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Computation of a summation formula associated with certain special functions

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Abstract

The main aim of the present paper is to establish a summation formula involving certain special functions.

Keywords: Gauss second summation theorem, Recurrence relation, Prudnikov

2010 MSC: 33C05 , 33C20 , 33D15 , 33D50 , 33D60.

1 Introduction

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{ll} a_1,a_2,\cdots,a_A & ; \\ & z \\ b_1,b_2,\cdots,b_B & ; \end{array}\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_A)_k z^k}{(b_1)_k(b_2)_k\cdots(b_B)_k k!} \tag{1}$$

.

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers and $|z| = 1$

Contiguous Relation is defined by

[Andrews p.363(9.16), E. D. p.51(10)]

$$ (a-b)\ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \\ c\ \ ; & z \end{array}\right] = a\ {}_2F_1\left[\begin{array}{ll} a+1,\ b\ ; & \\ c\ \ ; & z \end{array}\right] - b\ {}_2F_1\left[\begin{array}{ll} a,\ b+1\ ; & \\ c\ ; & z \end{array}\right] \tag{2}$$

Gauss second summation theorem is defined by [Prudnikov., 491(7.3.7.5)]

$$ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \frac{1}{2} \\ \frac{a+b+1}{2}\ ; & 2 \end{array}\right] = \frac{\Gamma(\frac{a+b+1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} \tag{3}$$

$$ = \frac{2^{(b-1)}\,\Gamma(\frac{b}{2})\,\Gamma(\frac{a+b+1}{2})}{\Gamma(b)\,\Gamma(\frac{a+1}{2})} \tag{4}$$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov., p.491(7.3.7.8)]

$$ {}_2F_1\left[\begin{array}{ll} a,\ b\ \ ; & \frac{1}{2} \\ \frac{a+b-1}{2}\ ; & 2 \end{array}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} + \frac{2\,\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\,\Gamma(b)}\right] \tag{5}$$

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Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$$ {}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b-1}{2} & ; \end{array} \ \frac{1}{2}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-1}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{6} $$

Recurrence relation is defined by

$$ \Gamma(\zeta+1) = \zeta\,\Gamma(\zeta) \tag{7} $$

2 Main formula

$$ {}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b+46}{2} & ; \end{array} \ \frac{1}{2}\right] = $$

$$ = \frac{2^b\,\Gamma(\frac{a+b+46}{2})}{(a-b)\,\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{1}{\left[\prod_{\zeta=0}^{21}\{a-b-2\zeta\}\right]\left[\prod_{\eta=1}^{22}\{a-b+2\eta\}\right]}\times\right.\right. $$

$$ \times\Big(4194304(-1071454715572847955148800000a + 19529183870862778957824000000a^2 $$
$$ -156569123088349991534592000a^3 + 7447335820376446567710720 0a^4 $$
$$ -23811192195736807158054912a^5 + 5481259447061368207835136a^6 $$
$$ -948292268763887952199680a^7 + 126888416217818346291200a^8 - 13393761871844011671552a^9 $$
$$ +1130574271590544777216a^{10} - 77005895857888757760a^{11} + 4254539623864857600a^{12} $$
$$ -191027711898895872a^{13} + 6960284638689536a^{14} - 204763953757440a^{15} + 4818538806400a^{16} $$
$$ -89349365952a^{17} + 1275548736a^{18} - 13518120a^{19} + 100100a^{20} - 462a^{21} + a^{22} $$
$$ +1071454715572847955148800000b + 544099662756275407552512000ab $$
$$ -214707088455270681437798400a^2b + 558803648319188167242547200a^3b $$
$$ -124626488196420160060391424a^4b + 76589682258781485165182976a^5b $$
$$ -10211457792319715925295104a^6b + 2878977486669978884112384a^7b $$
$$ -246395819137011656949760a^8b + 38995817187683205431296a^9b - 2221801102545787496448a^{10}b $$
$$ +216261270555291906048a^{11}b - 8291822969736024576a^{12}b + 517746324868286976a^{13}b $$
$$ -13186658427534592a^{14}b + 533571656983552a^{15}b - 8636653092672a^{16}b + 221751421056a^{17}b $$
$$ -2057016456a^{18}b + 31308816a^{19}b - 125818a^{20}b + 946a^{21}b + 19529183870862778957824000000b^2 $$
$$ +214707088455270681437798400ab^2 + 101382073742102896903716864 0a^2b^2 $$
$$ -108314139708425338286505984a^3b^2 + 302158850748929274166640640a^4b^2 $$
$$ -29771284120854232910266368a^5b^2 + 19694196117372618265329664a^6b^2 $$
$$ -1388623739871132154593280a^7b^2 + 417983263466651432951808a^8b^2 $$
$$ -20622266111066467088384a^9b^2 + 3449288077905817511936a^{10}b^2 - 117885481520131060736a^{11}b^2 $$
$$ +11998017793499063040a^{12}b^2 - 277636257039660288a^{13}b^2 + 17919113392649216a^{14}b^2 $$
$$ -267151285637632a^{15}b^2 + 11020924611136a^{16}b^2 - 95167656760a^{17}b^2 + 2443673144a^{18}b^2 $$
$$ -9231068a^{19}b^2 + 135751a^{20}b^2 + 156569123088349991534592000b^3 $$

$$+5588036483191881672425472 00ab^3 + 108314139708425338286505984a^2b^3$$
$$+466153606552294291044040704a^3b^3 - 209008432499740289390346624a^4b^3$$
$$+576786653594585802431528 96a^5b^3 - 300662268238765047467212 8a^6b^3$$
$$+200146386868233196594790 4a^7b^3 - 825490792702889396695 04a^8b^3$$
$$+250684986574785668505 60a^9b^3 - 7565757313807975659 52a^{10}b^3 + 1273121493704801495 04a^{11}b^3$$
$$-268348363208607052 8a^{12}b^3 + 273162140985789440a^{13}b^3 - 3785489362783744a^{14}b^3$$
$$+242209382992896a^{15}b^3 - 1973241008024a^{16}b^3 + 79630140304a^{17}b^3 - 287404260a^{18}b^3$$
$$+7059052a^{19}b^3 + 744733582037644656771072 00b^4 + 124626488196420160060391424ab^4$$
$$+302158850748929274166640640a^2b^4 + 209008432499740289390346624a^3b^4$$
$$+816375797657450563837624 32a^4b^4 - 186134531249085253471436 8a^5b^4$$
$$+492133252502986911306547 2a^6b^4 - 148592519721237065650176a^7b^4$$
$$+964224654069314862315 52a^8b^4 - 243519465678521421875 2a^9b^4 + 724681220181884469504a^{10}b^4$$
$$-135682532398459537 92a^{11}b^4 + 22363917101054397 44a^{12}b^4 - 28469082137658624a^{13}b^4$$
$$+2826281258958080a^{14}b^4 - 21619108507040a^{15}b^4 + 1337751868596a^{16}b^4 - 4608048302a^{17}b^4$$
$$+177232627a^{18}b^4 + 238111921957368071580549 12b^5 + 765896822587814851651829 76ab^5$$
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$$+186134531249085253471436 8a^4b^5 + 660353916138283085519257 6a^5b^5$$
$$-846733527132977741537 28a^6b^5 + 211310139502169635479552a^7b^5$$
$$-386385070406840993945 6a^8b^5 + 240237173249729251481 6a^9b^5 - 37522170157907452160a^{10}b^5$$
$$+107569892136589076 48a^{11}b^5 - 1217449877683719 68a^{12}b^5 + 19331973916462592a^{13}b^5$$
$$-136278256884000a^{14}b^5 + 12978881608000a^{15}b^5 - 42181365226a^{16}b^5 + 2481256778a^{17}b^5$$
$$+548125944706136820783513 6b^6 + 102114577923197159252951 04ab^6$$
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$$-543431444391229954 56a^9b^6 + 320452100287182223 36a^{10}b^6 - 3012517444392135 68a^{11}b^6$$
$$+822915047519685 12a^{12}b^6 - 515206630456672a^{13}b^6 + 77925205174432a^{14}b^6 - 233619868944a^{15}b^6$$
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$$+138886237398711321545932 80a^2b^7 + 200146386868233196594790 4a^3b^7$$
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$$+610059693501516544 00a^9b^7 - 407431732173193728a^{10}b^7 + 226372726335788032a^{11}b^7$$
$$-11720830175454 40a^{12}b^7 + 302753027024448a^{13}b^7 - 804690659696a^{14}b^7 + 114955808528a^{15}b^7$$
$$+126888416217818346291200b^8 + 246395819137011656949760ab^8$$
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$$-1715884494940a^{13}b^8+416714805914a^{14}b^8+1339376187184401167155 2b^9$$
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$$+25068498657478566850560a^3b^9+24351946567852142187 52a^4b^9+24023717324972925148 16a^5b^9$$
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$$+50221380563788262 4a^9b^9-730579753229040a^{10}b^9+1377434664214752a^{11}b^9$$
$$-2113247220084a^{12}b^9+1029530696964a^{13}b^9+11305742715905447772 16b^{10}$$
$$+22218011025457874964 48ab^{10}+34492880779058175119 36a^2b^{10}+75657573138079756595 2a^3b^{10}$$
$$+7246812201818844695 04a^4b^{10}+3752217015790745216 0a^5b^{10}+3204521002871822233 6a^6b^{10}$$
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$$+4608048302a^4b^{17}+2481256778a^5b^{17}+1275548736b^{18}+2057016456ab^{18}+2443673144a^2b^{18}$$
$$+287404260a^3b^{18}+177232627a^4b^{18}+13518120b^{19}+31308816ab^{19}+9231068a^2b^{19}+7059052a^3b^{19}$$
$$+100100b^{20}+125818ab^{20}+135751a^2b^{20}+462b^{21}+946ab^{21}+b^{22})\Big)$$
$$+\frac{1}{\left[\prod\limits_{\mu=0}^{22}\left\{a-b-2\mu\right\}\right]\left[\prod\limits_{\xi=1}^{21}\left\{a-b+2\xi\right\}\right]}\Big(16777216b(116835417521691373338624000a$$
$$+281256994666286659141632 00a^2+588304873033123070214144 00a^3$$
$$+835763038131117634250342 4a^4+541464636360475451326464 0a^5$$
$$+498187083349892413784064a^6+152686290382212699521024a^7$$
$$+95199641617513747578 88a^8+1637773645456507142144a^9$$

$$+70525027615268548608a^{10} + 7408115142042620928a^{11}$$
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$$+294659633454592a^{14} + 12598920966272a^{15} + 161915465856a^{16}$$
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$$-2819259057729680309452 8a^4b^6+64311349472759628675072a^5b^6+144733981291211337625 6a^7b^6$$

$$+103363825855863372 80a^8b^6+116761885568972483 84a^9b^6+84237361688480000a^{10}b^6$$

$$+35936621746305280a^{11}b^6+185491443545408a^{12}b^6+40413183472816a^{13}b^6+103831052864a^{14}b^6$$

$$+12978881608a^{15}b^6+15268629038221269952 1024b^7-11377726146150486533 7344ab^7$$

$$+245681304324011271913472a^2b^7-245738748688585535651 84a^3b^7$$

$$+381666009054034916823 04a^4b^7-73430272700286334976 0a^5b^7+144733981291211337625 6a^6b^7$$

$$+179132211101988601 60a^8b^7+77535200126182656a^9b^7+80316848819494144a^{10}b^7$$

$$+323795278490112a^{11}b^7+127895988844624a^{12}b^7+284008468128a^{13}b^7+57477904264a^{14}b^7$$

$$-95199641617513747578 88b^8+33718970664166741245952ab^8-87457614222128313180 16a^2b^8$$

$$+131229665895201364142 08a^3b^8-70582772875201291161 6a^4b^8+90664135571548680627 2a^5b^8$$

$$-103363825855863372 80a^6b^8+179132211101988601 60a^7b^8+119567125189072704a^9b^8$$

$$+286328226518048a^{10}b^8+272441173357496a^{11}b^8+469610493352a^{12}b^8+171588449494a^{13}b^8$$

$$+163777364545650714214 4b^9-134603349022623127961 6ab^9+24691546583117063936 00a^2b^9$$

$$-30213767059672314060 8a^3b^9+35023502369246590284 8a^4b^9-10657379160455346176a^5b^9$$

$$+116761885568972483 84a^6b^9-77535200126182656a^7b^9+119567125189072704a^8b^9$$

$$+396284169175752a^{10}b^9+402523280016a^{11}b^9+352207870014a^{12}b^9-70525027615268548608b^{10}$$

$$+225590733333209074688ab^{10}-642470767311955845 12a^2b^{10}+80539981502597540352a^3b^{10}$$

$$-51835857626534558 72a^4b^{10}+49060702832195408 64a^5b^{10}-84237361688480000a^6b^{10}$$

$$+80316848819494144a^7b^{10}-286328226518048a^8b^{10}+396284169175752a^9b^{10}$$

$$+503154100020a^{11}b^{10}+7408115142042620928b^{11}-6339888946109069312ab^{11}$$

$$+10321203147577973248a^2b^{11}-13568285031833026 56a^3b^{11}+12970073166696239 36a^4b^{11}$$

$$-449767216639336 96a^5b^{11}+35936621746305280a^6b^{11}-323795278490112a^7b^{11}$$

$$+272441173357496a^8b^{11}-402523280016a^9b^{11}+503154100020a^{10}b^{11}-220112291277345792b^{12}$$

$$+648854692726186752ab^{12}-188375185237384704a^2b^{12}+206745582687160192a^3b^{12}$$

$$-13619908987053824a^4b^{12}+10518844238424704a^5b^{12}-185491443545408a^6b^{12}$$

$$+127895988844624a^7b^{12}-469610493352a^8b^{12}+352207870014a^9b^{12}+14690239437993728b^{13}$$

$$-12495420024887296ab^{13}+18473671918179968a^2b^{13}-2360282331414784a^3b^{13}$$

$$+1953639560494720a^4b^{13}-62529614167552a^5b^{13}+40413183472816a^6b^{13}-284008468128a^7b^{13}$$

$$+171588449494a^8b^{13}-294659633454592b^{14}+808459848960384ab^{14}-222601677688064a^2b^{14}$$

$$+219242056144640a^3b^14-12668736904640a^4b^{14}+8393796705392a^5b^{14}-103831052864a^6b^{14}$$

$$+57477904264a^7b^{14}+12598920966272b^{15}-10105741289728ab^{15}+13721151546112a^2b^{15}$$

$$-1502397796864a^3b^{15}+1104721948816a^4b^{15}-23597966560a^5b^{15}+12978881608a^6b^{15}$$

$$-161915465856b^{16}+414427759840ab^{16}-97389445776a^2b^{16}+87086974468a^3b^{16}$$

$$-3286859628a^4b^{16}+1917334783a^5b^{16}+4327664352b^{17}-3006978304ab^{17}+3756889532a^2b^{17}$$

$$-268243976a^3b^{17}+177232627a^4b^{17}-31831536b^{18}+75526748ab^{18}-11790944a^2b^{18}$$

$$+9580142a^3b^{18}+491876b^{19}-238392ab^{19}+271502a^2b^{19}-1540b^{20}+3311ab^{20}+11b^{21})\Big)\Bigg\}-$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\Bigg\{\frac{16777216a}{\Big[\prod\limits_{\zeta=0}^{21}\{a-b-2\zeta\}\Big]\Big[\prod\limits_{\eta=1}^{22}\{a-b+2\eta\}\Big]}\Big(116835417521691373338624000a$$

$$-28125699466628665914163200a^2+5883048730331230702141440a^3$$

$$-835763038131117634250342 4a^4+541464636360475451326464 0a^5$$

$$-49818708334989241378406 4a^6+15268629038221269952102 4a^7-9519964161751374757888a^8$$

$$+163777364545650714214 4a^9-705250276152685486 08a^{10}+740811514204262092 8a^{11}$$

$$-220112291277345792a^{12}+14690239437993728a^{13}-294659633454592a^{14}+12598920966272a^{15}$$

$$-161915465856a^{16}+4327664352a^{17}-31831536a^{18}+491876a^{19}-1540a^{20}+11a^{21}$$

$$+116835417521691373338624000b+22621635650505447233814528 0a^2b$$

$$-2308585338453349338867302 4a^3b+425966575657120571798323 20a^4b$$

$$-33170579299422639583395 84a^5b+20312297278713405571072 00a^6b$$

$$-11377726146150486533734 4a^7b+3371897066416674124595 2a^8b-134603349022623127961 6a^9b$$

$$+225590733333209074688a^{10}b-6339888946109069312a^{11}b+648854692726186752a^{12}b$$

$$-12495420024887296a^{13}b+808459848960384a^{14}b-10105741289728a^{15}b+414427759840a^{16}b$$

$$-3006978304a^{17}b+75526748a^{18}b-238392a^{19}b+3311a^{20}b+28125699466628665914163200b^2$$

$$+22621635650505447233814528 0ab^2+10631534500341300657232281 6a^3b^2$$

$$-5592201387587352898043904a^4b^2+9233971963059119562424320a^5b^2$$

$$-4285948822478712458444480a^6b^2+245681304324011271913472a^7b^2$$

$$-87457614222128313180 16a^8b^2+2469154658311706393600a^9b^2-64247076731195584512a^{10}b^2$$

$$+1032120314757797324 8a^{11}b^2-188375185237384704a^{12}b^2+18473671918179968a^{13}b^2$$

$$-222601677688064a^{14}b^2+13721151546112a^{15}b^2-97389445776a^{16}b^2+3756889532a^{17}b^2$$

$$-11790944a^{18}b^2+271502a^{19}b^2+5883048730331230702141440 0b^3$$

$$+23085853384533493388673024ab^3+10631534500341300657232281 6a^2b^3$$

$$+18881491334335208163639296a^4b^3-5685625626768801385676 80a^5b^3$$

$$+84754945129377441408614 4a^6b^3-24573874868858553566518 4a^7b^3$$

$$+1312296658952013641420 8a^8b^3-302137670596723140608a^9b^3+805399815025975403 52a^{10}b^3$$

$$-1356828503183302656a^{11}b^3+206745582687160192a^{12}b^3-2360282331414784a^{13}b^3$$

$$+219242056144640a^{14}b^3-1502397796864a^{15}b^3+87086974468a^{16}b^3-268243976a^{17}b^3$$

$$+9580142a^{18}b^3+8357630381311176342503424b^4+4259665756571205717983232 0ab^4$$

$$+5592201387587352898043904a^2b^4+18881491334335208163639296a^3b^4$$

$$+1542081407404976488054784a^5b^4-28192590577296803094528a^6b^4$$

$$+381666009054034916823 04a^7b^4-7058277287520129116 16a^8b^4+3502350236924659028 48a^9b^4$$

$$-5183585762653455872a^{10}b^4+1297007316669623936a^{11}b^4-13619908987053824a^{12}b^4$$

$$+1953639560494720a^{13}b^4-12668736904640a^{14}b^4+1104721948816a^{15}b^4-3286859628a^{16}b^4$$

$$+177232627a^{17}b^4+5414646363604754513264640b^5+3317057929942263958339584ab^5$$

$$+9233971963059119562424320a^2b^5+568562562676880138567680a^3b^5$$

$$+1542081407404976488054784a^4b^5+64311349472759628675072a^6b^5$$

$$
\begin{aligned}
&-73430272700286334976 0a^{7}b^{5}+906641355715486806272a^{8}b^{5}-1065737916045534617 6a^{9}b^{5}\\
&+490607028321954086 4a^{10}b^{5}-44976721663933696a^{11}b^{5}+10518844238424704a^{12}b^{5}\\
&-625296141675 52a^{13}b^{5}+8393796705392a^{14}b^{5}-23597966560a^{15}b^{5}+1917334783a^{16}b^{5}\\
&+4981870833498924137840 64b^{6}+203122972787134055710720 0ab^{6}\\
&+42859488224787124584448 0a^{2}b^{6}+84754945129377441408614 4a^{3}b^{6}\\
&+28192590577296803094528a^{4}b^{6}+6431134947275962867507 2a^{5}b^{6}\\
&+144733981291211337625 6a^{7}b^{6}-1033638258558633728 0a^{8}b^{6}+116761885568972483 84a^{9}b^{6}\\
&-84237361688480000a^{10}b^{6}+35936621746305280a^{11}b^{6}-185491443545408a^{12}b^{6}\\
&+40413183472816a^{13}b^{6}-103831052864a^{14}b^{6}+12978881608a^{15}b^{6}+1526862903822126995210 24b^{7}\\
&+1137772614615048653373 44ab^{7}+2456813043240112719134 72a^{2}b^{7}\\
&+2457387486885855356518 4a^{3}b^{7}+3816660090540349168230 4a^{4}b^{7}+7343027270028633497 60a^{5}b^{7}\\
&+144733981291211337625 6a^{6}b^{7}+179132211101988601 60a^{8}b^{7}-77535200126182656a^{9}b^{7}\\
&+80316848819494144a^{10}b^{7}-323795278490112a^{11}b^{7}+127895988844624a^{12}b^{7}-284008468128a^{13}b^{7}\\
&+57477904264a^{14}b^{7}+951996416175137475788 8b^{8}+3371897066416674124595 2ab^{8}\\
&+8745761422212831318016a^{2}b^{8}+1312296658952013641420 8a^{3}b^{8}+705827728752012911616a^{4}b^{8}\\
&+906641355715486806272a^{5}b^{8}+1033638258558633728 0a^{6}b^{8}+179132211101988601 60a^{7}b^{8}\\
&+11956712518907270 4a^{9}b^{8}-286328226518048a^{10}b^{8}+272441173357496a^{11}b^{8}-469610493352a^{12}b^{8}\\
&+171588449494a^{13}b^{8}+1637773645456507142144b^{9}+1346033490226231279616ab^{9}\\
&+2469154658311706393600a^{2}b^{9}+302137670596723140608a^{3}b^{9}+350235023692465902848a^{4}b^{9}\\
&+1065737916045534617 6a^{5}b^{9}+116761885568972483 84a^{6}b^{9}+77535200126182656a^{7}b^{9}\\
&+11956712518907270 4a^{8}b^{9}+396284169175752a^{10}b^{9}-402523280016a^{11}b^{9}+352207870014a^{12}b^{9}\\
&+70525027615268548608b^{10}+225590733333209074688ab^{10}+64247076731195584512a^{2}b^{10}\\
&+80539981502597540352a^{3}b^{10}+5183585762653455872a^{4}b^{10}+490607028321954086 4a^{5}b^{10}\\
&+84237361688480000a^{6}b^{10}+80316848819494144a^{7}b^{10}+286328226518048a^{8}b^{10}\\
&+396284169175752a^{9}b^{10}+503154100020a^{11}b^{10}+7408115142042620928b^{11}\\
&+6339888946109069312ab^{11}+10321203147577973248a^{2}b^{11}+1356828503183302656a^{3}b^{11}\\
&+1297007316669623936a^{4}b^{11}+44976721663933696a^{5}b^{11}+35936621746305280a^{6}b^{11}\\
&+323795278490112a^{7}b^{11}+272441173357496a^{8}b^{11}+402523280016a^{9}b^{11}+503154100020a^{10}b^{11}\\
&+220112291277345792b^{12}+648854692726186752ab^{12}+188375185237384704a^{2}b^{12}\\
&+206745582687160192a^{3}b^{12}+13619908987053824a^{4}b^{12}+10518844238424704a^{5}b^{12}\\
&+185491443545408a^{6}b^{12}+127895988844624a^{7}b^{12}+469610493352a^{8}b^{12}+352207870014a^{9}b^{12}\\
&+14690239437993728b^{13}+12495420024887296ab^{13}+18473671918179968a^{2}b^{13}\\
&+2360282331414784a^{3}b^{13}+1953639560494720a^{4}b^{13}+62529614167552a^{5}b^{13}+40413183472816a^{6}b^{13}\\
&+284008468128a^{7}b^{13}+171588449494a^{8}b^{13}+294659633454592b^{14}+808459848960384ab^{14}\\
&+222601677688064a^{2}b^{14}+219242056144640a^{3}b^{14}+12668736904640a^{4}b^{14}+8393796705392a^{5}b^{14}\\
&+103831052864a^{6}b^{14}+57477904264a^{7}b^{14}+12598920966272b^{15}+10105741289728ab^{15}\\
&+13721151546112a^{2}b^{15}+1502397796864a^{3}b^{15}+1104721948816a^{4}b^{15}+23597966560a^{5}b^{15}\\
&+12978881608a^{6}b^{15}+161915465856b^{16}+414427759840ab^{16}+97389445776a^{2}b^{16}
\end{aligned}
$$

$$+87086974468a^3b^{16}+3286859628a^4b^{16}+1917334783a^5b^{16}+4327664352b^{17}+3006978304ab^{17}$$
$$+3756889532a^2b^{17}+268243976a^3b^{17}+177232627a^4b^{17}+318315360b^{18}+75526748ab^{18}$$
$$+11790944a^2b^{18}+9580142a^3b^{18}+491876b^{19}+238392ab^{19}+271502a^2b^{19}+1540b^{20}+3311ab^{20}+11b^{21}\Big)$$
$$+\frac{4194304}{\left[\prod\limits_{\mu=0}^{22}\{a-b-2\mu\}\right]\left[\prod\limits_{\xi=1}^{21}\{a-b+2\xi\}\right]}\Big(107145471557284795514880000a$$
$$+195291838708627789578240000a^2+156569123088349991534592000a^3$$
$$+74473358203764465677107200a^4+23811192195736807158054912a^5$$
$$+5481259447061368207835136a^6+948292268763887952199680a^7$$
$$+126888416217818346291200a^8+13393761871844011671552a^9+1130574271590544777216a^{10}$$
$$+77005895857888757760a^{11}+4254539623864857600a^{12}+191027711898895872a^{13}$$
$$+6960284638689536a^{14}+204763953757440a^{15}+4818538806400a^{16}+89349365952a^{17}$$
$$+1275548736a^{18}+13518120a^{19}+100100a^{20}+462a^{21}+a^{22}-107145471557284795514880000b$$
$$+544099662756275407552512000ab+214707088455270681437798400a^2b$$
$$+55880364831918816724254720 0a^3b+124626488196420160060391424a^4b$$
$$+76589682258781485165182976a^5b+10211457792319715925295104a^6b$$
$$+2878977486669978884112384a^7b+246395819137011656949760a^8b$$
$$+38995817187683205431296a^9b+2221801102545787496448a^{10}b+216261270555291906048a^{11}b$$
$$+8291822969736024576a^{12}b+517746324868286976a^{13}b+13186658427534592a^{14}b$$
$$+533571656983552a^{15}b+8636653092672a^{16}b+221751421056a^{17}b+2057016456a^{18}b+31308816a^{19}b$$
$$+125818a^{20}b+946a^{21}b+195291838708627789578240000b^2-214707088455270681437798400ab^2$$
$$+1013820737421028969037168640a^2b^2+1083141397084253382865059 84a^3b^2$$
$$+302158850748929274166640640a^4b^2+29771284120854232910266368a^5b^2$$
$$+19694196117372618265329664a^6b^2+1388623739871132154593280a^7b^2$$
$$+417983263466651432951808a^8b^2+20622266111066467088384a^9b^2+3449288077905817511936a^{10}b^2$$
$$+117885481520131060736a^{11}b^2+11998017793499063040a^{12}b^2+277636257039660288a^{13}b^2$$
$$+17919113392649216a^{14}b^2+267151285637632a^{15}b^2+11020924611136a^{16}b^2+95167656760a^{17}b^2$$
$$+2443673144a^{18}b^2+9231068a^{19}b^2+135751a^{20}b^2-156569123088349991534592000b^3$$
$$+558803648319188167242547200ab^3-1083141397084253382865059 84a^2b^3$$
$$+466153606552294291044040704a^3b^3+209008432499740289390346 24a^4b^3$$
$$+57678665359458580243152896a^5b^3+30066226823876504746721 28a^6b^3$$
$$+2001463868682331965947904a^7b^3+825490792702889396695 04a^8b^3$$
$$+25068498657478566850560a^9b^3+756575731380797565952a^{10}b^3+127312149370480149504a^{11}b^3$$
$$+2683483632086070528a^{12}b^3+273162140985789440a^{13}b^3+3785489362783744a^{14}b^3$$
$$+242209382992896a^{15}b^3+1973241008024a^{16}b^3+79630140304a^{17}b^3+287404260a^{18}b^3+7059052a^{19}b^3$$
$$+74473358203764465677107200b^4-124626488196420160060391424ab^4$$
$$+302158850748929274166640640a^2b^4-20900843249974028939034624a^3b^4$$
$$+816375797657450563837624 32a^4b^4+18613453124908525347714368a^5b^4$$

$$+49213325250298691130654 72a^6b^4 + 148592519721237065650176a^7b^4$$
$$+96422465406931486231552a^8b^4 + 24351946567852142187 52a^9b^4 + 7246812201818844695 04a^{10}b^4$$
$$+135682532398459537 92a^{11}b^4 + 22363917101054397 44a^{12}b^4 + 284690821376586 24a^{13}b^4$$
$$+2826281258958080a^{14}b^4 + 216191085070 40a^{15}b^4 + 1337751868596a^{16}b^4 + 4608048302a^{17}b^4$$
$$+177232627a^{18}b^4 - 2381119219573680715805491 2b^5 + 765896822587814851651829 76ab^5$$
$$-297712841208542329102663 68a^2b^5 + 576786653594585802431528 96a^3b^5$$
$$-186134531249085253471436 8a^4b^5 + 66035391613828308551925 76a^5b^5$$
$$+846733527132977741537 28a^6b^5 + 211310139502169635479552a^7b^5$$
$$+3863850704068409939456a^8b^5 + 2402371732497292514816a^9b^5 + 375221701579074521 60a^{10}b^5$$
$$+107569892136589076 48a^{11}b^5 + 121744987768371968a^{12}b^5 + 19331973916462592a^{13}b^5$$
$$+136278256884000a^{14}b^5 + 12978881608000a^{15}b^5 + 42181365226a^{16}b^5 + 2481256778a^{17}b^5$$
$$+5481259447061368207835136b^6 - 102114577923197159252951 04ab^6$$
$$+196941961173726182653296 64a^2b^6 - 300662268238765047467212 8a^3b^6$$
$$+49213325250298691130654 72a^4b^6 - 846733527132977741537 28a^5b^6$$
$$+273482748886432393211904a^6b^6 + 207677195221415945625 6a^7b^6 + 485470975093667020057 6a^8b^6$$
$$+543431444391229954 56a^9b^6 + 320452100287182223 36a^{10}b^6 + 30125174443921356 8a^{11}b^6$$
$$+82291504751968512a^{12}b^6 + 515206630456672a^{13}b^6 + 77925205174432a^{14}b^6 + 233619868944a^{15}b^6$$
$$+21090682613a^{16}b^6 - 948292268763887952199680b^7 + 287897748666997888411238 4ab^7$$
$$-1388623739871132154593280a^2b^7 + 200146386868233196594790 4a^3b^7$$
$$-148592519721237065650176a^4b^7 + 211310139502169635479552a^5b^7$$
$$-207677195221415945625 6a^6b^7 + 612273222044057948774 4a^7b^7 + 279772924480472783 36a^8b^7$$
$$+610059693501516544 00a^9b^7 + 40743173217319372 8a^{10}b^7 + 22637272633578803 2a^{11}b^7$$
$$+1172083017545440a^{12}b^7 + 3027530270244 48a^{13}b^7 + 804690659696a^{14}b^7 + 114955808528a^{15}b^7$$
$$+126888416217818346291200b^8 - 246395819137011656949760ab^8 + 417983263466651432951808a^2b^8$$
$$-825490792702889396695 04a^3b^8 + 96422465406931486231552a^4b^8 - 3863850704068409939456a^5b^8$$
$$+485470975093667020057 6a^6b^8 - 279772924480472783 36a^7b^8 + 754758607484674158 08a^8b^8$$
$$+2030037607639092 48a^9b^8 + 41192300365093363 2a^{10}b^8 + 1503961148566448a^{11}b^8$$
$$+783779913404488a^{12}b^8 + 1715884494940a^{13}b^8 + 416714805914a^{14}b^8 - 13393761871844011671552b^9$$
$$+38995817187683205431296ab^9 - 20622266111066467088384a^2b^9 + 25068498657478566850560a^3b^9$$
$$-24351946567852142187 52a^4b^9 + 2402371732497292514816a^5b^9 - 543431444391229954 56a^6b^9$$
$$+610059693501516544 00a^7b^9 - 2030037607639092 48a^8b^9 + 5022138056378826 24a^9b^9$$
$$+730579753229040a^{10}b^9 + 1377434664214752a^{11}b^9 + 2113247220084a^{12}b^9 + 1029530696964a^{13}b^9$$
$$+1130574271590544777216b^{10} - 2221801102545787496448ab^{10} + 3449288077905817511936a^2b^{10}$$
$$-756575731380797565952a^3b^{10} + 7246812201818844695 04a^4b^{10} - 375221701579074521 60a^5b^{10}$$
$$+320452100287182223 36a^6b^{10} - 40743173217319372 8a^7b^{10} + 41192300365093363 2a^8b^{10}$$
$$-730579753229040a^9b^{10} + 1660408530066000a^{10}b^{10} + 1006308200040a^{11}b^{10} + 1761039350070a^{12}b^{10}$$
$$-77005895857888757760b^{11} + 216261270555291906048ab^{11} - 117885481520131060736a^2b^{11}$$
$$+127312149370480149504a^3b^{11} - 135682532398459537 92a^4b^{11} + 107569892136589076 48a^5b^{11}$$

$$
\begin{aligned}
&-301251744439213568a^6b^{11} + 226372726335788032a^7b^{11} - 1503961148566448a^8b^{11} \\
&+1377434664214752a^9b^{11} - 1006308200040a^{10}b^{11} + 2104098963720a^{11}b^{11} \\
&+4254539623864857600b^{12} - 8291822969736024576ab^{12} + 11998017793499063040a^2b^{12} \\
&-2683483632086070528a^3b^{12} + 2236391710105439744a^4b^{12} - 121744987768371968a^5b^{12} \\
&+82291504751968512a^6b^{12} - 1172083017545440a^7b^{12} + 783779913404488a^8b^{12} \\
&-2113247220084a^9b^{12} + 1761039350070a^{10}b^{12} - 191027711898895872b^{13} \\
&+517746324868286976ab^{13} - 277636257039660288a^2b^{13} + 273162140985789440a^3b^{13} \\
&-28469082137658624a^4b^{13} + 19331973916462592a^5b^{13} - 515206630456672a^6b^{13} \\
&+302753027024448a^7b^{13} - 1715884494940a^8b^{13} + 1029530696964a^9b^{13} + 6960284638689536b^{14} \\
&-13186658427534592ab^{14} + 17919113392649216a^2b^{14} - 3785489362783744a^3b^{14} \\
&+2826281258958080a^4b^{14} - 136278256884000a^5b^{14} + 77925205174432a^6b^{14} - 804690659696a^7b^{14} \\
&+416714805914a^8b^{14} - 204763953757440b^{15} + 533571656983552ab^{15} - 267151285637632a^2b^{15} \\
&+242209382992896a^3b^{15} - 21619108507040a^4b^{15} + 12978881608000a^5b^{15} - 233619868944a^6b^{15} \\
&+114955808528a^7b^{15} + 4818538806400b^{16} - 8636653092672ab^{16} + 11020924611136a^2b^{16} \\
&-1973241008024a^3b^{16} + 1337751868596a^4b^{16} - 42181365226a^5b^{16} + 21090682613a^6b^{16} \\
&-89349365952b^{17} + 221751421056ab^{17} - 95167656760a^2b^{17} + 79630140304a^3b^{17} - 4608048302a^4b^{17} \\
&+2481256778a^5b^{17} + 1275548736b^{18} - 2057016456ab^{18} + 2443673144a^2b^{18} - 287404260a^3b^{18} \\
&+177232627a^4b^{18} - 13518120b^{19} + 31308816ab^{19} - 9231068a^2b^{19} + 7059052a^3b^{19} + 100100b^{20} \\
&-125818ab^{20} + 135751a^2b^{20} - 462b^{21} + 946ab^{21} + b^{22}\Big)\Big\}\Big]
\end{aligned} \tag{8}
$$

3 Derivation of the Main Formula

Putting $c = \frac{a+b+46}{2}$ and $z = \frac{1}{2}$ in equation (2), we get

$$(a-b)\,{}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \frac{1}{2} \\ \frac{a+b+46}{2}\ ; & \end{array}\right] = a\,{}_2F_1\left[\begin{array}{ll} a+1,\ b\ ; & \frac{1}{2} \\ \frac{a+b+46}{2}\ ; & \end{array}\right] - b\,{}_2F_1\left[\begin{array}{ll} a,\ b+1\ ; & \frac{1}{2} \\ \frac{a+b+46}{2}\ ; & \end{array}\right]$$

Now involving the derived formula [Salahuddin et. al. p.12-41(8)], the summation formula is obtained.

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General energy decay for nonlinear wave equation of $\phi-$Laplacian type with a delay term in the internal feedback

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Abstract

Under conditions on the delay term, using the multiplier method and general weighted integral inequalities, we study the question of asymptotic behavior of solutions for a nonlinear wave equation with $\phi-$Laplacian operator and a delay term in the internal feedback.

Keywords: Nonlinear wave equation, Time delay term, Decay rate, Multiplier method, $\phi-$Laplacian.

2010 MSC: 35B40, 35L70.

1 Introduction

It is well known that the $\phi-$Laplacian operator degenerates equations in divergence form. It has been much studied during the last years and their results is by now rather developed, especially with delay. In the classical theory of the wave equations several main parts of mathematics are joined in a fruitful way, it is very remarkable that the $\phi-$Laplace wave equation occupies a similar position, when it comes to nonlinear problems. In recent years, the PDEs with time delay effects have become an active area of research and arise in many applied problems.

In this paper we investigate the decay properties of solutions for the initial boundary value problem of a nonlinear wave equation

$$\begin{cases} \left(|u'|^{l-2}u'\right)' - \Delta_\phi u + \mu_1 g(u'(x,t)) + \mu_2 g(u'(x,t-\tau)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \;\; u'(x,0) = u_1(x) & \text{in } \Omega, \\ u'(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau(0)[, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n\in\mathbb{N}^*$, with a smooth boundary $\partial\Omega=\Gamma$, $\tau>0$ is a time delay, μ_1 and μ_2 are positive real numbers and the initial data (u_0,u_1,f_0) belong to a suitable space. The operator Δ_ϕ is defined by

$$\Delta_\phi = \sum_{i=1}^{n} \partial_{x_i}(\phi(|\partial_{x_i}|^2)\partial_{x_i}). \tag{1.2}$$

For $\phi\sim 1$, when g is linear, it is well known that if $\mu_2=0$, that is, in the absence of a delay, the energy of problem (1.1) exponentially decays to zero (see for instance [5, 6, 12, 18]). On the contrary, if $\mu_1=0$, that is, there exists only the delay part in the interior, the system (1.1) becomes unstable (see for instance [8]). In [8], the authors showed that a small delay in a boundary control can turn such a well-behaved hyperbolic system into a wild one and therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [19, 20, 21]). In [19] the authors examined

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the problem (P) with $\phi \sim 1$ and determined suitable relations between μ_1 and μ_2, for which stability or, alternatively, instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution will be unstable if $\mu_2 \geq \mu_1$. The main approach used in [19], is an observability inequality obtained by means of a Carleman estimate. The same results were shown if both the damping and the delay act in the boundary domain. We also recall the result by Xu, Yung and Li in [21], where the authors proved the same result as in [19] for the one-dimension space by adopting the spectral analysis approach.

When g is nonlinear and in the case $\mu_2 = 0, \phi \sim 1$, the problem of existence and energy decay have been previously studied by several authors (see [1, 3, 11, 12, 13]) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay). The decay rate of a global solution depends on the growth near zero of $g(s)$ as it was proved in [11, 12, 13, 17].

In this article, we use some technique from [3] to give energy decay estimates of solutions to the problem (1.1) for a nonlinear damping and a delay term in the $\phi-$Laplace type. We use the multiplier method and some properties of convex functions. These arguments of convexity were introduced and developed in [4, 7, 13, 14, 15], and used by Liu and Zuazua [16], Eller et al. [9] and Alabau-Boussouira [1].

2 Preliminaries and Notations

We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$. We use familiar function spaces $W_0^{m,\Phi}$, where the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ si colled an N-function, in the sense of Definition 2.1 given in [3, pp 6-8].

We use the following hypotheses:

(hyp1) $g : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^0(\mathbb{R})$ such that there exist ϵ_1 (sufficiently small), $c_1,\ c_2, c_3, \alpha_1, \alpha_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0,\infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon_1]$ or ($H' > 0$ and $H' = 0$ on $]0, \epsilon_1]$), such that

$$c_1|s|^{l-1} \leq |g(s)| \leq c_2|s|^p \quad \text{if} \ \ |s| \geq \epsilon_1, \tag{2.3}$$

$$|s|^l + |g|^{(p+1)/p}(s) \leq H^{-1}(sg(s)) \quad \text{if } |s| \leq \epsilon_1, \tag{2.4}$$

with p satisfying

$$l-1 \leq p \leq \frac{n+2}{n-2}, \text{ if } n > 2$$
$$l-1 \leq p < \infty, \text{ if } n \leq 2$$

$$|g'(s)| \leq c_3, \tag{2.5}$$

$$\alpha_1\, sg(s) \leq G(s) \leq \alpha_2\, sg(s), \tag{2.6}$$

where

$$G(s) = \int_0^s g(r)\, dr$$

(hyp2) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $C^1(]0,+\infty[) \cap C(]0,+\infty[)$ satisfying $\phi(s) > 0$ on $]0,+\infty[$ and ϕ is non decreasing.

(hyp3)

$$\alpha_2\mu_2 < \alpha_1\mu_1. \tag{2.7}$$

We first state some lemmas which will be needed later.

Lemma 2.1 (Sobolev–Poincaré's inequality). *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2, ..., p)$ or $2 \leq q \leq pn/(n-p)$ $(n \geq p+1)$. Then there is a constant $c_* = c_*(\Omega, q, p)$ such that*

$$\|u\|_q \leq c_*\|\nabla u\|_p \quad \textit{for} \quad u \in W_0^{1,p}(\Omega).$$

The case $p = q = 2$ gives the known Poincare's inequality.

Lemma 2.2 ([9, 10]). *Let $E:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing differentiable function and $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ a convex and increasing function such that $\Psi(0)=0$. Assume that*

$$\int_s^T \Psi(E(t))\,dt \le E(s) \quad \forall 0\le s\le T.$$

Then E satisfies the following estimate:

$$E(t)\le \psi^{-1}\left(h(t)+\psi(E(0))\right) \quad \forall t\ge 0, \tag{2.8}$$

where $\psi(t)=\int_t^1 \frac{1}{\Psi(s)}\,ds$ for $t>0$, $h(t)=0$ for $0\le t\le \frac{E(0)}{\Psi(E(0))}$, and

$$h^{-1}(t)=t+\frac{\psi^{-1}\left(t+\psi(E(0))\right)}{\Psi\left(\psi^{-1}\left(t+\psi(E(0))\right)\right)} \quad \forall t\ge \frac{E(0)}{\Psi(E(0))}.$$

We introduce as in [19] the new variable

$$z(x,\rho,t)=u_t(x,t-\tau\rho),\quad x\in\Omega,\quad \rho\in(0,1),\quad t>0. \tag{2.9}$$

Then we have

$$\tau z'(x,\rho,t)+z_\rho(x,\rho,t)=0 \quad \text{in } \Omega\times(0,1)\times(0,+\infty). \tag{2.10}$$

Therefore problem (1.1) is equivalent to:

$$\begin{cases} \left(|u'|^{l-2}u'\right)'-\Delta_\phi u(x,t)+\mu_1 g(u'(x,t))+\mu_2 g(z(x,1,t))=0 & \text{in } \Omega\times]0,+\infty[,\\ \tau z'(x,\rho,t)+z_\rho(x,\rho,t)=0 & \text{in } \Omega\times]0,1[\times]0,+\infty[,\\ u(x,t)=0 & \text{on } \partial\Omega\times[0,+\infty[,\\ z(x,0,t)=u'(x,t) & \text{on } \Omega\times[0,+\infty[,\\ u(x,0)=u_0(x)\ u'(x,0)=u_1(x) & \text{in } \Omega\\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in } \Omega\times]0,1[. \end{cases} \tag{2.11}$$

Let ξ be a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1}<\xi<\tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}. \tag{2.12}$$

The energy of u at time t of the problem (2.11) is defined by

$$E(t)=\frac{l-1}{l}\|u'(t)\|_l^l+\int_\Omega\sum_{i=1}^n\tilde{\phi}(|\partial_{x_i}u|^2)dx+\xi\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx. \tag{2.13}$$

where $\tilde{\phi}(s)=\frac{1}{2}\int_0^s\phi(t)dt$. We give an explicit formula for the derivative of the energy.

Lemma 2.3. *Let (u,z) be a solution of the problem (2.11). Then, the energy functional defined by (2.13) satisfies*

$$\begin{aligned} E'(t) &\le -\left(\mu_1-\frac{\xi\alpha_2}{\tau}-\mu_2\alpha_2\right)\int_\Omega u'g(u')\,dx\\ &\quad -\left(\frac{\xi}{\tau}\alpha_1-\mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)g(z(x,1,t))\,dx\\ &\le 0. \end{aligned} \tag{2.14}$$

Proof. Multiplying the first equation in (2.11) by u', integrating over Ω, we get

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{(l-1)}{l}\|u'\|_l^l+\int_\Omega\sum_{i=1}^n\tilde{\phi}(|\partial_{x_i}u|^2)\right)dx\\ &+ \mu_1\int_\Omega u'g(u')\,dx+\mu_2\int_\Omega u'g(z(x,1,t))dx. \end{aligned} \tag{2.15}$$

We multiply the second equation in (2.11) by $\xi g(z)$ and integrate the result over $\Omega \times (0,1)$ to obtain

$$\begin{aligned} \xi \int_\Omega \int_0^1 z' g(z(x,\rho,t))\, d\rho\, dx &= -\frac{\xi}{\tau} \int_\Omega \int_0^1 \frac{\partial}{\partial \rho} G(z(x,\rho,t))\, d\rho\, dx \\ &= -\frac{\xi}{\tau} \int_\Omega (G(z(x,1,t)) - G(z(x,0,t)))\, dx. \end{aligned} \tag{2.16}$$

Then

$$\xi \frac{d}{dt} \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx = -\frac{\xi}{\tau} \int_\Omega G(z(x,1,t))\, dx + \frac{\xi}{\tau} \int_\Omega G(u')\, dx. \tag{2.17}$$

From (2.15), (2.17) and using the Young inequality we get

$$\begin{aligned} E'(t) &= -\left(\mu_1 - \frac{\xi \alpha_2}{\tau}\right) \int_\Omega u' g(u')\, dx \\ &- \frac{\xi}{\tau} \int_\Omega G(z(x,1,t))\, dx - \mu_2 \int_\Omega u'(t) g(z(x,1,t))\, dx. \end{aligned} \tag{2.18}$$

Let us denote G^* to be the conjugate function of the convex function G, i.e., $G^*(s) = \sup_{t \in \mathbb{R}^+}(st - G(t))$. Then G^* is the Legendre transform of G which is given by (see [2], [4], [7], [14], [15], [17])

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)] \quad \forall s \geq 0, \tag{2.19}$$

and satisfies the following inequality

$$st \leq G^*(s) + G(t) \quad \forall s, t \geq 0. \tag{2.20}$$

Then by the definition of G we get

$$G^*(s) = s g^{-1}(s) - G(g^{-1}(s)).$$

Hence

$$\begin{aligned} G^*(g(z(x,1,t))) &= z(x,1,t) g(z(x,1,t)) - G(z(x,1,t)) \\ &\leq (1-\alpha_1) z(x,1,t) g(z(x,1,t)). \end{aligned} \tag{2.21}$$

Making use of (2.18), (2.20) and (2.21), we have

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi \alpha_2}{\tau}\right) \int_\Omega u' g(u')\, dx - \frac{\xi}{\tau} \int_\Omega G(z(x,1,t))\, dx \\ &+ \mu_2 \int_\Omega \left(G(u') + G^*(g(z(x,1,t)))\right)\, dx \\ &\leq -\left(\mu_1 - \frac{\xi \alpha_2}{\tau} - \mu_2 \alpha_2\right) \int_\Omega u' g(u')\, dx - \frac{\xi}{\tau} \int_\Omega G(z(x,1,t))\, dx \\ &+ \mu_2 \int_\Omega G^*(g(z(x,1,t)))\, dx. \end{aligned} \tag{2.22}$$

Using (2.6) and (2.12), we obtain

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi \alpha_2}{\tau} - \mu_2 \alpha_2\right) \int_\Omega u' g(u')\, dx \\ &- \left(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right) \int_\Omega z(x,1,t) g(z(x,1,t))\, dx \\ &\leq 0. \end{aligned}$$

3 Main result

Our main result reads as.

Theorem 3.1. *Let* $(u_0, u_1, f_0) \in W^{2,\Phi} \cap W_0^{1,\Phi} \times W_0^{1,l}(\Omega) \times W_0^{1,l}(\Omega; W^{1,l}(0,1))$ *and assume that the hypotheses (hyp1)–(hyp3) hold. Then, for some constants* ω, ϵ_0 *we have*

$$E(t) \leq \psi^{-1}\left(h(t) + \psi(E(0))\right) \quad \forall t > 0, \tag{3.23}$$

where $\psi(t) = \int_t^1 \frac{1}{\omega\varphi(\tau)} d\tau$ *for* $t > 0$, $h(t) = 0$ *for* $0 \leq t \leq \frac{E(0)}{\omega\varphi(E(0))}$,

$$h^{-1}(t) = t + \frac{\psi^{-1}\left(t + \psi(E(0))\right)}{\omega\varphi\left(\psi^{-1}\left(t + \psi(E(0))\right)\right)} \quad \forall t > 0,$$

$$\varphi(s) = \{s \text{ if } H \text{ is linear on } [0, \epsilon_1], sH'(\epsilon_0 s) \text{ if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \epsilon_1].\}$$

Proof. Multiplying the first equation of (2.11) by $\frac{\varphi(E)}{E}u$, we obtain for all $0 \leq S \leq T$,

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E} \int_\Omega u\left(\left(|u'|^{l-2}u'\right)' - \Delta_\phi u + \mu_1 g(u'(x,t)) + \mu_2 g(z(x,1,t))\right) dx\, dt \\
&= \left[\frac{\varphi(E)}{E}\int_\Omega u|u'|^{l-2}u' dx\right]_S^T - \int_S^T \left(\frac{\varphi(E)}{E}\right)' \int_\Omega u|u'|^{l-2}u' dx dt \\
&- \int_S^T \frac{\varphi(E)}{E}\int_\Omega u'^l dx dt + \int_S^T \frac{\varphi(E)}{E}\int_\Omega \left(\sum_{i=1}^n \phi(|\partial_{x_i}u|^2)|\partial_{x_i}u|^2 dx dt\right. \\
&+ \mu_1 \int_S^T \frac{\varphi(E)}{E}\int_\Omega u g(u')\, dx\, dt + \mu_2 \int_S^T \frac{\varphi(E)}{E}\int_\Omega u g(z(x,1,t))\, dx dt
\end{aligned}$$

Similarly, we multiply the second equation of (2.11) by $\frac{\varphi(E)}{E}e^{-2\tau\rho}g(z(x,\rho,t))$, we have

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_0^1 e^{-2\tau\rho} g(z)(\tau z' + z_\rho)\, dx d\rho dt \\
&= \left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 \tau e^{-2\tau\rho} G(z)\, dx d\rho\right]_S^T \\
&- \tau\int_S^T \left(\frac{\varphi(E)}{E}\right)' \int_\Omega\int_0^1 e^{-2\tau\rho} G(z)\, dx d\rho dt \\
&+ \int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_0^1 \left(\frac{\partial}{\partial\rho}(e^{-2\tau\rho}G(z)) + 2\tau e^{-2\tau\rho}G(z)\right) dx d\rho dt \\
&= \left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 \tau e^{-2\tau\rho} G(z)\, dx d\rho\right]_S^T \\
&- \tau\int_S^T \left(\frac{\varphi(E)}{E}\right)' \int_\Omega\int_0^1 e^{-2\tau\rho} G(z)\, dx d\rho dt \\
&+ \int_S^T \frac{\varphi(E)}{E}\int_\Omega (e^{-2\tau}G(z(x,1,t)) - G(z(x,0,t)))\, dx dt \\
&+ 2\tau\int_S^T \frac{\varphi(E)}{E}\int_0^1\int_\Omega e^{-2\tau\rho} G(z)\, dx d\rho dt.
\end{aligned}$$

We have by (hyp2), $s\phi(s) \geq 2\tilde{\phi}(s)$, (note that $\tilde{\phi}$ is convex and defines a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$), summing to obtain, for $A=2\min\{1,\tau e^{-2\tau}/2\xi\}$

$$\begin{aligned}
A\int_S^T\varphi(E)\,dt &\leq -\left[\frac{\varphi(E)}{E}\int_\Omega u|u'|^{l-2}u'dx\right]_S^T+\int_S^T(\frac{\varphi(E)}{E})'\int_\Omega u|u'|^{l-2}u'dxdt\\
&+\frac{3l-2}{l}\int_S^T\frac{\varphi(E)}{E}\int_\Omega u'^l\,dxdt-\mu_1\int_S^T\frac{\varphi(E)}{E}\int_\Omega ug(u')\,dx\,dt\\
&-\mu_2\int_S^T\frac{\varphi(E)}{E}\int_\Omega ug(z(x,1,t))\,dxdt-\left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1\tau e^{-2\tau\rho}G(z)dxd\rho\right]_S^T\\
&+\tau\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)dxd\rho dt\\
&-\int_S^T\frac{\varphi(E)}{E}\int_\Omega(e^{-2\tau}G(z(x,1,t))-G(z(x,0,t)))\,dxdt. \qquad (3.24)
\end{aligned}$$

Using Lemma 2.1, since E is non-increasing, using the Holder, Cauchy–Schwartz, Poincare and Young's inequalities with exponents $\frac{l}{l-1},l$, to get

$$\begin{aligned}
|\int_\Omega u|u'|^{l-2}u'dx| &\leq (\int_\Omega|u|^ldx)^{1/l}(\int_\Omega|u'|^ldx)^{(l-1)/l}\\
&\leq c(\int_\Omega|\nabla u|^2dx)^{1/2}E^{(l-1)/l}(t)\\
&\leq cE^{(l-1)/l}(t)(\sum_{i=1}^n\tilde{\phi}^{-1}(\int_\Omega\sum_{i=1}^n\tilde{\phi}(|\partial_{x_i}|^2)dx))^{1/2}\\
&\leq cE^{(l-1)/l}(t)(\tilde{\phi}^{-1}(E(t)))^{1/2} \qquad (3.25)
\end{aligned}$$

For $l\geq 2$, $\tilde{\phi}^{-1}$ is non decreasing and φ is convex, increasing and of class $C^1(]0,+\infty[)$ such that $\varphi(0)=0$ (then $s\to s^{(l-1)/l}, s\to\tilde{\phi}^{-1}(s)$ and $s\to\frac{\varphi(s)}{s}$ are non decreasing), we deduce that

$$\begin{aligned}
-\left[\frac{\varphi(E)}{E}\int_\Omega u|u'|^{l-2}u'dx\right]_S^T &= \frac{\varphi(E(S))}{E(S)}\int_\Omega u(S)|u'(S)|^{l-2}u'(S)dx\\
&-\frac{\varphi(E(T))}{E(T)}\int_\Omega u(T)|u'(T)|^{l-2}u'(T)dx\\
&\leq C\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
\left|\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega u|u'|^{l-2}u'\,dxdt\right| &\leq c\int_S^T\left|\left(\frac{\varphi(E)}{E}\right)'\right|E\,dt\\
&\leq c\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
-\left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho\right]_S^T &= \frac{\varphi(E(S))}{E(S)}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,S))\,dxd\rho,\\
&-\frac{\varphi(E(T))}{E(T)}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,T))\,dxd\rho\\
&\leq C\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
\int_S^T\left(\left(\frac{\varphi(E)}{E}\right)'\right)\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho dt &\leq c\int_S^T\left(-\left(\frac{\varphi(E)}{E}\right)'\right)Edt\\
&\leq c\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
\int_S^T\frac{\varphi(E)}{E}\int_\Omega e^{-2\tau}G((x,1,t))\,dxdt &\leq c\int_S^T\frac{\varphi(E)}{E}(-E')\,dt\\
&\leq c\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_\Omega G(z(x,0,t))dxdt &= \int_S^T \frac{\varphi(E)}{E}\int_\Omega G(u'(x,t))\,dxdt\\ &\le c\int_S^T \frac{\varphi(E)}{E}(-E')\,dt\\ &\le c\varphi(E(S)),\end{aligned}$$

We conclude

$$\begin{aligned}A\int_S^T \varphi(E)dt &\le c\varphi(E(S))+\mu_1\int_S^T \frac{\varphi(E)}{E}\int_\Omega |u||g(u')|\,dx\,dt\\ &+ \frac{3l-2}{l}\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'^l dxdt+\mu_2\int_S^T \frac{\varphi(E)}{E}\int_\Omega |u||g(z(x,1,t))|\,dxdt.\end{aligned} \tag{3.26}$$

In order to apply the results of Lemma 2.2, we estimate the terms of the right-hand side of (3.26) .
We distinguish two cases.

1. H **is linear on** $[0,\epsilon_1]$. We have $c_1|s|^{l-1}\le|g(s)|\le c_2|s|^p$ for all $s\in\mathbb{R}$, and then, using (2.6) and noting that $s\mapsto\frac{\varphi(E(s))}{E(s)}$ is non-increasing,

$$\frac{3l-2}{l}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |u'|^l dxdt \le c\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt\le c\varphi(E(S)),$$

Using the Poincaré, Young inequalities and the energy inequality from Lemma 2.3, we obtain, for all $\epsilon>0$,

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |ug(u')|dxdt &\le \epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega u^{p+1}dxdt+c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega g^{1+\frac{1}{p}}(u')dxdt\\ &\le \epsilon c\int_S^T \varphi(E)dt+c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt\\ &\le \epsilon c\int_S^T \varphi(E)dt+c_\epsilon\varphi(E(S)),\end{aligned}$$

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |ug(z(x,1,t))|dxdt &\le \epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega u^{p+1}dxdt+c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega g^{1+\frac{1}{p}}(z(x,1,t))dxdt\\ &\le \epsilon c\int_S^T \varphi(E)dt+c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega z(x,1,t)g(z(x,1,t))dxdt\\ &\le \epsilon c\int_S^T \varphi(E)dt+c_\epsilon\varphi(E(S)).\end{aligned}$$

Inserting these two inequalities into (3.26), choosing $\epsilon>0$ small enough, we deduce that

$$\int_S^T \varphi(E(t))dt\le c\varphi(E(S)).$$

Using Lemma 2.2 for E in the particular case where $\varphi(s)=s$, we deduce from (2.8) that

$$E(t)\le ce^{-\omega t}.$$

2. $H'(0)=0$ **and** $H''>0$ **on** $]0,\epsilon_1]$. For all $t\ge 0$, we consider the following partition of Ω

$$\begin{aligned}\Omega_t^1 &= \{x\in\Omega:\ |u'|\ge\epsilon_1\}, & \Omega_t^2 &= \{x\in\Omega:\ |u'|\le\epsilon_1\},\\ \tilde\Omega_t^1 &= \{x\in\Omega:\ |z(x,1,t)|\ge\epsilon_1\}, & \tilde\Omega_t^2 &= \{x\in\Omega:\ |z(x,1,t)|\le\epsilon_1\}.\end{aligned}$$

Using (2.3), (2.6) and the fact that $s\mapsto\frac{\varphi(s)}{s}$ is non-decreasing, we obtain

$$c\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^1}(|u'|^l+g^{(p+1)/p}(u'))dxdt\le c\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt\le c\varphi(E(S)).$$

On the other hand, since H is convex and increasing, H^{-1} is concave and increasing. Therefore (2.4) and the reversed Jensen's inequality for a concave function imply that

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^l+g^{(p+1)/p}(u'))\,dxdt &\leq \int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2} H^{-1}(u'g(u'))\,dxdt\\ &\leq \int_S^T \frac{\varphi(E)}{E}|\Omega|H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega u'g(u')dx\Big)dt. \qquad (3.27)\end{aligned}$$

Let us assume H^* to be the conjugate function of the convex function H, i.e., $H^*(s)=\sup_{t\in\mathbb{R}^+}(st-H(t))$. Then H^* is the Legendre transform of H, which is given by (see Arnold [2, pp. 61–64] and [4, 7, 14, 15])

$$H^*(s)=s(H')^{-1}(s)-H[(H')^{-1}(s)] \quad \forall s\geq 0 \qquad (3.28)$$

and satisfies the following inequality

$$st\leq H^*(s)+H(t) \quad \forall s,t\geq 0. \qquad (3.29)$$

Due to our choice $\varphi(s)=sH'(\epsilon_0 s)$, we have

$$H^*\left(\frac{\varphi(s)}{s}\right)=\epsilon_0 sH'(\epsilon_0 s)-H(\epsilon_0 s)\leq \epsilon_0\varphi(s). \qquad (3.30)$$

Making use of (3.27), (3.29) and (3.30), we have

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^l+g^{(p+1)/p}(u'))\,dxdt &\leq c\int_S^T H^*\left(\frac{\varphi(E)}{E}\right)dt+c\int_S^T\int_\Omega u'g(u')dt\\ &\leq \epsilon_0\int_S^T \varphi(E)dt+cE(S),\end{aligned}$$

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\tilde{\Omega}_t^2} g^{(p+1)/p}(z(x,1,t))\,dxdt &\leq \int_S^T \frac{\varphi(E)}{E}\int_{\tilde{\Omega}_t^2} H^{-1}(z(x,1,t)g(z(x,1,t)))\,dxdt\\ &\leq \int_S^T \frac{\varphi(E)}{E}|\Omega|H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega z(x,1,t)g(z(x,1,t))dx\Big)dt\\ &\leq c\int_S^T H^*\left(\frac{\varphi(E)}{E}\right)dt+c\int_S^T\int_\Omega z(x,1,t)g(z(x,1,t))dt\\ &\leq \epsilon_0\int_S^T \varphi(E)dt+cE(S). \qquad (3.31)\end{aligned}$$

Then, choosing $\epsilon_0>0$ small enough and using (3.26), we obtain in both cases

$$\begin{aligned}\int_S^{+\infty}\varphi(E(t))dt &\leq c\Big(E(S)+\varphi(E(S))\Big)\\ &\leq c\left(1+\frac{\varphi(E(S))}{E(S)}\right)E(S)\\ &\leq cE(S) \quad \forall S\geq 0. \qquad (3.32)\end{aligned}$$

Using Lemma 2.2 in the particular case where $\Psi(s)=\omega\varphi(s)$, we deduce from (2.8) our estimate (3.23). The proof of Theorem 3.1 is now complete. □

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Mixed domination in an M-strong fuzzy graph

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Abstract

In this paper, mixed dominating set, mixed domination number , mixed strong domination number and mixed weak domination number of an M-strong fuzzy graph $G = (\sigma, \mu)$ are defined. Also these numbers are determined for various standard fuzzy graphs. The relationship between these numbers and other well known numbers are derived.

Keywords: Fuzzy graph, M-strong fuzzy graph, mixed domination number, mixed strong domination number, mixed weak domination number.

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1 Introduction

Zadeh [13] introduced the concept of Fuzzy sets in the year 1965. In 1975, Fuzzy graph was introduced by Rosenfeld [7]. Rosenfeld has obtained the fuzzy analogues of several basic graph-theoretic concepts like bridges, paths, cycles, trees, connectedness and established some of their properties. Fuzzy trees were characterized by Sunitha and Vijayakumar [11]. They have obtained a characterization for blocks in fuzzy graphs using the concept of strongest paths [12]. Bhutani and Rosenfeld have introduced the concepts of strong arcs , fuzzy end nodes and geodesics in fuzzy graphs [2]. Mordeson and Peng [6] introduced strong fuzzy graph using effective edges. Bhutani and Battou [1] consider the strong fuzzy graph of Mordeson and Peng as M-strong fuzzy graph.

The concept of domination in fuzzy graphs was defined by Somasundaram and Somasundaram [9]. The vertex neighbourhood number and edge neighbourhood number of an M-strong fuzzy graphs are introduced by S. Ismail Mohideen and A. Mohamed Ismayil [3, 4].

Mixed domination in crisp graph was introduced by E. Sampathkumar and S.S. Kamath [8]. In this paper, Mixed dominating set and mixed domination number in an M-strong fuzzy graph are defined. Mixed strong domination number and mixed weak domination number in an M-strong fuzzy graph are also defined. Theorems related to these mixed dominating sets and mixed domination numbers are stated and proved. The relation between these numbers and other well known parameters are derived.

2 Preliminaries

Definition 2.1. *Let V be a finite non empty set and E be the collection of two element subsets of V. A fuzzy graph $G = (\sigma, \mu)$ is a set with two functions $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ such that $\mu(u, v) \le \sigma(u) \wedge \sigma(v)$for all $u, v \in V$.*

Definition 2.2. *Let $G = (\sigma, \mu)$ be a fuzzy graph defined on V and $S \subseteq V$. Then the* scalar cardinality *of S is defined by $\sum_{u \in S} \sigma(u)$. The* order *(denoted by p) and* size *(denoted by q) of a fuzzy graph $G = (\sigma, \mu)$ are the scalar cardinality of σ and μ respectively.*

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Definition 2.3. *A fuzzy graph $G_1 = (\sigma_1, \mu_1)$ is called the fuzzy sub graph induced by V_1 if $\sigma_1(u) \leq \sigma(u)$ for all $u \in V_1$ and $\mu_1(u,v) \leq \sigma_1(u) \wedge \sigma_1(v) \wedge \mu(u,v)$ for all $u,v \in V_1$ and is denoted by $\langle V_1 \rangle$. A fuzzy graph $G_1 = (\sigma_1, \mu_1)$ is called the full fuzzy sub graph induced by V_1 if $\sigma_1(u) = \sigma(u)$ for all $u \in V_1$ and $\mu_1(u,v) = \mu(u,v)$ for all $u,v \in V_1$ and is denoted by $\langle\langle V_1 \rangle\rangle$.*

Definition 2.4. *An edge $e = (u,v)$ of a fuzzy graph is called an effective edge if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$. If $e = (u,v)$ is an effective edge, then u and v are adjacent vertices and e is incident with u and v. A fuzzy graph $G = (\sigma, \mu)$ is said to be M-strong fuzzy graph [1] if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $(u,v) \in E$. That is, In an M-strong fuzzy graph every edge is an effective edge.*

Definition 2.5. *A fuzzy graph $G = (\sigma, \mu)$ is said to be* complete fuzzy graph *if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$. That is, In a complete fuzzy graph every pair of verices should have an effective edge.*

Definition 2.6. *Let $u, v \in V$ and $e = (u,v) \in E$ then $N(u) = \{v \in V : \mu(u,v) = \sigma(u) \wedge \sigma(v)\}$ is called* open neighbourhood *of u and $N[u] = N(u) \cup \{u\}$ is called* closed neighbourhood *of u. $N[e] = N(u) \cup N(v)$ is called* closed neighbourhood *of e. If $N(u) = \phi$ then u is said to be* isolated vertex.

Definition 2.7. *The* neighbourhood degree *of a vertex u is defined to be the sum of the weights of the vertices adjacent to u and is denoted by $d_N(u)$, the* minimum neighbourhood degree *is $\delta_N(u) = min\{d_N(u) : u \in V\}$ and the* maximum neighbourhood degree *is $\Delta_N(G) = max\{d_N(u) : u \in V\}$.*

Definition 2.8. *A fuzzy graph $G = (\sigma, \mu)$ is said to be* bipartite *if the vertex set V can be partitioned into two sets V_1 defined on σ_1 and V_2 defined on σ_2 such that $\mu(v_1, v_2) = 0$ if $(v_1, v_2) \in V_1 \times V_1$ or $(v_1, v_2) \in V_2 X V_2$.*

Definition 2.9. *A bipatite fuzzy graph $G = (\sigma, \mu)$ is said to be* complete bipartite *if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u \in V_1$ defined on σ_1 and $v \in V_2$ defined on σ_2 and is denoted by K_{σ_1, σ_2}.*

Definition 2.10. *A* path *in a fuzzy graph G is a sequence of distinct vertices $u_0, u_1, u_2, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) = \sigma(u_{i-1}) \wedge \sigma(u_i), 1 \leq i \leq n,\ n > 0$ is called the length of the path. The path in a fuzzy graph is called a* fuzzy cycle *if $u_0 = u_n, n \geq 3$.*

Definition 2.11. *A fuzzy graph is said to be* cyclic *if it contains at least one cycle, otherwise it is called* acyclic.

Definition 2.12. *A fuzzy graph is said to be connected if there exists at least one path between every pair of vertices.*

Definition 2.13. *A connected acyclic fuzzy graph is said to be a* tree.

Definition 2.14. *A vertex in a fuzzy graph having only one neighbour is called a* pendent vertex. *Otherwise it is called non-pendent vertex.*

Definition 2.15. *An edge in a fuzzy graph incident with a pendent vertex is called a* pendent edge. *Otherwise it is called non-pendent edge.*

Definition 2.16. *A vertex in a fuzzy graph adjacent to the pendent vertices is called a* support *of the pendent edges.*

Definition 2.17. *[10] A* vertex covering *of fuzzy graph G is a subset K of V such that every effective edge of G has at least one end in K. The minimum scalar cardinality of vertices in K is called a vertex covering number of G and is denoted by α_0. α_0-set is a vertex cover with minimum scalar cardinality. Similarly edge cover number(α_1), vertex independence number(β_0) and edge independence number (β_1) can be defined.*

Theorem 2.1. *[10] For any fuzzy graph G, $\alpha_0 + \beta_0 = p$.*

Definition 2.18. *Let $G = (\sigma, \mu)$ be a fuzzy graph and let $u, v \in V$. If $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ then u dominates v (or v is dominated by u) in G. A subset D of V is called a* dominating set *in G if for every $v \notin D$ there exist $u \in D$ such that u dominates v. The minimum scalar cardinality taken over all dominating set is called* domination number *and is denoted by the symbol γ.*

Definition 2.19. *A set $S \subseteq V$ in an M-strong fuzzy graph $G(\sigma, \mu)$ is a* vertex neighbourhood set *of G if $G = \cup_{u \in S} \langle\langle N[u] \rangle\rangle$, where $\langle\langle N[u] \rangle\rangle$ is the full fuzzy sub graph of G induced by $N[u]$ and is denoted by n-set. The minimum scalar cardinality taken over all n-set of G is called* vertex neighbourhood number *and is denoted by n_0.*

Theorem 2.2. *[3] For any M-strong fuzzy graph G without isolated vertices. Then*

1. $\gamma(G) \leq n_0(G) \leq \alpha_0(G)$

2. $n_0(G) \leq \alpha_1(G)$

Corollary 2.1. *[3] If G is a M-strong fuzzy graph without isolated vertices and having no triangles, then* $n_0(G) = \alpha_0(G)$.

Definition 2.20. *Let* $e = (u, v)$ *be an edge in an M-strong fuzzy graph* $G(\sigma, \mu)$. *A set* $M \subseteq E$ *in G is an* edge neighbourhood set *of G if* $G = \cup_{u \in S} \langle\langle N[e] \rangle\rangle$ *, where* $\langle\langle N[e] \rangle\rangle$ *is a full induced fuzzy sub graph of G and is denoted by en-set. The minimum scalar cardinality taken over all en-set of G is called* edge neighbourhood number *and is denoted by* n_0.

Theorem 2.3. *[4] For any M-strong fuzzy graph G*

1. $\gamma - m \leq n_1 \leq n_0$ *, where m is the number of edges in minimum en-set.*

2. $n_1 \leq \gamma_1 \leq min(\alpha_0, \alpha_1, \beta_1)$.

3. $n_1 \leq \beta_0$.

4. $n_1 \leq p/2$*, where p is the order of G.*

3 Mixed Domination in an M-strong fuzzy graph

Definition 3.21. *Let* $G = (\sigma, \mu)$ *be an* $M-$*strong fuzzy graph defined on* V. *A vertex* $v \in V$ *dominates an edge* $e \in E$ *if* $e \in \langle\langle N[v] \rangle\rangle$ *Where* $\langle\langle N[v] \rangle\rangle$ *is a full induced fuzzy sub graph of G. An edge* $e = (u, v) \in E$ *dominates* $v \in V$ *if* $v \in N[e]$, *where* $N[e] = N(u) \cup N(v)$.

Note 1. *If v dominates e, then e dominates v but the converse is not true.*

Example 3.1. *In the fuzzy graph given in Figure 1. Here* e_1 *dominates* v_3 *but* v_3 *does not dominate* e_1 .

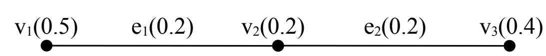

Figure 1:

Now, using this concept the vertex-edge dominating set , edge-vertex dominating set and mixed dominating sets in an M-strong fuzzy graphs are defined.

Definition 3.22. *A set* $S \subset V$ *in an M-strong fuzzy graph G is a* vertex-edge dominating set $(ved - set)$ *if every edge of G is dominated by a vertex in S. The minimum scalar cardinality taken over all ved-set is called* ve-domination number *and is denoted by the symbol* γ_{ve}. *A ved-set with minimum scalar cardinality is called* γ_{ve}*-set. The* Γ_{ve} *is called the maximum scalar cardinality of a minimal ved-set of G.*

Remark 3.1. *Every* $n_0 - set$ *in an M-strong fuzzy graph without isolated vertices is an* $\gamma_{ve} - set$ *and converse also true. That is* $\gamma_{ve} = n_0$.

Definition 3.23. *A set* $M \subseteq E$ *in an M-strong fuzzy graph G is an* edge-vertex dominating set $(evd - set)$ *if every vertex of G is dominated by an edge in M. The scalar cardinality taken over all evd-set is called* ev-domination number *and is denoted by the symbol* γ_{ev}. *An evd-set with minimum scalar cardinality is called* γ_{ev}*-set. The* Γ_{ev} *is called the maximum scalar cardinality of a minimal ved-set of G.*

Remark 3.2. *Every* $n_1 - set$ *in an M-strong fuzzy graph without isolated vertices is an* $evd - set$ *but converse not true. That is* $\gamma_{ev} \leq n_1$.

Definition 3.24. *A set* $D \subseteq V \cup E$ *in an M-strong fuzzy graph G is a* mixed dominating set (md-set) *if*

1. *every vertex* $v \notin D$ *is dominated by at least an edge* $e \in D$ *and*

2. *every edge $e \notin D$ is dominated by at least one vertex in $v \in D$.*

The minimum scalar cardinality taken over all md-set is called mixed domination number *and is denoted by the symbol γ_m. Γ_m is called the maximum scalar cardinality of a minimal md-set of G.*

Note 2. *A ved-set with minimum scalar cardinality is called γ_{ve}-set, similarly γ_{ev}-set and γ_m-set.*

Observation 1. *Let K_σ be a complete fuzzy graph with more than two vertices defined on V, $\gamma_{ve} = \gamma_{ev} = min_{u \in V}\sigma(u)$ and $\gamma_m = 2min_{u\in V}\sigma(u)$.*

Observation 2. *Let K_σ be a complete fuzzy graph with two vertices defined on V, $\gamma_{ve} = \gamma_{ev} = \gamma_m = min_{u\in V}\sigma(u)$.*

Observation 3. *Let K_{σ_1,σ_2} be a complete bipartite fuzzy graph, σ_1 defined on V_1 and σ_2 defined on V_2 respectively and $V = V_1 \cup V_2$. Then $\gamma_{ve} = min\{|\sigma_1|, |\sigma_2|\}$, $\gamma_{ev} = min_{u\in V}\sigma(u)$ and $\gamma_m = \gamma_{ve} + \gamma_{ev}$.*

Theorem 3.4. *For any M-strong fuzzy graph without isolated vertices G*
$\gamma_{ev} \leq \gamma_{ve} \leq \gamma_m \leq \gamma_{ev} + \gamma_{ve}$.
proof: *The first inequality follows from the fact that an evd-set is obtained by choosing one edge incident at each vertex v in $\gamma_{ve} -$ set. The second inequality follows from the fact that by replacing each of the edges in γ_m-set by one of its end vertices with minimum membership grade, we get a ved-set. The last inequality follows from the fact that the union of ved-set and an evd-set is an md-set.*

Note 3. *Let $\alpha_0, \alpha_1, \beta_0$ and β_1 are the vertex cover, edge cover, vertex independent and edge independent numbers of a fuzzy graph G.*

Theorem 3.5. *For any M-strong fuzzy graph G without isolated vertices , The following results are true:*

1. *$\gamma_{ve} \leq \alpha_0$, where α_0 is a vertex cover number of G.*
2. *$\gamma_{ev} \leq \beta_0$ where β_0 is a vertex independent number of G.*
3. *$\gamma_m \leq p$, where p is the order of G.*

proof: *(1)From the remark 3.1, $\gamma_{ve} = n_0$ and from the theorem 2.2(1), $n_0 \leq \alpha_0$. Hence $\gamma_{ve} \leq \alpha_0$.*
(2)From the remark 3.2, $\gamma_{ev} = n_1$ and from the theorem 2.3(3), $n_1 \leq \beta_0$. Hence $\gamma_{ev} \leq \beta_0$.
(3) By theorem 3.4, $\gamma_m \leq \gamma_{ev} + \gamma_{ve} \leq \alpha_0 + \beta_0$ and theorem 2.1 $\alpha_0 + \beta_0 = p$. Hence $\gamma_m \leq p$.

Theorem 3.6. *If G is an M-strong fuzzy graph without isolated vertices and no triangles. Then*

1. $\gamma_{ve} = \alpha_0$
2. $\gamma_{ev} \leq \gamma \leq \gamma_{ve}$
3. $\gamma_m \leq \gamma_{ve} + \frac{p}{2}$
4. $\gamma_m \leq \gamma + \alpha_0$.

proof:(1)*From the remark 3.1, $\gamma_{ve} = n_0$ and from the corrolery 2.1, $n_0 = \alpha_0$. Hence $\alpha_0 = \gamma_{ve}$.*
(2) Every dominating set is an evd-set, because a vertex v in a dominating set dominates only adjacent vertices but en edge e in $\gamma_{ev} -$ set dominates adjacent vertices of both the end vertices of e. Hence $\gamma_{ev} \leq \gamma$. Every ved-set is a dominating set, since G has no triangles. Hence $\gamma \leq \gamma_{ve}$.
(3) An edge will definitely dominate at least two vertices in an evd-set, therefore $\gamma_{ev} \leq \frac{p}{2} \Rightarrow \gamma_{ev} \leq \frac{p}{2} + \gamma_{ve} + \gamma_{ev} - \gamma_m$ by theorem 3.4. Hence $\gamma_m \leq \gamma_{ve} + \frac{p}{2}$.
(4) From theorem 3.4, $\gamma_m \leq \gamma_{ev} + \gamma_{ve} \leq \gamma + \alpha_0$

Theorem 3.7. *For any M-strong fuzzy graph G without isolated vertices, $\gamma_m \leq min\{p, q\}$. Where p and q be the order and size of G respectively.*
proof: *Let G be an M-strong fuzzy graph without isolated vertices and let p and q be the order and size of G.*
The vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ is an $md-$ set, since very edge of E is incident with at least two vertices in V.
Hence $\gamma_m \leq p - - - -(i)$.
The edge set $E = \{e_1, e_2, e_3, \ldots, e_m\}$ is also an $md-$ set, since every vertex of V is incident with at least one edge in E.
Hence $\gamma_m \leq q - - - -(ii)$.
From (i) and (ii) we obtain $\gamma_m \leq min\{p, q\}$.

Theorem 3.8. *Let T_σ be a tree in an M-strong fuzzy graph G. If r and s are the scalar cardinality of the pendent vetices and supports of the pendent edges of T_σ respectively. Then $\gamma_m(T_\sigma) \leq p + s - r - \sigma_0$, where $\sigma_0 = min_{u \in V}\sigma(u)$.*
proof: *Let T_σ be a tree in an M-strong fuzzy graph. Given r and s are the scalar cardinalitry of the pendent vertices and supports of the pendent edges of T_σ respectively.*

Let M, N and R be the set of all non-pendent edges, supports of the pendent edges and non-pendent vertices in T_σ respectively. Then the Union of M and N form an md-set. Therefore $\gamma_m \leq |M| + |N|$ ——(1)
The set of non-pendent edges of T_σ also form a tree. Therefore $|M| \leq |R| - \sigma_0$, where $\sigma_0 = min_{u \in V}\sigma(u)$. From (1) $\gamma_m(T_\sigma) \leq |R| - \sigma_0 + |N| \leq p - r + s - \sigma_0$.

Theorem 3.9. *Let G be an M-strong fuzzy graph without isolated vertices and $|N(v)| = \Delta_N$, if $e_i = (v, v_i), 1 \leq i \leq n$, $r = \sum_{i=1}^{n} \mu(e_i)$ and $s = min\{\mu(e_i)\}$, $i = 1$ to n. Then $\gamma_m \leq p + q - \Delta_N - r + s$.*
proof: *Let v be a vertex of an M-strong fuzzy graph G and $\{v_1, v_2, \ldots, v_n\}$ open neighbourhood set of v. Let Δ_N be the maximum neighbourhood degree of G. That is $|N(v)| = \Delta_N$. If $e_i = (v, v_i), 1 \leq i \leq k$, $r = \sum_{i=1}^{k} \mu(e_i)$ and $s = min\{\mu(e_i)\}$, $i = 1$ to k . Then the set $\{V - \{v_1, v_2, \ldots, v_k\}\} \cup \{E - \{e_1, e_2, \ldots, e_k\}\} \cup \{e_i\}$ such that e_i is the minimum of $\mu(e_i), \forall i$, is an md-set. Therefore $\gamma_m \leq p + q - \Delta_N - r + s$.*

4 Mixed strong(Weak) Domination in an M-strong fuzzy graph

Definition 4.25. *Let $v \in V$ and $e = (u, v) \in E$ in an M-strong fuzzy graph G. Then*

1. *v and e* strongly dominates *each other if $e \in \langle\langle N[v] \rangle\rangle$ and*
2. *v and e* weakly dominates *each other if $v \in N[e]$.*

Definition 4.26. *A set $D \subseteq V$ in an M-strong fuzzy graph G is a* vertex-edge strong dominating set *of G, if every edge in G is strongly dominated by at least one vertex in D. It is denoted by $vesd - set$. The minimum scalar cardinality taken over all vesd-set is called* vertex-edge strong domination number *and it is denoted by the symbol γ_{ves}.*
Similarly edge-vertex strong domination number (γ_{evs}), vertex-edge weak domination number (γ_{vew}) *and* edge-vertex weak domination number (γ_{evw}) *can be defined.*

Observation 4. *For any M-strong fuzzy graph G without isolated vertices:*

1. *a vertex v dominates an edge e $\Leftrightarrow$ a vertex v strongly dominates an edge e. Therefore $\gamma_{ves} = \gamma_{ve}$.*
2. *an edge e dominates a vertex v $\Leftrightarrow$ an edge e weakly dominates a vertex v. Therefore $\gamma_{evw} = \gamma_{ev}$.*
3. *a vertex v dominates an edge e $\Rightarrow$ a vertex v weakly dominates an edge e. Therefore $\gamma_{vew} \leq \gamma_{ve}$.*
4. *an edge e strongly dominates a vertex v $\Rightarrow$ an edge e dominates a vertex v. Therefore $\gamma_{ev} \leq \gamma_{evs}$.*

Remark 4.3. *For some of the M-strong fuzzy graph G that we considered, there is no relation exist between γ_{vew} and γ_{evw}.*

Example 4.2. *Consider the M-strong fuzzy graphs given in Figures 2 and Figure 3.*

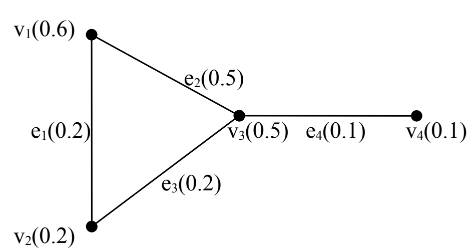

Figure 2:

From the Figure. 2 $\gamma_{vew} - set = \{v_2\} \Rightarrow \gamma_{vew} = 0.2$ *and*
$\gamma_{evw} = \{e_4\} \Rightarrow \gamma_{evw} = 0.1$

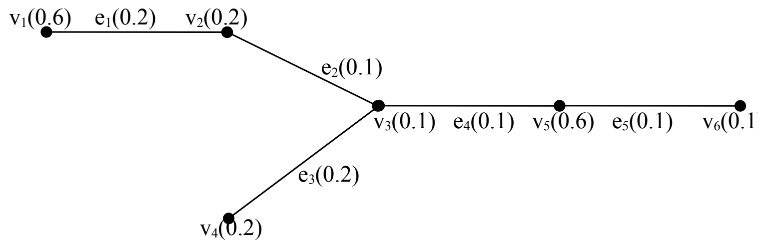

Figure 3:

Hence $\gamma_{evw} \leq \gamma_{vew}$.
From the Figure. 3 $\gamma_{vew} - set = \{v_3\} \Rightarrow \gamma_{vew} = 0.1$ *and*
$\gamma_{evw} = \{e_2, e_5\} \Rightarrow \gamma_{evw} = 0.2$
Hence $\gamma_{vew} \leq \gamma_{evw}$. *Therefore there is no relation exist between* γ_{vew} *and* γ_{evw}.

Remark 4.4. *Similarly, for some of the M-strong fuzzy graph G that we considered, there is no relation exist between* γ_{ves} *and* γ_{evs}.

Example 4.3. *Consider the M-strong fuzzy graphs given in Figure 4 and Figure 5.*

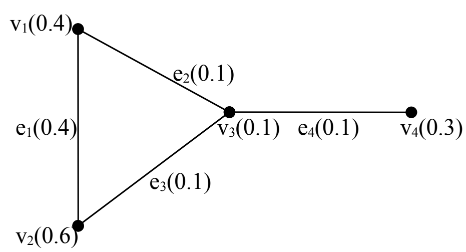

Figure 4:

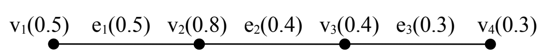

Figure 5:

From the Figure. 4 $\gamma_{ves} - set = \{v_3\} \Rightarrow \gamma_{ves} = 0.1$ *and*
$\gamma_{evs} = \{e_2, e_4\} \Rightarrow \gamma_{evs} = 0.2$
Hence $\gamma_{ves} \leq \gamma_{evs}$.

From the Figure. 5 $\gamma_{ves} - set = \{v_1, v_3\} \Rightarrow \gamma_{ves} = 0.9$ *and*
$\gamma_{evs} = \{e_1, e_3\} \Rightarrow \gamma_{evs} = 0.8$

Hence $\gamma_{evs} \leq \gamma_{ves}$. *Therefore there is no relation exist between* γ_{ves} *and* γ_{evs}.

Theorem 4.10. *For any M-strong fuzzy graph without isolated vertices G,*

1. $\gamma_{vew} \leq \gamma_{ves} = \gamma_{ve}$.
2. $\gamma_{ev} = \gamma_{evw} \leq \gamma_{evs}$.

Proof. (1) From the Observation 4(1) $\gamma_{ves} = \gamma_{ve}$ and the Observation 4(3)$\gamma_{vew} \leq \gamma ve$. Hence $\gamma_{vew} \leq \gamma_{ves} = \gamma_{ve}$.
(2) From the Observation 4(2) $\gamma_{evw} = \gamma_{ev}$ and the Observation 4(4)$\gamma_{ev} \leq \gamma evs$. Hence $\gamma_{ev} = \gamma_{evw} \leq \gamma_{evs}$. □

Definition 4.27. *A set* $D \subseteq V \cup E$ *in an M-strong fuzzy graph G is a* mixed strong(weak) dominating set *of G, if*

1. *every vertex* $v \in V$ *not in D is strongly(weakly) dominated by at least one edge in D and*

2. *every edge $e \in E$ not in D is strongly(weakly) dominated by at least one vertex in D.*

The mixed strong(weak) dominating set is denoted by msd-set(mwd-set). The minimum scalar cardinality taken over all msd-set(mwd-set) is called mixed strong(weak) domination number and it is denoted by the symbol $\gamma_{ms}(\gamma_{mw})$.

Theorem 4.11. *For any M-strong fuzzy graph without isolated vertices G,*

1. $\gamma_{vew} \leq \gamma_{mw} \leq \gamma_{vew} + \gamma_{evw}$.
2. $\gamma_{ves} \leq \gamma_{ms} \leq \gamma_{ves} + \gamma_{evs}$.

Proof. (1) Let $S = \{v_1, v_2, \ldots, v_m, e_{m+1}, e_{m+2}, \ldots, e_n\}$ be a $\gamma_{mw} - set$ in an M-strong fuzzy graph G. Replace each e_j of S by v_j such that $\sigma(v_j) = \mu(e_j), m+1 \leq j \leq n$ and form the s'. Therefore $s' = \{v_1, v_2, \ldots, v_m, v_{m+1}, v_{m+2}, \ldots, v_n\}$. Hence $\gamma_{vew} \leq \gamma_{mw}$. Also the union of vewd-set and evwd-set forms an mwd-set. Therefore $\gamma_{mw} \leq \gamma_{vew} + \gamma_{evw}$.
(2) Similarly we can prove $\gamma_{ves} \leq \gamma_{ms} \leq \gamma_{ves} + \gamma_{evs}$. □

Theorem 4.12. *For any M-strong fuzzy graph without isolated vertices G , $\gamma_{mw} \leq \gamma_m \leq \gamma_{ms}$.*

Proof. Let $D = \{v_1, v_2, \ldots, v_m, e_1, e_2, \ldots, e_n\} - - - -(i)$ be any $\gamma_m - set$ in an M-strong fuzzy graph G. Let $v \in D = \gamma_m - set$. Then v dominates at least one edge $e \in E - D$. By Observation 4 (3), v weakly dominates at least one edge $e \in E - D. - - - (ii)$. Also, let $e \in D$. Then e dominates at least one vertex $v \in V - D$. By Observation 4 (2), e weakly dominates at least one vertex $v \in V - D. - - - -(iii)$ Hence by (i), (ii) and (iii) every $\gamma_m - set$ is a mixed weak dominating set. The scalar cardinality of mixed dominating set $\leq \gamma_m. - - - -(iv)$ Hence $\gamma_{mw} \leq \gamma_m$.

Let $S = \{v_1, v_2, \ldots, v_m, e_1, e_2, \ldots, e_n\} - - - -(v)$ be any $\gamma_{ms} - set$. Let $v \in S = \gamma_{ms} - set$. Then v strongly dominates at least one edge $e \in E - S$. By Observation 4 (1), v dominates at least one edge $e \in E - S. - - - -(vi)$ Also, let $e \in S$. Then e strongly dominates at least one vertex $v \in V - S$. By Observation 4 (4), e dominates at least one vertex $v \in V - S. - - - -(vii)$ Hence by (v), (vi) and (vii) every $\gamma_{ms} - set$ is a mixed dominating set . The scalar cardinality of mixed dominating set $\leq \gamma_{ms}. - - - -(viii)$ Hence $\gamma_{mw} \leq \gamma_m \leq \gamma_{ms}$. □

Example 4.4. *Consider an M-strong fuzzy graph given in Figure 6*
$\gamma_{ve} - set = \{v_1, v_3, v_5\} \Rightarrow \gamma_{ve} = 1.2.$

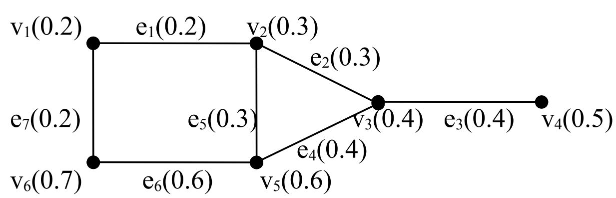

Figure 6:

$\gamma_{ev} - set = \{e_1, e_2\} \Rightarrow \gamma_{ev} = 0.5.$
$\gamma_m - set = \{v_3, v_5, e_1, e_2\} \Rightarrow \gamma_m = 1.5.$
$\gamma_{vew} - set = \{v_2\} \Rightarrow \gamma_{vew} = 0.3.$
$\gamma_{evw} - set = \{e_1, e_2\} \Rightarrow \gamma_{evw} = 0.5.$
$\gamma_{mw} - set = \{v_2, e_1, e_2\} \Rightarrow \gamma_{mw} = 0.8.$
$\gamma_{ves} - set = \{v_1, v_3, v_5\} \Rightarrow \gamma_{ves} = 1.2.$
$\gamma_{evs} - set = \{e_1, e_3, e_6, e_7\} \Rightarrow \gamma_{evs} = 1.4.$
$\gamma_{ms} - set = \{v_3, e_1, e_2, e_6, e_7\} \Rightarrow \gamma_{ms} = 1.8.$

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Some vector-valued statistical convergent sequence spaces

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Abstract

In the present paper we introduce some vector-valued statistical convergent sequence spaces defined by a sequence of modulus functions associated with multiplier sequences and we also make an effort to study some topological properties and inclusion relation between these spaces.

Keywords: Modulus function, paranorm space, difference sequence space, statistical convergence.

2010 MSC: 40A05, 46A45, 40C05.

1 Introduction and Preliminaries

The study on vector-valued sequence spaces was exploited by Kamthan [11], Ratha and Srivastava [18], Leonard [14], Gupta [9], Tripathy and Sen [26] and many others. The scope for the studies on sequence spaces was extended on introducing the notion of associated multiplier sequences. Goes and Goes [8] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, with the help of multiplier sequences (k^{-1}) and (k) respectively. Kamthan used the multiplier sequence $(k!)$ see [11]. The study on multiplier sequence spaces were carried out by Colak [2], Colak et al. [3], Srivastava and Srivastava [25], Tripathy and Mahanta [28] and many others. Let w be the set of all sequences of real or complex numbers and let l_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ respectively with the usual norm $||x|| = \sup |x_k|$, where $k \in \mathbb{N}$, is the set of positive integers.
Throughout the paper, for all $k \in \mathbb{N}$, E_k are seminormed spaces seminormed by q_k and X is a seminormed space seminormed by q. By $w(E_k)$, $c(E_k)$, $l_\infty(E_k)$ and $l_p(E_k)$ we denote the spaces of all, convergent, bounded and p-absoluetly summable E_k-valued sequences. In the case $E_k = \mathbb{C}$ (the field of complex numbers) for all $k \in \mathbb{N}$, one has the scalar valued sequence spaces respectively. The zero element of E_k is denoted by θ_k and the zero sequence is denoted by $\bar{\theta} = (\theta_k)$.
The notion of difference sequence spaces was introduced by Kizmaz [12], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [4] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, s be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_s^m) = \{x = (x_k) \in w : (\Delta_s^m x_k) \in Z\},$$

where $\Delta_s^m x = (\Delta_s^m x_k) = (\Delta_s^{m-1} x_k - \Delta_s^{m-1} x_{k+1})$ and $\Delta_s^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_s^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+sv}.$$

Taking $s = 1$, we get the spaces which were studied by Et and Colak [4]. Taking $m = s = 1$, we get the spaces which were introduced and studied by Kizmaz [12].

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Definition 1.1. *A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that*

1. $f(x) = 0$ *if and only if* $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ *for all* $x \geq 0$, $y \geq 0$,
3. f *is increasing,*
4. f *is continuous from right at* 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([1], [16], [19], [20], [23]) and many others.

Definition 1.2. *Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if*

1. $p(x) \geq 0$, *for all* $x \in X$,
2. $p(-x) = p(x)$, *for all* $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, *for all* $x, y \in X$,
4. *if* (λ_n) *is a sequence of scalars with* $\lambda_n \to \lambda$ *as* $n \to \infty$ *and* (x_n) *is a sequence of vectors with* $p(x_n - x) \to 0$ *as* $n \to \infty$, *then* $p(\lambda_n x_n - \lambda x) \to 0$ *as* $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [29], Theorem 10.4.2, P-183).

Let $p = (p_k)$ be a bounded sequence of positive real numbers, let $F = (f_k)$ be a sequence of modulus function. Also let $t = t_k = p_k^{-1}$ and suppose $u = (u_k)$ is a sequence of strictly positive real numbers. In this paper we define the following sequence spaces:

$$W_0(\Delta_s^m, F, Q, p, u, t) = \Big\{(x_k) : x_k \in E_k \text{ for all } k \in \mathbb{N} \text{ and there exists } r > 0 \text{ such that } \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m x_k r\big)\big)\Big]^{p_k} \to 0 \text{ as } n \to \infty\Big\},$$

$$W_1(\Delta_s^m, F, Q, p, u, t) = \Big\{(x_k) : x_k \in E_k \text{ for all } k \in \mathbb{N} \text{ and there exists } r > 0 \text{ such that } \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m x_k r - l\big)\big)\Big]^{p_k} \to 0 \text{ as } n \to \infty, l \in E_k\Big\}$$

and

$$W_\infty(\Delta_s^m, F, Q, p, u, t) = \Big\{(x_k) : x_k \in E_k \text{ for all } k \in \mathbb{N} \text{ and there exists } r > 0 \text{ such that } \sup_n \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m x_k r\big)\big)\Big]^{p_k} < \infty\Big\}.$$

In the case $f_k = f$ and $q_k = q$ for all $k \in \mathbb{N}$, we write $W_0(\Delta_s^m, f, q, p, u, t)$, $W_1(\Delta_s^m, f, q, p, u, t)$ and $W_\infty(\Delta_s^m, f, q, p, u, t)$ instead of $W_0(\Delta_s^m, F, Q, p, u, t)$, $W_1(\Delta_s^m, F, Q, p, u, t)$ and
$W_\infty(\Delta_s^m, F, Q, p, u, t)$ respectively.
Throughout the paper Z denotes any of the values $0, 1$ and ∞. If $x = (x_k) \in W_1(\Delta_s^m, f, q, p, u, t)$, we say that x is strongly $u_{q,t}$ Cesaro summable with respect to the modulus function f and write $x_k \to l \;\; W_1(\Delta_s^m, f, q, p, u, t)$; l is called the $u_{q,t}$ limit of x with respect to the modulus function f.
The main aim of this paper is to introduced the sequence spaces $W_Z(\Delta_s^m, F, Q, p, u, t)$, $\; Z = 0, 1$ and ∞. We also make an effort to study some topological properties and inclusion relations between these spaces.

2 Main Results

Theorem 2.1. *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the spaces $W_Z(\Delta_s^m, F, Q, p, u, t)$, $Z = 0, 1, \infty$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. We shall prove the result for $Z = 0$. Let $x = (x_k) \in W_0(\Delta_s^m, F, Q, p, u, t)$. Then there exists $r > 0$ such that $\frac{1}{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m x_k r\big)\big)\Big]^{p_k} \to 0$ as $n \to \infty$. Let $\lambda \in \mathbb{C}$. Without loss of generality we can take $\lambda \neq 0$. Let $\rho = r(|\lambda|)^{-1} > 0$, then we have

$$\frac{1}{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m \lambda x_k\big)\rho\big)\Big]^{p_k} = \frac{1}{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m x_k r\big)\big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Therefore $\lambda x \in W_0(\Delta_s^m, F, Q, p, u, t)$, for all $\lambda \in \mathbb{C}$ and for all $x = (x_k) \in W_0(\Delta_s^m, F, Q, p, u, t)$. Next, suppose that $x = (x_k)$, $y = (y_k) \in W_0(\Delta_s^m, F, Q, p, u, t)$. Then there exists $r_1, r_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m x_k r_1\big)\big)\Big]^{p_k} \to 0 \text{ as } n \to \infty$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m y_k r_2\big)\big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Thus given $\varepsilon > 0$, there exists $k_1, k_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m x_k r_1\big)\big)\Big]^{p_k} < \varepsilon p_k, \text{ for all } k \geq k_1$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m y_k r_2\big)\big)\Big]^{p_k} < \varepsilon p_k, \text{ for all } k \geq k_2.$$

Let $r = r_1 r_2 (r_1 + r_2)^{-1}$ and $k_0 = \max(k_1, k_2)$. Then we have for all $k \geq k_0$,

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m (x_k + y_k) r\big)\big)\Big]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m x_k r_1\big) r_2 (r_1 + r_2)^{-1} + f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m y_k r_2\big) r_1 (r_1 + r_2)^{-1}\Big]^{p_k} < \varepsilon p_k.$$

Hence $x + y \in W_0(\Delta_s^m, F, Q, p, u, t)$. Thus $W_0(\Delta_s^m, F, Q, p, u, t)$ is a linear space. Similarly we can prove that $W_1(\Delta_s^m, F, Q, p, u, t)$ and $W_\infty(\Delta_s^m, F, Q, p, u, t)$ are linear spaces. □

Theorem 2.2. *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the space $W_0(\Delta_s^m, F, Q, p, u, t)$ is a complete paranormed space with paranorm defined by*

$$g(x) = \sup_n \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\big(q_k\big(p^{-t_k} u_k \Delta_s^m x_k r\big)\big)\Big]^{p_k}\Big)^{\frac{1}{M}},$$

where $M = \max\{1, \sup p_k\}$.

Proof. Let $(x^{(i)})$ be a Cauchy sequence in $W_0(\Delta_s^m, F, Q, p, u, t)$. Then for a given $\varepsilon > 0$, there exists n_0 such that $g(x^i - x^j) < \varepsilon$, for all $i, j \geq n_0$. Thus, we have

$$\Big[\sum_{k=1}^{\infty}\big(f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m (x_k^i - x_k^j) r\big)\big)\big)^{p_k}\Big]^{\frac{1}{M}} < \varepsilon, \text{ for all } i, j \geq n_0. \tag{2.1}$$

$$\implies \Big(f_k\big(q_k\big(p_k^{-t_k} u_k \Delta_s^m (x_k^i - x_k^j) r\big)\big)\Big) < \varepsilon, \text{ for all } i, j \geq n_0.$$
$$\implies \Delta_s^m (x_k^i - x_k^j) < \varepsilon, \text{ for all } i, j \geq n_0, \text{ for all } k \in \mathbb{N}.$$

Hence $(x_k^i)_{i=1}^{\infty}$ is a Cauchy sequence in E_k, for each $k \in \mathbb{N}$. Since $E_k{}'s$ are complete for each $k \in \mathbb{N}$, so $(x_k^i)_{i=1}^{\infty}$ converges in E_k, for each $k \in \mathbb{N}$. On taking limit as $j \to \infty$ in (2.1), we have

$$\Big[\sum_{k=1}^{\infty}\big(f_k\big(q_k\big(p_k^{-t_k}u_k\Delta_s^m(x_k^i - x_k)r\big)\big)\big)^{p_k}\Big]^{\frac{1}{M}} < \varepsilon, \quad \text{for all} \quad i \geq n_0.$$

$$\implies \Delta_s^m(x_k^i - x) \in W_0(\Delta_s^m, F, Q, p, u, t).$$

Since $W_0(\Delta_s^m, F, Q, p, u, t)$ is a linear space, so we have $x = x^{(i)} - (x^{(i)} - x) \in W_0(\Delta_s^m, F, Q, p, u, t)$. Thus $W_0(\Delta_s^m, F, Q, p, u, t)$ is a complete paranormed space. This completes the proof of the theorem. □

Theorem 2.3. *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then*

$$W_0(\Delta_s^m, F, Q, p, u, t) \subset W_1(\Delta_s^m, F, Q, p, u, t) \subset W_\infty(\Delta_s^m, F, Q, p, u, t).$$

Proof. It is easy to prove so we omit the details. □

Theorem 2.4. *Let $F = (f_k)$ and $G = (g_k)$ be any two sequences of modulus functions. For any bounded sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers and any two sequences of seminorms $Q = (q_k)$, $V = (v_k)$, the following are true:*
(i) $W_Z(\Delta_s^m, f, Q, u, t) \subset W_Z(\Delta_s^m, f \circ g, Q, u, t)$,
(ii) $W_Z(\Delta_s^m, F, Q, p, u, t) \cap W_Z(\Delta_s^m, F, V, p, u, t) \subset W_Z(\Delta_s^m, F, Q + V, p, u, t)$,
(iii) $W_Z(\Delta_s^m, F, Q, p, u, t) \cap W_Z(\Delta_s^m, G, Q, p, u, t) \subset W_Z(\Delta_s^m, F + G, Q, p, u, t)$,
(iv) if q is stronger than v, then $W_Z(\Delta_s^m, F, Q, p, u, t) \subset W_Z(\Delta_s^m, F, V, p, u, t)$,
(v) if q is equivalent v, then $W_Z(\Delta_s^m, F, Q, p, u, t) = W_Z(\Delta_s^m, F, V, p, u, t)$,
(vi) $W_Z(\Delta_s^m, F, Q, p, u, t) \cap W_Z(\Delta_s^m, F, V, p, u, t) \neq \varphi$.

Proof. We shall prove (i) for the case $Z = 0$. Let $\varepsilon > 0$. We choose δ, $0 < \delta < 1$, such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$ and all $k \in \mathbb{N}$. We write $y_k = g\big(q_k\big(p_k^{-t_k}u_k\Delta_s^m x_k r\big)\big)$ and consider

$$\sum_{k=1}^{n}\big[f(y_k)\big] = \sum_{1}\big[f(y_k)\big] + \sum_{2}\big[f(y_k)\big],$$

where the first summation is over $y_k \leq \delta$ and the second summation is over $y_k > \delta$. Since f is continuous, we have

$$\sum_{1}\big[f(y_k)\big] < n\varepsilon. \tag{2.2}$$

By the definition of f, we have the following relation for $y_k > \delta$:

$$f(y_k) < 2f(1)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{n}\sum_{2}\big[f(y_k)\big] \leq 2\delta^{-1}f(1)\frac{1}{n}\sum_{k=1}^{n} y_k. \tag{2.3}$$

It follows from (2.2) and (2.3) that $W_Z(\Delta_s^m, f, Q, u, t) \subset W_Z(\Delta_s^m, f \circ g, Q, u, t)$. Similarly, we can prove the result for other cases. □

Theorem 2.5. *Let f be a modulus function. Then $W_Z(\Delta_s^m, Q, u, t) \subset W_Z(\Delta_s^m, f, Q, u, t)$.*

Proof. It is easy to prove in view of Theorem 2.4(i). □

Theorem 2.6. *Let $0 < p_k < r_k$ and $\left(\frac{r_k}{p_k}\right)$ be bounded. Then*

$$W_Z(\Delta_s^m, F, Q, r, u, t) \subset W_Z(\Delta_s^m, F, Q, p, u, t).$$

Proof. By taking $y_k = \Big[f_k\big(q_k\big(p_k^{-t_k}u_k\Delta_s^m x_k r\big)\big)\Big]^{r_k}$ for all k and using the same technique as in Theorem 5 of Maddox [15], one can easily prove the theorem. □

Theorem 2.7. *Let f be a modulus function. If $\lim_{m\to\infty}\dfrac{f(m)}{m} = \beta > 0$, then $W_1(\Delta_s^m, Q, p, u, t) \subset W_1(\Delta_s^m, f, Q, p, u, t)$.*

Proof. It is easy to prove so we omit the details. □

3 $u_{q,t}$-Statistical Convergence

The notion of statistical convergence was introduced by Fast [6] and Schoenberg [24] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [7], Connor [5], Salat [21], Murasaleen [17], Isik [10], Savas [22], Malkowsky and Savas [16], Kolk [13], Maddox [15], Tripathy and Sen [27] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

Definition 3.3. *A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists:*

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k),$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

Definition 3.4. *A sequence $x = (x_k)$ is said to be $u_{q,t}$-statistical convergent to l if for every $\varepsilon > 0$,*

$$\delta\Big(\big\{k \in \mathbb{N} : q\big(p_k^{-t_k} u_k \Delta_s^m x_k r - l\big) \geq \varepsilon\big\}\Big) = 0.$$

In this case we write $x_k - l\big(S^q_{u,t}\big)$. The set of all $u_{q,t}$-statistical convergent sequences is denoted by $S^q_{u,t}$. By S, we denote the set of all statistically convergent sequences.

If $q(x) = |x|$, $u_k = p_k = t_k = 1$ for all $k \in \mathbb{N}$ and $r = 1$, then $S^q_{u,t}$ is same as S. In case $l = 0$ we write $S^q_{0u,t}$ instead of $S^q_{u,t}$.

Theorem 3.8. *Let $p = (p_k)$ be a bounded sequence and $0 < h = \inf p_k \leq p_k \leq \sup p_k = H < \infty$ and let f be a modulus function. Then*

$$W_1(\Delta_s^m, f, q, p, u, t) \subset S^q_{u,t}.$$

Proof. Let $x \in W_1(\Delta_s^m, f, q, p, u, t)$ and let $\varepsilon > 0$ be given. Let $\sum_1$ and $\sum_2$ denote the sums over $k \leq n$ with $q(p_k^{-t_k} u_k \Delta_s^m x_k r - l) \geq \varepsilon$ and $q(p_k^{-t_k} u_k \Delta_s^m x_k r - l) < \varepsilon$, respectively. Then

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\Big[f\big(q\big(p_k^{-t_k} u_k \Delta_s^m x_k r - l\big)\big)\Big]^{p_k} \\
&\geq \frac{1}{n}\sum_1 \Big[f\big(q\big(p_k^{-t_k} u_k \Delta_s^m x_k r - l\big)\big)\Big]^{p_k} \\
&\geq \frac{1}{n}\sum_1 [f(\varepsilon)]^{p_k} \\
&\geq \frac{1}{n}\sum_1 \min\Big([f(\varepsilon)]^h, [f(\varepsilon)]^H\Big) \\
&\geq \frac{1}{n}\Big|\{k \leq n : q(p_k^{-t_k} u_k \Delta_s^m x_k - l) \geq \varepsilon\}\Big| \min\Big([f(\varepsilon)]^h, [f(\varepsilon)]^H\Big).
\end{aligned}$$

Hence, $x \in S^q_{u,t}$. □

Theorem 3.9. *Let f be a bounded modulus function. Then $S^q_{u,t} \subset W_1(\Delta_s^m, f, q, p, u, t)$.*

Proof. Suppose that f is bounded. Let $\epsilon > 0$ and let $\sum_1$ and $\sum_2$ be the sums introduced in the Theorem 3.1. Since f is bounded, there exists an integer K such that $f(x) < K$ for all $x \geq 0$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f\big(q\big(p_k^{-t_k}u_k\Delta_s^m x_k r-l\big)\big)\Big]^{p_k}$$

$$\begin{aligned}
&\leq \frac{1}{n}\Big(\sum_1\Big[f\big(q\big(p_k^{-t_k}u_k\Delta_s^m x_k r-l\big)\big)\Big]^{p_k}+\sum_2\Big[f\big(q\big(p_k^{-t_k}u_k\Delta_s^m x_k r-l\big)\big)\Big]^{p_k}\Big)\\
&\leq \frac{1}{n}\sum_1\max(K^h,K^H)+\frac{1}{n}\sum_2\big[f(\varepsilon)\big]^{p_k}\\
&\leq \max(K^h,K^H)\frac{1}{n}\Big|\{k\leq n: q(p_k^{-t_k}u_k\Delta_s^m x_k-l)\geq\varepsilon\}\Big|+\max(f(\varepsilon)^h,f(\varepsilon)^H).
\end{aligned}$$

Hence, $x\in W_1(\Delta_s^m,f,q,p,u,t)$. □

Theorem 3.10. *$S_{u,t}^q=W_1(\Delta_s^m,f,q,p,u,t)$ if and only if f is bounded.*

Proof. Let f be bounded. By Theorems 3.1 and 3.2, we have $S_{u,t}^q=W_1(\Delta_s^m,f,q,p,u,t)$.
Conversely, suppose that f is unbounded. Then there exists a sequence (t_k) of positive numbers with $f(t_k)=k^2$ for $k=1,2,\cdots$. If we choose

$$p_k^{-t_k}u_i\Delta_s^m x_i r=\begin{cases}t_k, & i=k^2,\ k=1,2,\cdots\\ 0, & \text{otherwise.}\end{cases}$$

Then we have

$$\frac{1}{n}|\{k\leq n:|p_k^{-t_k}u_k\Delta_s^m x_k r|\geq\epsilon\}|\leq\frac{\sqrt{n}}{n}$$

for all n, and so $x\in S_{u,t}^q$ but $x\notin W_1(\Delta_s^m,f,q,p,u,t)$ for $X=\mathbb{C}$, $q(x)=|x|$ and $p_k=1$ for all $k\in\mathbb{N}$. This contradicts the assumption that $S_{u,t}^q=W_1(\Delta_s^m,f,q,p,u,t)$. □

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Global nonexistence of solutions for a system of viscoelastic wave equations with weak damping terms

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Abstract

This paper deals with the initial boundary value problem for the viscoelastic wave equations

$$\begin{cases} u_{tt}-\triangle u+\int_0^t g_1(t-\tau)\triangle u(\tau)\,d\tau+u_t=f_1(u,v), \\ v_{tt}-\triangle v+\int_0^t g_2(t-\tau)\triangle v(\tau)\,d\tau+v_t=f_2(u,v) \end{cases}$$

in a bounded domain. We obtain the global nonexistence of solutions by applying a lemma due to Y. Zhou [Global existence and nonexistence for a nonliear wave equation with damping and source terms, Math. Nacht, 278 (2005) 1341–1358].

Keywords: Global nonexistence, viscoelastic wave equation.

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1 Introduction

In this paper we consider the following initial boundary value problem

$$\begin{cases} u_{tt}-\triangle u+\int_0^t g_1(t-\tau)\triangle u(\tau)\,d\tau+u_t=f_1(u,v), & (x,t)\in\Omega\times(0,T), \\ v_{tt}-\triangle v+\int_0^t g_2(t-\tau)\triangle v(\tau)\,d\tau+v_t=f_2(u,v), & (x,t)\in\Omega\times(0,T), \\ u(x,t)=v(x,t)=0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n, $n=1,2,3$; and $g_i(.):R^+\longrightarrow R^+$, $f_i(.,.):R^2\longrightarrow R$ are given functions to be specified later.

The single viscoelastic wave equation of the form

$$u_{tt}-\triangle u+\int_0^t g(t-\tau)\triangle u(\tau)\,d\tau+h(u_t)=f(u),\ x\in\Omega,\ t>0, \tag{1.2}$$

has been extensively studied and many results concerning nonexistence have been proved. See in this regard [5, 8, 9, 17].

The equation (1.2) without the viscoelastic term (i.e., $g=0$) can be written in the following form

$$u_{tt}-\triangle u+h(u_t)=f(u),\ x\in\Omega,\ t>0. \tag{1.3}$$

The local existence, global existence, and blow up in finite time of solution for (1.3) were established (see [3, 6, 7, 10, 11] and references therein).

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Agre and Rammaha [2] studied the global existence and blow up of the solution of the problem (1.1) with $g_i = 0$ $(i = 1, 2)$ using the same techniques as in [3]. After that, the blow up result was improved by Said-Houari [15]. Also, he showed that the local solution obtained in [2] is global and decay of solutions [14].

Recently, Han and Wang [4] obtained the local existence, global existence and blow up of the solution of the problem (1.1) under some suitable conditions. Messaoudi and Houari [12] considered problem (1.1) and improved the blow up result in [4], for positive initial energy, using the some techniques as in [15]. Also, Houari et. al. [16] studied the general decay of the solution of the problem (1.1) by using the Lyapunov functional method.

In this paper, we consider the problem (1.1) and prove a global nonexistence result of solutions.

This paper is organized as follows. In section 2, we present some lemmas. In section 3, we state the local existence result. In section 4, we show the global nonexistence of solutions.

2 Preliminaries

In this section, we shall give some assumptions and lemmas which will be used throughout this work. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. Firstly, we make the following assumptions:

(A1) $g_i : R^+ \longrightarrow R^+$ $(i = 1, 2)$ nonincreasing differentiable function satisfying

$$1 - \int_0^\infty g_i(s)\, ds = l_i > 0.$$

(A2) $g_i(t) \geq 0,\ \forall t \geq 0.$

Concerning the functions $f_1(u, v)$ and $f_2(u, v)$, we take

$$f_1(u, v) = a|u + v|^{2(p+1)}(u + v) + b|u|^p |v|^{p+2} u,$$

$$f_2(u, v) = a|u + v|^{2(p+1)}(u + v) + b|u|^{p+2} |v|^p v,$$

where $a, b > 0$ are constants and p satisfies

$$\begin{cases} -1 < p & \text{if } n = 1, 2, \\ -1 < p \leq 1 & \text{if } n = 3. \end{cases} \tag{2.1}$$

According to the above equalities we can easily verify that

$$u f_1(u, v) + v f_2(u, v) = 2(p + 2) F(u, v),\ \forall (u, v) \in R^2, \tag{2.2}$$

where

$$F(u, v) = \frac{1}{2(p + 2)} \left[a|u + v|^{2(p+2)} + 2b|uv|^{p+2} \right]. \tag{2.3}$$

We have the following result.

Lemma 2.1 [12]. There exist two positive constants c_0 and c_1 such that

$$\frac{c_0}{2(p + 2)} \left(|u|^{2(p+2)} + |v|^{2(p+2)} \right) \leq F(u, v) \leq \frac{c_1}{2(p + 2)} \left(|u|^{2(p+2)} + |v|^{2(p+2)} \right) \tag{2.4}$$

is satisfied.

Lemma 2.2 [13]. For any $\phi \in C^1(R)$ we have

$$\begin{aligned} \int_\Omega \int_0^t g(t - \tau) \triangle \phi(\tau) \phi'(t)\, d\tau dx &= -\frac{1}{2}\left(g' \circ \nabla\phi\right)(t) + \frac{1}{2} g(t) \|\nabla\phi\|^2 \\ &+ \frac{1}{2}\frac{d}{dt}\left[\left(g \circ \nabla\phi\right)(t) - \int_0^t g(\tau) \|\nabla\phi\|^2 d\tau \right]. \end{aligned}$$

Lemma 2.3 (Sobolev-Poincare inequality) [1]. Let p be a number with $2 \leq p < \infty$ $(n = 1, 2)$ or $2 \leq p \leq 2n/(n - 2)$ $(n \geq 3)$, then there is a constant $C_* = C_*(\Omega,\ p)$ such that

$$\|u\|_p \leq C_* \|\nabla u\| \ \text{ for } u \in H_0^1(\Omega).$$

Lemma 2.4 [18]. Suppose that $\psi(t)$ is a twice continuously differentiable function satisfying

$$\begin{cases} \psi''(t) + \psi'(t) \geq C_0 \psi^{1+\alpha}(t), & t > 0, \\ \psi(0) > 0, \quad \psi'(0) \geq 0, \end{cases}$$

where $C_0 > 0$, $\alpha > 0$ are constants. Then, $\psi(t)$ blows up in finite time.

3 Local existence

In this section we state local existence and the uniqueness of the solution of the problem (1.1).

Definition 3.1. A pair of functions (u, v) is said to be a weak solution of (1.1) on $[0, T]$ if

$$u, v \in C\left([0,T); H_0^1(\Omega)\right) \cap C^1\left([0,T); L^2(\Omega)\right),$$
$$u_t \in L^2(\Omega \times (0,T)), v_t \in L^2(\Omega \times (0,T)),$$
$$u'' \in L^2\left(0,T; H^{-1}(\Omega) + L^2(\Omega)\right),$$
$$v'' \in L^2\left(0,T; H^{-1}(\Omega) + L^2(\Omega)\right),$$

where $H^{-1}(\Omega) + L^2(\Omega)$ is the dual space of $H_0^1(\Omega) \cap L^2(\Omega)$. In additon, (u, v) satisfies

$$\begin{aligned} &\int_\Omega u'(t)\phi dx - \int_\Omega u_1(t)\phi dx + \int_\Omega \nabla u \nabla \phi dx \\ &-\int_0^t \int_\Omega (g_1 * \nabla u)\nabla \phi dx d\tau + \int_0^t \int_\Omega u' \phi dx d\tau \\ = &\int_0^t \int_\Omega f_1(u(\tau), v(\tau))\phi dx d\tau, \end{aligned}$$

$$\begin{aligned} &\int_\Omega v'(t)\varphi dx - \int_\Omega v_1(t)\varphi dx + \int_\Omega \nabla v \nabla \varphi dx \\ &-\int_0^t \int_\Omega (g_2 * \nabla v)\nabla \varphi dx d\tau + \int_0^t \int_\Omega v' \varphi dx d\tau \\ = &\int_0^t \int_\Omega f_2(u(\tau), v(\tau))\varphi dx d\tau, \end{aligned}$$

for all test functions $\phi \in H_0^1(\Omega) \cap L^2(\Omega)$, $\varphi \in H_0^1(\Omega) \cap L^2(\Omega)$ and for almost all $t \in [0, T]$.

Now, we state the local existence theorem that is proved in [4].

Theorem 3.1 (Local existence)**.** Assume that (2.1), (A1) and (A2) hold. Then for any initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique local weak solution (u, v) of problem (1.1) (in the sense of Definition 3.1) defined on $[0, T]$ for some $T > 0$, and satisfies the energy identity

$$\begin{aligned} &E(t) + \int_0^t \left(\|u_\tau(\tau)\|^2 + \|v_\tau(\tau)\|^2\right) d\tau - \frac{1}{2}\int_0^t \left(\left(g_1' \circ \nabla u\right)(\tau) + \left(g_2' \circ \nabla v\right)(\tau)\right) d\tau \\ &\frac{1}{2}\int_0^t \left(g_1(\tau)\|\nabla u(\tau)\|^2 + g_2(\tau)\|\nabla v(\tau)\|^2\right) d\tau \\ = &E(0) \end{aligned}$$

where $E(t)$ is defined in (4.3).

4 Global nonexistence result

In this section, we prove the global nonexistence of the solution of the problem (1.1). In order to do so, let us first introduce the following functionals,

$$\begin{aligned} J(t) = &\frac{1}{2}\left(1 - \int_0^t g_1(\tau)d\tau\right)\|\nabla u\|^2 + \frac{1}{2}\left(1 - \int_0^t g_2(\tau)d\tau\right)\|\nabla v\|^2 \\ &+\frac{1}{2}\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] - \int_\Omega F(u, v)dx, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} I(t) &= \left(1-\int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|^2+\left(1-\int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|^2 \\ &\quad +(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)-(p+1)\int_\Omega F(u,v)\,dx. \end{aligned} \tag{4.2}$$

We also define the energy function as follows

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\left(1-\int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|^2+\frac{1}{2}\left(1-\int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|^2 \\ &\quad +\frac{1}{2}\left[(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right]-\int_\Omega F(u,v)\,dx, \end{aligned} \tag{4.3}$$

where

$$(\Phi\circ\Psi)(t)=\int_0^t\Phi(t-\tau)\int_\Omega|\Psi(t)-\Psi(\tau)|\,dxd\tau.$$

Finally, we define

$$W=\left\{(u,v):(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)\,,\ I(u,v)>0\right\}\cup\{(0,0)\}\,. \tag{4.4}$$

The next lemma shows that our energy functional (4.3) is a nonincreasing function along the solution of the problem (1.1).

Lemma 4.1. $E(t)$ is a decreasing function for $t\geq 0$ and

$$\begin{aligned} E'(t) &\leq -\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\left[\left(g_1'\circ\nabla u\right)(t)+\left(g_2'\circ\nabla v\right)(t)\right] \\ &\leq 0,\ \forall t\geq 0. \end{aligned} \tag{4.5}$$

Proof. Multiplying the first equation of (1.1) by u_t and the second equation by v_t, integrating over Ω, and using (2.5) and the assumption (A1)-(A2), we obtain (4.5).

Theorem 4.1. Under the conditions of Theorem 3.1, assume that initial conditions satisfies

$$E(0)\leq 0,\quad \int_\Omega(u_0u_1+v_0v_1)\,dx\geq 0,$$

and

$$\max\left\{\int_0^t g_1(s)\,ds,\ \int_0^t g_2(s)\,ds\right\}\leq\frac{p+1}{p+3-\frac{1}{4(p+2)}}$$

then the corresponding solution blows up in finite time. In other words, there exists a positive constant T^* such that $\lim\limits_{t\longrightarrow T^*}\left(\|u\|^2+\|v\|^2\right)=\infty$.

Proof. To apply Lemma 2.4, we define

$$\psi(t)=\frac{1}{2}\int_\Omega\left(|u|^2+|v|^2\right)dx. \tag{4.6}$$

Therefore,

$$\psi'(t)=\int_\Omega(uu_t+vv_t)\,dx, \tag{4.7}$$

and

$$\psi''(t)=\int_\Omega\left(u_t^2+v_t^2\right)dx+\int_\Omega(uu_{tt}+vv_{tt})\,dx. \tag{4.8}$$

Then, eq (1.1) is used to estimate (4.8) as follows

$$\begin{aligned} \psi''(t) &= \int_\Omega\left(u_t^2+v_t^2\right)dx-\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ &\quad +\int_\Omega\int_0^t g_1(t-s)\nabla u(t)\nabla u(s)\,dsdx \\ &\quad +\int_\Omega\int_0^t g_2(t-s)\nabla v(t)\nabla v(s)\,dsdx \\ &\quad -\int_\Omega(uu_t+vv_t)\,dx+2(p+2)\int_\Omega F(u,v)\,dx. \end{aligned} \tag{4.9}$$

We then use Young's inequality to estimates third and fiveth terms in (4.9);

$$\begin{aligned} & \int_\Omega \int_0^t g_1(t-s)\nabla u(t)\nabla u(s)\,dsdx \\ = & \int_\Omega \int_0^t g_1(t-s)\nabla u(t)\left[\nabla u(s)-\nabla u(t)\right]dsdx + \left(\int_0^t g_1(s)\,ds\right)\|\nabla u\|^2 \\ \leq & \ \delta\|\nabla u\|^2 + \frac{1}{4\delta}\left(\int_0^t g_1(s)\,ds\right)(g_1\circ\nabla u)(t) + \left(\int_0^t g_1(s)\,ds\right)\|\nabla u\|^2 \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} & \int_\Omega \int_0^t g_2(t-s)\nabla v(t)\left[\nabla v(s)-\nabla v(t)\right]dsdx \\ \leq & \ \delta\|\nabla v\|^2 + \frac{1}{4\delta}\left(\int_0^t g_2(s)\,ds\right)(g_2\circ\nabla v)(t) + \left(\int_0^t g_2(s)\,ds\right)\|\nabla v\|^2. \end{aligned} \tag{4.11}$$

Inserting (4.10), (4.11) into (4.9) to get

$$\begin{aligned} \psi''(t)+\psi'(t) \geq & \left(\|u_t\|^2+\|v_t\|^2\right) - \left(1+\int_0^t g_1(s)\,ds+\delta\right)\|\nabla u\|^2 \\ & -\left(1+\int_0^t g_2(s)\,ds+\delta\right)\|\nabla v\|^2 + 2(p+2)\int_\Omega F(u,v)\,dx \\ & -\frac{1}{4\delta}\left(\int_0^t g_1(s)\,ds\right)(g_1\circ\nabla u)(t) - \frac{1}{4\delta}\left(\int_0^t g_2(s)\,ds\right)(g_2\circ\nabla v)(t). \end{aligned} \tag{4.12}$$

From the definition of $E(t)$, it follows that

$$\begin{aligned} \|\nabla u\|^2+\|\nabla v\|^2 \leq & \ \frac{2}{1-l}E(t) + \frac{2}{1-l}\int_\Omega F(u,v)\,dx - \frac{1}{1-l}\left(\|u_t\|^2+\|v_t\|^2\right) \\ & -\frac{1}{1-l}\left[(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right], \end{aligned} \tag{4.13}$$

where $l=\max\left\{\int_0^t g_1(s)\,ds,\ \int_0^t g_2(s)\,ds\right\}$. Substituting (4.13) into (4.12), we have

$$\begin{aligned} \psi''(t)+\psi'(t) \geq & \left(\frac{2+\delta}{1-l}\right)\left(\|u_t\|^2+\|v_t\|^2\right) - 2\left(\frac{1+l+\delta}{1-l}\right)E(t) \\ & + \left[2(p+2)-2\left(\frac{1+l+\delta}{1-l}\right)\right]\int_\Omega F(u,v)\,dx \\ & + \left[\frac{1+l+\delta}{1-l} - \frac{1}{4\delta}\left(\int_0^t g_1(s)\,ds\right)\right](g_1\circ\nabla u)(t) \\ & + \left[\frac{1+l+\delta}{1-l} - \frac{1}{4\delta}\left(\int_0^t g_2(s)\,ds\right)\right](g_2\circ\nabla v)(t). \end{aligned} \tag{4.14}$$

At this point we choose $\delta>0$, so that

$$\frac{1+l+\delta}{1-l} - \frac{1}{4\delta}\left(\int_0^t g_1(s)\,ds\right) \geq 0, \quad \frac{1+l+\delta}{1-l} - \frac{1}{4\delta}\left(\int_0^t g_2(s)\,ds\right) \geq 0.$$

Therefore, (4.14) becomes

$$\begin{aligned} \psi''(t)+\psi'(t) \geq & \left[2(p+2)-2\left(\frac{1+l+\delta}{1-l}\right)\right]\int_\Omega F(u,v)\,dx \\ \geq & \ \frac{c_0}{2(p+2)}\left[2(p+2)-2\left(\frac{1+l+\delta}{1-l}\right)\right]\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right) \\ \geq & \ \gamma\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right) \end{aligned} \tag{4.15}$$

where $\gamma=\frac{c_0}{2(p+2)}\left[2(p+2)-2\left(\frac{1+l+\delta}{1-l}\right)\right]$. Also, from assumption of the theorem $\gamma\geq 0$.

Now, Hölder inequality are used to estimates $\|u\|_{2(p+2)}^{2(p+2)}$ and $\|v\|_{2(p+2)}^{2(p+2)}$ as follows

$$\int_{\Omega} |u|^2 \, dx \leq \left(\int_{\Omega} |u|^{2(p+2)} \, dx \right)^{\frac{1}{p+2}} \left(\int_{\Omega} 1 dx \right)^{\frac{p+1}{p+2}}.$$

W_n is called the volume of the domain Ω, then

$$\|u\|_{2(p+2)}^{2(p+2)} \geq \left(\int_{\Omega} |u|^2 \, dx \right)^{p+2} (W_n)^{-(p+1)},$$

and similarly, we have

$$\|v\|_{2(p+2)}^{2(p+2)} \geq \left(\int_{\Omega} |v|^2 \, dx \right)^{p+2} (W_n)^{-(p+1)}.$$

Consequently, we have

$$\psi''(t) + \psi'(t) \geq \gamma (W_n)^{-(p+1)} \left[\left(\int_{\Omega} |u|^2 \, dx \right)^{p+2} + \left(\int_{\Omega} |v|^2 \, dx \right)^{p+2} \right]. \tag{4.16}$$

In order to estimate the right-hand side in (4.16), we make use of the following inequality

$$(X+Y)^{\rho} \leq 2^{\rho-1} \left(X^{\rho} + Y^{\rho} \right),$$

$X, Y \geq 0$, $1 \leq \rho < \infty$, applying the above inequality we have

$$2^{-(p+1)} \left(\int_{\Omega} |u|^2 \, dx + \int_{\Omega} |v|^2 \, dx \right)^{p+2} \leq \left(\int_{\Omega} |u|^2 \, dx \right)^{p+2} + \left(\int_{\Omega} |v|^2 \, dx \right)^{p+2}.$$

Consequently, (4.16) becomes

$$\begin{aligned} \psi''(t) + \psi'(t) &\geq 2^{-(p+1)} \gamma (W_n)^{-(p+1)} \left(\int_{\Omega} |u|^2 \, dx + \int_{\Omega} |v|^2 \, dx \right)^{p+2} \\ &= 2\gamma (W_n)^{-(p+1)} \psi^{p+2}(t). \end{aligned}$$

It is easy to verify that the requirements of Lemma 2.4 are satisfied by

$$C_0 = 2\gamma (W_n)^{-(p+1)} > 0, \quad \alpha = p+1 > 0.$$

Therefore $\psi(t)$ blows up in finite. The proof of Theorem 4.1 is completed.

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Further Results on Sum Cordial Graphs

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Abstract

In this paper, we prove that wheel, closed helm, quadrilateral snake, double quadrilateral snake and gear graphs are sum cordial graphs.

Keywords: Cordial labeling, Sum cordial labeling, Sum cordial graph.

2010 MSC: 54C05, 54C10.

1 Introduction

All graphs $G = (V(G), E(G))$ in this paper are finite, connected and undirected. For any undefined notations and terminology we follow [3]. If the vertices or edges or both of the graph are assigned valued subject to certain conditions it is known as graph labeling. A dynamic survey on graph labeling is regularly updated by Gallian [4]. Labeled graphs have variety of applications in graph theory, particularly for missile guidance code, design good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graphs plays vital role in the study of X-ray crystallography, communication network and to determine optimal circuit layouts. A detailed study on variety of applications on graph labeling is carried out in Bloom and Golomb [1].

Definition 1.1. *A mapping $f : V(G) \longrightarrow \{0,1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

The induced edge labeling $f^* : E(G) \longrightarrow \{0,1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let us denote $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f ad $e_f(0)$, $e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^*.

Definition 1.2. *A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called cordial if it admits labeling.*

The concept of cordial labeling was introduced by Cahit [2] in which he investigated several results on this newly defined concept. Also, some new graphs are investigated as product cordial graphs by Vaidya [6].

Definition 1.3. *A binary vertex labeling of a graph G with induce edge labeling $f^* : E(G) \longrightarrow \{0,1\}$ defined by $f^*(uv) = (f(u) + f(v))(mod2)$ is called sum cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is sum cordial if it admits sum cordial labeling.*

Shiama [5] investigated the sum cordial labeling and proved that path P_n, cycle C_n, star $K_{1,n}$ etc are some cordial graphs.

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Definition 1.4. *The wheel graph W_n is defined as the join of $K_1 + C_n$. The vertex corresponding to K_1 is said to be apex vertex, the vertices corresponding to cycle are called rim vertices. The edges corresponding to cycle are called the rim edges and edges joining apex and vertices of the cycle are called spoke edges.*

Definition 1.5. *The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.*

Definition 1.6. *The closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to each rim vertex.*

Definition 1.7. *The quadrilateral snake Q_n is obtained from the path P_n by replacing every edge of a path by a cycle C_n.*

Definition 1.8. *The double quadrilateral snake DQ_n consists of two quadrilateral snakes that have a common path.*

Definition 1.9. *Let $e = uv$ be an edge of a graph G and w is not a vertex of G. The edge e is sub divided when it is replaced by the edges $e' = uw$ and $e'' = wv$.*

Definition 1.10. *The gear graph G_n is obtained from the wheel W_n by sub dividing each of its rim edge.*

2 Main Results

Theorem 2.1. *The wheel W_n is a sum cordial graph except $n \equiv 3(mod4)$.*

Proof: Let v be an apex vertex and $v_1, v_2, \ldots, v_n$ are rim vertices for wheel W_n. Then $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$.
To define $f : V(W_n) \longrightarrow \{0, 1\}$, we consider the following cases,

For $n \equiv 0, 1, 2(mod4)$

$$f(v) = 0;$$
$$f(v_i) = \begin{cases} 1, & i \equiv 1 \text{ or } 2(mod4); \\ 0, & i \equiv 3 \text{ or } 4(mod4). \end{cases} \quad ; 1 \leq i \leq n$$

Therefore,

$$v_f(0) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil, & n \equiv 0(mod4); \\ \frac{n+1}{2}, & n \equiv 1(mod4); \\ \left\lfloor \frac{n+1}{2} \right\rfloor, & n \equiv 2(mod4). \end{cases}$$
$$v_f(1) = \begin{cases} \left\lfloor \frac{n+1}{2} \right\rfloor, & n \equiv 0(mod4); \\ \frac{n+1}{2}, & n \equiv 1(mod4); \\ \left\lceil \frac{n+1}{2} \right\rceil, & n \equiv 2(mod4). \end{cases}$$
$$e_f(0) = e_f(1) = n$$

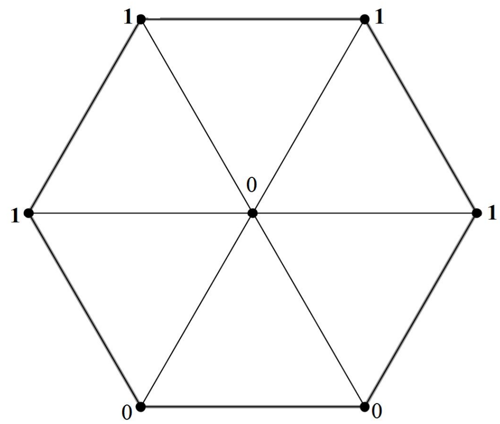
Therefore,

$$v_f(0) - v_f(1) = \begin{cases} 1, & n \equiv 0(mod4); \\ 0, & n \equiv 1(mod4); \\ -1, & n \equiv 2(mod4). \end{cases}$$

Hence, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. So, wheel W_n is a sum cordial for $n \equiv 0, 1$ or $2(mod4)$.

For $n \equiv 3(mod4)$ In order to satisfy the vertex condition for the sum cordial graph it is necessary to assign 0 to $\frac{n+1}{2}$ vertices out of $n + 1$ vertices. The vertices having label 1 will give rise at least $\left\lceil \frac{2n+1}{2} \right\rceil$ edges with label 1 and at most $\left\lfloor \frac{2n-1}{2} \right\rfloor$ edges with label 0 out of $2n$ edges. Therefore, $|e_f(0) - e_f(1)| \geq 2$. Hence the edge condition for the sum cordial graph is not satisfied. So wheel W_n is not sum cordial for $n \equiv 3(mod4)$.

Example 2.1. *The wheel W_6 is a sum cordial graph.*

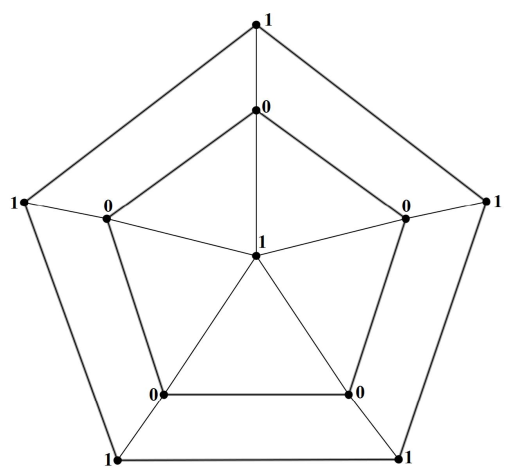


Sum cordial labeling of Wheel W_6

Theorem 2.2. *The closed Helm CH_n is a sum cordial graph.*

Proof: Let v be an apex vertex and $v_1, v_2, \dots, v_n$ are rim vertices. We denote the pendant vertices by $v_1^{'}, v_2^{'}, \dots, v_n^{'}$. Then $|V(CH_n)| = 2n + 1$ and $|E(CH_n)| = 4n$.
Define $f : V(CH_n) \to \{0, 1\}$ by $f(v) = 1,\ f(v_i) = 0,\ f(v_i^{'}) = 1$ for $1 \leq i \leq n$.
In view of the above labeling pattern, we have $v_f(0) = n,\ v_f(1) = n + 1,\ e_f(0) = 2n = e_f(1)$. Thus, we get $|v_f(0) - v_f(1)| \leq 1,\ |e_f(0) - e_f(1)| \leq 1$.
Hence, CH_n is a sum cordial graph.

Example 2.2. *The Closed helm CH_5 is a sum cordial graph.*



Sum cordial labeling of Closed helm CH_5

Theorem 2.3. *The quadrilateral snake Q_n is a sum cordial graph.*

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices and $e_1, e_2, \dots, e_{n-1}$ be the edges of a path P_n. To construct a quadrilateral snake Q_n from the path P_n, we join v_i and v_{i+1} to new vertices w_i and $w_i^{'}$ by edges $e_{2i-1}^{'} = v_i w_i,\ e_{2i}^{'} = v_{i+1} w_i^{'}$ and $e_i^{''} = w_i w_i^{'}$ for $i = 1, 2, \dots, n-1$. Then $|V(Q_n)| = 3n - 2$ and $|E(Q_n)| = 4n - 4$.
To define $f : V(Q_n) \to \{0, 1\}$, we consider the following cases,

n **is even**

$$f(v_i) = 1:\ 1 \le i \le n$$
$$f(w_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{2}; \\ 1, & \frac{n}{2} < i \le n-1. \end{cases}$$
$$f(w_i^{'}) = 0:\ 1 \le i \le n-1$$

Therefore, $v_f(0) = \frac{3n-2}{2} = v_f(1)$ and $e_f(0) = 2n-2 = e_f(1)$.
Therefore, $|v_f(0) - v_f(1)| = 0 = |e_f(0) - e_f(1)|$.

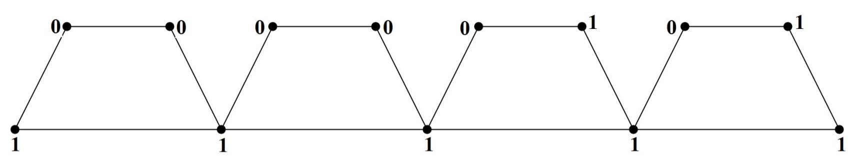
n **is odd**

$$f(v_i) = 1;\ 1 \le i \le n$$
$$f(w_i) = 0;\ 1 \le i \le n-1$$
$$f(w_i^{'}) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2}; \\ 1, & \frac{n-1}{2} < i \le n-1. \end{cases}$$

Therefore, $v_f(0) = \left\lfloor \frac{3n-2}{2} \right\rfloor$, $v_f(1) = \left\lceil \frac{3n-2}{2} \right\rceil$ and $e_f(0) = 2n-2 = e_f(1)$.
Therefore, $|v_f(0) - v_f(1)| = 1$ and $|e_f(0) - e_f(1)| = 0$.

Hence, Q_n is a sum cordial graph.

Example 2.3. *The quadrilateral snake Q_5 is a sum cordial graph.*



Sum cordial labeling of Quadrilateral snake Q_5

Theorem 2.4. *The double quadrilateral snake DQ_n is a sum cordial graph.*

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices and $e_1, e_2, \dots, e_{n-1}$ be the edges of the path P_n. To construct a double quadrilateral snake DQ_n from the path P_n, we join v_i and v_{i+1} to new vertices u_i, $u_i^{'}$, w_i and $w_i^{'}$ by edges $e^u_{2i-1} = v_i u_i$, $e^u_{2i} = v_{i+1} u_i^{'}$, $e^{uu}_i = u_i u_i^{'}$, $e^w_{2i-1} = v_i w_i$, $e^w_{2i} = v_{i+1} w_i^{'}$ and $e^{ww}_i = w_i w_i^{'}$ for $i = 1, 2, \dots, n-1$. Then $|V(DQ_n)| = 5n-4$ and $|E(DQ_n)| = 7n-7$.
Define $f : V(DQ_n) \to \{0, 1\}$ such that

$$f(v_i) = \begin{cases} 1, & i \equiv 1 \text{ or } 2(mod4); \\ 0, & i \equiv 0 \text{ or } 3(mod4). \end{cases} \quad 1 \le i \le n$$
$$f(u_i) = f(u_i^{'}) = \begin{cases} 1, & i \equiv 3(mod4); \\ 0, & \text{otherwise}. \end{cases} \quad 1 \le i \le n$$
$$f(w_i) = 1;\ 1 \le i \le n$$
$$f(w_i^{'}) = \begin{cases} 0, & i \equiv 1 \text{ or } 3(mod4); \\ 1, & i \equiv 0 \text{ or } 2(mod4). \end{cases} \quad 1 \le i \le n$$

Therefore,

For even n $v_f(0) = \frac{5n-4}{2} = v_f(1)$ and $e_f(0) = \left\lfloor \frac{7(n-1)}{2} \right\rfloor$, $e_f(1) = \left\lceil \frac{7(n-1)}{2} \right\rceil$.
Therefore, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

For odd n

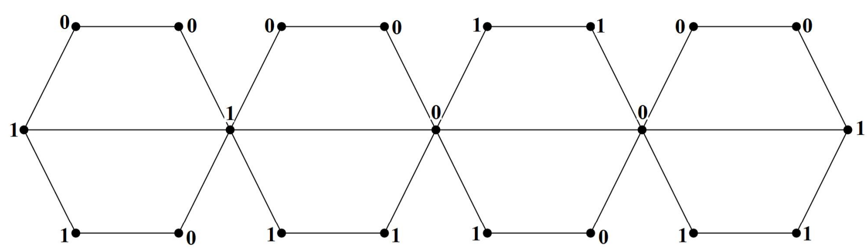
$$v_f(0) = \begin{cases} \left\lfloor \dfrac{5n-4}{2} \right\rfloor, & n \equiv 1(mod4); \\ \left\lceil \dfrac{5n-4}{2} \right\rceil, & n \equiv 3(mod4). \end{cases}$$

$$v_f(1) = \begin{cases} \left\lceil \dfrac{5n-4}{2} \right\rceil, & n \equiv 1(mod4); \\ \left\lfloor \dfrac{5n-4}{2} \right\rfloor, & n \equiv 3(mod4). \end{cases}$$

Also, $e_f(0) = \dfrac{7(n-1)}{2} = e_f(1)$.
Therefore, $|v_f(0) - v_f(1)| = 1$ and $|e_f(0) - e_f(1)| = 0$.

Hence, DQ_n is a sum cordial graph.

Example 2.4. *The double quadrilateral snake DQ_5 is a sum cordial graph.*



Sum cordial labeling of Double quadrilateral snake DQ_5

Theorem 2.5. *The gear graph G_n is a sum cordial graph.*

Proof: Let W_n be the wheel with an apex vertex v and rim vertices be $v_1, v_2, \ldots, v_n$. To obtain the gear graph G_n, subdivide each rim edge of wheel by the vertices $u_1, u_2, \ldots, u_n$, where each u_i sub divides the edge $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$ and u_n subdivides the edge $v_1 v_n$. Then $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$.
To define $f : V(G_n) \longrightarrow \{0, 1\}$, we consider the following two cases,

For even n Define

$$f(v) = 1$$

$$f(v_i) = \begin{cases} 1, & 1 \leq i \leq \dfrac{n}{2}; \\ 0, & \dfrac{n}{2} < i \leq n. \end{cases}$$

$$f(u_i) = \begin{cases} 1, & i \text{ is odd}; \\ 0, & i \text{ is even}. \end{cases}$$

Therefore, $v_f(0) = \left\lfloor \dfrac{2n+1}{2} \right\rfloor$, $v_f(1) = \left\lceil \dfrac{2n+1}{2} \right\rceil$, $e_f(0) = \dfrac{3n}{2} = e_f(1)$. Thus, we get $|v_f(0) - v_f(1)| \leq 1$, $|e_f(0) - e_f(1)| \leq 1$.

For odd n Define

$$f(v) = 1$$

$$f(v_1) = 1$$

$$f(v_i) = f(v_{n+2-i}) = \begin{cases} 1, & \text{if } i \text{ is odd}; \\ 0, & \text{if } i \text{ is even}. \end{cases} ; 2 \leq i \leq \dfrac{n+1}{2}$$

$$f(u_i) = \begin{cases} 1, & \text{if } i \text{ is odd except } i = \dfrac{n+1}{2}; \\ 0, & \text{otherwise}. \end{cases} ; 1 \leq i \leq n$$

Therefore, $v_f(0) = \left\lfloor \frac{2n+1}{2} \right\rfloor$, $v_f(1) = \left\lceil \frac{2n+1}{2} \right\rceil$ and

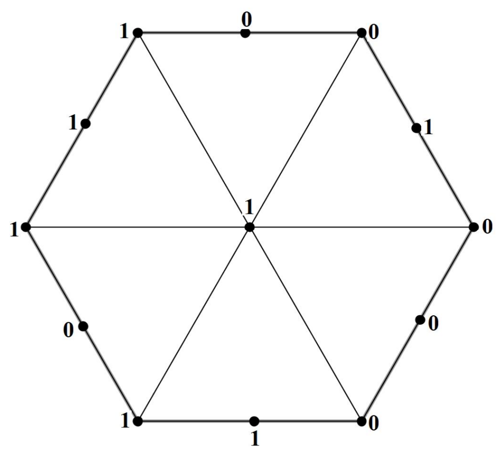
$$e_f(0) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } n \equiv 1(mod4); \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \equiv 3(mod4). \end{cases}$$

$$e_f(1) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \equiv 1(mod4); \\ \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } n \equiv 3(mod4). \end{cases}$$

Therefore, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, the gear G_n is a sum cordial graph.

Example 2.5. *The Gear G_6 is a sum cordial graph.*



Sum cordial labeling of Gear G_6

3 Conclusion

We contribute some new results on sum cordial labeling. The labeling pattern is demonstrated by means of examples. To derive similar results for other graph families and in the context of different labeling problems is an open area of research.

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A note on Civin-Yood Theorem for locally C*-algebras

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Abstract

In the present note we establish Civin-Yood Theorem for locally C^*-algebras, i.e. we show that if A be a locally C^*-algebra, and J be its closed Jordan ideal, then J is as well a closed two-sided *-ideals in A.

Keywords: C^*-algebras, locally C^*-algebras, projective limit of projective family of C^*-algebras.

2010 MSC: 46K05, 46K10

1 Introduction

Let A be a C^*-algebra, and J be a closed Jordan ideal in A. In 1965 in their paper [2] Civin and Yood proved among other things that J is a two-sided *-ideal in A.

The Hausdorff projective limits of projective families of Banach algebras as natural locally-convex generalizations of Banach algebras have been studied sporadically by many authors since 1952, when they were first introduced by Arens [1] and Michael [8]. The Hausdorff projective limits of projective families of C^*-algebras were first mentioned by Arens [1]. They have since been studied under various names by many authors. Development of the subject is reflected in the monograph of Fragoulopoulou [4]. We will follow Inoue [6] in the usage of the name **locally C^*-algebras** for these algebras.

The purpose of the present notes is to extend the aforementioned result of Civin and Yood from [2] to locally C^*-algebras.

2 Preliminaries

First, we recall some basic notions on topological *-algebras. A *-algebra (or involutory algebra) is an algebra A over $\mathbb{C}$ with an involution

$$^*: A \to A,$$

such that

$$(a + \lambda b)^* = a^* + \overline{\lambda} b^*,$$

and

$$(ab)^* = b^* a^*,$$

for every $a, b \in A$ and $\lambda \in \mathbb{C}$.

A seminorm $\|.\|$ on a *-algebra A is a C^*-seminorm if it is submultiplicative, i.e.

$$\|ab\| \leq \|a\| \, \|b\| \,,$$

and satisfies the C^*-condition, i.e.

$$\|a^* a\| = \|a\|^2 \,,$$

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for every $a, b \in A$. Note that the C^*-condition alone implies that $\|.\|$ is submultiplicative, and in particular

$$\|a^*\| = \|a\|,$$

for every $a \in A$ (cf. for example [4]).

When a seminorm $\|.\|$ on a *-algebra A is a C^*-norm, and A is complete in in the topology generated by this norm, A is called a C^***-algebra**.

A topological *-algebra is a *-algebra A equipped with a topology making the operations (addition, multiplication, additive inverse, involution) jointly continuous. For a topological *-algebra A, one puts $N(A)$ for the set of continuous C^*-seminorms on A. One can see that $N(A)$ is a directed set with respect to pointwise ordering, because

$$\max\{\|.\|_{\alpha}, \|.\|_{\beta}\} \in N(A)$$

for every $\|.\|_{\alpha}, \|.\|_{\beta} \in N(A)$, where $\alpha, \beta \in \Lambda$, with Λ being a certain directed set.

For a topological *-algebra A, and $\|.\|_{\alpha} \in N(A)$, $\alpha \in \Lambda$,

$$\ker \|.\|_{\alpha} = \{a \in A : \|a\|_{\alpha} = 0\}$$

is a *-ideal in A, and $\|.\|_{\alpha}$ induces a C^*-norm (we as well denote it by $\|.\|_{\alpha}$) on the quotient $A_{\alpha} = A/\ker \|.\|_{\alpha}$, and A_{α} is automatically complete in the topology generated by the norm $\|.\|_{\alpha}$, thus is a C^*-algebra (see [4] for details). Each pair $\|.\|_{\alpha}, \|.\|_{\beta} \in N(A)$, such that

$$\beta \succeq \alpha,$$

$\alpha, \beta \in \Lambda$, induces a natural (continuous) surjective *-homomorphism

$$g_{\alpha}^{\beta} : A_{\beta} \to A_{\alpha}.$$

Let, again, Λ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) " $\preceq$ ". Let

$$\{A_{\alpha}, \alpha \in \Lambda\}$$

be a family of C^*-algebras, and g_{α}^{β} be, for

$$\alpha \preceq \beta,$$

the continuous linear *-mappings

$$g_{\alpha}^{\beta} : A_{\beta} \longrightarrow A_{\alpha},$$

so that

$$g_{\alpha}^{\alpha}(x_{\alpha}) = x_{\alpha},$$

for all $\alpha \in \Lambda$, and

$$g_{\alpha}^{\beta} \circ g_{\beta}^{\gamma} = g_{\alpha}^{\gamma},$$

whenever

$$\alpha \preceq \beta \preceq \gamma.$$

Let Γ be the collections $\{g_{\alpha}^{\beta}\}$ of all such transformations. Let A be a *-subalgebra of the direct product algebra

$$_{\alpha \in \Lambda} A_{\alpha},$$

so that for its elements

$$x_{\alpha} = g_{\alpha}^{\beta}(x_{\beta}),$$

for all

$$\alpha \preceq \beta,$$

where

$$x_{\alpha} \in A_{\alpha},$$

and

$$x_{\beta} \in A_{\beta}.$$

The *-algebra A constructed above is called a **Hausdorff projective limit** of the projective family

$$\{A_\alpha, \alpha \in \Lambda\},$$

relatively to the collection

$$\Gamma = \{g_\alpha^\beta : \alpha, \beta \in \Lambda : \alpha \preceq \beta\},$$

and is denoted by

$$\varprojlim A_\alpha,$$

$\alpha \in \Lambda$, and is called the Arens-Michael decomposition of A.

It is well known (see, for example [11]) that for each $x \in A$, and each pair $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$, there is a natural projection

$$\pi_\beta : A \longrightarrow A_\beta,$$

defined by

$$\pi_\alpha(x) = g_\alpha^\beta(\pi_\beta(x)),$$

and each projection π_α for all $\alpha \in \Lambda$ is continuous.

A topological *-algebra (A, τ) over $\mathbb{C}$ is called a **locally C^*-algebra** if there exists a projective family of C^*-algebras

$$\{A_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\},$$

so that

$$A \cong \varprojlim A_\alpha,$$

$\alpha \in \Lambda$, i.e. A is topologically *-isomorphic to a projective limit of a projective family of C^*-algebras, i.e. there exits its Arens-Michael decomposition of A composed entirely of C^*-algebras.

A topological *-algebra (A, τ) over $\mathbb{C}$ is a locally C^*-algebra iff A is a complete Hausdorff topological *-algebra in which the topology τ is generated by a saturated separating family $\digamma$ of C^*-seminorms (see [4] for details).

Every C^*-algebra is a locally C^*-algebra.

A closed *-subalgebra of a locally C^*-algebra is a locally C^*-algebra.

The product ${}_{\alpha\in\Lambda}A_\alpha$ of C^*-algebras A_α, with the product topology, is a locally C^*-algebra.

Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$). Then the algebra $C(X)$ of all continuous, not necessarily bounded complex-valued functions on X, with the topology of uniform convergence on compact subsets, is a locally C^*-algebra. It is well known that all metrizable spaces and all locally compact Hausdorff spaces are compactly generated (see [7] for details).

Let A be a locally C^*-algebra. Then an element $a \in A$ is called **bounded**, if

$$\|a\|_\infty = \{\sup \|a\|_\alpha, \alpha \in \Lambda : \|.\|_\alpha \in N(A)\} < \infty.$$

The set of all bounded elements of A is denoted by $b(A)$.

It is well-known that for each locally C^*-algebra A, its set $b(A)$ of bounded elements of A is a locally C^*-subalgebra, which is a C^*-algebra in the norm $\|.\|_\infty$, such that it is dense in A in its topology (see for example [4]).

3 Civin-Yood Theorem for locally C*-algebras

Let us recall that a subspace J of an associative algebra A is called a **Jordan ideal** of A, if for each $a \in J$ and $b \in A$,

$$\frac{ab + ba}{2} = a \circ b \in J,$$

where the multiplication $a \circ b$ thus defined is called symmetric (see [5] for details).

Now we are ready to present the main theorem of the current notes.

Theorem 3.1. *Let (A, τ_A) be a locally C^*-algebra, and (J, τ_J) be a closed Jordan ideals in A, such that*

$$\tau_J = \tau_A | J.$$

Then (J, τ_J) is a closed two-sided *-ideal of A.

Proof. Let now (A, τ_A) be a locally C^*-algebra, and let

$$A = \varprojlim A_\alpha,$$

$\alpha \in \Lambda$, be its Arens-Michael decomposition into a projective limit of a projective family of C^*-algebras $A_\alpha, \alpha \in \Lambda$, built using the family of seminorms $\|.\|_\alpha, \alpha \in \Lambda$, that defines the topology τ_A. Let

$$\pi_\alpha : A \to A_\alpha,$$

$\alpha \in \Lambda$, be a projection from A onto A_α, for each $\alpha \in \Lambda$. Each π_α is an surjective *-homomorphism from A onto $A_\alpha, \alpha \in \Lambda$. Let

$$g_\alpha^\beta : A_\beta \to A_\alpha,$$

be a surjective *-homomorphism from A_β onto A_α, for each pair $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$. Such family g_α^β, $\alpha, \beta \in \Lambda$ does exist because the family $A_\alpha, \alpha \in \Lambda$ is projective. Let

$$J_\alpha = \pi_\alpha(J),$$

for each $\alpha \in \Lambda$. One can see now that

$$g_\alpha^\beta(J_\beta) = J_\alpha,$$

because

$$\pi_\alpha = g_\alpha^\beta \circ \pi_\beta,$$

for all $\alpha \preceq \beta, \alpha, \beta \in \Lambda$.

From the fact that J is a closed in τ_J topology subspace of A it follows that J_α is a closed in $\|.\|_\alpha$ subspace of A_α for all $\alpha \in \Lambda$.

We show now that J_α is a Jordan ideal of A_α for each $\alpha \in \Lambda$. In fact, let $a_\alpha \in J_\alpha$, and $b_\alpha \in A_\alpha$ be arbitrary, and $\alpha \in \Lambda$. We select arbitrary $a \in \pi_\alpha^{-1}(a_\alpha)$ which is obviously in J, and $b \in \pi_\alpha^{-1}(b_\alpha)$, which is obviously in A. Because J is a Jordan ideal of A it follows that

$$a \circ b = \frac{ab + ba}{2} \in J.$$

One can see that

$$\pi_\alpha(a) = a_\alpha \text{ and } \pi_\alpha(b) = b_\alpha.$$

Thus,

$$\begin{aligned} J_\alpha &\ni \pi_\alpha(a \circ b) = \pi_\alpha\left(\frac{ab + ba}{2}\right) = \frac{\pi_\alpha(ab + ba)}{2} = \frac{\pi_\alpha(ab) + \pi_\alpha(ba)}{2} \\ &= \frac{\pi_\alpha(a)\pi_\alpha(b) + \pi_\alpha(b)\pi_\alpha(a)}{2} = \frac{a_\alpha b_\alpha + b_\alpha a_\alpha}{2} = a_\alpha \circ b_\alpha. \end{aligned}$$

Now, applying to each J_α, $\alpha \in \Lambda$ Civin-Yood theorem from [2] we conclude that each J_α, $\alpha \in \Lambda$ is a two-sided *-ideal of A_α, i.e. for arbitrary $a_\alpha \in J_\alpha$ and $b_\alpha \in A_\alpha$ it follows that $a_\alpha b_\alpha, b_\alpha a_\alpha, a_\alpha^* \in J_\alpha$.

Let now $a \in J$ and $b \in A$ be arbitrary elements from J and A respectively. Then for each $\alpha \in \Lambda$,

$$J_\alpha \ni \pi_\alpha(a)\pi_\alpha(b) = \pi_\alpha(ab),$$

which implies that there exists a unique element $ab \in J$. Similarly we obtain that $ba \in J$.

At the same time for each $\alpha \in \Lambda$, even though generally speaking a^* exists in A, because

$$(\pi_\alpha(a))^* = \pi_\alpha(a^*) = a_\alpha^* \in J_\alpha,$$

which implies that $a^* \in J$. □

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Numerical solution of weakly singular integro-differential equations

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Abstract

In this work, we prove the existence and uniqueness of the solution of weakly singular integro-differential equations. After some transformations direct numerical schemes using collocation methods are proposed for any peicewise closed contours.

Keywords: Weakly singular integral equation, singular integral equation, approximation theory, collocation methods.

2010 MSC: 45D05, 45E05, 45L05, 45L10 and 65R20

1 Introduction

Singular integro-differential equations with logarithmic kernel arise in different problems of elasticity theory, aerodynamics, mechanics, elasticity, this kind of equations has gained a lot of interest in many application fields, in particular their numerical treatment is asked [1]. While several numerical methods for approximating the solution of Volterra integro-differential equations and Fredholm integro-differential equations are known [2, 4]. On the other hand, the singular integro-differential equations have poor documentation.

It is known that, the most effective methods for the approximate solution of weakly singular integro-differential equations consists in their reduction to a system of linear algebraic equations by the replacement of the integral with a proper quadrature sum [5, 6, 7].

Consider the weakly singular integro-differential equation of the form

$$\varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \ln(t - t_0)\varphi'(t)dt = f(t_0), \tag{1.1}$$

where Γ designates a smooth-oriented contour; t and t_0 are points on Γ and $f(t)$ is a given function on Γ, the density $\varphi(t)$ is the desired function has to satisfy the Holder condition $H(\mu)$ [6].

The equation (1) can be put in the form of functional equation

$$\varphi(t_0) + AD\varphi(t_0) = f(t_0), \tag{1.2}$$

with the linear mappings A and D respectively given by

$$A\varphi(t_0) = \frac{1}{\pi i}\int_\Gamma \ln(t - t_0)\varphi(t)dt, \qquad D\varphi(t) = \varphi'(t). \tag{1.3}$$

In this work we prove the existence and the uniqueness of the solution of the equation (1) and solve it numerically.

Let $\varepsilon > 0$ be a sufficiently small number and describe around t_0 a circle centred at t_0 with a radius ε this circle intersects the curve Γ in the two points t_1 and t_2 such that the arc lenghts t_1t_0 and t_0t_2 are equal to ε and denoting by Γ_ε this part of Γ limited by t_1t_2.

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2 Main results

Theorem: *Suppose that the function $\varphi(t) \in W^1(\Gamma)$, t and t_0 are points on the smooth-oriented contour Γ then, the equation (1) given by*

$$\varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \ln(t-t_0)\varphi'(t)dt = f(t_0),$$

admits a unique solution for all $f(t_0)$ in the given space.

Proof

The integration by parts for the operator $AD\varphi(t_0)$ in (2) gives

$$\begin{aligned}
\pi i AD\varphi(t_0) &= \int_{\Gamma-\Gamma_\varepsilon} \ln(t-t_0)\varphi'(t)dt \\
&= \varphi(t_1)\ln(t_1-t_0) - \varphi(t_2)\ln(t_2-t_0) - \int_{\Gamma-\Gamma_\varepsilon} \frac{\varphi(t)}{t-t_0}dt \\
&= \varphi(t)\left[\ln(t_1-t_0) - \ln(t_2-t_0)\right] + (\varphi(t_1)-\varphi(t_0))\ln(t_1-t_0) \\
&\quad + (\varphi(t_2)-\varphi(t_0))\ln(t_2-t_0) - \int_{\Gamma-\Gamma_\varepsilon} \frac{\varphi(t)}{t-t_0}dt.
\end{aligned}$$

The expansion $\varphi(t)\left[\ln(t_1-t_0) - \ln(t_2-t_0)\right]$ converges to $\pi i\varphi(t_0)$ as ε converges to zero, on the other hand the expansions

$(\varphi(t_1)-\varphi(t_0))\ln(t_1-t_0)$ and $(\varphi(t_2)-\varphi(t_0))\ln(t_2-t_0)$ converge to zero as ε goes to zero. Hence the integral becomes

$$\begin{aligned}
AD\varphi(t_0) &= \frac{1}{\pi i}\int_{\Gamma-\Gamma_\varepsilon} \ln(t-t_0)\varphi'(t)dt \\
&= \varphi(t_0) - \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt.
\end{aligned}$$

Therefore the equation (1)

$$\varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \ln(t-t_0)\varphi'(t)dt = f(t_0),$$

is transformed to the following equation

$$2\varphi(t_0) - \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt = f(t_0). \tag{2.4}$$

The equation (4) admits a unique solution for all second member, that is to say, the equation (1) admits a unique solution or all second member.

3 Numerical Experiments

In this section we describe some of the numerical experiments performed in solving the weakly singular integro-differential equations (1), using collocation methods with the approximation technical in [5,7]. In all cases, the curve is taking the unit circle and we chose the right hand side $f(t)$ in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with the method is correct.

In each table, φ represents the given exact solution of the weakly singular integro-differential equations and $\widetilde{\varphi}$ corresponds to the approximate solution of the equation produced by the approximation method for singular integral with logarithmic kernel in [5,7].

Example 1

Consider the weakly singular integro-differential equation on the unit circle Γ

$$t_0\varphi(t_0) + \int_\Gamma \ln(t-t_0)\varphi'(t)dt = t_0^3 - t_0^2,$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = t^2.$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the solution of a system of linear algebraic equations by the replacement of the integral with a proper quadrature sum.

Points of t	Exact solution	Approx solution	Error
1.0000	1.0000e+000	1.00e+000 +2.07e-014i	3.15e-014
3.68e-001 +9.29e-001i	-7.28e-001 +6.84e-001i	-7.28e-001 +6.84e-001i	3.37e-014
-7.70e-001 +6.37e-001i	1.87e-001 -9.82e-001i	1.87e-001 -9.82e-001i	3.19e-014
-8.44e-001 -5.35e-001i	4.25e-001 +9.04e-001i	4.25e-001 +9.04e-001i	2.86e-014
3.09e-001 -9.51e-001i	-8.09e-001 -5.87e-001i	-8.09e-001 -5.87e-001i	2.75e-014
9.98e-001 -6.27e-002i	9.92e-001 -1.25e-001i	9.92e-001 -1.25e-001i	3.25e-014

Table 1. The exact and approximate solutions of example 1 in some arbitrary points and the error

Example 2

Consider the weakly singular integro-differential equation on the unit circle Γ

$$\varphi(t_0) + \int_\Gamma \ln(t - t_0)\varphi'(t)dt = \frac{1}{t_0 + 2},$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t + 2}.$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the solution of a system of linear algebraic equations by the replacement of the integral with a proper quadrature sum.

Points of t	Exact solution	Approx solution	Error
1.0000	3.3333e-001	3.33e-001 -3.70e-007i	5.13e-007
3.68e-001 +9.29e-001i	3.65e-001 -1.43e-001i	3.65e-001 -1.43e-001i	1.64e-006
-7.70e-001 +6.37e-001i	6.41e-001 -3.32e-001i	6.41e-001 -3.32e-001i	1.73e-005
-8.44e-001 -5.35e-001i	7.12e-001 +3.30e-001i	7.12e-001 +3.30e-001i	2.64e-005
3.09e-001 -9.51e-001i	3.70e-001 +1.52e-001i	3.70e-001 +1.52e-001i	1.00e-006
9.98e-001 -6.27e-002i	3.33e-001 +6.98e-003i	3.33e-001 +6.98e-003i	8.75e-007

Table 2. The exact and approximate solutions of example 2 in some arbitrary points and the error

Example 3

Consider the weakly singular integro-differential equation on the unit circle Γ

$$\varphi(t_0) + \int_\Gamma \ln(t - t_0)\varphi'(t)dt = \frac{3}{t_0},$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t}.$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the solution of a system of linear algebraic equations by the replacement of the integral with a proper quadrature sum.

Points of t	Exact solution	Approx solution	Error
1.0000	1.000000e+000	9.99e-001 -1.61e-005i	5.82e-005
3.68e-001 +9.29e-001i	3.68e-001 -9.29e-001i	3.68e-001 -9.29e-001i	5.09e-005
-7.70e-001 +6.37e-001i	-7.70e-001 -6.37e-001i	-7.70e-001 -6.37e-001i	5.09e-005
-8.44e-001 -5.35e-001i	-8.44e-001 +5.35e-001i	-8.44e-001 +5.35e-001i	5.09e-005
3.09e-001 -9.51e-001i	3.09e-001 +9.51e-001i	3.09e-001 +9.51e-001i	5.82e-005
9.98e-001 -6.27e-002i	9.98e-001 +6.27e-002i	9.98e-001 +6.27e-002i	5.09e-005

Table 3. The exact and approximate solutions of example 2 in some arbitrary points and the error

4 Conclusion

In this work we remark the convergence of the method to the exact solution with a considerable accuracy for the weakly singular integro-differential equations. This numerical results show that the accuracy improves with increasing of the number of points on the curve. Finally we confirm that this method lead us to the good approximation of the exact solution.

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On local attractivity of nonlinear functional integral equations via measures of noncompactness

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Abstract

In this paper, we prove the local attractivity of solutions for a certain nonlinear Volterra type functional integral equations. We rely on a measure theoretic fixed point theorem of Dhage (2008) for nonlinear $\mathcal{D}$-set-contraction in Banach spaces. Finally, we furnish an example to validate all the hypotheses of our main result and to ensure the existence and ultimate attractivity of solutions for a numerical nonlinear functional integral equation.

Keywords: Measure of noncompactness, fixed point theorem, functional integral equation, attractivity of solutions.

2010 MSC: 45G10, 45G99.

1 Introduction

The last three decades witnessed the active area of research in the connotation of measure theoretic fixed point theory and its applications to the problems of nonlinear differential and integral equations. The novelty of this approach lies in the advantage that along with existence we obtain some additional information about some characterizations of the solutions automatically. Local and global stability of the solutions of certain functional integral equations is discussed via measures of noncompactness by many researchers (see, for instance, Banas and Goebel [3], Banas and Rzepka [4], Dhage [8, 9], Dhage and Ntouyas [13] and the references therein). Very recently, Dhage [8] derived an abstract fixed point result more general than Darbo [5] fixed point theorem using the notion of measures of noncompactness and applied to stability problem of certain nonlinear functional integral equations. See Dhage and Lakshmikantham [12] and the references therein. Inspired or motivated by the idea of $\mathcal{D}$-functions that given in the examples of Dhage [10, 11], we prove in this paper the local attractivity of solutions for a certain nonlinear Volterra type functional integral equations via Dhage's measure theoretic fixed point theorem.

We now consider the following generalized nonlinear functional integral equation (in short GNFIE)

$$x(t) = u(t, x(t)) + p\Big(\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds, \int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds\Big) \tag{1.1}$$

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for $t \in \mathbb{R}^+ = [0, \infty) \subset \mathbb{R}$, where $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $f, g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\gamma, \theta, \sigma, \eta : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions.

Notice that the functional integral equation (1.1) is "general" in the sense that it includes several classes of known integral equations discussed in the literature (see Banas and Rzepka [4], Dhage [8], Dhage and Ntouyas [13], O'Regan and Meehan [15], Krasnoselskii [14], Väth [16], Dhage [7, 8] and the references therein). In this paper, we intend to obtain solution of GNFIE (1.1) in the space $BC(\mathbb{R}^+, \mathbb{R})$ of all bounded and continuous real-valued functions on $\mathbb{R}^+$. We use a fixed point theorem of Dhage [8] involving general measures of noncompactness to prove the existence and ultimate attractivity of solutions of GNFIE (1.1) under certain new conditions. The results of this paper are new to the theory of nonlinear differential and integral equations.

2 Auxiliary Results

This section is devoted to presenting a few auxiliary results needed in the sequel. Assume that E is a Banach space with the norm $\|\cdot\|$ and the zero element θ. Denote by $B[x, r]$ the closed ball centered at x and with radius r. If X, Y are arbitrary subsets of E then the symbols λX and $X + Y$ stand for the usual algebraic operations on those sets. Moreover, we write $\overline{X}$, $\overline{\text{co}}\, X$ to denote the closure and the closed convex hull of X, respectively.

Further, let $\mathcal{P}_p(E)$ denote the class of all nonempty subsets of E with a property p. Here p may be p =closed (cl, in short), p =bounded (bd, in short), p =relatively compact (rcp, in short) etc. Thus, $\mathcal{P}_{cl}(E), \mathcal{P}_{bd}(E), \mathcal{P}_{cl,bd}(E)$ and $\mathcal{P}_{rcp}(E)$ denote respectively the classes of closed, bounded, closed and bounded and relatively compact subsets of E.

The axiomatic way of defining the concept of the measure of noncompactness has been adopted in several papers in the literature. See Akhmerov *et al.* [2], Deimling [6], Väth [16] and Zeidler [17]. In this paper, we adopt the following axiomatic definition of the measure of noncompactness in a Banach space given in Banas and Goebel [3] and Dhage [7, 8].

Definition 2.1. *A mapping $\mu : \mathcal{P}_{bd}(E) \to \mathbb{R}^+$ is called the measure of noncompactness in E if it satisfies the following conditions:*

1^o *The family* $\ker \mu = \{X \in \mathcal{P}_{bd}(E) : \mu(X) = 0\}$ *is nonempty and* $\ker \mu \subset \mathcal{P}_{rcp}(E)$.

2^o $\mu(\overline{X}) = \mu(X)$.

3^o $\mu(\overline{co}\, X) = \mu(X)$.

4^o $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

5^o *If $\{X_n\}$ is a decreasing sequence of sets in $\mathcal{P}_{cl,bd}(E)$ such that $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $\overline{X}_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.*

The family $\ker \mu$ described in 1^o is said to be *the kernel of the measure of noncompactness* μ. We refer to [2, 3, 4, 6, 16, 17] for further facts concerning the measures of noncompactness and their properties. Let us only observe that the intersection set X_∞ is a member of the family $\ker \mu$. Indeed, since $\mu(X_\infty) \le \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$. In view of 1^o this yields that $X_\infty \in \ker \mu$.

A measure μ of noncompactness is said to be sublinear if

6^o $\mu(X + Y) \le \mu(A) + \mu(B)$ for all $X, Y \in \mathcal{P}_{bd}(E)$, and

7^o $\mu(\lambda X) \le |\lambda| \mu(X)$ for all $\lambda \in \mathbb{R}$ and $X \in \mathcal{P}_{bd}(E)$.

Let $E = BC(\mathbb{R}^+, \mathbb{R})$ be the space of all continuous and bounded functions on $\mathbb{R}^+$ and define a norm $\|\cdot\|$ in E by

$$\|x\| = \sup\{|x(t)| : t \ge 0\}.$$

Clearly E is a Banach space with this supremum norm. Let us fix a bounded subset A of E and a positive real number T. For any $x \in A$ and $\epsilon \geq 0$, denote by $\omega^T(x,\epsilon)$, the modulus of continuity of x on the interval $[0,T]$ defined by

$$\omega^T(x,\epsilon) = \sup\{|x(t) - x(s)| : t,s \in [0,T], |t-s| \leq \epsilon\}.$$

Moreover, let

$$\omega^T(A,\epsilon) = \sup\{\omega^T(x,\epsilon) : x \in A\},$$

$$\omega_0^T(A) = \lim_{\epsilon \longrightarrow 0} \omega^T(A,\epsilon),$$

$$\omega_0(A) = \lim_{T \longrightarrow \infty} \omega_0^T(A).$$

By $A(t)$ we mean a set in $\mathbb{R}$ defined by $A(t) = \{x(t) | x \in A\}$ for $t \in \mathbb{R}^+$. We denote diam $(A(t)) = \sup\{|x(t) - y(t)| : x,y \in A\}$. Finally we define a function μ on $\mathcal{P}_{bd}(E)$ by the formula

$$\mu(A) = \omega_0(A) + \limsup_{t \longrightarrow \infty} \operatorname{diam}(A(t)). \tag{2.2}$$

It has been shown in Banas and Goebel [3] that μ is a sublinear measure of noncompactness in E. From the definition of the measure μ, it is clear that the thickness of the bundle of functions $A(t)$ tends to zero as t tends to ∞. This particular characteristic of μ has been utilized in formulating the main existence and attractiivity result of this paper.

Before going to the key tool used in this paper, we recall the following useful definition introduced by Dhage [8].

Definition 2.2. *A mapping $\mathcal{T} : E \to E$ is called $\mathcal{D}$-set-Lipschitz if there exists a upper semi-continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(\mathcal{T}(A)) \leq \varphi(\mu(A)$ for all $A \in \mathcal{P}_{bd}(E)$ with $\mathcal{T}(A) \in P_{bd}(E)$, where $\varphi(0) = 0$. The function φ is sometimes called a $\mathcal{D}$-function of $\mathcal{T}$ on E. Especially when $\varphi(r) = kr, k > 0$, U is called a k-set-Lipschitz mapping and if $k < 1$, then $\mathcal{T}$ is called a k-set-contraction on E. Further, if $\varphi(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a nonlinear $\mathcal{D}$-set-contraction on E.*

Lemma 2.1 (Dhage [8]). *If φ is a $\mathcal{D}$-function with $\varphi(r) < r$ for $r > 0$, then $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in [0,\infty)$ and vice-versa.*

Using Lemma 2.1, Dhage [8] proved the following important result.

Theorem 2.1. *Let C be a closed, convex and bounded subset of a Banach space E and let $\mathcal{T} : C \to C$ be a continuous and nonlinear $\mathcal{D}$-set-contraction. Then $\mathcal{T}$ has a fixed point.*

Remark 2.1. Let us denote by $\mathrm{Fix}(\mathcal{T})$ the set of all fixed points of the operator $\mathcal{T}$ which belong to C. It can be shown that the set $\mathrm{Fix}(\mathcal{T})$ existing in Theorem 2.1 belongs to the family $\ker \mu$. Indeed, if $\mathrm{Fix}(\mathcal{T}) \notin \ker \mu$, then $\mu(\mathrm{Fix}(\mathcal{T})) > 0$ and $\mathcal{T}(\mathrm{Fix}(\mathcal{T})) = \mathrm{Fix}(\mathcal{T})$. Now from nonlinear set-contractivity it follows that $\mu(\mathcal{T}(\mathrm{Fix}(\mathcal{T}))) \leq \phi(\mu(\mathrm{Fix}(\mathcal{T})))$ which is a contradiction since $\phi(r) < r$ for $r > 0$. Hence $\mathrm{Fix}\,(\mathcal{T}) \in \ker \mu$. This particular characteristic has been utilized in our study of local attractivity of the solutions of nonlinear integral equations.

3 Local Attractivity Results

In this section we prove our main existence and attractivity results for the GNFIE (1.1) under some suitable conditions. We need the following definition in what follows. Let us assume that $E = BC(\mathbb{R}^+, \mathbb{R})$ and let Ω be a subset of E. Let $\mathcal{T} : E \to E$ be an operator and consider the operator equation in E,

$$\mathcal{T}x(t) = x(t) \ \text{ for all } \ t \in \mathbb{R}^+. \tag{3.3}$$

Below we give an attractivity characterizations of the solutions for the operator equation (3.3) on $\mathbb{R}^+$.

Definition 3.3. *We say that solutions of the equation (3.3) are locally ultimately attractive if there exists a closed ball $B[x_0, r_0]$ in the space $BC(\mathbb{R}^+, \mathbb{R})$ for some $x_0 \in BC(\mathbb{R}^+, \mathbb{R})$ such that, for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (3.3) belonging to $B[x_0, r_0] \cap \Omega$, we have*

$$\lim_{t\to\infty} (x(t) - y(t)) = 0. \tag{3.4}$$

In case the limit (3.2) is uniform with respect to the set $B[x_0, r_0] \cap \Omega$, i.e., for each $\epsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \le \epsilon \tag{3.5}$$

for all solutions $x, y \in B[x_0, r_0] \cap \Omega$ of (3.3) and for $t \ge T$, we will then say that solutions of equation (3.3) are uniformly locally ultimately attractive on $\mathbb{R}^+$.

We consider the following set of hypotheses in the sequel.

(H_0) The functions $\gamma, \theta, \sigma, \eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous and $\lim_{t\longrightarrow\infty} \gamma(t) = 0$.

(H_1) The function $u : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|u(t, x) - u(t, y)| \le \varphi(|x - y|)$$

for each $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$. Moreover, we assume $\varphi(r) < r$ for $r > 0$.

(H_2) The function $U : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined by $U(t) = |u(t, 0)|$ is bounded with $c_1 = \sup_{t\ge0} U(t)$.

(H_3) The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist a constant $k > 0$ and a function φ as appears in (H_1) such that

$$|f(t, s, x) - f(t, s, y)| \le k\, \varphi(|x - y|)$$

for $t, s \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$.

(H_4) The function $F : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined by $F(t) = \displaystyle\int_0^{\gamma(t)} |f(t, s, 0)|ds$ is bounded with $c_2 = \sup_{t\ge0} F(t)$.

(H_5) The function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist functions $a, b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$|g(t, s, x)| \le a(t)b(s)$$

for $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Moreover, $\displaystyle\lim_{t\longrightarrow\infty} a(t) \int_0^{\sigma(t)} b(s)ds = 0$.

(H_6) The function $p : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following condition

$$|p(t_1, t_2) - p(t_1', t_2')| \le |t_1 - t_1'| + |t_2 - t_2'|$$

for all $t_1, t_2, t_1', t_2' \in \mathbb{R}$. Moreover, $p(0, 0) = 0$.

Remark 3.2. *Since the hypothesis (H_0) holds, there exists a constant $c_0 > 0$ such that $c_0 = \sup_{t\ge0} \gamma(t)$. Similarly, since (H_5) holds, the function $v : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $v(t) = a(t) \int_0^{\sigma(t)} b(s)ds$ is continuous and the number $c_3 = \sup_{t\ge0} v(t)$ exists.*

Theorem 3.2. *Assume that the hypotheses $(H_0) - (H_6)$ hold. Further if there exists a positive solution r_0 of the inequality*

$$(1 + c_0 k)\varphi(r) + q \le r, \tag{3.6}$$

where q is the constant defined by $q = \sum_{i=1}^{3} c_i$, then the GNFIE (1.1) has a solution and the solutions are uniformly locally ultimately attractive on $\mathbb{R}^+$.

Proof. Now consider the closed ball $B[0,r_0]$ in E centered at origin of radius r_0. Define the mapping $\mathcal{T}$ on E by

$$\mathcal{T}x(t) = u(t,x(t)) + p\Big(\int_0^{\gamma(t)} f(t,s,x(\theta(s)))ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\Big) \tag{3.7}$$

for $t \in \mathbb{R}^+$. We shall show that the map $\mathcal{T}$ satisfies all the conditions of Theorem 3.1 on E.

Step I: First we show that $\mathcal{T}$ defines a mapping $\mathcal{T} : E \longrightarrow E$. Since p,q,γ,σ are continuous, $\mathcal{T}x$ is continuous and hence it is measurable on $\mathbb{R}^+$ for each $x \in E$. As $\theta(\mathbb{R}^+) \subseteq \mathbb{R}^+$, we have $\max_{t\geq 0}|x(\theta(t))| \leq \max_{t\geq 0}|x(t)|$. On the other hand, hypotheses $(H_0)-(H_3)$ and (H_5) imply that

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq |u(t,x(t))| + \Big|p\Big(\int_0^{\gamma(t)} f(t,s,x(\theta(s)))ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\Big) - p(0,0)\Big| \\
&\leq |u(t,x(t)) - u(t,0)| + |u(t,0)| + \Big|\int_0^{\gamma(t)} f(t,s,x(\theta(s)))ds\Big| + \Big|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\Big| \\
&\leq \varphi(|x(t)|) + |u(t,0)| + \int_0^{\gamma(t)} |f(t,s,x(\theta(s)))|ds + \int_0^{\sigma(t)} |g(t,s,x(\eta(s)))|ds \\
&\leq \varphi(|x(t)|) + |u(t,0)| + \int_0^{\gamma(t)} |f(t,s,x(\theta(s))) - f(t,s,0)|ds + \int_0^{\gamma(t)} |f(t,s,0)|ds + \int_0^{\sigma(t)} a(t)b(s)ds \\
&\leq \varphi(\|x\|) + U(t) + k\gamma(t)\varphi(\|x\|) + c_2 + v(t) \\
&\leq (1 + c_0\, k)\varphi(\|x\|) + q,
\end{aligned}$$

for all $t \in \mathbb{R}^+$. Taking supemum over t, we obtain,

$$\|\mathcal{T}x\| \leq (1 + c_0\, k)\varphi(\|x\|) + q \leq r. \tag{3.8}$$

From (3.7), we deduce that $\mathcal{T}x \in E$ and $\mathcal{T}$ defines a mapping $\mathcal{T} : B[0,r_0] \to B[0,r_0]$.

Step II: We show that $\mathcal{T}$ is continuous on $B[0,r_0]$. Let $\epsilon > 0$ be given and let $x,y \in B[0,r_0]$ be such that $\|x-y\| \leq \epsilon$. Then by hypotheses $(H_1)-(H_5)$

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}y(t)| &\leq |u(t,x(t)) - u(t,y(t))| + \Big|p\Big(\int_0^{\gamma(t)} f(t,s,x(\theta(s)))ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\Big) \\
&\qquad - p\Big(\int_0^{\gamma(t)} f(t,s,y(\theta(s)))ds, \int_0^{\sigma(t)} g(t,s,y(\eta(s)))ds\Big)\Big| \\
&\leq \varphi(|x(t)-y(t)|) + \Big|\int_0^{\gamma(t)} [f(t,s,x(\theta(s))) - f(t,s,y(\theta(s)))]ds\Big| \\
&\quad + \Big|\int_0^{\sigma(t)} [g(t,s,x(\eta(s))) - g(t,s,y(\eta(s)))]ds\Big| \\
&\leq \varphi(|x(t)-y(t)|) + \int_0^{\gamma(t)} |f(t,s,x(\theta(s))) - f(t,s,y(\theta(s)))|ds \\
&\quad + \int_0^{\sigma(t)} |g(t,s,x(\eta(s))) - g(t,s,y(\eta(s)))|ds \\
&\leq \varphi(\|x-y\|) + k\gamma(t)\varphi(\|x-y\|) + 2\int_0^{\sigma(t)} a(t)b(s)ds \\
&\leq (1 + c_0 k)\varphi(\epsilon) + 2v(t) \\
&\leq (1 + c_0 k)\epsilon + 2v(t).
\end{aligned} \tag{3.9}$$

Since $v(t) \to 0$ as $t \to \infty$, there exists $T > 0$ such that $v(t) \leq \epsilon$, $\forall t > T$. Thus if $t > T$, then from (3.8) we have that

$$|\mathcal{T}x(t) - \mathcal{T}y(t)| \leq (3 + c_0 k)\epsilon.$$

If $t < T$, then define a function $\omega = \omega(\epsilon)$ by the formula

$$\omega(\epsilon) = \sup\{|g(t,s,x) - g(t,s,y)| : t,s \in [0,T], x,y \in [-r_0,r_0], |x-y| \leq \epsilon\}. \tag{3.10}$$

Now $g(t,s,x)$ is continuous and hence uniformly continuous on $[0,T] \times [0,T] \times [-r_0,r_0]$. As a result we have $\omega(\epsilon) \to 0$ as $\epsilon \to 0$. Therefore, from (3.10),

$$|\mathcal{T}x(t) - \mathcal{T}y(t)| \leq (1 + c_0 k)\epsilon + \sigma^* \omega(\epsilon)$$

for all $t \in \mathbb{R}^+$, where $\sigma^* = \max\{\sigma(t) : t \in [0, T]\}$. Hence, it follows that

$$\begin{aligned} \|\mathcal{T}x - \mathcal{T}y\| &\leq \max\{(3 + c_0 k)\epsilon, (1 + c_0 k)\epsilon + \sigma^* \omega(\epsilon)\} \\ &\to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

Hence $\mathcal{T}$ is a continuous mapping from $B[0, r_0]$ into itself.

Step III: Here we show that $\mathcal{T}$ is a nonlinear set-contraction on $B[0, r_0]$. This will be done in the following two cases:

Case I : Let $A \subset B[0, r_0]$ be non-empty. Further fix the number $T > 0$ and $\epsilon > 0$. Since the functions f and g are continuous on compact $[0, T] \times [0, T] \times [-r_0, r_0]$, there are constants $c_4 > 0$ and $c_5 > 0$ such that $|f(t, s, x)| \leq c_4$ and $|g(t, s, x)| \leq c_5$ for all $t, s \in [0, T]$ and $x \in [-r_0, r_0]$. Then choosing $t, \tau \in [0, T]$ such that $|t - \tau| \leq \epsilon$ and taking into account our hypotheses, we obtain

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}x(\tau)| &\leq |u(t, x(t)) - u(\tau, x(\tau))| + \Big| p\Big(\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds, \int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds\Big) \\ &\quad - p\Big(\int_0^{\gamma(\tau)} f(\tau, s, x(\theta(s)))ds, \int_0^{\sigma(\tau)} g(\tau, s, x(\eta(s)))ds\Big)\Big| \\ &\leq |u(t, x(t)) - u(t, x(\tau))| + |u(t, x(\tau)) - u(\tau, x(\tau))| \\ &\quad + \left|\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds - \int_0^{\gamma(\tau)} f(\tau, s, x(\theta(s)))ds\right| \\ &\quad + \left|\int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds - \int_0^{\sigma(\tau)} g(\tau, s, x(\eta(s)))ds\right| \\ &\leq \varphi(|x(t) - x(\tau)|) + |u(t, x(\tau)) - u(\tau, x(\tau))| \\ &\quad + \left|\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds - \int_0^{\gamma(t)} f(\tau, s, x(\theta(s)))ds\right| \\ &\quad + \left|\int_0^{\gamma(t)} f(\tau, s, x(\theta(s)))ds - \int_0^{\gamma(\tau)} f(\tau, s, x(\theta(s)))ds\right| \\ &\quad + \left|\int_{\sigma(t)}^{\sigma(\tau)} g(t, s, x(\eta(s)))ds\right| \\ &\quad + \left|\int_0^{\sigma(\tau)} [g(t, s, x(\eta(s)))ds - g(\tau, s, x(\eta(s)))]ds\right| \\ &\leq \varphi(|x(t) - x(\tau)|) + |u(t, x(\tau)) - u(\tau, x(\tau))| \\ &\quad + \int_0^{\gamma(t)} |f(t, s, x(\theta(s))) - f(\tau, s, x(\theta(s)))|ds \\ &\quad + \left|\int_{\gamma(\tau)}^{\gamma(t)} |f(\tau, s, x(\theta(s)))|\, ds\right| + \left|\int_{\sigma(t)}^{\sigma(\tau)} |g(t, s, x(\eta(s)))|\, ds\right| \\ &\quad + \int_0^{\sigma(\tau)} |g(t, s, x(\eta(s)))ds - g(\tau, s, x(\eta(s)))|ds \\ &\leq \varphi(|x(t) - x(\tau)|) + \omega^T(u, \epsilon) + k\omega^T(f, \epsilon) + c_4 \omega^T(\gamma, \epsilon) \\ &\quad + T\omega^T(g, \epsilon) + c_5 \omega^T(\sigma, \epsilon), \end{aligned}$$

where

$$\begin{aligned} \omega^T(\gamma, \epsilon) &= \sup\{|\gamma(t) - \gamma(\tau)| : t, \tau \in [0, T], |t - \tau| \leq \epsilon\}, \\ \omega^T(\sigma, \epsilon) &= \sup\{|\sigma(t) - \sigma(\tau)| : t, \tau \in [0, T], |t - \tau| \leq \epsilon\}, \end{aligned}$$

$$\begin{aligned}\omega^T(u,\epsilon) &= \sup\{|u(t,x)-u(\tau,x)| : t,\tau\in[0,T], |t-\tau|\le\epsilon, |x|\le r_0\},\\ \omega^T(f,\epsilon) &= \sup\{|f(t,s,x)-f(\tau,s,x)| : t,\tau\in[0,T], |t-\tau|\le\epsilon, |x|\le r_0\},\\ \omega^T(g,\epsilon) &= \sup\{|g(t,s,x)-g(\tau,s,x)| : t,\tau\in[0,T], |t-\tau|\le\epsilon, |x|\le r_0\}.\end{aligned}$$

The above inequality further implies that

$$\begin{aligned}\omega^T(\mathcal{T}x,\epsilon) &\le \varphi(\omega^T(x,\epsilon)) + \omega^T(u,\epsilon) + c_0k\omega^T(f,\epsilon)\\ &\quad + c_4\omega^T(\gamma,\epsilon) + T\omega^T(g,\epsilon) + c_5\omega^T(\sigma,\epsilon).\end{aligned} \tag{3.11}$$

Since by hypotheses, the functions u,φ,γ,σ and f,g are continuous respectively on $[0,T]$ and $[0,T]\times[0,T]\times[-r_0,r_0]$, we infer that they are uniformly continuous there. Hence we deduce that $\varphi(\omega^T(x,\epsilon))\to 0, \omega^T(u,\epsilon)\to 0, \omega^T(\gamma,\epsilon)\to 0, \omega^T(f,\epsilon)\to 0, \omega^T(g,\epsilon)\to 0$ as $\epsilon\to 0$. Hence from the above estimate (3.11), we obtain

$$\omega_0^T(\mathcal{T}(A)) = 0,$$

and consequently

$$\omega_0(\mathcal{T}(A)) = 0. \tag{3.12}$$

Case II: Now for any $x,y\in A$ one has:

$$\begin{aligned}|\mathcal{T}x(t)-\mathcal{T}y(t)| &\le |u(t,x(t))-u(t,y(t))| + \Big|p\Big(\int_0^{\gamma(t)} f(t,s,x(\theta(s)))ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\Big)\\ &\quad - p\Big(\int_0^{\gamma(t)} f(t,s,y(\theta(s)))ds, \int_0^{\sigma(t)} g(t,s,y(\eta(s)))ds\Big)\Big|\\ &\le \varphi(|x(t)-y(t)|) + \Big|\int_0^{\gamma(t)} f(t,s,x(\theta(s))) - f(t,s,y(\theta(s)))ds\Big|\\ &\quad + \Big|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds - g(t,s,y(\eta(s)))ds\Big|\\ &\le \varphi(\text{diam}\,(A(t))) + \int_0^{\gamma(t)} |f(t,s,x(\theta(s))) - f(t,s,y(\theta(s)))|ds\\ &\quad + \int_0^{\sigma(t)} |g(t,s,x(\eta(s)))ds - g(t,s,y(\eta(s)))|ds\\ &\le \varphi(\text{diam}\,(A(t))) + k\int_0^{\gamma(t)} |x(\theta(s))-y(\theta(s))|\,ds + 2v(t)\\ &\le \varphi(\text{diam}\,(A(t))) + k\int_0^{\gamma(t)} \text{diam}\,A(\theta(s))ds + 2v(t)\\ &\le \varphi(\text{diam}\,(A(t))) + k\int_0^{\gamma(t)} \text{diam}\,(A)\,ds + 2v(t)\\ &\le \varphi(\text{diam}\,(A(t))) + k\gamma(t)\text{diam}\,(A)\,ds + 2v(t).\end{aligned}$$

As a result of the above inequality we obtain

$$\text{diam}\,(\mathcal{T}(A(t))) \le \varphi(\text{diam}\,(A(t))) + k\gamma(t)\text{diam}\,(A) + 2v(t).$$

Taking the limit superior as $t\to\infty$ in the above inequality yields

$$\begin{aligned}\limsup_{t\to\infty} \text{diam}\,(\mathcal{T}(A(t))) &\le \varphi\left(\limsup_{t\to\infty} \text{diam}\,(A(t))\right)\\ &\quad + k\limsup_{t\to\infty}\gamma(t)\,\text{diam}\,A + 2\limsup_{t\to\infty} v(t).\end{aligned}$$

Since both the limits, namely $\lim_{t\to\infty} v(t)$ and $\lim_{t\to\infty}\gamma(t)$ exist and each one is equal to 0, it follows that $\limsup_{t\to\infty} v(t) = 0$ and $\limsup_{t\to\infty}\gamma(t) = 0$. Hence, from the above inequality, we have

$$\limsup_{t\to\infty} \text{diam}\,(\mathcal{T}(A(t))) \le \varphi\left(\limsup_{t\to\infty} \text{diam}\,(A(t))\right). \tag{3.13}$$

Now from the inequalities (3.12), (3.13) and the definition of μ it follows that

$$\begin{aligned}\mu(\mathcal{T}(A)) &= \omega_0(\mathcal{T}(A)) + \limsup_{t\to\infty} \operatorname{diam}(\mathcal{T}(A(t)))\\ &\leq \varphi\left(0 + \limsup_{t\to\infty} \operatorname{diam}(A(t))\right)\\ &\leq \varphi\left(\omega_0(A) + \limsup_{t\to\infty} \operatorname{diam}(A(t))\right),\end{aligned}$$

or, equivalently,

$$\mu(\mathcal{T}(A)) \leq \varphi(\mu(A)), \tag{3.14}$$

where μ is the measure of noncompactness defined in the space $BC(\mathbb{R}^+, \mathbb{R})$. This shows that $\mathcal{T}$ is a nonlinear $\mathcal{D}$-set-contraction on $B[0, r_0]$. Thus, the map $\mathcal{T}$ satisfies all the conditions of Theorem 2.2 with $C = B[0, r_0]$ and an application of it yields that $\mathcal{T}$ has a fixed point in $B[0, r_0]$. This further by definition of $\mathcal{T}$ implies that the GNFIE (1.1) has a solution in $B[0, r_0]$. Moreover, taking into account that the image of $B[0, r_0]$ under the operator $\mathcal{T}$ is again contained in the ball $B[0, r_0]$ we infer that the set $\mathcal{F}(\mathcal{T})$ of all fixed points of $\mathcal{T}$ is contained in $B[0, r_0]$. If the set $\mathcal{F}(\mathcal{T})$ contains all solutions of the equation (1.1), then we conclude from Remark 2.1 that the set $\mathcal{F}(\mathcal{T})$ belongs to the family $\ker \mu$. Now, taking into account the description of sets belonging to $\ker \mu$ (given in Section 2) we deduce that all solutions of the equation (1.1) are uniformly locally ultimately attractive on $\mathbb{R}^+$. This completes the proof. □

4 An Example

As an application, we consider the following nonlinear functional integral equation

$$\begin{aligned}x(t) = \frac{1}{1+t}\ln\left(1+\frac{1}{2}|x(t)|\right) &+ \int_0^{\frac{t^2}{t^3+1}} \left(\frac{1+t}{1+t+t^2}\right)\ln\left(1+\frac{1}{2}|x|\right)ds\\ &+ \int_0^{\frac{t}{1+t}} \exp(-t^2)\frac{s^2\cos x(s)}{1+|\sin x(s)|}ds,\end{aligned} \tag{4.15}$$

for all $t \in \mathbb{R}^+$.

Let

$$p(t,t') = t+t',\ \varphi(t) = \ln\left(1+\frac{1}{2}t\right),\ \theta(t) = t^2+1,\ \sigma(t) = \frac{t}{1+t},\ \eta(t) = t,$$

$$\gamma(t) = \frac{t^2}{t^3+1},\ u(t,x) = \frac{1}{1+t}\ln\left(1+\frac{1}{2}|x(t)|\right),\ a(t) = (1+t)^3\exp(-t),\ b(s) = s^2,$$

for all $t, t', s \in \mathbb{R}^+$, and

$$f(t,s,x) = \left(\frac{1+t}{1+t+t^2}\right)\ln\left(1+\frac{1}{2}|x|\right),$$

$$g(t,s,x) = (1+t)^3\exp(-t)\frac{s^2\cos x(s)}{1+|\sin x(s)|}$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Notice that:

(i) The functions $\gamma, \theta, \sigma, \eta$ are obviously continuous and $\lim_{t\to\infty}\gamma(t) = \lim_{t\to\infty}\frac{t^2}{1+t^3} = 0$. Also $c_0 = \sup_{t\geq 0}\gamma(t) = \sup_{t\geq 0}\frac{t^2}{1+t^3} \approx 0.441$.

(ii) For arbitrary fixed $x, y \in \mathbb{R}$ we have

$$\begin{aligned} |u(t,x)-u(t,y)| &= \frac{1}{1+t}\left|\ln\left(1+\frac{1}{2}|x|\right)-\ln\left(1+\frac{1}{2}|y|\right)\right| \\ &\leq \ln\frac{1+\frac{1}{2}|x|}{1+\frac{1}{2}|y|} \leq \ln\left(1+\frac{1}{2}\cdot\frac{|x|-|y|}{1+\frac{1}{2}|y|}\right) \\ &< \ln\left(1+\frac{1}{2}|x-y|\right) \\ &= \varphi(|x-y|). \end{aligned}$$

Therefore, hypothesis (H_1) is satisfied with $\varphi(r) = \ln\left(1+\frac{1}{2}r\right) < r$, for $r > 0$.

(iii) (H_2) is satisfied since $U(t) = |u(t,0)| = 0$ and $c_1 = \sup_{t\geq 0}|u(t,0)| = 0$.

(iv) For arbitrary fixed $x, y \in \mathbb{R}$ such that $|x| \geq |y|$ and for $t > 0$ we obtain

$$|f(t,s,x)-f(t,s,y)| = \left(\frac{1+t}{1+t+t^2}\right)\ln\frac{1+\frac{1}{2}|x|}{1+\frac{1}{2}|y|} \leq \varphi(|x-y|),$$

as in (ii). The case is similar when $|y| \geq |x|$. Thus (H_3) is satisfied with $k = 1$ and $\varphi(r) = \ln\left(1+\frac{1}{2}r\right) < r$, for $r > 0$.

(v) Next, hypothesis (H_4) is satisfied, since the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$F(t) = \int_0^{\gamma(t)} |f(t,s,0)|ds = \int_0^{\frac{t^2}{t^3+1}} 0\, ds = 0$$

is bounded with $c_2 = \sup_{t\geq 0} F(t) = 0$.

(vi) The function g acts continuously from the set $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ into $\mathbb{R}$. Moreover, we have

$$|g(t,s,x)| \leq (1+t)^3 \exp(-t)\, s^2 = a(t)b(s),$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$, then we can see that hypothesis (H_5) is satisfied. Indeed, we have

$$\begin{aligned} \lim_{t\to\infty} a(t)\int_0^{\frac{t}{1+t}} b(s)ds &= \lim_{t\to\infty}(1+t)^3\exp(-t)\int_0^{\frac{t}{1+t}} s^2 ds \\ &= \frac{1}{3}\lim_{t\to\infty} t^3\exp(-t) = 0. \end{aligned}$$

Also we have $c_3 = \sup_{t\geq 0}\frac{1}{3}t^3\exp(-t) \approx 0.37$.

(vii) Obviously, hypothesis (H_6) is satisfied.

The inequality

$$(1+c_0k)\varphi(r) + q \leq r$$

reduces to the form

$$(1+0.441)\ \ln\left(1+\frac{1}{2}r\right) + 0.37 \leq r.$$

It is easily seen that each number $r \geq 0.6$ satisfies the above inequality. Thus, as the number r_0 we can take $r_0 = 0.6$. Note that this estimate of r_0 can be improved.

Keeping in view the above observations, we find that the functions γ, φ, θ, σ, η, u, f, g,a and b satisfy all the conditions of Theorem 3.2 and hence the GNFIE (4.1) has at least one solution in the space $BC(\mathbb{R}^+,\mathbb{R})$ and the solutions of the equation (4.1) are uniformly locally ultimately attractive on $\mathbb{R}^+$ located in the ball $B[0,0.6]$.

Remark 4.3. We remark that:

(i) Taking $u(t, x(t)) = q(t), p(t, t') = t + t'$ for all $t, t' \in \mathbb{R}^+$ and for any $x \in \mathbb{R}$ the generalized nonlinear functional integral equation (4.1) reduces to the nonlinear functional integral equation considered by Dhage [8] which, in turn, includes several classes of known integral equations discussed in the literature.

(ii) Taking $p(t, t') = t'$, $\gamma(t) = t$ and $\theta(s) = s$ for all $t, s \in \mathbb{R}^+$, we retrieve the functional integral equation studied by Aghajani, Banas and Sabzali [1].

(iii) The authors in [1] generalized Theorem 2.2 under the weaker upper semi-continuity of the $\mathcal{D}$-function ψ and the requirement of the condition that $\lim_{n\to\infty} \psi^n(r) = 0$ for all $t > 0$, however to hold this condition, they needed an additional condition on the function ψ that $\psi(r) < r$ for $r > 0$. But in actual practice, it is very difficult to verify this condition and the authors in [1] did not provide any example of the function ψ illustrating the comparison between two conditions in applications. Moreover, for applications to the existence result, they assumed an additional condition on the function ψ, namely, supperadditivity which automatically yields the upper semi-continuity together with the monotone characterization of the function ψ and so, the existence theorem for the nonlinear integral equation considered in Aghajani *et.al.* [1] follows by a direct application of Theorem 2.2 of Dhage [8].

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Some Results for the Bessel transform

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Abstract

In this paper, using a Bessel generalized translation , we prove the estimates for the Bessel transform in the space $L^2_p(\mathbb{R}_+)$ on certain classes of functions.

Keywords: Bessel operator; Bessel transform; Bessel generalized translation .

2010 MSC: 42B37.

1 Introduction and preliminaries

Integral transforms and their inverses (e.g., the Bessel transform) are widely used to solve various problems in calculus, mechanics, mathemtical physics, and computational mathematics (see, e.g.,[3, 8]).

In [7], E.C. Titchmarsh characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition for the Fourier transform, namely we have

Theorem 1.1. *Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents*

1. $\|f(x+h) - f(x)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$,
2. $\int_{|\lambda|\geq r} |\mathcal{F}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *as* $r \longrightarrow +\infty$,

where $\mathcal{F}$ stands for the Fourier transform of f.

The main aim of this paper is to establish a generalization of Theorem 1.1 in the Bessel transform setting by means of the Bessel generalized translation. We point out that similar results have been established in the context of noncompact rank 1 Riemannian symmetric spaces and of Jacobi transform (see [2, 6]).

In this section, we give some definition and preliminaries concerning the Bessel transform. Everywhere below p is a real number, $p \geq -\frac{1}{2}$.

Let

$$D = \frac{d^2}{dx^2} + \frac{(2p+1)}{x}\frac{d}{dx}$$

be the Bessel differential operator. We introduce the normalized Bessel function of the first kind j_p defined by

$$j_p(z) = \Gamma(p+1)\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+p+1)}\left(\frac{z}{2}\right)^{2n}, z \in \mathbb{C}, \tag{1.1}$$

where $\Gamma(x)$ is the gamma-function (see[4]). The function $y = j_p(x)$ satisfies the differential equation

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$$\mathrm{D}y + y = 0$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. The function $j_p(x)$ is infinitely differentiable, even, and, moreover entire analytic.

From (1.1) we see that

$$\lim_{z \longrightarrow 0} \frac{j_p(z) - 1}{z^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Longrightarrow |j_p(z) - 1| \geq c|z|^2 \tag{1.2}$$

From [1], we have

$$|j_p(x)| \leq 1. \tag{1.3}$$

and

$$1 - j_p(x) = O(x^2),\ 0 \leq x \leq 1. \tag{1.4}$$

Assume that $\mathrm{L}^2_p(\mathbb{R}_+)$, $p \geq -\frac{1}{2}$, is the Hilbert space of measurable functions $f(x)$ on $\mathbb{R}_+$ with the finite norm

$$\|f\| = \|f\|_{2,p} = \left(\int_0^\infty |f(x)|^2 x^{2p+1} dx \right)^{1/2}$$

Given $f \in \mathrm{L}^2_p(\mathbb{R}_+)$, the Bessel transform is defined by

$$\widehat{f}(\lambda) = \int_0^\infty f(t) j_p(\lambda t) t^{2p+1} dt,\ \lambda \in \mathbb{R}_+.$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^p \Gamma(p+1))^{-2} \int_0^\infty \widehat{f}(\lambda) j_p(\lambda t) \lambda^{2p+1} d\lambda.$$

From [3], we have the Parseval's identity

$$\int_0^\infty |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda = 2^{2p} \Gamma^2(p+1) \int_0^\infty |f(t)|^2 t^{2p+1} dt.$$

In $\mathrm{L}^2_p(\mathbb{R}_+)$, consider the Bessel generalized translation T_h (see [3, p. 121])

$$\mathrm{T}_h f(x) = c_p \int_0^\pi f(\sqrt{x^2 + h^2 - 2xhcost}) sin^{2p} t dt,\ \ p \geq -\frac{1}{2},\ h > 0,$$

where

$$c_p = \left(\int_0^\pi sin^{2p} t dt \right)^{-1} = \frac{\Gamma(p+1)}{\Gamma(\frac{1}{2})\Gamma(p+\frac{1}{2})}$$

From [5], we note importants properties of Bessel transform

$$\widehat{(\mathrm{D}f)}(\lambda) = (-\lambda^2) \widehat{f}(\lambda). \tag{1.5}$$

and

$$\widehat{(\mathrm{T}_h f)}(\lambda) = j_p(\lambda h) \widehat{f}(\lambda). \tag{1.6}$$

We define the differences of first and higher orders as

$$\Delta_h f(x) = \mathrm{T}_h f(x) - f(x) = (\mathrm{T}_h - \mathrm{E}) f(x)$$

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (\mathrm{T}_h - \mathrm{E})^k f(x) = \sum_{i=1}^{\infty} (-1)^{k-i} \binom{k}{i} \mathrm{T}_h^i f(x), \tag{1.7}$$

where $\mathrm{T}_h^0 f(x) = f(x)$, $\mathrm{T}_h^i f(x) = \mathrm{T}_h(\mathrm{T}_h^{i-1} f(x))$, $i = 1, 2, .., k$; k=1,2,.... and E is the unit operator in the space $\mathrm{L}_p^2(\mathbb{R}_+)$.

2 Main results

Lemma 2.1. *For $f \in \mathrm{L}_p^2(\mathbb{R}_+)$. Then*

$$\|\Delta_h^k \mathrm{D}^r f(x)\|^2 = \int_0^{\infty} t^{4r} |j_p(th) - 1|^{2k} |\widehat{f}(t)|^2 t^{2p+1} dt$$

Proof From formula (1.5), we have

$$\widehat{(\mathrm{D}^r f)}(t) = (-1)^r t^{2r} \widehat{f}(t); \; r = 0, 1, 2, ... \tag{2.8}$$

We use formulas (1.6) and (2.8), we conclude that

$$\widehat{\mathrm{T}_h^i \mathrm{D}^r f}(t) = (-1)^r j_p^i(th) t^{2r} \widehat{f}(t); \; 1 \le i \le k. \tag{2.9}$$

Or, from formulas (1.7) and (2.9) the image $\Delta_h^k \mathrm{D}^r f(x)$ under the Bessel transform has the form

$$\widehat{\Delta_h^k \mathrm{D}^r f}(t) = (-1)^r (j_p(th) - 1)^k t^{2r} \widehat{f}(t).$$

By Parseval's identity, we have the result.

Our main result is as follows

Theorem 2.2. *Let $f \in \mathrm{L}_p^2(\mathbb{R}_+)$. Then the following are equivalents*

1. $\|\Delta_h^k \mathrm{D}^r f(x)\| = O(h^{\alpha})$ *as* $h \longrightarrow 0$, $(0 < \alpha < k)$
2. $\int_s^{\infty} t^{4r} |\widehat{f}(t)|^2 t^{2p+1} dt = O(s^{-2\alpha})$ *as* $s \longrightarrow +\infty$

Proof 1) $\Longrightarrow$ 2) Suppose that

$$\|\Delta_h^k \mathrm{D}^r f(x)\| = O(h^{\alpha}) \; as \; h \longrightarrow 0$$

From Lemma 2.1, we have

$$\|\Delta_h^k \mathrm{D}^r f(x)\|^2 = \int_0^{\infty} t^{4r} |j_p(th) - 1|^{2k} |\widehat{f}(t)|^2 t^{2p+1} dt.$$

By formula (1.2), we obtain

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} t^{4r} |j_p(th) - 1|^{2k} |\widehat{f}(t)|^2 t^{2p+1} dt \ge \frac{c^{2k} \eta^{4k}}{2^{4k}} \int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} t^{4r} |\widehat{f}(t)|^2 t^{2p+1} dt.$$

There exists then a positive constant C such that

$$\begin{aligned} \int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} t^{4r} |\widehat{f}(t)|^2 t^{2p+1} dt &\le C \int_0^{\infty} t^{4r} |j_p(th) - 1|^{2k} |\widehat{f}(t)|^2 t^{2p+1} dt \\ &\le C h^{2\alpha} \end{aligned}$$

Then

$$\int_s^{2s} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt = O(s^{-2\alpha})$$

for all $s > 0$.

Moreover, we have

$$\begin{aligned}\int_s^{\infty} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt &= \sum_{j=0}^{\infty}\int_{2^js}^{2^{j+1}s} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt\\ &\leq C\sum_{j=0}^{\infty}(2^js)^{-2\alpha}\\ &\leq Cs^{-2\alpha}.\end{aligned}$$

This proves that

$$\int_s^{\infty} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt = O(s^{-2\alpha})\ as\ s \longrightarrow +\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_s^{\infty} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt = O(s^{-2\alpha})\ as\ s \longrightarrow +\infty.$$

We have to show that

$$\int_0^{\infty} t^{4r}|j_p(th)-1|^{2k}|\widehat{f}(t)|^2 t^{2p+1}dt = O(h^{2\alpha})\ as\ h \longrightarrow 0.$$

We write

$$\int_0^{\infty} t^{4r}|j_p(th)-1|^{2k}|\widehat{f}(t)|^2 t^{2p+1}dt = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_0^{1/h} t^{4r}|j_p(th)-1|^{2k}|\widehat{f}(t)|^2 t^{2p+1}dt$$

and

$$\mathrm{I}_2 = \int_{1/h}^{\infty} t^{4r}|j_p(th)-1|^{2k}|\widehat{f}(t)|^2 t^{2p+1}dt$$

From formula (1.3), we obtain

$$\mathrm{I}_2 \leq 4^k\int_{1/h}^{\infty} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt = O(h^{2\alpha})\ as\ h \longrightarrow 0.$$

Set

$$\psi(t) = \int_t^{\infty} x^{4r}|\widehat{f}(x)|^2 x^{2p+1}dx$$

From formula (1.4) and integration by parts, we have

$$\begin{aligned}\mathrm{I}_1 &= -\int_0^{1/h}|j_p(th)-1|^{2k}|\psi'(t)dt\\ &\leq -h^{2k}\int_0^{1/h} t^{2k}\psi'(t)dt\\ &\leq -\psi(\frac{1}{h}) + 2kh^{2k}\int_0^{1/h} t^{2k-1-2\alpha}dt\end{aligned}$$

Or, we see that $\alpha < k$ the integral exists. Then

$$\begin{aligned} I_1 &\leq \frac{2k}{2k-2\alpha} h^{2k} h^{-2k+2\alpha} \\ &\leq Ch^{2\alpha} \end{aligned}$$

and this ends the proof.

Corollary 2.1. *Let* $f \in L^2_p(\mathbb{R}_+)$, $(p \geq -\frac{1}{2})$, *and let*

$$\|\Delta_h^k D^r f(x)\| = O(h^\alpha) \text{ as } h \longrightarrow 0.$$

Then

$$\int_s^\infty |\widehat{f}(t)|^2 t^{2p+1} dt = O(s^{-4r-2\alpha}) \text{ as } s \longrightarrow +\infty$$

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Remarks on rg-compact, gpr-compact and gpr-connected spaces

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Abstract

We give some characterizations of rg-compact, gpr-compact and gpr-connected spaces by utilizing rg-open, gpr-open and gpr-closed sets. The paper is closely related to [A.M.Al-Shibani, rg-compact spaces and rg-connected spaces, Mathematica Pannonica, 17/1 (2006), 61-68], [Y.Gnanambal and K.Balachandran, On gpr-continuous functions in topological spaces,Indian .J.Pure appl.Math., 30(6) (1999),581-593] and [P.Gnanachandra et. al., Ultra Scientist, 24(1) A (2012), 185-191]

Keywords: rg-closed, rg-open, gpr-closed, gpr-open, gpr-compact, rg-compact and gpr-connected.

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1 Introduction

In 1993, N.Palaniappan and K.Chandrasekhara Rao[8], introduced the concept of regular generalized closed(briefly, rg-closed) sets and regular generalized open (briefly, rg-open) sets in a topological space. They are also defined regular generalized continuous(briefly, rg-continuous) map and regular generalized irresolute(briefly, rg-irresolute) map between topological spaces and studied some of their properties. In 1999, Y.Gnanambal and K.Balachandran [5], introduced and investigated the concept of generalized pre-regular closed (briefly, gpr-closed)sets and generalized pre-regular open (briefly, gpr-open) sets in topological spaces.Further they introduced gpr-continuous functions, gpr-connected spaces and gpr-compact spaces[6]. A.M.Ai-Shibani[1] introduced and investigated rg-compact spaces and rg-connected spaces using rg-open sets.
The purpose of this paper is to characterize these spaces using the well known fact that " every singleton is rg-open and hence gpr-open"[3].

Throughout this paper, space X mean topological space (X, τ). For a subset A of X, the closure, rg-closure,gpr-closure, interior and the complement of A are denoted by cl(A), rg-cl(A),gpr-cl(A), int(A) and A^c respectively.

2 Definitions and Basic Properties

Definition 2.1. *(i) A subset A of a space X is said to be regular open if A= int(cl(A))and regular closed if A=cl(int(A))[9].*
(ii) A subset A of a space X is said to be pre-open if A⊆ int(cl(A)) and pre-closed if cl(int(A))⊆A[7].
The pre-closure of a subset A of X is the intersection of all pre-closed sets containing A and is denoted by pcl(A).

Definition 2.2. *A subset A of a space X is said to be regular generalized closed (briefly. rg-closed)[8] if cl(A)⊆U whenever A⊆U, where U is regular open.It is said to be be regular generalized open (briefly. rg-open) if A^c is rg-closed.(equivalently F⊆int(A) whenever F⊆A and F is regular closed.*

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Definition 2.3. *The intersection of all rg-closed sets containing a set A is called the regular generalized closure of A and is denoted by rg-cl(A).*

Definition 2.4. *A subset A of a space X is said to be generalized pre-regular closed (briefly. gpr-closed)[5] if pcl(A)⊆U whenever A⊆U, where U is regular open.It is said to be be generalized pre-regular open (briefly. gpr-open) if A^c is gpr-closed.*
The intersection of all gpr-closed sets containing a set A is called the generalized pre-regular closure of A and is denoted by gpr-cl(A).

Definition 2.5. *Let f : X → Y be a function. Then f is*
(i). rg-continuous[8] if $f^{-1}(V)$ is rg-closed for every closed set V of Y.
(ii). rg-irresolute[8] if $f^{-1}(G)$ is rg-closed in X for every rg-closed set G of Y.
(iii).gpr-continuous[6] if $f^{-1}(V)$ is gpr-closed for every closed set V of Y.

Definition 2.6. *A collection $\{A_\alpha: \alpha \in \nabla\}$ of rg-open sets in a topological space X is called rg-open cover[1] of a subset B of X if B⊆ $\cup\{A_\alpha: \alpha \in \nabla\}$ holds.*

Definition 2.7. *A topological space X is called regular generalized compact(briefly. rg-compact)[1] if every rg-open cover of X has a finite subcover.*

Definition 2.8. *A subset B of X is called rg-compact relative to X [1] if for every collection $\{A_\alpha: \alpha \in \nabla\}$ of rg-open subsets of X such that B⊆ $\cup\{A_\alpha: \alpha \in \nabla\}$, there exist a finite subset$\nabla_\circ$ of ∇ such that B⊆ $\cup\{A_\alpha: \alpha \in \nabla_\circ\}$*

Definition 2.9. *A collection $\{A_\alpha: \alpha \in \nabla\}$ of gpr-open sets in a topological space X is called gpr-open cover[6] of a subset B of X if B⊆ $\cup\{A_\alpha: \alpha \in \nabla\}$ holds.*

Definition 2.10. *A topological space X is called generalized pre-regular compact(briefly. gpr-compact)[6] if every gpr-open cover of X has a finite subcover.*

Definition 2.11. *A subset B of X is called gpr-compact relative to X[6] if for every collection $\{A_\alpha: \alpha \in \nabla\}$ of gpr-open subsets of X such that B⊆ $\cup\{A_\alpha: \alpha \in \nabla\}$, there exist a finite subset$\nabla_\circ$ of ∇ such that B⊆ $\cup\{A_\alpha: \alpha \in \nabla_\circ\}$*

Lemma 2.13. *(i). If A⊆X, then A⊆rg-cl(A)⊆cl(A).*
(ii). If A⊆B, then rg-cl(A)⊆rg-cl(B).
(iii).If A is rg-closed an A⊆B⊆cl(A), then B is rg-closed.

Lemma 2.14. *In a topological space X, the following hold:[3]*
(i). $\{x\}$is rg-open for every x∈X.
(ii). rg-cl(A)=gpr-cl(A)=A, for every subset A of X.

Lemma 2.15. *For a topological space, the following are equivalent:[6]*
(i) X is gpr-connected.
(ii) The only subsets of X which are both gpr-open and gpr-closed are the empty set ϕ and X.
(iii) Each gpr-continuous mp of X into a discrete space Y with atleast two points is a constant map.

Lemma 2.16. *In a topological space X, $\{x\}$ is open or pre-closed for every x∈X.[4]*

3 rg-compact spaces

A.M.Al-Shibani[Theorem 3.4[1]] established the equivalence of the following statements in any topological space (X, τ).
(i). For each x∈X and each open set V in Y with f(x)∈V, there exists an rg-open set U in X such that x∈U, f(U)⊆V.
(ii).For every subset A of X, f(rg-cl(A))⊆ cl(f(A)).
(iii). For every subset B of Y, rg-cl(f^{-1}(B))⊆f^{-1}(cl(B)). However the above statements are always true in any topological space as shown in the next proposition.

Proposition 3.1. *If (X, τ) is a topological space, then the following hold: (1). For each x∈X and each open set V in Y with f(x)∈V, there exists an rg-open set U in X such that x∈U, f(U)⊆V.*
(2).For every subset A of X, f(rg-cl(A))⊆ cl(f(A)).
(3). For every subset B of Y, rg-cl(f^{-1}(B))⊆f^{-1}(cl(B)).

Proof. (1). Take U={x}, then by lemma 2.13, U is rg-open and f(U)=f({x}) ⊆V. (2) and (3) follows from the fact that rg-cl(A)=A, for any set A. □

Theorem 3.2. *A topological space X is rg-compact if and only if X is finite.*

Proof. Let X be a rg-compact space. Since {x} is rg-open for all x∈X, {{x}: x∈X} is an rg-open cover of X. Since X is rg-compact, there exists a finite subset $X_\circ$ of X such that X⊆∪{{x} : x∈$X_\circ$}=$X_\circ$⊆X. Hence X=$X_\circ$, which is finite. Converse is obivious. □

Remark 3.3. *A.M.Al-Shibani established that*
(1) If X is rg-compact and f:X→Y is rg-continuous and bijective, then Y is compact.
(2) If f:X →Y is rg-irresolute and B is rg-compact relative to X, then f(B) is rg-compact relative to Y.
But the conditions f:X→Y is rg-continuous, bijective in (1) and f:X→Y is rg-irresolute in (2) are not necessary as shown in the following theorem.

Theorem 3.4. *Let f:X→Y be a map.*
(1). If X is rg-compact and f is surjective, then Y is compact.
(2). If B is rg-compact relative to X, then f(B) is rg-compact relative to Y.

Proof. (1) Let f:X→Y be a surjective map. If X is rg-compact, then by theorem 3.2, X is finite. Since f is surjective, Y=f(X), which is also finite and hence Y is compact.
(2) If B is rg-compact relative to X, then B is a finite subset of X, by Theorem 3.2. Therefore f(B) is also a finite subset of Y and hence f(B) is rg-compact relative to Y. □

4 gpr-compact spaces

Theorem 4.1. *A topological space X is gpr-compact if and only if X is finite.*

Proof. Let X be a gpr-compact space. Since {x} is gpr-open for all x∈X, {{x}: x∈X} is an gpr-open cover of X. Since X is gpr-compact, there exists a finite subset $X_\circ$ of X such that X⊆∪{{x} : x∈$X_\circ$}=$X_\circ$⊆X. Hence X=$X_\circ$, which is finite. Converse is obivious. □

5 gpr-connected spaces

A topological space (X, τ) is said to be gpr-connected [2] if X cannot be written as the disjoint union of two non empty gpr-open sets.

Theorem 5.1. *No topological space is gpr-connected.*

Proof. Let (X, τ) be topological space.
Case(1): Suppose {x} is open for all x∈X. In this case,(X, τ) is a discrete space and hence every subset of X is both gpr-open and gpr-closed. Therefore by lemma 2.14, (X, τ) cannot be gpr-connected.
Case (2): Suppose {x} is not open for all x∈X. Then {y} is not open for some y∈X. By lemma 2.15,{y} is pre-closed and hence {y} is gpr-closed. Also by lemma 2.13, {y} is gpr-open. Hence {y}is both gpr-closed and gpr-open.Therefore by using lemma 2.14, (X, τ) is not gpr-connected. □

6 Conclusion

In this paper the following results are established:
1.A topological space X is rg-compact if and only if X is finite.
2.A topological space X is gpr-compact if and only if X is finite.
3.No topological space is gpr-connected.

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Milovanović bounds for minimum dominating energy of a graph

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Abstract

Recently Milovanović et.al gave a sharper lower bounds for energy of a graph. In this paper similar bounds for minimum dominating energy and Laplacian minimum dominating energy of a graph are established.

Keywords: Minimum dominating energy, Minimum covering energy, Laplacian minimum dominating energy, Laplacian Minimum covering energy

2010 MSC: 05C50, 05C69.

1 Introduction

The concept of energy of a graph was introduced by I. Gutman [3] in the year 1978. Let G be a graph with n vertices $\{v_1, v_2, ..., v_n\}$ and m edges. Let $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|$ be the eigenvalues of adjacency matrix $A = (a_{ij})$ of the graph. Then the energy of a graph is defined by $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

For details on the mathematical aspects of theory of graph energy see the papers [4, 5] and the references cited there in. The basic properties including various upper and lower bounds for energy of a graph have been established in [7, 8] and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [2, 6].

Let $|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq ... \geq |\mu_n|$ denotes eigenvalues of Laplacian matrix $L = (l_{ij})$ of a graph G. Then Laplacian energy is defined by $LE(G)=\sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$

Recently Milovanović [9] et.al gave a sharper lower bounds for energy of a graph. In this paper similar bounds for minimum dominating energy and Laplacian minimum dominating energy of a graph are established. Similar bounds for minimum covering energy and Laplacian minimum covering energy of a graph can also be derived.

2 Preliminaries

Definition 2.1. ***Minimum Dominating Energy of a Graph**: Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. A subset D of V is called a dominating set of G if every vertex of $V - D$ is incident to some vertex of D. Any dominating set with minimum cardinality is called a minimum dominating set. For the graph G with minimum dominating set D, the minimum dominating matrix is defined by*

$$A_D(G) := (a_{ij}^D), \text{ where } a_{ij}^D = \begin{cases} 1 & if \;\; v_i v_j \in E \\ 1 & if \;\; i = j \;\; and \;\; v_i \in D \\ 0 & otherwise \end{cases}$$

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If $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|$ are the eigenvalues of adjacency matrix $A_D(G)$ of the graph, then the minimum Dominating energy of the graph G is defined by $E_D(G) := \sum_{i=1}^{n} |\lambda_i|$.

Definition 2.2. ***Laplacian Minimum Dominating Energy of a Graph***: *If $D(G)$ denotes the diagonal matrix of vertex degree of the graph G, then $L_D(G)$=$D(G) - A_D(G)$ is called Laplacian dominating matrix of G. If $|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq ... \geq |\mu_n|$ denotes eigenvalues of matrix $L_D(G)$, then Laplacian minimum dominating energy is defined by $LE_D(G) := \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.*

For the basic properties on minimum covering energy, Laplacian minimum covering energy, minimum dominating energy, Laplacian minimum dominating energy, see the papers [1, 10, 11, 12] and the references cited there in.

3 Milovanović bounds for minimum dominating energy of a graph

Theorem 3.1. *Let G be a graph with n vertices and m edges. Let $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ be a non-increasing order of eigenvalues of $A_D(G)$ and D is minimum dominating set then $E_D(G) \geq \sqrt{n(2m+|D|) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$ where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$ and $[x]$ denotes the integral part of a real number*

Proof. Let $a, a_1, a_2, \ldots a_n, A$ and $b, b_1, b_2, \ldots b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ $\forall\ i = 1, 2, \ldots n$ then the following inequality is valid. $\left|n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i\right| \leq \alpha(n)(A-a)(B-b)$ where α(n)= $n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$ and equality holds if and only if $a_1 = a_2 = \ldots = a_n$ and $b_1 = b_2 = \ldots = b_n$.
If $a_i = |\lambda_i|$, $b_i = |\lambda_i|$, $a = b = |\lambda_n|$ and $A = B = |\lambda_1|$, then

$$\left|n\sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} |\lambda_i|\right)^2\right| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$$

But $\sum_{i=1}^{n} |\lambda_i|^2$ =$2m + |D|$ and $E_D(G) \leq \sqrt{n(2m+|D|)}$ [10] then the above inequality becomes

$$n(2m + |D|) - (E_D(G))^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$$

$$\text{i,e., } E_D(G) \geq \sqrt{n(2m+|D|) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$$

□

The above theorem is also true for the minimum covering energy of a graph. Hence we have the following result.
Let G be a graph with n vertices and m edges. Let$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ be a non-increasing order of eigenvalues of $A_C(G)$ and C is minimum covering set, then $E_C(G) \geq \sqrt{n(2m+|C|) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$ where α(n)= $n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$ and $[x]$ denotes integral part of a real number

Theorem 3.2. *Let G be a graph with n vertices and m edges. Let $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| > 0$ be a non-increasing order of eigenvalues of $A_D(G)$ then $E_D(G) \geq \dfrac{2m + |D| + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)}$*

Proof. Let $a_i \neq 0$, b_i, r and R be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds.[Theorem 2, [9]]

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i \leq (r+R)\sum_{i=1}^{n} a_i b_i$$

Put $b_i = |\lambda_i|$, a_i =1, $r = |\lambda_n|$ and $R = |\lambda_1|$ then

$$\sum_{i=1}^{n}|\lambda_i|^2 + |\lambda_1||\lambda_n|\sum_{i=1}^{n}1 \leq (|\lambda_1| + |\lambda_n|)\sum_{i=1}^{n}||\lambda_i|$$

$$i.e., \ 2m + |D| + |\lambda_1||\lambda_n|n \leq (|\lambda_1| + |\lambda_n|)E_D(G)$$

$$E_D(G) \geq \frac{2m + |D| + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)}$$

□

This bound is similar for minimum covering energy of a graph.

4 Milovanović bounds for laplacian minimum dominating energy

Theorem 4.3. *Let G be a graph with n vertices and m edges. Let* $|\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_n|$ *be a non-increasing order of eigenvalues of* $L_D(G)$*. If D is minimum dominating set then* $LE_D(G) \geq \sqrt{2nM - \alpha(n)(|\mu_1| - |\mu_n|)^2} - 2m$*, where* $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$*,* $[x]$ *denotes greatest integer part of real number and* $M = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - c_i)^2$*.*

Here $c_i = \begin{cases} 1 & \text{if } v_i \in D \\ 0 & \text{if } v_i \notin D \end{cases}$

Proof. Let $a, a_1, a_2, \ldots a_n, A$ and $b, b_1, b_2, \ldots b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ $\forall\ i = 1, 2, \ldots n$ then the following inequality is valid.

$$\Big|n\sum_{i=1}^{n}a_ib_i - \sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i\Big| \leq \alpha(n)(A - a)(B - b)$$

If $a_i = |\mu_i|$, $b_i = |\mu_i|$, $a = b = |\mu_n|$ and $A = b = |\mu_1|$

$$\Big|n\sum_{i=1}^{n}|\mu_i|^2 - \Big(\sum_{i=1}^{n}|\mu_i|\Big)^2\Big| \leq \alpha(n)(|\mu_1| - |\mu_n|)^2$$

But $(\sum_{i=1}^{n}|\mu_i|)^2 \leq 2nM \quad \Rightarrow n2M - \Big(\sum_{i=1}^{n}|\mu_i|\Big)^2 \leq \alpha(n)(|\mu_1| - |\mu_n|)^2$

$$\Big(\sum_{i=1}^{n}|\mu_i|\Big) \geq \sqrt{2Mn - \alpha(n)(|\mu_1| - |\mu_n|)^2}$$

Since $LE_D(G) = \sum_{i=1}^{n}\Big|\mu_i - \dfrac{2m}{n}\Big| \ \geq \sum_{i=1}^{n}|\mu_i| - \Big|\dfrac{2m}{n}\Big|$

Hence $LE_D(G) \geq \sqrt{2nM - \alpha(n)(|\mu_1| - |\mu_n|)^2}$ -2m □

Theorem 4.4. *Let G be a graph with n vertices and m edges. Let* $|\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_n| > 0$ *be a non-increasing order of eigenvalues of* $LE_D(G)$ *and D is minimum dominating set then* $LE_D(G) \geq \dfrac{2M + n|\mu_1||\mu_n|}{(|\mu_1| + |\mu_n|)} - 2m$ *where* $M = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - c_i)^2$*. Here* $c_i = \begin{cases} 1 & \text{if } v_i \in D \\ 0 & \text{if } v_i \notin D \end{cases}$

Proof. Let $a_i \neq 0$, b_i, r and R be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then we have the following inequality

$$\sum_{i=1}^{n}b_i^2 + rR\sum_{i=1}^{n}a_i \leq (r + R)\sum_{i=1}^{n}a_ib_i$$

Put $b_i = |\mu_i|$, a_i =1, $r = |\mu_n|$ and $R = |\mu_1|$

$$\sum_{i=1}^{n}|\mu_i|^2 + |\mu_1||\mu_n|\sum_{i=1}^{n}1 \leq (|\mu_1| + |\mu_n|)\sum_{i=1}^{n}|\mu_i|$$

$$i.e., 2M + |\mu_1||\mu_n|n \leq (|\mu_1| + |\mu_n|)\sum_{i=1}^{n}|\mu_i|$$

$$\Rightarrow \sum_{i=1}^{n}|\mu_i| \geq \frac{2M + n|\mu_1||\mu_n|}{(|\mu_1| + |\mu_n|)}$$

We know that $LE_D(G) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| \quad LE_D(G) \geq \sum_{i=1}^{n}\left|\mu_i\right| - \left|\frac{2m}{n}\right|$

$$\Rightarrow LE_D(G) \geq \frac{2M + n|\mu_1||\mu_n|}{(|\mu_1| + |\mu_n|)} - 2m$$

□

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Oscillation Criteria of Third Order Nonlinear Neutral Difference Equations

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Abstract

In this paper we consider the third order nonlinear neutral difference equation of the form

$$\Delta(r_n(\Delta^2(x_n \pm p_n x_{\sigma(n)}))^{\alpha}) + f(n,\ x_{\tau(n)}) = 0,$$

we establish some sufficient conditions which ensure that every solution of this equation are either oscillatory or converges to zero. Examples are provided to illustrate the main results.

Keywords: Third order, oscillation, neutral difference equations.

2010 MSC: 39A10.

1 Introduction

In this paper, we establish oscillation criteria for third order nonlinear neutral difference equation of the form

$$\Delta\Big(r_n(\Delta^2(x_n \pm p_n x_{\sigma(n)}))^{\alpha}\Big) + f(n,\ x_{\tau(n)}) = 0, n \in \mathbb{N}_0 \tag{1}$$

where $\mathbb{N}_0 = \{n_0,\ n_0+1,\ n_0+2,\ \ldots\}$, and n_0 is a nonnegative integer subject to the following conditions:

(C_1) $\{r_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{r_n^{1/\alpha}} = \infty$ and α is a ratio of odd positive integers;

(C_2) $\{p_n\}$ is a nonnegative real sequence with $-\mu \leq p_n \leq 1$ for $\mu \in (0,1)$;

(C_3) $\{\sigma(n)\}$ is a nonnegative sequence of integers with $\sigma(n) \leq n$ such that $\lim_{n\to\infty} \sigma(n) = \infty$;

(C_4) $\{\tau(n)\}$ is a nonnegative sequence of integers with $\tau(n) \leq n$ such that $\lim_{n\to\infty} \tau(n) = \infty$;

(C_5) $f : \mathbb{N}_0 \times \mathbb{R} \to [0,\infty)$ and there is a nonnegative real sequence $\{q_n\}$ such that $\frac{f(n,\ u)}{u^{\alpha}} \geq Lq_n$, for $u \neq 0$ where $L > 0$.

By a solution of equation (1) we mean a real sequence $\{x_n\}$ and satisfying equation (1) for all $n \in \mathbb{N}_0$. We consider only those solution $\{x_n\}$ of equation (1) which satisfy $\sup\{|x_n| : n \geq N\} > 0$ for all $N \in \mathbb{N}_0$. A solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years, much research has been done on the oscillatory behavior of solutions of third order difference equations, see for example ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]) and the references cited therein.
In ([13], [14]), the authors consider the following third order neutral difference equations of the form

$$\Delta(r_n(\Delta^2(x_n \pm p_n x_{n-\sigma}))^{\alpha}) + q_n x_{n+1-\tau}^{\alpha} = 0, \tag{2}$$

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and

$$\Delta(r_n(\Delta^2(x_n \pm p_n h(x_{n-\sigma})))^\alpha) + q_n f(x_{n+1-\tau}) = 0, \tag{3}$$

and established some criteria for the oscillation and asymptotic behavior of all solutions of equations (2) and (3).

In [12], the authors studied the following third order difference equation

$$\Delta(c_n \Delta(d_n \Delta(x_n + p_n x_{n-k}))) + q_n f(x_{n-m}) = e_n \tag{4}$$

and established some criteria for the oscillation and asymptotic behavior of all solutions of equation (4).

In [15], the authors considered the following third order difference equation

$$\Delta(a_n(\Delta^2(x_n + p_n x_{n-\sigma}))^\alpha) + q_n x^\alpha_{n-\tau} = 0 \tag{5}$$

and established some criteria for the oscillation and asymptotic behavior of all solutions of equation (5).

The oscillatory properties of oscillation of equation (1) was studied by the authors in [7], when $p_n \equiv 0$. Following this trend, in this paper, we establish some new sufficient conditions for the oscillation of all solutions of equation (1). In Section 2, we present some oscillation theorems and in Section 3, we provide examples to illustrate the main results.

2 Oscillation Theorems

First we consider the following difference equation

$$\Delta\Big(r_n(\Delta^2(x_n + p_n x_{\sigma(n)}))^\alpha\Big) + f(n,\ x_{\tau(n)}) = 0, n \in \mathbb{N}_0, \tag{6}$$

and establish some sufficient conditions for the oscillation and asymptotic behavior of its solutions. We begin with the following lemma.

Lemma 2.1. *Let $\{x_n\}$ be a positive solution of equation (6), then the corresponding function $z_n = x_n + p_n x_{\sigma(n)}$ satisfies only of the following two cases:*

(I) $z_n > 0,\ \Delta z_n > 0,\ \Delta^2 z_n > 0;$

(II) $z_n > 0,\ \Delta z_n < 0,\ \Delta^2 z_n > 0$

for $n \geq n_1 \in \mathbb{N}_0$, where n_1 is sufficiently large.

Proof. The proof can be found in [13, 14], and hence the details are omitted. □

Lemma 2.2. *Let $\{x_n\}$ be a positive solution of equation (6), and let the corresponding function $\{z_n\}$ satisfies the Case (II) of Lemma 2.1. If*

$$\sum_{n=n_0}^{\infty}\sum_{s=n}^{\infty}\left[\frac{1}{r_s}\sum_{t=s}^{\infty} q_t\right]^{1/\alpha} = \infty, \tag{7}$$

then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = 0.$

Proof. The proof is similar to that of Lemma 2.2 in [13], and hence the details are omitted. □

Before stating the next lemma, we define

$$A_n = \sum_{s=n_0}^{\infty} r_s^{-1/\alpha},$$
$$Q_n = (1 - p_{\tau(n)})^\alpha L q_n,$$

and

$$R_n = \sum_{s=n_0}^{n-1} Q_s \ \text{ for all } \ n \in \mathbb{N}_0.$$

Lemma 2.3. *Let $\{x_n\}$ be a positive solution of equation (6) and the corresponding z_n satisfies Case(I) of Lemma 2.1. Then there exists a positive real sequence $\{w_n\}$ such that*

$$w_n \geq R_n + \sum_{s=n}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha}, \tag{8}$$

$$\lim_{n\to\infty} \sup[w_{n+1} A_{\tau(n)}^{\alpha/(\alpha+1)}] \leq c, \tag{9}$$

for some constant $c > 0$, and

$$\sum_{n=n_0}^{\infty} Q_n < \infty, \quad \sum_{n=n_0}^{\infty} A_{\tau(n)} R_{n+1}^{1+1/\alpha} < \infty. \tag{10}$$

Proof. Let $\{x_n\}$ be a positive solution of equation (6). Assume that $x_n > 0$, $x_{\sigma(n)} > 0$ and $x_{\tau(n)} > 0$ for all $n \geq n_1 \geq n_0$. Then $z_n > x_n > 0$ and satisfies Case(I) of Lemma 2.1, for all $n \geq N \geq n_1$. From (6), we have

$$\begin{aligned} \Delta(r_n(\Delta z_n)^\alpha) &\leq -f(n, x_{\tau(n)}) \\ &\leq -x_{\tau(n)}^\alpha L q_n, \quad n \geq n_1. \end{aligned} \tag{11}$$

From the monotone nature of z_n, we have

$$x_n = z_n - p_n x_{\sigma(n)}$$

or

$$x_{\tau(n)} \geq (1 - p_{\tau(n)}) z_{\tau(n)}. \tag{12}$$

From (11) and (12), we have

$$\Delta(r_n(\Delta^2 z_n)^\alpha) \leq -(1 - p_{\tau(n)}) z_{\tau(n)}^\alpha L q_n$$

or

$$\frac{\Delta(r_n(\Delta^2 z_n)^\alpha)}{z_{\tau(n)}^\alpha} \leq -(1 - p_{\tau(n)}) L q_n. \tag{13}$$

Define

$$w_n = \frac{r_n(\Delta^2 z_n)^\alpha}{z_{\tau(n)}^\alpha}. \tag{14}$$

Then $w_n > 0$ for all $n \geq n_1$, and

$$\Delta w_n = \frac{\Delta(r_n(\Delta^2 z_n)^\alpha)}{z_{\tau(n)}^\alpha} - \frac{r_{n+1}(\Delta^2 z_{n+1})^\alpha}{z_{\tau(n)}^\alpha \, z_{\tau(n+1)}^\alpha} \Delta(z_{\tau(n)}^\alpha).$$

Using (13) and (14) in the last inequality, we obtain

$$\Delta w_n \leq -(1 - p_{\tau(n)})^\alpha L q_n - w_{n+1} \frac{\Delta(z_{\tau(n)}^\alpha)}{z_{\tau(n)}^\alpha}. \tag{15}$$

By Mean Value Theorem

$$\Delta z_{\tau(n)}^\alpha = \alpha t^{\alpha-1} \Delta z_{\tau(n)},$$

where $z_{\tau(n)} \leq t \leq z_{\tau(n+1)}$. Since $\alpha \geq 1$, we have

$$\Delta z_{\tau(n)}^\alpha \geq \alpha z_{\tau(n)}^{\alpha-1} \Delta z_{\tau(n)}. \tag{16}$$

Using (16) in the inequality (15), we obtain

$$\Delta w_n \leq -Q_n - \alpha w_{n+1} \frac{\Delta z_{\tau(n)}}{z_{\tau(n)}}. \tag{17}$$

From the monotonicity property of $\{\Delta^2 z_n\}$, we have

$$\Delta z_n = \Delta z_{n_0} + \sum_{s=n_0}^{n-1} \Delta^2 z_s \geq \sum_{s=n_0}^{n-1} \Delta^2 z_s$$

or

$$\begin{aligned}\Delta z_n &\geq \sum_{s=n_0}^{n-1} r_s^{-1/\alpha}(r_s(\Delta^2 z_s)^\alpha)^{1/\alpha}\\ &\geq (r_n(\Delta^2 z_n)^\alpha)^{1/\alpha} A_n.\end{aligned}$$

Then

$$\Delta z_{\tau(n)} \geq (r_{\tau(n)}(\Delta^2 z_{\tau(n)})^\alpha)^{1/\alpha} A_{\tau(n)}. \tag{18}$$

Using (18) in the inequality (17), we get

$$\Delta w_n \leq -Q_n - \alpha w_{n+1}^{1+1/\alpha} A_{\tau(n)}$$

or

$$\Delta w_n + Q_n + \alpha w_{n+1}^{1+1/\alpha} A_{\tau(n)} \leq 0,\ n \geq N. \tag{19}$$

Summing the last inequality from N to $n-1$, we have

$$w_n \leq w_N - \sum_{s=N}^{n-1} Q_s - \sum_{s=N}^{n-1} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha} \ \text{ for } \ n \geq N. \tag{20}$$

We claim that $\sum_{n=N}^{\infty} Q_n < \infty$ for all $n \geq N$. Otherwise from the inequality (21), we obtain

$$w_n \leq w_N - \sum_{s=N}^{n-1} Q_s,$$

and letting limit $n \to \infty$ we obatin $w_n \to -\infty$, which contradicts the positivity of w_n. Similarly we can show that

$$\sum_{s=N}^{\infty} A_{\tau(s)} w_{s+1}^{1+1/\alpha} < \infty. \tag{21}$$

Now, letting limit as $n \to \infty$ in (20) we have

$$w_\infty - w_N + \sum_{s=N}^{\infty} Q_s + \sum_{s=N}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha} \leq 0$$

or

$$w_n \geq R_n + \sum_{s=n}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha} \quad \text{for} \ \ n \geq N. \tag{22}$$

Since $Q_n > 0$ and $w_n > 0$ for $n \geq N$, we have from (19) that $\Delta w_n < 0$ and $\lim_{n\to\infty} w_n = M$, for some constant $M > 0$. Now from (19), we have

$$\Delta w_n \leq -\alpha A_{\tau(n)} w_{n+1}^{1+1/\alpha}$$

or

$$-\frac{\Delta w_n}{w_{n+1}^{1+1/\alpha}} \geq \alpha A_{\tau(n)}$$

or

$$\frac{w_n}{\alpha w_{n+1}^{1+1/\alpha}} \geq A_{\tau(n)}.$$

Taking limit supreme, we obtain

$$M \geq \lim_{n\to\infty} \sup(w_{n+1}^{1+1/\alpha} A_{\tau(n)})$$

or

$$\lim_{n\to\infty} \sup(w_{n+1} A_{\tau(n)}^{\alpha/(\alpha+1)}) \leq c,$$

for some constant $c > 0$. This completes the proof. □

Theorem 2.1. *Assume that*

$$\liminf_{n\to\infty} \frac{1}{R_n} \sum_{s=n}^{\infty} P_s R_{s+1}^{1+1/\alpha} > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{23}$$

where $P_n = \alpha A_{\tau(n)}$ then every solution of equation (6) is either oscillatory or converges to zero as $n \to \infty$.

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (6). Without loss of generality we may assume that $x_n > 0$, $x_{\sigma(n)} > 0$ and $x_{\tau(n)} > 0$ for all $n \geq n_1 \geq n_0$ and the corresponding $\{z_n\}$ satisfies two cases of Lemma 2.1.

Case(I). Let $\{z_n\}$ satisfies Case (I) of Lemma 2.1. From Lemma 2.3, we obtain (8), then

$$\begin{aligned} \frac{w_n}{R_n} &\geq 1 + \frac{1}{R_n} \sum_{s=n}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha} \\ &\geq 1 + \frac{1}{R_n} \sum_{s=n}^{\infty} P_s R_{s+1}^{1+1/\alpha} \left(\frac{w_{s+1}}{R_{s+1}}\right)^{1+1/\alpha}. \end{aligned} \tag{24}$$

From the assumption of the theorem, there exists a $\beta > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}$,

$$\liminf_{n\to\infty} \frac{1}{R_n} \sum_{s=n}^{\infty} P_s R_{s+1}^{1+1/\alpha} > \beta \tag{25}$$

and let

$$\lambda = \inf_{n \geq n_0} \frac{w_n}{R_n}, \tag{26}$$

then $\lambda \geq 1$. Using (25) and (26) in the inequality (24) we have

$$\lambda \geq 1 + \beta \lambda^{1+1/\alpha}.$$

Therefore

$$\lambda - \beta\lambda^{1+1/\alpha} \leq \frac{\alpha}{(\alpha+1)^{\alpha}} \frac{1}{\beta^{\alpha}}.$$

Then, we get $\beta \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{(\alpha+1)/\alpha}}$, which is a contradicts to our assumption.

If $\{z_n\}$ satisfies Case(II) of Lemma 2.1, then by the condition (7) we have $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Theorem 2.2. *Assume that*

$$\limsup_{n\to\infty} \left[A_{\tau(n)}^{\alpha/(\alpha+1)} \left(R_{n+1} + \sum_{s=n+1}^{\infty} \alpha A_{\tau(s)} R_{s+1}^{1+1/\alpha} \right) \right] = \infty \tag{27}$$

then every solution of equation (6) is either oscillatory or converges to zero as $n \to \infty$.

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (6). Without loss of generality we may assume that $x_n > 0$, $x_{\sigma(n)} > 0$ and $x_{\tau(n)} > 0$ for all $n \geq n_1 \geq n_0$ and the corresponding $\{z_n\}$ satisfies two cases of Lemma 2.1.

Case(I). Let $\{z_n\}$ satisfies Case (I) of Lemma 2.1. From Lemma 2.3, we obtain (8), then

$$w_n \geq R_n + \sum_{s=n}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha}.$$

Since $w_n \geq R_n$, we have

$$w_n \geq R_n + \sum_{s=n}^{\infty} \alpha A_{\tau(s)} R_{s+1}^{1+1/\alpha}.$$

Using this in (9), we have

$$\limsup_{n\to\infty} \left[A_{\tau(n)}^{\alpha/(\alpha+1)} \left(R_{n+1} + \sum_{s=n+1}^{\infty} \alpha A_{\tau(s)} R_{s+1}^{1+1/\alpha} \right) \right] \leq c,$$

which is a contradiction. If $\{z_n\}$ satisfies Case(II) of Lemma 2.1, then by the condition (7) we have $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Next, we consider the case $-\mu \le p_n \le 0$, and the equation (1) takes the form

$$\Delta\Big(r_n(\Delta^2(x_n - p_n x_{\sigma(n)}))^{\alpha}\Big) + f(n,\ x_{\tau(n)}) = 0, n \in \mathbb{N}_0. \tag{28}$$

Lemma 2.4. *Let $\{x_n\}$ be a positive solution of equation (28) and the corresponding $\{z_n\}$ satisfies Case(I) of Lemma 2.1. Then there exists a positive function $\{w_n\}$ such that*

$$w_n \ge Q_n + \sum_{s=n}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha}, \tag{29}$$

$$\lim_{n\to\infty} \sup[w_{n+1} A_{\tau(n)}^{\alpha/(\alpha+1)}] \le c, \tag{30}$$

for some constant $c > 0$, and

$$Q_n < \infty, \quad \sum_{s=n}^{\infty} A_{\tau(s)} Q_{s+1}^{1+1/\alpha} < \infty. \tag{31}$$

Proof. Let $\{x_n\}$ be a positive solution of equation (28). Assume that $x_n > 0$, $x_{\sigma(n)} > 0$ and $x_{\tau(n)} > 0$ for all $n \ge n_1 \ge n_0$. Then $z_n > x_n > 0$ and satisfies Case(I) of Lemma 2.1, for all $n \ge N \ge n_1$. We have (11)

$$\Delta(r_n(\Delta z_n)^{\alpha}) \le -x_{\tau(n)}^{\alpha} L q_n, \quad n \ge n_1. \tag{32}$$

We have two possible cases for z_n:

(i) $z_n > 0$

(ii) $z_n < 0$.

Case (i). $z_n > 0$, the proof is similar to that of Lemma 2.3 and hence the details are omitted.
Case (ii). $z_n < 0$ eventually for all $n \ge n_2 \ge n_1 \ge n_0$, then we have two cases for x_n:

(a) x_n is unbounded,

(b) x_n is bounded.

Case (a). Assume that x_n is unbounded, then

$$x_n = z_n - p_n x_{\sigma(n)} < -p_n x_{\sigma(n)} < x_{\sigma(n)}. \tag{33}$$

Since $\{x_n\}$ is unbounded, we can choose a sequence $\{x_{n_k}\}$ satisfying $\lim_{k\to\infty} x_k = \infty$ from which $\lim_{k\to\infty} x_{N_k} = \infty$ and $\max x_n = x_{N_n}$ by choosing N large such that $\sigma(N_k) > N_1$ for all $N_k > n_2$. Thus $\max x_n = x_{N_n}$. This contradicts with (33).
Case (b). Assume that $\{x_n\}$ is bounded, and we show that $x_n \to 0$ as $n \to \infty$. Since

$$\lim_{n\to\infty} \sup z_n \le 0,$$

then we have

$$\begin{aligned} \lim_{n\to\infty} \sup(x_n + p_n x_{\sigma(n)}) &\le 0 \\ \lim_{n\to\infty} \sup x_n + \lim_{n\to\infty} \sup p_n x_{\sigma(n)} &\le 0 \\ \lim_{n\to\infty} \sup x_n - \mu \lim_{n\to\infty} \sup x_{\sigma(n)} &\le 0 \\ (1-\mu) \lim_{n\to\infty} \sup x_n &\le 0. \end{aligned}$$

This shows that $x_n \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 2.3. *Assume that*

$$\lim_{n\to\infty} \inf \frac{1}{Q_n} \sum_{s=n}^{\infty} P_s Q_{s+1}^{1+1/\alpha} > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{34}$$

where $P_n = \alpha A_{\tau(n)}$ then every solution of equation (28) is either oscillatory or converges to zero as $n \to \infty$.

Proof. The proof is similar to that of Theorem 2.1 and hence the details are omitted. □

Theorem 2.4. *Assume that*

$$\limsup_{n\to\infty}\left[A_{\tau(n)}^{\alpha/(\alpha+1)}\left(Q_{n+1}+\sum_{s=n+1}^{\infty}\alpha A_{\tau(s)}Q_{s+1}^{1+1/\alpha}\right)\right]=\infty \tag{35}$$

then every solution of equation (28) is either oscillatory or converges to zero as $n\to\infty$.

Proof. The proof is similar to that of Theorem 2.2 and hence the details are omitted. □

3 Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. *Consider the third order difference equation*

$$\Delta\Big(n(\Delta^2\Big(x_n+\frac{1}{2}x_{n-2}\Big))^3\Big)+\frac{1}{(n+1)(n+2)}x_{n-3}^3=0,\ \ n\geq 1. \tag{36}$$

Here $r_n=n$, $p_n=\frac{1}{2}$, $q_n=\frac{1}{(n+1)(n+2)}$, $\alpha=3$ $\sigma(n)=n-2$, $\tau(n)=n-3$ and $L=1$. It is easy to see that all conditions of Theorem 2.1 are satisfied. Hence every solution of equation (36) is either oscillatory or converges to zero as $n\to\infty$.

Example 3.2. *Consider the third order difference equation*

$$\Delta\Big(\frac{1}{n}\Delta^2\Big(x_n+\frac{1}{2}x_{n-2}\Big)\Big)+\frac{1}{(n+1)(n+2)}x_{n-1}=0,\ \ n\geq 1. \tag{37}$$

Here $r_n=\frac{1}{n}$, $p_n=\frac{1}{2}$, $q_n=\frac{1}{(n+1)(n+2)}$, $\alpha=1$ $\sigma(n)=n-2$, $\tau(n)=n-1$ and $L=1$. It is easy to see that all conditions of Theorem 2.2 are satisfied. Hence every solution of equation (37) is either oscillatory or converges to zero as $n\to\infty$.

Example 3.3. *Consider the third order difference equation*

$$\Delta^3\Big(x_n-\frac{1}{3}x_{n-1}\Big)+nx_{n-2}=0,\ \ n\geq 1. \tag{38}$$

Here $r_n=1$, $p_n=\frac{1}{3}$, $q_n=n$, $\alpha=1$ $\sigma(n)=n-1$, $\tau(n)=n-2$ and $L=1$. It is easy to see that all conditions of Theorem 2.3 are satisfied. Hence every solution of equation (38) is either oscillatory or converges to zero as $n\to\infty$.

Example 3.4. *Consider the third order difference equation*

$$\Delta^3\Big(x_n-\frac{1}{2}x_{n-1}\Big)+12x_{n-2}=0,\ \ n\geq 1. \tag{39}$$

Here $r_n=1$, $p_n=\frac{1}{2}$, $q_n=12$, $\alpha=1$ $\sigma(n)=n-1$, $\tau(n)=n-2$ and $L=1$. It is easy to see that all conditions of Theorem 2.4 are satisfied. Hence every solution of equation (39) is either oscillatory or converges to zero as $n\to\infty$. In fact $\{x_n\}=\{(-1)^n\}$ is one such oscillatory solution of equation (39) is oscillatory or converging to zero.

We conclude this paper with the following remark.

Remark 3.1. *It would be interesting to extend the results of this paper to the equation (1) when $\sum_{n=n_0}^{\infty}\frac{1}{r_n^{1/\alpha}}<\infty$.*

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