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On $\mathcal{I}_\sigma$ arithmetic convergence

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Abstract. In this paper, we introduce the concepts of $\mathcal{I}$-invariant arithmetic convergence, $\mathcal{I}^*$-invariant arithmetic convergence, strongly q-invariant arithmetic convergence for real sequences and give some inclusion relations.

AMS Subject Classifications: 40A05, 40A99, 46A70, 46A99.

Keywords: Ideal, invariant, arithmetic convergence.

Contents

1. Introduction and Background

Statistical convergence of a real number sequence was firstly defined by Fast [10]. It became a noteworthy topic in summability theory after the work of Fridy [11] and Šalát [12].

In the wake of the study of ideal convergence defined by Kostyrko et al. [13], there has been comprehensive research to discover applications and summability studies of the classical theories. A lot of development have been seen in area about ideal convergence of sequences after the work of [14–23]

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal iff (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ iff (i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ it is called the filter associated with the ideal.

An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$.

A sequence (x_k) is said to be $\mathcal{I}$-convergent to L if for each $\varepsilon > 0$,

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}.$$

If (x_k) is $\mathcal{I}$-convergent to L, then we write $\mathcal{I}-\lim x = L$.

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An admissible ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, ...\}$ of mutually disjoint sets of I, there is sequence $\{B_1, B_2, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$ $(i = 1, 2, ...)$ is finite and $\bigcup\limits_{i=1}^{\infty} B_i \in \mathcal{I}$.

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of all positive integers). A continuous linear functional Φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if it satisfies the following conditions:

(1) $\Phi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all $n \in \mathbb{N}$;

(2) $\Phi(e) = 1$, where $e = (1, 1, 1, ...)$;

(3) $\Phi\left(x_{\sigma(n)}\right) = \Phi(x_n)$ for all $(x_n) \in l_\infty$.

The mappings Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x_n) = \lim x_n$, for all $(x_n) \in c$.

In case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit.

The space V_σ, the set of bounded sequences whose invariant means are equal, can be shown that

$$V_\sigma = \left\{ (x_k) \in l_\infty : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L \right\}$$

uniformly in n.

Several authors studied invariant mean and invariant convergent sequence (for examples, see [24–33]).

Savaş and Nuray [26] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Nuray et al. [28] defined the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence for real sequences and investigated relationships between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Ulusu and Nuray [29] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers. Recently, the concept of strong σ-convergence was generalized by Savaş [30]. The concept of strongly σ-convergence was defined by Mursaleen [32].

Let $E \subseteq \mathbb{N}$ and

$$s_m := \min_n \left\{ \left| E \cap \left\{ \sigma(n), \sigma^2(n), ..., \sigma^m(n) \right\} \right| \right\}$$
$$S_m := \max_n \left\{ \left| E \cap \left\{ \sigma(n), \sigma^2(n), ..., \sigma^m(n) \right\} \right| \right\}.$$

If the following limits exist

$$\underline{V}(E) = \lim_{m \to \infty} \frac{s_m}{m}, \; \overline{V}(E) = \lim_{m \to \infty} \frac{S_m}{m},$$

then they are called a lower invariant uniform density and an upper invariant uniform density of the set E, respectively. If $\underline{V}(E) = \overline{V}(E)$, then $V(E) = \underline{V}(E) = \overline{V}(E)$ is called the invariant uniform density of E.

The idea of arithmetic convergence was firstly originated by Ruckle [1]. Then, it was further investigated by many authors (for examples, [2–8]).

A sequence $x = (x_m)$ is called arithmetically convergent if for each $\varepsilon > 0$, there is an integer n such that for every integer m we have $|x_m - x_{\langle m,n \rangle}| < \varepsilon$, where the symbol $\langle m, n \rangle$ denotes the greatest common divisior of two integers m and n. We denote the sequence space of all arithmetic convergent sequence by AC.

A sequence $x = (x_m)$ is said to be arithmetic statistically convergent if for $\varepsilon > 0$, there is an integer n such that

$$\lim_{t \to \infty} \frac{1}{t} |\{m \leq t : |x_m - x_{\langle m,n \rangle}| \geq \varepsilon\}| = 0.$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences. We shall write $ASC - \lim x_m = x_{\langle m,n \rangle}$ to denote the sequence (x_m) is arithmetic statistically convergent to $x_{\langle m,n \rangle}$.

Kişi [7] investigated the concepts of invariant arithmetic convergence, strongly invariant arithmetic convergence, invariant arithmetic statistically convergence, lacunary invariant arithmetic statistical convergence for real sequences and obtained interesting results.

In [8], arithmetic $\mathcal{I}$-statistically convergent sequence space $A\mathcal{I}SC$, $\mathcal{I}$-lacunary arithmetic statistically convergent sequence space $A\mathcal{I}SC_\theta$, strongly $\mathcal{I}$-lacunary arithmetic convergent sequence space $AN_\theta\left[\mathcal{I}\right]$ were investigated and some inclusion relations between these spaces were proved.

Kisi [9] gave the notion of lacunary $\mathcal{I}_\sigma$ arithmetic convergence for real sequences and examined relations between this new type convergence notion and the notions of lacunary invariant arithmetic summability, lacunary strongly q-invariant arithmetic summability and lacunary σ-statistical arithmetic convergence. Finally, giving the notions of lacunary $\mathcal{I}_\sigma$ arithmetic statistically convergence, lacunary strongly $\mathcal{I}_\sigma$ arithmetic summability, he proved the inclusion relation between them.

A sequence $x=(x_p)$ is said to be invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty}\frac{1}{m}\sum_{p=1}^{m}x_{\sigma^p(s)}=x_{\langle p,n\rangle}$$

uniformly in s. In this case we write $x_p\to x_{\langle p,n\rangle}\left(AV_\sigma\right)$ and the set of all invariant arithmetic convergent sequences will be demostrated by AV_σ.

A sequence $x=(x_p)$ is said to be strongly invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty}\frac{1}{m}\sum_{p=1}^{m}\left|x_{\sigma^p(s)}-x_{\langle p,n\rangle}\right|=0$$

uniformly in s. In this case we write $x_p\to x_{\langle p,n\rangle}\left[AV_\sigma\right]$ to denote the sequence (x_p) is strongly invariant arithmetic convergent to $x_{\langle p,n\rangle}$ and the set of all invariant arithmetic convergent sequences will be demostrated by $\left[AV_\sigma\right]$.

A sequence $x=(x_p)$ is said to be invariant arithmetic statistically convergent if for every $\varepsilon>0$, there is an integer n such that

$$\lim_{m\to\infty}\frac{1}{m}\left|\left\{p\leq m:\left|x_{\sigma^p(s)}-x_{\langle p,n\rangle}\right|\geq\varepsilon\right\}\right|=0$$

uniformly in s. We shall use $AS_\sigma C$ to denote the set of all invariant arithmetic statistical convergent sequences. In this case we write $AS_\sigma C-\lim x_p=x_{\langle p,n\rangle}$ or $x_p\to x_{\langle p,n\rangle}\left(AS_\sigma C\right)$.

2. Main Results

Definition 2.1. *A sequence $x=(x_p)$ is called to be $\mathcal{I}$-invariant arithmetic convergent if for every $\varepsilon>0$, there is an integer η such that*

$$A\left(\varepsilon\right):=\left\{p\in\mathbb{N}:\left|x_p-x_{\langle p,\eta\rangle}\right|\geq\varepsilon\right\}$$

belongs to $\mathcal{I}_\sigma$; i.e., $V\left(A\left(\varepsilon\right)\right)=0$. We can use $A\mathcal{I}_\sigma C$ to denote the set of all $\mathcal{I}_\sigma$ arithmetic convergent sequences. Thus, we define

$$A\mathcal{I}_\sigma C=\left\{x=(x_p):\text{for some } x_{\langle p,\eta\rangle},\ A\mathcal{I}_\sigma C-\lim x_p=x_{\langle p,\eta\rangle}\right\}.$$

In this case we write $A\mathcal{I}_\sigma C-\lim x_p=x_{\langle p,\eta\rangle}$ or $x_p\to x_{\langle p,\eta\rangle}\left(A\mathcal{I}_\sigma C\right)$.

Theorem 2.2. *Assume $x=(x_p)$ is a bounded sequence. If x is $\mathcal{I}$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$, then x is invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. Let $r,m\in\mathbb{N}$ be arbitrary and $\varepsilon>0$. We estimate

$$t\left(r,m\right):=\left|\frac{x_{\sigma(r)}+x_{\sigma^2(r)}+...+x_{\sigma^m(r)}}{m}-x_{\langle p,\eta\rangle}\right|$$

Then, we have

$$t(r,m) \leq t^1(r,m) + t^2(r,m),$$

where

$$t^1(r,m) := \frac{1}{m} \sum_{1\leq j\leq m,\ |x_{\sigma^j(r)}-x_{\langle p,\eta\rangle}|\geq\varepsilon} |x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}|$$

and

$$t^2(r,m) = \frac{1}{m} \sum_{1\leq j\leq m,\ |x_{\sigma^j(r)}-x_{\langle p,\eta\rangle}|<\varepsilon} |x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}|.$$

Therefore, we have $t^2(r,m) < \varepsilon$, for every $r = 1,2,....$ The boundedness of (x_p) implies that there is $K > 0$ such that

$$|x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}| \leq K,\ (j = 1,2,...; r = 1,2...),$$

then, this implies that

$$\begin{aligned} t^1(r,m) &\leq \tfrac{K}{m} \left|\left\{1 < j \leq m : |x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\}\right| \\ &\leq K.\frac{\max_r \left|\left\{1<j\leq m: |x_{\sigma^j(r)}-x_{\langle p,\eta\rangle}|\geq\varepsilon\right\}\right|}{m} \\ &= K.\tfrac{S_m}{m} \end{aligned}$$

Hence, (x_p) is invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$. ■

The converse of the previous theorem does not hold. For example, $x = (x_p)$ is the sequence defined by $x_p = 1$ if p is even and $x_p = 0$ if p is odd. When $\sigma(r) = r+1$, this sequence is invariant arithmetic convergent to $\frac{1}{2}$, but it is not $\mathcal{I}$-invariant arithmetic convergent.

Definition 2.3. *A sequence (x_p) is said to be strongly q-invariant arithmetic summable to $x_{\langle p,\eta\rangle}$, if for an integer η*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{p=1}^{m} |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}|^q = 0, \quad \textit{uniformly in } s = 1,2,...$$

where $0 < q < \infty$. In this case, we write $x_p \to x_{\langle p,\eta\rangle}([AV_\sigma]_q)$.

Theorem 2.4. *Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal and $0 < q < \infty$.*

(i) If $x_p \to x_{\langle p,\eta\rangle}([AV_\sigma]_q)$, then $x_p \to x_{\langle p,\eta\rangle}\ (A\mathcal{I}_\sigma C)$.

(ii) If $x = (x_p) \in l_\infty$ and $x_p \to x_{\langle p,\eta\rangle}\ (A\mathcal{I}_\sigma C)$, then $x_p \to x_{\langle p,\eta\rangle}([AV_\sigma]_q)$.

Proof. (i) Let $\varepsilon > 0$ and $x_p \to x_{\langle p,\eta\rangle}([AV_\sigma]_q)$. Then, we can write

$$\sum_{p=1}^{m} |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}|^q$$

$$\geq \sum_{\substack{1\leq p\leq m \\ |x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|\geq\varepsilon}} |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}|^q$$

$$\geq \varepsilon^q. \left|\left\{1 \leq p \leq m : |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\}\right|$$

$$\geq \varepsilon^q. \max_s \left|\left\{1 \leq p \leq m : |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\}\right|$$

and

$$\frac{1}{m}\sum_{p=1}^{m}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q$$

$$\geq \varepsilon^q.\frac{\max_s\left|\left\{p\leq m:|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|\geq\varepsilon\right\}\right|}{m}$$

$$=\varepsilon^q.\frac{S_m}{m}$$

for every $s=1,2,....$ This implies $\lim_{m\to\infty}\frac{S_m}{m}=0$ and so $x_p\to x_{\langle p,\eta\rangle}\ (A\mathcal{I}_\sigma C)$.

(ii) Presume that $x\in l_\infty$ and $x_p\to x_{\langle p,\eta\rangle}\ (A\mathcal{I}_\sigma C)$. Let $\varepsilon>0$. Since (x_p) is bounded, then there is $M>0$ such that

$$|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|\leq M,$$

for $p=1,2,...;\ s=1,2,....$ Observe that for every $s\in\mathbb{N}$ we have that

$$\frac{1}{m}\sum_{p=1}^{m}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q$$

$$=\frac{1}{m}\sum_{\substack{1\leq p\leq m\\|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|\geq\varepsilon}}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q$$

$$+\sum_{\substack{1\leq p\leq m\\|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|<\varepsilon}}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q$$

$$\leq M\frac{\max_s\left|\left\{1\leq p\leq m:\ |x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|\geq\varepsilon\right\}\right|}{m}+\varepsilon^q$$

$$\leq M\frac{S_m}{m}+\varepsilon^q.$$

Hence, we obtain

$$\lim_{m\to\infty}\frac{1}{m}\sum_{p=1}^{m}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q=0\quad\text{uniformly in } s=1,2,\ldots.$$

■

Definition 2.5. *A sequence $x=(x_p)$ is said to be $\mathcal{I}^*$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$, if there exists a set $M=\{m_1<m_2<...<m_p<...\}\in\mathcal{F}(\mathcal{I}_\sigma)$ and there is an integer η such that*

$$\lim_{p\to\infty}x_{m_p}=x_{\langle p,\eta\rangle}.$$

In this case, we write $x_p\to x_{\langle p,\eta\rangle}\ (A\mathcal{I}_\sigma^*C)$.

$A\mathcal{I}_\sigma^*$-convergence is better applicable in some situations.

Theorem 2.6. *Let $\mathcal{I}_\sigma$ be an admissible ideal. If a sequence (x_p) is $\mathcal{I}^*$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$, then this sequence is $\mathcal{I}$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. By assumption, there is a set $H\in\mathcal{I}_\sigma$ such that for

$$M=N\setminus H=\{m_1<m_2<...<m_p<...\}$$

we have

$$\lim_{p\to\infty}x_{m_p}=x_{\langle p,\eta\rangle}. \tag{2.1}$$

Let $\varepsilon > 0$. By (2.1), there is $p_0 \in \mathbb{N}$ such that $|x_{m_p} - x_{\langle p,\eta\rangle}| < \varepsilon$ for each $p > p_0$. Then, clearly

$$\{p \in \mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < ... < m_{p_0}\}. \tag{2.2}$$

Since $\mathcal{I}_\sigma$ is admissible, the set on the right-hand side of (2.2) belongs to $\mathcal{I}_\sigma$. Hence, $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma C)$. ∎

The converse of the Theorem 2.6 holds if $\mathcal{I}_\sigma$ has property (AP).

Theorem 2.7. *Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal with property (AP). If $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma C)$, then $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma^* C)$.*

Proof. Presume that $\mathcal{I}_\sigma$ satisfies condition (AP). Let $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma C)$. Then, we write

$$\left\{p \in \mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\} \in \mathcal{I}_\sigma$$

for each $\varepsilon > 0$. Put

$$E_1 = \left\{p \in \mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq 1\right\}$$

and

$$E_r = \left\{p \in \mathbb{N} : \frac{1}{r} \leq |x_p - x_{\langle p,\eta\rangle}| < \frac{1}{r-1}\right\}$$

for $r \geq 2$, and $r \in \mathbb{N}$. Clearly, $E_i \cap F_j = \emptyset$ for $i \neq j$. By condition (AP) there is a sequence of sets $\{F_r\}_{r\in\mathbb{N}}$ such that $E_j \Delta F_j$ are finite sets for $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_\sigma$. It is sufficient to demonstrate that for $M = \mathbb{N} \setminus F$,

$$M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I}_\sigma)$$

we have

$$\lim_{p\in M, p\to\infty} x_p = x_{\langle p,\eta\rangle}. \tag{2.3}$$

Let $\lambda > 0$. Select $r \in \mathbb{N}$ such that $\frac{1}{r+1} < \lambda$. Then

$$\left\{p \in \mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq \lambda\right\} \subset \bigcup_{j=1}^{r+1} E_j.$$

Since $E_j \Delta F_j$, $j = 1, 2, ..., r+1$ are finite sets, there is $p_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{r+1} F_j\right) \cap \{p \in \mathbb{N} : p > p_0\} = \left(\bigcup_{j=1}^{r+1} E_j\right) \cap \{p \in \mathbb{N} : p > p_0\} \tag{2.4}$$

If $p > p_0$ and $p \notin F$, then $p \notin \bigcup_{j=1}^{r+1} F_j$ and by (2.4) $p \notin \bigcup_{j=1}^{r+1} E_j$. But then $|x_p - x_{\langle p,\eta\rangle}| < \frac{1}{r+1} < \lambda$; so (2.3) holds and we obtain $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma^* C)$. ∎

Now, we shall state a theorem that gives a relation between S_σ arithmetic convergence and $\mathcal{I}$-invariant arithmetic convergence.

Theorem 2.8. *A sequence $x = (x_p)$ is S_σ arithmetic convergent to $x_{\langle p,\eta\rangle}$ iff it is $\mathcal{I}$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

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Approximation of common fixed points of finite family of nonexpansive and asymptotically generalized Φ–hemicontractive mappings

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Abstract. In this paper, we propose a modified hybrid S-iteration scheme for finite family of nonexpansive and asymptotically generalized Φ–hemicontractive mappings in the frame work of real Banach spaces. We remark that the iteration process of Kang et al. [14] can be obtained as a special case of our iteration process. A different approach is used to obtain our result and the necessity of condition (C3) is not required to prove our strong convergence theorem. Our result mainly extends and complements the result of [14] and several other related results in the literature.

AMS Subject Classifications: 40A05, 40A99, 46A70, 46A99.

Keywords: Fixed point, Banach space, hybrid S-iteration process, nonexpansive mapping, asymptotically generalized Φ-hemicontractive mapping.

Contents

1. Introduction and Background

Let E be an arbitrary real Banach space with dual E^*. We denote by J the *normalized duality* mapping from E into 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing.

In the sequel, we give the following definitions which will be useful in this study.

Definition 1.1. Let K be a nonempty subset of real Banach space E. A mapping $T : K \to K$ is said to be:

(1) *nonexpansive* if,

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in K; \tag{1.2}$$

(2) *strongly pseudocontractive* (Kim et al. [18]) if for all $x, y \in K$, there exists a constant $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2; \tag{1.3}$$

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(3) *ϕ-strongly pseudocontractive* (Kim et al. [18]) if for all $x, y \in K$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|; \tag{1.4}$$

It has been proved (see [21]) that the class of ϕ-strongly pseudocontractive mappings properly contains the class of strongly pseudocontractive mappings. By taking $\Phi(s) = s\phi(s)$, where $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$. However, the converse is not true.

(3) *generalized Φ-pseudocontractive* (Albert et al. [1], Chidume and Chidume [4]) if for all $x, y \in K$, there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|); \tag{1.5}$$

The class of generalized Φ-pseudocontractive mappings is also called uniformly pseudocontractive mappings (see [4]). Clearly, the class of generalized Φ-pseudocontractive mappings properly contains the class of ϕ-pseudocontractive mappings.

(4) *generalized Φ-hemicontractive* if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$, and there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$, such that for all $x \in K$, $p \in F(T)$, there exists $j(x-p) \in J(x-p)$ such that the following inequality holds:

$$\langle Tx - p, j(x-p)\rangle \leq \|x-p\|^2 - \Phi(\|x-p\|); \tag{1.6}$$

Clearly, the class of generalized Φ-hemicontractive mappings includes the class of generalized Φ-pseudocontractive mappings in which the fixed points set $F(T)$ is nonempty.

(5) *asymptotically generalized Φ-pseudocontractive* (Kim et al. [18]) with sequence $\{h_n\} \subset [1, \infty)$ and $\lim_{n\to} h_n = 1$, if for each $x, y \in K$, there exist a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ satisfying

$$\langle T^n x - T^n y, j(x-y)\rangle \leq h_n\|x-y\|^2 - \Phi(\|x-y\|). \tag{1.7}$$

The class of asymptotically generalized Φ-pseudocontractive mappings is a generalization of the class of strongly pseudocontractive maps and the class of ϕ-strongly peudocontractive maps. The class of asymptotically generalized Φ-pseudocontractive mappings was introduced by Kim et al. [18] in 2009.

(6) *asymptotically generalized Φ–hemicontractive* with sequence $\{h_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} h_n = 1$ if there exist a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$, such that for each $x \in K,\ p \in F(T)$, there exists $j(x-p) \in J(x-p)$ such that the following inequality holds:

$$\langle T^n - p, j(x-p)\rangle \leq h_n\|x_n - p\|^2 - \Phi(\|x-p\|). \tag{1.8}$$

Clearly, every asymptotically generalized Φ–pseudocontractive mapping with a nonempty fixed point set is an asymptotically generalized Φ–hemicontractive mapping. It follows that the class of asymptotically generalized Φ–hemicontractive mapping is most general of all the class of mappings mentioned above.

On the other hand, the class of asymptotically generalized Φ-hemicontractive has been studied by several Authors (see for example, [3–5, 12, 13, 17, 20, 26, 30]).

The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \end{cases} \quad \forall n \geq 1, \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in [0,1].

Further, the Ishikawa iteration process is defined by the sequence $\{x_n\}$

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad \forall n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \tag{1.10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1]. This iteration process reduces to Mann iteration when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Argawal et al. [2] introduced the following iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \quad \forall n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in [0,1]. They showed that their iteration process is independent of Mann and Ishikawa and converges faster than both for contractions.

In 2007, Sahu et al. [22], [23] introduced the following S-iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = T y_n, \qquad \forall n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \tag{1.12}$$

where $\{\beta_n\}$ is the sequence in [0,1].

In 1991, Schu [27] considered the modified Mann iteration process which is a generalization of the Mann iteration process as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \end{cases} \quad \forall n \geq 1, \tag{1.13}$$

where $\{\alpha_n\}$ is a sequence in [0,1].

In 1994, Tan and Xu [28] studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad \forall n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \tag{1.14}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1].

Again, in 2007 Argawal et al. [2] introduced the modified Argawal iteration process as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \quad \forall n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \tag{1.15}$$

The above processes deal with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debata [7] gave and studied a two mappings process. Also see, for example, [15] and [25]. The problem of approximating common fixed points of finitely many mappings plays an important role in applied mathematics, especially in the theory of evolution equations and the minimization problems; see [8–10, 24], for example.

The following Ishikawa-type iteration process for two mappings has aslo been studied by many authors including [7, 15, 25, 26].

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad \forall n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n S^n x_n \end{cases} \tag{1.16}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1].

In 2009, Khan et al. [16] modified the Argawal iteration process (1.15) to the case of two mappings as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \quad \forall n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \tag{1.17}$$

$\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1].

In 2013, Kang et al. [14] considered the following iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = S y_n, \qquad \forall n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \tag{1.18}$$

where $\{\beta_n\}$ is the sequence in [0,1]. They proved the following results.

Theorem 1.2 (see [14]). *Let K be a nonempty closed convex subset of a real Banach space E, let $S : K \to K$ be a nonexpansive mapping, and let $T : K \to K$ be a Lipschitz strongly pseudocontractive mapping such that $p \in F(S) \bigcap F(T) = \{x \in K : Sx = Tx = x\}$ and*

$$\|x - Sy\| \leq \|Sx - Sy\|, \quad \|x - Ty\| \leq \|Tx - Ty\| \tag{1.19}$$

for all $x, y \in K$. Let $\{\beta_n\}$ be sequence in [0,1] satisfying

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\lim_{n\to\infty} \beta_n = 0$.

For arbitrary $x_1 \in K$, the iteration process defined by (1.18) *converges strongly to a fixed point p of S and T.*

In 2016, Gopinath et al. [11] considered the following modified S-iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = S y_n, \qquad \forall n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \tag{1.20}$$

where $\{\beta\}$ is the sequence in [0,1]. They proved the following result.

Theorem 1.3 (see [11]). *Let K be a nonempty closed convex subset of a real Banach space E, let $S : K \to K$ be a nonexpansive mapping, and let $T : K \to K$ be a uniform L-Lipschitzian, asymptotically demicontractive mapping with sequence $\{h_n\} \subset [0,1)$, $\lim_{n\to\infty} h_n = 1$ such that*

$$\|x - Sy\| \leq \|Sx - Sy\|, \;\; x, y \in K \tag{1.21}$$

$$\|x - Ty\| \leq \|Tx - Ty\|, \;\; x, y \in K. \tag{1.22}$$

Assume that $F(S)\bigcap F(T)=\{x\in K: Sx=Tx=x\}\neq\emptyset$. *Let* $p\in F(S)\bigcap F(T)$ *and* $\{\beta_n\}$ *be sequences in [0,1] satisfying*

(i) $\sum_{n=1}^{\infty}\beta_n=\infty$;

(ii) $\lim_{n\to\infty}\beta_n=0$.

For arbitrary $x_1\in K$, *the iteration process defined by* (1.20) *converges strongly to a fixed point* p *of* S *and* T.

In [14], Kang et al. introduced the following condition.

Remark 1.4. Let $S,T: K\to K$ be two mappings. The mappings S and T are said to satisfy condition $(C3)$ if

$$\|x-Sy\|\leq\|Sx-Sy\|,\quad \|x-Ty\|\leq\|Tx-Ty\| \tag{1.23}$$

for all $x,y\in K$.

Inspired and motivated by the above results, we modify (1.20) for finite families of nonexpansive and asymptotically generalized Φ-hemicontractive mappings in Banach spaces. The result in this paper can be view as generalization and extension of the corresponding results of Kang et al. [14], Gopinath et al. [11] and several others in the literature.

Definition 1.5. Let $\{S_i\}_{i=1}^N: K\to K$ be finite family of nonexpansive mappings and $\{T_i\}_{i=1}^N: K\to K$ be finite family of asymptotically generalized Φ–hemicontractive mappings. Define the sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1\in K, \\ x_{n+1}=S_{i(n)}y_n, & \forall n\geq 1, \\ y_n=(1-\alpha_n)x_n+\alpha_n T_{i(n)}^{k(n)}x_n \end{cases} \tag{1.24}$$

where $\{\alpha_n\}$ is a sequence in [0,1] and $n=(k-1)N+i$, $i=i(n)\in\{1,2,...,N\}$, $k=k(n)\geq 1$ is some positive integers and $k(n)\to\infty$ as $n\to\infty$.

Remark 1.6. If we take $N=1$, then (1.24) reduces to (1.20). Again, if we take $N=1$ and $T^n=T$ for all $n\geq 1$, then (1.24) reduces to (1.18).

The purpose of this paper is to study the strong convergence of the new modified hybrid S-iteration process (1.24) for the finite families of nonexpansive and asymptotically generalized Φ-hemicontractive mappings in Banach space.

2. Preliminaries

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 (see [3]). *Let* $J: E\to 2^{E^*}$ *be the normalized duality mapping. Then for any* $x,y\in E$, *one has*

$$\|x+y\|^2\leq\|x\|^2+2\langle y,j(x+y)\rangle,\ \ \forall j(x+y)\in J(x+y). \tag{2.1}$$

Lemma 2.2 (see [29]). *Let* $\{\rho_n\}$ *and* $\{\theta_n\}$ *be nonnegative sequences satisfying*

$$\rho_{n+1}\leq(1-\theta_n)\rho_n+\omega_n \tag{2.2}$$

where $\theta_n\in[0,1]$, $\sum_{n\geq1}\theta_n=\infty$ *and* $\omega_n=o(\theta_n)$. *Then* $\lim_{n\to\infty}\rho_n=0$.

3. Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Banach space E. Let $\{S_i\}_{i=1}^N : K \to K$ be finite family of nonexpansive mappings and let $\{T_i\}_{i=1}^N : K \to K$ be finite family of asymptotically generalized Φ–hemicontractive mappings with $\{T_i(K)\}_{i=1}^N$ bounded and the sequence $\{h_{in}\} \subset [1, \infty)$, where $\lim_{n\to\infty} h_{in} = 1$ for each $1 \le i \le N$. Furthermore, let $\{T_i\}_{i=1}^N$ be uniformly continuous. Assume that $p \in \boldsymbol{F} = \bigcap_{i=1}^N F(S_i) \bigcap \bigcap_{i=1}^N F(T_i)$ $= \{x \in K : S_i x = T_i x = x\} \neq \emptyset$, for each $1 \le i \le N$. Let $h_n = \sup\{h_{in} : 1 \le i \le N\}$ and $\{\alpha_n\}$ be a sequence in [0,1] satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence iteratively defined by (1.24)*. Then the sequence $\{x_n\}$ converges strongly at common fixed p of S_i and T_i for each $1 \le i \le N$.*

Proof. Let $p \in \mathbf{F}$ and since $T_i(K)$ is bounded, we set

$$M_1 = \|x_0 - p\| + \sup_{n\ge1} \|T_{i(n)}^{k(n)} x_n - p\|, \;\; 1 \le i \le N.$$

It is clear that $\|x_0 - p\| \le M_1$. Let $\|x_n - p\| \le M_1$. Next we will prove that $\|x_{n+1} - p\| \le M_1$. From (1.24), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|S_{n(i)} y_n - p\| \\
&= \|S_{i(n)} y_n - S_{i(n)} p\| \\
&\le \|y_n - p\| \\
&= \|(1-\alpha_n)x_n + \alpha_n T_{i(n)}^{k(n)} x_n - p\| \\
&= \|(1-\alpha_n)(x_n - p) + \alpha_n (T_{i(n)}^{k(n)} x_n - p)\| \\
&\le (1-\alpha_n)\|x_n - p\| + \alpha_n \|T_{i(n)}^{k(n)} x_n - p\| \\
&\le (1-\alpha_n)M_1 + \alpha_n M_1 = M_1.
\end{aligned}$$

This implies that $\{\|x_n - p\|\}$ is bounded.

Let

$$M_2 = \sup_{n\ge1} \|x_n - p\| + M_1. \tag{3.1}$$

From (1.24) and condition (ii), we obtain

$$\begin{aligned}
\|x_n - y_n\| &= \|x_n - (1-\alpha_n)x_n - \alpha_n T_{i(n)}^{k(n)} x_n\| \\
&= \alpha_n \|x_n - T_{i(n)}^{k(n)} x_n\| \\
&\le \alpha_n (\|x_n - p\| + \|T_{i(n)}^{k(n)} x_n - p\|) \\
&\le \alpha_n (M_2 + M_1) \to 0 \;\text{ as }\; n \to \infty,
\end{aligned} \tag{3.2}$$

which implies that $\{\|x_n - y_n\|\}$ is bounded.
Again, let

$$M_3 = \sup_{n\ge1} \|x_n - y_n\| + M_2.$$

Since,

$$\begin{aligned}\|y_n - p\| &= \|y_n - x_n + x_n - p\| \\ &\leq \|x_n - y_n\| + \|x_n - p\| \\ &\leq M_3\end{aligned}$$

therefore, $\{\|y_n - p\|\}$ is bounded.

Set

$$M_4 = \sup_{n\geq 1} \|y_n - p\| + \sup_{n\geq 1} \|T_{i(n)}^{k(n)} y_n - p\|.$$

Denote

$$M = M_1 + M_2 + M_3 + M_4, \text{ obviously, } M < \infty.$$

Now from (1.24) for all $n \geq 1$, we obtain

$$\|x_{n+1} - p\|^2 = \|S_{i(n)}y_n - p\|^2 = \|S_{i(n)}y_n - S_{i(n)}p\|^2 \leq \|y_n - p\|^2, \tag{3.3}$$

thus by Lemma 2.1 and (1.8), we get

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n T_{i(n)}^{k(n)} x_n - p\|^2 \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n (T_{i(n)}^{k(n)} x_n - p)\|^2 \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n \langle T_{i(n)}^{k(n)} x_n - p, j(y_n - p)\rangle \\ &= (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n \langle T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n + T_{i(n)}^{k(n)} y_n - p, j(y_n - p)\rangle \\ &= (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n \langle T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n, j(y_n - p)\rangle \\ &\quad + 2\alpha_n \langle T_{i(n)}^{k(n)} y_n - p, j(y_n - p)\rangle \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n \|T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n\| \|y_n - p\| \\ &\quad + 2\alpha_n \{h_n\|y_n - p\|^2 - \Phi(\|y_n - p\|)\} \\ &= (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n \delta_{in} \\ &\quad + 2\alpha_n h_n \|y_n - p\|^2 - 2\alpha_n \Phi(\|y_n - p\|),\end{aligned} \tag{3.4}$$

where

$$\delta_{in} = M\|T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n\|, \ (1 \leq i \leq N).$$

From (3.2), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

From the uniform continuity of T_i , $(1 \leq i \leq N)$ leads to

$$\lim_{n\to\infty} \|T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n\| = 0,$$

thus, we have

$$\lim_{n\to\infty} \delta_{in} = 0.$$

Also,

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n T_{i(n)}^{k(n)} x_n - p\|^2 \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n (T_{i(n)}^{k(n)} x_n - p)\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|T_{i(n)}^{k(n)} x_n - p\|^2 \\ &\leq \|x_n - p\|^2 + M^2\alpha_n, \end{aligned} \tag{3.5}$$

where the first inequality holds by the convexity of $\|\cdot\|^2$.

Now substituting (3.5) into (3.4), we obtain

$$\begin{aligned}\|y_n - p\|^2 &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\delta_{in} \\ &\quad +2\alpha_n h_n(\|x_n - p\|^2 + M^2\alpha_n) - 2\alpha_n\Phi(\|y_n - p\|) \\ &= (1 - 2\alpha_n + \alpha_n^2)\|x_n - p\|^2 + 2\alpha_n h_n\|x_n - p\|^2 + 2h_n M^2\alpha_n^2 \\ &\quad +2\alpha_n\delta_{in} - 2\alpha_n\Phi(\|y_n - p\|) \\ &= (1-2\alpha_n)\|x_n - p\|^2 + (\alpha_n^2 + 2\alpha_n h_n)\|x_n - p\|^2 + 2h_n M^2\alpha_n^2 \\ &\quad +2\alpha_n\delta_{in} - 2\alpha_n\Phi(\|y_n - p\|) \\ &\leq (1-2\alpha_n)\|x_n - p\|^2 + (\alpha_n^2 + 2\alpha_n h_n)M^2 + 2h_n M^2\alpha_n^2 \\ &\quad +2\alpha_n\delta_{in} - 2\alpha_n\Phi(\|y_n - p\|) \\ &\leq (1-2\alpha_n)\|x_n - p\|^2 + \alpha_n[M^2(\alpha_n + 2h_n + 2\alpha_n h_n) + 2\delta_{in}]. \end{aligned} \tag{3.6}$$

Hence, from (3.3) and (3.6) we obtain

$$\|x_{n+1} - p\|^2 \leq (1-2\alpha_n)\|x_n - p\|^2 + \alpha_n[M^2(\alpha_n + 2h_n + 2\alpha_n h_n) + \delta_{in}].$$

For all $n \geq 1$, put

$$\begin{aligned}\rho_n &= \|x_n - p\|, \\ \theta_n &= 2\alpha_n, \\ \omega_n &= \alpha_n[M^2(\alpha_n + 2h_n + 2\alpha_n h_n) + \delta_{in}], \end{aligned}$$

then according to Lemma 2.2, we obtain that

$$\lim_{n\to\infty} \|x_n - p\| = 0. \tag{3.7}$$

Completing the proof of Theorem 3.1.

Corollary 3.2. *Let K be a nonempty closed convex subset of a real Banach space E. Let $S : K \to K$ be a nonexpansive mapping and let $T : K \to K$ be an asymptotically generalized Φ–hemicontractive mappings with $T(K)$ bounded and the sequence $\{h_n\} \subset [1,\infty)$, where $\lim_{n\to\infty} h_n = 1$. Furthermore, let T be uniformly continuous. Assume that $p \in \boldsymbol{F} = F(S)\bigcap F(T) = \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in [0,1] satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence iteratively defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = Sy_n, & \forall n \geq 1. \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n \end{cases} \tag{3.8}$$

Then the sequence $\{x_n\}$ converges strongly at common fixed p of S and T.

Proof. Taking $N = 1$ and $T^n = T$ in Theorem 3.1, the conclusion can be obtained immediately.

Remark 3.4.

(i) Corollary 3.3 recaptures the results of Kang et al. [14]. It follows that the result Kang et al. [14] is a special case of our result. Hence, our result extends and improves the results of Kang et al [14] and many others in the literature.

(ii) In our result the necessity of condition (C3) as considered by [14] and [11] is not required to prove our strong convergence theorem.

The above results are also valid for Lipschitz asymptotically generalized Φ-hemicontractive mappings.

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Binary soft locally closed sets

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Abstract. This work focuses on the concept of binary soft locally closed sets. Binary soft submaximal spaces are defined via binary soft locally closed sets. In addition, a new class of binary soft functions namely, BSLC-continuous, BSLC-irresolute, BScoLC-continuous and BScoLC-irresolute functions are defined and their characterizations are investigated.

AMS Subject Classifications: 54A05, 54A99, 54C08.

Keywords: Binary soft locally closed sets, binary soft contra locally closed sets, binary soft submaximal spaces.

Contents

1. Introduction and Background

In 1989, Ganster and Reilly [4] studied the notion of locally closed sets in topological spaces, which is defined by Bourbaki [3] as a subset of a topological space (X, τ) is locally closed if it is the intersection of an open and a closed set of X.

In this paper we have extend the notion of locally closed sets in the area of binary soft topological spaces. The notion of binary soft topological spaces is one of the latest topics, which is a combination of two popular ideas, binary topological spaces and soft topological spaces. Jothi and Thangavelu [5] introduced the concept of topology between two sets, known as binary topology. Binary topology is a structure which carries the subsets of two universal sets. The pioneer work of Molodtsov [7] on soft sets act as a successful mathematical tool over fuzzy mathematics, interval mathematics and theory of probability. In 2016, Acikgoz and Tas [1] defined binary soft sets as, (A, ρ) is a binary soft set over the two universal sets U_1, U_2 if $A : \rho \rightarrow P(U_1) \times P(U_2)$, $A(\varrho) = (X, Y)$ for every $\varrho \in \rho$ and $X \subseteq U_1$, $Y \subseteq U_2$, where $P(U_1)$ and $P(U_2)$ represents the power sets of U_1 and U_2 respectively and ρ is a set of constraints. Further, some set operations on binary soft sets namely, complement of a binary soft set, union, intersection and difference of binary soft sets are defined, and also, the notions of binary soft subset, binary absolute and null soft sets are initiated by [1].

1. (G, ρ) is called a binary soft subset of (H, ρ) over U_1, U_2 if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ where $G(\varrho) = (X_1, Y_1)$ and $H(\varrho) = (X_2, Y_2)$ for all $\varrho \in \rho$ and is denoted by $(G, \rho) \subseteq (H, \rho)$.

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2. (G, ρ) over U_1, U_2 is called a binary absolute soft set if $G(\varrho) = (U_1, U_2)$ for all $\varrho \in \rho$ and is denoted by $\widetilde{\widetilde{\rho}}$.

3. (H, ρ) over U_1, U_2 is called a binary null soft set if $H(\varrho) = (\emptyset, \emptyset)$ for all $\varrho \in \rho$ and is denoted by $\widetilde{\widetilde{\emptyset}}$.

In 2017, Benchalli et al. [2] coined the concept of binary soft topological spaces and stated the definition of binary soft topology as, a collection τ of binary soft subsets over U_1, U_2 is a binary soft topology over U_1, U_2 if $\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{\rho}} \in \tau$ and τ is closed under arbitrary union and finite intersection of binary soft sets. The members of τ are binary soft open sets and their family is denoted by $BSO(U_1, U_2)$. The complements of binary soft open sets are binary soft closed sets, and the structure (U_1, U_2, τ, ρ) is a binary soft topological space. Also the notions of binary soft interior and binary soft closure of binary soft sets are introduced by [2]. Let (A, ρ) be a binary soft subset in (U_1, U_2, τ, ρ), then:

1. binary soft interior of (A, ρ) is denoted by $(A, \rho)^{\circledcirc}$ and is given by the union of all binary soft open sets contained in (A, ρ).

2. binary soft closure of (A, ρ) is denoted by $\overline{\overline{(A, \rho)}}$ and is given by the intersection of all binary soft closed sets containing (A, ρ).

Patil et al. [9], [10] studied new separation axioms in binary soft topological spaces as well as introduced and investigated binary soft functions in binary soft topological spaces. A binary soft function $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ is called binary soft continuous [10], if $f^{-1}(V, \omega) \in BSO(U_1, U_2)$ for all $(V, \omega) \in BSO(V_1, V_2)$. In continuation, Patil et al. [11] studied the compactness and connectedness properties of binary soft topological spaces by introducing the notion of infiniteness in binary soft sets.

The main aim of this work is to study the concept of binary soft locally closed sets and hence to define binary soft submaximal spaces. A new type of binary soft functions namely, BSLC-continuous and BSLC-irresolute functions are defined, which are the generalizations of binary soft continuity. Further, the notions of binary soft contra locally closed sets are introduced and via these ideas BScoLC-continuous and BScoLC-irresolute functions are introduced.

2. Binary Soft Locally Closed Sets

Definition 2.1. *A binary soft set (L, ρ) over U_1, U_2 is said to be a binary soft locally closed set in (U_1, U_2, η, ρ) if $(L, \rho) = (O, \rho) \cap (C, \rho)$, where (O, ρ), $(C, \rho)' \in \eta$.*

The family of all binary soft locally closed sets in (U_1, U_2, η, ρ) is denoted by $BSLC(U_1, U_2)$.

Example 2.2. *Let $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$ and $\rho = \{\varrho_1, \varrho_2\}$ with*

$\eta = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{\rho}}, \{(\varrho_1, (\{a_1\}, \{b_1\})), (\varrho_2, (\{a_1\}, \{b_1\}))\}, \{(\varrho_1, (\{a_2\}, \{b_1\})), (\varrho_2, (\{a_2\}, \{b_1\}))\},$
$\{(\varrho_1, (\{a_1, a_2\}, \{b_1\})), (\varrho_2, (\{a_1, a_2\}, \{b_1\}))\}, \{(\varrho_1, (\emptyset, \{b_1\})), (\varrho_2, (\emptyset, \{b_1\}))\}\}.$

Then, (U_1, U_2, η, ρ) is a binary soft topological space. Here,

$$\begin{aligned} BSLC(U_1, U_2) = \{&\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{\rho}}, \{(\varrho_1, (\{a_1\}, \{b_1\})), (\varrho_2, (\{a_1\}, \{b_1\}))\}, \{(\varrho_1, (\{a_2\}, \{b_1\})), (\varrho_2, (\{a_2\}, \{b_1\}))\},\\ &\{(\varrho_1, (\{a_1, a_2\}, \{b_1\})), (\varrho_2, (\{a_1, a_2\}, \{b_1\}))\}, \{(\varrho_1, (\emptyset, \{b_1\})), (\varrho_2, (\emptyset, \{b_1\}))\},\\ &\{(\varrho_1, (\{a_2\}, \{b_2\})), (\varrho_2, (\{a_2\}, \{b_2\}))\}, \{(\varrho_1, (\{a_1\}, \{b_2\})), (\varrho_2, (\{a_1\}, \{b_2\}))\},\\ &\{(\varrho_1, (\emptyset, \{b_2\})), (\varrho_2, (\emptyset, \{b_2\}))\}\{(\varrho_1, (\{a_1, a_2\}, \{b_2\})), (\varrho_2, (\{a_1, a_2\}, \{b_2\}))\},\\ &\{(\varrho_1, (\{a_1\}, \emptyset)), (\varrho_2, (\{a_1\}, \emptyset))\}, \{(\varrho_1, (\{a_2\}, \emptyset)), (\varrho_2, (\{a_2\}, \emptyset))\},\\ &\{(\varrho_1, (\{a_1, a_2\}, \emptyset)), (\varrho_2, (\{a_1, a_2\}, \emptyset))\}\}. \end{aligned}$$

Remark 2.3. *Every binary soft open and binary soft closed sets are binary soft locally closed.*

Remark 2.4. *A binary soft subset (L, ρ) over U_1, U_2 is binary soft locally closed in (U_1, U_2, η, ρ) if and only if $(L, \rho)'$ is the union of binary soft open and binary soft closed set.*

Remark 2.5. *From Example 2.2, it is clear that the complement of a binary soft locally closed set need not be binary soft locally closed.*

Theorem 2.6. *For a binary soft subset (L,ρ) of (U_1,U_2,η,ρ), the following statements are equivalent:*

1. *(L,ρ) is binary soft locally closed.*
2. *$(L,\rho)=(O,\rho)\cap\overline{\overline{(L,\rho)}}$ for some $(O,\rho)\in\eta$.*
3. *$\overline{\overline{(L,\rho)}}\setminus(L,\rho)$ is binary soft closed.*
4. *$(L,\rho)\cup\left(\overline{\overline{(L,\rho)}}\right)'$ is binary soft open.*

Proof. $(1)\Rightarrow(2)$
Let $(L,\rho)=(O,\rho)\cap(C,\rho)$ for some (O,ρ), $(C,\rho)'\in\eta$.
Since $(L,\rho)\subseteq(O,\rho)$ and $(L,\rho)\subseteq\overline{\overline{(L,\rho)}}$, we have $(L,\rho)\subseteq(O,\rho)\cap\overline{\overline{(L,\rho)}}$.
Again, $\overline{\overline{(L,\rho)}}\subseteq\overline{\overline{(C,\rho)}}=(C,\rho)$. Therefore, $(O,\rho)\cap\overline{\overline{(L,\rho)}}\subseteq(O,\rho)\cap(C,\rho)=(L,\rho)$.
Hence, $(L,\rho)=(O,\rho)\cap\overline{\overline{(L,\rho)}}$.
$(2)\Rightarrow(1)$
$\overline{\overline{(L,\rho)}}$ is binary soft closed. Therefore, $(O,\rho)\cap\overline{\overline{(L,\rho)}}=(L,\rho)$ is binary soft locally closed.
$(2)\Rightarrow(3)$
$\overline{\overline{(L,\rho)}}\setminus(L,\rho)=\overline{\overline{(L,\rho)}}\setminus\left[(O,\rho)\cap\overline{\overline{(L,\rho)}}\right]=\left[\overline{\overline{(L,\rho)}}\setminus(O,\rho)\right]\cup\widetilde{\widetilde{\emptyset}}=\overline{\overline{(L,\rho)}}\cap(O,\rho)'$.
Therefore, $\overline{\overline{(L,\rho)}}\setminus(L,\rho)$ is binary soft closed.
$(3)\Rightarrow(2)$
Let $(O,\rho)=\left[\overline{\overline{(L,\rho)}}\setminus(L,\rho)\right]'$. Therefore, $(O,\rho)\in\eta$.
Now,
$(O,\rho)\cap\overline{\overline{(L,\rho)}}=\left[\overline{\overline{(L,\rho)}}\setminus(L,\rho)\right]'\cap\overline{\overline{(L,\rho)}}=\left[\overline{\overline{(L,\rho)}}\cap(L,\rho)'\right]'\cap\overline{\overline{(L,\rho)}}=\left[\overline{\overline{(L,\rho)}}'\cup(L,\rho)\right]\cap\overline{\overline{(L,\rho)}}=$
$\widetilde{\widetilde{\emptyset}}\cup\left[\overline{\overline{(L,\rho)}}\cap(L,\rho)\right]=(L,\rho)$.
$(3)\Rightarrow(4)$
Let $(C,\rho)=\overline{\overline{(L,\rho)}}\setminus(L,\rho)$. Therefore, $(C,\rho)'\in\eta$.
That is, $(C,\rho)'=\left[\overline{\overline{(L,\rho)}}\setminus(L,\rho)\right]'=\left[\overline{\overline{(L,\rho)}}\cap(L,\rho)'\right]'=(L,\rho)\cup\left(\overline{\overline{(L,\rho)}}\right)'$ is binary soft open.
$(4)\Rightarrow(3)$
Let $(O,\rho)=(L,\rho)\cup\left(\overline{\overline{(L,\rho)}}\right)'$ be binary soft open.
Then, $(O,\rho)'=\left[(L,\rho)\cup\left(\overline{\overline{(L,\rho)}}\right)'\right]'=\overline{\overline{(L,\rho)}}\cap(L,\rho)'=\overline{\overline{(L,\rho)}}\setminus(L,\rho)$ is binary soft closed.

■

Remark 2.7. *The family $BSLC(U_1,U_2)$ is closed under finite intersection.*

Remark 2.8. *Union of two binary soft locally closed sets need not be binary soft locally closed.*

In Example 2.2,
$\{(\varrho_1,(\{a_1\},\{b_1\})),(\varrho_2,(\{a_1\},\{b_1\}))\},\{(\varrho_1,(\{a_1\},\{b_2\})),(\varrho_2,(\{a_1\},\{b_2\}))\}\in BSLC(U_1,U_2)$. But $\{(\varrho_1,(\{a_1\},\{b_1,b_2\})),(\varrho_2,(\{a_1\},\{b_1,b_2\}))\}\notin BSLC(U_1,U_2)$.

Theorem 2.9. *If (A,ρ), $(B,\rho)\in BSLC(U_1,U_2)$ are binary soft separated, then, $(A,\rho)\cup(B,\rho)\in BSLC(U_1,U_2)$.*

Proof. Since, $(A,\rho),(B,\rho)\in BSLC(U_1,U_2)$, there exist $(U,\rho),(V,\rho)\in\eta$ such that
$(A,\rho)=(U,\rho)\cap\overline{\overline{(A,\rho)}}$ and $(B,\rho)=(V,\rho)\cap\overline{\overline{(B,\rho)}}$.
Therefore, $(A,\rho)\cup(B,\rho)=[(U,\rho)\cup(V,\rho)]\cap\overline{\overline{[(A,\rho)\cup(B,\rho)]}}\in BSLC(U_1,U_2)$. ■

Theorem 2.10. *In a binary soft topological space (U_1,U_2,η,ρ), let $(A,\rho)\in BSLC(U_1,U_2)$. If $(B,\rho)\subseteq(A,\rho)$ and $(B,\rho)\in BSLC(A,\eta_A,\rho)$, then, $(B,\rho)\in BSLC(U_1,U_2)$.*

Proof. Since $(B,\rho)\in BSLC(A,\eta_A,\rho)$, $(B,\rho)=({}^{A}O,\rho)\cap({}^{A}C,\rho)$ for some $({}^{A}O,\rho),({}^{A}C,\rho)'\in\eta_A$, where $(O,\rho)\cap(C,\rho)'\in\eta$.
Therefore, $(B,\rho)=[(A,\rho)\cap(O,\rho)]\cap[(A,\rho)\cap(C,\rho)]=(A,\rho)\cap[(O,\rho)\cap(C,\rho)]$.
By Remark 2.7, $(B,\rho)\in BSLC(U_1,U_2)$. ■

Theorem 2.11. *If $(A,\rho)\subseteq(B,\rho)$ in (U_1,U_2,η,ρ) and $(B,\rho)\in BSLC(U_1,U_2)$, then there exist $(U,\rho)\in BSLC(U_1,U_2)$ such that $(A,\rho)\subseteq(U,\rho)\subseteq(B,\rho)$.*

Proof. $(B,\rho)=(O,\rho)\cap\overline{\overline{(B,\rho)}}$ for some $(O,\rho)\in\tau$. Since $(A,\rho)\subseteq(B,\rho)\subseteq(O,\rho)$ and $(A,\rho)\subseteq\overline{\overline{(A,\rho)}}$, we get $(A,\rho)\subseteq(O,\rho)\cap\overline{\overline{(A,\rho)}}=(U,\rho)$, say. Thus, $(U,\rho)\in BSLC(U_1,U_2)$ and $(A,\rho)\subseteq(U,\rho)\subseteq(B,\rho)$. ■

Definition 2.12. *A binary soft topological space (U_1,U_2,η,ρ) is binary soft submaximal if and only if every binary soft subset over U_1, U_2 is binary soft locally closed.*

Definition 2.13. *A binary soft subset (A,ρ) of a binary soft topological space (U_1,U_2,η,ρ) is said to be*

1. *binary soft semi-open if $(A,\rho)\subseteq\overline{\overline{((A,\rho)^{\circledcirc})}}$.*
2. *binary soft pre-open if $(A,\rho)\subseteq\left(\overline{\overline{(A,\rho)}}\right)^{\circledcirc}$ [6].*
3. *binary soft α-open if $(A,\rho)\subseteq\left(\overline{\overline{((A,\rho)^{\circledcirc})}}\right)^{\circledcirc}$.*
4. *binary soft β-open or binary soft semi-pre-open if $(A,\rho)\subseteq\overline{\overline{\left(\overline{\overline{(A,\rho)}}\right)^{\circledcirc}}}$.*

The respective complements of above binary soft sets are known as binary soft semi-closed, binary soft pre-closed [6], binary soft α-closed and binary soft β-closed or binary soft semi-pre-closed sets.

Definition 2.14. *A binary soft function $f:(U_1,U_2,\tau,\rho)\to(V_1,V_2,\eta,\omega)$ is said to be a binary soft semi-continuous (binary soft pre-continuous, binary soft α-continuous, binary soft β-continuous respectively) if the binary soft inverse image of any binary soft open set in (V_1,V_2,η,ω) is binary soft semi-open (binary soft pre-open, binary soft α-open, binary soft β-open respectively).*

Theorem 2.15. *For a binary soft subset (A,ρ) of (U_1,U_2,τ,ρ), the following statements are equivalent:*

1. *$(A,\rho)\in BSO(U_1,U_2)$.*
2. *$(A,\rho)\in BSLC(U_1,U_2)$ and binary soft α-open.*
3. *$(A,\rho)\in BSLC(U_1,U_2)$ and binary soft pre-open.*

Proof. $(1)\Rightarrow(2)$ and $(2)\Rightarrow(3)$ are obvious.
$(3)\Rightarrow(1)$
$(A,\rho)\subseteq\left(\overline{\overline{(A,\rho)}}\right)^{\circledcirc}$ and $(A,\rho)=(U,\rho)\cap\overline{\overline{(A,\rho)}}$ for some $(U,\rho)\in BSO(U_1,U_2)$.
Now, $(A,\rho)\subseteq(U,\rho)\cap\left(\overline{\overline{(A,\rho)}}\right)^{\circledcirc}=(U,\rho)^{\circledcirc}\cap\left(\overline{\overline{(A,\rho)}}\right)^{\circledcirc}=\left((U,\rho)\cap\overline{\overline{(A,\rho)}}\right)^{\circledcirc}=(A,\rho)^{\circledcirc}$.
Therefore, $(A,\rho)\in BSO(U_1,U_2)$. ■

3. Binary Soft Locally Closed Continuous Maps

Definition 3.1. *A binary soft function* $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ *is said to be a binary soft locally closed continuous (briefly, BSLC-continuous) map if* $f^{-1}(V, \omega) \in BSLC(U_1, U_2)$ *for every* $(V, \omega) \in BSO(V_1, V_2)$.

Definition 3.2. *A binary soft function* $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ *is said to be a binary soft locally closed irresolute (briefly, BSLC-irresolute) map if* $f^{-1}(V, \omega) \in BSLC(U_1, U_2)$ *for every* $(V, \omega) \in BSLC(V_1, V_2)$.

Theorem 3.3. *Every binary soft continuous function is BSLC-irresolute, but not conversely.*

Proof. Let $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ be binary soft continuous. Then, for any $(V, \omega) \in BSO(V_1, V_2) \subseteq BSLC(V_1, V_2)$, $f^{-1}(V, \omega) \in BSO(U_1, U_2) \subseteq BSLC(U_1, U_2)$. Thus, f is BSLC-irresolute.

Example 3.4. *Let* $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$, $\rho = \{1, 2\}$ *and* $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $\omega = \{i, ii\}$ *with* $\tau = \{\tilde{\tilde{\emptyset}}, \tilde{\tilde{\rho}}, \{(1, (\{a_1\}, \{b_1\})), (2, (\{a_1, a_2\}, \{b_2\}))\}\}$ *and* $\eta = \{\tilde{\tilde{\emptyset}}, \tilde{\tilde{\omega}}, \{(i, (\{x_1\}, \{y_2\})), (ii, (\emptyset, \{y_1\}))\}\}$.
Define $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ *as* $u_1 : U_1 \to V_1$, $u_2 : U_2 \to V_2$ *and* $p : \rho \to \omega$ *so that* $u_1(a_1) = x_2$, $u_1(a_2) = x_1$, $u_2(b_1) = y_1$, $u_2(b_2) = y_2$, $p(1) = i$, $p(2) = ii$.
Then, $BSLC(U_1, U_2) = \{\tilde{\tilde{\emptyset}}, \tilde{\tilde{\rho}}, \{(1, (\{a_1\}, \{b_1\})), (2, (\{a_1, a_2\}, \{b_2\}))\}, \{(1, (\{a_2\}, \{b_2\})), (2, (\emptyset, \{b_1\}))\}\}$ *and* $BSLC(V_1, V_2) = \{\tilde{\tilde{\emptyset}}, \tilde{\tilde{\omega}}, \{(i, (\{x_1\}, \{y_2\})), (ii, (\emptyset, \{y_1\}))\}, \{(i, (\{x_2\}, \{y_1\})), (ii, (\{x_1, x_2\}, \{y_2\}))\}\}$.
Now, f *is BSLC-irresolute but not binary soft continuous.*

■

Theorem 3.5. *Every BSLC-irresolute map is BSLC-continuous, but not conversely.*

Proof. Let $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ be BSLC-irresolute. Then, for any $(V, \omega) \in BSO(V_1, V_2) \subseteq BSLC(V_1, V_2)$, $f^{-1}(V, \omega) \in BSLC(U_1, U_2)$. Thus, f is BSLC-continuous.

Example 3.6. *Let* $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$, $\rho = \{1, 2\}$ *and* $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2, y_3\}$, $\omega = \{i, ii\}$ *with*
$\tau = \{\tilde{\tilde{\emptyset}}, \tilde{\tilde{\rho}}, \{(1, (\{a_1\}, \{b_1\})), (2, (\{a_1\}, \{b_1\}))\}, \{(1, (\{a_2\}, \{b_1\})), (2, (\{a_2\}, \{b_1\}))\}, \{(1, (\{a_1, a_2\}, \{b_1\})), (2, (\{a_1, a_2\}, \{b_1\}))\}, \{(1, (\emptyset, \{b_1\})), (2, (\emptyset, \{b_1\}))\}\}$ *and*
$\eta = \{\tilde{\tilde{\emptyset}}, \tilde{\tilde{\omega}}, \{(i, (\{x_1\}, \emptyset)), (ii, (\{x_1\}, \emptyset))\}\}$.
Define $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ *as* $u_1 : U_1 \to V_1$, $u_2 : U_2 \to V_2$ *and* $p : \rho \to \omega$ *so that* $u_1(a_1) = x_1$, $u_1(a_2) = x_2$, $u_2(b_1) = y_1$, $u_2(b_2) = y_2$, $p(1) = i$, $p(2) = ii$.
Now, f *is BSLC-continuous but not BSLC-irresolute.*
Because, $\{(i, (\{x_2\}, \{y_1, y_2\})), (ii, (\{x_2\}, \{y_1, y_2\}))\} \in BSLC(V_1, V_2)$ *but* $f^{-1}(\{(i, (\{x_2\}, \{y_1, y_2\})), (ii, (\{x_2\}, \{y_1, y_2\}))\}) = \{(1, (\{a_2\}, \{b_1, b_2\})), (2, (\{a_2\}, \{b_1, b_2\}))\} \notin BSLC(U_1, U_2)$

■

Theorem 3.7. *A binary soft topological space* (U_1, U_2, τ, ρ) *is binary soft submaximal if and only if every binary soft function* $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ *is BSLC-continuous, where* (V_1, V_2, η, ω) *is any binary soft topological space.*

Proof. Consider any $(V, \omega) \in BSO(V_1, V_2)$. Then, for any $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$, $f^{-1}(V, \omega) \in BSLC(U_1, U_2)$ as (U_1, U_2, τ, ρ) is binary soft submaximal. Therefore, f is BSLC-continuous.
Conversely, take any binary soft subset (A, ρ) over U_1, U_2. Then, there exist some $(V, \omega) \in BSO(V_1, V_2)$ and BSLC-continuous function f so that $f^{-1}(V, \omega) = (A, \rho) \in BSLC(U_1, U_2)$. Hence, (U_1, U_2, τ, ρ) is binary soft submaximal. ■

Theorem 3.8. *Let $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ and $g : (V_1, V_2, \eta, \omega) \to (W_1, W_2, \mu, \sigma)$ be two binary soft functions. Then,*

1. *$g \circ f$ is BSLC-irresolute if both f and g are BSLC-irresolute.*
2. *$g \circ f$ is BSLC-continuous if f is BSLC-irresolute and g is BSLC-continuous.*

Proof. (1) Let f and g are BSLC-irresolute and $(W, \sigma) \in BSLC(W_1, W_2)$. Since g is BSLC-irresolute, $g^{-1}(W, \sigma) \in BSLC(V_1, V_2)$. Further, since f is BSLC-irresolute, $f^{-1}(g^{-1}(W, \sigma)) = (g \circ f)^{-1}(W, \sigma) \in BSLC(U_1, U_2)$. Hence, $g \circ f$ is BSLC-irresolute.

(2) Let f be BSLC-irresolute, g is BSLC-continuous and $(W, \sigma) \in BSO(W_1, W_2)$. Since g is BSLC-continuous, $g^{-1}(W, \sigma) \in BSLC(V_1, V_2)$. Further, since f is BSLC-irresolute, $f^{-1}(g^{-1}(W, \sigma)) = (g \circ f)^{-1}(W, \sigma) \in BSLC(U_1, U_2)$. Hence, $g \circ f$ is BSLC-continuous. ■

Proposition 3.9. *Let $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ be a binary soft function. Then f is binary soft continuous if and only if*

1. *f is both BSLC-continuous and binary soft α-continuous.*
2. *f is both BSLC-continuous and binary soft pre-continuous.*

Definition 3.10. *If $(A, \rho) \in BSLC(U_1, U_2)$ in (U_1, U_2, τ, ρ) then $(A, \rho)'$ is called a binary soft contra locally closed set.*

The family of all binary soft contra locally closed sets is denoted by $BSLC'(U_1, U_2)$.

Remark 3.11. *If $(B, \rho) \in BSLC'(U_1, U_2)$, then there exist some (G, ρ), $(H, \rho)' \in BSO(U_1, U_2)$ so that $(B, \rho) = (G, \rho) \cup (H, \rho)$.*

Example 3.12. *In Example 2.2,*

$$\begin{aligned} BSLC'(U_1, U_2) = \{&\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{\rho}}, \{(\varrho_1, (\{a_1\}, \{b_1\})), (\varrho_2, (\{a_1\}, \{b_1\}))\}, \{(\varrho_1, (\{a_2\}, \{b_1\})), (\varrho_2, (\{a_2\}, \{b_1\}))\}, \\ &\{(\varrho_1, (\{a_1, a_2\}, \{b_1\})), (\varrho_2, (\{a_1, a_2\}, \{b_1\}))\}, \{(\varrho_1, (\emptyset, \{b_1\})), (\varrho_2, (\emptyset, \{b_1\}))\}, \\ &\{(\varrho_1, (\{a_2\}, \{b_2\})), (\varrho_2, (\{a_2\}, \{b_2\}))\}, \{(\varrho_1, (\{a_1\}, \{b_2\})), (\varrho_2, (\{a_1\}, \{b_2\}))\}, \\ &\{(\varrho_1, (\emptyset, \{b_2\})), (\varrho_2, (\emptyset, \{b_2\}))\}\{(\varrho_1, (\{a_1, a_2\}, \{b_2\})), (\varrho_2, (\{a_1, a_2\}, \{b_2\}))\}, \\ &\{(\varrho_1, (\{a_1\}, \{b_1, b_2\})), (\varrho_2, (\{a_1\}, \{b_1, b_2\}))\}, \{(\varrho_1, (\{a_2\}, \{b_1, b_2\})), (\varrho_2, (\{a_2\}, \{b_1, b_2\}))\}, \\ &\{(\varrho_1, (\emptyset, \{b_1, b_2\})), (\varrho_2, (\emptyset, \{b_1, b_2\}))\}\}. \end{aligned}$$

Remark 3.13. *Every binary soft open and binary soft closed sets are binary soft contra locally closed sets.*

Theorem 3.14. *A binary soft set $(A, \rho) \in BSLC'(U_1, U_2)$ if and only if $(A, \rho) = (A, \rho)^{\circledcirc} \cup (C, \rho)$ for some binary soft closed set (C, ρ).*

Proof. We have, $(A, \rho) = (O, \rho) \cup (C, \rho)$ for some (O, ρ), $(C, \rho)' \in BSO(U_1, U_2)$. Since, $(C, \rho) \subseteq (A, \rho)$ and $(A, \rho)^{\circledcirc} \subseteq (A, \rho)$, we get $(A, \rho)^{\circledcirc} \cup (C, \rho) \subseteq (A, \rho)$.
Also, $(O, \rho) \subseteq (A, \rho)^{\circledcirc}$ as $(O, \rho) \subseteq (A, \rho)$. Now, $(O, \rho) \cup (C, \rho) = (A, \rho) \subseteq (A, \rho)^{\circledcirc} \cup (C, \rho)$.
Hence, $(A, \rho) = (A, \rho)^{\circledcirc} \cup (C, \rho)$. ■

Theorem 3.15. *If a binary soft contra locally closed set is binary soft pre-closed, then it is binary soft closed.*

Proof. We have, $(A, \rho) = (A, \rho)^{\circledcirc} \cup (C, \rho)$ for some binary soft closed set (C, ρ) as well as $\overline{\overline{(A, \rho)^{\circledcirc}}} \subseteq (A, \rho)$. Now, $\overline{\overline{(C, \rho)}} \cup \overline{\overline{(A, \rho)^{\circledcirc}}} \subseteq (A, \rho)$, since (C, ρ) is binary soft closed. Therefore, $\overline{\overline{(C, \rho) \cup (A, \rho)^{\circledcirc}}} = \overline{\overline{(A, \rho)}} \subseteq (A, \rho)$. Hence, (A, ρ) is binary soft closed. ■

Remark 3.16. *The family $BSLC'(U_1, U_2)$ is closed under union.*

Remark 3.17. *Intersection of two binary soft contra locally closed sets need not be binary soft contra locally closed.*

In Example 2.2,
$\{(\varrho_1, (\{a_1\}, \{b_1\})), (\varrho_2, (\{a_1\}, \{b_1\}))\}, \{(\varrho_1, (\{a_1\}, \{b_2\})), (\varrho_2, (\{a_1\}, \{b_2\}))\} \in BSLC'(U_1, U_2)$. But $\{(\varrho_1, (\{a_1\}, \emptyset)), (\varrho_2, (\{a_1\}, \emptyset))\} \notin BSLC'(U_1, U_2)$.

Definition 3.18. *A binary soft function $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ is called a binary soft contra locally closed continuous (briefly, BScoLC-continuous) function if for any $(V, \omega) \in BSO(V_1, V_2)$, $f^{-1}(V, \omega) \in BSLC'(U_1, U_2)$.*

Definition 3.19. *A binary soft function $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ is called a binary soft contra locally closed irresolute (briefly, BScoLC-irresolute) function if for any $(V, \omega) \in BSLC(V_1, V_2)$, $f^{-1}(V, \omega) \in BSLC'(U_1, U_2)$.*

Theorem 3.20. *Every binary soft continuous function is BScoLC-irresolute.*

Theorem 3.21. *Every BScoLC-irresolute function is BScoLC-continuous.*

Remark 3.22. *The binary soft functions BSLC-continuous and BScoLC-continuous are independent.*

Example 3.23. *In Example 3.6, f is BSLC-continuous but not BScoLC-continuous.*
Further, consider $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$, $\rho = \{1, 2\}$ and
$V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $\omega = \{i, ii\}$ with

$$\tau = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{\rho}}, \{(1, (\{a_1\}, \{b_1\})), (2, (\{a_1\}, \{b_1\}))\}, \{(1, (\{a_2\}, \{b_1\})), (2, (\{a_2\}, \{b_1\}))\}, \{(1, (\{a_1, a_2\}, \{b_1\})), (2, (\{a_1, a_2\}, \{b_1\}))\}, \{(1, (\emptyset, \{b_1\})), (2, (\emptyset, \{b_1\}))\}\}$$

and $\eta = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{\omega}}, \{(i, (\{x_1\}, \{y_1, y_2\})), (ii, (\{x_1\}, \{y_1, y_2\}))\}\}$.
Define $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ as $u_1 : U_1 \to V_1$, $u_2 : U_2 \to V_2$ and $p : \rho \to \omega$ so that
$u_1(a_1) = x_1$, $u_1(a_2) = x_2$, $u_2(b_1) = y_1$, $u_2(b_2) = y_1$, $p(1) = i$, $p(2) = ii$.
Now, f is BScoLC-continuous but not BSLC-continuous.

Remark 3.24. *The binary soft functions BSLC-irresolute and BScoLC-irresolute are independent.*

Example 3.25. *Let $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$, $\rho = \{1, 2\}$;*
$V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $\omega = \{i, ii\}$ and
$W_1 = \{c_1, c_2\}$, $W_2 = \{d_1, d_2\}$, $\sigma = \{i, ii\}$ with

$$\tau = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{\rho}}, \{(1, (\{a_1\}, \{b_1\})), (2, (\{a_1\}, \{b_1\}))\}, \{(1, (\{a_2\}, \{b_1\})), (2, (\{a_2\}, \{b_1\}))\}, \{(1, (\{a_1, a_2\}, \{b_1\})), (2, (\{a_1, a_2\}, \{b_1\}))\}, \{(1, (\emptyset, \{b_1\})), (2, (\emptyset, \{b_1\}))\}\};$$

$\eta = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{\omega}}, \{(i, (\{x_1\}, \emptyset)), (ii, (\{x_1\}, \emptyset))\}\}$ *and*

$\mu = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{\sigma}}, \{(i, (\{c_1\}, \{d_1, d_2\})), (ii, (\{c_1\}, \{d_1, d_2\}))\}\}$.
Define $f : (U_1, U_2, \tau, \rho) \to (V_1, V_2, \eta, \omega)$ as $u_1 : U_1 \to V_1$, $u_2 : U_2 \to V_2$ and $p : \rho \to \omega$ so that $u_1(a_1) = x_2$, $u_1(a_2) = x_1$, $u_2(b_1) = y_1$, $u_2(b_2) = y_2$, $p(1) = ii$, $p(2) = i$.
Then, f is BSLC-irresolute but not BScoLC-irresolute.
Define $g : (U_1, U_2, \tau, \rho) \to (W_1, W_2, \mu, \sigma)$ as $u_1 : U_1 \to W_1$, $u_2 : U_2 \to W_2$ and $p : \rho \to \sigma$ so that $u_1(a_1) = c_1$, $u_1(a_2) = c_2$, $u_2(b_1) = d_2$, $u_2(b_2) = d_1$, $p(1) = i$, $p(2) = ii$.
Then, f is BScoLC-irresolute but not BSLC-irresolute.

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Results of ω-order reversing partial contraction mapping generating a differential operator

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Abstract. In this paper, we presents some partial differential operators defined on suitably chosen function spaces such as $H^{-1}(\Omega)$, $L^p(\Omega)$, with $p \in [1, +\infty)$. Laplace operator on a domain Ω in $\mathbb{R}^n$ subject to the Dirichlet boundary condition was established by generating a C_0-semigroup, which is generated by an infinitesimal generator ω-order reversing partial contraction (ω-$ORCP_n$).

AMS Subject Classifications: 40A05, 40A99, 46A70, 46A99.

Keywords: ω-$ORCP_n$, C_0-semigroup, C_0-Semigroup of Contraction, Differential Operator.

Contents

1. Introduction and Background

Considering heat equation in a domain Ω in $\mathbb{R}^3$

$$\begin{cases} v_s = \Delta v & (s,x) \in Q_\infty \\ v = 0 & (s,x) \in \Sigma_\infty \\ v(0,x) = v_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator, $Q_\infty = \mathbb{R}_+ \times \Omega$ and $\Sigma_\infty = \mathbb{R}_+ \times \Gamma$. We rewrite this partial differential equation as an ordinary differential equation of the form

$$\begin{cases} v' = Av \\ v(0) = v_0 \end{cases} \tag{1.2}$$

in an infinite-dimensional Banach space X which is chosen suitably, so that the unbounded linear operator $A : D(A) \subseteq X \to X$ generate a C_0-Semigroup of contractions.

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Let X be a Banach space, $X_n \subseteq X$ be a finite set, H is a Hibert space, $(T(s))_{s\geq 0}$ is a C_0-semigroup, $\omega - ORCP_n$ is the ω-order reversing partial contraction mapping, M_m be matrix, P_n is a partial transformation semigroup,$L(X)$ is a bounded linear operator on X, $\rho(A)$ is a resolvent set, $\sigma(A)$ is the spectrum and $A \in \omega - ORCP_n$ is a generator of C_0-semigroup.

Akinyele *et al.* [1], introduced some results on perturbation of infinitesimal generator in semigroup and also in [2], Akinyele *et al.* obtained infinitesimal generator of Mean Ergodic theorem in semigroup of linear operator. Amann [3], established and solved some linear quasilinear parabolic problems and also in [4], Amann introduced measures to a linear parabolic problems. Arendt [5], introduced some Laplace transform in vector-valued and Cauchy problems. Balakrishnan [6], obtained an operator in infinitesimal generator of semigroup. Banach [7], established and introduced the concept of Banach spaces. Barbu [8], deduced some boundary problems for partial differential equation. Carja and Vrabie [9], obtained some results on new viability for semilinear differential insertion. Rauf and Akinyele [10], obtained ω-order-preserving partial contraction mapping and established the properties, also in [11], Rauf *et al.* established some stability and spectra properties on semigroup of linear operator. Vrabie [12], deduced some results of C_0-semigroup and its applications. Yosida [13], established made a representation and differentiability of one-parameter semigroup.

2. Preliminaries

Definition 2.1 (ω-$ORCP_n$) [10]
A transformation $\alpha \in P_n$ is called ω-order-reversing partial contraction mapping if $\forall x, y \in$Dom$\alpha : x \leq y \implies \alpha x \geq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(s+t) = T(s)T(t)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.
Definition 2.2 (C_0-semigroup) [12]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.3 (C_0-semigroup of contraction)[12]
A C_0-semigroup $\{T(s); s \geq 0\}$ is called of type (ζ, ω) with $\zeta \geq 1$ and $\omega \in \mathbb{R}$, if for each $t \geq 0$, we have
$\|T(s)\|_{L(X)} \leq \zeta e^{t\omega}$.
A C_0-semigroup $\{T(s); s \geq 0\}$ is called a C_0-semigroup of contraction or non expansive operator, if it is of type $1 < \alpha < 0$ for all $\alpha \in \mathbb{R}$, and for each $s \geq 0$, we have
$\|T(s)\|_{L(X)} \leq 1$.
Definition 2.4 (Differential operator) [8]
A differential operator is an operator defined as a function of the differentiation operator.

Example 1
Consider the 3×3 matrix $[M_m(\mathbb{C})]$, and for each $\beta > 0$ such that $\beta \in \rho(A)$, where $\rho(A)$ is a resolvent set on X. Suppose

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

and assume $T(t) = e^{tA_\beta}$, then

$$e^{tA_\beta} = \begin{pmatrix} e^{3t\beta} & e^{2t\beta} & e^{t\beta} \\ e^{2t\beta} & e^{2t\beta} & e^{t\beta} \\ e^{3t\beta} & e^{2t\beta} & e^{2t\beta} \end{pmatrix}.$$

Example 2
In the $H^{-1}(\Omega)$ setting, assume Ω be a nonempty and open subset in $\mathbb{R}^n$, let $X = H^{-1}(\Omega)$, and suppose we

define $A : D(A) \subseteq X \to X$ by

$$\begin{cases} D(A) = H_0^1(\Omega) \\ \quad Av = \Delta v, \end{cases} \tag{2.1}$$

for each $v \in D(A)$ and $A \in \omega - ORCP_n$. It follows that $H_0^1(\Omega)$ is equipped with the usual norm on $H^{-1}(\Omega)$ defined by

$$\|v\|_{H^1(\Omega)} = (\|v\|^2_{L^2(\Omega)} + \|\nabla\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Example 3

In the $L^2(\Omega)$ setting, suppose Ω be a nonempty and open subset in $\mathbb{R}^n$ and assume $X = L^2(\Omega)$. Consider the operator A on X, defined by

$$\begin{cases} D(A) = \{x \in H_0^1(\Omega); \Delta v \in L^2(\Omega)\} \\ \quad Av = \Delta v, \end{cases} \tag{2.2}$$

for each $x \in D(A)$ and $A \in \omega - ORCP_n$.

Theorem 2.1

Suppose Ω is a nonempty, open and bounded subset in $\mathbb{R}^n$ whose boundary is of class C^1, $r \in \mathbb{N}$ and $p, q \in [1, +\infty)$. Then,

i. if $rp < n$ and $q < \frac{np}{n-rp}$, we have that $W^{r,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$;

ii. if $rp = n$ and $q \in [1, +\infty)$ is compactly imbedded in $L^q(\Omega)$; and

iii. if $rp > n$, then $W^{r,p}(\Omega)$ is compactly imbedded in $C(\overline{\Omega})$.

Theorem 2.2

Assume H is a Hibert space and $\{A, D(A)\}$ a densely defined operator. Then we have,

i. if $(I - A)^{-1} \in \mathcal{L}(H)$, then A is self-adjoint if and only if A is symmetric; and

ii. if $(I \pm A)^{-1} \in \mathcal{L}(H)$, then A is skew - adjoint if and only if A is skew - symmetric.

Theorem 2.3

For any $\beta > 0$ and $f \in H^{-1}(\Omega)$, the equation $\beta_v - \Delta v = f$ has a unique solution $v \in H^1_{0(\Omega)}$.

Theorem 2.4

Suppose Ω is a nonempty open and bounded subset in $\mathbb{R}^n$ whose boundary Γ is of class C^1. Then $\|.\| : H^1(\Omega) \to \mathbb{R}_+$. defined by

$$\|v\| = (\|\nabla v\|^2_{L^2(\Omega)} + \|v_\Gamma\|^2_{L^2(\Gamma)})^{\frac{1}{2}}$$

for each $v \in H^1(\Omega)$, is a norm on $H^1(\Omega)$ and equivalent with the usual one. In particular, the restriction of this norm to $H_0^1(\Omega)$, i.e. $\|.\| : H_0^1(\Omega) \to \mathbb{R}_+$ defined by

$$\|v\|_0 = \|\nabla v\|_{L^2(\Omega)},$$

for each $v \in H_0^1(\Omega)$, is a norm on $H_0^1(\Omega)$ (called the gradient norm) equivalent with the usual one. In respect with this norm the application $D : H_0^1(\Omega) \to H^{-1}(\Omega)$, defined by

$$< u, \Delta v >_{H_0^1(\Omega), H^{-1}(\Omega)} = \int_\Omega \nabla v \nabla u dw,$$

is a canonical isomorphism between $H_0^1(\Omega)$ and its dual H^{-1}. The restriction of this application to H^2 coincides with $-\Delta$, where Δ is the Laplace operator in the sense of distributions over $\Delta(\Omega)$.

Theorem 2.5

The application $I - \Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the canonical isomorphism between $H_0^1(\Omega)$, endowed with the usual norm on $H^1(\Omega)$ and its dual $H^{-1}(\Omega)$, equipped with the usual dual norm. In addition, for each $v \in H_0^1$ and each $u \in L^2(\Omega)$, we have

$$< v, u >_{L^2(\Omega)} = < v, u >_{H_0^1(\Omega), H^{-1}(\Omega)} .$$

Theorem 2.6(Hille-Yoshida)[12]
A linear operator $\{A, D(A)\}$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed; and

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\beta > 0$, we have

$$\|R(\beta, A)\|_{L(X)} \leq \frac{1}{\beta}. \tag{2.3}$$

Theorem 2.7
Assume $\{A, D(A)\}$ is the infinitesimal generator of a C_0-semigroup and let $\|.\|_{D(A)} : D(A) \to \mathbb{R}_+$ and $|.|_{D(A)} : D(A) \to \mathbb{R}_+$ be defined by $\|x\|_{D(A)} = \|x\| + \|Ax\|$, and respectively by $|x|_{D(A)} = \|X - Ax\|$, for each $x \in D(A)$. Then:

i. $\|.\|_{D(A)}$ is a norm on $D(A)$, called the graph norm, with respect to which $D(A)$ is a Banach space;

ii. $D(A)$ endowed with the norm $\|.\|_{D(A)}$ is continuously imbedded in X;

iii. $A \in L(D(A), X)$ where $D(A)$ is endowed with $\|.\|_{D(A)}$;

iv. $|.|_{D(A)}$ is a norm on $D(A)$ equivalent with $\|.\|_{D(A)}$;

v. $I - A$ is an isometry from $(D(A), |.|_{D(A)})$ to $(X, \|.\|)$; and

vi. for each $x \in D(A)$, $S(.)x \in C[0, +\infty);\ D(A) \cup C^1([0, +\infty); X)^1$.

3. Main Results

This section section presents results of ω-$ORCP_n$ on Laplace operator with respect to the Dirichlet boundary condition by generating a C_0-semigroup of contractions:

Theorem 3.1
The operator $A \in \omega - ORCP_n$ defined by

$$\|v\|_{H^1(\Omega)} = (\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

is the generator of a C_0-semigroup of contractions. In addition, A is self-adjoint and $\|.\|_{D(A)}$ is equivalent with the norm of the space $H^{-1}(\Omega)$.

Proof:
By virtue of Theorem 2.5, we know that $I - \Delta$ is the canonical isomorphism between $H_0^1(\Omega)$, endowed with usual norm of $H^1(\Omega)$, and its dual $H^{-1}(\Omega)$. Let us denote that $F = (I - \Delta)^{-1}$ is an isometry joining $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Consequently

$$< v, u >_{H^{-1}(\Omega)} = < Fv, Fu >_{H_0^1(\Omega)} \tag{3.1}$$

for each $u, v \in H^{-1}(\Omega)$. Let $u, v \in H_0^1(\Omega)$, then we have

$$\begin{aligned} < v, Fu >_{H_0^1(\Omega)} &= \int_\Omega \nabla v \nabla (Fu) dw + \int_\Omega uFvdw \\ &= \int_\Omega v(-\Delta(Fu))dw + \int_\Omega vFudw \\ &= \int_\Omega v(I - \Delta)F(u)dw = < v, u >_{L^2(\Omega)} . \end{aligned} \tag{3.2}$$

From (3.1), taking into account that $F(I-\Delta)=I$, we deduce

$$\begin{aligned} < -\Delta v, u >_{H^{-1}(\Omega)} &= < v-\Delta v, u >_{H^{-1}(\Omega)} - < v, u >_{H^{-1}(\Omega)} \\ &= < F(v-\Delta v), Fu >_{H_0^1(\Omega)} - < v, u >_{H^{-1}(\Omega)} \\ &= < v, Fv >_{H^{-1}(\Omega)} - < v, u >_{H^{-1}(\Omega)} . \end{aligned}$$

From (3.2), we have

$$< \Delta v, u >_{H^{-1}(\Omega)} = < v, u >_{H^{-1}(\Omega)} - < v, u >_{L^2(\Omega)} . \tag{3.3}$$

Therefore A is symmetric. But $(I-A)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, and therefore by Theorem 2.2, it follows that A is self-adjoint. Taking $u=v$ in (3.3), we obtain

$$< Av, v >_{H^{-1}(\Omega)} = \|v\|^2_{H^{-1}(\Omega)} - \|v\|^2_{L^2(\Omega)} \leq 0. \tag{3.4}$$

Theorem 2.3 shows that, for $\beta>0$, we have $(\beta I - A)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, while (3.4) implies that, for $\beta>0$, we have

$$< \lambda v - Av, v >_{H^{-1}(\Omega)} \geq \lambda \|v\|^2_{H^{-1}(\Omega)}.$$

Hence $\|R(\beta; A)\|_{L(H^{-1}(\Omega))} \leq \frac{1}{\beta}$. Since $H_0^1(\Omega)$ is dense in $H^{-1}(\Omega)$, we are in the hypothesis of Theorem 2.6, from where it follows that A generates a C_0-semigroup of contractions on $H^{-1}(\Omega)$. Finally by (iv) in Theorem 2.7 and (3.4), it follows that $\|.\|_{D(A)}$ is equivalent with the norm of the space H^{Ω} and this complete the proof.

Theorem 3.2

The linear operator $A \in \omega - ORCP_n$ defined by

$$\begin{cases} D(A) = \{v \in H_0^1(\Omega); \Delta v \in L^2(\Omega)\} \\ \quad Av = \Delta v, \end{cases} \tag{3.5}$$

for each $v \in D(A)$ is the infinitesimal generator of a C_0-semigroup of contractions. Moreover, A is self-adjoint and $(D(A), \|.\|_{D(A)})$ is continuously included in $H_0^1(\Omega)$. Suppose Ω is bounded with C^1 boundary, then $(D(A), \|.\|_{D(A)})$ is compactly imbedded in $L^2(\Omega)$.

Proof:

Assume $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, and $C_0^\infty(\Omega) \subseteq D(A)$, it follows that A is densely defined. Let $\lambda>0$ and $f \in L^2(\Omega)$. Since $L^2(\Omega)$ is continuously imbedded in $H^{-1}(\Omega)$, and $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the duality mapping with respect to the gradient norm on $H_0^1(\Omega)$, we have

$$< Av, u >_{L^2(\Omega)} = < \nabla v, \nabla u >_{L^2(\Omega)} = < u, \Delta v >_{H_0^1(\Omega),\ H^1(\Omega)} . \tag{3.6}$$

By Theorem 3.1, we know that for any $\lambda>0$ and $f \in L^2(\Omega)$ (notice that $L^2(\Omega) \subset H^{-1}(\Omega)$), the equation

$$\lambda v - \Delta v = f \tag{3.7}$$

has a unique solution $v_\lambda \in H_0^1(\Omega) \subset L^2(\Omega)$. So, $\Delta v_\lambda = \lambda v_\lambda - f$ is in $L^2(\Omega)$, which shows that $v_\lambda \in D(A)$ and $\lambda v_\lambda - Av_\lambda = f$. Taking the L^2 inner product on both sides of (3.7) above by v_λ and taking into account that by (3.6), we have $< Av, v >_{L^2(\Omega)} \leq 0$ for each $v \in D(A)$, then we deduce that

$$\lambda \|v_\lambda\|^2_{L^2(\Omega)} \leq \ < f, v_\lambda >_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v_\lambda\|_{L^2(\Omega)},$$

which shows that $\|R(\lambda; A)\|_{L(X)} \leq \frac{1}{\lambda}$. Finally from (3.6) and Theorem 2.2, it follows that A is self-adjoint. Considering both inclusions, then $D(A) \subset H_0^1 \subset L^2(\Omega)$ are continuous, and the latter is compact whenever Ω is bounded by Theorem 2.1. Hence the proof is achieved.

Theorem 3.3

Let $A \in \omega - ORCP_n$ be the Laplace operator with the Dirichlet boundary condition in $H^{-1}(\Omega)$, let $\lambda > 0$ and $1 \leq p < +\infty$. Then:

(1.) There exists a unique $\mathcal{R}_\lambda \in \mathcal{L}(L^p(\Omega))$ so that $\mathcal{R}_\lambda u = R(\lambda; A)u$ for all $u \in H^{-1}(\Omega) \cap L^p(\Omega)$ and $\mathcal{R}_\lambda$ satisfies:

i. $\|\mathcal{R}_\lambda u\|_{L^p(\Omega)} \leq \frac{1}{\lambda}\|u\|_{L^p(\Omega)}$;

ii. for each $f \in L^p(\Omega)$, $A\mathcal{R}_\lambda f \in L^p(\Omega)$ and $\lambda\mathcal{R}_\lambda f - A\mathcal{R}_\lambda f = f$; and

iii. for each $\lambda > 0$ and $\mu > 0$, $\mathcal{R}_\lambda(L^p(\Omega)) = \mathcal{R}_\mu(L^p(\Omega))$.

(2.) Let $\mathcal{R}_1 \in \mathcal{L}(L^p(\Omega))$ for each $u \in \mathcal{R}(L^p(\Omega))$, we have $\Delta u \in L^p(\Omega)$, and the operator $A : D(A) \subseteq L^p(\Omega) \to L^p(\Omega)$, defined by

$$\begin{cases} D(A) = \mathcal{R}_1(L^p(\Omega)) \\ \quad Au = \Delta u \quad for\, u \in D(A), \end{cases}$$

is the generator of a C_0-semigroup of contractions.

Proof:

Since $H^{-1}(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$, then it follows that $R(\lambda; A)$ has a unique extension $\mathcal{R}_\lambda \in \mathcal{L}(L^p(\Omega))$ satisfying (i). Next, let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $D(\Omega)$ convergent to f in $L^p(\Omega)$. As $\mathcal{R}_\lambda f_k - \lambda A\mathcal{R}_\lambda f_k = f_k$ in $H^{-1}(\Omega)$, we have $\mathcal{R}_\lambda f - \lambda A\mathcal{R}_\lambda f = f$ in $D^1(\Omega)$, from there we get (ii). Finally, let $f \in H^{-1}(\Omega) \cap L^p(\Omega)$, and $u = \mathcal{R}_\lambda f \in H^{-1}(\Omega) \cap L^p(\Omega)$. For each $\mu > 0$, we have

$$\mu u - \Delta u = f + (\mu - \lambda)\mathcal{R}_\lambda f \tag{3.8}$$

Let us denote by g the right-hand side of (3.8), i.e.

$$g = f + (\mu - \lambda)\mathcal{R}_\lambda f$$

and let us observe that $\mathcal{R}_\lambda f = u = \mathcal{R}_\mu g \in H^{-1}(\Omega) \cap L^p(\Omega)$ and therefore

$$\mathcal{R}_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)) \subseteq \mathcal{R}_\mu(H^{-1}(\Omega) \cap L^p(\Omega)).$$

Analogously

$$\mathcal{R}_\mu(H^{-1}(\Omega) \cap L^p(\Omega)) \subseteq (\mathcal{R}_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)),$$

and so

$$\mathcal{R}_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)) = (\mathcal{R}_\mu(H^{-1}(\Omega) \cap L^p(\Omega)).$$

Since $H^{-1}(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$, and $\mathcal{R}_\lambda$, $\mathcal{R}_\mu$ are linear and continuous operators in $L^p(\Omega)$, then we deduce (iii). And this complete the proof of (1). To prove (2), for each $u \in \mathcal{R}_1(L^p(\Omega))$ and $A \in \omega - ORCP_n$, we have $\Delta u \in L^p(\Omega)$, follows from (ii) in (1) above. So let $u \in D(A)$, $\lambda > 0$, $A \in \omega - ORCP_n$ and denote that $f = \lambda u - \Delta u$. From (iii) in (1), there exists $g \in L^p(\Omega)$ such that $u = \mathcal{R}_\lambda g$. We then conclude that $g = \lambda u - \Delta u$ and so $f = g$. Then $\lambda \in \rho(A)$ and $R(\lambda; A) = (\lambda I - A)^{-1} = \mathcal{R}_\lambda$. This relation in (i) from (1) above show that $\|R(\lambda; A)\|_{L^p(\Omega)} \leq \frac{1}{\lambda}\|u\|_{L^p(\Omega)}$. Thus A satisfies (ii) in Theorem 2.6. To complete the proof, we have to merely to show that $D(A)$ is dense in $L^p(\Omega)$. To this aim, let $u \in D(\Omega)$ and $f = u - \Delta u \in D(\Omega)$. Obviously $u = \mathcal{R}_1 f$ and therefore $D(\Omega) \subseteq D(A)$. Hence $D(A)$is dense in $L^P(\Omega)$ which complete the proof.

Theorem 3.4

Let Ω be a nonempty and open subset in $\mathbb{R}^n$ with C^1 boundary Γ, let $X = [H^{-1}(\Omega)]^*$ then:

(i.) operator $A : D(A) \subseteq X \to X$, defined by

$$\begin{cases} D(A) = H^1(\Omega) \\ < Au, v >_{H^1(\Omega),[H^1(\Omega)]^*} =< \nabla u, \nabla v >_{L^2(\Omega)} \end{cases}$$

for each $u, v \in H^1(\Omega)$ and $A \in \omega - ORCP_n$ is the generator of a C_0-semigroup of contraction on X; and (ii.) the operator $\{B, D(B)\}$, defined by

$$\begin{cases} D(B) = \{u \in H^2(\Omega); u_v = 0 \; on \; \Gamma \\ \quad Bu = \Delta, \; for \; u \in D(B) \end{cases}$$

is the generator of a C_0-semigroup of contraction on X.
Proof:
Since $H^1(\Omega)$ is densely imbedded in $[H^1(\Omega)]^*$, in view of Theorem 2.6, we have merely to show that for each $\lambda > 0$, the operator $\lambda I - A : D(A) \subseteq X \to X$, where A is defined as above is one to one onto and

$$\|(\lambda I - A)^{-1}\|_{L(X)} \leq \frac{1}{\lambda}. \tag{3.9}$$

But this simply follows from the obvious identity

$$< \lambda u - Au, u >_{[H^1(\Omega)]^*, H^1(\Omega)} = \lambda \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}$$

and this achieves the proof of (i). To prove (ii), let $u \in D(B)$. Then, for each $v \in H^1(\Omega)$, we have

$$< Au, v >_{H^1(\Omega), [H^1(\Omega)]^*} = < \nabla u, \nabla v >_{L^2(\Omega)} = < \Delta u, v >_{L^2(\Omega)}$$

and thus, $Au = Bu$ for each $u \in D(B)$ and $A, B \in \omega - ORCP_n$. In addition

$$< Bu, v >_{L^2(\Omega)} = - < \nabla u, \nabla v >_{L^2(\Omega)}$$

for each $u, v \in D(B)$ and $B \in \omega - ORCP_n$. Thus B is symmetric and for each $\lambda > 0$, $\lambda I - B$ is bijective from $D(B)$ to $L^2(\Omega)$ and

$$\|(\lambda I - B)^{-1}\|_{L(X)} \leq \frac{1}{\lambda}.$$

If $D(B)$ is dense in $X = L^2(\Omega)$, then we are in the hypothesis of the Theorem 2.6 and this complete the proof.

4. Conclusion

This paper have established that $\omega - ORCP_n$ generates a C_0-semigroup of contractions which was obtained by a Laplace operator with Dirichlet boundary condition.

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A study on dual hyperbolic generalized Pell numbers

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Abstract. In this paper, we introduce the generalized dual hyperbolic Pell numbers. As special cases, we deal with dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers. We present Binet's formulas, generating functions and the summation formulas for these numbers. Moreover, we give Catalan's, Cassini's, d'Ocagne's, Gelin-Cesàro's, Melham's identities and present matrices related with these sequences.

Keywords: Pell numbers, Pell-Lucas numbers, dual hyperbolic numbers, dual hyperbolic Pell numbers, Cassini identity.

Contents

1. Introduction

A generalized Pell sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$V_n = 2V_{n-1} + V_{n-2}; \quad V_0 = a, \; V_1 = b, \quad (n \geq 2) \tag{1.1}$$

with the initial values V_0, V_1 not all being zero. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -2V_{-(n-1)} + V_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Pell numbers with positive subscript and negative subscript are given in the following Table 1.

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Table 1. A few generalized Pell numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$-2V_0 + V_1$
2	$V_0 + 2V_1$	$5V_0 - 2V_1$
3	$2V_0 + 5V_1$	$-12V_0 + 5V_1$
4	$5V_0 + 12V_1$	$29V_0 - 12V_1$
5	$12V_0 + 29V_1$	$-70V_0 + 29V_1$

If we set $V_0 = 0, V_1 = 1$ then $\{V_n\}$ is the well-known Pell sequence and if we set $V_0 = 2, V_1 = 2$ then $\{V_n\}$ is the well-known Pell-Lucas sequence. In other words, Pell sequence $\{P_n\}_{n\geq 0}$ and Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, P_1 = 1 \tag{1.2}$$

and

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad Q_0 = 2, Q_1 = 2. \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -2P_{-(n-1)} + P_{-(n-2)}$$

and

$$Q_{-n} = -2Q_{-(n-1)} + Q_{-(n-2)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [3, 8, 9, 11, 13, 16, 19, 20, 29]. For higher order Pell sequences, see [17, 18, 24, 25].

We can list some important properties of generalized Pell numbers that are needed.

- Binet formula of generalized Pell sequence can be calculated using its characteristic equation which is given as

  $$t^2 - 2t - 1 = 0.$$

  The roots of characteristic equation are

  $$\alpha = 1 + \sqrt{2}, \ \ \beta = 1 - \sqrt{2}$$

  and the roots satisfy the following

  $$\alpha + \beta = 2, \ \alpha\beta = -1, \ \alpha - \beta = 2\sqrt{2}.$$

  Using these roots and the recurrence relation, Binet formula can be given as

  $$V_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.4}$$

  where $A = V_1 - V_0\beta$ and $B = V_1 - V_0\alpha$.

- Binet formula of Pell and Pell-Lucas sequences are

  $$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

  and

  $$Q_n = \alpha^n + \beta^n$$

  respectively.

- The generating function for generalized Pell numbers is

$$g(t)=\frac{W_0+(W_1-2W_0)\,t}{1-2t-t^2}. \tag{1.5}$$

- The Cassini identity for generalized Pell numbers is

$$V_{n+1}V_{n-1}-V_n^2=(2V_0V_1-V_1^2-V_0^2). \tag{1.6}$$

-

$$A\alpha^n=\alpha V_n+V_{n-1}, \tag{1.7}$$

$$B\beta^n=\beta V_n+V_{n-1}. \tag{1.8}$$

The hypercomplex numbers systems [15], are extensions of real numbers. Complex numbers,

$$\mathbb{C}=\{z=a+ib:a,b\in\mathbb{R},i^2=-1\},$$

hyperbolic (double, split-complex) numbers [23],

$$\mathbb{H}=\{h=a+jb:a,b\in\mathbb{R},j^2=1,j\neq\pm1\},$$

and dual numbers [10],

$$\mathbb{D}=\{d=a+\varepsilon b:a,b\in\mathbb{R},\varepsilon^2=0,\varepsilon\neq0\}.$$

are some commutative examples of hypercomplex number systems. Quaternions [12],

$$\mathbb{H}_{\mathbb{Q}}=\{q=a_0+ia_1+ja_2+ka_3\},$$

where $a_0,a_1,a_2,a_3\in\mathbb{R},i^2=j^2=k^2=ijk=-1$, octonions [2] and sedenions [26] are some non-commutative examples of hypercomplex number systems.

The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions, (see for example [4, 14, 21]).

- Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [12] as an extension to the complex numbers.
- Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848 [7].
- H. H. Cheng and S. Thompson [5] introduced dual numbers with complex coefficients.
- Akar, Yüce and Şahin [1] introduced dual hyperbolic numbers.

A dual hyperbolic number is a hyper-complex number and is defined by

$$q=(a_0+ja_1)+\varepsilon(a_2+ja_3)=a_0+ja_1+\varepsilon a_2+\varepsilon ja_3$$

where a_0,a_1,a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}}=\{a_0+ja_1+\varepsilon a_2+\varepsilon ja_3\}$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, $j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0$. The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$1.\varepsilon = \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1$$
$$\varepsilon.j = j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

The product of two dual hyperbolic numbers $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ is
$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$
and addition of dual hyperbolic numbers is defined as componentwise.

For more information on the dual hyperbolic numbers, see [1].

In this paper, we define the dual hyperbolic generalized Pell numbers in the next section and give some properties of them.

2. Dual Hyperbolic Generalized Pell Numbers, Generating Functions and Binet's Formulas

In this section, we define dual hyperbolic generalized Pell numbers and present generating functions and Binet formulas for them.

In [6], the authors defined dual hyperbolic Pell and Pell-Lucas numbers and in [28], the author introduced dual hyperbolic generalized Fibonacci numbers.We now define dual hyperbolic generalized Pell numbers over $\mathbb{H}_{\mathbb{D}}$. The nth dual hyperbolic generalized Pell number is

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}. \tag{2.1}$$

As special cases, the nth dual hyperbolic Pell numbers and the nth dual hyperbolic Pell-Lucas numbers are given as

$$\widehat{P}_n = P_n + jP_{n+1} + \varepsilon P_{n+2} + j\varepsilon P_{n+3}$$

and

$$\widehat{Q}_n = Q_n + jQ_{n+1} + \varepsilon Q_{n+2} + j\varepsilon Q_{n+3}$$

respectively. It can be easily shown that

$$\widehat{V}_n = 2\widehat{V}_{n-1} + \widehat{V}_{n-2}. \tag{2.2}$$

The sequence $\{\widehat{V}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{V}_{-n} = -2\widehat{V}_{-(n-1)} + \widehat{V}_{-(n-2)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

The first few dual hyperbolic generalized Pell numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few dual hyperbolic generalized Pell numbers

n	$\widehat{V}_n$	$\widehat{V}_{-n}$
0	$\widehat{V}_0$	...
1	$\widehat{V}_1$	$-2\widehat{V}_0 + \widehat{V}_1$
2	$\widehat{V}_0 + 2\widehat{V}_1$	$5\widehat{V}_0 - 2\widehat{V}_1$
3	$2\widehat{V}_0 + 5\widehat{V}_1$	$-12\widehat{V}_0 + 5\widehat{V}_1$
4	$5\widehat{V}_0 + 12\widehat{V}_1$	$29\widehat{V}_0 - 12\widehat{V}_1$
5	$12\widehat{V}_0 + 29\widehat{V}_1$	$-70\widehat{V}_0 + 29\widehat{V}_1$

Note that

$$
\begin{aligned}
\widehat{V}_0 &= V_0 + jV_1 + \varepsilon V_2 + j\varepsilon V_3 \\
&= V_0 + jV_1 + \varepsilon(V_0 + 2V_1) + j\varepsilon(2V_0 + 5V_1), \\
\widehat{V}_1 &= V_1 + jV_2 + \varepsilon V_3 + j\varepsilon V_4 \\
&= V_1 + j(V_0 + 2V_1) + \varepsilon(2V_0 + 5V_1) + j\varepsilon(5V_0 + 12V_1).
\end{aligned}
$$

For dual hyperbolic Pell numbers (taking $V_n = P_n, P_0 = 0, P_1 = 1$) we get

$$
\begin{aligned}
\widehat{P}_0 &= j + 2\varepsilon + 5j\varepsilon, \\
\widehat{P}_1 &= 1 + 2j + 5\varepsilon + 12j\varepsilon,
\end{aligned}
$$

and for dual hyperbolic Pell-Lucas numbers (taking $V_n = Q_n, Q_0 = 2, Q_1 = 2$) we get

$$
\begin{aligned}
\widehat{Q}_0 &= 2 + 2j + 6\varepsilon + 14j\varepsilon, \\
\widehat{Q}_1 &= 2 + 6j + 14\varepsilon + 34j\varepsilon.
\end{aligned}
$$

A few dual hyperbolic Pell numbers and dual hyperbolic Pell-Lucas numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4.

Table 3. Dual hyperbolic Pell numbers

n	$\widehat{P}_n$	$\widehat{P}_{-n}$
0	$j + 2\varepsilon + 5j\varepsilon$	...
1	$1 + 2j + 5\varepsilon + 12j\varepsilon$	$1 + \varepsilon + 2j\varepsilon$
2	$2 + 5j + 12\varepsilon + 29j\varepsilon$	$-2 + j + j\varepsilon$
3	$5 + 12j + 29\varepsilon + 70j\varepsilon$	$5 + \varepsilon - 2j$
4	$12 + 29j + 70\varepsilon + 169j\varepsilon$	$-12 + 5j - 2\varepsilon + j\varepsilon$
5	$29 + 70j + 169\varepsilon + 408j\varepsilon$	$29 + 5\varepsilon - 12j - 2j\varepsilon$

Table 4. Dual hyperbolic Pell-Lucas numbers

n	$\widehat{Q}_n$	$\widehat{Q}_{-n}$
0	$2 + 2j + 6\varepsilon + 14j\varepsilon$	...
1	$2 + 6j + 14\varepsilon + 34j\varepsilon$	$-2 + 2j + 2\varepsilon + 6j\varepsilon$
2	$6 + 14j + 34\varepsilon + 82j\varepsilon$	$6 + 2\varepsilon - 2j + 2j\varepsilon$
3	$14 + 34j + 82\varepsilon + 198j\varepsilon$	$-14 + 6j - 2\varepsilon + 2j\varepsilon$
4	$34 + 82j + 198\varepsilon + 478j\varepsilon$	$34 + 6\varepsilon - 14j - 2j\varepsilon$
5	$82 + 198j + 478\varepsilon + 1154j\varepsilon$	$-82 + 34j - 14\varepsilon + 6j\varepsilon$

Now, we will state Binet's formula for the dual hyperbolic generalized Pell numbers and in the rest of the paper, we fix the following notations:

$$
\begin{aligned}
\widehat{\alpha} &= 1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3, \\
\widehat{\beta} &= 1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3.
\end{aligned}
$$

Note that we have the following identities:

$$
\begin{aligned}
\widehat{\alpha} &= 1 + j\alpha + \varepsilon(2\alpha + 1) + j\varepsilon(5\alpha + 2), \\
\widehat{\beta} &= 1 + j\beta + \varepsilon(2\beta + 1) + j\varepsilon(5\beta + 2), \\
\widehat{\alpha}^2 &= 2 + 2\alpha + 2\alpha j + (12 + 28\alpha)\varepsilon + (8 + 20\alpha)j\varepsilon, \\
\widehat{\beta}^2 &= 2 + 2\beta + 2\beta j + (12 + 28\beta)\varepsilon + (8 + 20\beta)j\varepsilon, \\
\widehat{\alpha}\widehat{\beta} &= 2j + 12j\varepsilon, \\
\widehat{\alpha}^2\widehat{\beta} &= 2\alpha + 2j + (4 + 22\alpha)\varepsilon + (14 + 4\alpha)j\varepsilon, \\
\widehat{\alpha}\widehat{\beta}^2 &= 2\beta + 2j + (4 + 22\beta)\varepsilon + (14 + 4\beta)j\varepsilon, \\
\widehat{\alpha}^2\widehat{\beta}^2 &= 4 + 48\varepsilon.
\end{aligned}
$$

Theorem 2.1. *(Binet's Formula) For any integer n, the nth dual hyperbolic generalized Pell number is*

$$\widehat{V}_n = \frac{A\widehat{\alpha}\alpha^n - B\widehat{\beta}\beta^n}{\alpha - \beta}. \tag{2.3}$$

Proof. Using Binet's formula

$$V_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

of the generalized Pell numbers, we obtain

$$
\begin{aligned}
\widehat{V}_n &= V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3} \\
&= \frac{A\alpha^n - B\beta^n}{\alpha - \beta} + j\frac{A\alpha^{n+1} - B\beta^{n+1}}{\alpha - \beta} \\
&\quad + \varepsilon\frac{A\alpha^{n+2} - B\beta^{n+2}}{\alpha - \beta} + j\varepsilon\frac{A\alpha^{n+3} - B\beta^{n+3}}{\alpha - \beta} \\
&= \frac{A(1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3)\alpha^n}{\alpha - \beta} \\
&\quad - \frac{B(1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3)\beta^n}{\alpha - \beta}.
\end{aligned}
$$

This proves (2.3).

As special cases, for any integer n, the Binet's Formula of nth dual hyperbolic Pell number is

$$\widehat{P}_n = \frac{\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n}{\alpha - \beta} \tag{2.4}$$

and the Binet's Formula of nth dual hyperbolic Pell-Lucas number is

$$\widehat{Q}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n. \tag{2.5}$$

Next, we present generating function.

Theorem 2.2. *The generating function for the dual hyperbolic generalized Pell numbers is*

$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = \frac{\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)x}{1 - 2x - x^2}. \tag{2.6}$$

Proof. Let

$$g(x)=\sum_{n=0}^{\infty}\widehat{V}_n x^n$$

be generating function of the dual hyperbolic generalized Pell numbers. Then, using the definition of the dual hyperbolic generalized Pell numbers, and substracting $2xg(x)$ and $x^2g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1-2x-x^2)g(x) &= \sum_{n=0}^{\infty}\widehat{V}_n x^n - 2x\sum_{n=0}^{\infty}\widehat{V}_n x^n - x^2\sum_{n=0}^{\infty}\widehat{V}_n x^n \\
&= \sum_{n=0}^{\infty}\widehat{V}_n x^n - 2\sum_{n=0}^{\infty}\widehat{V}_n x^{n+1} - \sum_{n=0}^{\infty}\widehat{V}_n x^{n+2} \\
&= \sum_{n=0}^{\infty}\widehat{V}_n x^n - 2\sum_{n=1}^{\infty}\widehat{V}_{n-1} x^{n} - \sum_{n=2}^{\infty}\widehat{V}_{n-2} x^{n} \\
&= (\widehat{V}_0+\widehat{V}_1 x)-2\widehat{V}_0 x \\
&\quad +\sum_{n=2}^{\infty}(\widehat{V}_n-2\widehat{V}_{n-1}-\widehat{V}_{n-2})x^n \\
&= (\widehat{V}_0+\widehat{V}_1 x)-2\widehat{V}_0 x \\
&= \widehat{V}_0+(\widehat{V}_1-2\widehat{V}_0)x.
\end{aligned}$$

Note that we used the recurrence relation $\widehat{V}_n=2\widehat{V}_{n-1}+\widehat{V}_{n-2}$. Rearranging above equation, we get

$$g(x)=\frac{\widehat{V}_0+(\widehat{V}_1-2\widehat{V}_0)x}{1-2x-x^2}.$$

As special cases, the generating functions for the dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers are

$$\sum_{n=0}^{\infty}\widehat{P}_n x^n=\frac{(j+2\varepsilon+5j\varepsilon)+(1+\varepsilon+2j\varepsilon)x}{1-2x-x^2}$$

and

$$\sum_{n=0}^{\infty}\widehat{Q}_n x^n=\frac{(2+2j+6\varepsilon+14j\varepsilon)+(-2+2j+2\varepsilon+6j\varepsilon)x}{1-2x-x^2}$$

respectively.

3. Obtaining Binet Formula from Generating Function

We will next find Binet formula of dual hyperbolic generalized Pell number $\{\widehat{V}_n\}$ by the use of generating function for $\widehat{V}_n$.

Theorem 3.1. *(Binet formula of dual hyperbolic generalized Pell numbers)*

$$\widehat{V}_n=\frac{d_1\alpha^n}{(\alpha-\beta)}-\frac{d_2\beta^n}{(\alpha-\beta)} \tag{3.1}$$

where

$$\begin{aligned}
d_1 &= \widehat{V}_0\alpha+(\widehat{V}_1-2\widehat{V}_0), \\
d_2 &= \widehat{V}_0\beta+(\widehat{V}_1-2\widehat{V}_0).
\end{aligned}$$

Proof. Let

$$h(x) = 1 - 2x - x^2.$$

Then for some α and β we write

$$h(x) = (1 - \alpha x)(1 - \beta x)$$

i.e.,

$$1 - 2x - x^2 = (1 - \alpha x)(1 - \beta x) \tag{3.2}$$

Hence $\frac{1}{\alpha}$ ve $\frac{1}{\beta}$ are the roots of $h(x)$. This gives α and β as the roots of

$$h(\frac{1}{x}) = 1 - \frac{2}{x} - \frac{1}{x^2} = 0.$$

This implies $x^2 - 2x - 1 = 0$. Now, by (2.6) and (3.2), it follows that

$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = \frac{\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)x}{(1 - \alpha x)(1 - \beta x)}.$$

Then we write

$$\frac{\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)x}{(1 - \alpha x)(1 - \beta x)} = \frac{A_1}{(1 - \alpha x)} + \frac{A_2}{(1 - \beta x)}. \tag{3.3}$$

So

$$\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)x = A_1(1 - \beta x) + A_2(1 - \alpha x).$$

If we consider $x = \frac{1}{\alpha}$, we get $\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)\frac{1}{\alpha} = A_1(1 - \beta\frac{1}{\alpha})$. This gives

$$A_1 = \frac{\widehat{V}_0\alpha + (\widehat{V}_1 - 2\widehat{V}_0)}{(\alpha - \beta)} = \frac{d_1}{(\alpha - \beta)}.$$

Similarly, we obtain

$$\begin{aligned} \widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)\frac{1}{\beta} &= A_2(1 - \alpha\frac{1}{\beta}) \\ &\Rightarrow \widehat{V}_0\beta + (\widehat{V}_1 - 2\widehat{V}_0) = A_2(\beta - \alpha) \end{aligned}$$

and so

$$A_2 = -\frac{\widehat{V}_0\beta + (\widehat{V}_1 - 2\widehat{V}_0)}{(\alpha - \beta)} = -\frac{d_2}{(\alpha - \beta)}.$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = A_1(1 - \alpha x)^{-1} + A_2(1 - \beta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$\widehat{V}_n = A_1\alpha^n + A_2\beta^n$$

and then we get (3.1).

Note that from (2.3) and (3.1) we have

$$
\begin{aligned}
(V_1 - V_0\beta)\widehat{\alpha} &= \widehat{V}_0\alpha + (\widehat{V}_1 - 2\widehat{V}_0),\\
(V_1 - V_0\alpha)\widehat{\beta} &= \widehat{V}_0\beta + (\widehat{V}_1 - 2\widehat{V}_0).
\end{aligned}
$$

Next, using Theorem 3.1, we present the Binet formulas of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers.

Corollary 3.2. *Binet formulas of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers are*

$$\widehat{P}_n = \frac{\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n}{\alpha - \beta}$$

and

$$\widehat{Q}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n$$

respectively.

4. Some Identities

We now present a few special identities for the dual hyperbolic generalized Pell sequence $\{\widehat{V}_n\}$. The following theorem presents the Catalan's identity for the dual hyperbolic generalized Pell numbers.

Theorem 4.1. *(Catalan's identity) For all integers n and m, the following identity holds*

$\widehat{V}_{n+m}\widehat{V}_{n-m} - \widehat{V}_n^2 = \frac{(-1)^{n-m+1}((A+B)V_{2m-1}+(A\beta+B\alpha)V_{2m}-2(-1)^m AB)}{8}(2j + 12j\varepsilon).$

Proof. Using the Binet Formula

$$\widehat{V}_n = \frac{A\widehat{\alpha}\alpha^n - B\widehat{\beta}\beta^n}{\alpha - \beta}$$

and

$$
\begin{aligned}
A\alpha^n &= \alpha V_n + V_{n-1},\\
B\beta^n &= \beta V_n + V_{n-1},
\end{aligned}
$$

we get

$$
\begin{aligned}
&\widehat{V}_{n+m}\widehat{V}_{n-m} - \widehat{V}_n^2\\
&= \frac{(A\widehat{\alpha}\alpha^{n+m} - B\widehat{\beta}\beta^{n+m})(A\widehat{\alpha}\alpha^{n-m} - B\widehat{\beta}\beta^{n-m}) - (A\widehat{\alpha}\alpha^n - B\widehat{\beta}\beta^n)^2}{(\alpha-\beta)^2}\\
&= \frac{-AB\widehat{\alpha}\widehat{\beta}\alpha^{n+m}\beta^{n-m} - AB\widehat{\beta}\widehat{\alpha}\alpha^{n-m}\beta^{n+m} + 2AB\widehat{\alpha}\widehat{\beta}\alpha^n\beta^n}{(\alpha-\beta)^2}\\
&= \frac{-AB\widehat{\alpha}\widehat{\beta}\alpha^{n+m}\beta^{n-m} - AB\widehat{\alpha}\widehat{\beta}\alpha^{n-m}\beta^{n+m} + 2AB\widehat{\alpha}\widehat{\beta}\alpha^n\beta^n}{(\alpha-\beta)^2}\\
&= -AB\widehat{\alpha}\widehat{\beta}\frac{(\alpha^m - \beta^m)^2}{(\alpha-\beta)^2}\alpha^{n-m}\beta^{n-m}\\
&= \frac{(-1)^{n-m+1}AB(\alpha^m - \beta^m)^2}{8}\widehat{\alpha}\widehat{\beta}\\
&= \frac{(-1)^{n-m+1}((A+B)V_{2m-1}+(A\beta+B\alpha)V_{2m}-2(-1)^m AB)}{8}(2j + 12j\varepsilon)
\end{aligned}
$$

where $\alpha\beta = -1$ and $\widehat{\alpha}\widehat{\beta} = 2j + 12j\varepsilon$.

As special cases of the above theorem, we give Catalan's identity of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers. Firstly, we present Catalan's identity of dual hyperbolic Pell numbers.

Corollary 4.2. *(Catalan's identity for the dual hyperbolic Pell numbers) For all integers n and m, the following identity holds*

$$\widehat{P}_{n+m}\widehat{P}_{n-m} - \widehat{P}_n^2 = \frac{(-1)^{n-m+1}\left(P_{2m-1} + P_{2m} - (-1)^m\right)}{2}(j + 6j\varepsilon).$$

Proof. Taking $V_n = P_n$ in Theorem 4.1 we get the required result.

Secondly, we give Catalan's identity of dual hyperbolic Pell-Lucas numbers.

Corollary 4.3. *(Catalan's identity for the dual hyperbolic Pell-Lucas numbers) For all integers n and m, the following identity holds*

$$\widehat{Q}_{n+m}\widehat{Q}_{n-m} - \widehat{Q}_n^2 = (-1)^{n-m}\left(Q_{2m} - 2(-1)^m\right)(2j + 12j\varepsilon).$$

Proof. Taking $V_n = Q_n$ in Theorem 4.1, we get the required result.

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the dual hyperbolic generalized Pell sequence.

Corollary 4.4. *(Cassini's identity) For all integers n, the following identity holds*

$\widehat{V}_{n+1}\widehat{V}_{n-1} - \widehat{V}_n^2 = \frac{(-1)^n((A+B)V_1+(A\beta+B\alpha)V_2+2AB)}{4}(j + 6j\varepsilon).$

As special cases of Cassini's identity, we give Cassini's identity of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers. Firstly, we present Cassini's identity of dual hyperbolic Pell numbers.

Corollary 4.5. *(Cassini's identity of dual hyperbolic Pell numbers) For all integers n, the following identity holds*

$$\widehat{P}_{n+1}\widehat{P}_{n-1} - \widehat{P}_n^2 = 2(-1)^n(j + 6j\varepsilon).$$

Secondly, we give Cassini's identity of dual hyperbolic Pell-Lucas numbers.

Corollary 4.6. *(Cassini's identity of dual hyperbolic Pell-Lucas numbers) For all integers n, the following identity holds*

$$\widehat{Q}_{n+1}\widehat{Q}_{n-1} - \widehat{Q}_n^2 = 16(-1)^{n+1}(j + 6j\varepsilon).$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using the Binet Formula of the dual hyperbolic generalized Pell sequence:

$$\widehat{V}_n = \frac{A\widehat{\alpha}\alpha^n - B\widehat{\beta}\beta^n}{\alpha - \beta}.$$

The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of the dual hyperbolic generalized Pell sequence $\{\widehat{V}_n\}$.

Theorem 4.7. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$\widehat{V}_{m+1}\widehat{V}_n - \widehat{V}_m\widehat{V}_{n+1} = \left(V_nV_{m-1} - V_mV_{n-1}\right)(2j + 12j\varepsilon).$$

(b) *(Gelin-Cesàro's identity)*

$$\widehat{V}_{n+2}\widehat{V}_{n+1}\widehat{V}_{n-1}\widehat{V}_{n-2} - \widehat{V}_n^4 = \frac{AB(-1)^{n+1}}{2}(k_1 + k_2j + k_3\varepsilon + k_4j\varepsilon)$$

where

$$k_1 = 26(-1)^n AB + 6(V_{2n-1}(V_0+V_1) + V_{2n}(V_0+3V_1))$$

$$k_2 = 3(4V_{2n}(V_0+2V_1) + V_{2n-1}((A+B)+2(V_0+V_1)))$$

$$k_3 = 12(26(-1)^n AB + 2V_{2n}(5V_0+13V_1) + V_{2n-1}(A+B+8(V_0+V_1)))$$

$$k_4 = 12(V_{2n}(16V_0+36V_1) + V_{2n-1}(3(A+B)+10(V_1+V_0))).$$

(c) *(Melham's identity)*
$\widehat{V}_{n+1}\widehat{V}_{n+2}\widehat{V}_{n+6} - \widehat{V}_{n+3}^3 = 2(-1)^n AB((91V_n+38V_{n-1}) + (38V_n+15V_{n-1})j + (1077V_n+448V_{n-1})\varepsilon + (448V_n+181V_{n-1})j\varepsilon)$.

Proof.

(a) Using (1.7) and (1.8) we obtain

$$\begin{aligned}
&\widehat{V}_{m+1}\widehat{V}_n - \widehat{V}_m\widehat{V}_{n+1} \\
&= \frac{AB\widehat{\alpha}\widehat{\beta}(-\alpha^{m+1}\beta^n - \alpha^n\beta^{m+1} + \alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m)}{(\alpha-\beta)^2} \\
&= \frac{AB(\alpha^n\beta^m - \alpha^m\beta^n)}{(\alpha-\beta)}\widehat{\alpha}\widehat{\beta} \\
&= \frac{(\alpha V_n + V_{n-1})(\beta V_m + V_{m-1})}{(\alpha-\beta)}(2j+12j\varepsilon) \\
&\quad - \frac{(\alpha V_m + V_{m-1})(\beta V_n + V_{n-1})}{(\alpha-\beta)}(2j+12j\varepsilon) \\
&= (V_nV_{m-1} - V_mV_{n-1})(2j+12j\varepsilon).
\end{aligned}$$

(b) It requires lengthy and tedious work. So we omit the proof.

(c) Using (1.7), (1.8) and Binet formula of $\widehat{V}_n$, we get

$$\widehat{V}_{n+1}\widehat{V}_{n+2}\widehat{V}_{n+6} - \widehat{V}_{n+3}^3 = (-1)^{n+1} AB\left(-\tfrac{30+23\sqrt{2}}{4}A\widehat{\alpha}\alpha^n + \tfrac{-30+23\sqrt{2}}{4}B\widehat{\beta}\beta^n\right)\widehat{\alpha}\widehat{\beta}$$

and then using

$$\begin{aligned}
\widehat{\alpha}^2\widehat{\beta} &= 2\alpha + 2j + (4+22\alpha)\varepsilon + (14+4\alpha)j\varepsilon, \\
\widehat{\alpha}\widehat{\beta}^2 &= 2\beta + 2j + (4+22\beta)\varepsilon + (14+4\beta)j\varepsilon,
\end{aligned}$$

we obtain the required result.

As special cases of the above theorem, we give the d'Ocagne's, Gelin-Cesàro's and Melham' identities of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers. Firstly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of dual hyperbolic Pell numbers.

Corollary 4.8. *Let n and m be any integers. Then, for the dual hyperbolic Pell numbers, the following identities are true:*

(a) *(d'Ocagne's identity)*

$$\widehat{P}_{m+1}\widehat{P}_n - \widehat{P}_m\widehat{P}_{n+1} = (P_nP_{m-1} - P_mP_{n-1})(2j+12j\varepsilon).$$

(b) *(Gelin-Cesàro's identity)*

$\widehat{P}_{n+2}\widehat{P}_{n+1}\widehat{P}_{n-1}\widehat{P}_{n-2}-\widehat{P}_{n}^{4}=(-1)^{n+1}(13(-1)^{n}+3(3P_{2n}+P_{2n-1})+6(2P_{2n}+P_{2n-1})j+12(13(-1)^{n}+13P_{2n}+5P_{2n-1})\varepsilon+24(9P_{2n}+4P_{2n-1})j\varepsilon).$

(c) *(Melham's identity)*

$\widehat{P}_{n+1}\widehat{P}_{n+2}\widehat{P}_{n+6}-\widehat{P}_{n+3}^{3}=2(-1)^{n}((91P_{n}+38P_{n-1})+(38P_{n}+15P_{n-1})j+(1077P_{n}+448P_{n-1})\varepsilon+(448P_{n}+181P_{n-1})j\varepsilon).$

Secondly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of dual hyperbolic Pell-Lucas numbers.

Corollary 4.9. *Let n and m be any integers. Then, for the dual hyperbolic Pell-Lucas numbers, the following identities are true:*

(a) *(d'Ocagne's identity)*

$$\widehat{Q}_{m+1}\widehat{Q}_{n}-\widehat{Q}_{m}\widehat{Q}_{n+1}=(Q_{n}Q_{m-1}-Q_{m}Q_{n-1})(2j+12j\varepsilon).$$

(b) *(Gelin-Cesàro's identity)*

$\widehat{Q}_{n+2}\widehat{Q}_{n+1}\widehat{Q}_{n-1}\widehat{Q}_{n-2}-\widehat{Q}_{n}^{4}=32(-1)^{n}(26(-1)^{n+1}+3(2Q_{2n}+Q_{2n-1})+3(3Q_{2n}+Q_{2n-1})j+12(26(-1)^{n+1}+9Q_{2n}+4Q_{2n-1})\varepsilon+12(13Q_{2n}+5Q_{2n-1})j\varepsilon).$

(c) *(Melham's identity)*

$\widehat{Q}_{n+1}\widehat{Q}_{n+2}\widehat{Q}_{n+6}-\widehat{Q}_{n+3}^{3}=16(-1)^{n+1}((91Q_{n}+38Q_{n-1})+(38Q_{n}+15Q_{n-1})j+(1077Q_{n}+448Q_{n-1})\varepsilon+(448Q_{n}+181Q_{n-1})j\varepsilon).$

5. Linear Sums

In this section, we give the summation formulas of the dual hyperbolic generalized Pell numbers with positive and negatif subscripts. Now, we present the summation formulas of the generalized Pell numbers.

Proposition 5.1. *For the generalized Pell numbers, for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}V_{k}=\frac{1}{2}(V_{n+2}-V_{n+1}-V_{1}+V_{0}).$

(b) $\sum_{k=0}^{n}V_{2k}=\frac{1}{2}(V_{2n+1}-V_{1}+2V_{0}).$

(c) $\sum_{k=0}^{n}V_{2k+1}=\frac{1}{2}(V_{2n+2}-V_{2}+2V_{1}).$

Proof. For the proof, see Soykan [27].

Next, we present the formulas which give the summation of the first n dual hyperbolic generalized Pell numbers.

Theorem 5.2. *For $n\geq 0$, dual hyperbolic generalized Pell numbers have the following formulas:.*

(a) $\sum_{k=0}^{n}\widehat{V}_{k}=\frac{1}{2}(\widehat{V}_{n+2}-\widehat{V}_{n+1}-\widehat{V}_{1}+\widehat{V}_{0}).$

(b) $\sum_{k=0}^{n}\widehat{V}_{2k}=\frac{1}{2}(\widehat{V}_{2n+1}-\widehat{V}_{1}+2\widehat{V}_{0}).$

(c) $\sum_{k=0}^{n}\widehat{V}_{2k+1}=\frac{1}{2}(\widehat{V}_{2n+2}-\widehat{V}_{0}).$

Proof. Note that using Proposition 5.1 (a) we get

$$\begin{aligned}\sum_{k=0}^{n} V_{k+1} &= \frac{1}{2}(V_{n+3}-V_{n+2}-V_1-V_0),\\ \sum_{k=0}^{n} V_{k+2} &= \frac{1}{2}(V_{n+4}-V_{n+3}-3V_1-V_0),\\ \sum_{k=0}^{n} V_{k+3} &= \frac{1}{2}(V_{n+5}-V_{n+4}-7V_1-3V_0).\end{aligned}$$

Then it follows that

$$\sum_{k=0}^{n} \widehat{V}_k = \tfrac{1}{2}((V_{n+2}+jV_{n+3}+\varepsilon V_{n+4}+j\varepsilon V_{n+5})-(V_{n+1}+jV_{n+2}+\varepsilon V_{n+3}+j\varepsilon V_{n+4})$$
$$+(-V_1+V_0)+j(-V_1-V_0)+\varepsilon(-3V_1-V_0)+j\varepsilon(-7V_1-3V_0))$$
$$=\frac{1}{2}(\widehat{V}_{n+2}-\widehat{V}_{n+1}+((-V_1+V_0)+j(-V_2+V_1)+\varepsilon(-V_3+V_2)+j\varepsilon(-V_4+V_3))$$
$$=\frac{1}{2}(\widehat{V}_{n+2}-\widehat{V}_{n+1}-\widehat{V}_1+\widehat{V}_0).$$

This proves (a).

(b) Note that using Proposition 5.1 (b) and (c) we get

$$\begin{aligned}\sum_{k=0}^{n} V_{2k+2} &= \frac{1}{2}(V_{2n+3}-V_1),\\ \sum_{k=0}^{n} V_{2k+3} &= \frac{1}{2}(V_{2n+4}-2V_1-V_0).\end{aligned}$$

Then it follows that

$$\begin{aligned}&\sum_{k=0}^{n} \widehat{V}_{2k}\\ &= \frac{1}{2}((V_{2n+1}+jV_{2n+2}+\varepsilon V_{2n+3}+j\varepsilon V_{2n+4})\\ &\quad +((-V_1+2V_0)+j(-V_0)+\varepsilon(-V_1)+j\varepsilon(-2V_1-V_0)))\\ &= \frac{1}{2}((V_{2n+1}+jV_{2n+2}+\varepsilon V_{2n+3}+j\varepsilon V_{2n+4})\\ &\quad +((-V_1+2V_0)+j(-V_2+2V_1)+\varepsilon(-V_3+2V_2)+j\varepsilon(-V_4+2V_3))\\ &= \frac{1}{2}((V_{2n+1}+jV_{2n+2}+\varepsilon V_{2n+3}+j\varepsilon V_{2n+4})\\ &\quad -(V_1+jV_2+\varepsilon V_3+j\varepsilon V_4)+2(V_0+jV_1+\varepsilon V_2+j\varepsilon V_3))\\ &= \frac{1}{2}(\widehat{V}_{2n+1}-\widehat{V}_1+2\widehat{V}_0).\end{aligned}$$

(c) Note that using Proposition 5.1 (b) and (c) we get

$$\sum_{k=0}^{n} V_{2k+4} = \frac{1}{2}(V_{2n+5}-5V_1-2V_0).$$

Then it follows that

$$\begin{aligned}
&\sum_{k=0}^{n}\widehat{V}_{2k+1}\\
&=\frac{1}{2}((V_{2n+2}+jV_{2n+3}+\varepsilon V_{2n+4}+j\varepsilon V_{2n+5})\\
&\quad -(V_0+jV_1+\varepsilon(2V_1+V_0)+j\varepsilon(5V_1+2V_0)))\\
&=\frac{1}{2}(\widehat{V}_{2n+2}-(V_0+jV_1+\varepsilon V_2+j\varepsilon V_3))\\
&=\frac{1}{2}(\widehat{V}_{2n+2}-\widehat{V}_0).
\end{aligned}$$

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic Pell numbers:

Corollary 5.3. *For $n\geq 0$, dual hyperbolic Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n}\widehat{P}_k=\frac{1}{2}(\widehat{P}_{n+2}-\widehat{P}_{n+1}-\widehat{P}_1+\widehat{P}_0)=\frac{1}{2}(\widehat{P}_{n+2}-\widehat{P}_{n+1}-(1+j+3\varepsilon+7j\varepsilon))$.

(b) $\sum_{k=0}^{n}\widehat{P}_{2k}=\frac{1}{2}(\widehat{P}_{2n+1}-\widehat{P}_1+2\widehat{P}_0)=\frac{1}{2}(\widehat{P}_{2n+1}-(1+\varepsilon+2j\varepsilon))$.

(c) $\sum_{k=0}^{n}\widehat{P}_{2k+1}=\frac{1}{2}(\widehat{P}_{2n+2}-\widehat{P}_0)=\frac{1}{2}(\widehat{P}_{2n+2}-(j+2\varepsilon+5j\varepsilon))$.

As a second special case of the above theorem, we have the following summation formulas for dual hyperbolic Pell-Lucas numbers:

Corollary 5.4. *For $n\geq 0$, dual hyperbolic Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n}\widehat{Q}_k=\frac{1}{2}(\widehat{Q}_{n+2}-\widehat{Q}_{n+1}-\widehat{Q}_1+\widehat{Q}_0)=\frac{1}{2}(\widehat{Q}_{n+2}-\widehat{Q}_{n+1}-4\,(j+2\varepsilon+5j\varepsilon))$.

(b) $\sum_{k=0}^{n}\widehat{Q}_{2k}=\frac{1}{2}(\widehat{Q}_{2n+1}-\widehat{Q}_1+2\widehat{Q}_0)=\frac{1}{2}(\widehat{Q}_{2n+1}+2(1-j-\varepsilon-3j\varepsilon))$.

(c) $\sum_{k=0}^{n}\widehat{Q}_{2k+1}=\frac{1}{2}(\widehat{Q}_{2n+2}-\widehat{Q}_0)=\frac{1}{2}(\widehat{Q}_{2n+2}-(2+2j+6\varepsilon+14j\varepsilon))$.

Now, we present the formula which give the summation formulas of the generalized Pell numbers with negative subscripts.

Proposition 5.5. *For $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}V_{-k}=\frac{1}{2}(-3V_{-n-1}-V_{-n-2}+V_1-V_0)$.

(b) $\sum_{k=1}^{n}V_{-2k}=\frac{1}{2}(-V_{-2n-1}+V_1-2V_0)$.

(c) $\sum_{k=1}^{n}V_{-2k+1}=\frac{1}{2}(-V_{-2n}+V_0)$.

Proof. This is given in Soykan [27].

Next, we present the formulas which give the summation of the first n dual hyperbolic generalized Pell numbers with negative subscripts

Theorem 5.6. *For $n\geq 1$, dual hyperbolic generalized Pell numbers have the following formulas:*

(a) $\sum_{k=1}^{n}\widehat{V}_{-k}=\frac{1}{2}(-3\widehat{V}_{-n-1}-\widehat{V}_{-n-2}+\widehat{V}_1-\widehat{V}_0)$.

(b) $\sum_{k=1}^{n}\widehat{V}_{-2k}=\frac{1}{2}(-\widehat{V}_{-2n-1}+\widehat{V}_1-2\widehat{V}_0)$.

(c) $\sum_{k=1}^{n}\widehat{V}_{-2k+1}=\frac{1}{2}(-\widehat{V}_{-2n}+\widehat{V}_{0})$.

Proof. We prove (a). Note that using Proposition 5.1 (a) we get

$$\sum_{k=1}^{n}V_{-k+1}=\frac{1}{2}(-3V_{-n}-V_{-n-1}+V_{1}+V_{0}),$$
$$\sum_{k=1}^{n}V_{-k+2}=\frac{1}{2}(-3V_{-n+1}-V_{-n}+3V_{1}+V_{0}),$$
$$\sum_{k=1}^{n}V_{-k+3}=\frac{1}{2}(-3V_{-n+2}-V_{-n+1}+7V_{1}+3V_{0}).$$

Then it follows that

$\sum_{k=1}^{n}\widehat{V}_{-k}=\frac{1}{2}(3(V_{-n-1}+jV_{-n}+\varepsilon V_{-n+1}+j\varepsilon V_{-n+2})-(V_{-n-2}+jV_{-n-1}+\varepsilon V_{-n}+j\varepsilon V_{-n+1})$
$+(V_{1}-V_{0})+j(V_{1}+V_{0})+\varepsilon(3V_{1}+V_{0})+j\varepsilon(7V_{1}+3V_{0}))$
$=\dfrac{1}{2}(-3\widehat{V}_{-n-1}-\widehat{V}_{-n-2}+((V_{1}-V_{0})+j(V_{2}-V_{1})+\varepsilon(V_{3}-V_{2})+j\varepsilon(V_{4}-V_{3}))$
$=\dfrac{1}{2}(-3\widehat{V}_{-n-1}-\widehat{V}_{-n-2}+\widehat{V}_{1}-\widehat{V}_{0})$.

This proves (a). (b) and (c) can be proved similarly.

As a first special case of above theorem, we have the following summation formulas for dual hyperbolic Pell numbers:

Corollary 5.7. *For $n\geq 1$, dual hyperbolic Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n}\widehat{P}_{-k}=\frac{1}{2}(-3\widehat{P}_{-n-1}-\widehat{P}_{-n-2}+\widehat{P}_{1}-\widehat{P}_{0})=\frac{1}{2}(-3\widehat{P}_{-n-1}-\widehat{P}_{-n-2}+(1+j+3\varepsilon+7j\varepsilon))$.

(b) $\sum_{k=1}^{n}\widehat{P}_{-2k}=\frac{1}{2}(-\widehat{P}_{-2n-1}+\widehat{P}_{1}-2\widehat{P}_{0})=\frac{1}{2}(-\widehat{P}_{-2n-1}+(1+\varepsilon+2j\varepsilon))$.

(c) $\sum_{k=1}^{n}\widehat{P}_{-2k+1}=\frac{1}{2}(-\widehat{P}_{-2n}+\widehat{P}_{0})=\frac{1}{2}(-\widehat{P}_{-2n}+(j+2\varepsilon+5j\varepsilon))$.

Corollary 5.8. *For $n\geq 1$, dual hyperbolic Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n}\widehat{Q}_{-k}=\frac{1}{2}(-3\widehat{Q}_{-n-1}-\widehat{Q}_{-n-2}+\widehat{Q}_{1}-\widehat{Q}_{0})=\frac{1}{2}(-3\widehat{Q}_{-n-1}-\widehat{Q}_{-n-2}+(4j+8\varepsilon+20j\varepsilon))$.

(b) $\sum_{k=1}^{n}\widehat{Q}_{-2k}=\frac{1}{2}(-\widehat{Q}_{-2n-1}+\widehat{Q}_{1}-2\widehat{Q}_{0})=\frac{1}{2}(-\widehat{Q}_{-2n-1}+(-2+2j+2\varepsilon+6j\varepsilon))$.

(c) $\sum_{k=1}^{n}\widehat{Q}_{-2k+1}=\frac{1}{2}(-\widehat{Q}_{-2n}+\widehat{Q}_{0})=\frac{1}{2}(-\widehat{Q}_{-2n}+(2+2j+6\varepsilon+14j\varepsilon))$.

6. Matrices related with Dual Hyperbolic Generalized Pell Numbers

We define the square matrix M of order 2 as:

$$M=\begin{pmatrix}2&1\\1&0\end{pmatrix}$$

such that $\det M=-1$. Induction proof may be used to establish

$$M^{n}=\begin{pmatrix}P_{n+1}&P_{n}\\P_{n}&P_{n-1}\end{pmatrix}\tag{6.1}$$

and (the matrix formulation of V_n)

$$\begin{pmatrix}V_{n+1}\\V_{n}\end{pmatrix}=\begin{pmatrix}2&1\\1&0\end{pmatrix}^{n}\begin{pmatrix}V_{1}\\V_{0}\end{pmatrix}.\tag{6.2}$$

Now, we define the matrices M_V as

$$M_V = \begin{pmatrix} \widehat{V}_3 & \widehat{V}_2 \\ \widehat{V}_2 & \widehat{V}_1 \end{pmatrix}.$$

This matrice M_V is called dual hyperbolic generalized Pell matrix. As special cases, dual hyperbolic Pell matrix and dual hyperbolic Pell-Lucas matrix are

$$M_P = \begin{pmatrix} \widehat{P}_3 & \widehat{P}_2 \\ \widehat{P}_2 & \widehat{P}_1 \end{pmatrix} \text{ and}$$

$$M_Q = \begin{pmatrix} \widehat{Q}_3 & \widehat{Q}_2 \\ \widehat{Q}_2 & \widehat{Q}_1 \end{pmatrix}$$

respectively.

Theorem 6.1. *For $n \geq 0$, the following is valid:*

$$M_V \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{V}_{n+3} & \widehat{V}_{n+2} \\ \widehat{V}_{n+2} & \widehat{V}_{n+1} \end{pmatrix}. \tag{6.3}$$

Proof. We prove by mathematical induction on n. If $n = 0$, then the result is clear. Now, we assume it is true for $n = k$, that is

$$M_V M^k = \begin{pmatrix} \widehat{V}_{k+3} & \widehat{V}_{k+2} \\ \widehat{V}_{k+2} & \widehat{V}_{k+1} \end{pmatrix}.$$

If we use (2.1), then we have $\widehat{V}_{k+2} = 2\widehat{V}_{k+1} + \widehat{V}_k$. Then, by induction hypothesis, we obtain

$$\begin{aligned} M_V M^{k+1} = (M_V M^k)M &= \begin{pmatrix} \widehat{V}_{k+3} & \widehat{V}_{k+2} \\ \widehat{V}_{k+2} & \widehat{V}_{k+1} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2\widehat{V}_{k+3} + \widehat{V}_{k+2} & \widehat{V}_{k+3} \\ 2\widehat{V}_{k+2} + \widehat{V}_{k+1} & \widehat{V}_{k+2} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{V}_{k+4} & \widehat{V}_{k+3} \\ \widehat{V}_{k+3} & \widehat{V}_{k+2} \end{pmatrix}. \end{aligned}$$

Thus, (6.3) holds for all non-negative integers n.

Remark 6.2. *The above theorem is true for $n \leq -1$. It can also be proved by induction.*

Corollary 6.3. *For all integers n, the following holds:*

$$\widehat{V}_{n+2} = \widehat{V}_2 P_{n+1} + \widehat{V}_1 P_n.$$

Proof. The proof can be seen by the coefficient of the matrix M_V and (6.1).

Taking $V_n = P_n$ and $V_n = Q_n$, respectively, in the above corollary, we obtain the following results.

Corollary 6.4. *For all integers n, the followings are true.*

(a) $\widehat{P}_{n+2} = \widehat{P}_2 P_{n+1} + \widehat{P}_1 P_n$.

(b) $\widehat{Q}_{n+2} = \widehat{Q}_2 P_{n+1} + \widehat{Q}_1 P_n$.

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Existence of solutions of a second order equation defined on unbounded intervals with integral conditions on the boundary

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Abstract. In this paper we shall use the upper and lower solutions method to prove the existence of at least one solution for the second order equation defined on unbounded intervals with integral conditions on the boundary:

$$u''(t) - m^2 u(t) + f(t, e^{-mt} u(t), e^{-mt} u'(t)) = 0, \quad \text{for all } t \in [0, +\infty),$$

$$u(0) - \frac{1}{m} u'(0) = \int_0^{+\infty} e^{-2ms} u(s)\, ds, \ \lim_{t \to +\infty} \{e^{-mt} u(t)\} = B,$$

where $m > 0, m \neq \frac{1}{6}, B \in \mathbb{R}$ and $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying a suitable locally L^1 bounded condition and a kind of Nagumo's condition with respect to the first derivative.

AMS Subject Classifications: 34B40, 34B15, 74H20.

Keywords: Boundary value problems, Integral boundary conditions, Upper and lower solutions method, Existence of solution.

Contents

1. Introduction

Integral boundary conditions have been considered in many papers on the literature. They represent a nonlocal dependence of the solution at some points of the interval. For instance, Jankowski uses the method of lower and

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upper solutions in [15] to ensure the existence of the first order differential equation on a bounded interval with integral boundary condition

$$x'(t) = f(t, x(t)),\ t \in [0, T], \quad x(0) = \lambda \int_0^T x(s)\, ds + d.$$

This method have been used in second order differential equations on bounded intervals by A. Boucherif on [2], where the following problem is considered

$$x''(t) = f(t, x(t), x'(t)),\ t \in [0, 1],$$

coupled to the integral boundary conditions

$$x(0) - a x'(0) = \int_0^1 g_0(s)\, x(s)\, ds \quad x(1) + b x'(1) = \int_0^1 g_1(s)\, x(s)\, ds.$$

Many authors have deduced existence, uniqueness and multiplicity of solutions for different kind of differential equations defined on bounded intervals and coupled to suitable integral boundary conditions, see [10, 11, 13, 19–21, 26] and references therein. The used tools are related to continuation methods.

Equations defined on unbounded intervals have had a great attention in the literature.This is mainly due to the search of heteroclinic or homoclinic solutions of many evolution equations. It is important to note that there are many types of solutions defined on unbounded domains, see for instance, the monograph of Agarwal and O'Regan [1] or the paper of Rohleder, Burkotová, López-Somoza and Stryja [23]. Many results on this direction have been obtained for instance in [6, 7, 9, 12, 16–18, 22, 24].

We point out that in [14] it is considered the following equation

$$(q(t)u^{(n-1)}(t))' = f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)),\ \text{a.e. } t \in (0, +\infty),$$

subject to the integral boundary conditions

$$u^{(i)}(0) = 0,\ i = 1, 2, \ldots, n-3,$$

and

$$u^{(n-2)}(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t) dt, \quad \lim_{t \to +\infty} \{q(t)u^{(n-1)}(t)\} = 0.$$

The existence of solutions follows from degree theory.

The method of lower and upper solutions is a very well known tool that has been used in many different problems. We refer to the monograph [5] and the survey [4] and references therein.

In [25], Yan, Agarwal and O'Regan use the upper and lower solution method for the boundary value problem

$$y''(t) + \phi(t), f(t, y(t), y'(t) = 0; t \in [0, +\infty)$$

coupled to the boundary conditions

$$a, y(0) - b, y'(0) = y_0 \geq 0, \lim_{t \to +\infty} \{y'(t)\} = k > 0$$

In [17] this method has been applied to the same second order equation but with the following boundary conditions

$$y'(0) - a, y''(0) = B, \quad \lim_{t \to +\infty} \{y''(t)\} = C$$

Following the ideas developed in previous mentioned works, in this paper we are interested in to deduce existence of solutions via this method for a particular problem defined in an unbounded interval. The boundary conditions have functional dependence at the starting point and it is assumed an asymptotic behavior at $+\infty$.

More concisely, the considered problem is the following one:

$$u''(t) - m^2 u(t) + f\left(t, e^{-mt} u(t), e^{-mt} u'(t)\right) = 0, \quad \text{for all } t \in [0, +\infty), \tag{1.1}$$

$$u(0) - \frac{1}{m} u'(0) = \int_0^{+\infty} e^{-2ms} u(s)\, ds, \lim_{t \to +\infty} \left\{ e^{-mt} u(t) \right\} = B, \tag{1.2}$$

where $m > 0, m \neq \frac{1}{6}, B \in \mathbb{R}$ and $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying the following locally bounded condition

(F) For each $\rho > 0$, there exists a positive function φ_ρ, such that $\varphi_\rho \in L^1[0, +\infty)$ such that, for all $x, y \in (-\rho, \rho)$, it is satisfied that

$$|f(t, x, y)| \leq \varphi_\rho(t), \text{ for all } t \in [0, +\infty).$$

The paper is divided in four sections. After this introduction, it is given a section with preliminary results, where the expression of the Green's function is obtained. On next section, it is obtained an a priori bound by means of a Nagumo kind condition. Moreover, the method of lower and upper solutions is developed to deduce the existence of at least one solution of the considered problem. The last section is devoted to show an example of the applicability of the obtained results.

2. Preliminaries

First recall some notation, definitions and theorems which will be used later.

We will denote $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{R}_0^+ := (0, +\infty)$ and define the space

$$X = \left\{ x \in C^1[0, +\infty) : \lim_{t \to +\infty} e^{-mt} x(t) \in \mathbb{R} \right\}$$

endowed with the norm $\|x\|_1 = \max \{\|x\|, \|x'\|\}$, where

$$\|y\| = \sup_{t \in [0, +\infty)} \left\{ \left| e^{-mt} y(t) \right| \right\}.$$

Remark 2.1. *Notice that if $x \in X$ is such that*

$$\lim_{t \to +\infty} e^{-mt} x(t) = l \in \mathbb{R}$$

then

$$\lim_{t \to +\infty} e^{-mt} x'(t) = m\, l \in \mathbb{R}.$$

As a consequence, $\|\cdot\|_1$ is well defined on X.

It is not difficult to verify that $(X, \|\cdot\|_1)$ is a Banach space.

Next we introduce the concept of lower and upper solutions

Definition 2.2. *A function $\alpha \in C^2[0, +\infty) \cap X$ is a lower solution of the functional boundary value problem* (1.1)-(1.2) *if the following inequalities hold for some $B_1 \in \mathbb{R}$:*

(a) $\alpha(0) - \frac{1}{m} \alpha'(0) \leq \int_0^{+\infty} e^{-2ms} \alpha(s)\, ds, \lim_{t \to +\infty} \{e^{-mt} \alpha(t)\} = B_1 < B,$

(b) $\alpha''(t) - m^2 \alpha(t) + f(t, e^{-mt} \alpha(t), e^{-mt} \alpha'(t)) \geq 0,$ *for all* $t \in (0, +\infty)$.

A function $\beta \in C^2[0,+\infty) \cap X$ is an upper solution if it satisfies the reversed inequalities.

Next lemma gives the exact solution for the associated linear problem by using the Green's function technique.

Lemma 2.3. *Assume that $y : [0,+\infty) \to \mathbb{R}$ is such that $y \in L^1[0,+\infty)$, $m > 0$, $m \neq \frac{1}{6}$ and $B \in \mathbb{R}$. Then the linear functional boundary value problem*

$$\begin{cases} u''(t) - m^2 u(t) + y(t) = 0, \quad t \in (0,+\infty) \\ u(0) - \frac{1}{m}u'(0) = \int\limits_0^{+\infty} e^{-2ms}u(s)\,ds, \lim\limits_{t\to+\infty}\left\{e^{-mt}u(t)\right\} = B \end{cases} \tag{2.1}$$

has a unique solution $u \in X$, given by

$$u(t) = \int_0^{+\infty} G(t,s)\,y(s)\,ds + \frac{3B}{6m-1}e^{-mt} + Be^{mt} \tag{2.2}$$

where

$$G(t,s) = \frac{e^{-mt}}{2m^2(6m-1)}\left(3e^{-ms} - 2e^{-2ms}\right) + \frac{1}{2m}\begin{cases} e^{m(s-t)}, s \leq t \\ e^{m(t-s)}, s > t \end{cases}. \tag{2.3}$$

Proof. Firstly we solve the following boundary value problem

$$\begin{cases} u''(t) - m^2 u(t) + y(t) = 0, \quad t \in (0,+\infty) \\ u(0) - \frac{1}{m}u'(0) = A, \lim\limits_{t\to+\infty}\{e^{-mt}u(t)\} = B, \end{cases} \tag{2.4}$$

where $A \in \mathbb{R}$.

The general solution of the homogeneous equation

$$u''(t) - m^2 u(t) = 0, \quad t \in (0,+\infty),$$

follows the expression

$$u(t) = d_1 e^{-mt} + d_2 e^{mt},$$

with $d_1, d_2 \in \mathbb{R}$.

First, it is obvious that the unique solution on X of the homogeneous problem

$$\begin{cases} v''(t) - m^2\, v(t) = 0, \quad t \in (0,+\infty) \\ v(0) - \frac{1}{m}v'(0) = A, \lim\limits_{t\to+\infty}\{e^{-mt}v(t)\} = B. \end{cases}$$

is given by

$$v(t) = \frac{A}{2}e^{-mt} + Be^{mt}.$$

Then the solution of the boundary value problem (2.4) has the form

$$u(t) = \int_0^{+\infty} g(t,s)\,y(s)\,ds + \frac{A}{2}e^{-mt} + Be^{mt}, \tag{2.5}$$

where

$$g(t,s) = \begin{cases} C_1(s)\,e^{-mt} + C_2(s)\,e^{mt}, t < s \\ C_3(s)\,e^{-mt} + C_4(s)\,e^{mt}, t \geq s \end{cases}.$$

Using the fact that g is continuous and $\frac{\partial g}{\partial t}$ has a jump (which equals 1) at $t = s$ (see [3] for details), we get

$$g(t,s) = \frac{1}{2m}\begin{cases} e^{m(t-s)}, t < s \\ e^{m(s-t)}, t \geq s \end{cases}. \tag{2.6}$$

Now, in (2.5), putting $A=\int\limits_0^{+\infty}e^{-2ms}u\left(s\right)ds,$ it yields

$$\int_0^{+\infty}e^{-2ms}u\left(s\right)ds=\int_0^{+\infty}\left(e^{-2ms}\int_0^{+\infty}g\left(s,r\right)y\left(r\right)dr\right)ds \\ +\frac{A}{2}\int_0^{+\infty}e^{-3ms}ds+B\int_0^{+\infty}e^{-ms}ds.$$

So, by interchanging the order of integration we obtain

$$\begin{aligned} A&=\frac{6m}{6m-1}\int_0^{+\infty}\left(\int_0^{+\infty}e^{-2ms}g\left(s,r\right)ds\right)y\left(r\right)dr+\frac{6B}{6m-1}\\ &=\frac{3}{m^2\left(6m-1\right)}\int_0^{+\infty}\left(e^{-mr}-\frac{2}{3}e^{-2mr}\right)y(r)dr+\frac{6B}{6m-1}. \end{aligned} \tag{2.7}$$

Finally, replacing (2.7) in (2.5), we have

$$u\left(t\right)=\int_0^{+\infty}g\left(t,s\right)y\left(s\right)ds+\frac{e^{-mt}}{2m^2\left(6m-1\right)}\int_0^{+\infty}\left(3e^{-ms}-2e^{-2ms}\right)y(s)ds \\ +\frac{3Be^{-mt}}{6m-1}+Be^{mt},$$

which gives the result of the lemma. ■

In order to deduce the existence results, the following compactness criteria will be useful.

Lemma 2.4. *[8]*
A set $M\subset X$ is relatively compact if the following conditions hold:
(i) M is bounded in X.
(ii) The functions from M are equicontinuous on any compact sub-interval of $[0,+\infty)$.
(iii) The functions from M are equiconvergent at $+$, that is, for any $\varepsilon>0$, there exists a $T=T(\varepsilon)>0$ such that, $\left|e^{-mt}x^{(i)}\left(t\right)-\lim_{t\to+\infty}e^{-mt}x^{(i)}\left(t\right)\right|<\varepsilon$ for all $t\geq T$, $i=0,1$ and $x\in M$.

3. Main Result.

In this section we prove the existence and location of at least one solution for Problem (1.1)- (1.2).

In a first moment we introduce a kind of Nagumo's condition, that impose a growth restriction on the dependence with respect to the last variable of the nonlinear part of the equation.

Definition 3.1. *Consider α and $\beta\in X$ be such that $\alpha\leq\beta$ on $[0,+\infty)$. Define*

$$D=\left\{(t,x,y)\in[0,+\infty)\times\mathbb{R}^2:e^{-mt}\alpha\left(t\right)\leq x\leq e^{-mt}\beta\left(t\right)\right\},$$

and suppose that $f:D\to\mathbb{R}$ is a continuous function that satisfies:

$$|f(t,u,v)|\leq h(|v|)\quad\forall(t,u,v)\in D, \tag{3.1}$$

where $h:[0,+\infty)\to[0,+\infty)$ is a continuous and nondecreasing function such that

$$\lim_{s\to+\infty}\frac{s}{h(s)}>\left(\frac{2}{m^2|6m-1|}+\frac{1}{m}\right). \tag{3.2}$$

To guarantee the existence of solutions of (1.1)-(1.2) we have to find a priori bounds for the derivative of all the posible solutions of the considered problem. Hence, we need the following lemma.

Lemma 3.2. *Let α, β be a pair of lower and upper solutions for Problem* (1.1)–(1.2) *such that $\alpha \leq \beta$ on $[0, +\infty)$, and let $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying the conditions on Definition 3.1. Then there exists $b > 0$, such that for every solution u of* (1.1)-(1.2) *with $\alpha(t) \leq u(t) \leq \beta(t), \forall t \in [0, +\infty)$, we have*

$$\|u'\| \leq b.$$

Proof. From Lemma 2.3, we know that the solutions of Problem (1.1)–(1.2) are characterized as the solutions of the following integral equation:

$$u(t) = \int_0^{+\infty} G(t,s) f(s, e^{-ms} u(s), e^{-ms} u'(s)) ds. \tag{3.3}$$

Differentiating in (3.3), we obtain

$$e^{-mt} u'(t) = \int_0^{+\infty} e^{-mt} \frac{\partial G}{\partial t}(t,s) f(s, e^{-ms} u(s), e^{-ms} u'(s)) ds. \tag{3.4}$$

Now, we have that

$$e^{-mt} \frac{\partial G}{\partial t}(t,s) = -\frac{e^{-2mt}}{2m(6m-1)} \left(3e^{-ms} - 2e^{-2ms}\right) + \frac{1}{2} \begin{cases} -e^{m(s-2t)} & , s \leq t \\ e^{-ms} & , s > t \end{cases}. \tag{3.5}$$

Using (3.1), and the fact that h is nondecreasing, we get

$$\begin{aligned} |e^{-mt} u'(t)| &\leq \int_0^{+\infty} e^{-mt} \left| \frac{\partial G}{\partial t}(t,s) \right| |f(s, e^{-ms} u(s), e^{-ms} u'(s))| ds \\ &\leq \int_0^{+\infty} \frac{e^{-2mt}}{2m|6m-1|} \left(3e^{-ms} + 2e^{-2ms}\right) h(|e^{-ms} u'(s)|) ds \\ &\quad + \int_0^t \frac{e^{m(s-2t)}}{2} h(|e^{-ms} u'(s)|) ds + \int_t^{+\infty} \frac{e^{-ms}}{2} h(|e^{-ms} u'(s)|) ds \\ &\leq h(\|u'\|) \left(\frac{2e^{-2mt}}{m^2|6m-1|} + \frac{e^{-2mt}\left(2e^{mt} - 1\right)}{2m} \right) \\ &\leq h(\|u'\|) \left(\frac{2}{m^2|6m-1|} + \frac{1}{m} \right), \quad \text{for all } t \in [0, +\infty), \end{aligned}$$

which implies that

$$\frac{\|u'\|}{h(\|u'\|)} \leq \left(\frac{2}{m^2|6m-1|} + \frac{1}{m} \right).$$

Then, from (3.2), we deduce that there exists $b > 0$ such that $\|u'\| < b$.
This completes the proof. ■

Now, we are in a position to prove the main result of this paper.

Theorem 3.3. *Let α and β be a pair of lower and upper solutions for the functional boundary value problem* (1.1)-(1.2) *such that $\alpha(t) \leq \beta(t)$ for every $t \in [0, +\infty)$ and let $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying the conditions on Definition 3.1.. Then the functional boundary value problem* (1.1)–(1.2) *has at least one solution $u \in C^2[0, +\infty) \cap X$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \ \forall t \in [0, +\infty).$$

Proof. First, we define the truncated functions

$$p(t,x)=\max\left\{\alpha(t),\min\left\{x,\beta(t)\right\}\right\}$$

and

$$q(y)=\max\left\{-K,\min\left\{y,K\right\}\right\},$$

where $K=\max\{b,\|\alpha\|_1,\|\beta\|_1\}$ and b is the constant given in Lemma 3.2.

Consider now the following modified problem

$$\left\{\begin{array}{c} u''(t)-m^2u(t)+F\left(t,u(t),e^{-mt}u'(t)\right)=0, \quad t\in(0,+\infty) \\ u(0)-\frac{1}{m}u'(0)=\int\limits_0^{+\infty}e^{-2ms}p(s,u(s))ds, \quad \lim\limits_{t\to+\infty}\left\{e^{-mt}u(t)\right\}=B \end{array}\right. \tag{3.6}$$

with

$$F(t,x,y)=f(t,e^{-mt}p(t,x),q(y)).$$

We will show that the solutions of the modified problem (3.6) lie in a region where f is unmodified i.e. $\alpha(t)\le u(t)\le\beta(t)$, and $-b\le e^{-mt}u'(t)\le b$ for all $t\in[0,+\infty)$ and, hence, they will be solutions of problem (1.1)–(1.2). The proof will be done in two steps.

Step 1: Existence of solution.

By (2.5) it is clear that the solutions of the truncated problem (3.6) coincide with the fixed points of the operator $T:X\to X$ defined by

$$T\,u(t)=\int_0^{+\infty}g(t,s)\,F\left(s,u(s),e^{-ms}u'(s)\right)ds+\frac{e^{-mt}}{2}\int\limits_0^{+\infty}e^{-2ms}p(s,u(s))ds+Be^{mt}.$$

Let us see that operator T is well defined in X. Indeed, let $u\in X$, by definition of function p, α and β, we have that $e^{-2ms}p(s,u(s))\in L^1[0,+\infty)$. Moreover $e^{-ms}p(s,u(s))$ and $q(e^{ms}u'(s))$ are bounded in $[0,+\infty)$. So, we can use condition (F) to deduce that there is $R>0$ such that

$$|F(t,x,y)|\le\varphi_R(t),\ \text{ for all }\ t\in[0,+\infty).$$

with $\varphi_R\in L^1[0,+\infty)$.

As a direct consequence, we have that $\varphi_R(\cdot)\,g(t,\cdot)$ and $\varphi_R(\cdot)\,\frac{\partial g}{\partial t}(t,\cdot)$ are in $L^1[0,+\infty)$. So, we deduce that $Tu(t)\in C^1[0,+\infty)$. Moreover

$$\lim_{t\to+\infty}\left\{e^{-mt}Tu(t)\right\}=B$$

and, using Remark 2.1, that

$$\lim_{t\to+\infty}\left\{e^{-mt}\left(Tu\right)'(t)\right\}=m\,B,$$

That is: $T\,u\in X$.

Moreover, as a direct consequence, there is $\bar{R}>0$ such that

$$\|T\,u\|_1\le\bar{R}, \quad \text{for all }\ u\in X.$$

Consequently, $T(B)$ is uniformly bounded and maps the closed, bounded and convex set

$$B=\left\{u\in X:||u||\le\bar{R}\right\},$$

into itself.

Furthermore, for $C>0$ and $t_1,t_2\in[0,C]$, $t_1<t_2$, we have

$$\begin{aligned}\left|e^{-mt_1}Tu(t_1)-e^{-mt_2}Tu(t_2)\right| &\leq \frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2}\int_0^{+\infty} e^{-2ms}|p(s,u(s))|ds\\ &+\frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2\,m}\int_0^{t_1} e^{m\,s}|F(s,u(s),e^{-ms}u'(s))|ds\\ &+\frac{\left|1+e^{-2mt_2}\right|}{2\,m}\int_{t_1}^{t_2} e^{m\,s}|F(s,u(s),e^{-ms}u'(s))|ds\\ &\leq \frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s),e^{-ms}\beta(s)\}|ds\\ &+\frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2\,m}\int_0^{t_1} e^{m\,s}\varphi_{\bar R}(s)ds\\ &+\frac{\left|1+e^{-2mt_2}\right|}{2\,m}\int_{t_1}^{t_2} e^{m\,s}\varphi_{\bar R}(s)ds,\end{aligned}$$

which converges to 0 as $t_1\to t_2$, and it is independent of $u\in X$. (Notice that $e^{m\,s}\varphi_{\bar R}(s)\in L^1_{loc}[0,+\infty)$)

Analogously, we have

$$\begin{aligned}\left|e^{-mt_1}(Tu)'(t_1)-e^{-mt_2}(Tu)'(t_2)\right| &\leq m\,\frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s),e^{-ms}\beta(s)\}|ds\\ &+\frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2}\int_0^{t_1} e^{m\,s}\varphi_{\bar R}(s)ds\\ &+\frac{\left|1+e^{-2mt_2}\right|}{2}\int_{t_1}^{t_2} e^{m\,s}\varphi_{\bar R}(s)ds,\end{aligned}$$

and converges to 0 as $t_1\to t_2$ with independence of $u\in X$.

This shows that T is equicontinuous on compact subintervals of $[0,+\infty)$.

Finally, the fact that $T(B)$ is equiconvergent at infinity follows from the following inequalities:

$$\begin{aligned}\left|e^{-mt}Tu(t)-\lim_{t\to+\infty}\left\{e^{-mt}Tu(t)\right\}\right| &= \left|e^{-mt}Tu(t)-B\right|\\ &\leq \frac{e^{-2mt}}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s),e^{-ms}\beta(s)\}|ds\\ &+\frac{e^{-2mt}}{2\,m}\int_0^{t} e^{m\,s}\varphi_{\bar R}(s)ds\\ &+\frac{1}{2\,m}\int_t^{+\infty} e^{-m\,s}\varphi_{\bar R}(s)ds\end{aligned}$$

$$\leq \frac{e^{-2mt}}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s),e^{-ms}\beta(s)\}|ds + \frac{e^{-mt}}{2\,m}\|\varphi_{\bar{R}}\|_{L^1[0,+\infty)} + \frac{1}{2\,m}\int_t^{+\infty} e^{-m\,s}\varphi_{\bar{R}}(s)ds,$$

and

$$\left|e^{-mt}(Tu)'(t) - \lim_{t\to+\infty}\left\{e^{-mt}(Tu(t))'\right\}\right| = \left|e^{-mt}Tu(t) - m\,B\right| \leq \frac{m\,e^{-2mt}}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s),e^{-ms}\beta(s)\}|ds + \frac{e^{-mt}}{2}\|\varphi_{\bar{R}}\|_{L^1[0,+\infty)} + \frac{1}{2}\int_t^{+\infty} e^{-m\,s}\varphi_{\bar{R}}(s)ds,$$

Consequently, By lemma 2.4, the set $T(B)$ is relatively compact. In addition T is continuous via dominated convergence theorem. Therefore, the map T is completely continuous. Using Schauder's Theorem, we conclude that T has a fixed point in X, then, the BVP (3.6) has at least one solution $u \in C^2[0,+\infty) \cap X$.

Step 2: If u is a solution of the truncated problem (3.6), then

$$\alpha(t) \leq u(t) \leq \beta(t), \forall t \in [0,+\infty).$$

First, notice that, since $\lim_{t\to+\infty}\{e^{-mt}(\alpha - u)(t)\} < 0$, we have that there is $t_1 \geq 0$ such that $\alpha < u$ on $(t_1, +\infty)$.

Assuming that there exists $t_0 \in (0,+\infty)$ such that

$$\inf_{t\in[0,+\infty)}(u(t) - \alpha(t)) = u(t_0) - \alpha(t_0) < 0,$$

we have two cases to consider such as the following:

Case 1: If $t_0 \in (0,+\infty)$, we get $u'(t_0) = \alpha'(t_0)$ and

$$0 \leq u''(t_0) - \alpha''(t_0) \leq -\,f\left(t_0, e^{-mt_0}\alpha(t_0), e^{-mt_0}\alpha'(t_0)\right) + m^2u(t_0) + f\left(t_0, e^{-mt_0}\alpha(t_0), e^{-mt_0}\alpha'(t_0)\right) - m^2\alpha(t_0) < 0.$$

that is a contradiction, thus, the infimum of $u - \alpha$ is not achieved at the point t_0.

Case 2: If $t_0 = 0$, we have

$$\min_{t\in[0,+\infty)}(u(t) - \alpha(t)) = u(0) - \alpha(0) < 0.$$

and

$$u'(0) - \alpha'(0) \geq 0,$$

so, since $m > 0$ and the fact that α is a lower solution, it yields to the following contradiction

$$0 > u(0) - \alpha(0) - \frac{1}{m}(u'(0) - \alpha'(0)) \geq \int_0^{+\infty} e^{-2ms}\Big(p(s,u(s)) - \alpha(s)\Big)ds \geq 0.$$

To complete the proof, we apply Lemma 3.2 to F and we deduce that $\|u'\| \leq b$. ■

4. Example

Consider the following BVP

$$
\begin{aligned}
&u''(t)-u(t)=f\left(t,e^{-t}u(t),e^{-t}u'(t)\right), \quad t\in[0,+\infty)\\
&u(0)-u'(0)=\int_0^{+\infty}e^{-2s}u(s)\,ds,\ \lim_{t\to+\infty}e^{-t}u(t)=B,
\end{aligned}
$$

where $m=1$ and $f(t,x,y)=\frac{e^{-t/3}}{B}\sqrt[3]{x+y}-e^{-2t}$, with $B<0$.

Firstly, let $B_1<\min\{B,4B^3-1/6\}$ and $B_2\geq 0$.

Let us see that functions $\alpha(t)=\frac{11+12B_1}{20}e^{-t}-\frac{1}{3}e^{-2t}+B_1e^t$ and $\beta(t)=\frac{11+12B_2}{20}e^{-t}-\frac{1}{3}e^{-2t}+B_2e^t$ are a pair of lower and upper solutions of this BVP such that $\alpha(t)\leq\beta(t)\,,t\in[0,+\infty)$. Indeed,

$$
\frac{6}{5}B_2+\frac{1}{10}=\beta(0)-\beta'(0)=\int_0^{+\infty}e^{-2t}\beta(t)\,dt,\ \lim_{t\to+\infty}\left\{e^{-t}\beta(t)\right\}=B_2>B
$$

and, using that $B_2\geq 0$,

$$
\beta''(t)-\beta(t)+\frac{1}{B}\sqrt[3]{e^{-t}\beta(t)+e^{-t}\beta'(t)}=\frac{e^{-t/3}\sqrt[3]{6B_2+e^{-3t}}}{\sqrt[3]{3}B}-2e^{-2t}\leq 0.
$$

Moreover

$$
\frac{6}{5}B_1+\frac{1}{10}=\alpha(0)-\alpha'(0)=\int_0^{+\infty}e^{-2t}\alpha(t)\,dt,\quad \lim_{t\to+\infty}\left\{e^{-t}\alpha(t)\right\}=B_1<B
$$

and, since $B_1\leq 4B^3-1/6$,

$$
\alpha''(t)-\alpha(t)+\frac{1}{B}\sqrt[3]{e^{-t}\alpha(t)+e^{-t}\alpha'(t)}+e^{-2t}=\frac{e^{-t/3}\sqrt[3]{6B_1+e^{-3t}}}{\sqrt[3]{3}B}-2e^{-2t}\geq 0
$$

Moreover, the function f satisfy the condition (F).

For each $\rho>0,\ x,y\in(-\rho,\rho)$, we have

$$
\begin{aligned}
|f(t,x,y)|&\leq\frac{e^{-t/3}}{|B|}\sqrt[3]{|x|+|y|}+e^{-2t}\\
&\leq\frac{e^{-t/3}}{|B|}\sqrt[3]{2\,\rho}+e^{-2t}=:\varphi_\rho(t),\ \text{for all } t\in[0,+\infty)\,,
\end{aligned}
$$

with $\varphi_\rho\in L^1[0,+\infty)$.

Finally, for any $t\in[0,+\infty)$ and $e^{-t}\alpha(t)\leq x\leq e^{-t}\beta(t)$, we have that there is a positive constant C such that

$$
|f(t,x,y)|\leq\frac{1}{|B|}\sqrt[3]{C+|y|}+1=:h(|y|).
$$

Clearly, $h:[0,+\infty)\to[0,+\infty)$ is a continuous and nondecreasing function such that

$$
\lim_{s\to+\infty}\frac{s}{h(s)}=+\infty.
$$

As a consequence, all the assumptions of Theorem 3.3 are fulfilled and this problem admits at least one solution lying between α and β.

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An existence result of μ-pseudo almost automorphic solutions of Clifford-valued semi-linear delay differential equations

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Abstract. In this work we are concern with Clifford-valued semi-linear delay differential equations in a Banach space. By using the Banach fixed point theorem, we prove the existence and uniqueness of $\mu-$pseudo almost automorphic solution for Clifford-valued semi-linear delay differential equations.

AMS Subject Classifications: 15A66, 43A60.

Keywords: μ-pseudo almost automorphic functions; Clifford algebra, Semi-linear delay differential equations.

Contents

1. Introduction

The notion of almost automorphy was introduced in the early sixties by S. Bochner in [7–9] when studying a problem in differential geometry. It turns out to be a generalization of almost periodicity in the sense of Bohr. After the emergence of the concept of almost automorphy, many authors have produced extensive literature on the theory of almost automorphy with usefull generalizations. Veech [34] and Zaki [36] studied almost automorphic functions respectively on groups and the real number set. In his paper [28], N'Guérékata introduced the concept of asymptotically almost automorphic functions. For more informations on the concept of almost automorphy and its application to evolution equations, we refer the reader to [26, 29]. In [35], Xiao et al. introduced the notion of pseudo almost automorphy as suggested by N'Guérékata in [29]. Later on, the notion of weighted pseudo almost automorphy was introduced by J. Blot et al. in [6]. Recently, Blot et al. in [4] introduced the concept of $\mu-$pseudo almost automorphy which is more general than the class of weighted pseudo almost automorphic functions. Due to a lot of applications, the existence of pseudo almost automorphic, weighted pseudo almost automorphic and $\mu-$pseudo almost automorphic solutions of various differential equations has become an interesting field. Many authors have made important contributions on these topics [1, 2, 4, 6, 12, 16, 17, 20, 24, 35, 36].

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In 1878 Clifford [15] introduced Clifford algebra which includes real numbers, complex numbers, quaternions and Grassmann algebra. After the monographs of Chevalley [13] and Riesz [33] published in 1954 and 1958, respectively, Clifford algebra received more and more attention. Nowadays, Clifford algebra is used in many fields such as geometry, satellite navigation, neural network, theoretical physics, robotics, image processing and quantium computing [18, 19, 21]. In Neural network, Pearson first proposed a Clifford-valued neural network [32] described by Clifford-valued differential equations. In [11], Buchholz conclued tha Clifford-valued neural network have more advantages than real-valued ones. Since these works, Clifford-valued neural networks has become a very attractive field of research. In [22], by decomposing Clifford-valued system into real-valued systems, Li et al. prove the existence of almost periodic solution and the global asymptotic synchronization for a class of Clifford-valued neural networks. Recently in [23], by non-decomposing method, Li et al. studied the existence and global exponential stability of $\mu-$pseudo almost periodic solutions of Clifford-valued semi-linear delay equations.

Motivated by the above papers, we would like to study the existence and uniqueness of $\mu-$pseudo almost automorphic mild solutions for the following Clifford-valued semi-linear delay equations:

$$x'(t) = -D(t)\,x(t) + F(t, x(t), x(t-\tau(t)))\,;\; t \in \mathbb{R}, \tag{1.1}$$

where $D(\cdot) = diag\{d_1(\cdot),\, d_2(\cdot), ..., \, d_n(\cdot)\} \in \mathbb{R}^{n\times n}$, $F \in C\left(\mathbb{R}\times \mathcal{A}^{2n}, \mathcal{A}^n\right)$, $\tau \in C(\mathbb{R}, \mathbb{R}^+)$, $\mathcal{A}$ is a real Clifford algebra.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions and results about Clifford algebras and the notion of $\mu-$pseudo almost automorphic functions. Section 3 is devoted to our main results.

2. Preliminaries

In this section, we recall some basic definitions and preliminary results on Clifford algebras and $\mu-$pseudo almost automorphic functions.

Definition 2.1. *Let m be a natural number. The real Clifford algebra over $\mathbb{R}^m$ is defined as*

$$\mathcal{A} = \left\{ \sum_{A\subseteq\{1,2,...,m\}} a_A e_A,\, a_A \in \mathbb{R} \right\},$$

where $e_A = e_{h_1}e_{h_2}...e_{h_\nu}$ with $A = \{h_1, h_2, ..., h_\nu\}$, $1 \le h_1 < h_2 < ... < h_\nu \le m$. Moreover, $e_\emptyset = e_0 = 1$ and e_i, $i = 1, 2, ..., m$ are Clifford generators and satisfy $e_i^2 = -1$, $i = 1, 2, ..., m$ and $e_ie_j + e_je_i = 0$, $\forall i, j = 1, 2, ..., m$, $i \ne j$.

In the sequel, we will denote by $e_{h_1h_2...h_\nu}$ the product of Clifford generators $e_{h_1},\ e_{h_2}, ...,\ e_{h_\nu}$. Let $E = \{1, 2, ..., m\}$ and $\Pi = \mathcal{P}(E)$, then it is obvious that $\mathcal{A} = \left\{ \sum\limits_{A\in\Pi} a_Ae_A,\, a_A \in \mathbb{R}\right\}$ and $\dim(\mathcal{A}) = 2^m$.

Definition 2.2. *For $x = \sum\limits_{A\in\Pi} x_Ae_A \in \mathcal{A}$, the involution of x is defined as*

$$\overline{x} = \sum_{A\in\Pi} x_A\overline{e}_A$$

where $\overline{e}_A = (-1)^{\frac{n(A)(n(A)+1)}{2}} e_A$, if $A = \emptyset$, then $n(A) = 0$ and if $A = \{h_1, h_2, ..., h_\nu\} \in \Pi$, then $n(A) = \nu$.

It's clear that $e_A\overline{e}_A = 1$ and easy to verify that the involution has the property $\overline{xy} = \overline{y}\overline{x}$, $\forall x, y \in \mathcal{A}$. For $x, y \in \mathcal{A}$, we define the inner product of x and y by

$$(x, y)_0 = 2^m\,[x\overline{y}]_0 = 2^m \sum_{A\in\Pi} x^Ay^A,$$

where $[x\overline{y}]_0$ is the coefficient of e_0 conpoment of $x\overline{y}$. Then $\mathcal{A}$ with this inner product is a real Hilbert space and with the norm defined by $\|x\|_{\mathcal{A}} = \sqrt{(x,x)_0}$ is a Banach algebra since for all $x, y \in \mathcal{A}$

$$\|xy\|_{\mathcal{A}} \leq \|x\|_{\mathcal{A}} \|y\|_{\mathcal{A}}.$$

The derivative of $x(t) = \sum_{A\in\Pi} x_A(t) e_A$ is given by $\frac{dx(t)}{dt} = \sum_{A\in\Pi} \frac{dx_A(t)}{dt} e_A$. We refer the reader to [10] for more informations about Clifford algebra.

Now, let us recall some definitions and results on almost automorphic functions.

Let $\mathcal{B}$ be the Lebesgue σ-field of $\mathbb{R}$ and $\mathcal{M}$ the set of all positive mesures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$. Throughout the rest of this paper, $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ will stand for Banach spaces and

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|_{\mathbb{Z}},$$

where $\mathbb{Z} = \mathbb{X}$, or $\mathbb{Y}$. We also denote by $\mathcal{B}(\mathbb{R},\mathbb{Z})$, $\mathcal{C}(\mathbb{R},\mathbb{Z})$ and $\mathcal{BC}(\mathbb{R},\mathbb{Z})$ the collections of all bounded functions, all continuous functions and all continuous and bounded functions from $\mathbb{R}$ to $\mathbb{Z}$, respectively.

Definition 2.3. *([27]) A function $f \in \mathcal{C}(\mathbb{R},\mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(\tau'_n)_n$ there exists a subsequence $(\tau_n)_n$ such that*

$$g(t) = \lim_{n\to+\infty} f(t+\tau_n) \text{ exists for each } t \in \mathbb{R}$$

and

$$\lim_{n\to+\infty} g(t-\tau_n) = f(t) \text{ for each } t \in \mathbb{R}.$$

We denote by $AA(\mathbb{R},\mathbb{X})$ the space of the almost automorphic $\mathbb{X}$-valued functions.

Remark 2.4. *Note that in the above limit the function g is just mesurable. If the convergence in both limits is uniform in $t \in \mathbb{R}$, then f is almost periodic in the sense of Bohr. The concept of almost automorphy is then larger than almost periodicity. If f is almost automorphic, then its range is relatively compact, thus bounded in norm.*

Example 2.5. *([27]) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that*

$$f(t) = \sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right) \text{ for } t \in \mathbb{R}.$$

Then f is almost automorphic, but it is not uniformly continuous on $\mathbb{R}$. Therefore, it is not almost periodic.

Proposition 2.6. *([27]) $(AA(\mathbb{R},\mathbb{X}), \|.\|_\infty)$ is a Banach space.*

Definition 2.7. *A function $f \in \mathcal{C}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ is said to be almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$, if the following two conditions hold:*

i) for all $x \in \mathbb{X}$, $f(\cdot, x) \in AA(\mathbb{R},\mathbb{Y})$,

ii) f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x, namely, for each compact set K in $\mathbb{X}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in K$, one has

$$\|x_1 - x_2\| \leq \delta \Longrightarrow \sup_{t\in\mathbb{R}} \|f(t,x_1) - f(t,x_2)\| \leq \varepsilon$$

We denote by $AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ the set of all such functions.

Theorem 2.8. *([5]) Let $f \in AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and $x \in AA(\mathbb{R},\mathbb{X})$. Then $[t \longmapsto f(t,x(t))] \in AA(\mathbb{R},\mathbb{Y})$.*

Definition 2.9. *([4]) Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \longrightarrow \mathbb{X}$ is said to be μ-ergodic if*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| \, d\mu(t) = 0.$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R},\mathbb{X},\mu)$.

Proposition 2.10. *([4]) Let $\mu \in \mathcal{M}$. Then $(\mathcal{E}(\mathbb{R},\mathbb{X},\mu), \|.\|_\infty)$ is a Banach space.*

Definition 2.11. *([4]) Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form:*

$$f = \phi + \psi$$

where $\phi \in AA(\mathbb{R},\mathbb{X})$ and $\psi \in \mathcal{E}(\mathbb{R},\mathbb{X},\mu)$. We denote the space of all such functions by $PAA(\mathbb{R},\mathbb{X},\mu)$. Then, we have

$$AA(\mathbb{R},\mathbb{X}) \subset PAA(\mathbb{R},\mathbb{X},\mu) \subset \mathcal{BC}(\mathbb{R},\mathbb{X}).$$

Remark 2.12. *Without assumption on the measure μ, the decomposition in the above definition of the corresponding μ-pseudo almost automorphic function is not unique.*

Remark 2.13. *A pseudo almost automorphic function is μ-pseudo almost automorphic function in the particular case where the measure μ is the Lebesgue measure. For more details on pseudo almost automorphic functions, we refer to [24, 25].*

Remark 2.14. *The notion of μ-pseudo almost automorphic functions is a generalization of the weighted pseudo almost automorphic functions which is due to Blot et al. [6]. Following [6], a function f is so-called weighted pseudo almost automorphic if f is a μ-pseudo almost automorphic function in the particular case where the measure μ is defined by $\mu(A) = \int_A \rho(t)\,dt$ for $A \in \mathcal{B}$ with $\rho(t) > 0$ a.e on $\mathbb{R}$ for the Lebesgue measure and $\int_{-\infty}^{+\infty} \rho(t)\,dt = +\infty$.*

Proposition 2.15. *([4]) Let $\mu \in \mathcal{M}$. Then $PAA(\mathbb{R},\mathbb{X},\mu)$ is a vector space.*

Definition 2.16. *([4]) Let μ_1 and $\mu_2 \in \mathcal{M}$. μ_1 is said to be equivalent to μ_2 ($\mu_1 \sim \mu_2$) if there exist constants α, $\beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that*

$$\alpha\mu_1(A) \le \mu_2(A) \le \beta\mu_1(A), \text{ for } A \in \mathcal{B} \text{ satisfying } A \cap I = \emptyset.$$

Remark 2.17. *The relation $\sim$ is an equivalence relation on $\mathcal{M}$.*

Theorem 2.18. *([4]) Let μ_1, $\mu_2 \in \mathcal{M}$. If μ_1 and μ_2 are equivalent, then $\mathcal{E}(\mathbb{R},\mathbb{X},\mu_1) = \mathcal{E}(\mathbb{R},\mathbb{X},\mu_2)$ and $PAA(\mathbb{R},\mathbb{X},\mu_1) = PAA(\mathbb{R},\mathbb{X},\mu_2)$.*

For $\mu \in \mathcal{M}$, $\tau \in \mathbb{R}$ and $A \in \mathcal{B}$, we denote μ_τ the positive measure on $(\mathbb{R},\mathcal{B})$ defined by

$$\mu_\tau(A) = \mu(\{a+\tau, a \in A\}).$$

From $\mu \in \mathcal{M}$, we formulate the following hypothesis:

$$\textbf{(H0)} \begin{cases} \text{For all } \tau \in \mathbb{R}, \text{ there exist } \beta > 0 \text{ and a bounded interval } I \text{ such that} \\ \qquad \mu_\tau(A) \le \beta\mu(A), \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset. \end{cases}$$

Lemma 2.19. *([4]) Let $\mu \in \mathcal{M}$. Then μ satisfies (H0) if and only if the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 2.20. ***([4])** Hypothesis (H0) implies*

$$\text{for all } \sigma > 0, \limsup_{r \to +\infty} \frac{\mu([-r-\sigma, r+\sigma])}{\mu([-r,r])} < +\infty.$$

Theorem 2.21. ***([4])** Let $\mu \in \mathcal{M}$ satisfying (H0). Then $\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, therefore $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is also translation invariant.*

Theorem 2.22. ***([4, Theorem 3.9])** Let $\mu \in \mathcal{M}$ satisfy (H0). If $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ and $g \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$, then the convolution product $f * g$ is also μ-pseudo almost automorphic. In fact, if $f \in AA(\mathbb{R}, \mathbb{X})$, then $f * g \in AA(\mathbb{R}, \mathbb{X})$ and if $f \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$, then $f * g \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$.*

Theorem 2.23. ***([4])** Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = \phi + \psi$ where $\phi \in AA(\mathbb{R}, \mathbb{X})$ and $\psi \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$, is unique.*

Theorem 2.24. ***([4])** Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|.\|_\infty)$ is a Banach space.*

Definition 2.25. ***([4])** Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$ if the following two conditions are hold:*

i) for all $x \in \mathbb{X}$, $f(., x) \in AA(\mathbb{R}, \mathbb{Y})$

ii) f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x, namely, for each compact set K in $\mathbb{X}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in K$, one has

$$\|x_1 - x_2\| \leq \delta \Longrightarrow \sup_{t \in \mathbb{R}} \|f(t, x_1) - f(t, x_2)\| \leq \varepsilon.$$

Denote by $AAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ the set of all such functions.

Definition 2.26. ***([4])** Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be μ-ergodic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$ if the following two conditions are true:*

i) for all $x \in \mathbb{X}$, $f(., x) \in \mathcal{E}(\mathbb{R}, \mathbb{Y}, \mu)$

ii) f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x.

Denote by $\mathcal{E}U(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ the set of all such functions.

Definition 2.27. ***([4])** Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be μ-pseudo almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$ if f is written in the form $f = \phi + \psi$ where $\phi \in AAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ and $\psi \in \mathcal{E}U(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$.*

$PAAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ denote the set of all such functions.

Remark 2.28. *We have $AAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}) \subset PAAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$.*

Theorem 2.29. ***([4, Theorem 5.7])** Let $\mu \in \mathcal{M}$, $f \in PAAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ and $x \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. Assume that for all bounded subset B of $\mathbb{X}$, f is bounded on $\mathbb{R} \times B$. Then $[t \longmapsto f(t, x(t))] \in PAA(\mathbb{R}, \mathbb{Y}, \mu)$.*

In the sequel we assume that

(H1) $F = F_1 + F_2 \in PAAU\left(\mathbb{R} \times \mathcal{A}^{2n}, \mathcal{A}^n, \mu\right)$ is bounded function on $\mathbb{R} \times \Omega$ for any bounded subset Ω of $\mathcal{A}^{2n}$, and there exist real numbers $L_1, L_1' > 0$ and $L_2, L_2' > 0$ such that

$$\|F_1(t, x_1, y_1) - F_1(t, x_2, y_2)\|_{\mathcal{A}^n} \leq L_1 \|x_1 - y_1\| + L_1' \|x_2 - y_2\|, \forall t \in \mathbb{R}, \forall x_1, x_2, y_1, y_2 \in \mathcal{A}^n,$$

$$\|F_2(t,x_1,y_1)-F_2(t,x_2,y_2)\|_{\mathcal{A}^n}\le L_2\|x_1-y_1\|+L_2'\|x_2-y_2\|,\forall t\in\mathbb{R},\forall x_1,x_2,y_1,y_2\in\mathcal{A}^n.$$

(H2) For $i=1,2,...,n$; $d_i\in AA(\mathbb{R},\mathbb{R})$ with $\min\limits_{1\le i\le n}\left\{\inf\limits_{t\in\mathbb{R}}d_i(t)\right\}=d^*>0$, and $\tau\in AA(\mathbb{R},\mathbb{R}^+)$ with $\tau^*=\sup\limits_{t\in\mathbb{R}}|\tau(t)|$.

(H3) There exists $\lambda\in C(\mathbb{R},\mathbb{R}^+)$ such that $d\mu(\gamma(t))=\lambda(t)\,d\mu(t)$ for all $t\in\mathbb{R}$ and

$$\limsup_{r\to+\infty}\frac{M(r)\,\mu([-K(r),K(r)])}{\mu([-r,r])}<\infty,$$

where $\gamma(t)$ is the inverse function of $t\mapsto t-\tau(t)$, $K(r)=\sup\limits_{t\in[-r,r]}|t-\tau(t)|$ and $M(r)=\sup\limits_{t\in[-K(r),K(r)]}|\lambda(t)|$.

(H4) $\frac{L_1+L_2+L_1'+L_2'}{d^*}<1$, where $L_1,\ L_2,\ L_1',\ L_2'$ and d^* are defined in *(H1)* and *(H2)*.

3. Main results

From now on $\mathbb{X}=\mathcal{A}^{2n}$ and $\mathbb{Y}=\mathcal{A}^n$.

Lemma 3.1. ***[23, Lemma 3.1]*** *Function x solves the equation (1.1) if and only if x solves the following equation:*

$$x(t)=\int_{-\infty}^{t}e^{-\int_s^t D(u)du}F(s,x(s),x(s-\tau(s)))\,ds,\ \forall t\in\mathbb{R}. \tag{3.1}$$

We need the following lemma.

Lemma 3.2. *Suppose that (H3) holds and let $u=u_1+u_2\in PAA(\mathbb{R},\mathcal{A}^n,\mu)$ with $u_2\in\mathcal{E}AA(\mathbb{R},\mathcal{A}^n,\mu)$ and $u_1\in AA(\mathbb{R},\mathcal{A}^n)$. Then $t\mapsto u(t-\tau(t))\in PAA(\mathbb{R},\mathcal{A}^n,\mu)$.*

Proof. Let $(\alpha_n')_n$ be a sequence of real numbers. For a fixed $t\in\mathbb{R}$ we set $\beta_n'=\alpha_n'-\tau(t+\alpha_n')$ for all $n\in\mathbb{N}$. Since $(\beta_n')_n$ is a sequence of real numbers and $u_1\in AA(\mathbb{R},\mathcal{A}^n)$, there exists a subsequence $(\beta_n)_n$ of $(\beta_n')_n$ such that

$$\lim_{n\longrightarrow+\infty}u_1(t+\beta_n)=\overline{u}_1(t)\ \text{ exists for all } t\in\mathbb{R},$$

and

$$\lim_{n\longrightarrow+\infty}\overline{u}_1(t-\beta_n)=u_1(t)\ \text{ exists for all } t\in\mathbb{R}.$$

That is there exists a subsequence $(\alpha_n)_n$ of $(\alpha_n')_n$ such that $\beta_n=\alpha_n-\tau(t+\alpha_n)$ for all $n\in\mathbb{N}$, and

$$\lim_{n\longrightarrow+\infty}u_1(t+\alpha_n-\tau(t+\alpha_n))=\overline{u}_1(t)\ \text{ exists for all } t\in\mathbb{R},$$

$$\lim_{n\longrightarrow+\infty}\overline{u}_1(t+\alpha_n-\tau(t+\alpha_n))=u_1(t)\ \text{ exists for all } t\in\mathbb{R}.$$

So, $t\longmapsto u_1(t-\tau(t))\in AA(\mathbb{R},\mathcal{A}^n)$. On the other hand, from assumption (H3) we have

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t-\tau(t))\|_{\mathcal{A}^n}\,d\mu(t)\\&=\frac{\mu([-K(r),K(r)])}{\mu([-r,r])}\frac{1}{\mu([-K(r),K(r)])}\int_{-K(r)}^{K(r)}\|u_2(t)\|_{\mathcal{A}^n}\,\lambda(t)\,d\mu(t)\\&\le\frac{M(r)\,.\mu([-K(r),K(r)])}{\mu([-r,r])}\frac{1}{\mu([-K(r),K(r)])}\int_{-K(r)}^{K(r)}\|u_2(t)\|_{\mathcal{A}^n}\,d\mu(t)\end{aligned}$$

Since $u_2 \in \mathcal{E}AA(\mathbb{R}, \mathcal{A}^n, \mu)$, we obtain from Assumption (H3) and the above inequality that

$$\lim_{r \longrightarrow +\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \|u_2(t-\tau(t))\|_{\mathcal{A}^n} d\mu(t) = 0,$$

thus $t \longmapsto u_2(t-\tau(t)) \in \mathcal{E}AA(\mathbb{R}, \mathcal{A}^n, \mu)$. The proof is complet. ∎

Lemma 3.3. *Assume that assumptions (H1), (H3) hold and* $u \in PAA(\mathbb{R}, \mathcal{A}^n, \mu)$. *Then* $t \longmapsto F(t, u(t), u(t-\tau(t))) \in PAA(\mathbb{R}, \mathcal{A}^n, \mu)$.

Proof. Apply Lemma 3.2 and Theorem 2.29 with $\mathbb{X} = \mathcal{A}^{2n}$, $\mathbb{Y} = \mathcal{A}^n$, $f = F$ and $x(t) = (u(t), u(t-\tau(t)))$. ∎

Lemma 3.4. *Let* $u, v \in PAA(\mathbb{R}, \mathcal{A}^n, \mu)$. *Then* $uv \in PAA(\mathbb{R}, \mathcal{A}^n, \mu)$.

Proof. Since $u, v \in PAA(\mathbb{R}, \mathcal{A}^n, \mu)$ then there exist $u_1, v_1 \in AA(\mathbb{R}, \mathcal{A}^n)$ and $u_2, v_2 \in \mathcal{E}AA(\mathbb{R}, \mathcal{A}^n, \mu)$ such that $u = u_1 + u_2$ and $v = v_1 + v_2$. So, $uv = u_1v_1 + u_1v_2 + u_2v_1 + u_2v_2$. It obvious that $u_1v_1 \in AA(\mathbb{R}, \mathcal{A}^n)$. We have

$$\begin{aligned}
&\|u_1(t)v_2(t) + u_2(t)v_1(t) + u_2(t)v_2(t)\|_{\mathcal{A}^n} \\
&\leq \|u_1(t)\|_{\mathcal{A}^n}\|v_2(t)\|_{\mathcal{A}^n} + \|u_2(t)\|_{\mathcal{A}^n}\|v_1(t)\|_{\mathcal{A}^n} + \|u_2(t)\|_{\mathcal{A}^n}\|v_2(t)\|_{\mathcal{A}^n} \\
&\leq \|u_1\|_0\|v_2(t)\|_{\mathcal{A}^n} + \|v_1\|_0\|u_2(t)\|_{\mathcal{A}^n} + \|u_2\|_0\|v_2(t)\|_{\mathcal{A}^n}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{r \longrightarrow +\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \|u_1(t)v_2(t) + u_2(t)v_1(t) + u_2(t)v_2(t)\|_{\mathcal{A}^n} d\mu(t) \\
\leq &\lim_{r \longrightarrow +\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} (\|u_1\|_0\|v_2(t)\|_{\mathcal{A}^n} + \|v_1\|_0\|u_2(t)\|_{\mathcal{A}^n} \\
&+ \|u_2\|_0\|v_2(t)\|_{\mathcal{A}^n}) d\mu(t) \\
\leq &\lim_{r \longrightarrow +\infty} \frac{\|u_1\|_0}{\mu([-r,r])} \int_{-r}^{r} \|v_2(t)\|_{\mathcal{A}^n} d\mu(t) + \lim_{r \longrightarrow +\infty} \frac{\|v_1\|_0}{\mu([-r,r])} \int_{-r}^{r} \|u_2(t)\|_{\mathcal{A}^n} d\mu(t) \\
&+ \lim_{r \longrightarrow +\infty} \frac{\|u_2\|_0}{\mu([-r,r])} \int_{-r}^{r} \|v_2(t)\|_{\mathcal{A}^n} d\mu(t) \\
= &\ 0.
\end{aligned}$$

Hence,

$$\lim_{r \longrightarrow +\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \|u_1(t)v_2(t) + u_2(t)v_1(t) + u_2(t)v_2(t)\|_{\mathcal{A}^n} d\mu(t) = 0.$$

Therefore, $(u_1v_2 + u_2v_1 + u_2v_2) \in \mathcal{E}AA(\mathbb{R}, \mathcal{A}^n, \mu)$. This complet the proof. ∎

Theorem 3.5. *Assume that the assumptions (H0)-(H4) hold. Then system (1.1) has a unique* $\mu-$*pseudo almost automorphic solution.*

Proof. We define an operator $\Lambda : PAA(\mathbb{R}, \mathcal{A}^n, \mu) \longrightarrow PAA(\mathbb{R}, \mathcal{A}^n, \mu)$ as follows

$$\Lambda x\,(t) = \int_{-\infty}^{t} e^{-\int_s^t D(u)du} F\,(s, x\,(s)\,, x\,(s-\tau\,(s)))\,ds, \forall x \in PAA(\mathbb{R}, \mathcal{A}^n, \mu).$$

Since $F \in PAA(\mathbb{R}\times\mathcal{A}^{2n}, \mathcal{A}^n, \mu)$ and $x \in PAA(\mathbb{R}, \mathcal{A}^n, \mu)$, by Lemma 3.3,

$$s \longmapsto f\,(s) = F\,(s, x\,(s)\,, x\,(s-\tau\,(s))) \in PAA(\mathbb{R}, \mathcal{A}^n, \mu).$$

So, there exist $f_1 \in AA(\mathbb{R}, \mathcal{A}^n)$ and $f_2 \in \mathcal{E}(\mathbb{R}, \mathcal{A}^n, \mu)$ such that $f = f_1 + f_2$ and for any sequence of real numbers $(\alpha'_n)_n$, there exists a subsequence $(\alpha_n)_n$ sucht that

$$\lim_{n\longrightarrow+\infty} f_1\,(t+\alpha_n) = \overline{f}_1\,(t) \text{ exists for all } t \in \mathbb{R}, \tag{3.2}$$

$$\lim_{n\longrightarrow+\infty} D\,(t+\alpha_n) = \overline{D}\,(t) \text{ exists for all } t \in \mathbb{R}. \tag{3.3}$$

Fisrt step: We will prove that $\Lambda x\,(t)$ exists

We have

$$\Lambda x\,(t) = \int_{-\infty}^{t} e^{-\int_s^t D(u)du} f\,(s)\,ds = \int_{-\infty}^{0} e^{-\int_s^0 D(t+u)du} f\,(t+s)\,ds.$$

So, by assumption (H2)

$$\begin{aligned}
\|\Lambda x\,(t)\|_{\mathcal{A}^n} &= \left\|\int_{-\infty}^{0} e^{-\int_s^0 D(t+u)du} f\,(t+s)\,ds\right\|_{\mathcal{A}^n} \\
&\leq \int_{-\infty}^{0}\left(\left\|e^{-\int_s^0 D(t+u)du}\right\|_{M_n(\mathbb{R})} \|f\,(t+s)\|_{\mathcal{A}^n}\right)ds \\
&\leq \|f\|_0 \int_{-\infty}^{0} e^{-\int_s^0 d^* du} ds \\
&\leq \frac{\|f\|_0}{d^*}.
\end{aligned}$$

Hence, $\Lambda x\,(t)$ exists.

Step 2: We will prove that $\Lambda x \in PAA(\mathbb{R}, \mathcal{A}^n, \mu)$.

For a fixed $t \in \mathbb{R}$, we have $\Lambda x\,(t) = \Lambda f_1\,(t) + \Lambda f_2\,(t) = g_1\,(t) + g_2\,(t)$ where

$$g_1\,(t) = \int_{-\infty}^{t} e^{-\int_s^t D(u)du} f_1\,(s)\,ds = \int_{-\infty}^{0} e^{-\int_s^0 D(t+u)du} f_1\,(t+s)\,ds$$

and

$$g_2\,(t) = \int_{-\infty}^{t} e^{-\int_s^t D(u)du} f_2\,(s)\,ds = \int_{-\infty}^{0} e^{-\int_s^0 D(t+u)du} f_2\,(t+s)\,ds.$$

We have

$$\begin{aligned}
g_1\,(t+\alpha_n) &= \int_{-\infty}^{t+\alpha_n} e^{-\int_s^{t+\alpha_n} D(u)du} f_1\,(s)\,ds \\
&= \int_{-\infty}^{0} e^{-\int_s^0 D(t+\alpha_n+u)du} f_1\,(t+s+\alpha_n)\,ds.
\end{aligned}$$

Using (3.2) and (3.3) it is easy to check that

$$\lim_{n\longrightarrow+\infty} e^{-\int_s^0 D(t+\alpha_n+u)du} f_1\,(t+s+\alpha_n) = e^{-\int_s^0 \overline{D}(t+u)du}\overline{f}_1\,(t+s)\,.$$

On the over hand, we have

$$\begin{aligned}\left\|e^{-\int_s^0 D(t+\alpha_n+u)du} f_1(t+s+\alpha_n)\right\|_{\mathcal{A}^n} &\leq \left\|e^{-\int_s^0 D(t+\alpha_n+u)du}\right\|_{M_n(\mathbb{R})} \|f_1(t+s+\alpha_n)\|_{\mathcal{A}^n} \\ &\leq \|f_1\|_0 e^{d^* s}\end{aligned}$$

and $\int_{-\infty}^0 \|f_1\|_0 e^{d^* s} ds = \dfrac{\|f_1\|_0}{d^*} < +\infty$, it follows from Lebesgue dominated convergence theorem that

$$\lim_{n\longrightarrow+\infty} g_1(t+\alpha_n) = \overline{g}_1(t) = \int_{-\infty}^0 e^{-\int_s^0 \overline{D}(t+u)du} \overline{f}_1(t+s)\,ds \text{ exists.}$$

Using the same argument one can prove that $\lim\limits_{n\longrightarrow+\infty} \overline{g}_1(t-\alpha_n) = g_1(t)$. So, $t \longmapsto g_1(t) = \Lambda f_1(t) \in AA(\mathbb{R},\mathcal{A}^n)$.

By assumption (H2) we have

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{-r}^r \left\|\int_{-\infty}^t e^{-\int_s^t D(u)du} f_2(s)\,ds\right\|_{\mathcal{A}^n} d\mu(t) \\ =\ &\frac{1}{\mu([-r,r])}\int_{-r}^r \left\|\int_{-\infty}^0 e^{-\int_s^0 D(t+u)du} f_2(t+s)\,ds\right\|_{\mathcal{A}^n} d\mu(t) \\ \leq\ &\frac{1}{\mu([-r,r])}\int_{-r}^r \left\{\int_{-\infty}^0 \left\|e^{-\int_s^0 D(t+u)du}\right\|_{M_n(\mathbb{R})} \|f_2(t+s)\|_{\mathcal{A}^n}\,ds\right\} d\mu(t) \\ \leq\ &\int_{-\infty}^0 \left\{e^{-d^* s}\frac{1}{\mu([-r,r])}\int_{-r}^r \|f_2(t+s)\|_{\mathcal{A}^n}\,d\mu(t)\right\} ds.\end{aligned}$$

We also have

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{-r}^r \|f_2(t+s)\|_{\mathcal{A}^n}\,d\mu(t) \\ =\ &\frac{1}{\mu([-r,r])}\int_{-r+s}^{r+s} \|f_2(t)\|_{\mathcal{A}^n}\,d\mu_{-s}(t) \\ \leq\ &\frac{\mu([-r-s,r+s])}{\mu([-r,r])}\frac{1}{\mu([-r-s,r+s])}\int_{-r-s}^{r+s} \|f_2(t)\|_{\mathcal{A}^n}\,d\mu_{-s}(t).\end{aligned}$$

By Lemma 2.20, Lemma 2.19 and Theorem 2.18 we deduce that

$$\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r \|f_2(t+s)\|_{\mathcal{A}^n}\,d\mu(t) = 0,$$

therefore, the dominated convergence theorem allows us to say that

$$\begin{aligned}&\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r \|g_2(t)\|_{\mathcal{A}^n}\,d\mu(t) \\ =\ &\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r \left\|\int_{-\infty}^t e^{-\int_s^t D(u)du} f_2(s)\,ds\right\|_{\mathcal{A}^n} d\mu(t) = 0.\end{aligned}$$

Hence, $t \longmapsto g_2(t) = \Lambda f_2(t) \in \mathcal{E}(\mathbb{R},\mathcal{A}^n,\mu)$ and so $\Lambda x \in PAA(\mathbb{R},\mathcal{A}^n,\mu)$.

Third step: We will prove that Λ is a contraction:

By assumption (H1) we have

$$
\begin{aligned}
&\|\Lambda x(t)-\Lambda y(t)\|_0\\
&=\left\|\int_{-\infty}^{t} e^{-\int_s^t D(u)du}F(s,x(s),x(s-\tau(s)))\,ds-\int_{-\infty}^{t} e^{-\int_s^t D(u)du}F(s,y(s),y(s-\tau(s)))\,ds\right\|_0\\
&\le \sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t} e^{-d^*(t-s)}\|F_1(s,x(s),x(s-\tau(s)))\,ds-F_1(s,y(s),y(s-\tau(s)))\|_{\mathcal{A}^n}\,ds\right.\\
&\quad\left.+\int_{-\infty}^{t} e^{-d^*(t-s)}\|F_2(s,x(s),x(s-\tau(s)))\,ds-F_2(s,y(s),y(s-\tau(s)))\|_{\mathcal{A}^n}\,ds\right\}\\
&\le \sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}\left[e^{-d^*(t-s)}(L_1+L_2)\|x(s)-y(s)\|_{\mathcal{A}^n}+(L_1'+L_2')\|x(s-\tau(s))-y(s-\tau(s))\|_{\mathcal{A}^n}\right]ds\right\}\\
&\le (L_1+L_2+L_1'+L_2')\|x-y\|_0\sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t} e^{-d^*(t-s)}ds\right\}\\
&\le \frac{(L_1+L_2+L_1'+L_2')}{d^*}\|x-y\|_0\,.
\end{aligned}
$$

From assumption (H4) and the above inequality we can conclude that Λ is a contraction operator. Thus, by Banach fixed point theorem, system (1.1) has a unique μ-pseudo almost automorphic solution. The proof is complete. ∎

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Generalized δ-$s\bigwedge_{ij}$-sets in bitopological spaces

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Abstract. The concepts of $ij-\delta-$semi closed and $ij-\delta-$semi open sets in bitopological spaces are introduced and studied. Also, the notions of $\delta-s\bigwedge_{ij}-$sets and $g\delta-s\bigwedge_{ij}-$sets are investigated. Furthermore , a new closure operator called $Cl_{\delta}^{s\bigwedge_{ij}}$ on the bitopological space (X,τ_1,τ_2) is defined and associated topology $\tau_{\delta}^{s\bigwedge_{ij}}$ is given.

AMS Subject Classifications: 54A10, 54C05, 54E55

Keywords: Bitopological space, *ij*-δ-semi open set ,$\delta-s\bigwedge_{ij}$-set, $g\delta-s\bigwedge_{ij}$-set.

Contents

1. Introduction

In 1963, Kelly [4], initiated the definition of a bitopological space as a triple (X,τ_1,τ_2), where *X* is a nonempty set and τ_1 and τ_2 are topologies on *X*. In 1981, Bose [2], introduced the concept of ij-semi open sets in bitopological spaces. In 1987, Banerjee [1], gave the notion of $ij-\delta-$open sets in such spaces.Also, investigations of $ij-\delta-$open sets were found in [5, 6]. In this paper, we introduce and study $ij-\delta-$semi closed and $ij-\delta-$semi open sets in bitopological spaces. Also, we introduce and study the notions of $\delta-s\bigwedge_{ij}$-sets and $g\delta-s\bigwedge_{ij}-$sets in bitopological spaces by generalizing the results obtained in [3]. Furthermore, we define a closure operator $Cl_{\delta}^{s\bigwedge_{ij}}$ and associated topology $\tau_{\delta}^{s\bigwedge_{ij}}$ on the bitopological space (X,τ_1,τ_2).

Throughout this paper (X,τ_1,τ_2) (or briefly X) always mean a bitopological space on which no separation axioms are assumed unless explicitly stated. Let *A* be a subset of *X*, by $i-Cl(A)$ and $i-Int(A)$ we denote the closure and the interior of *A* in the topological space (X,τ_i). By *i*-open (or τ_i-open) and *i*-closed (or τ_i-closed) we mean open and closed in the topological space (X,τ_i). $X\backslash A=A^c$ will be denote the complement of *A* and *I* denote for an index set. Also $i,j=1,2$ and $i\neq j$. Let A be a subset of a bitopological space (X,τ_1,τ_2) . A point $x\in X$ is called an $ij-\delta-$cluster point [1] of *A* if $i-Int(j-Cl(U))\cap A\neq\phi$ for every τ_i-open set *U* containing *x*. The set of all $ij-\delta-$cluster points of *A* is called the $ij-\delta-$closure of *A* and is denoted by $ij-Cl_{\delta}(A)$. A subset *A* is said to be $ij-\delta-$closed if $ij-Cl_{\delta}(A)=A$. The complement of an $ij-\delta-$closed set is called $ij-\delta-$open. A subset *A* of *X* is called $ij-$semi open [2] if $A\subset j-Cl(i-Int(A))$.

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2. ij-δ-semi open sets

Definition 2.1. *A subset A of bitopological space (X, τ_1, τ_2) is called $ij-\delta-$semi open if there exists $ij-\delta$-open set U such that $U \subset A \subset j - Cl(U)$. The complement of an $ij - \delta-$semi open set is called $ij - \delta$-semi closed.*

A point $x \in X$ is called an $ij - \delta$-semi cluster point of A if $A \cap U \neq \phi$ for every $ij - \delta$-semi open set U of X containing x. The set of all $ij - \delta-$semi cluster points of A is called the $ij - \delta-$semi closure of A and is denoted by $ij - \delta sCl(A)$. The collection of all $ij - \delta$-semi open (resp. $ij - \delta$-semi closed) sets of X will be denoted by $ij - \delta SO(X)$ (resp. $ij - \delta SC(X)$).

A subset U of X is called $ij - \delta$-semi neighborhood (briefly, $ij - \delta$-semi nbd) of a point x if there exists an $ij - \delta$-semi open set V such that $x \in V \subseteq U$.

Lemma 2.2. *The union of arbitrary collection of $ij - \delta-$semi open sets in (X, τ_1, τ_2) is $ij - \delta$-semi open.*

Proof. Since arbitrary union of $ij - \delta-$open sets is $ij - \delta$-open [4, Lemma 2.2], the result follows. ■

Lemma 2.3. *The intersection of arbitrary collection of $ij-\delta$-semi closed sets in (X, τ_1, τ_2) is $ij-\delta$-semi closed.*

Proof. Follows from Lemma 2.1. ■

Corollary 2.4. *Let $A \subset X$, $ij - \delta sCl(A) = \bigcap\{F : A \subseteq F,\ F \in ij - \delta SC(X)\}$.*

Corollary 2.5. *$ij - \delta sCl(A)$ is $ij - \delta$-semi closed, that is $ij - \delta sCl(ij - \delta sCl(A)) = ij - \delta sCl(A)$.*

Lemma 2.6. *Let (X, τ_1, τ_2) be a bitopological space. For subsets A, B and $A_k (k \in \Lambda)$ of* X*, we have*

(1) $A \subseteq ij - \delta sCl(A)$.

(2) $A \subseteq B \Rightarrow ij - \delta sCl(A) \subseteq ij - \delta sCl(B)$.

(3) $ij - \delta sCl(\bigcap\limits_k A_k) \subseteq \bigcap\limits_k ij - \delta sCl(A_k)$.

(4) $ij - \delta sCl(\bigcup\limits_k A_k) = \mathop{\cup}\limits_k \{ij - \delta sCl(A_k)\}$.

(5) A is $ij - \delta$-semi closed if and only if $A = ij - \delta sCl(A)$

3. $\delta - s \bigwedge_{ij}$ -sets and $g\delta - s \bigwedge_{ij}$-sets.

Definition 3.1. *For a subset B of a bitopological space (X, τ_1, τ_2), we define*

$B_\delta^{s\bigwedge_{ij}} = \bigcap\{O \in ij - \delta SO(X), B \subseteq O\}$

$B_\delta^{s\bigvee_{ij}} = \bigcup\{F \in ij - \delta SC(X), F \subseteq B\}$.

Definition 3.2. *A subset B of a bitopological space (X, τ_1, τ_2) is called $\delta - s \bigwedge_{ij}$-set (resp. $\delta - s \bigvee_{ij}$-set) if $B = B_\delta^{s\bigwedge_{ij}}$(resp. $B = B_\delta^{s\bigvee_{ij}}$).*

Definition 3.3. *A subset B of a bitopological space (X, τ_1, τ_2) is called*

(1) generalized $\delta - s \bigwedge_{ij}$-set (briefly, $g\delta - s \bigwedge_{ij}$-set) if $B_\delta^{s\bigwedge_{ij}} \subseteq F$ whenever $B \subseteq F$ and $F \in ji - \delta SC(X)$.

(2) generalized $\delta - s \bigvee_{ij}$-set (briefly, $g\delta - s \bigvee_{ij}$-set) if B^c is $g\delta - s \bigwedge_{ij}$.

By $G_\delta^{s\bigwedge_{ij}}$(resp. $G_\delta^{s\bigvee_{ij}}$) we will denote the family of all $g\delta - s \bigwedge_{ij}$-sets (resp. $g\delta - s \bigvee_{ij}$-sets).

Theorem 3.4. *Let A, B and B_k, $k \in I$ be subsets of a bitopological space (X, τ_1, τ_2). The following properties hold:*

(1) $B \subseteq B_\delta^{s\bigwedge_{ij}}$.

(2) If $A \subseteq B$, *then* $A_\delta^{s\bigwedge_{ij}} \subseteq B_\delta^{s\bigwedge_{ij}}$.

(3) $\left(\left(B_\delta^{s\bigwedge_{ij}}\right)_\delta\right)^{s\bigwedge_{ij}} = B_\delta^{s\bigwedge_{ij}}$.

(4) $(\bigcup_{k\in I} B_\lambda)_\delta^{s\bigwedge_{ij}} = \bigcup_{k\in I}(B_k)_\delta^{s\bigwedge_{ij}}$.

(5) If $A \in ij-\delta SO(X)$, *then* $A = A_\delta^{s\bigwedge_{ij}}$.

(6) $(B^c)_\delta^{s\bigwedge_{ij}} = \left(B_\delta^{s\bigvee_{ij}}\right)^c$.

(7) $B_\delta^{s\bigvee_{ij}} \subseteq B$.

(8) If $B \in ij-\delta SC(X)$, *then* $B = B_\delta^{s\bigvee_{ij}}$.

(9) $(\bigcap_{k\in I} B_k)_\delta^{s\bigwedge_{ij}} \subseteq \bigcap_{k\in I} (B_k)_\delta^{s\bigwedge_{ij}}$.

(10) $(\bigcup_{k\in I} B_k)_\delta^{s\bigvee_{ij}} \supseteq \bigcup_{k\in I} (B_k)_\delta^{s\bigvee_{ij}}$.

Proof. (1) Clear.

(2) Suppose $x \notin B_\delta^{s\bigwedge_{ij}}$. Then there exists an $ij-\delta$-semi open set U such that $B \subseteq U$ and $x \notin U$. Since $A \subseteq B$, then $x \notin A_\delta^{s\bigwedge_{ij}}$ and therefore $A_\delta^{s\bigwedge_{ij}} \subseteq B_\delta^{s\bigwedge_{ij}}$.

(3) Follows from (2).

(4) Let $x \notin \left(\bigcup_{k\in I} B_k\right)_\delta^{s\bigwedge_{ij}}$. Then there exists an $ij-\delta-$semi open set U such that $\bigcup_{k\in I} B_k \subseteq U$ and $x \notin U$. Thus for each $k \in I$ we have $x \notin (B_k)_\delta^{s\bigwedge_{ij}}$. So, $x \notin \bigcup_{k\in I} (B_k)_\delta^{s\bigwedge_{ij}}$.

Conversely, suppose that $x \notin \bigcup_{k\in I} (B_k)_\delta^{s\bigwedge_{ij}}$. Then there exists an $ij-\delta$-semi open set U_k (for each $k \in I$) such that $x \notin U_k$, $B_k \subseteq U_k$. Let $U = \bigcup_{k\in I} U_k$. Then, $x \notin U = \bigcup_{k\in I} U_k$, $\bigcup_{k\in I} B_k \subseteq U$ and U is $ij-\delta$-semi open. So, $x \notin \left(\bigcup_{k\in I} B_k\right)_\delta^{s\bigwedge_{ij}}$. This completes the proof of (4).

(5) Since A is an $ij-\delta-$semi open set, then $A_\delta^{s\bigwedge_{ij}} \subseteq A$. By (1) , we have $A_\delta^{s\bigwedge_{ij}} = A$.

(6) $(B_\delta^{s\bigvee_{ij}})^c = \bigcap F^c : F^c \supseteq B^c, F^c \in ij-\delta SO(X) = (B^c)_\delta^{s\bigwedge_{ij}}$.

(7) Clear.

(8) If $B \in ij-\delta SC(X)$, $B^c \in ij-\delta SO(X)$. By (5) and (6) $B^c = (B^c)_\delta^{s\bigwedge_{ij}} = \left(B_\delta^{s\bigvee_{ij}}\right)^c$. Hence $B = B_\delta^{s\bigvee_{ij}}$.

(9) Let $x \notin \bigcap_{k\in I} (B_k)_\delta^{s\bigwedge_{ij}}$. Then there exists $k \in I$ such that $x \notin (B_k)_\delta^{s\bigwedge_{ij}}$. Hence there exists $U \in ij-\delta SO(X)$ such that $B_k \subseteq U$ and $x \notin U$. Therefore $x \notin \left(\bigcap_{k\in I} B_k\right)_\delta^{s\bigwedge_{ij}}$.

(10)$(\bigcup_{k\in I} B_k)_\delta^{s\bigvee_{ij}} = \left(((\bigcup_{k\in I} B_k)^c)_\delta^{s\bigvee_{ij}}\right)^c = \left((\bigcap_{k\in I} B_k^c)_\delta^{s\bigvee_{ij}}\right)^c \supseteq \left(\bigcap_{k\in I}\left((B_k)_\delta^{s\bigvee_{ij}}\right)^c\right)^c = \bigcup_{k\in I} (B_k)_\delta^{s\bigvee_{ij}}$. ■

Theorem 3.5. *Let B be a subset of a bitopological space (X, τ_1, τ_2). Then*

(1) ϕ *and* X *are* $\delta - s\bigwedge_{ij}$*-sets and* $\delta - s\bigvee_{ij}$*-sets.*

(2) Every union of $\delta - s\bigwedge_{ij}$*-sets (resp.* $\delta - s\bigvee_{ij}$*-sets) is* $\delta - s\bigwedge_{ij}$*-sets (resp.* $\delta - s\bigvee_{ij}$*-sets).*

(3) Every intersection of $\delta - s\bigwedge_{ij}$*-sets (resp.* $\delta - s\bigvee_{ij}$*-sets) is* $\delta - s\bigwedge_{ij}$*-sets (resp.* $\delta - s\bigvee_{ij}$*-sets).*

(4) B is a $\delta - s\bigwedge_{ij}$ *−set if and only if* B^c *is a* $\delta - s\bigvee_{ij}$*-set.*

Proof. (1) and (4) are obvious.

(2) Let $\{B_k : k \in I\}$ be a family of $\delta - s\bigwedge_{ij}$-sets in (X, τ_1, τ_2). Then by Theorem 3.1(4) we have $\bigcup_{k\in I} B_k = \bigcup_{k\in I}(B_k)_\delta^{s\bigwedge_{ij}} = (\bigcup_{k\in I} B_k)_\delta^{s\bigwedge_{ij}}$.

(3) Let $\{B_k : k \in I\}$ be a family of $\delta - s\bigwedge_{ij}$ $-$sets in (X, τ_1, τ_2). Then, by Theorem 3.1(9), we have $(\bigcap_{k\in I} B_k)_\delta^{s\bigwedge_{ij}} \subseteq \bigcap_{k\in I}(B_k)_\delta^{s\bigwedge_{ij}} = \bigcap_{k\in I} B_k$. Hence, by Theorem 3.1, $\bigcap_{k\in I} B_k = (\bigcap_{k\in I} B_k)_\delta^{s\bigwedge_{ij}}$. ■

Remark 3.6. *By Theorem 3.2, the family of all $\delta - s\bigwedge_{ij}$-sets (resp. $\delta - s\bigvee_{ij}$ $-$sets), denoted by $\lambda_\delta^{s\bigwedge_{ij}}$ (resp. $\lambda_\delta^{s\bigvee_{ij}}$) in (X, τ_1, τ_2) is a topology on X containing all $ij - \delta$-semi open (resp. $ij - \delta$-semi closed) sets. Clearly $(X, \lambda_\delta^{s\bigwedge_{ij}})$ and $(X, \lambda_\delta^{s\bigvee_{ij}})$ are Alexandroff spaces.*

Theorem 3.7. *Let (X, τ_1, τ_2) be a bitopological space. Then*

(1) Every $\delta - s\bigwedge_{ij}$-set is a $g\delta - s\bigwedge_{ij}$-set.

(2) Every $\delta - s\bigvee_{ij}$-set is a $g\delta - s\bigvee_{ij}$-set.

(3) If B_k is a $g\delta - s\bigwedge_{ij}$ $-$set for all $k \in I$ then $\bigcup_{k\in I} B_k$ is a $g\delta - s\bigwedge_{ij}$-set.

(4) If B_k is a $g\delta - s\bigvee_{ij}$-set for all $k \in I$ then $\bigcap_{k\in I} B_k$ is a $g\delta - s\bigvee_{ij}$-set.

Proof. (1) Obvious.

(2)Let *B* be a $\delta - s\bigvee_{ij}$-subset of *X*. Then $B = B_\delta^{s\bigvee_{ij}}$. By Theorem 3.1(6), $(B^c)_\delta^{s\bigwedge_{ij}} = \left(B_\delta^{s\bigvee_{ij}}\right)^c = B^c$. Therefore, by (1), *B* is a $g\delta - s\bigvee_{ij}$-set.

(3)Let B_k is a $g\delta - s\bigwedge_{ij}$-subset of *X* for all $k \in I$. Then by Theorem 3.1 (4), $(\bigcup_{k\in I} B_k)_\delta^{s\bigwedge_{ij}} = \bigcup_{k\in I}(B_k)_\delta^{s\bigwedge_{ij}}$. Hence, by hypothesis, $\bigcup_{k\in I} B_k$ is a $g\delta - s\bigwedge_{ij}$-set.

(4)Follows from (3). ■

Theorem 3.8. *A subset B of a bitopological space $(X, \tau_1, \tau_2)$is a $g\delta - s\bigvee_{ij}$-set if and only if $U \subseteq B_\delta^{s\bigvee_{ij}}$, whenever $U \subseteq B$ and U is an $ij - \delta$-semi open subset of X.*

Proof. Let *U* be an $ij - \delta$-semi open subset of *X* such that $U \subseteq B$. Then, since U^c is $ij - \delta$-semi closed and $B^c \subseteq U^c$, we have $(B^c)_\delta^{s\bigwedge_{ij}} \subseteq U^c$. Hence, by Theorem 3.1(6), $\left(B_\delta^{s\bigvee_{ij}}\right)^c \subseteq U^c$. Thus $U \subseteq B_\delta^{s\bigvee_{ij}}$. On the other hand, let *F* be an $ij - \delta$-semi closed subset of X such that $B^c \subseteq F$. Since F^c is $ij - \delta$-semi open and $F^c \subseteq B$, by assumption we have $F^c \subseteq B_\delta^{s\bigvee_{ij}}$. Then $F \supseteq \left(B_\delta^{s\bigvee_{ij}}\right)^c = (B^c)_\delta^{s\bigvee_{ij}}$. Thus B^c is a $g\delta - s\bigwedge_{ij}$-set, i.e., *B* is a $g\delta - s\bigvee_{ij}$-set. ■

4. $Cl_\delta^{s\bigwedge_{ij}}$ closure operator and associated $\tau_\delta^{s\bigwedge_{ij}}$

In this section, we define a closure operator $Cl_\delta^{s\bigwedge_{ij}}$ and the associated topology $\tau_\delta^{s\bigwedge_{ij}}$ on the bitopological space (X, τ_1, τ_2) using the family of $g\delta - s\bigwedge_{ij}$-sets.

Definition 4.1. *For any subset B of a bitopological space (X, τ_1, τ_2), define $Cl_\delta^{s\bigwedge_{ij}}(B) = \bigcap\{U : B \subseteq U, U \in G_\delta^{s\bigwedge_{ij}}\}$ and $Int_\delta^{s\bigwedge_{ij}}(B) = \bigcup\{F : B \supseteq F, F^c \in G_\delta^{s\bigwedge_{ij}}\}$.*

Theorem 4.2. *Let A, B and $B_k : k\epsilon I$ be subsets of a bitopological space (X, τ_1, τ_2). Then the following statements are true:*

(1) $B \subseteq Cl_\delta^{s\bigwedge_{ij}}(B)$.

(2) $Cl_\delta^{s\bigwedge_{ij}}(B^c) = \left(Int_\delta^{s\bigwedge_{ij}}(B)\right)^c$.

(3) $Cl_\delta^{s\bigwedge_{ij}}(\phi) = \phi$.

(4) $\bigcup_{k\in I} Cl_\delta^{s\bigwedge_{ij}}(B_k) = Cl_\delta^{s\bigwedge_{ij}}(\bigcup_{k\in I} B_k)$.

(5) $Cl_\delta^{s\bigwedge_{ij}}\left(Cl_\delta^{s\bigwedge_{ij}}(B)\right) = Cl_\delta^{s\bigwedge_{ij}}(B)$.

(6) *If* $A \subseteq B$, *then* $Cl_\delta^{s\bigwedge_{ij}}(A) \subseteq Cl_\delta^{s\bigwedge_{ij}}(B)$.

(7) *If B is* $g\delta - s\bigwedge_{ij}$ *−set, then* $Cl_\delta^{s\bigwedge_{ij}}(B) = B$.

(8) *If B is* $g\delta - s\bigvee_{ij}$*-set, then* $Int_\delta^{s\bigwedge_{ij}}(B) = B$.

Proof. (1), (2) and (3) are clear.

(4) Let $x \notin Cl_\delta^{s\bigwedge_{ij}}(\bigcup_{k\in I} B_k)$. Then, there exists $U \in G_\delta^{s\bigwedge_{ij}}$ such that $\bigcup_{k\in I} B_k \subseteq U$ and $x \notin U$. Thus for each $k \in I$ we have $x \notin Cl_\delta^{s\bigwedge_{ij}}(B_k)$. This implies that $x \notin \bigcup_{k\in I} Cl_\delta^{s\bigwedge_{ij}}(B_k)$. Conversely, suppose $x \notin \bigcup_{k\in I} Cl_\delta^{s\bigwedge_{ij}}(B_k)$. Then there exist subsets $U_k \in G_\delta^{s\bigwedge_{ij}}$ for all $k \in I$ such that $x \notin U_k$ and $B_k \subseteq U_k$. Let $U = \bigcup_{k\in I} U_k$. Then $x \notin U$, $\bigcup_{k\in I} B_k \subseteq U$ and $U \in G_\delta^{s\bigwedge_{ij}}$. Thus, $x \notin Cl_\delta^{s\bigwedge_{ij}}(\bigcup_{k\in I} B_k)$.

(5) Suppose that $x \notin Cl_\delta^{s\bigwedge_{ij}}(B)$. Then there exists a subset $U \in G_\delta^{s\bigwedge_{ij}}$ such that $x \notin U$ and $B \subseteq U$. Since $U \in G_\delta^{s\bigwedge_{ij}}$ we have $Cl_\delta^{s\bigwedge_{ij}}(B) \subseteq U$. Thus we have $x \notin Cl_\delta^{s\bigwedge_{ij}}(Cl_\delta^{s\bigwedge_{ij}}(B))$. Therefore $Cl_\delta^{s\bigwedge_{ij}}\left(Cl_\delta^{s\bigwedge_{ij}}(B)\right) \subseteq Cl_\delta^{s\bigwedge_{ij}}(B)$. But by (6) $Cl_\delta^{s\bigwedge_{ij}}(B) \subseteq Cl_\delta^{s\bigwedge_{ij}}\left(Cl_\delta^{s\bigwedge_{ij}}(B)\right)$. Then the result follows.

(6) It is clear.

(7) Follows from (1).

(8) Follows from (7) and (2). ■

Theorem 4.3. $Cl_\delta^{s\bigwedge_{ij}}$ *is a Kuratowski closure operator on X.*

Definition 4.4. *Let* $\tau_\delta^{s\bigwedge_{ij}}$ *be the topology on X generated by* $Cl_\delta^{s\bigwedge_{ij}}$ *in the usual manner, i.e.,* $\tau_\delta^{s\bigwedge_{ij}} = \{B \subseteq X,\ Cl_\delta^{s\bigwedge_{ij}}(B^c) = B^c\}$.
We define a family $\rho_\delta^{s\bigwedge_{ij}}$ *by* $\rho_\delta^{s\bigwedge_{ij}} = \{B \subseteq X,\ Cl_\delta^{s\bigwedge_{ij}}(B) = B\}$, *equivalently* $\rho_\delta^{s\bigwedge_{ij}} = \{B \subseteq X,\ B^c \in \tau_\delta^{s\bigwedge_{ij}}\}$.

Theorem 4.5. *Let* (X, τ_1, τ_2) *be a bitopological space. Then*

(1) $\tau_\delta^{s\bigwedge_{ij}} = \{B \subseteq X,\ Int_\delta^{s\bigwedge_{ij}}(B) = B\}$.

(2) $ij - \delta SO(X) \subseteq G_\delta^{s\bigwedge_{ij}} \subseteq \rho_\delta^{s\bigwedge_{ij}}$.

(3) $ij - \delta SC(X) \subseteq G_\delta^{s\bigwedge_{ij}} \subseteq \tau_\delta^{s\bigwedge_{ij}}$.

(4) *If* $ij - \delta SC(X) = \tau_\delta^{s\bigwedge_{ij}}$, *then every* $g\delta - s\bigwedge_{ij}$*-set of X is* $ij - \delta$*-semi open.*

(5) *If every* $g\delta - s\bigwedge_{ij}$*-set of X is* $ij - \delta$*-semi open (i.e., if* $G_\delta^{s\bigwedge_{ij}} \subseteq ij - \delta SO(X)$*), then* $\tau_\delta^{s\bigwedge_{ij}} = \{B \subseteq X,\ B = B_\delta^{s\bigwedge_{ij}}\}$.

(6) *If every* $g\delta - s\bigwedge_{ij}$*-set of X is* $ij - \delta$*-semi closed (i.e., if* $G_\delta^{s\bigwedge_{ij}} \subseteq ij - \delta SC(X)$*), then* $ij - \delta SO(X) = \tau_\delta^{s\bigwedge_{ij}}$.

Proof. (1) By Theorem 4.1 (2), we have: If $A \subseteq X$, then $A \in \tau_\delta^{s\bigwedge_{ij}}$ if and only if $Cl_\delta^{s\bigwedge_{ij}}(A^c) = A^c$ if and only if $\left(Int_\delta^{s\bigvee_{ij}}(A)\right)^c = A^c$ if and only if $Int_\delta^{s\bigvee_{ij}}(A) = A$. Thus, $\tau_\delta^{s\bigwedge_{ij}} = \{B \subseteq X,\ Int_\delta^{s\bigvee_{ij}}(B) = B\}$.

(2) Let $B \in ij - \delta SO(X)$. By Theorem 3.1(5) *B* is a $\delta - s\bigwedge_{ij}$-set. By Theorem 3.3, *B* is a $g\delta - s\bigwedge_{ij}$ $-$set, i.e., $B \in G_\delta^{s\bigwedge_{ij}}$. Suppose *B* any element of $G_\delta^{s\bigwedge_{ij}}$. By Theorem 3.1, $B = Cl_\delta^{s\bigwedge_{ij}}(B)$, i.e., $B \in \rho_\delta^{s\bigwedge_{ij}}$. Therefore $ij - \delta SO(X) \subseteq G_\delta^{s\bigwedge_{ij}} \subseteq \rho_\delta^{s\bigwedge_{ij}}$.

(3) Let $B \in ij - \delta SC(X)$. By Theorem 3.3, $B = B_\delta^{s\bigvee_{ij}}$. Thus *B* is a $\delta - s\bigvee_{ij}$ $-$set. By Theorem 3.1, *B* is a $g\delta - s\bigvee_{ij}$ $-$set. Hence $B \in G_\delta^{s\bigvee_{ij}}$. Now, if $B \in G_\delta^{s\bigvee_{ij}}$, then by (1) and Theorem 3.4(8), $B \in \tau_\delta^{s\bigwedge_{ij}}$.

(4) Let *B* be any $g\delta - s\bigwedge_{ij}$-set, i.e., $B \in G_\delta^{s\bigwedge_{ij}}$. By (2), $B \in \rho_\delta^{s\bigwedge_{ij}}$. Thus, $B^c \in \tau_\delta^{s\bigwedge_{ij}}$. From assumption, we have $B^c \in ij - \delta SC(X)$. Hence $B \in ij - \delta SO(X)$.

(5) Let $A \subseteq X$ and $A \in \tau_\delta^{s\bigwedge_{ij}}$. Then, $A^c = Cl_\delta^{s\bigwedge_{ij}}(A^c) = \bigcap\{U : U \supseteq A, U \in G_\delta^{s\bigwedge_{ij}}\} = \bigcap\{U : U \supseteq A^c, U \in ij - \delta SO(X)\} = (A^c)_\delta^{s\bigwedge_{ij}}$. Using Theorem 3.1, we have $A = A_\delta^{s\bigvee_{ij}}$, i.e., $A \in \{B \subseteq X : B = B_\delta^{s\bigvee_{ij}}\}$.

Conversely, if $A \in \{B \subseteq X : B = B_\delta^{s\bigvee_{ij}}\}$, then by Theorem 3.3, A is a $g\delta - s\bigvee_{ij}$ $-$set. Thus $A \in G_\delta^{s\bigvee_{ij}}$. By using (3), $A \in \tau_\delta^{s\bigwedge_{ij}}$.

(6) Let $A \subseteq X$ and $A \in \tau_\delta^{s\bigwedge_{ij}}$. Then $A = \left(Cl_\delta^{s\bigwedge_{ij}}(A^c)\right)^c = \left(\bigcap\{U : A^c \subseteq U, U \in G_\delta^{s\bigwedge_{ij}}\}\right)^c = \bigcup\{U^c : U^c \in ij - \delta SO(X)\} \in ij - \delta SO(X)$.

Conversely, if $A \in ij - \delta SO(X)$, then by Theorems 3.1 and 3.3, $A \in G_\delta^{s\bigwedge_{ij}}$. By assumption $A \in ij - \delta SC(X)$. Using (3), $A \in \tau_\delta^{s\bigwedge_{ij}}$. ■

Lemma 4.6. *Let (X, τ_1, τ_2) be a bitopological space.*
(1)For each $x \in X$, $\{x\}$ is an $ij - \delta-$semi open set or $\{x\}^c$ is a $g\delta - s\bigwedge_{ij}$ $-$set of X.
(2)For each $x \in X$, $\{x\}$ is an $ij - \delta-$semi open set or $\{x\}$ is a $g\delta - s\bigvee_{ij}$ $-$set of X.

Proof. (1) Suppose that $\{x\}$ is not $ij - \delta-$semi open. Then the only $ij - \delta$-semi closed set *F* containing $\{x\}^c$ is *X*. Thus $(\{x\}^c)_\delta^{s\bigwedge_{ij}} \subseteq F = X$ and $\{x\}^c$ is a $g\delta - s\bigwedge_{ij}$ $-$set of *X*.

(2) Follows from (1). ■

Theorem 4.7. *If $ij - \delta SO(X) = \tau_\delta^{s\bigwedge_{ij}}$, then every singleton $\{x\}$ is $\tau_\delta^{s\bigwedge_{ij}}$-open.*

Proof. Suppose that $\{x\}$ is not $ij - \delta$-semi open. Then by Lemma 4.1, $\{x\}^c$ is a $g\delta - s\bigwedge_{ij}$-set. Thus $\{x\} \in \tau_\delta^{s\bigwedge_{ij}}$. Suppose that $\{x\}$ is $ij - \delta$-semi open. Then $\{x\} \in ij - \delta SO(X) = \tau_\delta^{s\bigwedge_{ij}}$. Therefore, every singleton $\{x\}$ is $\tau_\delta^{s\bigwedge_{ij}}$-open. ■

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Stability and convergence of new random approximation algorithms for random contractive-type operators in separable Hilbert spaces

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Abstract. In this paper, new iterative schemes called Jungck-DI-Noor random iterative scheme and Jungck-DI-SP random iterative scheme are introduced and studied. Also, stability and convergence results were obtained without necessarily imposing sum conditions on the countably finite family of the control sequences and injectivity condition on the operators, which makes our schemes to be more desirable in applications than the ones studied in this paper and several others currently in literature.

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Keywords: Strong convergence, Jungck-DI-Noor random iterative scheme, Jungck-DI-SP random iterative scheme, Stability, Contractive-type operator, fixed point, separable Hilbert space.

Contents

1. Introduction

Let (Y,ρ) be a complete metric space and $\Gamma : Y \longrightarrow Y$ a selfmap of Y. Suppose that $F_\Gamma = \{q \in Y : \Gamma q = q\}$ is the set of fixed points of Γ.

Over the years, different iterative schemes have been succesfully employed in approximating fixed points (or common fixed point) of different contractive operators in different spaces (see for example, [1], [4] , [12] and [16] -[44] and the references therein for more details). In 1971, Kirk [20] introduced the following iterative scheme:

Let X be a normed linear space and $\Gamma : X \longrightarrow X$ be a self-map on X. For arbitrarily chosen $y_0 \in X$, define the sequence $\{y_n\}_{n=0}^{\infty}$ iteratively as follows:

$$y_{n+1} = \sum_{j=0}^{\ell} \alpha_j \Gamma^j y_n, \sum_{j=0}^{\ell} \alpha_j = 1, n \geq 0. \tag{1.1}$$

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Since its emergence, different researchers have modified and generalised (1.1) in different spaces, see for example, [11],[15] and [29] and the reference therein.

In [29], Olatinwo introduced the iterative schemes below: Let Y be a Banach space and $\Gamma : Y \longrightarrow Y$ be a selfmap of Y.

(i) For an arbitrary point $y_0 \in Y$, $\alpha_{n,t} \geq 0, \alpha_{n,0} \neq 0, \alpha_{n,t} \in [0,1]$ *and* ℓ as a fixed integer, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$y_{n+1} = \sum_{t=0}^{\ell} \alpha_{n,t}\Gamma^t y_n, \sum_{t=0}^{\ell} \alpha_{n,t} = 1, n \geq 0 \tag{1.2}$$

(ii) For an arbitrary point $y_0 \in Y$, $\ell \geq m, \alpha_{n,t}, \beta_{n,t} \geq 0 \; with \; \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,t}, \beta_{n,t} \in [0,1] \; and \; \ell, m$ as fixed integers, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \alpha_{n,0} y_n + \sum_{t=0}^{\ell} \alpha_{n,t}\Gamma^j z_n, \sum_{t=0}^{\ell} \alpha_{n,t} = 1; \\ z_n &= \sum_{t=0}^{m} \beta_{n,t}\Gamma^t y_n, \sum_{t=0}^{\ell} \beta_{n,t} = 1, n \geq 0, \end{aligned} \tag{1.3}$$

and called them Kirk-Mann and Kirk-Ishikawa algorithms, respectively.

Chugh and Kumar [12] introduced and studied the iterative scheme below: Let Y be a Banach space and $\Gamma : Y \longrightarrow Y$ be a selfmap of Y. For an arbitrary point $y_0 \in Y$ and for $\ell \geq m \geq p, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \geq 0, \gamma_{n,0}, \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \in [0,1] \; and \; \ell, m, p$ as fixed integers, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \gamma_{n,0} y_n + \sum_{r=1}^{\ell} \gamma_{n,r}\Gamma^r z_n, \sum_{r=0}^{\ell} \gamma_{n,r} = 1; \\ z_n &= \alpha_{n,0} y_n + \sum_{s=1}^{m} \alpha_{n,s}\Gamma^s z_n, \sum_{s=0}^{m} \alpha_{n,s} = 1; \\ z_n &= \sum_{t=0}^{p} \beta_{n,t}\Gamma^t y_n, \sum_{t=0}^{p} \beta_{n,t} = 1, n \geq 0, \end{aligned} \tag{1.4}$$

In 1976, Jungck[19] introduced and studied the iterative scheme below: Let Z be a Banach space, Y an arbitrary set and $S, \Gamma : Y \longrightarrow Z$ such that $\Gamma(Y) \subseteq S(Y)$. For arbitrary $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty}$ as follows

$$Sx_{n+1} = \Gamma x_n, n = 1, 2, \cdots \tag{1.5}$$

The iterative sequence defined by (1.5) is called Jungck iterative scheme and becomes Picard iterative scheme if $S = I_d$ (identity mapping) and $Y = Z$. It is worthy to note that (1.5) has been studied and generalised by different authors in different nonlinear spaces. Interested readers should see [2], [23], [24], [27] and [41] for more details.

In [12], the following iterative scheme was introduced and studied as a generalisation of (1.4): Let Z be a Banach space, Y an arbitrary set and $S, \Gamma : Y \longrightarrow Z$ a nonself operator such that $\Gamma(Y) \subseteq S(Y)$. For arbitrary

$y_0 \in Y$, define the sequence $\{Sy_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} Sy_{n+1} &= \gamma_{n,0}Sy_n + \sum_{r=1}^{\ell}\gamma_{n,r}\Gamma^r z_n, \sum_{r=0}^{\ell}\gamma_{n,r} = 1; \\ Sz_n &= \alpha_{n,0}Sy_n + \sum_{s=1}^{m}\alpha_{n,s}\Gamma^s z_n, \sum_{s=0}^{m}\alpha_{n,s} = 1; \\ Sz_n &= \beta_{n,0}Sy_n + \sum_{t=1}^{p}\beta_{n,t}\Gamma^t y_n, \sum_{t=0}^{p}\beta_{n,t} = 1, n \geq 0, \end{aligned} \tag{1.6}$$

where $\ell \geq m \geq p, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \geq 0, \gamma_{n,0}, \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \in [0,1]$ and ℓ, m, p as fixed integers.

Remark 1.1. *Notably,* (1.6) *reduces to* (1.4) *if $S = I_d$ (identity).*

Following the introduction of random fixed point theorems by Prague school of probability in 1950, considerable efforts have been devoted toward developing this theory. This unwavering interest stem from the priceless stance of fixed point theorems in probabilistic functional analysis and probabilistic model along with their diverse applications. It is worthwhile mentioning that problems relating to measurability of solutions, probabilistic and statistical aspect of random solutions found their way in the current literature due to the introduction of randomness. Also, it is of interest to note that random fixed point theorems are stochastic generalization of classical fixed point theorems and are usually needed in the theory of random equations, random matrices, random differential equations, and different classes of random operators emanating in physical systems (see, for example, [10] for details). In 1976, a paper by Bharucha-Reid [6], which provided sufficient conditions for a stochastic analogue of Schauder's fixed point theorem for random operators, prompted various mathematicians to construct varying degree of fixed point iteration procedures for approximating fixed point of nonlinear random operators. In [14] and [42], Hans and Spacek initiated the idea of random fixed point theorems for contraction self mappings, Subsequently, Itoh [7] extended the result to multivalued random operators. In [43], using mappings that satisfied inward or the Leray Schauder condition, Xu [43] generalised the results in [7] to the case of nonself random operators. Further results in this direction could be found in [10] and the refrence therein

Definition 1.2. *Let (Ω, Σ) be a measurable space $(\Omega -$ $a\ set\ and$ $\Sigma -$ $sigma\ algebra)$, D a nonempty closed and convex subset of a real separable Banach space E and $\Gamma : \Omega \longrightarrow D$ a given mapping. Then,*

1. *Γ is said to be measurable if $\Gamma^{-1}(B \cap D) \in \Sigma$ for each Borel subset B of H;*
2. *$\Gamma : \Omega \times D \longrightarrow D$ is called random operator if $\Gamma(., \omega) : \Omega \longrightarrow D$ is measurable for every $\omega \in D$ and*
3. *Γ is siad to be continuous if for any given $\xi \in \Omega, \Gamma(\xi, .) : \Omega \times D \longrightarrow D$ is continuous.*

Definition 1.3. *Let (Ω, Σ) be a measurable space $(\Omega -$ $a\ set\ and$ $\Sigma -$ $sigma\ algebra)$, D a nonempty closed and convex subset of a real separable Banach space E and $\Gamma : \Omega \longrightarrow D$ a given mapping. A measurable function $g : \Omega \longrightarrow D$ is called a fixed point for the operator $\Gamma : \Omega \times D \longrightarrow D$ if $\Gamma(\xi, g(\xi)) = g(\xi)$ and it is referred to as a coincidence point for two random operators $S, \Gamma : \Omega \times D \longrightarrow D$ if $\Gamma(\xi, g(\xi)) = S(\xi, g(\xi)), \forall \xi \in \Omega$. The operators S, Γ are called random weakly compatible if they commute at the random coincidence point; i.e., if $\Gamma(\xi, g(\xi)) = S(\xi, g(\xi))$ for every $\xi \in \Omega$, then $\Gamma(S(\xi, g(\xi))) = S(\Gamma((\xi, g(\xi))))$ or $\Gamma(\xi, S(\xi, g(\xi))) = S(\xi, \Gamma(\xi, g(\xi)))$. The set of random common fixed points of the random mappings $S, \Gamma : \Omega \times D \longrightarrow D$ shall be denoted by $F(S, \Gamma) = \{g(\xi) \in D : S(\xi, g(\xi)) = \Gamma(\xi, g(\xi)) = g(\xi), \xi \in \Omega\}$.*

To approximate the fixed point of random mappings, different fixed point iterative schemes have been used by different authors (see, for example, [14], [40], [42], [43] and the reference therein).

Recently, Rashwan and Hammad [40] introduced the following random version of Jungck-Kirk-Noor iterative scheme defined in [24]: Let $\Gamma, S : \Omega \times Z \longrightarrow Y$ be two random mappings defined on a nonempty closed and convex subset D of a separable Banach space Y. Let $x_0 : \Omega \longrightarrow D$ be an arbitrary measurable mapping. For $\xi \in \Omega, n = 0, 1, 2, \cdots$, with $\Gamma(\xi, Z) \subseteq S(\xi, Z)$, then

$$\begin{cases} S(\xi, y_{n+1}(\xi)) = \alpha_{n,0}S(\xi, y_n(\xi)) + \sum_{i=1}^{\ell_1} \alpha_{n,i}\Gamma^i(\xi, z_n(\xi)), \sum_{i=1}^{\ell_1} \alpha_{n,i} = 1; \\ S(\xi, z_n(\xi)) = \delta_{n,0}S(\xi, y_n(\xi)) + \sum_{j=1}^{\ell_2} \delta_{n,j}\Gamma^j(\xi, t_n(\xi)), \sum_{j=1}^{\ell_2} \delta_{n,j} = 1; \\ S(\xi, t_n(\xi)) = \sum_{k=0}^{\ell_3} \gamma_{n,k}\Gamma^k(\xi, y_n(\xi)), \sum_{k=1}^{\ell_3} \gamma_{n,k} = 1, \end{cases} \tag{1.7}$$

where $\ell_1, \ell_2 \; and \; \ell_3$ are fixed integers with $\ell_1 \geq \ell_2 \geq \ell_3, \alpha_{n,i} \geq 0, \alpha_{n,0} \neq 0, \delta_{n,j} \geq 0, \delta_{n,0} \neq 0$ and $\gamma_{n,k} \geq 0, \gamma_{n,0} \neq 0$ are measurable sequences in $[0, 1]$. They called (1.7) Jungck-Kirk-Noor random iterative scheme.

Remark 1.4. *If $\ell_3 = 0 \; and \; \ell_2 = \ell_3 = 0$ in* (1.7)*, then we have the following random iterative schemes:*

$$\begin{cases} S(\xi, y_{n+1}(\xi)) = \alpha_{n,0}S(\xi, y_n(\xi)) + \sum_{i=1}^{\ell_1} \alpha_{n,i}\Gamma^i(\xi, z_n(\xi)), \sum_{i=1}^{\ell_1} \alpha_{n,i} = 1; \\ S(\xi, z_n(\xi)) = \delta_{n,0}S(\xi, y_n(\xi)) + \sum_{j=1}^{\ell_2} \delta_{n,j}\Gamma^j(\xi, y_n(\xi)) \end{cases} \tag{1.8}$$

and

$$S(\xi, y_{n+1}(\xi)) = \alpha_{n,0}S(\xi, y_n(\xi)) + \sum_{i=1}^{\ell_1} \alpha_{n,i}\Gamma^i(\xi, y_n(\xi)), \sum_{i=1}^{\ell_1} \alpha_{n,i} = 1, \tag{1.9}$$

respectively. (1.8) *and* (1.9) *are called Jungk-Kirk-Ishikawa and Jungck-Kirk-Man iterative schemes respectively.*

In real life applications, the workability of the various iterative schemes studied in this paper would be questionable if their stability is not guaranteed. In [32], Ostrowski initiated the notion of stability of iterative schemes and started investigation on this using Banach contractive conditions. Subsequently, different researchers have continued this investigation using more general contractive-type mappings than the one studied in [32]. Some recent works in this direction could be seen in [33], [34],[30],[28],[13],[32],[8],[31], [17],[11],[4] and the references therein.

Remark 1.5. *To obtain stability and convergence results in the papers studied using* (1.1)*,* (1.4)*,* (1.6)*,* (1.7)*,* (1.8)*,* (1.9) *and their variants required that the finite sum of the countably finite sequences of the measurable control parameters be unity (i.e., $\sum_{k=0}^{\ell} \gamma_{n,k} = 1, \sum_{i=0}^{m} \alpha_{n,i} = 1, \sum_{i=0}^{p} \delta_{n,j} = 1$, etc.). However, in real life applications, if $\ell, m \; and \; p$ are very large, it would be very difficult or almost impossible to generate a family of such measurable control parameters. Again, the computational cost of generating such a family of measurable control parameters (if possible) is quite enormous and also takes a very long process.*

In an attempt to overcome these challenges mentioned in Remark 1.3 for the case of a nonrandom operators, Agwu and Igbokwe introduced alternative iterative schemes in [1]. To the best of our knowledge, the problem of 'sum conditions' is still unresolved for the case of random iterative schemes. Consequently, the following question becomes necessary:

Question 1.1. *Is it possible to construct alternative random iterative schemes that would address the problems generated by the sum conditions $\left(\sum_{k=0}^{\ell_3} \gamma_{n,k} = 1, \sum_{j=0}^{\ell_2} \delta_{n,j} = 1 \; and \; \sum_{i=0}^{\ell_1} \alpha_{n,i} = 1 \right)$ imposed on the control parameters $\{\{\gamma_{n,k}\}_{n=1}^{\infty}\}_{k=1}^{\ell_3}, \{\{\alpha_{n,i}\}_{n=1}^{\infty}\}_{i=1}^{\ell_1}$ $and \; \{\{\delta_{n,j}\}_{n=1}^{\infty}\}_{j=1}^{\ell_2}$, respectively while maintaining the convergence and stability results in [40]?*

Following the same argument as in [18] regarding the linear combination of the products of countably finite family of control parameters and the problems identified in each of the iterative schemes studied, the aim of this paper is to provide an affirmative answer to Question 1.1.

2. Preliminary

The following definitions, lemmas and propositions will be needed to prove our main results.

Definition 2.1. *(see [32]) Let (Y, d) be a metric space and let $\Gamma : Y \longrightarrow Y$ be a self-map of Y. Let $\{x_n\}_{n=0}^{\infty} \subseteq Y$ be a sequence generated by an iteration scheme*

$$x_{n+1} = g(\Gamma, x_n), \tag{2.1}$$

where $x_0 \in Y$ is the initial approximation and g is some function. Suppeose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point q of Γ. Let $\{t_n\}_{n=0}^{\infty} \subseteq Y$ be an arbitrary sequence and set $\epsilon_n = d(t_n, g(\Gamma, t_n)), n = 1, 2, \cdots$ Then, the iteration scheme (2.1) *is called Γ-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = q$.*

Note that in practice, the sequence $\{t_n\}_{n=0}^{\infty}$ could be obtained in the following manner: let $x_0 \in Y$. Set $x_{n+1} = g(\Gamma, x_n)$ and let $t_0 = x_0$. Now, $x_1 = g(\Gamma, x_0)$ because of rounding in the function Γ, and a new value t_1 (approximately equal to x_1) might be calculated to give t_2, an approximate value of $g(\Gamma, t_1)$. The procedure is continued to yield the sequence $\{t_n\}_{n=0}^{\infty}$, an approximate sequence of $\{x_n\}_{n=0}^{\infty}$.

Definition 2.2. *(see, e.g., [40]) For two random operators $S, \Gamma : \Omega \times D \longrightarrow E$ with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and C is a nonempty closed and convex subset of a separable Banach space E, there exist real numbers $\eta \in [0, 1], \delta \in [0, 1)$ and a monotone increasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\phi(0) = 0$ and $\forall x, y \in C$, we get*

$$\|\Gamma(\xi, x) - \Gamma(\xi, y)\| \leq \frac{\phi(\|S(\xi, x) - \Gamma(\xi, x)\|) + \delta\|S(\xi, x) - S(\xi, y)\|}{1 + \eta\|S(\xi, x) - \Gamma(\xi, x)\|} \tag{2.2}$$

Lemma 2.3. *Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\tau_n \to 0$ as $n \to \infty$. For $0 \leq \delta < 1$, let $\{w_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $w_{n+1} \leq \delta w_n + \tau_n, n = 0, 1, 2, \cdots$ Then, $w_n \to 0$ as $n \to \infty$.*

Lemma 2.4. *(see, e.g., [40]) Let $(E, \|, \|)$ be a normed linear space and S, Γ random commuting mappings on an arbitrary set D with values in E satisfying* (2.2) *such that $\forall x, y \in D, \xi \in \Omega$,*

$$\begin{cases} \Gamma(\xi, D) \subseteq S(\xi, D); \\ \|S(\xi, S(\xi, x)) - \Gamma(\xi, S(\xi, x))\| \leq \|S(\xi, x) - \Gamma(\xi, x)\| \\ \|S(\xi, S(\xi, x)) - S(\xi, S(\xi, x))\| \leq \|S(\xi, x) - S(\xi, y)\| \end{cases} \tag{2.3}$$

Consider $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, a sublinear monotone increasing function such that $\phi(0 = 0)$ and $\phi(u) = (1 - \delta)u, \forall \delta \in [0, 1), u \in \mathbb{R}^+$. Then, $\forall i \in \mathbb{N}$ and $\forall x, y \in D$, we get

$$\|\Gamma^i(\xi, x) - \Gamma^i(\xi, y)\| \leq \frac{\sum_{j=1}^{i} \binom{i}{j} \nu^{i-1} \phi^j(\|S(\xi, x) - \Gamma(\xi, x)\|) + \nu^i\|S(\xi, x) - S(\xi, y)\|}{1 + \eta^i\|S(\xi, x) - \Gamma(\xi, x)\|} \tag{2.4}$$

Proposition 2.5. *(see,e.g., [18]) Let $\{\alpha_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ be a countable subset of the set of real numbers $\mathbb{R}$, where k is a fixed nonnegative integer and $\mathbb{N}$ is any integer with $k + 1 \leq N$. Then, the following holds:*

$$\alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1} (1 - \alpha_j) + \prod_{j=k}^{N} (1 - \alpha_j) = 1. \tag{2.5}$$

Proposition 2.6. *(see,e.g., [18]) Let t, u and v be arbitrary elements of a real Hilbert space H. Let k be any fixed nonnegetive integer and $N \in \mathbb{N}$ be such that $k + 1 \leq N$. Let $\{v_i\}_{i=1}^{N-1} \subseteq H$ and $\{\alpha_i\}_{i=1}^{N} \subseteq [0, 1]$ be a countable finite subset of H and $\mathbb{R}$, respectively. Define*

$$y = \alpha_k t + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1} (1 - \alpha_j) v_{i-1} + \prod_{j=k}^{N} (1 - \alpha_j) v.$$

Then,

$$\|y-u\|^2 = \alpha_k\|t-u\|^2 + \sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2 + \prod_{j=k}^{N}(1-\alpha_j)\|v-u\|^2$$
$$-\alpha_k\Big[\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2 + \prod_{j=k}^{i-1}(1-\alpha_j)\|t-v\|^2\Big]$$
$$-(1-\alpha_k)\Big[\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-(\alpha_{i+1}+w_{i+1})\|^2$$
$$+\alpha_N\prod_{j=k}^{i-1}(1-\alpha_j)\|v-v_{N-1}\|^2\Big],$$

where $w_k = \sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{i-1}(1-\alpha_j)v, k = 1,2,\cdots,N$ *and* $w_n = (1-c_n)v$.

3. Main Results I

Let $\Gamma, S : \Omega\times D \longrightarrow H$ be two random mappings defined on a nonempty closed convex subset of a separable Hilbert space, H. Let $x_0 : \Omega \longrightarrow C$ be an arbitrary measurable mapping. For $\xi\in\Omega, n = 1,2,\cdots$, with $\Gamma(\xi, D)\subseteq S(xi, D)$, then

$$\begin{cases} S(\xi,x_{n+1}(\xi)) = \alpha_{n,1}S(\xi,x_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + A;\\ S(\xi,y_n(\xi)) = \gamma_{n,1}S(\xi,x_n(\xi)) + \sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) + B;\\ S(\xi,z_n(\xi)) = \delta_{n,1}S(\xi,x_n(\xi)) + \sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,s})\Gamma^{s-1}(\xi,x_n(\xi)) + C, n\geq 0,1,2,.., \end{cases} \tag{3.1}$$

and

$$\begin{cases} S(\xi,x_{n+1}(\xi)) = \alpha_{n,1}S(\xi,y_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + A;\\ S(\xi,y_n(\xi)) = \gamma_{n,1}S(\xi,z_n(\xi)) + \sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) + B;\\ S(\xi,z_n(\xi)) = \delta_{n,1}S(\xi,x_n(\xi)) + \sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,s})\Gamma^{s-1}(\xi,x_n(\xi)) + C, n\geq 0,1,2,.., \end{cases} \tag{3.2}$$

where $A = \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Gamma^{\ell_1}(\xi,y_n(\xi)), B = \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\Gamma^{\ell_2}(\xi,z_n(\xi)), C = \prod_{c=1}^{\ell_3}(1-\delta_{n,s})\Gamma^{\ell_3}(\xi,x_n(\xi)), \{\{\delta_{n,s}\}_{n=0}^{\infty}\}_{s=1}^{a}, \{\{\gamma_{n,t}\}_{n=0}^{\infty}\}_{t=1}^{b}, \{\{\alpha_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{c}$ are countable finite family of measurable real sequences in $[0,1]$ and $\ell_1,\ell_2,\ell_3\in\mathbb{N}$. We shall call the iterative schemes defined by (3.1) and (3.2) the Jungck-DI-Noor random iterative scheme and Jungck-DI-SP random iterative scheme, respectively.

Remark 3.1. $1(a)$ *If* $\ell_3 = 0$ *in* (3.1)*, we obtain the following remarkable iterative schemes:*

$$\begin{cases} S(\xi,x_{n+1}(\xi)) = \alpha_{n,1}S(\xi,x_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + A;\\ S(\xi,y_n(\xi)) = \gamma_{n,1}S(\xi,x_n(\xi)) + \sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) + B, n\geq 0,1,2,.., \end{cases} \tag{3.3}$$

(b) *if* $\ell_2 = \ell_3 = 0$ *in* (3.1)*, we have the following important algorithm:*

$$S(\xi,x_{n+1}(\xi)) = \alpha_{n,1}S(\xi,x_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + A, \tag{3.4}$$

where A and B *are as defined above. The iterative schemes defined by* (3.3) *and* (3.4) *are called Jungck-DI-ishikawa and Jungck-DI-Mann random iterative schemes respectively.*

2. *If Ω is a singleton in* (3.1) *and* (3.2)*, we obtain the nonrandom version of* (3.1) *and* (3.2)*, respectively.*

3. *If S is an identity mapping in* (3.1) *and* (3.2)*, we get the following iterative algorithms:*

$$\begin{cases} x_{n+1}(\xi) = \alpha_{n,1}x_n(\xi) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, y_n(\xi)) + A; \\ y_n(\xi) = \gamma_{n,1}x_n(\xi) + \sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi, z_n(\xi)) + B; \\ z_n(\xi) = \sum_{s=1}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,s})\Gamma^{s-1}(\xi, x_n(\xi)) + C \\ , n \geq 0, 1, 2, .., \end{cases} \tag{3.5}$$

and

$$\begin{cases} x_{n+1}(\xi) = \alpha_{n,1}y_n(\xi) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, y_n(\xi)) + A; \\ y_n(\xi) = \gamma_{n,1}z_n(\xi) + \sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi, z_n(\xi)) + B; \\ z_n(\xi) = \sum_{s=1}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,s})\Gamma^{s-1}(\xi, x_n(\xi)) + C \\ , n \geq 0, 1, 2, .., \end{cases} \tag{3.6}$$

where $A, B, C, \{\{\delta_{n,s}\}_{n=0}^{\infty}\}_{s=1}^{a}, \{\{\gamma_{n,t}\}_{n=0}^{\infty}\}_{t=1}^{b}, \{\{\alpha_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{c}$ are and ℓ_1, ℓ_2, ℓ_3 are as defined in (3.1)*. We shall call the iterative schemes defined by* (3.5) *and* (3.6) *the the modified DI-Noor random iterative scheme and the modified DI-SP random iterative scheme, respectively.*

4(a). If $\ell_3 = 0$ in (3.5)*, we obtain the following remarkable iterative schemes:*

$$\begin{cases} x_{n+1}(\xi) = \alpha_{n,1}x_n(\xi) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, y_n(\xi)) + A; \\ y_n(\xi) = \gamma_{n,1}x_n(\xi) + \sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi, z_n(\xi)) + B, n \geq 0, 1, 2, .., \end{cases} \tag{3.7}$$

(b) *if $\ell_2 = \ell_3 = 0$ in* (3.5)*, we have the following important algorithm:*

$$x_{n+1}(\xi) = \alpha_{n,1}x_n(\xi) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, y_n(\xi)) + A, \tag{3.8}$$

Theorem 3.2. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.1) *converges to $q(\xi)$, then the random Jungck-DI-Noor iterative scheme is S, Γ-stable.*

Proof. Let $q(\xi) : \Omega \longrightarrow D$ be a measurable mapping and $z(\xi) : \Omega \longrightarrow D$ a random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. Let $\{S(\xi, t_n(\xi))\}_{n=0}^{\infty} \subset H$ and

$$\begin{aligned} \epsilon_n = & \|S(\xi, t_{n+1}(\xi)) - \alpha_{n,1}S(\xi, t_n(\xi)) - \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, g_n(\xi)) \\ & - \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Gamma^{\ell_1}(\xi, g_n(\xi))\|, \end{aligned} \tag{3.9}$$

where, for every $\xi \in \Omega$,

$$S(\xi, g_n(\xi)) = \gamma_{n,1} S(\xi, t_n(\xi)) + \sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1} (1-\gamma_{n,b}) \Gamma^{t-1}(\xi, f_n(\xi)) + \prod_{b=1}^{\ell_2} (1-\gamma_{n,b}) \Gamma^{\ell_2}(\xi, f_n(\xi)), \tag{3.10}$$

and

$$S(\xi, f_n(\xi)) = \delta_{n,1} S(\xi, t_n(\xi)) + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1} (1-\delta_{n,c}) \Gamma^{s-1}(\xi, t_n(\xi)) + \prod_{c=1}^{\ell_3} (1-\delta_{n,c}) \Gamma^{\ell_3}(\xi, t_n(\xi)). \tag{3.11}$$

Let $\epsilon_n \to 0 \ as \ n \to \infty$, then by lemma 2.2 and Proposition 2.4, with $S(\xi, t_n(\xi)) = t, \Gamma^{i-1}(\xi, g_n(\xi)) = v_{j-1}, \Gamma^{\ell_1}(\xi, g_n(\xi)) = v \ and \ k = 1$, we get the following estimates:

$$\begin{aligned}
\|S(\xi, t_{n+1}(\xi)) - q(\xi)\|^2 &= \|\alpha_{n,1} S(\xi, t_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) - q(\xi) - \Big[\alpha_{n,1} S(\xi, t_n(\xi)) \\
&\quad + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) \\
&\quad - S(\xi, t_n(\xi))\Big]\|^2 \\
&\leq \|\alpha_{n,1} S(\xi, t_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) - q(\xi)\|^2 + \| - \Big[\alpha_{n,1} S(\xi, t_n(\xi)) \\
&\quad + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) \\
&\quad - S(\xi, t_n(\xi))\Big]\|^2 \\
&= \|\alpha_{n,1} S(\xi, t_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) - q(\xi)\|^2 + \epsilon_n
\end{aligned}$$

$$\begin{aligned}\leq\ &\epsilon_n\\ &+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2\\ &+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\|\Gamma^{i-1}(\xi,g_n(\xi))-q(\xi)\|^2\\ &+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\|\Gamma^{\ell_1}(\xi,g_n(\xi))-q(\xi)\|^2 \end{aligned}\tag{3.12}$$

But,

$$\|\Gamma^{i-1}(\xi,g_n(\xi))-\Gamma^{i-1}(\xi,z(\xi))\|\leq H, \tag{3.13}$$

where

$$H=\frac{\sum_{j=1}^{i}\binom{i}{j}\nu^{i-1}\phi^j(\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|)+\nu^i\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\|}{1+\eta^i\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|}$$

(3.13) implies

$$\|\Gamma^{i-1}(\xi,g_n(\xi))-\Gamma^{i-1}(\xi,z(\xi))\|\leq\frac{\sum_{j=1}^{i}\binom{i}{j}\nu^{i-1}\phi^j(0)+\nu^i\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\|}{1+\eta^i\|0\|}$$

Since $\phi^i(0)=0$, it follows from the last inequality above that

$$\|\Gamma^{i-1}(\xi,g_n(\xi))-\Gamma^{i-1}(\xi,z(\xi))\|\leq\nu^i\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\| \tag{3.14}$$

(3.12) and (3.14)

$$\begin{aligned}\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\leq\ &\epsilon_n\\ &+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2\\ &+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\|^2\\ &+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\|^2 \end{aligned}\tag{3.15}$$

Also, using (3.10) and Proposition 2.4, with $S(\xi,t_n(\xi))=t,\Gamma^{i-1}(\xi,f_n(\xi))=v_{j-1},\Gamma^{\ell_2}(\xi,f_n(\xi))=v\ and\ k=1$, we obtain the following estimaes:

$$\begin{aligned}&\|S(\xi,g_n(\xi))-q(\xi)\|\\ &=\|\gamma_{n,1}S(\xi,t_n(\xi))\\ &\quad+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,f_n(\xi))\\ &\quad+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\Gamma^{\ell_2}(\xi,f_n(\xi))-q(\xi)\|^2\\ &\leq\|\gamma_{n,1}S(\xi,t_n(\xi))-q(\xi)\|^2\\ &\quad+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\|\Gamma^{t-1}(\xi,f_n(\xi))-q(\xi)\|^2\\ &\quad+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\|\Gamma^{\ell_2}(\xi,f_n(\xi))-q(\xi)\|^2 \end{aligned}\tag{3.16}$$

Since $\phi(0)=0$, it follows from Lemma 2.2 that

$$\|\Gamma^{t-1}(\xi, g_n(\xi)) - \Gamma^{t-1}(\xi, z(\xi))\| \leq H^\star, \tag{3.17}$$

where

$$H^\star = \frac{\sum_{j=1}^{t}\binom{t}{j}\nu^{t-j}\phi^j(\|S(\xi,z(\xi)) - \Gamma(\xi,z(\xi))\|) + \nu^t\|S(\xi,z(\xi)) - S(\xi,f_n(\xi))\|}{1+\eta^t\|S(\xi,z(\xi)) - \Gamma(\xi,z(\xi))\|}$$

(3.17) implies

$$\begin{aligned}\|\Gamma^{t-1}(\xi, g_n(\xi)) - \Gamma^{i-1}(\xi, z(\xi))\| &\leq \frac{\sum_{j=1}^{t}\binom{t}{j}\nu^{t-j}\phi^j(0) + \nu^t\|S(\xi,z(\xi)) - S(\xi,f_n(\xi))\|}{1+\eta^t\|0\|} \\ &= \nu^t\|S(\xi,z(\xi)) - S(\xi,f_n(\xi))\| \end{aligned}\tag{3.18}$$

Again, using (3.11) and Proposition 2.4, with

$$S(\xi,t_n(\xi)) = t, \Gamma^{i-1}(\xi,t_n(\xi)) = v_{j-1}, \Gamma^{\ell_2}(\xi,t_n(\xi)) = v \text{ and } k=1,$$

we obtain the following estimaes:

$$\begin{aligned}\|S(\xi,f_n(\xi)) - q(\xi)\|^2 &= \|\delta_{n,1}S(\xi,t_n(\xi)) + \sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})\Gamma^{s-1}(\xi,t_n(\xi)) \\ &\quad + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})\Gamma^{\ell_3}(\xi,t_n(\xi)) - q(\xi)\|^2 \\ &\leq \delta_{n,1}\|S(\xi,t_n(\xi)) - q(\xi)\|^2 + \sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})\|\Gamma^{s-1}(\xi,t_n(\xi)) - q(\xi)\|^2 \\ &\quad + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})\|\Gamma^{\ell_3}(\xi,t_n(\xi)) - q(\xi)\|^2 \end{aligned}\tag{3.19}$$

Since $z(\xi)$ is the coincidence point of $S, \Gamma, \phi(0)=0$ and

$$\|\Gamma^{s-1}(\xi, g_n(\xi)) - \Gamma^{s-1}(\xi, z(\xi))\| \leq W^\star,$$

where

$$W^\star = \frac{\sum_{j=1}^{s}\binom{s}{j}\nu^{s-j}\phi^j(\|S(\xi,z(\xi)) - \Gamma(\xi,z(\xi))\|) + \nu^s\|S(\xi,z(\xi)) - S(\xi,t_n(\xi))\|}{1+\eta^s\|S(\xi,z(\xi)) - \Gamma(\xi,z(\xi))\|},$$

it follows that

$$\begin{aligned}\|\Gamma^{s-1}(\xi, t_n(\xi)) - \Gamma^{s-1}(\xi, z(\xi))\| &\leq \frac{\sum_{j=1}^{s}\binom{s}{j}\nu^{s-j}\phi^j(0) + \nu^s\|S(\xi,z(\xi)) - S(\xi,t_n(\xi))\|}{1+\eta^s\|0\|} \\ &= \nu^s\|S(\xi,z(\xi)) - S(\xi,t_n(\xi))\|. \end{aligned}\tag{3.20}$$

Since (3.16) and (3.18) imply

$$\begin{aligned}\|S(\xi,g_n(\xi)) - q(\xi)\| &\leq \gamma_{n,1}\|S(\xi,t_n(\xi)) - q(\xi)\|^2 \\ &\quad + \Big(\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2 + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big) \\ &\quad \times\|S(\xi,z(\xi)) - S(\xi,f_n(\xi))\|^2 \end{aligned}\tag{3.21}$$

and (3.19) and (3.20) imply

$$\|S(\xi, f_n(\xi)) - q(\xi)\|^2 \leq \Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2 + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big) \times \|S(\xi, z(\xi)) - S(\xi, t_n(\xi))\|^2,$$

we have (using (3.15)) that

$$\begin{aligned}\|S(\xi, t_{n+1}(\xi)) - q(\xi)\|^2 \leq \Big\{&\alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2 + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2 + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2 + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\Big]\Big\}\\ &\times\|S(\xi, z(\xi)) - S(\xi, t_n(\xi))\|^2 + \epsilon_n \end{aligned} \tag{3.22}$$

Let

$$\begin{aligned}\delta_n = \Big\{&\alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2 + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2 + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2 + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\Big]\Big\},\end{aligned}$$

so that from Proposition 2.3 and the fact that $\nu^i \in [0,1)$, we obtain

$$\begin{aligned}\delta_n = \Big\{&\alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2 + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2 + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2 + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\Big]\Big\},\\ < \Big\{&\alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a}) + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Big)\\ &\times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b}) + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\Big)\\ &\times\Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c}) + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})\Big)\Big]\Big\} = 1\end{aligned} \tag{3.23}$$

Using Lemma 2.1, we obtain from (3.22) and (3.23) that $S(\xi, t_n(\xi)) \to q(\xi)$ *as* $n \to \infty$.

Conversely, let $S(\xi, t_n(\xi)) \to 0$ *as* $n \to \infty$. Then, we show that $\epsilon_n \to 0$ *as* $n \to \infty$. Now, by using (3.9), (3.22), Proposition 2.4 and Lemma 2.2, we estimate as follows:

$$\begin{aligned}
\epsilon_n &= \|S(\xi,t_{n+1}(\xi)) - q(\xi) - \Big[\alpha_{n,1}S(\xi,t_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Gamma^{\ell_1}(\xi,g_n(\xi)) - q(\xi)\Big]\|^2 \\
&\le \|S(\xi,t_{n+1}(\xi)) - q(\xi)\|^2 + \|\alpha_{n,1}S(\xi,t_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Gamma^{\ell_1}(\xi,g_n(\xi)) - q(\xi)\|^2 \\
&\le \|S(\xi,t_{n+1}(\xi)) - q(\xi)\|^2 + \alpha_{n,1}\|S(\xi,t_n(\xi)) - q(\xi)\|^2 \\
&\quad + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\|\Gamma^{i-1}(\xi,g_n(\xi)) - q(\xi)\|^2 + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\|\Gamma^{\ell_1}(\xi,g_n(\xi)) - q(\xi)\|^2 \\
&\le \|S(\xi,t_{n+1}(\xi)) - q(\xi)\|^2 + \alpha_{n,1}\|S(\xi,t_n(\xi)) - q(\xi)\|^2 \\
&\quad + \Big(\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2 + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\|S(\xi,g_n(\xi)) - q(\xi)\|^2 \\
&\le \|S(\xi,t_{n+1}(\xi)) - q(\xi)\|^2 + \Bigg\{\alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2 + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big) \\
&\quad \times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2 + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big) \\
&\quad \times\Big(\delta_{n,1} + \sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2 + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\Big]\Bigg\}\|S(\xi,z(\xi)) - S(\xi,t_n(\xi))\|^2
\end{aligned}$$

Observe that the right hand side of the last inequality tends to 0 as $n \to \infty$, hence $\epsilon_n \to 0$ as $n \to \infty$. The completes the proof.

If $\ell_3 = 0$ *and* $\ell_2 = \ell_3 = 0$, then Theorem 3.1 yields the following corollaries: ■

Corollary 3.3. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators S, Γ, S^i, Γ^i(i.e., $S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi)$). For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.3) *converges to $q(\xi)$, then the random Jungck-DI-Ishikawa iterative scheme is S, Γ-stable.*

Corollary 3.4. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators S, Γ, S^i, Γ^i(i.e., $S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi)$). For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.4) *converges to $q(\xi)$, then the random Jungck-DI-Mann iterative scheme is S, Γ-stable.*

If S is an identity in (3.1), (3.3) and (3.4), we obtain the following corollaries:

Corollary 3.5. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{x_n(\xi)\}_{n=0}^{\infty}$ generated by* (3.5) *converges to $q(\xi)$, then the random DI-Noor iterative scheme is S, Γ-stable.*

Corollary 3.6. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.7) *converges to $q(\xi)$, then the random DI-Ishikawa iterative scheme is S, Γ-stable.*

Corollary 3.7. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.8) *converges to $q(\xi)$, then the random DI-Mann iterative scheme is S, Γ-stable.*

Theorem 3.8. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.2) *converges to $q(\xi)$, then the random Jungck-DI-SP iterative scheme is S, Γ-stable.*

Proof. Using similar argument as in Theorem 3.1, the proof of Theorem 3.4 follows immediately. ■

Again, if S is an identity in (3.2), we obtain the following corollary from Theorem 3.7:

Corollary 3.9. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.6) *converges to $q(\xi)$, then the random DI-SP iterative scheme is S, Γ-stable.*

4. Main Result II

Theorem 4.1. *Let H be a real separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators for an arbitrary set D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, let $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ be the random Jungck-DI-SP iterative scheme generated by* (3.2)*. Then,*

(i) *q is the unique common fixed point of Γ^{i-1} and $S^{i-1} (i = 2, 3, \cdots)$ if $D = H$ and Γ, S commute at q (i.e., Γ, S are weakly comprtible);*

(ii) *the Jungck-DI-SP iteration scheme converges strongly to $q(\xi) \in \Gamma(\xi)$.*

Proof. Assume that $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ has a pointwise limit ($i.e., \lim_{n\to\infty} S(\xi, x_n(\xi)) = q(\xi), \forall \xi \in \Omega$). Since H is a separable Hilbert space, it follows that $S(\xi, g(\xi)) = q(\xi)$ is a measurable mapping for any random operator $S : \Omega \times K \longrightarrow K$ and any measurable mapping $g : \Omega \longrightarrow K$. Thus, the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by the random Jungck-DI-SP iterative scheme (3.2) is a sequence of measurable mappings. Also, since K is convex and $q(\xi)$ is measurable, then $q : \omega \longrightarrow K$ (being limit of measurable mapping) is as well measurable.

Now, we show that $S, \Gamma, S^i \; and \; \Gamma^i$ have a unique coincidence point $z(\xi)$. Let $K(S, \Gamma, S^i, \Gamma^i)$ be the set of all coincidence points of $S, \Gamma, S^i \; and \; \Gamma^i$; and suppose there exists another coincidence point $q' \in K(S, \Gamma, S^i, \Gamma^i) \; with \; q' \neq q$. Then, we can find $z^\star(\xi) \neq z(\xi)$ such that $S(\xi, z^\star(\xi)) = \Gamma(\xi, z^\star(\xi)) = S^i(\xi, z^\star(\xi)) = S^i(\xi, z^\star(\xi)) = q'(\xi)$. Using (2.4) and the fact that $\phi(0) = 0$, we get

$$\|q(\xi) - q'(\xi)\| = \|\Gamma^{i-1}(\xi, z(\xi)) - \Gamma^{i-1}(\xi, z^\star(\xi))\| \leq Q^\star, \tag{4.1}$$

where

$$Q^\star = \frac{\sum_{j=1}^{i} \binom{s}{j} \nu^{i-j} \phi^j(\|S(\xi, z(\xi)) - \Gamma(\xi, z(\xi))\|) + \nu^i \|S(\xi, z(\xi)) - S(\xi, z^\star(\xi))\|}{1 + \eta^i \|S(\xi, z(\xi)) - \Gamma(\xi, z(\xi))\|}.$$

From (4.1), we obtain

$$\begin{aligned}\|q(\xi) - q'(\xi)\| &\leq \frac{\sum_{j=1}^{i} \binom{i}{j} \nu^{i-j} \phi^j(0) + \nu^i \|S(\xi, z_1(\xi)) - S(\xi, z_2(\xi))\|}{1 + \eta^i \|0\|} \\ &= \nu^s \|S(\xi, z(\xi)) - S(\xi, z^\star(\xi))\| = \nu^i \|q(\xi) - q'(\xi)\|.,\end{aligned}$$

which yields $(1 - \nu^i)\|q(\xi) - q'(\xi)\| \leq 0$. Since $\nu^i \in [0, 1)$ and the norm is a nonnegative function, it follows that $q(\xi) = q'(\xi)$, which is a contradiction to our earlier assumption that $q(\xi) \neq q'(\xi)$. Hence, $S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = S^i(\xi, z(\xi)) = q(\xi)$. Therefore, $q(\xi)$ is unique. Further, since $\Gamma(\xi) \; and \; S(\xi)$ are weakly compartible, we have $\Gamma(\xi, S(\xi, z(\xi))) = S(\xi, \Gamma(\xi, z(\xi))) \; and \; \Gamma^i(\xi, S(\xi, z(\xi))) = S^i(\xi, \Gamma^i(\xi, z(\xi)))$. Hence, $\Gamma(\xi, q(\xi)) = S(\xi, q(\xi)) = \Gamma^i(\xi, q(\xi)) = S^i(\xi, q(\xi))$ so that $q(\xi)$ is the coimcidence point of $\Gamma, S, \Gamma^i \; and \; S^i$. Also, since the coincidence point is unique, we get $q(\xi) = z(\xi)$. Thus, $\Gamma(\xi, z(\xi)) = S(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = S^i(\xi, z(\xi)) = q(\xi)$.

Next, we show that $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ converges to $q(\xi)$. Using (3.2), lemma 2.2 and Proposition 2.4, with $S(\xi, y_n(\xi)) = t, \Gamma^{i-1}(\xi, y_n(\xi)) = v_{j-1}, \Gamma^{\ell_1}(\xi, y_n(\xi)) = v \; and \; k = 1$, we get the following estimates:

$$\begin{aligned}\|S(\xi, x_{n+1}(\xi)) - q(\xi)\|^2 &= \|\alpha_{n,1} S(\xi, y_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1 - \alpha_{n,a}) \Gamma^{i-1}(\xi, y_n(\xi)) \\ &\quad + \prod_{a=1}^{\ell_1} (1 - \alpha_{n,a}) \Gamma^{\ell_1}(\xi, y_n(\xi)) - q(\xi)\|^2 \\ &\leq \alpha_{n,1} \|S(\xi, y_n(\xi)) - q(\xi)\|^2 \\ &\quad + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1 - \alpha_{n,a}) \|\Gamma^{i-1}(\xi, y_n(\xi)) - q(\xi)\|^2 \\ &\quad + \prod_{a=1}^{\ell_1} (1 - \alpha_{n,a}) \|\Gamma^{\ell_1}(\xi, y_n(\xi)) - q(\xi)\|^2.\end{aligned} \tag{4.2}$$

Since $z(\xi)$ is the coincidence point of $S, \Gamma, \phi(0) = 0$ and

$$\|\Gamma^{i-1}(\xi, y_n(\xi)) - \Gamma^{i-1}(\xi, z(\xi))\| \leq P^\star,$$

where

$$P^{\star}=\frac{\sum_{j=1}^{s}\binom{i}{j}\nu^{i-j}\phi^{j}(\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|)+\nu^{i}\|S(\xi,z(\xi))-S(\xi,y_{n}(\xi))\|}{1+\eta^{i}\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|},$$

it follows that

$$\begin{aligned}\|\Gamma^{i-1}(\xi,y_{n}(\xi))-\Gamma^{i-1}(\xi,z(\xi))\| &\leq \frac{\sum_{j=1}^{i}\binom{i}{j}\nu^{s-j}\phi^{j}(0)+\nu^{s}\|S(\xi,z(\xi))-S(\xi,y_{n}(\xi))\|}{1+\eta^{i}\|0\|}\\ &=\nu^{i}\|S(\xi,z(\xi))-S(\xi,y_{n}(\xi))\|. \qquad (4.3)\end{aligned}$$

(4.2) and (4.3) imply

$$\begin{aligned}\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^{2} \leq &\Big(\alpha_{n,1}+\sum_{i=2}^{\ell_{1}}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i})^{2}+\prod_{a=1}^{\ell_{1}}(1-\alpha_{n,a})(\nu^{i})^{2}\Big)\\ &\times\|S(\xi,y_{n}(\xi))-q(\xi)\|^{2}. \qquad (4.4)\end{aligned}$$

Again, from (3.2), lemma 2.2 and Proposition 2.4, with $S(\xi,z_{n}(\xi))=t,\Gamma^{i-1}(\xi,z_{n}(\xi))=v_{j-1},\Gamma^{\ell_{1}}(\xi,z_{n}(\xi))=v\ and\ k=1$, we get the following estimates:

$$\begin{aligned}\|S(\xi,y_{n}(\xi))-q(\xi)\|^{2} &= \|\gamma_{n,1}S(\xi,z_{n}(\xi))+\sum_{t=2}^{\ell_{2}}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_{n}(\xi))\\ &\quad+\prod_{b=1}^{\ell_{2}}(1-\gamma_{n,b})\Gamma^{\ell_{2}}(\xi,z_{n}(\xi))-q(\xi)\|^{2}\\ &\leq \gamma_{n,1}\|S(\xi,z_{n}(\xi))-q(\xi)\|^{2}\\ &\quad+\sum_{t=2}^{\ell_{2}}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\|\Gamma^{i-1}(\xi,z_{n}(\xi))-q(\xi)\|^{2}\\ &\quad+\prod_{b=1}^{\ell_{2}}(1-\gamma_{n,b})\|\Gamma^{\ell_{2}}(\xi,z_{n}(\xi))-q(\xi)\|^{2}. \qquad (4.5)\end{aligned}$$

Since $z(\xi)$ is the coincidence point of $S,\Gamma,\phi(0)=0$ and

$$\|\Gamma^{t-1}(\xi,z_{n}(\xi))-\Gamma^{t-1}(\xi,z(\xi))\|\leq P^{\star\star},$$

where

$$P^{\star\star}=\frac{\sum_{j=1}^{t}\binom{t}{j}\nu^{t-j}\phi^{j}(\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|)+\nu^{t}\|S(\xi,z(\xi))-S(\xi,z_{n}(\xi))\|}{1+\eta^{t}\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|},$$

it follows that

$$\begin{aligned}\|\Gamma^{t-1}(\xi,z_{n}(\xi))-\Gamma^{t-1}(\xi,z(\xi))\| &\leq \frac{\sum_{j=1}^{i}\binom{i}{j}\nu^{t-j}\phi^{j}(0)+\nu^{t}\|S(\xi,z(\xi))-S(\xi,z_{n}(\xi))\|}{1+\eta^{t}\|0\|}\\ &=\nu^{t}\|S(\xi,z(\xi))-S(\xi,z_{n}(\xi))\|. \qquad (4.6)\end{aligned}$$

(4.5) and (4.6) imply that

$$\begin{aligned}\|S(\xi,y_{n}(\xi))-q(\xi)\|^{2} \leq &\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_{2}}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t})^{2}+\prod_{b=1}^{\ell_{2}}(1-\gamma_{n,b})(\nu^{t})^{2}\Big)\\ &\times\|S(\xi,z_{n}(\xi))-q(\xi)\|^{2}. \qquad (4.7)\end{aligned}$$

Further, using (3.2) and similar argument as above, we obtain

$$\begin{aligned}\|S(\xi,z_n(\xi))-q(\xi)\|^2 \le &\Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\\ &\times\|S(\xi,x_n(\xi))-q(\xi)\|^2. \end{aligned} \tag{4.8}$$

(4.4), (4.7) and (4.9) imply

$$\begin{aligned}\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2 \le &\Big(\alpha_{n,1}+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\\ &\times\|S(\xi,x_n(\xi))-q(\xi)\|^2. \end{aligned} \tag{4.9}$$

Let

$$\begin{aligned}\delta_n^\star = &\Big(\alpha_{n,1}+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big) \end{aligned} \tag{4.10}$$

Since $\nu^i \in [0,1)$, w obtain from (4.10) and Proposition 2.3 that

$$\begin{aligned}\delta_n^\star = &\Big(\alpha_{n,1}+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\\ < &\Big(\alpha_{n,1}+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Big)\\ &\times\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\Big)\\ &\times\Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})\Big)=1 \end{aligned} \tag{4.11}$$

From (4.9), (4.11) and Lemma 2.1, we get that $S(\xi,x_n(\xi)) \to q(\xi)\ as\ n \to \infty$. The proof is completed. ■

If s is an identity in (3.1), then the following corollary from Theorem 4.1:

Corollary 4.2. *Let H be a real separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting operators for an arbitrary set D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, let $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ be the random DI-SP iterative scheme generated by* (3.6)*. Then,*

(i) *q is the unique common fixed point of Γ^{i-1} and $S^{i-1}(i = 2, 3, \cdots)$ if $D = H$ and Γ, S commute at q (i.e., Γ, S are weakly comprtible);*

(ii) *the DI-SP iteration scheme converges strongly to $q(\xi) \in \Gamma(\xi)$.*

Theorem 4.3. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, let $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ be the random Jungck-DI-Noor iterative scheme generated by* (3.1)*. Then,*

(i) *$\Gamma(\xi)$ defined by* (2.4) *has a unique fixed point q;*

(ii) *the Jungck-DI-SP iteration scheme converges strongly to $q\xi \in \Gamma(\xi)$.*

Proof. Using similar argument as in Theorem 4.1, the proof of Theorem 4.2 follows immediately. ■

Also, if S is an identity in (3.1), we obtain the following corollary from Theorem 4.3:

Corollary 4.4. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, let $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ be the random DI-Noor iterative scheme generated by* (3.5)*. Then,*

(i) *$\Gamma(\xi)$ defined by* (2.4) *has a unique fixed point q;*

(ii) *the DI-SP iteration scheme converges strongly to $q\xi \in \Gamma(\xi)$.*

Remark 4.5. *The following areas are still open:*

(i) to reconstruct, approximate the fixed points and the stability results of some existing random iterative schemes in the current literature, other than the ones under study, for finite family of certain class of contractive-type map;

(ii) to compare convergent rates of the iterative schemes defined by (3.1) *and* (3.2) *with those of* (1.7)*;*

(iii) to prove Proposition [2.3 and 2.4] in more general spaces so as to extend the results in this paper to such spaces.

5. Conclusion

An affirmative answer has been provided for Question 1.1. The results obtained in this paper improve the corresponding results in [10], [19], [40] and several others currrently announced in literature.

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