ISSN 2319-3786
VOLUME 3, ISSUE 3, JULY 2015

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Malaya Journal of Matematik

an international journal of mathematical sciences

UNIVERSITY PRESS
5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),
Coimbatore- 641114, Tamil Nadu, India.
www.malayajournal.org

The Malaya Journal of Matematik is published quarterly in single volume annually and four issues constitute one volume appearing in the months of January, April, July and October.

Subscription

The subscription fee is as follows:

USD 350.00 For USA and Canada

Euro 190.00 For rest of the world

Rs. 4000.00 In India. (For Indian Institutions in India only)

Prices are inclusive of handling and postage; and issues will be delivered by Registered Air-Mail for subscribers outside India.

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Coimbatore- 641114, Tamil Nadu, India.

Contact No. : +91-9585408402

E-mail : info@mkdpress.com; editorinchief@malayajournal.org; publishingeditor@malayajournal.org

Website : https://mkdpress.com/index.php/index/index

Vol. 3 No. 03 (2015): Malaya Journal of Matematik (MJM)

Malaya J. Mat. 3(3)(2015) 224–232

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

An adaptive integration scheme using a mixed quadrature of three different quadrature rules

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Abstract

In the present work,a mixed quadrature rule of precision seven is constructed blending Gauss-Legendre 2-point rule, Fejer's first and second 3-point rules each having precision three.The error analysis of the mixed rule is incorporated.An algorithm is designed for adaptive integration scheme using the mixed quadrature rule.Through some numerical examples,the effectiveness of adopting mixed quadrature rule in place of their constituent rules in the adaptive integration scheme is discussed.

Keywords: Gauss-Legendre quadrature, Fejer's quadrature, mixed quadrature and adaptive integration scheme

2010 MSC: 65D30, 65D32.

1 Introduction

In this article, we consider the following problem. Given a continuous function $f(x)$ over a bounded interval $[a,b]$ and a prescribed tolerance ϵ, we seek to find an approximation $Q(f)$ using a mixed quadrature rule to the integral

$$I(f) = \int_a^b f(x)dx \tag{1.1}$$

so that

$$|Q(f) - I(f)| \leq \epsilon \tag{1.2}$$

This can be done following adaptive integration scheme (AIS)[1] [2] [3].

Conte and Boor[3] evaluated real definite integral (1.1) in the adaptive integration scheme using Simpson's $\frac{1}{3}$ rule as a base rule.They fix a termination criterion for adaptive integration scheme using Simpon's $\frac{1}{3}$ two panel rule and Simpson's $\frac{1}{3}$ four panel rule (composite rule). Recently, R.B.Dash and D.Das[7] [8] [9] constructed some mixed quadrature rules and fix the termination criterion for adaptive integration using the mixed quadrature rule and evaluated successfully various real definite integrals. Mixed quadrature [5] [6] [7][8] [9] [10] [11] means a quadrature of higher precision which is formed by taking the linear/ convex combination of two or more quadrature rules of equal lower precision.

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The idea of mixed quadrature was first given by R.N. Das and G. Pradhan (1996) [5], who constructed a mixed quadrature rule of precision 5 blending Simpson's $\frac{1}{3}$ rule with Gauss- Legendre 2-point rule, each having precision 3. Evaluating some real definite integrals on the whole interval, they showed the superiority of the mixed quadrature rule over their constituent rules. N. Das and S.K. Pradhan(2004)[6] derived a mixed quadrature rule of precision 7 by taking a linear combination of Simpson's $\frac{1}{3}$ rule,Simpson's $\frac{3}{8}$ rule and Gauss-Legendre 2-point rule, each having precision 3. They also showed the superiority of the mixed quadrature rule over their constituent rules by evaluating some real definite integrals in the whole interval method.

In this paper, we have constructed a mixed quadrature rule of precision 7 by mixing Gauss-Legendre 2-point rule[4] with Fejer's first and second 3-point rules[2] [10] each having equal precision (i.e. precision 3) for approximating some real definite integrals in the adaptive integration scheme. The construction of mixed quadrature rule is outlined in the following section.

2 Construction of the mixed quadrature rule of precision seven

A mixed quadrature rule of precision seven is constructed by using the following three well-known quadrature rules.

(i) Gauss- Legendre 2-point rule

(ii) Fejer's first 3-point rule

(iii) Fejer's second 3- point rule

The Gauss-Legendre 2-point rule $(R_{GL_2}(f))$ is

$$I(f)=\int_a^b f(x)dx=\int_{-1}^{1} f(x)dx \approx R_{GL_2}(f)=f(-\frac{1}{\sqrt{3}})+f(\frac{1}{\sqrt{3}}) \tag{2.3}$$

The Fejer's first 3-point rule $(R_{1_{F_3}}(f))$ is

$$I(f)=\int_a^b f(x)dx=\int_{-1}^{1} f(x)dx \approx R_{1_{F_3}}(f)=\frac{1}{9}[4f(\frac{-\sqrt{3}}{2})+10f(0)+4f(\frac{\sqrt{3}}{2})] \tag{2.4}$$

The Fejer's second 3-point rule $(R_{2_{F_3}}(f))$ is

$$I(f)=\int_a^b f(x)dx=\int_{-1}^{1} f(x)dx \approx R_{2_{F_3}}(f)=\frac{2}{3}[f(\frac{-1}{\sqrt{2}})+f(0)+f(\frac{1}{\sqrt{2}})] \tag{2.5}$$

Each of these rules (2.1), (2.2) and (2.3) is of precision 3. Let $E_{GL_2}(f), E_{1_{F_3}}(f), E_{2_{F_3}}(f)$ denote the errors in approximating the integral $I(f)$ by the rules (2.1), (2.2) and (2.3) respectively.

Then,

$$I(f)=R_{GL_2}(f)+E_{GL_2}(f) \tag{2.6}$$

$$I(f) = R_{1_{F_3}}(f) + E_{1_{F_3}}(f) \tag{2.7}$$

$$I(f) = R_{2_{F_3}}(f) + E_{2_{F_3}}(f) \tag{2.8}$$

Assuming $f(x)$ to be sufficiently differentiable in $-1 \leq x \leq 1$, and using Maclaurin's expansion of function $f(x)$, we can express the errors associated with the quadrature rules under reference as

$E_{GL_2}(f) = \frac{8}{5!\times 9} f^{(iv)}(0) + \frac{40}{7!\times 27} f^{(vi)}(0) + \frac{16}{9!\times 9} f^{(viii)}(0) + ...$

$E_{1_{F_3}}(f) = -\frac{1}{5!\times 2} f^{(iv)}(0) - \frac{5}{8!} f^{(vi)}(0) - \frac{17}{9!\times 32} f^{(viii)}(0) - ...$

$E_{2_{F_3}}(f) = \frac{1}{3\times 5!} f^{(iv)}(0) + \frac{5}{6\times 7!} f^{(vi)}(0) + \frac{5}{4\times 9!} f^{(viii)}(0) + ...$

Now multiplying the Eqs (2.4), (2.5) and (2.6) by 27, 32 and -24 respectively, then adding the results we obtain,

$I(f) = \frac{1}{35}(27R_{GL_2}(f) + 32R_{1_{F_3}}(f) - 24R_{2_{F_3}}(f)) + \frac{1}{35}(27E_{GL_2}(f) + 32E_{1_{F_3}}(f) - 24E_{2_{F_3}}(f))$

$$I(f) = R_{GL_2 1_{F_3} 2_{F_3}}(f) + E_{GL_2 1_{F_3} 2_{F_3}}(f) \tag{2.9}$$

Where

$$R_{GL_2 1_{F_3} 2_{F_3}}(f) = \frac{1}{35}(27R_{GL_2}(f) + 32R_{1_{F_3}}(f) - 24R_{2_{F_3}}(f)) \tag{2.10}$$

And

$$E_{GL_2 1_{F_3} 2_{F_3}}(f) = \frac{1}{35}(27E_{GL_2}(f) + 32E_{1_{F_3}}(f) - 24E_{2_{F_3}}(f)) \tag{2.11}$$

Eq.(2.8) expresses the desired mixed quadrature rule for the approximate evaluation of $I(f)$ and Eq (2.9) expresses the error generated in this approximation.

Hence,

$$E_{GL_2 1_{F_3} 2_{F_3}}(f) = \frac{1}{9! \times 35} f^{(viii)}(0) + ... \tag{2.12}$$

As the first term of $E_{GL_2 1_{F_3} 2_{F_3}}(f)$ contains 8^{th} order derivative of the integrand, the degree of precision of the mixed quadrature rule is 7. It is called a mixed type rule as it is constructed from three different types of rules of equal precision.

3 Error analysis of the mixed quadrature rule

An asymptotic error estimate and an error bound of the rule (2.8) are given in theorems 3.1 and 3.2 respectively.

Theorem-3.1

Let $f(x)$ be a sufficiently differentiable function in the closed interval $[-1,1]$. Then the error $E_{GL_2 1_{F_3} 2_{F_3}}(f)$ associated with the mixed quadrature rule $R_{GL_2 1_{F_3} 2_{F_3}}(f)$ is given by

$$|E_{GL_2 1_{F_3} 2_{F_3}}(f)| \approx \frac{1}{9! \times 35} |f^{(viii)}(0)|$$

Proof The proof follows from the Eq (2.10).

Theorem 3.2

The bound for the truncation error $E_{GL_2 1_{F_3} 2_{F_3}}(f) = I(f) - R_{GL_2 1_{F_3} 2_{F_3}}(f) \ \ is \ \ given \ \ by$

$$E_{GL_2 1_{F_3} 2_{F_3}}(f) \leq \frac{2M}{175}$$

$$where M = max_{-1 \leq x \leq 1} |f^{(v)}(x)|$$

Proof

$$E_{GL_2}(f) = \frac{8}{5! \times 9} f^{(iv)}(\eta_1), \qquad \eta_1 \in [-1,1]$$

$$E_{1_{F_3}}(f) = -\frac{1}{5! \times 2} f^{(iv)}(\eta_2), \qquad \eta_2 \in [-1,1]$$

$$E_{2_{F_3}}(f) = \frac{1}{5! \times 3} f^{(iv)}(\eta_3), \qquad \eta_3 \in [-1,1]$$

$$E_{GL_2 1_{F_3} 2_{F_3}}(f) = \frac{1}{35}[27 E_{GL_2}(f) + 32 E_{1_{F_3}}(f) - 24 E_{2_{F_3}}(f)]$$

$$= \frac{24}{5! \times 35} f^{(iv)}(\eta_1) - \frac{16}{5! \times 35} f^{(iv)}(\eta_2) - \frac{8}{5! \times 35} f^{(iv)}(\eta_3)$$

Let $K = max_{x \in [-1,1]} |f^{(iv)}(x)| \ \ and \ \ k = min_{x \in [-1,1]} |f^{(iv)}(x)|$. As $f^{(iv)}(x)$ is continuous and $[-1,1]$ is compact, there exist points b and a in the interval $[-1,1]$ such that $K = f^{(iv)}(b) \ \ and \ \ k = f^{(iv)}(a)$. Thus

$$E_{GL_2 1_{F_3} 2_{F_3}}(f) \leq \frac{24}{5! \times 35} f^{(iv)}(b) - \frac{16}{5! \times 35} f^{(iv)}(a) - \frac{8}{5! \times 35} f^{(iv)}(a)$$

$$= \frac{24}{5! \times 35} [f^{(iv)}(b) - f^{(iv)}(a)]$$

$$= \frac{1}{175} \int_a^b f^{(v)}(x) dx$$

$= \frac{1}{175}(b-a) f^{(v)}(\xi) \ \ for \ \ some \ \ \xi \in [-1,1]$ by mean value theorem.

Hence by choosing $|(b-a)| \leq 2$

we have $E_{GL_2 1_{F_3} 2_{F_3}}(f) \leq \frac{1}{175} |(b-a)| |f^{(v)}(\xi)| \leq \frac{2M}{175}$

Where $M = max_{-1\leq x\leq 1}|f^{(v)}(x)|$

4 Algorithm for adaptive quadrature routine

Applying the constituent rules $(R_{GL_2}(f), R_{1_{F_3}}(f), R_{2_{F_3}}(f))$ and the mixed quadrature rule $(R_{GL_2 1_{F_3} 2_{F_3}}(f))$, one can evaluate real definite integrals of the type $I(f) = \int_a^b f(x)dx$ in adaptive integration scheme. In the adaptive integration scheme, the desired accuracy is sought by progressively subdividing the interval of integration according to the computed behavior of the integrand, and applying the same formula over each subinterval. A simple adaptive strategy is outlined using the mixed quadrature rule $(R_{GL_2 1_{F_3} 2_{F_3}}(f))$ in the following four step algorithm.

Input: Function $F : [a, b] \longrightarrow R$ and the prescribed tolerance ϵ .

Output: An approximation $Q(f)$ *to the integral* $I(f) = \int_a^b f(x)dx$ *such that* $|Q(f) - I(f)| \leq \epsilon$.

Step-1: The mixed quadrature rule $(R_{GL_2 1_{F_3} 2_{F_3}}(f))$ is applied to approximate the integral $I(f) = \int_a^b f(x)dx$.

The approximate value is denoted by $(R_{GL_2 1_{F_3} 2_{F_3}}[a, b])$.

Step-2 : The interval of integration $[a, b]$ is divided into two equal pieces, $[a, c]$ *and* $[c, b]$. The mixed

quadrature rule $(R_{GL_2 1_{F_3} 2_{F_3}}(f))$ is applied to approximate the integral $I_1(f) = \int_a^c f(x)dx$ and the approximate value is denoted by $(R_{GL_2 1_{F_3} 2_{F_3}}[a, c])$. Similarly, the mixed quadrature rule $(R_{GL_2 1_{F_3} 2_{F_3}}(f))$ is applied to approximate the integral $I_2(f) = \int_c^b f(x)dx$ and the approximate value is denoted by $(R_{GL_2 1_{F_3} 2_{F_3}}[c, b])$.

Step-3:$(R_{GL_2 1_{F_3} 2_{F_3}}[a, c] + (R_{GL_2 1_{F_3} 2_{F_3}}[c, b])$ is compared with $(R_{GL_2 1_{F_3} 2_{F_3}}[a, b])$ to estimate the error in

$(R_{GL_2 1_{F_3} 2_{F_3}}[a, c] + (R_{GL_2 1_{F_3} 2_{F_3}}[c, b])$.

Step-4: If $|estimated\ \ error| \leq \frac{\epsilon}{2}$ (termination criterion) then $(R_{GL_2 1_{F_3} 2_{F_3}}[a, c] + R_{GL_2 1_{F_3} 2_{F_3}}[c, b])$ is accepted as

an approximation to $I(f) = \int_a^b f(x)dx$. Otherwise the same procedure is applied to $[a, c]$ *and* $[c, b]$, allowing each pieces a tolerance of $\frac{\epsilon}{2}$. If the termination criterion is not satisfied on one or more of the sub intervals, then those sub-intervals must be further subdivided and the entire process repeated. When the process stops, the addition of all accepted values yields the desired approximate value $Q(f)$ of the integral $I(f)$ such that $|Q(f) - I(f)| \leq \epsilon$.

N:B: In this algorithm we can use any quadrature rule to evaluate real definite integrals in adaptive integration scheme.

5 Numerical verification

Table 5.1: Comparative study among the quadrature rule $R_{GL_2}(f), R_{1_{F_3}}(f)$ *and* $R_{2_{F_3}}(f)$ for approximation of some real definite integrals without using adaptive integration scheme

Integrals	**Exact Value** $I(f)$	**Approximate Value (**$Q(f)$**) by**		
		$R_{GL_2}(f)$	$R_{1_{F_3}}(f)$	$R_{2_{F_3}}(f)$
$I_1(f)=\int_0^1 \frac{4}{1+x^2}dx$	$\pi \approx 3.14159265358$	3.14754	3.1379	3.14336
$I_2(f)=\int_0^3 \frac{sin2x}{1+x^2}dx$	0.4761463020	0.7939	0.2752	0.5673
$I_3(f)=\int_0^3 (sin4x)e^{-2x}dx$	0.1997146621	0.2398	0.2955	0.3898
$I_4(f)=\int_{0.04}^1 \frac{1}{\sqrt{x}}dx$	1.6	1.5116	1.620	1.5419
$I_5(f)=\int_0^2 \frac{1}{x^2+\frac{1}{10}}dx$	4.4713993943	3.9753	4.9022	4.4155
$I_6(f)=\int_{\frac{1}{2\pi}}^2 sin(\frac{1}{x})dx$	1.1140744942	1.4263	0.8665	1.2698
$I_7(f)=\int_0^{\frac{\pi}{2}} (x^2+x+1)cosxdx$	2.038197427067	2.0366	2.0389	2.0375
$I_8(f)=\int_0^5 \frac{x^3}{e^x-1}dx$	4.8998922	4.6016	5.0588	4.7760
$I_9(f)=\int_0^1 e^{-x^2}dx$	0.7468241328	0.7465	0.7469	0.7467
$I_{10}(f)=\int_0^4 13(x-x^2)e^{-\frac{3x}{2}}dx$	-1.5487883725279	-0.5999	-1.7966	-0.8318
$I_{11}(f)=\int_0^2 \sqrt{4x-x^2}dx$	π	3.1844	3.1312	3.1683
$I_{12}(f)=\int_1^6 [2+sin(2\sqrt{x})]dx$	8.1834792077	8.2627	8.1420	8.2171
$I_{13}(f)=\int_0^1 \frac{1}{1+x^4}dx$	0.8669729870	0.8595	0.8715	0.8646
$I_{14}(f)=\int_0^1 sin(\sqrt{x})dx$	0.6023373578	0.6097	0.6005	0.6069

Table 5.2: Comparative study among the quadrature/mixed quadrature rules ($R_{GL_3}(f)$,$R_{2_{F_5}}(f)$ and $R_{GL_2 1_{F_3} 2_{F_3}}(f)$) for approximation of integrals (table 5.1) without using adaptive integration scheme

Integrals	**Exact Value** $I(f)$	**Approximate Value (**$Q(f)$**) by**		
		$R_{GL_3}(f)$	$R_{2_{F_5}}(f)$	$R_{GL_2 1_{F_3} 2_{F_3}}(f)$
$I_1(f)=\int_0^1 \frac{4}{1+x^2}dx$	$\pi \approx 3.14159265358$	3.14106	3.14147	3.1415979
$I_2(f)=\int_0^3 \frac{sin2x}{1+x^2}dx$	0.4761463020	0.4415	0.4659	0.4751
$I_3(f)=\int_0^3 (sin4x)e^{-2x}dx$	0.1997146621	0.3913	0.2326	0.1878
$I_4(f)=\int_{0.04}^1 \frac{1}{\sqrt{x}}dx$	1.6	1.5667	1.5844	1.5905
$I_5(f)=\int_0^2 \frac{1}{x^2+\frac{1}{10}}dx$	4.4713993943	4.6629	4.5628	4.5209
$I_6(f)=\int_{\frac{1}{2\pi}}^2 sin(\frac{1}{x})dx$	1.1140744942	1.1304	1.0498	1.0219
$I_7(f)=\int_0^{\frac{\pi}{2}} (x^2+x+1)cosxdx$	2.038197427067	2.03810	2.03817	2.03819762
$I_8(f)=\int_0^5 \frac{x^3}{e^x-1}dx$	4.8998922	4.8862	4.8968	4.90003
$I_9(f)=\int_0^1 e^{-x^2}dx$	0.7468241328	0.746814	0.746822	0.74682421
$I_{10}(f)=\int_0^4 13(x-x^2)e^{-\frac{3x}{2}}dx$	-1.5487883725279	-1.1196	-1.43307	-1.5350
$I_{11}(f)=\int_0^2 \sqrt{4x-x^2}dx$	π	3.1560	3.1492	3.1468
$I_{12}(f)=\int_1^6 [2+sin(2\sqrt{x})]dx$	8.1834792077	8.1882	8.1847	8.1836
$I_{13}(f)=\int_0^1 \frac{1}{1+x^4}dx$	0.8669729870	0.8675	0.8670	0.866965
$I_{14}(f)=\int_0^1 sin(\sqrt{x})dx$	0.6023373578	0.6048	0.6036	0.6032

$R_{GL_3}(f)$**:Gauss-Legendre 3-point rule**

$R_{2_{F_5}}(f)$**: Fejer's second 5-point rule**

Table 5.3: Comparison of the results following from the Gauss-Legendre 2-point rule, Fejer's first 3-point rule and Fejer's second 3-point rule for approximating integrals using the adaptive integration scheme

Integrals	Approximate value ($Q(f)$) by					
	$(R_{GL_2}(f))$	#steps	$(R1_{F_3}(f))$	#steps	$(R2_{F_3}(f))$	#steps
$I_1(f) = \int_0^1 \frac{4}{1+x^2}dx$	3.141592690	17	3.141592653573	15	3.14159265359	15
$I_2(f) = \int_0^3 \frac{sin2x}{1+x^2}dx$	0.47614627	41	0.476146256	35	0.476146332	35
$I_3(f) = \int_0^3 (sin4x)e^{-2x}dx$	0.199714693	51	0.199714686	43	0.19971459	39
$I_4(f) = \int_{0.04}^1 \frac{1}{\sqrt{x}}dx$	1.59999986	39	1.6000001	35	1.59999986	31
$I_5(f) = \int_0^2 \frac{1}{x^2+\frac{1}{10}}dx$	4.471399346	53	4.471399461	49	4.471399326	43
$I_6(f) = \int_{\frac{1}{2\pi}}^2 sin(\frac{1}{x})dx$	1.114074589	51	1.114074448	43	1.114074503	41
$I_7(f) = \int_0^{\frac{\pi}{2}} (x^2+x+1)cosxdx$	2.0381974132	23	2.0381974183	17	2.0381974106	15
$I_8(f) = \int_0^5 \frac{x^3}{e^x-1}dx$	4.899892102	43	4.899892237	39	4.899892026	29
$I_9(f) = \int_0^1 e^{-x^2}dx$	0.7468241276	15	0.746824114	13	0.746824120	11
$I_{10}(f) = \int_0^4 13(x-x^2)e^{-\frac{3x}{2}}dx$	-1.5487882018	57	-1.5487884508	51	-1.5487882663	47
$I_{11}(f) = \int_0^2 \sqrt{4x-x^2}dx$	3.1415929475	45	3.141592395	37	3.141592855	39
$I_{12}(f) = \int_1^6 [2+sin(2\sqrt{x})]dx$	8.1834793329	31	8.18347908	27	8.183479317	25
$I_{13}(f) = \int_0^1 \frac{1}{1+x^4}dx$	0.8669729661	15	0.86697299	15	0.866972942	13
$I_{14}(f) = \int_0^1 sin(\sqrt{x})dx$	0.602337696	29	0.602337112	25	0.602337592	25

N:B:The prescribed tolerance(ϵ)=0.000001

 # Steps: No. of Steps

Table 5.4: Comparison of the results following from the Gauss-Legendre 3-point rule, Fejer's second 5-point rule and mixed quadrature rule $R_{GL_2 1_{F_3} 2_{F_3}}(f)$ for approximating integrals (given in table 5.3) using the adaptive integration scheme

Integrals	Approximate Value ($Q(f)$) by					
	($R_{GL_3}(f)$)	#steps	($R_{2_{F_5}}(f)$)	# steps	($R_{GL_2 1_{F_3} 2_{F_3}}(f)$)	#steps
$I_1(f) = \int_0^1 \frac{4}{1+x^2}dx$	3.14159265347	7	3.141592651	3	3.141592653589621	3
$I_2(f) = \int_0^3 \frac{sin2x}{1+x^2}dx$	0.4761463032	15	0.4761463085	11	0.4761463008	5
$I_3(f) = \int_0^3 (sin4x)e^{-2x}dx$	0.1997146667	19	0.1997146587	13	0.1997146616	9
$I_4(f) = \int_{0.04}^1 \frac{1}{\sqrt{x}}dx$	1.599999987	17	1.599999985	13	1.599999998	9
$I_5(f) = \int_0^2 \frac{1}{x^2+\frac{1}{10}}dx$	4.4713993946	17	4.471399387	15	4.471399396	11
$I_6(f) = \int_{\frac{1}{2\pi}}^2 sin(\frac{1}{x})dx$	1.114074506	21	1.114074477	19	1.114074495	11
$I_7(f) = \int_0^{\frac{\pi}{2}} (x^2+x+1)cosxdx$	2.0381974267	7	2.0381974227	3	2.03819742776	1
$I_8(f) = \int_0^5 \frac{x^3}{e^x-1}dx$	4.8998921534	13	4.8998921579	7	4.899892158	3
$I_9(f) = \int_0^1 e^{-x^2}dx$	0.7468241324	3	0.7468241327	3	0.7468241329	1
$I_{10}(f) = \int_0^4 13(x-x^2)e^{-\frac{3x}{2}}dx$	-1.5487883665	21	-1.548788353	13	-1.5487883721	9
$I_{11}(f) = \int_0^2 \sqrt{4x-x^2}dx$	3.1415928159	25	3.1415928990	19	3.141592813	19
$I_{12}(f) = \int_1^6 [2+sin(2\sqrt{x})]dx$	8.1834792212	9	8.1834792108	9	8.1834792081	5
$I_{13}(f) = \int_0^1 \frac{1}{1+x^4}dx$	0.8669729873	7	0.886972987	7	0.8669729873	3
$I_{14}(f) = \int_0^1 sin(\sqrt{x})dx$	0.602337586	17	0.602337475	17	0.60233758	15

N:B:The prescribed tolerance(ϵ)=0.000001

All the computations are done using 'C' Program[8].

6 Conclusion

We observe from Tables-5.1 and 5.2, that the mixed quadrature rule gives more accurate result in comparison to their constituent rules. Gauss-Legendre 3-point rule and Fejer's second 5-point rule when integrals ($I_1 - I_{14}$) are evaluated without using adaptive integration scheme. Tables-5.3 and 5.4, reveal that when these integrals are evaluated using the adaptive integration scheme, the mixed qudrature rule reduces the number of steps to achieve the prescribed accuracy and gives more accurate result in comparison to the their constituent rules, Gauss-Legendre 3-point rule and Fejer's second 5-point rule.

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Received: October 10, 2014; *Accepted*: May 23, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 233–240

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

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ISSN : 2319-3786

Line gracefulness in the context of switching of a vertex

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Abstract

We investigate line graceful labeling of graphs obtained by switching of vertex operation.

Keywords: Edge graceful labeling, line graceful labeling, graceful labeing, switching of vertex.

2010 MSC: 05C78, 05C38, 05C76.

1 Introduction

Labeling of discrete structures is a one of the potential area of research due to its potential applications. The optimal linear arrangement concern to network problems in electrical engineering and placement problems in production engineering can be formalized as a graph labeling problems as stated by Yegnanaryanan and Vaidhyanathan [13]. A dynamic survey on different graph labeling schemes with an extensive bibliography can be found in Gallian [2].
In this paper, the term "graph" means finite, connected, undirected and simple graph $G = (V(G), E(G))$ with p vertices and q edges. For standard terminology and notation we refer to Balakrishnan and Ranganathan [1].

Definition 1.1. *A* graph labeling *is an assignment of numbers to the vertices or edges or both subject to certain condition(s).*

Definition 1.2. *A function f is called* graceful labeling *of graph if $f : V(G) \rightarrow \{0, 1, 2, 3, ..., q\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 2, ..., q\}$ defined as $f^*(e = uv) = |f(x) - f(y)|$ is bijective. A graph which admits graceful labeling is called a graceful graph.*

Most of the graph labeling techniques trace their origin with graceful labeling which was introduced independently by Rosa [7] and Golomb [4]. A variant of graceful labeling termed as edge graceful labeling is introduced by Lo [6].

Definition 1.3. *A graph $G = (V(G), E(G))$ is said to be* edge graceful *if there exists a bijection $f : E(G) \rightarrow \{1, 2, 3, ..., q\}$ such that the induced mapping $f^* : V(G) \rightarrow \{0, 1, ..., p-1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i) \pmod p$ is bijection.*

Lo [6] derived a necessary condition for a graph to be edge graceful and also investigate edge graceful labeling of many graph families. Wilson and Risking [12] proved that the cartesian product of any number of odd cycle is edge graceful. All trees of odd order are edge graceful was conjunctured by Lee [5]. Shiu, Lee and Schaffer [8] investigated the edge gracefulness of multigraphs. Gnanajothi [3] introduced and studied line graceful labeling in her Ph.D. thesis which is little weaker than edge graceful labeling.

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Definition 1.4. *A mapping $f : E(G) \rightarrow \{0, 1, 2, ..., p\}$ is called* line graceful *of graph with p vertices, if induced function $f^* : V(G) \rightarrow \{0, 1, 2, ..., p-1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i) \pmod p$ is bijective.*

Definition 1.5. *The triangular snake T_n is obtained from the path P_n by replacing every edge of a path by a triangle C_3 .*

Definition 1.6. *A vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges to v and adding edges joining v to every other vertex which are not adjacent to v in G.*

Definition 1.7. *The* helm *H_n is the graph obtained from a wheel W_n by attaching a pendant edge to every rim vertex.*

Definition 1.8. *The* fan *f_n is a graph on $n+1$ vertices obtained by joining all the vertices of P_n to a new vertex called the center.*

Vaidya and Kothari [9, 10, 11] have investigated many results on line gracefulness of graphs in various contexts while this paper is focus on line gracefulness on the graph obtained by switching of a vertex.

2 Main results

Proposition 2.1. *[3] If the graph is line graceful then its order is not congruent to* $2 \pmod 4$.

Theorem 2.1. *Switching of a pendant vertex in path P_n is line graceful except* $n \equiv 2 \pmod 4$.

Proof. Let $v_1, v_2, \ldots, v_n$ be vertices of path P_n. Let G_v be the graph obtained by switching pendant vertex v of P_n. Without loss of generality let the switched vertex be v_n. We note that $|V(G_v)| = n$ and $|E(G_v)| = 2n-4$. Define edge labeling $f : E(G_v) \rightarrow \{0, 1, \ldots, n-1\}$ as follows.

Case 1: $n \equiv 0 \pmod 4$

<u>for odd i</u>

$$f(v_i v_{i+1}) = \begin{cases} \frac{i+1}{2} & \text{for } 1 \leq i \leq \frac{n}{2} \\ \frac{i+1}{2} + 1 & \text{for } \frac{n}{2} \leq i \leq n-3 \end{cases}$$

<u>for even i</u>

$$\begin{aligned} f(v_i v_{i+1}) &= \tfrac{i+2}{2} && \text{for } 2 \leq i \leq n-3 \\ f(v_{n-2} v_{n-1}) &= \tfrac{n}{2} + 2 \\ f(v_n v_i) &= 0 && \text{for } 1 \leq i \leq n-2 \end{aligned}$$

Case 2: $n \equiv 1 \pmod 4$

<u>for odd i</u>

$$f(v_i v_{i+1}) = \begin{cases} \frac{i+1}{2} & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ \frac{i+3}{2} & \text{for } \lceil \frac{n}{2} \rceil \leq i \leq n-3 \end{cases}$$

<u>for even i</u>

$$\begin{aligned} f(v_i v_{i+1}) &= \tfrac{i+2}{2} && \text{for } 2 \leq i \leq n-3 \\ f(v_{n-2} v_{n-1}) &= 2 \\ f(v_n v_i) &= 0 && \text{for } 1 \leq i \leq n-2 \end{aligned}$$

Case 3: $n \equiv 3 \pmod 4$

<u>for odd i</u>

$$f(v_i v_{i+1}) = \begin{cases} \frac{i+1}{2} & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ \frac{i+3}{2} & \text{for } \lceil \frac{n}{2} \rceil \leq i \leq n-2 \end{cases}$$

<u>for even i</u>

$$\begin{aligned} f(v_iv_{i+1}) &= \tfrac{i+2}{2} && \text{for } 2 \le i \le n-3 \\ f(v_{n-2}v_{n-1}) &= \left\lfloor \tfrac{n}{2} \right\rfloor + 3 \\ f(v_nv_i) &= 0 && \text{for } 1 \le i \le n-2 \end{aligned}$$

Case 4: $n \equiv 2\,(\mathrm{mod}\ 4)$
In this case $|V(G_v)| = n \equiv 2\,(\mathrm{mod}\ 4)$.
Then according to Proposition 2.1 G_v is not line graceful.
In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_v) \to \{0, 1, ..., n-1\}$ such that $f^*(v) = \sum_{e \in E(G_v)} f(e) \pmod{(n)}$ for $n \equiv 0, 1, 3\,(\mathrm{mod}\ 4)$. Hence we proved that graph G_v obtained from switching of pendant vertex in path P_n is line graceful except $n \equiv 2\,(\mathrm{mod}\ 4)$.

□

Illustration 2.1. *Switching of vertex v_8 in path P_8 and its line graceful labeling is shown in figure* 2.

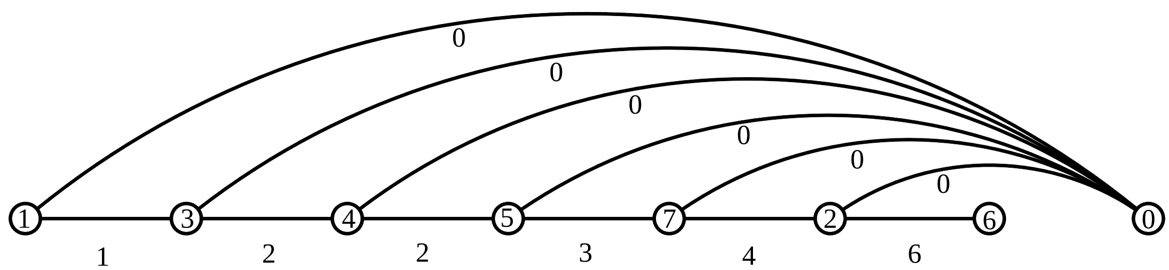

figure 2

Theorem 2.2. *Switching of vertex in cycle C_n is line graceful except $n \equiv 2\,(\mathrm{mod}\ 4)$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and G_{v_1} be the graph obtained by switching of vertex v_1 of cycle C_n. Here without loss of generality, we have switched the vertex v_1. Note that $|V(G_{v_1})| = n$ and $|E(G_{v_1})| = 2n - 5$. Define edge labeling $f : E(G_{v_1}) \to \{0, 1, \ldots, n-1\}$ as follows.
Case 1: $n \equiv 0\,(\mathrm{mod}\ 4)$

$$f(v_1v_i) = 0 \quad \text{for } 3 \le i \le n-1$$

<u>for odd i</u>

$$f(v_iv_{i+1}) = \tfrac{i+1}{2} \quad \text{for } 3 \le i \le n-3$$

<u>for even i</u>

$$f(v_iv_{i+1}) = \begin{cases} \frac{i}{2} & \text{for } 2 \le i \le \frac{n}{2} \\ \frac{i}{2} + 1 & \text{for } \frac{n}{2} < i \le n-2 \end{cases}$$

$$f(v_{n-1}v_n) = f(v_{n-2}v_{n-1}) + 2$$

Case 2: $n \equiv 1\,(\mathrm{mod}\ 4)$

$$f(v_1v_i) = 0 \quad \text{for } 3 \le i \le n-1$$

<u>for odd i</u>

$$f(v_iv_{i+1}) = \tfrac{i+1}{2} \quad \text{for } 3 \le i \le n-2$$

<u>for even i</u>

$$f(v_iv_{i+1}) = \begin{cases} \frac{i}{2} & \text{for } 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ \frac{i}{2} + 1 & \text{for } \left\lceil \frac{n}{2} \right\rceil \le i \le n-3 \end{cases}$$

$$f(v_{n-1}v_n) = 2$$

Case 3: $n \equiv 3\,(\mathrm{mod}\ 4)$

$$f(v_1v_i) = 0 \quad \text{for } 3 \le i \le n-1$$

for odd i

$$f(v_i v_{i+1}) = \frac{i+1}{2} \quad \text{for } 3 \le i \le n-2$$

for even i

$$f(v_i v_{i+1}) = \begin{cases} \frac{i}{2} & \text{for } 2 \le i \le \lceil \frac{n}{2} \rceil \\ \frac{i}{2}+1 & \text{for } \lceil \frac{n}{2} \rceil < i \le n-3 \end{cases}$$

$$f(v_{n-1}v_n) = \lceil \tfrac{n}{2} \rceil + 2$$

Case 4: $n \equiv 2 \,(\text{mod } 4)$
In this case $|V(G_{v_1})| = n \equiv 2 \,(\text{mod } 4)$.
Then according to Proposition 2.1 G_{v_1} is not line graceful.
In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_{v_1}) \to \{0, 1, ..., n-1\}$ such that $f^*(v) = \sum_{e \in E(G_{v_1})} f(e) \pmod{(n)}$ for $n \equiv 0, 1, 3 \,(\text{mod } 4)$. Hence we proved that graph G_{v_1} obtained from switching of vertex v_1 in cycle C_n is line graceful except $n \equiv 2 \,(\text{mod } 4)$. □

Illustration 2.2. *Switching of vertex v_1 in cycle C_9 and its line graceful labeling is shown in figure 3.*

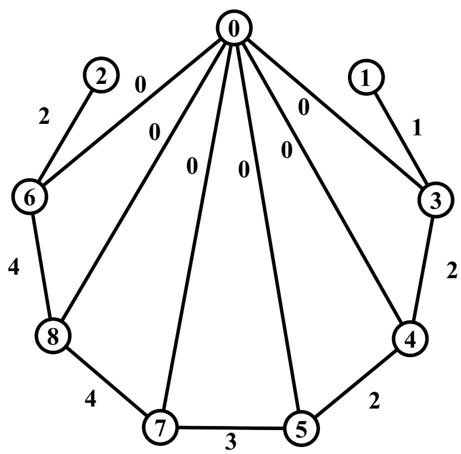

figure 3

Theorem 2.3. *For $n > 3$, switching of a rim vertex in wheel W_n is line graceful except $n \equiv 1 \,(\text{mod } 4)$.*

Proof. Let v be a apex vertex, $v_1, v_2, \ldots, v_n$ be rim vertices of W_n and G_{v_1} be the graph obtained by switching a rim vertex v_1 of W_n. Here without loss of generality, we have switched vertex v_1. Observe that $|V(G_{v_1})| = n+1$ and $|E(G_{v_1})| = 3n-5$. Define edge labeling $f : E(G_{v_1}) \to \{0, 1, \ldots, n\}$. as follows.
Case 1: $n = 4$
The graph G_{v_1} and its line graceful labeling is shown in figure 4.

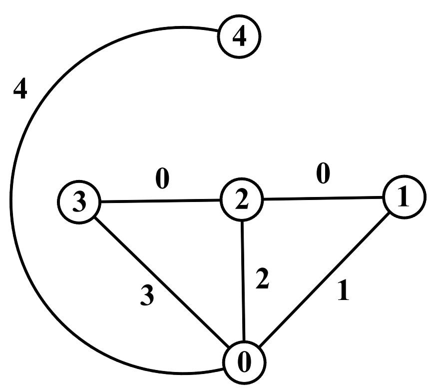

figure 4

Case 2: $n = 6$
The graph G_{v_1} and its line graceful labeling is shown in figure 5.

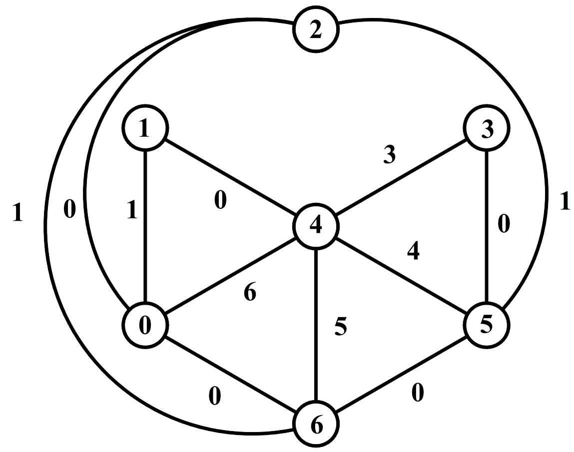

figure 5

Case 3: $n \equiv 0, 2 \pmod 4$

$$\begin{aligned} f(vv_i) &= i+1 \quad \text{for } 2 \le i \le n \\ f(v_iv_{i+1}) &= \begin{cases} 0 & \text{for } 2 \le i \le n-2 \\ 1 & \text{for } i = n-1 \end{cases} \\ f(v_1v_i) &= \begin{cases} 0 & \text{for } 3 \le i \le n-4,\ i = n-1 \\ 1 & n-3 \le i \le n-2 \end{cases} \end{aligned}$$

Case 4: $n \equiv 3 \pmod 4$

$$\begin{aligned} f(vv_i) &= i-1 \quad \text{for } 2 \le i \le n \\ f(v_1v_i) &= 0 \qquad \text{for } 3 \le i \le n-1 \end{aligned}$$

<u>for $2 \le i \le \left\lceil \frac{n}{2} \right\rceil + 1$</u>

$$f(v_iv_{i+1}) = 0$$

<u>for $\left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n-1$</u>

$$f(v_iv_{i+1}) = \begin{cases} 1 & \text{for even } i \\ 0 & \text{for odd } i \end{cases}$$

Case 5: $n \equiv 1 \pmod 4$
In this case $|V(G_{v_1})| = n + 1 \equiv 2 \pmod 4$.
Then according to Proposition 2.1 G_{v_1} is not line graceful.
In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_{v_1}) \to \{0, 1, ..., n\}$ such that $f^*(v) = \sum_{e \in E(G_{v_1})} f(e) \pmod{(n+1)}$ for $n \equiv 0, 2, 3 \pmod 4$. Hence we proved that for $n > 3$, the graph G_{v_1} obtained from switching of a rim vertex v_1 in wheel W_n is line graceful except $n \equiv 1 \pmod 4$. □

Illustration 2.3. *Switching of vertex v_1 in cycle W_{11} and its line graceful labeling is shown in figure 6.*

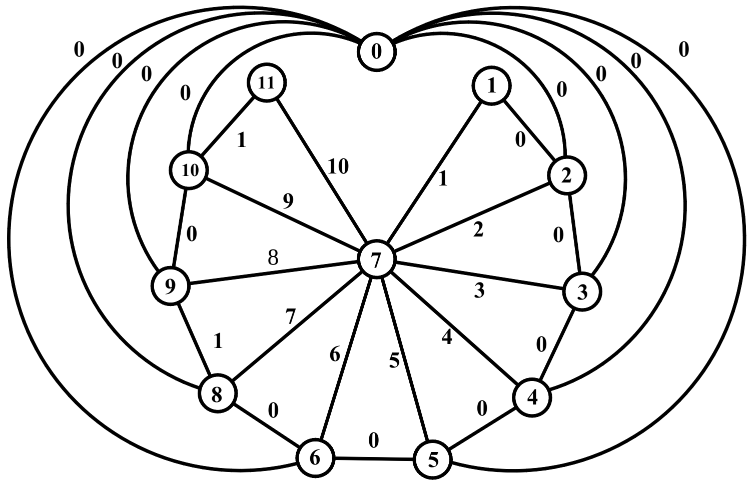

figure 6

Theorem 2.4. *Switching of apex vertex in helm H_n is line graceful for all n.*

Proof. Let v be a apex vertex, $v_1, v_2, \ldots, v_n$ be rim vertices and $u_1, u_2, \ldots, u_n$ be pendant vertices of helm H_n. G_v be the graph obtained from switching apex vertex v of helm. Observe that $|V(G_v)| = 2n+1$ and $|E(G_v)| = 3n$. Define edge labeling $f : E(G_v) \rightarrow \{0, 1, \ldots, 2n\}$ as follows.

$$\begin{aligned} f(vu_i) &= n+1 \\ f(v_iv_{i+1}) &= 0 \quad \text{for } 1 \leq i \leq n-1 \\ f(v_nv_1) &= 0 \end{aligned}$$

for odd n

$$f(v_iu_i) = \left\lfloor \frac{n}{2} \right\rfloor + i \quad \text{for } 1 \leq i \leq n$$

for even n

$$f(v_iu_i) = \frac{3n}{2} + i \quad \text{for } 1 \leq i \leq n$$

In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_v) \rightarrow \{0, 1, ..., 2n\}$ such that $f^*(v) = \sum_{e \in E(G_v)} f(e) \pmod{(2n+1)}$. Thus we proved that graph G_v obtained by switching apex vertex of helm admits line graceful labeling for all n. □

Illustration 2.4. *Switching of apex vertex v in helm H_7 and its line graceful labeling is shown in figure 7.*

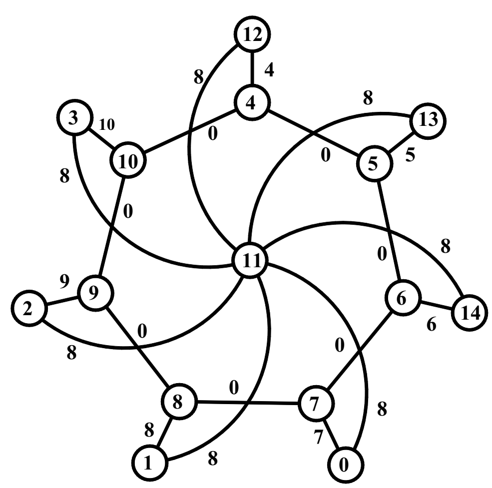

figure 7

Theorem 2.5. *Switching of vertex having degree 2 in fan f_n is line graceful except $n \equiv 1 \pmod 4$.*

Proof. Let v be the apex vertex and $v_1, v_2, \ldots, v_n$ be the vertices of f_n. Let G_{v_1} denotes graph obtained by switching of a vertex v_1 having degree 2 of f_n. Note that $|V(G_{v_1})| = n+1$ and $|E(G_{v_1})| = 3n-5$.
We define $f : E(G_{v_1}) \rightarrow \{1, 2, \ldots, n+1\}$ as follows.
Case 1: $n \equiv 0, 2 \pmod 4$

$$\begin{aligned} f(vv_i) &= n \quad \text{for } 2 \leq i \leq n \\ f(v_i v_{i+1}) &= 0 \quad \text{for } 2 \leq i \leq n-1 \\ f(v_1 v_i) &= \begin{cases} i-2 & \text{for } i = 3, 4 \\ i-1 & \text{for } 5 \leq i \leq n \end{cases} \end{aligned}$$

Case 2: $n \equiv 3 \pmod 4$

$$\begin{aligned} f(vv_i) &= \left\lfloor \tfrac{n}{4} \right\rfloor \quad \text{for } 2 \leq i \leq n \\ f(v_i v_{i+1}) &= 0 \quad \text{for } 2 \leq i \leq n-1 \\ f(v_1 v_i) &= \begin{cases} i-2 & \text{for } 3 \leq i \leq 5 + \left\lfloor \tfrac{n}{4} \right\rfloor \\ i-1 & \text{for } 6 + \left\lfloor \tfrac{n}{4} \right\rfloor \leq i \leq n \end{cases} \end{aligned}$$

Case 3: $n \equiv 1 \pmod 4$
In this case $|V(G_{v_1})| = n+1 \equiv 2 \pmod 4$.
Then according to Proposition 2.1 G_{v_1} is not line graceful.
In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_{v_1}) \rightarrow \{0, 1, ..., n\}$ such that $f^*(v) = \sum_{e \in E(G_{v_1})} f(e) \pmod{(n+1)}$ for $n \equiv 0, 2, 3 \pmod 4$. Hence we proved that the graph G_{v_1} obtained by switching a vertex of degree 2 in fan f_n is line graceful except $n \equiv 1 \pmod 4$.

□

Illustration 2.5. *Switching of vertex v_1 in fan f_{11} and its line graceful labeling is shown in figure* 8.

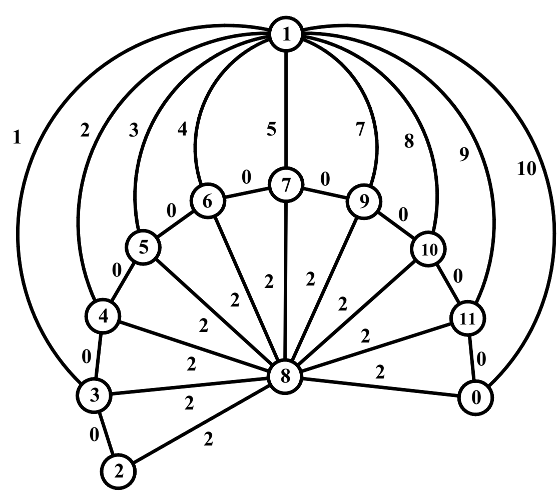

figure 8

3 Concluding Remarks

Edge gracefulness and line gracefulness of a graph are independent concepts. A graph may posses one or both of these or neither as mentioned below.

- C_{2n+1} is edge graceful as well as line graceful.
- P_n is neither edge graceful nor line graceful for $n \equiv 2 \pmod 4$.
- C_{4n} is not edge graceful but line graceful.
- Triangular snake T_n is edge graceful only for $n = 3$ while it is line graceful for all n.

4 Acknowledgment

The authors are highly thankful to the anonymous referee for kind suggestions and comments.

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Received: October 10, 2014; *Accepted*: May 10, 2015

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Malaya J. Mat. 3(3)(2015) 241–249

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

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ISSN : 2319-3786

Hermite Hadamard-Fejer type inequalities for quasi convex functions via fractional integrals

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Abstract

In this paper, Hermite-Hadamard-Fejer type inequalities for quasi-convex via fractional integrals are obtained.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, quasi convex functions.

2010 MSC: 26D07, 26D15.

1 Introduction

The following definition for convex functions is well know in the mathematical literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f : [a,b] \to \mathbb{R}$, is known in the literature as Hermite-Hadamard's inequality. More details, one can consult ([1]-[11]).

In [3], Fejer established the following Hermite-Hadamard Fejer inequality which is the weighted generalization of Hermite-Hadamard inequality.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \tag{1.2}$$

holds, where $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f : [a,b] \to \mathbb{R}$, is said quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \sup\{f(x), f(y)\}, \forall x, y \in [a,b]$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$(see [10]).

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Furthermore, there exist quasi-convex functions which are not convex (see [5]).

In [8] Özdemir et. al. represented Hermite-Hadamard's inequalities for quasi-convex functions in fractional integral forms as follows:

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is quasi convex on $[a,b]$ and $\alpha>0$, then the following inequality for fractional integrals holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \\ \leq\;& \frac{b-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}.\end{aligned} \tag{1.3}$$

In [9] Set et. al. obtained the following lemma.

Lemma 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and let $g:[a,b]\to\mathbb{R}$. If $f',g\in L[a,b]$, then the following identity for fractional integrals holds:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}g(a)+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}g(b)\right]-\left[J_{\left(\frac{a+b}{2}\right)_-}^{\alpha}(fg)(a)+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}(fg)(b)\right] \\ =\;& \frac{1}{\Gamma(\alpha)}\int_a^b k(t)f'(t)\,dt\end{aligned} \tag{1.4}$$

where

$$k(t)=\begin{cases}\displaystyle\int_a^t(s-a)^{\alpha-1}g(s)ds, & t\in\left[a,\frac{a+b}{2}\right) \\ \displaystyle-\int_t^b(b-s)^{\alpha-1}g(s)ds, & t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

In [11] İşcan proved the following lemma.

Lemma 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) and $a<b$ with $f'\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to $(a+b)/2$ then the following equality for fractional integrals holds*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right] \\ =\;& \frac{1}{\Gamma(\alpha)}\int_a^b\left[\int_a^t(b-s)^{\alpha-1}g(s)\,ds-\int_t^b(s-a)^{\alpha-1}g(s)\,ds\right]f'(t)\,dt\end{aligned} \tag{1.5}$$

with $\alpha>0$.

We give some neccessary definitions and mathematical preliminiaries of fractional calculus theory which are used throughout this paper.

Lemma 1.3. *([6],[7])For $0<\alpha\leq 1$ and $0\leq a<b$, we have*

$$|a^{\alpha}-b^{\alpha}|\leq(b-a)^{\alpha}.$$

Definition 1.1. *Let $f\in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}f(x)$ and $J_{b^-}^{\alpha}f(x)$ of oder $\alpha>0$ with $\alpha\geq 0$ are defined by*

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\ x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma functions by $\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt$ and $J_{a^+}^0f(x)=J_{b^-}^0f(x)=f(x)$.

In this paper, motivated by the recent results given in [11], [9] , we established Hermite-Hadamard-Fejer type inequalities for quasi convex functions via fractional integral.

2 Main result

Throughout this paper, let I be an interval on $\mathbb{R}$ and let $\|g\|_{[a,b],\infty} = \sup_{t\in[a,b]} g(t)$, for the continuous function $g:[a,b]\to\mathbb{R}$.

Theorem 2.3. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|^q$ is quasi convex on $[a,b]$, $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|\\ \le\;& \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^\alpha(\alpha+1)\Gamma(\alpha+1)}\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\end{aligned} \tag{2.6}$$

with $\alpha>0$.

Proof. Since is $|f'|^q$ is quasi-convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f'(t)|^q=\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\le\sup\left\{|f'(a)|^q,|f'(b)|^q\right\} \tag{2.7}$$

Using Lemma 1.1, Power mean inequality and the quasi-convex of $|f'|^q$, it follows that

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|\\
\le\;&\frac{1}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}g(s)\,ds\right||f'(t)|^q\,dt\right)^{\frac{1}{q}}\\
&+\frac{1}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}g(s)\,ds\right||f'(t)|^q\,dt\right)^{\frac{1}{q}}\\
\le\;&\frac{\|g\|_{[a,\frac{a+b}{2}],\infty}}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}ds\right||f'(t)|^q\,dt\right)^{\frac{1}{q}}\\
&+\frac{\|g\|_{[\frac{a+b}{2},b],\infty}}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}ds\right||f'(t)|^q\,dt\right)^{\frac{1}{q}}\\
\le\;&\frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\\
&\times\left\{\|g\|_{[a,\frac{a+b}{2}],\infty}\left(\int_a^{\frac{a+b}{2}}(t-a)^\alpha\,dt\right)^{\frac{1}{q}}+\|g\|_{[\frac{a+b}{2},b],\infty}\left(\int_{\frac{a+b}{2}}^b(b-t)^\alpha\,dt\right)^{\frac{1}{q}}\right\}\\
\le\;&\frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{\frac{1}{q}}\\
&\times\left(\|g\|_{[a,\frac{a+b}{2}],\infty}+\|g\|_{[\frac{a+b}{2},b],\infty}\right)\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\\
\le\;&\frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{\Gamma(\alpha+1)2^\alpha(\alpha+1)}\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\end{aligned}$$

where it is easily seen that

$$\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} ds \right| dt = \int_{\frac{a+b}{2}}^b \left| \int_t^b (b-s)^{\alpha-1} ds \right| dt = \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)\alpha}.$$

Hence, the proof is completed.

Corollary 2.1. *If we choose $g(x)=1$ and $\alpha=1$ in the inequality (2.6), then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| \le \frac{b-a}{4} \left(\sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}.$$

We can state another inequality for $q>1$ as follows:

Theorem 2.4. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a<b$ and $g:[a,b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is quasi convex on $[a,b]$, $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} g(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} g(b) \right] - \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} (fg)(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} (fg)(b) \right] \right| \\ &\le \frac{(b-a)^{\alpha+1} \|g\|_\infty}{2^\alpha (\alpha p+1)^{\frac{1}{p}} \Gamma(\alpha+1)} \left(\sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \end{aligned} \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1.1, Hölder's inequality and the quasi convexity of $|f'|^q$, it follows that

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} g(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} g(b) \right] - \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} (fg)(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} (fg)(b) \right] \right| \\ &\le \frac{1}{\Gamma(\alpha)} \left\{ \int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} g(s)\,ds \right| |f'(t)|\,dt \right. \\ &\quad \left. + \int_{\frac{a+b}{2}}^b \left| \int_t^b (b-s)^{\alpha-1} g(s)\,ds \right| |f'(t)|\,dt \right\} \\ &\le \frac{1}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} g(s)\,ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q\,dt \right)^{\frac{1}{q}} \\ &\quad + \frac{1}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b \left| \int_t^b (b-s)^{\alpha-1} g(s)\,ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q\,dt \right)^{\frac{1}{q}} \\ &\le \frac{\|g\|_{\infty,[a,\frac{a+b}{2}]}}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} ds \right|^p \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q\,dt \right)^{\frac{1}{q}} \\ &\quad + \frac{\|g\|_{\infty,[\frac{a+b}{2},b]}}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b \left| \int_t^b (b-s)^{\alpha-1} ds \right|^p \right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q\,dt \right)^{\frac{1}{q}} \\ &\le \frac{\|g\|_\infty}{\Gamma(\alpha)} \left(\frac{(b-a)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)\alpha^p} \right)^{\frac{1}{p}} \left[\left(\int_a^{\frac{a+b}{2}} \sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\int_{\frac{a+b}{2}}^b \sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}} \right] \\ &= \frac{\|g\|_\infty (b-a)^{\alpha+1}}{2^\alpha (\alpha p+1)^{1/p} \Gamma(\alpha+1)} \left(\sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}. \end{aligned}$$

Here we use

$$\int_a^{\frac{a+b}{2}} \left|\int_a^t (s-a)^{\alpha-1} ds\right|^p dt = \int_{\frac{a+b}{2}}^b \left|\int_t^b (b-s)^{\alpha-1} ds\right|^p dt = \frac{(b-a)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)\alpha^p}$$

$$\begin{aligned}\int_a^{\frac{a+b}{2}} \left|f'(t)\right|^q dt &\leq \frac{b-a}{2} \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}\\ \int_{\frac{a+b}{2}}^b \left|f'(t)\right|^q dt &\leq \frac{b-a}{2} \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}.\end{aligned}$$

Hence the inequality (2.8) is proved.

Corollary 2.2. *If we choose $g(x) = 1$ and $\alpha = 1$ in the inequality (2.8), then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left(\sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 2.5. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is quasi convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{(a+b)}{2}$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] - \left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right]\right| \qquad (2.9)\\ \leq\ &\frac{2(b-a)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\sup\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\}\end{aligned}$$

with $\alpha > 0$.

Proof. From Lemma 1.2, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] - \left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right]\right| \qquad (2.10)\\ \leq\ &\frac{1}{\Gamma(\alpha)}\int_a^b \left|\int_a^t (b-s)^{\alpha-1} g(s)\,ds - \int_t^b (s-a)^{\alpha-1} g(s)\,ds\right| \left|f'(t)\right| dt.\end{aligned}$$

Since $|f'|$ is quasi convex on $[a,b]$, we know that for $t \in [a,b]$

$$\left|f'(t)\right| = \left|f'\left(\frac{b-t}{b-a}a + \frac{t-b}{b-a}b\right)\right| \leq \sup\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\} \qquad (2.11)$$

and since $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $(a+b)/2$ we write

$$\begin{aligned}\int_t^b (s-a)^{\alpha-1} g(s)\,ds &= \int_a^{a+b-t} (b-s)^{\alpha-1} g(a+b-s)\,ds\\ &= \int_a^{a+b-t} (b-s)^{\alpha-1} g(s)\,ds.\end{aligned}$$

Then we get

$$\begin{aligned}&\left|\int_a^t (b-s)^{\alpha-1} g(s)\,ds - \int_t^b (s-a)^{\alpha-1} g(s)\,ds\right|\\ =\ &\left|\int_t^{a+b-t} (b-s)^{\alpha-1} g(s)\,ds\right|\end{aligned}$$

$$\leq \begin{cases}\int_t^{a+b-t} \left|(b-s)^{\alpha-1} g(s)\right| ds, & t \in \left[a, \frac{a+b}{2}\right]\\ \int_{a+b-t}^t \left|(b-s)^{\alpha-1} g(s)\right| ds, & t \in \left[\frac{a+b}{2}, b\right]\end{cases} \qquad (2.12)$$

A combination of (2.10), (2.11) and(2.12), we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \qquad (2.13)\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left(\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right)dt\\
&+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt\\
\leq\ & \frac{\|g\|_{\infty}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}}{\Gamma(\alpha)}\\
&\times\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}\right|ds\right)dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}\right|ds\right)dt\right]\\
=\ & \frac{\|g\|_{\infty}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}}{\Gamma(\alpha+1)}\\
&\times\left[\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]dt+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]dt\right].
\end{aligned}$$

Since

$$\int_a^{\frac{a+b}{2}}(b-t)^{\alpha}dt=\int_{\frac{a+b}{2}}^{b}(t-a)^{\alpha}dt=\frac{(b-a)^{\alpha+1}\left(2^{\alpha+1}-1\right)}{2^{\alpha+1}(\alpha+1)} \qquad (2.14)$$

and

$$\int_a^{\frac{a+b}{2}}(t-a)^{\alpha}dt=\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha}dt=\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}. \qquad (2.15)$$

Hence, if we use (2.14) and (2.15) in (2.13), we obtaion the desired result. This completes the proof.

Remark 2.1. *In Theorem 1.5, if we take $g(x)=1$, then inequality (2.9), becomes inequality (1.3) of Theorem 1.2.*

Theorem 2.6. *Let $f: I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^q$, $q\geq 1$, is quasi convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $\frac{(a+b)}{2}$, then the following inequality for fractional integrals holds*

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \qquad (2.16)\\
\leq\ & \frac{2(b-a)^{\alpha+1}\|g\|_{\infty}}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\alpha>0$.

Proof. Using Lemma 1.2, Power mean inequality, (2.12) and the quasi convexity of $|f'|^q$, it follows that

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \tag{2.17}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)\,ds\right|\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\right.$$

$$\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\right]^{1-\frac{1}{q}}$$

$$\times\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left|f'(t)\right|^q dt\right.$$

$$\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left|f'(t)\right|^q dt\right]^{\frac{1}{q}}$$

$$\leq \frac{2(b-a)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}$$

where it is easily seen that

$$\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}\right|ds\right)dt+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}\right|ds\right)dt$$

$$=\frac{2(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right).$$

Hence if we use (2.14) and (2.15) in (2.17), we obtain the desired result. This completes the proof.

We can state another inequality for $q>1$ as follows:

Theorem 2.7. *Let $f: I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^q$, $q>1$, is quasi convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequality for fractional integrals holds*

(i)

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq\frac{2^{\frac{1}{p}}\|g\|_\infty(b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} \tag{2.18}$$

with $\alpha>0$.

(ii)

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq\frac{\|g\|_\infty(b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} \tag{2.19}$$

for $0<\alpha\leq 1$, where $1/p+1/q=1$.

Proof. (i) Using Lemma 1.2, Hölder's inequality, (2.12) and the quasi convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha} g(b)+J_{b-}^{\alpha} g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \qquad (2.20)\\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1} g(s)\, ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]^p dt+\int_{\frac{a+b}{2}}^b\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_a^b \sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\\
=\ & \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]^p dt+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]^p dt\right)^{\frac{1}{p}}\\
&\times\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\\
\leq\ & \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha p}-t^{\alpha p}\right] dt+\int_{\frac{1}{2}}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right] dt\right)^{\frac{1}{p}}\\
&\times\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\\
\leq\ & \frac{2^{\frac{1}{p}}\|g\|_{\infty}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)(\alpha p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Here we use

$$\left[(1-t)^{\alpha}-t^{\alpha}\right]^p \leq (1-t)^{\alpha p}-t^{\alpha p}$$

for $t \in \left[0, \frac{1}{2}\right]$ and

$$\left[t^{\alpha}-(1-t)^{\alpha}\right]^p \leq t^{\alpha p}-(1-t)^{\alpha p}$$

for $t \in \left[\frac{1}{2}, 1\right]$ which follows from $(A-B)^q \leq A^q-B^q$ for any $A \geq B \geq 0$ and $q \geq 1$. Hence the inequality (2.18) is proved.

(ii) The inequality (2.19) is easily proved using the inequality (2.20) and Lemma 1.3. □

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Received: December 18, 2014; *Accepted*: April 25, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 250–267

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Solution and stability of system of quartic functional equations

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Abstract

In this paper, the authors introduced and investigated the general solution of system of quartic functional equations

$$\begin{aligned}
&f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z)\\
&\quad=2[f(x+y)+f(x-y)+f(x+z)+f(x-z)+f(y+z)+f(y-z)]\\
&\qquad-4[f(x)+f(y)+f(z)],\\
&f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z)\\
&\quad=72[f(x+y)+f(x-y)]+18[f(x+z)+f(x-z)]+8[f(y+z)+f(y-z)]\\
&\qquad+144f(x)-96f(y)-48f(z),\\
&f(x+2y+3z)+f(x+2y-3z)+f(x-2y+3z)+f(x-2y-3z)\\
&\quad=8[f(x+y)+f(x-y)]+18[f(x+z)+f(x-z)]+72[f(y+z)+f(y-z)]\\
&\qquad-48f(x)-96f(y)+144f(z).
\end{aligned}$$

Its generalized Hyers-Ulam stability using Hyers direct method and fixed point method are discussed. Counter examples for non stable cases are also given.

Keywords: Quartic functional equation, Generalized Hyers-Ulam stability, fixed point

2010 MSC: 39B52,39B72,39B82.

1 Introduction

One of the interesting questions in the theory of functional analysis concerning the stability problem of functional equations had been first raised by S.M. Ulam [28] as follows: When is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation? For very general functional equations, the concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

In 1941, D. H. Hyers [9] gave an affirmative answer to the question of S.M. Ulam for Banach spaces. In 1950, T. Aoki [2] was the second author to treat this problem for additive mappings. In 1978, Th.M. Rassias [20] succeeded in extending Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $||x||^p + ||y||^p,\ p \in [0, 1)$, to be unbounded.

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In 1982, J.M. Rassias [18] replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in \mathbb{R}$. A generalization of all the above stability results was obtained by P. Gavruta [8] in 1994 by replacing the unbounded Cauchy difference by a general control function $\varphi(x, y)$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et. al., [25] by considering the summation of both the sum and the product of two $p-$ norms. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 5, 6, 7, 10, 11, 12, 15, 21, 23]) and reference cited there in.

The quartic functional equation

$$F(x+2y) + F(x-2y) + 6F(x) = 4[F(x+y) + F(x-y) + 6F(y)] \tag{1.1}$$

was first introduced by J.M. Rassias [19], who solved its Ulam stability problem. Later P.K. Sahoo and J.K. Chung [26], S.H. Lee et. al., [13] remodified J.M. Rassias' equation as

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y) \tag{1.2}$$

and obtained its general solution.

Also the generalized Hyers-Ulam-Rassias stability for a 3 dimensional quartic functional equation

$$\begin{aligned} &g(2x+y+z) + g(2x+y-z) + g(2x-y+z) + g(-2x+y+z) + 16g(y) + 16g(z) \\ &= 8[g(x+y) + g(x-y) + g(x+z) + g(x-z)] + 2[g(y+z) + g(y-z)] + 32g(x) \end{aligned} \tag{1.3}$$

in fuzzy normed space was discussed by M. Arunkumar [3]. Several other types of quartic functional equations were introduced and investigated in [4, 16, 22, 24, 27].

In this paper, the authors introduced and investigated the general solution of system of quartic functional equations

$$\begin{aligned} &f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) \\ &= 2[f(x+y) + f(x-y) + f(x+z) + f(x-z) + f(y+z) + f(y-z)] \\ &\quad - 4[f(x) + f(y) + f(z)], \end{aligned} \tag{1.4}$$

$$\begin{aligned} &f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z) \\ &= 72[f(x+y) + f(x-y)] + 18[f(x+z) + f(x-z)] + 8[f(y+z) + f(y-z)] \\ &\quad + 144f(x) - 96f(y) - 48f(z), \end{aligned} \tag{1.5}$$

$$\begin{aligned} &f(x+2y+3z) + f(x+2y-3z) + f(x-2y+3z) + f(x-2y-3z) \\ &= 8[f(x+y) + f(x-y)] + 18[f(x+z) + f(x-z)] + 72[f(y+z) + f(y-z)] \\ &\quad - 48f(x) - 96f(y) + 144f(z). \end{aligned} \tag{1.6}$$

Its generalized Ulam - Hyers stability using Hyers direct method and fixed point method are discussed. Counter examples for non stable cases are also given.

In Section 2, we proved the general solutions of (1.4), (1.5) and (1.6) are provided.

In Section 3, the generalized Ulam - Hyers stability of the functional equation (1.5) using Hyers direct method is investigated.

In Section 4, Counter examples of non stable cases are provided.

The generalized Ulam - Hyers stability of the functional equation (1.5) using another substitutions is given in Section 5.

Also, the generalized Ulam - Hyers stability of the functional equation (1.5) using fixed point method is present in Section 6.

2 General Solutions of (1.4), (1.5) and (1.6)

In this section, the general solutions of (1.4), (1.5) and (1.6) are given. Throughout this section, let X and Y be real vector spaces.

Lemma 2.1. *[13] If a mapping $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$, then $f : X \to Y$ is quartic.*

Proof. Let $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$. Setting (x, y) by $(0, 0)$ in (1.2), we get $f(0) = 0$. Again setting x by 0 in (1.2), we reach

$$f(-y) = f(y)$$

for all $y \in X$. Therefore f is an even function. Replacing y by 0 and y by $2x$ in (1.2), we obtain

$$f(2x) = 2^4 f(x) \qquad and \qquad f(3x) = 3^4 f(x)$$

respectively, for all $x \in X$. In general for any positive integer a, we have

$$f(ax) = a^4 f(x)$$

for all $x \in X$. Hence f is quartic. □

Theorem 2.0. *If the mapping $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$, then $f : X \to Y$ satisfies the functional equation (1.4) for all $x, y, z \in X$.*

Proof. Let $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$. Replacing (x, y) by (y, z) in (1.2), we get

$$f(2y+z) + f(2y-z) = 4f(y+z) + 4f(y-z) - 6f(z) + 24f(y) \tag{2.1}$$

for all $y, z \in X$. Replacing z by $x + z$ in (2.1)and using evenness of f , we obtain

$$f(x+2y+z) + f(x-2y+z) = 4[f(x+y+z) + f(x-y+z)] - 6f(x+z) + 24f(y) \tag{2.2}$$

for all $x, y, z \in X$. Replacing z by $-z$ in (2.2), we get

$$f(x+2y-z) + f(x-2y-z) = 4[f(x+y-z) + f(x-y-z)] - 6f(x-z) + 24f(y) \tag{2.3}$$

for all $x, y, z \in X$. Adding (2.2) and (2.3), we reach

$$\begin{aligned} f(x+2y+z) &+ f(x-2y+z) + f(x+2y-z) + f(x-2y-z) \\ &= 4[f(x+y+z) + f(x-y+z) + f(x+y-z) + f(x-y-z)] \\ &\quad - 6[f(x+z) + f(x-z)] + 48f(y) \end{aligned} \tag{2.4}$$

for all $x, y, z \in X$. Interchanging y and z in (2.4), we get

$$\begin{aligned} f(x+y+2z) &+ f(x-y+2z) + f(x+y-2z) + f(x-y-2z) \\ &= 4[f(x+y+z) + f(x-y+z) + f(x+y-z) + f(x-y-z)] \\ &\quad - 6[f(x+y) + f(x-y)] + 48f(z) \end{aligned} \tag{2.5}$$

for all $x, y, z \in X$. Interchanging x and z in (2.5) and using evenness of f, we have

$$\begin{aligned} f(2x+y+z) &+ f(2x-y+z) + f(2x+y-z) + f(2x-y-z) \\ &= 4[f(x+y+z) + f(x-y+z) + f(x+y-z) + f(x-y-z)] \\ &\quad - 6[f(y+z) + f(y-z)] + 48f(x) \end{aligned} \tag{2.6}$$

for all $x, y, z \in X$. Replacing y by $2y$ in (2.6) and using (2.4) and (2.1), we arrive

$$\begin{aligned} f(2x+2y+z) &+ f(2x-2y+z) + f(2x+2y-z) + f(2x-2y-z) \\ &= 4[f(x+2y+z) + f(x-2y+z) + f(x+2y-z) + f(x-2y-z)] \\ &\quad - 6[f(2y+z) + f(2y-z)] + 48f(x) \\ &= 16[f(x+y+z) + f(x-y+z) + f(x+y-z) + f(x-y-z)] \\ &\quad - 24[f(x+z) + f(x-z)] - 24[f(y+z) + f(y-z)] \\ &\quad + 48f(x) + 48f(y) + 36f(z) \end{aligned} \tag{2.7}$$

for all $x, y, z \in X$. Replacing z by $2z$ in (2.7) and using (2.5) and (2.1), we obtain

$$\begin{aligned} &f(2x+2y+2z)+f(2x-2y+2z)+f(2x+2y-2z)+f(2x-2y-2z)\\ &=16[f(x+y+2z)+f(x-y+2z)+f(x+y-2z)+f(x-y-2z)]\\ &\quad-24[f(x+2z)+f(x-2z)]-24[f(y+2z)+f(y-2z)]\\ &\quad+48f(x)+48f(y)+36f(2z) \end{aligned} \tag{2.8}$$

for all $x, y, z \in X$. With the help of Lemma 2.1, we desired our result. □

Theorem 2.0. *A mapping $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$ if, and only if, $f : X \to Y$ satisfies the functional equation (1.5) for all $x, y, z \in X$.*

Proof. Let $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$. Replacing y by $x + z$ in (1.2) and evenness of f, we obtain

$$f(3x+z)+f(x-z)=4[f(2x+z)+f(z)]+24f(x)-6f(x+z) \tag{2.9}$$

for all $x, z \in X$. Replacing z by $-z$ in (2.9) and using evenness of f, we get

$$f(3x-z)+f(x+z)=4[f(2x-z)+f(z)]+24f(x)-6f(x-z) \tag{2.10}$$

for all $x, z \in X$. Adding (2.9) and (2.10) and using (1.2), we arrive

$$f(3x+z)+f(3x-z)=9[f(x+z)+f(x-z)]+144f(x)-16f(z) \tag{2.11}$$

for all $x, z \in X$. Replacing z by $y + z$ in (2.11), we have

$$f(3x+y+z)+f(3x-y-z)=9[f(x+y+z)+f(x-y-z)]+144f(x)-16f(y+z) \tag{2.12}$$

for all $x, y, z \in X$. Replacing z by $-z$ in (2.12), we get

$$f(3x+y-z)+f(3x-y+z)=9[f(x+y-z)+f(x-y+z)]+144f(x)-16f(y-z) \tag{2.13}$$

for all $x, y, z \in X$. Adding (2.12) and (2.13) and using Theorem 2.0, we have

$$\begin{aligned} &f(3x+y+z)+f(3x-y-z)+f(3x+y-z)+f(3x-y+z)\\ &=9[f(x+y+z)+f(x-y-z)+f(x+y-z)+f(x-y+z)]\\ &\quad-16[f(y+z)+f(y-z)]+288f(x)\\ &=18[f(x+y)+f(x-y)]+18[f(x+z)+f(x-z)]+2[f(y+z)+f(y-z)]\\ &\quad+252f(x)-36f(y)-36f(z) \end{aligned} \tag{2.14}$$

for all $x, y, z \in X$. Replacing y by $2y$ in (2.14) and using (2.1), we arrive (1.5) as desired.

Conversely, assume that $f : X \to Y$ satisfies the functional equation (1.5) for all $x, y, z \in X$. Setting $x = y = z = 0$ in (1.5), we obtain $f(0) = 0$. Replacing (x, y, z) by $(0, 0, x)$ in (1.5), we reach $f(-x) = f(x)$ for all $x \in X$. Setting $x = z = 0$ in (1.5), we have $f(2y) = 2^4 f(y)$ for all $y \in X$. Setting $y = z = 0$ in (1.5), we get $f(3x) = 3^4 f(x)$ for all $x \in X$. In general for any positive integer a, we obtain $f(ax) = a^4 f(x)$ for all $x \in X$. Replacing (x, y, z) by $(0, x, y)$ in (1.5) and using evenness of f, we reach (1.2) as desired. □

Theorem 2.0. *If $f : X \to Y$ satisfies the functional equation (1.5), then there exists a unique symmetric multi - additive function $Q : X \times X \times X \times X \to Y$ such that*

$$f(x) = Q(x, x, x, x)$$

for all $x \in X$.

Proof. By Theorem 2.0, if $f : X \to Y$ satisfies the functional equation (1.5), then $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$. By Theorem 2.1 of [13], we desired our result. □

Corollary 2.0. *If the mapping $f : X \to Y$ satisfies the functional equation (1.5) for all $x, y, z \in X$, then $f : X \to Y$ satisfies the functional equation (1.4) for all $x, y, z \in X$.*

Corollary 2.0. *If the mapping $f : X \to Y$ satisfies the functional equation (1.5) for all $x, y, z \in X$, then $f : X \to Y$ satisfies the functional equation (1.6) for all $x, y, z \in X$.*

Hereafter, through out this paper, let we consider G be a normed space and H be a Banach space. Define a mapping $Df : G \to H$ by

$$\begin{aligned} Df(x,y,z) = {} & f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z) \\ & -72[f(x+y)+f(x-y)] - 18[f(x+z)+f(x-z)] \\ & -8[f(y+z)+f(y-z)] - 144f(x) + 96f(y) + 48f(z) \end{aligned}$$

for all $x, y, z \in G$.

3 Stability results of (1.2): Direct method

In this section, the generalized Ulam - Hyers stability of the quartic functional equation (1.5) is given.

Theorem 3.0. *Let $j = \pm 1$ and $\psi : G^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{6^{4nj}} = 0 \tag{3.1}$$

for all $x, y, z \in G$. Let $f : G \to H$ be a function satisfying the inequality

$$\|Df(x,y,z)\| \le \psi(x,y,z) \tag{3.2}$$

for all $x, y, z \in G$. Then there exists a unique quartic mapping $Q : G \to H$ which satisfies (1.5) and

$$\|f(x) - Q(x)\| \le \frac{1}{6^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi\left(6^{kj}x\right)}{6^{4kj}} \tag{3.3}$$

where $\xi(x)$ and $Q(x)$ are defined by

$$\xi(x) = \psi(x,x,x) + \frac{1}{2}\psi(x,0,x) + \frac{89}{4}\psi(0,x,0) \tag{3.4}$$

and

$$Q(x) = \lim_{n\to\infty} \frac{f(6^{nj}x)}{6^{4nj}} \tag{3.5}$$

for all $x \in G$, respectively.

Proof. Replacing (x, y, z) by (x, x, x) in (3.2), we get

$$\|f(6x) + f(4x) - 97f(2x)\| \le \psi(x,x,x) \tag{3.6}$$

for all $x \in G$. Again, replacing (x, y, z) by $(x, 0, x)$ in (3.2), we obtain

$$\|f(4x) - 8f(2x) - 128f(x)\| \le \frac{1}{2}\psi(x,0,x) \tag{3.7}$$

for all $x \in G$. Finally, replacing (x, y, z) by $(0, x, 0)$ in (3.2), we have

$$\|f(2x) - 16f(x)\| \le \frac{1}{4}\psi(0,x,0) \tag{3.8}$$

for all $x \in G$. It follows from (3.6), (3.7), and (3.8) that

$$\begin{aligned} & \|f(6x) - 1296f(x)\| \\ & = \|f(6x) + f(4x) - 97f(2x) - f(4x) + 8f(2x) + 128f(x) + 89f(2x) - 1424f(x)\| \\ & \le \|f(6x) + f(4x) - 97f(2x)\| + \|f(4x) - 8f(2x) - 128f(x)\| + 89\,\|f(2x) - 16f(x)\| \\ & \le \psi(x,x,x) + \frac{1}{2}\psi(x,0,x) + \frac{89}{4}\psi(0,x,0) \end{aligned} \tag{3.9}$$

for all $x \in G$. Dividing the above inequality by 1296, we obtain

$$\left\| \frac{f(6x)}{6^4} - f(x) \right\| \leq \frac{\xi(x)}{6^4} \tag{3.10}$$

where

$$\xi(x) = \psi(x,x,x) + \frac{1}{2}\psi(x,0,x) + \frac{89}{4}\psi(0,x,0)$$

for all $x \in G$. Now replacing x by $6x$ and dividing by 6^4 in (3.10), we get

$$\left\| \frac{f(6^2x)}{6^8} - \frac{f(6x)}{6^4} \right\| \leq \frac{\xi(6x)}{6^8} \tag{3.11}$$

for all $x \in G$. From (3.10) and (3.11), we obtain

$$\begin{aligned} \left\| \frac{f(6^2x)}{6^8} - f(x) \right\| &\leq \left\| \frac{f(6x)}{6^4} - f(x) \right\| + \left\| \frac{f(6^2x)}{6^8} - \frac{f(6x)}{6^4} \right\| \\ &\leq \frac{1}{6^4}\left[\xi(x) + \frac{\xi(6x)}{6^4} \right] \end{aligned} \tag{3.12}$$

for all $x \in G$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} \left\| \frac{f(6^nx)}{6^{4n}} - f(x) \right\| &\leq \frac{1}{6^4} \sum_{k=0}^{n-1} \frac{\xi(6^kx)}{6^{4k}} \\ &\leq \frac{1}{6^4} \sum_{k=0}^{\infty} \frac{\xi(6^kx)}{6^{4k}} \end{aligned} \tag{3.13}$$

for all $x \in G$. In order to prove the convergence of the sequence $\left\{ \frac{f(6^nx)}{6^{4n}} \right\}$, replace x by 6^mx and dividing by 6^{4m} in (3.13), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| \frac{f(6^{n+m}x)}{6^{4(n+m)}} - \frac{f(6^mx)}{6^{4m}} \right\| &= \frac{1}{6^{4m}} \left\| \frac{f(6^n \cdot 6^mx)}{6^{4n}} - f(6^mx) \right\| \\ &\leq \frac{1}{6^4} \sum_{k=0}^{n-1} \frac{\xi(6^{k+m}x)}{6^{4(k+m)}} \\ &\leq \frac{1}{6^4} \sum_{k=0}^{\infty} \frac{\xi(6^{k+m}x)}{6^{4(k+m)}} \\ &\to 0 \quad as \quad m \to \infty \end{aligned}$$

for all $x \in G$. Hence the sequence $\left\{ \frac{f(6^nx)}{6^{4n}} \right\}$ is a Cauchy sequence. Since H is complete, there exists a mapping $Q : G \to H$ such that

$$Q(x) = \lim_{n\to\infty} \frac{f(6^nx)}{6^{4n}}, \quad \forall\ x \in G.$$

Letting $n \to \infty$ in (3.13), we see that (3.3) holds for all $x \in G$. To prove that Q satisfies (1.5), replacing (x, y, z) by $(6^nx, 6^ny, 6^nz)$ and dividing by 6^{4n} in (3.2), we obtain

$$\begin{aligned} \frac{1}{6^{4n}} \Big\| & f(6^n(3x+2y+z)) + f(6^n(3x+2y-z)) + f(6^n(3x-2y+z)) + f(6^n(3x-2y-z)) \\ & - 72[f(6^n(x+y)) + f(6^n(x-y))] - 18[f(6^n(x+z)) + f(6^n(x-z))] \\ & - 8[f(6^n(y+z)) + f(6^n(y-z))] - 144f(6^nx) \\ & + 96f(6^ny) + 48f(6^nz) \Big\| \leq \frac{1}{6^{4n}} \psi(6^nx, 6^ny, 6^nz) \end{aligned}$$

for all $x, y, z \in G$. Letting $n \to \infty$ in the above inequality and using the definition of $Q(x)$, we see that

$$\begin{aligned} & Q(3x+2y+z) + Q(3x+2y-z) + Q(3x-2y+z) + Q(3x-2y-z) \\ & = 72[Q(x+y) + Q(x-y)] + 18[Q(x+z) + Q(x-z)] + 8[Q(y+z) + Q(y-z)] \\ & \quad + 144Q(x) - 96Q(y) - 48Q(z). \end{aligned}$$

Hence Q satisfies (1.5) for all $x, y, z \in G$. To prove that Q is unique, let $R(x)$ be another quartic mapping satisfying (1.5) and (3.3), then

$$\begin{aligned}\|Q(x) - R(x)\| &= \frac{1}{6^{4n}} \|Q(6^n x) - R(6^n x)\| \\ &\leq \frac{1}{6^{4n}} \{\|Q(6^n x) - f(6^n x)\| + \|f(6^n x) - R(6^n x)\|\} \\ &\leq \frac{2}{6^4} \sum_{k=0}^{\infty} \frac{\xi(6^{k+n} x)}{6^{4(k+n)}} \\ &\to 0 \quad as \quad n \to \infty\end{aligned}$$

for all $x \in G$. Thus Q is unique. Hence for $j = 1$ the theorem holds.

Now, replacing x by $\frac{x}{6}$ in (3.9), we reach

$$\left\| f(x) - 1296 f\left(\frac{x}{6}\right) \right\| \leq \psi\left(\frac{x}{6}, \frac{x}{6}, \frac{x}{6}\right) + \frac{1}{2}\psi\left(\frac{x}{6}, 0, \frac{x}{6}\right) + \frac{89}{4}\psi\left(0, \frac{x}{6}, 0\right) \tag{3.14}$$

for all $x \in G$. The rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem holds. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.0 concerning the Ulam-Hyers [9], Ulam-TRassias [20], Ulam-GRassias [18] and Ulam-JRassias [25] stabilities of (1.5).

Corollary 3.0. *Let ρ and s be nonnegative real numbers. Let $f : G \to H$ be a function satisfying the inequality*

$$\|Df(x, y, z)\| \leq \begin{cases} \rho, & \\ \rho \left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 4; \\ \rho \left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{3.15}$$

for all $x, y, z \in G$. Then there exists a unique quartic function $Q : G \to H$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{\rho}{4|3^3 - 1|}, \\ \dfrac{\rho||x||^s}{4|3^3 - 3^s|}, \\ \dfrac{\rho||x||^{3s}}{4|3^3 - 3^{3s}|} \end{cases} \tag{3.16}$$

for all $x \in G$.

4 Counter examples for non stable cases of (1.5)

Now, we will provide an example to illustrate that the functional equation (1.5) is not stable for $s = 4$ in condition (ii) of Corollary 3.0. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\psi(x) = \begin{cases} \mu x^4, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(6^n x)}{6^{4n}} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$|Df(x, y, z)| \leq \frac{(488 \times 6^8)\mu}{1295} \left(|x|^4 + |y|^4 + |z|^4\right) \tag{4.1}$$

for all $x, y, z \in \mathbb{R}$. Then there do not exist a quartic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f(x) - Q(x)| \leq \kappa |x|^4 \qquad for\ all \quad x \in \mathbb{R}. \tag{4.2}$$

Proof. Now

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\psi(6^n x)|}{|6^{4n}|} \leq \sum_{n=0}^{\infty} \frac{\mu}{6^{4n}} = \frac{1296\mu}{1295}.$$

Therefore, we see that f is bounded. We are going to prove that f satisfies (4.1).

If $x = y = z = 0$ then (4.1) is trivial. If $|x|^4 + |y|^4 + |z|^4 \geq \frac{1}{6^4}$ then the left hand side of (4.1) is less than $\frac{488 \times 6^8 \mu}{1295}$. Now suppose that $0 < |x|^4 + |y|^4 + |z|^4 < \frac{1}{6^4}$. Then there exists a positive integer k such that

$$\frac{1}{6^{4(k+1)}} \leq |x|^4 + |y|^4 + |z|^4 < \frac{1}{6^{4k}}, \tag{4.3}$$

so that $6^{k-1}x < \frac{1}{6}, 6^{k-1}y < \frac{1}{6}, 6^{k-1}z < \frac{1}{6}$ and consequently

$$\begin{aligned}
&6^{k-1}(3x+2y+z), 6^{k-1}(3x-2y+z), 6^{k-1}(3x+2y-z), 6^{k-1}(3x-2y-z),\\
&\quad 6^{k-1}(x+y), 6^{k-1}(x-y), 6^{k-1}(x+z), 6^{k-1}(x-z),\\
&\qquad 6^{k-1}(y+z), 6^{k-1}(y-z), 6^{k-1}(x), 6^{k-1}(y), 6^{k-1}(z) \in (-1,1).
\end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned}
&6^{n}(3x+2y+z), 6^{n}(3x-2y+z), 6^{n}(3x+2y-z), 6^{n}(3x-2y-z),\\
&\quad 6^{n}(x+y), 6^{n}(x-y), 6^{n}(x+z), 6^{n}(x-z),\\
&\qquad 6^{n}(y+z), 6^{n}(y-z), 6^{n}(x), 6^{n}(y), 6^{n}(z) \in (-1,1).
\end{aligned}$$

and

$$\begin{aligned}
&\psi(6^n(3x+2y+z)) + \psi(6^n(3x+2y-z)) + \psi(6^n(3x-2y+z)) + \psi(6^n(3x-2y-z))\\
&\quad - 72[\psi(6^n(x+y)) + \psi(6^n(x-y))] - 18[\psi(6^n(x+z)) + \psi(6^n(x-z))]\\
&\qquad - 8[\psi(6^n(y+z)) + \psi(6^n(y-z))] - 144\psi(6^n(x)) + 96\psi(6^n(y)) + 48\psi(6^n(z)) = 0
\end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f and (4.3), we obtain that

$$\begin{aligned}
&\Big| f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z)\\
&\quad - 72[f(x+y) + f(x-y)] - 18[f(x+z) + f(x-z)] - 8[f(y+z) + f(y-z)]\\
&\qquad - 144f(x) + 96f(y) + 48f(z)\Big|\\
&\leq \sum_{n=0}^{\infty} \frac{1}{6^{4n}} \Big| \psi(6^n(3x+2y+z)) + \psi(6^n(3x+2y-z)) + \psi(6^n(3x-2y+z)) + \psi(6^n(3x-2y-z))\\
&\quad - 72[\psi(6^n(x+y)) + \psi(6^n(x-y))] - 18[\psi(6^n(x+z)) + \psi(6^n(x-z))]\\
&\qquad - 8[\psi(6^n(y+z)) + \psi(6^n(y-z))] - 144\psi(6^n(x)) + 96\psi(6^n(y)) + 48\psi(6^n(z))\Big|\\
&= \sum_{n=k}^{\infty} \frac{1}{6^{4n}} \Big| \psi(6^n(3x+2y+z)) + \psi(6^n(3x+2y-z)) + \psi(6^n(3x-2y+z)) + \psi(6^n(3x-2y-z))\\
&\quad - 72[\psi(6^n(x+y)) + \psi(6^n(x-y))] - 18[\psi(6^n(x+z)) + \psi(6^n(x-z))]\\
&\qquad - 8[\psi(6^n(y+z)) + \psi(6^n(y-z))] - 144\psi(6^n(x)) + 96\psi(6^n(y)) + 48\psi(6^n(z))\Big|\\
&\leq \sum_{n=k}^{\infty} \frac{1}{6^{4n}} 488\mu = 488\,\mu \times \frac{1296}{1295 \cdot 6^{4k}} \leq \frac{6^8 \times 488\mu}{1295}(|x|^4 + |y|^4 + |z|^4).
\end{aligned}$$

Thus f satisfies (4.1) for all $x \in \mathbb{R}$ with $0 < |x|^4 + |y|^4 + |z|^4 < \frac{1}{6^4}$.

We claim that the quartic functional equation (1.5) is not stable for $s = 4$ in condition (ii) of Corollary 3.0. Suppose on the contrary that there exist a quartic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (4.2). Since f is bounded and continuous for all $x \in \mathbb{R}$, Q is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.0, Q must have the form $Q(x) = cx^4$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|f(x)| \leq (\kappa + |c|)\, |x|^4. \tag{4.4}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{6^{m-1}}\right)$, then $6^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(6^n x)}{6^{4n}} \geq \sum_{n=0}^{m-1} \frac{\mu(6^n x)^4}{6^{4n}} = m\mu x^4 > (\kappa + |c|)\, x^4$$

which contradicts (4.4). Therefore the quartic functional equation (1.5) is not stable in sense of Ulam, Hyers and Rassias if $s = 4$, assumed in the inequality condition (ii) of (3.16). □

A counter example to illustrate the non stability in Condition (iii) of Corollary 3.0. Let s be such that $0 < s < \frac{4}{3}$. Then there is a function $f : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying

$$|Df(x,y,z)| \leq \lambda |x|^{\frac{4s}{3}}\, |y|^{\frac{4s}{3}}\, |z|^{\frac{4-8s}{3}} \tag{4.5}$$

for all $x, y, z \in \mathbb{R}$ and

$$\sup_{x \neq 0} \frac{|f(x) - Q(x)|}{|x|^4} = +\infty \tag{4.6}$$

for every quartic mapping $Q : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f(x) = \begin{cases} x^4 \ln|x|, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (4.6), it follows that

$$\begin{aligned} \sup_{x \neq 0} \frac{|f(x) - Q(x)|}{|x|^4} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|f(n) - Q(n)|}{|n|^4} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|n^4 \ln|n| - n^4\, Q(1)\right|}{|n|^4} = \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \left|\ln|n| - Q(1)\right| = \infty. \end{aligned}$$

We have to prove (4.5) is true.

Case (i): If $x, y, z > 0$ in (4.5) then,

$$\begin{aligned} &|f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z) \\ &\quad - 72[f(x+y) + f(x-y)] - 18[f(x+z) + f(x-z)] - 8[f(y+z) + f(y-z)] \\ &\quad -144f(x) + 96f(y) + 48f(z)| \\ &= |(3x+2y+z)\ln|3x+2y+z| + (3x+2y-z)\ln|3x+2y-z| \\ &\quad + (3x-2y+z)\ln|3x-2y+z| + (3x-2y-z)\ln|3x-2y-z| \\ &\quad - 72[(x+y)\ln|x+y| + (x-y)\ln|x-y|] \\ &\quad - 18[(x+z)\ln|x+z| + (x-z)\ln|x-z|] \\ &\quad - 8[(y+z)\ln|y+z| + (y-z)\ln|y-z|] \\ &\quad -144(x)\ln|x| + 96(y)\ln|y| + 48(z)\ln|z|| \end{aligned}$$

Set $x = u, y = v, z = w$ it follows that

$$\begin{aligned}
&|f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z)\\
&\quad -72[f(x+y)+f(x-y)]-18[f(x+z)+f(x-z)]-8[f(y+z)+f(y-z)]\\
&\qquad -144f(x)+96f(y)+48f(z)|\\
&=|(3x+2y+z)\ln|3x+2y+z|+(3x+2y-z)\ln|3x+2y-z|\\
&\quad +(3x-2y+z)\ln|3x-2y+z|+(3x-2y-z)\ln|3x-2y-z|\\
&\qquad -72[(x+y)\ln|x+y|+(x-y)\ln|x-y|]\\
&\qquad\quad -18[(x+z)\ln|x+z|+(x-z)\ln|x-z|]\\
&\qquad\qquad -8[(y+z)\ln|y+z|+(y-z)\ln|y-z|]\\
&\qquad\qquad\quad -144(x)\ln|x|+96(y)\ln|y|+48(z)\ln|z||\\
&=|(3u+2v+w)\ln|3u+2v+w|+(3u+2v-w)\ln|3u+2v-w|\\
&\quad +(3u-2v+w)\ln|3u-2v+w|+(3u-2v-w)\ln|3u-2v-w|\\
&\qquad -72[(u+v)\ln|u+v|+(u-v)\ln|u-v|]\\
&\qquad\quad -18[(u+w)\ln|u+w|+(u-w)\ln|u-w|]\\
&\qquad\qquad -8[(v+w)\ln|v+w|+(v-w)\ln|v-w|]\\
&\qquad\qquad\quad -144(u)\ln|u|+96(v)\ln|v|+48(w)\ln|w||\\
&|f(3u+2v+w)+f(3u+2v-w)+f(3u-2v+w)+f(3u-2v-w)\\
&\quad -72[f(u+v)+f(u-v)]-18[f(u+w)+f(u-w)]-8[f(v+w)+f(v-w)]\\
&\qquad -144f(u)+96f(v)+48f(w)|\\
&\le \lambda|u|^{\frac{4s}{3}}\,|v|^{\frac{4s}{3}}\,|w|^{\frac{4-8s}{3}}=\lambda|x|^{\frac{4s}{3}}\,|y|^{\frac{4s}{3}}\,|z|^{\frac{4-8s}{3}}.
\end{aligned}$$

For the ***Cases:***

$$\begin{aligned}
&(ii): x,y,z<0\\
&(iii): x>0, y,z<0\\
&(iv): x<0, y,z>0\\
&(v): x=y=z=0
\end{aligned}$$

the proof is similar to that of Case (i). □

Now, we will provide an example to illustrate that the functional equation (1.5) is not stable for $s = \frac{4}{3}$ in condition (iv) of Corollary 3.0. The proof of the following example is similar to that of Example 4. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\psi(x) = \begin{cases} \mu x^4, & \text{if } |x| < \frac{4}{3} \\ \dfrac{4\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(6^n x)}{6^{4n}} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$|Df(x,y,z)| \le \frac{488 \times 6^8 \times 4\mu}{3 \cdot 1295}\left(|x|^{\frac{4}{3}}|y|^{\frac{4}{3}}|z|^{\frac{4}{3}} + |x|^4 + |y|^4 + |z|^4\right) \tag{4.7}$$

for all $x, y, z \in \mathbb{R}$. Then there do not exist a quartic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f(x) - Q(x)| \le \kappa |x|^4 \qquad \text{for all} \quad x \in \mathbb{R}. \tag{4.8}$$

5 Stability results of (1.2) using various substitutions

In this section, the generalized Ulam-Hyers stability of (1.5) using various substitutions is investigated. The proofs of the following theorems and corollaries are similar to that Theorem 3.0 and Corollary 3.0. Hence the details of the proofs are omitted.

Theorem 5.0. *Let $j = \pm 1$ and $\psi : G^3 \rightarrow [0, \infty)$ be a function such that*

$$\lim_{n\rightarrow\infty} \frac{\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{4^{4nj}} = 0 \tag{5.1}$$

for all $x, y, z \in G$. Let $f : G \rightarrow H$ be a function satisfying the inequality

$$\|Df(x, y, z)\| \leq \psi(x, y, z) \tag{5.2}$$

for all $x, y, z \in G$. Then there exists a unique quartic mapping $Q : G \rightarrow H$ which satisfies (1.5) and

$$\|f(x) - Q(x)\| \leq \frac{1}{2 \cdot 4^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta\left(4^{kj}x\right)}{4^{4kj}} \tag{5.3}$$

where $\zeta(x)$ and $Q(x)$ are defined by

$$\zeta(x) = \frac{1}{2}\psi(x, 0, x) + 4\psi(0, x, 0) \tag{5.4}$$

and

$$Q(x) = \lim_{n\rightarrow\infty} \frac{f(4^{nj}x)}{4^{4nj}} \tag{5.5}$$

for all $x \in G$, respectively.

Corollary 5.0. *Let ρ and s be nonnegative real numbers. Let $f : G \rightarrow H$ be a function satisfying the inequality*

$$\|Df(x, y, z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{5.6}$$

for all $x, y, z \in G$. Then there exists a unique quartic function $Q : G \rightarrow H$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{5\rho}{2|4^4 - 1|}, \\ \dfrac{5\rho||x||^s}{2|4^4 - 4^s|}, \\ \dfrac{5\rho||x||^{3s}}{2|4^4 - 4^{3s}|} \end{cases} \tag{5.7}$$

for all $x \in G$.

Theorem 5.0. *Let $j = \pm 1$ and $\psi : G^3 \rightarrow [0, \infty)$ be a function such that*

$$\lim_{n\rightarrow\infty} \frac{\psi\left(3^{nj}x, 3^{nj}y, 3^{nj}z\right)}{3^{4nj}} = 0 \tag{5.8}$$

for all $x, y, z \in G$. Let $f : G \rightarrow H$ be a function satisfying the inequality

$$\|Df(x, y, z)\| \leq \psi(x, y, z) \tag{5.9}$$

for all $x, y, z \in G$. Then there exists a unique quartic mapping $Q : G \rightarrow H$ which satisfies (1.5) and

$$\|f(x) - Q(x)\| \leq \frac{1}{4 \cdot 3^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\psi\left(3^{kj}x, 0, 0\right)}{3^{4kj}} \tag{5.10}$$

where $Q(x)$ is defined by

$$Q(x) = \lim_{n\rightarrow\infty} \frac{f(3^{nj}x)}{3^{4nj}} \tag{5.11}$$

for all $x \in G$.

Corollary 5.0. *Let ρ and s be nonnegative real numbers. Let $f : G \to H$ be a function satisfying the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{5.12}$$

for all $x, y, z \in G$. Then there exists a unique quartic function $Q : G \to H$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{\rho}{4|3^4 - 1|}, \\ \dfrac{\rho||x||^s}{4|3^4 - 3^s|}, \\ \dfrac{\rho||x||^{3s}}{4|3^4 - 3^{3s}|} \end{cases} \tag{5.13}$$

for all $x \in G$.

Theorem 5.0. *Let $j = \pm 1$ and $\psi : G^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{4nj}} = 0 \tag{5.14}$$

for all $x, y, z \in G$. Let $f : G \to H$ be a function satisfying the inequality

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{5.15}$$

for all $x, y, z \in G$. Then there exists a unique quartic mapping $Q : G \to H$ which satisfies (1.5) and

$$\|f(x) - Q(x)\| \leq \frac{1}{4 \cdot 2^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\psi\left(0, 2^{kj}x, 0\right)}{2^{4kj}} \tag{5.16}$$

where $Q(x)$ is defined by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{4nj}} \tag{5.17}$$

for all $x \in G$.

Corollary 5.0. *Let ρ and s be nonnegative real numbers. Let $f : G \to H$ be a function satisfying the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{5.18}$$

for all $x, y, z \in G$. Then there exists a unique quartic function $Q : G \to H$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{\rho}{4|2^4 - 1|}, \\ \dfrac{\rho||x||^s}{4|2^4 - 2^s|}, \\ \dfrac{\rho||x||^{3s}}{4|2^4 - 2^{3s}|} \end{cases} \tag{5.19}$$

for all $x \in G$.

6 Stability results of (1.2): fixed point method

In this section, we apply a fixed point method for achieving stability of the functional equation (1.5) is present. Now, first we will recall the fundamental results in fixed point theory.

Theorem 6.0. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant)* $L < 1$. *Then,*
(i) The mapping T *has one and only fixed point* $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* *is globally attractive, that is*

(A2) $lim_{n\to\infty} T^n x = x^*$, *for any starting point* $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X;$

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X.$

Theorem 6.0. *[14] Suppose that for a complete generalized metric space* (Ω, δ) *and a strictly contractive mapping* $T : \Omega \to \Omega$ *with Lipschitz constant* L. *Then, for each given* $x \in \Omega$, *either*

$$d(T^n x, T^{n+1}x) = \infty \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 *such that*
(FP1) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(FP2) The sequence $(T^n x)$ *is convergent to a fixed point* y^* *of* T
(FP3) y^* *is the unique fixed point of* T *in the set* $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Delta$.

Hereafter throughout this section, let us consider $\mathcal{G}$ and $\mathcal{H}$ to be a normed space and a Banach space, respectively.

Theorem 6.0. *Let* $f : \mathcal{G} \to \mathcal{H}$ *be a mapping for which there exists a function* $\psi : \mathcal{G}^3 \to [0, \infty)$ *with the condition*

$$\lim_{k\to\infty} \frac{1}{\tau_i^{4k}} \psi(\tau_i^k x, \tau_i^k y, \tau_i^k z) = 0 \tag{6.1}$$

where

$$\tau_i = \begin{cases} 6 & if \quad i = 0; \\ \frac{1}{6} & if \quad i = 1, \end{cases} \tag{6.2}$$

such that the functional inequality

$$\|Df(x, y, z)\| \leq \psi(x, y, z) \tag{6.3}$$

for all $x, y, z \in \mathcal{G}$. *If there exists* $L = L(i)$ *such that the function* $\Phi : \mathcal{G} \to [0, \infty)$ *defined by*

$$\Phi(x) = \xi\left(\frac{x}{6}\right)$$

where

$$\xi(x) = \psi(x, x, x) + \frac{1}{2}\psi(x, 0, x) + \frac{89}{4}\psi(0, x, 0)$$

has the property

$$\Phi(x) = \frac{L}{\tau_i^4}\Phi(\tau_i x). \tag{6.4}$$

for all $x \in \mathcal{G}$. *Then there exists a unique quartic mapping* $Q : \mathcal{G} \to \mathcal{H}$ *satisfying the functional equation (1.5) and*

$$\|f(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}\Phi(x) \tag{6.5}$$

for all $x \in \mathcal{G}$.

Proof. Consider the set

$$\Gamma = \{p/p : \mathcal{G} \to \mathcal{H},\ p(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(p, q) = \inf\{K \in (0, \infty) : \| p(x) - q(x) \| \leq K\Phi(x), x \in \mathcal{G}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon p(x) = \frac{1}{\tau_i^4} p(\tau_i x),$$

for all $x \in \mathcal{G}$. Now $p, q \in \Gamma$,

$$\begin{aligned} d(p,q) \leq K \Rightarrow & \parallel p(x) - q(x) \parallel \leq K\Phi(x), x \in \mathcal{G}, \\ & \Rightarrow \left\| \frac{1}{\tau_i^4} p(\tau_i x) - \frac{1}{\tau_i^4} q(\tau_i x) \right\| \leq \frac{1}{\tau_i^4} K\Phi(\tau_i x), x \in \mathcal{G}, \\ & \Rightarrow \left\| \frac{1}{\tau_i^4} p(\tau_i x) - \frac{1}{\tau_i^4} q(\tau_i x) \right\| \leq LK\Phi(x), x \in \mathcal{G}, \\ & \Rightarrow \parallel \Upsilon p(x) - \Upsilon q(x) \parallel \leq LK\Phi(x), x \in \mathcal{G}, \\ & \Rightarrow d(\Upsilon p, \Upsilon q) \leq LK. \end{aligned}$$

This implies $d(\Upsilon p, \Upsilon q) \leq Ld(p,q)$, for all $p, q \in \Gamma$. i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

It follows from (3.9), we arrive

$$\|f(6x) - 1296f(x)\| \leq \xi(x) \tag{6.6}$$

where

$$\xi(x) = \psi(x,x,x) + \frac{1}{2}\psi(x,0,x) + \frac{89}{4}\psi(0,x,0)$$

for all $x \in \mathcal{G}$. It follows from (6.6) that

$$\left\| \frac{f(6x)}{6^4} - f(x) \right\| \leq \frac{\xi(x)}{6^4} \tag{6.7}$$

for all $x \in \mathcal{G}$. Using (6.4) for the case $i = 0$ it reduces to

$$\left\| \frac{f(6x)}{6^4} - f(x) \right\| \leq L\Phi(x)$$

for all $x \in \mathcal{G}$,

$$\text{i.e.,} \quad d(\Upsilon f, f) \leq L \Rightarrow d(\Upsilon f, f) \leq L = L^{1-i} < \infty. \tag{6.8}$$

Again replacing $x = \frac{x}{6}$ in (6.6), we get

$$\left\| f(x) - 1296f\left(\frac{x}{6}\right) \right\| \leq \xi\left(\frac{x}{6}\right) \tag{6.9}$$

for all $x \in \mathcal{G}$. Using (6.4) for the case $i = 1$ it reduces to

$$\left\| f(x) - 1296f\left(\frac{x}{6}\right) \right\| \leq \Phi(x)$$

for all $x \in \mathcal{G}$,

$$\text{i.e.,} \quad d(f, \Upsilon f) \leq 1 \Rightarrow d(f, \Upsilon f) \leq 1 = L^{1-i} < \infty. \tag{6.10}$$

From (6.8) and (6.10), we arrive

$$d(f, \Upsilon f) \leq L^{1-i}.$$

Therefore (FP1) holds.

By (FP2), it follows that there exists a fixed point Q of Υ in Γ such that

$$Q(x) = \lim_{k\to\infty} \frac{f(\tau_i^k x)}{\tau_i^{4k}}, \qquad \forall \ x \in \mathcal{G}. \tag{6.11}$$

We have to prove $Q : \mathcal{G} \to \mathcal{H}$ is quartic. Replacing (x,y,z) by $(\tau_i^k x, \tau_i^k y, \tau_i^k z)$ in (6.3) and dividing by τ_i^{4k}, it follows from (6.1) that

$$\frac{1}{\tau_i^{4k}} \left\| Df(\tau_i^k x, \tau_i^k y, \tau_i^k z) \right\| \leq \frac{1}{\tau_i^{4k}} \psi(\tau_i^k x, \tau_i^k y, \tau_i^k z)$$

for all $x \in \mathcal{G}$. Letting $k \to \infty$ in the above inequality and using the definition of $Q(x)$, we see that

$$DQ(x,y,z) = 0$$

i.e., Q satisfies the functional equation (1.5) for all $x,y,z \in \mathcal{G}$.

By (FP3), Q is the unique fixed point of Υ in the set

$$\Delta = \{Q \in \Gamma : d(f,Q) < \infty\},$$

such that

$$\|f(x) - Q(x)\| \leq K\Phi(x)$$

for all $x \in \mathcal{G}$ and $K > 0$. Finally by (FP4), we obtain

$$d(f,Q) \leq \frac{1}{1-L} d(f,\Upsilon f)$$

this implies

$$d(f,Q) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\|f(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}\Phi(x)$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 6.0 concerning the stability of (1.5).

Corollary 6.0. *Let $f : \mathcal{G} \to \mathcal{H}$ be a mapping and there exist real numbers ρ and s such that*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{6.12}$$

for all $x \in \mathcal{G}$. Then there exists a unique quartic function $Q : \mathcal{G} \to \mathcal{H}$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{95\rho}{4|6^4-1|}, \\ \dfrac{105\rho||x||^s}{4|6^4-6^s|}, \\ \dfrac{\rho||x||^{3s}}{|6^4-6^{3s}|}, \\ \dfrac{109\rho||x||^{3s}}{4|6^4-6^{3s}|} \end{cases} \tag{6.13}$$

for all $x \in \mathcal{G}$.

Proof. Setting

$$\psi(x,y,z) = \begin{cases} \rho, \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, \\ \rho||x||^s||y||^s||z||^s, \\ \rho\left\{||x||^s||y||^s||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}. \end{cases}$$

for all $x \in \mathcal{G}$. Now,

$$\frac{1}{\tau_i^{4k}}\psi(\tau_i^k x, \tau_i^k y, \tau_i^k z) = \begin{cases} \frac{\rho}{\tau_i^{4k}}, \\ \frac{\rho}{\tau_i^{4k}}\left\{||\tau_i^k x||^s + ||\tau_i^k y||^s + ||\tau_i^k z||^s\right\}, \\ \frac{\rho}{\tau_i^{4k}}||\tau_i^k x||^s||\tau_i^k y||^s||\tau_i^k z||^s, \\ \frac{\rho}{\tau_i^{4k}}\left\{||\tau_i^k x||^s||\tau_i^k y||^s||\tau_i^k z||^s + ||\tau_i^k x||^{3s} + ||\tau_i^k y||^{3s} + ||\tau_i^k z||^{3s}\right\}. \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (6.1) is holds.

But we have $\Phi(x) = \xi\left(\frac{x}{6}\right)$ where $\xi(x) = \psi(x, x, x) + \frac{1}{2}\psi(x, 0, x) + \frac{89}{4}\psi(0, x, 0)$ has the property $\Phi(x) = \frac{L}{\tau_i^4}\Phi(\tau_i x)$ for all $x \in \mathcal{G}$. Hence

$$\Phi(x) = \xi\left(\frac{x}{6}\right) = \begin{cases} \frac{95\rho}{4}, \\ \frac{105\rho}{4\cdot 6^s}||x||^s, \\ \frac{\rho}{6^{3s}}||x||^{3s}, \\ \frac{109\rho}{4\cdot 6^{3s}}||x||^{3s}. \end{cases}$$

Now,

$$\frac{1}{\tau_i^4}\Phi(\tau_i x) = \begin{cases} \frac{95\rho}{4\cdot\tau_i^4} \\ \frac{105\rho}{4\cdot 6^s\cdot\tau_i^4}||\tau_i x||^s, \\ \frac{\rho}{6^{3s}\cdot\tau_i^4}||\tau_i x||^{3s}, \\ \frac{109\rho}{4\cdot 6^{3s}\cdot\tau_i^4}||\tau_i x||^{3s}. \end{cases} = \begin{cases} \tau_i^{-1}\Phi(x), \\ \tau_i^{s-4}\Phi(x), \\ \tau_i^{3s-4}\Phi(x), \\ \tau_i^{3s-4}\Phi(x). \end{cases}$$

Hence the inequality (6.4) holds either, $L = 6^{-4}$ if $i = 0$ and $L = 6^4$ if $i = 1$. Now from (6.5), we prove the following cases for condition (i).

Case:1 $L = 6^{-4}$ if $i = 0$

$$\|f(x) - Q(x)\| \le \frac{(6^{-4})^{1-0}}{1-6^{-4}}\Phi(x) = \frac{95\rho}{4(6^4-1)}.$$

Case:2 $L = 6^4$ if $i = 1$

$$\|f(x) - Q(x)\| \le \frac{(6^4)^{1-1}}{1-6^4}\Phi(x) = \frac{-95\rho}{4(1-6^4)}.$$

Also the inequality (6.4) holds either, $L = 6^{s-4}$ for $s < 4$ if $i = 0$ and $L = 6^{4-s}$ for $s > 4$ if $i = 1$. Now from (6.5), we prove the following cases for condition (ii).

Case:3 $L = 6^{s-4}$ for $s < 4$ if $i = 0$

$$\|f(x) - Q(x)\| \le \frac{\left(6^{(s-4)}\right)^{1-0}}{1-6^{(s-4)}}\Phi(x) = \frac{105\rho||x||^s}{6^4-6^s}.$$

Case:4 $L = 6^{4-s}$ for $s > 4$ if $i = 1$

$$\|f(x) - Q(x)\| \le \frac{(6^{4-s})^{1-1}}{1-6^{4-s}}\Phi(x) = \frac{105\rho||x||^s}{6^s-6^4}.$$

Again the inequality (6.4) holds either, $L = 6^{3s-4}$ for $3s < 4$ if $i = 0$ and $L = 6^{4-3s}$ for $3s > 4$ if $i = 1$. Now from (6.5), we prove the following cases for condition (iii).

Case:5 $L = 6^{3s-4}$ for $3s < 4$ if $i = 0$

$$\|f(x) - Q(x)\| \leq \frac{\left(6^{(3s-4)}\right)^{1-0}}{1 - 6^{(3s-4)}}\Phi(x) = \frac{\rho||x||^{3s}}{6^4 - 6^{3s}}.$$

Case:6 $L = 6^{4-3s}$ for $3s > 4$ if $i = 1$

$$\|f(x) - Q(x)\| \leq \frac{\left(6^{4-3s}\right)^{1-1}}{1 - 6^{4-3s}}\Phi(x) = \frac{\rho||x||^{3s}}{6^{3s} - 6^4}.$$

The proof of condition (iv) is similar to that of condition (iii). Hence the proof is complete. □

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Received: October 10, 2014; *Accepted*: March 23, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 268–276

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On $\tilde{\mu}$-open sets in generalized topological spaces

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Abstract

In this paper, we introduce the notion of $\tilde{\mu}$-open sets in generalized topological spaces. Further, we introduce the notions of interior, closure, boundary, exterior and study some of their properties. In addition, we introduce the concepts of $\tilde{\mu}$-T_i $(i = 0, \frac{1}{2}, 1, 2)$ spaces are characterized them using $\tilde{\mu}$-open and $\tilde{\mu}$-closed sets.

Keywords: $\tilde{\mu}$-open, $\tilde{\mu}$-closed, $\tilde{\mu}$-interior, $\tilde{\mu}$-closure, $\tilde{\mu}$-boundary, $\tilde{\mu}$-exterior and $\tilde{\mu}$-T_i $(i = 0, \frac{1}{2}, 1, 2)$.

2010 MSC: 34G20.

1 Introduction

Generalized topologies were introduced by A. Csaszar. Further, he defined the concepts of μ-open sets and their corresponding interior and closure operators in generalized topological spaces. Also, he obtained and studied the notions of μ-semi-open sets, μ-preopen sets, μ-α-open sets and μ-β-open sets in generalized topological spaces. In this paper in section 3, we introduced the concept of $\tilde{\mu}$-open sets, which is analogous to μ-semi-open sets and introduced the notion $\tilde{\mu}O(X)$ which is the set of all $\tilde{\mu}$-open sets in a generalized topological space (X, μ). Further, we introduced the concepts of $\tilde{\mu}$-interior, $\tilde{\mu}$-closure, $\tilde{\mu}$-boundary and $\tilde{\mu}$-exterior operators and studied some of their fundamental properties. In section 4, we introduced the notion of $\tilde{\mu}$-T_i spaces $(i = 0, \frac{1}{2}, 1, 2)$ and characterized $\tilde{\mu}$-T_i spaces using $\tilde{\mu}$-closed and $\tilde{\mu}$-open sets.

2 Preliminaries

We recall some basic definitions and notations. Let X be a nonempty set and $\exp(X)$ the power set of X. We called a class $\mu \subseteq \exp(X)$ a generalized topology (briefly, GT) if $\emptyset \in \mu$ and the arbitrary union of elements of μ belongs to μ [4]. We called the pair (X, μ) a generalized topological space (briefly, GTS). For a generalized topological space (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets [4]. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A [7]; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A [7]. It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : exp(X) \to exp(X)$ is said to be idempotent iff $A \subseteq B \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic iff $\gamma(A) \subseteq \gamma(B)$ [2]. According to [9], let μ be a generalized topology on X, $A \subseteq X$ and $x \in X$, then (1) $x \in c_\mu(A)$ if and only if $M \cap A \neq \emptyset$ for each $M \in \mu$ containing x; (2) $c_\mu(X \setminus A) = X \setminus i_\mu(A)$ and (3) $c_\mu(c_\mu(A)) = c_\mu(A)$. A subset A of a generalized topological space (X, μ) is said to be μ-semi-open (resp. μ-preopen, μ-α-open, μ-β-open) if $A \subseteq c_\mu(i_\mu(A))$ (resp. $A \subseteq i_\mu(c_\mu(A))$, $A \subseteq i_\mu(c_\mu(i_\mu(A)))$, $A \subseteq c_\mu(i_\mu(c_\mu(A)))$. The complement of

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a μ-semi-open (resp. μ-preopen, μ-α-open, μ-β-open) set is said to be μ-semi-closed (resp. μ-preclosed, μ-α-closed, μ-β-closed) [4]. For $A \subseteq X$, we denote by $c_{s_\mu}(A)$ the intersection of all μ-semi-closed sets containing A, i.e., the smallest μ-semi-closed set containing A [7]; and by $i_{s_\mu}(A)$ the union of all μ-semi-open sets contained in A, i.e., the largest μ-semi-open set contained in A [7]. According to [9], let μ be a generalized topology on X, $A \subseteq X$ and $x \in X$, then (1) $x \in c_{s_\mu}(A)$ if and only if $M \cap A \neq \emptyset$ for each μ-semi-open set M containing x; (2) $c_{s_\mu}(X \setminus A) = X \setminus i_{s_\mu}(A)$ and (3) $c_{s_\mu}(c_{s_\mu}(A)) = c_{s_\mu}(A)$.

3 $\tilde{\mu}$-open sets

Definition 3.1. *Let (X, μ) be a generalized topological space. A subset A of X is said to be a $\tilde{\mu}$-open set, if there exists a μ-open set U of X such that $U \subseteq A \subseteq c_{s_\mu}(U)$. The set of all $\tilde{\mu}$-open sets is denoted by $\tilde{\mu}O(X)$.*

Example 3.1. *Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}\}$. Then $\tilde{\mu}$-open sets are $\{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$.*

Theorem 3.1. *Let (X, μ) be a generalized topological space, A be a subset of X. If A is $\tilde{\mu}$-open in (X, μ) if and only if $A \subseteq c_{s_\mu}(i_\mu(A))$.*

Proof. If A is a $\tilde{\mu}$-open of X, then there exists a μ-open set U such that $U \subseteq A \subseteq c_{s_\mu}(U)$. Since U is $\tilde{\mu}$-open, we have that $U = i_\mu(U) \subseteq i_\mu(A)$. Therefore $A \subseteq c_{s_\mu}(U) \subseteq c_{s_\mu}(i_\mu(A))$ and hence $A \subseteq c_{s_\mu}(i_\mu(A))$. Conversely, assume that $A \subseteq c_{s_\mu}(i_\mu(A))$. To prove that A is a $\tilde{\mu}$-open set in (X, μ). Take $U = i_\mu(A)$. Then $i_\mu(A) \subseteq A \subseteq c_{s_\mu}(i_\mu(A))$. Hence A is $\tilde{\mu}$-open in (X, μ). □

Theorem 3.2. *Let (X, μ) be a generalized topological space, A be a subset of X. If A is a μ-open set in (X, μ), then A is $\tilde{\mu}$-open in (X, μ).*

Proof. If A is a μ-open set in (X, μ), then $A = i_\mu(A)$. Since $A \subseteq c_{s_\mu}(A)$, we have that $A \subseteq c_{s_\mu}(i_\mu(A))$. Then by Theorem 3.1 A is $\tilde{\mu}$-open in (X, μ). □

Remark 3.1. *The following example shows that the converse of the above theorem need not be true.*

Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $A = \{a, b, d\}$ is a $\tilde{\mu}$-open set in (X, μ) but not μ-open.

Theorem 3.3. *Let (X, μ) be a generalized topological space, A be a subset of X. If A is a $\tilde{\mu}$-open set in (X, μ), then A is μ-semi-open in (X, μ).*

Proof. If A is a $\tilde{\mu}$-open set in (X, μ), then by Theorem 3.1 $A \subseteq c_{s_\mu}(i_\mu(A))$. Since every μ-closed set is μ-semi-closed and $c_{s_\mu}(i_\mu(A))$ is a least μ-semi-closed set containing $i_\mu(A)$, this implies that $c_{s_\mu}(i_\mu(A)) \subseteq c_\mu(i_\mu(A))$. Therefore $A \subseteq c_\mu(i_\mu(A))$ and hence A is a μ-semi-open set in (X, μ). □

Remark 3.2. *The following example shows that the converse of the above theorem need not be true.*

Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Then $A = \{a, d\}$ is a μ-semi-open set in (X, μ) but not $\tilde{\mu}$-open.

Theorem 3.4. *Let $\{A_\alpha : \alpha \in J\}$ be the collection of $\tilde{\mu}$-open sets in a generalized topological space (X, μ). Then $\bigcup_{\alpha \in J} A_\alpha$ is also a $\tilde{\mu}$-open set in (X, μ).*

Proof. Since A_α is $\tilde{\mu}$-open, then there exists a μ-open set U_α of X such that $U_\alpha \subseteq A \subseteq c_{s_\mu}(U_\alpha)$. This implies that $\bigcup_{\alpha \in J} U_\alpha \subseteq \bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in J} c_{s_\mu}(U_\alpha) \subseteq c_{s_\mu}(\bigcup_{\alpha \in J} U_\alpha)$ since union of all μ-open sets is μ-open. Therefore $\bigcup_{\alpha \in J} A_\alpha$ is a $\tilde{\mu}$-open set in (X, μ). □

Remark 3.3. *If A and B are two $\tilde{\mu}$-open sets in (X, μ), then $A \cap B$ need not be $\tilde{\mu}$-open in (X, μ).*

(i) Let $X = \{1, 2, 3, ..., n\}$ and $\mu = \{\emptyset, X\} \cup \{M \subseteq X \mid M = X - \{i\}$ for some $i \in X\}$. Take $A = X - \{1\}$ and $B = X - \{2\}$. Then A and B are $\tilde{\mu}$-open sets in (X, μ) but $A \cap B = X - \{1, 2\}$ is not $\tilde{\mu}$-open in (X, μ).

(ii) Let $X = R$, with the usual topology. If A = [-1, 0] and B = [0, 1], then A and B are $\tilde{\mu}$-open sets in (X, μ) but $A \cap B = \{0\}$ is not $\tilde{\mu}$-open in (X, μ).

Theorem 3.5. *Let A be a $\tilde{\mu}$-open set in (X, μ) and B be any set such that $A \subseteq B \subseteq c_{s_\mu}(i_\mu(A))$. Then B is also a $\tilde{\mu}$-open set in (X, μ).*

Proof. If A is a $\tilde{\mu}$-open set in (X, μ), then by Theorem 3.1 $A \subseteq c_{s_\mu}(i_\mu(A))$. Since $A \subseteq B$, this implies that $c_{s_\mu}(i_\mu(A)) \subseteq c_{s_\mu}(i_\mu(B))$. By hypothesis $B \subseteq c_{s_\mu}(i_\mu(A)) \subseteq c_{s_\mu}(i_\mu(B))$ and hence $B \subseteq c_{s_\mu}(i_\mu(B))$. This shows that B is a $\tilde{\mu}$-open set in (X, μ). □

Definition 3.2. *Let (X, μ) be a generalized topological space. A subset A of X is called $\tilde{\mu}$-closed if its complement $X \setminus A$ is $\tilde{\mu}$-open.*

Theorem 3.6. *Let (X, μ) be a generalized topological space, A be a subset of X. Then A is $\tilde{\mu}$-closed in (X, μ) if and only if $i_{s_\mu}(c_\mu(A)) \subseteq A$.*

Proof. If A is a $\tilde{\mu}$-closed set in (X, μ), then $X \setminus A$ is $\tilde{\mu}$-open. Therefore $X \setminus A \subseteq c_{s_\mu}(i_\mu(X \setminus A))$ (by Theorem 3.1) $= c_{s_\mu}(X \setminus c_\mu(A)) = X \setminus i_{s_\mu}(c_\mu(A))$. This implies that $i_{s_\mu}(c_\mu(A)) \subseteq A$. Conversely, suppose that $i_{s_\mu}(c_\mu(A)) \subseteq A$. Then $X \setminus A \subseteq X \setminus i_{s_\mu}(c_\mu(A)) = c_{s_\mu}(X \setminus c_\mu(A)) = c_{s_\mu}(i_\mu(X \setminus A))$. Therefore $X \setminus A$ is $\tilde{\mu}$-open set in (X, μ) and this shows that A is $\tilde{\mu}$-closed set in (X, μ). □

Theorem 3.7. *Let (X, μ) be a generalized topological space, A be a subset of X. If $i_{s_\mu}(F) \subseteq A \subseteq F$, then A is $\tilde{\mu}$-closed in (X, μ) for any μ-closed set F of (X, μ).*

Proof. Let $i_{s_\mu}(F) \subseteq A \subseteq F$ where F is μ-closed subset of X. Then $X \setminus F \subseteq X \setminus A \subseteq X \setminus i_{s_\mu}(F) = c_{s_\mu}(X \setminus F)$. Let $U = X \setminus F$. Then U is μ-open and $U \subseteq X \setminus A \subseteq c_{s_\mu}(U)$. This implies that $X \setminus A$ is a $\tilde{\mu}$-open set in (X, μ) and hence A is a $\tilde{\mu}$-closed set in (X, μ). □

Remark 3.4. *The converse of the above theorem need not be true.*

In Example 3.1 for the $\tilde{\mu}$-closed set $\{b\}$, does not exist any μ-closed set in (X, μ).

Theorem 3.8. *Let (X, μ) be a generalized topological space, A be a subset of X. Then (i) $i_{s_\mu}(c_\mu(A))$ is $\tilde{\mu}$-closed; (ii) $c_{s_\mu}(i_\mu(A))$ is $\tilde{\mu}$-open.*

Proof. (i) Obviously $i_{s_\mu}(c_\mu(i_{s_\mu}(c_\mu(A)))) \subseteq i_{s_\mu}(c_\mu(c_\mu(A))) = i_{s_\mu}(c_\mu(A))$. Hence $i_{s_\mu}(c_\mu(A))$ is $\tilde{\mu}$-closed.
(ii) Follows from (i) and Theorem 3.1. □

Theorem 3.9. *Let $\{A_\alpha : \alpha \in J\}$ be the collection of $\tilde{\mu}$-closed sets in a generalized topological space (X, μ). Then $\bigcap_{\alpha \in J} A_\alpha$ is also a $\tilde{\mu}$-closed set in (X, μ).*

Proof. Let A_α be $\tilde{\mu}$-closed in (X, μ). Then $X \setminus A_\alpha$ is $\tilde{\mu}$-open. By Theorem 3.4 $\bigcup_{\alpha \in J}(X \setminus A_\alpha)$ is also $\tilde{\mu}$-open. This implies that $\bigcup_{\alpha \in J}(X \setminus A_\alpha) = X \setminus \bigcap_{\alpha \in J} A_\alpha$ is $\tilde{\mu}$-open and hence $\bigcap_{\alpha \in J} A_\alpha$ is $\tilde{\mu}$-closed in (X, μ). □

Definition 3.3. *Let (X, μ) be a generalized topological space, A be a subset of X. Then $\tilde{\mu}$-interior of A is defined as union of all $\tilde{\mu}$-open sets contained in A. Thus $i_{\tilde{\mu}}(A) = \cup\{U : U \in \tilde{\mu}O(X)$ and $U \subseteq A\}$.*

Definition 3.4. *Let (X, μ) be a generalized topological space, A be a subset of X. Then $\tilde{\mu}$-closure of A is defined as intersection of all $\tilde{\mu}$-closed sets containing A. Thus* $c_{\tilde{\mu}}(A) = \cap\{F : X \setminus F \in \tilde{\mu}O(X)$ *and* $A \subseteq F\}$.

Theorem 3.10. *Let (X, μ) be a generalized topological space, A be a subset of X. Then (i)* $i_{\tilde{\mu}}(A)$ *is a* $\tilde{\mu}$*-open set contained in A;*
(ii) $c_{\tilde{\mu}}(A)$ *is a* $\tilde{\mu}$*-closed set containing A;*
(iii) A is $\tilde{\mu}$*-closed if and only if* $c_{\tilde{\mu}}(A) = A$;
(iv) A is $\tilde{\mu}$*-open if and only if* $i_{\tilde{\mu}}(A) = A$;
(v) $i_{\tilde{\mu}}(i_{\tilde{\mu}}(A)) = i_{\tilde{\mu}}(A)$;
(vi) $c_{\tilde{\mu}}(c_{\tilde{\mu}}(A)) = c_{\tilde{\mu}}(A)$;
(vii) $i_{\tilde{\mu}}(A) = X \setminus c_{\tilde{\mu}}(X \setminus A)$;
(viii) $c_{\tilde{\mu}}(A) = X \setminus i_{\tilde{\mu}}(X \setminus A)$.

Proof. (i) Follows from Definition 3.3 and Theorem 3.4.
(ii) Follows from Definition 3.4 and Theorem 3.9.
(iii) and (iv) Follows from Definition 3.5, (ii) and Definition 3.4, (i) respectively.
(v) and (vi) Follows from (i), (iv) and (ii), (iii) respectively.
(vii) and (viii) Follows from Definitions 3.2, 3.4 and 3.5. □

Theorem 3.11. *Let (X, μ) be a generalized topological space. If A and B are two subsets of X, then the following are hold:*
(i) If $A \subseteq B$*, then* $i_{\tilde{\mu}}(A) \subseteq i_{\tilde{\mu}}(B)$;
(ii) If $A \subseteq B$*, then* $c_{\tilde{\mu}}(A) \subseteq c_{\tilde{\mu}}(B)$;
(iii) $i_{\tilde{\mu}}(A \cup B) = i_{\tilde{\mu}}(A) \cup i_{\tilde{\mu}}(B)$;
(iv) $c_{\tilde{\mu}}(A \cap B) = c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(B)$;
(v) $i_{\tilde{\mu}}(A \cap B) \subseteq i_{\tilde{\mu}}(A) \cap i_{\tilde{\mu}}(B)$.
(vi) $c_{\tilde{\mu}}(A \cup B) \supseteq c_{\tilde{\mu}}(A) \cup c_{\tilde{\mu}}(B)$.

Proof. (i) Follows from Definition 3.3 and 3.4 respectively.
(ii) Follows from (i), Theorem 3.4 and (ii), Theorem 3.9 respectively.
(iii) Follows from (i) and (ii) respectively. □

Theorem 3.12. *Let (X, μ) be a generalized topological space, A be a subset of X. (i) If* $A \subseteq i_{s_\mu}(c_\mu(A))$*, then* $c_{\tilde{\mu}}(A) \subseteq i_{s_\mu}(c_\mu(A))$;
(ii) If $c_{s_\mu}(i_\mu(A)) \subseteq A$*, then* $i_{\tilde{\mu}}(A) \supseteq c_{s_\mu}(i_\mu(A))$.

Proof. (i) Since $c_{\tilde{\mu}}(A)$ is the least $\tilde{\mu}$-closed set containing A and Theorem 3.8(i) shows that $i_{s_\mu}(c_\mu(A))$ is $\tilde{\mu}$-closed. Therefore $c_{\tilde{\mu}}(A) \subseteq i_{s_\mu}(c_\mu(A))$.

(ii) Since $i_{\tilde{\mu}}(A)$ is the greatest $\tilde{\mu}$-open set containing A and Theorem 3.8(ii) shows that $c_{s_\mu}(i_\mu(A))$ is $\tilde{\mu}$-open. Therefore $i_{\tilde{\mu}}(A) \supseteq c_{s_\mu}(i_\mu(A))$. □

Definition 3.5. *Let (X, μ) be a generalized topological space. A subset A of X is called* $\tilde{\mu}$*-regular if it is both* $\tilde{\mu}$*-open and* $\tilde{\mu}$*-closed. The class of all* $\tilde{\mu}$*-regular set of X is denoted by* $\tilde{\mu}R(X)$.

Remark 3.5. *If A is a* $\tilde{\mu}$*-regular set in (X, μ), then* $X \setminus A$ *is* $\tilde{\mu}$*-regular in (X, μ).*

Proof. Follows from Definition 3.5. □

Definition 3.6. *Let (X, μ) be a generalized topological space and A be a subset of X. Then* $\tilde{\mu}$*-boundary of A is denoted by* $bd_{\tilde{\mu}}(A)$ *and is defined as* $bd_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A)$.

Theorem 3.13. *For a subset A of X, $bd_{\tilde{\mu}}(A) = \emptyset$ if and only if A is $\tilde{\mu}$-regular in (X, μ).*

Proof. Let $bd_{\tilde{\mu}}(A) = \emptyset$. Then $c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A) = \emptyset$. This implies that $c_{\tilde{\mu}}(A) \subseteq X \setminus c_{\tilde{\mu}}(X \setminus A) = i_{\tilde{\mu}}(A)$ (by Theorem 3.10(vii)). Therefore $c_{\tilde{\mu}}(A) = A = i_{\tilde{\mu}}(A)$ and hence A is both $\tilde{\mu}$-closed and $\tilde{\mu}$-open in (X, μ). Conversely, assume that A is $\tilde{\mu}$-regular. Then A is both $\tilde{\mu}$-closed and $\tilde{\mu}$-open. This implies that $c_{\tilde{\mu}}(A) = A = i_{\tilde{\mu}}(A) = X \setminus c_{\tilde{\mu}}(X \setminus A)$ (by Theorem 3.10(vii)). Since $X \setminus c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(X \setminus A) = \emptyset$, we have that $c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A) = \emptyset$. This shows that $bd_{\tilde{\mu}}(A) = \emptyset$. □

Theorem 3.14. *In any generalized topological space (X, μ), the following are equivalent:*
(i) $X \setminus bd_{\tilde{\mu}}(A) = i_{\tilde{\mu}}(A) \cup i_{\tilde{\mu}}(X \setminus A)$;
(ii) $c_{\tilde{\mu}}(A) = i_{\tilde{\mu}}(A) \cup bd_{\tilde{\mu}}(A)$;
(iii) $bd_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A) = c_{\tilde{\mu}}(A) \setminus i_{\tilde{\mu}}(A)$.

Proof. (i) ⇒ (ii). From (i) $X \setminus bd_{\tilde{\mu}}(A) = i_{\tilde{\mu}}(A) \cup i_{\tilde{\mu}}(X \setminus A)$ implies that $bd_{\tilde{\mu}}(A) = [X \setminus i_{\tilde{\mu}}(A)] \cap [X \setminus i_{\tilde{\mu}}(X \setminus A)]$. Therefore $i_{\tilde{\mu}}(A) \cup bd_{\tilde{\mu}}(A) = [i_{\tilde{\mu}}(A) \cup (X \setminus i_{\tilde{\mu}}(A))] \cap [i_{\tilde{\mu}}(A) \cup c_{\tilde{\mu}}(A)] = X \cap c_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(A)$. Hence $c_{\tilde{\mu}}(A) = i_{\tilde{\mu}}(A) \cup bd_{\tilde{\mu}}(A)$.

(ii) ⇒ (iii). From (ii) $c_{\tilde{\mu}}(A) \setminus i_{\tilde{\mu}}(A) = [i_{\tilde{\mu}}(A) \cup bd_{\tilde{\mu}}(A)] \setminus i_{\tilde{\mu}}(A) = bd_{\tilde{\mu}}(A)$ ....(*1). Also from (ii) $X \cap c_{\tilde{\mu}}(A) = i_{\tilde{\mu}}(A) \cup bd_{\tilde{\mu}}(A)$ implies that $[i_{\tilde{\mu}}(A) \cup (X \setminus i_{\tilde{\mu}}(A))] \cap [i_{\tilde{\mu}}(A) \cup c_{\tilde{\mu}}(A)] = i_{\tilde{\mu}}(A) \cup bd_{\tilde{\mu}}(A)$ implies that $i_{\tilde{\mu}}(A) \cup [c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(A)] = i_{\tilde{\mu}}(A) \cup bd_{\tilde{\mu}}(A)$. Therefore $bd_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A)$ ....(*2). From (*1) and (*2), we have that $bd_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A) = c_{\tilde{\mu}}(A) \setminus i_{\tilde{\mu}}(A)$.

(iii) ⇒ (i). From (iii), we have that $X \setminus bd_{\tilde{\mu}}(A) = X \setminus [c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(A)] = [X \setminus c_{\tilde{\mu}}(X \setminus A)] \cup [X \setminus c_{\tilde{\mu}}(A)] = i_{\tilde{\mu}}(A) \cup i_{\tilde{\mu}}(X \setminus A)$. Therefore $X \setminus bd_{\tilde{\mu}}(A) = i_{\tilde{\mu}}(A) \cup i_{\tilde{\mu}}(X \setminus A)$. □

Theorem 3.15. *For a subset A of generalized topological space (X, μ), we have the following conditions hold:*
(i) $bd_{\tilde{\mu}}(A) = bd_{\tilde{\mu}}(X \setminus A)$;
(ii) $bd_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(A) \setminus i_{\tilde{\mu}}(A)$;
(iii) $bd_{\tilde{\mu}}(A) \cap i_{\tilde{\mu}}(A) = \emptyset$;
(iv) $c_{\tilde{\mu}}(A) = i_{\tilde{\mu}}(A) \cup bd_{\tilde{\mu}}(A)$;
(v) $bd_{\tilde{\mu}}(i_{\tilde{\mu}}(A)) \subseteq bd_{\tilde{\mu}}(A)$;
(vi) $bd_{\tilde{\mu}}(c_{\tilde{\mu}}(A)) \subseteq bd_{\tilde{\mu}}(A)$;
(vii) $X \setminus bd_{\tilde{\mu}}(A) = i_{\tilde{\mu}}(A) \cup i_{\tilde{\mu}}(X \setminus A)$;
(viii) $X = i_{\tilde{\mu}}(A) \cup i_{\tilde{\mu}}(X \setminus A) \cup bd_{\tilde{\mu}}(A)$.

Proof. (i) By Definition 3.6, we have that $bd_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(X \setminus (X \setminus A)) = bd_{\tilde{\mu}}(X \setminus A)$. Therefore $bd_{\tilde{\mu}}(A) = bd_{\tilde{\mu}}(X \setminus A)$.

(ii) By Definition 3.6, we have that $bd_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A) = c_{\tilde{\mu}}(A) \setminus (X \setminus c_{\tilde{\mu}}(X \setminus A)) = c_{\tilde{\mu}}(A) \setminus i_{\tilde{\mu}}(A)$ (by Theorem 3.10 (vii)). Therefore $bd_{\tilde{\mu}}(A) = c_{\tilde{\mu}}(A) \setminus i_{\tilde{\mu}}(A)$.

(iii) By Definition 3.6, we have that $bd_{\tilde{\mu}}(A) \cap i_{\tilde{\mu}}(A) = (c_{\tilde{\mu}}(A) \setminus i_{\tilde{\mu}}(A)) \cap i_{\tilde{\mu}}(A)$ (by (ii)) $= \emptyset$. Hence $bd_{\tilde{\mu}}(A) \cap i_{\tilde{\mu}}(A) = \emptyset$.

(iv) Follows from (ii) and Theorem 3.14.

(v) By Definition 3.6, we have that $bd_{\tilde{\mu}}(i_{\tilde{\mu}}(A)) = c_{\tilde{\mu}}(X \setminus i_{\tilde{\mu}}(A)) \cap c_{\tilde{\mu}}(i_{\tilde{\mu}}(A)) = c_{\tilde{\mu}}(c_{\tilde{\mu}}(X \setminus A)) \cap c_{\tilde{\mu}}(i_{\tilde{\mu}}(A)) = c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(i_{\tilde{\mu}}(A))$ (by Theorem 3.10 (vi)) $\subseteq c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(A) = bd_{\tilde{\mu}}(A)$. This shows that $bd_{\tilde{\mu}}(i_{\tilde{\mu}}(A)) \subseteq bd_{\tilde{\mu}}(A)$.

(vi) By Definition 3.6, we have that $bd_{\tilde{\mu}}(c_{\tilde{\mu}}(A)) = c_{\tilde{\mu}}(X \setminus c_{\tilde{\mu}}(A)) \cap c_{\tilde{\mu}}(c_{\tilde{\mu}}(A)) = c_{\tilde{\mu}}(i_{\tilde{\mu}}(X \setminus A)) \cap c_{\tilde{\mu}}(A)$ (by Theorem 3.10 (vi)) $\subseteq c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(A) = bd_{\tilde{\mu}}(A)$. Therefore $bd_{\tilde{\mu}}(c_{\tilde{\mu}}(A)) \subseteq bd_{\tilde{\mu}}(A)$.

(vii) Follows from (iv) and Theorem 3.14.

(viii) Using (vii) $(X \setminus bd_{\tilde{\mu}}(A)) \cup bd_{\tilde{\mu}}(A) = [i_{\tilde{\mu}}(A) \cup i_{\tilde{\mu}}(X \setminus A)] \cup bd_{\tilde{\mu}}(A)$. This implies that $X = i_{\tilde{\mu}}(A) \cup i_{\tilde{\mu}}(X \setminus A) \cup bd_{\tilde{\mu}}(A)$. □

Theorem 3.16. *Let A be a subset of generalized topological space (X, μ). Then (i) A is $\tilde{\mu}$-open if and only if $A \cap bd_{\tilde{\mu}}(A) = \emptyset$;*
(ii) A is $\tilde{\mu}$-closed if and only if $bd_{\tilde{\mu}}(A) \subseteq A$.

Proof. Let A be a $\tilde{\mu}$-open set in (X, μ). Then $X \setminus A$ is $\tilde{\mu}$-closed and $c_{\tilde{\mu}}(X \setminus A) = X \setminus A$. Also $A \neq c_{\tilde{\mu}}(A)$. By Definition 3.6 $A \cap bd_{\tilde{\mu}}(A) = A \cap (c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A)) = A \cap c_{\tilde{\mu}}(A \cap (X \setminus A)) = A \cap \emptyset = \emptyset$. Thus $A \cap bd_{\tilde{\mu}}(A) = \emptyset$. Conversely, assume that $A \cap bd_{\tilde{\mu}}(A) = \emptyset$. Then $A \cap (c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A)) = \emptyset$. This implies that $A \cap c_{\tilde{\mu}}(X \setminus A) = \emptyset$ and hence $c_{\tilde{\mu}}(X \setminus A) \subseteq X \setminus A$. Therefore $c_{\tilde{\mu}}(X \setminus A) = X \setminus A$. This shows that $X \setminus A$ is $\tilde{\mu}$-closed in (X, μ) and hence A is $\tilde{\mu}$-open in (X, μ).

(ii) Let A be a $\tilde{\mu}$-closed set in (X, μ). Then $A = c_{\tilde{\mu}}(A)$. Since $bd_{\tilde{\mu}}(A) = (c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A)) \subseteq c_{\tilde{\mu}}(A) = A$. Therefore $bd_{\tilde{\mu}}(A) \subseteq A$. Conversely, let $bd_{\tilde{\mu}}(A) \subseteq A$. Then $bd_{\tilde{\mu}}(A) \cap (X \setminus A) = \emptyset$. By Theorem 3.15 (i) $bd_{\tilde{\mu}}(X \setminus A) \cap (X \setminus A) = \emptyset$. By (i) $X \setminus A$ is $\tilde{\mu}$-open in (X, μ). Hence A is $\tilde{\mu}$-closed in (X, μ). □

Definition 3.7. *Let (X, μ) be a generalized topological space and A be a subset of X. Then $\tilde{\mu}$-exterior of A is denoted by $ext_{\tilde{\mu}}(A)$ and is defined as $ext_{\tilde{\mu}}(A) = i_{\tilde{\mu}}(X \setminus A)$.*

Theorem 3.17. *Let A and B be two subsets of generalized topological space (X, μ). Then (i) $ext_{\tilde{\mu}}(A \cup B) \subseteq ext_{\tilde{\mu}}(A) \cup ext_{\tilde{\mu}}(B)$;*
(ii) $bd_{\tilde{\mu}}(A \cup B) = c_{\tilde{\mu}}(A \cup B) \cap c_{\tilde{\mu}}(X - A) \cap c_{\tilde{\mu}}(X \setminus B)$;
(iii) $bd_{\tilde{\mu}}(A \cap B) = (bd_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(B)) \cup (bd_{\tilde{\mu}}(B) \cap c_{\tilde{\mu}}(A))$.

Proof. By Definition 3.7, we have that $ext_{\tilde{\mu}}(A \cup B) = i_{\tilde{\mu}}(X \setminus (A \cup B)) = i_{\tilde{\mu}}((X \setminus A) \cap (X \setminus B)) \subseteq i_{\tilde{\mu}}(X \setminus A) \cap i_{\tilde{\mu}}(X \setminus B)$ (by Theorem 3.11 (v)) $= ext_{\tilde{\mu}}(A) \cup ext_{\tilde{\mu}}(B)$. Hence $ext_{\tilde{\mu}}(A \cup B) \subseteq ext_{\tilde{\mu}}(A) \cup ext_{\tilde{\mu}}(B)$.

(ii) By Definition 3.6, we have that $bd_{\tilde{\mu}}(A \cup B) = c_{\tilde{\mu}}(A \cup B) \cap c_{\tilde{\mu}}(X \setminus (A \cup B)) = c_{\tilde{\mu}}(A \cup B) \cap c_{\tilde{\mu}}((X \setminus A) \cap (X \setminus B)) = c_{\tilde{\mu}}(A \cup B) \cap c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(X \setminus B)$ (by Theorem 3.11 (iv)). Hence $bd_{\tilde{\mu}}(A \cup B) = c_{\tilde{\mu}}(A \cup B) \cap c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(X \setminus B)$.

(iii) By Definition 3.6, we have that $bd_{\tilde{\mu}}(A \cap B) = c_{\tilde{\mu}}(A \cap B) \cap c_{\tilde{\mu}}(X \setminus (A \cap B)) = (c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(B)) \cap (c_{\tilde{\mu}}(X \setminus A) \cup c_{\tilde{\mu}}(X \setminus B)) = ((c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(B)) \cap c_{\tilde{\mu}}(X \setminus A)) \cup ((c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(B)) \cap c_{\tilde{\mu}}(X \setminus B)) = ((c_{\tilde{\mu}}(A) \cap c_{\tilde{\mu}}(X \setminus A)) \cap c_{\tilde{\mu}}(B)) \cup (c_{\tilde{\mu}}(A) \cap (c_{\tilde{\mu}}(B) \cap c_{\tilde{\mu}}(X \setminus B))) = c_{\tilde{\mu}}(A \cup B) \cap c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(X \setminus B)$. Hence $bd_{\tilde{\mu}}(A \cup B) = c_{\tilde{\mu}}(A \cup B) \cap c_{\tilde{\mu}}(X \setminus A) \cap c_{\tilde{\mu}}(X \setminus B)$. □

Theorem 3.18. *For any two subsets A and B of generalized topological space (X, μ), we have the following conditions hold:*
(i) $ext_{\tilde{\mu}}(X \setminus ext_{\tilde{\mu}}(A)) = ext_{\tilde{\mu}}(A)$;
(ii) $ext_{\tilde{\mu}}(A \cap B) = ext_{\tilde{\mu}}(A) \cup ext_{\tilde{\mu}}(B)$.

Proof. (i) By Definition 3.7, we have that $ext_{\tilde{\mu}}(X \setminus ext_{\tilde{\mu}}(A)) = i_{\tilde{\mu}}(X \setminus (X \setminus ext_{\tilde{\mu}}(A))) = i_{\tilde{\mu}}(ext_{\tilde{\mu}}(A)) = ext_{\tilde{\mu}}(A)$. Hence $ext_{\tilde{\mu}}(X \setminus ext_{\tilde{\mu}}(A)) = ext_{\tilde{\mu}}(A)$.

(ii) By Definition 3.7, we have that $ext_{\tilde{\mu}}(A \cap B) = i_{\tilde{\mu}}(X \setminus (A \cap B)) = i_{\tilde{\mu}}((X \setminus A) \cup (X \setminus B)) = i_{\tilde{\mu}}(X \setminus A) \cup i_{\tilde{\mu}}(X \setminus B)$ (by Theorem 3.11 (iii) $= ext_{\tilde{\mu}}(A) \cup ext_{\tilde{\mu}}(B)$. Hence $ext_{\tilde{\mu}}(A \cap B) = ext_{\tilde{\mu}}(A) \cup ext_{\tilde{\mu}}(B)$. □

4 Separation axioms

Definition 4.8. *A generalized topological space (X, μ) is called a $\tilde{\mu}$-T_0 space if for each pair of distinct points $x, y \in X$, there exists a $\tilde{\mu}$-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.*

Definition 4.9. *A generalized topological space (X, μ) is called a $\tilde{\mu}$-T_1 space if for each pair of distinct points $x, y \in X$, there exists a $\tilde{\mu}$-open sets U and V contain x and y respectively such that $y \notin U$ and $x \notin V$.*

Definition 4.10. *A generalized topological space (X, μ) is called a $\tilde{\mu}$-T_2 space if for each pair of distinct points $x, y \in X$, there exists a $\tilde{\mu}$-open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.*

Definition 4.11. *Let (X, μ) be a generalized topological space and A be a subset of X. Then A is called a $\tilde{\mu}$-generalizedized closed (briefly $\tilde{\mu}$-g.closed) set if $c_{\tilde{\mu}}(A) \subseteq U$ whenever $A \subseteq U$ and U is a $\tilde{\mu}$-open set in (X, μ).*

Remark 4.6. *From Definition 4.4, every $\tilde{\mu}$-closed set is $\tilde{\mu}$-g.closed set. But, the converse need not be true.*

Definition 4.12. *A generalized topological space (X, μ) is called a $\tilde{\mu}$-$T_{\frac{1}{2}}$ space each $\tilde{\mu}$-g.closed set of (X, μ) is $\tilde{\mu}$-closed.*

Theorem 4.19. *Let (X, μ) be a generalized topological space. Then for a point $x \in X$, $x \in c_{\tilde{\mu}}(A)$ if and only if $V \cap A \neq \emptyset$ for any $V \in \tilde{\mu}O(X)$ such that $x \in V$.*

Proof. Let F_0 be the set of all $y \in X$ such that $V \cap A \neq \emptyset$ for any $V \in \tilde{\mu}O(X)$ and $y \in V$. Now, we prove that $c_{\tilde{\mu}}(A) = F_0$. Let us assume $x \in c_{\tilde{\mu}}(A)$ and $x \notin F_0$. Then there exists a $\tilde{\mu}$-open set U of x such that $U \cap A = \emptyset$. This implies that $A \subseteq X \setminus U$. Therefore $c_{\tilde{\mu}}(A) \subseteq X \setminus U$. Hence $x \notin c_{\tilde{\mu}}(A)$. This is a contradiction. Hence $c_{\tilde{\mu}}(A) \subseteq F_0$. Conversely, let F be a set such that $A \subseteq F$ and $X \setminus F \in \tilde{\mu}O(X)$. Let $x \notin F$. Then we have that $x \in X \setminus F$ and $(X \setminus F) \cap A = \emptyset$. This implies that $x \notin F_0$. Therefore $F_0 \subseteq F$. Hence $F_0 \subseteq c_{\tilde{\mu}}(A)$. □

Definition 4.13. *Let (X, μ) be a generalized topological space and A be a subset of X. Then A is $\tilde{\mu}$-g.closed if and only if $c_{\tilde{\mu}}(\{x\}) \cap A \neq \emptyset$ holds for every $x \in c_{\tilde{\mu}}(A)$.*

Proof. Let U be any $\tilde{\mu}$-open set in (X, μ) such that $A \subseteq U$. Let $x \in c_{\tilde{\mu}}(A)$. By assumption there exists a point $z \in c_{\tilde{\mu}}(\{x\})$ and $z \in A \subseteq U$. Therefore from Theorem 5.1, we have that $U \cap \{x\} \neq \emptyset$. This implies that $x \in U$. Hence A is a $\tilde{\mu}$-g.closed set in X. Conversely, suppose there exists a point $x \in c_{\tilde{\mu}}(A)$ such that $c_{\tilde{\mu}}(\{x\}) \cap A = \emptyset$. Since $c_{\tilde{\mu}}(\{x\})$ is a $\tilde{\mu}$-closed set implies that $X \setminus c_{\tilde{\mu}}(\{x\})$ is a $\tilde{\mu}$-open set. Since $A \subseteq X \setminus c_{\tilde{\mu}}(\{x\})$ and A is $\tilde{\mu}$-g.closed set, implies that $c_{\tilde{\mu}}(A) \subseteq X \setminus c_{\tilde{\mu}}(\{x\})$. Hence $x \notin c_{\tilde{\mu}}(A)$. This is a contradiction. □

Theorem 4.20. *Let (X, μ) be a generalized topological space and A be a subset of X. Then $c_{\tilde{\mu}}(\{x\}) \cap A \neq \emptyset$ for every $x \in c_{\tilde{\mu}}(A)$ if and only if $c_{\tilde{\mu}}(A) \subseteq ker_{\tilde{\mu}}(A)$ holds, where $ker_{\tilde{\mu}}(E) = \cap\{V : V \in \tilde{\mu}O(X)$ and $E \subseteq V\}$ for any subset E of X.*

Proof. Let $x \in c_{\tilde{\mu}}(A)$. By hypothesis, there exists a point z such that $z \in c_{\tilde{\mu}}(\{x\})$ and $z \in A$. Let $U \in \tilde{\mu}O(X)$ be a subset of X such that $A \subseteq U$. Since $z \in U$ and $z \in c_{\tilde{\mu}}(\{x\})$. By Theorem 4.2, we have that $U \cap \{x\} \neq \emptyset$, this implies that $x \in ker_{\tilde{\mu}}(A)$. Hence $c_{\tilde{\mu}}(A) \subseteq ker_{\tilde{\mu}}(A)$. Conversely, let U ba any $\tilde{\mu}$-open set such that $A \subseteq U$. Let x be a point such that $x \in c_{\tilde{\mu}}(A)$. By hypothesis, $x \in ker_{\tilde{\mu}}(A)$ holds. Namely, we have that $x \in U$, because $A \subseteq U$ and U is $\tilde{\mu}$-open set. Therefore $c_{\tilde{\mu}}(A) \subseteq U$. By Definition 4.4 A is $\tilde{\mu}$-g.closed. Then by Theorem 4.2 $c_{\tilde{\mu}}(\{x\}) \cap A \neq \emptyset$ holds for every $x \in c_{\tilde{\mu}}(A)$. □

Theorem 4.21. *Let (X, μ) be a generalized topological space and A be the $\tilde{\mu}$-g.closed set in (X, μ). Then $c_{\tilde{\mu}}(A) \setminus A$ does not contain a non empty $\tilde{\mu}$-closed set.*

Proof. Suppose there exists a non empty $\tilde{\mu}$-closed set F such that $F \subseteq c_{\tilde{\mu}}(A) \setminus A$. Let $x \in F$. Then $x \in c_{\tilde{\mu}}(A)$, implies that $F \cap A = c_{\tilde{\mu}}(A) \cap A \supseteq c_{\tilde{\mu}}(\{x\}) \cap A \neq \emptyset$ and hence $F \cap A \neq \emptyset$. This is a contradiction. □

Theorem 4.22. *For each $x \in X$, $\{x\}$ is $\tilde{\mu}$-closed or $X \setminus \{x\}$ is $\tilde{\mu}$-g.closed.*

Proof. Suppose that $\{x\}$ is not $\tilde{\mu}$-closed. Then $X \setminus \{x\}$ is not $\tilde{\mu}$-open. This implies that X is the only $\tilde{\mu}$-open set containing $X \setminus \{x\}$ and hence $X \setminus \{x\}$ is $\tilde{\mu}$-g.closed. □

Theorem 4.23. *A generalized topological space (X, μ) is a $\tilde{\mu}$-$T_{\frac{1}{2}}$ space if and only if for each $x \in X$, $\{x\}$ is $\tilde{\mu}$-open or $\tilde{\mu}$-closed.*

Proof. Suppose that $\{x\}$ is not $\tilde{\mu}$-closed. Then it follows from the assumption and Theorem 4.5, $\{x\}$ is $\tilde{\mu}$-open. Conversely, let F be a $\tilde{\mu}$-g.closed set in (X, μ). Let $x \in c_{\tilde{\mu}}(F)$. Then by the assumption $\{x\}$ is either $\tilde{\mu}$-open or $\tilde{\mu}$-closed.

Case(i): Suppose that $\{x\}$ is $\tilde{\mu}$-open. Then by Theorem 4.1, $\{x\} \cap F \neq \emptyset$. This implies that $c_{\tilde{\mu}}(F) = F$. Therefore (X, μ) is a $\tilde{\mu}$-$T_{\frac{1}{2}}$ space.

Case(ii): Suppose that $\{x\}$ is $\tilde{\mu}$-closed. Let us assume $x \notin F$. Then $x \in c_{\tilde{\mu}}(F) \setminus F$. This is a contradiction. Hence $x \in F$. Therefore (X, μ) is a $\tilde{\mu}$-$T_{\frac{1}{2}}$ space. □

A space (X, μ) is $\tilde{\mu}$-T_1 if and only if for any $x \in X$, $\{x\}$ is $\tilde{\mu}$-closed.

Proof. Follows from Definitions 2.14 and 4.2. □

Remark 4.7. *(i) From the Theorems 4.5, 4.6 and 4.7, we have that every $\tilde{\mu}$-$T_{\frac{1}{2}}$ space is $\tilde{\mu}$-T_0, every $\tilde{\mu}$-T_1 space is $\tilde{\mu}$-$T_{\frac{1}{2}}$ and every $\tilde{\mu}$-T_2 space is $\tilde{\mu}$-T_1.*

(ii) Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, X, \{a\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. For the distinct points $a, k \in X$, where $k \in \{b, c, d\}$, there exists a $\tilde{\mu}$-open set $\{a\}$ but which is not contain k; the pair $b, c \in X$, there exists a $\tilde{\mu}$-open set $\{b, d\}$ but which is not contain c; the points $b, d \in X$, there exists a $\tilde{\mu}$-open set $\{c, d\}$ but which is not contain b; more over for the points $c, d \in X$, there exists a $\tilde{\mu}$-open set $\{a, b, d\}$ but which is not contain c. This implies that (X, μ) is a $\tilde{\mu}$-T_0 space. Also $\{b, c\}, \{a, b, c\}$ are $\tilde{\mu}$-g.closed sets but not $\tilde{\mu}$-closed. Then by Definition 4.5 (X, μ) is not a $\tilde{\mu}$-$T_{\frac{1}{2}}$ space.

(iii) Let $X = \{a, b, c\}$, $\mu = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then the $\tilde{\mu}$-g.closed sets $\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}$ are all $\tilde{\mu}$-closed. This implies that (X, μ) is a $\tilde{\mu}$-$T_{\frac{1}{2}}$ space. Also for the point $c \in X$, we have that two $\tilde{\mu}$-open sets $\{a, c\}$ and X but these sets containing the distinct point a. By Definition 4.2 (X, μ) is not a $\tilde{\mu}$-T_1 space. Then (X, μ) is a $\tilde{\mu}$-$T_{\frac{1}{2}}$ space but not $\tilde{\mu}$-T_1.

(iv) Let $X = \{a, b, c\}$, $\mu = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. For the distinct points $a, k \in X$, where $k \in \{b, c\}$, there exists $\tilde{\mu}$-open sets $\{a\}$ and $\{b, c\}$ containing a and k respectively such that $a \notin \{b, c\}$ and $k \notin \{a\}$. Also for the distinct points $b, c \in X$, there exists $\tilde{\mu}$-open sets $\{a, b\}$ and $\{a, c\}$ containing b and c respectively such that $b \notin \{a, c\}$ and $c \notin \{a, b\}$. This implies that (X, μ) is a $\tilde{\mu}$-T_1 space. More over for the distinct points $b, c \in X$, there does not exist disjoint $\tilde{\mu}$-open sets. By Definition 4.3 (X, μ) is not a $\tilde{\mu}$-T_2 space. Then (X, μ) is a $\tilde{\mu}$-T_1 space but not $\tilde{\mu}$-T_2.

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Received: November 16, 2014; *Accepted*: March 15, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 277–288

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Mild solutions for semi-linear fractional order functional stochastic differential equations with impulse effect

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Abstract

This paper is concerned with the existence results of mild solution for an impulsive fractional order stochastic differential equation with infinite delay subject to nonlocal conditions. The results are obtained by using the fixed point techniques and solution operator generated by sectorial operator on a Hilbert space.

Keywords: Fractional order differential equation, nonlocal conditions, existence and uniqueness, impulsive conditions, stochastic differential equations.

2010 MSC: 26A33,34B10,34A12,34A37,34K50.

1 Introduction

Recently, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economics and science. Fractional models have various applications such as nonlinear oscillations of earthquakes, viscoelasticity, electrochemistry, seepage flow in porous media, and electromagnetic, etc. There has been a significant development in fractional differential equations since last few years for more details one can see the papers ([7],[8],[9],[11],[14],[15],[19]) and references cited therein.

The deterministic systems often fluctuate due to environmental noise due to this reason it is important and necessary for researcher to study these systems. These systems are modeled as stochastic differential systems. In many evolution processes impulsive effects exist in which states are changed abruptly at certain moments of time. Therefore the stochastic differential equations with impulsive effects exist in real systems and provide a more accurate mathematical model. For more details one can see the papers ([16],[17],[18]) and references therein.

Further, if we combine the stochastic differential equation with a nonlocal initial condition strengthens the model even further. These fact motivate us to study such model in this paper. The basic tools are used in this paper including fixed-point techniques, the theory of linear semi-groups, results for probability measures, and results for infinite dimensional stochastic differential equations. The results are important from the viewpoint of applications since they cover nonlocal generalizations of integro-differential stochastic differential equation arising in various fields such as electromagnetic theory, population dynamics, and heat conduction in materials with memory, for more detail one can see the papers ([6],[13],[16],[23],[24],[25]) and references therein.

In [4] Bahuguna, considered the following problem

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), u(b_1(t)), u(b_2(t)), \dots, u(b_m(t))), t \in (0, T], \\ h(u) = \phi_0 \ \text{ on } [-\tau, 0], \end{cases}$$

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and found the existence, uniqueness and continuation of a mild solution on the maximal interval of existence. The author also proved some regularity results under various conditions. Chauhan et al. [5] considered the following semi-linear fractional order differential equations with nonlocal condition

$$\frac{d^\alpha}{dt^\alpha}x(t)+Ax(t)=f(t,x(t),x(a_1(t)),\dots,x(a_m(t))),t\in[0,T],t\neq t_i,$$
$$x(0)+g(x)=x_0,\ \ \Delta x(t_i)=I_i(x(t_i^-)),$$

and discussed the existence and uniqueness results of solutions using the applications of classical fixed point theorems.

Balasubramaniam et al. [2] studied the existence of solutions for the the following semi-linear neutral stochastic functional differential equations

$$\begin{aligned} d[x(t)+F(t,x(t),x(b_1(t)),\dots,x(b_m(t)))] &= Ax(t)dt+G(t,x(t),x(a_1(t)),\dots,x(a_n(t)))dw(t),t\in J=[0,b],\\ x(0) &= x_0+g(x), \end{aligned}$$

where A is a infinitesimal generator of an analytic semigroup of bounded linear operators $T(t),t\geq 0$, on a separable Hilbert space. By using fractional power of operators and Sadovskii fixed point theorem, the authors established the existence of mild and strong solutions.

Sakthivel et al. [22] considered the following impulsive fractional stochastic differential equations with infinite delay in the form

$$\begin{cases} D_t^\alpha x(t)=Ax(t)+f(t,x_t,B_1x(t))+\sigma(t,x_t,B_2x(t))\frac{dw(t)}{dt},t\in[0,T],t\neq t_k,\\ \Delta x(t_k)=I_k(x(t_k)),\ k=1,2,\dots,m\ x(t)=\phi(t),\ \ \phi(t)\in\mathfrak{B}_h, \end{cases}$$

and studied the existence results of mild solutions and established the sufficient conditions for the existence of mild solutions by using fixed point techniques.

Motivated by the works of these author's ([2],[4],[5],[22]), we study the existence of mild solutions of the following semi-linear stochastic fractional functional differential equation of the form:

$$\begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t)+f(t,x_t,x(a_1(t)),\dots,x(a_m(t)))\\ &\quad+\sigma(t,x_t,x(a_1(t)),\dots,x(a_m(t)))\frac{dw(t)}{dt},t\in J,t\neq t_k, \qquad (1.1)\\ \Delta x(t_k) &= I_k(x(t_k^-)),k=1,2,\dots,p, \qquad (1.2)\\ x(t)+g(x) &= \phi(t),\ t\in(-\infty,0], \qquad (1.3) \end{aligned}$$

where $J=[0,T]$ and ${}^cD_t^\alpha$ denotes the Caputo's fractional derivative of order $\alpha\in(0,1)$. $A:D(A)\subset\mathbb{H}\to\mathbb{H}$ is a closed linear sectorial operator defined on a Hilbert space $(\mathbb{H},\|\cdot\|)$. The functions f,σ are given and satisfy some assumptions to be defined later. We assume that $x_t:(-\infty,0]\to\mathbb{H},x_t(s)=x(t+s),s\leq 0$, belong to an abstract phase space $\mathfrak{B}_h$. Here $0\leq t_0<t_1<\cdots<t_p<t_{p+1}\leq T,I_k\in C(\mathbb{H},\mathbb{H}),(k=1,2,\dots,p)$, are bounded functions, $\Delta x(t_k)=x(t_k^+)-x(t_k^-),x(t_k^+)=\lim_{h\to 0}x(t_k+h)$ and $x(t_k^-)=\lim_{h\to 0}x(t_k-h)$ represent the right and left-hand limits of x(t) at $t=t_k$, respectively, also we take $x(t_i^-)=x(t_i)$.

The nonlocal condition $g:\mathbb{H}\to\mathbb{H}$ is defined as $g(x)=\sum_{k=1}^p c_kx(t_k)$ where $c_k,k=1,\dots,p$, are given constants and $0<t_1<t_2<\cdots<t_p<T$. Such nonlocal conditions were first introduced by Deng [10]. The initial data $\phi=\{\phi(t),t\in(-\infty,0]\}$ is an $\mathcal{F}_0$-measurable, $\mathfrak{B}_h$-valued random variable independent of $w(t)$ with finite second moments.

To the best of our knowledge, the existence and uniqueness of mild solution for the system (1.1) − (1.3) with non local condition is an untreated topic yet in the literature and this fact is the motivation of the present work.

Our work is divided in four sections, Second section provides the basic definitions and preliminaries results which are used in proving our main results. In the third section, we state and prove the existence results of the considered problem in this the paper. The fourth section includes examples.

2 Preliminaries

Let $\mathbb{H},\mathbb{K}$ be two separable Hilbert spaces and $\mathcal{L}(\mathbb{K},\mathbb{H})$ be the space of bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathbb{H},\mathbb{K}$ and $\mathcal{L}(\mathbb{K},\mathbb{H})$, and use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. $\mathbb{W}=(\mathbb{W}_t)_{t\geq 0}$ be a $\mathbb{Q}$-Wiener process defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ with the covariance operator $\mathbb{Q}$ such that $Tr\mathbb{Q}<\infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $\mathbb{Q}e_k=\lambda_k e_k,\ k=1,2,\ldots,$ and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(w(t),e)_{\mathbb{K}}=\sum_{k=1}^{\infty}\sqrt{\lambda_k}(e_k,e)_{\mathbb{K}}\beta_k(t), e\in\mathbb{K}, t\geq 0.$$

Let $\mathcal{L}_2^0=\mathcal{L}_2(\mathbb{Q}^{\frac{1}{2}}\mathbb{K},\mathbb{H})$ be the space of all Hilbert Schmidt operators from $\mathbb{Q}^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $<\varphi,\psi>_{\mathcal{L}_2^0}=Tr[\varphi\mathbb{Q}\psi*]$.

Now, we introduce abstract space phase $\mathfrak{B}_h$. Assume that $h:(-\infty,0]\to(0,\infty)$ with $l=\int_{-\infty}^{0}h(t)dt<\infty$, a continuous function. An abstract phase $\mathfrak{B}_h$ defined by

$$\begin{aligned}\mathfrak{B}_h \quad = \quad & \{\phi:(-\infty,0]\to\mathbb{H},\ \text{for any } a>0,\ (E|\phi(\theta)|^2)^{1/2} \text{ is bounded and measurable function on}[-a,0] \text{ with} \\ & \phi(0)=0 \text{ and } \int_{-\infty}^{0}h(s)\sup_{s\leq\theta\leq 0}(E|\phi(\theta)|^2)^{1/2}ds<\infty\}.\end{aligned}$$

If $\mathfrak{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathfrak{B}_h}=\int_{-\infty}^{0}h(s)\sup_{s\leq\theta\leq 0}(E|\phi(\theta)|^2)^{1/2}ds,\ \phi\in\mathfrak{B}_h,$$

then $(\mathfrak{B}_h,\|\cdot\|_{\mathfrak{B}_h})$ is a Banach space ([20],[21]).

Now we consider the space

$$\begin{aligned}\mathfrak{B}'_h \quad = \quad & \{x:(-\infty,T]\to\mathbb{H} \text{ such that } x|_{J_k}\in C(J_k,\mathbb{H}) \text{ and there exist} \\ & x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k)=x(t_k^-), x_0=\phi\in\mathfrak{B}_h, k=1,2,\ldots,p\},\end{aligned}$$

where $x|_{J_k}$ is the restriction of x to $J_k=(t_k,t_{k+1}], k=0,1,2,\ldots,p$. The function $\|\cdot\|_{\mathfrak{B}'_h}$ to be a semi-norm in $\mathfrak{B}'_h$, it is defined by

$$\|x\|_{\mathfrak{B}'_h}=\|\phi\|_{\mathfrak{B}_h}+\sup_{s\in[0,T]}(E\|x(s)\|^2)^{1/2}, x\in\mathfrak{B}'_h.$$

Lemma 2.1. ([2]) Assume that $x\in\mathfrak{B}'_h$, then for $t\in J, x_t\in\mathfrak{B}_h$. Moreover,

$$l(E\|x(t)\|^2)^{1/2}\leq l\sup_{s\in[0,t]}(E\|x(s)\|^2)^{1/2}+\|x_0\|_{\mathfrak{B}_h},\ \text{where } l=\int_{-\infty}^{0}h(s)ds<\infty.$$

Definition 2.1. The Reimann-Liouville fractional integral operator for order $\alpha>0$, of a function $f:\mathbb{R}^+\to\mathbb{R}$ and $f\in L^1(\mathbb{R}^+,X)$ is defined by

$$\mathbb{J}_t^0 f(t)=f(t),\ \mathbb{J}_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\quad \alpha>0,\ t>0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. Caputo's derivative of order $\alpha>0$ for a function $f:[0,\infty)\to\mathbb{R}$ is defined as

$$D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds=\mathbb{J}^{n-\alpha}f^{(n)}(t),$$

for $n-1<\alpha<n,\ n\in N$. If $0<\alpha<1$, then

$$D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}f^{(1)}(s)ds.$$

Obviously, Caputo's derivative of a constant is equal to zero.

Definition 2.3. A two parameter function of the Mittag Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi\iota}\int_c\frac{\mu^{\alpha-\beta}e^{\mu}}{\mu^{\alpha}-z}d\mu,\alpha,\beta>0,z\in\mathbb{C},$$

where c is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu|\leq|z|^{\frac{1}{\alpha}}$ counter clockwise. The most interesting properties of the Mittag Lefller functions are associated with their Laplace integral

$$\int_0^{\infty}e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^{\alpha})dt=\frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}-\omega},Re\lambda>\omega^{\frac{1}{\alpha}},\omega>0.$$

Definition 2.4. [12] A closed and linear operator A is said to be sectorial if there are constants $\omega\in\mathbb{R},\ \theta\in[\frac{\pi}{2},\pi],\ M>0$, such that the following two conditions are satisfied:

(1) $\sum_{(\theta,\omega)}=\{\lambda\in C:\lambda\neq\omega,|arg(\lambda-\omega)|<\theta\}\subset\rho(A)$,

(2) $\|R(\lambda,A)\|_{L(X)}\leq\frac{M}{|\lambda-\omega|},\lambda\in\sum_{(\theta,\omega)}$.

Definition 2.5. [1] Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega\geq0$ and a strongly continuous function $T_\alpha:\mathbb{R}_+\to\mathcal{L}(X)$, where $\mathcal{L}(X)$ is a Banach space of all bounded linear operators from X into X and the corresponding norm is denoted by $\|.\|$, such that $\{\lambda^{\alpha}:Re\lambda>\omega\}\subset\rho(A)$ and

$$(\lambda^{\alpha}I-A)^{-1}=\int_0^{\infty}e^{\lambda t}T_\alpha(t)xdt,\quad Re\lambda>\omega,x\in X,$$

where $T_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.6. [11] Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X and $\alpha>0$. We say that A is the generator of a solution operator if there exist $\omega\geq0$ and a strongly continuous function $S_\alpha:\mathbb{R}_+\to\mathcal{L}(X)$, such that $\{\lambda^{\alpha}:Re\lambda>\omega\}\subset\rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^{\alpha}I-A)^{-1}=\int_0^{\infty}e^{\lambda t}S_\alpha(t)xdt,\quad Re\lambda>\omega,x\in X,$$

where $S_\alpha(t)$ is called the solution operator generated by A.

Theorem 2.1. [26](Schauder fixed point theorem) If U is a closed , bounded, convex subset of a Banach space X and the mapping $T:U\to U$ is completely continuous, then T has a fixed point in U.

Definition 2.7. A measurable $\mathcal{F}_t-$ adapted stochastic process $x:(-\infty,T]\to\mathbb{H}$ is called a mild solution of the system (1.1)-(1.3) if $x(0)=\phi(0)-g(x)\in\mathfrak{B}_h$ on $(-\infty,0],\Delta x|_{t=t_k}=I_k(x(t_k^-)),k=1,2,\ldots,p$, the restriction of $x(\cdot)$ to the interval $[0,T)\backslash t_1,\ldots,t_p$, is continuous and $x(t)$ satisfies the following fractional integral equation

$$x(t)=\begin{cases}S_\alpha(t)(\phi(0)-g(x))+\int_0^tT_\alpha(t-s)f(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))ds & \\ +\int_0^tT_\alpha(t-s)\sigma(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))dw(s), & t\in[0,t_1],\\ S_\alpha(t-t_1)[x(t_1^-)+I_1(x(t_1^-))]+\int_{t_1}^tT_\alpha(t-s)f(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))ds & \\ +\int_{t_1}^tT_\alpha(t-s)\sigma(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))dw(s), & t\in(t_1,t_2],\\ \cdots & \\ S_\alpha(t-t_p)[x(t_p^-)+I_p(x(t_p^-))]+\int_{t_p}^tT_\alpha(t-s)f(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))ds & \\ +\int_{t_p}^tT_\alpha(t-s)\sigma(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))dw(s), & t\in(t_p,T],\end{cases}\tag{2.4}$$

where

$$S_\alpha(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^{\alpha}I-A)^{-1}d\lambda,\ T_\alpha(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda^{\alpha}I-A)^{-1}d\lambda,$$

are called analytic solutions operator and $\alpha-$resolvent family and Γ is a suitable path lying on $\sum_{\theta,\omega}$ for more details one can see [11].

Further we introduce the following assumptions to establish our results:

(H0) If $\alpha \in (0,1)$ and $A \in \mathbb{A}^{\alpha}(\theta_0, \omega_0)$ then for any $x \in \mathbb{H}$ and $t > 0$ we have $\|T_\alpha(t)\| \leq Me^{\omega t}$ and $\|S_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1})$, $\omega > \omega_0$. Thus we have

$$\|T_\alpha(t)\| \leq \widetilde{M}_T \text{ and } \|S_\alpha(t)\| \leq t^{\alpha-1}\widetilde{M}_S,$$

where $\widetilde{M}_T = \sup_{0 \geq t \geq T} \|T_\alpha(t)\|$ and $\widetilde{M}_S = \sup_{0 \geq t \geq T} Ce^{\omega t}(1 + t^{1-\alpha})$(for more details, see [12]).

(H1) There exist a constants $L_g > 0$, such that $E\|g(x) - g(y)\|^2_{\mathbb{H}} \leq L_g\|x - y\|^2_{\mathbb{H}}$.

(H2) The nonlinear maps $f : J \times \mathfrak{B}_h \times \mathbb{H}^m \to \mathbb{H}$ and $\sigma : J \times \mathfrak{B}_h \times \mathbb{H}^m \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ are continuous and there exist constants L_f, L_σ, such that

$$\begin{aligned} E\|f(t, \varphi, x_1, x_2, \ldots, x_m) - f(t, \psi, y_1, y_2, \ldots, y_m)\|^2_{\mathbb{H}} &\leq L_f[\|\varphi - \psi\|^2_{\mathfrak{B}_h} + \sum_{i=1}^{m} E\|x_i - y_i\|^2_{\mathbb{H}}], \\ E\|\sigma(t, \varphi, x_1, x_2, \ldots, x_m) - \sigma(t, \psi, y_1, y_2, \ldots, y_m)\|^2_{\mathcal{L}(\mathbb{K},\mathbb{H})} &\leq L_\sigma[\|\varphi - \psi\|^2_{\mathfrak{B}_h} + \sum_{i=1}^{m} E\|x_i - y_i\|^2_{\mathbb{H}}], \end{aligned}$$

for all $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_m) \in \mathbb{H}^m$, $t \in J$ and $\varphi, \psi \in \mathfrak{B}_h$.

(H3) The functions $I_k : \mathbb{H} \to \mathbb{H}$ are continuous and there exists $L_k > 0$, such that

$$E\|I_k(x) - I_k(y)\|^2_{\mathbb{H}} \leq L_k E\|x - y\|^2_{\mathbb{H}},$$

$x, y \in \mathbb{H},\ k = 1, 2, \ldots, p,\ L = \max\{L_k\} > L_g$.

3 Existence and uniqueness of solutions

Theorem 3.2. Let the assumptions (H0)-(H3) are satisfied and

$$\Theta = \left[3\widetilde{M}_S^2(1 + L) + 3\widetilde{M}_T^2\frac{T^{2\alpha}}{\alpha^2}L_f(l + m) + 3\widetilde{M}_T^2\frac{T^{2\alpha-1}}{2\alpha - 1}L_\sigma(l + m)\right] < 1,$$

then the problem (1.1)-(1.3) has a unique mild solution $x \in \mathbb{H}$ on J.

Proof. First we convert the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $P : \mathfrak{B}'_h \to \mathfrak{B}'_h$ defined by

$$(Px)(t) = \begin{cases} S_\alpha(t)(\phi(0) - g(x)) + \int_0^t T_\alpha(t - s)f(s, x_s, x(a_1(s)), \ldots, x(a_m(s)))ds & \\ + \int_0^t T_\alpha(t - s)\sigma(s, x_s, x(a_1(s)), \ldots, x(a_m(s)))dw(s), & t \in [0, t_1], \\ S_\alpha(t - t_1)[x(t_1^-) + I_1(x(t_1^-))] + \int_{t_1}^t T_\alpha(t - s)f(s, x_s, x(a_1(s)), \ldots, x(a_m(s)))ds & \\ + \int_{t_1}^t T_\alpha(t - s)\sigma(s, x_s, x(a_1(s)), \ldots, x(a_m(s)))dw(s), & t \in (t_1, t_2], \\ \cdots & \\ S_\alpha(t - t_p)[x(t_p^-) + I_p(x(t_p^-))] + \int_{t_p}^t T_\alpha(t - s)f(s, x_s, x(a_1(s)), \ldots, x(a_m(s)))ds & \\ + \int_{t_p}^t T_\alpha(t - s)\sigma(s, x_s, x(a_1(s)), \ldots, x(a_m(s)))dw(s), & t \in (t_p, T]. \end{cases}$$

Let $y(.) : (-\infty, T] \to \mathbb{H}$ be the function defined by

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ 0, & t \in J, \end{cases} \quad \text{then } \ y_0 = \phi.$$

For each $z : J \to \mathbb{H}$ with $z\mid_{t_k} \in C(J_k, \mathbb{H}), k = 1, \ldots, p$ and $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z} = \begin{cases} 0, & t \in (-\infty, 0] \\ z(t), & t \in J. \end{cases}$$

If $x(\cdot)$ satisfies the system (2.4), then we can decompose $x(\cdot)$ as $x(t) = y(t) + \bar{z}(t)$, which implies $x_t = y_t + \bar{z}_t$ for $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = \begin{cases} S_\alpha(t)(\phi(0) - g(y+\bar{z})) + \int_0^t T_\alpha(t-s) f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \\ \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds + \int_0^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, \\ y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in [0, t_1], \\ S_\alpha(t-t_1)[y(t_1^-) + \bar{z}(t_1^-) + I_1(y(t_1^-) + \bar{z}(t_1^-))] \\ + \int_{t_1}^t T_\alpha(t-s) f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds \\ + \int_{t_1}^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in (t_1, t_2], \\ \ldots \\ S_\alpha(t-t_p)[y(t_p^-) + \bar{z}(t_p^-) + I_p(y(t_p^-) + \bar{z}(t_p^-))] \\ + \int_{t_p}^t T_\alpha(t-s) f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds \\ + \int_{t_p}^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in (t_p, T]. \end{cases}$$

Set $\mathfrak{B}_h''$, such that $z_0 = 0$ and for any $z \in \mathfrak{B}_h''$, we have

$$\|z\|_{\mathfrak{B}_h''} = \|z_0\|_{\mathfrak{B}_h} + \sup_{t \in J}(E\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t \in J}(E\|z(t)\|^2)^{\frac{1}{2}}.$$

Thus $(\mathfrak{B}_h'', \|\cdot\|_{\mathfrak{B}_h''})$ is a Banach space. Define an operator $N : \mathfrak{B}_h'' \to \mathfrak{B}_h''$ by

$$(Nz)(t) = \begin{cases} S_\alpha(t)(\phi(0) - g(y+\bar{z})) + \int_0^t T_\alpha(t-s) f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \\ \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds + \int_0^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, \\ y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in [0, t_1], \\ S_\alpha(t-t_1)[y(t_1^-) + \bar{z}(t_1^-) + I_1(y(t_1^-) + \bar{z}(t_1^-))] \\ + \int_{t_1}^t T_\alpha(t-s) f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds \\ + \int_{t_1}^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in (t_1, t_2], \\ \ldots \\ S_\alpha(t-t_p)[y(t_p^-) + \bar{z}(t_p^-) + I_p(y(t_p^-) + \bar{z}(t_p^-))] \\ + \int_{t_p}^t T_\alpha(t-s) f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds \\ + \int_{t_p}^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in (t_p, T]. \end{cases}$$

In order to prove existence results, it is enough to show that N has a unique fixed point. Let $z, z^* \in \mathfrak{B}_h''$ then for $t \in [0, t_1]$, we have

$$\begin{aligned} E\|(Nz)(t) - (Nz^*)(t)\|_{\mathbb{H}}^2 \leq\; & 3E\|S_\alpha(t)[g(y+\bar{z}) - g(y+\bar{z}^*)]\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s))) \\ & -f(s, y_s + \bar{z}_s^*, y(a_1(s)) + \bar{z}^*(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^*(a_m(s)))]ds\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s))) \\ & -\sigma(s, y_s + \bar{z}_s^*, y(a_1(s)) + \bar{z}^*(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^*(a_m(s)))]dw(s)\|_{\mathbb{H}}^2, \end{aligned}$$

by applying assumptions, we have

$$E\|(Nz)(t) - (Nz^*)(t)\|_{\mathbb{H}}^2 \leq (3\widetilde{M}_S^2 L_g + 3\widetilde{M}_T^2 \frac{T^{2\alpha}}{\alpha^2} L_f(l+m) + 3\widetilde{M}_T^2 \frac{T^{2\alpha-1}}{2\alpha-1} L_\sigma(l+m))\|z - z^*\|_{\mathfrak{B}_h''}^2.$$

For $t \in (t_1, t_2]$, we have

$$\begin{aligned} E\|(Nz)(t) - (Nz^*)(t)\|^2_{\mathbb{H}} \leq\ & 3E\|S_\alpha(t-t_1)[\bar{z}(t_1^-) - \bar{z}^*(t_1^-) + I_1(y(t_1^-) + \bar{z}(t_1^-)) - I_1(y(t_1^-) + \bar{z}^*(t_1^-))]\|^2_{\mathbb{H}} \\ & +3E\|\int_{t_1}^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s))) \\ & -f(s, y_s + \bar{z}^*_s, y(a_1(s)) + \bar{z}^*(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^*(a_m(s)))]ds\|^2_{\mathbb{H}} \\ & +3E\|\int_{t_1}^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s))) \\ & -\sigma(s, y_s + \bar{z}^*_s, y(a_1(s)) + \bar{z}^*(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^*(a_m(s)))]dw(s)\|^2_{\mathbb{H}}. \end{aligned}$$

by applying assumptions, we obtain

$$E\|(Nz)(t) - (Nz^*)(t)\|^2_{\mathbb{H}} \leq (3\widetilde{M}^2_S(1 + L_1) + 3\widetilde{M}^2_T \frac{T^{2\alpha}}{\alpha^2} L_f(l+m) + 3\widetilde{M}^2_T \frac{T^{2\alpha-1}}{2\alpha-1} L_\sigma(l+m))\|z - z^*\|^2_{\mathfrak{B}''_h}.$$

Similarly, for $t \in (t_p, T]$, we have

$$\begin{aligned} E\|(Nz)(t) - (Nz^*)(t)\|^2_{\mathbb{H}} \leq\ & 3E\|S_\alpha(t-t_p)[\bar{z}(t_p^-) - \bar{z}^*(t_p^-) + I_p(y(t_p^-) + \bar{z}(t_p^-)) - I_p(y(t_p^-) + \bar{z}^*(t_p^-))]\|^2_{\mathbb{H}} \\ & +3E\|\int_{t_p}^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_n(s)) + \bar{z}(a_n(s))) \\ & -f(s, y_s + \bar{z}^*_s, y(a_1(s)) + \bar{z}^*(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^*(a_m(s)))]ds\|^2_{\mathbb{H}} \\ & +3E\|\int_{t_1}^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s))) \\ & -\sigma(s, y_s + \bar{z}^*_s, y(a_1(s)) + \bar{z}^*(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^*(a_m(s)))]dw(s)\|^2_{\mathbb{H}}, \end{aligned}$$

by applying assumptions, we have

$$E\|(Nz)(t) - (Nz^*)(t)\|^2_{\mathbb{H}} \leq (3\widetilde{M}^2_S(1 + L_p) + 3\widetilde{M}^2_T \frac{T^{2\alpha}}{\alpha^2} L_f(l+m) + 3\widetilde{M}^2_T \frac{T^{2\alpha-1}}{2\alpha-1} L_\sigma(l+m))\|z - z^*\|^2_{\mathfrak{B}''_h}.$$

Thus for all $t \in [0, T]$, we estimate

$$\begin{aligned} E\|(Nz)(t) - (Nz^*)(t)\|^2_{\mathbb{H}} &\leq \left\{3\widetilde{M}^2_S(1 + L) + 3\widetilde{M}^2_T \frac{T^{2\alpha}}{\alpha^2} L_f(l+m)\ + 3\widetilde{M}^2_T \frac{T^{2\alpha-1}}{2\alpha-1} L_\sigma(l+m)\right\}\|z - z^*\|^2_{\mathfrak{B}''_h}, \\ &\leq \Theta\|z - z^*\|^2_{\mathfrak{B}''_h}. \end{aligned}$$

Since $\Theta < 1$ as in the Theorem 3.2, therefore N is a contraction. Hence N has a unique fixed point by Banach contraction principle. This completes the proof of the theorem. □

The second result is proved by using the Schauder fixed point theorem. For this we take the following assumptions

(H4) There exist a constants $M_1 > 0$, such that $E\|g(x)\|^2_{\mathbb{H}} \leq M_1$.

(H5) The functions $I_k : \mathbb{H} \to \mathbb{H}$ are continuous and there exists $M_2 > 0$, such that $E\|I_k(x)\|^2_{\mathbb{H}} \leq M_2$.

(H6) $f, \sigma : J \times \mathfrak{B}_h \times \mathbb{H}^m \to \mathbb{H}$ are continuous and there exits constants M_3, M_4, such that

$$E\|f(t, \varphi, x_1, x_2, \ldots, x_m)\|^2_{\mathbb{H}} \leq M_3,\ E\|\sigma(t, \varphi, x_1, x_2, \ldots, x_m)\|^2_{\mathbb{H}} \leq M_4.$$

Theorem 3.3. Let the assumptions (H3)-(H6) are satisfied then the impulsive stochastic differential equation (1.1)-(1.3) has at least one mild solution.

Proof. let us consider the space $B_r = \{y \in \mathfrak{B}_h'' : \|y\| \leq r\}$. It is obvious that B_r is closed convex and bounded subset of $\mathfrak{B}_h''$. Consider the operator $N : B_r \to B_r$ defined by

$$(Nz)(t) = \begin{cases} S_\alpha(t)(\phi(0) - g(y+\bar{z})) + \int_0^t T_\alpha(t-s)f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \\ \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds + \int_0^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, \\ y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in [0, t_1], \\ S_\alpha(t-t_1)[y(t_1^-) + \bar{z}(t_1^-) + I_1(y(t_1^-) + \bar{z}(t_1^-))] \\ + \int_{t_1}^t T_\alpha(t-s)f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds \\ + \int_{t_1}^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in (t_1, t_2], \\ \ldots \\ S_\alpha(t-t_p)[y(t_p^-) + \bar{z}(t_p^-) + I_p(y(t_p^-) + \bar{z}(t_p^-))] \\ + \int_{t_p}^t T_\alpha(t-s)f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds \\ + \int_{t_p}^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in (t_p, T]. \end{cases}$$

First we shall show that N is continuous, for this let $\{z^n\}_{n=1}^\infty$ be a sequence in B_r such that $\lim z^n \to z \in B_r$. When $t \in [0, t_1]$, we have

$$\begin{aligned} E\|(Nz^n)(t) - (Nz)(t)\|_{\mathbb{H}}^2 \leq\ & 3E\|S_\alpha(t)[g(y+\bar{z}^n) - g(y+\bar{z})\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s^n, y(a_1(s)) + \bar{z}^n(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^n(a_m(s))) \\ & -f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s^n, y(a_1(s)) + \bar{z}^n(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^n(a_m(s))) \\ & -\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2. \end{aligned}$$

Then for $t \in (t_i, t_{i+1}]$, where $i = 1, 2, \ldots, p$, then we have

$$\begin{aligned} E\|(Nz^n)(t) - (Nz)(t)\|_{\mathbb{H}}^2 \leq\ & 3E\|S_\alpha(t-t_i)[\bar{z}^n(t_i^-) - \bar{z}(t_i^-) + I_i(y(t_i^-) + \bar{z}^n(t_i^-)) - I_i(y(t_i^-) + \bar{z}(t_i^-))]\|_{\mathbb{H}}^2 \\ & +3E\|\int_{t_i}^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s^n, y(a_1(s)) + \bar{z}^n(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^n(a_m(s))) \\ & -f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2 \\ & +3E\|\int_{t_i}^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s^n, y(a_1(s)) + \bar{z}^n(a_1(s)), \ldots, y(a_m(s)) + \bar{z}^n(a_m(s))) \\ & -\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2. \end{aligned}$$

Since the functions f, σ, g and $I_i, i = 1, 2, \ldots, p$, are continuous, hence $\lim_{n\to\infty} E\|(Nz^n)(t) - (Nz)(t)\|_{\mathbb{H}}^2 \to 0$. This implies that the mapping N is continuous on B_r.

Now we show that N maps bounded set into bounded sets in B_r. Let $z \in B_r$ then we have $E\|(Nz)(t)\|_{\mathbb{H}}^2 \leq \hat{M}$, for $t \in (t_i, t_{i+1}], i = 0, 1, 2, \ldots, p$. Then for $t \in [0, t_1]$, we have

$$\begin{aligned} E\|(Nz)(t)\|_{\mathbb{H}}^2 \leq\ & 3E\|S_\alpha(t)[\phi(0) + g(y+\bar{z})]\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2, \\ & \leq 3\widetilde{M}_S^2[r + M_1] + 3\widetilde{M}_T^2 \frac{T^{2\alpha}}{\alpha^2} M_3 + 3\widetilde{M}_T^2 \frac{T^{2\alpha-1}}{2\alpha-1} M_4. \end{aligned}$$

For $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, p$, then we have

$$
\begin{aligned}
E\|(Nz)(t)\|_{\mathbb{H}}^2 \leq\ & 3E\|S_\alpha(t-t_i)[y(t_i^-)+\bar{z}(t_i^-)+I_i(y(t_i^-)+\bar{z}(t_i^-))]\|_{\mathbb{H}}^2 \\
& +3E\|\int_{t_i}^t T_\alpha(t-s)[f(s, y_s+\bar{z}_s, y(a_1(s))+\bar{z}(a_1(s)), \ldots, y(a_m(s))+\bar{z}(a_m(s)))ds\|_{\mathbb{H}}^2 \\
& +3E\|\int_{t_i}^t T_\alpha(t-s)[\sigma(s, y_s+\bar{z}_s, y(a_1(s))+\bar{z}(a_1(s)), \ldots, y(a_m(s))+\bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2 \\
& \leq 3\widetilde{M}_S^2[r+M_2]+3\widetilde{M}_T^2\frac{T^{2\alpha}}{\alpha^2}M_3+3\widetilde{M}_T^2\frac{T^{2\alpha-1}}{2\alpha-1}M_4=\hat{M}.
\end{aligned}
$$

It proves that N maps bounded set into bounded sets in B_r for all sub interval $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, p$. Finally, we show that N maps bounded set into equi-continuous sets in B_r. let $l_1, l_2 \in (t_i, t_{i+1}], t_i \leq l_1 < l_2 \leq t_{i+1}, i = 0, 1, 2, \ldots, p, z \in B_r$, we obtain for $t \in [0, t_1]$

$$
\begin{aligned}
E\|(Nz)(l_2)-(Nz)(l_1)\|_{\mathbb{H}}^2 \leq\ & 3E\|[S_\alpha(l_2)-S_\alpha(l_1)][\phi_0+g(y+\bar{z})]\|_{\mathbb{H}}^2 \\
& +3E\|\int_0^t [T_\alpha(l_2-s)-T_\alpha(l_1-s)][f(s, y_s+\bar{z}_s, y(a_1(s))+\bar{z}(a_1(s)), \ldots, \\
& y(a_m(s))+\bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2+3E\|\int_0^t [T_\alpha(l_2-s)-T_\alpha(l_1-s)] \\
& \times[\sigma(s, y_s+\bar{z}_s, y(a_1(s))+\bar{z}(a_1(s)), \ldots, y(a_m(s))+\bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2, \\
\leq\ & 3[M_1+r]E\|[S_\alpha(l_2)-S_\alpha(l_1)]\|_{\mathbb{H}}^2+3M_3E\|\int_0^t [T_\alpha(l_2-s)-T_\alpha(l_1-s)]\|_{\mathbb{H}}^2 \\
& +3M_4E\|\int_0^t [T_\alpha(l_2-s)-T_\alpha(l_1-s)]\|_{\mathbb{H}}^2.
\end{aligned}
$$

For $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, p$, we have

$$
\begin{aligned}
E\|(Nz)(l_2)-(Nz)(l_1)\|_{\mathbb{H}}^2 \leq\ & 3E\|[S_\alpha(l_2-t_i)-S_\alpha(l_1-t_i)][y(t_i^-)+\bar{z}(t_i^-)+I_i(y(t_i^-)+\bar{z}(t_i^-))]\|_{\mathbb{H}}^2 \\
& +3E\|\int_{t_i}^t [T_\alpha(l_2-s)-T_\alpha(l_1-s)][f(s, y_s+\bar{z}_s, y(a_1(s))+\bar{z}(a_1(s)), \ldots, \\
& y(a_m(s))+\bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2+3E\|\int_{t_i}^t [T_\alpha(l_2-s)-T_\alpha(l_1-s)] \\
& \times[\sigma(s, y_s+\bar{z}_s, y(a_1(s))+\bar{z}(a_1(s)), \ldots, y(a_m(s))+\bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2, \\
\leq\ & 3[M_2+r]E\|[S_\alpha(l_2-t_i)-S_\alpha(l_1-t_i)]\|_{\mathbb{H}}^2+3M_3E\|\int_{t_i}^t [T_\alpha(l_2-s)-T_\alpha(l_1-s)]\|_{\mathbb{H}}^2 \\
& +3M_4E\|\int_{t_i}^t [T_\alpha(l_2-s)-T_\alpha(l_1-s)]\|_{\mathbb{H}}^2.
\end{aligned}
$$

Since $T_\alpha(t)$ and $S_\alpha(t)$ are strongly continuous its implies that $\lim_{l_2 \to l_1} \|[S_\alpha(l_2-t_i)-S_\alpha(l_1-t_i)]\|_{\mathbb{H}}^2 = 0$ and $\lim_{l_2 \to l_1} \|[T_\alpha(l_2-t_i)-T_\alpha(l_1-t_i)]\|_{\mathbb{H}}^2 = 0$ This implies that N is equi-continuous on all subintervals $(t_i, t_{i+1}], i = 1, 2, \ldots, p$. Thus by Arzela -Ascoli theorem, it follows that N is a compact operator. Hence N is completely continuous operator. Therefore, by Schauder fixed point theorem, the operator N has a fixed point, which in turns implies that (1.1)-(1.3) has at least one solution on $[0, T]$. This completes the proof of the theorem. □

4 Example

Example 4.1. *Consider the following nonlocal impulsive fractional partial differential equation of the form*

$$\begin{aligned}
\frac{\partial^q}{\partial t^q}u(t,x) &= \frac{\partial^2}{\partial y^2}u(t,x)+\frac{1}{25}\int_{-\infty}^{t}H(t,x,s-t)Q_1(u(s,x),u(a_1(s),\dots,u(a_m(s)))ds \\
&\quad+\left[\frac{1}{25}\int_{-\infty}^{t}V(t,x,s-t)Q_2(u(s,x),u(a_1(s),\dots,u(a_m(s)))ds\right]\frac{dw(t)}{dt}, \qquad (4.5)\\
u(t,0) &= u(t,\pi)=0,\ \ t\geq 0, \qquad (4.6)\\
u(t,x)+\sum_{k=1}^{m}c_ku(x,t_k) &= \phi(t,x),t\in(-\infty,0],x\in[0,\pi], \qquad (4.7)\\
\Delta u(t_i)(x) &= \frac{1}{9}\int_{-\infty}^{t_i}q_i(t_i-s)u(s,x)ds,x\in[0,\pi], \qquad (4.8)
\end{aligned}$$

where $\frac{\partial^q}{\partial t^q}$ *is Caputo's fractional derivative of order* $0<q<1, 0<t_1<t_2<\cdots<t_n\leq T$ *are prefixed numbers,* $\phi\in\mathfrak{B}_h$. *Let* $\mathbb{H}=L^2[0,\pi]$ *and define the operator* $A:D(A)\subset\mathbb{H}\to\mathbb{H}$ *by* $A\omega=\omega''$ *with the domain* $D(A):=\{\omega\in X:\omega,\omega'\text{are absolutely continuous},\omega''\in\mathbb{H},\omega(0)=0=\omega(\pi)\}$. *Then*

$A\omega=\sum_{n=1}^{\infty}n^2(\omega,\omega_n)\omega_n,\omega\in D(A)$, *where* $\omega_n(x)=\sqrt{\frac{2}{\pi}}sin(nx),n\in N$ *is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup* $(T(t))_{t\geq0}$ *in* $\mathbb{H}$ *and is given by*

$$T(t)\omega=\sum_{n=1}^{\infty}e^{-n^2t}(\omega,\omega_n)\omega_n,\ \text{for all}\ \omega\in\mathbb{H},\ \text{and every}\ t>0.$$

The subordination principle of solution operator (Theorem 3.1 in [3]) implies that A is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\geq0}$. *Since* $S_\alpha(t)$ *is strongly continuous on* $[0,\infty)$, *by uniformly bounded theorem, there exists a constant* $M>0$, *such that* $\|S_\alpha(t)\|_{L(\mathbb{H})}\leq M$, *for* $t\in[0,T]$. *Let* $h(s)=e^{2s},s<0$ *then* $l=\int_{-\infty}^{0}h(s)ds=\frac{1}{2}$ *and define*

$$\|\phi\|_{\mathfrak{B}_h}=\int_{-\infty}^{0}h(s)\sup_{\theta\in[s,0]}\|\phi(\theta)\|_{L^2}ds.$$

Hence for $(t,\phi)\in[0,T]\times\mathfrak{B}_h$, *where* $\phi(\theta)(x)=\phi(\theta,x),(\theta,x)\in(-\infty,0]\times[0,\pi]$. *Set* $u(t)(x)=u(t,x)$,

$$\begin{aligned}
f(t,\phi,u(a_1(t)),\dots,u(a_m(t))))(x) &= \frac{1}{25}\int_{-\infty}^{0}H(t,x,\theta)Q_1(\phi(\theta,u(a_1(t)),\dots,u(a_m(t)))(x))d\theta,\\
\sigma(t,\phi,u(a_1(t)),\dots,u(a_m(t))))(x) &= \frac{1}{25}\int_{-\infty}^{0}V(t,x,\theta)Q_2(\phi(\theta,u(a_1(t)),\dots,u(a_m(t)))(x))d\theta,\\
I_i(\phi)(x) &= \frac{1}{9}\int_{-\infty}^{0}q_i(-\theta)\phi(\theta)(x)d\theta,\\
g(x) &= \sum_{k=1}^{m}c_ku(x,t_k).
\end{aligned}$$

Then with these settings the equations (4.5)-(4.8) can be written in the abstract form of equations (1.1)-(1.3). Further we have here $L_f=\frac{1}{25},L_\sigma=\frac{1}{25},L=\frac{1}{9},T=1,l=\frac{1}{2},\widetilde{M}_T=1,\widetilde{M}_S=\frac{1}{5}$ *and* $m=2$. *In this formulation of the problem we can verify the assumptions of Theorem (3.2). We get the value of condition in Theorem (3.2) as* $\Theta=.73<1$. *This implies that there exists a unique mild solution u on* $[0,1]$.

Example 4.2. *Here we consider the following non-trivial problem*

$$\begin{aligned}
\frac{\partial^q}{\partial t^q}u(t,x) &= \frac{\partial^2}{\partial y^2}u(t,x)+\frac{e^{-t}}{25+e^t}\int_{-\infty}^{t}H(t,x,s-t)[Q_1(u(s,x),u(a_1(s),\dots,u(a_m(s)))+\frac{t}{7}]ds\\
&\quad+\frac{e^{-t}}{25+e^t}\int_{-\infty}^{t}V(t,x,s-t)[Q_2(u(s,x),u(a_1(s),\dots,u(a_m(s)))+\frac{t}{7}]dw(s) \qquad (4.9)\\
u(t,x) &= u(t,\pi)=0,t\geq0, \qquad (4.10)\\
u(t,x)+\sum_{k=1}^{m}c_ku(x,t_k) &= \phi(t,x),\ t\in(-\infty,0],\ x\in[0,\pi], \qquad (4.11)\\
\Delta u\big|_{t=\frac{1}{2}^-} &= \sin(\frac{1}{9}\|u(\frac{1}{2}^-,x)\|),\ 0\leq t\leq 1, 0\leq x\leq\pi, \qquad (4.12)
\end{aligned}$$

where $q \in (0,1)$. *In the perspective of Example 1 we set*

$$f(t,\phi,u(a_1(t)),\ldots,u(a_m(t))))(x) = \frac{e^{-t}}{25+e^t}\int_{-\infty}^{0} H(t,x,\theta)[Q_1(\phi(\theta,u(a_1(t)),\ldots,u(a_m(t)))(x)) + \frac{t}{7}]d\theta,$$

$$\sigma(t,\phi,u(a_1(t)),\ldots,u(a_m(t))))(x) = \frac{e^{-t}}{25+e^t}\int_{-\infty}^{0} V(t,x,\theta)[Q_2(\phi(\theta,u(a_1(t)),\ldots,u(a_m(t)))(x)) + \frac{t}{7}]d\theta.$$

Then with these settings the equations (4.9)-(4.12) can be written in the abstract form of equations (1.1)-(1.3). Hence the our problem (4.9)-(4.12) have a unique mild solution on $[0,1]$.

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Received: October 24, 2014; *Accepted*: March 09, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 289–295

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

A curious summation formula in the light of Gamma function and contiguous relation

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Abstract

The main aim of the present paper is to establish a curious summation formula involving recurrence relation of Gamma function .

Keywords: Gauss second summation theorem ,Recurrence relation, Prudnikov.

2010 MSC: 33C05 , 33C20 , 33D15 , 33D50 , 33D60.

1 Introduction

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{ll} a_1, a_2, \cdots, a_A & ; \\ & z \\ b_1, b_2, \cdots, b_B & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} \tag{1} $$

.

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers and $| z |= 1$

Contiguous Relation is defined by

[Andrews p.363(9.16), E. D. p.51(10)]

$$ (a-b)\ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \\ c\ \ ; & z \end{array}\right] = a\ {}_2F_1\left[\begin{array}{ll} a+1,\ b\ ; & \\ c\ \ ; & z \end{array}\right] - b\ {}_2F_1\left[\begin{array}{ll} a,\ b+1\ ; & \\ c\ ; & z \end{array}\right] \tag{2} $$

Gauss second summation theorem is defined by [Prudnikov., 491(7.3.7.5)]

$$ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \\ \frac{a+b+1}{2}\ ; & \frac{1}{2} \end{array}\right] = \frac{\Gamma(\frac{a+b+1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} \tag{3} $$

$$ = \frac{2^{(b-1)}\,\Gamma(\frac{b}{2})\,\Gamma(\frac{a+b+1}{2})}{\Gamma(b)\,\Gamma(\frac{a+1}{2})} \tag{4} $$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov., p.491(7.3.7.8)]

$$ {}_2F_1\left[\begin{array}{ll} a,\ b\ \ ; & \\ \frac{a+b-1}{2}\ ; & \frac{1}{2} \end{array}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} + \frac{2\,\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\,\Gamma(b)}\right] \tag{5} $$

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Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$$ {}_2F_1\left[\begin{array}{ll} a,\ b & ; \\ \frac{a+b-1}{2} & ; \end{array} \begin{array}{l} \frac{1}{2} \\ 2 \end{array}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{6} $$

Recurrence relation is defined by

$$ \Gamma(\zeta+1) = \zeta\,\Gamma(\zeta) \tag{7} $$

2 Main summation formula

$$ {}_2F_1\left[\begin{array}{ll} a,\ b & ; \\ \frac{a+b+48}{2} & ; \end{array} \begin{array}{l} \frac{1}{2} \\ 2 \end{array}\right] = \frac{2^b\,\Gamma(\frac{a+b+48}{2})}{(a-b)\,\Gamma(b)\Big[\prod\limits_{\Lambda=1}^{23}\{a-b-2\Lambda\}\Big]\Big[\prod\limits_{\Psi=1}^{23}\{a-b+2\Psi\}\Big]} $$

$$ \Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\Big\{8388608(-21686243443194426122117120000 + a^{23} + 5598433692632772048574218240000b $$
$$ -569545783776841112218710835200b^2 + 35020096178299422697806888960 0b^3 $$
$$ -130274290623536732525341704192b^4 + 38258159328821814810743144448b^5 $$
$$ -7392547502167306440045232128b^6 + 1278328901424788437956820992b^7 $$
$$ -147385136152123509508145152b^8 + 16541594137308004947263488b^9 $$
$$ -1217098252828610199584768b^{10} + 93319246373147360817152b^{11} - 4520782665435130478592b^{12} $$
$$ +242909449904204187648b^{13} - 7797274383016572928b^{14} + 295577121333620992b^{15} $$
$$ -6177274611310592b^{16} + 163279973416448b^{17} - 2112944531328b^{18} + 37538840592b^{19} $$
$$ -268641472b^{20} + 2948968b^{21} - 8648b^{22} + 47b^{23} + 23a^{22}(-24+47b) + 253a^{21}(568-752b+705b^2) $$
$$ +1771a^{20}(-13248+28952b-10152b^2+6063b^3) + 253a^{19}(10651312-18378880b+18307440b^2 $$
$$ -2910240b^3+1242915b^4) + 4807a^{18}(-48160896+111412560b-59317760b^2+36701360b^3 $$
$$ -3314440b^4+1077193b^5) + 437a^{17}(35335680512-68886941952b+71068775952b^2-18381075840b^3 $$
$$ +8004372600b^4-465347376b^5+119568423b^6) + 7429a^{16}(-110280732672+264410732032b $$
$$ -172039726464b^2+108639162192b^3-16625231040b^4+5446950696b^5-220741704b^6 $$
$$ +459987855b^7) + 14858a^{15}(2365174520192-4965191440384b+5236842909696b^2 $$
$$ -1725798706176b^3+758786681184b^4-76810158336b^5+19719592224b^6-588644544b^7+101173281b^8) $$
$$ +874a^{14}(-1409491500899328+3463171135496576b-2549281676298240b^2+1628606520320000b^3 $$
$$ -326241761038080b^4+106997067614688b^5-7699470635520b^6+1590811880160b^7-36422381160b^8 $$
$$ +5227286185b^9) + 46a^{13}(772492387032024064-1706132584134203392b+1825151909639889024b^2 $$
$$ -700251511732561920b^3+308859506811100160b^4-41720602570934784b^5+10644260621618400b^6 $$
$$ -572668731075840b^7+96968519441640b^8-1756368158160b^9+2122278191111b^{10}) $$
$$ +598a^{12}(-1415643061687443456+3539464620490595328b-2837813317389384704b^2 $$
$$ +1823283877135639168b^3-433637769393797120b^4+141648230090454016b^5 $$
$$ -13801291695856896b^6+2815337632560096b^7-117555258659520b^8+16518224344600b^9 $$
$$ -242546078984b^{10}+24805848987b^{11}) + 2a^{11}(833614654875150237900 8 $$

$$-19110879531730263293952b + 20623641739160852121600b^{2} - 8803708000865147375616b^{3}$$
$$+3877040498846657697152b^{4} - 630933569726210070528b^{5} + 159290392776689860608b^{6}$$
$$-11738492743825376256b^{7} + 1950421219324385808b^{8} - 65053869300351360b^{9}$$
$$+7649418238997392b^{10} - 92873098607328b^{11} + 8061900920775b^{12})$$
$$+22a^{10}(-12318426342334540333056 + 31189294080015037855744b - 26643664822555271921664b^{2}$$
$$+17140623788483946063872b^{3} - 4605072082300887901184b^{4} + 1492188460507590418560b^{5}$$
$$-177180979433782351872b^{6} + 35548416593321988096b^{7} - 2052494989469688192b^{8}$$
$$+281304925678327408b^{9} - 7668336833158144b^{10} + 758463638626512b^{11} - 7739424883944b^{12}$$
$$+570534526701b^{13}) + 22a^{9}(164573150471956455718912 - 388241091264007637073920b$$
$$+420943316891085447997440b^{2} - 194683884826863646556160b^{3} + 85278443565011756866560b^{4}$$
$$-15861465348367839070208b^{5} + 3948145039255526734976b^{6} - 357922456741054965760b^{7}$$
$$+58142553453809144320b^{8} - 2703140922913516800b^{9} + 308127143096167824b^{10}$$
$$-6994146932156800b^{11} + 583323319956520b^{12} - 5071418015120b^{13} + 316963625945b^{14})$$
$$+22a^{8}(-1794419261680900851892224 + 4584366935603738318766080b$$
$$-4114251545048432952852480b^{2} + 2640794540104311940581376b^{3} - 778226828495277913989120b^{4}$$
$$+249266337772945624175616b^{5} - 34151179297906370239488b^{6} + 6714896930011659811968b^{7}$$
$$-480769634811446323200b^{8} + 64026820139834140160b^{9} - 2449850955905578368b^{10}$$
$$+233377227411710928b^{11} - 4479970542016320b^{12} + 315039984628920b^{13} - 2360832524280b^{14}$$
$$+124599494337b^{15}) + a^{7}(347649718060675616799195136 - 838882082577028634028015616b$$
$$+910822905671321377263845376b^{2} - 448673507687430886271483904b^{3}$$
$$+194828024586480427411873792b^{4} - 40139945352469260322013184b^{5}$$
$$+9816269459458694551068672b^{6} - 1035093445173598250778624b^{7} + 163773463835797810765056b^{8}$$
$$-9507758473057537904640b^{9} + 1046292644614417620992b^{10} - 33543815778321598464b^{11}$$
$$+2675398056697299648b^{12} - 44004970534863360b^{13} + 2602500541819200b^{14} - 16977685938048b^{15}$$
$$+751616304549b^{16}) + a^{6}(-2436468094409369161115369472 + 6263040594757311926167928832b$$
$$-5848486988767053581851295744b^{2} + 3733541278446785126654017536b^{3}$$
$$-1184188234508519054509670400b^{4} + 373651019226936518470082560b^{5}$$
$$-57174017531600230727811072b^{6} + 10975982455244081771618304b^{7}$$
$$-920520158677719502915584b^{8} + 118626675019529327475968b^{9} - 5703526377919620050944b^{10}$$
$$+520823675966042701824b^{11} - 14201827298789125632b^{12} + 947443730648350656b^{13}$$
$$-13503082015288320b^{14} + 668046217895616b^{15} - 3826410277704b^{16} + 140676848445b^{17})$$
$$+a^{5}(13311549776672286362560364544 - 32711559967534938612352155648b$$
$$+35464011035200609536911081472b^{2} - 18392078988954334205143154688b^{3}$$
$$+7891021782941046181484429312b^{4} - 1764877986073715914181836800b^{5}$$
$$+422468597935738118375137280b^{6} - 50111992196873469323575296b^{7}$$
$$+7690160291401572696950784b^{8} - 526300747131570952679424b^{9} + 55643641282334619534592b^{10}$$
$$-2254723506020816523264b^{11} + 171002161902726342656b^{12} - 4017063029686619136b^{13}$$
$$+223281440905959744b^{14} - 2784340493839872b^{15} + 114245678291448b^{16} - 578784747888b^{17}$$

$$+17417133617b^{18}) + a^4(-5505302436559844959037738188 8+ 14204497723875060906320304537 6b$$

$$-13707799071906280086102343680 0b^2 + 8676071172415318401564855500 8b^3$$

$$-2924040171669361887116001280 0b^4 + 905563901884059693153425817 6b^5$$

$$-151548175625849761449541632 0b^6 + 28284127665108161482099097 6b^7$$

$$-268595725694170393463193 60b^8 + 33326602669360436030566 40b^9$$

$$-190033682028660149368832b^{10} + 16535976570423823189248b^{11} - 573130879234289725440b^{12}$$

$$+36015944570559027200b^{13} - 737043756023662080b^{14} + 33885424088126016b^{15}$$

$$-372890329765440b^{16} + 12527300511720b^{17} - 56488000920b^{18} + 1362649145b^{19})$$

$$+a^3(16502058992107940555815649280 0 - 41161009850519180937974828236 8b$$

$$+44432034290186287158692256153 6b^2 - 24064052226392211093345376665 6b^3$$

$$+10164307271513241760242440601 6b^4 - 2434758036887616713334666035 2b^5$$

$$+567983627038415528709980160 0b^6 - 74192011537043310062154547 2b^7$$

$$+109848794149157432560799744b^8 - 8566222273855029320908800b^9$$

$$+864659016160210118774784b^{10} - 41800643016639985786880b^{11} + 2991812117724256942336b^{12}$$

$$-89845526485916487680b^{13} + 4650290892979978240b^{14} - 83728552007841792b^{15}$$

$$+3147554863570800b^{16} - 30798996193920b^{17} + 831387500720b^{18} - 3354213280b^{19} + 62891499b^{20})$$

$$+a^2(-33457931608615416872357068800 0 + 86454808042502636755432243200 0b$$

$$-85801456964756272716465360076 8b^2 + 53654343773068382198216785920 0b^3$$

$$-19047789584188685248144343040 0b^4 + 5763735450704138764179052953 6b^5$$

$$-1040612877589000469656004198 4b^6 + 187758232292779490876352102 4b^7$$

$$-19766144583168119181557760 0b^8 + 2345308642014470658729984 0b^9$$

$$-1533547764773194725654528b^{10} + 126112680824475795320832b^{11} - 5247383586349278087168b^{12}$$

$$+307487912205751326976b^{13} - 8093782952810557440b^{14} + 341481867848146944b^{15}$$

$$-5458828464945024b^{16} + 164912278071984b^{17} - 1444736263680b^{18} + 30474254800b^{19}$$

$$-110443608b^{20} + 1533939b^{21}) + a(40491377398641827770269696000 0$$

$$-102253195562254993614522286080 0b + 109550501147261529056857817088 0b^2$$

$$-61623125429118827559163999027 2b^3 + 25505297299416120437139465830 4b^4$$

$$-6486579549927146227238122291 2b^5 + 146576735600848558113758904 32b^6$$

$$-208050168703685725021169254 4b^7 + 29496257232068168605217587 2b^8$$

$$-25677211257278696652472320b^9 + 2451531478102328114786304b^{10}$$

$$-136847793704700345286656b^{11} + 9138388985836193208320b^{12} - 331765554383021232128b^{13}$$

$$+15765373619992310016b^{14} - 367721195189587968b^{15} + 12439930291235328b^{16}$$

$$-178105159009536b^{17} + 4212252152528b^{18} - 33249138560b^{19} + 524821176b^{20} - 1712304b^{21}$$

$$+16215b^{22}))\Big\} - \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\Big\{8388608(-21686243443194442612211712000 0 + 47a^{23}$$

$$+40491377398641827770269696000 0b - 33457931608615416872357068800 0b^2$$

$$+16502058992107940555815649280 0b^3 - 5505302436559844959037738188 8b^4$$

$$+1331154977667228636256036454 4b^5 - 2436468094409369161115369472b^6$$

$$+34764971806067561679919513 6b^7 - 39477223756979818741628928b^8$$
$$+3620609310383042025816064b^9 - 2710053795313598873272 32b^{10} + 1667229309750300475801 6b^{11}$$
$$-846554550889091186688b^{12} + 35534649803473106944b^{13} - 1231895571786012672b^{14}$$
$$+35141763021012736b^{15} - 819275563020288b^{16} + 15441692383744b^{17} - 231509427072b^{18}$$
$$+2694781936b^{19} - 23462208b^{20} + 143704b^{21} - 552b^{22} + b^{23} + 1081a^{22}(-8 + 15b)$$
$$+11891a^{21}(248 - 144b + 129b^2) + 11891a^{20}(-22592 + 44136b - 9288b^2 + 5289b^3)$$
$$+11891a^{19}(3156912 - 2796160b + 2562800b^2 - 282080b^3 + 114595b^4) + 20539a^{18}(-102874752$$
$$+205085552b - 70341120b^2 + 40478480b^3 - 2750280b^4 + 848003b^5) + 20539a^{17}(7949752832$$
$$-8671559424b + 8029226256b^2 - 1499537280b^3 + 609927480b^4 - 28179792b^5 + 6849255b^6)$$
$$+349163a^{16}(-17691664384 + 35627859456b - 15634040448b^2 + 9014571600b^3 - 1067954880b^4$$
$$+327198696b^5 - 10958808b^6 + 2152623b^7) + 41078a^{15}(7195509064064 - 8951779424256b$$
$$+8313011048448b^2 - 2038282097664b^3 + 824904427872b^4 - 67781793024b^5 + 16262871072b^6$$
$$-413303616b^7 + 66731313b^8) + 2162a^{14}(-3606509890386944 + 7292032201661568b$$
$$-3743655389829120b^2 + 2150920857067520b^3 - 340908305283840b^4 + 103275412074912b^5$$
$$-6245643855360b^6 + 1203746781600b^7 - 24023272680b^8 + 3225346795b^9)$$
$$+2162a^{13}(112354047134229504 - 153453077882988544b + 142223826182123648b^2$$
$$-41556672750192640b^3 + 16658623760665600b^4 - 1858031003555328b^5 + 438225592344288b^6$$
$$-20353825409280b^7 + 3205772276520b^8 - 51605548720b^9 + 5805624231b^{10})$$
$$+1222a^{12}(-3699494816231694336 + 7478223392664642560b - 4294094587847199744b^2$$
$$+2448291422032943488b^3 - 469010539471595520b^4 + 139936302702722048b^5$$
$$-11621789933542656b^6 + 2189360111863584b^7 - 80654134144320b^8 + 10501729164520b^9$$
$$-139334981544b^{10} + 13194600525b^{11}) + 94a^{11}(992757940139865540608 - 1455827592603195162624b$$
$$+1341624264090168035328b^2 - 444687691666382827520b^3 + 175914644366210884992b^4$$
$$-23986420276817197056b^5 + 5540677403894071296b^6 - 356849104024697856b^7$$
$$+54620202160187664b^8 - 1636928005398400b^9 + 177512766487056b^{10} - 1976023374624b^{11}$$
$$+157807422279b^{12}) + 1034a^{10}(-11770776139541684 71552 + 23709201915883250626 56b$$
$$-14831216293744629841 92b^2 + 8362272883561026293 76b^3 - 18378499229077383884 8b^4$$
$$+53813966423921295488b^5 - 5515982957369071616b^6 + 1011888437731545088b^7$$
$$-52124488423522944b^8 + 6555896661620592b^9 - 163156102833152b^{10} + 14795779959376b^{11}$$
$$-140273264248b^{12} + 9441469709b^{13}) + 1034a^9(1599767324691296416563 2$$
$$-2483289289872214376448 0b + 2268190176029468722176 0b^2 - 828454765363155640320 0b^3$$
$$+3223075693361744296960b^4 - 508994919856451598336b^5 + 114725991314825268352b^6$$
$$-9195124248604968960b^7 + 1362272768932641280b^8 - 57513636657734400b^9 + 5985211184645264b^{10}$$
$$-125829534430080b^{11} + 9553092996200b^{12} - 78136301040b^{13} + 4418421785b^{14})$$
$$+94a^8(-156792698034173946285 2608 + 313789970553916687289 5488b$$
$$-2102781338634906295910400b^2 + 1168604193076142899582976b^3 - 285740133717202546237440b^4$$
$$+81810215865974177627136b^5 - 9792767645507654286336b^6 + 1742270891870189476224b^7$$
$$-112520552828210841600b^8 + 13607831659402140160b^9 - 480371167748224896b^{10}$$

$$+414983238154412464b^{11} - 747851539131840b^{12} + 47452679726760b^{13} - 338650650360b^{14}$$
$$+15991836267b^{15}) + 47a^7(2719848726435720080759193\,6 - 44265993341209728727908352b$$
$$+39948560062293508697096192b^2 - 15785534369583682991947776b^3$$
$$+6017899503214502442999808b^4 - 1066212599933478070714368b^5 + 233531541600937910034432b^6$$
$$-22023264790927622356992b^7 + 3143143243835245018368b^8 - 167538171240493813760b^9$$
$$+16639684362831568896b^{10} - 499510329524484096b^{11} + 35820678814275264b^{12}$$
$$-560484289989120b^{13} + 29582331558720b^{14} - 186086822016b^{15} + 7269016485b^{16})$$
$$+47a^6(-15728824472696396680947302\,4 + 31186539489542246407182745\,6b$$
$$-22140699523170222758638387\,2b^2 + 120847580220939474193612800b^3$$
$$-32244292686351013074370560b^4 + 8988693573100811029258240b^5$$
$$-1216468458119153845272576b^6 + 208856797009759458533376b^7 - 15985658394764683941888b^8$$
$$+1848067890715352939776b^9 - 82935777607302377472b^{10} + 6778314586242121728b^{11}$$
$$-175599413491966464b^{12} + 10417786991371200b^{13} - 143177390115840b^{14} + 6233908537536b^{15}$$
$$-34891279128b^{16} + 1111731933b^{17}) + 47a^5(81400338997493223001581158\,4$$
$$-13801233084951374951570472\,96b + 12263266916391784604636282\,88b^2$$
$$-518033624869705683688226816b^3 + 192673170613629721947537408b^4$$
$$-37550595448376934344294400b^5 + 7950021685679500392980480b^6 - 854041390478069368553472b^7$$
$$+116677860234144760252416b^8 - 7424515694980690628608b^9 + 698471194280148706560b^{10}$$
$$-26848237009625960448b^{11} + 1802247693491308544b^{12} - 40832930175808512b^{13}$$
$$+1989690150962496b^{14} - 24281815586304b^{15} + 860965887672b^{16} - 4326740496b^{17} + 110171633b^{18})$$
$$+47a^4(-27717934175220581388370575\,36 + 54266589998757703057743544\,32b$$
$$-40527211881252521804562432\,00b^2 + 2162618568407072714945200128b^3$$
$$-622136206738162103641702400b^4 + 167894080488107365563498496b^5$$
$$-25195494351245086266163200b^6 + 4145277118861285689614336b^7 - 364276387806300300165120b^8$$
$$+39917569328303375554560b^9 - 2155565655545096464384b^{10} + 164980446759432242432b^{11}$$
$$-5517348640372142080b^{12} + 302288027942778880b^{13} - 6066708492495360b^{14} + 239873457638976b^{15}$$
$$-2627847689280b^{16} + 74423634600b^{17} - 338989640b^{18} + 6690585b^{19})$$
$$+47a^3(74510842932551963186823168\,00 - 13111303282791239906205106\,176b$$
$$+11415817824057102595365273600b^2 - 5120011111998342785818165248b^3$$
$$+1845972589875599659907416064b^4 - 391320829552219876705173504b^5$$
$$+79437048477591172907532288b^6 - 9546244844413423112159232b^7$$
$$+1236116593240316227506176b^8 - 91128626940234047324160b^9 + 8023270709503123689472b^{10}$$
$$-374625872377240313856b^{11} + 23198377841002387712b^{12} - 685352543397826560b^{13}$$
$$+30285151037440000b^{14} - 545572705880064b^{15} + 17171922040944b^{16} - 170904896640b^{17}$$
$$+3753690160b^{18} - 15665760b^{19} + 228459b^{20}) + 47a^2(-12117995399507257706781081600$$
$$+23308617265374793416352727040b - 18255629141437504833290502144b^2$$
$$+9453624317060912161423884288b^3 - 2916552994022612784277094400b^4$$
$$+754553426280864032700235776b^5 - 124435893378022416635133952b^6$$

$$
\begin{gathered}
+1937921075896428462263500 8b^7 - 1925819872150330318356480b^8 + 1970372972681676565509440b^9 \\
-1247150268289821238886 4b^{10} + 877601776134504345600b^{11} - 36106646038273447936b^{12} \\
+1786318890285848832b^{13} - 47405791172014080b^{14} + 1655510892601344b^{15} - 27193258040448b^{16} \\
+660788406192b^{17} - 6066818560b^{18} + 98548560b^{19} - 382536b^{20} + 3795b^{21}) \\
+47a(1191156104815483414590259200 0 - 2175599905579893481160048640 0b \\
+1839464000904311420328345600 0b^2 - 875766167032322998680315494 4b^3 \\
+302223355827128955453623500 8b^4 - 69599063760712635345430118 4b^5 \\
+1332561828671768494929346 56b^6 - 178485549484474177452769 28b^7 \\
+21458738847506860215500 80b^8 - 1817298725065567662899 20b^9 + 145992440374538475069 44b^{10} \\
-8132289162438409912 32b^{11} + 450340392139016171 52b^{12} - 16698318908547522 56b^{13} \\
+644002462217873 92b^{14} - 156963434938777 6b^{15} + 4179377294182 4b^{16} - 64050199219 2b^{17} \\
+11394897360b^{18} - 98933120b^{19} + 1090936b^{20} - 4048b^{21} + 23b^{22}))\Big\}\Bigg] \qquad (8)
\end{gathered}
$$

3 Derivation of the summation Formula

Putting $c = \frac{a+b+48}{2}$ and $z = \frac{1}{2}$ in equation (2), we get

$$
(a-b)\ {}_2F_1\left[\begin{matrix} a,\ b\ ; \\ \frac{a+b+48}{2}\ ; \end{matrix}\ \ \frac{1}{2}\right] = a\ {}_2F_1\left[\begin{matrix} a+1,\ b\ ; \\ \frac{a+b+48}{2}\ ; \end{matrix}\ \ \frac{1}{2}\right] - b\ {}_2F_1\left[\begin{matrix} a,\ b+1\ ; \\ \frac{a+b+48}{2}\ ; \end{matrix}\ \ \frac{1}{2}\right]
$$

Now involving the derived formula [Salahuddin et. al. p.12-41(8)], the summation formula is obtained.

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Received: October 12, 2014; *Accepted*: May 16, 2015

Malaya J. Mat. 3(3)(2015) 296–302

Malaya Journal of Matematik
www.malayajournal.org

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ISSN : 2319-3786

Invariant solutions and conservation laws for a three-dimensional K-S equation

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Abstract

In this paper, we study three-dimensional Kudryashov-Sinelshchikov (K-S) equation, which describes long nonlinear pressure waves in a liquid containing gas bubbles. Firstly, We find the symmetry groups of the K-S equation. Secondly, using the symmetry groups, exact solutions which are invariant under a three-dimensional subalgebra of the symmetry Lie algebra are derived. Finally, by adding Bluman-Anco homotopy formula to the direct method local conservation laws of the K-S equation are obtained.

Keywords: Three-dimensional Kudryashov-Sinelshchikov equation, Lie symmetry analysis, Invariant solution, Conservation laws.

2010 MSC: 58J70, 76M60, 35L65.

1 Introduction

A liquid with gas bubbles has many applications in nature, technology and medicine. An extended equation for the description of nonlinear waves in a liquid with gas bubbles was introduced in [1]. Extended models of nonlinear waves in bubbly liquid were considered in [2]. In this study we consider the following equation

$$u_{tx} + u_x^2 + uu_{xx} - \lambda u_{xxx} + u_{xxxx} + \frac{1}{2}(u_{yy} + u_{zz}) = 0, \tag{1.1}$$

where λ is parameter. This equation was introduced by Kudryashov-Sinelshchikov in [3]. This nonlinear equation is for a description of long nonlinear pressure waves. By using Painlevé test, it is shown that the K-S equation is not Painlevé integrable. Bifurcations and phase portraits for the equation were discussed in [4].

To find solutions to nonlinear partial differential equations, the study of their symmetry groups is one of the powerful methods in the theory of nonlinear partial differential equations. Then, the corresponding symmetry groups will be used in construction of exact solutions and mapping solutions to other solutions.

In the study of partial differential equations, the concept of a conservation law plays a very important role in the analyze of essential properties of the solutions, particularly, investigation of existence, uniqueness and stability of the solutions.

This work is organized as follows. In Section 2, we present group classification of the K-S equation. Section 3 is devoted to reductions to ordinary differential equations and exact solutions. In Section 4, the conservation laws associated to the equation are obtained via direct method. The conclusions are presented in Section 5.

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2 Group classification of the K-S equation

In this section we completely classify the Lie point symmetries of the K-S equation in terms of λ. For the non-extended transformations group of equation (1.1) the infinitesimal generator X is given by

$$X = \xi^t(t,x,y,z,u)\partial t + \xi^x(t,x,y,z,u)\partial x + \xi^y(t,x,y,z,u)\partial y + \xi^z(t,x,y,z,u)\partial z + \eta(t,x,y,z,u)\partial u. \tag{2.2}$$

The fourth prolongation of X is

$$X^{(4)} = X + \eta_i^{(1)}\partial u_i + \cdots + \eta^{(4)}_{i_1 i_2 i_3 i_4}\partial u_{i_1 i_2 i_3 i_4}, \tag{2.3}$$

where

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j, \qquad i,j = 1,...,4 \tag{2.4}$$

and for $l = 1,2,\cdots,k$ with $k \geq 2$, $i_l = 1,2,\cdots,4$

$$\eta^{(k)}_{i_1 i_2\cdots i_k} = D_{i_k}\eta^{(k-1)}_{i_1 i_2\cdots i_{k-1}} - (D_{i_k}\xi_j)u_{i_1 i_2\cdots i_{k-1}j}, \tag{2.5}$$

where D_i is the total derivative operator defined by

$$D_i = \partial x_i + u_i\partial u + u_{ij}\partial u_j + ..., \qquad i = 1,...,4 \tag{2.6}$$

with summation over a repeated index.

The vector field X generates a one parameter symmetry group of K-S equation if and only if

$$\begin{aligned}&\Big(X^{(4)}[u_{tx} + u_x^2 + uu_{xx} - \lambda u_{xxx} + u_{xxxx} + \frac{1}{2}(u_{yy} + u_{zz})]\Big)\Big|_{(1.1)} = \\ &\Big(\eta u_{xx} + 2u_x\eta_x^{(1)} + \eta_{tx}^{(2)} + u\eta_{xx}^{(2)} + \frac{1}{2}(\eta_{yy}^{(2)} + \eta_{zz}^{(2)}) - \lambda\eta_{xxx}^{(3)} + \eta_{xxxx}^{(4)}\Big)\Big|_{(1.1)} = 0.\end{aligned} \tag{2.7}$$

For more details see [5], [6].

Calculating the needed terms in (2.7) and spliting with respect to partial derivatives with respect to t,x,y, and z and various power of u, we can find the determining equations for the symmetry group of the equation (1.1). We study two cases: $\lambda = 0, \lambda \neq 0$.

Case A. $\lambda \neq 0$

Here, we find the following determining equations:

$$\begin{aligned}&\xi^t_t = \xi^t_x = \xi^t_y = \xi^t_z = \xi^t_u = \xi^x_x = \xi^x_{yy} = \xi^x_{zy} = \xi^x_{zz} = \xi^x_u = \xi^y_x = \xi^y_y = \xi^y_{zz} = \xi^y_u = 0,\\ &\xi^z_x = \xi^z_z = \xi^z_u = \eta_x = \eta_{yy} = \eta_{zy} = \eta_{zz} = \eta_u = 0, \xi^x_t = \eta, \xi^y_t = -\xi^x_y, \xi^z_t = -\xi^x_z, \xi^z_y = -\xi^y_z.\end{aligned} \tag{2.8}$$

So we have

$$\begin{aligned}&\xi^t = c_1, \quad \xi^x = -f_1'(t)y - f_2'(t)z + f_3(t), \quad \xi^y = f_1(t) + c_2 z,\\ &\xi^z = f_2(t) - c_2 y, \quad \eta = -f_1''(t)y - f_2''(t)z + f_3'(t),\end{aligned} \tag{2.9}$$

with $f_1(t), f_2(t), f_3(t)$ arbitrary functions and c_1, c_2 arbitrary constants. Thus the K-S equation admits an infinite-dimensional symmetry Lie algebra spanned by

$$\begin{aligned}&X_1 = \partial t, \quad X_2 = -y\partial z + z\partial y, \quad X_\infty = f_3'(t)\partial u + f_3(t)\partial x,\\ &X_\infty = -yf_1''(t)\partial u - yf_1'(t)\partial x + f_1(t)\partial y, \quad X_\infty = -zf_2''(t)\partial u - zf_2'(t)\partial x + f_2(t)\partial z,\end{aligned} \tag{2.10}$$

where $f_1(t), f_2(t), f_3(t)$ are arbitrary functions.

Case B. $\lambda = 0$

Here, we find the following determining equations:

$$\xi^t_x = \xi^t_y = \xi^t_z = \xi^t_u = \xi^x_{yy} = \xi^x_{zy} = \xi^x_{zz} = \xi^x_u = 0,$$
$$\xi^y_x = \xi^y_{zz} = \xi^y_u = \xi^z_x = \xi^z_u = \eta_x = \eta_{tu} = \eta_{yy} = \eta_{yu} = \eta_{zy} = \eta_{zz} = \eta_{zu} = \eta_{uu} = 0,$$
$$\xi^t_t = -\frac{3}{2}\eta_u, \xi^x_t = -\eta_u u + \eta, \xi^x_x = -\frac{1}{2}\eta_u, \xi^y_t = -\xi^x_y, \xi^y_y = \xi^z_z = -\eta_u, \xi^z_t = -\xi^x_z, \xi^z_y = -\xi^y_z. \tag{2.11}$$

So we have

$$\xi^t = c_1 t + c_2, \quad \xi^x = \frac{c_1}{3}x - f_1'(t)y - f_2'(t)z + f_3(t), \quad \xi^y = \frac{2c_1}{3}y + c_3 z + f_1(t),$$
$$\xi^z = -c_3 y + \frac{2c_1}{3}z + f_2(t), \quad \eta = -f_1''(t)y - f_2''(t)z - \frac{2c_1}{3}u + f_3'(t), \tag{2.12}$$

with $f_1(t), f_2(t), f_3(t)$ arbitrary functions and c_1, c_2, c_3 arbitrary constants. Thus the K-S equation admits an infinite-dimensional symmetry Lie algebra spanned by

$$X_1 = \partial t, \quad X_\infty = -zf_2''(t)\partial u - zf_2'(t)\partial x + f_2(t)\partial z, \quad X_\infty = u\partial u - \frac{3t}{2}\partial t - \frac{x}{2}\partial x - y\partial y - z\partial z,$$
$$X_2 = -y\partial z + z\partial y, \quad X_\infty = f_3'(t)\partial u + f_3(t)\partial x, \quad X_\infty = -yf_1''(t)\partial u - yf_1'(t)\partial x + f_1(t)\partial y, \tag{2.13}$$

where $f_1(t), f_2(t), f_3(t)$ are arbitrary functions.

3 Invariant solutions

Here, we use the results of the group classification in the previous section for the construction of exact solutions of the K-S equation. We search for solutions invariant under a three-dimensional subalgebra of the Lie algebra (2.13). Then equation (1.1) is reduced to a fourth-order ordinary differential equation. Solving this equation we find exact solution for the K-S equation[6, 7, 8, 9]. We choose the following three vector fields:

$$X_1 = 2y\partial u + 2ty\partial x - t^2\partial y, \quad X_2 = 2z\partial u + 2tz\partial x - t^2\partial z,$$
$$X_3 = u\partial u - \frac{3t}{2}\partial t - \frac{x}{2}\partial x - y\partial y - z\partial z. \tag{3.14}$$

These vector fields generate a three-dimensional subalgebra of the symmetry Lie algebra (2.13). We construct an exact solution of equation (1) which is invariant under these three vector fields: $X_1(I) = X_2(I) = X_3(I) = 0$. From $X_1(I) = 0$, we obtain four invariants $J_1 = t, J_2 = z, J_3 = u - x/t, J_4 = y^2 + tx$. Now, we rewrite X_2 and X_3 in terms of J_1,J_2,J_3 and J_4:

$$X_2 = -J_1^2\partial J_2 + 2J_1^2 J_2\partial J_4, \quad X_3 = -\frac{3}{2}J_1\partial J_1 - J_2\partial J_2 + J_3\partial J_3 - 2J_4\partial J_4. \tag{3.15}$$

Since the common solution $I(t, x, y, z, u)$ is defined as a function of the invariants J_1,J_2,J_3 and J_4 of X_1, it must be a solution to the differential equations

$$X_2(I) = -J_1^2\frac{\partial I}{\partial J_2} + 2J_1^2 J_2\frac{\partial I}{\partial J_4} = 0, \quad X_3(I) = -\frac{3}{2}J_1\frac{\partial I}{\partial J_1} - J_2\frac{\partial I}{\partial J_2} - J_3\frac{\partial I}{\partial J_3} - 2J_4\frac{\partial I}{\partial J_4} = 0. \tag{3.16}$$

The equation $X_2(I) = 0$ gives the three invariants $K_1 = J_1, K_2 = J_3, K_3 = J_4 + J_2^2$. Again we express these invariants as new variables. Writing X_3 in terms of K_1, K_2, and K_3, we obtain

$$X_3 = -\frac{3}{2}K_1\partial K_1 + K_2\partial K_2 - 2K_3\partial K_3. \tag{3.17}$$

From $X_3(I) = 0$ two invariants $I_1 = K_1^{2/3}K_2, I_2 = K_1^{-4/3}K_3$ are found.

The invariant solution is given by $I_1 = \Phi(I_2)$, where Φ is a function to be determined [7], [8]. Thus

$$t^{\frac{2}{3}}(u - \frac{x}{t}) = \Phi(\frac{y^2 + tx + z^2}{t^{\frac{4}{3}}}). \tag{3.18}$$

From (3.18) we have

$$u = t^{\frac{-2}{3}}\Phi(\delta) + \frac{x}{t}, \tag{3.19}$$

where $\delta = \dfrac{y^2 + tx + z^2}{t^{4/3}}$. Substituting u in the K-S equation (with $\lambda = 0$) we obtain

$$\Phi'''' + \Phi''(\Phi + \frac{2}{3}\delta) + \Phi'^2 + 3\Phi' = 0. \tag{3.20}$$

A solution which arises from the above equation is

$$\Phi = -3\delta = -3(\frac{y^2 + tx + z^2}{t^{\frac{4}{3}}}). \tag{3.21}$$

Therefore the exact solution

$$u = \frac{-3(y^2 + z^2 + \frac{2}{3}tx)}{t^2}, \tag{3.22}$$

for the K-S equation is obtained.

4 Conservation laws

There are many methods to investigate conservation laws, such as Noether's method, the direct method, etc. Here, we present the direct method [10, 11, 12, 13].

Consider a differential equation $P\{x; u\}$ of order k with n independent variables $x = (x^1, \ldots, x^n)$ and one dependent variable u, given by

$$P[u] = P(x, u, \partial u, \ldots, \partial^k u) = 0. \tag{4.23}$$

A multiplier $\Lambda(x, u, \partial u, \ldots, \partial^l u)$ provides a conservation law $\Lambda[u]P[u] = D_i\phi^i[u] = 0$ for the differential equation $P\{x; u\}$ if and only if

$$E_U\Big(\Lambda(x, U, \partial U, \ldots, \partial^l U)P(x, U, \partial U, \ldots, \partial^k U)\Big) \equiv 0, \tag{4.24}$$

for arbitrary functions $U(x)$, where E_U is the Euler operator with respect to U defined as

$$E_U = \partial U - D_i\partial U + \ldots + (-1)^s D_{i_1}\ldots D_{i_s}\partial U_{i_1\ldots i_s}. \tag{4.25}$$

Since the K-S equation is of *Cauchy-Kovalevskaya form* with respect to x, y, and z, it follows that multipliers providing local conservation laws for equation (1.1) are in the form $\Lambda = \xi(t, x, y, z, U, \partial_t U, \ldots, \partial_t^l U), l = 1, 2, \ldots$ and we can obtain all of its nontrivial local conservation laws from multipliers. Consequently, $\Lambda = \xi(t, x, y, z, U, \partial_t U, \ldots, \partial_t^l U)$ is a conservation law multiplier for the equation(1.1) if and only if

$$E_U\Big[\xi(t, x, y, z, U, \partial_t U, \ldots, \partial_t^l U)\,(U_{tx} + U_x^2 + UU_{xx} - \lambda U_{xxx} + U_{xxxx} + \frac{1}{2}(U_{yy} + U_{zz}))\Big] \equiv 0 \tag{4.26}$$

for an arbitrary function $U(t, x, y, z)$.

We look for all multipliers in the form $\Lambda = \xi(t, x, y, z, U, \partial U_t, \partial U_{tt}, \partial U_{ttt}, \partial U_{tttt})$ for the equation(1). Thus, the Euler operator is taken to be

$$E_U = \partial U - D_i\partial U_i + \ldots + (-1)^4 D_{i_1}\ldots D_{i_4}\partial U_{i_1\ldots i_4}, \tag{4.27}$$

and the determining equations become

$$E_U[\xi(t, x, y, z, U, \partial U_t, \ldots, \partial U_{tttt})(U_{tx} + U_x^2 + UU_{xx} - \lambda U_{xxx} + U_{xxxx} + \frac{1}{2}(U_{yy} + U_{zz}))] \equiv 0 \tag{4.28}$$

where $U(t, x, y, z)$ is arbitrary function. Equation (4.28) split with respect to $U_x, U_{tx}, \ldots, U_{xxxx}$ to provide the over-determined equations:

$$\xi_{yyyy} = -(2\xi_{zyyz} + \xi_{zzzz}), \quad \xi_{yxy} = -\xi_{zxz}, \quad \xi_{tx} = -\frac{\xi_{yy} + \xi_{zz}}{2}, \quad \xi_{xx} = \xi_U = \xi_{U_t} = \xi_{U_{tt}} = \xi_{U_{ttt}} = \xi_{U_{tttt}} = 0. \tag{4.29}$$

Solving the equations (4.29), we find the infinite set of local multipliers

$$\xi = (f_1(t,z-iy) + f_2(t,z+iy))x + f_3(t,z-iy) + f_4(t,z+iy) - \\ 2\int^y \int^b (D_1(f_1)(t,-2ib+iy+z) + D_1(f_2)(t,2ia-2ib+iy+z))dadb, \quad (4.30)$$

where f_1, f_2, f_3 and f_4 are arbitrary functions. We study two cases: $f_1(r,s) = f_2(r,s) = f_3(r,s) = f_4(r,s) = r+s$ and $f_1(r,s) = f_2(r,s) = f_3(r,s) = f_4(r,s) = \exp(r+s)$.

Case A

By setting $f_1(r,s) = f_2(r,s) = f_3(r,s) = f_4(r,s) = r+s$ into (4.30), we have $\xi = 2(t+z)(x+1) - 2y^2$. Applying Bluman-Anco homotopy formula [10, 11, 12], we find conserved components Φ^t, Φ^x, Φ^y, and Φ^z with respect to multiplier ξ:

$$\begin{aligned}
\Phi^t &= 2\Big[(t+z)(x+1) - y^2\Big]u_x,\\
\Phi^x &= 3\Big[(t+z)(x+1) - y^2\Big]uu_x - \Big[(t+z)u + ((t+z)(x+1) - y^2)u_x\Big]u - 2\Big[x+1\Big]u - \\
&\quad 2\Big[(t+z)(x+1) - y^2\Big]\lambda u_{xx} + 2\Big[(t+z)(x+1) - y^2\Big]u_{xxx} + 2\Big[t+z\Big]\lambda u_x - 2\Big[t+z\Big]u_{xx},\\
\Phi^y &= 2\Big[y\Big]u + \Big[(t+z)(x+1) - y^2\Big]u_y,\\
\Phi^z &= -\Big[x+1\Big]u + \Big[(t+z)(x+1) - y^2\Big]u_z. \qquad (4.31)
\end{aligned}$$

So we obtain the following local conservation law of the K-S equation:

$$\begin{aligned}
&D_t\Big(2[(t+z)(x+1) - y^2]u_x\Big) + D_x\Big(3[(t+z)(x+1) - y^2]uu_x - 2[(t+z)(x+1) - y^2]\lambda u_{xx} - \\
&[(t+z)u + ((t+z)(x+1) - y^2)u_x]u - 2[x+1]u + 2[(t+z)(x+1) - y^2]u_{xxx} + \\
&2[t+z]\lambda u_x - 2[t+z]u_{xx}\Big) + D_y\Big(2[y]u + [(t+z)(x+1) - y^2]u_y\Big) + \\
&D_z\Big(-[x+1]u + [(t+z)(x+1) - y^2]u_z\Big) = 0. \qquad (4.32)
\end{aligned}$$

Case B

By setting $f_1(r,s) = f_2(r,s) = f_3(r,s) = f_4(r,s) = \exp(r+s)$ into (4.30), we have:

$$\xi = (x - iy + \frac{1}{2})\exp(t+z-iy) + (x+iy+1)\exp(t+z+iy), \quad (4.33)$$

Applying Bluman-Anco homotopy formula, we find conserved components Φ^t, Φ^x, Φ^y, and Φ^z with respect to multiplier ξ:

$$\begin{aligned}
\Phi^t &= \Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u_x,\\
\Phi^x &= \Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]uu_x-\\
&\quad \frac{1}{2}\Big[\exp(t+z-iy)+\exp(t+z+iy)\Big]u^2+\lambda\Big[\exp(t+z-iy)+\exp(t+z+iy)\Big]u_x-\\
&\quad \Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u-\\
&\quad \Big[(\lambda x-\lambda iy+\frac{\lambda}{2}+1)\exp(t+z-iy)+(\lambda x+\lambda iy+\lambda+1)\exp(t+z+iy)\Big]u_{xx}+\\
&\quad \Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u_{xxx},\\
\Phi^y &= \frac{i}{2}\Big[(x-iy+\frac{3}{2})\exp(t+z-iy)-(x+iy+2)\exp(t+z+iy)\Big]u+\\
&\quad \frac{1}{2}\Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u_y,\\
\Phi^z &= -\frac{1}{2}\Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u+\\
&\quad \frac{1}{2}\Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u_z.
\end{aligned} \tag{4.34}$$

So we find the following local conservation law of the equation (1.1):

$$D_t\Phi^t + D_x\Phi^x + D_y\Phi^y + D_z\Phi^z = 0. \tag{4.35}$$

5 Conclusions

In the present paper, we investigated the Lie point symmetries, exact solutions and conservation laws of the K-S equation. We derived exact solutions which are invariant under a three-dimensional subalgebra of the symmetry Lie algebra. We obtained the conservation laws of the K-S equation by adding Bluman-Anco homotopy formula to the direct method.

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Received: November 18, 2014; *Accepted*: June 15, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 303–311

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On semi-invariant submanifolds of a nearly trans-hyperbolic Sasakian manifold

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Abstract

Semi-invariant submanifold of a trans Sasakian manifold has been studies. In the present paper we study semi invariant submanifolds of a nearly trans hyperbolic Sasakian manifold. Nejenhuis tensor in a nearly trans hyperbolic Sasakian manifold is calculated. Integrability conditions for some distributions on a semi invariant submanifold of a nearly trans hyperbolic Sasakian manifold are investigated.

Keywords: Semi-invariant submanifolds, nearly trans hyperbolic Sasakian manifold, Gauss and Weingarten equations, integrability conditions, distributions.

2010 MSC: 53D12,53C05.

1 Introduction

The study of geometry of semi invariant submanifold of a Sasakian manifold has been studied by Bejancu [1] and Bejancu and Papaghuic [4]. After that a number of authors have studied these submanifolds ([3],[5],[12]). Latter on, Oubina [8] introduced a new class of almost contact Riemannian manifold known as trans Sasakian manifold. Upadhyay and Dube [13] have studied almost contact hyperbolic (f, g, η, ξ)-structure. Shahid studied on semi invariant submanifolds of a nearly Sasakian manifold [14]. Matsumoto, Shahid, and Mihai [10] have also worked on semi invariant submanifolds of certain almost contact manifolds. Joshi and Dube [15] studied on Semi-invariant submanifold of an almost r-contact hyperbolic metric manifold. Gill and Dube have worked on CR submanifolds of trans-hyperbolic Sasakian manifolds [7].

2 Preliminaries

Nearly trans hyperbolic Sasakian Manifolds: Let $\bar{M}$ be an n dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (ϕ, ξ, η, g) where a tensor ϕ of type $(1,1)$, a vector field ξ, called structure vector field and η, the dual 1-form of is a 1-form ξ satisfying the following

$$\phi^2 X = X - \eta(X)\xi, \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = -1 \tag{2.2}$$

$$g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any X,Y tangents to $\bar{M}$ [6]. In this case

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.4}$$

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An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called trans-hyperbolic Sasakian [7] if and only if

$$(\bar{\nabla}_X\phi)Y = \alpha[g(X,Y)\xi - \eta(Y)\phi X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi X] \tag{2.5}$$

for all X,Y tangents to $\bar{M}$ and α, β are functions on $\bar{M}$. On a trans-hyperbolic Sasakian manifold M, we have

$$\bar{\nabla}_X\xi = -\alpha(\phi X) + \beta[X - \eta(X)\xi] \tag{2.6}$$

a Riemannian metric g and Riemannian connection $\bar{\nabla}$. Further, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly trans-hyperbolic Sasakian if [9]

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \alpha[2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y] - \beta[\eta(X)\phi Y + \eta(Y)\phi X] \tag{2.7}$$

Semi-invariant submanifolds: Let M be a submanifold of a Riemannian manifold $\bar{M}$ endowed with a Riemannian metric g. Then Gauss and Wiengarten formulae are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (X, Y \epsilon TM) \tag{2.8}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \quad (N \epsilon T^{\perp}M) \tag{2.9}$$

where $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\bar{M}$, M and the normal bundle of $T^{\perp}M$ of M respectively, and h is the second fundamental form related to A by

$$g(h(X,Y), N) = g(A_N X, Y) \tag{2.10}$$

Moreover, if ϕ is a $(1,1)$ tensor field on $\bar{M}$, for $X \epsilon TM$ and $N \epsilon T^{\perp}M$ we have

$$(\bar{\nabla}_X\phi)Y = ((\nabla_X P)Y - A_{FY}X - th(X,Y)) + ((\nabla_X F)Y + h(X, PY) - fh(X,Y)) \tag{2.11}$$

$$(\bar{\nabla}_X\phi)N = ((\nabla_X t)Y - A_{fN}X - PA_N X)) + ((\nabla_X f)N + h(X, tN) - FA_N X)) \tag{2.12}$$

where

$$\phi X \equiv PX + FX \quad (PX \epsilon TM, FX \epsilon T^{\perp}M) \tag{2.13}$$

$$\phi N \equiv tN + fN \quad (tN \epsilon TM, fN \epsilon T^{\perp}M) \tag{2.14}$$

$$(\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y, \quad (\nabla_X F)Y \equiv \nabla_X^{\perp} FY - F\nabla_X Y$$

$$(\nabla_X t)N \equiv \nabla_X tN - t\nabla_X^{\perp} N, \quad (\nabla_X f)N \equiv \nabla_X^{\perp} fN - f\nabla_X^{\perp} N$$

The submanifold M is known to be totally geodesic in $\bar{M}$ if $h = 0$, minimal in $\bar{M}$ if $H = trace(h)/dim(M) = 0$, and totally umbilical in $\bar{M}$ if $h(X,Y) = g(X,Y)H$.

For a distribution D on M, M is said to be D-totally geodesic if for all $X, Y \epsilon D$ we have $h(X,Y) = 0$. If for all $X, Y \epsilon D$ we have $h(X,Y) = g(X,Y)K$ for some normal vector K, then M is called D-totally umbilical. For two distributions D and ε defined on M, M is said to be (D, ε)-mixed totally geodesic if for all $X \epsilon D$ and $Y \epsilon \varepsilon$ we have $h(X,Y) = 0$.

Let D and ε be two distributions defined on a manifold M. We say that D is ε -parallel if for all $X \epsilon \varepsilon$ and $Y \epsilon D$ we have $\nabla_X Y \epsilon D$. If D is D-parallel then it is called autoparallel. D is called X-parallel for some $X \epsilon TM$ if for all $Y \epsilon D$ we have $\nabla_X Y \epsilon D$. D is said to be parallel if for all $X \epsilon TM$ and $Y \epsilon D, \nabla_X Y \epsilon D$.

If a distribution D on M is autoparallel, then it is clearly integrable, and by Gauss formula D is totally geodesic in M. If D is parallel then the orthogonal complementary distribution $D^{\perp}$ is also parallel, which implies that D is parallel if and only if $D^{\perp}$ is parallel. In this case M is locally the product of the leaves of D and $D^{\perp}$.

Let M be a submanifold of an almost contact metric manifold. If $\xi \epsilon TM$ then we write $TM = \{\xi\} \oplus \{\xi\}^{\perp}$, where (ξ) is the distribution spanned by ξ and $\{\xi\}^{\perp}$ is the complementary orthogonal distribution of $\{\xi\}$ in M. Then one gets

$$P\xi = 0 = F\xi, \quad \eta o P = 0 = \eta o F, \tag{2.15}$$

$$P^2 + tF = -I + \eta \otimes \xi, \quad FP + fF = 0, \tag{2.16}$$

$$f^2 + Ft = -I, \quad tf + Pt = 0 \tag{2.17}$$

A submanifold M of an almost contact metric manifold $\bar{M}$ with $\xi \epsilon TM$ is called a semi-invariant submanifold (Bejancu, [1]) of $\bar{M}$ if there exists two differentiable distributions D^1 and D^0 on M such that

(1) $TM = D^1 \oplus D^0 \oplus \{\xi\}$,

(2) the distribution D^1 is invariant by ϕ, that is, $\phi(D^1) = D^1$ and

(3) the distribution D^0 is anti-invariant by ϕ, that is, $\phi(D^0) \subseteq T^{\perp}M$.

For $X \epsilon TM$ we can write

$$X = D^1 X + D^0 X + \eta(X)\xi \tag{2.18}$$

where D^1 and D^0 are the projection operators of TM on D^1 and D^0, respectively. A semi-invariant submanifold of an almost contact metric manifold becomes an invariant submanifold ([2], [11]) (resp. anti-invariant submanifold ([2], [11]) if $D^0 = \{0\}$ (resp. $D^1 = \{0\}$).

3 The Nijenhuis tensor

A hyperbolic contact metric manifold is said to be normal ([6]) if the torsion tensor N^1 vanishes:

$$N^1 \equiv [\phi, \phi] + d\eta \otimes \xi = 0 \tag{3.19}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ and d denotes the exterior derivatives operatoer. In this section we obtain expression for Nijenhuis tensor $[\phi, \phi]$ of the structure tensor field ϕ given by

$$[\phi, \phi](X, Y) = ((\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X) - \phi((\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X) \tag{3.20}$$

in a nearly trans hyperbolic Sasakian manifold. First, we need the following lemma.

Lemma 3.1. *In an almost hyperbolic contact metric manifold we have*

$$(\bar{\nabla}_Y\phi)\phi X = -\phi(\bar{\nabla}_Y\phi)X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(X)\bar{\nabla}_Y\xi \tag{3.21}$$

Proof. For $X, Y \epsilon T\bar{M}$, we have

$$\begin{aligned}(\bar{\nabla}_Y\phi)\phi X &= -\phi^2\bar{\nabla}_Y X - \phi(\bar{\nabla}_Y\phi)X + \bar{\nabla}_Y X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(\bar{\nabla}_Y X)\xi - \eta(X)\bar{\nabla}_Y\xi \\ &= -\bar{\nabla}_Y X + \eta(\bar{\nabla}_Y X)\xi - \phi(\bar{\nabla}_Y\phi)X + \bar{\nabla}_Y X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(\bar{\nabla}_Y X)\xi - \eta(X)\bar{\nabla}_Y\xi\end{aligned}$$

which gives the equation (3.21). □

Now, we prove the following theorem

Theorem 3.1. *In a nearly trans-hyperbolic Sasakian manifold the Nijenhuis tensor $[\phi, \phi]$ of ϕ is given by*

$$\begin{aligned}[\phi, \phi](X, Y) &= 4\phi(\bar{\nabla}_Y\phi)X + 2d\eta(X, Y)\xi + \eta(X)\bar{\nabla}_Y\xi - \eta(Y)\bar{\nabla}_X\xi \\ &+ 4\alpha g(\phi X, Y)\xi + (\alpha + \beta)\eta(Y)\phi^2 X + 3(\alpha + \beta)\eta(X)\phi^2 Y\end{aligned} \tag{3.22}$$

Proof. Using Lemma 3.1 and $\eta o \phi = 0$ in (2.7) we get

$$(\bar{\nabla}_{\phi X}\phi)Y = \phi(\bar{\nabla}_Y\phi)X + ((\bar{\nabla}_Y\eta)X)\xi + \eta(X)\bar{\nabla}_Y\xi + 2\alpha g(\phi X, Y)\xi - (\alpha + \beta)\eta(Y)\phi^2 X \tag{3.23}$$

Thus

$$
\begin{aligned}
[\phi,\phi](X,Y) &= ((\bar{\nabla}_{\phi X}\phi)Y-(\bar{\nabla}_{\phi Y}\phi)X)-\phi((\bar{\nabla}_X\phi)Y-(\bar{\nabla}_Y\phi)X)\\
&= 2\phi(\bar{\nabla}_Y\phi)X-2\phi(\bar{\nabla}_X\phi)Y+[((\bar{\nabla}_X\eta)Y)\xi-((\bar{\nabla}_Y\eta)X)\xi]+\eta(X)\bar{\nabla}_Y\xi\\
&\quad -\eta(Y)\bar{\nabla}_X\xi+4\alpha g(\phi X,Y)\xi-(\alpha+\beta)[\eta(Y)\phi^2X-\eta(X)\phi^2Y]\\
&= 4\phi(\bar{\nabla}_Y\phi)X-2\phi[\alpha(2g(X,Y)\xi\\
&\quad -\eta(Y)\phi X-\eta(X)\phi Y-\beta(\eta(X)\phi Y+\eta(Y)\phi X)]\\
&\quad +2d\eta(X,Y)\xi+\eta(X)\bar{\nabla}_Y\xi-\eta(Y)\bar{\nabla}_X\xi\\
&\quad +4\alpha g(\phi X,Y)\xi-(\alpha+\beta)[\eta(Y)\phi^2X-\eta(X)\phi^2Y]\\
&= 4\phi(\bar{\nabla}_Y\phi)X+2\alpha\eta(Y)\phi^2X+2\alpha\eta(X)\phi^2Y-\beta[\eta(X)\phi Y+\eta(Y)\phi X]\\
&\quad +2d\eta(X,Y)\xi+\eta(X)\bar{\nabla}_Y\xi-\eta(Y)\bar{\nabla}_X\xi\\
&\quad +4\alpha g(\phi X,Y)\xi-(\alpha+\beta)[\eta(Y)\phi^2X-\eta(X)\phi^2Y]\\
&= 4\phi(\bar{\nabla}_Y\phi)X+2(\alpha+\beta)\eta(Y)\phi^2X+2(\alpha+\beta)\eta(X)\phi^2Y+2d\eta(X,Y)\xi\\
&\quad +\eta(X)\bar{\nabla}_Y\xi-\eta(Y)\bar{\nabla}_X\xi+4\alpha g(\phi X,Y)\xi\\
&\quad -(\alpha+\beta)\eta(Y)\phi^2X+(\alpha+\beta)\eta(X)\phi^2Y]\\
[\phi,\phi](X,Y) &= 4\phi(\bar{\nabla}_Y\phi)X+2d\eta(X,Y)\xi+\eta(X)\bar{\nabla}_Y\xi-\eta(Y)\bar{\nabla}_X\xi\\
&\quad +4\alpha g(\phi X,Y)\xi+(\alpha+\beta)\eta(Y)\phi^2X+3(\alpha+\beta)\eta(X)\phi^2Y
\end{aligned}
$$

which implies the equation (3.22). From Equation (3.22), we get

$$\eta(N^1(X,Y)) = 3d\eta(X,Y)-4\alpha g(X,\phi Y) \tag{3.24}$$

In particular, if X and Y are perpendicular to ξ, then (3.22) gives

$$[\phi,\phi](X,Y) = 4\phi(\bar{\nabla}_Y\phi)X-2(\eta[X,Y])\xi \tag{3.25}$$

□

4 Some basic results

Let M be a submanifold of a nearly trans-hyperbolic Sasakian manifold. Using (2.11), (2.13) in (2.7) for $X, Y \epsilon TM$, we get

$$
\begin{aligned}
&(\nabla_X P)Y+(\nabla_Y P)X-A_{FY}X-A_{FX}Y-2th(X,Y)+(\nabla_X F)Y\\
&+(\nabla_Y F)X+h(X,PY)+h(Y,PX)-2fh(X,Y)\\
&=\alpha[2g(X,Y)\xi-\eta(Y)PX-\eta(Y)FX-\eta(X)PY-\eta(X)FY]\\
&-\beta[\eta(X)PY+\eta(X)FY+\eta(Y)PX+\eta(Y)FX]
\end{aligned} \tag{4.26}
$$

Consequently, we have

Proposition 4.1. *Let M be a submanifold of a nearly trans-hyperbolic Sasakian manifold. Then for all $X, Y \epsilon TM$ we have*

$$
\begin{aligned}
&(\nabla_X P)Y+(\nabla_Y P)X-A_{FY}X-A_{FX}Y-2th(X,Y)\\
&=2\alpha g(X,Y)\xi-(\alpha+\beta)(\eta(Y)PX+\eta(X)PY)
\end{aligned} \tag{4.27}
$$

and

$$
\begin{aligned}
&(\nabla_X F)Y+(\nabla_Y F)X+h(X,PY)+h(Y,PX)-2fh(X,Y)\\
&=-(\alpha+\beta)[\eta(X)FY+\eta(Y)FX]
\end{aligned} \tag{4.28}
$$

for all $X, Y \epsilon TM$.

Now we state the following proposition.

Proposition 4.2. *Let M be a submanifold of a nearly trans-hyperbolic Sasakian manifold. Then*

$$\begin{aligned}\bar{\nabla}_X\phi Y+\bar{\nabla}_Y\phi X-\phi[X,Y] &= 2((\nabla_X P)Y-A_{FY}X-th(X,Y))\\ &+2((\nabla_X F)Y+h(X,PY)-fh(X,Y))+2\alpha g(X,Y)\xi)\\ &-(\alpha+\beta)(\eta(Y)PX+\eta(X)PY)-(\alpha+\beta)(\eta(Y)FX+\eta(X)FY)\end{aligned} \tag{4.29}$$

Consequently,

$$\begin{aligned}P[X,Y] &= A_{FY}X+A_{FX}Y+2th(X,Y)-2\alpha g(X,Y)\xi\\ &-(\alpha+\beta)(\eta(Y)PX+\eta(X)PY-\nabla_X PY-\nabla_Y PX+2P\nabla_X Y\end{aligned} \tag{4.30}$$

$$\begin{aligned}F[X,Y] &= -\nabla^{\perp}_X FY-\nabla^{\perp}_Y FY-h(X,PY)-h(Y,PY)+2fh(X,Y)\\ &-(\alpha+\beta)(\eta(Y)FX+\eta(X)FY)+2F\nabla_X Y\end{aligned} \tag{4.31}$$

The proof is straightforward and hence omitted.

Proposition 4.3. *Let M be a semi invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then (P,ξ,η,g) is a nearly trans-hyperbolic Sasakian structure on the distribution $D^1\oplus\{\xi\}$ if $th(X,Y)=0$ for all $X,Y\epsilon D^1\oplus\{\xi\}$.*

Proof. From $D^1\oplus\{\xi\}=ker(F)$ and (2.16) we have $P^2=I-\eta\otimes\xi$ on $D^1\oplus\{\xi\}$. We also get $P\xi=0,\eta(\xi)=2,\eta\circ P=0$.Using $D^1\oplus\{\xi\}=ker(F)$ and $th(X,Y)=0$ in 4.27 we get

$$(\nabla_X P)Y+(\nabla_Y P)X=2\alpha g(X,Y)\xi-(\alpha+\beta)(\eta(Y)PX+\eta(X)PY), \tag{4.32}$$

for all X, Y $\epsilon D^1\oplus\{\xi\}$. □

This completes the proof.

Theorem 4.2. *Let M be a semi invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. We have (i) if $D^0\oplus\{\xi\}$ is autoparallel then*

$$A_{FX}Y+A_{FY}X+2th(X,Y)=0,\quad \forall X,Y\epsilon D^0\oplus\{\xi\} \tag{4.33}$$

(ii) if $D^1\oplus\{\xi\}$ is autoparallel then

$$h(X,PY)+h(PX,Y)=2fh(X,Y)\quad \forall X,Y\epsilon D^1\oplus\{\xi\}. \tag{4.34}$$

Proof. In view of (4.27) and autoparallelness of $D^0\oplus\{\xi\}$ we get (1), while in view of (4.28) and appropriateness of $D^1\oplus\{\xi\}$ we get (ii). In view of Proposition 4.3 and Theorem 4.2(ii), we get □

Theorem 4.3. *Let M be a submanifold of a nearly trans-hyperbolic Sasakian manifold with $\xi\epsilon TM$. If M is invariant then M is nearly trans-hyperbolic Sasakian. Moreover*

$$h(X,PY)+h(PX,Y)-2fh(X,Y)=0,\quad X,Y\epsilon TM.$$

5 Integrability Conditions

Integrability of the distribution $D^1\oplus\{\xi\}$: We begin with a lemma

Lemma 5.2. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. For $X,Y\epsilon D^1\oplus\{\xi\}$ we get*

$$F[X,Y]=-h(X,PY)-h(PX,Y)+2F\nabla_X Y+2fh(X,Y) \tag{5.35}$$

or equivalently

$$-h(X,PX)+F\nabla_X X+fh(X,X)=0 \tag{5.36}$$

Proof. Equation (5.1) follows from $D^1 \oplus \{\xi\} = ker(F)$ and (4.6). Equivalence of (5.1) and (5.2) is obvious. In view of (5.1) and $D^1 \oplus \{\xi\} = ker(F)$ we can state the following theorem. □

Theorem 5.4. *The distribution $D^1 \oplus \{\xi\}$ on a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold is integrable if and only if*

$$h(X, PY) + h(PX, Y) = 2(F\nabla_X Y + fh(X, Y)) \tag{5.37}$$

Now, we need the following

Definition 5.1. *([16]) Let M be a Riemannian manifold with the Riemannian connection ∇ . A distribution D on M will be called nearly autoparallel if for all $X, Y \epsilon D$ we have $(\nabla_X Y + \nabla_Y X) \epsilon D$ or equivalently $\nabla_X X \epsilon D$.*

Thus, we have the following flow chart ([16]):
Parallel ⇒ Autoparallel ⇒ Nearly autoparallel,
Parallel ⇒ Integrable,
Autoparallel ⇒ Integrable, and
Nearly autoparallel + Integrable ⇒ Autoparallel.

Theorem 5.5. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then the following four statements*
(a) the distribution $D^1 \oplus \{\xi\}$ is autoparallel,
(b) $h(X, PY) + h(PX, Y) = 2fh(X, Y), \quad X, Y \epsilon D^1 \oplus \{\xi\}$,
(c) $h(X, PX) = fh(X, X), \quad X \epsilon D^1 \oplus \{\xi\}$,
(d) the distribution $D^1 \oplus \{\xi\}$ is nearly autoparallel,
are related by (a) ⇒ (b) ⇔ (c) ⇒ (d). In particular, if $D^1 \oplus \{\xi\}$ is integrable then the above four statements are equivalent.

The proof is similar to that Theorem 4.4 of [16].
Let $X, Y \epsilon D^1 \oplus \{\xi\}$. Using (2.1) and (2.13) in (3.19) and we get

$$\begin{aligned} N^{(1)}(X, Y) = [\phi X, \phi Y] - P[\phi X, Y] - F[\phi X, Y] - P[X, \phi Y] \\ -F[X, \phi Y] + [X, Y] + \eta([X, Y])\xi + 2d\eta \otimes \xi \end{aligned} \tag{5.38}$$

On the other hand from equation (3.23) we have

$$(\bar{\nabla}_{\phi X}\phi)Y = \phi(\bar{\nabla}_Y\phi)X + ((\bar{\nabla}_Y\eta)X)\xi + \eta(X)\bar{\nabla}_Y\xi + 2\alpha g(\phi X, Y)\xi - (\alpha+\beta)\eta(Y)\phi^2 X$$

which implies that

$$\begin{aligned} (\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X = \phi((\bar{\nabla}_Y\phi)X - (\bar{\nabla}_X\phi)Y) + 2d\eta(X, Y)\xi + \eta(X)U^1\nabla_Y\xi \\ +\eta(X)U^0\nabla_Y\xi + \eta(X)h(Y, \xi) - \eta(Y)U^1\nabla_X\xi - \eta(Y)U^0\nabla_X\xi \\ -\eta(Y)h(X, \xi) - (\alpha+\beta)(\eta(Y)\phi^2 X - \eta(X)\phi^2 Y) \end{aligned} \tag{5.39}$$

Next we easily can get

$$\begin{aligned} \phi(\bar{\nabla}_Y\phi)X = \phi(\bar{\nabla}_Y\phi X) - \phi^2(\bar{\nabla}_Y X) \\ = \phi(\nabla_Y\phi X + h(Y, \phi X)) - (\bar{\nabla}_Y X + \eta\bar{\nabla}_Y X)\xi \end{aligned} \tag{5.40}$$

so that

$$\begin{aligned} \phi((\bar{\nabla}_Y\phi)X - (\bar{\nabla}_X\phi)Y) = (\nabla_Y\phi X - \nabla_X\phi Y) + [X, Y] - \eta([X, Y])\xi \\ +F(\nabla_Y\phi X - \nabla_X\phi Y) + \phi(h(Y, \phi X) - h(X, \phi Y)) \end{aligned} \tag{5.41}$$

In view of (5.39) and (5.41) we get

$$\begin{aligned} N^{(1)}(X, Y) = 4d\eta \otimes \xi + 2[X, Y] - 2\eta([X, Y])\xi + 2P[\nabla_Y\phi X - \nabla_X\phi Y] \\ +2F[\nabla_Y\phi X - \nabla_X\phi Y] + 2\phi(h(Y, \phi X) - h(X, \phi Y)) + \eta(X)U^1\nabla_Y\xi \\ +\eta(X)U^0\nabla_Y\xi + \eta(X)h(Y, \xi) - \eta(Y)U^1\nabla_X\xi - \eta(Y)U^0\nabla_X\xi \\ -\eta(Y)h(X, \xi) - (\alpha+\beta)(\eta(Y)\phi^2 X - \eta(X)\phi^2 Y) \end{aligned} \tag{5.42}$$

Theorem 5.6. *The distribution $D^1 \oplus \{\xi\}$ is integrable on a semi-invariant submanifold M of a nearly trans-hyperbolic Sasakian manifold if and only if for all $X, Y \epsilon D^1 \oplus \{\xi\}$*

$$N^1(X,Y)\epsilon D^1 \oplus (\xi) \tag{5.43}$$

$$2(h(Y,\phi X) - h(X,\phi Y)) = -\eta(X)(\phi U^0\nabla_Y\xi + fh(Y,\xi)) + \eta(Y)(\phi U^0\nabla_X\xi + fh(X,\xi)) \tag{5.44}$$

Proof. Let $X, Y\epsilon D^1 \oplus \{\xi\}$. If $D^1 \oplus \{\xi\}$ is integrable, then (5.43) is true and from (5.42) we get

$$0 = 2F(\nabla_Y\phi X - \nabla_X\phi Y) + 2\phi(h(Y,\phi X) - h(X,\phi Y) + \eta(X)U^0\nabla_Y\xi$$
$$+\eta(X)h(Y,\xi) - \eta(Y)U^0\nabla_X\xi - \eta(Y)h(X,\xi)$$

Applying ϕ to the above equation, we get

$$0 = -2U^0(\nabla_Y\phi X - \nabla_X\phi Y) + 2(h(Y,\phi X) - h(X,\phi Y) + \eta(X)\phi U^0\nabla_Y\xi$$
$$+\eta(X)th(Y,\xi) + \eta(X)fh(Y,\xi) - \eta(Y)\phi U^0\nabla_X\xi - \eta(Y)th(X,\xi) - \eta(Y)fh(X,\xi)$$

Hence taking the normal part we get (5.44).

Conversely, let (5.43) and (5.44) be true. Using (5.44) in (5.42) we get

$$0 = 2U^0[X,Y] + 2F(\nabla_Y\phi X - \nabla_X\phi Y) + 2\phi(h(Y,\phi X) - h(X,\phi Y) + \eta(X)U^0\nabla_Y\xi$$
$$+\eta(X)h(Y,\xi) - \eta(Y)U^0\nabla_X\xi - \eta(Y)h(X,\xi)$$

Applying ϕ to the above equation and using (5.44) we get $\phi U^0[X,Y] = 0$, from which we get $U^0[X,Y] = 0$, and hence $D^1 \oplus \{\xi\}$ is integrable.

If $\bar{M}$ is a trans-hyperbolic Sasakian manifold then for all $X\epsilon D^1 \oplus \{\xi\}$ it is known that $h(X,\xi) = 0$ and $U^0\nabla_X\xi = 0$. Hence in view of the previous theorem we have □

Corollary 5.1. . *If M is a semi-invariant submanifold of a trans-hyperbolic Sasakian manifold, then the distribution $D^1 \oplus \{\xi\}$ is integrable if and only if for all $X, Y\epsilon D^1 \oplus \{\xi\}$*

$$h(X,\phi Y) = h(Y,\phi X)$$

Integrability of the distribution $D^0 \oplus \{\xi\}$:

Lemma 5.3. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then*

$$3(A_{FX}Y - A_{FY}X) = P[X,Y], \quad X, Y\epsilon D^0 \oplus (\xi) \tag{5.45}$$

Proof. Let $X, Y\epsilon D^0 \oplus \{\xi\}$ and $Z\epsilon TM$. We have

$$-A_{\phi X}Z + \nabla^{\perp}_Z\phi X = \bar{\nabla}_Z\phi X = (\bar{\nabla}_Z\phi)X + \phi(\bar{\nabla}_Z X)$$
$$= -(\bar{\nabla}_X\phi)Z - \eta(X)\phi Z - \eta(Z)\phi X + \phi\nabla_Z X + \phi h(Z,X)$$

so that

$$\phi h(Z,X) = -A_{\phi X}Z + \nabla^{\perp}_Z\phi X + (\bar{\nabla}_X\phi)Z + \eta(X)\phi Z + \eta(Z)\phi X - \phi\nabla_Z X$$

and hence we have

$$g(\phi h(Z,X),Y) = -g(A_{\phi X}Y,Z) - g((\bar{\nabla}_X\phi)Y,Z)$$

On the other hand

$$g(\phi h(Z,X),Y) = -g(h(Z,X),\phi Y) = -g(A_{\phi Y}X,Z)$$

Thus from the above two relations we get

$$g(A_{\phi Y}X,Z) = g(A_{\phi X}Y,Z) + g((\bar{\nabla}_X\phi)Y,Z) \tag{5.46}$$

For $X, Y \epsilon D^0 \oplus \{\xi\}$ we calculate $(\bar{\nabla}_X \phi)Y$ as follows. In view of

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = A_{\phi X} Y - A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X$$

and

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X, Y]$$

we have

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = A_{\phi X} Y - A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X - \phi[X, Y]$$

which gives

$$(\bar{\nabla}_X \phi)Y = 1/2(A_{\phi X} Y - A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X \\ -\phi[X, Y] - \eta(Y)\phi X - \eta(X)\phi Y)$$

□

Using this equation in the equation (5.46) we get (5.45).

In view of $D^0 \oplus \{\xi\} = ker(P)$, this lemma leads to the following

Theorem 5.7. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then the distribution $D^0 \oplus \{\xi\}$ is integrable if and only if*

$$A_{FX} Y = A_{FY} X \quad for all X, Y \epsilon D^0 \oplus \{\xi\}$$

Integrability of the distribution D^0: We calculate the torsion tensor $N^1(Y, X)$ for $Y, X \epsilon D^0$. It can be verified that

$$\phi((\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X) = -[X, Y] + \eta([X, Y])\xi + \phi(A_{\phi X} Y - A_{\phi Y} X) + \phi(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X) \tag{5.47}$$

$$(\bar{\nabla}_{\phi X} \phi)Y - (\bar{\nabla}_{\phi Y} \phi)X = [X, Y] - \phi(A_{\phi X} Y - A_{\phi Y} X) - \phi(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X \tag{5.48}$$

Using (5.13), (5.14) and (5.11) we get for $Y, X \epsilon D^0$

$$N^1(Y, X) = -2[X, Y] + 2/3\phi P[X, Y] + 2\phi(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X) \tag{5.49}$$

Theorem 5.8. *The distribution D^0 is integrable on a semi-invariant submanifold M of a nearly trans-hyperbolic Sasakian manifold if and only if*

$$N^{(1)}(Y, X) \epsilon D^0 \oplus \bar{D}^1 \quad X, Y \epsilon D^0 \tag{5.50}$$

$$A_{FX} Y = A_{FY} X \quad X, Y \epsilon D^0 \tag{5.51}$$

Proof. If D^0 is integrable, then in view of (5.48) and (5.49), the relation (5.50) and (5.51) follow easily. Conversely, let $X, Y \epsilon D^0$ and let the relation (5.50) and (5.51) be true. Then in view (5.48), we get $P[X, Y] = 0$ and in view of (5.49), we get

$$0 = g(\xi, N^1(Y, X)) = g(\xi, 2[Y, X]).$$

Thus $[X, Y] \epsilon D^0$. □

Non-integrability of the distribution D^1:

Theorem 5.9. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold with $\alpha \neq 0$. Then the non-zero invariant distribution D^1 is not integrable.*

Proof. If D^1 is integrable then for $X, Y \epsilon D^1$ it follows that $d\eta(X, Y) = 0$ and $[\phi, \phi](X, Y) \epsilon D^1$. Therefore, for $X \epsilon D^1$ in view of (3.24), we get

$$0 = \eta([\phi, \phi](X, PX) + 2d\eta(X, PX)\xi) \\ = \eta(N^1(X, PX) = 4\alpha g(\phi X, PX) = 4\alpha g(PX, PX),$$

which is a contradiction. □

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Received: October 10, 2014; *Accepted*: May 23, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 312–317

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

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ISSN : 2319-3786

Super Edge-antimagic Graceful labeling of Graphs

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Abstract

For a graph $G = (V, E)$, a bijection g from $V(G) \cup E(G)$ into $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called (a, d)-edge-antimagic graceful labeling of G if the edge-weights $w(xy) = |g(x) + g(y) - g(xy)|, xy \in E(G)$, form an arithmetic progression starting from a and having a common difference d. An (a, d)-edge-antimagic graceful labeling is called super (a, d)-edge-antimagic graceful if $g(V(G)) = \{1, 2, \ldots, |V(G)|\}$. Note that the notion of super (a, d)-edge-antimagic graceful graphs is a generalization of the article "G. Marimuthu and M. Balakrishnan, Super edge magic graceful graphs, Inf.Sci.,287(2014)140–151", since super $(a, 0)$-edge-antimagic graceful graph is a super edge magic graceful graph.We study super (a, d)-edge-antimagic graceful properties of certain classes of graphs, including complete graphs and complete bipartite graphs.

Keywords: Edge-antimagic graceful labeling, Super edge-antimagic graceful labeling.

2010 MSC: 34G20.

1 Introduction

We consider finite undirected nontrivial graphs without loops and multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively. Let $|V(G)| = p$ and $|E(G)| = q$ be the number of vertices and the number of edges of G respectively. General references for graph-theoretic notions are [2, 24].

A labeling of a graph is any map that carries some set of graph elements to numbers. Kotzig and Rosa [15, 16] introduced the concept of edge-magic labeling. For more information on edge-magic and super edge-magic labelings, please see [10].

Hartsfield and Ringel [11] introduced the concept of an antimagic labeling and they defined an antimagic labeling of a (p, q) graph G as a bijection f from $E(G)$ to the set $\{1, 2, \ldots, q\}$ such that the sums of label of the edges incident with each vertex $v \in V(G)$ are distinct. (a, d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [22]. This labeling is the extension of the notions of edge-magic labeling, see [15, 16].

For a graph $G = (V, E)$, a bijection g from $V(G) \cup E(G)$ into $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called a (a, d)-edge-antimagic total labeling of G if the edge-weights $w(xy) = g(x) + g(y) + g(xy), xy \in E(G)$, form an arithmetic progression starting from a and having a common difference d. The $(a, 0)$-edge-antimagic total labelings are usually called edge-magic in the literature (see [8, 9, 15, 16]). An (a, d)-edge antimagic total labeling is called super if the smallest possible labels appear on the vertices.

All cycles and paths have a (a, d)-edge antimagic total labeling for some values of a and d, see [22]. In [1], Baca et al. proved the (a, d)-edge-antimagic properties of certain classes of graphs. Ivanco and Luckanicova [13] described some constructions of super edge-magic total (super $(a, 0)$-edge-antimagic total) labelings for

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disconnected graphs, namely, $nC_k \cup mP_k$ and $K_{1,m} \cup K_{1,n}$. Super (a,d)-edge-antimagic labelings for $P_n \cup P_{n+1}, nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been described by Sudarsana et al. in [23].

In [7], Dafik et al. proved super edge-antimagicness of a disjoint unionof m copies of C_n. For most recent research in the subject, refer to [3, 14, 17, 19, 20, 21].

We look at a computer network as a connected undirected graph. A network designer may want to know which edges in the network are most important. If these edges are removed from the network, there will be a great decrease in its performance. Such edges are called the most vital edges in a network [5, 6, 12]. However, they are only concerned with the effect of the maximum flow or the shortest path in the network. We can consider the effect of a minimum spanning tree in the network. Suppose that $G = (V, E)$ is a weighted graph with a weight $w(e)$ assigned to every edge e in G. In the weighted graph G, the weight of a spanning tree $T, w(T)$ is defined to be $\sum w(e)$ for all $e \in E(T)$. A spanning tree T in G is called a minimum spanning tree if $w(T) \leq w(T')$ for all spanning trees T' in G. Let $g(G)$ denote the weight of a minimum spanning tree of G if G is connected; otherwise, $g(G) = \infty$. An edge e is called a most vital edge (MVE) in G if $g(G-e) \geq g(G-e')$ for every edge e' of G. We have a question : Is there any possibility to label the vertices and edges of a network G in such a way that every spanning tree of G is minimum and every edge is a most vital edge in G? The answer is 'yes'.

To solve this problem Marimuthu and Balakrishnan [18] introduced an edge magic graceful labeling of a graph.

They presented some properties of super edge magic graceful graphs and proved some classes of graphs are super edge magic graceful.

A (p,q) graph G is called edge magic graceful if there exists a bijection $g : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that $|g(x) + g(y) - g(xy)| = k$, a constant for any edge xy of G. G is said to be super edge magic graceful if $g(V(G)) = \{1, 2, \ldots, p\}$.

An (a,d)-edge-antimagic graceful labeling is defined as a bijective mapping from $V(G) \cup E(G)$ into the set $\{1, 2, 3, \ldots, p + q\}$ so that the set of edge-weights of all edges in G is equal to $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, for two integers $a \geq 0$ and $d \geq 0$.

An (a,d)-edge-antimagic graceful labeling g is called super (a,d)-edge-antimagic graceful if $g(V(G)) = \{1, 2, \ldots, p\}$ and $g(E(G)) = \{p+1, p+2, \ldots, p+q\}$. A graph G is called (a,d)-edge-antimagic graceful or super (a,d)-edge-antimagic graceful if there exists an (a,d)-edge-antimagic graceful or a super (a,d)-edge-antimagic graceful labeling of G.

Note that the notion of super (a,d)-edge-antimagic graceful graphs is a generalization of the article 'G. Marimuthu and M. Balakrishnan, Super edge magic graceful graphs, Inf.Sci.,287(2014)140–151", since super $(a,0)$-edge-antimagic graceful graph is a super edge magic graceful graph.

In this paper, we study super (a,d)-edge-antimagic graceful properties of certain classes of graphs, including complete graphs and complete bipartite graphs.

2 Complete graphs

Theorem 2.1. *If the complete graph $K_n, n \geq 3$, is super (a,d)-edge-antimagic graceful, then $d \leq 1$.*

Proof. Assume that a one-to-one mapping $f : V(K_n) \cup E(K_n) \rightarrow \{1, 2, \ldots, |V(K_n)| + |E(K_n)|\}$ is a super (a,d)-edge-antimagic graceful labeling of complete graph K_n, where the set of edge-weights of all edges in K_n is equal to $\{a, a+d, \ldots, a+(|E(K_n)|-1)d\}$.

The maximum edge-weight $a + (|E(K_n)| - 1)d$ is no more than $\left|1 + (n-1) - \left(\frac{n^2+n}{2} - 1\right)\right|$.

Thus, $a + (|E(K_n)| - 1)d \leq \frac{n^2-n-2}{2}$.

$$a + \left(\frac{n^2 - n - 2}{2}\right) d \leq \frac{n^2 - n - 2}{2} \tag{2.1}$$

The minimum edge-weight is $|1+n-(n+1)| = 0$.

Therefore,

$$a = 0 \tag{2.2}$$

From (1) and (2) we get $0 + d\left(\frac{n^2-n-2}{2}\right) \le \frac{n^2-n-2}{2}$. Hence $d \le 1$. □

Theorem 2.2. *Every complete graph $K_n, n \ge 3$ is super $(a,1)$-edge-antimagic graceful.*

Proof. For $n \ge 3$, let K_n be the complete graph with $V(K_n) = \{x_i : 1 \le i \le n\}$ and $E(K_n) = \bigcup_{i=1}^{n-1} \{x_i x_{i+j} : 1 \le j \le n-i\}$. Construct the one-to-one mapping $f : V(K_n) \cup E(K_n) \to \left\{1, 2, \ldots, \frac{n^2}{2} + \frac{n}{2}\right\}$ as follows:

If $1 \le i \le n$, then $f(x_i) = i$. If $1 \le j \le n-1$ and $1 \le i \le n-j$, then $f(x_i x_{i+j}) = nj + i + \sum_{k=1}^{j}(1-k)$. It is a routine procedure to verify that the set of edge-weights consists of the consecutive integers $\left\{0, 1, 2, \ldots, \frac{n(n-1)}{2} - 1\right\}$ which implies that f is a super $(0,1)$-edge-antimagic graceful labeling of K_n. □

An example to illustrate Theorem 2.2 is given in Fig. 1

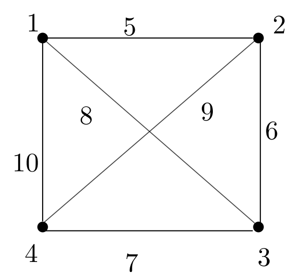

Fig. 1 A $(0,1)$-super edge-antimagic graceful completegraph.

3 Complete bipartite graphs

Let $K_{n,n}$ be the complete bipartite graph with $V(K_{n,n}) = \{x_i : 1 \le i \le n\} \cup \{y_j : 1 \le j \le n\}$ and $E(K_{n,n}) = \{x_i y_j : 1 \le i \le n \text{ and } 1 \le j \le n\}$.

Our first result in this section provides an upper bound for the parameter d for a super (a,d)-edge-antimagic graceful labeling of the complete bipartite graph $K_{n,n}$.

Theorem 3.1. *If a complete bipartite graph $K_{n,n}$ $n \ge 2$, is super (a,d)-edge-antimagic graceful, then $d = 1$.*

Proof. Let $K_{n,n}, n \ge 2$ be a super (a,d)-edge-antimagic graceful graph with a super (a,d)-edge-antimagic graceful lableing $g : V(K_{n,n}) \cup E(K_{n,n}) \to \{1, 2, \ldots, 2n + n^2\}$ and $W = \{w(xy) : xy \in E(K_{n,n})\} = \{a, a+d, a+2d, \ldots, a+(n^2-1)d\}$ be the set of edge-weights.

The sum of all vertex labels and edge labels used to calculate the edge-weight is equal to

$$\left| n\sum_{i=1}^{n} g(x_i) + n\sum_{j=1}^{n} g(y_j) - \sum_{i=1}^{n}\sum_{j=1}^{n} g(x_i y_j) \right| = \frac{n^4 - n^2}{2} \tag{3.3}$$

The sum of edge-weights in the set W is

$$\sum_{xy \in E(K_{n,n})} w(xy) = \frac{n^2}{2}(2a + d(n^2-1)) \tag{3.4}$$

The minimum edge-weight $a = |1 + 2n - (2n+1)| = 0$. Therefore $a = 0$.

Combining (3) and (4) we get, $\frac{n^4-n^2}{2} = \frac{n^2}{2}(2a + d(n^2-1))$.

Hence $d = 1$ for $n \geq 2$. □

Theorem 3.2. *Every complete bipartite graph $K_{n,n}, n \geq 2$ is super $(a,1)$-edge-antimagic graceful.*

Proof. Define the bijective function $g : V(K_{n,n}) \cup E(K_{n,n}) \rightarrow \{1, 2, \ldots, |V(K_{n,n})| + |E(K_{n,n})|\}$ of $K_{n,n}$ in the following way:

$$\begin{aligned}
g(x_i) &= i \text{ for } 1 \leq i \leq n \\
g(y_j) &= n + j \text{ for } 1 \leq j \leq n \\
g(x_i y_j) &= (j - i + 2)n + i - 1 + \sum_{k=0}^{j-i}(1-k) \text{ for } 1 \leq i \leq n \text{ and } i \leq j \leq n \\
g(x_i y_j) &= \frac{n^2+n}{2} + (i - j + 1)n + j - 1 + \sum_{k=0}^{i-j}(1-k) \text{ for } 1 \leq j \leq n-1 \\
&\qquad \text{and } j + 1 \leq i \leq n.
\end{aligned}$$

Let $A = (a_{ij})$ be a square matrix, where $a_{ij} = g(x_i) + g(y_j), 1 \leq i \leq n$ and $1 \leq j \leq n$.

The matrix A is formed from the edge-weights of $K_{n,n}$ under the vertex labeling:

$$A = \begin{bmatrix}
n+2 & n+3 & n+4 & n+5 & \ldots & 2n & 2n+1 \\
n+3 & n+4 & n+5 & n+6 & \ldots & 2n+1 & 2n+2 \\
n+4 & n+5 & n+6 & n+7 & \ldots & 2n+2 & 2n+3 \\
n+5 & n+6 & n+7 & n+8 & \ldots & 2n+3 & 2n+4 \\
\vdots & & & & & & \\
2n & 2n+1 & 2n+2 & 2n+3 & \ldots & 3n-2 & 3n-1 \\
2n+1 & 2n+2 & 2n+3 & 2n+4 & \ldots & 3n-1 & 3n
\end{bmatrix}$$

It is not difficult to see that the labels of the edges $x_i y_j$ form the square matrix $B = (b_{ij})$, where $b_{ij} = g(x_i y_j)$, for $1 \leq i \leq n, 1 \leq j \leq n$ and $t = \frac{n^2+5n}{2}, r = n^2 + 2n$:

$$B = \begin{bmatrix}
2n+1 & 3n+1 & 4n & 5n-2 & \ldots & t-2 & t \\
\frac{n^2+5n}{2}+1 & 2n+2 & 3n+2 & 4n+1 & \ldots & t-4 & t-1 \\
\frac{n^2+7n}{2} & \frac{n^2+5n}{2}+2 & 2n+3 & 3n+3 & \ldots & t-7 & t-3 \\
\frac{n^2+9n}{2}-2 & \frac{n^2+7n}{2}+1 & \frac{n^2+5n}{2}+3 & 2n+4 & \ldots & t-11 & t-6 \\
\vdots & & & & & & \\
r-2 & r-4 & r-7 & r-11 & \ldots & 3n-1 & 4n-1 \\
r & r-1 & r-3 & r-6 & \ldots & n+t-1 & 3n
\end{bmatrix}$$

The vertex labeling and the edge labeling of $K_{n,n}$ combine to give a total labeling where the edge-weights of edges $x_i y_j, 1 \leq i \leq n$ and $1 \leq j \leq n$ are given by the square matrix $C = (c_{ij})$ which is $|A - B|$.

We are setting $p = \frac{n^2+n}{2}$ and $q = n^2$.

$$C = \begin{bmatrix}
n-1 & 2n-2 & 3n-4 & 4n-7 & \ldots & p-2 & p-1 \\
\frac{n^2+3n-4}{2} & n-2 & 2n-3 & 3n-5 & \ldots & p-5 & p-3 \\
\frac{n^2+5n-8}{2} & \frac{n^2+3n-6}{2} & n-3 & 2n-4 & \ldots & p-9 & p-6 \\
\frac{n^2+7n-14}{2} & \frac{n^2+5n-10}{2} & \frac{n^2+3n-8}{2} & n-4 & \ldots & p-14 & p-10 \\
 & & & \vdots & & & \\
q-2 & q-5 & q-9 & q-14 & \ldots & 1 & n \\
q-1 & q-3 & q-6 & q-10 & \ldots & \frac{n^2+n}{2} & 0
\end{bmatrix}$$

We can see that the matrix C is formed from consecutive integers $0, 1, 2, \ldots, n^2 - 1$. This implies that the labeling $g : V(K_{n,n}) \cup E(K_{n,n}) \to \{1, 2, \ldots, n^2 + 2n\}$ is super $(0, 1)$-edge-antimagic graceful.

□

Figure 2 illustrates the proof of the above theorem.

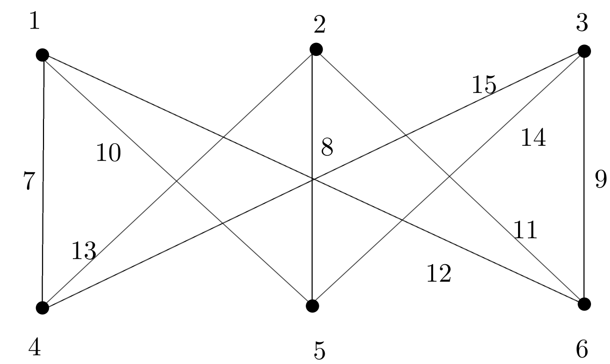

Fig. 2 A (0,1)- super edge-antimagic graceful completebipartite graph.

4 Conclusion

In the foregoing sections we studied super (a, d)-edge-antimagic graceful labeling for complete graphs and complete bipartite graphs. We have shown a bound for the feasible values of the parameter d and observed that for every super (a, d)-edge-antimagic graceful graph,$d < 2$.There are many research avenues on super (a, d)-edge-antimagic gracefullness of graphs.

If a graph G is super (a, d)-edge-antimagic graceful,is the disjoint union of multiple copies of the graph G super (a, d)-edge-antimagic graceful as well?An example of super (a, d)-edge-antimagic graceful disconnected graph is given in Figure 3.

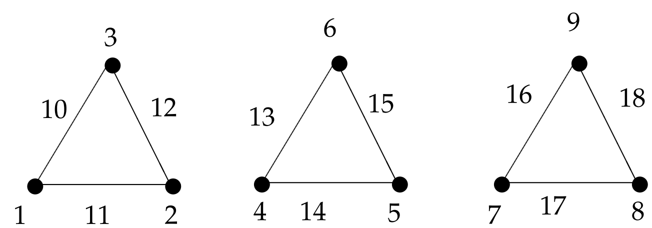

Figure 3. A super edge-antimagic gracefulness of disconnected graph.

To find the solution for the above question, We propose the following open problem.

Open Problem 4.1. *Discuss the super (a, d)-edge-antimagic gracefulness of disconnected graphs.*

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Received: February 10, 2015; *Accepted*: June 23, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 318–326

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On Quasi-weak Commutative Boolean-like Near-Rings

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Abstract

In this paper we establish a result that every quasi-weak commutative Boolean-like near-ring can be imbedded into a quasi-weak commutative Boolean-like commutative semi-ring with identity. Key words: Quasi-weak commutative near-ring, Boolean-like near-ring.

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2010 MSC: 16Y30, 16Y60.

1 Introduction

The concept of Boolean-like ring was coined by A.L.Foster[1]. Foster proved that if R is a Boolean ring with identity then ab(1-a)(1-b) = 0 for all a,b ϵ R. He generalized the concept of Boolean ring as Boolean-like ring as a ring R with identity satisfying (i) ab(1-a)(1-b) = 0 and (ii) 2a = 0 for all a,bϵ R. He also observed that the equation ab(1-a)(1-b) = 0 can be re-written as $(ab)^2$ - ab^2 a^2b +ab =0. He re-defined a Boolean-like ring as a commutative ring with identity satisfying (i) $(ab)^2$ - ab^2 a^2b +ab =0 and (ii) 2a = 0 for all a,b ϵ R. In 1962 Adil Yaqub [8] proved that the condition 'commutativity 'is not necessary in the definition of Boolean-like rings. He proved that any ring R with the conditions (i) $(ab)^2$ - ab^2 a^2b +ab =0 and (ii) 2a = 0 for all a,b ϵ R is necessarily commutative.

Ketsela Hailu and others [4] have constructed the Boolean-like semi-ring of fractions of a weak commutative Boolean-like semi-ring. We have coined and studied the concept of quasi-weak commutative near-ring in [2]. In this paper we define Boolean-like near ring (right) and prove that every quasi-weak commutative. Boolean-like near ring can be imbedded into a quasi weak commutative semi ring with identity.

2 Preliminaries

In this section we recal some definitions and results which we use in the sequal.

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2.1. Definition

A non empty set R together with two binary operations + and · satisfying the following axioms is called a right near-ring
(i) (R,+) is a group
(ii) · is associative
(iii) · is right distributive w.r.to +
(ie) (a+b) c = a · c + b · c ∀ a,b,c ϵ R

2.2. Note

In a right near-ring R, 0 a = 0 ∀ a ϵ R.
If (R,+) is an abelian group, then (R,+, ·) is called a semi-ring.

2.3. Definition

A right near-ring (R,+, ·) is called a Boolean-like near ring if
(i) 2a = 0 ∀ a ϵ R and
(ii) (a+b-ab)ab = ab ∀ a,b ϵ R

2.4.Remark

If (R,+, ·) is a Boolean-like near ring,then (R,+) is always an abelian group for 2x = 0 ∀ x ϵ R implies x = -x ∀ x ϵ R. We know, a group in which every element is its own inverse is always commutative.

2.5. Definition [5]

A right near ring R is said to be weak commutative if xyz = xzy ∀ x,y,z ϵ R

2.6. Definition [8]

A right near ring R is said to be pseudo commutative if xyz = zyx ∀ x,y,z ϵ R

2.7. Definition [2]

A right near ring R is said to be quasi-weak commutative if xyz = yxz ∀ x,y,z ϵ R

2.8. Definition

Let R be a right near ring. A subset B R is said to be multiplicatively closed if a,b ϵ B implies ab ϵ B.

3.Main results

3.1. Lemma

In a Boolean-like near ring (right) R a · 0 = 0 ∀ a ϵ R

Proof:

Since R is Boolean-like near ring, (a+b-ab)ab = ab $\forall$ a,b ϵ R
Taking a=0, we get
(0 + b - 0b) 0b = 0b
(ie) b $\cdot$ 0 = 0
Thus a$\cdot$ 0 = 0 $\forall$ a ϵ R.

3.2. Lemma

Let R be a quasi-weak commutative right near ring R. Then $(ab)^n = a^n b^n$ $\forall$ a,b ϵ R and for all n $\geq$1.

Proof:

Let a,b ϵ R.
Then $(ab)^2$ = (ab) (ab) = a (bab)
= a (abb) (quasi weak)
$(ab)^2 = a^2b^2$
Assume $(ab)^m = a^m\, b^m$
Now $(ab)^{(m+1)} = (ab)^m$ ab
$= a^m\, b^m$ ab
$= a^m\, (ab^m b)$
$= a^{m+1} b^{m+1}$
Thus $(ab)^m = a^m\, b^m$ $\forall$ a,b ϵ R and for all integer m$\geq$ 1.

3.3 lemma

Let R be a quasi-weak commutative Boolean like near-ring.Then
$a^2b + ab^2 = ab + (ab)^2$ $\forall$ a,b ϵ R.

Proof:

$a^2b + ab^2$ = aab + abb
= aab + bab
= (a + b)ab
= (a + b ab + ab)ab
$= (a + b\ \ ab)ab + (ab)^2$
$a^2b + ab^2 = ab + (ab)^2$ (R is Boolean-like near-ring)

3.4 Lemma

In a quasi-weak commutative Boolean like near ring (R,+, .),
$(a + a^2)(b + b^2)c = 0$ $\forall$ a,b,c ϵ R.

Proof:

$(a + a^2)(b + b^2)c = \{a(b + b^2) + a^2(b + b^2)\}$ c
$= a(b + b^2)c + a^2(b + b^2)c$

= (b + b^2)ac + (b + b^2) a^2c (R is quasi-weak commutative)
= $\{(b+b^2)a+(b+b^2)a^2\}$ c
=$\{ba+b^2a+ba^2+b^2a^2\}$c
=$\{ba+ba+(ba)^2+b^2a^2\}$ (using Lemma 3.3)
= $\{ba+ba+b^2a^2+b^2a^2\}$ (using Lemma 3.2)
=$\{2ba+2b^2a^2\}$
= 0 (R is Boolean-like near-ring).

3.5 Lemma

In a quasi-weak commutative Boolean like near ring R, (a - a^2) (b - b^2)c = 0 $\forall$ a,b,c ϵ R.

Proof:

(a - a^2)(b - b^2)c =$\{a(b-b^2)-a^2(b-b^2)\}$ c
= a(b - b^2)c - a^2(b - b^2)c
= (b - b^2)ac - (b - b^2) a^2c (quasi-weak commutative)
= $\{(b-b^2)a-(b-b^2)a^2\}$ c
=$\{ba-b^2a-ba^2-b^2a^2\}$c
=$\{ba-ba-(ba)^2-b^2a^2\}$
= $\{ba-ba-b^2a^2-b^2a^2\}$ (using Lemma 3.3)
= 0

3.6 Lemma

Let R be a quasi commutative Boolean like near-ring.Let S be a commutative subset of R which is multiplicatively closed.Define a relation N on R$\times$ S by (r_1 ,s_1) $\sim$ (r_2 ,s_2)if and only if there exists an element s ϵ S such that (r_1s_2 - r_2s_1)s = 0.Then N is an equivalence relation.

Proof:

(i) Let (r,s) ϵ R$\times$ S. Since rs-rs = 0,
we get (rs-rs)t=0 for all t ϵ S.
Hence $\sim$ is reflexive.

(ii) Let (r_1,s_1)$\sim$(r_2,s_2).Then there exists an element sϵS such that
(r_1,s_1-r_2,s_1)s=0. So (r_2,s_1-r_1,s_2)s = 0.
This proves $\sim$ is symmetric.

(iii) Let (r_1,s_1)$\sim$(r_2,s_2) and (r_2,s_2)$\sim$(r_3,s_3).
Then there exists p,qϵ S such that
(r_1s_2-r_2s_1)p=0 and (r_2s_3-r_3s_2)q = 0.
So s_3(r_1s_2-r_2s_1)p=0=s_1(r_2s_3-r_3s_2)q (By Lemma 3.1)
$\Longrightarrow$ (r_1s_2-r_2s_1)s_3p=0=(r_2s_3-r_3s_2)s_1q(R is quasi-weak commutative)
$\Longrightarrow$ (r_1s_2- r_2s_1)$s_3$pq=0=p(r_2s_3-r_3s_2)s_1q
$\Longrightarrow$(r_1s_2-r_2s_1)$s_3$pq=0=(r_2s_3-r_3s_2)ps_1q(R is quasi-weak commutative)
$\Longrightarrow$(r_1s_2-r_2s_1)$s_3$pq=0=(r_2s_3-r_3s_2)$s_1$pq(R is quasi-weak commutative)
$\Longrightarrow$ ($r_1s_2s_3$-$r_2s_1s_3$)pq=0=($r_2s_3s_1$-$r_3s_2s_1$)pq
$\Longrightarrow$ ($r_1s_2s_3$- $r_2s_1s_3$ +$r_2s_3s_1$-$r_3s_2s_1$) pq = 0.

$\Longrightarrow$ (r_1s_3 s_2-$r_2s_1s_3$+$r_2s_1s_3$-$r_3s_1s_2$)pq=0.(S is commutative)
$\Longrightarrow$ (r_1s_3-r_3s_1)$s_2$pq=0
$\Longrightarrow$ (r_1s_3-r_3s_1)r=0 where r = s_2 pqϵ S.
This implies (r_1,s_1)$\sim$(r_3,s_3).
Hence $\sim$ is transitive.
Hence the Lemma.

3.6 Remark

We denote the equivalence class containing (r,s)ϵ R $\times$ S by $\frac{r}{s}$ and the set of all equivalence classes by $S^{-1}R$.

3.8 Lemma

Let R be a quasi weak commutative Boolean like near-ring. Let S be a commutative subset of R which is also multiplicatively closed. If 0$\notin$ S and R has no zero divisors,then
(r_1,s_1)$\sim$ (r_2,s_2) if and only if r_1s_2=r_2s_1.

Proof:

Assume (r_1,s_1)$\sim$ (r_2,s_2).Then there exists an element sϵS such that (r_1s_2-r_2s_1)s=0.
Since 0$\notin$S and R has zero divisors,we get(r_1s_2-r_2s_1)= 0.
(i.e) r_1s_2 = r_2s_1
Conversely assume r_1s_2 = r_2s_1.
Then r_1s_2 - r_2s_1 = 0 and so (r_1s_2-r_2s_1) s = 0 for all sϵS.
Hence (r_1s_1) $\notin$ (r_2s_2).

3.9 Lemma:

Let R be a quasi weak commutative Boolean like near-ring. Let S be a commutative subset of R,which is also multiplicatively closed.
Then (i) $\frac{r}{s}$ = $\frac{rt}{st}$ = $\frac{tr}{st}$= $\frac{tr}{ts}$ for all rϵR and for all s,tϵS.
(ii) $\frac{rs}{s}$ = $\frac{rs'}{s'}$ for all rϵR and for all s,s' ϵ S.
(iii) $\frac{s}{s}$ = $\frac{s'}{s'}$ for all s,s$'$ ϵ S.
(iv) If 0ϵS,then $S^{-1}R$ contains exactly one element.

Proof:

The proof of (i),(ii) and (iii) are routine.
(iv) Since 0ϵS , (r_1s_2 - r_2s_1)0 = 0 $\forall\frac{r_1}{s_1},\frac{r_2}{s_2}$ ϵ $S^{-1}R$.
and so $\frac{r_1}{s_1}$= $\frac{r_2}{s_2}$.
Then $S^{-1}R$ contains exactly one-element.

3.10 Theorem:

Let R be a quasi weak commutative Boolean like near ring.Let S be a commutative subset of R which is also multiplicatively closed. Define binary operation + and on $S^{-1}R$ as follows :
$\frac{r_1}{s_1}$ +$\frac{r_2}{s_2}$ =$\frac{r_1s_2+r_2s_1}{s_1s_2}$ and

$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$
Then $S^{-1}R$ is a commutative Boolean like semi-ring with identity.

Proof:

Let us first prove that + and · are well defined. Let $\frac{r_1}{s_1} = \frac{r_1'}{r_1'}$ and $\frac{r_2}{s_2} = \frac{r_2'}{r_2'}$ Then there exists t_1, $t_2 \epsilon S$ such that $(r_1s_1'-r_1's_1)t=0$ ....(1)
and $(r_2s_2'-r_2's_2)t=0$ ....(2)
Now$[(r_1s_2+r_2s_1)s_1's_2'-(r_1's_2'+r_2's_1')s_1s_2]t_1t_2$
$=[r_1s_2s_1's_2'+r_2s_1s_1's_2'-r_1' s_2's_1s_2-r_2's_1's_1s_2]\ t_1t_2$
$=[r_1s_1's_2s_2'-r_1's_1s_2s_2'+r_2s_2's_1s_1'-r_2's_2s_1s_1']t_1t_2$
$=[(r_1s_1'-r_1's_1)s_2s_2'+(r_2s_2'-r_2's_2)s_1s_1']t_1t_2$
$=(r_1s_1'-r_1's_1)t_1s_2s_2't_2+(r_2s_2'-r_2's_2)t_2s_1s_1't_1$
$=0 \cdot s_2s_2't_2 + 0 \cdot s_1s_1't_1$
$=0$
Hence $\frac{r_1s_2+r_2s_1}{s_1s_2} = \frac{r_1's_2'+r_2's_1'}{s_1's_2'}$
(i.e) $\frac{r_1}{s_1}+\frac{r_2}{s_2}=\frac{r_1'}{s_1'}+\frac{r_2'}{s_2'}$
Hence + is well defined.
From (1) we get
$(r_1s_1'-r_1's_1)t_1t_2r_2s_2'=0$
$t_1t_2(r_1s_1'-r_1's_1)r_2s_2'=0$ (quasi weak commutative)
$t_1t_2(r_1s_1'r_2-r_1's_1r_2)s_2'=0$
$(r_1s_1'r_2-r_1's_1r_2)s_2't_1t_2=0$ (S is commutative subset)
$(r_1s_1'r_2s_2'-r_1's_1r_2s_2')t_1t_2=0$
$(r_1r_2s_1's_2'-r_1'r_2s_1s_2')t_1t_2=0$
$r_1r_2s_1's_2't_1t_2-r_1'r_2s_1s_2't_1t_2=0$ ....(3)
From (2) we get
$(r_2s_2'-r_2's_2)t_2t_1r_1's_1=0$
$(r_2s_2'-r_2's_2)t_1t_2r_1's_1=0$ (S is commutative subset)
$t_1t_2(r_2s_2'-r_2's_2)r_1's_1=0$ (quasi weak commutative)
$t_1t_2(r_2s_2'r_1'-r_2's_2r_1')s_1=0$
$(r_2s_2'r_1'-r_2's_2r_1')t_1t_2s_1=0$ (quasi weak commutative)
$(r_2s_2'r_1'-r_2's_2r_1')s_1t_1t_2=0$ (S is commutative subset)
$(r_2s_2'r_1's_1-r_2's_2r_1's_1)t_1t_2=0$
$(r_2r_1's_2's_1-r_2'r_1's_2s_1)t_1t_2=0$ (quasi weak commutative)
$(r_1'r_2s_2's_1-r_1'r_2's_2s_1)t_1t_2=0$ (quasi weak commutative)
$r_1'r_2s_1s_2't_1t_2-r_1'r_2's_1s_2t_1t_2=0$(S is commutative subset).........(4)
(3) + (4) gives
$r_1r_2s_1's_2't_1t_2-r_1'r_2's_1s_2t_1t_2=0$
$(r_1r_2s_1's_2'-r_1'r_2's_1s_2)t_1t_2=0$
This means $\frac{r_1r_2}{s_1s_2}=\frac{r_1'r_2'}{s_1's_2'}$
Hence is well-defined.
We note that $\frac{r_1}{s_1}+\frac{r_2}{s_2}=\frac{r_1s_2+r_2s_1}{s_1s_2}=\frac{(r_1+r_3)s}{s^2}$
$=\frac{r_1+r_2}{s}$ (by lemma 3.9).........(5)
Claim:1$(S^{-1}R,+)$is an abelian group.
Let $\frac{r_1}{s_1},\frac{r_2}{s_2},\frac{r_3}{s_3}\ \epsilon\ S^{-1}R$.
Then

$\frac{r_1}{s_1}+(\frac{r_2}{s_2}+\frac{r_3}{s_3})=\frac{r_1}{s_1}+(\frac{r_2s_3+r_3s_2}{s_2s_3})$
$=\frac{r_1s_2s_3+(r_2s_3+r_3s_2)s_1}{s_1s_2s_3}$
$=\frac{r_1s_2s_3+r_2s_3s_1+r_3s_2s_1}{s_1s_2s_3}$

Also $(\frac{r_1}{s_1}+\frac{r_2}{s_2})+\frac{r_3}{s_3}=(\frac{r_1s_2+r_2s_1}{s_1s_2})+\frac{r_3}{s_3}$
$=\frac{(r_1s_2+r_2s_1)s_3+r_3s_1s_2}{s_1s_2s_3}$
$=\frac{r1s_2s_3+r_2s_3s_1+r_3s_1s_2}{s_1s_2s_3}$

$\frac{r_1}{s_1}+(\frac{r_2}{s_2}+\frac{r_3}{s_3})=(\frac{r_1}{s_1}+\frac{r_2}{s_2})+\frac{r_3}{s_3}$

So + is associative.

For any $\frac{r}{s}\epsilon$ R,we have

$\frac{r}{s}+\frac{0}{s}=\frac{r+0}{s}=\frac{r}{s}$

Also $\frac{0}{s}+\frac{r}{s}=\frac{0+r}{s}=\frac{r}{s}$

Hence $\frac{0}{s}$ is the additive identity of $\frac{r}{s}\ \epsilon S^{-1}R$ for all r ϵ R

Clearly + is commutative.

Thus (R,+) is an abelian group.

Claim:2 $\cdot$ is associative.

Now $\frac{r_1}{s_1}\cdot(\frac{r_2}{s_2}\cdot\frac{r_3}{s_3})=\frac{r_1}{s_1}\cdot(\frac{r_2r_3}{s_2s_3})=\frac{r_1(r_2r_3)}{s_1(s_2s_3)}$
$=\frac{(r_1r_2)r_3}{(s_1s_2)s_3}$
$=(\frac{r_1}{s_1}\cdot\frac{r_2}{s_2})\cdot\frac{r_3}{s_3}$

So $\cdot$ is associative.

Claim:3 $\cdot$ is right distributive with respect to +.

Let $\frac{r_1}{s_1},\frac{r_2}{s_2},\frac{r_3}{s_3}\ \epsilon\ S^{-1}R$.

Now $(\frac{r_1}{s_1}+\frac{r_2}{s_2})\cdot\frac{r_3}{s_3}=(\frac{r_1s_2+r_2s_1}{s_1s_2})\cdot\frac{r_3}{s_3}$
$=\frac{r_1s_2r_3+r_2s_1r_3}{s_1s_2s_3}$
$=\frac{s_2r_1r_3+s_1r_2r_3}{s_1s_2s_3}$ (quasi weak commutative)
$=\frac{s_2r_1r_3}{s_1s_2s_3}+\frac{s_1r_2r_3}{s_1s_2s_3}$ (using (5))
$=\frac{s_2r_1r_3}{s_2s_1s_3}+\frac{s_1r_2r_3}{s_1s_2s_3}$
$=\frac{r_1r_3}{s_1s_3}+\frac{r_2r_3}{s_2s_3}$
$=\frac{r_1}{s_1}\cdot\frac{r_3}{s_3}+\frac{r_2}{s_2}\cdot\frac{r_3}{s_3}$

This proves right - distributive law.

Claim:4 $S^{-1}R$ is a Boolean-like ring.

It is already proved in **claim 1** that

$2(\frac{r}{s})=0$ for all $\frac{r}{s}\ \epsilon S^{-1}R$

Let a = $\frac{r_1}{s_1}$ and b = $\frac{r_2}{s_2}$ be any two elements of $S^{-1}R$ Let t ϵ S be any element.

Now by Lemma 3.5

$(a - a^2)(b - b^2)\ t = 0$

$\Rightarrow (\frac{r_1}{s_1}-\frac{r_1^2}{s_1^2})(\frac{r_2}{s_2}-\frac{r_2^2}{s_2^2})t = 0$

$[\frac{r_1}{s_1}(\frac{r_2}{s_2}-\frac{r_2^2}{s_2^2})-\frac{r_1^2}{s_1^2}(\frac{r_2}{s_2}-\frac{r_2^2}{s_2^2})]t = 0$

$\frac{r_1}{s_1}(\frac{r_2}{s_2}-\frac{r_2^2}{s_2^2})t-\frac{r_1^2}{s_1^2}(\frac{r_2}{s_2}\frac{r_2^2}{s_2^2})t = 0$

$(\frac{r_2}{s_2}-\frac{r_2^2}{s_2^2})\frac{r_1}{s_1}t-(\frac{r_2}{s_2}-\frac{r_2^2}{s_2^2})\frac{r_1^2}{s_1^2}t=0$(quasi weak commutative)

$[(\frac{r_2}{s_2}-\frac{r_2^2}{s_2^2})\frac{r_1}{s_1}-(\frac{r_2}{s_2}-\frac{r_2^2}{s_2^2})\frac{r_1^2}{s_1^2}]t=0$

$[(\frac{r_2s_2-r_2^2}{s_2^2})\frac{r_1}{s_1}-(\frac{r_2s_2-r_2^2}{s_2^2})\frac{r_1^2}{s_1^2}]t=0$

$[(\frac{r_2s_2-r_2^2}{s_2^2})\frac{r_1s_1}{s_1^2}-(\frac{r_2s_2-r_2^2}{s_2^2})\frac{r_1^2}{s_1^2}]t=0$ (using Lemma 3.9)

$[(\frac{r_2s_2r_1s_1-r_2^2r_1s_1}{s_2^2s_1^2})-\frac{r_2s_2r_1^2-r_2^2r_1^2}{s_2^2s_1^2}]t=0$

$[(\frac{r_2r_1s_2s_1-r_2^2r_1s_1}{s_2^2s_1^2})$-$\frac{s_2r_2r_1^2-r_2^2r_1^2}{s_2^2s_1^2}]$t=0(quasi weak commutative)

$[(\frac{r_2r_1s_2s_1}{s_2^2s_1^2}$-$\frac{r_2^2r_1s_1}{s_2^2s_1^2}$ -$\frac{s_2r_2r_1^2}{s_2^2s_1^2}+\frac{r_2^2r_1^2}{s_2^2s_1^2}]$t=0

$[\frac{r_2r_1}{s_2s_1}$-$\frac{r_2^2}{s_2^2}\frac{r_1}{s_1}$-$\frac{r_2}{s_2}\frac{r_1^2}{s_1^2}+\frac{r_2^2r_1^2}{s_2^2s_1^2}]$t=0

$[ba-b^2a-ba^2+b^2a^2]$t=0

$\Rightarrow$ba = b^2a - ba^2 + b^2a^2

= b^2a + ba^2 - $(ba)^2$ (using Lemma 3.2)

ba = ba(b+a-ba)

This proves $S^{-1}R$ is Boolean-like near ring.

Claim :5 multiplication in $S^{-1}R$ is commutative

Let $\frac{r_1}{s_1},\frac{r_2}{s_2}$ be any two elements of $S^{-1}R$.

Then $\frac{r_1}{s_1}\cdot\frac{r_2}{s_2}=\frac{r_1r_2}{s_1s_2}=\frac{r_1r_2s}{s_1s_2s}$ $\forall$ sϵS $\frac{r_2r_1s}{s_1s_2s}$(quasi weak commutative)

$=\frac{r_2r_1s}{s_2s_1s}$(S is commutative subset)

$=\frac{r_2}{s_2}\frac{r_1}{s_1}$(using Lemma 3.9)

Hence multiplication in $S^{-1}R$ is commutative.

Claim:6 Existence of multiplicative identity in $S^{-1}R$

Let $\frac{r}{s}S^{-1}R$ be any element.

Then $\frac{r}{s}\cdot\frac{s}{s}=\frac{rs}{ss}=\frac{r}{s}$

Also $\frac{s}{s}\cdot\frac{r}{s}=\frac{sr}{ss}=\frac{r}{s}$

Hence $\frac{s}{s}\ \epsilon\ S^{-1}$ R is the multiplicative identity of $S^{-1}R$

Thus $S^{-1}R$ is a commutative Boolean-like near-ring with identity.

3.11 Theorem

$S^{-1}R$ is quasi-weak commutative near-ring.

Proof:

Let a = $\frac{r_1}{s_1}$, b = $\frac{r_2}{s_2}$, c = $\frac{r_3}{s_3}$be any three elements of $S^{-1}R$

Now abc = $\frac{r_1}{s_1}\cdot\frac{r_2}{s_2}\cdot\frac{r_3}{s_3}=\frac{r_1r_2r_3}{s_1s_2s_3}$

$=\frac{r_2r_1r_3}{s_1s_2s_3}$(R is quasi-weak commutative)

$=\frac{r_2r_1r_3}{s_2s_1s_3}$ (S is commutative)

$=\frac{r_2}{s_2}\frac{r_1}{s_1}\frac{r_3}{s_3}$

Then abc = bac $\forall$a,b,c ϵ $S^{-1}R$.

This proves $S^{-1}R$ is quasi-weak commutative near-ring.

3.12 Theorem

Let R be a quasi-weak commutative Boolean-like near ring.Let S be a commutative subset of R which is multiplicatively closed. Let 0$\neq$ s ϵS. Define a map f_s: R$\rightarrow$ S^{-1} R as $f_s(r)=\frac{rs}{s}$ $\forall$ rϵ R. Then f_s is a near-ring monomorphism.

Proof:

Let r_1,r_2 ϵ R.

Then f_s $(r_1+r_2)=\frac{(r_1+r_2)s}{s}=\frac{r_1s+r_2s}{s}$

$=\frac{r_1s}{s}+\frac{r_2s}{s}$(By (5) of Theorem 3.11)

= $f(r_1)$ + $f(r_2)$

Also f_s $(r_1\cdot r_2)$ = $\frac{(r_1r_2)s}{s}$

$=\frac{r_1r_2s^2}{s^2}$

$=\frac{r_1r_2ss}{s^2}$

$=\frac{r_1(sr_2s)}{s^2}$
$=\frac{r_1s}{s}\cdot\frac{r_2s}{s}$(quasi weak commutative)
$=f_s(r_1)\cdot f_s(r_2)$

Also $f_s(r_1) = f_s(r_2) \Rightarrow \frac{r_1s}{s}=\frac{r_2s}{s}$
$\Rightarrow \frac{r_1s}{s}-\frac{r_2s}{s}=0$
$\Rightarrow \frac{(r_1s-r_2s)}{s}=0$
$\Rightarrow \frac{(r_1-r_2)s}{s}=0$
$\Rightarrow (\frac{r_1}{s}-\frac{r_2}{s})=0$
$\Rightarrow \frac{r_1}{s}=\frac{r_2}{s}$

Hence f_s is a monomorphism

3.13 Theorem

Let R be a quasi-weak commutative Boolean-like near-ring. Then R be embedded into a quasi-weak commutative. Boolean like commutative semi ring with identity.

Proof:

Follows from Theorem 3.11 and 3.12.

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Received: October 10, 2014; *Accepted*: March 23, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 327–334

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

On Quasi Weak Commutative Near-rings-II

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Abstract

A right near-ring N is called weak Commutative,(Definition 9.4 Pilz [9]) if xyz = xzy for every x,y,z ε N. A right near-ring N is called pseudo commutative (Definition 2.1, S.Uma and others [10]) if xyz = zyx for all x,y,z ε N. A right near-ring N is called quasi weak commutative near-ring if xyz = yxz for every x,y,z ε N [4]. In [4], we have obtained some interesting results of quasi-weak commutative near-rings. In this paper we obtain some more results of quasi weak commutative near-rings.

Keywords: Quasi-weak commutative near-ring, Boolean-like near-ring.

2010 MSC: 16Y30, 16Y60.

1 Introduction

Through out this paper, N denotes a right near-ring (N,+,.) with atleast two elements.For any non-empty subset A of N,we denote A - {0} = A*.The following definitions and results are well known.

Definition:1.1

An element a ε N is said to be
1.Idempotent if a^2 = a.
2.Nilpotent, if there exists a positive integer k such that a^k = 0.

Result: 1.2 (Theorem 1.62 Pilz [9])

Each near-ring N is isomorphic to a subdirect product of subdirectly irreducible near-rings.

Definition: 1.3

A near-ring N is said to be zero symmetric if ab = 0 implies ba = 0,where a,b ε N.

Result: 1.4

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If N is zero symmetric, then
Every left ideal A of N is an N-subgroup of N.
Every ideal I of N satisfies the condition NIN $\subseteq$ I. (i.e) every ideal is an N-subgroup. N* I* N* $\subseteq$ I*.

Result: 1.5

Let N be a near-ring. Then the following are true.
If A is an ideal of N and B is any subset of N,then (A:B) = $\{n\varepsilon$ N such that nB $\subseteq A\}$ is always a left ideal.
If A is an ideal of N and B is an N-subgroup,then (A : B) is an ideal.
In particular if A and B are ideals of a zero-symmetric near-ring, then
(A : B) is an ideal.

Result: 1.6

1. Let N be a regular near-ring, a ε N and a = axa,then ax,xa are idempotents and so the set of idempotent elements of N is non-empty.
2. axN = aN and Nxa = Na.
3. N is S and S'near-rings.

Result: 1.7 (Lemma 4 Dheena [1])

Let N be a zero-symmetric reduced near-ring. For any a,b ε N and for any idempotent element e ε N, abe = aeb.

Result: 1.8 (Gratzer [6] and Fain [3])

A near-ring N is sub-directly irreducible if and only if the intersection of all non-zero ideals of N is not zero.

Result: 1.9 (Gratzer [6])

Each simple near-ring is sub directly irreducible.

Result: 1.10 (Pilz [9])

A non-zero symmetric near-ring N has IFP if and only if (O : S) is an ideal for any subset S of N.

Result: 1.11 (Oswald [8])

An N-subgroup A of N is essential if A$\cap$ B = $\{0\}$,where B is any N subgroup of N,implies B = $\{0\}$.

Definition: 1.12

A near-ring N is said to be reduced if N has no non-zero nilpotent elements.

Definition: 1.13

A near-ring N is said to be an integral near-ring,if N has no non-zero divisors.

Lemma: 1.14

Let N be a near-ring.If for all a ε N,$a^2 = 0 \Rightarrow a = 0$,then N has no non-zero nilpotent elements.

Definition: 1.15

Let N be a near-ring. N is said to satisfy intersection of factors property (IFP) if ab = o anb = 0 for all n ε N, where a,b ε N.

Definition: 1.16

1. An ideal I of N is called a prime ideal if for all ideals A,B of N, AB is subset of I ⇒ A is subset of I or B is subset of I.
2. I is called a semi-prime ideal if for all ideals A of N, A^2 is subset of I implies A is subset of I.
3. I is called a completely semi-prime-ideal,if for any x ε N, x2 ε I ⇒ x ε I.
4. A completely prime ideal,if for any x,y ε N, xy ε I ⇒ x ε I or y ε I.
5. N is said to have strong IFP,if for all ideals I of N, ab ε I implies anb ε I for all n ε N.

Result: 1.17 (Proposition 2.4[10])

Let N be a Pseudo commutative near-ring. Then every idempotent element is central.

Result: 1.18[4]

Let N be a regular quasi weak commutative near-ring. Then
1. A = $\sqrt{A}$, for every N sub-group A of N
2. N is reduced
3. N has (*IFP)

Result: 1.19[4]

Let N be a regular quasi weak commutative near-ring. Then every N sub group is an ideal N = Na = Na^2 = aN = aNa for all a ε N

Result: 1.20[4]

Let N be a quasi weak commutative near-ring. For every ideal I of N, (I:S) is an ideal of N where S is any subset of N.

Result: 1.21[4]

Every quasi weak commutative near-ring N is isomorphic to a sub-direct product of Sub-directly irreducible quasi weak commutative near-rings.

2. Main Results:

Lemma: 2.1

Let N be a regular quasi weak commutative near-ring.
Then
(i) P∩Q = PQ for any two N-subgroups P,Q of N.
(ii) P = P^2 for every N-sub group(ideal) P of N.
(iii) If P is a proper N-subgroup of N,then each element of P is a zero divisor.
(iv) Na Nb = Na ∩ Nb = Nab for all a,b ε N.
(v) Every N-subgroup of N is essential if N is integral.

Proof:

(i) Let P and Q be two N-subgroups of N.

Then by Result1.19[4] they are ideals.

Hence PQ $\subset$ P and PQ $\subseteq$ Q.So PQ $\subseteq$ P$\cap$Q.

Let a ε P $\cap$ Q.Since N is regular,there exists b ε N such that

a = aba = (ab) a ε (PN)Q $\subseteq$ PQ.

Hence P$\cap$Q = PQ.This completes (i).

(ii) Taking Q = P in (i) we get P = P^2.

(iii) Let P be a proper N-subgroup of N.

Let 0 $\neq$ a ε P.Now by(ii) Na = $(Na)^2$ = NaNa.

Therefore for every n ε N,there exists x,y ε N such that na = xaya.

Hence (n-xay)a = 0.If a is not a zero divisor,then n-xay = 0.

(i.e) n = xay ε NPN $\subseteq$P.

Hence N = P, contradicting P is a proper ideal of N.So a is a zero divisor of N. This proves (iii).

(iv) Since Na and Nb are N-subgroups,

Na $\cap$ Nb = Na Nb. (by(i))

Since Na $\subseteq$ N, Na $\cap$ N = Na = Na$\cap$ Na =Na Na

$\subseteq$ Na N = Na N.

and Na is an ideal implies Na N = (Na)N $\subseteq$ Na

= Na $\cap$ N.

Therefore Na = Na $\cap$ N = Na N.

This implies that Nab = (Na)b = (Na N)b = Na Nb = Na $\cap$ Nb.

This proves (iv).

(v) Let P be a non-zero N-subgroup of N.

Suppose there exists an N-subgroup Q of N such that P$\cap$Q = $\{0\}$.

Then by (i) PQ = $\{0\}$ and since N is an integral near-ring Q = $\{0\}$.

This proves (v).

Theorem:2.2

Let N be a regular quasi weak commutative near-ring and P be a proper N-subgroup of N.Then the following are equivalent

(i) P is a prime ideal.

(ii) P is a completely prime ideal.

(iii) P is a primary ideal.

(iv) P is a maximal ideal.

Proof:

(i)$\Rightarrow$ (ii)

Let P be a proper N-subgroup of N.

Assume P is prime.Let ab ε P.

By Lemma 2.1(iv)

Na Nb = Nab $\subseteq$ NP $\subseteq$ P.

Also by Result1.19[4],Na and Nb are ideals of N.

Since P is prime, Na Nb $\subseteq$ P implies Na $\subseteq$ P (or) Nb $\subseteq$ P.

Since N is regular,there exists x,y ε N such that a = axa and b = byb.

If Na ⊆ P,then a = axa ε Na⊆P or if Nb ⊆ P,then b = byb ε Nb ⊆ P.

(i.e) aεP or bεP and hence P is completely prime.

(ii) ⇒ (i) is obvious.

(ii)⇒ (iii)

Let a,bε N.By Lemma 2.1(iv) Nab = Na∩ Nb.

Since Na ∩ Nb = Nb ∩ Na, Nab = Nba for all a,b ε N.

Hence for all a,b,c ε N.

Nabc = Nacb = Nbca = Nbac = Ncab = Ncba.

Suppose abc ε P and ab ∉ P,by (ii) cεP.

Again suppose abc ε P and ac ∉ P.

Since N is regular,acb ε Nacb ⊆ NP ⊆ P.

Thus acb = (ac)b εP implies bεP (by(ii)).

Continuing in the sameway, we can easily prove that if abcεP and if the product of any two of a,b,c doesnot belong to P,then the third belongs to P:

This proves (iii).

(iii)⇒ (i)

Let abεP and a ∉ P.

Since N is regular a = axa for some xε N.

We shall first prove that xa ∉ P.

Suppose xa ε P, then a = axa = a(xa) ε NP ⊆ P,which is a contradiction.

Therefore xa ∉ P.

Also x(ab) ε NP ⊆P.Thus xab ε P and xa ∉ P.

As P is a primary ideal of N,bk εP for some integer k.Now bk εP

implies bε$\sqrt{P}$ P.But by Result1.18[4] $\sqrt{P}$ = P.So bε P.

This proves (ii).

(i) ⇒ (iv)

Let J be an ideal of N such that P ⊆ J ⊆ N.

Suppose P = J,there is nothing to prove.

So,assume P ⊂ J.We shall prove that J = N.

Let a ε J\ P.Since N is regular there exists x ε N such that a = axa.

Then a = (xa)a = xa^2 (quasi weak commutative).

So, for all n ε N, na = nxa^2 and this implies (n - nxa) a = 0.

Since N has I ⊂ P,we get n - nxa) ya = 0 for all y ε N.

Consequently, N(n-nxa) Na = N0 ={0}.

If b = (n-nxa) then Na Nb = Nab = {0} ⊆ P.

Since P is a prime ideal and Na and Nb are ideals in N, Na ⊆ P or Nb ⊂P.

If Na ⊆ P, then a = axa ε P which is a contradiction.

Hence Nb ⊆ P ⊆ J.

Since N is regular,there exists y ε N such that b = byb, (i.e) b = (by)b ε Nb⊆ J.

(i.e) b = n-nxa ε J.Since a ε J, nxa ε nJ ⊆ J. (By Lemma 1.4)

Therefore nε J.Hence J = N.So P is maximal.

(v) ⇒ (i) is obvious.

This completes the proof of the theorem.

Theorem:2.3

Any quasi-weak commutative near-ring N with left identity is commutative.

Proof:

Let a,b ε N and e ε N be the identity.

Then ab = abe = bae (quasi weak commutative).

= ba

Hence N is commutative.

Theorem : 2.4

Let N be a subdirectly irreducible quasi weak commutative near-ring.

Then either N is simple with each non-zero idempotent element is an identity or the intersection of the non-zero ideals of N has no non-zero idempotents.

Proof:

Let N be a subdirectly irreducible quasi weak commutative near-ring.

Suppose that N is simple.

Let e ε N be a non-zero idempotent element.

Then by Result1.8[4] N has IFP.By Theorem1.20 [4], (0:e) is an ideal.

Since e $\notin$ (0:e) and N is simple, we get (0:e) = $\{0\}$.

Hence (ene - en)e = ene^2 - ene = ene - ene = 0 for all nε N.

This implies (ene - en) ε (0: e) = $\{0\}$.

Hence ene - en = 0.

(i.e) ene = en $\cdots\cdots$ (1)

Also since N is quasi weak commutative,

ene = nee = ne^2 = ne $\cdots\cdots$ (2)

(1) and (2) gives ne = en $\cdots\cdots$ (3)

Also (ne - n)e = ne^2 - ne = ne - ne = 0 for all n ε N.

This implies ne - n = 0 $\cdots\cdots$ (4)

(3) and (4) gives

ne = en = n. Hence e is an identity of N.

Suppose N is not simple.

Let I be the intersection of non-zero ideals of N.Since N is subdirectly irreducible, we have I $\neq \{0\}$.

Suppose that I contains a non-zero idempotent e.

We claim that e is a right identity.

If not,there exists nεN such that ne $\neq$ n.

Hence ne - n $\neq$ 0.Since (ne - n)e = 0.

We have ne - n ε (0:e) and hence (0:e) is a non-zero ideal of N.

Therefore I $\subseteq$ (0:e).Hence eεI $\subseteq$ (0:e)

(i.e) e ε (0:e).This contradiction leads to conclude that e is a right identity of N. Hence for all nεN, n = ne ε NI $\subseteq$ I.

This implies that I = N,again a contradiction.Hence the intersection of the non-zero ideals of N has no non-zero idempotents.

This proves the theorem.

Theorem:2.5

Let N be a regular quasi weak commutative near-ring.

Then the following are equivalent

(i) N is subdirectly irreducible.

(ii) Non-zero idempotents of N are not zero divisors.

(iii) N is simple.

Proof:

(i) $\Rightarrow$ (ii)

Let J be the set of all non-zero idempotents in N which are zero divisor too.We shall prove that J is empty.If J is not empty, let I = $\cap \{(0 : e)/e\varepsilon J\}$.

Since N is sub-directly irreducible, I $\neq$ 0 by Result1.8([6],[3])

Let 0 $\neq$ a εI.

Since N is regular,there exists an element bεN such that a = aba $\cdots\cdots$ (1)

Also ab,ba are idempotents.Since 0 $\neq$ a εI, ae = 0 for all e εJ $\cdots\cdots$ (2)

Then (ae)b = 0.

Since N is zero symmetric b(ae) = 0.

(i.e) (ba)e = 0.Hence ba is a zero divisor and so ba εJ.

So by (2) a(ba) = 0.

This is a contradiction as a $\neq$ 0.Hence J is empty.

(ii) $\Rightarrow$ (iii)

Let I be a non-zero ideal of N and 0$\neq$ x εI.

Since N is regular,there exists yεN such that x = xyx $\cdots\cdots$ (3)

Also yx is an idempotent element of N.

Therefore for every nεN, nx = nxyx.

(i.e) (n-nxy)x = 0.Since N has IFP, (n-nyx)yx = 0.By (ii) n-nxy = 0

(i.e) for every nεN, n = nxy ε NIN $\subset$ I.

Thus N $\subseteq$ I.This proves that N has no non-trivial ideal of N.

So N is Simple.

(iv) $\Rightarrow$ (i)

This follows from the Result 1.9.

Corollary:2.6

Let N be a regular quasi weak commutative near-ring.Then N is subdirectly irreducible if and only if N is a field.

Proof: By theorem 2.4 and 2.5 every non-zero idempotent is an identity.

Since N is regular,

a = aba for some b ε N $\cdots\cdots$ (1)

a = (ba)a

That is inverse exists for every a ε N.

Hence N is a field.The converse is obvious.

Theorem:2.7

Let N be a regular quasi weak commutative near-ring.Then N is isomorphic to a subdirect product of fields.

Proof:

By Result1.21[4] N is isomorphic to a subdirect product of subdirectly irreducible quasi weak commutative near-rings N_k's, each N_k is regular and quasi weak commutative. Then the proof follows from the above corollary.

Corollary:2.8

Let N be a regular quasi weak commutative near-ring.Then N has no non-zero zero divisors if and only if N is a field.

Proof:

Follows from the theorem.

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Received: October 10, 2014; *Accepted*: March 23, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 3(1)(2015) 335–345

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Effect of Magnetic field on Herschel-Bulkley fluid through multiple stenoses

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Abstract

A mathematical model for electrically conducting flow of Herschel-Bulkley fluid through a uniform tube of multiple stenoses has been studied. Analytical solutions of resistance to the flow and wall shear stress have been calculated. It is found that the resistance to the flow increases with the heights of the stenoses, power law index, volumetric flow rate, radius of the plug core-region and yield stress, but decreases with induced magnetic field and shear stress. It is also observed that the wall shear stress is increasing with the heights of the stenoses and radius of the plug core-region.

Keywords: Multiple stenoses, Herschel-Bulkley fluid, Magnetic field.

2010 MSC: 92C10,92C30,76S05.

1 Introduction

Diseases in the blood vessels and in the heart, such as heart attacks and strokes, are the major mortality worldwide. The underlying cause for these events is the formation of lesions, known as atherosclerosis. These lesions and plaques can grow and occlude the artery and hence prevent blood supply to the distal bed. Plaques with calcium in them can also rupture and initiate formation of blood clots (thrombus). The clots can form as emboli and occlude the smaller vessels that can also result in interruption of blood supply to the distal bed. Plaques formed in coronary arteries can lead to heart attacks and clots in the risk factors for the presence of atherosclerotic lesions.

Hence the formation of stenosis/ atherosclerosis is found to be largely responsible for the cause of several vascular diseases. Thus a proper knowledge of the flow characteristics of blood in such blood vessels may lead to better understanding of the development of these diseases. This in turn may help in proper diagnosis of such diseases and design and development of improvised artificial organs.

In view of this, a number of researchers have studied different aspects of blood flow analysis in arteries. Young [1], Lee and Fung [2],Padmanabhan [3] have studied the flow of blood in stenosed artery by considering blood as a Newtonian fluid. Blood behaves cerebral circulation can result in a stroke. There are number of differently when flowing in large vessels, in which Newtonian behavior is expected and in small vessels where non-Newtonian effects appear Buchanan *et al.*[4], Mandal[5],Ismail *et al.* [6], Radhakrishnamacharya [7]. In small vessels blood behaves like a Herschel-Bulkley fluid rather than Power law and Bingham fluids

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Chaturani and Samy [8]. But the blood can be regarded as magnetic fluid in which red blood cells are magnetic in nature. Liquid carries in the blood contain magnetic suspension of the particle Tzirtzilakis [9].

The MHD principles may be used to de accelerate the flow of blood in a human arterial system and is useful in the treatment of certain cardiovascular disorders and in the diseases which accelerate blood circulation like hemorrhage and hypertension etc. Das and Saha [10].

The effect of magnetic field on blood flow has been analyzed by treating blood as an electrically conductive fluid Chen [11]. Ogulua and Abbey [12] studied the effects of heat and magnetic field on blood through constricted artery. Shaw *et al.* [13] have shown the influence of the externally imposed body acceleration on the flow of blood through an asymmetric stenosed artery by considering blood as Casson fluid. Bali and Awasti [14] Studied the effect of an externally applied uniform magnetic field on the multi-stenosed artery. Sankar and Lee [15] have shown the effect of magnetic field in the pulsatile flow of blood through narrow arteries treating blood as Casson fluid. Recently Lokendra Pramar *et al.* [16] studied the role of magnetic field intensity through overlapping stenosis. Bhargva *et al.* [17] Showed that the magnetic field can be used as a flow control mechanism in medical applications.

With this motivation, a mathematical model on the effect of magnetic field on Herschel-Bulkley fluid through a uniform tube with two stenoses is developed. Expressions for the velocity, resistance to the flow and wall shear stress have been calculated by assuming that the stenosis to be mild. The effects of various parameters on these variables have been investigated

2 Mathematical Formulation

Consider the steady flow of an electrically conducting Herschel Bulkley fluid through a tube of uniform cross section with two stenoses. Assuming that the flow is axi-symmetric and the stenosis over a length of the artery have been developed in axi-symmetric manner. Let the length of the tube is L, the magnitude of the distance along the artery over which the stenosis is spread out be L_i, the locations of the stenosis be indicated by d_i and the maximum heights of the stenosis δ_i (where i=1,2). Here we consider the transverse magnetic field since the bio-magnetic fluid (blood) is subjected to a magnetic field. The schematic diagram is shown in Figure -1. The cylindrical polar coordinates (z,r,θ) is chosen so that the z -axis coincides with the axis of the tube

The radius of the cylindrical tube is given as

$$\overline{h} = \frac{\overline{R}(z)}{\overline{R}_0} = \begin{cases} 1 & 0 \leq \overline{z} \leq d_1 \\ 1 - \frac{\overline{\delta}_1}{2}\left(1 + Cos\frac{2\pi}{L_1}\left(\overline{z} - d_1 - \frac{L_1}{2}\right)\right) & d_1 \leq \overline{z} \leq d_1 + L_1 \\ 1 & d_1 + L_1 \leq \overline{z} \leq d_2 \\ 1 - \frac{\overline{\delta}_2}{2}\left(1 + Cos\frac{2\pi}{L_2}\left(\overline{z} - d_2 - \frac{L_2}{2}\right)\right) & d_2 \leq \overline{z} \leq d_2 + L_2 \\ 1 & d_2 + L_2 \leq \overline{z} \leq L \end{cases} \tag{2.1}$$

Where R(z) is the radius of the tube with stenosis ,$R_0(z)$ is the radius of the tube without stenosis, r_o is the radius of the plug flow region.
The basic momentum equation governing the flow is (Rekha Bali *et al.* [13])

$$-\frac{\partial\overline{p}}{\partial\overline{z}} + \frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}(\overline{r}\overline{\tau}_{rz}) + \overline{\mu_0}\overline{M}(\frac{\partial\overline{H}}{\partial\overline{z}}) = 0 \tag{2.2}$$

Where $\overline{\tau}_{rz}$ is the shear stress for Herschel Bulkley fluid, is given by

$$\overline{\tau}_{rz} = (\frac{-\partial\overline{u}}{\partial\overline{r}})^n + \overline{\tau}_0, \text{ if } \overline{\tau}_{rz} \geq \overline{\tau}_0 \tag{2.3}$$

$$\frac{\partial\overline{u}}{\partial\overline{r}} = 0, \; if \; \overline{\tau}_{rz} < \overline{\tau}_0 \tag{2.4}$$

where $\overline{r}, \overline{z}$ denote the radial and axial coordinates respectively, $\overline{\mu_0}$ magnetic permeability, $\overline{M}$ magnetization,

$\overline{H}$ magnetic field intensity, $\overline{p}$ pressure,$\overline{u}$ is the velocity of the fluid, $\overline{\tau}_{rz}$ stress,$\overline{\tau}_0$ yield stress.

When $\overline{\tau}_{rz} < \overline{\tau}_0$ i.e. the shear stress is less than yield stress, there is a core region which flows as a plug (FIG.1), and Eq. (2.4) corresponds to vanishing velocity gradient in that region. However the fluid behavior is indicated whenever $\overline{\tau}_{rz} > \overline{\tau}_0$.
The boundary conditions are

$$\overline{\tau}_{rz}\ is\ finite\ at\ \overline{r} = 0 \tag{2.5}$$

$$\overline{u} = 0,\ at\ \overline{r} = \overline{h}(z) \tag{2.6}$$

Introducing the following non-dimensional quantities

$$\overline{z} = \frac{z}{L}, \overline{\delta} = \frac{\delta}{R_0}, \overline{R}(z) = \frac{R(z)}{R_0}, \overline{P} = \frac{P}{\mu UL/R_0^2}, \overline{\tau_0} = \frac{\tau_0}{\mu(U/R_0)}, \overline{\tau}_{rz} = \frac{\tau_{rz}}{\mu(U/R_0)}, \overline{Q} = \frac{Q}{(\pi R_0^2 U)}, \overline{H} = \frac{H}{H_0} \tag{2.1}$$

Where H_0 is external transerverse uniform constant magnetic field.

Using the non-dimensional scheme the governing equations from (2.1)-(2.6) can be written as
The radius of the cylindrical tube is given as

$$\overline{h} = \frac{R(z)}{R_0} = \begin{cases} 1 & 0 \le z \ \le d_1 \\ 1 - \frac{\delta_1}{2}\left(1 + Cos\frac{2\pi}{L_1}\left(z - d_1 - \frac{L_1}{2}\right)\right) & d_1 \le z \le d_1 + L_1 \\ 1 & d_1 + L_1 \le z \le d_2 \\ 1 - \frac{\delta_2}{2}\left(1 + Cos\frac{2\pi}{L_2}\left(z - d_2 - \frac{L_2}{2}\right)\right) & d_2 \le z \le d_2 + L_2 \\ 1 & d_2 + L_2 \le z \le L \end{cases} \tag{2.7}$$

$$-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \mu_0 M(\frac{\partial H}{\partial z}) = 0 \tag{2.8}$$

Where

$$\tau_{rz} = (\frac{-\partial u}{\partial r})^n + \tau_0,\ if\ \tau_{rz} \ge \tau_0$$

$$\frac{\partial u}{\partial r} = 0,\ if\ \tau_{rz} < \tau_0 \tag{2.9}$$

The following restrictions for mild stenoses (MARUTHI PRASAD et al. [7]), are supposed to be satisfied.

$$\delta_i \ll min(R_0, R_out)$$

$$\delta_i \ll L_i,$$

where R_out=R(z) at $z = L$
Here L_i,δ_i (i=1, 2) are the lengths and maximum heights of the two stenoses. (The suffixes 1 and 2 refer to the first and second stenosis respectively).
The boundary conditions (2.5) and (2.6) becomes

$$\tau_{rz} is finite at r = 0 \tag{2.10}$$

$$u = 0, at r = h(z) \tag{2.11}$$

3 Solution

The solution of equation (2.8) under the boundary conditions (2.10) and (2.11), the velocity is obtained as

$$U=\frac{h^{k+1}(p-F)^k}{2^k(k+1)}\{(1-\frac{2\tau_0}{h(p-f)})^{k+1}-(\frac{r}{h}-\frac{2\tau_0}{h(p-f)})^{k+1}\},\ for\ \ r_0\leq r\leq h \tag{3.1}$$

Where $\frac{\partial p}{\partial z}=P,\ f_1(\frac{\partial H}{\partial z})=F,\ k=\frac{1}{n}$ Using the boundary condition (2.9), the upper limit of the plug flow region (i.e. the region $0\leq r\leq r_0$) for which $\tau_r z<\tau_0$ is obtained as

$$r_0=\frac{2\tau_0}{(P-F)} \tag{3.2}$$

Using the condition that $\tau_r z=\tau_h$, at r=h ,

$$\frac{r_0}{h}=\frac{\tau_0}{\tau_h}=\tau,\ \ for\ 0<\tau<1 \tag{3.3}$$

Taking r=r_0 in Eq. (3.1), the plug core velocity

$$u_p=\frac{h^{k+1}(P-F)^k}{2^k(k+1)}(1-\frac{r_0}{h})^{k+1},\ for\,0\leq r\leq r_0 \tag{3.4}$$

The volume flow rate is defined by

$$Q=2[\int_0^{r_0}u_p r d_r+\int_{r_0}^{h}u r d_r] \tag{3.5}$$

On integrating,

$$Q=A((k+2)(k+3)(1-\frac{r_0}{h})^{k+1}-2(k+3)(1-\frac{r_0}{h})^{k+2}+2(1-\frac{r_0}{h})^{k+3}) \tag{3.6}$$

where $A=\frac{h^{(k+3)}(P-F)^k}{2^k(k+1)(k+2)(k+3)}$

$$From Eq.(3.6),P-F=\frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{h^{1+\frac{3}{k}}\{(k+2)(k+3)(1-\frac{r_0}{h})^{k+1}-2(k+3)(1-\frac{r_0}{h})^{k+2}+2(1-\frac{r_0}{h})^{k+3}\}^{\frac{1}{k}}} \tag{3.7}$$

$$\frac{dp}{dz}=\frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{h^{1+\frac{3}{k}}\{(k+2)(k+3)(1-\frac{r_0}{h})^{k+1}-2(k+3)(1-\frac{r_0}{h})^{k+2}+2(1-\frac{r_0}{h})^{k+3}\}^{\frac{1}{k}}}+F \tag{3.8}$$

When k=1,H=0 and $\tau_0\rightarrow 0$ Eq. (3.8) reduces to the results of YOUNG [1] .
The pressure drop Δp across the stenosis between z=0 to z=1 is obtained by integrating Eq. (3.8), as

$$\Delta p=\int_0^1(\frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{h^{1+\frac{3}{k}}\{(k+2)(k+3)(1-\frac{r_0}{h})^{k+1}-2(k+3)(1-\frac{r_0}{h})^{k+2}+2(1-\frac{r_0}{h})^{k+3}\}^{\frac{1}{k}}}+F)dz \tag{3.9}$$

The resistance to the flow,λ, is defined by

$$\lambda=\frac{\Delta p}{Q}=\frac{1}{Q}\int_0^1(\frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{h^{1+\frac{3}{k}}\{(k+2)(k+3)(1-\frac{r_0}{h})^{k+1}-2(k+3)(1-\frac{r_0}{h})^{k+2}+2(1-\frac{r_0}{h})^{k+3}\}^{\frac{1}{k}}}+F)dz \tag{3.10}$$

the pressure drop in the absence of stenosis (h=1) is denoted by ΔP_N, is obtained from Eq. (3.9).

$$\Delta P_N=\int_0^1(\frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{\{(k+2)(k+3)(1-r_0)^{k+1}-2(k+3)(1-r_0)^{k+2}+2(1-r_0)^{k+3}\}^{\frac{1}{k}}}+F)dz \tag{3.11}$$

The resistance to the flow in the absence of stenosis is denoted by λ_N is obtained from Eq. (3.10) as

$$\lambda_N = \frac{\Delta P_N}{Q} = \frac{1}{Q}\int_0^1 \left(\frac{2Q^{\frac{1}{k}}\left[(k+1)(k+2)(k+3)\right]^{\frac{1}{k}}}{\left\{(k+2)(k+3)(1-r_0)^{k+1} - 2(k+3)(1-r_0)^{k+2} + 2(1-r_0)^{k+3}\right\}^{\frac{1}{k}}} + F\right) dz \quad (3.12)$$

The normalized resistance to the flow denoted by

$$\overline{\lambda} = \frac{\lambda}{\lambda_N} \quad (3.13)$$

And the wall shear stress

$$\tau_h = \frac{h}{2}\frac{dp}{dz} \quad (3.14)$$

4 Results

The expressions for velocity (u), core velocity(u_p), volumetric flow rate (Q) ,resistance to the flow ($\overline{\lambda}$) and wall shear stress(τ_h)are given by the equations (3.1,3.4,3.6,3.13,3.14). The effects of various parameters on the resistance to the flow (($\overline{\lambda}$)) , wall shear stress (τ_h) have been computed numerically by using mathematica 8.1 and results are shown graphically in Fig.2-14, by taking d_1=0.2,d_2=0.6,L_1=L_2=0.2,L=1.

It is observed that the resistance to the flow increases with the heights of the stenosis (δ_1,δ_2) (fig.2-10).It can be seen from the fig 2-3 that, the resistance to the flow increases with the power law index (k=1/n) along with the heights of the primary and secondary stenosis (δ_1,δ_2).It is interesting to note that the increase in resistance is significant only when the height of the second stenosis exceeds the value 0.02.

From, Fig.7 & 8 it is observed that the resistance to the flow increases with volumetric flow rate (Q), radius of plug core region (r_0) (Fig.9 & 10) and yield stress (τ_0) (Fig.11).

It is interesting to observed that the resistance to the flow decreases with the increase of the magnetic field (**H**) (Figs.4 & 5), and it is also seen that resistance to the flow is more in non-Newtonian fluid than the Newtonian fluid (Fig.6).

The effects various parameters on shear stress are shown in (Figs. 12-14). It is noted that the wall shear stress is increasing with the heights of the stenoses and the radius of the plug-core region.

5 Conclusion

A mathematical model for electrically conducting flow of Herschel-Bulkley fluid through a uniform tube of multiple stenoses has been studied. It is observed that the resistance to the flow increases with the heights of the stenoses, power law index, volumetric flow rate, radius of the plug core-region and yield stress, but decreases with induced magnetic field and shear stress. It is also observed that the wall shear stress is increasing with the heights of the stenoses and radius of the plug core-region.

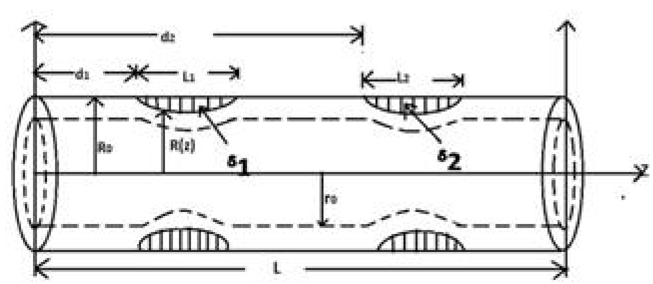

Figure 1: Schematic diagram of multiple stenosed artery

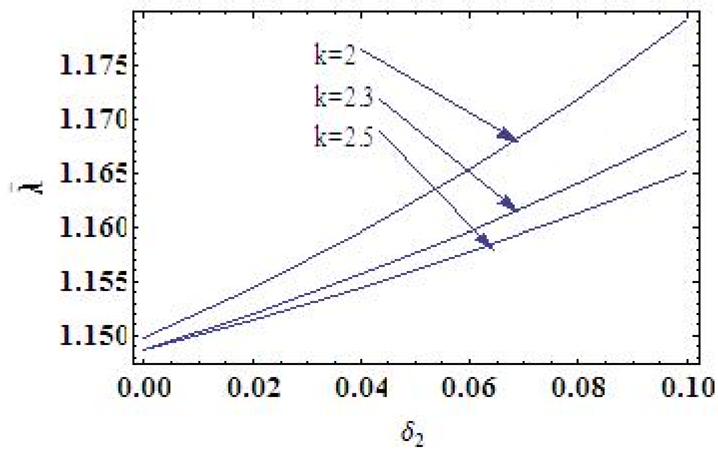

Figure 2: Variation of impedance $\overline{\lambda}$ with δ_2 for different k (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,Q=0.1,δ_1=0.0,r_0=0.2)

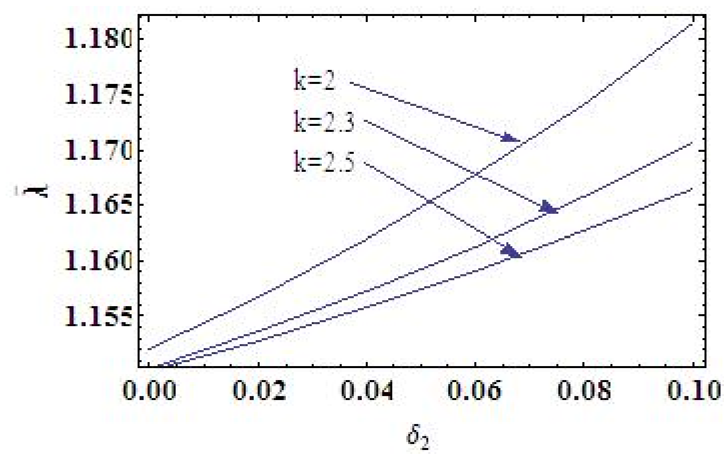

Figure 3: Variation of impedance $\overline{\lambda}$ with δ_2 for different k (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,Q=0.1,δ_1=0.01,r_0=0.2)

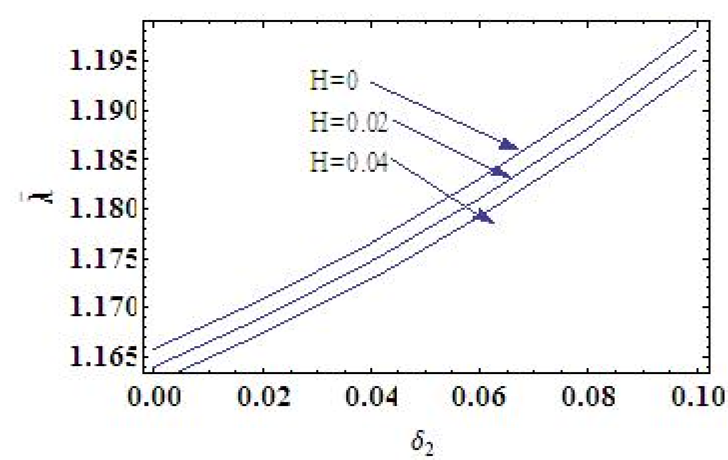

Figure 4: Variation of impedance $\overline{\lambda}$ with δ_2 for different H (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,Q=0.1,k=2,δ_1=0.0,r_0=0.2)

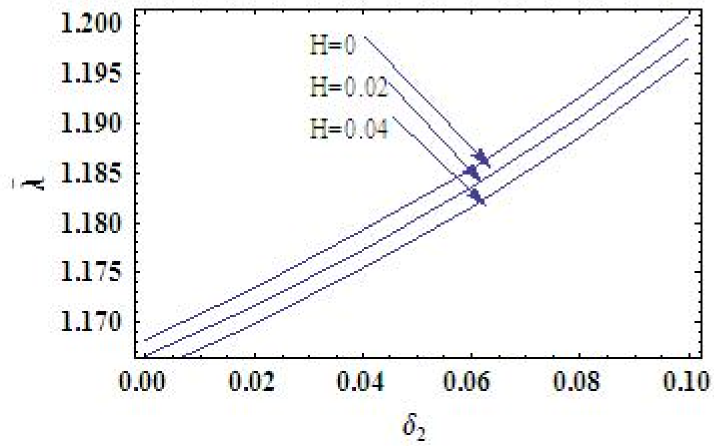

Figure 5: Variation of impedance $\overline{\lambda}$ with δ_2 for different H (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,Q=0.1,k=2,δ_1=0.01,r_0=0.2)

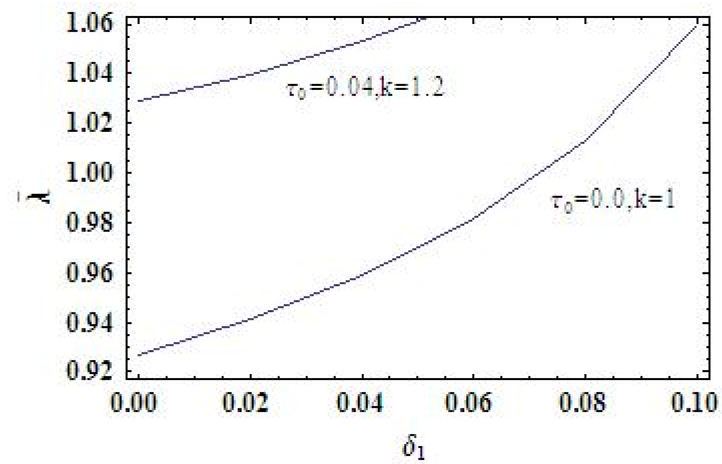

Figure 6: Comparison of magnetic field effect on Newtonian and non-Newtonian fluids. (d_1=0.2,d_2=0.6,L_1=L_2=0.2,L=1,Q=0.1,H=0.2,δ_1=0.01)

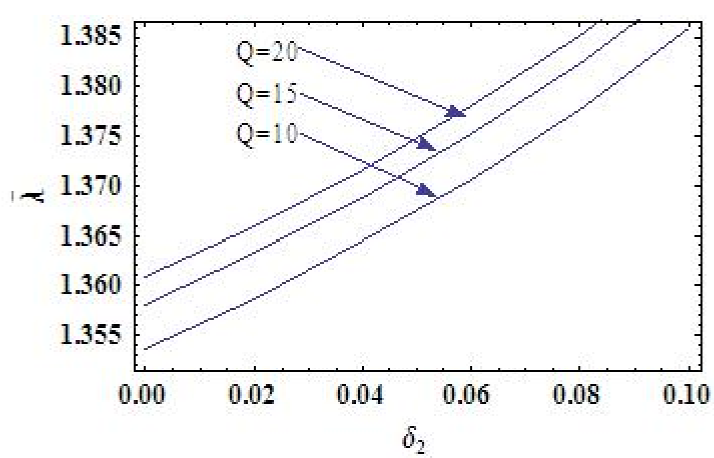

Figure 7: Variation of impedance $\overline{\lambda}$ with δ_2 for different Q (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,k=2,r_0=0.2,δ_1=0.0)

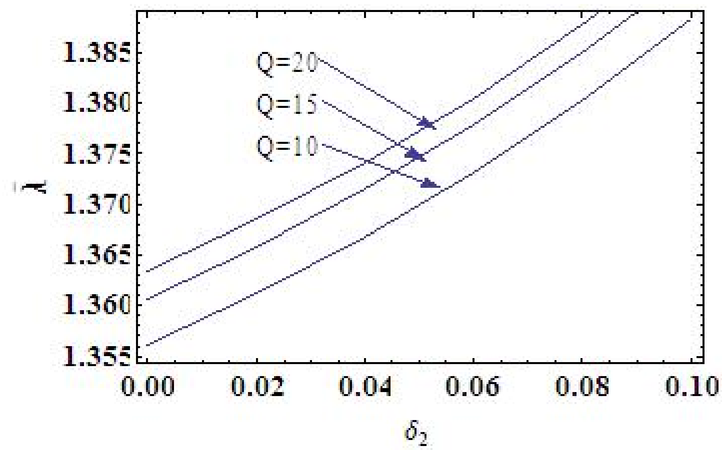

Figure 8: Variation of impedance $\overline{\lambda}$ with δ_2 for different Q (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,k=2,r_0=0.2,δ_1=0.01)

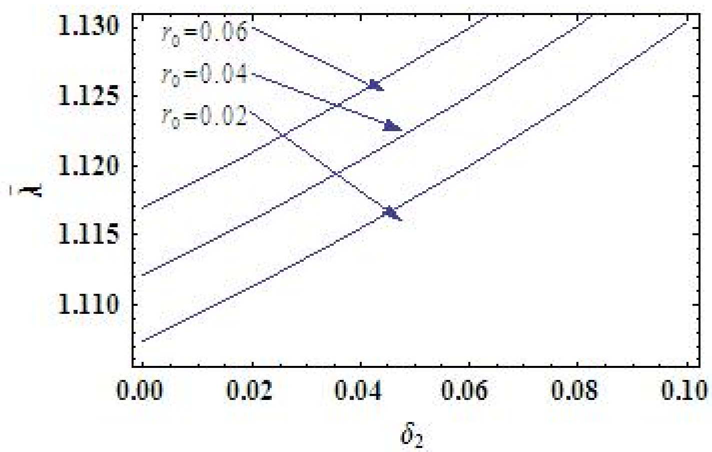

Figure 9: Variation of impedance $\overline{\lambda}$ with δ_2 for different r_0 (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,k=2,δ_1=0.0,Q=0.1)

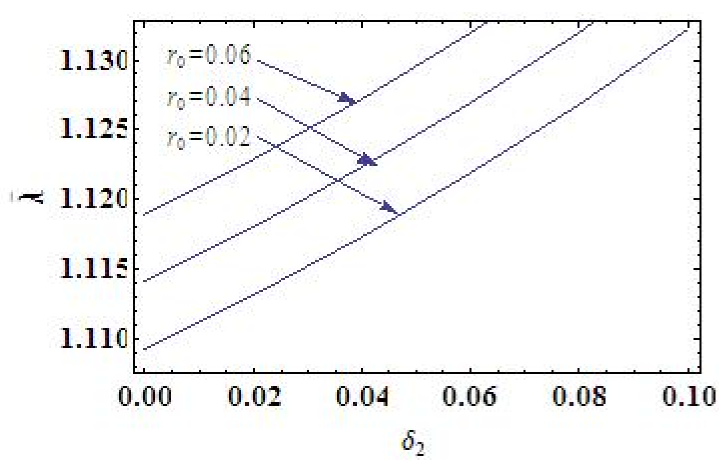

Figure 10: Variation of impedance $\overline{\lambda}$ with δ_2 for different r_0 (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,k=2,δ_1=0.01,Q=0.1)

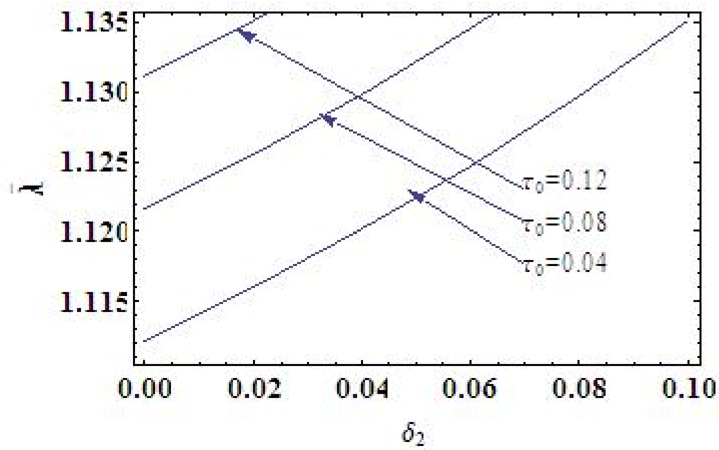

Figure 11: Variation of impedance $\overline{\lambda}$ with δ_2 for different τ_0 (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,k=2,δ_1=0.0,Q=0.1,τ_h=1)

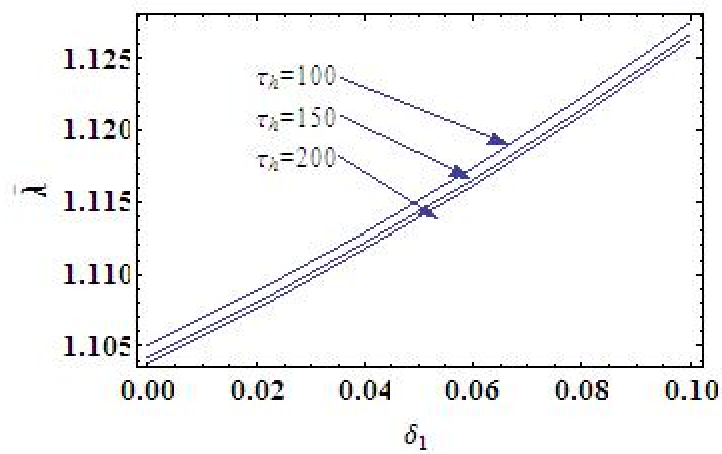

Figure 12: Variation of impedance $\overline{\lambda}$ with δ_1 for different τ_h (d_1=0.2,d_2=0.6,H=0.2,L_1=L_2=0.2,L=1,k=2,δ_2=0.0,Q=0.1,τ_0=1)

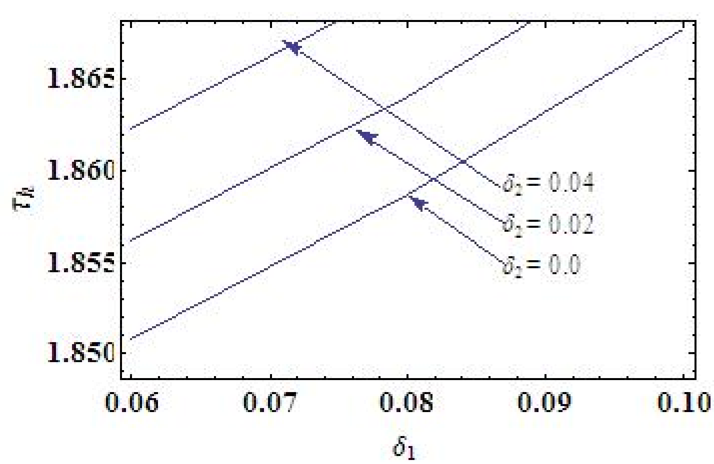

Figure 13: Variation of wall shear stress τ_h with δ_1 for different δ_2 (r_0=0.02,k=2,Q=0.01)

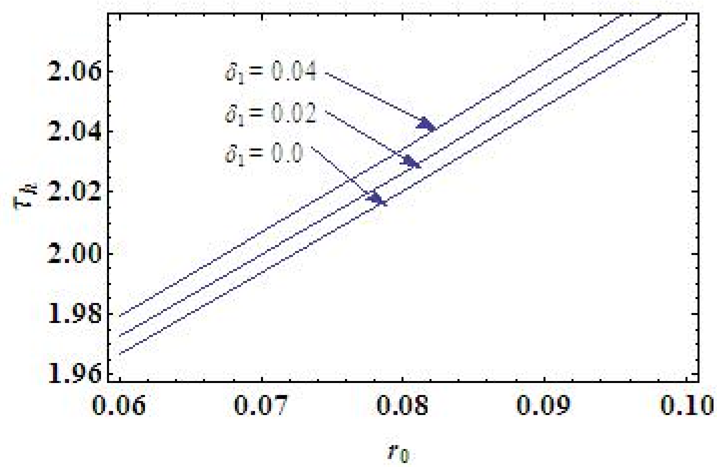

Figure 14: Variation of wall shear stress τ_h with r_0 for different δ_1 (δ_2=0.1,k=2,Q=0.01)

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Malaya J. Mat. 3(3)(2015) 346-359

Malaya Journal of Matematik

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an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Hermite-Hadamard Inequalities for L(j)-convex Functions and S(j)-convex Functions

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Abstract

In this article, Hermite-Hadamard Inequalities for L(j)-convex functions are analyzed. S(j)-convex functions which is founded upon $\mathbb{B}^{-1}-$convexity concept, are defined and for this functions, Hermite-Hadamard Inequalities are investigated. On some special domains, concrete form of inequalities are denoted.

Keywords: Hermite-Hadamard inequalities, L(j)-convex functions, S(j)-convex functions, abstract convexity.

2010 MSC: 26B25, 26D15, 26D07, 52A40, 52A20.

1 Introduction

Integral inequalities have played an important role in the development of all branches of mathematics. Also, Hermite-Hadamard inequalities are one of the integral inequalities. Recently, Hermite-Hadamard inequalities and their applications have attracted considerable interest. Hence the Hermite-Hadamard inequalities have been studied for varied families of functions which are obtained by many authors. (e.g. [1], [5], [6], etc.)

In this paper, we examine Hermite-Hadamard Type Inequalities for L(j)-convex functions. L(j)-convex functions are founded upon the $\mathbb{B}$-convexity concept in $\mathbb{R}^n_+$ [2] (Section 3). In section 4, S(j)-convex functions which is related to $\mathbb{B}^{-1}-$convexity concept are defined. After, for this family of functions, Hermite-Hadamard Type Inequalities are analyzed (Section 5). Additionally, different examples about both cases are discussed and studied.

2 L(j)-convex Functions

The sets which are given the following forms, are discussed to define the L(j)-convex functions [2]. For all $z \in \mathbb{R}^n_{++}$

$$N_0(z) = \left\{x \in \mathbb{R}^n_{++} : 0 < x_i \leq z_i, \quad i = \overline{1,n}\right\}$$
$$N_j(z) = \left\{x \in \mathbb{R}^n_{++} : z_j \leq x_j \quad \text{and} \quad x_i z_j \leq z_i x_j, \forall i = \overline{1,n}\right\}, j = \overline{1,n}.$$

$N_0(z)$ is closed, convex and radiant set, $N_j(z)$ $\left(j = \overline{1,n}\right)$ are closed, convex and co-radiant sets [4].

Using these sets, $(n+1)$ relations are defined as follows ([2]): for $x, y \in \mathbb{R}^n_{++}$

$$x \preceq_0 y \Leftrightarrow x \in N_0(y)$$
$$x \preceq_j y \Leftrightarrow y \in N_j(x), \qquad j = \overline{1,n}.$$

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$\preceq_j, j = \overline{0,n}$ are partial order relation on $\mathbb{R}^n_{++}$ (see [4]).

We can write Minkowski functions according to $N_j(y)$ $\left(y \in \mathbb{R}^n_{++},\ j = \overline{0,n}\right)$ sets and $\preceq_j$ order relations. For $y \in \mathbb{R}^n_{++}$, $N_0(y)$ is radiant set and $\preceq_0$ is coordinate-wise order relation hence Minkowski gauge is

$$\mu_{N_0(y)}(x) := \inf\{\alpha > 0 : x \in \alpha N_0(y)\} = \inf\{\alpha > 0 : x \preceq_0 \alpha y\}.$$

Let us show this function with $l_{0,y}$, namely

$$l_{0,y}(x) := \mu_{N_0(y)}(x), \qquad x \in \mathbb{R}^n_{++}.$$

For $j = \overline{1,n}$ and $y \in \mathbb{R}^n_{++}$, the sets $N_j(y)$ are co-radiant, thus Minkowski co-gauges are defined by

$$\upsilon_{N_j(y)}(x) := \sup\{\alpha : x \in \alpha N_j(y)\} = \sup\left\{\alpha : \alpha y \preceq_j x\right\}$$

we denote these functions with $l_{j,y}$, namely

$$l_{j,y}(x) := \upsilon_{N_j(y)}(x), \qquad x \in \mathbb{R}^n_{++}.$$

Remark 2.1. *Let $y \in \mathbb{R}^n_{++}$ and $j = \overline{1,n}$. Then the sets $N_j(y)$ coincides with the intersection of the cone*

$$V_j(y) = \left\{x \in \mathbb{R}^n_+ : \frac{x_i}{y_i} \le \frac{x_j}{y_j} \quad \left(i = \overline{1,n}\right)\right\}$$

and the half-space

$$H_j(y) = \{x \in \mathbb{R}^n : x_j \ge y_j\}.$$

Using the cone $V_j(y)$, $l_{j,y}$ can be shown another form. If $x \in V_j(y)$, then

$$l_{j,y}(x) = \sup\left\{\alpha : \alpha y \preceq_j x\right\} = \sup\{\alpha : \alpha y_j \le x_j\} = \frac{x_j}{y_j}.$$

If $x \notin V_j(y)$, then for all $\alpha > 0$ the inequality $\alpha y \preceq_j x$ does not hold therefore $l_{j,y}(x) = 0$. Consequently,

$$l_{j,y}(x) = \begin{cases} \frac{x_j}{y_j}, & x \in V_j(y) \\ 0, & x \notin V_j(y) \end{cases}.$$

For $j = \overline{0,n}$, let us analyze the convexity with respect to the family of functions $L(j) = \left\{l_{j,y} : y \in \mathbb{R}^n_{++}\right\}$.

Definition 2.1. *Let $j = \overline{0,n}$. A function $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ is an IPH(j) function if f is positively homogeneous of degree one and increasing according to order relation $\preceq_j$.*

Theorem 2.1. *For all $j = \overline{0,n}$ and $y \in \mathbb{R}^n_{++}$, $l_{j,y}$ functions are IPH(j) functions.*

Proof. For $j = 0$

$$\begin{aligned} l_{0,y}(\lambda x) &= \inf\{\alpha > 0 : \lambda x \in \alpha N_0(y)\} = \inf\{\alpha > 0 : \lambda x \preceq_0 \alpha y\} \\ &= \inf\left\{\alpha > 0 : x \preceq_0 \frac{\alpha}{\lambda} y\right\} = \lambda \inf\{\alpha' > 0 : x \preceq_0 \alpha' y\} = \lambda l_{0,y}(x). \end{aligned}$$

For $j = \overline{1,n}$

$$\begin{aligned} l_{j,y}(\lambda x) &= \sup\{\alpha : \lambda x \in \alpha N_j(y)\} = \sup\left\{\alpha : \alpha y \preceq_j \lambda x\right\} \\ &= \sup\left\{\alpha : \frac{\alpha}{\lambda} y \preceq_j x\right\} = \lambda \sup\left\{\alpha' : \alpha' y \preceq_j x\right\} = \lambda l_{j,y}(x). \end{aligned}$$

Namely, $l_{j,y}$ $\left(j = \overline{0,n}\right)$ are positively homogeneous of degree one.

Now, let us prove that the functions $l_{j,y}$ $\left(j = \overline{0,n}\right)$ are increasing. Let $j = 0$. If $x_1 \preceq_0 x_2$, then $\{\alpha > 0 : x_2 \preceq_0 \alpha y\} \subset \{\alpha > 0 : x_1 \preceq_0 \alpha y\}$ and hence $l_{0,y}(x_1) \le l_{0,y}(x_2)$. For $j = \overline{1,n}$, if $x_1 \preceq_j x_2$, then $\left\{\alpha > 0 : \alpha y \preceq_j x_1\right\} \subset \left\{\alpha > 0 : \alpha y \preceq_j x_2\right\}$ and thus $l_{j,y}(x_1) \le l_{j,y}(x_2)$. □

Following theorem can be proved using Corollary 2.6 in [2].

Theorem 2.2. *The function $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is L(j)-convex function $\left(j = \overline{0,n}\right)$ if and only if f is IPH(j) function.*

Moreover, some important properties of IPH(j) functions are given, in [2].

3 Hermite-Hadamard Type Inequalities for L(j)-convex Functions

We begin with the following theorem which has an important role in Hermite-Hadamard Type Inequalities for L(j)-convex functions [2].

Theorem 3.3. *For $j=\overline{1,n}$ and $p:\mathbb{R}^n_{++}\to\mathbb{R}_{+\infty}$, the following statements are equivalent:*
(i) p is an IPH(j) function.
(ii) $p(x)\geq \lambda p(y)$ for all $\forall x,y\in\mathbb{R}^n_{++}$ and $\lambda>0$ such that $\lambda y\preceq_j x$.
(iii) $p(x)\geq l_{j,y}(x)\,p(y)$ for all $\forall x,y\in\mathbb{R}^n_{++}$.

We can obtain Hermite-Hadamard Type Inequalities for L(j)-convex functions as a corollary of the above theorem.

Corollary 3.1. *Let $D\subset\mathbb{R}^n_{++}$, $p:D\to\mathbb{R}_{+\infty}$ be a L(j)-convex function and integrable function on D. Then, for all $y\in D$, we have*

$$p(y)\int_D l_{j,y}(x)\,dx\leq\int_D p(x)\,dx. \tag{3.1}$$

Let us investigate Hermite-Hadamard Type Inequalities via $Q(D)$ sets given in [6].

Let $D\subset\mathbb{R}^n_{++}$ be bounded and hold condition of $cl\,(intD)=D$. We denote by $Q(D)$ the sets of all $x^*\in D$ such that

$$\frac{1}{A(D)}\int_D l_{j,x^*}(x)\,dx=1 \tag{3.2}$$

where $A(D)=\int_D dx$

Theorem 3.4. *Let p be L(j)-convex function defined on D and integrable on D. If $Q(D)$ is nonempty, then one has the inequality:*

$$\sup_{x^*\in Q(D)}p(x^*)\leq\frac{1}{A(D)}\int_D p(x)\,dx \tag{3.3}$$

Proof. If $p(x^*)=+\infty$, then by using $p(x)\geq l_{j,y}(x)\,p(y)$, it can be shown that p cannot be integrable. It conflicts integrable of p. So $p(x^*)<+\infty$. From Theorem 3.3 (iii), for all $x\in D$

$$p(x)\geq l_{j,x^*}(x)\,p(x^*)\,.$$

Since $x^*\in Q(D)$, by (3.2)

$$\begin{aligned}p(x^*)&=p(x^*)\frac{1}{A(D)}\int_D l_{j,x^*}(x)\,dx\\&=\frac{1}{A(D)}\int_D p(x^*)\,l_{j,x^*}(x)\,dx\leq\frac{1}{A(D)}\int_D p(x)\,dx.\end{aligned}$$

□

Remark 3.2. *As it is clear that, for each $x^*\in Q(D)$, inequality*

$$p(x^*)\leq\frac{1}{A(D)}\int_D p(x)\,dx \tag{3.4}$$

is hold. If we get $p(x)=l_{j,x^*}(x)$, (3.4) is an equality.

Let p be a L(j)-convex function defined on $D\subset\mathbb{R}^n_{++}$ and be integrable on D. For all $x,y\in D$, the inequality

$$p(x)\geq l_{j,y}(x)\,p(y)$$

is hold. Hence,

$$p(y)\leq\varphi_{j,x}(y)\,p(x) \tag{3.5}$$

where

$$\varphi_{j,x}(y)=\frac{1}{l_{j,y}(x)}=\begin{cases}\frac{y_j}{x_j}, & x\in V_j(y)\\ \infty, & x\notin V_j(y)\end{cases}=\begin{cases}\frac{y_j}{x_j}, & y\notin intV_j(x)\\ \infty, & y\in intV_j(x)\end{cases}.$$

The following theorem can be proved, using the inequality (3.5).

Theorem 3.5. *Let $D \subset \mathbb{R}^n_{++}$, $p : D \to \mathbb{R}_{+\infty}$ be an integrable, L(j)-convex function and $D \cap intV_j(y) = \emptyset$. Then, the following inequality holds:*

$$\int_D p(x)\,dx \le p(y) \int_D \varphi_{j,y}(x)\,dx \tag{3.6}$$

for all $y \in D$.

Examples:

On some special domains of $\mathbb{R}^2_{++}$, Hermite-Hadamard Type Inequalities for L(j)-convex functions have been implied with concrete form.

Firstly, for $D \subset \mathbb{R}^2_{++}$ and every $y \in D$, let us derive computation formula of the integral $\int_D l_{j,y}(x)\,dx$.

Let $D \subset \mathbb{R}^2_{++}$ and $y = (y_1, y_2) \in D$. Then, on $\mathbb{R}^2_{++}$

$$V_1(y) = \left\{ x \in \mathbb{R}^2_{++} : \frac{x_2}{y_2} \le \frac{x_1}{y_1} \right\}, \quad V_2(y) = \left\{ x \in \mathbb{R}^2_{++} : \frac{x_1}{y_1} \le \frac{x_2}{y_2} \right\}$$

and

$$l_{1,y}(x) = \begin{cases} \frac{x_1}{y_1}, & x \in V_1(y) \\ 0, & x \notin V_1(y) \end{cases}, \quad l_{2,y}(x) = \begin{cases} \frac{x_2}{y_2}, & x \in V_2(y) \\ 0, & x \notin V_2(y) \end{cases}.$$

Let $V_j^c(y)\,(j = 1, 2)$ be the complement of $V_j(y)\,(j = 1, 2)$. Therefore, with the above assumptions, we can separate the region D into two regions: $D_j(y) = D \bigcap V_j(y)$ and $D \setminus D_j(y) = D \bigcap V_j^c(y)$. Thus, we have

$$\begin{aligned} \int_D l_{j,y}(x)\,dx &= \int_{D_j(y)} l_{j,y}(x)\,dx + \int_{D \setminus D_j(y)} l_{j,y}(x)\,dx \\ &= \int_{D_j(y)} \frac{x_j}{y_j} dx + \int_{D \setminus D_j(y)} 0 dx = \frac{1}{y_j} \int_{D_j(y)} x_j dx. \end{aligned}$$

Example 3.1. *Consider the triangle D defined as*

$$D = \left\{ (x_1, x_2) \in \mathbb{R}^2_{++} : 0 < x_1 \le a, 0 < x_2 \le v x_1 \right\}.$$

For $y \in D$, $D_j(y)$ would be as follows:

$$\begin{aligned} D_1(y) &= \left\{ x \in D : 0 < x_1 \le a,\ 0 < x_2 \le \frac{y_2}{y_1} x_1 \right\} \\ D_2(y) &= \left\{ x \in D : 0 < x_1 \le a,\ \frac{y_2}{y_1} x_1 < x_2 \le v x_1 \right\}. \end{aligned}$$

For $j = 1$; we deduce that:

$$\int_D l_{1,y}(x)\,dx = \frac{1}{y_1} \int_{D_1(y)} x_1 dx = \frac{1}{y_1} \int_0^a \int_0^{\frac{y_2 x_1}{y_1}} x_1 dx_2 dx_1 = \frac{y_2}{y_1^2} \frac{a^3}{3}.$$

Hence, for the given region D, the inequality (3.1) will be as follows:

$$p(y_1, y_2) \le \frac{3y_1^2}{a^3 y_2} \int_D p(x_1, x_2)\,dx_1 dx_2.$$

For $j = 2$; we have

$$\begin{aligned} \int_D l_{2,y}(x) dx &= \frac{1}{y_2} \int_{D_2(y)} x_2 dx = \frac{1}{y_2} \int_0^a \int_{\frac{y_2 x_1}{y_1}}^{v x_1} x_2 dx_2 dx_1 \\ &= \frac{1}{2y_2} \int_0^a \left[v^2 - \left(\frac{y_2}{y_1} \right)^2 \right] x_1^2 dx_1 = \frac{v^2 y_1^2 - y_2^2}{2 y_2 y_1^2} \frac{a^3}{3}. \end{aligned}$$

Then, for the same region D, the inequality (3.1) is as follows:

$$p(y_1, y_2) \le \frac{6 y_1^2 y_2}{a^3 \left(v^2 y_1^2 - y_2^2 \right)} \int_D p(x_1, x_2)\,dx_1 dx_2.$$

Let's derive the set $Q(D)$ for the given triangular domain D. Since $A(D) = \frac{va^2}{2}$, $y^* \in D$ is element of $Q(D)$ if and only if, for $j = 1$;

$$\frac{2}{va^2}\frac{y_2^*}{(y_1^*)^2}\frac{a^3}{3} = 1 \Leftrightarrow y_2^* = \frac{3v\,(y_1^*)^2}{2a}$$

for $j = 2$;

$$\frac{2}{va^2}\frac{\left(v^2\,(y_1^*)^2 - (y_2^*)^2\right)a^3}{6\,(y_1^*)^2\,y_2^*} = 1 \Leftrightarrow y_1^* = \left(\frac{a\,(y_2^*)^2}{av^2 - 3y_2^* v}\right)^{\frac{1}{2}}.$$

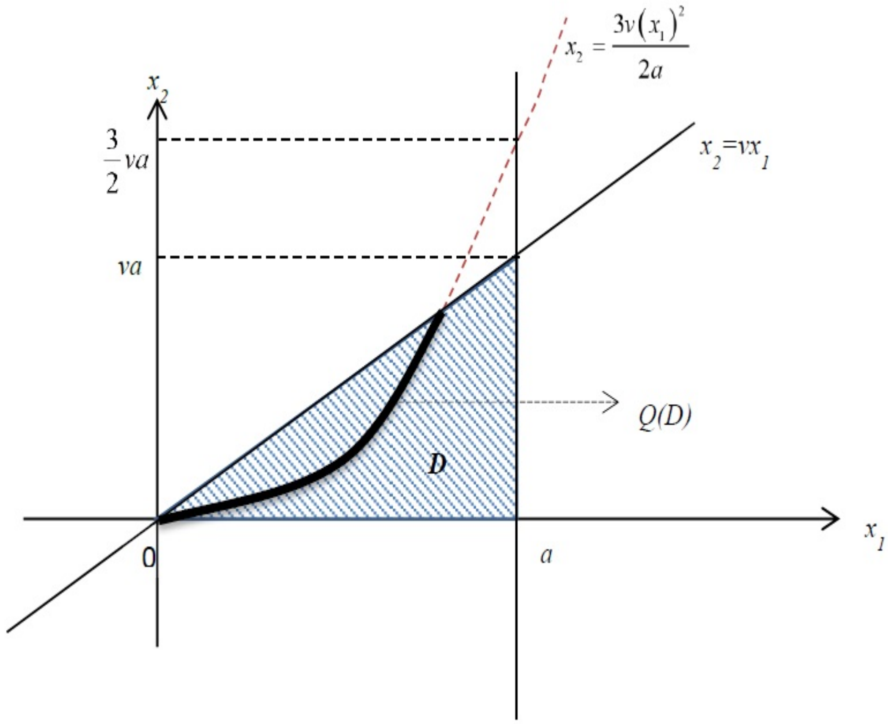

Figure 1. In case of $j = 1$, the set $Q(D)$ for triangular domain D

Example 3.2. *Let the triangular region D be as follows:*

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} : \frac{x_1}{a} + \frac{x_2}{b} \leq 1\right\}.$$

In this region, for $y \in D$, the sets $D_j(y)$ $(j = 1, 2)$ are as following forms:

$$D_1(y) = \left\{x \in D : 0 < x_2 \leq \frac{aby_2}{ay_2 + by_1}, \quad \frac{y_1}{y_2}x_2 \leq x_1 \leq a - \frac{a}{b}x_2\right\}$$
$$D_2(y) = \left\{x \in D : 0 < x_1 \leq \frac{aby_1}{ay_2 + by_1}, \quad \frac{y_2}{y_1}x_1 \leq x_2 \leq b - \frac{b}{a}x_1\right\}.$$

If $j = 1$, then we have

$$\int_D l_{1,y}(x)\,dx = \frac{1}{y_1}\int_{D_1(y)} x_1 dx = \frac{1}{y_1}\int_0^{\frac{aby_2}{ay_2+by_1}}\int_{\frac{y_1 x_2}{y_2}}^{a-\frac{ax_2}{b}} x_1 dx_1 dx_2$$
$$= \frac{1}{2y_1}\int_0^{\frac{aby_2}{ay_2+by_1}}\left[\left(a - \frac{a}{b}\right)^2 - \left(\frac{y_1}{y_2}\right)^2\right]x_2^2 dx_2 = \frac{a^3 by_2\left[(ab-a)^2 y_2^2 - b^2 y_1^2\right]}{6y_1\,(ay_2 + by_1)^3}.$$

For $j=2$; we get

$$\int_D l_{2,y}(x)\,dx = \frac{1}{y_2}\int_{D_2(y)} x_2 dx = \frac{1}{y_2}\int_0^{\frac{aby_1}{ay_2+by_1}}\int_{\frac{y_2x_1}{y_1}}^{b-\frac{bx_1}{a}} x_2 dx_2 dx_1$$
$$= \frac{1}{2y_2}\int_0^{\frac{aby_1}{ay_2+by_1}}\left[\left(b-\frac{b}{a}\right)^2-\left(\frac{y_2}{y_1}\right)^2\right]x_1^2 dx_1 = \frac{b^3ay_1\left[(ba-b)^2y_1^2-a^2y_2^2\right]}{6y_2\left(ay_2+by_1\right)^3}.$$

Thereby, in D, to $j=1$; the inequality is

$$p(y_1,y_2) \le \frac{6y_1\left(ay_2+by_1\right)^3}{a^3by_2\left[(ab-a)^2y_2^2-b^2y_1^2\right]}\int_D p(x_1,x_2)\,dx_1dx_2$$

for $j=2$; the inequality (3.1) is

$$p(y_1,y_2) \le \frac{6y_2\left(ay_2+by_1\right)^3}{b^3ay_1\left[(ba-b)^2y_1^2-a^2y_2^2\right]}\int_D p(x_1,x_2)\,dx_1dx_2.$$

Let us construct $Q(D)$ for the given region D. Since $A(D)=\frac{ab}{2}$, if we get $j=1$, then we obtain

$$y^* \in Q(D) \Leftrightarrow \frac{a^2y_2^*\left[(ab-a)^2(y_2^*)^2-b^2(y_1^*)^2\right]}{3y_1^*\left(ay_2^*+by_1^*\right)^3}=1$$

also, if we get $j=2$, then we have

$$y^* \in Q(D) \Leftrightarrow \frac{b^2y_1^*\left[(ba-b)^2(y_1^*)^2-a^2(y_2^*)^2\right]}{3y_2^*\left(ay_2^*+by_1^*\right)^3}=1.$$

Example 3.3. *Now, let us get a rectangular region D which is defined as follows:*

$$D=\left\{(x_1,x_2)\in\mathbb{R}^2_{++} : x_1 \le a,\ x_2 \le b\right\}.$$

In this type region, it can be two cases: For $y \in D$

1) $\frac{y_2}{y_1} \le \frac{b}{a}$
2) $\frac{y_2}{y_1} \ge \frac{b}{a}$

1) *Let $\frac{y_2}{y_1} \le \frac{b}{a}$. Under this condition, the sets $D_j(y)$ will be:*

$$D_1(y)=\left\{x\in D : 0<x_1\le a,\quad 0<x_2\le \tfrac{y_2}{y_1}x_1\right\}$$
$$D_2(y)=\left\{x\in D : 0<x_1\le a,\quad \tfrac{y_2}{y_1}x_1<x_2\le b\right\}.$$

Hence, for $j=1$; we have

$$\int_D l_{1,y}(x)\,dx = \int_{D_1(y)} l_{1,y}(x)\,dx = \frac{1}{y_1}\int_{D_1(y)} x_1 dx$$
$$= \frac{1}{y_1}\int_0^a\int_0^{\frac{y_2x_1}{y_1}} x_1 dx_2 dx_1 = \frac{1}{y_1}\int_0^a\left(\frac{y_2}{y_1}\right)x_1^2 dx_1 = \frac{a^3y_2}{3y_1^2}$$

for $j=2$; we obtain

$$\int_D l_{2,y}(x)\,dx = \frac{1}{y_2}\int_{D_2(y)} x_2 dx = \frac{1}{y_2}\int_0^a\int_{\frac{y_2x_1}{y_1}}^b x_2 dx_2 dx_1$$
$$= \frac{1}{2y_2}\int_0^a\left[b^2-\left(\frac{y_2}{y_1}\right)^2x_1^2\right]dx_1 = \frac{3y_1^2b^2a-y_2^2a^3}{6y_1^2y_2}.$$

By taking into account these, (3.1) becomes following inequalities: for $j = 1$;

$$p(y_1, y_2) \leq \frac{3y_1^2}{a^3 y_2} \int_D p(x_1, x_2)\, dx_1 dx_2$$

for $j = 2$;

$$p(y_1, y_2) \leq \frac{6y_1^2 y_2}{3y_1^2 b^2 a - y_2^2 a^3} \int_D p(x_1, x_2)\, dx_1 dx_2.$$

Let us derive the set $Q(D)$. Since $A(D) = ab$, then while $j = 1$;

$$y^* \in Q(D) \Leftrightarrow y_2^* = \frac{3b\,(y_1^*)^2}{a^2}$$

while $j = 2$;

$$y^* \in Q(D) \Leftrightarrow y_1^* = \left(\frac{(y_2^*)^2\, a^4 b}{3b^3 a^2 - 6y_2^*} \right)^{\frac{1}{2}}.$$

2) Now, let us consider the second case. Namely, let $\frac{y_2}{y_1} \geq \frac{b}{a}$. Therefore, we have that

$$D_1(y) = \left\{ x \in D : \tfrac{y_1}{y_2} x_2 \leq x_1 \leq a, \quad 0 < x_2 \leq b \right\}$$
$$D_2(y) = \left\{ x \in D : 0 < x_1 \leq \tfrac{y_1}{y_2} x_2, \quad 0 < x_2 \leq b \right\}.$$

To $j = 1$; we have

$$\int_D l_{1,y}(x)\, dx = \frac{1}{y_1} \int_{D_1(y)} x_1 dx = \frac{1}{y_1} \int_0^b \int_{\frac{y_1 x_2}{y_2}}^a x_1 dx_1 dx_2$$
$$= \frac{1}{2y_1} \int_0^b \left(a^2 - \left(\frac{y_1}{y_2} \right)^2 x_2^2 \right) dx_2 = \frac{3y_2^2 a^2 b - b^3 y_1^2}{6y_1 y_2^2}.$$

Thereby, in this case, the inequality (3.1) is

$$p(y_1, y_2) \leq \frac{6y_1 y_2^2}{3y_2^2 a^2 b - b^3 y_1^2} \int_D p(x_1, x_2)\, dx_1 dx_2.$$

In case $j = 2$, we get

$$\int_D l_{2,y}(x)\, dx = \int_{D_2(y)} l_{2,y}(x)\, dx = \frac{1}{y_2} \int_{D_2(y)} x_2 dx$$
$$= \frac{1}{y_2} \int_0^b \int_0^{\frac{y_1 x_2}{y_2}} x_2 dx_1 dx_2 = \frac{1}{y_2} \int_0^b \frac{y_1}{y_2} x_2^2 dx_2 = \frac{b^3 y_1}{3y_2^2}.$$

Thus, the inequality (3.1) will be as follows:

$$p(y_1, y_2) \leq \frac{3y_2^2}{b^3 y_1} \int_D p(x_1, x_2)\, dx_1 dx_2.$$

By taking into account both cases, $Q(D)$ becomes as follows: for $j = 1$;

$$Q(D) = \left\{ y^* \in D : \frac{y_2^*}{y_1^*} \leq \frac{b}{a},\ y_2^* = \frac{3b\,(y_1^*)^2}{a^2} \right\} \bigcup$$
$$\left\{ y^* \in D : \frac{y_2^*}{y_1^*} \geq \frac{b}{a},\ y_2^* = \left(\frac{b^2\,(y_1^*)^2}{3a^2 - 6y_1^* a} \right)^{\frac{1}{2}} \right\}$$

for $j = 2$;

$$Q(D) = \left\{ y^* \in D : \frac{y_2^*}{y_1^*} \leq \frac{b}{a},\ y_1^* = \left(\frac{(y_2^*)^2\, a^4 b}{3b^3 a^2 - 6y_2^*} \right)^{\frac{1}{2}} \right\} \bigcup$$
$$\left\{ y^* \in D : \frac{y_2^*}{y_1^*} \geq \frac{b}{a},\ y_1^* = \frac{3a\,(y_2^*)^2}{b^2} \right\}.$$

Example 3.4. *We shall now consider the case where the set D is part of the disk defined as*

$$D=\left\{(x_1,x_2)\in\mathbb{R}^2_{++}:x_1^2+x_2^2\leq r^2\right\}.$$

For $y\in D$, the set $D_1(y)$ is combination of

$$D_1^*(y)=\left\{x\in D:0<x_1\leq\frac{ry_1}{\sqrt{y_1^2+y_2^2}},0<x_2\leq\frac{y_2}{y_1}x_1\right\}$$

and

$$D_1^{**}(y)=\left\{x\in D:\frac{ry_1}{\sqrt{y_1^2+y_2^2}}\leq x_1\leq r\,,0<x_2\leq\sqrt{r^2-x_1^2}\right\}.$$

Namely, $D_1(y)=D_1^*(y)\bigcup D_1^{**}(y)$. The set $D_2(y)$ will be as follows:

$$D_2(y)=\left\{x\in D:0<x_1\leq\frac{ry_1}{\sqrt{y_1^2+y_2^2}},\frac{y_2}{y_1}x_1\leq x_2\leq\sqrt{r^2-x_1^2}\right\}.$$

To $j=1$; we have

$$\begin{aligned}\int_D l_{1,y}(x)\,dx&=\frac{1}{y_1}\int_{D_1(y)}x_1dx=\frac{1}{y_1}\int_{D_1^*(y)}x_1dx+\frac{1}{y_1}\int_{D_1^{**}(y)}x_1dx\\&=\frac{1}{y_1}\int_0^{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}\int_0^{\frac{y_2}{y_1}x_1}x_1dx_2dx_1+\frac{1}{y_1}\int_{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}^{r}\int_0^{\sqrt{r^2-x_1^2}}x_1dx_2dx_1\\&=\frac{1}{y_1}\int_0^{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}\left(\frac{y_2}{y_1}x_1^2\right)dx_1+\frac{1}{y_1}\int_{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}^{r}x_1\sqrt{r^2-x_1^2}dx_1=\frac{r^3y_2}{3y_1\sqrt{y_1^2+y_2^2}}.\end{aligned}$$

In this case, for the given region D, the inequality (3.1) will be following form:

$$p(y_1,y_2)\leq\frac{3y_1\sqrt{y_1^2+y_2^2}}{r^3y_2}\int_D p(x_1,x_2)\,dx_1dx_2.$$

To $j=2$; we obtain that

$$\begin{aligned}\int_D l_{2,y}(x)\,dx&=\frac{1}{y_2}\int_{D_2(y)}x_2dx=\frac{1}{y_2}\int_0^{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}\int_{\frac{y_2}{y_1}x_1}^{\sqrt{r^2-x_1^2}}x_2dx_2dx_1\\&=\frac{1}{2y_2}\int_0^{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}\left(r^2-\left(1+\frac{y_2^2}{y_1^2}\right)x_1^2\right)dx_1=\frac{r^3y_1}{3y_2\sqrt{y_1^2+y_2^2}}\end{aligned}$$

and by using the equality above, the inequality (3.1) will be as follows:

$$p(y_1,y_2)\leq\frac{3y_2\sqrt{y_1^2+y_2^2}}{r^3y_1}\int_D p(x_1,x_2)\,dx_1dx_2.$$

Since $A(D)=\frac{\pi r^2}{4}$, let us give the conditions for becoming elements of the set $Q(D)$. For $j=1$; we have

$$y^*\in Q(D)\Leftrightarrow\frac{4r\left(y_2^*\right)^2}{3\pi\left(y_1^*\right)^2\left(\left(y_1^*\right)^2+\left(y_2^*\right)^2\right)^{\frac{1}{2}}}=1.$$

For $j=2$; we get

$$y^*\in Q(D)\Leftrightarrow\frac{4r\left(y_1^*\right)^2}{3\pi\left(y_2^*\right)^2\left(\left(y_1^*\right)^2+\left(y_2^*\right)^2\right)^{\frac{1}{2}}}=1.$$

Remark 3.3. *From Theorem 3.5, the right hand side of Hermite-Hadamard Inequalities can be also analyzed for concrete domains. But, in this case, $D\cap intV_j(y)=\emptyset$ is required because of integrability of the function $\varphi_{j,y}$ on D.*

Example 3.5. *As in the Example 3.1, we discuss the triangle*

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} : 0 < x_1 \leq a, 0 < x_2 \leq vx_1\right\}.$$

If $j = 1$, then $D \cap intV_1(y) \neq \emptyset$ for $\forall y \in \mathbb{R}^2_{++}$. Thus, from Theorem 3.5, the right hand side of Hermite-Hadamard Inequalities for L(1)-convex functions is not obtained.

Let $j = 2$. It is obvious that $D \cap intV_2(y) = \emptyset \Leftrightarrow y_2 \geq vy_1$. From Theorem 3.5, we have

$$\int_D p(x_1, x_2)\, dx_1 dx_2 \leq p(y_1, y_2) \int_D \frac{x_2}{y_2} dx_1 dx_2.$$

Since

$$\int_D \frac{x_2}{y_2} dx_1 dx_2 = \frac{1}{y_2} \int_0^a \int_0^{vx_1} x_2 dx_2 dx_1 = \frac{v^2}{2y_2} \int_0^a x_1^2 dx_1 = \frac{a^3 v^2}{6y_2}.$$

for all $y \in D$ which satisfy the condition $y_2 \geq vy_1$ (namely, y on the long side of the triangle) and all p that are L(2)-convex, integrable on D, the inequality

$$\int_D p(x_1, x_2)\, dx_1 dx_2 \leq \frac{v^2 a^3}{6y_2} p(y_1, y_2)$$

is hold, or since $A(D)$ is area of triangular domain, we obtain the inequality

$$\frac{1}{A(D)} \int_D p(x_1, x_2)\, dx_1 dx_2 \leq \frac{va}{3y_2} p(y_1, y_2).$$

4 S(j)-convex Functions

Firstly, let us recall the definition of $\mathbb{B}^{-1}-$convex set [3]:

Definition 4.2. *A subset M of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}-$convex if for all $x_1, x_2 \in M$ and all $t \in [1, \infty)$ one has $tx_1 \wedge x_2 \in M$.*

Here, $\wedge$ is the greatest lower bound of x_1, x_2, that is,

$$x_1 \wedge x_2 = (\min\{x_{1,1}, x_{2,1}\}, ..., \min\{x_{1,n}, x_{2,n}\}).$$

For every $z \in \mathbb{R}^n_{++}$, $\mathbb{R}^n_{++}$ can be written as the combination of $(n+1)$-parts which are given with the following forms:

$$M_0(z) = \left\{x \in \mathbb{R}^n_{++} : z_i \leq x_i, \quad i = \overline{1, n}\right\}$$
$$M_j(z) = \left\{x \in \mathbb{R}^n_{++} : x_j \leq z_j \quad \text{and} \quad x_j z_i \leq z_j x_i, \forall i = \overline{1, n}\right\}.$$

The sets $M_j(z)\left(j = \overline{0, n}\right)$ are closed and convex sets. The following theorem gives construction of the sets $M_j(z)\left(j = \overline{0, n}\right)$.

Theorem 4.6. *$M_0(z)$ is co-radiant, $\mathbb{B}^{-1}-$convex set and $M_j(z)\left(j = \overline{1, n}\right)$ are radiant, $\mathbb{B}^{-1}-$convex sets.*

Proof. Let us show that $M_0(z)$ is co-radiant, namely $x \in M_0(z)$, $\lambda \geq 1 \Rightarrow \lambda x \in M_0(z)$. Since $x \in M_0(z)$, then $z_i \leq x_i\left(i = \overline{1, n}\right)$. $\lambda \geq 1$, so $z_i \leq x_i \leq \lambda x_i\left(i = \overline{1, n}\right)$. Consequently, we have $\lambda x \in M_0(z)$.

Now, let us prove that $M_0(z)$ is $\mathbb{B}^{-1}-$convex. Let $x, y \in M_0(z)$, $t \in [1, \infty)$. Hence, for $\forall i = \overline{1, n}$, we have $z_i \leq x_i$ and $z_i \leq y_i$. By using these inequalities; since $z_i \leq x_i \leq tx_i$ and $z_i \leq y_i$, we obtain $z_i \leq tx_i \wedge y_i$, $i = \overline{1, n}$. We have shown that $tx \wedge y \in M_0(z)$.

And now, we have to see that $M_j(z)\left(j = \overline{1, n}\right)$ are radiant. Let $x \in M_j(z)$ and $0 < \lambda \leq 1$. Since $x \in M_j(z)$, we have $x_j \leq z_j$ and $x_j z_i \leq z_j x_i$, $i = \overline{1, n}$. $0 < \lambda \leq 1$ so that $\lambda x_j \leq x_j \leq z_j$ then $\lambda x_j \leq z_j$. Also, $\lambda > 0$, hence we can derive $\lambda x_j z_i \leq z_j \lambda x_i$, $i = \overline{1, n}$. By taking into account both cases, $\lambda x \in M_j(z)$.

Finally, let us show that $M_j(z)$ are $\mathbb{B}^{-1}-$convex. Let $x, y \in M_j(z)$, $t \in [1, \infty)$.

$$x \in M_j(z) \Leftrightarrow x_j \leq z_j \quad \text{and} \quad x_j z_i \leq z_j x_i, \ i = \overline{1, n}$$
$$y \in M_j(z) \Leftrightarrow y_j \leq z_j \quad \text{and} \quad y_j z_i \leq z_j y_i, \ i = \overline{1, n}.$$

There are two possible cases: for $t \in [1, \infty)$

I) it can be $tx_j \leq z_j$. In this case, from $y_j \leq z_j$, we obtain $tx_j \wedge y_j \leq z_j$.

II) let $tx_j > z_j$. Again, since $y_j \le z_j$, we have $tx_j \wedge y_j \le z_j$. Hence, we deduce that $tx_j \wedge y_j \le z_j$.
In second part, for $z \in \mathbb{R}^n_{++}$

$$(tx_j \wedge y_j)\, z_i = tx_j z_i \wedge y_j z_i \le tx_i z_j \wedge y_i z_j = (tx_i \wedge y_i)\, z_j.$$

Thus, we have shown that $tx \wedge y \in M_j(z)$. □

The $(n+1)$-relations according to $M_j(z)$ $\left(j = \overline{0,n}\right)$ can be given by

$$\begin{array}{l} x \preccurlyeq_0 y \Leftrightarrow y \in M_0(x) \\ x \preccurlyeq_j y \Leftrightarrow x \in M_j(y), \qquad j = \overline{1,n}. \end{array}$$

Let us see that $\preccurlyeq_j$, $\left(j = \overline{0,n}\right)$ are partial order relations.

Theorem 4.7. $\preccurlyeq_j$, $\left(j = \overline{0,n}\right)$ *are partial order relations.*

Proof. Let $j = 0$. $\preccurlyeq_0$ is coordinate-wise order relation, namely,

$$x \preccurlyeq_0 y \Leftrightarrow y - x \in \mathbb{R}^n_{++}.$$

So that $\preccurlyeq_0$ is a partial order relation.

Let $j = \overline{1,n}$.

Firstly, we show that $\preccurlyeq_j$ $\left(j = \overline{1,n}\right)$ are reflexivity. For all $x \in \mathbb{R}^n_{++}$ and all $j = \overline{1,n}$, then $x_j \le x_j$. Also, for all $i = \overline{1,n}$, we have $x_j x_i \le x_j x_i$. Consequently, $x \preccurlyeq_j x$.

Let us show that $\preccurlyeq_j$ $\left(j = \overline{1,n}\right)$ are antisymmetric: Let $x, z \in \mathbb{R}^n_{++}$, $x \preccurlyeq_j z$ and $z \preccurlyeq_j x$. We deduce that

$$\begin{array}{lll} x \preccurlyeq_j z \;\Leftrightarrow\; x_j \le z_j \text{ and } x_j z_i \le z_j x_i, & & i = \overline{1,n} \\ z \preccurlyeq_j x \;\Leftrightarrow\; z_j \le x_j \text{ and } z_j x_i \le x_j z_i, & & i = \overline{1,n}. \end{array}$$

From the first part, for $j = \overline{1,n}$, we get $x_j = z_j$.

By using this equality and the second part, for all $i = \overline{1,n}$, since

$$\begin{array}{l} x_j z_i \le z_j x_i \Rightarrow z_i \le x_i \\ z_j x_i \le x_j z_i \Rightarrow x_i \le z_i \end{array}$$

thus, it is $x_i = z_i$.

Accordingly, we obtain $x = z$.

Now, we have to prove that $\preccurlyeq_j$ $\left(j = \overline{1,n}\right)$ are transitive. Let $x, y, z \in \mathbb{R}^n_{++}$ $x \preccurlyeq_j y$ and $y \preccurlyeq_j z$. Hence, we have that

$$\begin{array}{lll} x \preccurlyeq_j y \;\Leftrightarrow\; x_j \le y_j \text{ and } x_j y_i \le y_j x_i, & & i = \overline{1,n} \\ y \preccurlyeq_j z \;\Leftrightarrow\; y_j \le z_j \text{ and } y_j z_i \le z_j y_i, & & i = \overline{1,n}. \end{array}$$

Since $x_j \le y_j \le z_j$, then we obtain

$$x_j \le z_j. \tag{4.7}$$

Taking into account that the above inequalities hold, we have that

$$\begin{aligned} x_j y_i \le y_j x_i \;\Rightarrow\; & x_j y_i (y_j z_i) \le y_j x_i (y_j z_i) \le y_j x_i (z_j y_i) \\ & x_j z_i (y_i y_j) \le x_i z_j (y_j y_i) \\ & x_j z_i \le z_j x_i. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we have $x \preccurlyeq_j z$. The theorem is proved. □

Now, we can write Minkowski functions according to $M_j(z)$ $\left(z \in \mathbb{R}^n_{++}, j = \overline{0,n}\right)$ sets and $\preccurlyeq_j$ partial order relations. For $z \in \mathbb{R}^n_{++}$, since that $M_0(z)$ is co-radiant;

$$v_{M_0(z)}(x) := \sup\{\alpha : x \in \alpha M_0(z)\} = \sup\{\alpha : \alpha z \preccurlyeq_0 x\}$$

then, we denote this function with $s_{0,z}$,

$$s_{0,z}(x) := v_{M_0(z)}(x), \qquad x \in \mathbb{R}^n_{++}.$$

For $z \in \mathbb{R}^n_{++}$ and $j = \overline{1,n}$; by taking into account that $M_j(z)$ are radiant sets; Minkowski gauge of $M_j(z)$ are

$$\mu_{M_j(z)}(x) := \inf\{\alpha > 0 : x \in \alpha M_j(z)\} = \inf\{\alpha > 0 : x \preccurlyeq_j \alpha z\}.$$

Let us denote this function with the following notation

$$s_{j,z}(x) := \mu_{M_j(z)}(x), \qquad x \in \mathbb{R}^n_{++}.$$

The sets $M_j(z)\left(j = \overline{1,n}, z \in \mathbb{R}^n_{++}\right)$ can be written as the intersection of the cone

$$U_j(z) = \left\{x \in \mathbb{R}^n_{++} : \frac{x_j}{z_j} \le \frac{x_i}{z_i} \ i = \overline{1,n}\right\}$$

and the half-space

$$H_j(z) = \{x \in \mathbb{R}^n : x_j \le z_j\}.$$

The functions $s_{j,z}$ can be denoted the following form, if we use the cone $U_j(z)$.

$$s_{j,z}(x) = \begin{cases} \frac{x_j}{z_j}, & x \in U_j(z) \\ \infty, & x \notin U_j(z). \end{cases} \tag{4.9}$$

Let us analyze convexity with respect to the family of functions $S(j) = \{s_{j,z} : z \in \mathbb{R}^n_{++}\}, j = \overline{0,n}$.

Definition 4.3. *Let $j = \overline{0,n}$. A function $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is an IPH[j] function if f is positively homogeneous of degree one and increasing according to order relation $\preccurlyeq_j$.*

Theorem 4.8. *$\forall j = \overline{0,n}$ and $\forall z \in \mathbb{R}^n_{++}$, $s_{j,z}$ are IPH[j] functions.*

Proof. Let us show that $s_{j,z}$ are positively homogeneous of degree one.

For $j = 0$, we have that

$$\begin{aligned} s_{0,z}(\lambda x) &= \sup\{\alpha : \lambda x \in \alpha M_0(z)\} = \sup\{\alpha : \alpha z \preccurlyeq_0 \lambda x\} \\ &= \sup\left\{\alpha : \alpha z_i \le \lambda x_i, i = \overline{1,n}\right\} = \sup\left\{\lambda\alpha' : \alpha' z_i \le x_i, i = \overline{1,n}\right\} \\ &= \lambda \sup\{\alpha' : \alpha' z \preccurlyeq_0 x\} = \lambda s_{0,z}(x). \end{aligned}$$

For $j = \overline{1,n}$, we get

$$\begin{aligned} s_{j,z}(\lambda x) &= \inf\{\alpha > 0 : \lambda x \in \alpha M_j(z)\} = \inf\{\alpha > 0 : \lambda x \preccurlyeq_j \alpha z\} \\ &= \lambda \inf\{\alpha' > 0 : x \preccurlyeq_j \alpha' z\} = \lambda s_{j,z}(x). \end{aligned}$$

Let us prove that $s_{j,z}$ are increasing according to $\preccurlyeq_j \left(j = \overline{0,n}\right)$.

Let $j = 0$ and $x_1 \preccurlyeq_0 x_2$. Then, we have $\{\alpha : \alpha z \preccurlyeq_0 x_1\} \subset \{\alpha : \alpha z \preccurlyeq_0 x_2\}$. From properties of supremum, we obtain that $s_{0,z}(x_1) \le s_{0,z}(x_2)$.

Let $j = \overline{1,n}$ and $x_1 \preccurlyeq_j x_2$. Hence, we have $\{\alpha > 0 : x_2 \preccurlyeq_j \alpha z\} \subset \{\alpha > 0 : x_1 \preccurlyeq_j \alpha z\}$. Consequently, we obtain $s_{j,z}(x_1) \le s_{j,z}(x_2)$. □

Now, let us give the following theorem which can be easily proved via Corollary 2.6 in [2].

Theorem 4.9. *For $j = \overline{0,n}$, $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is S(j)-convex function if and only if f is IPH[j] function.*

The following theorem implies some properties of IPH[j] functions.

Theorem 4.10. *Let $j = \overline{1,n}$ and $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ be an IPH[j] function. Then following statements are hold:*

(i) $f(x) \ge 0$ for all $x \in \mathbb{R}^n_{++}$.

(ii) If $f(x^*) = +\infty$ where $x^* \in \mathbb{R}^n_{++}$ then $f(x) = +\infty$ on the set $\left\{x \in \mathbb{R}^n_{++} : \exists\lambda > 0 \ \text{ such that } \lambda x^* \preccurlyeq_j x\right\}$.

(iii) If $f(x^*) = 0$ where $x^* \in \mathbb{R}^n_{++}$ then for all $x \in \left\{x \in \mathbb{R}^n_{++} : \exists\lambda > 0,\ x \preccurlyeq_j \lambda x^*\right\}$, $f(x) = 0$.

Proof. (i) Let $x \in \mathbb{R}^n_{++}$. Because $\frac{1}{2}x \preccurlyeq_j x$, we have $\frac{1}{2}f(x) = f\left(\frac{x}{2}\right) \le f(x)$. Therefore $f(x) \ge 0$.

(ii) Let $x \in \mathbb{R}^n_{++}$ be a point such that there exists $\lambda > 0$ with the property $\lambda x^* \preccurlyeq_j x$. Then $f(x) \ge f(\lambda x^*) = \lambda f(x^*) = +\infty$.

(iii) Let $x \in \mathbb{R}^n_{++}$ and let there be $\lambda > 0$ such that $x \preccurlyeq_j \lambda x^*$. Thus, we have that $0 \le f(x) \le f(\lambda x^*) = \lambda f(x^*) = 0$. □

5 Hermite-Hadamard Type Inequalities for S(j)-convex Functions

Let us prove the following theorem which has an important role in Hermite-Hadamard Type Inequalities for S(j)-convex functions.

Theorem 5.11. *For $j = \overline{1, n}$ and $p : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$, the following statements are equivalent:*
(i) p is an IPH[j] function.
(ii) For all $x, z \in \mathbb{R}^n_{++}$ and $\lambda > 0$ such that $x \preccurlyeq_j \lambda z$, we have $p(x) \leq \lambda p(z)$.
(iii) For all $x, z \in \mathbb{R}^n_{++}$, we have $p(x) \leq s_{j,z}(x)\, p(z)$.

Proof. $i \Rightarrow ii$) Since p is an IPH[j] function, for all $\lambda > 0$, we get $x \preccurlyeq_j \lambda z$. Hence $p(x) \leq p(\lambda z) = \lambda p(z)$.

$ii \Rightarrow i$) The monotonicity of p follows from (ii) with $\lambda = 1$. We now show that p is positively homogeneous. Let $x = \lambda z$ with $\lambda > 0$. Then by (ii), we have $p(x) = p(\lambda z) \leq \lambda p(z)$. Because $z = \lambda^{-1} x$, we conclude that $p(z) \leq \lambda^{-1} p(x)$. Thus $p(\lambda z) = \lambda p(z)$.

$ii \Rightarrow iii$) If $p(z) = 0$, we have $0 \leq p(x) \leq s_{j,z}(x)\, p(z) = 0$ for all x. Let $p(z) > 0$ and $\lambda > 0$ be a number such that $x \preccurlyeq_j \lambda z$. Applying (ii), we conclude that $\frac{p(x)}{p(z)} \leq \lambda$. It follows from the definition of $s_{j,z}$ that $s_{j,z}(x) = \inf\{\lambda > 0 : x \preccurlyeq_j \lambda z\}$, therefore $\frac{p(x)}{p(z)} \leq s_{j,z}(x)$.

$iii \Rightarrow ii$) follows directly from the definition of $s_{j,z}$. □

If we use the above theorem, then we can deduce the Hermite-Hadamard Type Inequalities for S(j)-convex functions.

Corollary 5.2. *Let $p : D \to \mathbb{R}_{+\infty}, D \subset \mathbb{R}^n_{++}$ be a S(j)-convex function and integrable function on D where $D \subset U_j(z)$. Then, for all $z \in D$, the following inequality holds:*

$$\int_D p(x)\, dx \leq p(z) \int_D s_{j,z}(x)\, dx. \tag{5.10}$$

Proof. It is proven from Theorem 5.11 (iii) and (4.9). □

Let's analyze the inequality (5.10) via sets $Q(D)$.

Let $D \subset \mathbb{R}^n_{++}$ be bounded and satisfy condition $cl\,(intD) = D$. $Q(D)$ consist of all point $x^* \in D$ such that

$$\frac{1}{A(D)} \int_D s_{j,x^*}(x)\, dx = 1,$$

here $A(D) = \int_D dx$.

We can give a theorem about the set $Q(D)$ and Hermite-Hadamard Type Inequalities of S(j)-convex functions.

Theorem 5.12. *Let p be a S(j)-convex function defined and integrable on D. If $Q(D) \neq \emptyset$, then one has the inequality:*

$$\frac{1}{A(D)} \int_D p(x)\, dx \leq \inf_{x^* \in Q(D)} p(x^*)$$

Proof. If $p(x^*) = 0$, from $p(x) \leq s_{j,x^*}(x)\, p(x^*)$ we have $p(x) = 0$. Thus, let $p(x^*) > 0$. For all $x \in D$,

$$p(x) \leq s_{j,x^*}(x)\, p(x^*)$$

is hold. Because $x^* \in Q(D)$, we have

$$\begin{aligned} p(x^*) &= p(x^*) \frac{1}{A(D)} \int_D s_{j,x^*}(x)\, dx \\ &= \frac{1}{A(D)} \int_D p(x^*)\, s_{j,x^*}(x)\, dx \geq \frac{1}{A(D)} \int_D p(x)\, dx. \end{aligned}$$

□

For every $x^* \in Q(D)$, the inequality

$$\frac{1}{A(D)} \int_D p(x)\, dx \leq p(x^*) \tag{5.11}$$

is hold. If we take $p(x) = s_{j,x^*}(x)$, the inequality (5.11) will be turn equality.

Let p be a S(j)-convex function defined and integrable on D which is closed, bounded and connected set. For all $x, z \in D$, we have

$$p(x) \leq s_{j,z}(x)\, p(z).$$

Hence, below inequality is obtained:

$$p(x)\, \psi_{j,x}(z) \leq p(z)$$

where

$$\psi_{j,x}(z) = \frac{1}{s_{j,z}(x)} = \begin{cases} \frac{z_j}{x_j}, & x \in U_j(z) \\ 0, & x \notin U_j(z) \end{cases} = \begin{cases} \frac{z_j}{x_j}, & z \notin intU_j(x) \\ 0, & z \in intU_j(x) \end{cases} \tag{5.12}$$

In this case, we can write second part of the Hermite-Hadamard Type Inequality for S(j)-convex functions.

Theorem 5.13. *Let $D \subset \mathbb{R}^n_{++}$, $p : D \to \mathbb{R}_{+\infty}$ be S(j)-convex and integrable on D. Then, for all $z \in D$, we have the inequality:*

$$p(z) \int_D \psi_{j,z}(x)\, dx \leq \int_D p(x)\, dx \tag{5.13}$$

Examples:

On the same domains in previous section, Hermite-Hadamard Inequalities for S(j)-convex functions can be also considered. For example, let us discuss triangular domain in Example 3.1.

Example 5.6. *Let*

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} : 0 < x_1 \leq a,\ \ 0 < x_2 \leq vx_1\right\}.$$

$D \subset U_j(z)$ is necessary in order to the inequality (5.10) can be written on this region.

When $j = 1$, for all $z \in \mathbb{R}^2_{++}$ it is $D \not\subset U_1(z)$. Hence, from Corollary 5.2, for S(1)-convex functions, the right part of Hermite-Hadamard Inequalities can not computed on this domain.

Let $j = 2$. This is obvious that $D \subset U_2(z) \Leftrightarrow z_2 \geq vz_1$. From (5.10), we obtain

$$\int_D p(x_1, x_2)\, dx_1 dx_2 \leq p(z_1, z_2) \int_D \frac{x_2}{z_2} dx_1 dx_2.$$

When the right integral is calculated, for all $z \in D$ satisfying the condition $z_2 \geq vz_1$ (thus, z is on the hypotenuse of the triangle) and for all p that is S(2)-convex, integrable on D, we have

$$\frac{1}{A(D)} \int_D p(x_1, x_2)\, dx_1 dx_2 \leq \frac{va}{3z_2} p(z_1, z_2)$$

where $A(D)$ is area of the triangular domain.

For the same domain, if we apply the Theorem 5.13, then we can estimate the left part of Hermite-Hadamard Inequality.

Let $j = 1$. From (5.13), we have

$$p(z_1, z_2) \int_D \psi_{1,z}(x_1, x_2)\, dx_1 dx_2 \leq \int_D p(x_1, x_2)\, dx_1 dx_2$$

and from (5.12), we obtain

$$\int_D \psi_{1,z}(x_1, x_2)\, dx_1 dx_2 = \frac{a^3 z_2}{3z_1^2}.$$

Thereby, the inequality is

$$p(z_1, z_2) \frac{a^3 z_2}{3z_1^2} \leq \int_D p(x_1, x_2)\, dx_1 dx_2.$$

Let $j = 2$. The left part of the Hermite-Hadamard Inequality is

$$p(z_1, z_2) \int_D \psi_{2,z}(x_1, x_2)\, dx_1 dx_2 \leq \int_D p(x_1, x_2)\, dx_1 dx_2.$$

Since, with a simple calculation, we obtain

$$\int_D \psi_{2,z}(x_1, x_2)\, dx_1 dx_2 = \frac{a^3\left(z_1^2 v^2 - z_2^2\right)}{6 z_1^2 z_2}$$

and from above inequality, we have

$$p(z_1, z_2)\, \frac{a^3\left(z_1^2 v^2 - z_2^2\right)}{6 z_1^2 z_2} \leq \int_D p(x_1, x_2)\, dx_1 dx_2.$$

Acknowledgements

The authors wish to thank Akdeniz University, Mersin University and TUBITAK (The Scientific and Technological Research Council of Turkey).

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Received: December 03, 2014; *Accepted*: May 10, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 3(3)(2015) 360–367

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Inclusion properties for certain subclasses of analytic functions defined by using the generalized Bessel functions

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Abstract

By making use of the operator B_ν^c defined by the generalized Bessel functions of the first kind, the authors introduce and investigate several new subclasses of starlike, convex, close-to-convex and quasi-convex functions. The authors establish inclusion relationships associated with the aforementioned operator. Some interesting corollaries and consequences of the main inclusion relationships are also considered.

Keywords: Analytic functions; Starlike functions; Convex functions; Close-to-convex functions; Quasi-convex functions; Generalized Bessel functions.

2010 MSC: Primary 30C45; Secondary 33C10, 33C90.

1 Introduction, Definitions and Preliminaries

Let

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

be the unit disk in the complex z-plane. Also let $\mathcal{A}$ be the class of functions f of the form:

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}, \tag{1.1}$$

which are analytic in $\mathbb{U}$ and satisfy the following normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, $\mathcal{C}(\beta, \alpha)$ and $\mathcal{C}^*(\beta, \alpha)$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike of order α in $\mathbb{U}$, convex of order α in $\mathbb{U}$, close-to-convex of order β and type α in $\mathbb{U}$ and quasi-convex of order β and type α in $\mathbb{U}$. Thus, by definition, we have (for details, see [4, 6, 7, 11])

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (0 \leqq \alpha < 1;\ z \in \mathbb{U}) \right\},$$

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (0 \leqq \alpha < 1;\ z \in \mathbb{U}) \right\},$$

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$$\mathcal{C}(\beta,\alpha) := \left\{ f : f \in \mathcal{A},\ g \in \mathcal{S}^*(\alpha) \quad \text{and} \quad \Re\left(\frac{zf'(z)}{g(z)}\right) > \beta \quad (0 \leqq \alpha, \beta < 1;\ z \in \mathbb{U}) \right\}$$

and

$$\mathcal{C}^*(\beta,\alpha) := \left\{ f : f \in \mathcal{A},\ g \in \mathcal{K}(\alpha) \quad \text{and} \quad \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \beta \quad (0 \leqq \alpha, \beta < 1;\ z \in \mathbb{U} \right\}.$$

It is easily observed from the above definitions that

$$f(z) \in \mathcal{K}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha)$$

and

$$f(z) \in \mathcal{C}^*(\beta,\alpha) \Longleftrightarrow zf'(z) \in \mathcal{C}(\beta,\alpha).$$

For $f \in \mathcal{A}$ given by (1.1) and $g(z)$ given by $g(z) = z + \sum_{n=1}^{\infty} b_{n+1} z^{n+1}$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=1}^{\infty} a_{n+1} b_{n+1} z^{n+1} =: (g * f)(z) \qquad (z \in \mathbb{U}).$$

The generalized Bessel function of the first kind of order p is defined as a particular solution of the following second-order differential equation (see, for details, [1]):

$$z^2 w''(z) + bzw'(z) + [cz^2 - p^2 + (1-b)p]w(z) = 0 \qquad (b, c, p \in \mathbb{C}) \tag{1.2}$$

and has the familiar representation given by

$$\omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \cdot \Gamma(p + n + \frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+p} \qquad (z \in \mathbb{C}). \tag{1.3}$$

The series in (1.3) permits a unified study of the Bessel, the modified Bessel and the spherical Bessel functions. The following cases are worthy of note here.

1. Taking $b = c = 1$ in (1.3), we obtain the familiar Bessel function of the first kind of order p defined by (see [1, 8, 12])
$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot \Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \qquad (z \in \mathbb{C}). \tag{1.4}$$
2. Putting $b = 1$ and $c = -1$ in (1.3), we get the modified Bessel function of the first kind of order p defined by (see [1, 12])
$$I_p(z) = \sum_{n=0}^{\infty} \frac{1}{n! \cdot \Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \qquad (z \in \mathbb{C}). \tag{1.5}$$
3. Letting $b = 2$ and $c = 1$ in (1.3), we have the spherical Bessel function of the first kind of order p defined by (see [1])
$$j_p(z) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot \Gamma(p+n+3/2)} \left(\frac{z}{2}\right)^{2n+p} \qquad (z \in \mathbb{C}). \tag{1.6}$$

Recently, Deniz *et al.* [3] considered the function $\varphi_{p,b,c}(z)$ defined, in terms of the generalized Bessel function $\omega_{p,b,c}(z)$, by

$$\begin{aligned}\varphi_{p,b,c}(z) &= 2^p\, \Gamma\left(p + \frac{b+1}{2}\right)\, z^{1-p/2}\, \omega_{p,b,c}(\sqrt{z}) \\ &= z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n \cdot (\nu)_n} \frac{z^{n+1}}{n!} \qquad \left(\nu = p + \frac{b+1}{2} \notin \mathbb{Z}_0^- := \{0, -1, -2, \cdots\}\right),\end{aligned} \tag{1.7}$$

where $(\lambda)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_0 = 1 \qquad \text{and} \qquad (\lambda)_n = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) \quad (n \in \mathbb{N} := \{1, 2, 3, \cdots\}).$$

Subsequently, by using the function $\varphi_{p,b,c}(z)$, Deniz [2] introduced the operator B_ν^c as follows:

$$B_\nu^c f(z) = \varphi_{p,b,c}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n a_{n+1}}{4^n \cdot (\nu)_n} \frac{z^{n+1}}{n!} \qquad (z \in \mathbb{C}). \tag{1.8}$$

It is easy to verify from (1.8) that

$$z\left(B_{\nu+1}^c f(z)\right)' = \nu B_\nu^c f(z) - (\nu - 1) B_{\nu+1}^c f(z), \tag{1.9}$$

where

$$\nu = p + \frac{b+1}{2} \notin \mathbb{Z}_0^- .$$

In fact, the operator B_ν^c given by (1.8) provides an elementary transform of the generalized hypergeometric function, that is, we have

$$B_\nu^c f(z) = z\,{}_0F_1\left(\text{—}\,; \nu; -\frac{c}{4}\,z\right) * f(z)$$

and

$$\varphi_{\nu,c}\left(-\frac{c}{4}\,z\right) = z\,{}_0F_1(\text{—}\,; \nu; z).$$

In the present article, we investigate various inclusion relationships for each of the following subclasses of the normalized analytic function class $\mathcal{A}$, which are defined by means of the generalized Bessel function of the first kind (see also [9] and [10] for inclusion relationships for various other function classes). Indeed, for $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $0 \leqq \alpha < 1$, we write

$$\mathcal{S}_{\nu,c}^*(\alpha) := \{f : f \in \mathcal{A} \quad \text{and} \quad B_\nu^c f(z) \in \mathcal{S}^*(\alpha) \quad (z \in \mathbb{U})\},$$

$$\mathcal{K}_{\nu,c}(\alpha) := \{f : f \in \mathcal{A} \quad \text{and} \quad B_\nu^c f(z) \in \mathcal{K}(\alpha) \quad (z \in \mathbb{U})\},$$

$$\mathcal{C}_{\nu,c}(\beta,\alpha) := \{f : f \in \mathcal{A} \quad \text{and} \quad B_\nu^c f(z) \in \mathcal{C}(\beta,\alpha) \quad (z \in \mathbb{U})\}$$

and

$$\mathcal{C}_{\nu,c}^*(\beta,\alpha) := \{f : f \in \mathcal{A} \quad \text{and} \quad B_\nu^c f(z) \in \mathcal{C}^*(\beta,\alpha) \quad (z \in \mathbb{U})\}.$$

We also note that

$$f(z) \in \mathcal{K}_{\nu,c}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}_{\nu,c}^*(\alpha) \tag{1.10}$$

and

$$f(z) \in \mathcal{C}_{\nu,c}^*(\beta,\alpha) \Longleftrightarrow zf'(z) \in \mathcal{C}_{\nu,c}(\beta,\alpha). \tag{1.11}$$

In our investigation of the inclusion relationships involving the function classes $\mathcal{S}_{\nu,c}^*(\alpha)$, $\mathcal{K}_{\nu,c}(\alpha)$, $\mathcal{C}_{\nu,c}(\beta,\alpha)$ and $\mathcal{C}_{\nu,c}^*(\beta,\alpha)$ given by the above definitions, we shall make use of the following Miller-Mocanu lemma.

Lemma 1.1. *(see Miller and Mocanu [5]) Let $\Theta(u,v)$ be a complex-valued function, such that*

$$\Theta : \mathbb{D} \to \mathbb{C} \qquad (\mathbb{D} \subset \mathbb{C} \times \mathbb{C}),$$

$\mathbb{C}$ being the complex plane. Also let

$$u = u_1 + iu_2 \quad \textit{and} \quad v = v_1 + iv_2.$$

Suppose that the function $\Theta(u,v)$ satisfies each of the following conditions:

(i) $\Theta(u,v)$ is continuous in $\mathbb{D}$;

(ii) $(1,0) \in \mathbb{D}$ and $\Re\big(\Theta(1,0)\big) > 0$;

(iii) $\Re\big(\Theta(iu_2, v_1)\big) \leqq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leqq -\frac{1}{2}(1 + u_2^2)$.

Let

$$\phi(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1.12}$$

be analytic (regular) in $\mathbb{U}$ *such that*

$$\phi(z) \not\equiv 1 \quad \textit{and} \quad (\phi(z), z\phi'(z)) \in \mathbb{D} \qquad (z \in \mathbb{U}).$$

If

$$\Re\Big(\Theta\big(\phi(z), z\phi'(z)\big)\Big) > 0 \qquad (z \in \mathbb{U}),$$

then

$$\Re\big(\phi(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

2 Inclusion Relationships

Our first set of inclusion relationships is given by Theorem 2.1 below.

Theorem 2.1. *Let* $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ *and* $\alpha + \nu > 1 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{S}^*_{\nu,c}(\alpha) \Longrightarrow f \in \mathcal{S}^*_{\nu+1,c}(\alpha)$$

or, equivalently,

$$\mathcal{S}^*_{\nu,c}(\alpha) \subset \mathcal{S}^*_{\nu+1,c}(\alpha).$$

Proof. Let $f \in \mathcal{S}^*_{\nu,c}(\alpha)$ and set

$$\frac{z\big(B^c_{\nu+1} f(z)\big)'}{B^c_{\nu+1} f(z)} - \alpha = (1-\alpha)\phi(z), \tag{2.13}$$

where $\phi(z)$ is given by (1.12). From (1.9) we get

$$\nu \frac{B^c_{\nu} f(z)}{B^c_{\nu+1} f(z)} = \frac{z\big(B^c_{\nu+1} f(z)\big)'}{B^c_{\nu+1} f(z)} + (\nu - 1). \tag{2.14}$$

By combining (2.13) and (2.14), we obtain

$$\frac{B^c_{\nu} f(z)}{B^c_{\nu+1} f(z)} = \frac{1}{\nu}\big[(1-\alpha)\phi(z) + \alpha + \nu - 1\big]. \tag{2.15}$$

Now, by applying the logarithmic differentiation on both sides of (2.15) and multiplying the resulting equation by z, we have

$$\frac{z\big(B^c_{\nu} f(z)\big)'}{B^c_{\nu} f(z)} = \frac{z\big(B^c_{\nu+1} f(z)\big)'}{B^c_{\nu+1} f(z)} + \frac{(1-\alpha) z\phi'(z)}{(1-\alpha)\phi(z) + \alpha + \nu - 1},$$

which, in view of (2.13), yields

$$\frac{z\big(B^c_{\nu} f(z)\big)'}{B^c_{\nu} f(z)} - \alpha = (1-\alpha)\phi(z) + \frac{(1-\alpha) z\phi'(z)}{(1-\alpha)\phi(z) + \alpha + \nu - 1}. \tag{2.16}$$

Upon taking

$$u = \phi(z) = u_1 + iu_2 \quad \text{and} \quad \nu = z\phi'(z) = v_1 + iv_2,$$

if we define the function $\Theta(u, v)$ by

$$\Theta(u, v) = (1-\alpha)u + \frac{(1-\alpha)v}{(1-\alpha)u + \alpha + \nu - 1},$$

then we observe that $\Theta(u, v)$ is continuous in

$$\mathbb{D} = \left(\mathbb{C} \backslash \left\{\frac{\alpha + \nu - 1}{\alpha - 1}\right\}\right) \times \mathbb{C}$$

and $(1,0) \in \mathbb{D}$, with $\Re(\Theta(1,0)) > 0$. Also, for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leqq -\frac{1}{2}(1+u_2^2)$, we have

$$\begin{aligned}\Re(\Theta(iu_2, v_1)) &= \Re\left(\frac{(1-\alpha)v_1}{(1-\alpha)iu_2+\alpha+\nu-1}\right)\\ &= \frac{(1-\alpha)(\alpha+\nu-1)v_1}{(\alpha+\nu-1)^2+(1-\alpha)^2u_2^2}\\ &\leqq \frac{-1}{2}\cdot\frac{(1-\alpha)(\alpha+\nu-1)(1+u_2^2)}{(\alpha+\nu-1)^2+(1-\alpha)^2u_2^2}\\ &< 0,\end{aligned}$$

which shows that $\Theta(u, v)$ satisfies the hypotheses of the above Miller-Mocanu Lemma. Therefore, we have

$$\Re(\phi(z)) > 0 \qquad (z \in \mathbb{U}).$$

Thus, by making use of (2.13) and (2.16), we find that $f \in \mathcal{S}^*_{\nu+1,c}(\alpha)$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let* $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ *and* $\alpha + \nu > 1 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{K}_{\nu,c}(\alpha) \Longrightarrow f \in \mathcal{K}_{\nu+1,c}(\alpha)$$

or, equivalently,

$$\mathcal{K}_{\nu,c}(\alpha) \subset \mathcal{K}_{\nu+1,c}(\alpha).$$

Proof. Applying (1.10) and Theorem 2.1, we observe that

$$\begin{aligned} f \in \mathcal{K}_{\nu,c}(\alpha) &\Longleftrightarrow B_\nu^c f(z) \in \mathcal{K}(\alpha)\\ &\Longleftrightarrow z\left(B_\nu^c f(z)\right)' \in \mathcal{S}^*(\alpha)\\ &\Longleftrightarrow B_\nu^c\left(zf'(z)\right) \in \mathcal{S}^*(\alpha)\\ &\Longleftrightarrow zf'(z) \in \mathcal{S}^*_{\nu,c}(\alpha)\\ &\Longrightarrow zf'(z) \in \mathcal{S}^*_{\nu+1,c}(\alpha)\\ &\Longleftrightarrow B_{\nu+1}^c\left(zf'(z)\right) \in \mathcal{S}^*(\alpha)\\ &\Longleftrightarrow z\left(B_{\nu+1}^c f(z)\right)' \in \mathcal{S}^*(\alpha)\\ &\Longleftrightarrow B_{\nu+1}^c f(z) \in \mathcal{K}(\alpha)\\ &\Longleftrightarrow f \in \mathcal{K}_{\nu+1,c}(\alpha),\end{aligned}$$

which evidently proves Theorem 2.2. □

Theorem 2.3. *Let* $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ *and* $\alpha + \nu > 1 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{C}_{\nu,c}(\beta,\alpha) \Longrightarrow f \in \mathcal{C}_{\nu+1,c}(\beta,\alpha) \quad (0 \leqq \beta < 1)$$

or, equivalently,

$$\mathcal{C}_{\nu,c}(\beta,\alpha) \subset \mathcal{C}_{\nu+1,c}(\beta,\alpha).$$

Proof. Let $f \in \mathcal{C}_{\nu,c}(\beta,\alpha)$. Then, in view of the definition of the class $\mathcal{C}_{\nu,c}(\beta,\alpha)$, there exists a function $g \in \mathcal{S}^*_{\nu,c}(\alpha)$ such that

$$\Re\left(\frac{z\left(B_\nu^c f(z)\right)'}{B_\nu^c g(z)}\right) > \beta, \quad (0 \leqq \beta < 1;\ z \in \mathbb{U}).$$

We now let

$$\frac{z\left(B_{\nu+1}^c f(z)\right)'}{B_{\nu+1}^c g(z)} - \beta = (1-\beta)\phi(z), \tag{2.17}$$

where the function $\phi(z)$ is given by (1.12). Now, making use of the identity (1.9), we also have

$$\begin{aligned}\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} &= \frac{B_\nu^c\big(zf'(z)\big)}{B_\nu^c g(z)}\\ &= \frac{z\Big(B_{\nu+1}^c\big(zf'(z)\big)\Big)' + (\nu-1)B_{\nu+1}^c\big(zf'(z)\big)}{z\big(B_{\nu+1}^c g(z)\big)' + (\nu-1)B_{\nu+1}^c g(z)}\\ &= \left(\frac{z\Big(B_{\nu+1}^c\big(zf'(z)\big)\Big)'}{B_{\nu+1}^c g(z)} + (\nu-1)\frac{B_{\nu+1}^c\big(zf'(z)\big)}{B_{\nu+1}^c g(z)}\right)\cdot\left(\frac{z\big(B_{\nu+1}^c g(z)\big)'}{B_{\nu+1}^c g(z)} + \nu - 1\right)^{-1}. \qquad (2.18)\end{aligned}$$

By Theorem 2.1, we know that

$$g \in \mathcal{S}_{\nu,c}^*(\alpha) \Longrightarrow g \in \mathcal{S}_{\nu+1,c}^*(\alpha),$$

so that we can set

$$\frac{z\big(B_{\nu+1}^c g(z)\big)'}{B_{\nu+1}^c g(z)} = (1-\alpha)q(z) + \alpha, \qquad (2.19)$$

where

$$\Re\big(q(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

Upon substituting from (2.17) and (2.19) into (2.18), we have

$$\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} = \frac{\left[z\Big(B_{\nu+1}^c\big(zf'(z)\big)\Big)'\right]\cdot\big[B_{\nu+1}^c g(z)\big]^{-1} + (\nu-1)[(1-\beta)\phi(z)+\beta]}{(1-\alpha)q(z)+\alpha+\nu-1}. \qquad (2.20)$$

By logarithmically differentiating both sides of (2.17) with respect to z, we have

$$\frac{z\Big(B_{\nu+1}^c\big(zf'(z)\big)\Big)'}{B_{\nu+1}^c g(z)} = (1-\beta)z\phi'(z) + [(1-\alpha)q(z)+\alpha]\cdot[(1-\beta)\phi(z)+\beta],$$

which, in conjunction with (2.20), yields

$$\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} - \beta = (1-\beta)\phi(z) + \frac{(1-\beta)z\phi'(z)}{(1-\alpha)q(z)+\alpha+\nu-1}. \qquad (2.21)$$

The remaining part of our proof of Theorem 2.3 is much akin to that of Theorem 2.1. Therefore, we choose to omit the analogous details involved. □

Theorem 2.4. *Let* $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R}\setminus\mathbb{Z}_0^-$ *and* $\alpha + \nu > 1 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{C}_{\nu,c}^*(\beta,\alpha) \Longrightarrow f \in \mathcal{C}_{\nu+1,c}^*(\beta,\alpha)$$

or, equivalently,

$$\mathcal{C}_{\nu,c}^*(\beta,\alpha) \subset \mathcal{C}_{\nu+1,c}^*(\beta,\alpha).$$

Proof. Applying (1.11) and Theorem 2.3, we observe that

$$\begin{aligned} f \in \mathcal{C}_{\nu,c}^*(\alpha) &\Longleftrightarrow B_\nu^c f(z) \in \mathcal{C}^*(\beta,\alpha)\\ &\Longleftrightarrow z\left(B_\nu^c f(z)\right)' \in \mathcal{C}(\beta,\alpha)\\ &\Longleftrightarrow B_\nu^c\left(zf'(z)\right) \in \mathcal{C}(\beta,\alpha)\\ &\Longleftrightarrow zf'(z) \in \mathcal{C}_{\nu,c}(\beta,\alpha)\\ &\Longrightarrow zf'(z) \in \mathcal{C}_{\nu+1,c}(\beta,\alpha)\\ &\Longleftrightarrow B_{\nu+1}^c\left(zf'(z)\right) \in \mathcal{C}(\beta,\alpha)\\ &\Longleftrightarrow z\left(B_{\nu+1}^c f(z)\right)' \in \mathcal{C}(\beta,\alpha)\\ &\Longleftrightarrow B_{\nu+1}^c f(z) \in \mathcal{C}^*(\beta,\alpha)\\ &\Longleftrightarrow f \in \mathcal{C}_{\nu+1,c}^*(\beta,\alpha),\end{aligned}$$

which evidently proves Theorem 2.4. □

3 Remarks and Observations

As already discussed in Section 1, the study of the generalized Bessel function of the first kind permits a unified study of the Bessel, the modified Bessel and the spherical Bessel functions. By specializing the parameters in the operator B_ν^c, we obtain the following new operators associated with the Bessel, the modified Bessel and the spherical Bessel functions (see, for details, [2]):

- Choosing $b = c = 1$ in (1.8), we obtain the operator $\mathcal{J}_p : \mathcal{A} \longrightarrow \mathcal{A}$ associated with the Bessel function, which is defined by

$$\mathcal{J}_p f(z) = \varphi_{p,1,1}(z) * f(z) = \left[2^p \Gamma(p+1) z^{1-p/2}\, J_p(\sqrt{z})\right] * f(z)$$
$$= z + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n (p+1)_n} \frac{z^{n+1}}{n!}. \tag{3.22}$$

- Taking $b = 1$ and $c = -1$ in (1.8), we obtain the operator $\mathcal{I}_p : \mathcal{A} \longrightarrow \mathcal{A}$ associated with the modified Bessel function, which is defined by

$$\mathcal{I}_p f(z) = \varphi_{p,1,-1}(z) * f(z) = \left[2^p \Gamma(p+1) z^{1-p/2}\, I_p(\sqrt{z})\right] * f(z)$$
$$= z + \sum_{n=1}^{\infty} \frac{a_{n+1}}{4^n (p+1)_n} \frac{z^{n+1}}{n!}. \tag{3.23}$$

- Letting $b = 2$ and $c = 1$ in (1.8), we obtain the operator $\mathcal{Q}_p : \mathcal{A} \longrightarrow \mathcal{A}$ associated with the spherical Bessel function, which is defined by

$$\mathcal{Q}_p f(z) = \varphi_{p,2,1}(z) * f(z) = \left[\pi^{-1/2} 2^{p+1/2} \Gamma\left(p + \frac{3}{2}\right) z^{1-p/2}\, j_p(\sqrt{z})\right] * f(z)$$
$$= z + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n (p+3/2)_n} \frac{z^{n+1}}{n!}. \tag{3.24}$$

Our main results (Theorems 2.1 to 2.4) can thus be applied with a view of deducing the following consequences.

Corollary 3.1. *Let $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\alpha + p > 0 \quad (0 \leqq \alpha < 1)$. Then*

$$f \in \mathcal{S}^*_{p+1,1}(\alpha) \Longrightarrow f \in \mathcal{S}^*_{p+2,1}(\alpha)$$

or, equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{S}^*(\alpha) \quad \Longrightarrow \quad f(z) \in \mathcal{S}^*_{p+n,1}(\alpha) \quad (n \in \mathbb{N} \setminus \{1\}).$$

Corollary 3.2. *Let $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\alpha + p > 0 \quad (0 \leqq \alpha < 1)$. Then*

$$f \in \mathcal{K}_{p+1,1}(\alpha) \Longrightarrow f \in \mathcal{K}_{p+2,1}(\alpha)$$

or, equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{K}(\alpha) \quad \Longrightarrow \quad f(z) \in \mathcal{K}_{p+n,1}(\alpha) \quad (n \in \mathbb{N} \setminus \{1\}).$$

Corollary 3.3. *Let $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\alpha + p > 0 \quad (0 \leqq \alpha < 1)$. Then*

$$f \in \mathcal{C}_{p+1,1}(\beta, \alpha) \Longrightarrow f \in \mathcal{C}_{p+2,1}(\beta, \alpha) \quad (0 \leqq \beta < 1)$$

or, equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{C}(\beta, \alpha) \quad \Longrightarrow \quad f(z) \in \mathcal{C}_{p+n,1}(\beta, \alpha) \quad (n \in \mathbb{N} \setminus \{1\}).$$

Corollary 3.4. *Let $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\alpha + p > 0 \quad (0 \leqq \alpha < 1)$. Then*

$$f \in \mathcal{C}^*_{p+1,1}(\beta, \alpha) \Longrightarrow f \in \mathcal{C}^*_{p+2,1}(\beta, \alpha) \quad (0 \leqq \beta < 1)$$

or, equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{C}^*(\beta, \alpha) \quad \Longrightarrow \quad f(z) \in \mathcal{C}^*_{p+n,1}(\beta, \alpha) \quad (n \in \mathbb{N} \setminus \{1\}).$$

Finally, we remark that similar results can be obtained involving the operators $\mathcal{I}_p$ and $\mathcal{Q}_p$ by specializing the parameter in Theorems 2.1 to 2.4. Numerous other applications and consequences of our main results (Theorems 2.1 to 2.4) and their aforementioned consequences (Corollaries 3.1 to 3.4) can indeed be derived similarly.

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Received: January 23, 2015; *Accepted*: June 12, 2015

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