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Existence of mild solution result for fractional neutral stochastic integro-differential equations with nonlocal conditions and infinite delay

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Abstract

We investigate in this paper the existence of mild solutions for the fractional differential equations of neutral type with nonlocal conditions and infinite delay in Hilbert spaces by employing fractional calculus and Krasnoselski-Schaefer fixed point theorem. Finally an example is provided to illustrate the application of the obtained results.

Keywords: Infinite delay, Stochastic fractional differential equations, mild solution, fixed point theorem.

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1 Introduction

The main purpose of this paper is to prove the Existence of the mild solution for fractional differential equations of neutral type with infinite delay in Hilbert spaces of the form.

$$\begin{cases} {}^cD_t^\alpha[x(t)-h(t,x_t)] = A[x(t)-h(t,x_t)] + f(t,x_t) + \int_{-\infty}^{t}\sigma(t,s,x_s)dW(s) & t\in J=[0,b] \\ x(0)+\mu(x) = x_0 = \phi(t) & t\in(-\infty,0], \end{cases} \tag{1.1}$$

Here, $x(.)$ takes value in a real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}$. The fractional derivative ${}^cD^\alpha, \alpha\in(0,1)$, is understood in the Caputo sense. The operator A generates a strongly continuous semigroup of bounded linear operators $S(t), t\geq 0$, on $\mathbb{H}$. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(.,.)_{\mathbb{K}}$ and the norm $\|\cdot\|_{\mathbb{K}}$. W is a given $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q\geq 0$ defined on a filtered complete probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. The histories $x_t:\Omega\to\mathcal{C}_v$ defined by $x_t=\{x(t+\theta),\theta\in(-\infty,0]\}$ belong to the phase space $\mathcal{C}_v$, which will be defined in section 2. The initial data $\phi=\{\phi(t),t\in(-\infty,0]\}$ is an $\mathcal{F}_0$- measurable, $\mathcal{C}_v$-valued random variable independent of W with finite second moments, and $h: J\times\mathcal{C}_\vartheta\to\mathbb{H}$, $h: J\times\mathcal{C}_v\to\mathbb{H}$, $\sigma: J\times J_1\times\mathcal{C}_v\to\mathcal{L}_2^0(\mathbb{K},\mathbb{H})$ are appropriate functions, where $J_1=(-\infty,b]$ and $\mathcal{L}_2^0(\mathbb{K},\mathbb{H})$ denotes the space of all Q-Hilbert Schmidt operators from $\mathbb{K}$. into $\mathbb{H}$. $\mu: C(J,\mathbb{H})\to\mathbb{H}$ is bounded and the initial data x_0 is an $\mathcal{F}$ adapted $\mathbb{H}$-valued random variable independent of Wiener process W.

The fractional differential equations arise in many engineering and scientific disciplines as the mathematica modeling of systems and processes in the fields of physics,
chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology,etc.,
involves derivatives of fractional order. It is worthwhile mentioning that several important problems of the theory of ordinary and delay differential equations lead to investigations of functional differential equations

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of various types (see the books by Hale and Verduyn Lunel [16] , Wu [31] , Liang et al [17], Liang and Xiao [18], and the references therein).

In particular the nonlocal condition problems for some fractional differential equations have been attractive to many researchers Mophou et al [23] studied existence of mild solution for some fractional differential equations with nonlocal condition. Chang et al [7] investigate the fractional order integro-differential equations with nonlocal conditions in the Riemann-Liouville fractional derivative sense.

In this paper, we prove the existence theorem of mild solution for neutral differential equation with nonlocal conditions and infinite delay by using the Krasnoselski-Schaefer fixed point theorem. An example is provided to illustrate the application of the obtained results.

2 Preliminaries

Next we mention a few results and notations needed to establish our results.
Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ be two real separable Hilbert spaces. We denote by $\mathcal{L}(\mathbb{K}, \mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|.\|$. In this article, we use the symbol $\|.\|$ to denote norms of operators regardless of the spaces involved when no confusion possibly arises.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $W = (W_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, ...$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(W(t), e)_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathbb{K}}\beta_k(t) \qquad e \in \mathbb{K} \qquad t \geq 0$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $\langle\varphi, \phi\rangle_{\mathcal{L}_2^0} = tr[\varphi Q\phi^*]$.

The semigroup $S(\cdot)$ is uniformly bounded. That is to say, $\|S(t)\| \leq M$ for some constant $M \geq 1$ and every $t \geq 0$.

Assume that $\upsilon : (-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^{0} \upsilon(t)dt < +\infty$ a continuous function.
Recall that the abstract phase space $\mathcal{C}_\upsilon$ is defined by

$$\mathcal{C}_\upsilon = \{\varphi : (-\infty, 0] \to \mathbb{H}, \text{ for any } a > 0, (\mathbb{E}\,|\varphi(\theta)|^2)^{1/2} \text{ is bounded and measurable}$$

$$\text{function on } [-a, 0] \text{ and } \int_{-\infty}^{0} \upsilon(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}\,|\varphi(\theta)|^2)^{1/2} ds < +\infty\}.$$

If $\mathcal{C}_\upsilon$ is endowed with the norm

$$\|\varphi\|_{\mathcal{C}_\upsilon} = \int_{-\infty}^{0} \upsilon(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}\,|\varphi(\theta)|^2)^{\frac{1}{2}} ds, \qquad \varphi \in \mathcal{C}_\upsilon$$

then $(\mathcal{C}_\upsilon, \|.\|_{\mathcal{C}_\upsilon})$ is a Banach space (see [20]).

Let us now recall some basic definitions and results of fractional calculus.

Definition 2.1. [21] *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^\alpha f = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \qquad t > 0 \qquad \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(.)$ is the gamma function.

Definition 2.2. *The Caputo derivative of order α with the lower limit 0 for a function f can be written as*

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \; t > 0, \; 0 \leq n-1 < \alpha < n$$

The Caputo derivative of a constant equal to zero. If f is an abstract function with values in $\mathbb{H}$, then the integrals appearing in the above definitions are taken in Bochner's sense (see [21]).

Lemma 2.1. [5] *Let H be a Hilbert space and Φ_1, Φ_2 two operators on H such that*

i) *Φ_1 is a contraction and*

ii) *Φ_2 is completely continuous.*

Then either

a) *the operator equation $\Phi_1 x + \Phi_2 x = x$ has a solution or*

b) *$G = \{x \in \mathbb{H} : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ is unbounded for $\lambda \in (0, 1)$.*

Lemma 2.2. [15] *Let $v(.), w(.) : [0, b] \to [0, \infty)$ be continuous function. If $w(.)$ is nondecreasing and there exist two constants $\theta \geq 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \quad t \in J$$

then

$$v(t) \leq e^{\theta^n (\Gamma(\alpha))^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha}\right)^j w(t),$$

for every $t \in [0, b]$ and every $n \in N$ such that $n\alpha > 1$.

3 Existence results

Definition 3.3. *An $\mathbb{H}$- valued stochastic process $\{x(t), t \in (-\infty, b]\}$ is a mild solution of the system 1.1 if $x(0)+\mu(x) = x_0 = \phi(t)$ on $(-\infty, 0]$ satisfying $\|\phi\|^2_{\mathcal{C}_v} < +\infty$, the process x satisfies the following integral equation*

$$\begin{aligned} x(t) = S_\alpha(t)[\phi(0) - \mu(x) - h(0, \phi)] + h(t, x_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, x_\tau) dW(\tau)\right] ds \end{aligned}$$

where

$$S_\alpha(t)x = \int_0^\infty \zeta_\alpha(\theta) S(t^\alpha \theta) x d\theta, \quad T_\alpha(t)x = \alpha \int_0^\infty \theta\zeta_\alpha(\theta) S(t^\alpha\theta) x d\theta$$

and ζ_α is a probability density function defined on $(0, \infty)$

The following properties of $S_\alpha(t)$ and $T_\alpha(t)$ appeared in [34] are useful.

Lemma 3.3. *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties*

i) *For any fixed $t \geq 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators such that for any $x \in \mathbb{H}$*

$$\|S_\alpha(t)x\|_{\mathbb{H}} \leq M \|x\|_{\mathbb{H}} \text{ and } \|T_\alpha(t)x\|_{\mathbb{H}} \leq \frac{M_\alpha}{\Gamma(1+\alpha)} \|x\|_{\mathbb{H}}$$

ii) *$S_\alpha(t)$ and $T_\alpha(t)$ are strongly continuous and compact.*

To study existence of mild solutions of 1.1, we introduce the following hypotheses.

(H_1) : The function $h, f : J \times \mathcal{C}_v \to \mathbb{H}$ are continuous and there exist some constants M_h, M_f, such that

$$E\|h(t,x) - h(t,y)\|^2_{\mathbb{H}} \leq M_h \|x - y\|^2_{\mathcal{C}_v}, x, y \in \mathcal{C}_v, \quad t \in J$$

$$E\|h(t,x)\|^2_{\mathbb{H}} \leq M_h(1 + \|x\|^2_{\mathcal{C}_v})$$

$$E\|f(t,x) - f(t,y)\|^2_{\mathbb{H}} \leq M_f \|x - y\|^2_{\mathcal{C}_v}, x, y \in \mathcal{C}_v, \quad t \in J$$

$$E\|f(t,x)\|^2_{\mathbb{H}} \leq M_f(1 + \|x\|^2_{\mathcal{C}_v})$$

(H_2) : μ is continuous and there exists some positive constants M_μ such that

$$E\left\|\mu(x)-\mu(y)\right\|_{\mathbb{H}}^2 \le M_\mu \left\|x-y\right\|_{\mathcal{C}_\upsilon}^2, x,y\in\mathcal{C}_\upsilon, \quad t\in J$$

$$E\left\|\mu(x)\right\|_{\mathbb{H}}^2 \le M_\mu(1+\left\|x\right\|_{\mathcal{C}_\upsilon}^2)$$

(H_3) : For each $\varphi\in\mathcal{C}_\upsilon$,

$$k(t)=\lim_{a\to\infty}\int_{-a}^{0}\sigma(t,s,\varphi)dW(s)$$

exists and is continuous. Further, there exists a positive constant M_k such that

$$E\left\|k(t)\right\|_{\mathbb{H}}^2 \le M_k$$

(H_4) The function $\sigma: J\times J_1\times\mathcal{C}_\upsilon\to L(\mathbb{K},\mathbb{H})$ satisfies the following:

i) for each $(t,s)\in J\times J\times, \sigma(t,s,.):\mathcal{C}_\upsilon\to L(\mathbb{K},\mathbb{H})$ is continuous and for each $x\in\mathcal{C}_\upsilon$, $\sigma(.,.,x): J\times J\to L(\mathbb{K},\mathbb{H})$ is strongly measurable;

ii) there is a positive integrable function $m\in L^1([0,b])$ and a continuous nondecreasing function M_σ : $[0,\infty)\to(0,\infty)$ such that for every $(t,s,x)\in J\times J\times\mathcal{C}_\upsilon$, we have

$$\int_0^t E\left\|\sigma(t,s,x)\right\|_{\mathcal{L}_2^0}^2 ds \le m(t)M_\sigma(\left\|x\right\|_{\mathcal{C}_\upsilon}^2), \quad \liminf_{r\to\infty}\frac{M_\sigma(r)}{r}ds=\Delta<\infty$$

iii) For any $x,y\in\mathcal{C}_\upsilon, t\ge 0$, there exists a positive constant L_σ such that

$$\int_0^t E\left\|\sigma(t,s,x)-\sigma(t,s,y)\right\|_{\mathcal{L}_2^0}^2 ds \le L_\sigma\left\|x-y\right\|_{\mathcal{C}_\upsilon}^2$$

(H_5) :

$$N_0=2l^2\{12M^2M_\mu+4M_h\} \tag{3.1}$$

$$N_1=2\left\|\phi\right\|_{\mathcal{C}_\upsilon}^2+2l^2\bar{F} \tag{3.2}$$

$$N_2=8l^2\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}M_f \tag{3.3}$$

$$N_3=16bl^2\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}Tr(Q) \tag{3.4}$$

$$K_1=\frac{N_1}{1-N_0},\quad K_2=\frac{N_2}{1-N_0},\quad K_3=\frac{N_3}{1-N_0} \tag{3.5}$$

$$\bar{F}=12M^2(C_1+C_2)+12M^2M_\mu+4M_h+4\left(\tfrac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\tfrac{b^{2\alpha}}{\alpha^2}M_f+8b\left(\tfrac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\tfrac{b^{2\alpha}}{\alpha^2}M_k \tag{3.6}$$

Now, we consider the space,

$$\mathcal{C}_\upsilon^{'}=\{x:(-\infty,b]\to\mathbb{H}, x_0=\phi\in\mathcal{C}_\upsilon\}$$

Set $\left\|.\right\|_b$ be a seminorm defined by

$$\left\|x\right\|_b=\left\|x_0\right\|_{\mathcal{C}_\upsilon}+\sup_{s\in[0,b]}(E\left|x(s)\right|^2)^{\frac{1}{2}},\quad x\in\mathcal{C}_\upsilon^{'}$$

We have the following useful lemma appeared in [20].

Lemma 3.4. [6] *Assume that $x \in \mathcal{C}_v^{'}$, then for all $t \in J$, $x_t \in \mathcal{C}_v$, Moreover,*

$$l(E|x(t)|^2)^{\frac{1}{2}} \leq \|x_t\|_{\mathcal{C}_v} \leq l \sup_{s\in[0,t]} (E|x(s)|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{C}_v}$$

where $l = \int_{-\infty}^{0} v(s)ds < \infty$

The main object of this paper is to explain and prove the following theorem.

Theorem 3.1. *Assume that assumptions $(H_0) - (H_5)$ hold.Then there exists a mild solution*

Proof Consider the map $\Pi : \mathcal{C}_v^{'} \to \mathcal{C}_v^{'}$ defined by

$$(\Pi x)(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ \\ S_\alpha(t)[\phi(0) - \mu(x) - h(0,\phi)] + h(t,x_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s,x_s)ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^{s} \sigma(s,\tau,x_\tau)dW(\tau)\right] ds \quad t \in J \end{cases} \tag{3.7}$$

In what follows, we shall show that the operator Π has a fixed point, which is then a mild solution for system 1.1.
For $\phi \in \mathcal{C}_v$, define

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ \\ S_\alpha(t)\phi(0) & t \in J \end{cases} \tag{3.8}$$

Then $\tilde{\phi} \in \mathcal{C}_v^{'}$. Let $x(t) = \tilde{\phi}(t) + z(t)$, $-\infty < t \leq b$. It is easy to see that x satisfies 1.1 if and only if z satisfies $z_0 = 0$ and

$$z(t) = S_\alpha(t)\left[-\mu(\tilde{\phi} + z) - h(0,\phi)\right] + h(t, \tilde{\phi}_t + z_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s)ds$$
$$+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^{s} \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)\right] ds$$

Let

$$\mathcal{C}_v^{''} = \{z \in \mathcal{C}_v^{'}, z_0 = 0 \in \mathcal{C}_v\}$$

For any $z \in \mathcal{C}_v^{''}$, we have

$$\|z\|_b = \|z_0\|_{\mathcal{C}_v} + \sup_{s\in[0,b]} (E\|z(s)\|^2)^{\frac{1}{2}} = \sup_{s\in[0,b]} (E\|z(s)\|^2)^{\frac{1}{2}}$$

Thus $(\mathcal{C}_v^{''}, \|.\|_b)$ is a Banach space, set

$$B_q = \{z \in \mathcal{C}_v^{''}, \|z\|_b^2 \leq q\}, \text{ for some } q \geq 0$$

then, $B_q \subset \mathcal{C}_v^{''}$ is uniformly bounded.

then, for each q, B_q is clearly a bounded closed convex set in $\mathcal{C}^{''}$. For $z \in B_q$, from Lemma 3.3, we have

$$\begin{aligned} \left\|z_t + \tilde{\phi}_t\right\|_{\mathcal{C}_v}^2 &\leq 2(\|z_t\|_{\mathcal{C}_v}^2 + \left\|\tilde{\phi}_t\right\|_{\mathcal{C}_v}^2) \\ &\leq 4(l^2 \sup_{s\in[0,t]} E\|z(s)\|^2 + \|z_0\|_{\mathcal{C}_v}^2 + l^2 \sup_{s\in[0,t]} E\left\|\tilde{\phi}(s)\right\|^2 + \left\|\tilde{\phi}_0\right\|_{\mathcal{C}_v}^2) \\ &\leq 4l^2(q + M^2 E\|\phi(0)\|_{\mathbb{H}}^2) + 4\|\phi\|_{\mathcal{C}_v}^2 \\ &= \grave{q} \end{aligned}$$

Define the operator $\Phi : C_v^{''} \to C_v^{''}$ by

$$(\Phi z)(t) = \begin{cases} 0 \quad t \in (-\infty, 0] \\ \\ S_\alpha(t)[-\mu(\tilde{\phi}+z) - h(0,\phi)] + h(t, \tilde{\phi}_t + z_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau) dW(\tau) \right] ds \;\; t \in J \end{cases}$$

Observe that Φ is well defined on B_q for each $q > 0$.

Now we will show that the operator Φ has a fixed point on B_q, which implies that E.q 1.1 has a mild solution. To this end, we decompose Φ as $\Phi = \Phi_1 + \Phi_2$, where the operators Φ_1 and Φ_2 are defined on B_q, respectively, by

$$(\Phi_1 z)(t) = S_\alpha(t)[-\mu(\tilde{\phi}+z) - h(0,\phi)] + h(t, \tilde{\phi}_t + z_t)$$

$$\begin{aligned}(\Phi_2 z)(t) = & \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds \\ & + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau) dW(\tau) \right] ds\end{aligned}$$

Thus, the theorem follows from the next theorem

Theorem 3.2. *If assumption $(H_1) - (H_5)$ hold, then Φ_1 is a contraction and Φ_2 is completely continuous.*

Proof To prove that Φ_1 is a contraction on $\mathcal{C}_v^{''}$, we take $u, v \in \mathcal{C}_v^{''}$. Then for each $t \in J$ we have

$$\begin{aligned} E\left\|\Phi_1 u(t) - \Phi_1 v(t)\right\|_{\mathbb{H}}^2 \leq & \, 2E\left\|S_\alpha(t)(\mu(\tilde{\phi}+u) - \mu(\tilde{\phi}+v))\right\|_{\mathbb{H}}^2 \\ & + 2E\left\|h(t,\tilde{\phi}_t + u_t) - h(t,\tilde{\phi}_t + v_t)\right\|_{\mathbb{H}}^2 \\ \leq & \, 2M^2 M_\mu \left\|u - v\right\|_{\mathcal{C}_v}^2 + 2M_h \left\|u_t - v_t\right\|_{\mathcal{C}_v}^2 \\ \leq & \, 2(M^2 M_\mu + M_h) \left\|u_t - v_t\right\|_{\mathcal{C}_v}^2 \\ \leq & \, 2(M^2 M_\mu + M_h) \\ & [2l^2 \sup_{s\in[0,t]} E\left\|u(s) - v(s)\right\|^2 + 2\left\|u_0\right\|_{\mathcal{C}_v}^2 + 2\left\|v_0\right\|_{\mathcal{C}_v}^2] \\ \leq & \, 4l^2(M^2 M_\mu + M_h) E\left\|u(s) - v(s)\right\|^2 \\ \leq & \sup_{s\in[0,b]} L_0 E\left\|u(s) - v(s)\right\|^2 \end{aligned}$$

where we have used the fact that $\|u_0\|_{\mathcal{C}_v}^2 = 0, \|v_0\|_{\mathcal{C}_v}^2 = 0$.
Thus,

$$\|\Phi_1 u - \Phi_1 v\| \leq L_0 \|u - v\|$$

and by assumption $0 \leq L_0 \leq 1$ it is clear that Φ_1 is contraction.

Now, we show that the operator Φ_2 is completely continuous, firstly we prove that $\Phi_2 : \mathcal{C}_h^{''} \to \mathcal{C}_h^{''}$ is continuous.

Let $\{z^n(t)\}_{n=0}^\infty$,with $z^n \to z$ in $\mathcal{C}_h^{''}$. Then, there is a number $q \geq 0$ such that $|z^n(t)| \leq q$, for all n and a.e. $t \in J$. So $z^{(n)} \in B_q$ and $z \in B_q$.

$$f(t, z_t^{(n)} + \tilde{\phi}_t) \to f(t, z_t + \tilde{\phi}_t)$$

$$\sigma(s,\tau, z_\tau^{(n)} + \tilde{\phi}_\tau) \to \sigma(s,\tau, z_\tau + \tilde{\phi}_\tau)$$

for $t \in J$, and since

$$E\left\|[f(t, z_t^{(n)} + \tilde{\phi}_t) - f(t, z_t + \tilde{\phi}_t)]\right\|^2 \leq 2M_{q'}(t)$$

$$E\left\|[\sigma(s, \tau, z_\tau^{(n)} + \tilde{\phi}_\tau) - \sigma(s, \tau, z_\tau^{(n)} + \tilde{\phi}_\tau)]\right\|^2 \leq 2m(t)M_\sigma(q^{'})$$

By the dominated convergence theorem we obtain continuity of Φ_2

$$\begin{aligned}
E\left\|\Phi z_t^{(n)} - \Phi_t z\right\|^2 \leq & 2\sup_{t\in J} E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(t, z_s^{(n)}) - f(t, z_s)]ds\right\|^2 \\
& + 2b\sup_{t\in J} E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s [\sigma(s,\tau,z_\tau^{(n)}) - \sigma(s,\tau,z_\tau)]dw(\tau)\right]ds\right\|^2 \\
& \leq 2\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2}\int_0^t E\left\|[f(t, z_s^{(n)}) - f(t, z_s)]\right\|^2 ds \\
& + 2b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2}\int_0^t E\left\|\left[\int_{-\infty}^s [\sigma(s,\tau,z_\tau^{(n)}) - \sigma(s,\tau,z_\tau)]dw(\tau)\right]ds\right\|^2 \\
& \to 0 \text{ as } n \to \infty
\end{aligned}$$

Next, we prove that Φ_2 maps bounded sets into bounded sets in $\mathcal{C}_v^{''}$.
For each $z \in B_q$ from [3.4], we have

$$\left\|z_t + \tilde{\phi}_t\right\|_{\mathcal{C}_v}^2 \leq 4l^2(q + M^2 E\left\|\phi(0)\right\|_{\mathbb{H}}^2) + 4\left\|\phi\right\|_{\mathcal{C}_v}^2 = q^{'}$$

$$\begin{aligned}
E\left\|\Phi_2 z(t)\right\|_{\mathbb{H}}^2 \leq & 2E\left\|(t-s)^{\alpha-1}T_\alpha(t-s)f(s, \tilde{\phi}_s + z_s)\right\|_{\mathbb{H}}^2 \\
& + 2E\left\|(t-s)^{\alpha-1}T_\alpha(t-s)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)]ds\right\|_{\mathbb{H}}^2 \\
& \leq 2\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}M_f(1 + \left\|\tilde{\phi}_s + z_s\right\|_{\mathcal{C}_v}^2)ds \\
& + \left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}(2M_k + 2Tr(Q)m(s)M_\sigma(\left\|\tilde{\phi}_s + z_s\right\|_{\mathcal{C}_v}^2)ds. \\
& \leq 2\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \frac{b^{2\alpha}}{\alpha^2}M_f(1 + q^{'}) \\
& + 2\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \frac{b^{2\alpha}}{\alpha^2}(M_k + Tr(Q)M_\sigma(q^{'})\sup_{t\in J} m(s)) \\
& \leq r
\end{aligned}$$

Which implies that for each $z \in B_q$, $|\Phi_2 z\|_b^2 \leq r$.

Next, we establish the compactness of Φ_2. We employ the Arzela-Ascoli theorem to show the set $V(t) = \{(\Phi_2 z)(t), z \in B_q\}$ is relatively compact in $\mathbb{H}$. Le $0 < t \leq b$ be fixed and ϵ be a real number satisfying $0 < \epsilon \leq t$. For $\delta > 0$, for $z \in B_q$, We define

$$
\begin{aligned}
(\Phi_2^{\epsilon,\delta}z)(t) &= \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\alpha\theta)f(s,\tilde{\phi}_s+z_s)ds \\
&+ \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\alpha\theta)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)]ds \\
&= S(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\theta-\epsilon^\alpha\delta)f(s,\tilde{\phi}_s+z_s)ds \\
&+ S(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\theta-\epsilon^\alpha\delta)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)]ds
\end{aligned}
$$

Since $S(t), t>0$, is a compact operator, the set $V_{\epsilon,\delta}=\{\Phi_2^{\epsilon,\delta}(t), z\in B_q\}$ is relatively compact in $\mathbb{H}$ for every $\epsilon\in(0,t)$, $\delta>0$. Moreover, for each $z\in B_q$, we have

$$
\begin{aligned}
& E\left\|(\Phi_2 z)(t)-(\Phi_2^{\epsilon,\delta}z)(t)\right\|_{\mathbb{H}}^2 \\
& \le 4\alpha^2 E\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)f(s,z_s+\tilde{\phi}_s)dt\theta ds\right\|_{\mathbb{H}}^2 \\
& + 4\alpha^2 E\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)f(s,z_s+\tilde{\phi}_s)d\theta ds\right\|_{\mathbb{H}}^2 \\
& + 4\alpha^2 E\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\left[\int_{-\infty}^s \sigma(s,\tau,z_\tau+\tilde{\phi}_\tau)dW(\tau)\right]d\theta ds\right\|_{\mathbb{H}}^2 \\
& + 4\alpha^2 E\left\|\int_{t-\epsilon}^t\int_\infty^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\left[\int_{-\infty}^s \sigma(s,\tau,z_\tau+\tilde{\phi}_\tau)dW(\tau)\right]d\theta ds\right\|_{\mathbb{H}}^2 \\
& \le 4M^2b^{2\alpha}M_f(1+q^{'})\left(\int_0^\delta \theta\eta_\alpha(\theta)d\theta\right)^2+\frac{4M^2\epsilon^{2\alpha}M_f(1+q^{'})}{\Gamma^2(1+\alpha)} \\
& + 4\alpha M^2b^\alpha\int_0^t (t-s)^{\alpha-1}(2M_k+2Tr(Q)M_\sigma(q^{'})m(s))ds\left(\int_0^\delta \theta\eta_\alpha(\theta)d\theta\right)^2 \\
& + \frac{4\alpha M^2\epsilon^\alpha}{\Gamma^2(1+\alpha)}\int_{t-\epsilon}^t (t-s)^{\alpha-1}(2M_k+2Tr(Q)M_\sigma(q^{'})m(s))ds
\end{aligned}
$$

where we have used the equality (see [22, 29])

$$
\int_0^\infty \theta^\varsigma\eta_\alpha(\theta)=\frac{\Gamma(1+\varsigma)}{\Gamma(1+\alpha\varsigma)}
$$

We see that for each $z\in B_q$

$$
E\left\|(\Phi_2 z)(t)-(\Phi_2^{\epsilon,\delta})\right\|_{\mathbb{H}}^2 \to 0 \text{ as } \epsilon^+\to 0, \delta\longrightarrow 0.
$$

Since the right-hand side of the above inequality can be made arbitrarily small, there is relatively compact $V_{\epsilon,\delta}$ arbitrarily close to the set $V(t)$. Hence, the set $V(t)$ is relatively compact in B_q. It remains to showt hat Φ_2 maps is bounded set into equicontinuous sets of $\mathcal{C}_v^{''}$.
Let $0<\epsilon<t<b$ and $\delta>0$ such that $\|T_\alpha(s_1)-T_\alpha(s_2)\|\le\epsilon$, for every $s_1,s_2\in J$.

with $|s_1 - s_2| < \delta$. For $z \in B_q$, we have

$$
\begin{aligned}
& E\left\|\Phi_2 z(t+h)-\Phi_2 z(t)\right\|_{\mathbb{H}}^2 \\
& \leq 6E\left\|\int_0^t [(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]T_\alpha(t+h-s)f(s,\tilde{\phi}_s+z_s)ds\right\|_{\mathbb{H}}^2 \\
& +6E\left\|\int_t^{t+h}(t+h-s)^{\alpha-1}T_\alpha(t+h-s)f(s,\tilde{\phi}_s+z_s)ds\right\|_{\mathbb{H}}^2 \\
& +6E\left\|\int_0^t (t-s)^{\alpha-1}[T_\alpha(t+h-s)-T_\alpha(t-s)]f(s,\tilde{\phi}_s+z_s)ds\right\|_{\mathbb{H}}^2 \\
& +6E\left\|\int_0^t [(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]T_\alpha(t+h-s)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)ds]\right\|_{\mathbb{H}}^2
\end{aligned}
$$

$$
\begin{aligned}
& +6E\left\|\int_t^{t+h}(t+h-s)^{\alpha-1}T_\alpha(t+h-s)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)ds]\right\|_{\mathbb{H}}^2. \\
& +6E\left\|\int_0^t (t-s)^{\alpha-1}[T_\alpha(t+h-s)-T_\alpha(t-s)][\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)ds]\right\|_{\mathbb{H}}^2 \\
& \leq 6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_0^t \left|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\right|^2 M_f(1+q^{'})ds \\
& +6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_t^{t+h} \left|(t+h-s)^{\alpha-1}\right|^2 M_f(1+q^{'})ds \\
& +6\epsilon^2\int_0^t \left|(t-s)^{\alpha-1}\right|^2 M_f(1+q^{'})ds+6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_0^t \left|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\right|^2 \\
& \times(2M_k+2Tr(Q)m(s)M_\sigma(q^{'}))ds \\
& +6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_0^t \left|(t+h-s)^{\alpha-1}\right|^2 (2M_k+2Tr(Q)m(s)M_\sigma(q^{'}))ds \\
& +6\epsilon^2\int_0^t \left|(t-s)^{\alpha-1}\right|(2M_k+2Tr(Q)m(s)M_\sigma(q^{'}))ds
\end{aligned}
$$

It is known that the compactness of $T_\alpha(t), t > 0$ implies the continuity in the uniform operator topology. Therefore, for ϵ sufficiently small, the right-hand side of the above inequality tends to zero as $h \to 0$. Thus, the set $\{\Phi_2 z, z \in B_q\}$ is equicontinuous.
This completes the proof that Φ_2 is completely continuous.

To apply the Krasnoselski-Schaefer theorem, it remains to show that the set

$$G=\{x\in\mathbb{H}:\lambda\Phi_1(\frac{x}{\lambda})+\lambda\Phi_2 x=x\} \text{ is bounded for } \lambda\in(0,1)$$

.

We consider the following nonlinear operator equation,

$$
\begin{aligned}
x(t) &= \lambda\left(S_\alpha(t)[\phi(0)-\mu(x)-h(0,\phi)]\right)+\lambda h(t,x_t) \\
&+\lambda\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x_s)ds \\
&+\lambda\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau)dW(\tau)\right]ds
\end{aligned}
$$

$$
\begin{aligned}
E\|x(t)\|^2 \le\ & E\|S_\alpha(t)(\phi(0)-\mu(x)-h(0,\phi))\|^2_{\mathbb{H}} + 4\|h(t,x_t)\|^2_{\mathbb{H}} \\
& + 4E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x_s)ds\right\|^2_{\mathbb{H}} \\
& + 4E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau)dW(\tau)\right]ds\right\|^2_{\mathbb{H}} \\
& \le 12M^2(C_1+C_2+M_\mu) + 12M^2(1+\|x\|^2_{\mathcal{C}_\upsilon}) \\
& + 4\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2} M_f(1+\|x\|^2_{\mathcal{C}_\upsilon}) \\
& + 4b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}(2M_k + 2Tr(Q)m(s)M_\sigma(\|x_s\|^2_{\mathcal{C}_\upsilon})ds
\end{aligned}
$$

Now, we consider the function ν defined by

$$\vartheta(t) = \sup\{E\|x(s)\|^2, 0 \le s \le t\}, 0 \le t \le b$$

From lemma [3.4] and the above inequality, we have

$$E\|x(t)\|^2 = 2\|\phi\|^2_{\mathcal{C}_\upsilon} + 2l^2 \sup_{0\le s\le t}(E\|x(s)\|^2)$$

Therefore, we get

$$
\begin{aligned}
\vartheta(t) \le\ & 2\|\phi\|^2_{\mathcal{C}_\upsilon} + 2l^2\{\bar{F} + 12M^2\vartheta(t) + 4\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2}M_f\vartheta(t) \\
& + 8b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}Tr(Q)m(s)M_\sigma(\vartheta(s))ds\}
\end{aligned}
$$

where $\bar{F}$ is given in (3.6). Thus, we have

$$\vartheta(t) \le K_1 + K_2\int_0^t \frac{\vartheta(s)}{(t-s)^{1-\alpha}}ds + K_3\int_0^t m(s)M_\sigma(\vartheta(s))ds$$

where K_1, K_2, K_3 are given in (3.5). By Lemma [2.2], we have

$$\vartheta \le B_0(K_1 + K_3\int_0^t m(s)M_\sigma(\vartheta(s))ds)$$

Where

$$B_0 = e^{K_2^n(\Gamma(\alpha))^n b^{n\alpha}/\Gamma(n\alpha)}\sum_{j=0}^{n-1}\left(\frac{K_2 b^\alpha}{\alpha}\right)^j$$

Denoting by $\nu(t)$ the right hand side of the last inequality, we have $\nu(0) = B_0K_1$

$$\grave{\nu}(t) \le B_0K_3m(t)M_\sigma\vartheta(t)$$

$$\grave{\nu}(t) \le B_0K_3m(t)M_\sigma(\vartheta(t))$$

This implies

$$\int_{\nu(0)}^{\nu(t)}\frac{ds}{M_\sigma(s)} \le \int_0^b \pi(s)ds < \int_{B_0K_1}^\infty \frac{ds}{M_\sigma(s)}$$

This inequality implies that there is a constant ρ such that $\nu(t) \le \rho$, $t \in J$ and hence $\vartheta(t) \le \rho$, $t \in J$. Furthermore, we get $\|x_t\|^2_{\mathcal{C}_\upsilon} \le \vartheta(t) \le \nu(t) \le \rho$, $t \in J$, where ρ depends only on b and on the functions $\pi(s)$ and $M_\sigma(s)$.

Theorem 3.3. *Assume that the hypotheses $(H_1)-(H_5)$ hold. Then problem has at least one mild solution on J.*

Proof. Let us take the set

$$D(\Phi) = \{z \in \mathcal{C}_v^{''} : z = \lambda\Phi_1(\frac{z}{x}) + \lambda\Phi_2 z \text{ for some } \lambda \in [0,1]\} \tag{3.9}$$

Then, for any $z \in D(\Phi)$, we have by theorem ... that $\|x\|_{\mathcal{C}_v}^2 \leq K, t \in J$, and hence

$$\begin{aligned}
\|z\|_b^2 &= \|z_0\|_{\mathcal{C}_v}^2 + \sup\{E\|z(t)\|^2 ; 0 \leq t \leq b\} \\
&= \sup\{E\|z(t)\|^2 : 0 \leq t \leq b\} \\
&\leq \sup\{E\|x(t)\|^2 : 0 \leq t \leq b\} + \sup\{E\left\|\tilde{\phi}(t)\right\|^2 : 0 \leq t \leq b\} \\
&\leq \sup\{l^- \|x(t)\|_{\mathcal{C}_v}^2 : 0 \leq t \leq b\} + \sup\{\|s_\alpha(t)\phi(0)\| : 0 \leq t \leq b\} \\
&\leq l^- \rho + M_1\|\phi(0)\|^2
\end{aligned}$$

□

This implies that D is bounded on J. Consequently by Lemma 2.1 , the operator Φ has a fixed point $z \in \mathcal{C}_h^{''}$. So Eq.(1.1) has a mild solution.Theorem is proved.

Example 3.1. *As an application of the above result,consider the following fractional order neutral stochastic partial differential system with non local conditions and infinite delay in Hilbert space.*

$$\begin{cases}
{}^cD_t^\alpha[z(t,x) - \int_{-\infty}^t e^{4(s-t)}z(s,x)ds] = \frac{\partial^2}{\partial x^2}[z(t,x) - \int_{-\infty}^t e^{4(s-t)}z(s,x)ds] + \eta(t,x) \\
+\int_{-\infty}^0 \hat{a}(s)\sin z(t+s,x)ds + \int_{-\infty}^t \int_{-\infty}^t \sigma(t,x,s-t)dsd\beta(s,x) \quad t \in J = [0,b] \\
z(t,0) = z(t,\pi) = 0 \quad t \in J \\
z(0,x) + \int_0^\pi k_1(x,y)z(t,y)dy = x_0 = \varphi(t,x) \quad t \in (-\infty, 0],
\end{cases} \tag{3.10}$$

Where ${}^cD^\alpha$ is a Caputo fractional partial derivative of order $\alpha \in (0,1)$, and $K_1(x,y) \in \mathbb{H} = L^2([0,\pi] \times [0,\pi])$ and $\int_{-\infty}^0 |\hat{a}(s)|\, ds < +\infty$. $\beta(t)$ is a one-dimensional standard Wiener process on filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. To rewrite this system into the abstract form (1.1), let $\mathbb{H} = L^2([0,\pi])$ with the norm $\|.\|$. Define $A : \mathbb{H} \to \mathbb{H}$ by $A(t)z = z^{''}$ with the domain

$$\mathcal{D}(A) = \left\{z \in \mathbb{H} : z, z^{'} \text{ are absolutely continuous }, z^{''} \in \mathbb{H}, z(0) = z(\pi) = 0\right\}$$

It is well known that A generates a strongly continuous semigroup T(.),which is compact,analytic and self adjoint. Then

$$Az = \sum_{n=1}^\infty n^2 \langle z, z_n\rangle z_n, \quad z \in D(A)$$

where $z_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns), n = 1,2,....$ is the orthonormal set of eigenvector of A. It is well known that A is thein infinitesimal generator of ananalytic semigroup T(t)in H and is given by

$$T(t)z = \sum_{n=1}^\infty e^{-n^2t} \langle z, z_n\rangle z_n$$

Then the operator $A^{-\frac{1}{2}}$ is given by

$$A^{-\frac{1}{2}}z = \sum_{n=1}^\infty n \langle z, z_n\rangle z_n$$

on the space $\mathcal{D}(A^{-\frac{1}{2}}) = \{z(.) \in \mathbb{H} : \sum_{n=1}^\infty n\langle \zeta, z_n\rangle z_n \in \mathbb{H}\}$.

Now, we present a special $\mathcal{C}_{\upsilon}$ space. Let $\vartheta(s) = e^{2s}, s < 0$, then $l = \int_{-\infty}^{0} \vartheta(s)ds = \frac{1}{2}$.
Let

$$\|\varphi\|_{\mathcal{C}_{\upsilon}} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} E\left(\|\varphi(\theta)\|^2\right)^{\frac{1}{2}} ds$$

Then $(\mathcal{C}_{\upsilon}, \|.\|_{\mathcal{C}_{\upsilon}})$ is a Banach space.

For $(t,\varphi) \in J \times \mathcal{C}_{\upsilon}$ where $\varphi(\theta)(x) = \varphi(\theta, x), (\theta, x) \in (-\infty, 0] \times [0, \pi]$, and define the Lipschitz continuous functions $h, f : J \times \mathcal{C}_{\upsilon} \to H, \sigma : J \times \mathcal{C}_{\upsilon} \to L_Q(\mathbb{H})$, for the infinite delay as follows

$$h(t,\varphi)(x) = \int_{-\infty}^{0} e^{-4\theta}\varphi(\theta)(x)d\theta$$

$$f(t,\varphi)(x) = \int_{-\infty}^{0} \hat{a}(\theta)\sin(\varphi(\theta)(x))d\theta$$

$$\sigma(t,\varphi)(x) = \int_{-\infty}^{0} \varsigma(t,x,\theta)\sigma(\varphi(\theta)(x))d\theta$$

Then, the equation (3.10) can be rewritten as the abstract form as the system 1.1. Thus, under the appropriate condition so the functions h,f, and σ are satisfies the hypotheses $(H_1) - (H_5)$. All conditions of the Theorem 3.2 are satisfied, therefore the system (3.10) has a mild solution.

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Concept Lattice: A rough set approach

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Abstract

Concept lattice is an efficient tool for knowledge representation and knowledge discovery and is applied to many fields successfully. However, in many real life applications, the problem under investigation cannot be described by formal concepts. Such concepts are called the non-definable concepts. The hierarchical structure of formal concept (called concept lattice) represents a structural information which obtained automatically from the input data table. We deal with the problem in which how further additional information be supplied to utilize the basic object attribute data table. In this paper , we provide rough concept lattice to incorporate the rough set into the concept lattice by using equivalence relation. Some results are established to illustrate the paper.

Keywords: Rough Set, Formal Concept lattice, Equivalence Relation, Lattice.

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1 Introduction

The formal concept analysis (FCA) is a mathematical framework, developed by Wille [23] and his colleagues at Darmstadt University, which is very useful for representation and analysis of data [18].The concept lattice is also called Galois lattice, was proposed by Wille in 1982 [23]. A concept lattice is an ordered hierarchy that is defined by a binary relationship between objects and attributes in a data set. As an efficient tool of data analysis and knowledge processing, the concept lattice has been applied in many fields, such as knowledge engineering, data mining, information searches, and software engineering [6]. Most of the researchers have concentrated on their attention to the concept lattice and defined on such topics as: construction of the concept lattice, pruning of the concept lattice, acquisition of rules, relationship between the concept lattice and rough set [7] and approximation. The basic formal concept analysis deals with input data in the form of a table with rows corresponding to objects and columns corresponding to attributes. The data table is formally represented by a so called formal context which is a triplet (A, B, I) where A and B are sets and I is subset of $A \times B$ (i.e, $I \subseteq A \times B$) and defined a binary relation between A and B. The elements of A are called objects while the elements of B are called attributes or simply considered as the characteristics of objects. For a object and b characteristic , $(a, b) \in I$ or aIb shall indicate the following: a object owns the b attribute. Let us assume that (A, B, I) is a formal context. The knowledge about a considered universe is the starting point. Using two operations, a lower and an upper approximations, we can describe every subset of the universe. The concepts of the lower and upper approximations in rough set theory are fundamental to the examination of granularity in knowledge.

In this paper, we discuss the rough properties of concept lattice in rough set. FCA and rough set theory are two kinds of complementary mathematical tools for data analysis and data processing ([26], [27]). Up to now, many efforts have been made to combine these two theories ([26],[27],[3],[18],[22],[8],[24],[4]), in which the

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concept lattices based on rough set theory, including the attribute oriented concept lattice [2], lattice for covering rough approximation[14], rough set approach on lattice [12] and distributive lattice [16], rough modular lattice [13], lattice for rough intervals [11] and the object oriented concept lattice ([26],[27]), are perspective concept lattices for knowledge representation and knowledge discovery. However, the concept lattices usually contain redundant attributes and objects. In this paper , we provide rough concept lattice to incorporate rough set into the concept lattice by using equivalence relation.

2 Preliminaries

In this section, we present some definitions and fundamental concept on covering lattice.

Definition 2.1. *If $X \subseteq A$, $Y \subseteq B$, then two operators α and β can be defined as $\alpha : 2^A \rightarrow 2^B, \alpha(Y) = \{a \in B : aIb, \forall b \in B\}$.*
$\beta : 2^A \rightarrow 2^B, \beta(X) = \{b \in B : aIb, \forall a \in A\}$
$\beta(X)$ will take us to the set of attributes that are common in the entire objects in X set. Similarly $\alpha(Y)$ will take us to the attribute set of A that owns the entire attributes of Y. In other word $\beta(X)$, shall give the maximum object set that it hired by the entire objects in X while $\alpha(Y)$ still give the maximum object set that it owns by the entire objects in Y. (β, α) shall form a Galois connection between 2^A and 2^B.
Definition : *Let $K = (A, B, I)$ be a formal context, $X \in P(A)$ and $Y \in P(B)$, where $P(A)$ and $P(B)$ are the power set of A and B respectively. (X, Y) is called a concept, if $\alpha(X) = Y$ and $\beta(Y) = X$ hold for X and Y, where X is called the extent of the concept and Y is called the intent of the concept. $L(K)$ denotes the set of all concepts in the formal context..*

Definition 2.2. *Let (A, B, I) be a formal context. If there exists an attribute set $D \subseteq B$ such that $Latt(A, B, I_D) \cong Latt(A, B, I)$, then D is called a consistent set of (A, B, I). And further, if $\forall d \in D$, $Latt(A, D - \{d\}, I_{D-\{d\}}) \neq Latt(A, B, I)$. Then D is called a reduct of (A, B, I). The intersection of all the reducts is called the core of (A, B, I).*
Definition : *For the formal context $K = (A, B, I)$, let $H_1 = (X_1, Y_1)$ and $H_2 = (X_2, Y_2)$ be two elements of $Latt(K)$. If there exists $H_1 \leq H_2 \Leftrightarrow Y_2 \leq Y_1$, then $\leq$ is a partial order of $Latt(K)$, which produce a lattice structure in $Latt(K)$, called concept lattice of formal context $K = (A, B, I)$, also denoted by $Latt(K)$ Table 1 is a formal context, and Figure 1 shows its Hasse diagram..*

Lemma 2.1. *Let (A, B, I) be a context. Then the following assertions hold:*

- *$X_1 \subseteq X_2$ implies $\beta(A_1) \supseteq \beta(A_2)$for every $X_1, X_2 \subseteq A$, and $Y_1 \subseteq Y_2$ implies $\alpha(Y_1) \supseteq \alpha(Y_2)$for every $Y_1, Y_2 \subseteq B$.*
- *$X \subseteq \alpha(\beta(X))$ and $\beta(X) = \beta(\alpha(\beta(X)))$for all $X \subseteq A$, and $Y \subseteq \beta(\alpha(Y))$ and $\alpha(Y) = \alpha(\beta(\alpha(X)))$for all $Y \subseteq B$.*

.

3 Fundamental Theorem of FCA

The fundamental theorem of FCA states that the set of all formal concepts on a given context with the ordering $(X_1, Y_1) \leq (X_2, Y_2)$ if and only if $X_1 \subseteq X_2$ is a complete lattice called the concept lattice, in which the infima and suprema are given by $\bigwedge_{i \in J}(X_j, Y_j) = (\bigcap_{i \in J} X_j, \beta(\alpha(\bigcup_{i \in J} Y_j))) = (\bigcap_{i \in J} X_j, \beta(\bigcap_{i \in J} X_j)))$
$\bigvee_{i \in J}(X_j, Y_j) = (\alpha(\beta(\bigcup_{i \in J} X_j)), \bigcap_{i \in J} Y_j) = (\alpha(\bigcap_{i \in J} X_j)), \bigcap_{i \in J} X_j)$.

Theorem 3.1. *For two elements $H_1 = (X_1, Y_1)$ and $H_2 = (X_2, Y_2)$ of concept lattice $(Latt(K), \cap, \cup)$,if $\alpha(X_1 \cap X_2) = Y_1 \cap Y_2$ and $\beta(Y_1 \cap Y_2) = X_1 \cup X_2$ then $(L(K), \cap, \cup)$ is a distributive lattice.*

Proof. Since $(X_1, Y_1) \cup (X_2, Y_2) = (X_1 \cup X_2, Y_1 \cap Y_2)$ and $(X_1, Y_1) \cap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2)$, therefore

$$\begin{aligned}
H_1 \cap (H_2 \cap H_3) &= (X_1, Y_1) \cap (X_2, Y_2) \cap (X_3, Y_3) \\
&= (X_1, Y_1)(X_2 \cup X_3, Y_2 \cup Y_3) \\
&= (X_1 \cap (X_2 \cup X_3), Y_1 \cup (Y_2 \cap Y_3)) \\
&= ((X_1 \cap X_2)(X_1 \cap X_3), (Y_1 \cup Y_2)(Y_1 \cup Y_3)).
\end{aligned}$$

and $(H_1 \cap H_2)(H_1 \cup H_3) = (X_1 \cap X_2, Y_1 \cap Y_2)(X_1 \cap X_3, Y_1 \cap Y_3) = ((X_1 \cap X_2)(X_1 \cap X_3,), (Y_1 \cup Y_2)(Y_1 \cup Y_3))$ i.e., $H_1 \cap (H_2 \cup H_3) = (H_1 \cup H_2)(H_1 \cap H_3)$. Similarly , we can prove $H_1 \cup (H_2 \cap H_3) = (H_1 \cap H_2)(H_1 \cup H_3)$. Thus $(Latt(K), \cap, \cup)$ is a distributive lattice. □

Theorem 3.2. *Let $Latt(K)$ be a concept lattice in formal context $K = (A, B, I)$. Let $H_1 = (X_1, Y_1)$ and $H_2 = (X_2, Y_2)$ be elements of $L(K)$. The following propositions are equivalent.*
i) $H_1 \leq H_2$ ii) $H_1 \cap H_2 = H_1; H_1 \cup H_2 = H_2$.

Proof. Suppose that i) is true . Since $Y_2 \subseteq Y_1$ and $X_1 \subseteq X_2$, we have $(X_1, Y_1) \cap (X_2, Y_2) = (X_1 \cap X_2, \alpha(X_1 \cap X_2)) = (X_1, \alpha(X_1)) = (X_1, Y_1) = H_1$ and $H_1 \cup H_2 = (X_1, Y_1) \cup (X_2, Y_2) = (\beta(Y_1 \cap Y_2), Y_1 \cap Y_2)) = (\beta(Y_2), Y_2) = (X_2, Y_2) = H_2$, Hence, ii) is true. Suppose that ii) is true. Since $H_1 \cap H_2 = H_1 \Leftrightarrow (X_1, Y_1) = (X_1 \cap X_2, \alpha(X_1 \cap X_2))$, from definition 3 , it follows that $Y_1 \cup Y_2 \cap \alpha(X_1 \cap X_2) = Y_1$ and $Y_1 \cup Y_2 \subseteq Y_1$ .Thus $Y_2 \subseteq Y_1$, *i.e.*, $H_1 \leq H_2$. Hence i) is true. □

Example-1: Table-1 shows a formal context (A, B, I), in which $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$. The concepts are $(\{1\}, \{a, b, d, e\}); (\{2, 4\}, \{a, b, c\}); (\{1, 3\}, \{d\}); (\{1, 2, 4\}, \{a, b\}); (U, \emptyset)$. The concept lattice is shown in Figure-1.

	a	b	c	d	e
1	1	1	0	1	1
2	1	1	1	0	0
3	0	0	0	1	0
4	1	1	1	0	0

Table - 1

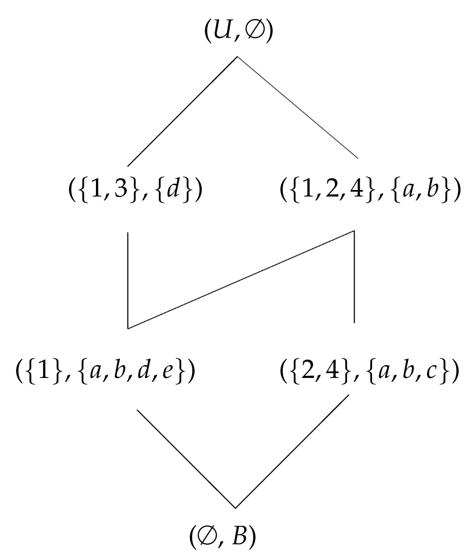

Figure-1: $Latt(A, B, I)$ in example 1

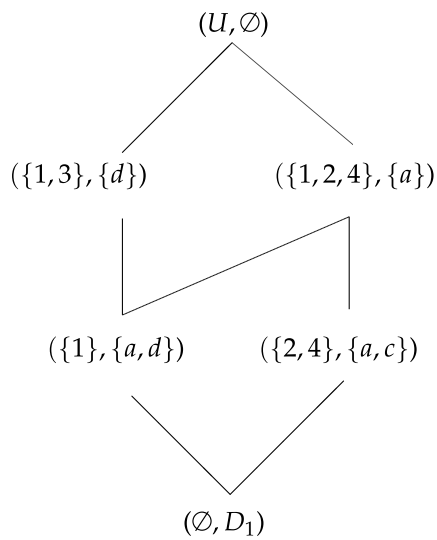

Figure-2: $Latt(A, B, I_{D_1})$ in example 1

If we consider another example then it will able to understand how far the approximation causes for concept approximation in rough set.

Example-2: Table-2 shows a formal context (A, B, I), in which $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$. The concepts are $(\{1\}, \{a, b, e\})$; $(\{4\}, \{a, c, e\})$; $(\{2\}, \{a, b, c\})$; $(\{2, 4\}, \{a, c\})$; $(\{1, 2\}, \{a, b\})$; $(\{1, 2, 4\}, \{a\})$; $(\{1, 3, 4\}, \{e\})$; $(U, \emptyset)$. The concept lattice is shown in Figure-3.

	a	b	c	d	e
1	1	1	0	0	1
2	1	1	1	0	0
3	0	0	0	1	1
4	1	0	1	0	1

Table - 2

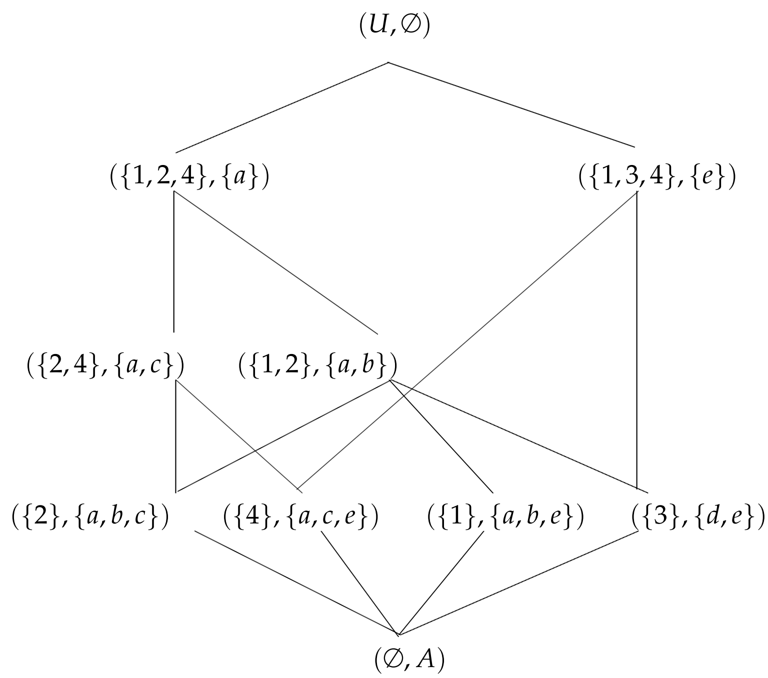

Figure-3: Concept lattice for table-2

Let (A, B, I) be a formal context. Clearly we have the following results:

Proposition 3.1. *The core the formal context is a reduct* $\Leftrightarrow$ *There is only on reduct in the formal context.*

Proposition 3.2. $a \in B$ *is an unnecessary attribute* $\Leftrightarrow$ $A - \{a\}$ *is a consistent set.*

Proposition 3.3. $a \in B$ *is an element of the core* $\Leftrightarrow$ $A - \{a\}$ *is not a consistent set.*

If we consider another example then it will able to understand how far the approximation causes for concept approximation in rough set.

4 Rough Concept Lattice

Let (A, B, I, E) be an information system, where $A = \{a_1, a_2, \ldots, a_m\}$ is an object set, $B = \{x_1, x_2, x_3, x_4, ..., x_n\}$ is an attribute (property) set, E is an equivalent relation on A, $\forall X \subseteq A$, we can define the upper and lower approximations of X about E

$$X \downarrow_E \quad = \quad \bigcup\{M \in A/E : M \subseteq X\} = \{x \in A : E_A(x) \subseteq X\} \tag{4.1}$$

$$X \uparrow_E \quad = \quad \bigcup\{M \in A/E : M \cap X \neq \emptyset\} = \{x \in A : E_X(x) \cap X \neq \emptyset\}. \tag{4.2}$$

$X \downarrow_E$, $X \uparrow_E$ are called E- lower approximation and E- upper approximation of X respectively. If $X \downarrow_E = X \uparrow_E$, we say that X is definable, otherwise, X is rough. Similarly, we can define the upper and lower approximations of attributes set $Y \subseteq B$ about an equivalent relation on B.

Definition 4.3. *For all* $Y \subseteq B$, *we denote* $E_Y = \{(a_i, a_j) \in X \times X : f_p(a_i) = f_p(a_j), p \in Y\}$, *where* $f_p : A \to \{0, 1\}$ *is defined by* $f_p(a_i) = 1$ *if and only if the object* a_i *possesses property* $p, (p \in A, a_i \in A)$, *and* E_Y *is an equivalent relation, and* E_Y *can generate a partition of* A, $X(Y) = \{Y(x) : x \in A\} = X/E_Y$, *where* $Y(x) = \{y \in A : yE_Yx\} = \{y \in A : f_p(y) = f_p(x), a_p \in Y\}$, *that is,* $Y(x) = \wedge\{(a_p, f_p(x)) : a_p \in Y\}, \bigcup Y(x) = \bigwedge(\wedge\{(a_p, f_p(x)) : a_p \in Y\})$.

Definition 4.4. *Let* (A, B, I, E) *is a rough formal context* $I \subseteq A \times B$ *for a set* $Y \subseteq B$ *of attributes , we define function* $\downarrow: 2^A \rightarrow 2^B, Y\downarrow = \{a \in A : (a, b) \in E, \forall b \in Y\}$ *(the set objects which have all attributes in Y). Correspondingly, for a set* $X \subseteq Y$ *of objects ,we define:* $\uparrow: 2^A \rightarrow 2^B, X\downarrow = \{b \in Y : (a, b) \in I, \forall a \in X\}$ *(the set objects which have all attributes in X).*

Example-3: The above table 1 is rough formal context (A, B, I, E) where $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d, e\}$ by rough theory $X = \{1, 2\} \subseteq A, Y = \{a, e\} \subseteq B$ then $f_a(1) = f_a(2) = f_a(4) = 1; f_a(3) = 0.f_b(1) = f_b(2) = f_b(4) = 1; f_b(3) = 0.f_c(2) = f_c(4) = 1; f_c(1) = f_c(3) = 0.f_d(1) = f_3(1) = 1; f_d(2) = f_d(4) = 0.f_e(1) = 1; f_e(2) = f_e(3) = f_e(4) = 0$, and the partition of X is : $B/X(1) = \{\{a, b, d, e\}, \{c\}\}; B/X(2) = \{\{a, b, c\}, \{d, e\}\}$. So $B/X = \{\{a, b\}, \{d, e\}, \{c\}, \emptyset\}$; and the partition of Y is $A/Y(a) = \{\{1, 2, 4\}, \{3\}\}.A/Y(e) = \{\{1\}, \{2, 3, 4\}\}$. So $A/Y = \{\{1\}, \{2, 4\}, \{3\}, \emptyset\}$. Under $B/X = E_X$ be the lower approximation of $Y = Y\downarrow_{E_X} = \emptyset$, the upper approximation of $Y = Y\uparrow_{E_X} = \{a, b, d, e\}$. Under $B/Y = E_Y$ be the lower approximation of $X = X\downarrow_{E_Y} = \{1\}$, the upper approximation of $X = Y\uparrow_{E_Y} = \{1, 2, 4\}$.
Here $f_p : A \rightarrow \{0, 1\}$, set $\{0, 1\}$ can extent to $[0, 1]$, and the condition of equivalent relation can also substitute for other relations, for example, the similar relation (that is, it satisfies reflexivity, symmetry, and transitivity). In essential, those functions are the same. The information system (A, B, I, E) which has lower and upper approximations and partition is called a rough formal context. $\forall a \in A, b \in B$, object a has attribute B then $(a, b) \in I, \quad aIb$. For the rough formal context in Table-1, the Hasse diagram is shown in Figure 4:

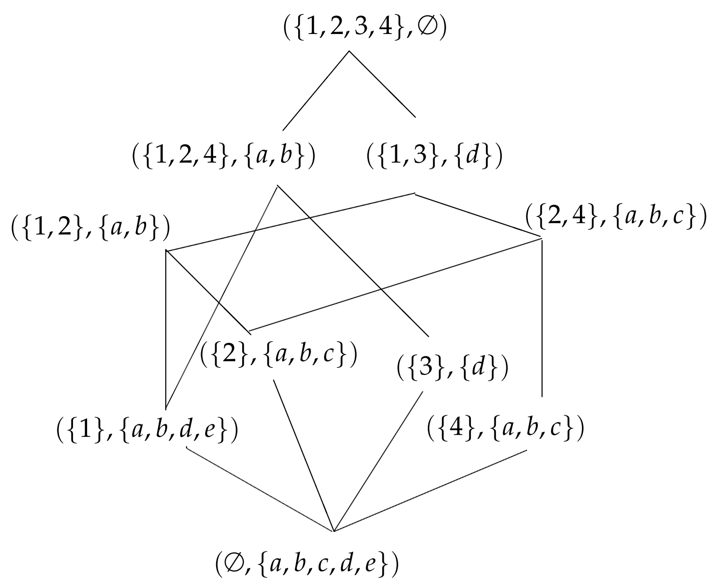

Figure-4: Rough concept lattice for the rough formal context in Table-1

Now we consider another example of a rough formal context and its corresponding concept lattice
Example-4: The following Table-3 is a rough formal context (A, B, I, E) where $A = \{1, 2, 3, 4, 5\}, B = \{h_1, h_2, ..., h_9\}$,

where 1,2,3,4,5 stands for "big", "beautiful","wooden", "cheap" and the green surroundings respectively. Also here $h_1, h_2, ..., h_9$ are nine houses.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
1	1	0	1	0	0	1	0	1	0
2	1	0	1	0	0	0	1	0	1
3	1	0	0	1	0	0	1	0	1
4	0	1	1	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0

Table -3

By rough set theory $X = \{1,2\} \subseteq A, Y = \{h_1, h_5\} \subseteq B$ then $f_{h_1}(1) = f_{h_1}(2) = f_{h_1}(3) = 1; f_{h_1}(4) = f_{h_1}(5) = 0. f_{h_2}(1) = f_{h_2}(2) = f_{h_2}(3) = 1; f_{h_2}(4) = f_{h_2}(5) = 0. f_{h_3}(1) = f_{h_3}(2) = f_{h_3}(4) = 1; f_{h_3}(3) = f_{h_3}(5) = 0$....and the partition of X are : $B/X(1) = \{\{h_1, h_3, h_6, h_8\}, \{h_2, h_4, h_5, h_7, h_9\}\}; B/X(2) = \{\{h_1, h_3, h_7\}, \{h_2, h_4, h_5, h_6, h_8\}\}$ so $B/X = \{\{h_1, h_3\}, \{h_6, h_8\}, \{h_7, h_9\}, \{h_4, h_5\}\}$; and the partition of Y are $A/Y(h_1) = \{\{1,2,3\}, \{4,5\}\}$, $A/Y(h_9) = \{\{2,3\}, \{1,4,5\}\}$. So $A/Y = \{\{1\}, \{2,3\}, \{4,5\}\}$. Under $B/X = E_X$ be the lower approximation of $Y = Y \downarrow_{E_X} = \emptyset$, the upper approximation of $Y = Y \uparrow_{E_X} = \{h_1, h_3, h_7, h_9\}$.Under $B/Y = E_Y$ be the lower approximation of $X = X \downarrow_{E_Y} = \{1\}$, the upper approximation of $X = Y \uparrow_{E_Y} = \{1,2,3\}$.
Here $I = \{0,1\}$ i.e, we only consider $\{a, b\} \in I$ or not. The following figure represents rough concept lattice, based on the information system described in Table-3.

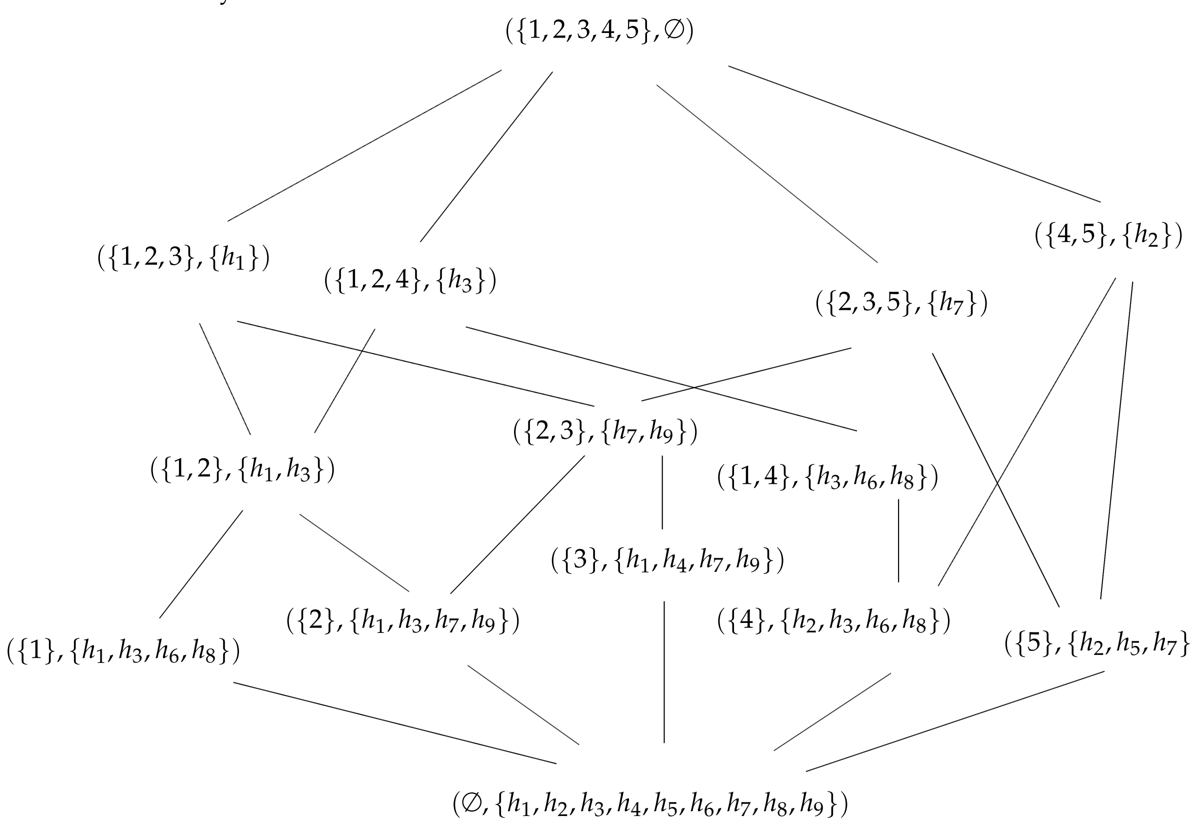

Figure-5: Rough concept lattice for the rough formal context in Table-3

One can find the extension of their meet by the set intersection of their extensions, and the intension of their join by the set intersection of their intensions. One cannot find directly the intension of their meet and the extension of their join by simply applying set-theoretic operators.

5 Conclusion

This paper presents the approach to approximate concepts in the framework of the formal concept analysis. The main focus is to show how rough set techniques can be employed as an approach to the problem of knowledge extraction. The approaches show how to approximate single sets of objects, single sets of features, and non-definable concepts. we use both the set of objects and the set of features for approximating non-definable concepts, whose results in the fact that non-definable concepts with the same set of objects have different and more accurate concept approximations. Rough lattice combines the advantages of concept lattice and rough set, so it is widely used in Information Retrieval, Data Mining, Software Engineering and other fields [27].

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Radial symmetry of positive solutions for nonlinear elliptic boundary value problems

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Abstract

The aim of this paper is to study the symmetry properties of positive solutions of nonlinear elliptic boundary value problems of type

$$\Delta u + f(|x|, u, \nabla u) = 0 \ \text{ in } \ R^n.$$

$$u(x) \to 0 \text{ as } |x| \to \infty$$

We employ the moving plane method based on maximum principle on unbounded domains to obtain the result on symmetry.

Keywords: Maximum principle; Moving plane method; Semilinear elliptic boundary value problems.

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1 Introduction

The moving plane method is a clever way of using the maximum principle to obtain the qualitative properties of positive solutions of some elliptic equations, notably the symmetry of solutions. It was introduced by Alexandroff [6] in his study of surfaces of constant mean curvature. In 1971, Serrin [13] first proved the symmetry properties of some overdetermined elliptic problems. It has become welknown through the work of Gidas, Ni and Nirenberg [3],[4] where it was used to obtain the symmetry results for positive solutions of nonlinear elliptic equations. Since then, this method has been further developed and used in variety of problems by many researchers. Pucci, Sciunzi and Serrin [12] studied symmetry of solutions of degenerate quasilinear elliptic problems by applying comparison principle. Farina, Montoro and Sciunzi [2] obtained symmetry results for semilinear p-Laplacian equation. In this paper we present an approach based on the maximum principle in unbounded domains together with the method of moving planes. Recently Dhaigude and Patil [1] proved the symmetry result for same equation in unit ball. Naito [10] obtained symmetry result for semilinear elliptic equations in R^2. Further Naito [9] studied the semilinear elliptic problem $\Delta u + f(|x|, u) = 0$ in R^n where $u(x) \to 0$ as $|x| \to \infty$.

In this paper we study the radial symmetry of positive solutions for nonlinear elliptic boundary value problems for second order elliptic equations in R^n. We consider the problem of the form

$$\Delta u + f(|x|, u, \nabla u) = 0 \ \text{ in } \ R^n \tag{1.1}$$

$$u(x) \to 0 \text{ as } |x| \to \infty$$

where $n \geq 3$. We organise the paper as follows: In section 2 the preliminary results and some useful lemmas are proved. The symmetry result and corollaries are proved in the last section.

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2 Preliminaries

In this section, first we state some lemmas and theorem which are useful to prove our main result.

Lemma 2.1. *Hopf Boundary lemma[5] : Let Ω be closed subset of R^n. Suppose that Ω satisfies the interior sphere condition at $x_0 \in \partial\Omega$. Let L be strictly elliptic with $c \leq 0$ where*

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial_{x_i}\partial_{x_j}} + b_i(x)\sum_{j=1}^{n}\frac{\partial}{\partial x_j} + c(x)$$

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $L(u) \geq 0$ and $\max_{\bar{\Omega}} u(x) = u(x_0)$ then either $u = u(x_0)$ on Ω or

$$\liminf_{t\to 0} \frac{u(x) - u(x_0 + t\nu)}{t} > 0$$

for every direction ν, pointing into an interior sphere. If $u \in C^1 \subset \Omega \cup \{0\}$ then

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

where $\frac{\partial}{\partial \nu}$ is any outward directional derivative.

Lemma 2.2. *[8] Let Ω be unbounded domain in R^n . Suppose that $u \neq 0$ satisfies*

$$L(u) \leq 0 \quad in \quad \Omega \quad and \quad u \geq 0 \quad on \quad \partial\Omega.$$

Suppose furthermore that there exist a function w such that $w > 0$ on $\Omega \cup \partial\Omega$ and $L(w) \leq 0$ in Ω. If

$$\frac{u(x)}{w(x)} \to 0 \quad as \quad |x| \to \infty, \quad x \in \Omega$$

then $u > 0$ in Ω.

Theorem 2.1. *[11] Let $u(x)$ satisfies differential inequality*

$$(L)(u) \geq 0,$$

in a domain Ω where L is uniformly elliptic. If there exist a function w(x) such that, $w(x) > 0$ on $\Omega \cup \partial\Omega$

$$(L)(w) \leq 0 \quad in \quad \Omega$$

then $\frac{u(x)}{w(x)}$ can not attain a non negative maximum at a point p on $\partial\Omega$, which lies on the boundary of a ball in Ω and if $\frac{u}{w}$ is not constant then,

$$\frac{\partial}{\partial \nu}\left(\frac{u}{v}\right) > 0 \ \ at\ P$$

where $\frac{\partial}{\partial \nu}$ is any outward directional derivative.

3 Main Results

We define following,

Let $\lambda > 0$ a real number. Define the plane $T_\lambda = \{x : x = (x_1, x_2, x_3, ..., x_n), x_1 = \lambda\}$, which is the plane perpendicular to x_1-axis. We will move this plane continuously normal to itself to new position till it begins to intersect Ω. After that point the plane advances in Ω along x_1- axis and cut of cap Σ_λ; which is the portion of Ω and lies in the same side of the plane T_λ as the original plane T.

$\Sigma_\lambda = \{x : x_1 < \lambda, x \in \Omega\}$.

Let $x^\lambda = (2\lambda - x_1, x_2, x_3, ...x_n)$ be the reflection of the point $x = (x_1, x_2, x_3, ..., x_n)$, about the plane T_λ.

Define $V_\lambda(x) = u(x) - u(x^\lambda)$. We have $|x^\lambda| \geq |x|$ and $u(x^\lambda) = u(2\lambda - x_1, x_2, x_3, ..., x_n)$.

Define $\Lambda = \{\lambda \in (0, \infty) : V_\lambda(x) > 0\}$ for $x \in \Sigma_\lambda$.

In (1.1), we assume that $f(|x|, u(x), \nabla u(x))$ is continuous and C^1 in $u \geq 0$. Also assume that $f(|x|, u(x), \nabla u(x))$ is nonincreasing in $|x| = r > 0$, for each fixed $u \geq 0$.

Our main result is the following

Theorem 3.2. *Let $u \in C^2(R^n)$ be a positive solution of (1.1) with following conditions*

1. *f is continuous in all of its variables and Lipschitz in u*
2. *$f(|x|, u, (p_1, p_2, p_3, ..., p_{i-1}, -p_i, p_{i+1}, ..., p_n)) = f(|x|, u, (p_1, p_2, p_3, ..., p_n))$ for all $1 \leq i \leq n$*
3. *f is nonincreasing in $|x| = r > 0$, for each fixed $u \geq 0$.*

Define U and Φ as

$$U(r) = Sup\{u(x) : |x| \geq r\} \tag{3.1}$$

$$\Phi(r) = Sup\Big\{\frac{\partial f}{\partial u}(|x|, u(x), \nabla u(x)) : 0 \leq u(x) \leq U(r)\Big\} \tag{3.2}$$

respectively. Assume that there exist a positive function w on $|x| \geq R_0$ for some $R_0 > 0$ satisfying

$$\Delta w + \phi(|x|)w \leq 0 \ in \ |x| > R_0 \tag{3.3}$$

$$\lim_{|x| \to \infty} \frac{u(|x|)}{w(x)} = 0, \tag{3.4}$$

then u must be radially symmetric about some point $x_0 \in R^n$ and $u_r < 0$ for $r > 0$.

Before proceeding to the proof of main result we shall state and prove some lemmas.

Lemma 3.1. *Let $\lambda \geq 0$ then*

$$\Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x) \leq 0 \ in \ \Sigma_\lambda, \tag{3.5}$$

where

$$C_\lambda(x) = \int_0^1 f_u(|x|, u(x) + t(u(x^\lambda) - u(x)), \nabla u(x))dt.$$

Proof. Let u be the positive solution of (1.1), $u(x^\lambda)$ satisfies the same equation that u does.

$$\Delta u(x^\lambda) + f(|x^\lambda|, u(x^\lambda), \nabla u(x^\lambda)) = 0 \ \text{ in } \ R^n. \tag{3.6}$$

Since $u(x) = u(x_1, x_2, x_3, ..., x_n)$

$$\begin{aligned}\nabla u(x) &= \hat{i_1}\frac{\partial u}{\partial x_1} + \hat{i_2}\frac{\partial u}{\partial x_2} + \hat{i_3}\frac{\partial u}{\partial x_3} + ... + \hat{i_n}\frac{\partial u}{\partial x_n}\\ &= (p_1, p_2, p_3, ..., p_n).\end{aligned}$$

Since $u(x^\lambda) = u(2\lambda - x_1, x_2, x_3, ..., x_n)$

$$\begin{aligned}\nabla u(x^\lambda) &= \hat{i_1}\frac{\partial u}{\partial x_1}(-1) + \hat{i_2}\frac{\partial u}{\partial x_2} + \hat{i_3}\frac{\partial u}{\partial x_3} + ... + \hat{i_n}\frac{\partial u}{\partial x_n}\\ &= (-p_1, p_2, p_3, ..., p_n).\end{aligned}$$

Subtracting equation (3.6) from equation (1.1) we get

$$\begin{aligned}0 &= [\Delta u(x) + f(|x|, u(x), \nabla u(x))] - [\Delta u(x^\lambda) + f(|x^\lambda|, u(x^\lambda), \nabla u(x^\lambda))]\\ &= \Delta u(x) - \Delta u(x^\lambda) + f(|x|, u(x), \nabla u(x)) - f(|x^\lambda|, u(x^\lambda), \nabla u(x^\lambda))\\ &= \Delta V_\lambda(x) + f(|x|, u(x), (p_1, p_2, p_3, ..., p_n)) - f(|x^\lambda|, u(x^\lambda), (-p_1, p_2, p_3, ..., p_n))\\ &= \Delta V_\lambda(x) + f(|x|, u(x), (p_1, p_2, p_3, ..., p_n)) - f(|x^\lambda|, u(x^\lambda), (p_1, p_2, p_3, ..., p_n))\\ &\geq \Delta V_\lambda(x) + f(|x|, u(x), \nabla u(x)) - f(|x|, u(x^\lambda), \nabla u(x))\\ &\geq \Delta V_\lambda(x) + \frac{f(|x|, u(x), \nabla u(x)) - f(|x|, u(x^\lambda), \nabla u(x))}{u(x) - u(x^\lambda)}(u(x) - u(x^\lambda))\\ &\geq \Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x))\end{aligned}$$

where

$$C_\lambda(x) = \frac{f(|x|, u(x), \nabla u(x)) - f(|x|, u(x^\lambda), \nabla u(x))}{u(x) - u(x^\lambda)}$$
$$= \int_0^1 f_u(|x|, u(x) + t(u(x^\lambda) - u(x)), \nabla u(x))dt.$$

□

Before the next lemma we shall define, $B_0 = \{x \in R^n : |x| < R_0\}$ and $\bar{B_0} = \{x \in R^n : |x| \le R_0\}$.

Lemma 3.2. *Let $\lambda > 0$, If $V_\lambda > 0$ on $\partial\Sigma_\lambda \cap \bar{B_0}$ then $\lambda \in \Lambda$.*

Proof. Let $\lambda > 0$. Suppose $V_\lambda > 0$ on $\partial\Sigma_\lambda \cap \bar{B_0}$ then from lemma 3.1 and assumption we have

$$\Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x) \le 0 \text{ in } \Sigma_\lambda \setminus \bar{B_0}$$
$$V_\lambda(x) \ge 0 \text{ on } \partial(\Sigma_\lambda \setminus \bar{B_0})$$

As $U(r) = sup\{u(x) : |x| \ge r\}$ and $\Phi(r) = sup\{f_u(|x|, u(x), \nabla u(x))\}$
$U(r)$ is nonincreasing,
$0 < u(x) + t(u(x^\lambda) - u(x)) \le u(|x|), \quad 0 \le t \le 1.$
Then by lemma 3.1,

$$C_\lambda(x) = \int_0^1 f_u(|x|, u(x) + t(u(x^\lambda) - u(x)), \nabla u(x))dt$$
$$\le \int_0^1 f_u(|x|, U(x), \nabla u(x))dt$$
$$\le \int_0^1 \Phi(|x|)dt \le \Phi(|x|) \text{ in } \Sigma_\lambda.$$

From

$$\Delta w + \phi(|x|)w \ge 0 \text{ in } |x| \ge R_0$$
$$\lim_{|x|\to\infty} \frac{u(|x|)}{w(x)} = 0.$$

The positive function w satisfies

$$\Delta w + \phi(|x|)w \le 0 \text{ in } \Sigma_\lambda \setminus \bar{B_0}$$

and

$$\frac{V_\lambda(x)}{w(x)} \le \frac{U(|x|)}{w(x)} \to 0 \text{ in } x \in \Sigma_\lambda \setminus \bar{B_0} \text{ as } |x| \to \infty.$$

Hence by maximum principle, $V_\lambda(x) > 0$ in $\Sigma_\lambda \setminus \bar{B_0}$.
By assumption $V_\lambda(x) > 0$ in Σ_λ. Therefore $\lambda \in \Lambda$. □

Lemma 3.3. *If $\lambda \in \Lambda$ then $\frac{\partial u}{\partial x_1} < 0$ on T_λ*

Proof. Let $\lambda \in \Lambda$. Hence $\lambda > 0$. By lemma 3.1

$$\Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x) \le 0 \text{ in } \Sigma_\lambda$$
$$V_\lambda(x) \ge 0 \text{ on } \partial(\Sigma_\lambda).$$

On T_λ we have,

$$u(x) = u(x^\lambda).$$

Hence $V_\lambda(x) = 0$ on T_λ.
By Hopf boundary lemma, $\frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ. Therefore $\frac{\partial u}{\partial x_1} = \frac{1}{2}\frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ □

Now we shall prove the main theorem 3.2

Proof. Since $u(x)$ is positive solution of (1.1) such that

$$\lim_{|x|\to\infty} u(x) = 0,$$

then we can find $R_1 > R_0$ such that

$$\max\{u(x) : |x| > R_1\} < \min\{u(x) : |x| \le R_0\}$$

where R_0 is constant. We shall prove the theorem in following three steps.
Step-I: Define $\bar{B_0} = \{x \in R^n : |x| < R_0\}$. Clearly $\bar{B_0} \subset \Sigma_{\bar{\lambda}}$. Let $\lambda \ge R_1$. Also $V_\lambda(x) \ge 0$ in $\bar{B_0}$. Therefore $V_\lambda(x) \ge 0$ in $\Sigma_\lambda \cap \bar{B_0}$. Hence $\lambda \in \Lambda$. Thus we can conclude that $[R_1, \infty) \subset \Lambda$.
Step-II: To prove,If $\lambda_0 \in \Lambda$ then there exist $\epsilon > 0$ such that $(\lambda_0 - \epsilon, \lambda_0) \subset \Lambda$. We use contradiction method to prove this. Suppose there exist increasing sequence $\{\lambda_i\}, i = 1, 2, 3, ...$ such that $\lambda_i \notin \Lambda$ and $\lambda_i \to \lambda_0$ as $i \to \infty$ then by contradiction to lemma 3.2 we have a sequence $\{x_i\}, i = 1, 2, 3, ...$ such that $x_i \in \Sigma_{\lambda_i} \cap \bar{B_0}$ and $V_{\lambda_i}(x_i) \le 0$. It has a subsequence which converges to $x_0 \in \bar{\Sigma_{\lambda_0}} \cap \bar{B_0}$. Then $V_{\lambda_0}(x_0) \le 0$ but in Σ_{λ_0} we have $V_{\lambda_0}(x_0) > 0$, therefore $x_0 \in T_{\lambda_0}$.
Using mean value theorem we can find y_i satisfying

$$\frac{\partial u}{\partial x_i}(y_i) \ge 0$$

on the line segment joining $x_i \to x_i^{\lambda_i}$ for each $i = 1, 2, 3, ...$ also $y_i \to x_0$ as $i \to \infty$. So $\frac{\partial u}{\partial x_1}(x_0) \ge 0$. But by lemma 3.3 we have

$$\frac{\partial u}{\partial x_i}(x_0) \le 0.$$

This is a contradiction.Hence the step-II is proved.
Thus if $\lambda_0 \in \Lambda$ then there exist $\epsilon > 0$ such that $(\lambda_0 - \epsilon, \lambda_0) \subset \Lambda$.
Step-III: In this step we shall prove that either statement (A) or statement (B) happens.
(A)$V_{\lambda_1}(x) = 0$ for some $\lambda_1 > 0$ and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_1$
or
(B)$V_{\lambda_1}(x) > 0$ in Σ_{λ_0} and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_0$
We have $\lambda_1 = \inf\{\lambda > 0 | (\lambda, \infty) \subset \Lambda\}$. Therefore $\lambda_1 > 0$ or $\lambda_1 = 0$ be the two distinct cases.
Case - 1: If $\lambda_1 > 0$, since u is continuous function, $V_{\lambda_1} \ge 0$ in Σ_{λ_1}. Therefore by lemma 3.1

$$\Delta V_\lambda(x) + C_\lambda(x) V_\lambda(x) \le 0 \;\; in \;\; \Sigma_{\lambda_1}$$

By strong maximum principle we have either $V_\lambda(x) \ge 0$ or $V_\lambda(x) = 0$ in Σ_{λ_1}. If $V_\lambda(x) = 0$, then statement (A) occurs, i.e. $u(x) = u(x^{\lambda_1})$ holds and by lemma 3.3 $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_1$. Now suppose $V_{\lambda_1}(x) > 0$ in Σ_{λ_1}, then $\lambda_1 \in \Lambda$ As we have already proved, there exist $\epsilon > 0$ such that $(\lambda_1 - \epsilon, \lambda_1) \subset \Lambda$. This contradicts to the fact that λ_1 is infimum. Therefore $V_{\lambda_1}(x) = 0$ in Σ_λ. This implies (A) holds.
Case - 2: $\lambda_1 = 0$
Since $u(x)$ is continuous we have $V_{\lambda_1}(x) > 0$ in Σ_0.
Therefore $V_{\lambda_1}(x) = u(x) - u(x^0) \ge 0$ in Σ_0. Therefore $u(x) \ge u(x^0)$ in Σ_0.
By lemma 3.3 $\frac{\partial u}{\partial x_1} < 0$ on T_λ for$\lambda > 0$. Thus (B) holds. If statement (B) occurs in step- III, we can repeat the previous steps I-III for the negative x_1- direction to conclude that either u is symmetric in the x_1- direction about some plane $x_1 = \lambda_1 < 0$ or $u(x) \le u(x^0)$ in Σ_0. Thus $u(x) = u(x^0)$ in Σ_0. Thus u must be radially symmetric in x_1- direction about some plane and strictly decreasing away from the plane. Since we can place x_1- axis along any direction we conclude that $u(x)$ is radially symmetric in every direction about some plane. Therefore u is radially symmetric about some point $x_0 \in R^n$ and $u_r < 0$. □

Corollary 3.1. *Assume that $f_u(r, u, \nabla u) \le 0$ for $r \ge r_0, 0 \le u \le u_0$ with some constants $r_0 \ge 0$ and $u_0 \ge 0$. Let u be the positive solution of (1.1). Then u must be radially symmetric about some point $x_0 \in R^n$ and $u_r < 0$ for $r = |x - x_0| > 0$.*

Remark 3.1. *. If gradient term is absent in f, related results have been obtained in [4, 7, 9].*

Proof. We see that the function U defined by (3.1) satisfies $U(r) \to 0$ as $|x| \to \infty$.

Take $R_0 > r_0$ so large that $U(r) < u_0$ for $r \geq R_0$. Define w as $w(x) \equiv 1$ on $|x| > R_0$, then w satisfies (3.4). Since $\Phi(r) = \max\{\frac{\partial f}{\partial u}(|x|, u(x), \nabla u(x)) | 0 \leq u(x) \leq U(r)\} \leq 0$ for $r \geq R_0$ we have (3.3). Thus all the conditions of theorem are satisfied so we can apply the theorem for conclusion. □

For simplicity we consider the equation of the form

$$\Delta u + \phi(|x|) f(u(x), \nabla u(x)) = 0 \ \text{in} \ R^n \tag{3.7}$$

with the assumption that $\phi \in C[0, \infty)$ satisfies $\phi(r) \geq 0$ for $r \geq 0$ and $\phi(r)$ is nonincreasing in $r > 0$, and that $f \in C^1[0, \infty)$ with $f(u, \nabla u)) > 0$ for $u > 0$.

Corollary 3.2. *In equation (3.7),suppose that*

1. *f is continuous in all of its variables and Lipschitz in u*
2. *$f(|x|, u, p_1, p_2, p_3, ..., p_{i-1}, -p_i, p_{i+1}, ..., p_n) = f(|x|, u, p_1, p_2, p_3, ..., p_n)$ for all $1 \leq i \leq n$*
3. *f is nonincreasing in $|x|$, for each fixed $u \geq 0$*

we furthermore assume that $\phi \not\equiv 0$ and

$$\int_0^\infty r\phi(r) dr < \infty. \tag{3.8}$$

Let u be positive solution of (3.7) , satisfying $u(x) \to c$ as $|x| \to \infty$ for some constant $C \geq 0$, then u must be radially symmetric about the origin and $u_r < 0$ for $r > 0$.

Remark 3.2. *. If gradient term is absent in f, related results have been obtained by [4, 7, 9].*

Proof. Consider $V(x) = u(x) - C$. Then we have $\nabla V(x) = \nabla u(x)$ and hence $\Delta V(x) = \Delta u(x)$. Then V satisfies

$$\Delta V(x) + \phi(|x|) h(V) = 0 \ \text{in} \ R^n$$
$$V(x) \to 0 \ \text{as} \ |x| \to \infty$$

where $h(V) = f(V + C, \nabla V)$

Since $-\Delta V = \phi h \geq 0$, we have $V > 0$ in R^n, by the maximum principle. We apply theorem 3.2 to the problem (3.7). We define U and Φ as $U(r) = sup\{V(x) : |x| \geq r\}$ and $\Phi(r) = sup\{\phi(r) h'(s) : 0 < s \leq U(r)\}$ respectively. Since $\Phi(r) \leq M\phi(r)$ for some constant $M > 0$. and (3.8) holds. Then there exist a positive function w on $|x| > R_0$ for some $R_0 > 0$ satisfying

$$\Delta w(x) + \phi(|x|) w(x) = 0 \ \text{and}$$
$$w(x) \to 0 \ \text{as} \ |x| \to \infty$$

Then w satisfies the conditions of the theorem 3.2. Therefore theorem 3.2 can be applied to conclude the assertion. □

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Some Results for a Four-Point Boundary Value Problems for a Coupled System Involving Caputo Derivatives

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Abstract

Motivated by the problem (1.1) in [5], in this paper, we prove the existence and uniqueness of solutions for the following system of fractional differential equations with four point boundary conditions:

$$\begin{cases} D^{\alpha}x(t)+f\left(t,y(t),D^{\delta}y(t)\right)=0, t\in J,\\ D^{\beta}y(t)+g\left(t,x(t),D^{\sigma}x(t)\right)=0, t\in J,\\ x(0)=y(0)=0, x(1)-\lambda_1 x(\eta)=0, y(1)-\lambda_1 y(\eta)=0,\\ x''(0)=y''(0)=0, x''(1)-\lambda_2 x''(\xi)=0, y''(1)-\lambda_2 y''(\xi)=0, \end{cases}$$

where $3<\alpha,\beta\leq 4, \alpha-2<\sigma\leq\alpha-1, \beta-2<\delta\leq\beta-1, 0<\xi,\eta<1$, and $D^{\alpha}, D^{\beta}, D^{\delta}$ and D^{σ}, are the Caputo fractional derivatives, $J=[0,1], \lambda_1, \lambda_2$ are real constants with $\lambda_1\eta\neq 1, \lambda_2\xi\neq 1$ and f, g continuous functions on $[0,1]\times\mathbb{R}^2$.

Keywords: Caputo derivative; Boundary Value Problem; fixed point theorem.

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1 Introduction

Differential equations of fractional order have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as electrochemistry, physics, chemistry, viscoelasticity, control, image and signal processing, biophysics. For more details, we refer the reader to [4, 7, 10, 12, 13, 15, 17, 18] and references therein. There has been a significant progress in the investigation of these equations in recent years, see [6, 8, 9, 15, 16, 27]. More recently, some basic theory for the initial boundary value problems of fractional differential equations has been discussed in [1, 14, 15]. Recently, existence

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and uniqueness of solutions to boundary value problems for fractional differential equations had attracted the attention of many authors, see for example, [4, 6, 8, 9, 15, 16, 19, 27] and the references therein. The study of coupled system of fractional order is also important as such systems occur in various problems of applied science [3, 11, 20, 21, 24, 26]. In the last decade, many authors have established the existence and uniqueness for solutions of some systems of nonlinear fractional differential equations, one can see [20, 23, 24, 25] and references cited therein. For example in [2, 5, 21, 26] the authors established sufficient conditions for the existence of solutions for a two-point and three-point boundary value problem for a coupled system of fractional differential equations.

In [2, 5, 21, 22, 26], the existence and uniqueness of solutions was investigated for a nonlinear coupled system for fractional differential equations with two-point and three-point boundary conditions by using Schauder's fixed point theorem.

Motivated by the problem (1.1) in [5], this paper deals with the existence of solution for the following fractional differential problem:

$$\begin{cases} D^{\alpha}x(t) + f\left(t, y(t), D^{\delta}y(t)\right) = 0, t \in J, \\ D^{\beta}y(t) + g(t, x(t), D^{\sigma}x(t)) = 0, t \in J, \\ x(0) = y(0) = 0, x(1) - \lambda_1 x(\eta) = 0, y(1) - \lambda_1 y(\eta) = 0, \\ x''(0) = y''(0) = 0, x''(1) - \lambda_2 x''(\xi) = 0, y''(1) - \lambda_2 y''(\xi) = 0, \end{cases} \tag{1.1}$$

where $3 < \alpha, \beta \le 4, \alpha - 2 < \sigma \le \alpha - 1, \beta - 2 < \delta \le \beta - 1, 0 < \xi, \eta < 1$, and $D^{\alpha}, D^{\beta}, D^{\delta}$ and D^{σ}, are the Caputo fractional derivatives, $J = [0,1], \lambda_1, \lambda_2$ are real constants with $\lambda_1\eta \neq 1, \lambda_2\xi \neq 1$ and f, g are continuous functions on $[0,1] \times \mathbb{R}^2$.

The rest of this paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solution of problem (1.1). In section 4 examples are treated illustrating our results.

2 Preliminaries

The following notations, definitions, and preliminary facts will be used throughout this paper.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as:*

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0, \tag{2.2}$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2.2. *The fractional derivative of $f \in C^n([0, \infty[)$ in the Caputo's sense is defined as:*

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, n-1 < \alpha, n \in N^*. \tag{2.3}$$

For more details about fractional calculus, we refer the reader to [15, 18].

We will consider the following spaces:

$$X = \{x : x \in C([0,1]), D^{\sigma} x \in C([0,1])\},$$

and

$$Y = \{y : y \in C([0,1]), D^{\delta} y \in C([0,1])\},$$

endowed with the norms:

$$\|x\|_X = \|x\| + \|D^{\sigma} x\|, \ \|x\| = \sup_{t\in J} |x(t)|, \|D^{\sigma} x\| = \sup_{t\in J} |D^{\sigma} x(t)|,$$

and

$$\|y\|_Y = \|y\| + \left\|D^{\delta} y\right\|, \ \|y\| = \sup_{t\in J} |y(t)|, \left\|D^{\delta} y\right\| = \sup_{t\in J} \left|D^{\delta} y(t)\right|.$$

We know that $(X, \| \, . \, \|_X)$ and $(Y, \| \, . \, \|_Y)$, is a Banach space. The product space $\left(X \times Y, \|(x,y)\|_{X\times Y}\right)$ is also a Banach space, with norm $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$.

We recall the following important lemmas [13]:

Lemma 2.1. *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha} x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1, n = [\alpha] + 1$.

Lemma 2.2. *Let $\alpha > 0$. Then*

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.5}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

We prove the following result:

Lemma 2.3. *Let $g \in C([0,1])$, the solution of the equation*

$$D^{\alpha} x(t) + g(t) = 0, t \in J, \ 3 < \alpha \leq 4, \tag{2.6}$$

subject to the conditions

$$\begin{gathered} x(0) = 0, \ x(1) - \lambda_1 x(\eta) = 0, \\ x''(0) = 0, \ x''(1) - \lambda_2 x''(\xi) = 0, \end{gathered} \tag{2.7}$$

is given by:

$$\begin{aligned} x(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds \\ & + \frac{\lambda_1 t}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} g(s)\, ds \\ & - \frac{t}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s)\, ds \\ & + \frac{\left(\lambda_2 - \lambda_2 \lambda_1 \eta^3\right) t + (\lambda_2 \lambda_1 \eta - \lambda_2)\, t^3}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^{\xi} (\xi - s)^{\alpha-3} g(s)\, ds \\ & - \frac{\left(1 - \lambda_1 \eta^3\right) t + (\lambda_1 \eta - 1)\, t^3}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^1 (1-s)^{\alpha-3} g(s)\, ds. \end{aligned} \tag{2.8}$$

Proof. We use the same technics as in [5]. For $c_i \in \mathbb{R}, i = 0,1,2,3$, and by Lemmas 2.1, 2.2, we have

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds - c_0 - c_1 t - c_2 t^2 - c_3 t^3 \tag{2.9}$$

Using (2.7), we get $c_0 = c_2 = 0$, and

$$\begin{aligned} c_1 = &-\frac{\lambda_1}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} g(s)\, ds \\ &+ \frac{1}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s)\, ds \\ &- \frac{\lambda_2(1-\lambda_1\eta^3)}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)} \int_0^\xi (\xi-s)^{\alpha-3} g(s)\, ds \\ &+ \frac{(1-\lambda_1\eta)}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} g(s)\, ds \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} c_3 = &-\frac{\lambda_2}{6(\lambda_2\xi-1)\Gamma(\alpha-2)} \int_0^\xi (\xi-s)^{\alpha-3} g(s)\, ds \\ &+ \frac{1}{6(\lambda_2\xi-1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} g(s)\, ds \end{aligned} \tag{2.11}$$

Substituting the value of c_1 and c_3 in (2.9), we obtain the desired quantity in Lemma. □

3 Main Results

Let us set:

$$\begin{aligned} M_1 = &\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ &+ \frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}, \\ M_2 = &\frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ &+ \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}, \\ M_3 = &\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\beta+1}{|\lambda_1\eta-1|\Gamma(\beta+1)} + \frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}, \\ M_4 = &\frac{1}{\Gamma(\beta-\delta+1)} + \frac{|\lambda_1|\eta^\beta+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)} \\ &+ \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}, \\ L_1 = &\frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}, \\ L_2 = &\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}, \\ L_3 = &\frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}, \\ L_4 = &\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}. \end{aligned} \tag{3.12}$$

Let us also consider the following hypotheses:

$(H1)$: There exist two constants k_1 and k_2 such that for all $t \in [0,1]$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq k_1 (|x_1 - x_2| + |y_1 - y_2|), \quad (3.13)$$
$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq k_2 (|x_1 - x_2| + |y_1 - y_2|).$$

$(H2))$: The functions f , g : $[0,1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous

$(H3)$: There exists positive constants N_1 and N_2 such that

$$|f(t,x,y)| \leq N_1,\ |g(t,x,y)| \leq N_2 \text{ for each } t \in J \text{ and all } x,y \in \mathbb{R}.$$

We prove the following theorem:

Theorem 3.1. *Assume that* $(H1)$ *holds.*

If

$$k_1 (M_1 + M_2) + k_2 (M_3 + M_4) < 1, \quad (3.14)$$

then the problem (1.1) *has a unique solution.*

Proof. The proof is similarly to that of Theorem 3.1 in [5] by taking $k_1 = \omega_1 + \omega_2$ and $k_2 = \varpi_1 + \varpi_2$.

Now, we prove the following result:

Theorem 3.2. *Assume that the hypotheses* $(H1) - (H2)$ *and* $(H3)$ *are satisfied, such that*

$$k_1\theta_1 + k_2\theta_2 < 1, \quad (3.15)$$

where

$$\theta_1 = \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)},$$
$$\theta_2 = \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\beta+1}{|\lambda_1\eta-1|\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta-\delta+1)} + \frac{|\lambda_1|\eta^\beta+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)}.$$

If there exists $\mu \in \mathbb{R}$ *such that*

$$N_1 (M_1 + M_2) + N_2 (M_3 + M_4) \leq \mu, \quad (3.16)$$

then, the problem (1.1) *has at least a solution.*

Proof. We shall use Krasnselskii's fixed point theorem to prove that ϕ has at least a fixed point on $X \times Y$.

Suppose that (3.16) holds and let us take

$$\phi(x,y)(t) := T(x,y)(t) + R(x,y)(t), \quad (3.17)$$

where

$$T(x,y)(t) := (T_1 y(t), T_2 x(t)), \quad (3.18)$$

$$T_1 y(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds \quad (3.19)$$
$$+ \frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds$$
$$- \frac{t}{(\lambda_1\eta-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds,$$

$$T_2x(t) = -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,x(s),D^\sigma x(s))\,ds$$
$$+\frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^\eta (\eta-s)^{\alpha-1} g(s,x(s),D^\sigma x(s))\,ds$$
$$-\frac{t}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^1 (1-s)^{\alpha-1} g(s,x(s),D^\sigma x(s))\,ds, \tag{3.20}$$

and

$$R(x,y)(t) := (R_1y(t), R_2x(t)), \tag{3.21}$$

where,

$$R_1y(t) = \frac{(\lambda_2-\lambda_2\lambda_1\eta^3)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3} f\left(s,y(s),D^\delta y(s)\right)ds$$
$$-\frac{(1-\lambda_1\eta^3)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3} f\left(s,y(s),D^\delta y(s)\right)ds, \tag{3.22}$$

$$R_2x(t) = \frac{(\lambda_2-\lambda_2\lambda_1\eta^3)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^\xi (\xi-s)^{\alpha-3} g(s,x(s),D^\sigma x(s))\,ds$$
$$-\frac{(1-\lambda_1\eta^3)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^1 (1-s)^{\alpha-3} g(s,x(s),D^\sigma x(s))\,ds. \tag{3.23}$$

The proof will be given in several steps.

Step1: We shall prove that for any $(x,y),(x_1,y_1) \in B_\mu$, then $T(x,y)+R(x_1,y_1) \in B_\mu$, Such that $B_\mu = \{(x,y) \in X\times Y; \|(x,y)\|_{X\times Y} \le \mu\}$.

For any $(x,y),(x_1,y_1) \in B_\mu$ and for each $t \in J$ we have:

$$|T_1y(t)+R_1y_1(t)| = |-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s,y(s),D^\delta y(s)\right)ds$$
$$+\frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1} f\left(s,y(s),D^\delta y(s)\right)ds$$
$$-\frac{t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f\left(s,y(s),D^\delta y(s)\right)ds$$
$$+\frac{(\lambda_2-\lambda_2\lambda_1\eta^3)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3} f\left(s,y(s),D^\delta y(s)\right)ds$$
$$-\frac{(1-\lambda_1\eta^3)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^1 (1-s)^{\alpha-3} f\left(s,y(s),D^\delta y(s)\right)ds\,|$$

then,

$$|T_1y(t)+R_1y_1(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds$$
$$+\frac{|\lambda_1|}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds$$
$$+\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds$$
$$+\frac{|\lambda_2-\lambda_2\lambda_1\eta^3|+|\lambda_2\lambda_1\eta-\lambda_2|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds$$
$$+\frac{|1-\lambda_1\eta^3|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds.$$

Using $(H3)$, we obtain

$$|T_1 y(t) + R_1 y_1(t)| \leq N_1 \left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}\right] + N_1 \left[\frac{\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right].$$

Consequently,

$$|T_1 y(t) + R_1 y_1(t)| \leq N_1 M_1.$$

Thus,

$$\|T_1(y) + R_1(y_1)\| \leq N_1 M_1, \tag{3.24}$$

and

$$\begin{aligned} |D^\sigma T_1 y(t) + D^\sigma R_1 y_1(t)| \leq{}& \frac{1}{\Gamma(\alpha-\sigma)} \int_0^t (t-s)^{\alpha-\sigma-1} \left|f\left(s, y(s), D^\delta y(s)\right)\right| ds \\ &+ \frac{|\lambda_1|}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)} \int_0^\eta (\eta-s)^{\alpha-1} \left|f\left(s, y(s), D^\delta y(s)\right)\right| ds \\ &+ \frac{1}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)} \int_0^1 (1-s)^{\alpha-1} \left|f\left(s, y(s), D^\delta y(s)\right)\right| ds \\ &+ \left[\begin{array}{c} \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)} \end{array}\right] \int_0^\xi (\xi-s)^{\alpha-3} \left|f\left(s, y_1(s), D^\delta y_1(s)\right)\right| ds \\ &+ \left[\begin{array}{c} \frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)} \end{array}\right] \int_0^1 (1-s)^{\alpha-3} \left|f\left(s, y_1(s), D^\delta y_1(s)\right)\right| ds. \end{aligned}$$

By $(H3)$, we have

$$\begin{aligned} |D^\sigma T_1 y(t) + D^\sigma R_1 y_1(t)| \leq{}& N_1 \left[\frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right] \\ &+ N_1 \left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right]. \end{aligned}$$

Consequently, we obtain

$$|D^\sigma T_1 y(t) + D^\sigma R_1 y_1(t)| \leq N_1 M_2.$$

Hence,

$$\|D^\sigma T_1(y) + D^\sigma R_1(y_1)\| \leq N_1 M_2. \tag{3.25}$$

Combining (3.24) and (3.25) , yields

$$\|T_1(y) + R_1(y_1)\|_X \leq N_1(M_1 + M_2). \tag{3.26}$$

Analogously, we have

$$\|T_2(x) + R_2(x_1)\|_Y \leq N_2(M_3 + M_4). \tag{3.27}$$

Hence, it follows from (3.26) and (3.27) that

$$\|T(x,y) + R(x_1,y_1)\|_{X\times Y} \leq N_1(M_1+M_2) + N_2(M_3+M_4) < \mu. \tag{3.28}$$

Step2: We shall prove that R is continuous and compact.

$[1^{*}]$: The continuity of f and g implies that the operator R is continuous.

$[2^{*}]$: Now, we prove that R maps bounded sets into bounded sets of $X \times Y$.

For $(x,y) \in B_{\mu}$ and for each $t \in J$, we have:

$$|R_1 y(t)| \leq \frac{|\lambda_2 - \lambda_2\lambda_1\eta^3| t + |\lambda_2\lambda_1\eta - \lambda_2| t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)} \int_0^{\xi} (\xi - s)^{\alpha-3} \left| f\left(s, y(s), D^{\delta}y(s)\right)\right| ds$$
$$+ \frac{|1-\lambda_1\eta^3| t + |\lambda_1\eta - 1| t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)} \int_0^{1} (1 - s)^{\alpha-3} \left| f\left(s, y(s), D^{\delta}y(s)\right)\right| ds.$$

Using $(H3)$, we obtain

$$|R_1 y(t)| \leq \frac{N_1\left[\left(|\lambda_2 - \lambda_2\lambda_1\eta^3| + |\lambda_2\lambda_1\eta - \lambda_2|\right)\xi^{\alpha-2} + |1-\lambda_1\eta^3| + |\lambda_1\eta - 1|\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}$$
$$\leq N_1 \left(\frac{\left(|\lambda_2 - \lambda_2\lambda_1\eta^3| + |\lambda_2\lambda_1\eta - \lambda_2|\right)\xi^{\alpha-2} + |1-\lambda_1\eta^3| + |\lambda_1\eta - 1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)} \right).$$

Thus,

$$|R_1 y(t)| \leq N_1 L_1, t \in J,$$

Therefore,

$$\|R_1(y)\| \leq N_1 L_1. \tag{3.29}$$

On the other hand,

$$|D^{\sigma} R_1 y(t)| \leq \frac{1}{\Gamma(\alpha-2)} \left(\begin{array}{c} \frac{|\lambda_2 - \lambda_2\lambda_1\eta^3| t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\sigma)} + \\ \frac{|\lambda_2\lambda_1\eta - \lambda_2| t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)} \end{array} \right) \int_0^{\xi} (\xi - s)^{\alpha-3} \left| f\left(s, y(s), D^{\delta}y(s)\right)\right| ds$$
$$+ \frac{1}{\Gamma(\alpha-2)} \left(\begin{array}{c} \frac{|1-\lambda_1\eta^3| t^{1-\sigma}}{6\Gamma(2-\sigma)|\lambda_1\eta-1||\lambda_2\xi-1|} + \\ \frac{|\lambda_1\eta - 1| t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)} \end{array} \right) \int_0^{1} (1 - s)^{\alpha-3} \left| f\left(s, y(s), D^{\delta}y(s)\right)\right| ds.$$

By $(H3)$, we have

$$|D^{\sigma}\phi_1 y(t)| \leq N_1 \left[\frac{|\lambda_2 - \lambda_2\lambda_1\eta^3|\xi^{\alpha-2} + |1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta - \lambda_2|\xi^{\alpha-2} + |\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)} \right]$$
$$\leq N_1 \left[\frac{|\lambda_2 - \lambda_2\lambda_1\eta^3|\xi^{\alpha-2} + |1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta - \lambda_2|\xi^{\alpha-2} + |\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)} \right].$$

Consequently, we obtain

$$|D^{\sigma} R_1 y(t)| \leq N_1 L_2, t \in J.$$

Therefore,

$$\|D^{\sigma} R_1(y)\| \leq N_1 L_2. \tag{3.30}$$

Hence, from (3.29) and (3.30), we have

$$\|R_1(y)\|_X \leq N_1 (L_1 + L_2). \tag{3.31}$$

Similarly, it can be shown that

$$\|R_2(x)\|_Y \leq N_2 (L_3 + L_4). \tag{3.32}$$

It follows from (3.31) and (3.32) that

$$\|R(x,y)\|_{X\times Y} \leq N_1 (L_1 + L_2) + N_2 (L_3 + L_4). \tag{3.33}$$

Consequently,

$$\|R\left(x,y\right)\|_{X\times Y}<\infty.$$

[3*] : We show that R is equi-continuous:

Let $t_1,t_2\in J$, such that $t_1<t_2$ and $(x,y)\in B_\mu$. Then, we have:

$$\begin{aligned}|R_1y\left(t_2\right)-R_1y\left(t_1\right)|\leq{}&\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|(t_2-t_1)+|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds\\&+\frac{\left|1-\lambda_1\eta^3\right|(t_1-t_2)+|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds.\end{aligned}$$

Using $(H3)$, we obtain

$$\begin{aligned}|R_1y\left(t_2\right)-R_1y\left(t_1\right)|\leq{}&\frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_2-t_1\right)\\&+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1-t_2\right)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_2^3-t_1^3\right)+\frac{N_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right),\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}|D^{\sigma}R_1y\left(t_2\right)-D^{\sigma}R_1y\left(t_1\right)|\leq{}&\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\\+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds\\&+\left[\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\\+\frac{|\lambda_1\eta-1|\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds.\end{aligned}$$

By $(H3)$, we have:

$$\begin{aligned}|D^{\sigma}R_1y\left(t_2\right)-D^{\sigma}R_1y\left(t_1\right)|\leq{}&\frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)\\&+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)\\&+\frac{N_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).\end{aligned}\tag{3.35}$$

Hence, by (3.34) and (3.35), we obtain

$$\begin{aligned}\|R_1y\left(t_2\right)-R_1y\left(t_1\right)\|_X\leq{}&\frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_2-t_1\right)+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1-t_2\right)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_2^3-t_1^3\right)+\frac{N_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right)\\&+\frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)\\&+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)\\&+\frac{N_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).\end{aligned}\tag{3.36}$$

Analogously, we can obtain

$$\begin{aligned}\left\|R_2x\left(t_2\right)-R_2x\left(t_1\right)\right\|_Y \leq\ & \tfrac{N_2\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_2-t_1\right)+\tfrac{N_2\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_1-t_2\right)\\ &+\tfrac{N_2|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_2^3-t_1^3\right)+\tfrac{N_2|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_1^3-t_2^3\right)\\ &+\tfrac{N_2\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\left(t_2^{1-\delta}-t_1^{1-\delta}\right)\\ &+\tfrac{N_2\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\left(t_1^{1-\delta}-t_2^{1-\delta}\right)\\ &+\tfrac{N_2|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_2^{3-\delta}-t_1^{3-\delta}\right)\\ &+\tfrac{N_2|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_1^{3-\delta}-t_2^{3-\delta}\right).\end{aligned} \tag{3.37}$$

Thanks to (3.36) and (3.37), we can state that $\|\phi\left(x,y\right)\left(t_2\right)-\phi\left(x,y\right)\left(t_1\right)\|\to 0$ as $t_1\to t_2$. Then, as a consequence of steps $\left([1^*],[2^*],[3^*]\right)$, we can conclude that R is continuous and compact.

Step3: Now, we prove that T is contractive.

Let $\left(x,y\right),\left(x_1,y_1\right)\in X\times Y$. Then, for each $t\in J$, we have

$$\begin{aligned}\left|T_1y\left(t\right)-T_1y_1\left(t\right)\right| \leq\ & \tfrac{1}{\Gamma(\alpha)}\int_0^t\left(t-s\right)^{\alpha-1}\left|\begin{array}{c}f\left(s,y\left(s\right),D^\delta y\left(s\right)\right)\\ -f\left(s,y_1\left(s\right),D^\delta y_1\left(s\right)\right)\end{array}\right|ds\\ &+\tfrac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^\eta\left(\eta-s\right)^{\alpha-1}\left|\begin{array}{c}f\left(s,y\left(s\right),D^\delta y\left(s\right)\right)\\ -f\left(s,y_1\left(s\right),D^\delta y_1\left(s\right)\right)\end{array}\right|ds\\ &+\tfrac{t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^1\left(1-s\right)^{\alpha-1}\left|\begin{array}{c}f\left(s,y\left(s\right),D^\delta y\left(s\right)\right)\\ -f\left(s,y_1\left(s\right),D^\delta y_1\left(s\right)\right)\end{array}\right|ds.\end{aligned}$$

Thanks to $(H1)$, we can write

$$\begin{aligned}\left|T_1y\left(t\right)-T_1y_1\left(t\right)\right| \leq\ & \tfrac{k_1}{\Gamma(\alpha+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)\\ &+\tfrac{k_1(|\lambda_1|\eta^\alpha+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right).\end{aligned}$$

Consequently,

$$\left\|T_1\left(y\right)-T_1\left(y_1\right)\right\| \leq \tfrac{k_1[|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1]}{|\lambda_1\eta-1|\Gamma(\alpha+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right), \tag{3.38}$$

and

$$\begin{aligned}\left|D^\sigma T_1y\left(t\right)-D^\sigma T_1y_1\left(t\right)\right| \leq\ & \tfrac{1}{\Gamma(\alpha-\sigma)}\int_0^t\left(t-s\right)^{\alpha-\sigma-1}\left|\begin{array}{c}f\left(s,y\left(s\right),D^\delta y\left(s\right)\right)\\ -f\left(s,y_1\left(s\right),D^\delta y_1\left(s\right)\right)\end{array}\right|ds\\ &+\tfrac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta\left(\eta-s\right)^{\alpha-1}\left|\begin{array}{c}f\left(s,y\left(s\right),D^\delta y\left(s\right)\right)\\ -f\left(s,y_1\left(s\right),D^\delta y_1\left(s\right)\right)\end{array}\right|ds\\ &+\tfrac{t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1\left(1-s\right)^{\alpha-1}\left|\begin{array}{c}f\left(s,y\left(s\right),D^\delta y\left(s\right)\right)\\ -f\left(s,y_1\left(s\right),D^\delta y_1\left(s\right)\right)\end{array}\right|ds.\end{aligned}$$

By $(H1)$, yields

$$\begin{aligned}\left|D^{\sigma}T_1y(t)-D^{\sigma}T_1y_1(t)\right| &\leq \tfrac{k_1}{\Gamma(\alpha-\sigma+1)}\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)\\ &+\tfrac{k_1|\lambda_1|\eta^{\alpha}}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)\\ &+\tfrac{k_1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right).\end{aligned}$$

Hence,

$$\left\|D^{\sigma}T_1(y)-D^{\sigma}T_1(y_1)\right\|\leq k_1\left[\begin{array}{c}\frac{1}{\Gamma(\alpha-\sigma+1)}\\ +\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\end{array}\right]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right). \tag{3.39}$$

By (3.38) and (3.39), we can write

$$\left\|T_1(y)-T_1(y_1)\right\|_X\leq k_1\left[\begin{array}{c}\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\sigma+1)}\\ +\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\end{array}\right]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right).$$

Thus,

$$\left\|T_1(y)-T_1(y_1)\right\|_X\leq k_1\theta_1\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right). \tag{3.40}$$

Analogously, we can get

$$\left\|T_2(x)-T_2(x_1)\right\|_Y\leq k_2\theta_2\left(\|x-x_1\|+\left\|D^{\sigma}x-D^{\sigma}x_1\right\|\right). \tag{3.41}$$

It follows from (3.40) and (3.41) that

$$\left\|T(x,y)-T(x_1,y_1)\right\|_{X\times Y}\leq\left[k_1\theta_1+k_2\theta_2\right]\left(\|(x-x_1,y-y_1)\|_{X\times Y}\right).$$

Using (3.15), we conclude that T is a contraction mapping.

As a consequence of Krasnoselskii's fixed point theorem we deduce that ϕ has a fixed point which is a solution of (1.1). □

4 Examples

In this section we give some examples to illustrate our main results.

Example 4.1. *Let us consider the following system:*

$$\begin{aligned}&D^{\frac{7}{2}}x(t)+\frac{\sqrt{\pi}e^{-\pi t^2}\cos(\pi t)\left(y(t)+D^{\frac{5}{2}}y(t)\right)}{\left(5\sqrt{\pi}+7e^{t}\right)\left(1+y(t)+D^{\frac{5}{2}}y(t)\right)}+\ln\left(1+t^2\right)=0,t\in J,\\ &D^{\frac{11}{3}}y(t)+\frac{\sqrt{\pi}e^{-\pi t^2}\cos(\pi t)\left(x(t)+D^{\frac{9}{4}}x(t)\right)}{\left(5\sqrt{\pi}+7e^{t}\right)\left(1+x(t)+D^{\frac{9}{4}}x(t)\right)}+\ln\left(1+t^2\right)=0,t\in J,\\ &x(0)=0,x(1)-\tfrac{3}{4}x\left(\tfrac{1}{3}\right)=0,y(0)=0,y(1)-\tfrac{3}{4}y\left(\tfrac{1}{3}\right)=0,\\ &x''(0)=0,x''(1)-\tfrac{4}{5}x''\left(\tfrac{2}{3}\right)=0,y''(0)=0,y''(1)-\tfrac{4}{5}y''\left(\tfrac{2}{3}\right)=0.\end{aligned}$$

Set

$$f(t,x,y)=g(t,x,y)=\frac{\sqrt{\pi}e^{-\pi t}\left|\cos(\pi t)\right|(|x|+|y|)}{\left(5\sqrt{\pi}+7e^{t}\right)^2(1+|x|+|y|)}+\ln\left(1+t^2\right),t\in[0,1],x,y\in[0,\infty).$$

For $t \in J = [0,1]$ and $x_1, y_1, x_2, y_2 \in [0,\infty)$, we have:

$$\begin{aligned}
|f(t,x,y) - f(t,x_1,y_1)| &= \frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|}{\left(5\sqrt{\pi}+7e^t\right)^2}\left|\frac{x+y}{(1+|x|+|y|)} - \frac{x_1+y_1}{(1+|x_1|+|y_1|)}\right| \\
&\leq \frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|(|x-x_1|+|y-y_1|)}{\left(5\sqrt{\pi}+7e^t\right)^2(1+|x|+|y|)(1+|x_1|+|y_1|)} \\
&\leq \frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|(|x-x_1|+|y-y_1|)}{\left(5\sqrt{\pi}+7e^t\right)^2} \\
&\leq \frac{\sqrt{\pi}}{\left(5\sqrt{\pi}+7\right)^2}(|x-x_1|+|y-y_1|).
\end{aligned}$$

The condition $(H1)$ holds with $k_1 = k_2 = \frac{\sqrt{\pi}}{(5\sqrt{\pi}+7)^2}$.
For $\alpha = \frac{7}{2}, \beta = \frac{11}{3}, \sigma = \frac{9}{4}, \delta = \frac{5}{2}$ and $\lambda_1 = \frac{3}{4}, \lambda_2 = \frac{4}{5} = \eta = \frac{1}{3}, \xi = \frac{2}{3}$, we have:

$$M_1 = 1,089, M_2 = 3,503, M_3 = 0,909, M_4 = 3,089,$$

and,

$$k_1(M_1+M_2) + k_2(M_3+M_4) = 0,0605075.$$

Therefore,

$$k_1(M_1+M_2) + k_2(M_3+M_4) < 1.$$

Hence, by Theorem 3.1, the problem (1.1) has a unique solution.

Example 4.2. *Consider the following system:*

$$\begin{cases}
D^{\frac{7}{2}}x(t) + \frac{\left|D^{\frac{7}{3}}y(t)\right|}{5\pi\left(\sqrt{\pi}+2e^t\right)} + \frac{e^{-t^2}|y(t)|}{5\pi\left(\sqrt{\pi}e^t+2\right)^2(1+|y(t)|)} = 0, t \in J, \\
D^{\frac{11}{3}}y(t) + \frac{|x(t)|}{14\sqrt{\pi}(1+|x(t)|)} + \frac{|\cos(\pi t)|\left|D^{\frac{5}{2}}x(t)\right|}{7\sqrt{\pi}(t+1)^2} = 0, t \in J, \\
x(0) = 0, x(1) - \frac{2}{3}x\left(\frac{1}{5}\right) = 0, y(0) = 0, y(1) - \frac{2}{3}y\left(\frac{1}{5}\right) \\
x''(0) = 0, x''(1) - \frac{1}{2}x''\left(\frac{1}{4}\right) = 0, y''(0) = 0, y''(1) - \frac{1}{2}y''\left(\frac{1}{4}\right) = 0.
\end{cases}$$

For this example, we have

$$\begin{aligned}
f(t,x,y) &= \frac{|x|}{5\pi\left(\sqrt{\pi}+2e^t\right)} + \frac{e^{-t^2}|y|}{5\pi\left(\sqrt{\pi}e^t+2\right)^2(1+|y|)}, t \in [0,1], x,y \in [0,\infty), \\
g(t,x,y) &= \frac{|x|}{14\sqrt{\pi}(1+|x|)} + \frac{|\cos(\pi t)||y|}{7\sqrt{\pi}(t+1)^2}, t \in [0,1], x,y \in [0,\infty).
\end{aligned}$$

For $t \in J = [0,1]$ and $x, y, x_1, y_1 \in [0,\infty)$, we have:

$$\begin{aligned}
|f(t,x,y) - f(t,x_1,y_1)| &= \frac{e^{-t^2}|x-x_1|}{5\left(\sqrt{\pi}e^t+2\right)^2(1+|x|)(1+|x_1|)} + \frac{|y-y_1|}{5\pi\left(\sqrt{\pi}+2e^t\right)} \\
&\leq \frac{e^{-t^2}}{5\pi\left(\sqrt{\pi}e^t+2\right)^2}|x-x_1| + \frac{1}{5\pi\left(\sqrt{\pi}+2e^t\right)}|y-y_1| \\
&\leq \frac{1}{5\pi\left(\sqrt{\pi}+2\right)^2}(|x-x_1|+|y-y_1|),
\end{aligned}$$

and

$$\begin{aligned}|g(t,x,y)-g(t,x_1,y_1)| &= \frac{|x-x_1|}{14\sqrt{\pi}\,(1+|x|)\,(1+|x_1|)}+\frac{|\cos(\pi t)|\,|y-y_1|}{7\sqrt{\pi}\,(t+1)^2}\\ &\le \frac{1}{14\sqrt{\pi}}\,|x-x_1|+\frac{|\cos(\pi t)|}{7\sqrt{\pi}\,(t+1)^2}\,|y-y_1|\\ &\le \frac{1}{14\sqrt{\pi}}\,(|x-x_1|+|y-y_1|)\,.\end{aligned}$$

By Theorem 3.2, *the problem* (1.1) *has at leat one solution.*

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Some curvature properties of (κ, μ) contact space forms

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Abstract

The object of the present paper is to find Ricci tensor of (k, μ) space forms. In particular we prove that a three dimensional (k, μ) space forms is $\eta-$Einstein for $\mu = \frac{1}{2}$. We also study three dimensional (k, μ) space forms with $\eta-$ parallel and cyclic parallel Ricci tensor for $\mu = \frac{1}{2}$. We also prove that every (k, μ) space forms is locally $\phi-$ symmetric.

Keywords: (k, μ) contact space forms, $\eta-$Einstein, $\eta-$ parallel and cyclic parallel Ricci tensor, locally $\phi-$ symmetric.

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1 Introduction

Now a days, a good number of contact geometers have worked on (k, μ) contact metric manifold. The notion of (k, μ) contact metric manifold was introduced by D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [2]. The notion of (k, μ) space forms was introduced by T. Koufogiorgos [8]. The Ricci curvature and the Riemannian curvature are two key objects regarding symmetry of a manifolds. The notion of local symmetry has been weakened by several authors in several ways. As a weaker version of local symmetry T. Takahashi [10] introduced the notion of local $\phi-$ symmetry in Sasakian manifolds. The notion of $\eta-$ parallel and cyclic parallel Ricci tensor was introduced in the paper [7] and [9]. In this regard we mention that $\eta-$ parallel and cyclic parallel Ricci tensor have been studied by the present authors in the paper[1]. Again $\eta-$ parallel and cyclic parallel Ricci tensor was studied by the authors in the paper [5]. The present paper is organized by the following way:

After introduction in Section 1 we give some preliminaries in Section 2. In Section 3 we study Ricci tensor of (k, μ) space forms. η-parallel, cyclic parallel Ricci tensors and Ricci operator of (k, μ) space forms of dimension three have been studied in Section 4. In Section 5 we have proved that every (2n+1) dimensional (k, μ) space forms is locally ϕ- symmetric.

2 Preliminaries

A differentiable manifold M^{2n+1} is said to be a contact manifold if it admits a global differentiable 1-form η such that $\eta \wedge (d\eta)^n \neq 0$, everywhere on M^{2n+1}.

Given a contact form η, one has a unique vectoe field, satisfying

$$\eta(\xi) = 1, \quad d\eta(\xi, X) = 0, \tag{2.1}$$

for any vector field X.

It is well-known that, there exists a Riemannian metric g and a (1,1) tensor field ϕ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.2}$$

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where X and Y are vector fields on M.
From (2.2) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

A differentiable manifold M^{2n+1} equipped with the structure tensors (ϕ, ξ, η, g) satisfying (2.3) is said to be a contact metric manifold.
On a contact metric manifold $M(\phi, \xi, \eta, g)$, we define a $(1, 1)$ tensor field h by $h = \frac{1}{2}L_\xi\phi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h, \quad \nabla_X\xi = -\phi X - \phi hX \tag{2.4}$$

where ∇ is levi-Civita connection[2].
For a contact metric manifold M one may define naturally an almost complex structure on the product $M \times \mathbb{R}$. If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. A Sasakian manifold is characterized by the condition

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.5}$$

for all vector fields X and Y on the manifold [4]. Equivalently, a contact metric manifold is said to be Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

holds for all X, Y on M [4].
For a contact manifold we have [3]

$$(\nabla_X h)(Y) = \{(1 - k)g(X, \phi Y) + g(X, h\phi Y)\}\xi + \eta(Y)\{h(\phi X + \phi hX)\}. \tag{2.6}$$

The $(k, \mu)-$nullity distribution of a contact metric manifold $M(\phi, \xi, \eta, g)$ is a distribution [8]

$$\begin{aligned} N(k, \mu) \quad &: \quad p \to N_p(k, \mu) \\ &= \quad \{Z \in T_p(M) : R(X, Y)Z \\ &= \quad k(g(Y, Z)X - g(X, Z)Y) + \mu(g(Y, Z)hX - g(X, Z)hY)\}, \end{aligned} \tag{2.7}$$

for any $X, Y \in T_pM$ and $\kappa, \mu \in R$. If $k = 1$, then $h = 0$ and M is a Sasakian manifold [8]. Also one has $trh = 0$, $trh\phi = 0$ and $h^2 = (k - 1)\phi^2$. So if the characteristic vector field ξ belongs to the $(k, \mu)-$nullity distribution, then we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.8}$$

Moreover, if M has constant $\phi-$sectional curvature c then it is called a (k, μ) space forms and is denoted by $M(c)$.
The curvature tensor of $M(c)$ is given by[8]:

$$\begin{aligned} 4R(X, Y)Z \quad &= \quad (c + 3)\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \quad (c + 3 - 4k)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \quad g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \\ &+ \quad (c - 1)\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &- \quad 2\{g(hX, Z)hY - g(hY, Z)hX + g(X, Z)hY \\ &- \quad 2g(Y, Z)hX - 2\eta(X)\eta(Z)hY + 2\eta(Y)\eta(Z)hX \\ &+ \quad 2g(hX, Z)Y - 2g(hY, Z)X + 2g(hY, Z)\eta(X)\xi \\ &- \quad 2g(hX, Z)\eta(Y)\xi - g(\phi hX, Z)\phi hY + g(\phi hY, Z)\phi hX\} \\ &+ \quad 4\mu\{\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY \\ &+ \quad g(hY, Z)\eta(X)\xi - g(hX, Z)\eta(Y)\xi, \end{aligned} \tag{2.9}$$

for any vector fields X, Y, Z on M.
If $k \neq 1$, then $\mu = \kappa + 1$ and $c = -2k - 1$.

Definition 2.1. *If an almost contact Riemannian manifold M satisfies the condition $S = ag + b\eta \otimes \eta$, for some functions a, b in $C^\infty(M)$ and S is the Ricci tensor, then M is said to be an η-Einstein manifold. If, in particular, a=0 then this manifold will be called a special type of η-Einstein manifold.*

3 Ricci tensor of (κ, μ) space forms

In this section we study Ricci tensor of (k, μ) space forms. Taking inner product on both side of (2.9) with W we obtain

$$\begin{aligned}
& 4g(R(X,Y)Z,W) \\
= & (c+3)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\
+ & (c+3-4k)\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\
+ & g(X,Z)\eta(Y)\eta(\xi,W) - g(Y,Z)\eta(X)g(\xi,W)\} \\
+ & (c-1)\{g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\} \\
- & 2\{g(hX,Z)g(hY,W) - g(hY,Z)g(hX,W) + g(X,Z)g(hY,W) \\
- & 2g(Y,Z)g(hX,W) - 2\eta(X)\eta(Z)g(hY,W) + 2\eta(Y)\eta(Z)g(hX,W) \\
+ & 2g(hX,Z)g(Y,W) - 2g(hY,Z)g(X,W) + 2g(hY,Z)\eta(X)g(\xi,W) \\
- & 2g(hX,Z)\eta(Y)g(\xi,W) - g(\phi hX,Z)g(\phi hY,W) + g(\phi hY,Z)g(\phi hX,W)\} \\
+ & 4\mu\{\eta(Y)\eta(Z)g(hX,W) - \eta(X)\eta(Z)g(hY,W) \\
+ & g(hY,Z)\eta(X)g(\xi,W) - g(hX,Z)\eta(Y)g(\xi,W).
\end{aligned} \tag{3.1}$$

Putting $X = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, $i = 1, 2, ..., 2n+1$, we get from (3.1)

$$\begin{aligned}
& 4S(X,W) \\
= & (c+3)\{(2n+1)g(X,W) - g(X,W)\} \\
+ & (c+3-4k)\{\textstyle\sum_i \eta(X)\eta(e_i)g(e_i,W) - \sum_i \eta(e_i)\eta(e_i)g(X,W) \\
+ & \textstyle\sum_i g(X,e_i)\eta(e_i)\eta(\xi,W) - \sum_i g(e_i,e_i)\eta(X)g(\xi,W)\} \\
+ & (c-1)\{\textstyle\sum_i g(X,\phi e_i)g(\phi e_i,W) - \sum_i g(e_i,\phi e_i)g(\phi X,W) + 2\sum_i g(X,\phi e_i)g(\phi_i,W)\} \\
- & 2\textstyle\sum_i\{g(hX,e_i)g(he_i,W) - \sum_i g(he_i,e_i)g(hX,W) + \sum_i g(X,e_i)g(he_i,W) \\
- & 2\textstyle\sum_i g(e_i,e_i)g(hX,W) - 2\sum_i \eta(X)\eta(e_i)g(he_i,W) + 2\sum_i \eta(e_i)\eta(e_i)g(hX,W) \\
+ & 2\textstyle\sum_i g(hX,e_i)g(e_i,W) - 2\sum_i g(he_i,e_i)g(X,W) + 2\sum_i g(he_i,e_i)\eta(X)g(\xi,W) \\
- & 2\textstyle\sum_i g(hX,e_i)\eta(e_i)g(\xi,W) - \sum_i g(\phi hX,e_i)g(\phi he_i,W) + \sum_i g(\phi he_i,e_i)g(\phi hX,W)\} \\
+ & 4\mu\{\textstyle\sum_i \eta(e_i)\eta(e_i)g(hX,W) - \sum_i \eta(X)\eta(e_i)g(he_i,W) \\
+ & \textstyle\sum_i g(he_i,e_i)\eta(X)g(\xi,W) - \sum_i g(hX,e_i)\eta(e_i)g(\xi,W).
\end{aligned} \tag{3.2}$$

or,

$$\begin{aligned}
& 4S(X,W) \\
= & 2n(c+3)g(X,W) \\
+ & (c+3-4k)\{\eta(X)\eta(W) - (2n+1)g(X,W) \\
+ & \eta(X)\eta(W) - (2n+1)\eta(X)\eta(W)\} \\
+ & (c-1)\{g(\phi X,\phi W) + 2g(\phi X,\phi W)\} \\
- & 2\{g(hX,hW) + g(X,hW) - 2(2n+1)g(hX,W) \\
- & \eta(X)\eta(hW) + 2(2n+1)g(hX,W) + 2g(hX,W) \\
- & \eta(hX)g(\xi,W) + g(\phi hX,h\phi W)\} \\
+ & 4\mu\{(2n+1)g(hX,W) - \eta(X)\eta(hW) - \eta(hX)g(\xi,W).
\end{aligned} \tag{3.3}$$

Since $h\xi = 0$, therefore $\eta(hX) = g(hX,\xi) = g(X,h\xi) = g(X,0) = 0$.
Using above result we obtain from (3.3)

$$\begin{aligned}
& 4S(X,W) \\
= & 2n(c+3)g(X,W) \\
+ & (c+3-4k)\{(1-2n)\eta(X)\eta(W) - (2n+1)g(X,W) \\
+ & 3(c-1)\{g(\phi X,\phi W)\} \\
- & 2\{g(h^2X,W) + 3g(X,hW) + g(\phi hX,h\phi W)\} \\
+ & 4\mu\{(2n+1)g(hX,W).
\end{aligned} \tag{3.4}$$

Using $h\phi = -\phi h$, $\phi^2(X) = -X + \eta(X)\xi$, in relation (3.4) we get

$$\begin{aligned} & 4S(X,W) \\ = \; & 2n(c+3)g(X,W) \\ + \; & (c+3-4k)\{(1-2n)\eta(X)\eta(W) - (2n+1)g(X,W) \\ + \; & 3(c-1)\{g(X,W) - \eta(X)\eta(W)\} \\ - \; & 2\{g(h^2X,W) + 3g(X,hW) - g(h^2X,hW)\} \\ + \; & 4\mu(2n+1)g(hX,W). \end{aligned} \tag{3.5}$$

or,

$$\begin{aligned} & 4S(X,W) \\ = \; & (8nk+4k+2c-3)g(X,W) + (8nk+4k-2nc-6n-2c+6)\eta(X)\eta(W) \\ + \; & \{4\mu(2n+1)-6\}g(hX,W). \end{aligned} \tag{3.6}$$

If we take $\mu = \frac{1}{2}$ and $n = 1$, then (3.6) becomes

$$4S(X,W) = (12k+2c-3)g(X,W) + (12k-4c)\eta(X)\eta(W) \tag{3.7}$$

i.e.

$$S(X,W) = (3k + \frac{c}{2} - \frac{3}{4})g(X,W) + (3k-c)\eta(X)\eta(W) \tag{3.8}$$

Thus we are in a position to state the following result:

Theorem 3.1. *A (k, μ) space forms of dimension three is η -Einstein for $\mu = \frac{1}{2}$.*
Again we know that $S(X,W) = g(QX,W)$, where Q is the Ricci operator. Thus using this in (3.8) we get

$$QX = (3k + \frac{c}{2} - \frac{3}{4})X + (3k-c)\eta(X)\xi, \tag{3.9}$$

where Q is the Ricci operator of (k, μ) space forms of dimension three for $\mu = \frac{1}{2}$. Again we have from (3.8) that

$$r = \sum_{i=1}^{3} S(e_i, e_i) \quad = 3(6k - \frac{c}{2} - \frac{3}{4}), \tag{3.10}$$

where r is the scalar curvature of (k, μ) space forms of dimension three for $\mu = \frac{1}{2}$.

4 η-parallel, cyclic parallel Ricci tensors and Ricci operator of (k, μ) space forms of dimension three

Definition 4.1. *The Ricci tensor S of (k, μ) space forms of dimension three will be called η-parallel if it satisfies,*

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{4.1}$$

for any vector fields X, Y, Z.
From (3.8) we get

$$(\nabla_W S)(X,Y) = (3k-c)\{\nabla_W \eta)(X)\eta(Y) + (\nabla_W \eta)(Y)\eta(X)\}. \tag{4.2}$$

From above it is clear that

$$(\nabla_X S)(\phi Y, \phi Z) = 0. \tag{4.3}$$

Now we are in a position to state the following:

Theorem 4.1. *The Ricci tensor of a (k, μ) space forms of dimension three is η-parallel for $\mu = \frac{1}{2}$.*

Definition 4.2. *The Ricci tensor of (k, μ) space forms of dimension three will be called cyclic parallel if*

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{4.4}$$

From (3.8) *we get*

$$\begin{aligned}&(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)\\ =\ &(3k-c)\{(\nabla_X\eta)(Y)\eta(Z)+(\nabla_X\eta)(Z)\eta(Y)\\ +\ &(\nabla_Y\eta)(Z)\eta(X)+(\nabla_Y\eta)(X)\eta(Z)\\ +\ &(\nabla_Z\eta)(X)\eta(Y)+(\nabla_Z\eta)(Y)\eta(X)\}.\end{aligned}\tag{4.5}$$

If we take X, Y, Z *orthogonal to* ξ, *then we obtain from above,*

$$(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)=0.\tag{4.6}$$

Now we are in a position to state the following:

Theorem 4.2. *The Ricci tensor of (k, μ) space forms of dimension three is cyclic parallel for* $\mu=\frac{1}{2}$.

Definition 4.3. *A (k, μ) space forms of dimension three is called locally* ϕ*-Ricci symmetric if,*

$$\phi^2(\nabla_W Q)X=0\tag{4.7}$$

, where the vector fields X *and* W *are orthogonal to* ξ. *The notion of locally* ϕ*-Ricci symmetry was introduced by U. C. De and A. Sarkar [6].*
Again from (3.8) *we obtain*

$$(\nabla_W Q)X=(3k-c)\{(\nabla_W\eta)(X)\xi+\eta(X)\nabla_W\xi\}\tag{4.8}$$

Taking X *orthogonal to* ξ *and applying* ϕ^2 *on both side of above we get*

$$\phi^2(\nabla_W Q)X=0.\tag{4.9}$$

Now we are in a position to state the following:

Theorem 4.3. *A (k, μ) space forms of dimension three is locally* ϕ*-Ricci symmetric for* $\mu=\frac{1}{2}$.

5 Locally ϕ- symmetric (k, μ) space forms

Definition 5.1. *A (2n+1)-dimensional (k, μ) space forms will be called locally* ϕ*-symmetric if* $\phi^2(\nabla_W R)(X,Y)Z=0$, *for any vector fields* X, Y, Z *and* W *orthogonal to* ξ.

In this connection it should be mentioned that the notion of locally ϕ- symmetric manifolds was introduced by T. Takahashi [10] in the context of Sasakian geometry.
First, we suppose that X, Y, Z and W orthogonal to ξ. Then relation (2.9) reduces to

$$\begin{aligned}4R(X,Y)Z\ =\ &(c+3)\{g(Y,Z)X-g(X,Z)Y\}\\ +\ &(c-1)\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}\\ -\ &2\{g(hX,Z)hY-g(hY,Z)hX+g(X,Z)hY\\ -\ &2g(Y,Z)hX+2g(hX,Z)Y\\ -\ &2g(hY,Z)X-g(\phi hX,Z)\phi hY+g(\phi hY,Z)\phi hX\}.\end{aligned}\tag{5.1}$$

Differentiating (5.1) covariantly with respect to a horizontal vector field W we get,

$$\begin{aligned}4(\nabla_W R)(X,Y)Z\ =\ &(c-1)\{g(X,(\nabla_W\phi)Z)\phi Y+g(X,\phi Z)(\nabla_W\phi)Y\\ -\ &g(Y,(\nabla_W\phi)Z)\phi X-g(Y,\phi Z)(\nabla_W\phi)X\\ +\ &2g(X,(\nabla_W\phi)Y)\phi Z+g(X,\phi Y)(\nabla_W\phi)Z\}\\ -\ &2\{g((\nabla_W h)X,Z)hY+g(hX,Z)(\nabla_W h)Y\\ -\ &g((\nabla_W h)Y,Z)hX-g(hY,Z)(\nabla_W h)X\\ +\ &g(X,Z)(\nabla_W h)Y-2g(Y,Z)(\nabla_W h)X\\ +\ &2g((\nabla_W h)X,Z)Y-2g((\nabla_W h)Y,Z)X\\ -\ &g((\nabla_W\phi)hX,Z)\phi hY-g(\phi hX,Z)(\nabla_W\phi)hY\\ +\ &g((\nabla_W\phi)hY,Z)\phi hX+g(\phi hY,Z)(\nabla_W\phi)hX\}.\end{aligned}\tag{5.2}$$

Again, as X, Y are orthogonal to ξ, so (2.5) and (2.6) reduces to

$$(\nabla_X\phi)Y=g(X,Y)\xi,\tag{5.3}$$

$$(\nabla_X h)(Y) = \{(1-k)g(X, \phi Y) + g(X, h\phi Y)\}\xi. \tag{5.4}$$

After using (5.3) and (5.4) in (5.2) and then applying ϕ^2 on both side we obtain

$$\phi^2(\nabla_W R)(X, Y)Z = 0. \tag{5.5}$$

Thus we are in a position to state the following result:

Theorem 5.1. *Every (2n+1) dimensional (k, μ) space forms is locally φ- symmetric.*

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A spline method for solving fourth order singularly perturbed boundary value problem

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Abstract

In this paper, singularly perturbed boundary value problem of fourth order ordinary differential equation with a small positive parameter multiplying with the highest derivative of the form

$$\varepsilon u^{(4)}(x) + p(x)u''(x) + q(x)u(x) = r(x), \;\; 0 \le x \le 1,$$

$$u(0) = \gamma_0, u(1) = \gamma_1, u''(0) = \eta_0, u''(1) = \eta_1, 0 \le \varepsilon \le 1$$

is considered. We have developed a numerical technique for the above problem using parametric and polynomial septic spline method. The method is shown to have second and fourth order convergent depending on the choice of parameters involved in the method. Truncation error and boundary equations are obtained. The method is tested on an example and the results are found to be in agreement with the theoretical analysis.

Keywords: Parametric septic splines, Polynomial septic splines, Boundary value problems, Boundary equations.

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1 Introduction

Singular perturbation problems appear in many branches of applied mathematics, and for more than two decades quite a large number of research papers on the qualitative and quantitative analysis of these problems for both ordinary differential equations (ODEs) and partial differential equations (PDEs) have been reported in the literature. Most of the papers connected with computational aspects are confined to second order equations. But only few authors have developed numerical methods for singularly perturbed higher order differential equations. These problems are classified on the basis of how the order of original differential equation is affected if sets $\varepsilon = 0$ [8]. Here, ε is a small positive parameter multiplying with the highest derivative of the differential equation. The singularly perturbed problem is of convection-diffusion type if the order of the differential equation is reduced by 1, whereas it is called reaction-diffusion type if order is reduced by 2. The objective of the present paper is to develop a computational method to solve singularly perturbed boundary value problems of fourth order ordinary differential equations of the form:

$$\left.\begin{aligned} &\varepsilon u^{(4)}(x) + p(x)u''(x) + q(x)u(x) = r(x), \;\; 0 \le x \le 1, \\ &u(0) = \gamma_0, u(1) = \gamma_1, u''(0) = \eta_0, u''(1) = \eta_1, 0 \le \varepsilon \le 1, \end{aligned}\right\} \tag{1.1}$$

where $p(x),\; q(x)$ and $r(x)$ are smooth, bounded, real functions $p(x) : R \to R,\; q(x) : R \to R,\; r(x) : R \to R$ satisfying the following conditions

$$-p \ge \beta > 0, 0 \ge q \ge -\gamma, \gamma > 0,$$

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$$\beta - 2\gamma \geq \eta > 0, \ for\ some\ \ \eta,$$
$$D = (0,1), \overline{D} = [0,1] \ and\ u\ \epsilon C^4(D) \bigcap C^2(\overline{D}). \tag{1.2}$$

Analytical and numerical treatment of these equations have drawn much attention of many authors [5,22-25]. The analytical treatment of singularly perturbed boundary value problems for higher order nonlinear ordinary differential equations, which have important applications in Fluid Dynamics, can be found in [1,2,6-9,13,20]. Semper [2] and Roos and Stynes [8] considered fourth order equations and applied a standard finite element method. Garland [3] has shown that uniform stability of discrete boundary value problem follows from uniform stability of the discrete initial value problem and uniform consistency of the scheme. Some results connected with the exponentially fitted higher order differences with identity expansion method [3] and defect corrections are available in the literature. Loghmani and Ahmadinia [5] have developed a numerical technique for solving singularly perturbed boundary value problems based on optimal control strategy by using B-spline functions and least square method. Also finite element method is reported in [6,7]. In [9], an iterative method is described. In [10,11,16,17,20], the authors have applied boundary value technique to find the numerical solution for singularly perturbed second order boundary value problems. Niederdrenk and Yserentant [12] considered convection diffusion type problems and derived conditions for the uniform stability of discrete and continuous problems. Feckan [13] considered higher order problems and his approach is based on the nonlinear analysis involving fixed point theory, Leray-Schauder theory etc. In [15], authors have given a brief survey on computational techniques for the different classes of singularly perturbed problems. Bawa [19] and Aziz and Khan [21] have solved second order singularly perturbed boundary value problem using spline technique. Shanthi and Ramanujam [22-25] have developed numerical methods for singularly perturbed higher order boundary value problems.

In this problem, we take $p(x) = p =$ constant and $q(x) = q =$ constant. In the present paper, parametric septic spline is described for fourth order boundary value problems. In section 2, a brief description of the method is given. Development of the boundary equations are given in section 3. In section 4, truncation error and class of methods are discussed. We established the convergence of our method in section 5 and section 6 contains the numerical results and discussions.

2 Derivation of the method

In order to develop the numerical method for approximating the solution of singularly perturbed fourth order boundary value problem, the interval $[0,1]$ is divided into N equal subintervals using the grid points $x_j = jh, \ \ j = 0(1)N$, where

$$x_0 = 0, \quad x_N = 1, \quad and \quad h = \frac{1}{N}. \tag{2.1}$$

A function $S_\Delta(x,\tau)$ of class $C^6[0,1]$ which interpolates $u(x)$ at the mesh point x_j depends on a parameter τ, and as $\tau \to 0$ it reduces to septic spline $S_\Delta(x)$ in $[0,1]$ is termed as parametric septic spline function. Since the parameter τ can occur in $S_\Delta(x)$ in many ways such a spline is not unique.

If $S_\Delta(x,\tau) = S_\Delta(x)$ is a piecewise function satisfying the following differential equation in the interval $[x_{j-1}, x_j]$

$$\begin{aligned} S_\Delta^{(6)}(x) - \tau^2 S_\Delta^{''}(x) &= (Q_j - \tau^2 M_j)\frac{x - x_{j-1}}{h} + (Q_{j-1} - \tau^2 M_{j-1})\frac{x_j - x}{h} \\ &= A_j z + A_{j-1}\bar{z}, \end{aligned} \tag{2.2}$$

where

$$z = \frac{x - x_{j-1}}{h}, \ \bar{z} = 1 - z, \ A_k = Q_k - \tau^2 M_k, S_\Delta^{''}(x_k,\tau) = M_k, \ \ S_\Delta^{(6)}(x_k,\tau) = Q_k, \ \ k = j-1, j; \ \ \tau > 0,$$

then it is termed as parametric septic spline II. Solving equation (2.2), we get

$$\begin{aligned} S_\triangle(x) &= B_1 + B_2 x + B_3 \cosh\sqrt{\tau}x + B_4 \sinh\sqrt{\tau}x + B_5 \cos\sqrt{\tau}x + B_6 \sin\sqrt{\tau}x \\ &\quad - \frac{1}{\tau^2}\left\{(Q_j - \tau^2 M_j)\frac{(x - x_{j-1})^3}{6h} + (Q_{j-1} - \tau^2 M_{j-1})\frac{(x_j - x)^3}{6h}\right\}. \end{aligned} \tag{2.3}$$

To develop the consistency relations between the value of spline and its derivatives at knots, let

$$\left.\begin{aligned} S_\Delta(x_j) &= u_j, \quad S_\Delta(x_{j+1}) = u_{j+1}, \\ S''_\Delta(x_j) &= M_j, \quad S''_\Delta(x_{j+1}) = M_{j+1}, \\ S^{(4)}_\Delta(x_j) &= F_j, \quad S^{(4)}_\Delta(x_{j+1}) = F_{j+1}. \end{aligned}\right\} \tag{2.4}$$

To define spline in terms of u_j's, M_j's and F_j's, the coefficients introduced in Eq.(2.3) are calculated as

$$\begin{aligned} B_1 &= u_{j-1} + \frac{h^2}{6\tau^2}(Q_{j-1} - \tau^2 M_{j-1}) - \frac{F_{j-1}}{\tau^2} \\ &\quad - \frac{x_{j-1}}{h}\left[(u_j - u_{j-1}) - \frac{h^2}{6\tau^2}(Q_{j-1} - \tau^2 M_{j-1}) + \frac{h^2}{6\tau^2}(Q_j - \tau^2 M_j) + \frac{1}{\tau^2}(F_{j-1} - F_j)\right], \\ B_2 &= \frac{1}{h}(u_j - u_{j-1}) + \frac{h}{6\tau^2}\left[-(Q_{j-1} - \tau^2 M_{j-1}) + (Q_j - \tau^2 M_j)\right] + \frac{1}{\tau^2 h}(F_{j-1} - F_j), \\ B_3 &= \frac{1}{\tau^2 \sinh\sqrt{\tau}h}\left[\frac{1}{2}\sinh\sqrt{\tau}x_j\left(F_{j-1} - \frac{Q_{j-1}}{\tau}\right) - \frac{1}{2}\sinh\sqrt{\tau}x_{j-1}\left(F_j - \frac{Q_j}{\tau}\right)\right. \\ &\quad \left. - \frac{1}{\tau}\sinh\sqrt{\tau}x_{j-1}Q_j + \frac{1}{\tau}\sinh\sqrt{\tau}x_j Q_{j-1}\right], \\ B_4 &= \frac{1}{\tau^2 \sinh\sqrt{\tau}h}\left[-\frac{1}{2}\cosh\sqrt{\tau}x_j\left(F_{j-1} - \frac{Q_{j-1}}{\tau}\right) + \frac{1}{2}\cosh\sqrt{\tau}x_{j-1}\left(F_j - \frac{Q_j}{\tau}\right)\right. \\ &\quad \left. + \frac{1}{\tau}\cosh\sqrt{\tau}x_{j-1}Q_j - \frac{1}{\tau}\cosh\sqrt{\tau}x_j Q_{j-1}\right], \\ B_5 &= \frac{1}{2\tau^2 \sinh\sqrt{\tau}h}\left[\sin\sqrt{\tau}x_j\left(F_{j-1} - \frac{Q_{j-1}}{\tau}\right) - \sin\sqrt{\tau}x_{j-1}\left(F_j - \frac{Q_j}{\tau}\right)\right], \\ B_6 &= \frac{1}{2\tau^2 \sinh\sqrt{\tau}h}\left[-\cos\sqrt{\tau}x_j\left(F_{j-1} - \frac{Q_{j-1}}{\tau}\right) + \cos\sqrt{\tau}x_{j-1}\left(F_j - \frac{Q_j}{\tau}\right)\right]. \end{aligned} \tag{2.5}$$

Substituting these values in (2.3), we get

$$S_\Delta(x) = zu_j + \bar{z}u_{j-1} + \frac{h^2}{6}\left[p(z)M_j + p(\bar{z})M_{j-1}\right] + \frac{h^4}{2}\left[r(z)F_j + r(\bar{z})F_{j-1}\right] + \frac{h^6}{6}\left[q(z)Q_j + q(\bar{z})Q_{j-1}\right], \tag{2.6}$$

where

$$\begin{aligned} p(z) &= z^3 - z, \quad q(z) = \frac{z}{\omega^4} - \frac{z^3}{\omega^4} + \frac{3\sinh\omega z}{\omega^6 \sinh\omega} - \frac{3\sin\omega z}{\omega^6 \sin\omega}, \\ r(z) &= \frac{-2z}{\omega^4} + \frac{\sinh\omega z}{\omega^4 \sinh\omega} + \frac{\sin\omega z}{\omega^4 \sin\omega}, \quad \omega = \sqrt{\tau}h. \end{aligned} \tag{2.7}$$

Applying the first, third and fifth derivative continuities at the knots, i.e. $S^{(\mu)}_\Delta(x_j^-) = S^{(\mu)}_\Delta(x_j^+)$, $\mu = 1, 3$ *and* 5, the following consistency relations are derived:

$$\begin{aligned} M_{j+1} + 4M_j + M_{j-1} &= \frac{6}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + 3h^2(\alpha_2 F_{j+1} + 2\beta_2 F_j + \alpha_2 F_{j-1}) \\ &\quad + h^4(\alpha_1 Q_{j+1} + 2\beta_1 Q_j + \alpha_1 Q_{j-1}), \quad j = 1(1)N-1, \end{aligned} \tag{2.8}$$

$$\begin{aligned} M_{j+1} - 2M_j + M_{j-1} &= \frac{h^2}{6}[(1 - \omega^4\alpha_1)F_{j+1} + 2(2 - \omega^4\beta_1)F_j + (1 - \omega^4\alpha_1)F_{j-1}] \\ &\quad - \frac{h^4}{2}(\alpha_2 Q_{j+1} + 2\beta_2 Q_j + \alpha_2 Q_{j-1}), \quad j = 1(1)N-1, \end{aligned} \tag{2.9}$$

$$h^2[(1-\omega^4\alpha_1)Q_{j+1}+2(2-\omega^4\beta_1)Q_j+(1-\omega^4\alpha_1)Q_{j-1}] =$$
$$3[(\omega^4\alpha_2+2)F_{j+1}+2(\omega^4\beta_2-2)F_j+(\omega^4\alpha_2+2)F_{j-1}],\ \ j=1(1)N-1, \tag{2.10}$$

where

$$\alpha_1 = \frac{1}{\omega^4}+\frac{3}{\omega^5\sinh\omega}-\frac{3}{\omega^5\sin\omega},\quad \beta_1=\frac{2}{\omega^4}-\frac{3}{\omega^5}\coth\omega+\frac{3}{\omega^5}\cot\omega,$$
$$\alpha_2 = \frac{-2}{\omega^4}+\frac{1}{\omega^3\sinh\omega}+\frac{1}{\omega^3\sin\omega},\quad \beta_2=\frac{2}{\omega^4}-\frac{1}{\omega^3}\coth\omega-\frac{1}{\omega^3}\cot\omega. \tag{2.11}$$

As $\tau\to 0$ that is $\omega\to 0$ then $(\alpha_1,\beta_1,\alpha_2,\beta_2)\to(\frac{-31}{2520},\frac{-4}{315},\frac{7}{180},\frac{2}{45})$.
Using equations (2.8)-(2.10), we obtain the following scheme

$$(e_1u_{j-3}+e_2u_{j-2}+e_3u_{j-1}+e_4u_j+e_3u_{j+1}+e_2u_{j+2}+e_1u_{j+3})$$
$$=\frac{h^4}{6}(p_1F_{j-3}+p_2F_{j-2}+p_3F_{j-1}+p_4F_j+p_3F_{j+1}+p_2F_{j+2}+p_1F_{j+3}), j=3(1)N-3, \tag{2.12}$$

where the coefficients (e_1,e_2,e_3,e_4) and (p_1,p_2,p_3,p_4) of the developed scheme are given by

$$\begin{aligned}
e_1 &= 1-3\omega^4\alpha_1+3\omega^8\alpha_1^2-\omega^{12}\alpha_1^3,\\
e_2 &= 4\omega^4\alpha_1-2\omega^4\beta_1-8\omega^8\alpha_1^2+4\omega^8\alpha_1\beta_1-2\omega^{12}\alpha_1^2\beta_1,\\
e_3 &= 7(1-\omega^4\alpha_1)^3-8(1-\omega^4\alpha_1)^2(2-\omega^4\beta_1),\\
e_4 &= 12(1-\omega^4\alpha_1)^2(2-\omega^4\beta_1)-8(1-\omega^4\alpha_1)^3,\\
p_1 &= c_1(1-\omega^4\alpha_1)^2,\\
p_2 &= 2c_1(1-\omega^4\alpha_1)(2-\omega^4\beta_1)+c_2(1-\omega^4\alpha_1)^2-3d_1(1-\omega^4\alpha_1)(2+\omega^4\alpha_2),\\
p_3 &= (c_1+c_3)(1-\omega^4\alpha_1)^2+6d_1(1-\omega^4\alpha_1)(2-\omega^4\beta_2)+2c_2(1-\omega^4\alpha_1)(2-\omega^4\beta_1)\\
&\quad -3d_2(1-\omega^4\alpha_1)(2+\omega^4\alpha_2),\\
p_4 &= 2c_2(1-\omega^4\alpha_1)^2-6d_1(1-\omega^4\alpha_1)(2+\omega^4\alpha_2)-6d_1(2-\omega^4\beta_1)(2-\omega^4\beta_2)\\
&\quad +2c_3(1-\omega^4\alpha_1)(2-\omega^4\beta_1)+6d_2(1-\omega^4\alpha_1)(2-\omega^4\beta_2).
\end{aligned} \tag{2.13}$$

Also

$$\begin{aligned}
c_1 &= \frac{1}{6}\omega^8\alpha_1^2-\frac{3}{2}\omega^4\alpha_2^2-\frac{1}{3}\omega^4\alpha_1-6\alpha_1-6\alpha_2+\frac{1}{6},\\
c_2 &= \frac{2}{3}\omega^8\alpha_1^2+\frac{1}{3}\omega^8\alpha_1\beta_1-18\omega^4\alpha_1\alpha_2-3\omega^4\alpha_2\beta_2-6\omega^4\alpha_2^2-2\omega^4\alpha_1-\frac{1}{3}\omega^4\beta_1-12\alpha_1-6\beta_2+\frac{4}{3},\\
c_3 &= \frac{1}{3}\omega^8\alpha_1^2+\frac{4}{3}\omega^8\alpha_1\beta_1-36\omega^4\alpha_1\beta_2-12\omega^4\alpha_2\beta_2-3\omega^4\alpha_2^2-\frac{10}{3}\omega^4\alpha_1-\frac{4}{3}\omega^4\beta_1+36\alpha_1\\
&\quad +12\alpha_2+12\beta_2+3,\\
d_1 &= \omega^4\alpha_2\beta_1-\omega^4\alpha_1\beta_2+6\omega^4\alpha_1^2-10\alpha_1-2\alpha_2+2\beta_1+\beta_2,\\
d_2 &= 4\omega^4\alpha_2\beta_1-4\omega^4\alpha_1\beta_2+12\omega^4\alpha_1\beta_1-16\alpha_1-18\alpha_2-4\beta_1+4\beta_2.
\end{aligned} \tag{2.14}$$

As $\tau\to 0$ that is $\omega\to 0$, we have
(i)$(e_1,e_2,e_3,e_4)\longrightarrow(1,0,-9,16)$,
(ii)$(c_1,c_2,c_3,d_1,d_2)\longrightarrow\left(\frac{1}{140},\frac{17}{14},\frac{249}{70},\frac{9}{140},\frac{4}{35}\right)$,
(iii)$(p_1,p_2,p_3,p_4)\longrightarrow\left(\frac{1}{140},\frac{6}{7},\frac{1191}{140},\frac{604}{35}\right)$.
[Remarks:] For these values our scheme reduces to the polynomial septic spline for fourth order boundary value problem which is given as equation (7) in G. Akram and S. S. Siddiqi [4].
We have taken $(e_1,e_2,e_3,e_4)=(1,0,-9,16)$ in Eq. (2.12) and obtained

$$p_1(F_{j-3}+F_{j+3})+p_2(F_{j-2}+F_{j+2})+p_3(F_{j-1}+F_{j+1})+p_4F_j=\frac{6}{h^4}[(u_{j-3}+u_{j+3})-9(u_{j-1}+u_{j+1})+16u_j]. \tag{2.15}$$

Eq. (1.1) can be written in the following form by taking $p(x) = p$ and $q(x) = q$ as

$$\varepsilon F_j + pM_j + qu_j = r_j. \tag{2.16}$$

Operate Λ_x on the both side of Eq. (2.16), we get

$$\varepsilon\Lambda_x F_j + p\Lambda_x M_j + q\Lambda_x u_j = \Lambda_x r_j \tag{2.17}$$

where operator Λ_x is defined as follows for any function $'W'$ evaluated at mesh point

$$\Lambda_x W_j = p_1(W_{j-3} + W_{j+3}) + p_2(W_{j-2} + W_{j+2}) + p_3(W_{j-1} + W_{j+1}) + p_4 W_j. \tag{2.18}$$

For second derivative of u, we take relation from [Eq. (5), Ref. [4]]

$$\Lambda_x M_j = \frac{42}{h^2}[(u_{j-3} + u_{j+3}) + 24(u_{j-2} + u_{j+2}) + 15(u_{j-1} + u_{j+1}) - 80u_j]. \tag{2.19}$$

Here, $(p_1, p_2, p_3, p_4) = (1, 120, 1191, 2416)$ for second derivative. Using (2.15-2.19), we get

$$(6\varepsilon + 42ph^2 + qh^4 p_1)(u_{j-3} + u_{j+3}) + (1008ph^2 + qh^4 p_2)(u_{j-2} + u_{j+2})$$

$$+(-54\varepsilon + 630ph^2 + qh^4 p_3)(u_{j-1} + u_{j+1}) + (96\varepsilon - 3360ph^2 + qh^4 p_4)u_j$$

$$= h^4[p_1(r_{j-3} + r_{j+3}) + p_2(r_{j-2} + r_{j+2}) + p_3(r_{j-1} + r_{j+1}) + p_4 r_j],\ j = 3(1)(N-3). \tag{2.20}$$

3 Development of boundary equations

The relation (2.20) gives $(N-5)$ algebraic equations in $(N-1)$ unknowns $u_j, j = 1(1)N-1$. We require four more equations, two at each end of range of integartion in order to have closed form solution for u_j. For the discretization of the boundary conditions, we have developed boundary equations for second and fourth order methods as follows:

3.1 Second order method

(i) $-5u_1 + 4u_2 - u_3 = -2\gamma_0 + h^2\eta_0 - \frac{11h^4}{12\varepsilon}(r_0 - q\gamma_0 - p\eta_0), j = 1,$

(ii) $\frac{52}{5}u_1 - \frac{57}{5}u_2 + \frac{28}{5}u_3 - \frac{11}{10}u_4 = \frac{7}{2}\gamma_0 - \frac{6}{5}h^2\eta_0, j = 2,$

(iii) $-\frac{11}{10}u_{N-4} + \frac{28}{5}u_{N-3} - \frac{57}{5}u_{N-2} + \frac{52}{5}u_{N-1} = \frac{7}{2}\gamma_1 - \frac{6}{5}h^2\eta_1, j = N-2,$

(iv) $-u_{N-3} + 4u_{N-2} - 5u_{N-1} = -2\gamma_1 + h^2\eta_1 - \frac{11h^4}{12\varepsilon}(r_N - q\gamma_1 - p\eta_1), j = N-1.$

3.2 Fourth order method

(i) $-\frac{1322}{35}u_1 + \frac{11066}{245}u_2 - \frac{6684}{245}u_3 + \frac{2171}{245}u_4 - \frac{302}{245}u_5 = -\frac{429}{35}\gamma_0 + \frac{204}{49}h^2\eta_0 - \frac{274h^4}{245\varepsilon}(r_0 - q\gamma_0 - p\eta_0), j = 1,$

(ii) $-\frac{1342}{97}u_1 + \frac{7213}{375}u_2 - \frac{2049}{146}u_3 + \frac{3799}{691}u_4 - \frac{11059}{12203}u_5 + \frac{12}{8551}u_6 = -\frac{9898}{2449}\gamma_0 + \frac{348}{323}h^2\eta_0, j = 2,$

(iii) $\frac{12}{8551}u_{N-6} - \frac{11059}{12203}u_{N-5} + \frac{3799}{691}u_{N-4} - \frac{2049}{146}u_{N-3} + \frac{7213}{375}u_{N-2} - \frac{1342}{97}u_{N-1} = -\frac{9898}{2449}\gamma_1 + \frac{348}{323}h^2\eta_1, j = N-2,$

(iv) $-\frac{1322}{35}u_{N-1} + \frac{11066}{245}u_{N-2} - \frac{6684}{245}u_{N-3} + \frac{2171}{245}u_{N-4} - \frac{302}{245}u_{N-5} = -\frac{429}{35}\gamma_1 + \frac{204}{49}h^2\eta_1 - \frac{274h^4}{245\varepsilon}(r_N - q\gamma_1 - p\eta_1), j = N-1.$

4 Truncation error

To obtain the local truncation error t_j, $j = 3(1)(N-3)$, associated with the scheme (2.20), substitute $r_j = \varepsilon u_j^{(4)} + p u_j'' + q u_j$ in eq. (2.20) and expanding it by Taylor series about x_j, we obtain the following local truncation error

$$\begin{aligned} t_j \;=\; & (864 - 48p_1 - 48p_2 - 48p_3 - 24p_4)\frac{\varepsilon h^4}{4!}u_j^{(4)} \\ & +(8640 - 6480p_1 - 2880p_2 - 720p_3)\frac{\varepsilon h^6}{6!}u_j^{(6)} \\ & +(78624 - 272160p_1 - 53760p_2 - 3360p_3)\frac{\varepsilon h^8}{8!}u_j^{(8)} - \frac{6720}{8!}ph^{10}u_j^{(8)} \\ & +(708480 - 7348320p_1 - 645120p_2 - 10080p_3)\frac{\varepsilon h^{10}}{10!}u_j^{(10)} \\ & +\frac{10080}{10!}ph^{12}u_j^{(10)} + O(h^{12}). \end{aligned} \tag{4.1}$$

By using the above equation and eliminating the coefficients of various powers of h, we can obtain class of the methods. For arbitrary choices of p_1, p_2, p_3 and p_4, we obtain the following methods:

4.1 Second order methods

By equating the coefficient of h^4 equal to zero in (4.1), we get second order methods. Therefore,

$$p_4 = 36 - 2p_1 - 2p_2 - 2p_3, \tag{4.2}$$

where p_1, p_2, p_3 and p_4, are arbitrary. The truncation error is given by

$$t_j = \frac{\varepsilon h^6}{6!}u_j^{(6)}(8640 - 6480p_1 - 2880p_2 - 720p_3). \tag{4.3}$$

The local truncation error at $j = 1, 2, N-2, N-1$ for second order methods is

$$t_j = \begin{cases} (\frac{-239}{360})h^6 u_j^{(6)} + O(h^7), & j = 1, N-1, \\ (\frac{-119}{75})h^6 u_j^{(6)} + O(h^7), & j = 2, N-2. \end{cases} \tag{4.4}$$

4.2 Fourth order methods

By equating the coefficient of h^4 and h^6 equal to zero in (4.1), we get fourth order methods. Therefore,

$$p_3 = 12 - 9p_1 - 4p_2 \; and \; p_4 = 12 + 16p_1 + 6p_2, \tag{4.5}$$

where p_1 and p_2 are arbitrary. The truncation error is given by

$$t_j = \frac{\varepsilon h^8}{8!}u_j^{(8)}(38304 - 241920p_1 - 40320p_2). \tag{4.6}$$

The local truncation error at $j = 1, 2, N-2, N-1$ for fourth order method is

$$t_j = \begin{cases} (\frac{-204737}{120960})h^8 u_j^{(8)} + O(h^9), & j = 1, N-1, \\ (\frac{-14323}{10080})h^8 u_j^{(8)} + O(h^9), & j = 2, N-2. \end{cases} \tag{4.7}$$

5 Convergence analysis

Now, we investigate the convergence analysis of the spline method described in section 2 for problem (1.1). To do so, we let, $U=(u_j)$, $\overline{U}=(\overline{u}_j)$, $V=(v_j)$, $T=(t_j)$ and $E=(e_j)=U-\overline{U}$, be N-dimensional column vectors. Here, $U,\overline{U},T$ denotes the exact solution, the approximate solution and truncation error respectively and e_j is the discretization error for $j=1(1)(N-1)$. Thus we can write the system (2.20) in the matrix form:

$$A\overline{U}-h^4BR=V \text{ and } A=A_1+h^2pA_2+h^4qB \tag{5.1}$$

where $A_1,\ A_2,\ B,\ R$ and V are defined by

$$A_1=\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & & & & \\ a_1^* & a_2^* & a_3^* & a_4^* & a_5^* & a_6^* & & & \\ 0 & -54\varepsilon & 96\varepsilon & -54\varepsilon & 0 & 6\varepsilon & & & \\ 6\varepsilon & 0 & -54\varepsilon & 96\varepsilon & -54\varepsilon & 0 & 6\varepsilon & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & 6\varepsilon & 0 & -54\varepsilon & 96\varepsilon & -54\varepsilon & 0 & 6\varepsilon \\ & & 6\varepsilon & 0 & -54\varepsilon & 96\varepsilon & -54\varepsilon & 0 \\ & & a_{N-6}^* & a_{N-5}^* & a_{N-4}^* & a_{N-3}^* & a_{N-2}^* & a_{N-1}^* \\ & & & a_{N-5} & a_{N-4} & a_{N-3} & a_{N-2} & a_{N-1} \end{bmatrix}, \tag{5.2}$$

$$A_2=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ 1008 & 630 & -3360 & 630 & 1008 & 42 & & \\ 42 & 1008 & 630 & -3360 & 630 & 1008 & 42 & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & 42 & 1008 & 630 & -3360 & 630 & 1008 & 42 \\ & & 42 & 1008 & 630 & -3360 & 630 & 1008 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.3}$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ p_2 & p_3 & p_4 & p_3 & p_2 & p_1 & & \\ p_1 & p_2 & p_3 & p_4 & p_3 & p_2 & p_1 & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & p_1 & p_2 & p_3 & p_4 & p_3 & p_2 & p_1 \\ & & p_1 & p_2 & p_3 & p_4 & p_3 & p_2 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.4}$$

$R=[r_j]^T$ and $V=[v_j]^T,\ j=1(1)N-1$.
Moreover,

$$v_j=\begin{cases} a_0\gamma_0+b_1h^2\eta_0+d_0\frac{h^4}{\varepsilon}(r_0-q\gamma_0-p\eta_0), & j=1, \\ a_0^*\gamma_0+b_2h^2\eta_0, & j=2, \\ -(6\varepsilon+42ph^2+qp_1h^4)\gamma_0+h^4p_1r_0, & j=3, \\ 0, & j=4(1)N-4, \\ -(6\varepsilon+42ph^2+qp_1h^4)\gamma_1+h^4p_1r_N, & j=N-3, \\ a_N^*\gamma_0+b_{N-2}h^2\eta_1, & j=N-2, \\ a_N\gamma_1+b_{N-1}h^2\eta_1+d_N\frac{h^4}{\varepsilon}(r_N-q\gamma_1-p\eta_1), & j=N-1, \end{cases} \tag{5.5}$$

where for second order method, we have

(i) $(a_0, a_1, a_2, a_3, a_4, a_5, b_1, d_0) = \left(-2, -5, 4, -1, 0, 0, 1, -\frac{11}{12} \right)$,

(ii) $(a_0^*, a_1^*, a_2^*, a_3^*, a_4^*, a_5^*, a_6^*, b_2) = \left(\frac{7}{2}, \frac{52}{5}, -\frac{57}{5}, \frac{28}{5}, -\frac{11}{10}, 0, 0, -\frac{6}{5} \right)$,

(iii) $(a_N^*, a_{N-6}^*, a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*, b_{N-2}) = \left(\frac{7}{2}, 0, 0, -\frac{11}{10}, \frac{28}{5}, -\frac{57}{5}, \frac{52}{5}, -\frac{6}{5} \right)$,

(iv) $(a_N, a_{N-5}, a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, b_{N-1}, d_N) = \left(-2, 0, 0, -1, 4, -5, 1, -\frac{11}{12} \right)$.

and for fourth order method, we have

(i) $(a_0, a_1, a_2, a_3, a_4, a_5, b_1, d_0) = \left(-\frac{429}{35}, -\frac{1322}{35}, \frac{11066}{245}, -\frac{6684}{245}, \frac{2171}{245}, -\frac{302}{245}, \frac{204}{49}, -\frac{274}{245} \right)$,

(ii) $(a_0^*, a_1^*, a_2^*, a_3^*, a_4^*, a_5^*, a_6^*, b_2) = \left(-\frac{9898}{2449}, -\frac{1342}{97}, \frac{7213}{375}, -\frac{2049}{146}, \frac{3799}{691}, -\frac{11059}{12203}, \frac{12}{8551}, \frac{348}{323} \right)$,

(iii) $(a_N^*, a_{N-6}^*, a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*, b_{N-2}) = \left(-\frac{9898}{2449}, \frac{12}{8551}, -\frac{11059}{12203}, \frac{3799}{691}, -\frac{2049}{146}, \frac{7213}{375}, -\frac{1342}{97}, \frac{348}{323} \right)$,

(iv) $(a_N, a_{N-5}, a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, b_{N-1}, d_N) = \left(-\frac{429}{35}, -\frac{302}{245}, \frac{2171}{245}, -\frac{6684}{245}, \frac{11066}{245}, -\frac{1322}{35}, \frac{204}{49}, -\frac{274}{245} \right)$.

Also, we have

$$AU - h^4 BR = T(h) + V. \tag{5.6}$$

From Eq. (5.1) and Eq. (5.6), we get

$$A(U - \overline{U}) = T(h),$$

$$AE = T(h). \tag{5.7}$$

Clearly

$$S_j = \begin{cases} a_1 + a_2 + a_3 + a_4 + a_5, & j = 1, \\ a_1^* + a_2^* + a_3^* + a_4^* + a_5^* + a_6^*, & j = 2, \\ -(6\varepsilon + 42ph^2) + q(p_1 + 2p_2 + 2p_3 + p_4)h^4, & j = 3, \\ q(2p_1 + 2p_2 + 2p_3 + p_4)h^4, & j = 4(1)N-4, \\ -(6\varepsilon + 42ph^2) + q(p_1 + 2p_2 + 2p_3 + p_4)h^4, & j = N-3, \\ a_{N-6}^* + a_{N-5}^* + a_{N-4}^* + a_{N-3}^* + a_{N-2}^* + a_{N-1}^*, & j = N-2, \\ a_{N-5} + a_{N-4} + a_{N-3} + a_{N-2} + a_{N-1}, & j = N-1. \end{cases} \tag{5.8}$$

We can choose h sufficiently small so that the matrix A is irreducible and monotone [18]. It follows that A^{-1} exists and its elements are non negative. Hence, from Eq. (5.7), we have

$$E = A^{-1}T(h). \tag{5.9}$$

Also, from the theory of the matrices, we have

$$\sum_{j=1}^{N-1} \overline{a}_{k,j} S_j = 1, \ k = 1(1)N-1, \tag{5.10}$$

where $\overline{a}_{k,j}$ is the $(k, j)th$ element of the matrix A^{-1}. Therefore

$$\sum_{j=1}^{N-1} \overline{a}_{k,j} \leq \frac{1}{\min_{1 \leq j \leq N-1} S_0} = \frac{1}{q(2p_1 + 2p_2 + 2p_3 + p_4)h^4} \tag{5.11}$$

From Eq.(5.9) and (5.11), we have

$$e_j = \sum_{j=1}^{N-1} \overline{a}_{k,j} T_j(h), \ k = 1(1)N-1$$

and therefore

$$| e_j | \leq \frac{KT_j}{h^4}, j = 1(1)N-1,$$

where K is a constant indepent of h. It follows that
(i) For second order methods the truncation error is $\| T \| = O(h^6)$. It follows that $\| E \| = O(h^2)$.
(ii) For fourth order methods the truncation error is $\| T \| = O(h^8)$. It follows that $\| E \| = O(h^4)$.

6 Numerical results and discussion

Example 1: Consider the boundary value problem

$$\varepsilon u^{(4)}(x) - 4u''(x) - u(x) = -\frac{x(1-x)}{8} - \frac{5\varepsilon}{16} + \frac{5\varepsilon}{16}\left[\frac{e^{-\frac{2x}{\sqrt{\varepsilon}}} - e^{-\frac{2(1+x)}{\sqrt{\varepsilon}}} + e^{-\frac{2(1-x)}{\sqrt{\varepsilon}}} - e^{-\frac{2(2-x)}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{4}{\sqrt{\varepsilon}}}}\right],$$

$$u(0) = u(1) = 1,\ u''(0) = u''(1) = -1.$$

We have solved this example by scheme (2.20) and have obtained approximate solution at $x = 0.001(0.001)0.009$ for the sake of comparison with references. The obtained numerical results are tabulated in table 1 and 2 for second and fourth order methods respectively. The comparison is also made in table 2 with the obatined results of [23].

Table 1. Maximum absolute errors for example 1

Second order method,$\varepsilon = 0.01, h = 0.001$

x	Present method $(p_1, p_2, p_3, p_4) = (0, 0, 0, 36)$	Exact [23]	Errors of present method
0.001	1.0000033	1.000063	5.97(−5)
0.002	1.0000058	1.000127	1.21(−4)
0.003	1.0000074	1.000192	1.84(−4)
0.004	1.0000084	1.000258	2.50(−4)
0.005	1.0000087	1.000324	3.15(−4)
0.006	1.0000084	1.000392	3.84(−4)
0.007	1.0000074	1.000461	4.54(−4)
0.008	1.0000058	1.000530	5.24(−4)
0.009	1.0000033	1.000600	5.97(−4)

Conclusion

We have developed a numerical method for the solution of fourth order singularly pertubed boundary value problem using parametric septic spline. It is a computationally efficient method and the algorithm can easily be implemented on a computer. The method has been analysed for convergence and proved that the method is second and fourth order convergent. Also, the errors at nodal points are compared with the errors of [23] and observed to be better.

Table 2. Maximum absolute errors for example 1

Fourth order method,$\varepsilon = 0.01, h = 0.001$

x	Present method $(p_1, p_2, p_3, p_4) = (0, 0, 12, 12)$	Exact [23]	Errors of present method	Errors [23]
0.001	1.000016	1.000063	4.70(−5)	6.19(−5)
0.002	1.000028	1.000127	9.90(−5)	1.22(−4)
0.003	1.000036	1.000192	1.56(−4)	1.82(−4)
0.004	1.000041	1.000258	2.17(−4)	2.40(−4)
0.005	1.000042	1.000324	2.82(−4)	2.97(−4)
0.006	1.000041	1.000392	3.51(−4)	3.53(−4)
0.007	1.000056	1.000461	4.05(−4)	4.08(−4)
0.008	1.000028	1.000530	5.02(−4)	4.62(−4)
0.009	1.000090	1.000600	5.10(−4)	5.15(−4)

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Nonlinear $\mathcal{D}$-set contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations

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Abstract

In this paper the author introduces the notion of partially nonlinear $\mathcal{D}$-set-contraction mappings in a partially ordered normed linear space and prove some hybrid fixed point theorems under certain mixed conditions from algebra, analysis and topology. The applications of abstract results presented here are given to perturbed nonlinear hybrid functional integral equations for proving the existence as well as global attractivity of the comparable solutions under certain monotonic conditions. The abstract theory developed in this paper is also useful to develop the algorithms for the solutions of some nonlinear problems of analysis and allied areas of mathematics.

Keywords: Partial measure of noncompactness; Partially nonlinear $\mathcal{D}$-set-contraction mappings; Fixed points; Functional integral equation; Existence theorem; Attractivity of solutions.

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1 Introduction

The topological methods or fixed point theorems involving the compactness arguments are useful tools to prove the existence of the solutions of some nonlinear equations but these, do not give any computational scheme or algorithm for solving such problems. See Burton [4], Burton and Zhang [5], Krasnoselskii [20] and the references therein. However if we combine the compactness (or its generalizations in terms of measure of noncompactness) with some algebraic arguments then it is possible to develop an algorithm for solutions of the nonlinear problems under some suitable conditions. The work along this line is of great interest and which is the main motivation of the present paper. Here, we combine the topological arguments with some order related hypotheses to prove some hybrid fixed point theorems for the mappings in partially ordered spaces and apply newly developed abstract results to obtain an algorithm for the solutions of a certain nonlinear functional integral equation under some mixed compactness and monotonic conditions. The results of this paper seem to be new in the literature.

The topological concept of compactness is very much useful in the development of nonlinear analysis to derive some far reaching conclusions. It is interesting to measure numerically the degree of noncompactness of sets in a normed linear space in terms of certain characteristics of the compactness property. The Kuratowskii [21] and Hausdorff [18] measures of noncompactness are well-known tools for discussing different aspects of the theory of nonlinear equations in the literature. However, an axiomatic way of approach of the measures of noncompactness is sometimes useful in the study of various qualitative properties of the dynamical systems in nonlinear analysis (cf. Appell [1] and Banas and Goebel [2]). In this article, we follow the axiomatic way of approach to define a new partial measure of noncompactness in a partially ordered normed linear space and which is subsequently exploited to derive some interesting consequences. We continue the study presented in Dhage [11], Nieto and Lopez [22] and Ran and Reurings [23] and prove some new hybrid fixed point theorems (FPTs) for partially condensing mappings in a partially ordered complete normed linear space and apply our abstract results to a certain nonlinear hybrid functional integral equation for proving the existence as well

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as global attractivity results on the unbounded intervals of real line. The abstract theory developed here is useful to develop the algorithms for the solutions of some nonlinear problems of analysis and allied areas of mathematics and mathematical sciences.

2 Partially Ordered Linear Spaces

Let E be a real vector or linear space. We introduce a partial order $\preceq$ in E as follows. A relation $\preceq$ in E is said to be a partial order if it satisfies the following properties: For any $a, b, c, d \in E$ and $\lambda \in \mathbb{R}$,

1. Reflexivity: $a \preceq a$ for all $a \in E$,
2. Antisymmetry: $a \preceq b$ and $b \preceq a$ implies $a = b$,
3. Transitivity: $a \preceq b$ and $b \preceq c$ implies $a \preceq c$, and
4. Order linearity: $a \preceq b$ and $c \preceq d \Longrightarrow a + c \preceq b + d$;
   and $a \preceq b \Longrightarrow \lambda a \preceq \lambda b$ for $\lambda \geq 0$.

The linear space E together with a partial order $\preceq$ becomes a **partially ordered linear or vector space**. Two elements x and y in a partially ordered linear space E are called **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. We introduce a norm $\|\cdot\|$ in a partially ordered linear space E so that E becomes now a **partially ordered normed linear space**. If E is complete with respect to the metric d defined through the above norm, then it is called a **partially ordered complete normed linear space**. We frequently need the concept of regularity of E in what follows. It is known that E is **regular** if $\{x_n\}$ is a nondecteasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The details of an ordered Banach space and operator theoretic techniques are given in Heikkilä and Lakshmikantham [19] and Dhage [11] and Carl and Hekkilä [6] and the references therein.

The following definitions have been introduced in Dhage [11] and are frequently used in the subsequent part of this paper.

Definition 2.1. *A subset S of a partially ordered normed linear space E is called* **partially bounded** *if every chain in S is bounded. S is called* **uniformly partially bounded** *if all chains in S are bounded with a unique bound.*

It is noted that every bounded set S in a partially ordered normed linear space E is uniformly partially bounded and every uniformly partially bounded set S is partially bounded, however the reverse implication may not hold.

Definition 2.2. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **monotone nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called a monotone nonincreasing if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. A monotone mapping $\mathcal{T}$ is one which is either monotone nondecreasing or monotone nonincreasing on E.*

Definition 2.3. *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called a partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is a partially continuous on E, then it is continuous on every chain C contained in E.*

The following terminologies may be found in any book on nonlinear operators, equations and applications.

An operator $\mathcal{T}$ on a normed linear space E into itself is called **compact** if $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called **totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is continuous and totally bounded, then it is called a **completely continuous** on E. The details of completely continuous operators on Banach spaces appear in Zeidler [24].

Definition 2.4. *A mapping $T : E \to E$ is called* **partially bounded** *if $\mathcal{T}(C)$ is a bounded subset of E for all totally ordered sets or chains C in E. Finally, $\mathcal{T}$ is called* **uniformly partially bounded** *if $\mathcal{T}(C)$ is a bounded by a unique constant for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **partially compact** *if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **uniformly partially compact** *if $\mathcal{T}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. Finally, if $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **a partially completely continuous** *on E.*

Remark 2.1. Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

Definition 2.5 (Dhage [11]). *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function has compatibility property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ possesses compatibility property w.r.t. the usual componentwise order relation and the usual standard norm in it.

The following applicable hybrid fixed point theorem in partially ordered normed linear spaces is proved in Dhage [11].

Theorem 2.1 (Dhage [11]). *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered linear space and suppose that the norm in E is such that E is a complete normed linear space. Let $\mathcal{T} : E \to E$ be a nondecreasing, partially compact and continuous mapping. Further if the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible and if there is an element $x_0 \in E$ satisfying $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{T^n x_0\}$ of successive iterations converges to x^*.*

Remark 2.2 (Dhage [12]). The assertion of above Theorem 2.1 also remains true if we replace the compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in E with the compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain C of E. The later condition holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$.

We note that Theorem 2.1 is very much useful for proving the existence theorems for several dynamical systems in nonlinear analysis modeled on nonlinear differential and integral equations (cf. Dhage and Dhage [14]). Here, in the following section we generalize above hybrid fixed point theorem under weaker conditions via partial measure of noncompactness and apply it to obtain the existence of the solutions of a certain nonlinear functional integral equation in a constructive way.

3 Nonlinear $\mathcal{D}$-set-contraction Mappings

Assume that $(E, ||\cdot||)$ is an infinite dimensional partially ordered Banach space with zero element θ. If C is a chain in E, then C' denotes the set of all limit points of C in E. The symbol $\overline{C}$ stands for the closure of C in E defined by $\overline{C} = C \cup C'$. The set $\overline{C}$ is called a closed chain in E. Thus, $\overline{C}$ is the intersection of all closed chains containing C. Clearly, $\inf C, \sup C \in \overline{C}$ provided $\inf C$ and $\sup C$ exist. The $\sup C$ is an element $z \in E$ such that for every $\epsilon > 0$ there exists a $c \in C$ such that $d(c, z) < \epsilon$ and $x \leq z$ for all $x \in C$. Similarly, $\inf C$ is defined in the same way.

In what follows, we denote by $\mathcal{P}_{cl}(E)$, $\mathcal{P}_{bd}(E)$, $\mathcal{P}_{rcp}(E)$, $\mathcal{P}_{ch}(E)$, $\mathcal{P}_{bd,ch}(E)$, $\mathcal{P}_{rcp,ch}(E)$ the family of all nonempty and closed, bounded, relatively compact, chains, bounded chains and relatively compact chains of E respectively. Now we introduce the concept of a partial measure of noncompactness in E on the lines of usual classical theory.

Definition 3.6. *A mapping $\mu^p : \mathcal{P}_{bd,ch}(E) \to \mathbb{R}_+ = [0, \infty)$ is said to be a partial measure of noncompactness in E if it satisfies the following conditions:*

1^o $\emptyset \neq (\mu^p)^{-1}(\{0\}) \subset \mathcal{P}_{rcp,ch}(E)$,

2^o $\mu^p(\overline{C}) = \mu^p(C)$

3^o *μ^p is nondecreasing, i.e., if $C_1 \subset C_2 \Rightarrow \mu^p(C_1) \leq \mu^p(C_2)$*

4^o *If $\{C_n\}$ is a sequence of closed chains from $\mathcal{P}_{bd,ch}(E)$ such that $C_{n+1} \subset C_n$ $(n = 1, 2, ...)$ and if $\lim\limits_{n\to\infty} \mu^p(C_n) = 0$, then the intersection set $\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty.*

The family of sets described in 1^o is said to be the *kernel of the partial measure of noncompactness* μ^p and is defined as

$$\ker \mu^p = \{C \in \mathcal{P}_{bd,ch}(E) \mid \mu^p(C) = 0\}.$$

Clearly, $\ker \mu^p \subset \mathcal{P}_{rcp,ch}(E)$. Observe that the intersection set C_∞ from condition 4^o is a member of the family $\ker \mu^p$. In fact, since $\mu^p(C_\infty) \le \mu^p(C_n)$ for any n, we infer that $\mu^p(C_\infty) = 0$. This yields that $C_\infty \in \ker \mu^p$. This simple observation will be essential in our further investigations.

The partial measure μ^p of noncompactness is called **sublinear** if it satisfies

5^o $\mu^p(C_1 + C_2) \le \mu^p(C_1) + \mu^p(C_2)$ for all $C_1, C_2 \in \mathcal{P}_{bd,ch}(E)$, and

6^o $\mu^p(\lambda C) = |\lambda| \mu^p(C)$ for $\lambda \in \mathbb{R}$.

Again, μ^p is said to satisfy **maximum property** if

7^o $\mu^p\Big(C_1 \bigcup C_2\Big) = \max\{\mu^p(C_1), \mu^p(C_2)\}$.

Finally, μ^p is said to be **full** if

8^o $\ker \mu^p = \mathcal{P}_{rcp,ch}(E)$.

Example 3.1. Define two functions $\alpha^p, \beta^p : \mathcal{P}_{bd,ch}(E) \to \mathbb{R}_+$ by

$$\alpha^p(C) = \inf\left\{ r > 0 \;\Big|\; C = \bigcup_{i=1}^n C_i,\ \operatorname{diam}(C_i) \le r \,\forall\, i \right\},$$

where $C \in \mathcal{P}_{bd,ch}(E)$ and $\operatorname{diam}(C_i) = \sup\{\|x-y\| : x, y \in C_i\}$, and

$$\beta^p(C) = \inf\left\{ r > 0 \;\Big|\; C \subset \bigcup_{i=1}^n \mathcal{B}(x_i, r) \text{ for some } x_i \in E \right\},$$

where $\mathcal{B}(x_i, r) = \{x \in E : \|x_i - x\| < r\}$. It is easy to prove that α^p and β^p are partial measures of noncompactness called respectively the partial Kuratowskii and partial ball or Hausdorff measures of noncompactness in E.

The above two partially Kuratowskii and Hausdorff measures of noncompactness α^p and β^p are sublinear, full and enjoy the maximum property in E. The verification of this claim is similar to classical Kuratowskii and Hausdorff measures of noncompactness given in Appell [1] and Banas and Goebel [2]. So we omit the details.

Definition 3.7. *A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a **dominating function** or, in short, $\mathcal{D}$-**function** if it is an upper semi-continuous and monotonic nondecreasing function satisfying $\psi(0) = 0$.*

There do exist $\mathcal{D}$-functions and commonly used $\mathcal{D}$-functions are

$$\begin{aligned}
\psi(r) &= k\,r, \quad \text{for some constant } k > 0,\\
\psi(r) &= \frac{L\,r}{K + r}, \quad \text{for some constants } L > 0, K > 0,\\
\psi(r) &= \tan^{-1} r,\\
\psi(r) &= e^r - 1,\\
\psi(r) &= \tanh r,\\
\psi(r) &= \sinh r,\\
\psi(r) &= \log(1 + r), \quad \text{and}\\
\psi(r) &= r - \log(1 + r).
\end{aligned}$$

The above defined $\mathcal{D}$-functions have been widely used in the existence theory of nonlinear differential and integral equations. See Dhage [7, 8, 9, 10, 11, 12] and the references therein.

Remark 3.3. *If $\phi, \psi\ \mathbb{R}_+ \to \mathbb{R}_+$ are two $\mathcal{D}$-functions, then i) $\phi + \psi$, ii) $\lambda\,\phi$, $\lambda > 0$, and iii) $\phi \circ \psi$ are also $\mathcal{D}$-functions on $\mathbb{R}_+$. The set of all $\mathcal{D}$-functions is denoted by $\mathfrak{D}$.*

Definition 3.8. *A nondecreasing mapping $\mathcal{T} : E \to E$ is called* **partially nonlinear $\mathcal{D}$-set-Lipschitz** *if there exists a $\mathcal{D}$-function ψ such that*

$$\mu^p(\mathcal{T}C) \le \psi(\mu^p(C))$$

for all bounded chain C in E. $\mathcal{T}$ is called **partially k-set-Lipschitz** *if $\psi(r) = k\,r$, $k > 0$. $\mathcal{T}$ is called* **partially k-set-contraction** *if it is a partially k-set-Lipschitz with $k < 1$. Finally, $\mathcal{T}$ is called a* **partially nonlinear $\mathcal{D}$-set-contraction** *in E if it is a partially nonlinear $\mathcal{D}$-Lipschitz with $\psi(r) < r$ for $r > 0$.*

The following lemma (see Dhage [11, page 159]) is frequently used in the analytical fixed point theory of metric spaces. See also Dhage [7, 8] and the references cited therein.

Lemma 3.1 (Dhage [11]). *If φ is a $\mathcal{D}$-function with $\varphi(r) < r$ for $r > 0$, then $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in [0, \infty)$ and vice-versa.*

Our main result of this section is as follows.

Theorem 3.2. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially nonlinear $\mathcal{D}$-set-contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Proof. Define a sequence $\{x_n\}$ of points in S by

$$x_{n+1} = \mathcal{T}x_n, \quad n = 0, 1, 2, \dots . \tag{3.1}$$

Since $\mathcal{T}$ is nondecreasing and $x_0 \preceq \mathcal{T}x_0$, we have that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots . \tag{3.2}$$

Denote

$$C_n = \{x_n, x_{n+1}, \dots\}$$

for $n = 0, 1, 2, \dots$. By the construction, each C_n is a bounded chain in S and

$$C_n = T(C_{n-1}), \quad n = 1, 2, \dots .$$

From the definition of C_n's, it follows that

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset \cdots ,$$

and so

$$\overline{C_0} \supset \overline{C_1} \supset \cdots \supset \overline{C_n} \supset \cdots . \tag{3.3}$$

Therefore, by nondecreasing nature of μ^p we obtain

$$\begin{aligned} \mu^p(C_n) &= \mu^p(\mathcal{T}(C_{n-1})) \\ &\le \psi(\mu^p(C_{n-1})) \\ &\le \psi^2(\mu^p(C_{n-2})) \\ &\;\;\vdots \\ &\le \psi^n(\mu^p(C_0)). \end{aligned} \tag{3.4}$$

Taking the limit superior as $n \to \infty$ in the above equality (3.4), in view of Lemma 3.1 we obtain,

$$\mu^p(\overline{C_n}) = \lim_{n\to\infty} \mu^p(C_n) \le \limsup_{n\to\infty} \psi^n(\mu^p(C_0)) = \lim_{n\to\infty} \psi^n(\mu^p(C_0)) = 0. \tag{3.5}$$

Hence, by condition (4^o) of μ^p,

$$\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n \ne \emptyset \quad \text{and} \quad C_\infty \in \mathcal{P}_{rcp,ch}(E).$$

From (3.5) it follows that for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\mu^p(C_n) < \epsilon \quad \forall\, n \geq n_0.$$

This shows that $\overline{C}_{n_0}$ and consequently $\overline{C}_0$ is a compact chain in E. Hence, $\{x_n\}$ has a convergent subsequence. Further since the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain of S, the whole sequence $\{x_n\} = \{\mathcal{T}^n x_0\}$ is convergent and converges monotonically to a point, say $x^* \in \overline{C}_0$. Since the ordered space space E is regular, we have that $x_n \leq x^*$. Finally, from the partial continuity of $\mathcal{T}$, we get

$$\mathcal{T}x^* = \mathcal{T}\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \mathcal{T}x_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

This completes the proof. □

Remark 3.4. The regularity of E and the partial continuity of $\mathcal{T}$ in above Theorem 3.2 may be replaced with the continuity of the operator $\mathcal{T}$ on E.

Remark 3.5. If the set $\mathcal{F}_{\mathcal{T}}$ of solutions to the above operator equation $\mathcal{T}x = x$ is a chain, then all the solutions belonging to $\mathcal{F}_{\mathcal{T}}$ are comparable. Further, if $\mu^p(\mathcal{F}_{\mathcal{T}}) > 0$, then $\mu^p(\mathcal{F}_{\mathcal{T}})) = \mu^p(\mathcal{T}(\mathcal{F}_{\mathcal{T}})) \leq \psi(\mu^p(\mathcal{F}_{\mathcal{T}})) < \mu^p(\mathcal{F}_{\mathcal{T}})$ which is a contradiction. Consequently, $\mathcal{F}_{\mathcal{T}} \in \ker \mu^p$. This simple fact has been utilized in the study of different qualitative properties of the comparable solutions of the dynamic systems under consideration.

Remark 3.6. Suppose that the order relation $\preceq$ is introduced in E with the help of an order cone $\mathcal{K}$ which is a non-empty closed set $\mathcal{K}$ in E satisfying (i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, (ii) $\lambda\,\mathcal{K} \subseteq \mathcal{K}$ and (iii) $\{-\mathcal{K}\} \cap \mathcal{K} = \{0\}$ (cf. [19]). Then the order relation $\preceq$ in E is defined as $x \preceq y \iff y - x \in \mathcal{K}$. The element $x_0 \in E$ satisfying $x_0 \preceq \mathcal{T}x_0$ in above Theorem 3.2 is called a lower solution of the operator equation $x = \mathcal{T}x$. If the operator equation $x = \mathcal{T}x$ has more than one lower solution and set of all these lower solutions are comparable, then the corresponding set $\mathcal{F}(Q)$ of solutions to the above operator equation is a chain and hence all solutions in $\mathcal{F}(Q)$ are comparable. To see this, let x_0 and y_0 be any two lower solutions of the above operator equation such that $x_0 \preceq y_0$ and let x^* and y^* respectively be the corresponding solutions under the conditions of Theorem 3.2. Then, by definition of $\preceq$, one has $y_0 - x_0 \in \mathcal{K}$ and from monotone nondecreasing nature of $\mathcal{T}$ it follows that $\mathcal{T}^n y_0 - \mathcal{T}^n x_0 \in \mathcal{K}$. Since $\mathcal{K}$ is closed, we have that $y^* - x^* \in \mathcal{K}$ or $x^* \preceq y^*$.

Theorem 3.3. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially nonlinear $\mathcal{D}$-set-contraction mapping. If there exists an element $x_0 \in S$ such that $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Proof. The proof is similar to Theorem 3.2 and so we omit the details. □

Observe that Theorems 3.2 and 3.3 improve and generalize the hybrid measure theoretic fixed point theorems of Dhage [11, 12], Dhage and Dhage [15] and Dhage *et.al.* [16] under weaker compatibility condition. As a consequence of Theorems 3.2 and 3.3, we derive some interesting corollaries.

Corollary 3.1. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially k-set-contraction with $k < 1$. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

We remark that Corollary 3.1 is very much useful for proving the existence results in the theory of nonlinear differential and integral equations. See Dhage [12] and the references therein. Before giving a further generalization of Theorem 3.2, we state a useful definition.

Definition 3.9. *A nondecreasing mapping $\mathcal{T} : E \to E$ is called partially condensing if for any bounded chain C in E, $\mu^p(\mathcal{T}(C)) < \mu^p(C)$ for $\mu^p(C) > 0$.*

We remark that every partially compact and partially nonlinear $\mathcal{D}$-set-contraction mappings are partially condensing, however the reverse implications may not hold.

Remark 3.7. It is clear tat every partially k-set-contraction is a partially nonlinear $\mathcal{D}$-set-contraction and every partially nonlinear $\mathcal{D}$-set-contraction is partially condensing, however, the converse implications may not be true. Actually, it is very difficult to prove practically a selfmaping of a normed linear space is partially condensing and we rarely come across a mapping of this kind. But the mappings with nonlinear $\mathcal{D}$-set-contraction and k-$\mathcal{D}$-set-contraction are easily available in the literature.

Theorem 3.4. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially condensing mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Proof. The proof is standard and hence we omit the details. □

Corollary 3.2. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially compact mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

We remark that the hybrid fixed point theorems proved above improve and generalize the hybrid fixed point theorems of Dhage and Dhage [14] and Dhage *et.al.* [16] under weaker compatibility condition.

Remark 3.8. We note that the proof of Theorems 3.2, 3.3 and 3.4 do not make any use of the linear structure of underlined space E, and therefore, Theorems 3.2, 3.3 and 3.4 also remain true in the setting of a partially ordered complete metric space E.

In view of above Remark 3.8, the slight generalizations of Theorems 3.2, 3.3 and 3.4 are as follows.

Theorem 3.5. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete metric space $(E, \preceq, d)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially condensing mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Corollary 3.3. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete metric space $(E, \preceq, d)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially nonlinear $\mathcal{D}$-set-contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Corollary 3.4. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete metric space $(E, \preceq, d)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially k-set-contraction with $k < 1$. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Again, the regularity of E in above Theorem 3.5 and Corollaries 3.3 and 3.4 may also be replaced with a stronger condition of continuity of the operator $\mathcal{T}$ on E. The following hybrid fixed point theorems are employed for proving the existence and uniqueness of the solutions of nonlinear equations. Before stating these results, we consider the following definition in what follows.

Definition 3.10. *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called partially nonlinear $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{3.6}$$

for all comparable elements $x, y \in E$. If $\psi(r) = kr$, $k > 0$, then $\mathcal{T}$is called a partially Lipschitz with a Lipschitz constant k. If $k < 1$, $\mathcal{T}$ is called a partially contraction with contraction constant k. Finally, $\mathcal{T}$ is called partially nonlinear $\mathcal{D}$-contraction if it is a partially nonlinear $\mathcal{D}$-Lipschitz with $\psi(r) < r$ for $r > 0$.

Lemma 3.2. *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered complete normed linear space. If $\mathcal{T} : E \to E$ is a nondecreasing and partially nonlinear $\mathcal{D}$-Lipschitz mapping, then for any bounded chain C in E,*

$$\alpha^p(\mathcal{T}C) \leq \psi(\alpha^p(C)) \tag{3.7}$$

where α^p is a partial Kurotowskii measure of noncompactness and ψ is a associated $\mathcal{D}$-function of $\mathcal{T}$ on E.

Proof. The proof is similar to standard result for usual nonlinear $\mathcal{D}$-Lipschitz mapping with the classical Kurotowskii measure of noncomactness α in the Banach space E. We omit the details. □

Corollary 3.5 (Dhage [11]). *Let S be a non-empty, closed and partially bounded subset of the partially ordered complete metric space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing and partially nonlinear $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.*

Proof. by Lemma 3.2, $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. We simply show that the order relation $\preceq$ and the metric d are compatible in every compact chain C of S. Let C be arbitrary compact chain in S. Assume that $\{x_n\}$ is monotone nondecreasing or monotone nonincreasing and that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent. Now it can be shown as in Dhage [11] that $\{x_n\}$ is a Cauchy sequence in S. Hence the whole sequence $\{x_n\}$ is convergent and converges to a point in S. Consequently, $\preceq$ and the metric d are compatible in C. Now the desired conclusion follows by an application of Theorem 3.2. This completes the proof. □

We notice that Corollary 3.5 includes a well-known hybrid fixed point theorems of Nieto and Lopez [22] and Dhage [11] for partially contraction and monotone mappings in a partially ordered complete metric space.

4 FPTs of Krasnoselskii and Dhage Type

The study of hybrid fixed oint theorems for the sum of two operators is attributed to Krasnoselskii [20] whereas the study involving the product of two operators in Banach algebra is attributed to Dhage [7]. Again the study of fixed point theorems in a Banach algebras involving the sum as well as product of operators is credited to Dhage [11]. Here, we prove the analogous results for the sum and the product of operators in a partially ordered complete normed linear space which are useful in applications to perturbed nonlinear differential and integral equations for proving the existence and attractivity of the solutions under weaker mixed partially compact, partially Lipschitz and monotonic conditions (cf. Dhage *et.al.* [17]).

4.1 FPTs of Krasnoselskii type

Theorem 4.6. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{A} : E \to E$ and $\mathcal{B} : S \to E$ be two nondecreasing operators such that*

(a) $\mathcal{A}$ is partially nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{B}$ is partially completely continuous,

(c) $\mathcal{A}x + \mathcal{B}x \in S$ for all $x \in S$, and

(d) there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the operator equation

$$\mathcal{A}x + \mathcal{B}x = x \tag{4.8}$$

has a solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,...; converges monotonically to x^*.

Proof. Define a mapping $\mathcal{T}$ on S by

$$\mathcal{T}x = \mathcal{A}x + \mathcal{B}x.$$

By hypothesis (c), $\mathcal{T}$ defines a mapping $\mathcal{T} : S \to S$. Since $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing, $\mathcal{T}$ is nondecreasing on S. From the partial continuity of the operators $\mathcal{A}$ and $\mathcal{B}$ and the continuity of the binary composition addition, it follows that the operator $\mathcal{T}$ is partially continuous on S. Again, by hypothesis (c), $x_0 \preceq \mathcal{T}x_0$. Next, we show that $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. Let C be a chain in S. Then by definition of $\mathcal{T}$, we have

$$\mathcal{T}(C) \subseteq \mathcal{A}(C) + \mathcal{B}(C).$$

Since $\mathcal{T}$ is nondecreasing and partially continuous, $\mathcal{T}(C)$ is again a chain in S which is bounded. By properties (5^o) and (8^o), α^p is subadditive and full. As a result,

$$\alpha^p(\mathcal{T}C) \leq \alpha^p(\mathcal{A}(C)) + \alpha^p(\mathcal{B}(C)) \leq \psi(\alpha^p(C))$$

which shows that $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. Next, the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S, so the desired conclusion follows by an application of Theorem 3.2. This completes the proof. □

Theorem 4.7. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $\big(E, \preceq, \|\cdot\|\big)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{A} : E \to E$ and $\mathcal{B} : S \to E$ be two nondecreasing operators such that*

(a) $\mathcal{A}$ is partially nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{B}$ is partially completely continuous,

(c) $\mathcal{A}x + \mathcal{B}x \in S$ for all $x \in S$, and

(d) there exists an element $x_0 \in S$ such that $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the operator equation (4.8) has a solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,...; converges monotonically to x^*.

4.2 FPTs of Dhage type

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \big\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\big\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}. \tag{4.9}$$

The elements of the set $\mathcal{K}$ are called the positive vectors in the normed linear algebra E. Then the following lemma is immediate and is useful in the hybrid fixed point theory in Banach algebras and applications.

Lemma 4.3 (Dhage [9]). *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 4.11. *An operator $\mathcal{T} : E \to E$ is said to be positive if the range $R(\mathcal{T})$ of $\mathcal{T}$ is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.*

For any two chains C_1 and C_2 in E, denote

$$C_1C_2 = \{x \in E \mid x = c_1c_2,\ c_1 \in C_1 \text{ and } c_2 \in C_2\}.$$

Then we have the following lemma.

Lemma 4.4. *If C_1 and C_2 are two bounded chains in a partially ordered normed linear algebra E, then*

$$\alpha^p(C_1C_2) \leq \|C_2\|\alpha^p(C_1) + \|C_1\|\alpha^p(C_2) \tag{4.10}$$

where $\|C\| = \sup\{\|c\| \mid c \in C\}$.

Theorem 4.8. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear algebra $\big(E, \preceq, \|\cdot\|\big)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{A}, \mathcal{B} : S \to \mathcal{K}$ and $\mathcal{C} : E \to E$ be three nondecreasing operators such that* *(a)* $\mathcal{A}$ *and* $\mathcal{C}$ *are partially nonlinear* $\mathcal{D}$*-Lipschitz with* $\mathcal{D}$*-functions* $\psi_{\mathcal{A}}$ *and* $\psi_{\mathcal{C}}$ *respectively,*

(b) $\mathcal{B}$ *is partially continuous and compact,*

(d) $\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x \in S$ *for all* $x \in S$*,*

(c) $M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$*,* $r > 0$*, where* $M = \|\mathcal{B}(S)\|$*, and*

(e) there exists an element $x_0 \in S$ *such that* $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$ *or* $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$*.*

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x \tag{4.11}$$

has a solution x^* *in* S *and the sequence* $\{x_n\}$ *of successive approximations defined by* $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n + \mathcal{C}x_n$*, n=0,1,...; converges monotonically to* x^**.*

Proof. Suppose that there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$. Define a mapping $\mathcal{T}$ on S by

$$\mathcal{T}x = \mathcal{A}x\,\mathcal{B}x + \mathcal{C}x.$$

By hypothesis (c), $\mathcal{T}$ defines a mapping $\mathcal{T} : S \to S$. Since $\mathcal{A}$ and $\mathcal{B}$ are positive and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are nondecreasing, T is nondecreasing on S. From the partial continuity of the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and the continuity of the binary compositions addition and multiplication, it follows that the operator $\mathcal{T}$ is partially continuous on S. Again, by hypothesis (d), $x_0 \preceq \mathcal{T}x_0$. Next, we show that $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. Let C be a chain in S. Then by definition of $\mathcal{T}$, we have

$$\mathcal{T}(C) \subseteq \mathcal{A}(C)\,\mathcal{B}(C) + \mathcal{C}(C).$$

Since $\mathcal{T}$ is nondecreasing and partially continuous, $\mathcal{T}(C)$ is again a chain in S. By properties (5^o) and (8^o), α^p is a subadditive and full partial measure of noncompactness. As a result, we have

$$\begin{aligned}
\alpha^p(\mathcal{T}C) &\leq \|\mathcal{A}(C)\|\alpha^p(\mathcal{B}(C)) + \|\mathcal{B}(C)\|\alpha^p(\mathcal{A}(C)) + \alpha^p(\mathcal{C}(C)) \\
&\leq \|\mathcal{B}(E)\|\alpha^p(\mathcal{A}(C)) + \alpha^p(\mathcal{C}(C)) \\
&\leq M\psi_{\mathcal{A}}(\alpha^p(C)) + \psi_{\mathcal{C}}(\alpha^p(C)) \\
&= \psi(\alpha^p(C)),
\end{aligned} \tag{4.12}$$

where $\psi(r) = M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, $r > 0$ and the constant M exists in view of the fact that $\mathcal{B}$ is compact operator on S. This shows that $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. Next, the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S, so the desired conclusion follows by an application of Theorem 3.2. Similarly, if $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$, then using Theorem 3.3 it can be proved that $\mathcal{T}$ has a fixed point. This completes the proof. □

Remark 4.9. If we take $\psi_{\mathcal{A}}(r) = \frac{L_1 r}{K+r}$ and $\psi_{\mathcal{C}}(r) = L_2\,r$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $\frac{L_1 M}{K+r} + L_2 < 1$ for each real number $r > 0$. Similarly, if $\psi_{\mathcal{A}}(r) = L_1\,r$, and $\psi_{\mathcal{C}}(r) = \frac{L_2 r}{K+r}$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $M\,L_1 + \frac{L_2 M}{K+r} < 1$ for each real number $r > 0$.

Corollary 4.6. *Let* S *be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear algebra* $(E, \preceq, \|\cdot\|)$ *such that* $S \cap \mathcal{K} \neq \emptyset$ *and the order relation* $\preceq$ *and the norm* $\|\cdot\|$ *are compatible in every compact chain* C *of* S*. Let* $\mathcal{A}, \mathcal{B} : S \to \mathcal{K}$ *be two nondecreasing operators such that*

(a) $\mathcal{A}$ *is partially nonlinear* $\mathcal{D}$*-Lipschitz with* $\mathcal{D}$*-function* $\psi_{\mathcal{A}}$*,*

(b) $\mathcal{B}$ *is partially continuous and compact,*

(c) $\mathcal{A}x\,\mathcal{B}x \in S$ *for all* $x \in S$*,*

(d) $M\psi_{\mathcal{A}}(r) < r$*,* $r > 0$*, where* $M = \|\mathcal{B}(S)\|$*, and*

(e) there exists an element $x_0 \in X$ *such that* $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0$ *or* $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0$*.*

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x = x \tag{4.13}$$

has a positive solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n$, n=0,1,...; converges monotonically to x^*.

Remark 4.10. The hypotheses (b) and (c) of Theorem 4.8 may be replaced with the weaker hypotheses as follows:

(b′) $\mathcal{B}$ is partially continuous and uniformly partially compact, and

(c′) $M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C \in \mathcal{P}_{ch}(S)\}$.

The proof of Theorem 4.8 under these new hypotheses is essentially the same as that given in the theorem. Similarly, the conclusion of Corollary 4.6 also remains true under the corresponding changes in the hypotheses (b) and (c) thereof. We mention that Theorem 4.8 and Corollary 4.6 are useful in the study of quadratic nonlinear differential and integral equations for discussing the qualitative aspects of the solutions.

5 Functional Integral Equations

In this section, we are going to prove a result on the existence and uniform global attractivity of the solutions of a nonlinear functional integral equation. Our investigations will be carried out in the Banach space of real functions which are defined, continuous and bounded on the right half real axis $\mathbb{R}_+$. The integral equation in question has rather general form and contains as particular cases a few of other functional equations and nonlinear integral equations of Volterra type. The main tool used in our considerations is the technique of partial measures of noncompactness and the measure theoretic hybrid fixed point result established in Theorem 3.2. The partial measure of noncompactness used in this paper allows us not only to obtain the existence of solutions of the mentioned functional integral equation but also to characterize the comparable solutions in terms of uniform global ultimate attractivity. This assertion means that all the possible comparable solutions of the functional integral equation in question are globally uniformly attractive in the sense of notion defined in the following section.

5.1 Notation, Definitions and Auxiliary facts

As mentioned earlier, our considerations will be placed in the function space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions $x = x(t)$ defined, continuous and bounded on $\mathbb{R}_+$. We place the HFIE (3.1) in the space $E = BC(\mathbb{R}_+, \mathbb{R})$. Define a norm $\|\cdot\|$ and the order relation $\le$ in E by

$$\|x\| = \sup\{|x(t)| : t \ge 0\}. \tag{5.14}$$

and

$$x \le y \iff x(t) \le y(t) \tag{5.15}$$

for all $t \in \mathbb{R}_+$. Clearly, E is a partially ordered Banach space with respect to the above norm $\|\cdot\|$ and the order relation $\le$. The following lemma follows immediately by an application of Arzellá-Ascoli theorem.

Lemma 5.5. *Let $\big(BC(\mathbb{R}_+, \mathbb{R}), \le, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\le$ defined by (5.14) and (5.15) respectively. Then the norm $\|\cdot\|$ and the order relation $\le$ are compatible in every partially compact subset of $BC(\mathbb{R}_+, \mathbb{R})$.*

Proof. Let S be a partially compact subset of $BC(\mathbb{R}_+, \mathbb{R})$ and let $\{x_n\}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \le x_2(t) \le \cdots \le x_n(t) \le \cdots \tag{$*$}$$

for each $t \in \mathbb{R}_+$.

Suppose that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}$ is convergent and converges to the point $x(t)$ in $\mathbb{R}$ for each $t \in \mathbb{R}_+$. This shows that the sequence $\{x_n(t)\}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\le$ are compatible in S. This completes the proof. □

For our purposes we introduce a partial measure of noncompactness which is a handy tool of the partial Hausdorff measure of noncompactness in the study of the solutions of certain nonlinear integral equations. To define this partial measure, let us fix a nonempty and bounded chain X of the space $BC(\mathbb{R}_+,\mathbb{R})$ and a positive real number T. For $x \in X$ and $\epsilon \geq 0$ denote by $\omega^T(x,\epsilon)$ *the modulus of continuity* of the function x on the interval $[0,T]$ defined by

$$\omega^T(x,\epsilon) = \sup\{|x(t)-x(s)| : \ t,s \in [0,T], \ |t-s| \leq \epsilon\} .$$

Next, let us put

$$\omega^T(X,\epsilon) = \sup\{\omega^T(x,\epsilon) : \ x \in X\} ,$$

$$\omega_0^T(X) = \lim_{\epsilon\to 0} \omega^T(X,\epsilon) ,$$

and

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X) .$$

The partial Hausdorff measure of noncompactness β^p in the function space $C([0,T],\mathbb{R})$ of continuous real-valued functions defined on a closed and bounded interval $[0,T]$ in $\mathbb{R}$, is very much useful in the applications to nonlinear differential and integral equations and it can be shown that

$$\beta^p(X) = \frac{1}{2}\omega_0^T(X)$$

for all bounded chain X in $C([0,T],\mathbb{R})$. The proof of this fact follows from the arguments that given in Banas and Goebel [2] and the references therein. Similarly, ω_0 is a handy tool of partial measure of noncompactness in the ordered Banach space $BC(\mathbb{R}_+,\mathbb{R})$ useful for some practical applications to nonlinear differential and integral equations.

Now, for a fixed number $t \in \mathbb{R}_+$ and a fixed bounded chain X in $BC(\mathbb{R}_+,\mathbb{R})$, let us denote

$$X(t) = \{x(t) : x \in X\}.$$

Let

$$\delta_a(X(t)) = |X(t)| = \sup\{|x(t)| : \ x \in X\} ,$$
$$\delta_a^T(X(t)) = \sup_{t\geq T} \delta_a(X(t)) = \sup_{t\geq T} |X(t)|$$

and

$$\delta_a(X) = \lim_{T\to\infty} \delta_a^T(X(t)) = \limsup_{t\to\infty} \ |X(t)|.$$

Again, for a fixed real number c, denote

$$X(t) - c = \{x(t) - c \ : \ x \in X\}$$

and

$$\delta_b(X(t)) = |X(t)-c| = \sup\{|x(t)-c| : \ x \in X\} .$$

Define

$$\delta_b^T(X(t)) = \sup_{t\geq T} \delta_b(X(t)) = \sup_{t\geq T} |X(t)-c|$$

and

$$\delta_b(X) = \lim_{T\to\infty} \delta_b^T(X(t)) = \limsup_{t\to\infty} \ |X(t)-c|.$$

Similarly, let

$$\delta_c(X(t)) = \operatorname{diam} X(t) = \sup\{|x(t)-y(t)| : \ x,y \in X\} ,$$

$$\delta_c^T(X(t)) = \sup_{t\geq T} \delta(X(t)) = \sup_{t\geq T} \operatorname{diam} X(t)$$

and

$$\delta_c(X) = \lim_{T\to\infty} \delta^T(X(t)) = \limsup_{t\to\infty} \text{diam } X(t).$$

The details of the functions δ_a, δ_b and δ_c appear in Dhage [12]. Finally, let us consider the functions μ_a^p, μ_b^p and μ_c^p defined on the family of bounded chains in $BC(\mathbb{R}_+,\mathbb{R})$ by the formula

$$\mu_a^p(X) = \max\left\{\omega_0(X)\,,\,\delta_a(X)\right\}, \tag{5.16}$$

$$\mu_b^p(X) = \max\left\{\omega_0(X)\,,\,\delta_b(X)\right\} \tag{5.17}$$

and

$$\mu_c^p(X) = \max\left\{\omega_0(X)\,,\,\delta_c(X)\right\}. \tag{5.18}$$

It can be shown that the function μ_a^p, μ_b^p and μ_c^p are partially measures of noncompactness in the space $BC(\mathbb{R}_+,\mathbb{R})$. The components ω_0 and δ_a are called the characteristic values of the partial measure of noncompactness μ_a^p. Similarly, ω_0, δ_b and ω_0, δ_c are respectively the characteristic values of the partial measures of noncompactness μ_b^p and μ_c^p in $BC(\mathbb{R}_+,\mathbb{R})$.

Remark 5.11. The kernel ker μ_c^p consists of nonempty and bounded chains X of $BC(\mathbb{R}_+,\mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by the functions from X tends to zero at infinity. Similarly, the kernels ker μ_a^p, and ker μ_b^p consist of nonempty and bounded chains X of $BC(\mathbb{R}_+,\mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X around the lines respectively $x(t) = c$ and $x(t) = 0$ tends to zero at infinity. These particular characteristics of ker μ_a^p, ker μ_b^p and ker μ_c^p have been useful in establishing the global attractivity and global asymptotic stability of the comparable solutions of the considered functional integral equations.

In order to introduce further concepts used in the paper, let us assume that Ω is a nonempty chain of the space $BC(\mathbb{R}_+,\mathbb{R})$. Moreover, let Q be an operator defined on Ω with values in $BC(\mathbb{R}_+,\mathbb{R})$.

Consider the operator equation of the form

$$x(t) = Qx(t),\ t\in\mathbb{R}_+. \tag{5.19}$$

Definition 5.12. We say that comparable solutions of the equation (5.19) are **locally attractive** if there exists a ball $\overline{\mathcal{B}}(x_0,r)$ in the space $BC(\mathbb{R}_+,\mathbb{R})$ such that for arbitrary comparable solutions $x = x(t)$ and $y = y(t)$ of the equation (5.19) belonging to $\overline{\mathcal{B}}(x_0,r)\cap\Omega$ we have that

$$\lim_{t\to\infty}\left[x(t) - y(t)\right] = 0\,. \tag{5.20}$$

In the case when limit (5.20) is uniform with respect to the set $\overline{\mathcal{B}}(x_0,r)\cap\Omega$, i.e. when for each $\epsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \le \epsilon \tag{5.21}$$

for all $x,y\in\overline{\mathcal{B}}(x_0,r)\cap\Omega$ being the comparable solutions of (5.19) and for $t\ge T$, we will say that the comparable solutions of the operator equation (5.19) are **uniformly locally ultimately attractive** defined on $\mathbb{R}_+$.

Definition 5.13. A solution $x = x(t)$ of equation (5.19) is said to be **globally ultimately attractive** if (5.20) holds for each comparable solution $y = y(t)$ of (5.19) on $\mathbb{R}_+$. Other words we may say that the comparable solutions of the equation (5.19) are globally ultimately attractive if for arbitrary comparable solutions $x(t)$ and $y(t)$ of (5.19) the condition (5.20) is satisfied. In the case when condition (5.20) is satisfied uniformly with respect to the set $\overline{\mathcal{B}}(x_0,r)\cap\Omega$, i.e. if for every $\epsilon > 0$ there exists $T > 0$ such that the inequality (5.21) is satisfied for all $x,y\in\Omega$ being the comparable solutions of (5.19) and for $t\ge T$, we will say that the comparable solutions of the equation (5.19) are **uniformly globally ultimately attractive** on $\mathbb{R}_+$.

Let us mention that the concept of asymptotic stability may be found in Banas and Dhage [3] and references therein whereas the concept of global attractivity of solutions is introduced in Dhage [9] and proved attractivity results for certain nonlinear integral equations. We mention that the present approach is constructive and different from that given in the above stated papers.

5.2 Integral Equation and Attractivity Result

Now, we will investigate the following nonlinear hybrid functional integral equation (in short HFIE)

$$x(t) = f(t, x(\alpha_1(t)), x(\alpha_2(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \tag{5.22}$$

for all $t \in \mathbb{R}_+$, where the functions $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha_i, \beta, \gamma_i : \mathbb{R}_+ \to \mathbb{R}_+$ for $i = 1, 2$, are continuous.

By a *solution* of the HFIE (5.22) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies the equation (5.22), where $C(\mathbb{R}_+, \mathbb{R})$ is the space of continuous real-valued functions on $\mathbb{R}_+$.

Observe that the above integral equation (5.22) has been discussed in Dhage [9] under strong Lipschitz condition for the attractivity of solutions and includes several classes of functional, integral and functional-integral equations considered in the literature (cf. [1, 3, 9] and references therein). Let us also mention that the functional integral equation considered in [3, 9] is a special case of the equation (5.22), where $\alpha_1(t) = \alpha_2(t) = \beta(t) = \gamma(t) = t$.

The equation (5.22) will be considered under the following assumptions:

(H_1) The functions $\alpha_1, \alpha_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $\alpha_1(t) \geq t$ and $\alpha_2(t) \geq t$ for all $t \in \mathbb{R}_+$.

(H_2) The function $F : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $F(t) = |f(t, 0, 0)|$ is bounded on $\mathbb{R}_+$ with $F_0 = \sup_{t \geq 0} F(t)$.

(H_3) There exist constants $L > 0$ and $K > 0$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \frac{L \max\{x_1 - y_1\,,\, x_2 - y_2\}}{K + \max\{x_1 - y_1\,,\, x_2 - y_2\}}$$

for all $t \in \mathbb{R}_+$ and $(x, x_2), (y_1, y_2) \in v \times \mathbb{R}$ with $x_1 \geq y_1$ and $x_2 \geq y_2$. Moreover, $L \leq K$.

(H_4) $g(t, s, x, y)$ is nondecreasing in x and y for each $t, s \in \mathbb{R}_+$.

(H_5) There exists an element $u \in C(J, \mathbb{R})$ such that

$$u(t) \leq f(t, u(\alpha_1(t)), u(\alpha_2(t)) + \int_{t_0}^{\beta(t))} g(t, s, u(\gamma_1(s)), u(\gamma_2(s)))\, ds$$

for all $t \in \mathbb{R}_+$.

(H_6) There exist continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, s, x, y)| \leq a(t)b(s)$$

for $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$. Moreover, we assume that

$$\lim_{t\to\infty} a(t) \int_0^{\beta(t)} b(s)ds = 0\,.$$

(H_7) There exists a real number c such that $f(t, c, c) = c$ for all $t \in \mathbb{R}_+$.

The hypotheses (H_1)-(H_2) and (H_4), (H_6) have been widely used in the literature in the theory of nonlinear differential and integral equations. The special case of hypothesis (H_3) with $L < K$ is considered in Nieto and Lopez [22]. Now we formulate the main existence result for the integral equation (5.22) under above mentioned natural conditions.

Theorem 5.9. *Assume that the hypotheses* (H$_1$) *through* (H$_6$) *hold. Then the functional integral equation* (5.22) *has at least one solution* x^* *in the space* $BC(\mathbb{R}_+, \mathbb{R})$ *and the sequence* $\{x_n\}$ *of successive approximations defined by*

$$x_n(t) = f(t, x_{n-1}(\alpha_1(t)), x_{n-1}(\alpha_2(t))) + \int_0^{\beta(t)} g(t, s, x_{n-1}(\gamma_1(s)), x_{n-1}(\gamma_2(s)))\, ds,\ t \in \mathbb{R}_+, \tag{5.23}$$

for each $n \in \mathbb{N}$ *with* $x_0 = u$ *converges monotonically to* x^*. *Moreover, the comparable solutions of the equation* (5.22) *are uniformly globally ultimately attractive on* $\mathbb{R}_+$.

Proof. We seek the solutions of the HFIE (5.22) in the space $BC(\mathbb{R}_+, \mathbb{R})$ of continuous and bounded real-valued functions defined on $\mathbb{R}_+$. Set $E = BC(\mathbb{R}_+, \mathbb{R})$. Then, in view of Lemma 5.5, every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define the operator Q defined on the space $BC(\mathbb{R}_+, \mathbb{R})$ by the formula

$$Qx(t) = f(t, x(\alpha_1(t)), x(\alpha_2(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds,\ t \in \mathbb{R}_+. \tag{5.24}$$

Observe that in view of our assumptions, for any function $x \in E$ the function Qx is continuous on $\mathbb{R}_+$. As a result, Q defines a mapping $Q : E \to E$. Let $x_0 = u$ and define an open ball $\mathcal{B}(x_0, r)$ in E, where $r = \|x_0\| + L + F_0 + V$. We show that Q satisfies all the conditions of Theorem 3.2 on $S = \overline{\mathcal{B}}(x_0, r)$. This will be achieved in a series of following steps:

Step I: *Q is a nondecreasing on S.*

Let $x, y \in S$ be such that $x \leq y$. Then by hypothesis (H$_3$)-(H$_4$), we obtain

$$\begin{aligned} Qx(t) &= f(t, x(\alpha_1(t)), x(\alpha_2(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \\ &\leq f(t, y(\alpha_1(t)), y(\alpha_1(t)) + \int_0^{\beta(t)} g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))\, ds \\ &= Qy(t) \end{aligned}$$

for all $t \in \mathbb{R}_+$. This shows that Q is a nondecreasing operator on S.

Step II: *Q maps a closed and partially bounded set S into itself.*

Let X be a chain in S and let $x \in X$. Since the function $v : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$v(t) = \lim_{t\to\infty} a(t) \int_0^{\beta(t)} b(s)\, ds \tag{5.25}$$

is continuous and in view of hypothesis (H$_6$), the number $V = \sup_{t \geq 0} v(t)$ exists. Moreover if $x \geq 0$, then for arbitrarily fixed $t \in \mathbb{R}_+$ we obtain:

$$\begin{aligned} |Qx(t)| &\leq |f(t, x(\alpha_1(t)), x(\alpha_2(t)))| + \int_0^{\beta(t)} |g(t, s, x(s))|\, ds \\ &\leq |f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, 0, 0)| \\ &\quad + |f(t, 0, 0)| + a(t) \int_0^{\beta(t)} b(s)\, ds \\ &\leq \frac{L \max\{|x(\alpha_1(t))|, |x(\alpha_2(t))|\}}{K + \max\{|x(\alpha_1(t))|, |x(\alpha_2(t))|\}} + F(t) + v(t) \\ &\leq \frac{L\|x\|}{K + \|x\|} + F_0 + V \\ &= L + F_0 + V \end{aligned} \tag{5.26}$$

Similarly, if $x \leq 0$, then it can be shown that $|Qx(t)| \leq L + F_0 + V$ for all $t \in \mathbb{R}_+$. Taking the suremum over t, we obtain $||Qx|| \leq L + F_0 + V$ for all $x \in X$. This means that the operator Q transforms any chain X into a bounded chain in E. Moreover, we have

$$\|x_0 - Qx\| \leq \|x_0\| + \|Qx\| \leq \|x_0\| + L + F_0 + V$$

for all $x \in X$. More precisely, we infer that the operator Q transforms every chain X in $\overline{\mathcal{B}}(x_0, r)$ into the chain $Q(X)$ contained in the ball $\overline{\mathcal{B}}(x_0, r)$, where $r = \|x_0\| + L + F_0 + V$. As a result, Q defines a mapping $Q : \mathcal{P}_{ch}(\overline{\mathcal{B}}(x_0, r)) \to \mathcal{P}_{ch}(\overline{\mathcal{B}}(x_0, r))$ and so Q maps a closed and partially bounded set $S = \overline{\mathcal{B}}(x_0, r)$ into itself. Moreover, in view of Lemma 5.5, every compact chain in S possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Step III: *Q is a partially continuous on S.*

Now we show that the operator Q is partially continuous on S. To do this, let X be a chain in S and let us fix an arbitrary $\epsilon > 0$ and take $x, y \in X$ such that $x \geq y$ and $||x - y|| \leq \epsilon$. Then we get:

$$\begin{aligned}
|Qx(t) - Qy(t)| &\leq \big|f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, y(\alpha_1(t)), y(\alpha_2(t)))\big| \\
&\quad + \left| \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \right. \\
&\quad \left. - \int_0^{\beta(t)} g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))\, ds \right| \\
&\leq \big|f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, y(\alpha_1(t)), y(\alpha_2(t)))\big| \\
&\quad + \int_0^{\beta(t)} |g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))|\, ds \\
&\quad + \int_0^{\beta(t)} |g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))|\, ds \\
&\leq \frac{L \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}}{K + \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}} \\
&\quad + 2a(t) \int_0^{\beta(t)} b(s)\, ds \\
&\leq \frac{L\|x - y\|}{K + \|x - y\|} + 2v(t) \\
&< \epsilon + 2\, v(t).
\end{aligned}$$

Hence, by virtue of hypothesis (H_6) we infer that there exists $T > 0$ such tha $v(t) \leq \frac{\epsilon}{2}$ for $t \geq T$. Thus, for $t \geq T$ we derive that

$$|Qx(t) - Qy(t)| < 2\epsilon\ . \tag{5.27}$$

Further, let us assume that $t \in [0, T]$. Then, evaluating similarly as above we get:

$$\begin{aligned}
|Qx(t) - Qy(t)| &\leq |f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, y(\alpha_1(t)), y(\alpha_2(t)))\big| \\
&\quad + \int_0^{\beta(t)} [|g(t, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))|]\, ds \\
&\leq \frac{L \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}}{K + \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}} \\
&\quad + \int_0^{\beta_T} \big[|g(t, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))|\big]\, ds \\
&< \epsilon + \beta_T\, \omega_r^T(g, \epsilon)\ ,
\end{aligned} \tag{5.28}$$

where we have denoted

$$\beta_T = \sup\{\beta(t) :\ t \in [0, T]\}\ ,$$

and

$$\begin{aligned}
\omega_r^T(g, \epsilon) = \sup\big\{&|g(t, s, x, y) - g(t, s, w, z)| : \\
&t, s \in [0, T],\ x, y, w, z \in [-r, r],\ |x - w| \leq \epsilon,\ |y - z| \leq \epsilon\big\}\ .
\end{aligned}$$

Obviously, in view of the continuity of β, we have that $\beta_T < \infty$. Moreover, from uniform continuity of the function $g(t, s, x, y)$ on the compact $[0, T] \times [0, T] \times [-r, r] \times [-r, r]$ we derive that $\omega_r^T(g, \epsilon) \to 0$ as $\epsilon \to 0$. Now, linking (5.27), (5.28) and the above established facts we conclude that the operator Q maps partially continuously the closed ball $\overline{\mathcal{B}}(x_0, r)$ into itself.

Step IV: *Q is a nonlinear $\mathcal{D}$-set-contraction w.r.t. characteristic value* ω_0.

Further on let us take a bounded chain X in S with bound $r > 0$, i.e., the chain X belonging to the ball $\mathcal{B}(\theta, r)$. Next, fix arbitrarily $T > 0$ and $\epsilon > 0$. Let us choose $x \in X$ and $t_1, t_2 \in [0, T]$ with $|t_2 - t_1| \leq \epsilon$. Without loss of generality we may assume that $x(\alpha_1(t_1)) \geq x(\alpha_1(t_2))$ and $x(\alpha_2(t_1)) \geq x(\alpha_2(t_2))$. Then, taking into account our assumptions, we get:

$$
\begin{aligned}
|Qx(t_1) - Qx(t_2)| &\leq \big|f(t_1, x(\alpha_1(t_1)), x(\alpha_1(t_1))) - f(t_2, x(\alpha(t_2)), x(\alpha_2(t_1)))\big| \\
&\quad + \bigg| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \\
&\qquad - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \bigg| \\
&\leq \big|f(t_1, x(\alpha(t_1)), x(\alpha_2(t_1))) - f(t_2, x(\alpha(t_2)), x(\alpha_2(t_2)))\big| \\
&\quad + \bigg| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \\
&\qquad - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \bigg| \\
&\quad + \bigg| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \\
&\qquad - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \bigg| \\
&\leq \big|f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))) - f(t_2, x(\alpha_1(t_2)), x(\alpha_2(t_2)))\big| \\
&\quad + \int_0^{\beta(t_1)} |g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \\
&\quad + \bigg| \int_{\beta(t_2)}^{\beta(t_1)} |g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \bigg| \\
&\leq \big|f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))) - f(t_2, x(\alpha_1(t_2)), x(\alpha_2(t_2)))\big| \\
&\quad + \int_0^{\beta_T} |g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \\
&\quad + G_T^r\, |v(t_1) - v(t_2)|,
\end{aligned} \tag{5.29}
$$

where

$$G_T^r = \sup\{|g(t, s, x, y)| : \ t \in [0, T], \ s \in [0, \beta_T], \ x, y \in [-r, r]\}$$

which does exist in view of continuity of the function g on compact $[0, T] \times [0, \beta_T] \times [-r, r] \times [-r, r]$.

Now combining (5.28) and (5.29) we obtain,

$$
\begin{aligned}
|Qx(t_2) - Qx(t_1)| &\leq \big|f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))) - f(t_2, x(\alpha_1(t_1)), x(\alpha_2(t_1)))\big| \\
&\quad + \frac{L \max\{|x(\alpha_1(t_1)) - x(\alpha_2(t_2))|, |x(\alpha_1(t_1)) - x(\alpha_2(t_2))|\}}{K + \max\{|x(\alpha_1(t_1)) - x(\alpha_2(t_2))|, |x(\alpha_1(t_1)) - x(\alpha_2(t_2))|\}} \\
&\quad + \int_0^{\beta_T} |g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \\
&\quad + G_T^r\, |v(t_1) - v(t_2)| \\
&\leq \frac{L \max\left\{\omega^T\big(x, \omega^T(\alpha_1, \epsilon)\big), \omega^T\big(x, \omega^T(\alpha_2, \epsilon)\big)\right\}}{K + \max\left\{\omega^T\big(x, \omega^T(\alpha_1, \epsilon)\big), \omega^T\big(x, \omega^T(\alpha_2, \epsilon)\big)\right\}} + \omega_r^T(f, \epsilon) \\
&\quad + \int_0^{\beta_T} \omega_r^T(g, \epsilon)\, ds + G_T^r\, \omega^T(v, \epsilon),
\end{aligned} \tag{5.30}
$$

where we have denoted

$$\omega^T(\alpha_1,\epsilon) = \sup\{|\alpha_1(t_2)-\alpha_1(t_1)| : \ t_1,t_2 \in [0,T], \ |t_2-t_1| \leq \epsilon\},$$

$$\omega^T(\alpha_2,\epsilon) = \sup\{|\alpha_2(t_2)-\alpha_2(t_1)| : \ t_1,t_2 \in [0,T], \ |t_2-t_1| \leq \epsilon\},$$

$$\omega^T(v,\epsilon) = \sup\{|v(t_2)-v(t_1)| : \ t_1,t_2 \in [0,T], \ |t_2-t_1| \leq \epsilon\},$$

$$\begin{aligned}\omega_r^T(f,\epsilon) &= \sup\{|f(t_2,x,y)-f(t_1,x,y)| : \ t_1,t_2 \in [0,T], \\ &\qquad |t_2-t_1| \leq \epsilon, \ x,y \in [-r,r]\},\end{aligned}$$

and

$$\begin{aligned}\omega_r^T(g,\epsilon) = \sup\big\{&|g(t_2,s,x,y)-g(t_1,s,x,y)| : \ t_1,t_2 \in [0,T], \ |t_2-t_1| \leq \epsilon, \\ &s \in [0,\beta_T], \ x,y \in [-r,r]\big\}.\end{aligned}$$

From the above estimate we derive the following one:

$$\begin{aligned}\omega^T(Q(X),\epsilon) \leq{}& \frac{L \max\left\{\omega^T\left(x,\omega^T(\alpha_1,\epsilon)\right),\omega^T\left(x,\omega^T(\alpha_2,\epsilon)\right)\right\}}{K+\max\left\{\omega^T\left(x,\omega^T(\alpha_1,\epsilon)\right),\omega^T\left(x,\omega^T(\alpha_2,\epsilon)\right)\right\}} + \omega_r^T(f,\epsilon) \\ &+ \int_0^{\beta_T} \omega_r^T(g,\epsilon)\,ds + G_T^r\,\omega^T(v,\epsilon). \end{aligned} \tag{5.31}$$

Observe that $\omega_r^T(f,\epsilon) \to 0$ and $\omega_r^T(g,\epsilon) \to 0$ as $\epsilon \to 0$, which is a simple consequence of the uniform continuity of the functions f and g on the sets $[0,T]\times[-r,r]\times[-r,r]$ and $[0,T]\times[0,\beta_T]\times[-r,r]\times[-r,r]$ respectively. Moreover, from the uniform continuity of α_1,α_2,v on $[0,T]$, it follows that $\omega^T(\alpha_1,\epsilon)\to 0$, $\omega^T(\alpha_2,\epsilon)\to 0$, $\omega^T(v,\epsilon)\to 0$ as $\epsilon \to 0$. Thus, linking the established facts with the estimate (5.31) we get

$$\omega_0^T(Q(X)) \leq \frac{L\omega_0^T(X)}{K+\omega_0^T(X)}.$$

Consequently, we obtain

$$\omega_0(Q(X)) \leq \frac{L\,\omega_0(X)}{K+\omega_0(X)}. \tag{5.32}$$

Step V: *Q is a nonlinear $\mathcal{D}$-set-contraction w.r.t. the characteristic value δ_c.*

Now, taking into account our assumptions, for arbitrarily fixed $t \in \mathbb{R}_+$ and for $x,y \in X$ with $x \geq y$, we deduce the following estimate (cf. the estimate (5.28)):

$$\begin{aligned}|(Qx)(t)-(Qy)(t)| &\leq |f(t,x(\alpha_1(t)),x(\alpha_2(t)))-f(t,y(\alpha_1(t)),y(\alpha_2(t)))| \\ &\quad + 2\left(a(t)\int_0^{\beta(t)} b(s)\,ds\right) \\ &\leq \frac{L\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,|x(\alpha_2(t))-y(\alpha_2(t))|\}}{K+\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,|x(\alpha_2(t))-y(\alpha_2(t))|\}} \\ &\quad + 2v(t).\end{aligned}$$

From the above inequality it follows that

$$\operatorname{diam}(QX(t)) \leq \frac{L\ \max\{\operatorname{diam}(X(\alpha_1(t))),\operatorname{diam}(X(\alpha_2(t)))\}}{K+\max\{\operatorname{diam}(X(\alpha_1(t))),\operatorname{diam}(X(\alpha_2(t)))\}} + v(t)$$

for each $t \in \mathbb{R}_+$. Therefore, taking the limit superior over $t \to \infty$, we obtain

$$\begin{aligned}
\delta_c(QX) &= \limsup_{t\to\infty} \operatorname{diam}\left(Q(X(t))\right)\\
&\leq \frac{L\,\max\left\{\limsup\limits_{t\to\infty}\,\operatorname{diam}\left(X(\alpha_1(t))\right), \limsup\limits_{t\to\infty}\,\operatorname{diam}\left(X(\alpha_2(t))\right)\right\}}{K+\,\max\left\{\limsup\limits_{t\to\infty}\,\operatorname{diam}\left(X(\alpha_1(t))\right), \limsup\limits_{t\to\infty}\operatorname{diam}\left(X(\alpha_2(t))\right)\right\}}\\
&\leq \frac{L\,\limsup_{t\to\infty}\,\operatorname{diam}\left(X(t)\right)}{K+\limsup_{t\to\infty}\,\operatorname{diam}\left(X(t)\right)}\\
&= \frac{L\delta_c(X)}{K+\delta_c(X)} \qquad (5.33)
\end{aligned}$$

Step VI: *Q is a partially nonlinear $\mathcal{D}$-set-contraction on S.*

Further, using the measure of noncompactness μ_c^p defined by the formula (5.18) and keeping in mind the estimates (5.32) and (5.33), we obtain

$$\begin{aligned}
\mu_c^p(QX) &= \max\left\{\omega_0(QX)\,,\,\delta_c(QX)\right\}\\
&\leq \max\left\{\frac{L\,\omega_0(X)}{K+\omega_0(X)}\,,\,\frac{L\delta_c(X)}{K+\delta_c(X)}\right\}\\
&\leq \frac{L\,\max\left\{\omega_0(X)\,,\,\delta_c(X)\right\}}{K+\max\left\{\omega_0(X)\,,\,\delta_c(X)\right\}}\,.\\
&= \frac{L\mu_c^p(X)}{K+\mu_c^p(X)}
\end{aligned}$$

for all chains X in S. Since $L \leq K$, the operator Q is a partially nonlinear $\mathcal{D}$-set-contraction on S with $\mathcal{D}$-function $\psi(r) = \dfrac{Lr}{K+r}$. Again, by hypothesis ($H_5$), there exists an element $x_0 = u \in S$ such that $x_0 \leq Qx_0$, that is, x_0 is a lower solution of the HFIE (5.22) defined on $\mathbb{R}_+$.

Thus Q satisfies all the conditions of Theorem 3.2 on S. Hence we apply it to the operator equation $Qx = x$ and deduce that the operator Q has a fixed point x^* in S. Obviously x^* is a solution of the functional integral equation (5.22) and the sequence $\{x_n\}$ of successive approximations defined by

$$\begin{aligned}
x_n(t) \;&=\; f(t, x_{n-1}(\alpha_1(t)), x_{n-1}(\alpha_2(t)))\\
&\quad + \int_0^{\beta(t)} g(t,s,x_{n-1}(\gamma_1(s)), x_{n-1}(\gamma_2(s)))\,ds,\ \ t\in\mathbb{R}_+,
\end{aligned}$$

for each $n \in \mathbb{N}$ converges monotonically to x^*. Moreover, taking into account that the image of every chain X under the operator Q is again a chain $Q(X)$ contained in the ball $\overline{\mathcal{B}}(x_0, r)$ we infer that the set $\mathcal{F}(Q)$ of all fixed points of Q is contained in $\overline{\mathcal{B}}(x_0, r)$. If the set $\mathcal{F}(Q)$ contains all comparable solutions of the equation (5.22), then we conclude from Remark 3.5 that the set $\mathcal{F}(Q)$ belongs to the family ker μ_c^p. Now, taking into account the description of sets belonging to ker μ_c^p (given in Subsection 5.1) we deduce that all the comparable solutions of the equation (5.22) are uniformly globally ultimately attractive on $\mathbb{R}_+$. This completes the proof. □

Theorem 5.10. *Assume that the hypotheses (H_1) through (H_7) hold. Then the functional HFIE (5.22) has at least one solution x^* in the space $BC(\mathbb{R}_+, \mathbb{R})$ and the sequence $\{x_n\}$ of successive approximations defined by (5.22) converges monotonically to x^*. Moreover, the comparable solutions of the equation (5.21) are uniformly globally ultimately attractive and asymptotically stable to the line $x(t) = c$ defined on $\mathbb{R}_+$.*

Proof. As in Theorem 5.9, we seek the solutions of the HFIE (5.22) in the Banach space $E = BC(\mathbb{R}_+, \mathbb{R})$. Define the closed and bounded set $S = \overline{\mathcal{B}}(x_0, r)$, where $r = \|x_0\| + L + F_0 + V$ and define the operator Q on S into itself by (5.24). Then proceeding as in the Step IV of the proof of Theorem 5.9 it can be proved that Q is a nonlinear $\mathcal{D}$-set-contraction with respect to the characteristic value ω_c with $\mathcal{D}$-function $\psi(r) = \frac{Lr}{K+r}$, i.e., the inequality (5.32) holds on E.

Next, we show that Q is a partially nonlinear $\mathcal{D}$-set-contraction with respect to the characteristic value δ_b. Now, taking into account our assumptions, for arbitrarily fixed $t \in \mathbb{R}_+$ and for $x \in X$ with $x \geq c$ on $\mathbb{R}_+$, we deduce the following estimate:

$$
\begin{aligned}
|(Qx)(t) - c| &\leq |f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, c, c)| \\
&\qquad + \int_0^{\beta(t)} |g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \\
&\leq \frac{L \max\{|x(\alpha_1(t)) - c|, |x(\alpha_2(t)) - c|\}}{K + \max\{|x(\alpha_1(t)) - c|, |x(\alpha_2(t)) - c|\}} + v(t).
\end{aligned}
$$

From the above inequality it follows that

$$
|QX(t) - c| \leq \frac{L \max\{|X(\alpha_1(t)) - c|, |X(\alpha_2(t)) - c|\}}{K + \max\{|X(\alpha_1(t)) - c|, |X(\alpha_2(t)) - c|\}} + v(t)
$$

for each $t \in \mathbb{R}_+$. Therefore, taking the limit superior over $t \to \infty$, we obtain

$$
\begin{aligned}
\delta_b(QX) &= \limsup_{t\to\infty} |Q(X(t)) - c| \\
&\leq \frac{L \max\left\{ \limsup\limits_{t\to\infty} |X(\alpha_1(t)) - c|, \limsup\limits_{t\to\infty} |X(\alpha_2(t)) - c| \right\}}{K + \max\left\{ \limsup\limits_{t\to\infty} |X(\alpha_1(t)) - c|, \limsup\limits_{t\to\infty} |X(\alpha_2(t)) - c| \right\}} \\
&\leq \frac{L \limsup_{t\to\infty} |X(t) - c|}{K + \limsup_{t\to\infty} |X(t) - c|} \\
&= \frac{L\,\delta_b(X)}{K + \delta_b(X)}.
\end{aligned}
\tag{5.34}
$$

Further, using the partial measure of noncompactness μ_b^p defined by the formula (5.17) and keeping in mind the estimates (5.32) and (5.34), we obtain

$$
\begin{aligned}
\mu_b^p(QX) &= \max\left\{\omega_0(QX)\,,\, \delta_b(QX)\right\} \\
&\leq \max\left\{ \frac{L\,\omega_0(X)}{K + \omega_0(X)}\,,\, \frac{L\delta_b(X)}{K + \delta_b(X)} \right\} \\
&\leq \frac{L \max\left\{\omega_0(X)\,,\, \delta_b(X)\right\}}{K + \max\left\{\omega_0(X)\,,\, \delta_b(X)\right\}}\,. \\
&= \frac{L\mu_b^p(X)}{K + \mu_b^p(X)}
\end{aligned}
$$

for all chains X in S. Since $L \leq K$, the operator Q is a partially nonlinear $\mathcal{D}$-set-contraction on S with $\mathcal{D}$-function $\psi(r) = \frac{Lr}{K+r}$. Again, by hypothesis (H_5), there exists an element $x_0 = u \in S$ such that $x_0 \leq Qx_0$, that is, x_0 is a lower solution of the HFIE (5.22) defined on $\mathbb{R}_+$. The rest of the proof is similar to Theorem 5.9 and now we conclude from Remark 3.5 that the set $\mathcal{F}(Q)$ belongs to the family ker μ_b^p. Now, taking into account the description of sets belonging to ker μ_b^p (given in Section 5.1) we deduce that the equation (5.22) has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by (5.23) converges monotonically to x^*. Moreover, all the comparable solutions of the equation (5.22) are uniformly globally ultimately asymptotically stable to the line $x(t) = c$ on $\mathbb{R}_+$. This completes the proof. □

If $c = 0$ in hypothesis (H_7), we obtain the following existence result concerning the asymptotic stability of the comparable solutions defined on $\mathbb{R}_+$.

Theorem 5.11. *Assume that the hypotheses* (H_1) *through* (H_7) *hold with* $c = 0$. *Then the functional HFIE* (5.22) *has at least one solution* x^* *in the space* $BC(\mathbb{R}_+, \mathbb{R})$ *and the sequence* $\{x_n\}$ *of successive approximations defined by* (5.23) *converges monotonically to* x^*. *Moreover, the comparable solutions of the equation* (5.22) *are uniformly globally ultimately asymptotically stable to* 0 *defined on* $\mathbb{R}_+$.

Remark 5.12. We remark that if a nonlinear hybrid integral equation (5.22) has more than one lower solution, then they are comparable in view of the fact that E is a lattice. In consequence, it may have a number of comparable lower solutions. The case of upper solution is similar. Furthermore, the order relation $\leq$ in $C(\mathbb{R}_+, \mathbb{R})$ is same as the order relation induced by the order cone

$$\mathcal{K} = \{x \in C(\mathbb{R}_+, \mathbb{R}) \mid x(t) \geq 0 \text{ for all } t \in \mathbb{R}_+\}$$

in $C(\mathbb{R}_+, \mathbb{R})$. Hence, by virtue of Remark 3.5 the integral equation (5.22) has a number of comparable solutions defined on $\mathbb{R}_+$. As a result, under the given conditions of Theorem 5.9 all the comparable solutions of the nonlinear functional integral equation (5.22) are uniformly globally ultimately attractive on $\mathbb{R}_+$

Remark 5.13. The conclusion of Theorems 5.9, 5.10 and 5.11 remains true if we replace the hypothesis (H_5) with the following one:

(H'_5) There exists an element $v \in C(\mathbb{R}_+, \mathbb{R})$ such that

$$v(t) \geq f(t, v(\alpha_1(t)), v(\alpha_2(t)) + \int_{t_0}^{\beta(t))} g(t, s, v(\gamma_1(s)), v(\gamma_2(s)))\, ds$$

for all $t \in \mathbb{R}_+$.

The proof under this new hypothesis is similar to Theorems 5.9 and 5.10 and now, the desired conclusion follows by an application of Theorem 3.3.

Remark 5.14. The conclusion of Theorems 5.9, 5.10 and 5.11 also remains true if we replace the hypothesis (H_5) with the following one:

(H'_3) There exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \phi\big(\max\{x_1 - y_1\,,\, x_2 - y_2\}\big)$$

for all $t \in \mathbb{R}_+$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$. Moreover, $\phi(r) < r$ for $r > 0$.

Example 5.2. Consider the linearly perturbed nonlinear hybrid functional integral equation,

$$x(t) = \tan^{-1} x(2t) + \int_0^{3t} \frac{1}{t^2+1}\, g(s, x(s/2))\, ds \tag{5.35}$$

for all $t \in \mathbb{R}_+$, where $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$g(t,x) = \begin{cases} 1, & \text{if } \; x \leq 0, \\ x^2+1, & \text{if } \; 0 < x \leq 1, \\ \dfrac{4x}{x+1}, & \text{if } \; x > 1. \end{cases}$$

We shall show that all the hypotheses of Theorem 3.2 are satisfied by the functions involved in HFIE (5.35). Here, $\alpha(t) = 2t$, $\beta(t) = 3t$ and $\gamma(t) = t/2$ and so, α, β, γ are continuous on $\mathbb{R}_+$ into itself and $\alpha(t) \geq t$ for all $t \in \mathbb{R}_+$. Thus, hypothesis (H_0) is satisfied. Again, $f(t,x) = \tan^{-1} x$ so that f is continuous on $\mathbb{R}_+ \times \mathbb{R}$ and nondecreasing in x for each $t \in \mathbb{R}_+$. The kernel $k(t,s)$ is given by $k(t,s) = \frac{1}{t^2+1}$. Obviously k is continuous and nonnegative function on $\mathbb{R}_+ \times \mathbb{R}_+$ and so (H_2) holds. Next, $g(t,x)$ is defines a continuous and nondecreasing function in x for each $t \in \mathbb{R}_+$. Moreover, $f(t,0) = 0$. So the hypotheses (H_1), (H_2), (H_4) and (H_5) are held.

Next, we show that f satisfies hypothesis (H'_3) on $\mathbb{R}_+ \times \mathbb{R}$. Let $x, y \in \mathbb{R}$ with $x \geq y$. Then,

$$0 \leq f(t,x) - f(t,y) = \tan^{-1} x - \tan^{-1} y = \frac{1}{1+\xi^2}(x-y)$$

for all $y < \xi < x$, and so hypothesis (H_3') is satisfied with $\frac{r}{1+\xi^2}$, $0 < \xi < r$.

Furthermore, $g(t,x) \leq 4$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Therefore,

$$v(t) = \int_0^{3t} \frac{1}{t^2+1} \cdot 4\, ds = \frac{12t}{t^2+1}.$$

Therefore,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{12t}{t^2+1} = 0.$$

Finally, it is easy to prove that $u \equiv 0$ is a lower solution of the HFIE (5.35) defined on $\mathbb{R}_+$ and hence the hypothesis (H_6) is held. Thus all the conditions of Theorem 5.9 are satisfied and by a direct application, we conclude that the HFIE (5.35) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined

$$x_{n+1}(t) = \tan^{-1} x_n(2t) + \int_0^{3t} \frac{1}{t^2+1}\, g(s, x_n(s/2))\, ds$$

converges monotonically to x^*, where $x_0 = 0$. Moreover, the comparable solutions of the HFIE (5.35) are uniformly asymptotically attractive and stable to zero defined on $\mathbb{R}_+$.

Example 5.3. Consider the hybrid differential equation with a linear perturbation of first type, viz.,

$$x(t) = f(t, x(2t)) + \int_0^{3t} \frac{t}{t^3+1}\, \tanh x(s/2)\, ds \tag{5.36}$$

for all $t \in \mathbb{R}_+$, where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$f(t,x) = \begin{cases} 1, & \text{if} \quad x \leq 0, \\ 1 + \dfrac{x}{x+1}, & \text{if} \quad x > 0. \end{cases}$$

We shall show that all the hypotheses of Theorem 3.2 are satisfied by the functions involved in HFIE (5.36). Here, as before, $\alpha(t) = 2t$, $\beta(t) = 3t$ and $\gamma(t) = t/2$ and so, α, β, γ are continuous on $\mathbb{R}_+$ into itself and $\alpha(t) \geq t$ for all $t \in \mathbb{R}_+$. Thus, hypothesis (H_0) is satisfied. Again, $f(t,x)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$ and nondecreasing in x for each $t \in \mathbb{R}_+$. The kernel $k(t,s)$ is given by $k(t,s) = \frac{t}{t^3+1}$. Obviously k is continuous and nonnegative function on $\mathbb{R}_+ \times \mathbb{R}_+$ and so (H_2) holds. Next, $g(t,x) = \tanh x$ is a continuous and nondecreasing function in x for each $t \in \mathbb{R}_+$. So the hypotheses (H_1), (H_2), (H_4) and (H_5) are held.

Now, we show that f satisfies hypothesis (H_3') on $\mathbb{R}_+ \times \mathbb{R}$. Let $x, y \in \mathbb{R}$ with $x \geq y$. Then,

$$0 \leq f(t,x) - f(t,y) = \frac{x}{x+1} - \frac{y}{y+1} = \frac{x-y}{1+x+y+xy} \leq \frac{x-y}{1+x-y}$$

and so, the hypothesis (H_3') is satisfied with $\phi(r) = \dfrac{r}{1+r}$ for $r > 0$.

Furthermore, $|g(t,x)| = |\tanh x| \leq 1$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Therefore,

$$v(t) = \int_0^{3t} \frac{t}{t^3+1} \cdot 1\, ds = \frac{3t^2}{t^3+1}.$$

Therefore,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{3t^2}{t^3+1} = 0.$$

Finally, it is easy to prove that $u \equiv 0$ is a lower solution of the HFIE (5.36) defined on $\mathbb{R}_+$ and hence the hypothesis (H_6) is held. Thus all the conditions of Theorem 5.9 are satisfied and by a direct application, we conclude that the HFIE (5.36) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined

$$x_{n+1}(t) = f(t, x_n(2t)) + \int_0^{3t} \frac{t}{t^3+1}\, \tanh x_n(s/2))\, ds$$

converges monotonically to x^*, where $x_0 = 0$. Moreover, the comparable solutions of the HFIE (5.36) are uniformly globally attractive on $\mathbb{R}_+$.

Remark 5.15. In this paper we have been able to weaken the Lipschitz condition to nonlinear one-sided Lipschitz condition which otherwise is considered to be a very strong condition in the existence theory of nonlinear differential and integral equations. However, we needed an additional assumption of monotonicity on the nonlinearities involved in the integral equation in order to guarantee the required characterization of attractivity of the comparable solutions.

Remark 5.16. The existence theorems proved in Section 5 may be extended with appropriate modifications to the generalized nonlinear hybrid functional integral equation

$$x(t) = f(t, x(\alpha_1(t)), \ldots, x(\alpha_n(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), \ldots, \gamma_n(s)))\, ds \tag{5.37}$$

for all $t \in \mathbb{R}_+$, where $\alpha_i, \beta, \gamma_i : \mathbb{R}_+ \to \mathbb{R}_+,\ i = 1, 2, \ldots, n,\ f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, and $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions.

Remark 5.17. The study of the present paper may be extended to other types of nonlinear hybrid integral equations with different linear as well as quadratic perturbations of first and second type. The detials of different types of perturbations are given in Dhage [8] and the references therein.

6 Conclusion

Observe that the main measure theoretic hybrid fixed point theorems of this paper may be applied to other nonlinear equations like hybrid causal and fractional differential, integral and integro-differential equations for proving the existence results, however unlike existence theorem for nonlinear hybrid integral equations discussed in Dhage [11] we do not require the assumption that E to be a lattice. Again the continuity of the functions $f(t, x)$ and $g(t, s, x)$ in the variable x means that they are partially continuous on $\mathbb{R}$ since $\mathbb{R}$ is a totally ordered set, and therefore, the corresponding operators defined in the proof of above theorem are partially continuous on the domains of their definition which is contrary to the case considered in Dhage [9]. The advantage of the present approach over the previous ones lies in the fact that we have been able to develop an algorithm for the solutions of the considered integral equation which otherwise is not possible via classical approach of measure of noncompactness treated in Banas and Goebel [2]. Another interesting feature of our work is that we generally need the uniqueness of the solution for predicting the behavior of the dynamic system related to the considered nonlinear functional integral equation, however with the present approach it has become possible for us to discuss the qualitative behaviour of the systems even though there exist a number of comparable solutions. Finally, while concluding this paper, we mention that the existence theorem proved in this paper for the considered nonlinear integral equation may also be proved by using the Krasnoselskii type hybrid fixed point theorem, Theorem 4.6 under weaker Carathéodory condition than continuity of the function g with appropriate modifications. Some of the results in the above mentioned direction will be reported elsewhere.

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Classical and partial symmetries of the Benney equation

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Abstract

Lie symmetry group method is applied to study Benney equation. The symmetry group and its optimal system are given,and group invariant solutions associated to the symmetries are obtained. Also the structure of the Lie algebra symmetries is determined. Mainly, we have compared one of the resolved analitical solutions of the Benney equation with one of it's numerical solutions which is obtained via homotopy perturbation method in [4].

Keywords: Lie group analysis, Partial symmetry, Symmetry group, Optimal system, Invariant solution, Benney equation.

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1 Introduction

In the past decades, both mathematicians and physicists have made efforts in the study of exact solutions of partial differential equations. The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [1]. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and is of great importance to understand and to construct solutions of differential equations. Reduction of order of partial differential equations or transformed to ordinary differential equations. According to the standard definition partial symmetries of $\triangle = 0$ as Lie-point invertible transformations T such that there is a non-empty subset $P \subset S_\triangle$such that $T(P) = P$, i.e.such that there is a subset of solutions to $\triangle = 0$ which are transformed into one another. We discuss how to determine both partial symmetries and the invariant set $P \subset S_\triangle$, and show that our procedure is effective by Benney equation.

The homotopy perturbation method (HPM) was established by Ji-Huan He in 1999. In this method, the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution. Using the homotopy technique from topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$, which is considered as a small parameter. The approximations obtained by the homotopy perturbation method are uniformly valid not only for small parameters, but also for very large parameters.

Benney's Equation is

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + u_{xxxx} = 0, \tag{1.1}$$

where u is considered to be periodic in x as an apology for the infinite domain. This equation seems rich in character and has been derived in several physical contexts including the flow of thin liquid films (Benney 1966, Topper and Kawahara 1977).

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2 Lie Symmetry Methods

In this part , we use general for determining symmetries for any system PDE. For this method, use the general case of a nonlinear system of PDE of order n^{th} in p independent and q dependent variables is given as a system of equations:

$$\triangle_{\nu}(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, \ell \tag{2.2}$$

via $x = (x^1, \cdots, x^p), u = (u^1, \cdots, u^q)$ and the derivatives of u whit respect to x up to n, and u^n respect all derivatives of u of all orders from 0 to n.

We consider a one-parameter Lie group of infinitesimal transformationsacting on the Jet space of the system (2.2)

$$\tilde{x}^i = x^i + \varepsilon\xi^i(x,u) + O(\varepsilon^2), \qquad \tilde{u}^j = x^j + \varepsilon\eta^j(x,u) + O(\varepsilon^2) \tag{2.3}$$

where ε is the parameter of the transformation and ξ^i, η^j are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator v associated with the above group of transformations can be written as

$$v = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \eta_j(x,u)\frac{\partial}{\partial u^j} \tag{2.4}$$

A symmetry of a differential equation is a transformation which maps solutions of the equation to other solutions. (be a vector field on an open subset $M \subset X \times U$). The invariance of the system (2.2) under the infinitesimal transformations leads to the invariance condition

$$\mathrm{pr}^{(n)}v[\triangle_{\nu}(x, u^{(n)})] = 0, \quad \triangle_{\nu}(x, u^n) = 0, \qquad \nu = 1, \cdots, \ell, \tag{2.5}$$

where $\mathrm{pr}^{(n)}$ is the n^{th} order prolongation of the infinitesimal generator given by

$$pr^{(n)}v = v + \sum_{\alpha=1}^{q}\sum_{J} \phi_J^{\alpha}(x, u^{(n)})\frac{\partial}{\partial u_J^{\alpha}} \tag{2.6}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$. $(J = (j_1, \cdots, j_k), 1 \le j_k \le p, 1 \le k \le n)$ with coefficient

$$\phi_{\alpha}^{J}(x, u^{(n)}) = D_J(\phi_{\alpha} - \sum_{i=1}^{p}\xi^i u_i^{\alpha}) + \sum_{i=1}^{p}\xi^i u_{J,i}^{\alpha} \tag{2.7}$$

where $u_i^{\alpha} = \dfrac{\partial u_{\alpha}}{\partial x_i}$ and $u_{J,i}^{\alpha} = \dfrac{\partial u_J^{\alpha}}{\partial x_i}$.

3 Lie Symmetry of the Benney equation

We consider the equation of Benney (1) (with 2 independent variable (x,t) and 1 dependent $u(x,t)$) where x, t are variables, u is a function and δ is a constant.

Let $v = \xi(x,t,u)D_x + \tau(x,t,u)D_t + \phi(x,t,u)D_u$ be a vector field on $X \times U$,$(X = (x,t))$. We wish to determine all possible coefficient function ξ, τ and ϕ so that the corresponding one-parameter group $\exp(\varepsilon v)$ is a symmetry group of the Benney equation, we need to know the fourth prolongation

$$pr^{(4)}v = v + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \cdots + \phi^{tttt}\frac{\partial}{\partial u_{tttt}}, \tag{3.8}$$

Table 1: Commutation relations satisfied by infinitesimal generators

$[\,,\,]$	v_1	v_2	v_3
v_1	0	v_3	0
v_2	$-v_3$	0	0
v_3	0	0	0

Table 2: Adjoint relations satisfied by infinitesimal generators

$[\,,\,]$	v_1	v_2	v_3
v_1	v_1	$v_2-\varepsilon v_3$	v_3
v_2	$v_1+\varepsilon v_3$	v_2	v_3
v_3	v_1	v_2	v_3

of v, with the coefficients :

$$\begin{aligned}\phi^x &= D_x\phi + D_u\phi u_x - u_x D_x\xi - D_u\xi u_x^2 - u_t D_x\tau - u_t D_u\tau u_x,\\ \phi^t &= D_t\phi + D_u\phi u_t - u_x D_t\xi - u_x D_u\xi u_t - u_t D_t\tau - D_u\tau u_t^2,\\ \phi^{xx} &= D_x^2\phi + 2D_{ux}\phi u_x - \cdots - 2\,u_{xt} D_u\tau u_x, \quad \text{and so on.}\end{aligned} \tag{3.9}$$

Applying $pr^{(4)}v$ to Benney equation , we find the infinitesimal criterion. determining equations yields:

$$\begin{array}{lllll} D_x\phi = 0, & D_x\tau = 0, & D_x\xi = 0, & D_t\phi = 0, & D_t\tau = 0,\\ D_u\phi = 0, & D_u\tau = 0, & D_u\xi = 0, & D_t\xi = \phi, & (\delta \neq D_t\tau,\ \ \delta \neq 0)\end{array} \tag{3.10}$$

The solution of the above system gives the following coefficients of the vector field v:

$$\phi\,(x,t,u) = C_2, \quad \tau\,(x,t,u) = C_1, \quad \xi\,(x,t,u) = C_2\,t + C_3, \tag{3.11}$$

Where $\{C_1, C_2, C_3\}$ are arbitrary constants, thus the Lie algebra $\mathbf{g}$ of the Benney equation is spanned by the three vector fields

$$v_1 = D_t, \qquad v_2 = tD_x + D_u, \qquad v_3 = D_x, \tag{3.12}$$

The one-parameter groups G_i generated by the base of $\mathbf{g}$ are given in the following table

$$\begin{aligned} g_1 : (x,t,u) &\longmapsto (x, t+\varepsilon, u),\\ g_2 : (x,t,u) &\longmapsto (\varepsilon t + x, t, \varepsilon + u),\\ g_3 : (x,t,u) &\longmapsto (\varepsilon + x, t, u).\end{aligned} \tag{3.13}$$

Since each group G_i is a symmetry group and if $u = f(x,t)$ is a solution of the Benney equation, so are the functions

$$u^{(1)} = f(x, t-\varepsilon), \quad u^{(2)} = f(x - t\varepsilon, t) - \varepsilon, \quad u^{(3)} = f(x-\varepsilon, t), \tag{3.14}$$

where ε is a real number.

Also their commutator table is and also table Adjoint with (i,j)-th entry indicating $\mathrm{Ad}(\exp(\varepsilon v_i)v_j)$: where ε is a real number. Here we can find the general group of the symmetries by considering a general linear combination "$c_1v_1 + c_2v_2 + c_3v_3$" of the given vector fields. In particular if $\mathbf{g}$ is the action of the symmetry group near the identity, it can be represented in the form $\mathbf{g} = \exp(c_1v_1)\cdots..\exp(c_3v_3)$.

4 Optimal system of the Benney equation

This part using the adjoint representation for classifying group-invariant solutions. let G a Lie group . An *optimal system* of subgroup is a list of conjugate equivalent subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of subalgebras forms an*optimal system* if every subalgebra of **g** is equivalent to a unique member of the list under some element of the adjoint representation:

$$\tilde{h} = \mathrm{Ad}\ \ g(h), g \in G. \tag{4.15}$$

We finding exact solutions and performing symmetry reductions of differential equations. As any transformation in the symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system. For one- dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation.

The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon v_i))v_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(ad\ v_i)^n(v_j), \tag{4.16}$$

where $[v_i, v_j]$ is a commutator for the Lie algebra, ε is a parameter, and $i, j = 1, 2, 3$.

Let $F_i^\varepsilon : g \longrightarrow g$ defined by $v \longrightarrow \mathrm{Ad}(\exp(\varepsilon v_i)v)$ is a linear map, for $i = 1, 2, 3$. The matrices M_i^ε of F_i^ε, $i = 1, 2, 3$ with respect to basis $\{v_1, v_2, v_3\}$

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & \varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.17}$$

by acting above matrices on a vector field v alternatively we can show that a one-dimensional optimal system of **g** is given by

$$Y_1 = v_3, \quad Y_2 = a_1 v_1 + v_2, \quad (a_1 \in \mathbb{R}), \quad Y_3 = v_1, \tag{4.18}$$

5 partial symmetries of Benney equation

Let us consider a general differential problem, given in the form of a system of ℓ differential equations, and briefly denoted, as usual, by

$$\Delta = \Delta(x, u^{(m)}) = 0, \tag{5.19}$$

where $\Delta = (\Delta_1, \cdots, \Delta_\ell)$ are smooth functions involving p independent variables $x = (x_1, \cdots, x_p) \in \mathbb{R}^p$ and q dependent unknown variables $u = (u_1, \cdots, u_q) \in \mathbb{R}^q$ together with the derivatives of the u_α with respect to the $x_i(\alpha = 1, \cdots, q; i = 1, \cdots, p)$ up to some order m. Let

$$X = \xi_i \frac{\partial}{\partial x_i} + \phi_\alpha \frac{\partial}{\partial u_\alpha}, \qquad \xi_i = \xi_i(x, u), \qquad \phi_\alpha = \phi_{\alpha(x,u)}, \tag{5.20}$$

be a given vector field, where ξ_i and ϕ_α are p+q smooth functions. if X is a exact symmetry then $pr^{(m)}X\ \ \Delta = 0$, We now assume that the vector field X is not a symmetry of Δ, hence $pr^{(m)}X\ \ \Delta \neq 0$, let $\Delta^{(1)} = pr^{(m)}X\ \ \Delta$ with this define $\Delta^{(1)}$ of order m' $(m' \leq m)$,

We consider two vector field for this equation.

1) We consider the generic scaling vector fields

$$X_1 = auD_u + bxD_x + ctD_t, \qquad (a, b, c\ \varepsilon\mathbb{R}) \tag{5.21}$$

that is a not a symmetry for Δ.

$$\Delta = u_t + uu_x + u_{xx} + \delta u_{xxx} + u_{xxxx} = 0. \tag{5.22}$$

Applying the fourth prolongation of X_1 on Δ , we obtain

$$\begin{aligned} \Delta^{(1)} &= \mathrm{pr}^{(4)} X_1 \Delta, \\ \Delta^{(1)} &= (2b-a)uu_x + (a-c)u_t + (a-2b)u_{xx} + (a\delta - 3\delta b)u_{xxx} + (a-4b)u_{xxxx}. \end{aligned} \tag{5.23}$$

We rewrite this as

$$\Delta^{\hat{(1)}} = Auu_x + Bu_t - Au_{xx} + Cu_{xxx} + Eu_{xxxx}, \qquad (A, B, C, E \in \mathbb{R}) \tag{5.24}$$

reduction to ODE. If $\langle A = B = E = 0$ and $C \neq 0\rangle$ reduce to $u_{xxx} = 0$, solve this ODE we obtain

$$u(x,t) = \alpha(t) + \beta(t)x + \gamma(t)x^2 \tag{5.25}$$

substituting this into the Δ equation we obtain

$$\beta'(t) + \beta^2(t) = 0, \qquad \alpha'(t) + \alpha(t)\beta(t) = 0, \qquad \gamma(t) = 0, \tag{5.26}$$

from (26) we obtain :

$$\beta(t) = \frac{1}{t + C_1}, \tag{5.27}$$

and from (27)

$$\alpha(t) = \frac{C_2}{t + C_1}, \tag{5.28}$$

in this case we are

$$u(x,t) = \frac{C_2}{t + C_1} + \frac{1}{t + C_1}x. \tag{5.29}$$

2) Consider $X_2 = D_t + D_u$ a arbitrary vector field , Applying the fourth prolongation of X_2 on Δ, we obtain

$$\Delta^{(1)} = \mathrm{pr}^{(4)} X_2\, \Delta = u_x, \qquad u_x = 0, \tag{5.30}$$

from above equation we obtain :

$$u(x,t) = \alpha(t), \tag{5.31}$$

via substituting $u(x,t) = \alpha(t)$ to Δ we are

$$\alpha'(t) = 0, \tag{5.32}$$

via solving above equation, we obtain :

$$\alpha(t) = C_0, \tag{5.33}$$

In finally we are

$$u(x,t) = C_0. \tag{5.34}$$

6 Compare Numerical Solution and Analytical Solution

In this part we use Homotopy Perturbation Method (HPM), the homotopy perturbation method provides an effective procedure for exact and numerical solutions of partial differential equations. Now we use article's [4] as flows.

Numerical solution :

$$u^*(x,t) = \frac{x}{t} - \frac{1}{4} e^{\sqrt[3]{2}x - 2\,\delta\, t}, \tag{6.35}$$

Analytical Solution in generic scaling :

$$u(x,t) = \frac{C_2}{t+C_1} + \frac{x}{t+C_1}, \tag{6.36}$$

With the right choice C_1 and C_2:

$$C_1 = .004801455707 \quad \text{and} \quad C_2 = -.01233867027, \quad (t=1, \delta=1). \tag{6.37}$$

We have :

$$u^*(x,1) = x - \frac{1}{4} 2.718281828^{\sqrt[3]{2}x-2}, \tag{6.38}$$

$$u(x,1) = -.01227970978 + .9952214878\, x. \tag{6.39}$$

Now we plot $u(x,1), u^*(x,1)$ to compare

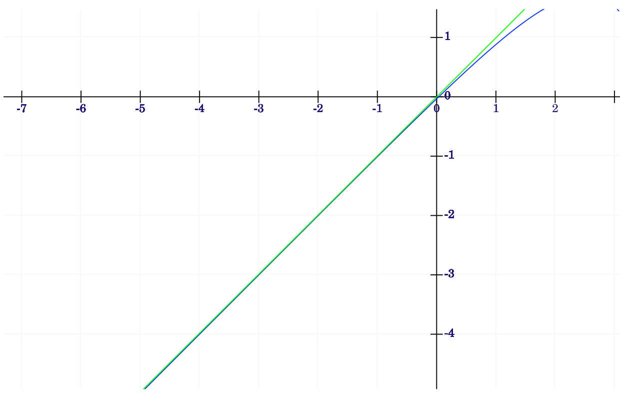

Figure 1: $u(x,1)$(green), $u^*(x,1)$(blue), $\delta = 1$

$$\forall x \in [-2,-1] \Longrightarrow \mid u^*(x,1) - u(x,1) \mid \leq .0131. \tag{6.40}$$

7 Conclusion

In this paper, by applying the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators, we find the most general Lie point symmetries group of the Benny equation. Also, we have constructed the optimal system of one-dimensional sub algebras of Benny equation. We find a analytic solution by partial symmetry and compare with numerical solution in order to calculation error.

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Fuzzy filters in $\Gamma-$semirings

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Abstract

We introduce the notion of fuzzy prime ideals and fuzzy filters in gamma semirings and study some of their properties.

Keywords: $\Gamma-$semiring , Fuzzy filter, fuzzy prime ideal.

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1 Introduction

The notion of semiring was introduced by H. S. Vandiver[6] in 1934. The notion of $\Gamma-$ ring was introduced by N. Nobusawa [4] as a generalization of ring in 1964. M. Murali Krishna Rao [3] introduced the notion of $\Gamma-$semiring which is a generalization of ring, ternary semiring and semiring. After the paper [3] was published , many mathematicians obtained interesting results on $\Gamma-$semirings. The theory of fuzzy sets was first introduced by L. A. Zadeh [7] in 1965, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. In this paper, we introduce the notion of fuzzy prime ideals and fuzzy filters in gamma semirings and study some of their properties. In this section we will recall some of the fundamental concepts and definitions, these are necessary for this paper.

Definition 1.1. *A set R together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided*

(i) Addition is a commutative operation.

(ii) Multiplication distributes over addition both from the left and from the right.

(iii) There exists $0 \in R$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for each $x \in R$.

Definition 1.2. *[1] Let M and Γ be additive abelian groups. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following conditions for all $x, y, z \in M, \alpha, \beta \in \Gamma$*

(i) $x\alpha(y\beta z) = (x\alpha y)\beta z$

(ii) $x\alpha(y+z) = x\alpha y + x\alpha z$

(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$

(iv) $(x+y)\alpha z = x\alpha z + y\alpha z.$

Then M is called a $\Gamma-$ ring.

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Definition 1.3. *[3] Let $(M,+)$ and $(\Gamma,+)$ be commutative semigroups. Then we call M as a $\Gamma-$semiring, if there exists a mapping $M\times\Gamma\times M\to M$ written as (x,α,y) as $x\alpha y$ such that it satisfies the following axioms for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$*

(i) $x\alpha(y+z)=x\alpha y+x\alpha z$

(ii) $(x+y)\alpha z=x\alpha z+y\alpha z$

(iii) $x(\alpha+\beta)y=x\alpha y+x\beta y$

(iv) $x\alpha(y\beta z)=(x\alpha y)\beta z.$

Definition 1.4. *A non empty subset A of $\Gamma-$semiring M is called a $\Gamma-$subsemiring M if $(A,+)$ is a sub semigroup of $(M,+)$ and $a\alpha b\in A$ for all $a,b\in A$ and $\alpha\in\Gamma$.*

Definition 1.5. *An additive sub semigroup I of a $\Gamma-$semiring M is said to be a left (right) ideal of M if $M\Gamma I\subseteq I$ $(I\Gamma M\subseteq I)$.*

If I is both left and right ideal then I is called an ideal of $\Gamma-$semiring M.

Definition 1.6. *[7] Let S be a non empty set, a mapping $f:S\to[0,1]$ is called a fuzzy subset of S.*

Definition 1.7. *Let f be a fuzzy subset of a nonempty set S, for $t\in[0,1]$ the set $f_t=\{x\in S\mid f(x)\geq t\}$ is called level subset of S with respect f.*

Definition 1.8. *A fuzzy subset $\mu:S\to[0,1]$ is nonempty if μ is not the constant function.*

Definition 1.9. *For any two fuzzy subsets λ and μ of S, $\lambda\subseteq\mu$ means $\lambda(x)\leq\mu(x)$ for all $x\in S$.*

Definition 1.10. *A fuzzy subset μ of M is a proper fuzzy subset if it is a non constant function.*

Definition 1.11. *A fuzzy subset μ is an improper if it is a constant function.*

Definition 1.12. *Let M be a $\Gamma-$semiring and f be a fuzzy subset of M. The mapping $f':M\to[0,1]$ is defined by $f'(x)=1-f(x)$ is a fuzzy subset of M, called complement of f.*

Definition 1.13. *Let M be a $\Gamma-$semiring . A fuzzy subset μ of M is said to be a fuzzy $\Gamma-$subsemiring of M if it satisfies the following conditions*
(i) $\mu(x+y)\geq min\{\mu(x),\mu(y)\}$
(ii) $\mu(x\alpha y)\geq min\{\mu(x),\mu(y)\}$ *for all* $x,y\in M,\alpha\in\Gamma$.

Definition 1.14. *A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy left(right) ideal of M if for all $x,y\in M,\alpha\in\Gamma$*
(i) $\mu(x+y)\geq min\{\mu(x),\mu(y)\}$
(ii) $\mu(x\alpha y)\geq\mu(y)\big(\mu(x)\big)$

Definition 1.15. *A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy ideal of M if for all $x,y\in M,\alpha\in\Gamma$*
(i) $\mu(x+y)\geq min\{\mu(x),\mu(y)\}$
(ii) $\mu(x\alpha y)\geq max\{\mu(x),\mu(y)\}$

2 Main results

In this section, we introduce the notion of fuzzy prime ideals, fuzzy filters in $\Gamma-$semirings and study some of their properties.

Definition 2.16. *Let M be a $\Gamma-$semiring. An ideal P of M is called a prime ideal of Mif for any $a,b\in M$ and $\gamma\in\Gamma, a\gamma b\in P\Rightarrow a$ or $b\in P$.*

Definition 2.17. *A proper fuzzy ideal μ of M is called fuzzy prime ideal if*

$$\mu(x\alpha y)=max\ \{\mu(x),\mu(y)\},\forall x,y\in M,\alpha\in\Gamma$$

Definition 2.18. *Let M be a $\Gamma-$semiring. A $\Gamma-$subsemiring F of M is called a filter of Mif for any $a,b\in M$ and $\gamma\in\Gamma, a\gamma b\in F\Rightarrow a$ and $b\in F$.*

Definition 2.19. *Let M be a $\Gamma-$semiring. A fuzzy $\Gamma-$subsemiring μ of M is called a fuzzy filter of M if*

$$\mu(x\alpha y) = min\{\mu(x), \mu(y)\}, \forall x, y \in M, \alpha \in \Gamma.$$

The following theorems are straight forward.

Theorem 2.1. *Let M be a $\Gamma-$semiring, f be a fuzzy subset of M and f' be the complement of f. Then the following statements are equivalent. Let $x, y \in M, \alpha \in \Gamma$.*

(1). $f(x+y) \geq min\{f(x), f(y)\}$
$f(x\alpha y) \geq max\{f(x), f(y)\}$

(2). $f'(x+y) \leq max\{f(x), f(y)\}$
$f'(x\alpha y) \leq min\{f(x), f(y)\}$

Theorem 2.2. *Let M be a $\Gamma-$semiring and f be a fuzzy subset of M. Then f is a fuzzy filter of M if and only if f', the complement of f is a fuzzy prime ideal of M.*

Theorem 2.3. *μ is a fuzzy filter of $\Gamma-$semiring M if and only if its level subset $\mu_t \neq \phi$, for any $t \in [0, 1]$ is a filter of M.*

Proof. Suppose μ is a fuzzy filter of $\Gamma-$semiring M. Let $t \in [0, 1]$ such that μ_t is a $\Gamma-$subsemiring of M. Let $a, b \in M, \alpha \in \Gamma$ and $a\alpha b \in \mu_t$

$$\begin{aligned} &\Rightarrow \mu(a\alpha b) \geq t \\ &\Rightarrow min\{\mu(a), \mu(b)\} \geq t \\ &\Rightarrow \mu(a) \geq t \text{ and } \mu(b) \geq t \\ &\Rightarrow a \in \mu_t \text{ and } b \in \mu_t. \end{aligned}$$

Hence μ_t is a filter of $\Gamma-$semiring M.

Conversely suppose that its level subset $\mu_t \neq \phi$, for any $t \in [0, 1]$ is a filter of M. Let $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned} \text{Suppose } \ t &= min\{\mu(x), \mu(y)\} \\ &\Rightarrow \mu(x) \geq t, \mu(y) \geq t \\ &\Rightarrow x, y \in \mu_t \\ &\Rightarrow x+y, x\alpha y \in \mu_t \\ &\Rightarrow \mu(x+y) \geq t = min\{\mu(x), \mu(y)\} \\ &\text{and } \mu(x\alpha y) \geq t = min\{\mu(x), \mu(y)\}. \end{aligned}$$

Hence μ is fuzzy $\Gamma-$subsemiring of M.
Let$x, y \in M, \gamma \in \Gamma$ and $\mu(xry) = t$.

$$\begin{aligned} &\Rightarrow xry \in \mu_t \\ &\Rightarrow x \in \mu_t \text{ and } y \in \mu_t \\ &\Rightarrow \mu(x) \text{ and } \mu(y) \geq t \\ &\Rightarrow min\{\mu(x), \mu(y)\} \geq t = \mu(xry) \\ &\Rightarrow min\{\mu(x), \mu(y)\} \geq \mu(xry) \geq min\{\mu(x), \mu(y)\} \\ \text{Hence } \ &\mu(xry) = min\{\mu(x), \mu(y)\}. \end{aligned}$$

Therefore μ is a fuzzy filter of $\Gamma-$semiring M. □

Theorem 2.4. *μ is a fuzzy prime ideal of $\Gamma-$semiring M if and only if for any $t \in [0, 1]$ such that μ_t is a prime ideal of M.*

Proof. Suppose μ is a fuzzy prime ideal of $\Gamma-$semiring M. Let $t \in [0, 1]$ such that μ_t is a proper ideal of $\Gamma-$semiring M. Let $x, y \in \mu_t$.
$\Rightarrow \ \mu(x) \geq t, \ \ \mu(y) \geq t \Rightarrow \mu(x+y) \geq min\{\mu(x), \mu(y)\} \geq t.$

Therefore $x+y \in \mu_t$ Let $x \in \mu_t, \alpha \in \Gamma, y \in M \setminus \mu_t \Rightarrow \mu(x) \geq t$ and $\mu(y) < t$
$\Rightarrow \mu(x\alpha y) = max\{\mu(x), \mu(y)\} \geq t \Rightarrow x\alpha y \in \mu_t$. similarly we can prove $y\alpha x \in \mu_t$. Hence μ_t is an ideal of $\Gamma-$semiring M.
Let $a, b \in M, \alpha \in \Gamma$ and $a\alpha b \in \mu_t$.

$$\begin{aligned} &\Rightarrow \mu(a\alpha b) \geq t \\ &\Rightarrow max\{\mu(a), \mu(b)\} \geq t \\ &\Rightarrow \mu(a) \geq t \text{ or } \mu(b) \geq t \\ &\Rightarrow a \in \mu_t \text{ or } b \in \mu_t. \end{aligned}$$

Hence μ_t is a prime ideal of M.

Conversely suppose that μ_t is a prime ideal, for any $t \in [0,1]$.
Let $x, y \in M, \alpha \in \Gamma$ and $min\{\mu(x), \mu(y)\} = t$

$$\begin{aligned} &\Rightarrow \mu(x) \geq t, \mu(y) \geq t \\ &\Rightarrow x, y \in \mu_t \\ &\Rightarrow x+y \in \mu_t \\ \text{Therefore} &\mu(x+y) \geq t = min\{\mu(x), \mu(y)\}. \\ \text{Let } s = &max\{\mu(x), \mu(y)\}. \\ &\Rightarrow \mu(x) = s \text{ or } \mu(y) = s \\ &\Rightarrow x \in \mu_s \text{ or } y \in \mu_s \\ &\Rightarrow x\alpha y \in \mu_s \\ &\Rightarrow \mu(x\alpha y) \geq s = max\{\mu(x), \mu(y)\}. \end{aligned}$$

Therefore μ is a fuzzy ideal of $\Gamma-$semiring. Let $x, y \in M, \gamma \in \Gamma$ and $\mu(xry) = t$.

$$\begin{aligned} &\Rightarrow xry \in \mu_t \\ &\Rightarrow x \in \mu_t \text{ or } y \in \mu_t \\ &\Rightarrow \mu(x) \geq t \text{ or } \mu(y) \geq t \\ &\Rightarrow max\{\mu(x), \mu(y)\} \geq t = \mu(xry) \\ &\Rightarrow max\{\mu(x), \mu(y)\} \geq \mu(xry) \geq max\{\mu(x), \mu(y)\}. \end{aligned}$$

Hence μ is a fuzzy prime ideal of M. □

Theorem 2.5. *Let M be a $\Gamma-$semiring. Then I is a prime ideal of M if and only if the fuzzy subsets χ_I is a fuzzy prime ideal of M.*

Proof. Let I be a prime ideal of $\Gamma-$semiring M. Obviously χ_I is a fuzzy ideal of M.. Let $x, y \in M, \alpha \in \Gamma$ and $x\alpha y \in I$. Since I is a prime ideal, we have $x \in I$ or $y \in I$.

$$\begin{aligned} &\Rightarrow \chi_I(x) = 1 \text{ or } \chi_I(y) = 1. \\ \text{Hence } &\chi_I(x\alpha y) = max\{\chi_I(x), \chi_I(y)\}. \end{aligned}$$

Let $x \in M \setminus I, y \in I, \alpha \in \Gamma$ then $x\alpha y \in I$

$$\Rightarrow \chi_I(x\alpha y) = max\{\chi_I(x), \chi_I(y)\}.$$

Hence χ_I is a fuzzy prime ideal of M.

Suppose χ_I is a fuzzy prime ideal of M. Let $x, y \in M, \alpha \in \Gamma$ such that $x\alpha y \in I$, we have

$$\begin{aligned} \chi_I(x\alpha y) &= max\{\chi_I(x), \chi_I(y)\} \\ \Rightarrow 1 &\leq max\{\chi_I(x), \chi_I(y)\} \\ \Rightarrow &\chi_I(x) = 1 \text{ or } \chi_I(y) = 1 \\ \Rightarrow &x \in I \text{ or } y \in I. \end{aligned}$$

Hence I is a prime ideal of $\Gamma-$semiring M. □

Theorem 2.6. *Let M be a $\Gamma-$semiring and $\phi \neq F \subseteq M$. Then F is a filter of M if and only if the fuzzy subset χ_F is a fuzzy filter of M.*

Proof. Suppose F is a filter of $\Gamma-$semiring M. Obviously χ_F is a non empty fuzzy subset of M and fuzzy $\Gamma-$subsemiring of M. Let $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned} \text{Suppose } x\alpha y \notin F &\Rightarrow x \notin F, y \in F \\ &\Rightarrow \chi_F(x\alpha y) = 0, \chi_F(x) = 0, \chi_F(y) = 1 \\ &\Rightarrow \chi_F(x\alpha y) = min\{\chi_F(x), \chi_F(y)\}. \\ &\text{If } x\alpha y \in F \Rightarrow x \in F \text{ and } y \in F \\ &\quad \chi_F(x\alpha y) = 1,\ \chi_F(x) = \chi_F(y) = 1. \\ \text{Hence } &\chi_F(x\alpha y) = min\{\chi_F(x), \chi_F(y)\}. \end{aligned}$$

Therefore χ_F is a fuzzy filter of M.

Conversely supposes that χ_F is a fuzzy filter of M. Obviously, F is a non empty $\Gamma-$subsemiring of M. Let $x\alpha y \in F, x, y \in M, \alpha \in \Gamma$. Since χ_F is a fuzzy filter of M. We have

$$\begin{aligned} &\chi_F(x\alpha y) = min\{\chi_F(x), \chi_F(y)\}. \\ \Rightarrow &\chi_F(x) = \chi_F(y) = 1 \\ \Rightarrow &x, y \in F. \end{aligned}$$

Hence F is a filter of $\Gamma-$semiring M. □

Theorem 2.7. *If μ is a proper and maximal fuzzy ideal of $\Gamma-$semiring M. Then μ is a fuzzy prime ideal of $\Gamma-$semiring M.*

Proof. Suppose μ is a proper and maximal fuzzy ideal of $\Gamma-$semiring M. Let $t \in [0,1]$ such that μ_t is a proper ideal of M. Let J be an ideal of M such that $\mu_t \subseteq J$. Suppose $J \neq M$. Then there exist $a \in M$ such that $a \notin J$. Therefore $a \notin \mu_t \Rightarrow \mu(a) < t$. Let γ be the fuzzy subset of M defined by $\gamma(x) = \mu(x)$ if $x \neq a, \gamma(a) = t$. Then $\mu \leq \gamma \leq \langle\gamma\rangle$ (The fuzzy ideal generated by γ). This is a contradiction to the fact that, μ is a maximal. Then $J = M$. Therefore μ_t is a maximal ideal $\Rightarrow$ μ_t is a prime ideal.
Let $x, y \in M, \alpha \in \Gamma$ such that $\mu(x\alpha y) = c$

$$\begin{aligned} \Rightarrow &x\alpha y \in \mu_c \\ \Rightarrow &x \in \mu_c \text{ or } y \in \mu_c \\ \Rightarrow &\mu(x) \geq c \text{ or } \mu(y) \geq c \\ \Rightarrow &max\{\mu(x), \mu(y)\} \geq \mu(x\alpha y) \\ \Rightarrow &\mu(x\alpha y) = max\{\mu(x), \mu(y)\} \end{aligned}$$

Hence μ is a fuzzy prime ideal of $\Gamma-$semiring M □

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Generalization of integral inequalities of the type of Hermite-Hadamard through invexity

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Abstract

In this paper, we obtain some inequalities of Hermite-Hadamard type for functions whose derivatives absolute values are prequasiinvex function. Applications to some special means are considered.

Keywords: Hermite-Hadamard inequality; quasi-convex function; power-mean inequality; Holder's integral inequality.

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1 Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamarad inequality, due to its rich geometrical significance and applications, which is stated as in [1]

Let $f : I \subset R \to R$ be a convex function defined on the interval $I$of real numbers and $a, b \in I$, with $a < b$.

Then f satisfies the following well-known Hermite Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

In many areas of analysis applications of Hermite-Hadamard inequality appear for different classes of functions with and without weights; see for convex functions [4,5], [7-10] [18-20].

In [5] Dragomir and Agarwal obtained the following inequalities for differentiable functions which estimate the difference between the middle and the rightmost terms in the above inequality.

Theorem 1. Let $f : I^0 \subseteq R \to R$ is a differentiable function on I^0 $a, b \in I^0$ with $a < b$,and If $|f'|$ is convex function on $[a, b]$,then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

Theorem 2. Let $f : I^0 \subseteq R \to R$ is a differentiable function on I^0 $a, b \in I^0$ with $a < b$,and If $|f'|^{p/(p-1)}$ is convex function on $[a, b]$,then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{1/p}}\left[\left|f'(a)\right|^{p/(p-1)} + \left|f'(b)\right|^{p/(p-1)}\right]^{(p-1)/p}.$$

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In [11], Pearce and J. Pecaric gave an improvement and simplication of the constants in Theorem 2and consolidated this results with Theorem 1. The following is the main result from [11]:

Theorem 3. Let $f : I^0 \subseteq R \to R$ is a differentiable function on I^0 $a, b \in I^0$ with $a < b$,and If $|f'|^q$ is convex function on $[a,b]$,for some fixed $q \geq 1$.then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.$$

If $|f'|^q$ is concave function on $[a,b]$, for some fixed $q \geq 1$.then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left| f'\left(\frac{a+b}{2}\right) \right|.$$

Now we recall that the notion of quasi-convex functions generalized the notion of convex functions. More precisely, a function $f : [a,b] \to R$ is said to be quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda) y) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in [a,b].$$

Clearly, any convex function is a quasi-convex function but the reverse are not true. Because there exist quasi-convex functions which are not convex, (see for example [8]).

Recently, D.A.Ion [8] obtained two inequalities of the right hand side of Hermite-Hadamard's type functions whose derivatives in absolute values are quasi-convex functions, as follows:

Theorem 4. Let $f : I^0 \subseteq R \to R$ is a differentiable function on I^0 $a, b \in I^0$ with $a < b$, and If $|f'|$ is quasi-convex function on $[a,b]$, then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \max\left[|f'(a)|, |f'(b)| \right].$$

Theorem 5. Let $f : I^0 \subseteq R \to R$ is a differentiable function on I^0 $a, b \in I^0$ with $a < b$, and If $|f'|^p$ is quasi-convex function on $[a,b]$, for some fixed $p > 1$. then the following inequality holds:

$$\begin{aligned} &\left| \tfrac{f(a)+f(b)}{2} \tfrac{1}{b-a} \int_a^b g(x)dx - \tfrac{1}{b-a} \int_a^b f(x)g(x)\,dx \right| \\ &\leq \tfrac{b-a}{2(p+1)^{1/p}} \left[\max\left\{ |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)} \right\} \right]^{(p-1)/p}. \end{aligned}$$

In [2] Alomari , Draus and Kirmaci established Hermite-Hadamard inequalities for quasi-convex functioss whose give refinements of those given above in Theorem 4 and Theorem 5.

Theorem 6. Let $f : I^0 \subset [0,\infty) \to R$ be differentiable mapping on I^0 $a, b \in I^0$ with $a < b$,and If $|f'|$ is quasi-convex on $[a,b]$,then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{8} \left[\begin{array}{c} \max\left\{ |f'(a)|, \left| f'\left(\frac{a+b}{2}\right) \right| \right\} \\ +\max\left\{ \left| f'\left(\frac{a+b}{2}\right) \right|, |f'(b)| \right\} \end{array} \right].$$

Theorem 7. Let $f : I^0 \subset [0,\infty) \to R$ be differentiable mapping on I^0 $a, b \in I^0$ with $a < b$,and If $|f'|^q$ is quasi-convex on$[a,b]$, $p > 1$.then the following inequality holds:

$$\begin{aligned} &\left| \tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a} \int_a^b f(x)dx \right| \leq \tfrac{(b-a)}{4} \left(\tfrac{1}{(1+p)} \right)^{1/p} \\ &\times \left[\begin{array}{c} \left(\max\left\{ \left| f'\left(\frac{a+b}{2}\right) \right|^{\frac{p}{p-1}}, |f'(a)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} \\ + \left(\max\left\{ \left| f'\left(\frac{a+b}{2}\right) \right|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} \end{array} \right]. \end{aligned}$$

Theorem 8. Let $f : I^0 \subset [0, \infty) \to R$ be a differentiable mapping on I^0 such that $f' \in L([a,b])$ for $a, b \in I$ with $a < b$,If $|f'(x)|$ is quasi-convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left[\max\left\{|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}^{\frac{1}{q}} + \left\{\max\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right\}^{\frac{1}{q}}\right].$$

Alomari, Darus and Dragomir in [3] introduced the following theorems for twice differentiable quasi-convex functions which are generalizations of Theorems 3, 4 and 5.

Theorem 9. Let $f : I^0 \subseteq R \to R$ is a differentiable function on $I^0 \quad a, b \in I^0$ with $a < b$,and If $|f''|$ is quasi-convex on $[a,b]$,then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\max\left\{\left|f''(a)\right| + \left|f''(b)\right|\right\}.$$

Theorem 10. Let $f : I^0 \subseteq R \to R$ be twice differentiable function on $I^0 \quad a, b \in I^0$ with $a < b$,and If $|f''|^{p/(p-1)}$ is quasi-convex on $[a,b]$,then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\left(\max\left\{|f''(a)|^q + |f''(b)|^q\right\}\right)^{1/q}.$$

Theorem 11. Let $f : I^0 \subseteq R \to R$ be twice differentiable function on $I^0 \quad a, b \in I^0$ with $a < b$,and If $|f''|^q$ is quasi-convex on$[a,b]$, $\quad q \geq 1$.then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left(\max\left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}\right)^{1/q}.$$

Let K be a closed set R^n and let $f : K \to \mathrm{R}$ and $\eta : K \times K \to \mathrm{R}$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(.,.)$,

If $x + t\eta(y,x) \in K, \forall x, y \in K, t \in [0,1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 12. The function f on the invex set $K$is said to be preinvex with respect to η, if

$$f(u + t\eta(v,u)) \leq +(1-t)f(u) + tf(v), \forall u, v \in K, t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x,y) = x - y$ but the converse is not true.

Definition 13. The function f on the invex set K is said to be preinvex with respect to η, if

$$f(u + t\eta(v,u)) \leq \max\{f(u), f(v)\}, \forall u, v \in K, t \in [0,1].$$

Also Every quasi-convex function is a prequasiinvex with respect to the map $\eta(u,v)$ but the converse does not holds, see for example [21].

In the recent paper, Noor [18] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 14. Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a open preinvex function on the interval of real numbers K^0 (the interior of K^0) and $a, b \in K^0$ with $a < a + \eta(b,a)$. the following inequality holds:

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Barani, Ghazanfari and Dragomir in [16], presented the following estimates of the right-side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

Theorem 15. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. If $|f'|$ is preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{\eta(b,a)}{8} \left\{ \left|f'(a)\right| + \left|f'(b)\right| \right\}.$$

Theorem 16. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. Assume $p \in R$ with $p > 1$. If $|f'|^{\frac{p}{(p-1)}}$ is preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right|$$
$$\leq \frac{\eta(b,a)}{2(1+p)^{1/p}} \left\{ \frac{|f'(a)|^{\frac{p}{(p-1)}} + |f'(b)|^{\frac{p}{(p-1)}}}{2} \right\}^{\frac{p-1}{p}}.$$

In [15] Barani, Ghazanfari and Dragomir gave similar results for quasi-preinvex functions as follows:

Theorem 17. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. If $|f'|$ is quasi-preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{\eta(b,a)}{8} \sup \left\{ \left|f'(a)\right|, \left|f'(b)\right| \right\}.$$

Theorem 18. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. Assume $p \in R$ with $p > 1$. If $|f'|^{\frac{p}{(p-1)}}$ is preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right|$$
$$\leq \frac{\eta(b,a)}{2(1+p)^{1/p}} \sup \left\{ |f'(a)|^{\frac{p}{(p-1)}}, |f'(b)|^{\frac{p}{(p-1)}} \right\}^{\frac{p-1}{p}}.$$

The main aim of this paper is to establish new refined inequalities of the right-hand side of Hermite-Hadamard result for the class of functions whose derivatives in absolute values are quasi-preinvex. Then we give some applications for some special means of real numbers.

2 Main results

Before proceeding towards our main theorem regarding generalization of the Hermite-Hadamard inequality using prequasinvex . We begin with the following Lemma.

Lemma 1. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$ and $a, b \in K$ with $a < a + \eta(b,a)$ suppose $f : K \to R$ is a differentiable mapping on K with $a, b \in I^0$ with $a < b, f'' \in L([a, a + \eta(b,a)])$. Then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx$$
$$= \frac{(\eta(b,a))^2}{16} \int_0^1 (1-\lambda^2) \left\{ f''\left(a + \left(\frac{1-\lambda}{2}\right)\eta(b,a)\right) d\lambda + f''\left(a + \left(\frac{1+\lambda}{2}\right)\eta(b,a)\right) d\lambda \right\}.$$

Proof. Integrating by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 (1-\lambda^2) f''\left(a+\left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right) d\lambda \\
&= \frac{2(1-\lambda^2) f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)}{-\eta(b,a)}\Bigg|_0^1 - \frac{2}{\eta(b,a)}\int_0^1 f'\left(a+\left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right) \\
&= \frac{2(1-\lambda^2) f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)}{-\eta(b,a)}\Bigg|_0^1 - \frac{4}{\eta(b,a)}\left[\begin{array}{c} \frac{2\lambda f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)}{-\eta(b,a)} \\ -\frac{2}{\eta(b,a)}\int_0^1 f\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right) \end{array}\right] \\
&= -\frac{2}{\eta(b,a)} f'\left(\tfrac{2a+\eta(b,a)}{2}\right) + \frac{8}{\eta(b,a)} f(a) - \frac{8}{\eta(b,a)}\int_0^1 f\left(a+\left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right)
\end{aligned}$$

Setting $x = a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)$ and $dx = \frac{-\eta(b,a)}{2}d\lambda$ which gives

$$I_1 = \frac{2}{\eta(b,a)} f'\left(\tfrac{2a+\eta(b,a)}{2}\right) + \frac{8}{(\eta(b,a))^2} f(a) - \frac{16}{(\eta(b,a))^3}\int_a^{a+\frac{1}{2}\eta(b,a)} f(x)\,dx$$

Similarly we can show that

$$\begin{aligned}
I_2 &= \int_0^1 (1-\lambda^2) f''\left(a+\left(\tfrac{1+\lambda}{2}\right)\eta(b,a)\right) d\lambda \\
&= -\frac{2}{\eta(b,a)} f'\left(\tfrac{2a+\eta(b,a)}{2}\right) + \frac{8}{(\eta(b,a))^2} f(a+\eta(b,a)) - \frac{16}{(\eta(b,a))^3}\int_{a+\frac{1}{2}\eta(b,a)}^{b} f(x)\,dx
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{(\eta(b,a))^2}{16}[I_1+I_2] \\
&= \frac{(\eta(b,a))^2}{16}\left[\frac{8}{(\eta(b,a))^2}(f(a)+f(a+\eta(b,a))) - \frac{16}{(\eta(b,a))^3}\int_a^b f(x)\,dx\right] \\
&= \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx
\end{aligned}$$

Which completes the proof.

In the following theorem, we shall propose some new upper bound for the right-hand side of Hermite-Hadamard inequality for functions whose second derivatives absolute values are prequasiinvex, which is better than the inequality had done in [3,6].

Theorem A. Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K\times K \to R$ and $a,b \in K$with $a < a+\eta(b,a)$suppose $f : K \to R$ is a differentiable mapping on Ksuch that $f'' \in L([a, a+\eta(b,a)])$. If $|f''|$ is preinvex on K, then for every $a,b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\
&\leq \frac{(\eta(b,a))^2}{24}\left[\begin{array}{c} \sup\left\{|f''(a)|, \left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\} \\ +\sup\left\{\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|, |f''(a+\eta(b,a))|\right\} \end{array}\right].
\end{aligned} \tag{2.1}$$

Proof. From Lemma 1, and Since $|f''|$ is prequasinvex, then we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\
&\leq \frac{(\eta(b,a))^2}{16}\left[\begin{array}{c} \int_0^1 (1-\lambda^2)\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right| d\lambda \\ +\int_0^1 (1-\lambda^2)\left|f''\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right| d\lambda \end{array}\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\eta(b,a))^2}{16}\int_0^1 (1-\lambda^2)\sup\left\{|f''(a)|, \left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right\} d\lambda \\
&+\frac{(\eta(b,a))^2}{16}\int_0^1 (1-\lambda^2)\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|, |f''(a+\eta(b,a))|\right\} d\lambda
\end{aligned}$$

$$\leq \frac{(\eta(b,a))^2}{16}\sup\left\{|f''(a)|,\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right\}\int_0^1(1-\lambda^2)\,d\lambda$$
$$+\frac{(\eta(b,a))^2}{16}\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,|f''(a+\eta(b,a))|\right\}\int_0^1(1-\lambda^2)d\lambda$$
$$=\frac{(\eta(b,a))^2}{24}\sup\left\{|f''(a)|,\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right\}$$
$$+\frac{(\eta(b,a))^2}{24}\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,|f''(a+\eta(b,a))|\right\}.$$

which completes the proof.

Corollary 1. Let f be defined as in Theorem A, if in addition

1. $|f''|$ is increasing, then we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leq \frac{(\eta(b,a))^2}{24}\left[|f''(a+\eta(b,a))|+\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right]. \tag{2.2}$$

2. $|f''|$ is decreasing, then we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leq \frac{(\eta(b,a))^2}{24}\left[|f''(a)|+\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right]. \tag{2.3}$$

Remark 2.1. we note that the inequalities (2.2) and (2.3) are two new refinements of the trapezoid inequality for quasipreinvex functions, and thus for convex functions.

Observation 1. If we take $\eta(b,a)=b-a$ in Theorem A, then inequality reduces to the [Theorem 2.1, 6]. If we take $\eta(b,a)=b-a$ in corollary 1, then (2.2) and (2.3) reduce to the related corollary of Theorem 2.1 from [6].

The corresponding version for powers of the absolute value of the second derivative is incorporated in the following result:

Theorem B. Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta: K\times K\to R$ and $a,b\in K$ with $a<a+\eta(b,a)$ suppose $f:K\to R$ is a differentiable mapping on Ksuch that $f''\in L([a,a+\eta(b,a)])$. If $|f''|^p$ is preinvex on K,from some $p>1$, then for every $a,b\in K$ with $\eta(b,a)\neq 0$ the following inequality holds:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq \frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}}\left(\sup\left\{|f''(a)|^{\frac{p}{p-1}},\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} \tag{2.4}$$
$$+\frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}}\left(\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}},|f''(a+\eta(b,a))|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$

Where $q=p/(p-1)$.

Proof . From Lemma 1, and using the well known Holder integral inequality, we

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq \frac{(\eta(b,a))^2}{16}\int_0^1(1-\lambda^2)\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda$$
$$+\frac{(\eta(b,a))^2}{16}\int_0^1(1-\lambda^2)\left|f''\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda$$

$$\leq \frac{(\eta(b,a))^2}{16}\left(\int_0^1(1-\lambda^2)^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^{\frac{p}{p-1}}d\lambda\right)^{\frac{p-1}{p}}$$
$$+\frac{(\eta(b,a))^2}{16}\left(\int_0^1(1-\lambda^2)^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^{\frac{p}{p-1}}d\lambda\right)^{\frac{p-1}{p}}$$

$$\leq \frac{(\eta(b,a))^2}{16}\left(\int_0^1 (1-\lambda^2)^p\right)^{\frac{1}{p}}\left(\int_0^1 \sup\left\{|f''(a)|^{\frac{p}{p-1}}, \left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}d\lambda\right)^{\frac{p-1}{p}}$$
$$+\frac{(\eta(b,a))^2}{16}\left(\int_0^1 (1-\lambda^2)^p\right)^{\frac{1}{p}}\left(\int_0^1 \sup\left\{\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}, |f''(a+\eta(b,a))|^{\frac{p}{p-1}}\right\}d\lambda\right)^{\frac{p-1}{p}}$$
$$=\frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}}\left(\sup\left\{|f''(a)|^{\frac{p}{p-1}}, \left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$
$$+\frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}}\left(\sup\left\{\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}, |f''(a+\eta(b,a))|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$

Which completes the proof.

Corollary 2. Let f be defined as in Theorem B, if in addition

1. $|f''|^{p/p-1}$ is increasing, then we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}} \times\left[|f''(a+\eta(b,a))|+\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right]. \tag{2.5}$$

2. $|f''|^{p/p-1}$ is decreasing, then we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}} \times\left[|f''(a)|+\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right]. \tag{2.6}$$

Observation 2. If we take $\eta(b,a)=b-a$ in Theorem B, then inequality reduces to the [Theorem 2.2, 6]. If we take $\eta(b,a)=b-a$ in corollary 2, then (2.5) and (2.6) reduce to the related corollary of Theorem 2.2 from [6].

Theorem C. Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta: K\times K\to R$and $a,b\in K$with $a<a+\eta(b,a)$suppose $f:K\to R$is a differentiable mapping on Ksuch that $f''\in L([a,a+\eta(b,a)])$. If $|f''|^q$ is preinvex on K, $q\geq 1$, then for every $a,b\in K$ with $\eta(b,a)\neq 0$the following inequality holds:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{(\eta(b,a))^2}{16}\left(\sup\left\{|f''(a)|^q, \left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}} +\frac{(\eta(b,a))^2}{16}\left(\sup\left\{\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q, |f''(a+\eta(b,a))|^q\right\}\right)^{\frac{1}{q}}. \tag{2.7}$$

Proof . Suppose that$q\geq 1$. From Lemma 1 and using the well known power mean inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \frac{(\eta(b,a))^2}{16}\int_0^1 (1-\lambda^2)\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda$$
$$+\frac{(\eta(b,a))^2}{16}\int_0^1 (1-\lambda^2)\left|f''\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda$$
$$\leq \frac{(\eta(b,a))^2}{16}\left(\int_0^1 (1-\lambda^2)d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^q d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(\eta(b,a))^2}{16}\left(\int_0^1 (1-\lambda^2)d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|f''\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|^q d\lambda\right)^{\frac{1}{q}}$$

Since $|f''|^q$ is quasi-preinvexity, we have

$$\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left(|f''(a)|^q, \left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right)$$

$$\left|f''\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left(\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q, |f''(a+\eta(b,a))|^q\right)$$

$$\begin{aligned}&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \tfrac{(\eta(b,a))^2}{16}\left(\sup\left\{|f''(a)|^q, \left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ &+\tfrac{(\eta(b,a))^2}{16}\left(\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q, |f''(a+\eta(b,a))|^q\right\}\right)^{\frac{1}{q}}.\end{aligned}$$

Which completes the proof.

Corollary 3. Let f be defined as in Theorem C, if in addition

1. $|f''|^{p/p-1}$ is increasing, then we have

$$\begin{aligned}&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \tfrac{(\eta(b,a))^2}{16}\left[|f''(a+\eta(b,a))| + \left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right].\end{aligned} \tag{2.8}$$

2. $|f''|^{p/p-1}$ is decreasing, then we have

$$\begin{aligned}&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \tfrac{(\eta(b,a))^2}{16}\left[|f''(a)| + \left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right].\end{aligned} \tag{2.9}$$

Observation 3. If we take $\eta(b,a) = b-a$ in Theorem C, then inequality reduces to the [Theorem 2.3, 6]. If we take $\eta(b,a) = b-a$ in corollary 3, then (2.8) and (2.9) reduce to the related corollary of Theorem 2.3 from [6].]

3 Application to some special means

In what follows we give certain generalization of some notions for a positive valued function of a positive variable.

Definition 3[14]. A function $M : R \to R$, is called a mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x,y)$,for all $a > 0$,
2. Symmetry: $M(x,y) = M(x,y)$,
3. Reflexivity: $M(x,x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x,y) = M(x',y')$,
5. Internality: $\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}$.

We consider some means for arbitrary positive real numbers a, b(see for instance [14]).

We now consider the applications of our theorem to the special means.

The Arithmetic Mean;

$$A := A(a,b) = \frac{a+b}{2}$$

The Geometic Mean;

$$G := G(a,b) = \sqrt{ab}$$

The Power Mean;

$$P_r := P_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}} \quad , \quad r \geq 1,$$

The Indentric Mean:

$$I = I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right) \ , & if a \neq b \\ a \ , & if a = b \end{cases}$$

The Harmonic Mean:

$$H := H(a,b) = \frac{2ab}{a+b} \quad ,$$

The Logarithmic Mean:

$$L = L(a,b) = \frac{a-b}{\ln|a| - \ln|b|}, \quad |a| \neq |b|$$

The *p*- Logarithmic Mean:

$$L_p \equiv L_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right], \quad a \neq b$$

$p \in \Re \setminus \{-1, 0\}$: $a, b > 0$.

It is well known that L_P is monotonic nondecreasing over $p \in$ R with $L_{-1} := L$and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A.$$

Now let a and b be positive real numbers such that $a < b$. consider the function $a < b$. $M : M(b,a) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \rightarrow R$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

$\eta(b,a) = M(b,a)$in (2.1), (2.4) and (2.7), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} &\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)dx\right| \\ &\leq \frac{(M(b,a))^2}{24} \left[\begin{array}{l} \sup\left\{|f''(a)|, \left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|\right\} \\ +\sup\left\{\left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|, |f''(a+M(b,a))|\right\} \end{array}\right]. \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)dx\right| \\ &\leq \frac{(M(b,a))^2}{16}\left(\frac{\sqrt{x}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}} \\ &\times\left(\sup\left\{|f''(a)|^{\frac{p}{p-1}}, \left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} \\ &+\frac{(M(b,a))^2}{16}\left(\frac{\sqrt{x}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}} \\ &\times\left(\sup\left\{\left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}}, |f''(a+M(a,b))|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)dx\right| \\ &\leq \frac{(M(b,a))^2}{16}\left(\sup\left\{|f''(a)|^q, \left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ &+\frac{(M(b,a))^2}{16}\left(\sup\left\{\left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|^q, |f''(a+M(b,a))|^q\right\}\right)^{\frac{1}{q}}. \end{aligned} \tag{3.12}$$

For $q \geq 1$.Letting $M = A, G, P_r, I, H, L, L_p$ in (3.10), (3.11) and (3.12), we can get the required inequalities, and the details are left to the interested reader.

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A new generalized vector-valued paranormed sequence space using modulus function

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Abstract

In this paper we introduce a new generalized vector-valued paranormed sequence spaces $N_p(E_k, \triangle_u^m, f, s)$ using modulus function f, where $p = (p_k)$ is a bounded sequence of positive real numbers such that $\inf_k p_k > 0$, (E_k, q_k) is a sequence of seminormed spaces with $E_{k+1} \subseteq E_k$ for each $k \in N$ and $s \geq 0$. We prove results regarding completeness, K-space, normality, inclusion relation are derived. These are more general than those of Ruckle [7], Maddox [5], Ozturk and Bilgin [6], Sahiner [8], Atlin *et al.* [1] and Srivastava and Kumar [9].

Keywords: Modulus function, paranormed space, normal sequence space, difference sequence space.

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1 Introduction

Let ω denote the space of all complex sequences. Kizmaz [4] studied the sequence space

$$X(\triangle) = \{x = (x_k) : \triangle x \in X\}, \text{ for } X = l_\infty, c, c_0,$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})$ and shown that these sequence spaces are Banach spaces with the norm

$$||x||_\triangle = |x|_1 + ||\triangle x||_\infty, \ x \in X(\triangle).$$

The sequence spaces $X(\triangle^m) = \{x = (x_k) : \ \triangle^m x \in X\}$ for $X = l_\infty, c$ and c_0 are introduced by Et. and Colak [2]. These sequence spaces are BK-spaces with norm

$$||x||_\triangle = \sum_{i=0}^{m} |x_i| + ||\triangle^m x||_\infty, x \in X(\triangle^m) \text{ where } m \in N.$$

Tripathy and Esi [10] introduced the difference operator $\triangle_u$, $u \geq 1$ and defined the sequence spaces

$$X(\triangle_u) = \{x = (x_k) : \triangle_u x \in X)\} \text{ for } X = l_\infty, c \text{ and } c_0 \text{ and } \triangle_u x = (\triangle_u x_k) = (x_k - x_{k+u}).$$

They proved that the above sequence spaces are Banach spaces and BK spaces with respect to the norm

$$\| x \|_{\triangle_u} = \sum_{r=1}^{u} | x_r | + \| \triangle^u x \|_\infty .$$

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Ruckle [7] constructed the sequence spaces $L(f) = \{x = (x_k) \in \omega : \sum_{k=1}^{\infty} f(| x_k |) < \infty\}$ using the idea of Modulus function f. He proved that $L(f)$ is BK space. Maddox [5] introduced the class of sequences which are strongly Cesaro summable with respect to the modulus function by

$$w_0(f) = \{x = (x_k) \in \omega : \frac{1}{n} \sum_{k=1}^{n} f(| x_k |) \longrightarrow 0 \ , as \ n \longrightarrow \infty\}.$$

Ozturk and Bligin [6] generalized the sequence spaces as

$$w_0(f, P) = \{x = (x_k) \in \omega : \frac{1}{n} \sum_{k=1}^{n} [f(| x_k |)]^{p_k} \longrightarrow 0 \ , as \ n \longrightarrow \infty\}.$$

where $p = (p_k)$ is a bounded sequence of positive real numbers.

Sahiner [8] introduced the sequence spaces

$$B_g(p, f, q, s) = \left\{ x = (x_k) \in \omega(X) : \sum_{k=1}^{\infty} k^{-s}[f(q(\triangle x_k))]^{p_k} < \infty, \ s \geq 0 \right\},$$

and

$$B_g(p, f^r, q, s) = \left\{ x = (x_k) \in \omega(X) : \sum_{k=1}^{\infty} k^{-s}[f^r(q(\triangle x_k))]^{p_k} < \infty, \ s \geq 0 \right\},$$

where $r \in N$ and (X, q) is a seminormed complex linear space.

Altin *etal*. [1] generalized the sequence space $B_g(p, f, q, s)$ as

$$l(\triangle^m, f, p, q, s) = \left\{ x = (x_k) \in \omega(X) : \frac{1}{n} \sum_{k=1}^{\infty} k^{-s}[f(q(\triangle^m x_k))]^{p_k} < \infty, \ s \geq 0 \right\}.$$

Srivastave and Kumar [9] introduced a new vector valued sequence space $N_p(E_k, \triangle^m, f, s)$ where

$$N_p(E_k, \triangle^m, f, s) = \{x = (x_k) \in \omega(E_k) : (| v_k |^{-(s/p_k)} \ f(q_k(\triangle^m x_k))) \in N_p, \ s \geq 0\},$$

where (E_k, q_k) is a sequence of seminormed spaces such that $E_{k+1} \subseteq E_k$ for each $k \in N$, $w(E_k) = \{x = (x_k) : \ x_k \in E_k, \ \text{for each } k \in N\}, v = (v_k)$ is a sequence of real complex numbers such that $1 \leq | v_k | < \infty$ for each $k \in N$ and N_p is normal AK sequence space with absolutely monotonic paranorm g_{N_p}.

Let $u, m \geq 0$ be fixed integers then we introduce the following new type of Generalized paranormed vector valued sequence space which unifies some earlier cases as particular cases:

$$N_p(E_k, \triangle_u^m, f, s) = \{x = (x_k) \in \omega(E_k) : (| v_k |^{-(s/p_k)} \ f(q_k(\triangle_u^m x_k))) \in N_p, \ s > 0\},$$

where $p = (p_k)$ is a bounded sequence of positive real numbers such that $\inf_k p_k > 0$ and

$$\triangle_u^m x_k = \sum_{\nu=0}^{m} (-1)^{\nu} \begin{pmatrix} m \\ v \end{pmatrix} x_k + u\nu, \ \ \text{for all} k \in N.$$

1.1.1 Particular Cases:

(i) For $E_k = C$ for each $k \in N$, $m = 0$, $u = 1$, $s = 0$ and $N_p = l_1$, (where $p_k = 1$ for each $k \in N$), space $N_p(E_k, \triangle_u^m, f, s)$ reduces to $L(f)$ of Ruckle [7].

(ii) For $E_k = C$ for each $k \in N$, $m = 0$, $u = 1$, $s = 0$ and $N_p = \omega_0$, (where $p_k = 1$ for each $k \in N$), space $N_p(E_k, \triangle_u^m, f, s)$ reduces to $\omega_0(f)$ of Maddox [5].

(iii) For $E_k = C$ for each $k \in N$, $m = 0$, $u = 1$, $s = 0$ and $N_p = \omega_0(p)$, space, $N_p(E_k, \triangle_u^m, f, s)$ reduces to $\omega_0(f, p)$ of Ozturk and Bilgin [6].

(iv) For $u = 1$, the space $N_p(E_k, \triangle_u^m, f, s)$ reduces to $N_p(E_k, \triangle^m, f, s)$ of Srivastave and Kumar [9].

(v) For $E_k = X$, for each $k \in N$, $v_k = k$, $m = 1$ and $u = 1$ and $N_p = l_p$, the space $N_p(E_k, \triangle_u^m, f, s)$ reduces to $B_g(p, f, q, s)$ of Sahiner [8].

(vi) For $E_k = X$, for each $k \in N$, $v_k = k$, $u = 1$ and $N_p = l_p$, the space $N_p(E_k, \triangle_u^m, f, s)$ reduces to $l(\triangle^m, f, q, s)$ of Altin *etal*. [1].

Thus study of the space $N_p(E_k, \triangle_u^m, f, s)$ gives a unified approach to many of the earlier known spaces.

2. Some Definitions and Lemmas

Definition 2.1[3]. A sequence space X is called normal space if $x = (x_k) \in X$ and $|\lambda_k| \leq 1$ for each $k \epsilon N$. This implies $\lambda x = (\lambda_k x_k) \epsilon X$.
For example, $l(p), c_0(p), \omega(p)$ are normal space.

Definition 2.2[3]. A sequence space X is called K space if the co-ordinate function $p_k : X \to K$ given by $p_k(x) = x_k$ is continuous for each $k \in N$.

Definition 2.3. A complete metric space is called Frechet space. An FK-space is a Frechet space with continuous co-ordinates.

Definition 2.4[9]. An FK-space X is said to be AK-space if $\Phi \subset X$ and $\{\delta^n\}$ is a basis for X, i.e., for each $x, x^{[n]} \to x$, where $x^{[n]}$ denotes the nth section of x. For example, $l(p), c_0(p), \omega(p)$ are AK-spaces.

Definition 2.5[3]. A paranorm g on a normal sequence space X is said to be absolutely monotone if

$$x = (x_k), y = (y_k) \epsilon X \text{ and } |x_k| \leq |y_k| \text{ for each } k \epsilon N \Longrightarrow g(x) \leq g(y).$$

Lemma 2.1[8]. If f is a modulus function, then f^r is also modulus function for each $r \epsilon N$, where $f^r = f \circ f \circ f \circ \cdots \circ f$($r$-times composition of f with itself).

Lemma 2.2[5]. There is a modulus function f such that $f(xy) \leq f(x) + f(y)$ for $x, y \geq 0$.

Lemma 2.3[5]. Let f_1 and f_2 be modulus functions and $0 < \delta < 1$. If $f_1(t) > \delta$ for $t \in [0, \infty)$, then

$$(f_2 \circ f_1)(t) < \left(\frac{2 f_2(1)}{\delta}\right) f_1(t).$$

3. Results on Sequence Space $N_p(E_K, \triangle_u^m, f, s)$.

Theorem:. $N_p(E_K, \triangle_u^m, f, s)$ is a linear space.

Proof. It is easy to show that $N_p(E_K, \triangle_u^m, f, s)$ is a linear space. So we omit proof.

Lemma 3.1. Let (E_k, q_k) be a sequence of seminormed spaces, and N_p be normal AK-sequence space with absolutely monotone paranaorm g_{N_p}. Then function defined by

$$\tilde{f}_n : [0, \infty) \to [0, \infty),\ \tilde{f}_n(t) = g_{N_p}[\sum_{k=1}^{n} |v_k|^{-(s/p_k)} f(t q_k(\triangle_u^m x_k) e_k]$$

is continuous function of t for each positive integer n, where $x = (x_k) \epsilon N_p(E_K, \triangle_u^m, f, s)$ and (e_k) is unit vector basis of N_p.

Proof. We define function $g_k : [0, \infty) \to N_p$ by

$$g_k(t) = |v_k|^{-(s/p_k)} f(t q_k(\triangle_u^m x_k) e_k.$$

Let $t_i \to 0$ as $i \to \infty$. Then for each $k = 1, 2, 3, ..., n$;

$$g_k(t_i) = |v_k|^{-(s/p_k)} f(t_i q_k(\triangle_u^m x_k))e_k \to (0, 0, \cdots) \ \ as \ \ i \to \infty.$$

Therefore,

$$\sum_{k=1}^{n} g_k(t_i) = \sum_{k=1}^{n} |v_k|^{-(s/p_k)} f(t_i q_k(\triangle_u^m x_k)e_k \to (0, 0, \cdots) \ \ as \ \ i \to \infty.$$

But paranorm g_{N_p} is continuous function, it follows that

$$g_{N_p}[\sum_{k=1}^{n} g_k(t_i)] \to 0 \text{as} i \to \infty.$$

Hence function $\tilde{f}_n$ is continuous function of t for each positive integer n.

Theorem 3.2. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is a paranormed space with paranorm

$$g(x) = \sum_{i=1}^{m} f(q_i(x_i)) + g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k)))\right], where \ \ x \in N_p(E_K, \triangle_u^m, f, s).$$

Proof: By definition of $g, g(x) \geq 0$ for any $x = (x_k) \in N_p(E_K, \triangle_u^m, f, s)$. It is clear that $g(0) = 0$, $g(x) = g(-x)$ and $g(x+y) \leq g(x) + g(y)$ for any $x, y \in N_p(E_K, \triangle_u^m, f, s)$. It is left to prove the continuity of scaler multiplication under g. Suppose $x^n \to x$ as $n \to \infty$ in $N_p(E_K, \triangle_u^m, f, s)$ and $\alpha_n \to \alpha$ as $n \to \infty$ in C. We have to show that $g(\alpha_n x^n - \alpha x) \to 0$ as $n \to \infty$. Consider

$$g(\alpha_n x^n - \alpha x) = \sum_{i=1}^{m} f(q_i(\alpha_n x_i^n - \alpha x_i)) + g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(\alpha_n x_k^n - \alpha x_k))))\right]$$

$$= \sum_{i=1}^{m} f(q_i(\alpha_n x_i^n - \alpha_n x_i + \alpha_n x_i - \alpha x_i))$$

$$+ g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(\alpha_n x_k^n - \alpha_n x_k + \alpha_n x_k - \alpha x_k))))\right]$$

$$\leq \sum_{i=1}^{m} f(|\alpha_n| q_i(x_i^n - x_i) + |\alpha_n - \alpha| q_i(x_i))$$

$$+ g_{N_p}\left[(|v_k|^{-(s/p_k)} f(|\alpha_n| q_k(\triangle_u^m(x_k^n - x_k)) + |\alpha_n - \alpha| q_k(\triangle_u^m x_k))))\right].$$

This gives,

$$g(\alpha_n x^n - \alpha x) \leq M\left(\sum_{i=1}^{m} f(q_i(x_i^n - x_i) + g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k)))\right.\right)$$

$$+ \sum_{i=1}^{m} f(|\alpha_n - \alpha| q_i(x_i)) + g_{N_p}\left[(|v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m x_k)))\right],$$

where $M = \sup_n(1 + [|\alpha_n|])$, this gives,

$$g(\alpha_n x^n - \alpha x) \leq Mg(x^n - x) + \sum_{i=1}^{m} f(|\alpha_n - \alpha| q_i(x_i))$$

$$+ g_{N_p}\left[(|v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m x_k)))\right]. \quad (3.1)$$

First and second expressions of R.H.S in (3.1) tend to zero as $x^n \to x$ as $n \to \infty$ in $N_p(E_K, \triangle_u^m, f, s)$ and $\alpha_n \to \alpha$ as $n \to \infty$. We must only show that

$$g_{N_p}\left[(|v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k^n - x_k)))\right] \to 0 \text{as} n \to \infty.$$

Since $(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)))\epsilon N_p$ is AK-sequence space, therefore

$$g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k))) - \sum_{k=1}^{m} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)e_k\right] \to 0 \;\; asm \to \infty.$$

That is $g_{N_p}\left[\sum_{k=m+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k))e_k\right] \to 0$ as $m \to \infty$.
Therefore, for every $\epsilon > 0$ there exists a positive integer m_0 such that

$$g_{N_p}\big[\sum_{k=m+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)))e_k\big] < \epsilon/2, \quad \text{for all } m \geq m_0.$$

In particular

$$g_{N_p}\big[\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)))e_k\big] < \epsilon/2. \tag{3.2}$$

Since $\alpha_n \to \alpha$ as $n \to \infty$, therefore, for $\epsilon = 1$, there exists a positive integer n_0' such that $|\alpha_n - \alpha| < 1$ for all $n \geq n_0'$. Consequently

$$\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k$$
$$\leq \sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k))e_k \text{ for all } n \geq n_0'.$$

But g_{N_p} is monotone paranorm, it follows that for all $n \geq n_0'$.

$$g_{N_p}\big[\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k\big]$$
$$\leq g_{N_p}\left[\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k))e_k\right].$$

Using inequality (3.2), for all $n \geq n_0'$

$$g_{N_p}\big[\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k\big] \leq \epsilon/2. \tag{3.3}$$

By Lemma 3.1, function

$$\tilde{f}_{m_0}(t) = g_{N_p}\big[\sum_{k=1}^{m_0} |v_k|^{-(s/p_k)} f(tq_k(\triangle_u^m(x_k))e_k\big], \;\; t \geq 0$$

is continuous function of t. Hence there exists $\delta \in (0,1)$ such that

$$\tilde{f}_{m_0}(t) < \epsilon/2, \text{ whenever } t < \infty.$$

Again, since $\alpha_n \to \alpha$ as $n \to \infty$, therefore for $\delta \in (0,1)$, there exist a positive integer n_0'' such that

$$|\alpha_n - \alpha| < \delta \text{ for all } n \geq n_0'' \text{ we have } \tilde{f}_{m_0}(|\alpha_n - \alpha|) < \epsilon/2, \text{ for all } n \geq n_0''$$

that is

$$g_{N_p}\left[\sum_{k=1}^{m_0} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k\right] < \epsilon/2 \text{ for all } n \geq n_0'' \tag{3.4}$$

We take $n_0 = \max(n_0', n_0'')$. Using inequality (3.3)and (3.4), we have

$$g_{N_p}\left[|v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))\right]$$
$$\leq g_{N_p}\left[\sum_{k=1}^{m_0} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k\right]$$

$$+ g_{N_p}\left[\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k)) e_k\right]$$
$$< \epsilon/2 + \epsilon/2 = \epsilon, \text{ for all } n \geq n_0.$$

From inequality (3.1), $g(\alpha_n x_n - \alpha x) \to 0$ as $n \to \infty$. Hence $N_p(E_K, \triangle_u^m, f, s)$ is a paranormed sequence space.

Remark 3.1. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is not totally paranormed space.
Let $g(x) = 0 \Longrightarrow \sum_{i=1}^{m} f(q_i(x_i)) + g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)))\right] = 0 \Longrightarrow q_i(x_i) = 0 \ \ foreach \ \ i = 1, 2, ..., m$ and

$$g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k))\right] = 0.$$

But

$$q_i(x_i) = 0$$

does not mean $x_i = 0$ as q_i is seminorm on E_i. Hence g is not total paranormed on space $N_p(E_K, \triangle_u^m, f, s)$.

Theorem 3.3. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is a K-space if N_p is a K-space.

Proof. We have to show the coordinate function $P_k : N_p(E_K, \triangle_u^m, f, s) \to E_k$ given by $P_k(x) = x_k$, where $x \in N_p(E_K, \triangle_u^m, f, s)$ is continuous for each $k \in N$.
Let (x^n) be any sequence in $N_p(E_K, \triangle_u^m, f, s)$ such that $x^n \to 0$ as $n \to \infty$ in $N_p(E_K, \triangle_u^m, f, s)$. That is

$$\sum_{i=1}^{mu} f(q_i(x_i^n)) + g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n)))\right] \to 0 \text{ as } n \to \infty.$$

This means that

$$f(q_i(x_i^n)) \to 0 \text{ as } n \to \infty \text{ for each } i = 1, 2, ..., m,$$

and

$$g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n)))\right] \to 0 \text{ as } n \to \infty \qquad (3.5).$$

Since N_p is a K-space, therefore for each k

$$|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n)) \to 0 \text{ as } n \to \infty,$$

that is $f(q_k(\triangle_u^m(x_k^n)) \to 0$ as $n \to \infty$. Thus for any $\delta > 0$, there exist $n_0 \in N$ such that $f(q_k(\triangle_u^m(x_k^n)) < \delta$ for all $n \geq n_0$. Let $\delta = f(\epsilon)$, where $\epsilon > 0$. Then

$$f(q_k(\triangle_u^m(x_k^n)) < f(\epsilon) \text{ for all } n \geq n_0 \to q_k(\triangle_u^m(x_k^n) < \epsilon \text{ for all } n \geq n_0.$$

This shows that for each $k, \triangle_u^m(x_k^n) \to 0$ in E_k as $n \to \infty$. By condition (3.5), $f(q_i(x_i^n)) \to 0$ as $n \to \infty$ for each $i = 1, 2, ..., m$. But f is modulus function, it follows that $x_i^n \to 0$ in E_i as $n \to \infty$ for each $i = 1, 2, ..., m$. Now $x_i^n \to 0$ in E_i as $n \to \infty$ for each $i = 1, 2, ..., m$ and $\triangle_u^m x_k^n \to 0$ in E_i as $n \to \infty$ for each $k \in N$. This implies that $x_k^n \to 0$ in E_k as $n \to \infty$ for each $k \in N$. Thus, coordinate wise function P_k is continuous for each $k \in N$. Hence $N_p(E_K, \triangle_u^m, f, s)$ is a K-space.

Theorem 3.4. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is a complete paranormed space under the paranorm g defined by

$$g(x) = \sum_{i=1}^{m} f(q_i(x_i)) + g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)))\right], \text{where } x \in N_p(E_K, \triangle_u^m, f, s),$$

if N_p is a K-space and (E_k, q_k) is a sequence of complete seminormed spaces.

Proof. Clearly $N_p(E_K, \triangle_u^m, f, s)$ is a paranormed space under g. To show that it is complete, Let $(x^n) = ((x_k^n)_k)$ be a Cauchy sequence in $N_p(E_K, \triangle_u^m, f, s)$.Then $g(x^n - x^t) \to 0$ as $n, t \to \infty$.

That is
$$\sum_{i=1}^{m} f(q_i(x_i^n - x_i^t)) + g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k^t)))\right] \to 0 \text{ as } n, t \to \infty.$$
This means that
$$f(q_i(x_i^n - x_i^t) \to 0 \text{ as } n, t \to \infty \text{ for each } i = 1, 2, ..., m,$$
and
$$g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k^t)))\right] \to 0 \text{ as } n, t \to \infty. \tag{3.6}$$
Since N_p is a K-space, therefore for each k,
$$|v_k|^{-(s/p_k)} f\left(q_k(\triangle_u^m(x_k^n - x_k^t))\right) \to 0 \text{ as } n, t \to \infty.$$
that is
$$f(q_k(\triangle_u^m(x_k^n - x_k^t))) \to 0 \text{ as } n, t \to \infty.$$
Thus for any δ positive, there exists $n_0 \in N$ such that
$$f(q_k(\triangle_u^m(x_k^n - x_k^t))) < \delta \text{ for all } n, t \geq n_0.$$
Let $\delta = f(\epsilon)$, where $\epsilon > 0$. Then
$$f(q_k(\triangle_u^m(x_k^n - x_k^t))) < f(\epsilon) \text{ for all } n, t \geq n_0.$$
This implies
$$(q_k(\triangle_u^m(x_k^n - x_k^t)) < \epsilon \text{ for all } n, t \geq n_0.$$
This shows that for each k, $(\triangle_u^m(x_k^n)$ is a Cauchy sequence in E_k. By condition (3.6), $f(q_i(x_i^n - x_i^t)) \to 0$ as $n, t \to \infty$, for each $i = 1, 2, ..., m$. But f is a modulus function, it follows that (x_i^n) is Cauchy sequence in E_i for each $i = 1, 2, ..., m$.

Now (x_i^n) is Cauchy sequence in E_i for each $i = 1, 2, ..., m$ and $(\triangle_u^m x_k^n)$ is Cauchy sequence in E_k for each $k \in N$. This implies that x_k^n is a cauchy sequence in E_k for each $k \in N$. Since each E_k is complete, so sequence (x_k^n) is convergent for each $k \in N$. Let $\lim_n x_k^N = x_k$ for each $k \in N$. Since (x^n) is Cauchy sequence therefore for each $\epsilon > 0$, there exists n_0 such that $g(x^n - x^t) < \epsilon$ for all $n, t \geq n_0$. So we have
$$\lim_t \sum_{i=1}^{m} f(q_i(x_i^n - x_i^t)) = \sum_{i=1}^{m} f(q_i(x_i^n - x_i)) < \epsilon$$
and
$$\begin{aligned}
&\lim_t g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k^t)))\right] \\
&= g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k)))\right] < \epsilon \text{ for all } n \geq n_0.
\end{aligned}$$
This implies that $g(x^n - x) < 2\epsilon$ for all $n \geq n_0$ that is $x^n \to x$ as $n \to \infty$ in $N_p(E_K, \triangle_u^m, f, s)$.

Next we will show that $x \in N_p(E_K, \triangle_u^m, f, s)$. Let $a_k^n = |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k))$. Then for each $k, a_k^n \to 0$ as $n \to \infty$, since f is continuous function. We choose δ_k^n with $0 < \delta_k^n < 1$ such that $a_k^n < \delta_k^n |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n))$. But $(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n))) \in N_p$ for each n. so for each $n, a^n = (a_k^n) \in N_p$. Again,
$$\begin{aligned}
|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k)) &= |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k - x_k^n))) \\
&\leq |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k))) + |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n)) \\
&< (1 + \delta_k^n)|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n)).
\end{aligned}$$
This implies,
$$|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k)) \leq M_n |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n)),$$
where $M_n = \sup_k(\delta_k^n + 1)$.
But N_p is normal sequence space, it follows that $|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k))) \in N_p$, that is $x \in N_p(E_K, \triangle_u^m, f, s)$.

Hence $N_p(E_K, \triangle_u^m, f, s)$ is a complete paranormed space.

Theorem 3.5. Let f, f_1, f_2 be modulus functions, (E_k, q_k) be a sequence of seminormed spaces and $s, s_1, s_2 \geq 0$.Then

$$(i) N_p(E_K, \triangle_u^m, f_1, s) \cap N_p(E_K, \triangle_u^m, f_2, s) \subseteq N_p(E_K, \triangle_u^m, f_1 + f_2, s),$$

$$(ii) N_p(E_K, \triangle_u^m, f, s_1) \subseteq N_p(E_K, \triangle_u^m, f, s_2), \;\; if \;\; s_1 \leq s_2$$

and

$$(iii) N_p(E_K, \triangle_u^m, f_1, s) \subseteq N_p(E_K, \triangle_u^m, f_2 \circ f_1, s), \;\; if \;\; (|v_k|^{-(s/p_k)}) \in N_p.$$

Proof. It is easy to prove (i) and (ii) part of the above theorem. So consider the third one,

(iii) Let $x \in N_p(E_K, \triangle_u^m, f_1, s)$. Then $(|v_k|^{-(s/p_k)} f_1(q_k(\triangle_u^m x_k))) \in N_p$. We choose δ such that $\delta \in (0,1)$ and define sets
$G_1 = \{k \in N : f_1(q_k(\triangle_u^m x_k)) \leq \delta\}$ and $G_2 = \{k \in N : f_1(q_k(\triangle_u^m x_k)) > \delta\}$.
If $k \in G_1$, then $(|v_k|^{-(s/p_k)}(f_2 \circ f_1)(q_k(\triangle_u^m x_k)) < |v_k|^{-(s/p_k)} f_2(\delta)$. Again if $k \in G_2$ then by Lemma 2.3

$$|v_k|^{-(s/p_k)}(f_2 \circ f_1)(q_k(\triangle_u^m x_k)) < |v_k|^{-(s/p_k)} \left(\frac{2f_2(1)}{\delta}\right) f_1(q_k(\triangle_u^m x_k)).$$

Therefore for any $k \in G_1 \cup G_2 = N$,

$$|v_k|^{-(s/p_k)}(f_2 \circ f_1)(q_k(\triangle_u^m x_k)) < |v_k|^{-(s/p_k)} f_2(\delta) + \left(\frac{2f_2(1)}{\delta}\right) |v_k|^{-(s/p_k)} f_1(q_k(\triangle_u^m x_k))$$

Above inequality is true for each $k \in N$. But N_p is normal sequence space and $(|v_k|^{-(s/p_k)}) \in N_p$, it follows that $(|v_k|^{-(s/p_k)}(f_2 \circ f_1)(q_k(\triangle_u^m x_k))) \in N_p$, that is $x \in N_p(E_K, \triangle_u^m, f_2 \circ f_1, s)$.

Theorem 3.6. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is a normal space if $m - 0$ and $u = 1$.
Proof. Let $x \in N_p(E_K, \triangle^0, f, s)$. Then $(|v_k|^{-(s/p_k)} f(q_k(x_k))) \in N_p$. Again, let $\lambda = (\lambda_k)$ be a sequence of scalars such that $|\lambda_k| \leq 1$ for each $k \in N$. We have

$$q_k(\lambda_k x_k) = |\lambda_k| q_k(x_k) \leq q_k(x_k) \text{ implies } |v_k|^{-(s/p_k)} f(q_k(\lambda_k x_k) \leq |v_k|^{-(s/p_k)} f(q_k(x_k).$$

But N_p is normal space, it follows that $|v_k|^{-(s/p_k)} f(q_k(\lambda_k x_k) \in N_p$. That is, $\lambda x \in N_p(E_k, \triangle^0, f, s)$. Hence $N_p(E_K, \triangle^0, f, s)$ is a normal space.

Remark 3.2. Above theorem does not hold for any $m, u \in N$.

To show that the space $N_p(E_K, \triangle_u^m, f, s)$ is not normal in general, consider the following example. Let $E_k = C$ for each $k \in N$, $f(x) = x$, $q_k(x) = |x_k|$, $m = 2$, $u = 1$, $s = 0$ and $N_p = l_1$ (where $p_k = 1$ for each $k \in N$). Then $x = (x_k) = (K) \in N_p(E_K, \triangle_u^m, f, s)$. But $\lambda x \in N_p(E_k, \triangle^m, f, s)$, where $\lambda(-1^k)$ for each $k \in N$.

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Generalized Mizoguchi-Takahashi contraction in consideration of common tripled fixed point theorem for hybrid pair of mappings

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Abstract

We establish a common tripled fixed point theorem for hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction. It is to be noted that to find tripled coincidence point, we do not employ the condition of continuity of any mapping involved therein. An example is also given to validate our result. We improve, extend and generalize several known results.

Keywords: Mizoguchi-Takahashi contraction, fixed point theorem.

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1 Introduction

Let (X, d) be a metric space and $CB(X)$ be the set of all non empty closed bounded subsets of X. Let $D(x, A)$ denote the distance from x to $A \subset X$ and H denote the Hausdorff metric induced by d, that is,

$$D(x,\ A) = \inf_{a \in A} d(x,\ a)$$

$$\text{and } H(A,\ B) = \max\left\{\sup_{a\in A} D(a,\ B),\ \sup_{b\in B} D(b,\ A)\right\},\ \text{for all } A, B \in CB(X).$$

Nadler [27] extended the famous Banach Contraction Principle [9] from single-valued mapping to multi-valued mapping. Then after several authors studied the existence of fixed points for various multi-valued contractive mappings under different conditions. For more details, see ([1],[2],[4],[15],[16],[19],[22],[23],[25],[26],[30]) and the reference therein. The theory of multi-valued mappings has application in control theory, convex optimization, differential inclusion and economics.

Bhaskar and Lakshmikantham [12] established some coupled fixed point theorems and applied these to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [24] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results of Bhaskar and Lakshmikantham [12].

Berinde and Borcut [10] introduced the concept of tripled fixed point for single valued mappings in partially ordered metric spaces. In [10], Berinde and Borcut established the existence of tripled fixed point of single-valued mappings in partially ordered metric spaces. Samet and Vetro [28] introduced the notion of fixed point of N order in case of single-valued mappings. In particular for N=3 (tripled case), we have the following definition:

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Definition 1.1. *Let X be a non-empty set and $F : X \times X \times X \to X$ be a given mapping. An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of the mapping F if*

$$F(x,\ y,\ z) = x,\ F(y,\ z,\ x) = y \text{ and } F(z,\ x,\ y) = z.$$

For more details on coupled and tripled fixed point theory, see ([3],[5],[6],[7],[8],[11],
[13],[14],[17],[18],[20]). Very recently Samet et al. [29] claimed that most of the coupled fixed point theorems in the setting of single-valued mappings on ordered metric spaces are consequences of well-known fixed point theorems.

Tripled fixed point theory to multi-valued mappings were extended by Deshpande et al. [19] and obtained tripled coincidence point and common tripled fixed point theorems involving hybrid pair of mappings under generalized nonlinear contraction. Very few authors established coupled and tripled fixed point theorems for hybrid pair of mappings including [1],[2],[19],[25].

In [19], Deshpande et al. introduced the following for multi-valued mappings:

Definition 1.2. *Let X be a non empty set, $F : X \times X \times X \to 2^X$ (a collection of all non empty subsets of X) and g be a self-mapping on X. An element $(x, y, z) \in X \times X \times X$ is called*

(1) *a tripled fixed point of F if $x \in F(x, y, z)$, $y \in F(y, z, x)$ and $z \in F(z, x, y)$.*

(2) *a tripled coincidence point of hybrid pair $\{F, g\}$ if $g(x) \in F(x, y, z)$, $g(y) \in F(y, z, x)$ and $g(z) \in F(z, x, y)$.*

(3) *a common tripled fixed point of hybrid pair $\{F, g\}$ if $x = g(x) \in F(x, y, z)$, $y = g(y) \in F(y, z, x)$ and $z = g(z) \in F(z, x, y)$.*

We denote the set of tripled coincidence points of mappings F and g by $C\{F, g\}$. Note that if $(x, y, z) \in C\{F, g\}$, then (y, z, x) and (z, x, y) are also in $C\{F, g\}$.

Definition 1.3. *Let $F : X \times X \times X \to 2^X$ be a multi-valued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called $w-$compatible if $g(F(x, y, z)) \subseteq F(gx, gy, gz)$ whenever $(x, y, z) \in C\{F, g\}$.*

Definition 1.4. *Let $F : X \times X \times X \to 2^X$ be a multi-valued mapping and g be a self-mapping on X. The mapping g is called $F-$weakly commuting at some point $(x, y, z) \in X \times X \times X$ if $g^2x \in F(gx, gy, gz)$, $g^2y \in F(gy, gz, gx)$ and $g^2z \in F(gz, gx, gy)$.*

Lemma 1.1. *Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in CB(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b \in B} d(a, b)$.*

Mizoguchi and Takahashi [26] proved the following generalization of Nadler's fixed point theorem for a weak contraction:

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued mapping. Assume that*

$$H(Tx,\ Ty) \le \psi(d(x,\ y))d(x,\ y),$$

for all $x, y \in X$, where ψ is a function from $[0, \infty)$ into $[0, 1)$ satisfying $\limsup_{s \to t+} \psi(s) < 1$ for all $t \ge 0$. Then T has a fixed point.

Amini-Harandi and O'Regan [4] obtained a generalization of Mizoguchi and Takahashi's fixed point theorem. Recently Ciric et al. [13] proved coupled fixed point theorems for mixed monotone mappings satisfying a generalized Mizoguchi-Takahashi's condition in the setting of ordered metric spaces. Main results of Ciric et al. [13] extended and generalized the results of Bhaskar and Lakshmikantham [12], Du [20] and Harjani et al. [21].

In this paper, we prove a common tripled fixed point theorem for hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction. We improve, extend and generalize the results of Amini-Harandi and O'Regan [4], Bhaskar and Lakshmikantham [12], Ciric et al. [13], Du [20], Harjani et al. [21] and Mizoguchi and Takahashi [26]. It is to be noted that to find tripled coincidence point, we do not employ the condition of continuity of any mapping involved therein. An example validate to our result has also been given.

2 Main results

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying
(i_φ) φ is non-decreasing,
(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$,
(iii_φ) $\limsup_{t\to 0+} \frac{t}{\varphi(t)} < \infty$.
Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, 1)$ which satisfies $\lim_{r\to t+} \psi(r) < 1$ for all $t \geq 0$.

Theorem 2.1. *Let (X, d) be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\varphi \in \Phi$ and some $\psi \in \Psi$ such that*

$$\begin{aligned} & \varphi\left(H(F(x, y, z), F(u, v, w))\right) \\ \leq\ & \psi\left(\varphi\left[\max\left\{d(gx, gu), d(gy, gv), d(gz, gw)\right\}\right]\right) \\ & \times\varphi\left[\max\left\{d(gx, gu), d(gy, gv), d(gz, gw)\right\}\right], \end{aligned} \tag{2.1}$$

for all $x, y, z, u, v, w \in X$. Furthermore, assume that $F(X \times X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a tripled coincidence point. Moreover, F and g have a common tripled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$ for some $(x, y, z) \in C\{F, g\}$ and for some $u, v, w \in X$ and g is continuous at u, v and w.

(b) g is $F-$weakly commuting for some $(x, y, z) \in C\{F, g\}$, gx, gy and gz are fixed points of g, that is, $g^2x = gx$, $g^2y = gy$ and $g^2z = gz$.

(c) g is continuous at x, y and z. $\lim_{n\to\infty} g^n u = x$, $\lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n w = z$ for some $(x, y, z) \in C\{F, g\}$ and for some $u, v, w \in X$.

(d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

Proof. Let $x_0, y_0, z_0 \in X$ be arbitrary. Then $F(x_0, y_0, z_0)$, $F(y_0, z_0, x_0)$ and $F(z_0, x_0, y_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0, z_0)$, $gy_1 \in F(y_0, z_0, x_0)$ and $gz_1 \in F(z_0, x_0, y_0)$, because $F(X \times X \times X) \subseteq g(X)$. Since $F : X \times X \times X \to CB(X)$, therefore by Lemma 1.1, there exist $u_1 \in F(x_1, y_1, z_1)$, $u_2 \in F(y_1, z_1, x_1)$ and $u_3 \in F(z_1, x_1, y_1)$ such that

$$\begin{aligned} d(gx_1, u_1) &\leq H(F(x_0, y_0, z_0), F(x_1, y_1, z_1)), \\ d(gy_1, u_2) &\leq H(F(y_0, z_0, x_0), F(y_1, z_1, x_1)), \\ d(gz_1, u_3) &\leq H(F(z_0, x_0, y_0), F(z_1, x_1, y_1)). \end{aligned}$$

Since $F(X \times X \times X) \subseteq g(X)$, there exist $x_2, y_2, z_2 \in X$ such that $u_1 = gx_2$, $u_2 = gy_2$ and $u_3 = gz_2$. Thus

$$\begin{aligned} d(gx_1, gx_2) &\leq H(F(x_0, y_0, z_0), F(x_1, y_1, z_1)), \\ d(gy_1, gy_2) &\leq H(F(y_0, z_0, x_0), F(y_1, z_1, x_1)), \\ d(gz_1, gz_2) &\leq H(F(z_0, x_0, y_0), F(z_1, x_1, y_1)). \end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that for all $n \in \mathbb{N}$, we have $gx_{n+1} \in F(x_n, y_n, z_n)$, $gy_{n+1} \in F(y_n, z_n, x_n)$ and $gz_{n+1} \in F(z_n, x_n, y_n)$ such that

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq H(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)), \\ d(gy_n, gy_{n+1}) &\leq H(F(y_{n-1}, z_{n-1}, x_{n-1}), F(y_n, z_n, x_n)), \\ d(gz_n, gz_{n+1}) &\leq H(F(z_{n-1}, x_{n-1}, y_{n-1}), F(z_n, x_n, y_n)), \end{aligned}$$

which implies, by (i_φ), we have

$$\begin{aligned} & \varphi\left(d(gx_n, gx_{n+1})\right) \\ \leq\ & \varphi\left(H(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n))\right) \\ \leq\ & \psi\left(\varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right]\right) \\ & \times\varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right], \end{aligned}$$

which, by the fact that $\psi < 1$, implies

$$\begin{aligned} & \varphi\left(d(gx_n, gx_{n+1})\right) \\ \leq\ & \varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right]. \end{aligned} \tag{2.2}$$

Similarly

$$\begin{aligned} & \varphi\left(d(gy_n, gy_{n+1})\right) \\ \leq\ & \varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right], \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} & \varphi\left(d(gz_n, gz_{n+1})\right) \\ \leq\ & \varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right]. \end{aligned} \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we get

$$\begin{aligned} & \max\left\{\varphi\left(d(gx_n, gx_{n+1})\right), \varphi\left(d(gy_n, gy_{n+1})\right), \varphi\left(d(gz_n, gz_{n+1})\right)\right\} \\ \leq\ & \varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right]. \end{aligned}$$

Since φ is non-decreasing, it follows that

$$\begin{aligned} & \varphi\left[\max\left\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\right\}\right] \\ \leq\ & \varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right], \end{aligned} \tag{2.5}$$

for all $n \geq 0$. Now (2.5) shows that $\{\varphi[\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\}]\}$ is a non-increasing sequence. Thus there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \varphi\left[\max\left\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\right\}\right] = \delta. \tag{2.6}$$

Since $\psi \in \Psi$, we have $\lim_{r\to\delta+} \psi(r) < 1$ and $\psi(\delta) < 1$. Then there exist $\alpha \in [0, 1)$ and $\varepsilon > 0$ such that $\psi(r) \leq \alpha$ for all $r \in [\delta, \delta + \varepsilon)$. From (2.6), we can take $n_0 \geq 0$ such that $\delta \leq \varphi[\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\}] \leq \delta + \varepsilon$ for all $n \geq n_0$. Then from (2.1) and (i_φ), for all $n \geq n_0$, we have

$$\begin{aligned} & \varphi\left(d(gx_n, gx_{n+1})\right) \\ \leq\ & \psi\left(\varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right]\right) \\ & \times\varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right] \\ \leq\ & \alpha\varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right]. \end{aligned}$$

Thus, for all $n \geq n_0$, we have

$$\begin{aligned} & \varphi\left(d(gx_n, gx_{n+1})\right) \\ \leq\ & \alpha\varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right]. \end{aligned} \tag{2.7}$$

Similarly, for all $n \geq n_0$, we have

$$\begin{aligned} & \varphi\left(d(gy_n, gy_{n+1})\right) \\ \leq\ & \alpha\varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right], \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} & \varphi\left(d(gz_n, gz_{n+1})\right) \\ \leq\ & \alpha\varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right]. \end{aligned} \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), for all $n \geq n_0$, we get

$$\begin{aligned} & \max\left\{\varphi\left(d(gx_n, gx_{n+1})\right), \varphi\left(d(gy_n, gy_{n+1})\right), \varphi\left(d(gz_n, gz_{n+1})\right)\right\} \\ \leq\ & \alpha\varphi\left[\max\left\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\right\}\right]. \end{aligned}$$

Since φ is non-decreasing, for all $n \geq n_0$, it follows that

$$\begin{aligned} & \varphi\left[\max\left\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\right\}\right] \\ \leq\ & \alpha\varphi\left[\max\left\{d(gx_{n-1},\, gx_n),\, d(gy_{n-1},\, gy_n),\, d(gz_{n-1},\, gz_n)\right\}\right], \end{aligned} \tag{2.10}$$

Letting $n \to \infty$ in (2.10) and using (2.6), we obtain that $\delta \leq \alpha\delta$. Since $\alpha \in [0, 1)$, therefore $\delta = 0$. Thus

$$\lim_{n\to\infty} \varphi\left[\max\left\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\right\}\right] = 0. \tag{2.11}$$

Since $\{\varphi[\max\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\}]\}$ is a non-increasing sequence and φ is non-decreasing, then $\{\max\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\}\}$ is also a non-increasing sequence of positive numbers. This implies that there exists $\theta \geq 0$ such that

$$\lim_{n\to\infty} \max\left\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\right\} = \theta.$$

Since φ is non-decreasing, we have

$$\varphi\left[\max\left\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\right\}\right] \geq \varphi[\theta]. \tag{2.12}$$

Letting $n \to \infty$ in (2.12), by using (2.11), we get $0 \geq \varphi[\theta]$, which, by (ii_φ), implies $\theta = 0$. Thus

$$\lim_{n\to\infty} \max\left\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\right\} = 0. \tag{2.13}$$

Suppose that $\max\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\} = 0$ for some $n \geq 0$. Then, we have $d(gx_n,\, gx_{n+1}) = 0$, $d(gy_n, gy_{n+1}) = 0$ and $d(gz_n, gz_{n+1}) = 0$ which implies that $gx_n = gx_{n+1} \in F(x_n, y_n, z_n)$, $gy_n = gy_{n+1} \in F(y_n, z_n, x_n)$ and $gz_n = gz_{n+1} \in F(z_n, x_n, y_n)$, that is, (x_n, y_n, z_n) is a tripled coincidence point of F and g. Now, suppose that $\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\} \neq 0$, for all $n \geq 0$. Denote

$$a_n = \varphi\left[\max\left\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\right\}\right], \text{ for all } n \geq 0.$$

From (2.10), we have $a_n \leq \alpha a_{n-1}$, for all $n \geq n_0$. Then, we have

$$\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{n_0} a_n + \sum_{n=n_0+1}^{\infty} \alpha^{n-n_0} a_{n_0} < \infty. \tag{2.14}$$

On the other hand, by (iii_φ), we have

$$\limsup_{n\to\infty} \frac{\max\left\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\right\}}{\varphi\left[\max\left\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\right\}\right]} < \infty. \tag{2.15}$$

Thus, by (2.14) and (2.15), we have $\sum \max\{d(gx_n,\, gx_{n+1}),\, d(gy_n,\, gy_{n+1}),\, d(gz_n,\, gz_{n+1})\} < \infty$. It means that $\{gx_n\}_{n=0}^{\infty}$, $\{gy_n\}_{n=0}^{\infty}$ and $\{gz_n\}_{n=0}^{\infty}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, so there exist x, y, $z \in X$ such that

$$\lim_{n\to\infty} gx_n = gx, \quad \lim_{n\to\infty} gy_n = gy \text{ and } \lim_{n\to\infty} gz_n = gz. \tag{2.16}$$

Now, since $gx_{n+1} \in F(x_n, y_n, z_n)$, $gy_{n+1} \in F(y_n, z_n, x_n)$ and $gz_{n+1} \in F(z_n, x_n, y_n)$, therefore by using condition (2.1), (i_φ) and by the fact that $\psi < 1$, we get

$$\begin{aligned} & \varphi\left(D(gx_{n+1},\, F(x,\, y,\, z))\right) \\ \leq\ & \varphi\left(H(F(x_n,\, y_n,\, z_n),\, F(x,\, y,\, z))\right) \\ \leq\ & \psi\left(\varphi\left[\max\left\{d(gx_n,\, gx),\, d(gy_n,\, gy),\, d(gz_n,\, gz)\right\}\right]\right) \\ & \times\varphi\left[\max\left\{d(gx_n,\, gx),\, d(gy_n,\, gy),\, d(gz_n,\, gz)\right\}\right] \\ \leq\ & \varphi\left[\max\left\{d(gx_n,\, gx),\, d(gy_n,\, gy),\, d(gz_n,\, gz)\right\}\right]. \end{aligned}$$

Since φ is non-decreasing, we have

$$D(gx_{n+1},\, F(x,\, y,\, z)) \leq \max\left\{d(gx_n,\, gx),\, d(gy_n,\, gy),\, d(gz_n,\, gz)\right\}. \tag{2.17}$$

Letting $n \to \infty$ in (2.17), by using (2.16), we obtain

$$D(gx,\ F(x,\ y,\ z)) = 0.$$

Similarly

$$D(gy,\ F(y,\ z,\ x)) = 0 \text{ and } D(gz,\ F(z,\ x,\ y)) = 0,$$

which implies that

$$gx \in F(x,\ y,\ z),\ gy \in F(y,\ z,\ x) \text{ and } gz \in F(z,\ x,\ y),$$

that is, (x, y, z) is a tripled coincidence point of F and g.

Suppose now that (a) holds. Assume that for some $(x, y, z) \in C\{F, g\}$,

$$\lim_{n\to\infty} g^n x = u,\ \lim_{n\to\infty} g^n y = v \text{ and } \lim_{n\to\infty} g^n z = w, \tag{2.18}$$

where $u, v, w \in X$. Since g is continuous at u, v and w. We have, by (2.18), that u, v and w are fixed points of g, that is,

$$gu = u,\ gv = v \text{ and } gw = w. \tag{2.19}$$

As F and g are $w-$compatible, so, for all $n \geq 1$,

$$\begin{aligned} g^n x &\in F(g^{n-1}x,\ g^{n-1}y,\ g^{n-1}z), \\ g^n y &\in F(g^{n-1}y,\ g^{n-1}z,\ g^{n-1}x), \\ g^n z &\in F(g^{n-1}z,\ g^{n-1}x,\ g^{n-1}y). \end{aligned} \tag{2.20}$$

Now, by using (2.1), (2.20), (i_φ) and by the fact that $\psi < 1$, we obtain

$$\begin{aligned} &\varphi\left(D(g^n x,\ F(u,\ v,\ w))\right) \\ \leq\ &\varphi\Big(H(F(g^{n-1}x,\ g^{n-1}y,\ g^{n-1}z),\ F(u,\ v,\ w))\Big) \\ \leq\ &\psi\left(\varphi\left[\max\left\{d(g^n x,\ gu),\ d(g^n y,\ gv),\ d(g^n z,\ gw)\right\}\right]\right) \\ &\times\varphi\left[\max\left\{d(g^n x,\ gu),\ d(g^n y,\ gv),\ d(g^n z,\ gw)\right\}\right] \\ \leq\ &\varphi\left[\max\left\{d(g^n x,\ gu),\ d(g^n y,\ gv),\ d(g^n z,\ gw)\right\}\right]. \end{aligned}$$

Since φ is non-decreasing, we have

$$D(g^n x,\ F(u,\ v,\ w)) \leq \max\left\{d(g^n x,\ gu),\ d(g^n y,\ gv),\ d(g^n z,\ gw)\right\}. \tag{2.21}$$

On taking limit as $n \to \infty$ in (2.21), by using (2.18) and (2.19), we get

$$D(gu,\ F(u,\ v,\ w)) = 0.$$

Similarly

$$D(gv,\ F(v,\ w,\ u)) = 0 \text{ and } D(gw,\ F(w,\ u,\ v)) = 0,$$

which implies that

$$gu \in F(u,\ v,\ w),\ gv \in F(v,\ w,\ u) \text{ and } gw \in F(w,\ u,\ v). \tag{2.22}$$

Now, from (2.19) and (2.22), we have

$$u = gu \in F(u,\ v,\ w),\ v = gv \in F(v,\ w,\ u) \text{ and } w = gw \in F(w,\ u,\ v),$$

that is, (u, v, w) is a common tripled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x, y, z) \in C\{F, g\}$, g is $F-$weakly commuting, that is, $g^2x \in F(gx, gy, gz)$, $g^2y \in F(gy, gz, gx)$, $g^2z \in F(gz, gx, gy)$ and $g^2x = gx$, $g^2y = gy$, $g^2z = gz$. Thus $gx = g^2x \in F(gx, gy, gz)$, $gy = g^2y \in F(gy, gz, gx)$ and $gz = g^2z \in F(gz, gx, gy)$, that is, (gx, gy, gz) is a common tripled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x, y, z) \in C\{F, g\}$ and for some $u, v, w \in X$,

$$\lim_{n\to\infty} g^n u = x,\ \lim_{n\to\infty} g^n v = y \text{ and } \lim_{n\to\infty} g^n w = z.$$

Since g is continuous at x, y and z. We have that x, y and z are fixed point of g, that is,

$$gx = x,\ gy = y \text{ and } gz = z.$$

Since $(x, y, z) \in C\{F, g\}$, therefore, we obtain

$$x = gx \in F(x,\ y,\ z),\ y = gy \in F(y,\ z,\ x) \text{ and } z = gz \in F(z,\ x,\ y),$$

that is, (x, y, z) is a common tripled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C\{F, g\}) = \{(x, x, x)\}$. Then $\{x\} = \{gx\} = F(x, x, x)$. Hence (x, x, x) is a common tripled fixed point of F and g. □

Example 2.1. *Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$. Let $F : X \times X \times X \to CB(X)$ be defined as*

$$F(x,\ y,\ z) = \begin{cases} \{0\},\ \text{for } x,\ y,\ z = 1, \\ \left[0,\ \frac{x^4}{4}\right],\ \text{for } x,\ y,\ z \in [0,\ 1), \end{cases}$$

and $g : X \to X$ be defined as

$$g(x) = x^2,\ \text{for all } x \in X.$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \begin{cases} \ln(t+1),\ \text{for } t \neq 1 \\ \frac{3}{4},\ \text{for } t = 1, \end{cases}$$

and $\psi : [0, +\infty) \to [0, 1)$ defined by

$$\psi(t) = \frac{\varphi(t)}{t},\ \text{for all } t \geq 0.$$

Now, for all $x, y, z, u, v, w \in X$ with $x, y, z, u, v, w \in [0, 1)$, we have

Case (a). If $x = u$, then

$$\begin{aligned} & H(F(x,\ y,\ z),\ F(u,\ v,\ w)) \\ = & \ \frac{u^4}{4} \\ \leq & \ \ln(u^2+1) \\ \leq & \ \ln(\max\{x^2,\ u^2\}+1) \\ \leq & \ \ln(d(gx,\ gu)\ +1) \\ \leq & \ \ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right], \end{aligned}$$

which implies that

$$\begin{aligned} & \varphi\left(H(F(x,\ y,\ z),\ F(u,\ v,\ w))\right) \\ = & \ \ln\left[H(F(x,\ y,\ z),\ F(u,\ v,\ w))+1\right] \\ \leq & \ \ln\left[\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right]+1\right] \\ \leq & \ \frac{\ln\left[\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right]+1\right]}{\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right]} \\ & \times\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right] \\ \leq & \ \psi\left(\varphi\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}\right]\right) \\ & \times\varphi\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}\right]. \end{aligned}$$

Case (b). If $x \neq u$ with $x < u$, then

$$\begin{aligned} & H(F(x,\ y,\ z),\ F(u,\ v,\ w)) \\ = & \ \frac{u^4}{4} \\ \leq & \ \ln(u^2+1) \\ \leq & \ \ln(\max\{x^2,\ u^2\}+1) \\ \leq & \ \ln(d(gx,\ gu)\ +1) \\ \leq & \ \ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right], \end{aligned}$$

which implies that

$$\begin{aligned}
& \varphi\left(H(F(x,\ y,\ z),\ F(u,\ v,\ w))\right) \\
= \ & \ln\left[H(F(x,\ y,\ z),\ F(u,\ v,\ w))+1\right] \\
\leq \ & \ln\left[\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right]+1\right] \\
\leq \ & \frac{\ln\left[\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right]+1\right]}{\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right]} \\
& \times\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right] \\
\leq \ & \psi\left(\varphi\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}\right]\right) \\
& \times\varphi\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}\right].
\end{aligned}$$

Similarly, we obtain the same result for $u < x$. Thus the contractive condition (2.1) is satisfied for all $x, y, z, u, v, w \in X$ with $x, y, z, u, v, w \in [0, 1)$. Again, for all $x, y, z, u, v, w \in X$ with $x, y, z \in [0, 1)$ and $u, v, w = 1$, we have

$$\begin{aligned}
& H(F(x,\ y,\ z),\ F(u,\ v,\ w)) \\
= \ & \frac{x^4}{4} \\
\leq \ & \ln(x^2+1) \\
\leq \ & \ln(\max\{x^2,\ u^2\}+1) \\
\leq \ & \ln(d(gx,\ gu)\ +1) \\
\leq \ & \ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
& \varphi\left(H(F(x,\ y,\ z),\ F(u,\ v,\ w))\right) \\
= \ & \ln\left[H(F(x,\ y,\ z),\ F(u,\ v,\ w))+1\right] \\
\leq \ & \ln\left[\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right]+1\right] \\
\leq \ & \frac{\ln\left[\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right]+1\right]}{\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right]} \\
& \times\ln\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}+1\right] \\
\leq \ & \psi\left(\varphi\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}\right]\right) \\
& \times\varphi\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv),\ d(gz,\ gw)\right\}\right].
\end{aligned}$$

Thus the contractive condition (2.1) is satisfied for all $x, y, z, u, v, w \in X$ with $x, y, z \in [0, 1)$ and $u, v, w = 1$. Similarly, we can see that the contractive condition (2.1) is satisfied for all $x, y, z, u, v, w \in X$ with $x, y, z, u, v, w = 1$. Hence, the hybrid pair $\{F, g\}$ satisfies the contractive condition (2.1), for all $x, y, z, u, v, w \in X$. In addition, all the other conditions of Theorem 2.1 are satisfied and $z = (0, 0, 0)$ is a common tripled fixed point of hybrid pair $\{F, g\}$. The function $F : X \times X \times X \to CB(X)$ involved in this example is not continuous at the point $(1, 1, 1) \in X \times X \times X$.

Remark 2.1. *We improve, extend and generalize the results of Ciric et al. [13] in the sense that*

(i) We prove our result for hybrid pair of mappings.

(ii) We prove our result in the framework of non complete metric space (X, d) and the product set $X \times X \times X$ is not empowered with any order.

(iii) We prove our result without the assumption of continuity and mixed g-monotone property for mapping $F : X \times X \times X \to CB(X)$.

(iv) The functions $\varphi : [0, +\infty) \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, 1)$ involved in our theorem and example are discontinuous.

If we put $g = I$ (the identity mapping) in the Theorem 2.1, we get the following result:

Corollary 2.1. *Let (X, d) be a complete metric space, $F : X \times X \times X \to CB(X)$ be a mapping. Suppose that there exist some $\varphi \in \Phi$ and some $\psi \in \Psi$ such that*

$$\begin{aligned}
& \varphi\left(H(F(x,\ y,\ z),\ F(u,\ v,\ w))\right) \\
\leq \ & \psi\left(\varphi\left[\max\left\{d(x,\ u),\ d(y,\ v),\ d(z,\ w)\right\}\right]\right) \\
& \times\varphi\left[\max\left\{d(x,\ u),\ d(y,\ v),\ d(z,\ w)\right\}\right],
\end{aligned}$$

for all $x, y, z, u, v, w \in X$. Then F has a tripled fixed point.

If we put $\psi(t) = 1 - \frac{\widetilde{\psi}(t)}{t}$ for all $t \geq 0$ in Theorem 2.1, then we get the following result:

Corollary 2.2. *Let (X, d) be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\varphi \in \Phi$ and some $\widetilde{\psi} \in \Psi$ such that*

$$\begin{aligned} & \varphi\left(H(F(x, y, z), F(u, v, w))\right) \\ \leq \; & \varphi\left[\max\left\{d(gx, gu), d(gy, gv), d(gz, gw)\right\}\right] \\ & -\widetilde{\psi}\left(\varphi\left[\max\left\{d(gx, gu), d(gy, gv), d(gz, gw)\right\}\right]\right), \end{aligned}$$

for all $x, y, z, u, v, w \in X$. Furthermore, assume that $F(X \times X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a tripled coincidence point. Moreover, F and g have a common tripled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$ for some $(x, y, z) \in C\{F, g\}$ and for some $u, v, w \in X$ and g is continuous at u, v and w.

(b) g is $F-$weakly commuting for some $(x, y, z) \in C\{F, g\}$, gx, gy and gz are fixed points of g, that is, $g^2 x = gx$, $g^2 y = gy$ and $g^2 z = gz$.

(c) g is continuous at x, y and z. $\lim_{n\to\infty} g^n u = x$, $\lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n w = z$ for some $(x, y, z) \in C\{F, g\}$ and for some $u, v, w \in X$.

(d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

If we put $g = I$ (the identity mapping) in the Corollary 2.2, we get the following result:

Corollary 2.3. *Let (X, d) be a complete metric space, $F : X \times X \times X \to CB(X)$ be a mapping. Suppose that there exist some $\varphi \in \Phi$ and some $\widetilde{\psi} \in \Psi$ such that*

$$\begin{aligned} & \varphi\left(H(F(x, y, z), F(u, v, w))\right) \\ \leq \; & \varphi\left[\max\left\{d(x, u), d(y, v), d(z, w)\right\}\right] \\ & -\widetilde{\psi}\left(\varphi\left[\max\left\{d(x, u), d(y, v), d(z, w)\right\}\right]\right), \end{aligned}$$

for all $x, y, z, u, v, w \in X$. Then F has a tripled fixed point.

If we put $\varphi(t) = 2t$ for all $t \geq 0$ in Theorem 2.1, then we get the following result:

Corollary 2.4. *Let (X, d) be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exists some $\psi \in \Psi$ such that*

$$\begin{aligned} & H(F(x, y, z), F(u, v, w)) \\ \leq \; & \psi\left(2\max\left\{d(gx, gu), d(gy, gv), d(gz, gw)\right\}\right) \\ & \times \max\left\{d(gx, gu), d(gy, gv), d(gz, gw)\right\}, \end{aligned}$$

for all $x, y, z, u, v, w \in X$. Furthermore, assume that $F(X \times X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a tripled coincidence point. Moreover, F and g have a common tripled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$ for some $(x, y, z) \in C\{F, g\}$ and for some $u, v, w \in X$ and g is continuous at u, v and w.

(b) g is $F-$weakly commuting for some $(x, y, z) \in C\{F, g\}$, gx, gy and gz are fixed points of g, that is, $g^2 x = gx$, $g^2 y = gy$ and $g^2 z = gz$.

(c) g is continuous at x, y and z. $\lim_{n\to\infty} g^n u = x$, $\lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n w = z$ for some $(x, y, z) \in C\{F, g\}$ and for some $u, v, w \in X$.

(d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

If we put $g = I$ (the identity mapping) in the Corollary 2.4, we get the following result:

Corollary 2.5. *Let (X, d) be a complete metric space, $F : X \times X \times X \to CB(X)$ be a mapping. Suppose that there exists some $\psi \in \Psi$ such that*

$$\begin{aligned} & H(F(x, y, z), F(u, v, w)) \\ \leq\ & \psi(2\max\{d(x, u), d(y, v), d(z, w)\}) \\ & \times \max\{d(x, u), d(y, v), d(z, w)\}, \end{aligned}$$

for all $x, y, z, u, v, w \in X$. Then F has a tripled fixed point.

If we put $\psi(t) = k$, where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.4, then we get the following result:

Corollary 2.6. *Let (X, d) be a metric space. Assume $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings satisfying*

$$\begin{aligned} & H(F(x, y, z), F(u, v, w)) \\ \leq\ & k\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}, \end{aligned}$$

for all $x, y, z, u, v, w \in X$, where $0 < k < 1$. Furthermore, assume that $F(X \times X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a tripled coincidence point. Moreover, F and g have a common tripled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$ for some $(x, y, z) \in C\{F, g\}$ and for some $u, v, w \in X$ and g is continuous at u, v and w.

(b) g is $F-$weakly commuting for some $(x, y, z) \in C\{F, g\}$, gx, gy and gz are fixed points of g, that is, $g^2x = gx$, $g^2y = gy$ and $g^2z = gz$.

(c) g is continuous at x, y and z. $\lim_{n\to\infty} g^n u = x$, $\lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n w = z$ for some $(x, y, z) \in C\{F, g\}$ and for some $u, v, w \in X$.

(d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

If we put $g = I$ (the identity mapping) in the Corollary 2.6, we get the following result:

Corollary 2.7. *Let (X, d) be a complete metric space. Assume $F : X \times X \times X \to CB(X)$ be a mapping satisfying*

$$\begin{aligned} & H(F(x, y, z), F(u, v, w)) \\ \leq\ & k\max\{d(x, u), d(y, v), d(z, w)\}, \end{aligned}$$

for all $x, y, z, u, v, w \in X$, where $0 < k < 1$. Then F has a tripled fixed point.

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