|=\left(\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial u}{\partial y}\right|^2\right)^{1/2}.$$

Let $\mathbf{Z} = \{\delta\mathbf{u} \in \mathbf{V}, b(q,\delta\mathbf{u}) = 0 \forall q \in \mathbf{H}\} = \{\delta\mathbf{u} \in \mathbf{H}, div\delta\mathbf{u} = 0\}$ denote the divergence-free subspace of $\mathbf{V}$. Then $\mathbf{u} \in \mathbf{V}$ is said to be a week solution of (1) with boundary conditions (2)-(5) if the following relations holds

$$\Big(\frac{1}{\tau}(\mathbf{u}-\mathbf{u}_0)-\mathbf{g},\delta\mathbf{u}\Big)+\nu a(\mathbf{u},\delta\mathbf{u})+a_1(\mathbf{u},\mathbf{u},\delta\mathbf{u})=0 \quad \text{for all } \delta\mathbf{u}\in\mathbf{Z}. \tag{8}$$

We remark that if a sufficiently smooth function $\mathbf{u}$, say in $(C^2(\bar{\Omega}))^2 \cap \mathbf{V}$, satisfies (8), then $\mathbf{u}$ should satisfy equation (1) and boundary condition (3) on Γ_2. In fact, let $\mathbf{u} \in (C^2(\bar{\Omega}))^2 \cap \mathbf{V}$ and let $\mathbf{f} = -\frac{1}{\nu}\left(\frac{1}{\tau}(\mathbf{u}-\mathbf{u}_0) - \mathbf{g} + P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u}\right) \in \mathbf{H}$, then (8) gives

$$a(\mathbf{u},\delta\mathbf{u}) = (\mathbf{f},\delta\mathbf{u}) \quad \text{for all } \delta\mathbf{u}\in\mathbf{V}.$$

Here we note that since $v \equiv 0$, $\operatorname{div}\mathbf{u} = u_x + v_y \equiv 0$ on Γ_3, $v_y \equiv 0$ and hence $u_x \equiv 0$ on Γ_3. Consequently, integration by parts yields

$$\begin{aligned} a(\mathbf{u},\delta\mathbf{u}) &= (\mathbf{f},\delta\mathbf{u}) \\ &= \int_\Omega(-\Delta u\delta u - \Delta v\delta v)dx + \int_{\Gamma_2}(u_Y\delta u + v_Y\delta v)dS + \int_{\Gamma_3} u_x\delta u dS \\ &= \int_\Omega(-\Delta u\delta u - \Delta v\delta v)dx + \int_{\Gamma_2}(U_Y\delta U + V_Y\delta V)dS, \end{aligned} \tag{9}$$

where δU and δV are $\mathbf{t}$ and $\mathbf{n}$ components of $\delta\mathbf{u}$ on Γ_2.

If we take $\delta\mathbf{u} \in (C_0^\infty(\Omega))^2 \cap \mathbf{V}$, then the term of the integration on Γ_2 in (9) vanishes, whence follows

$$(\mathbf{f},\delta\mathbf{u}) = (-\Delta\mathbf{u},\delta\mathbf{u}) \quad \text{for all } \delta\mathbf{u}\in(C_0^\infty(\Omega))^2\cap\mathbf{V}.$$

This says that $\mathbf{f} = -P_\sigma\Delta\mathbf{u}$ in the sense of distribution. Hence $\mathbf{f} = -P_\sigma\Delta\mathbf{u}$ holds a.e. in Ω, which implies that $\mathbf{u}$ gives a solution of (1). Furthermore, plugging this relation into (9), we get

$$\int_{\Gamma_2}(U_Y\delta U + V_Y\delta V)dS = 0 \quad \text{for any } \delta\mathbf{u}\in\mathbf{V},$$

whence easily follows that $\mathbf{u}$ should satisfy (3).

From [8], we have the following result.

Theorem 3.1. *Let $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{g} \in \mathbf{H}$. There exists a positive number $\tau_0 = \tau_0(|\mathbf{g}|, |\nabla\mathbf{u}_0|)$ such that for all $\tau \in (0,\tau_0]$, (8) admits a unique weak solution $\mathbf{u} \in \mathbf{V}$.*

4 Existence and Uniqueness of the Approximated Solution

We introduce the bilinear forms $a^h(.,.)$, $b^h(.,.)$ and trilinear form $a_1^h(.;.,.)$ as follows

$$a^h(\mathbf{u_h},\delta\mathbf{u_h}) := \sum_{K\in T_h}\int_K(\nabla u_h\cdot\nabla\delta u_h + \nabla v_h\cdot\nabla\delta v_h)\,d\mathbf{x}$$

for $\mathbf{u_h} = (u_h, v_h)$, $\delta\mathbf{u_h} = (\delta u_h, \delta v_h) \in \mathbf{V_h}$.

$$a_1^h(\mathbf{w_h},\mathbf{u_h},\delta\mathbf{u_h}) := \sum_{K\in T_h}\int_K(w_{1h}\frac{\partial u_h}{\partial x}\delta u_h + w_{1h}\frac{\partial v_h}{\partial x}\delta v_h + w_{2h}\frac{\partial u_h}{\partial y}\delta u_h + w_{2h}\frac{\partial v_h}{\partial y}\delta v_h)\,d\mathbf{x}$$

for $\mathbf{w_h} = (w_{1h}, w_{2h}) \in \mathbf{L}_4$, $\mathbf{u_h} = (u_h, v_h) \in \mathbf{V_h}$, $\delta\mathbf{u_h} = (\delta u_h, \delta v_h) \in \mathbf{L}_4$, where $\mathbf{x} = (x,y)$, respectively. Then the approximation of problem (1) reads as follows.

Find $(u_h, p_h) \in V_h \times H_h$, such that

$$\begin{aligned} (\frac{1}{\tau}(\mathbf{u_h}-\mathbf{u_{0h}})-\mathbf{g},\delta\mathbf{u_h}) + \nu a^h(\mathbf{u_h},\delta\mathbf{u_h}) & & \\ + a_1^h(\mathbf{u_h},\mathbf{u_h},\delta\mathbf{u_h}) - b^h(p_h,\delta\mathbf{u_h}) &= 0 & \forall\delta\mathbf{u_h}\in\mathbf{V_h} \\ b^h(q_h,\delta\mathbf{u_h}) &= 0 & \forall q_h\in\mathbf{H_h} \end{aligned} \tag{10}$$

The above forms $a^h(.,.)$, $b^h(.,.)$ and $a_1^h(.;.,.)$ have the following properties [2, 3]

$$\begin{aligned} a^h(\delta\mathbf{u}_h,\delta\mathbf{u}_h) &= \nu||\delta\mathbf{u}_h||_h^2, \forall\delta\mathbf{u}_h\in V_h, \\ |a^h(\mathbf{u}_h,\delta\mathbf{u}_h)| &\leq C||\mathbf{u}_h||_h||\delta\mathbf{u}_h||_h, \quad |b^h(p_h,\delta\mathbf{u}_h)|\leq C||p_h||_0||\delta\mathbf{u}_h||_h, \quad \forall\mathbf{u}_h,\delta\mathbf{u}_h\in V_h, \;\; p_h\in H_h, \\ a_1^h(\mathbf{u}_h;\delta\mathbf{u}_h,\delta\mathbf{u}_h) &= 0, \;\; a_1^h(\mathbf{u}_h;\delta\mathbf{u}_h,\mathbf{w}_h) = -a_1^h(\mathbf{u}_h;\mathbf{w}_h,\delta\mathbf{u}_h), \quad \forall\mathbf{u}_h,\delta\mathbf{u}_h,\mathbf{w}_h\in\delta\mathbf{u}_h, \\ |a_1^h(\mathbf{u}_h;\delta\mathbf{u}_h,\mathbf{w}_h)| &\leq N_h||\mathbf{u}_h||_h||\delta\mathbf{u}_h||_h||\mathbf{w}_h||_h, \quad \forall\mathbf{u}_h,\delta\mathbf{u}_h,\mathbf{w}_h\in H_h^1(\Omega)^2\cup V_h, \end{aligned} \tag{11}$$

where

$$N_h = \sup_{\mathbf{u_h},\delta\mathbf{u_h},\mathbf{w_h}\in\mathbf{V_h}} \frac{|a_1^h(\mathbf{u_h};\delta\mathbf{u_h},\mathbf{w_h})|}{||\nabla\mathbf{u_h}||_0||\nabla\delta\mathbf{u_h}||_0||\nabla\mathbf{w_h}||_0}$$

Now, we state the discrete embedding inequality over V_h, the same proof as one constructed by [17] ahows that

$$||\delta\mathbf{u}_h||_{0,2k,\Omega}\leq C(k)||\delta\mathbf{u}_h||_h, \quad \forall\delta\mathbf{u}_h\in X_h, \quad k=1,2,....$$

Using the discrete embedding inequality (12), we have

$$N_h\leq N_0, \quad N_0>1, \quad \forall\; 0<h\leq 1.$$

Therefore,

$$|a_1^h(\mathbf{u}_h;\delta\mathbf{u}_h,\mathbf{w}_h)|\leq N_0||\mathbf{u}_h||_h||\delta\mathbf{u}_h||_h||\mathbf{w}_h||_h, \quad \forall\mathbf{u}_h,\delta\mathbf{u}_h,\mathbf{w}_h\in H^1(\Omega)^2\cup V_h,$$

Let $Z_h=\{\delta\mathbf{u}\in V_h, \;\; b^h(q,\delta\mathbf{u})=0, \;\; \forall q\in H_h\}$ denote the divergence-free subspace of V_h. Then, the solution u_h of (10) lies in Z_h and satisfies

$$\left(\frac{1}{\tau}(\mathbf{u_h}-\mathbf{u}_0)_h-\mathbf{g},\delta\mathbf{u_h}\right)+\nu a(\mathbf{u_h},\delta\mathbf{u_h})+a_1^h(\mathbf{u_h},\mathbf{u_h},\delta\mathbf{u_h})=0 \quad \text{for all } \delta\mathbf{u_h}\in\mathbf{Z_h}. \tag{12}$$

Next, we discuss the existence and uniqueness of the solution to problem (10). To do this, from [9], the following assumption is necessary.

$$\frac{N_h||f||_h^*}{\nu^2}\leq 1-\delta_1, \quad 0<\delta_1<1, \tag{13}$$

where

$$||f||_h^* = \sup_{\delta\mathbf{u}_h\in V_h}\frac{(f,\delta\mathbf{u}_h)}{||\delta\mathbf{u}_h||_h}.$$

Lemma 4.1. *The space V_h and H_h satisfy the discrete inf-sup condition, that is,*

$$\sup_{\delta\mathbf{u}_h\in V_h}\frac{b^h(q_h,\delta\mathbf{u}_h)}{||\delta\mathbf{u}_h||_h}\geq\beta||q_h||_0, \;\; \forall\;\; q_h\in V_h,$$

where β is a positive constant independent of h.
Under condition (13), by (11) and lemma 4.1, the existence and uniqueness of the approximated solution is obvious (see [2] for details).

5 Numerical Examples

In this section, we present numerical example to conform our theoretical analysis with the algorithms described below.

We consider the following Navier-Stokes equations with slip-like boundary conditions,

$$\begin{gathered} \frac{1}{\tau}(\mathbf{u}-\mathbf{u}_0)+(\mathbf{u}\cdot\nabla)\mathbf{u}+\frac{1}{\rho}\nabla p-\nu\Delta\mathbf{u}=\mathbf{g} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u}=0 \quad \text{in } \Omega, \end{gathered} \tag{14}$$

with $\Omega = ([0, 2.2] \times [0, .41]$ - $\Omega_s)$ where Ω_s is semi circular and rectangular obstacle with diameter=0.1 and hight = 0.1 respectively for the fixed discretizing parameter $\tau > 0$ with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{15}$$

$$U_Y = V_Y = 0 \quad \text{on } \Gamma_2, \tag{16}$$

$$v = 0 \quad \text{on } \Gamma_3, \tag{17}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_4. \tag{18}$$

Navier-Stokes equations are difficult to be solved directly due to its nonlinearity. So many iteration algorithms, such as the Uzawa type algorithm, the Arrow Hurwicz algorithm, Schur complement pre-conditioners, and so on, are proposed in the literature [2, 3]. In this paper, we adopt the simpler and often more efficient method such as the Chorin-Teman Projection method with the following scheme.

5.1 Numerical Algorithm

This section describes the essential steps of the classical Chorin-Teman projection method
Step 1
Computing tentative velocity u^* by

$$(\frac{u^* - u^n}{\Delta t}, v) + ((u^*.\nabla)u^{**}, v) + (\nabla u^*, \nabla v) - (g, v) = 0$$

including boundary conditions for the velocity.
Step 2
Computing new pressure p^{n+1} by

$$(\nabla p^{n+1}, \nabla q) + \frac{1}{\Delta t}(\nabla.u^*, q) = 0$$

including boundary conditions for pressure,
Step 3
Compute corrected velocity by

$$(u^{n+1} - u^*, v) + \Delta t(\nabla p^{n+1}, v) = 0$$

including boundary conditions for the velocity.

Set the kinematic viscosity $\nu = 0.001 \ \ m^2/s$ and $\rho = 1.0 \ \ kg/m^2$. A do-nothing boundary condition is assumed at the outlet. Defining the inflow condition is given by

$$U = 4y(H - y)sin(\pi t/8)/H^2, \qquad V = 0 \tag{19}$$

and computing the flow on the time interval $[0, 8]$ with time-step $dt = 0.001$.

The figures of numerical solutions of problem (14) are shown in Figs $[1-8]$. The discrete velocity u_h, while the discrete pressure p_h are shown in Figs $[1-8]$, for different time interval for the Navier-Stokes equation with Slip-Like boundary condition.

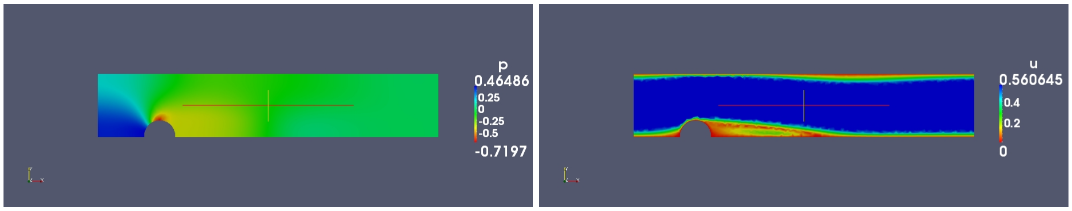

Figure 1: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=4

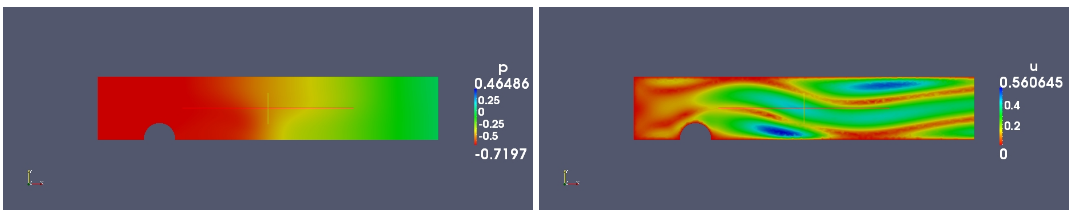

Figure 2: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=8

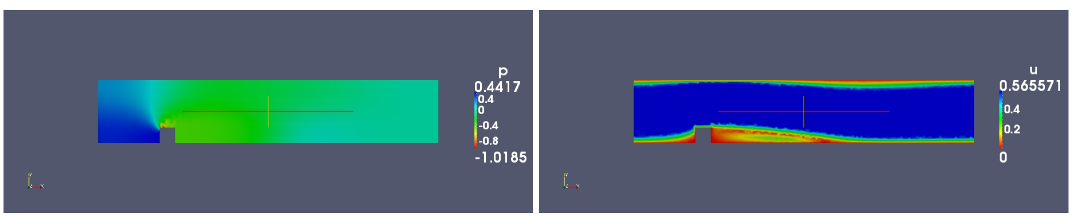

Figure 3: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=4

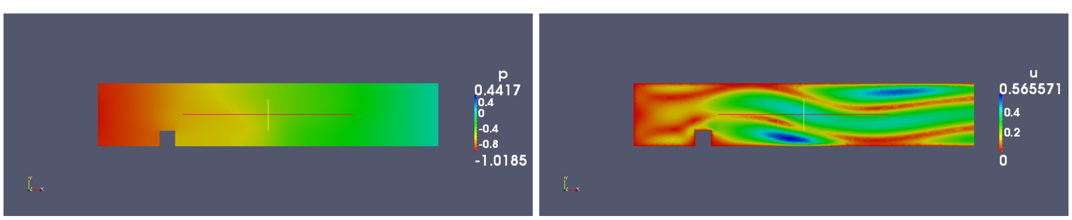

Figure 4: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=8

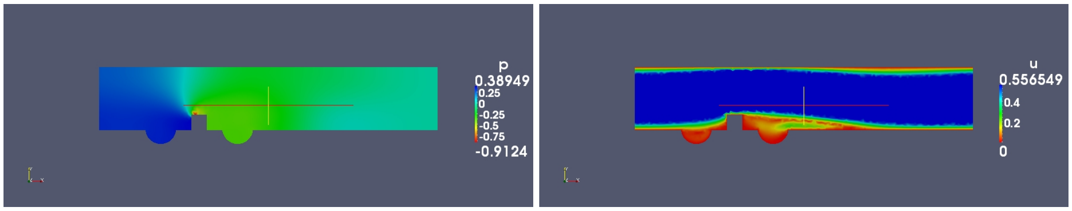

Figure 5: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=4

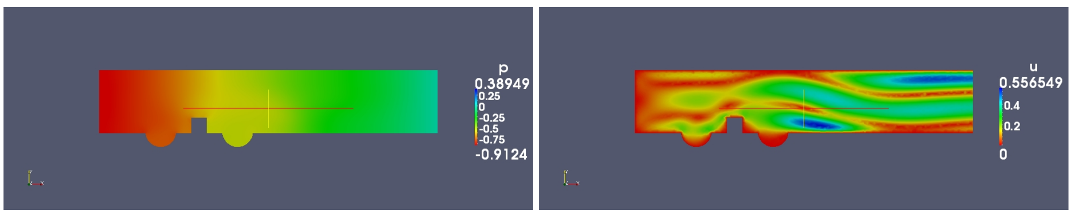

Figure 6: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=8

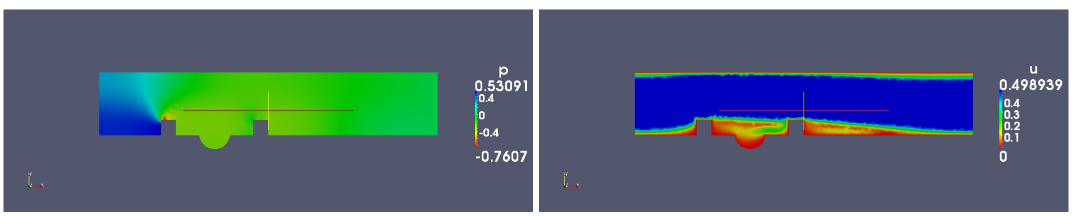

Figure 7: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=4

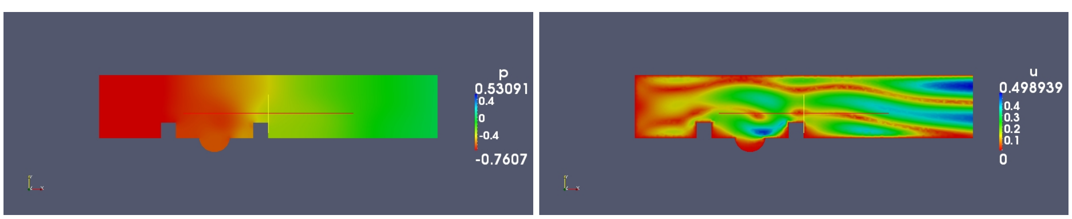

Figure 8: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=8

5.2 Conclusion

The flow around obstacle in the ice plate flow was simulated. The effects of four different sets of boundary conditions on a two-dimensional fluid flow across a fixed, rectangular, Semi Circular solid obstruction have been studied numerically using Continuous Galerkin method. Both wave structure far away the obstacle and boundary layer are well resolved. The number, position and wave length are practically identical and are in the good agreement with the theoretical prediction.

The Navier-Stokes Equation and the continuity equation, have been used under the approximations of incompressibility of the fluid, time independence and absence of external, potential dependent forces. The differentiations required have been carried out by the finite element method. The final flow field, the stream function, the vorticity and the velocity fields along at grid points upstream from the leading face of a fixed obstacle have been examined. The variation in the final solutions during the change in the Reynolds number and the size of the obstacle was investigated. The starting field was tested for higher values of the relaxation parameter to generate the reliable solutions.

The small differences are in the predicted maxima and minima of the computed quantities, which are higher in the multidomain approach. For the deeper understanding of the behavior of these models (e.g. dependency on the mesh density) further research is necessary.

For the kinematic viscosity $\nu = 0.001 \ \ m^2/s$ and density $\rho = 1.0 \ \ kg/m^2$, the region of almost dead flow behind the obstacle of size 1×1 was vanished before ending the domain length for the forced conditions at the downstream edge of free flow. The weak flow region remained up to the downstream edge for the conditions of no x derivatives at the downstream edge. The dead region was wider for the flow with higher Reynolds number. The region behind the bigger obstacle of size was also found to have a bigger dead region. The stream function drawn just ahead of the leading edge of the obstacle was such that it was bent to cross the height and went linearly. Vorticity was non zero only for a small region, near the leading corner of the obstacle. It was found to be more disturbed for higher Reynolds numbers for certain number of iterations. The curling of the fluid was more while using a taller obstacle. The velocity profiles along the x and y directions were found to obey the same nature for all the studied values of the kinamatic viscosity and density. They had higher higher peaks while using a taller obstacle.

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Received: May 12, 2014; *Accepted*: Jun 27, 2014

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