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Common fixed point theorems in intuitionistic menger spaces using CLR property

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Abstract

We use the notion of CLR property to prove some common fixed point theorems for weakly compatible mappings in intuitionistic Menger spaces. Our theorems generalize and improve theorems of [5], [6], [7], [8], [10], [20] and [28].

Keywords: Common fixed point, intuitionistic Menger space, weakly compatible mappings, CLR property, JCLR property.

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1 Introduction

There have been a number of generalizations of metric spaces. One of such generalization is a probabilistic metric space, briefly, PM-spaces, introduced in 1942 by Menger [21]. In the PM-space, we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. This space was developed by Schweizer and Sklar [26, 27]. Modifying the idea of Kramosil and Michalek [15], George and Veeramani [9] introduced fuzzy metric spaces which are very similar to Menger spaces. Recently, using the idea of intuitionistic fuzzy set, see Atanassovas [2] and [3], which is a generalization of a fuzzy set, see Zadeh [32], Park [24] introduced the notion of intuitionistic fuzzy metric spaces as a generalization of fuzzy metric spaces due to George and Veeramani [9]. Kutukcu et. al [17] introduced the notion of intuitionistic Menger spaces as a generalization of Menger spaces.

Jungck [13] introduced the notion of compatible mappings in metric spaces. Mishra [22] extended the notion of compatibility to probabilistic metric spaces and this condition has been weakened by introducing the notion of weak compatibility by Jungck [14].

Sintunavarat and Kumam [31] introduced the concept of CLR property. Very recentlly, Chauhan et. al [4] introduced the notion of JCLR property. The importance of these properties is that we don't require the closedness of subspaces for the existence of fixed points.

The purpose of this paper is to prove common fixed point theorems for weakly compatible mappings in intuitionistic Menger spaces using these propertirs. Our theorems generalize and improve theorems of [5], [6], [7], [8], [10], [20] and [28].

2 Preliminaries

Definition 2.1 ([26]). *A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions.*

a) $*$ is commutative and associative,

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b) $*$ *is continuous,*
c) $a * 1 = a$ *for all* $a \in [0,1]$ *,*
d) $a * bc * d$ *wherever* ac*,* bd *and* $a, b, c, d \in [0,1]$ *.*

Examples of t-norms are $a * b = \min\{a, b\}$ and $a * b = ab$.

Definition 2.2 ([26]). *A binary operation* $\Diamond : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-conorm if* $\Diamond$ *satisfies the following conditions.*
a) $\Diamond$ *is commutative and associative,*
b) $\Diamond$ *is continuous,*
c) $a \Diamond 0 = a$ *for all* $a \in [0,1]$ *,*
d) $a \Diamond b \geq c \Diamond d$ *wherever* $a \geq c$*,* $b \geq d$ *and* $a, b, c, d \in [0,1]$ *.*

Examples of t-conorms are $a \Diamond b = \max\{a, b\}$ and $a \Diamond b = \min\{1, a + b\}$.

Remark 2.1. *The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic sketlons that we use for characterizing fuzzy intersection and union respectively. These concepts were originally introduced by Menger [21] in his study of statistical metric spaces.*

Definition 2.3 ([26]). *A distance distribution function is a function* $F : \mathbb{R} \to \mathbb{R}_+$ *which is left continuous on* $\mathbb{R}$*, non-decreasing and* $\inf_{t \in \mathbb{R}} F(t) = 0$*,* $\sup_{t \in \mathbb{R}} F(t) = 1$*. We will denote by* D *the family of all distance distribution functions and by* H *a special element of* D *defined by* $H(t) = \begin{cases} 0, & if \quad t0 \\ 1, & if \quad t > 0 \end{cases}$ *.*

If X is a non-empty set, $F : X \times X \to D$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by F_{xy}.

Definition 2.4 ([17]). *A non-distance distribution function is a function* $L : \mathbb{R} \to \mathbb{R}_+$ *which is right continuous on* $\mathbb{R}$*, non-increasing and* $\inf_{t \in \mathbb{R}} L(t) = 1$*,* $\sup_{t \in \mathbb{R}} L(t) = 0$*. We will denote by* E *the family of all non-distance distribution functions and by* G *a special element of* E *defined by* $G(t) = \begin{cases} 1, & if \quad t0 \\ 0, & if \quad t > 0 \end{cases}$ *.*

If X is a non-empty set, $L : X \times X \to E$ is called a probabilistic non-distance on X and $L(x, y)$ is usually denoted by L_{xy}.

Definition 2.5 ([17]). *A triplet* (X, F, L) *is said to be an intuitionistic probabilistic metric space if* X *is an arbitrary set,* F *is a probabilistic distance and* L *is a probabilistic non-distance on* X *satisfying the following conditions for all* $x, y, z \in X$ *and* $t, s > 0$
1) $F_{xy}(t) + L_{xy}(t) 1$*,*
2) $F_{xy}(0) = 0$*,*
3) $F_{xy}(t) = 1$ *if and only if* $x = y$*,*
4) $F_{xy}(t) = F_{yx}(t)$*,*
5) If $F_{xy}(t) = 1$ *and* $F_{yz}(s) = 1$*, then* $F_{xz}(t + s) = 1$*,*
6) $L_{xy}(0) = 1$*,*
7) $L_{xy}(t) = 0$ *if and only if* $x = y$*,*
8) $L_{xy}(t) = L_{yx}(t)$*,*
9) If $L_{xy}(t) = 0$ *and* $L_{yz}(s) = 0$*, then* $L_{xz}(t + s) = 0$*.*

Definition 2.6 ([17]). *A 5-tuple* $(X, F, L, *, \Diamond)$ *is said to be an intuitionistic Menger metric space if* (X, F, L) *is an intuitionistic probabilistic metric space and in addition, the following inequalities hold for all* $x, y, z \in X$ *and* $t, s > 0$*,*
1) $F_{xy}(t) * F_{yz}(s) F_{xz}(t + s)$*,*
2) $L_{xy}(t) \Diamond L_{yz}(s) L_{xz}(t + s)$*,*
where $*$ *is a continous t-norm and* $\Diamond$ *is a continous t-conorm.*

The functions F_{xy} and L_{xy} denote the degree of nearness and the degree of non-nearness between x and y with respect to t respectively.

Remark 2.2. *In intuitionistic Menger space* $(X, F, L, *, \Diamond)$*,* F_{xy} *is non-decreasing and* L_{xy} *is non-increasing for all* $x, y \in X$*.*

Remark 2.3 ([17]). *Every Menger space $(X, F, *)$ is an intuitionistic Menger space of the form $(X, F, 1-F, *, \diamondsuit)$ such that the t-norm $*$ and the t-conorm $\diamondsuit$ are associated, see [19], that is $x \diamondsuit y = 1 - (1-x) * (1-y)$ for any $x, y \in X$.*

Remark 2.4. *Kutukcu et al. [17] proved that if the t-norm $*$ and the t-conorm of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ satisfy the conditions*

$$\sup_{t \in (0,1)} (t * t) = 1 \text{ and } \inf_{t \in (0,1)} ((1-t) \diamondsuit (1-t)) = 0,$$

then $(X, F, L, *, \diamondsuit)$ is a Hausdorff topological space in the (ϵ, λ) topology, i.e., the family of sets

$$\{U_x(\epsilon, \lambda), \epsilon > 0, \lambda \in (0,1], x \in X\}$$

is a basis of neighborhoods of point x for a Hausdorff topology $\tau_{(F,L)}$, or (ϵ, λ) topology on X, where

$$U_x(\epsilon, \lambda) = \left\{ y \in X : F_{xy}(\epsilon) > 1 - \lambda \text{ and } L_{xy}(\epsilon) < \lambda \right\}.$$

Example 2.1 ([17]). *Let (X, d) be a metric space. Then the metric d induces a distance distribution function F defined by $F_{xy}(t) = H(t - d(x,y))$ and a non-distance distribution function L defined by $L_{xy}(t) = G(t - d(x,y))$ for all $x, y \in X$ and $t > 0$. Therefore, (X, F, L) is an intuitionistic probabilistic metric space induced by a metric d. If the t-norm $*$ is defined by $a * b = \min\{a, b\}$ and the t-conorm $\diamondsuit$ is defined by $a \diamondsuit b = \min\{1, a+b\}$ for all $a, b \in [0,1]$, then $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space.*

Remark 2.5 ([17]). *Note that the above example holds even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \diamondsuit b = \max\{a, b\}$ and hence $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space with respect to any t-norm and t-conorm. Also note that, in the above example, t-norm $*$ and t-conorm $\diamondsuit$ are not associated.*

Remark 2.6. *Every an intuitionistic fuzzy metric space $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space by considering $F : X \times X \to D$ and $L : X \times X \to E$ defined by $F_{xy}(t) = M(x, y, t)$ and $L_{xy}(t) = N(x, y, t)$ for all $x, y \in X$.*

Throughout this paper, $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space with the following conditions:

$$\lim_{t \to +\infty} F_{xy}(t) = 1 \text{ and } \lim_{t \to +\infty} L_{xy}(t) = 0, \text{ for all } x, y \in X \text{ and } t > 0. \tag{2.1}$$

Definition 2.7 ([17]). *Let $(X, F, L, *, \diamondsuit)$ be an intuitionistic Menger space.*

(a) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be convergent to a point $x \in X$, if for each $t > 0$ and $\epsilon \in (0,1)$, there exists a positive integer $n_0 = n_0(t, \epsilon)$ such that for all $n \geq n_0$

$$F_{x_n x}(t) > 1 - \epsilon \text{ and } L_{x_n x}(t) < \epsilon.$$

(b) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is called a Cauchy sequence if for all $t > 0$ and $\epsilon \in (0,1)$, there exists a positive integer $n_0 = n_0(t, \epsilon)$ such that for all $n, m \geq n_0$

$$F_{x_n x_m}(t) > 1 - \epsilon \text{ and } L_{x_n x_m}(t) < \epsilon.$$

(c) An intuitionistic Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.7 ([17]). *An induced intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ is complete if (X, d) is complete.*

Theorem 2.1 ([17]). *Let $(X, F, L, *, \diamondsuit)$ be an intuitionistic Menger space.*

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be convergent to a point $x \in X$ if and only if

$$\lim_{n \to +\infty} F_{x_n x}(t) = 1 \text{ and } \lim_{n \to +\infty} L_{x_n x}(t) = 0, \text{ for all } t > 0.$$

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is called a Cauchy sequence if and only if

$$\lim_{n \to +\infty} F_{x_n x_m}(t) = 1 \text{ and } \lim_{n \to +\infty} L_{x_n x_m}(t) = 0, \text{ for all } t > 0.$$

Lemma 2.1 ([17]). *Let $(X, F, L, *, \diamondsuit)$ be an intuitionistic Menger space and $\{x_n\}$, $\{y_n\}$ be two sequences in X with $x_n \to x$ and $y_n \to y$, respectively. Then*

(a)

$$\lim_{n\to\infty}\inf F_{x_n y_n}(t) \geq F_{xy}(t) \text{ and } \lim_{n\to\infty}\sup L_{x_n y_n}(t) \leq L_{xy}(t) \text{ for all } t > 0.$$

(b) If $t > 0$ is a continuous point of F_{xy} and L_{xy}, then

$$\lim_{n\to\infty} F_{x_n y_n}(t) = F_{xy}(t) \text{ and } \lim_{n\to\infty} L_{x_n y_n}(t) = L_{xy}(t).$$

Lemma 2.2 ([23]). *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an intuitionistic Menger space with the condition (2.1). If there exists a number $k \in (0,1)$ such that for $x, y \in X$, $t > 0$ and $n = 0, 1, 2, ...$*

$$F_{x_{n+2},x_{n+1}}(kt) \geq F_{x_{n+1},x_n}(t) \text{ and } L_{x_{n+2},x_{n+1}}(kt) \leq L_{x_{n+1},x_n}(t),$$

then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.3 ([23]). *Let $(X, F, L, *, \diamondsuit)$ be an intuitionistic Menger space. If there exists a number $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$*

$$F_{xy}(kt) \geq F_{xy}(t) \text{ and } L_{xy}(kt) \leq L_{xy}(t),$$

then $x = y$.

Definition 2.8 ([23]). *Two self-mappings A and S of an intuitionistic Menger space are said to be compatible if*

$$\lim_{n\to+\infty} F_{ASx_n,SAx_n}(t) = 1 \text{ and } \lim_{n\to+\infty} L_{ASx_n,SAx_n}(t) = 0 \text{ for all } t > 0,$$

whenever $\{x_n\} \subset X$ such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = z \text{ for some } z \in X.$$

Definition 2.9. *Two self-mappings A and S of an intuitionistic Menger space are said to be non-compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} F_{Ax_n,z}(t) = \lim_{n\to+\infty} F_{Sx_n,z}(t) = 1$ for some $z \in X$, but for some $t > 0$, either $\lim_{n\to+\infty} F_{ASx_n,SAx_n}(t) \neq 1$ or $\lim_{n\to+\infty} L_{ASx_n,SAx_n}(t) \neq 0$ or one of the limits do not exist.*

Definition 2.10 ([14]). *Two self-mappings A and S of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.*

Remark 2.8. *Two compatible self-mappings are weakly compatible, however the converse is not true in general, see [30], example 1.*

Definition 2.11 ([1, 25]). *A pair of self-mappings A and S of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ is said to be tangential or satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that for some $z \in X$*

$$\lim_{n\to+\infty} F_{Ax_n,z}(t) = \lim_{n\to+\infty} F_{Sx_n,z}(t) = 1 \text{ and } \lim_{n\to+\infty} L_{Ax_n,z}(t) = \lim_{n\to+\infty} L_{Sx_n,z}(t) = 0 \text{ for all } t > 0. \tag{2.2}$$

Remark 2.9. *It is easy to see that two non-ncompatible self-mappings of an intuitionistic Menger space satisfy the property (E.A), but the converse is not true in general.*

Definition 2.12 ([18]). *Two pairs (A, S) and (B, T) of self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ are said to satisfy the common property E.A, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that some $z \in X$ and for all $t > 0$*

$$\begin{aligned}\lim_{n\to+\infty} F_{Ax_n,z}(t) &= \lim_{n\to+\infty} F_{Sx_n,z}(t) = \lim_{n\to+\infty} F_{By_n,z}(t) = \lim_{n\to+\infty} F_{Ty_n,z}(t) = 1 \text{ and} \\ \lim_{n\to+\infty} L_{Ax_n,z}(t) &= \lim_{n\to+\infty} L_{Sx_n,z}(t) = \lim_{n\to+\infty} L_{By_n,z}(t) = \lim_{n\to+\infty} L_{Ty_n,z}(t) = 0.\end{aligned}$$

If $B = A$ and $T = S$ in this definition we get the definition of the property (E.A).

Definition 2.13 ([31]). *A pair of self-mappings A and S of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ is said to satisfy the common limit range property with respect to the mapping S (briefly CLR_S property), if there exists a sequence $\{x_n\}$ in X such that (2.2) holds, where $z \in S(X)$.*

Now, we give an example of self-mappings A and S satisfying the CLR_S property.

Example 2.2. *Let* $(X, F, L, *, \Diamond)$ *be an intuitionistic Menger space, where* $X = [0, \infty)$*, the t-norm* $*$ *is defined by* $a * b = \min\{a, b\}$*, the t-conorm* $\Diamond$ *is defined by* $a \Diamond b = \max\{a, b\}$ *and*

$$F_{xy}(t) = H(t - |x - y|),\ L_{xy}(t) = G(t - |x - y|)$$

for all $x, y \in X$ *and* $t > 0$*. Define self-mappings* A *and* S *on* X *by:* $Ax = x + 4$, $Sx = 5x$*. Let a sequence* $\left\{x_n = 1 + \frac{1}{n}\right\}_{n \in \mathbb{N}^*}$ *in* X*. Since* $\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = 5$*, then*

$$\begin{aligned} \lim_{n \to +\infty} F_{Ax_n,5}(t) &= \lim_{n \to +\infty} F_{Sx_n,5}(t) = 1 \text{ and} \\ \lim_{n \to +\infty} L_{Ax_n,5}(t) &= \lim_{n \to +\infty} L_{Sx_n,5}(t) = 0 \text{ for all } t > 0, \end{aligned}$$

where $5 \in S(X)$*. Therefore, the mappings* A *and* S *satisfy the* CLR_S *property.*

From this example, it is clear that a pair (A, S) satisfying the property (E.A)with the closedness of the subspace $S(X)$ always verifies the CLR_S property.

Definition 2.14 ([12]). *Two pairs* (A, S) *and* (B, T) *of self-mappings of an intuitionistic Menger space* $(X, F, L, *, \Diamond)$ *are said to satisfy the common limit range property with respect to mappings S and T (briefly,* CLR_{ST} *property), if there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *such that for all* $t > 0$

$$\begin{aligned} \lim_{n \to +\infty} F_{Ax_n,z}(t) &= \lim_{n \to +\infty} F_{Sx_n,z}(t) = \lim_{n \to +\infty} F_{By_n,z}(t) = \lim_{n \to +\infty} F_{Ty_n,z}(t) = 1 \text{ and} \\ \lim_{n \to +\infty} L_{Ax_n,z}(t) &= \lim_{n \to +\infty} L_{Sx_n,z}(t) = \lim_{n \to +\infty} L_{By_n,z}(t) = \lim_{n \to +\infty} L_{Ty_n,z}(t) = 0, \end{aligned}$$

where $z \in S(X) \cap T(X)$.

Remark 2.10. *If* $B = A$ *and* $T = S$ *in this definition we get the definition of* CLR_S *property.*

Remark 2.11. *The* CLR_{ST} *property implies the common property (E.A.), but the converse is not true in general, see [5], example 21.*

Proposition 2.1 ([5]). *If the pairs* (A, S) *and* (B, T) *satisfy the common property (E.A.) and* $S(X)$ *and* $T(X)$ *are closed subsets of X, then the pairs satsfy also the* CLR_{ST} *property.*

Definition 2.15 ([4]). *Two pairs* (A, S) *and* (B, T) *of self-mappings of an intuitionistic Menger space* $(X, F, L, *, \Diamond)$ *are said to satisfy the joint common limit range property with respect to mappings S and T(briefly* JCLR_{ST} *property), if there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *such that for all* $t > 0$

$$\begin{aligned} \lim_{n \to +\infty} F_{Ax_n,z}(t) &= \lim_{n \to +\infty} F_{Sx_n,z}(t) = \lim_{n \to +\infty} F_{By_n,z}(t) = \lim_{n \to +\infty} F_{Ty_n,z}(t) = 1 \text{ and} \\ \lim_{n \to +\infty} L_{Ax_n,z}(t) &= \lim_{n \to +\infty} L_{Sx_n,z}(t) = \lim_{n \to +\infty} L_{By_n,z}(t) = \lim_{n \to +\infty} L_{Ty_n,z}(t) = 0, \end{aligned}$$

where $z = Su = Tu$, $u \in X$.

Remark 2.12. *If* $B = A$ *and* $T = S$ *in this definition we get the definition of* CLR_S *property.*

Definition 2.16 ([11]). *Two families of self-mappings* $\{A_i\}$ *and* $\{S_j\}$ *are said to be pairwise commuting if*

(1) $A_i A_j = A_j A_i$, $i, j \in \{1, 2, ..., m\}$,
(2) $S_k S_l = S_l S_k$, $k, l \in \{1, 2, ..., n\}$,
(3) $A_i S_k = S_k A_i$, $i \in \{1, 2, ..., m\}, k \in \{1, 2, ..., n\}$.

3 Main results

Lemma 3.4. *Let* A*,* B*,* S *and* T *be self-mappings of an intuitionistic Menger space* $(X, F, L, *, \Diamond)$ *satisfying the following conditions.*

1) The pair (A, S) *satisfies the* CLR_S *property or the pair* (B, T) *satisfies the* CLR_T *property,*
2) $A(X) \subseteq T(X)$ *or* $B(X) \subseteq S(X)$*,*

3) $T(X)$ or $S(X)$ is a closed subset of X.

4) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges or $A(x_n)$ converges for every sequence $\{x_n\}$ in X whenever $S(x_n)$ converges.

$$
\begin{aligned}
(1+\alpha F_{Sx,Ty}(t))\, F_{Ax,By}(t) &> \alpha \min\left\{F_{Ax,Sx}(t)\, F_{By,Ty}(t), F_{Sx,By}(t)\, F_{Ax,Ty}(t)\right\} \\
&+ \min\left\{\begin{array}{c} F_{Sx,Ty}(t), \sup_{t_1+t_2=\frac{2}{k}t} \min\left\{F_{Ax,Sx}(t_1), F_{By,Ty}(t_2)\right\}, \\ \sup_{t_3+t_4=2t} \min\left\{F_{Sx,By}(t_3), F_{Ax,Ty}(t_4)\right\} \end{array}\right\} \\
(1+\beta L_{Sx,Ty}(t))\, L_{Ax,By}(t) &< \beta \max\left\{L_{Ax,Sx}(t)\, L_{By,Ty}(t), L_{Sx,By}(t)\, L_{Ax,Ty}(t)\right\} \\
&+ \max\left\{\begin{array}{c} L_{Sx,Ty}(t), \inf_{t_1+t_2=\frac{2}{k}t} \max\left\{L_{Ax,Sx}(t_1), L_{By,Ty}(t_2)\right\}, \\ \inf_{t_3+t_4=2t} \max\left\{L_{Sx,By}(t_3), L_{Ax,Ty}(t_4)\right\} \end{array}\right\}
\end{aligned} \tag{3.1}
$$

for all $x, y \in X$, $t > 0$, for some α, $\beta \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. Suppose that the pair (A, S) satisfies the CLR_S property and $T(X)$ is a closed subset of X. Then, there exists a sequence $\{x_n\}$ in X such that

$$
\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = z, \text{ where } z \in S(X).
$$

Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$. So

$$
\lim_{n\to+\infty} Ty_n = \lim_{n\to+\infty} Ax_n = z, \text{ where } z \in S(X) \cap T(X).
$$

Thus, $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now, we show that $By_n \to z$.

Let $\lim_{n\to+\infty} F_{By_n,l}(t_0) = 1$ and $\lim_{n\to+\infty} L_{By_n,l}(t_0) = 0$. We assert that $l = z$. Assume that $l \neq z$. We prove that there exists $t_0 > 0$ such that

$$
F_{z,l}\left(\frac{2}{k}t_0\right) > F_{z,l}(t_0) \text{ and } L_{z,l}\left(\frac{2}{k}t_0\right) < L_{z,l}(t_0). \tag{3.2}
$$

Suppose the contrary. Therefore, for all $t > 0$ we have

$$
F_{z,l}\left(\frac{2}{k}t\right) \leq F_{z,l}(t) \text{ and } L_{z,l}\left(\frac{2}{k}t\right) \geq L_{z,l}(t). \tag{3.3}
$$

Using repeatedly (3.3), we obtain

$$
\begin{aligned}
F_{z,l}(t) &\geq F_{z,l}\left(\frac{2}{k}t\right) \geq \ldots \geq F_{z,l}\left(\left(\frac{2}{k}\right)^n t\right) \to 1 \text{ and} \\
L_{z,l}(t) &\leq L_{z,l}\left(\frac{2}{k}t\right) \leq \ldots \leq L_{z,l}\left(\left(\frac{2}{k}\right)^n t\right) \to 0;
\end{aligned}
$$

as $n \to +\infty$, this shows that $F_{z,l}(t) = 1$ and $L_{z,l}(t) = 0$ for all $t > 0$, which contradicts $l \neq z$ and hence (3.2) is proved.

Without loss of generality, we may assume that t_0 in (3.2) is a continuous point of $F_{z,l}$ and L_{zl}. Since every distance distribution function is left-continuous and every a non-distance distribution function is right continuous, (3.2) implies that there exists $\epsilon > 0$ such that (3.2) holds for all $t \in (t_0 - \epsilon, t_0)$. Since $F_{z,l}$ is non-decreasing and $L_{z,l}$ is non-increasing, the set of all discontinuous points of $F_{z,l}$ and $L_{z,l}$ is a countable set at most. Thus, when t_0 is a discontinuous point of $F_{z,l}$ and $L_{z,l}$, we can choose a continuous point t_1 of $F_{z,l}$ and $F_{z,l}$ in $(t_0 - \epsilon, t_0)$ to replace t_0. Using the inequality (3.1) with $x = x_n$, $y = y_n$, we get for some $t_0 > 0$

$$
\begin{aligned}
(1+\alpha F_{Sx_n,Ty_n}(t_0))\, F_{Ax_n,By_n}(t_0) &> \alpha \min\left\{\begin{array}{c} F_{Ax_n,Sx_n}(t_0)\, F_{By_n,Ty_n}(t_0), \\ F_{Sx_n,By_n}(t_0)\, F_{Ax_n,Ty_n}(t_0) \end{array}\right\} \\
&+ \min\left\{\begin{array}{c} F_{Sx_n,Ty_n}(t_0), \\ \min\left\{F_{Ax_n,Sx_n}(\varepsilon), F_{By_n,Ty_n}\left(\frac{2}{k}t_0 - \varepsilon\right)\right\}, \\ \min\left\{F_{Sx_n,By_n}(2t_0 - \varepsilon), F_{Ax_n,Ty_n}(\varepsilon)\right\} \end{array}\right\}
\end{aligned}
$$

and

$$\left(1+\beta L_{Sx_n,Ty_n}(t_0)\right)L_{Ax_n,By_n}(t_0) < \beta\max\left\{\begin{array}{c} L_{Ax_n,Sx_n}(t_0)\,L_{By_n,Ty_n}(t_0), \\ L_{Sx_n,By_n}(t_0)\,L_{Ax_n,Ty_n}(t_0) \end{array}\right\} + \max\left\{\begin{array}{c} L_{Sx_n,Ty_n}(t_0), \\ \max\left\{L_{Ax_n,Sx_n}(\varepsilon), L_{By_n,Ty_n}\left(\frac{2}{k}t_0-\varepsilon\right)\right\}, \\ \max\left\{L_{Sx_n,By_n}(2t_0-\varepsilon), L_{Ax_n,Ty_n}(\varepsilon)\right\} \end{array}\right\}$$

for all $\varepsilon\in\left(0,\frac{2}{k}t_0\right)$. Letting $n\to+\infty$, we have

$$F_{z,l}(t_0)+\alpha F_{z,l}(t_0) \geq \alpha F_{z,l}(t_0)+\min\left\{F_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right), F_{l,z}(2t_0-\varepsilon)\right\} \text{ and}$$
$$L_{z,l}(t_0) < \max\left\{L_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right), L_{l,z}(2t_0-\varepsilon)\right\}$$

As $\varepsilon\to 0$, we obtain

$$F_{z,l}(t_0)\geq F_{z,l}\left(\frac{2}{k}t_0\right) \text{ and } L_{z,l}(t_0)\leq L_{z,l}\left(\frac{2}{k}t_0\right)$$

which contradicts (3.2) and so we have $z=l$. Thus, the pairs (A,S) and (B,T) satisfy the CLR$_{ST}$ property. □

Remark 3.13. *The converse of lemma 3.4 is not true in general, see the example 3.3 below.*

Theorem 3.1. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X,F,L,*,\Diamond)$ satisfying the inequality (3.1) of lemma 3.4. If the pairs (A,S) and (B,T) satisfy the CLR$_{ST}$ property, then (A,S) and (B,T) have coincidence points. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. Since the pairs (A,S) and (B,T) satisfy the CLR$_{ST}$ property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty}Ax_n=\lim_{n\to+\infty}Sx_n=\lim_{n\to+\infty}By_n=\lim_{n\to+\infty}Ty_n=z,$$

where $z\in S(X)\cap T(X)$. Hence, there exist $u,v\in X$ such that $Su=Tv=z$. Now, we show that $Au=Su=z$. As in the proof of lemma 3.4, we can prove that $Au=Su=z$ by putting $x=u$ and $y=y_n$ in the inequality (3.1). Therefore, u is a coincidence point of the pair (A,S).

Now, we assert that $Bv=Tv=z$. If $z\neq Bv$, putting $x=u$ and $y=v$ in the inequality (3.1), we get for some $t_0>0$

$$\left(1+\alpha F_{Su,Tv}(t_0)\right)F_{Au,Bv}(t_0) > \alpha F_{Bv,z}(t_0)+\min\left\{F_{Bv,z}\left(\frac{2}{k}t_0-\epsilon\right), F_{Bv,z}(2t_0-\epsilon)\right\} \text{ and}$$
$$\left(1+\beta L_{Su,Tv}(t_0)\right)L_{Au,Bv}(t_0) < \max\left\{L_{Bv,z}\left(\frac{2}{k}t_0-\epsilon\right), L_{z,Bv}(2t_0-\epsilon)\right\}$$

for all $\varepsilon\in\left(0,\frac{2}{k}t_0\right)$. Letting $\varepsilon\to 0$, we have

$$F_{z,Bv}(t_0)\geq F_{z,Bv}\left(\frac{2}{k}t_0\right) \text{ and } L_{z,Bv}(t_0)\leq L_{z,Bv}\left(\frac{2}{k}t_0\right),$$

which contradicts (3.2) and so $Bv=Tv=z$. Therefore, v is a coincidence point of the pair (B,T).

Since the pair (A,S) is weakly compatible and $Au=Su$ we obtain $Az=Sz$. Now, we prove that z is a common fixed point of A and S. If $z\neq Az$, applying the inequality (3.1) with $x=z$ and $y=v$, we get for some $t_0>0$

$$\left(1+\alpha F_{Sz,Tv}(t_0)\right)F_{Az,Bv}(t_0) > \alpha(F_{Az,z}(t))^2+\min\left\{F_{Az,z}(t_0), F_{Az,z}(t_0)\right\}$$

and

$$\left(1+\beta L_{Sz,Tv}(t_0)\right)L_{Az,Bv}(t_0) < \beta(L_{Az,z}(t_0))^2+\max\left\{L_{Az,z}(t_0), L_{Az,z}(t_0)\right\}.$$

Hence

$$F_{Az,z}(t_0) > F_{Az,z}(t_0) \text{ and } L_{Az,z}(t_0) < L_{Az,z}(t_0),$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S.

Since the pair (B,T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T. Hence, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.1). □

Remark 3.14. *Theorem 3.1 improves and generalizes theorem 3.1 of [8].*

Now, we give an example to support our theorem 3.1.

Example 3.3. *Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space, where $X = [3,11[$, $a * b = \min\{a,b\}$ and $a \Diamond b = \max\{a,b\}$ with*

$$F_{xy}(t) = H(t - |x-y|), \quad L_{xy}(t) = G(t - |x-y|)$$

for all $x, y \in X$ and $t > 0$. Define the self-mappings A, B, S and T by

$$Ax = \begin{cases} 3 & x \in \{3\} \cup]5,11[\\ 10 & x \in]3,5] \end{cases}, \; Bx = \begin{cases} 3 & x \in \{3\} \cup]5,11[\\ 9 & x \in]3,5] \end{cases}$$

$$Sx = \begin{cases} 3 & \text{if} \quad x = 3 \\ 7 & \text{if} \quad x \in]3,5] \\ \dfrac{x+1}{2} & \text{if} \quad x \in]5,11[\end{cases}, \; Tx = \begin{cases} 3 & \text{if} \quad x = 3 \\ x+4 & \text{if} \quad x \in]3,5] \\ x-2 & \text{if} \quad x \in]5,11[\end{cases}.$$

We Take $\{x_n = 3\}$, $\left\{y_n = 5 + \frac{1}{n}\right\}$ or $\left\{x_n = 5 + \frac{1}{n}\right\}$, $\{y_n = 3\}$. Since

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = 3 \in S(X) \cap T(X),$$

then, the pairs (A,S) and (B,T) satisfy the property CLR_{ST}. Also,

$$A(X) = \{3,10\} \,]3,9[= T(X) \text{ and } B(X) = \{3,9\} \, (\{7\} \cup]3,6[) = S(X).$$

Thus, all the conditions of theorem 3.1 are satisfied and 3 is a unique common fixed point of the pairs (A,S) and (B,T).

Remark that all the mappings are even discontinuous at their unique common fixed 3. In this example $S(X)$ and $T(X)$ are not closed subsets of X.

Lemma 3.5. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the conditions 1,2,3,4 of lemma 3.4 and*

$$\begin{aligned} &(1 + \alpha F_{Sx,Ty}(t)) F_{Ax,By}(t) > \alpha \min\{F_{Ax,Sx}(t) F_{By,Ty}(t), F_{Sx,By}(t) F_{Ax,Ty}(t)\} \\ &\quad + \min\left\{ \begin{array}{c} F_{Sx,Ty}(t), \sup_{t_1+t_2=\frac{2}{k}t} \min\{F_{Ax,Sx}(t_1), F_{Sx,By}(t_2)\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \min\{F_{By,Ty}(t_3), F_{Ax,Ty}(t_4)\} \end{array} \right\} \\ &(1 + \beta L_{Sx,Ty}(t)) L_{Ax,By}(t) < \beta \max\{L_{Ax,Sx}(t) L_{By,Ty}(t), L_{Sx,By}(t) L_{Ax,Ty}(t)\} \\ &\quad + \max\left\{ \begin{array}{c} L_{Sx,Ty}(t), \inf_{t_1+t_2=\frac{2}{k}t} \max\{L_{Ax,Sx}(t_1), L_{Sx,By}(t_2)\}, \\ \inf_{t_3+t_4=\frac{2}{k}t} \max\{L_{By,Ty}(t_3), L_{Ax,Ty}(t_4)\} \end{array} \right\} \end{aligned} \tag{3.4}$$

for all $x, y \in X$, $t > 0$ for some α, $\beta \geq 0$ and $1 \leq k < 2$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Proof. As in the proof of lemma 3.4, there exists $t_0 > 0$ such that (3.2) holds. Using the inequality (3.4) with $x = x_n$, $y = y_n$ and letting $n \to +\infty$, we have for all $\varepsilon \in \left(0, \frac{2}{k} t_0\right)$.

$$F_{z,l}(t_0) \qquad \min\left\{1, F_{l,z}\left(\frac{2}{k}t_0 - \varepsilon\right)\right\} = F_{l,z}\left(\frac{2}{k}t_0 - \varepsilon\right),$$

$$L_{z,l}(t_0) \qquad \max\left\{0, L_{l,z}\left(\frac{2}{k}t_0 - \varepsilon\right)\right\} = L_{l,z}\left(\frac{2}{k}t_0 - \varepsilon\right)$$

As $\varepsilon \to 0$, we obtain

$$F_{z,l}(t_0) \geq F_{z,l}\left(\frac{2}{k}t_0\right) \text{ and } L_{z,l}(t_0) \leq L_{z,l}\left(\frac{2}{k}t_0\right)$$

which contradicts (3.2) and so we have $z = l$. Thus, the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. □

Theorem 3.2. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the inequality (3.4) of lemma 3.5. If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, then (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. As in the proof of theorem 3.1, there exist $u, v \in X$ such that $Su = Tv = z$. Now, we show that $Au = Su = z$. If $z \neq Au$, putting $x = u$ and $y = y_n$ in the inequality (3.4) and letting $n \to +\infty$, we have for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$.

$$F_{Au,z}(t_0)\, F_{Au,z}\left(\frac{2}{k}t_0 - \varepsilon\right) \text{ and } L_{Au,z}(t_0) \leq L_{Au,z}\left(\frac{2}{k}t_0 - \varepsilon\right).$$

Letting $\varepsilon \to 0$, we have

$$F_{Au,z}(t_0) \geq F_{Au,z}\left(\frac{2}{k}t_0\right) \text{ and } L_{Au,z}(t_0) \leq L_{Au,z}\left(\frac{2}{k}t_0\right),$$

which contradicts (3.2) and so $Au = Su = z$. Therefore, u is a coincidence point of the pair (A, S).

Now, we assert that $Bv = Tv = z$. If $z \neq Bv$, putting $x = u$ and $y = v$ in the inequality (3.4), we get for some $t_0 > 0$

$$\begin{aligned} (1 + \alpha F_{Su,Tv}(t_0))\, F_{Au,Bv}(t_0) &> \alpha F_{z,Bv}(t_0) + F_{z,Bv}\left(\frac{2}{k}t_0 - \varepsilon\right) \text{ and} \\ (1 + \beta L_{Su,Tv}(t_0))\, L_{Au,Bv}(t_0) &< L_{z,Bv}\left(\frac{2}{k}t_0 - \varepsilon\right) \end{aligned}$$

for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. As $\varepsilon \to 0$ we have

$$F_{z,Bv}(t_0) \geq F_{z,Bv}\left(\frac{2}{k}t_0\right) \text{ and } L_{z,Bv}(t_0) \leq L_{z,Bv}\left(\frac{2}{k}t_0\right),$$

which contradicts (3.2) and so $Bv = Tv = z$. Therefore, v is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now, we assert that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.4) with $x = z$ and $y = v$, we obtain for some $t_0 > 0$

$$(1 + \alpha F_{Sz,Tv}(t_0))\, F_{Az,Bv}(t_0) > \alpha (F_{Az,z}(t_0))^2 + \min\left\{F_{Az,z}(t_0), F_{Az,z}\left(\frac{2}{k}t_0\right)\right\}$$

and

$$(1 + \beta L_{Sz,Tv}(t_0))\, L_{Az,Bv}(t_0) < \beta (L_{Az,z}(t))^2 + \max\left\{L_{Az,z}(t_0), L_{Az,z}\left(\frac{2}{k}t_0\right)\right\}$$

Hence

$$\begin{aligned} F_{Az,z}(t_0) &> \min\left\{F_{Az,z}(t_0), F_{Az,z}\left(\frac{2}{k}t_0\right)\right\} = F_{Az,z}(t_0) \text{ and} \\ L_{Az,z}(t_0) &< \max\left\{L_{Az,z}(t_0), L_{Az,z}\left(\frac{2}{k}t_0\right)\right\} = L_{Az,z}(t_0). \end{aligned}$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S. Since the pair (B, T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T. Therefore, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.4). □

Remark 3.15. *Theorem 3.2 improves and generalizes theorem 3.2 of [8] and theorem 2.1 of [10].*

If $B = A$ and $T = S$ in theorems 3.1 and 3.2, we obtain a common fixed point for a pair of self-mappings.

Applying theorems 3.1, 3.2, 3.3 and 3.4, we deduce a common fixed point for four finite families of self-mappings given by the following corollary.

Corollary 3.1. *Let $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$, where $*$ is a continuous t-norm and $\Diamond$ is a continuous t-conorm with $A = A_1A_2....A_m$, $B = B_1B_2....B_n$, $S = S_1S_2....S_p$ and $T = T_1T_2....T_q$ satisfying the inequality (3.1) of lemma 3.4 or the the inequality (3.4) of lemma 3.5. Suppose that the pairs (A, S) and (B, T) verify the CLR$_{ST}$ property. Then $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ a unique common fixed point in X provided that the pairs of families $\left(\{A_i\}_{i=1}^m, \{S_k\}_{k=1}^p\right)$ and $\left(\{B_r\}_{r=1}^n, \{T_h\}_{h=1}^q\right)$ commute pairwise.*

By setting $A_1 = A_2 = ... = A_m = A$, $B_1 = B_2 = ... = B_n = B$, $S_1 = S_2 = ... = S_p = S$ and $T_1 = T_2 = ... = T_q = T$ in corollary 3.1, we get thai A, B, S and T have a unique common fixed point in X provided that the pairs (A^m, S^p) and (B^n, T^q) commute pairwise.

In the proof of the following lemma, we don't need to prove the inequality (3.2).

Lemma 3.6. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the conditions 1,2,3,4 of lemma 3.4 and*

$$\begin{aligned}
&(1 + \alpha F_{Sx,Ty}(t))\, F_{Ax,By}(t) > \alpha \min\{F_{Ax,Sx}(t)\, F_{By,Ty}(t), F_{Sx,By}(t)\, F_{Ax,Ty}(t)\} \\
&\quad + \min\left\{\begin{array}{c} F_{Sx,Ty}(t), \sup_{t_1+t_2=\frac{2}{k}t} \max\{F_{Ax,Sx}(t_1), F_{By,Ty}(t_2)\}, \\ \sup_{t_3+t_4=2t} \max\{F_{Sx,By}(t_3), F_{Ax,Ty}(t_4)\} \end{array}\right\} \\
&(1 + \beta L_{Sx,Ty}(t))\, L_{Ax,By}(t) < \beta \max\{L_{Ax,Sx}(t)\, L_{By,Ty}(t), L_{Sx,By}(t)\, L_{Ax,Ty}(t)\} \\
&\quad + \max\left\{\begin{array}{c} L_{Sx,Ty}(t), \inf_{t_1+t_2=\frac{2}{k}t} \min\{L_{Ax,Sx}(t_1), L_{By,Ty}(t_2)\}, \\ \inf_{t_3+t_4=2t} \min\{L_{Sx,By}(t_3), L_{Ax,Ty}(t_4)\} \end{array}\right\}
\end{aligned} \tag{3.5}$$

for all $x, y \in X$, $t > 0$, for some $\alpha, \beta \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR$_{ST}$ property.

Proof. As in the proof of lemma 3.4, $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now, we show that $By_n \to z$. We assert that $l = z$. Assume that $l \neq z$. Using the inequality (3.5) with $x = x_n$, $y = y_n$ and letting $n \to +\infty$ we have for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$

$$F_{z,l}(t_0)\, 1 \text{ and } L_{z,l}(t_0)\, 0$$

and so we have $z = l$. Thus, the pairs (A, S) and $B, T)$ satisfy the CLR$_{ST}$ property. □

Theorem 3.3. *Let A, B, S and T be self mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the inequality (3.5) of lemma 3.6 . If the pairs (A, S) and (B, T) satisfy the CLR$_{ST}$ property, then (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. As in the proof of theorem 3.1, there exist $u, v \in X$ such that $Au = Su = Bv = Tv = z$. Therefore, u is a coincidence point of the pair (A, S) and v is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now, we assert that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.5) with $x = z$ and $y = v$, we obtain for some $t_0 > 0$

$$(1 + \alpha F_{Sz,Tv}(t_0))\, F_{Az,Bv}(t_0) > \alpha (F_{Az,z}(t_0))^2 + \min\{F_{Az,z}(t_0), F_{Az,z}(t_0)\}$$

and

$$(1 + \beta L_{Sz,Tv}(t_0))\, L_{Az,Bv}(t_0)\, \beta (L_{Az,z}(t_0))^2 + \max\{L_{Az,z}(t_0), L_{Az,z}(t_0)\}.$$

Hence

$$F_{Az,z}(t_0) > F_{Az,z}(t_0) \text{ and } L_{Az,z}(t_0) < L_{Az,z}(t_0),$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S.

Since the pair (B, T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T by putting $x = y = z$ in the inequality (3.5). Therefore, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.5). □

Let Φ be the set of all non-decreasing and continuous functions $\varphi : (0,1] \to (0,1]$ such that $\varphi(t) > t$ for all $t \in (0,1]$ and Ψ be the set of all non-increasing and continuous functions $\psi : (0,1] \to (0,1]$ such that $\psi(t) < t$ for all $t \in (0,1]$.

Lemma 3.7. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the conditions 1,2,3,4 of lemma 3.4 and*

$$\begin{array}{l} (1+\alpha F_{Sx,Ty}(t)) F_{Ax,By}(t) \geq \alpha \min\{F_{Ax,Sx}(t) F_{By,Ty}(t), F_{Sx,By}(t) F_{Ax,Ty}(t)\} \\ \quad +\varphi\left(\min\left\{\begin{array}{c} F_{Sx,Ty}(t), \sup_{t_1+t_2=\frac{2}{k}t} \min\{F_{Ax,Sx}(t_1), F_{By,Ty}(t_2)\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \min\{F_{Sx,By}(t_3), F_{Ax,Ty}(t_4)\} \end{array}\right\}\right) \\ (1+\beta L_{Sx,Ty}(t)) L_{Ax,By}(t) \leq \beta \max\{L_{Ax,Sx}(t) L_{By,Ty}(t), L_{Sx,By}(t) L_{Ax,Ty}(t)\} \\ \quad \psi\left(\max\left\{\begin{array}{c} L_{Sx,Ty}(t), \inf_{t_1+t_2=\frac{2}{k}t} \max\{L_{Ax,Sx}(t_1), L_{By,Ty}(t_2)\}, \\ \inf_{t_3+t_4=\frac{2}{k}t} \max\{L_{Sx,By}(t_3), L_{Ax,Ty}(t_4)\} \end{array}\right\}\right) \end{array} \tag{3.6}$$

for all $x, y \in X$, $t > 0$, for some α, $\beta \geq 0$ and $1 \leq k < 2$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. It follows as in the proof of lemma 3.4. □

Remark 3.16. *Lemmas 3.4, 3.5, 3.6 and 3.7 remain true if we assume that the pair (B, T) satisfies the CLR_T property, $B(X) \subseteq S(X)$ and $S(X)$ is a closed subset of X.*

Theorem 3.4. *Let A, B, S and T be self mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the inequality (3.6) of lemma 3.7. If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, then (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. As in the proof of theorem 3.1, $z = Au = Su = Bv = Tv$. Since the pair (A, S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now, we assert that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.6) with $x = z$ and $y = v$, we obtain for some $t_0 > 0$

$$\begin{aligned} (1+\alpha F_{Sz,Tv}(t_0)) F_{Az,Bv}(t_0) &\geq \alpha (F_{Az,z}(t_0))^2 + \\ &\quad \varphi\left(\min\left\{F_{Az,z}(t_0), \min\left\{F_{Az,z}(\epsilon), F_{Az,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}\right\}\right) \end{aligned}$$

and

$$\begin{aligned} (1+\beta L_{Sz,Tv}(t_0)) L_{Az,Bv}(t_0) &\leq \beta (L_{Az,z}(t_0))^2 + \\ &\quad \psi\left(\max\left\{L_{Az,z}(t_0), \max\left\{L_{Az,z}(\epsilon), L_{Az,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}\right\}\right). \end{aligned}$$

for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $\varepsilon \to 0$, we get

$$\begin{aligned} F_{Az,z}(t_0) &\geq \varphi(\min F_{Az,z}(t_0)) > F_{Az,z}(t_0) \\ L_{Az,z}(t_0) &\leq \psi(\max L_{Az,z}(t_0)) < L_{Az,z}(t_0) \end{aligned}$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S.

Since the pair (B, T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T by putting $x = y = z$ in the inequality (3.6). Therefore, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.6). □

Remark 3.17. *Theorem 3.4 improves and generalizes theorem 26 of [5], theorem 3.2 of [6], theorem 3.1 of [7], theorems 3.1, 3.2 of [20] and theorem 1 of [28].*

Remark 3.18. *In theorems 3.1, 3.2, 3.3 and 3.4, by a similar manner, we can prove that A, B, S and T have a unique common fixed point in X if we assume that the pairs (A, S) and (B, T) verify $JCLR_{ST}$ property or CLR_{AB} property instead of CLR_{ST} property.*

Remark 3.19. *It is easy to see that theorem 3.1 remains true if we replace*

$$\sup_{t_3+t_4=2t} \min\left\{F_{Sx,By}\left(t_3\right), F_{Ax,Ty}\left(t_4\right)\right\} \text{ and } \inf_{t_3+t_4=2t} \max\left\{L_{Sx,By}\left(t_3\right), L_{Ax,Ty}\left(t_4\right)\right\}$$

in the inequality (3.1) by

$$\sup_{t_3+t_4=2t} \max\left\{F_{Sx,By}\left(t_3\right), F_{Ax,Ty}\left(t_4\right)\right\} \text{ and } \inf_{t_3+t_4=2t} \min\left\{L_{Sx,By}\left(t_3\right), L_{Ax,Ty}\left(t_4\right)\right\}$$

respectively. Also, theorems 3.2 and 3.4 remain true if we replace

$$\sup_{t_3+t_4=\frac{2}{k}t} \min\left\{F_{By,Ty}\left(t_3\right), F_{Ax,Ty}\left(t_4\right)\right\} \text{ and } \inf_{t_3+t_4=\frac{2}{k}t} \max\left\{L_{By,Ty}\left(t_3\right), L_{Ax,Ty}\left(t_4\right)\right\}$$

in the inequalities (3.4) and (3.6) by

$$\sup_{t_3+t_4=\frac{2}{k}t} \max\left\{F_{By,Ty}\left(t_3\right), F_{Ax,Ty}\left(t_4\right)\right\} \text{ and } \inf_{t_3+t_4=\frac{2}{k}t} \min\left\{L_{By,Ty}\left(t_3\right), L_{Ax,Ty}\left(t_4\right)\right\}$$

respectively.

Theorem 3.5. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the conditions of lemma 3.4 or lemma 3.5 or lemma 3.6 or lemma 3.7. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. In view of lemma 3.4, lemma 3.5, lemma 3.6 and lemma 3.7, the pairs (A, S) and (B, T) verify the CLR$_{ST}$ property, therefore there exist two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. The rest of the proof follows as in the proof of theorems 3.1, 3.2, 3.3 and 3.4. □

Remark 3.20. *Theorem 3.5 improves and generalizes theorem 28 of [5], theorem 3.3 of [7] and theorem 2.3 of [10].*

Example 3.4. *We retain A and B and replace S and T in the example 3.3 by the following mappings*

$$Sx = \begin{cases} 3 & \text{if} \quad x = 3 \\ 6 & \text{if} \quad x \in \left]3, 5\right] \\ \dfrac{x+1}{2} & \text{if} \quad x \in [5, 11) \end{cases}, \; Tx = \begin{cases} 3 & \text{if} \quad x = 3 \\ 9 & \text{if} \quad x \in \left]3, 5\right[\\ x-2 & \text{if} \quad x \in [5, 11) \end{cases}.$$

Therefore,

$$A(X) = \{3, 4\} \subset [3, 9] = T(X) \text{ and } B(X) = \{3, 5\} \subset [3, 6] = S(X).$$

Thus, all the conditions of theorem 3.3 are satisfied and 3 is a unique common fixed point of the pairs (A, S) and (B, T). Also, it is noted that theorem 3.1 can not be used in the context of this example as $S(X)$ and $T(X)$ are closed subsets of X.

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Graceful labeling of arrow graphs and double arrow graphs

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Abstract

In this paper we define arrow graph and double arrow graph. We also prove that arrow graphs A_n^2, A_n^3,A_n^5 are graceful and double arrow graphs DA_n^2 and DA_n^3 are graceful.

Keywords: Graceful labeling, arrow graph, double arrow graph.

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1 Introduction

The graceful labeling was introduced by A.Rosa[1] during 1967. Golomb[2] named such labeling as graceful labeling which was called earlier as β-valuation. We introduce new graphs which are called arrow graph(A_n^k) and double arrow graph(DA_n^k). Kaneria and Makadia[4] introduced step grid graph(St_n) and double step grid graph(DSt_n). These two graphs are graceful. They also proved that path union of finite copies of above graphs, star graph of above graphs, cycle graph of above graphs are graceful.

We begin with a simple, undirected finite graph G=(V,E), with $|V| = p$ vertices and $|E| = q$ edges. For all terminology and notations we follows Harary[3]. Here are some of the definitions, which are used in this paper.

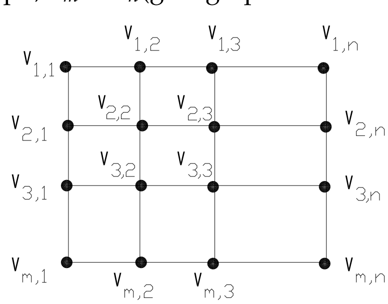
Definition 1.1 Graceful labeling

A function f is called graceful labeling of a graph G, if $f : V(G) \longrightarrow \{0, 1, ..., q\}$ is injective and the induced function $f^* : E(G) \longrightarrow \{1, 2, ..., q\}$ defined as $f^*(e) = |f(u) - f(v)|$ is bijective for every edge e=(u,v)$\in E(G)$, where $q = |E(G)|$.

1A graph G is called graceful if it admits a graceful labeling.

Definition 1.2 superior vertices

In Graph, $P_m \times P_n$(grid graph on mn vertices)



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vertices $v_{1,1}, v_{2,1}, v_{3,1}, ..., v_{m,1}$ and vertices $v_{1,n}, v_{2,n}, v_{3,n}, ..., v_{m,n}$ are known as superior vertices from both the

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ends .

Definition 1.3 Arrow Graph

An arrow graph A_n^t with width t and length n is obtained by joining a vertex v with superior vertices of $P_m \times P_n$ by m new edges from one end.

Definition 1.4 Double Arrow Graph

A Double arrow graph DA_n^t with width t and length n is obtained by joining two vertices v and w with superior vertices of $P_m \times P_n$ by $m + m$ new edges from both the ends.

In this paper we introduce gracefulness of arrow graph and double arrow graph. For detail survey of grpah labeling we refer Gallian[5].

2 Main results

Theorem$-$2.1 **:** A_n^2 is a graceful graph, where $n \in N$

Proof : Let G = A_n^2 be an arrow graph obtained by joining a vertex v with superior vertices of $P_2 \times P_n$ by 2 new edges.

Let $u_{i,j}$ (i= 1,2; j= 1,2,...,n) be vertices of $P_2 \times P_n$.

Join v with $u_{i,1}$ (i= 1,2) by two new edges to obtain G. Obviously $p = |V(G)| = 2n + 1$ and $q = |E(G)| = 3n$.

We define labeling function $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ as follows,

f(v) = 0, f($u_{2,1}$) = 1

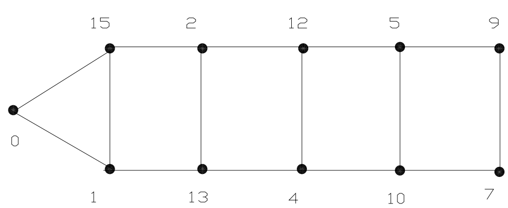
f($u_{1,j}$) = q-($\frac{j-1}{2}$)3 = q-$\frac{3}{2}$(j-1) ;when j $\equiv$ 1(mod 2)

= $\frac{3j}{2} - 1$;when j $\equiv$ 0(mod 2) $\forall j = 1, 2, \ldots, n;$

f($u_{2,j}$) = f($u_{1,j-1}$)-$(-1)^j$2, $\forall j = 2, 3, \ldots, n.$

ıAbove labeling pattern give rise a graceful labeling to given graph G.

Illustration$-$2.2**:** Arrow graph A_5^2 and its graceful labeling shown in figure$-$1



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Figure$-$1 Arrow Graph A_5^2 and its Graceful Labeling

Theorem$-$2.3 **:** A_n^3 is a graceful graph, where $n \geq 2$.

Proof : Let G = A_n^3 be an arrow graph obtained by joining a vertex v with superior vertices of $P_3 \times P_n$ by 3 new edges.

Let $u_{i,j}$ $(i = 1, 2, 3;\ t = 1, 2, \ldots, n)$ be vertices of $P_3 \times P_n$.

Join v with $u_{i,1}$ (i= 1,2,3) by three new edges to obtain G. Obviously $p = |V(G)| = 3n + 1$ and $q = |E(G)| = 5n$. We define labeling function $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ as follows,

f(v) = 0 f($u_{2,1}$) = 2;

f($u_{1,j}$) = q-$\frac{5}{2}$(j-1) ;when j $\equiv$ 1(mod 2)

= $\frac{5j}{2} - 1$;when j $\equiv$ 0(mod 2) $\forall j = 1, 2, \ldots, n-1;$

f($u_{2,j}$) = f($u_{1,j-1}$)-$(-1)^j \cdot 3$, $\forall j = 1, 2, \ldots, n-1;$

f($u_{3,j}$) = f($u_{1,j}$)+$(-1)^j$, $\forall j = 1, 2, \ldots, n-1;$

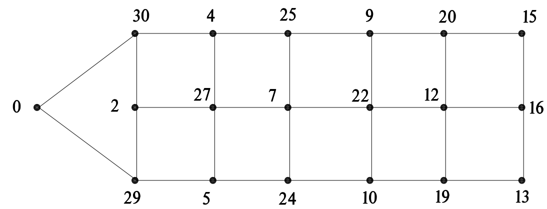
f($u_{1,n}$) = f($u_{1,n-1}$)-$(-1)^n \cdot 5$

f($u_{2,n}$) = f($u_{2,n-1}$)+$(-1)^n \cdot 4$

f($u_{3,n}$) = f($u_{3,n-1}$)-$(-1)^n \cdot 6$.

ıAbove labeling pattern give rise a graceful labeling to given graph G.

Illustration$-$2.4**:** Arrow graph A_6^3 and its graceful labeling shown in figure$-$2.



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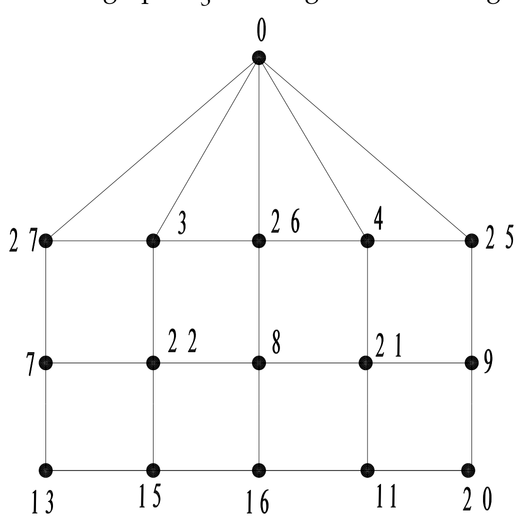
Figure−2 Arrow graph A_6^3 and its graceful labeling

Theorem−2.5 **:** A_n^5 is a graceful graph, where $n \geq 2$.
Proof : Let G = A_n^5 be an arrow graph obtained by joining a vertex v with superior vertices of $P_5 \times P_n$ by 5 new edges.
Let $u_{i,j}$ (i= 1,2,3,4,5; t= 1,2,. . . ,n) be vertices of $P_5 \times P_n$.
Join v with $u_{i,1}$ (i= 1,2,3,4,5) by five new edges to obtain G. Obviously $p = |V(G)| = 5n + 1$ and $q = |E(G)| = 9n$. We define labeling function $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ as follows,

f(v) = 0 f($u_{2,1}$) = 3 f($u_{4,1}$)=4;
f($u_{1,j}$) = q-$\frac{9}{2}$(j-1) ;when j $\equiv$ 1(mod 2)
= $\frac{9j}{2} - 2$;when j $\equiv$ 0(mod 2), $\forall j = 1, 2, \ldots, n-1$;
f($u_{i,j}$) = f($u_{1,j}$)+$(-1)^j$ $\frac{(i-1)}{2}$, $\forall i = 3, 5$; $\forall j = 1, 2, \ldots, n-1$;
f($u_{i,j}$) = f($u_{i-1,j-1}$)-$(-1)^j$ $\forall i = 2, 4$; $\forall j = 1, 2, \ldots, n-1$;
f($u_{1,n}$) = f($u_{1,n-1}$)-$(-1)^n \cdot 6$
f($u_{2,n}$) = f($u_{2,n-1}$)+$(-1)^n \cdot 7$
f($u_{3,n}$) = f($u_{3,n-1}$)-$(-1)^n \cdot 8$
f($u_{4,n}$) = f($u_{4,n-1}$)+$(-1)^n \cdot 10$
f($u_{5,n}$) = f($u_{5,n-1}$)-$(-1)^n \cdot 11$.

1Above labeling pattern give rise a graceful labeling to given graph G.
Illustration−2.6**:** Arrow graph A_3^5 and its graceful labeling shown in figure−3.

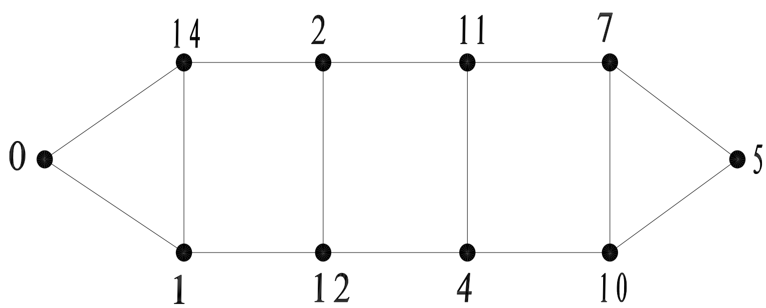


Figure−3 Arrow graph A_3^5 and its graceful labeling

Theorem−2.7 **:** DA_n^2 is a graceful graph, where $n \geq 2$
Proof : Let G = DA_n^2 be a double arrow graph obtained by joining two vertices v, w with $P_2 \times P_n$ by 2+2 new edges both sides.
Let $u_{i,j}$ (i= 1,2; j= 1,2,. . . ,n) be vertices of $P_2 \times P_n$.
Join v with $u_{i,1}$ (i= 1,2) and w with $u_{i,n}$ (i= 1,2) by 2+2 new edges to obtain G. Obviously $p = |V(G)| = 2n + 2$ and $q = |E(G)| = 3n + 2$. We define labeling function $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ as follows,

f(v) = 0 f($u_{2,1}$) = 1
f($u_{1,j}$) = q-($\frac{j-1}{2}$)3 = q-$\frac{3}{2}$(j-1) ;when j $\equiv$ 1(mod 2)
= $\frac{3j}{2} - 1$;when j $\equiv$ 0(mod 2), $\forall j = 1, 2, \ldots, n$;
f($u_{2,j}$) = f($u_{1,j-1}$)-$(-1)^j \cdot 2$, $\forall j = 2, 3, \ldots, n$.

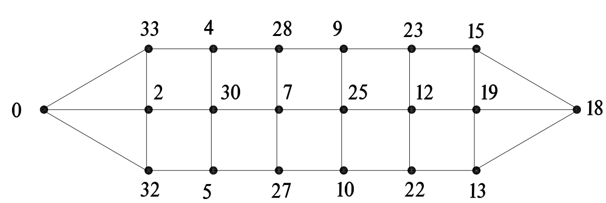
f($u_{1,n}$) = f($u_{1,n-1}$)-$(-1)^n \cdot 4$
f($u_{2,n}$) = f($u_{2,n-1}$)-$(-1)^n \cdot 6$
f(w) = f($u_{1,n}$)-$(-1)^n 2$.
1Above labeling patten give rise a graceful labeling to given graph G.
Illustration−2.8**:** Double arrow graph DA_4^2 and its graceful labeling shown in figure−4.



Figure−4 Double arrow graph DA_4^2 and its graceful labeling

Theorem−2.9 **:** DA_n^3 is a graceful graph, where $n \geq 2$
Proof : Let G = DA_n^3 be an arrow graph obtained by joining two vertices v and w with $P_3 \times P_n$ by 3+3 new edges both the sides.
Let $u_{i,j}$ (i= 1,2,3; t= 1,2,...,n) be vertices of $P_3 \times P_n$.
Join v with $u_{i,1}$ (i= 1,2,3) and w with $u_{i,n}$ by 3+3 new edges to obtain G. Obviously $p = |V(G)| = 3n + 2$ and $q = |E(G)| = 5n + 3$. We shall define labeling function $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ as follows,
f(v) = 0 f($u_{2,1}$) = 2;
f($u_{1,j}$) = q-$\frac{5}{2}$(j-1) ;when j $\equiv$ 1(mod 2)
= $\frac{5j}{2} - 1$;when j $\equiv$ 0(mod 2), $\forall j = 1, 2, \ldots, n-1$;
f($u_{2,j}$) = f($u_{1,j-1}$)-$(-1)^j \cdot 3$, $\forall j = 1, 2, \ldots, n-1$;
f($u_{3,j}$) = f($u_{1,j}$)+$(-1)^j$, $\forall j = 1, 2, \ldots, n-1$;
f($u_{1,n}$) = f($u_{1,n-1}$)-$(-1)^n \cdot 8$
f($u_{2,n}$) = f($u_{2,n-1}$)+$(-1)^n \cdot 7$
f($v_{3,n}$) = f($u_{3,n-1}$)-$(-1)^n \cdot 9$
f(w) = f($u_{2,n}$)-$(-1)^n$
1Above labeling patten give rise a graceful labeling to given graph G.

Illustration−2.10**:** Double arrow graph DA_6^3 and its graceful labeling shown in figure−5.



Figure−5 Double arrow graph DA_6^3 and its graceful labeling

3 Concluding Remarks

11We have introduced new graceful graphs namely arrow graph and double arrow graph. We also proved that arrow graphs A_n^2 where $n \in N$, A_n^3 ,A_n^5 where $n \geq 2$ are graceful and double arrow graphs DA_n^2 ,DA_n^3 where $n \geq 2$ are graceful. Present work contribute some new results to the theory of graceful graphs. The labelling pattern is demonstrated by suitable illustration.

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Numerical solution of time fractional nonlinear Schrödinger equation arising in quantum mechanics by cubic B-spline finite elements

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Abstract

In the present article, we are going to investigate the numerical solutions of time fractional nonlinear Schrödinger equation which is frequently encountered in quantum mechanics by using cubic B-spline collocation method. To be able to control the efficiency of the proposed method, some sample problems have been studied in this article. The outstanding purpose of the paper is to indicate that the finite element method based on the cubic B-spline collocation method approach can also be suitable for the handling of the fractional differential equations. At the end, the results of numerical experiments are compared with those of analytical solution to ensure the accuracy and efficiency of the presented scheme.

Keywords: Finite element method, collocation method, time fractional nonlinear Schrödinger equation, cubic B-Spline, fractional quantum mechanics.

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1 Introduction

The study of fractional calculus has gained more importance for the formulation of natural phenomena. This results from the fact that fractional equations instead of integer order differential equations may be used for a better modeling of natural physics process and dynamic system processes. Moreover, since they have the memory effects, fractional differential equations can more suitably describe natural processes involving memory and hereditary characteristics. However, generally speaking, derivation and application of the analytical solutions of the fractional differential equations is not so easy in most cases. Therefore, obtaining some reliable and effective methods to solve fractional differential equations has gained more importance in recent years. Recently, it has increasingly become clear that most of the phenomena in various fields of science such as engineering, physics, chemistry and many others can be accurately described by mathematical tools from fractional calculus, that is, the theory of derivatives and integrals of fractional (non-integer) order [1]. The concept of differentiation and integration to non-integer order dates back very early in history. In fact, this subject was evident almost as early as the ideas of the classical calculus were known [2]. Many authors have pointed out that derivatives and integrals of non-integer order are more suitable for the description of the behavior of various materials. It has also become clear that new fractional-order models are more adequate than previously used integer-order ones. The increasing number of fractional derivative applications in many fields of science and engineering clearly shows that there is a tremendous demand for better mathematical models of real objects, and that the fractional calculus provides one possible approach on the way to more adequate mathematical modeling of real objects and processes. Even though there are a few analytical techniques [3] for dealing with the fractional equations, as also

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happens with ordinary (non-fractional) partial differential equations, in many cases the initial condition, and/or the external force are such that the only reasonable option is to resort to numerical methods. However, although there have been an increasing number of works on this topic during the last few years [4-12], this field of applied mathematics is by far much less developed and understood than its non fractional counterpart [13]. Although there have been many methods applied to solve fractional partial differential equations, there is still a long way to go in this field. There are several definitions of a fractional derivative of order $\alpha > 0$ [14]. The two most widely utilized are the Riemann-Liouville and Caputo. The main difference between the two is in the order of evaluation. We have just started with recalling the essentials of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. Now, we give some basic definitions and properties of the fractional calculus theory.

Definition 1 [8]. For $\mu \in \mathbb{R}$ and $x > 0$, a real function $f(x)$, is said to be in the space C_μ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0,\infty)$, and for $m \in \mathbb{N}$ it is said to be in the space C_μ^m if $f^m \in C_\mu$.

Definition 2 [8]. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ for a function $f(x) \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\gamma-1} f(t)dt, \;\; \alpha > 0, \;\; x > 0, \;\; J^0 f(x) = f(x).$$

Also we have the following properties:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$$
$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$$
$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Definition 3 [8]. If m be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ is defined as

$$ {}_0^C D_t^\alpha U(x,t) = \frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\partial^m U(x,s)}{\partial s^m} (t-s)^{m-\alpha-1} ds, & m-1 < \alpha < m, m \in \mathbb{N} \\ \frac{\partial^m U(x,t)}{\partial t^m}, & m = \alpha. \end{cases}$$

The finite element method, especially, has been an important method for solving both ordinary and partial differential equations. Besides, in this paper, the finite element method is applied to solve fractional differential equation, namely time fractional telegraph equation. The main idea behind the finite element method is to divide the whole region of the given problem into an equivalent system of finite elements with associated nodes and to choose the most appropriate element type to model most closely the actual physical behavior. Thus, by means of the finite element method, a huge problem is converted into many solvable small ones. For easy implementation, those elements must be made small enough to give usable results and yet large enough to reduce computational effort [15]. In this paper, we will use cubic B-spline finite element method to obtain the numerical solutions of the time fractional nonlinear Schrödinger equation by using the $L1$ discretizaton formulae of the fractional derivative as used by Ref.[5].

The Schrödinger equation is one of the fundamental equations arising in physics for describing quantum mechanical behavior [16, 17]. It is also often called the Schrödinger wave equation and is a partial differential equation that describes how the wave function of a physical system evolves over time. The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics [18]. It was discovered by Nick Laskin [19, 20] as a result of extending the Feynman path integral,from the Brownian-like to Levy-like quantum mechanical paths. After that, he considered the fractional Schrödinger equation for some particular cases like fractional Bohratom and one-dimensional fractional oscillator [20]. Some other cases of the fractional Schrödinger equation were discussed in [8, 21–24].

2 Governing Equation

In this study, we will consider the time fractional nonlinear Schrödinger equation as a model given as follows

$$i\frac{\partial^\gamma U(x,t)}{\partial t^\gamma} + \frac{\partial^2 U(x,t)}{\partial x^2} + |U(x,t)|^2 U(x,t) = f(x,t) \qquad (2.1)$$

where

$$\frac{\partial^\gamma U(x,t)}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} \frac{\partial U(x,\tau)}{\partial \tau} d\tau \ ,0 < \gamma < 1$$

is the fractional derivatives given in the Caputo's sense [3, 14] and $i = \sqrt{-1}$. In this paper, for the time fractional nonlinear Schrödinger equation, we are going to take the boundary conditions of the model problem (2.1) given in the interval $a \le x \le b$ as

$$U(a,t) = h_1(t)\ , \quad U(b,t) = h_2(t), \quad t \ge 0 \tag{2.2}$$

and the initial condition as

$$U(x,0) = g(x)\ , \quad a \le x \le b. \tag{2.3}$$

In the numerical solution process, to be able to obtain a finite element scheme for solving time fractional nonlinear Schrödinger equation, we will also discretize the Caputo derivative by means of the so-called $L1$ formulae [5]:

$$\frac{\partial^\gamma f(t)}{\partial t^\gamma}|_{t_m} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{m-1} b_k^\gamma \left[f(t_{m-k}) - f(t_{m-1-k}) \right]$$

where

$$b_k^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma}.$$

Since $U(x,t)$ is complex valued function, we decompose $U(x,t)$ into its real and imaginary parts $R(x,t)$ and $S(x,t)$, respectively:

$$U(x,t) = R(x,t) + iS(x,t). \tag{2.4}$$

Substituting (2.4) into (2.1), the complex Eq. (2.1) can be rewritten as a system involving two time fractional partial differential equations:

$$\begin{aligned} &\tfrac{\partial^\gamma S(x,t)}{\partial t^\gamma} - \tfrac{\partial^2 R(x,t)}{\partial x^2} - \left(R(x,t)^2 + S(x,t)^2\right) R(x,t) = -f_r(x,t) \\ &\tfrac{\partial^\gamma R(x,t)}{\partial t^\gamma} + \tfrac{\partial^2 S(x,t)}{\partial x^2} + \left(R(x,t)^2 + S(x,t)^2\right) S(x,t) = f_l(x,t). \end{aligned} \tag{2.5}$$

where $-f_r(x,t)$ and $f_l(x,t)$ are, respectively, the real and imaginary parts of the $f(x,t)$. Also, we have initial and boundary conditions of Eq. (2.1) as follows:

$$\begin{aligned} &R(a,t) = h_{1_r}(t)\ , \quad R(b,t) = h_{2_r}(t), \quad t \ge 0 \\ &S(a,t) = h_{1_l}(t)\ , \quad S(b,t) = h_{2_l}(t), \quad t \ge 0 \end{aligned} \tag{2.6}$$

where $h_{1_r}(t)$ and $h_{1_l}(t)$ are, respectively, the real and imaginary parts of the $h_1(t)$ and $h_{2_r}(t)$ and $h_{2_l}(t)$ are, respectively, the real and imaginary parts of the $h_2(t)$. The initial conditions as

$$R(x,0) = g_r(x)\ , \quad S(x,0) = g_l(x)\ , \quad a \le x \le b. \tag{2.7}$$

where $g_r(x)$ and $g_l(x)$ are, respectively, the real and imaginary parts of the $g(x)$.

3 Cubic B-spline Finite Element Collocation Solutions

Before solving Eq. (2.5) with boundary conditions (2.6) and initial condition (2.7) by using collocation finite element method, first of all, we define cubic B-spline base functions. Let us assume that interval $[a,b]$ is partitioned into N finite elements of uniformly equal length by knots x_m, $m = 0,1,2,...,N$ such that $a = x_0 < x_1 \cdots < x_N = b$ and $h = x_{m+1} - x_m$. Cubic B-splines $\phi_m(x)$, $(m = -1(1)N+1)$, at knots x_m are defined over interval $[a,b]$ by [25]

$$\phi_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}], \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & x \in [x_{m-1}, x_m], \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & x \in [x_m, x_{m+1}], \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}], \\ 0 & \text{otherwise.} \end{cases}$$

The set of B-splines $\{\phi_{-1}(x), \phi_0(x), \ldots, \phi_{N+1}(x)\}$ forms a basis for the functions defined over $[a,b]$. Therefore, an approximation solutions $R_N(x,t)$ and $S_N(x,t)$ can be written in terms of the cubic B-splines trial functions as:

$$\begin{array}{l} R_N(x,t) = \sum_{m=-1}^{N+1} \delta_m(t)\phi_m(x) \\ S_N(x,t) = \sum_{m=-1}^{N+1} \sigma_m(t)\phi_m(x) \end{array} \tag{3.8}$$

where $\delta_m(t)$'s and $\sigma_m(t)$'s are unknown, time dependent quantities to be determined from the boundary and cubic B-spline collocation conditions. Each cubic B-spline covers four elements so that each element $[x_m, x_{m+1}]$ is covered by four cubic B-splines. For this problem, the finite elements are identified with the interval $[x_m, x_{m+1}]$ and the elements knots x_m, x_{m+1}. Using the nodal values R_m, $R_m^{'}$ and $R_m^{''}$ given in terms of the parameter $\delta_m(t)$

$$\begin{array}{l} R_m = R(x_m,t) = \delta_{m-1}(t) + 4\delta_m(t) + \delta_{m+1}(t), \\ R_m' = R'(x_m,t) = \frac{3}{h}(-\delta_{m-1}(t) + \delta_{m+1}(t)), \\ R_m'' = R''(x_m,t) = \frac{6}{h^2}(\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)), \end{array} \tag{3.9}$$

the variation of $R_N(x,t)$ over the typical element $[x_m, x_{m+1}]$ is given by

$$R_N(x,t) = \sum_{j=m-1}^{m+2} \delta_j(t)\phi_j(x).$$

Using the nodal values S_m, $S_m^{'}$ and $S_m^{''}$ given in terms of the parameter $\sigma_m(t)$

$$\begin{array}{l} S_m = S(x_m,t) = \sigma_{m-1}(t) + 4\sigma_m(t) + \sigma_{m+1}(t), \\ S_m' = S'(x_m,t) = \frac{3}{h}(-\sigma_{m-1}(t) + \sigma_{m+1}(t)), \\ S_m'' = S''(x_m,t) = \frac{6}{h^2}(\sigma_{m-1}(t) - 2\sigma_m(t) + \sigma_{m+1}(t)), \end{array} \tag{3.10}$$

the variation of $S_N(x,t)$ over the typical element $[x_m, x_{m+1}]$ is given by

$$S_N(x,t) = \sum_{j=m-1}^{m+2} \sigma_j(t)\phi_j(x).$$

Firstly, if we substitute the global approximations in (3.8) and its required derivatives (3.9) and (3.10) into Eq.(2.1), we easily obtain the following set of γ-th order fractional ordinary differential equations:

$$\begin{array}{l} (\dot{\sigma}_{m-1}(t) + 4\dot{\sigma}_m(t) + \dot{\sigma}_{m+1}(t)) - \frac{6}{h^2}\left(\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)\right) \\ -Z_m\left(\delta_{m-1}(t) + 4\delta_m(t) + \delta_{m+1}(t)\right) = -f_r(x,t) \\ \left(\dot{\delta}_{m-1}(t) + 4\dot{\delta}_m(t) + \dot{\delta}_{m+1}(t)\right) + \frac{6}{h^2}\left(\sigma_{m-1}(t) - 2\sigma_m(t) + \sigma_{m+1}(t)\right) \\ +Z_m\left(\sigma_{m-1}(t) + 4\sigma_m(t) + \sigma_{m+1}(t)\right) = f_l(x,t) \end{array} \tag{3.11}$$

where $\cdot$ denotes γ^{th} fractional derivative with respect to time and

$$Z_m = R^2 + S^2.$$

If time parameters $\delta_m(t)$'s and its fractional time derivatives $\dot{\delta}_m(t)$'s in Eq. (3.11) are discretized by the Crank-Nicolson formula, $L1$ formula, respectively:

$$\delta = \frac{1}{2}(\delta^n + \delta^{n+1}), \tag{3.12}$$

$$\dot{\delta} = \frac{d^{\gamma-1}\delta}{dt^{\gamma-1}} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{n-1} \left[(k+1)^{1-\gamma} - k^{1-\gamma}\right]\left[\delta^{n-k} - \delta^{n-k-1}\right], \tag{3.13}$$

and if time parameters $\sigma_m(t)$'s and its fractional time derivatives $\dot{\sigma}_m(t)$'s in Eq. (3.11) are discretized by the Crank-Nicolson formula, $L1$ formula, respectively:

$$\sigma = \frac{1}{2}(\sigma^n + \sigma^{n+1}), \tag{3.14}$$

$$\dot{\sigma} = \frac{d^{\gamma-1}\delta}{dt^{\gamma-1}} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{n-1} \left[(k+1)^{1-\gamma} - k^{1-\gamma}\right]\left[\sigma^{n-k} - \sigma^{n-k-1}\right], \tag{3.15}$$

we obtain a recurrence relationship between successive time levels relating unknown parameters $\delta_m^{n+1}(t)$ and $\sigma_m^{n+1}(t)$

$$\begin{aligned}
&\sigma_{m-1}^{n+1} + 4\sigma_m^{n+1} + \sigma_{m+1}^{n+1} + \left(-6\alpha - Z_m h^2\alpha\right)\delta_{m-1}^{n+1} + \left(12\alpha - 4Z_m h^2\alpha\right)\delta_m^{n+1} \\
&+ \left(-6\alpha - Z_m h^2\alpha\right)\delta_{m+1}^{n+1} = \sigma_{m-1}^{n} + 4\sigma_m^{n} + \sigma_{m+1}^{n} + \left(6\alpha + Z_m h^2\alpha\right)\delta_{m-1}^{n} \\
&+ \left(-12\alpha + 4Z_m h^2\alpha\right)\delta_m^{n} + \left(6\alpha + Z_m h^2\alpha\right)\delta_{m+1}^{n} - 2h^2\alpha f_r(x,t) \\
&- \sum_{k=1}^{n} b_k^{\gamma}\left[(\sigma_{m-1}^{n-k+1} - \sigma_{m-1}^{n-k}) + 4(\sigma_m^{n-k+1} - \sigma_m^{n-k}) + (\sigma_{m+1}^{n-k+1} - \sigma_{m+1}^{n-k})\right] \\
&\delta_{m-1}^{n+1} + 4\delta_m^{n+1} + \delta_{m+1}^{n+1} + \left(6\alpha + Z_m h^2\alpha\right)\sigma_{m-1}^{n+1} + \left(-12\alpha + 4Z_m h^2\alpha\right)\sigma_m^{n+1} \\
&+ \left(6\alpha + Z_m h^2\alpha\right)\sigma_{m+1}^{n+1} = \delta_{m-1}^{n} + 4\delta_m^{n} + \delta_{m+1}^{n} + \left(-6\alpha - Z_m h^2\alpha\right)\sigma_{m-1}^{n} \\
&+ \left(12\alpha - 4Z_m h^2\alpha\right)\sigma_m^{n} + \left(-6\alpha - Z_m h^2\alpha\right)\sigma_{m+1}^{n} + 2h^2\alpha f_l(x,t) \\
&- \sum_{k=1}^{n} b_k^{\gamma}\left[(\delta_{m-1}^{n-k+1} - \delta_{m-1}^{n-k}) + 4(\delta_m^{n-k+1} - \delta_m^{n-k}) + (\delta_{m+1}^{n-k+1} - \delta_{m+1}^{n-k})\right]
\end{aligned} \tag{3.16}$$

where

$$Z_m = (\delta_{m-1}(t) + 4\delta_m(t) + \delta_{m+1}(t))^2 + (\sigma_{m-1}(t) + 4\sigma_m(t) + \sigma_{m+1}(t))^2$$

and

$$\alpha = \frac{(\Delta t)^{\gamma}\Gamma(2-\gamma)}{2h^2}.$$

The iterative system (3.16) consists of $2N+2$ linear equations involving $2N+6$ unknown parameters $(\delta_{-1},\ldots,\delta_{N+1},\sigma_{-1},\ldots,\sigma_{N+1})^T$. In order to be able to obtain a unique solution to these systems, four additional constraints are needed. Those are obtained from the boundary conditions and their second derivatives and after that they are used to eliminate δ_{-1}, δ_{N+1}, σ_{-1}, σ_{N+1} from system (3.16). Using the relations

$$R_N(x,0) = \sum_{m=-1}^{N+1} \delta_m(0)\phi_m(x)$$

and

$$S_N(x,0) = \sum_{m=-1}^{N+1} \sigma_m(0)\phi_m(x)$$

together with extra conditions, which can easily be obtained from $R''(x_0,0) = R''_N(x_0,0)$ and $S''(x_0,0) = S''_N(x_0,0)$, since the second derivatives of the approximate initial conditions shall agree with those of the exact initial conditions to discard δ_{-1}, δ_{N+1}, σ_{-1}, σ_{N+1}, we obtain initial vectors δ_m^0 and σ_m^0 can be respectively obtained from the following matrix equations:

$$\begin{bmatrix} 6 & 0 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 0 & 6 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{N-1} \\ \delta_N \end{bmatrix} = \begin{bmatrix} R_0 - \frac{h^2}{6}R''_0 \\ R_1 \\ R_2 \\ \vdots \\ R_{N-1} \\ R_N - \frac{h^2}{6}R''_N \end{bmatrix}$$

and

$$\begin{bmatrix} 6 & 0 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 0 & 6 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{N-1} \\ \sigma_N \end{bmatrix} = \begin{bmatrix} S_0 - \frac{h^2}{6}S''_0 \\ S_1 \\ S_2 \\ \vdots \\ S_{N-1} \\ S_N - \frac{h^2}{6}S''_N \end{bmatrix},$$

which can be solved using an appropriate algorithm. Therefore, the approximate solution functions for $R(x,t)$ and $S(x,t)$ can be obtained from δ^n and σ^n using Eq. (3.16).

4 Numerical examples and results

Now, we are going to present a numerical example which support numerical results for time fractional nonlinear Schrödinger equation are obtained by collocation method using cubic B-spline base functions. The accuracy of the present method is measured by the error norm L_2

$$L_2 = \left\| U^{exact} - U_N \right\|_2 \simeq \sqrt{h \sum_{j=0}^{N} \left| U_j^{exact} - (U_N)_j \right|^2}$$

and the error norm L_∞

$$L_\infty = \left\| U^{exact} - U_N \right\|_\infty \simeq \max_j \left| U_j^{exact} - (U_N)_j \right| .$$

We are going to consider the time fractional nonlinear Schrödinger equation (2.1) with boundary conditions

$$U(0,t) = it^2 , \quad U(1,t) = it^2, \quad t \geq 0$$

and initial conditions as

$$U(x,0) = 0 , \quad 0 \leq x \leq 1.$$

The corresponding forcing term $f(x,t)$ is of the form

$$\begin{aligned} f(x,t) \qquad &= -\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} \cos(2\pi x) + (t^6 - 4\pi^2 t^2) \sin(2\pi x) \\ &+ i \left(\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} \sin(2\pi x) + (t^6 - 4\pi^2 t^2) \cos(2\pi x) \right). \end{aligned}$$

The exact solution of the problem is given by [8]

$$U(x,t) = t^2(\sin(2\pi x) + i\cos(2\pi x)).$$

A comparison of the analytical solution and numerical solutions obtained for the values of different values of γ is given in Tables 1-2. As it is clearly seen from the table, the analytical and numerical solutions obtained by the present scheme are in good agreement with each other. As the value of γ increases, the values of error norms L_2 and L_∞ decrease for $S(x,t)$ imaginary part of $U(x,t)$ and increase for $R(x,t)$ real part of $U(x,t)$. In Tables 3-4, we demonstrate the numerical results for $\gamma = 0.5$, $\Delta t = 0.002$ and $t_f = 1$ and for different number of divisions of the region. Tables 3-4 clearly show that as the number of division increases, the obtained numerical results become more accurate. We see this from the decreasing values of the error norms L_2 and L_∞. In Tables 5-6, we demonstrate the numerical results for $\gamma = 0.5$, $N = 40$ and $t_f = 0.25$ and for different number of Δt. Tables 5-6 clearly show that as the number of Δt decreases, the obtained numerical results become more accurate. We see this from the decreasing values of the error norms L_2 and L_∞. In Table 7, the error norm L_∞ of the present study are better than those in Ref. [8] at $t_f = 1$. In Figure 1, we demonstrate the graphs of numerical solutions obtained for $\gamma = 0.50$ and $N = 40$ at different time levels.

Table 1: The comparison of the exact solutions with the numerical solutions of $R(x,t)$ real part of $U(x,t)$ with $N=40$, $\Delta t=0.002$ and $t_f=1$ for different values of γ and the error norms L_2 and L_∞.

x	$\gamma=0.1$	$\gamma=0.3$	$\gamma=0.7$	$\gamma=0.9$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.587513	0.587497	0.587474	0.587460	0.587785
0.2	0.950666	0.950628	0.950542	0.950477	0.951057
0.3	0.950637	0.950580	0.950441	0.950328	0.951057
0.4	0.587431	0.587362	0.587185	0.587038	0.587785
0.5	-0.000224	-0.000299	-0.000491	-0.000651	0.000000
0.6	-0.587870	-0.587941	-0.588122	-0.588273	-0.587785
0.7	-0.951043	-0.951102	-0.951248	-0.951366	-0.951057
0.8	-0.951018	-0.951057	-0.951148	-0.951218	-0.951057
0.9	-0.587794	-0.587811	-0.587837	-0.587853	-0.587785
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2\times10^3$	0.244241	0.282432	0.388333	0.481627	
$L_\infty\times10^3$	0.421466	0.476307	0.622788	0.754269	

Table 2: The comparison of the exact solutions with the numerical solutions of $S(x,t)$ imaginary part of $U(x,t)$ with $N=40$, $\Delta t=0.002$ and $t_f=1$ for different values of γ and the error norms L_2 and L_∞.

x	$\gamma=0.1$	$\gamma=0.3$	$\gamma=0.7$	$\gamma=0.9$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.809065	0.809048	0.808989	0.808932	0.809017
0.2	0.309182	0.309153	0.309045	0.308944	0.309017
0.3	-0.308704	-0.308741	-0.308876	-0.309001	-0.309017
0.4	-0.808581	-0.808619	-0.808762	-0.808889	-0.809017
0.5	-0.999514	-0.999548	-0.999678	-0.999789	-1.000000
0.6	-0.808572	-0.808599	-0.808703	-0.808785	-0.809017
0.7	-0.308687	-0.308706	-0.308777	-0.308828	-0.309017
0.8	0.309202	0.309191	0.309150	0.309123	0.309017
0.9	0.809078	0.809073	0.809055	0.809045	0.809017
1.0	1.000000	1.000000	1.000000	1.000000	1.000000
$L_2\times10^3$	0.299659	0.276193	0.191137	0.132380	
$L_\infty\times10^3$	0.485920	0.451507	0.327280	0.232999	

Table 3: The comparison of the exact solutions with the numerical solutions of $R(x,t)$ imaginary part of $U(x,t)$ with $\gamma=0.5$, $\Delta t=0.002$ and $t_f=1$ for different values of N and the error norms L_2 and L_∞.

x	$N=10$	$N=20$	$N=40$	Exact
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.563396	0.582546	0.587484	0.587785
0.2	0.910919	0.942434	0.950586	0.951057
0.3	0.907609	0.941667	0.950514	0.951057
0.4	0.554170	0.580413	0.587279	0.587785
0.5	-0.014876	-0.003451	-0.000388	0.000000
0.6	-0.582572	-0.587006	-0.588025	-0.587785
0.7	-0.932052	-0.947350	-0.951170	-0.951057
0.8	-0.929195	-0.946693	-0.951101	-0.951057
0.9	-0.574077	-0.585053	-0.587825	-0.587785
1.0	0.000000	0.000000	0.000000	0.000000
$L_2\times10^3$	25.493140	5.451631	0.330630	
$L_\infty\times10^3$	43.447435	9.389941	0.542810	

Table 4: The comparison of the exact solutions with the numerical solutions of $S(x,t)$ imaginary part of $U(x,t)$ with $\gamma = 0.5$, $\Delta t = 0.002$ and $t_f = 1$ for different values of N and the error norms L_2 and L_∞.

x	$N = 10$	$N = 20$	$N = 40$	Exact
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.816537	0.810582	0.809024	0.809017
0.2	0.333872	0.314208	0.309109	0.309017
0.3	-0.262747	-0.299328	-0.308796	-0.309017
0.4	-0.745014	-0.795588	-0.808678	-0.809017
0.5	-0.928686	-0.985022	-0.999602	-1.000000
0.6	-0.743595	-0.795271	-0.808643	-0.809017
0.7	-0.260341	-0.298783	-0.308736	-0.309017
0.8	0.336396	0.314784	0.309173	0.309017
0.9	0.818142	0.810950	0.809065	0.809017
1.0	1.000000	1.000000	1.000000	1.000000
$L_2 \times 10^3$	44.135685	9.266138	0.240051	
$L_\infty \times 10^3$	71.314360	14.977938	0.399538	

Table 5: The comparison of the exact solutions with the numerical solutions of $R(x,t)$ imaginary part of $U(x,t)$ with $\gamma = 0.5$, $N = 40$ and $t_f = 0.25$ for different values of Δt and the error norms L_2 and L_∞.

	$\Delta t = 0.01$	$\Delta t = 0.005$	$\Delta t = 0.0025$	$\Delta t = 0.002$	$\Delta t = 0.001$
L_2	1.7777×10^{-3}	0.8377×10^{-3}	0.3677×10^{-3}	0.2738×10^{-3}	0.0869×10^{-3}
L_∞	2.9810×10^{-3}	1.3993×10^{-3}	0.6087×10^{-3}	0.4506×10^{-3}	0.1343×10^{-3}

Table 6: The comparison of the exact solutions with the numerical solutions of $S(x,t)$ imaginary part of $U(x,t)$ with $\gamma = 0.5$, $N = 40$ and $t_f = 0.25$ for different values of Δt and the error norms L_2 and L_∞.

	$\Delta t = 0.01$	$\Delta t = 0.005$	$\Delta t = 0.0025$	$\Delta t = 0.002$	$\Delta t = 0.001$
L_2	2.7965×10^{-3}	1.3264×10^{-3}	0.5915×10^{-3}	0.4445×10^{-3}	0.1506×10^{-3}
L_∞	4.5839×10^{-3}	2.1733×10^{-3}	0.9682×10^{-3}	0.7273×10^{-3}	0.2453×10^{-3}

Table 7: The results obtained of numerical solutions of $R(x,t)$ real part and $S(x,t)$ imaginary part of $U(x,t)$ for $N = 40$ and $\Delta t = 0.002$ by proposed method in comparison with the in Ref. [8] and exact solution at $t_f = 1$.

	$\gamma = 0.1$		$\gamma = 0.3$	
L_∞	Real Part	Imaginary Part	Real Part	Imaginary Part
Present	4.2147×10^{-4}	4.8592×10^{-4}	4.7631×10^{-4}	4.5151×10^{-4}
[8]	2.8536×10^{-3}	2.1753×10^{-3}	2.8610×10^{-3}	2.1771×10^{-3}

Figure 1: The comparison of the exact(lines) and numerical solutions for $\gamma = 0.5$, $N = 40$ and $\Delta t = 0.002$ at $t = 0.5$ (stars), $t = 0.75$ (squares), and $t = 1$ (triangles).

5 Conclusion

For last words, in the present study, numerical solutions of the time fractional nonlinear Schrödinger equation encountered in quantum mechanics based on the cubic B-spline finite element method have been calculated and presented. The time fractional derivative is considered in the form of the Caputo sense. In this study, the fractional derivative appearing in the time fractional nonlinear Schrödinger equation arising in quantum mechanics is approximated by means of the so-called $L1$ formulae. A test problem is worked out to examine the performance of the present algorithm. The performance and efficiency of the method are shown by calculating error norms L_2 and L_∞. The obtained results show that the error norms are sufficiently small during all computer runs. The obtained results also indicate that the present method is a particularly successful numerical scheme to solve the time fractional nonlinear Schrödinger equation arising in quantum mechanics. As a conclusion, in future studies, the method can efficiently be applied to this type of non-linear time fractional problems arising in physics and mathematics with success. Moreover, the method can also be applied and tested on a more wide range of other physically important equations.

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A common fixed point theorem for weakly subsequentially continuous mappings satisfying implicit relation

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Abstract

In this paper, we prove a common fixed point theorem for two weakly subsequentially continuous and compatible of type (E) for two pairs of self mappings, which satisfying implicit relation in metric spaces, an example is given to illustrate our results, also we give an application to solve a partial differential equations, and the study of its generalized Hyers-Ulam stability, our results improve and extend some previous results.

Keywords: Common fixed point, weakly subsequentially continuous, compatible of type (E), implicit relation.

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1 Introduction

The generalization of Banach principle, for more than one mapping was been initiated by Jungck[17], where he introduced the concept of commuting mappings to establish a common fixed point theorem for two mappings in metric spaces, Sessa[34] defined the weakly commuting mappings which is a generalization to the commuting mappings, later Jungck[18] introduced the concept of compatibility mappings in metric space, it is weaker than the last notions. After that many authors introduced various type of compatibility, compatibility of type (A),of type (B),of type (C) and of type (P) for two self mappings f and g on metric space (X,d) respectively in [19], [28],[30] and [29] as follows: the pair$\{S,T\}$ is compatible of type (A) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(TSx_n, S^2x_n) = 0,$$

S and T are compatible of type (B) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, S^2x_n)\Big] \text{ and}$$

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(TSx_n, gz) + \lim_{n\to\infty} d(Tz, T^2x_n)\Big],$$

they are compatible of type (C) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, T^2x_n) + \lim_{n\to\infty} d(Sz, T^2x_n)]$$

and

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(TSx_n, Tz) + \lim_{n\to\infty} d(Tz, T^2x_n) + \lim_{n\to\infty} d(Tz, S^2x_n)],$$

and said to be compatible of type(P) if

$$\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0,$$

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whenever in the all above definitions, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some $z \in X$.
Aamri and Moutawakil [1] defined two self maps S and T on a metric space (X, d) are said to be satisfy property (E,A), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z,$$

for some z in X.

2 Preliminaries

Pant[26] introduced the notion of reciprocal continuity as follows:

Definition 2.1. *Self maps S and T of a metric space (X, d) are said to be reciprocally continuous, if $\lim_{n\to\infty} STx_n = St$ and $\lim_{n\to\infty} TSx_n = Tt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.*

In 2009, Bouhadjera and Godet Thobie [9] introduced the concept of subcompatibility and subsequential continuity as follows:
Two self-mappings S and T on a metric space (X, d) are said to be subcompatible, if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \text{ and } \lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

for some $t \in X$

Definition 2.2. *The pair $\{S, T\}$ is called to be subsequentially continuous, if there exists a sequence $\{x_n\}$ in X, such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n) = Tt$.*

Now, as a generalization to the Definition 2.2, define:

Definition 2.3. *Let S and T to be two self mappings of a metric space (X, d), the pair $\{S, T\}$ is said to be weakly subsequentially continuous (shortly wsc), if there exists a sequence $\{x_n\}$, such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n) = Tt$.*

Notice that subsequentially continuous or, reciprocally continuous maps are weakly subsequentially continuous, but the converse may be not.

Definition 2.4. *The pair $\{S, T\}$ is said to be S-subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} STx_n = St$.*

Definition 2.5. *The pair $\{S, T\}$ is said to be S-subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} TSx_n = Tt$.*

Example 2.1. *Let $X = [0, 2]$ and d is the euclidian metric, we define S, T as follows:*

$$Sx = \begin{cases} 1 + x, & 0 \le x \le 1 \\ \frac{x+1}{2}, & 1 < x \le 2 \end{cases}, \quad Tx = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$$

Clearly that S and T are discontinuous at 1.
We consider a sequence $\{x_n\}$, which defined for each $n \ge 1$ by: $x_n = \frac{1}{n}$,
clearly that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sx_n = 1$, also we have:

$$\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S(2 - \frac{1}{n}) = 2 = S(1),$$

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T(2 + \frac{1}{n}) = 2 \neq T(1),$$

then $\{S,T\}$ is S-subsequentially continuous, so it is wsc .
On other hand, let $\{y_n\}$ be a sequence which defined or each $n \geq 1$ by: $y_n = 1 + \frac{1}{n}$, we have

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = 1,$$

but

$$\lim_{n\to\infty} STy_n = \lim_{n\to\infty} S(1+\frac{2}{n}) = 1 \neq S(1),$$

$$\lim_{n\to\infty} TSy_n = \lim_{n\to\infty} T(4+\frac{1}{n}) = 1 \neq T(1),$$

then S and T are never reciprocally continuous.

Singh and Mahendra Singh [36] introduced the notion of compatibility of type (E), and gave some properties about this type as follows:

Definition 2.6. *Self maps S and T on a metric space (X,d), are said to be compatible of type (E), if $\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = St$ and $\lim_{n\to\infty} S^2x_n = \lim_{n\to+\infty} STx_n = Tt$, whenever $\{x_n\}$ is a sequence in X, such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.*

Remark 2.1. *If $St = Tt$, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)), however the converse may be not true. Generally compatibility of type (E) implies compatibility of type (B).*

Definition 2.7. *Two self maps S and T of a metric space (X,d) are S-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} STx_n = Tt$, for some $t \in X$.*
The pair $\{S,T\}$ is said to be T-compatible of type (E), if $\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = St$, for some $t \in X$.

Notice that if S and T are compatible of type (E), then they are S-compatible and T-compatible of type (E), but the converse is not true.

Example 2.2. *Let $X = [0,\infty)$ endowed with the euclidian metric, we define S,T as follows:*

$$Sx = \begin{cases} 2, & 0 \leq x \leq 2 \\ x+1, & x > 2 \end{cases} \qquad Tx = \begin{cases} \frac{x+2}{2}, & 0 \leq x \leq 2 \\ 0, & x > 2 \end{cases}$$

Consider the sequence $\{x_n\}$ which defined by: $x_n = 2 - \frac{1}{n}$, for all $n \geq 1$.

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 2,$$

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} STxn) = 2 = T(2)$$

$$\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSxn) = 2 = S(2)$$

then $\{S,T\}$ is compatible of type (E).

Let $\mathcal{F}$ be the set of all continuous functions $F : \mathbb{R}^6_+ \to \mathbb{R}$, which are satisfying:

(F_1) : :F is non decreasing in t_1 and non increasing in t_2, t_3, t_4, t_5, t_6.

(F_2) : :For all $u > 0$, $F(u,u,0,0,u,u) > 0$.

Example 2.3.

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \alpha \max(t_2,t_3,t_4,t_5) - \beta(t_5+t_6),$$

where $\alpha, \beta \geq 0$, and $\alpha + 2\beta < 1$.

Example 2.4.

$$F(t_1,t_2,t_3,t_4,t_5) = t_1 - k \max(t_2,t_3,t_4,\frac{t_5+t_6}{2}),$$

where $0 \leq k < 1$.

Example 2.5.

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \psi(\max(t_2,t_3,t_4,t_5,t_6)),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, *is increasing function such* $\psi(0) = 0$ *and for all* $t > 0, \psi(t) < t$.

The aim of this paper is to prove the existence and the uniqueness of a common fixed point, for two pairs of self-mappings in metric space, which satisfying implicit relation, by using the weak subsequential continuity with compatibility of type (E), due to Singh et al.[36], also to support our results we give an example and an application, concerning the existence and uniqueness of a solution and the generalized Hyers-Ulam stability of a Dirichlet problem of partial differential equation, our results generalize and improve some previous results.

3 Main results

Theorem 3.1. *Let* (X,d) *be a metric space,* A,B,S *are four self mappings a on* X *such for all* $x,y \in X$ *we have:*

$$F(d(Sx,Ty),d(Ax,By),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(By,Sx)) \leq 0, \quad (3.1)$$

where $F \in \mathcal{F}$, *if the two pairs* $\{A,S\}$ *and* $\{B,T\}$ *are weakly subsequentially continuous (wsc) and compatible of type (E), then* A,B,S *and* T *have a unique common fixed point in* X.

Proof. Since $\{A,S\}$ is wsc, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az, \lim_{n\to\infty} SAx_n = Sz$ again $\{,S\}$ is compatible of type (E) implies that

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A^2x_n = Sz$$

and

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S^2x_n = Az,$$

consequently we obtain $Az = Sz$ and z is a coincidence point for A and S. Similarly for B and T, since $\{B,T\}$ is wsc (suppose that it is B-subsequentially continuous) there exists a sequence $\{yn\}$ such

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$ and

$$\lim_{n\to\infty} BTy_n = Bt,$$

again$\{B,T\}$ is compatible of type (E), we get

$$\lim_{n\to\infty} BTy_n = \lim_{n\to\infty} B^2y_n = Tt$$

$$\lim_{n\to\infty} TBy_n = \lim_{n\to\infty} T^2y_n = Bt,$$

so we have $Bt = Tt$.
We claim $Az = Bt$, if not by using (3.1) we get:

$$F(d(Sz,Tt),d(Az,Bt),d(Az,Sz),d(Bt,Tt),d(Az,Tt),d(Bt,Sz)) =$$

$$F(d(Az,Bt),d(Az,Bt),0,0,d(Az,Bt),d(Az,Bt)) \leq 0,$$

which is a contradiction with (F_2).
Now we will prove $z = Az$, if not by using(3.1) we get:

$$F(d(Sx_n,Tt),d(Ax_n,Bt),d(Ax_n,Sx_n),d(Bt,Tt),d(Ax_n,Tt),d(Bt,Sx_n)) \leq 0,$$

letting $n \to \infty$ we get:

$$F(d(z,Tt),d(z,Bt),0,0,d(z,Tt),d(Bt,z)) =$$

$$F(d(z,Az),d(z,Az),0,0,d(z,Az),d(z,Az)) \leq 0,$$

which is a contradiction, then $z = Az = Sz$.
Nextly we shall prove $z = t$, if not by using (3.1) we get:

$$F(d(Sx_n, Tyn), d(Ax_n, Byn), d(Ax_n, Sx_n), d(Byn, Tyn), d(Ax_n, Tyn), d(Byn, Sx_n)) \leq 0,$$

letting $n \to \infty$ we get:

$$F(d(z,t), d(z,t), 0, 0, d(z,t), d(t,z)) \leq 0,$$

which is a contradiction, then z is a fixed point for A, B, S and T. For the uniqueness suppose that there is another fixed point w and using (3.1) we get:

$$F(d(Sz, Tw), d(Az, Bw), d(Az, Sw), d(Bw, Tw), d(Az, Tw), d(Bw, Sz)) =$$

$$F(d(z,w), d(z,w), 0, 0, d(z,w), d(z,w)) \leq 0,$$

which contradicts $(F2)$, then z is unique. □

If $A = B$ and $S = T$, we obtain the following corollary:

Corollary 3.1. *Let (X, d) be a metric space and let $S, A : X \to X$ two self mappings such for all $x, y \in X$ we have:*

$$F(d(Sx, Sy), d(Ax, Ay), d(Ax, Sx), d(Ay, Sy), d(Ax, Sy), d(Ay, Sx)) \leq 0,$$

where $F \in \mathcal{F}$, assume that the pair $\{A, S\}$ is wsc A-subsequentially continuous and A-compatible of type (E), then A and S have a unique common fixed point in X.

If we combine Theorem 3.1 with Example 2.3, we obtain:

Corollary 3.2. *For four self mappings A, B, S and T on metric space (X, d) such for all $x, y \in X$ we have:*

$$d(Sx, Ty)) \leq \alpha \max\{(d(Ax, By), d(Ax, Sx), d(By, Ty), d(Ax, Ty)) + \beta d(By, Sx)),$$

where α, β are nonnegative numbers such $\alpha + 2\beta < 1$, assume that the following conditions hold:

1. *$\{A, S\}$ is A-subsequentially continuous and A-compatible of type (E),*
2. *$\{B, T\}$ is B-subsequentially continuous and B-compatible of type (E),*

then A, B, S and T have a unique common fixed point.

If we combine Theorem 3.1 with Example 2.4, we obtain:

Corollary 3.3. *For four self mappings A, B, S and T on metric space (X, d) such for all $x, y \in X$ we have:*

$$d(Sx, Ty)) \leq k \max\{(d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{d(Ax, Ty) + d(By, Sx)}{2}),$$

where $0 \leq k < 1$ if the four mappings satisfying:

1. *$\{A, S\}$ is A-subsequentially continuous and A-compatible of type (E),*
2. *$\{B, T\}$ is B-subsequentially continuous and B-compatible of type (E),*

then A, B, S and T have a unique common fixed point.

If we combine Example 2.5 with Theorem 3.1, we obtain the following corollary:

Corollary 3.4. *Let (X, d) be a space metric and let A, B, S and T self mappings on X such for all $x, y \in X$ we have:*

$$d(Sx, Ty)) \leq \varphi(\max\{(d(Ax, By), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, is increasing function such $\psi(0) = 0$ and for all $t > 0, \psi(t) < t$.
If the following conditions are satisfied:

1. *$\{A, S\}$ is S-subsequentially continuous and S-compatible of type (E),*

2. $\{B, T\}$ *is T-subsequentially continuous and T-compatible of type (E),*

then A, B, S *and* T *have a unique common fixed point.*

Now we can obtain the same result in Theorem **??**, by using the subsequential continuity with compatibility of type (E) as follows:

Theorem 3.2. *Let* (X, d) *be a space metric and let* A, B, S *and* T *be four self mappings on* X *such for all* $x, y \in X$ *we have:*

$$F(d(Sx, Ty)), d(Ax, By), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)) \leq 0,$$

where $F \in \mathcal{F}$ *if the four mappings satisfying:*

1. $\{A, S\}$ *is subsequentially continuous and S-compatible (or A-compatible) of type (E),*
2. $\{B, T\}$ *is subsequentially continuous and B-compatible (or T-compatible) of type (E),*

then A, B, S *and* T *have a unique common fixed point.*

Proof. It is similar as in proof of Theorem 3.1. □

Example 3.6. *Let* $X = [0, 1]$ *and* d *is the euclidian metric, we define* A, B, S *and* T *by*

$$Ax = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ \frac{3}{4}, & 2 < x \leq 1 \end{cases} \qquad Bx = \begin{cases} 1 - x, & 0 \leq x \leq \frac{1}{2} \\ 1, & 2 < x \leq 1 \end{cases}$$

$$Sx = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ 0, & \frac{1}{2} < x \leq 1 \end{cases} \qquad Tx = \begin{cases} \frac{x+1}{3}, & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} < x \leq 1 \end{cases}$$

We consider a sequence $\{x_n\}$ *which defined for each* $n \geq 1$ *by:*
$x_n = \frac{1}{2} - \frac{1}{n}$, *clearly that* $\lim\limits_{n\to\infty} Ax_n = \dfrac{1}{2}$ *and* $\lim\limits_{n\to\infty} Sx_n = \dfrac{1}{2}$, *also we have:*

$$\begin{aligned} \lim_{n\to\infty} ASx_n &= A(\frac{1}{2}) \\ &= S(\frac{1}{2}) = \frac{1}{2}, \end{aligned}$$

then $\{A, S\}$ *is A-subsequentially continuous and A-compatible of type (E), on the other hand consider a sequence defined by:* $y_n = \frac{1}{2} - e^{-n}$, *for all* $n > 1$.*It is clear that* $\lim\limits_{n\to\infty} By_n = \lim\limits_{n\to\infty} Tx_n = \dfrac{1}{2}$, *and* $\lim\limits_{n\to\infty} BTy_n = B(\dfrac{1}{2}) = T(\dfrac{1}{2}) = \dfrac{1}{2}$, *this yields that* $\{B, T\}$ *is B-subsequentially continuous and B-compatible of type (E).*
For the contractive condition, we have the following cases:

1. *For* $x, y \in [0, \frac{1}{2}]$, *we have*

$$d(Sx, Ty) = \frac{1}{6}|2y - 1| \leq \frac{1}{3}|2y - 1| = \frac{2}{3}d(By, Ty)$$

2. *For* $x \in [0, \frac{1}{2}]$ *and* $y \in (\frac{1}{2}, 1]$, *we have*

$$d(Sx, Ty) = \frac{1}{4} \leq \frac{1}{2} = \frac{2}{3}d(By, Ty)$$

3. *For* $x \in (\frac{1}{2}, 1]$ *and* $y \in [0, \frac{1}{2}]$, *we have*

$$d(Sx, Ty) = \frac{1}{4}(y + 1) \leq \frac{1}{2} = \frac{2}{3}d(Ax, Sx)$$

4. *For* $x, y \in (\frac{1}{2}, 1]$, *we have*

$$d(Sx, Ty) = \frac{1}{4} \leq \frac{1}{2} = \frac{2}{3}d(By, Ty)$$

Consequently all hypotheses of Corollary 3.3 with $k = \frac{2}{3}$ *satisfy, therefore* $\frac{1}{2}$ *is the unique common fixed for* A, B, S *and* T.

4 Application

In this section, we will use Corollary 3.4 to assert the existence of the solution for a Dirichlet problem of non linear partial differential equation, which has been studied by Lazer[23] in complete metric space where he applied theorem of Matkowski [25], also it has been studied on sobolev spaces in paper[7].
Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth border $\partial\Omega$, consider the following problem:

$$\begin{cases} -\Delta u = f(t, u(t)), \\ u_{\partial\Omega} = 0, \end{cases} \tag{4.2}$$

where f is continuous function on $\overline{\Omega} \times \mathbb{R}$, suppose that $\mathcal{C}(\overline{\Omega}, \mathbb{R})$ is the set of continuous functions from $\overline{\Omega}$ to $\mathbb{R}$. It is clear that the space $\mathcal{C}(\overline{\Omega}, \mathbb{R})$ endowed with the metric

$$\forall u, v \in \mathcal{C}(\overline{\Omega}, \mathbb{R}), d(u, v) = \max |u - v|,$$

is a complete metric space.
It is clear and is well know in the partial differential equations theory, that under the above conditions the problem (4.2) equivalent to the following integral equation:

$$u(t) = \int_{\Omega} G(t, u(s)) f(s, u(s) ds, \tag{4.3}$$

where G is the Green function associated to the Laplace operator.
We recall for the following definitions:

Definition 4.8. *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison one if it is increasing and $\varphi^n \to 0$ as $n \to \infty$,*

as a consequence for the last definitions, we have for all $t > 0, \varphi(t) < t, \varphi(0) = 0$ and φ is continuous at 0. Many authors study the stability in the sense of Hyers-Ulam and generalized Hyers-Ulam for the functional equations (for example see[16]) also for the integral equations and differential equations(see[3]).

Definition 4.9. *The equation (4.3) is said to be generalized Hyers-Ulam (Hyers-Ulam-Rassias) stable if there exists a comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such for each $\varepsilon > 0$ and each u satisfying the inequality*

$$|u(t) - \int_{\Omega} G(t,) f(s, u(s)) ds| \leq \varepsilon,$$

there exists a solution $u^* : \Omega \to X$ of (4.3), such that

$$\|u(t) - u^*(t)\| \leq \psi(\varepsilon), \forall t \in I,$$

If $\psi(t) = ct$, for each $t \geq 0$ with $c > 0$, then we said the equation (4.3) has a Hyers-Ulam stability.

Theorem 4.3. *Assume that:*

1. $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$
2. *there exists a continuous functions $\theta : \overline{\Omega} \to \mathbb{R}_+$ such*

$$\sup_{t \in \overline{\Omega}} \int_{\Omega} \theta(s) G(t, s) ds \leq 1,$$

3. *there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such for all $s \in \overline{\Omega}$ and $u, v \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$, we have*

$$|f(s, u(s)) - f(s, v(s))| \leq \theta(s) \varphi(|u - v|),$$

then the problem (4.2) have a unique solution.

In additionally if the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such $\psi(t) = t - \varphi(t)$ is bijective, then the problem (4.2) is generalized Hyers-Ulam stable.

Proof. . Consider the mapping:

$$Tx(t) = \int_{\Omega} G(s,u(s))f(s,u(s))ds,$$

since f is continuous on into ,then $T : \mathcal{C}(\overline{\Omega}, \mathbb{R}) \to \mathcal{C}(\overline{\Omega}, \mathbb{R})$, i.e T is a self mapping on $\mathcal{C}(\overline{\Omega}, \mathbb{R})$, and so the problem (4.3) have a solution if and only if the self-mapping T have a fixed point in $\mathcal{C}(\overline{\Omega}, \mathbb{R})$.
T is continuous, then $\{id_{\mathcal{C}(\overline{\Omega},\mathbb{R})}, S\}$ (id the identity in the space $\mathcal{C}(\overline{\Omega}, \mathbb{R})$) is subsequentially continuous and compatible of type (E), further we have:

$$|Tu(t) - Tv(t)| = |\int_{\Omega} G(t,s)(f(s,u(s)) - f(s,v(s)))ds| \leq$$

$$\int_{\Omega} |G(t,s)(f(s,u(s)) - f(s,v(s)))|ds \leq \varphi(|u-v|) \int_{\Omega} |G(t,s)\theta(s)|ds,$$

$$\leq \varphi(|u-v|) \leq \varphi(\max(d(u,v), 0, d(v,Tv), d(u,Tv), d(Tv,u)),$$

consequently all the hypotheses of Corollary 3.4 (with $A = B = S = id_{\mathcal{C}(\overline{\Omega},\mathbb{R})}$) hold, then T have a unique fixed point and so the problem (4.2) have a unique solution.
For the stability, putting $\psi(t) = t - \varphi(t)$, we get

$$d(u,u^*) \leq d(u,Tu) + d(Tu,Tu^*) \leq$$

$$\leq d(u,Tu) + \varphi(|u-u^*|)$$

then

$$d(u,u^*| \leq \psi^{-1}(d(u,Tu)) \leq \psi^{-1}(\varepsilon),$$

consequently the problem (4.2) has a generalized Hyers-Ulam stability. □

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A new public key encryption scheme based on two cryptographic assumptions

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Abstract

In this paper we present a new cryptosystem which is a combination of RSA variant namely rebalanced RSA-CRT and general formulation of DGDLP. Its security is depend upon integer factorization problem and general formulation of DGDLP.

Keywords: public key cryptography, integer factorization problem, discrete logarithm problem, generalized discrete logarithm problem, cryptosystem.

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1 Introduction

The public key cryptography has a major advantage over the symmetric key cryptography. In symmetric key cryptography, a prior communication of secret key is required. The public key cryptography eliminate the necessity of sharing secret key. It has own public key which is known by everybody and has corresponding private key which is known only by the intended recipient. The public key cryptography is based on one-way trapdoor function, where the encryption rule is easy to compute, but decryption rule is computationally infeasible without any additional information. Thus the security of public key cryptosystem are based on the intractability of hard mathematical problems such as integer factorization problem, discrete logarithm problem etc.

Up to now, most of the public key schemes are based on one cryptographic assumption. Although these schemes are secure but it is possible that in future efficient algorithms will be developed to break these assumptions. To enhance security is the major motivation for developing cryptosystems based on multiple cryptographic assumptions, since it is very unlikely that these assumptions would simultaneously become easy to solve. In 1988, K. S. McCurley [7] proposed the first key distribution scheme based on two hard dissimilar assumptions. The scheme is modification of ElGamal cryptosystem. Instated of using an arithmetic modulus a prime p, he used a modulus n that is a product of two primes. To break the scheme requires the prime factorization of n and ability to solve DLP. In [6], L.Harn proposed a cryptosystem based on two cryptographic assumption. To break the cryptosystem requires to solve simultaneously a Diffie-Hellman problem in a subgroup of $\mathbb{Z}_p^*$, where p is a large prime such that $p = 2p' \times q' + 1$ and p', q' are large primes which are part of the private key, and to factor $(p-1)/2$. After that many public key schemes was developed which are based on two cryptographic assumptions (for example [9], [4], [5] etc).

By this motivation, we proposed a public key scheme whose security is based on RSA variant namely rebalanced RSA-CRT [3] and generalization of GDLP. Rebalanced RSA-CRT is a variant of RSA that enables

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us to rebalanced the difficulty of encryption and decryption. It speed up the RSA decryption procedure. Generalization of GDLP stated that given a finite group of order n and elements $\alpha, \gamma \in G$, find an integer x modulo n such that $\alpha^x = \gamma$, provided that such an integer exists. In this formulation, it is not required that G be a cyclic group, and even it is, it is not required to consider generator of the group. Since α is not generator of the group so α^x is not unique, which makes the problem harder to solve than GDLP [8], [4]. In this proposed scheme, an attacker has to solve simultaneously two generalized GDLP (we call it generalization of DGDLP or general formulation of DGDLP) and IFP. One advantage of this scheme is that it include non-cyclic groups. Another advantage is that because of the use of Chinese remainder theorem (CRT), decryption process of the proposed cryptosystem is fast.

The remainder of this paper is organized as follows. In section 2, we present our cryptosystem. Section 3 is devoted to security of the proposed cryptosystem and in section 4 we deal with its performance. Section 5 is the conclusion of the paper. Throughout the paper all notations are usual. For example the multiplicative group of $\mathbb{Z}_n$ is denoted by $\mathbb{Z}_n^*$, the Eular's phi function of n is denoted by $\phi(n)$ etc.

2 The proposed public key cryptosystem

In this section we present our public key cryptosystem.

Public and private key generation:

A user $\mathcal{A}$, who wants to create a public and private key, have to do the following steps:

1. Choose two large prime p and q of almost same size such that $(p-1, q-1) = 2$.
2. Compute $n = pq$ and $\phi(n) = (p-1)(q-1)$.
3. Choose two integer d_p and d_q such that $(d_p, p-1) = 1$, $(d_q, q-1) = 1$ and $d_p \equiv d_q \ (mod\ 2)$.
4. Find d such that $d \equiv d_p \ (mod\ p-1)$ and $d \equiv d_q \ (mod\ q-1)$.
5. Compute $e \equiv d^{-1} \ (mod\ \phi(n))$.
6. Choose a, b such that $0 \leq a, b \leq \phi(n) - 1$.
7. Choose $\alpha, \beta \in \mathbb{Z}_n^*$ and compute $y_1 \equiv \alpha^a \ (mod\ n)$ and $y_2 \equiv \beta^b \ (mod\ n)$

The public key of $\mathcal{A}$ is (n, e, y_1, y_2) and the corresponding private key is $(a, b, \alpha, \beta, d_p, d_q)$.

Encryption:

The plaintext space is $\mathbb{Z}_n$. Suppose that another user $\mathcal{B}$ want to send a message $m \in \mathbb{Z}_n$ to $\mathcal{A}$ using $\mathcal{A}$'s public key. $\mathcal{B}$ have to do the following step:

1. Compute $c \equiv (my_1y_2)^e \ (mod\ n)$

$\mathcal{B}$ send to $\mathcal{A}$ the encrypted message c.

Decryption:

For the decryption of the message c, $\mathcal{A}$ should do the following steps:

1. Compute $d_1 \equiv \alpha^{\phi(n)-a} \ (mod\ n) \equiv \alpha^{-a} \ (mod\ n)$ and calculate $d_1^e \ (mod\ n)$.
2. Compute $d_2 \equiv \alpha^{\phi(n)-b} \ (mod\ n) \equiv \alpha^{-b} \ (mod\ n)$ and calculate $d_2^e \ (mod\ n)$.
3. Compute $M_p \equiv (d_1^e d_2^e c)^{d_p} \ (mod\ p)$ and $M_q \equiv (d_1^e d_2^e c)^{d_q} \ (mod\ q)$.
4. Then using CRT, $\mathcal{A}$ recover the plaintext m.

3 Security

The security of this proposed cryptosystem is based on factoring and discrete logarithm problem. A third party who intercepts the encrypt message c can recover m, by finding the primes factors p and q of n and so d and next by finding a and b from $y_1 \equiv \alpha^a \ (mod\ n)$ and $y_2 \equiv \beta^b \ (mod\ n)$ where α and β both are unknown. That is to break this scheme the attacker has to compute prime factorization of n and ability to solve DLP. The best way to factorized $n = pq$ is by using the number field sieve method. This method is just depend on the size of n and it is computationally infeasible to factor an integer of size 1024 bits and above. We consider the primes p and q in such a way that they resist factorization attack. Also, both d_p and d_q are atleast 160 bits long to prevent the attack proposed in [3]. Since d is large so it prevent the small-d attacks [2, 10]. The primes p and q are consider in such a way that DLP is intractable. To find a and b, a third party need to solve two generalization of GDLP (we called it general formulation of DGDLP or generalized DGDLP). The general formulation of DGDLP does not require that the multiplicative group $\mathbb{Z}_n^*$ be a cyclic group, and so, it is not required that α and β be generators of the group. Therefore the values of power of α and β are not unique and hence this problem is harder to solve than GDLP. As α and β are not public so it makes the problem more harder to solve in general.

Note that if an attacker finds easily a method to compute d or factoring n, then he has still to solve general formulation of DGDLP. Alternatively, if the attacker can easily solve the general formulation of DGDLP, then he also has to compute d by factoring n. Thus, in any case an attacker has to solve two hard problem.

Also, if $a = 0 = b$ then $y_1 = 0 = y_2$ and so Rebalanced RSA-CRT cryptosystem is a special case of the proposed cryptosystem. Hence if there is an oracle that can break the proposed cryptosystem then the oracle can break Rebalanced RSA-CRT. So, the proposed cryptosystem is atleast as secure as Rebalanced RSA-CRT.

4 Performance analysis

The encryption algorithm for our scheme requires two modular multiplications and one modular exponentiation. One modular multiplication can be done in advance. Thus, the encryption requires only one modular multiplication and one modular exponentiation. The decryption algorithm required four modular exponentiation viz. d_1^e, d_2^e, $(d_1^e d_2^e c)^{d_p}$ and $(d_1^e d_2^e c)^{d_q}$, two modular multiplication viz. $d_1^e d_2^e$ and $d_1^e d_2^e c$ and two applications of extended Euclidean algorithm for computation of $(\alpha^a)^{-1}$, $(\beta^b)^{-1}$ mod n. Thus the decryption algorithm required one modular multiplication viz. $d_1^e d_2^e c$ and two modular exponentiation viz. $(d_1^e d_2^e c)^{d_p}$ and $(d_1^e d_2^e c)^{d_q}$ and others can be done in advance. Hence, the encryption scheme is as efficient as the encryption scheme described in section iv of [5], if both the schemes have same e. Since e is large so encryption take more time than [5]. Since d_p and d_q are small so decryption scheme is fast compare to the decryption scheme in [5]. Also, plain-text and cipher-text is of same length.

5 Conclusion

In this paper we have proposed a public key cryptosystem which is based on rebalance RSA-CRT and general formulation DGDLP. Decryption of the proposed cryptosystem is faster than decryption of [5]. The proposed scheme is more secure than rebalanced RSA-CRT.

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Periodic boundary value problems for singular fractional differential equations with impulse effects

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Abstract

Firstly by using iterative method, we prove existence and uniqueness of solutions of Cauchy problems of differential equations involving Caputo fractional derivative, Riemann-Liouville and Hadamard fractional derivatives with order $q \in (0,1)$. Then we obtain exact expression of solutions of impulsive fractional differential equations, i.e., exact expression of piecewise continuous solutions. Finally, four classes of integral type periodic boundary value problems of singular fractional differential equations with impulse effects are proposed. Sufficient conditions are given for the existence of solutions of these problems. We allow the nonlinearity $p(t)f(t,x)$ in fractional differential equations to be singular at $t = 0, 1$ and be involved a super-linear and sub-linear term. The analysis relies on Schaefer's fixed point theorem.

Keywords: singular fractional differential system, impulsive boundary value problem, Riemann-Liouville fractional derivative, Caputo fractional derivative, Hadanard fractional derivative, Caputo type Hadamard fractional derivative, fixed point theorem.

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Contents

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1 Introduction

One knows that the fractional derivatives (Riemann-Liouville fractional derivative, Caputo fractional derivative and Hadamard fractional derivative and other type see [40]) are actually nonlocal operators because integrals are nonlocal operators. Moreover, calculating time fractional derivatives of a function at some time requires all the past history and hence fractional derivatives can be used for modeling systems with memory.

Fractional order differential equations are generalizations of integer order differential equations. Using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena. Fractional differential equations have many applications see Chapter 10 in [63], books [41, 63, 66].

In recent years, there have been many results obtained on the existence and uniqueness of solutions of initial value problems or boundary value problems for nonlinear fractional differential equations, see [16, 18, 58, 61, 62, 64, 71, 85, 88].

Dynamics of many evolutionary processes from various fields such as population dynamics, control theory, physics, biology, and medicine. undergo abrupt changes at certain moments of time like earthquake, harvesting, shock, and so forth. These perturbations can be well approximated as instantaneous change of states or impulses.These processes are modeled by impulsive differential equations. In 1960, Milman and Myshkis introduced impulsive differential equations in their paper [56]. Based on their work, several monographs have been published by many authors like Samoilenko and Perestyuk [67], Lakshmikantham et al. [50], Bainov and Simeonov [21, 22], Bainov and Covachev [23], and Benchohra et al. [24].

Fractional differential equation was extended to impulsive fractional differential equations, since Agarwal and Benchohra published the first paper on the topic [20] in 2008. Since then many authors such as in [8, 27, 30, 39, 42, 43, 46–49, 60, 64, 70, 71, 84] studied the existence or uniqueness of solutions of impulsive initial or boundary value problems for fractional differential equations. For examples, impulsive anti-periodic boundary value problems see [10, 11, 20, 44, 72, 73], impulsive periodic boundary value problems see [69, 79], impulsive initial value problems see [25, 29, 59, 68], two-point, three-point or multi-point impulsive boundary value problems see [9, 72, 87], impulsive boundary value problems on infinite intervals see [86].

In [31], Feckan and Zhou pointed out that the formula of solutions for impulsive fractional differential equations in [2, 7, 13, 19] is incorrect and gave their correct formula. In [76, 78], the authors established a general framework to find the solutions for impulsive fractional boundary value problems and obtained some sufficient conditions for the existence of the solutions to a kind of impulsive fractional differential equations respectively. In [75], the authors illustrated their comprehensions for the counterexample in [31] and criticized the viewpoint in [31, 76, 78]. Next, in [32], Feckan et al. expounded for the counterexample in [31] and provided further five explanations in the paper.

Recently, in [33, 78, 89], the authors studied the existence and uniqueness of solutions of the following boundary value problem of impulsive fractional differential equation

$$\begin{cases} {}^CD_{0^+}^q x(t) = f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \\ \Delta x|_{t=t_i} = I_i(x(t_i^-)), i \in \mathbb{N}, \\ ax(0) + bx(T) = x_0, \end{cases} \tag{1.1}$$

where $q \in (0, 1]$, ${}^CD_{0^+}^q$ is the standard Caputo fractional derivative of order q, $\mathbb{N}_0 = \{0, 1, \cdots, m\}$ and $\mathbb{N} = \{1, 2, \cdots, m\}$, $f : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ is is a jointly continuous function, $I_k : \mathbb{R} \mapsto \mathbb{R}(k = 1, 2, \cdots, m)$ are continuous functions, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = \lim_{t\to t_k^+} x(t) - \lim_{t\to t_k^-} x(t) = x(t_k^+) - x(t_k^-)$ and $x(t_k^+), x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively, a, b, x_0 a constant with $a + b \neq 0$. One knows that the boundary condition $ax(0) + bx(T) = x_0$ in (1.1) becomes $x(0) - x(T) = \frac{x_0}{a}$ when $a + b = 0$, that is so called nonhomogeneous periodic type boundary condition.

Wang and Bai [69] studied the existence and uniqueness of solutions of the following periodic boundary value problems for nonlinear impulsive fractional differential equation

$$\begin{cases} {}^{RL}D_{0^+}^\alpha x(t) - \lambda x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1\}, \\ \lim_{t\to 0} t^{1-\alpha} x(t) = x(1), \\ \lim_{t\to t_1^+} (t - t_1)^{1-\alpha}[x(t) - x(t_1)] = I(x(t_1)), \end{cases} \tag{1.2}$$

where $\alpha \in (0,1]$, ${}^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $\lambda \in \mathbb{R}$, $0 < t_1 < 1$, $I \in C(\mathbb{R},\mathbb{R})$, f is continuous at every point $(t,u) \in [0,1] \times \mathbb{R}$. We note that the impulse effects in (1.2) change to $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t) = I(x(t_1))$ when $\alpha \in (0,1)$. The assumptions imposed on f and I are as follows: (i) there exists a constant $M > 0$ such that $|f(t,u)| \le M$ and $|I(u)| \le M$ for all $t \in [0,1]$ and $u \in \mathbb{R}$; (ii) there exist positive constant k, and l such that $|f(t,u) - f(t,v)| \le k|u-v|$ and $|I(u) - I(v)| \le l|u-v|$ for all $t \in [0,1]$ and $u,v \in \mathbb{R}$.

One knows that $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}[x(t)-x(t_1)] = I(x(t_1))$ becomes $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t) = I(x(t_1))$ if $\alpha \in (0,1)$. So it is easy to know that the results can not be applied to solve the following problem

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) - \lambda x(t) = f(t,x(t)), t \in (0,1] \setminus \{t_1\}, \\ \lim_{t\to 0} t^{1-\alpha}x(t) = x(1), \\ \lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t) - x(t_1) = J(x(t_1)), \end{cases} \tag{1.3}$$

since $I(x) = x + J(x)$ in mentioned problem corresponding to (1.3) may be unbounded. Furthermore, it seems to be difficult to generalize the method in the proof of Lemma 2.1[69] to the following problem with multiple impulse point:

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) - \lambda x(t) = f(t,x(t)), t \in (t_i,t_{i+1}], i = 0,1,2,\cdots,m, \\ \lim_{t\to 0} t^{1-\alpha}x(t) = x(1), \\ \lim_{t\to t_i^+}(t-t_i)^{1-\alpha}x(t) - x(t_i) = I(t_i,x(t_i)), i = 1,2,\cdots,m. \end{cases} \tag{1.4}$$

In a fractional differential equation, there exist two cases: the firs case is $D^{\alpha} = D_{0^+}^{\alpha}$ in (1.1) or (1.3), i.e., the fractional derivative has a unique start point. Recently, Belmekki, Nieto and Rodriguez-Lopez [17] consider the second case in which D^{α} has multiple start points, i.e., $D^{\alpha} = D_{t_i^+}^{\alpha}$. They studied the existence and uniqueness of solutions of the following periodic boundary value problem of the impulsive fractional differential equation

$$\begin{cases} {}^{RL}D_{t_i^+}^{\alpha}u(t) - \lambda u(t) = f(t,u(t)), t \in (t_i,t_{i+1}], i = 0,1,2,\cdots,p, \\ \lim_{t\to 0^+} t^{1-\alpha}u(t) = u(1), \\ \lim_{t\to t_1^+}(t-t_i)^{1-\alpha}[u(t) - u(t_i)] = I_i(u(t_i)), i = 1,2,\cdots,p, \end{cases} \tag{1.5}$$

where $\alpha \in (0,1), 0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1, \lambda \in \mathbb{R}$, ${}^{RL}D_{t_i^+}^{\alpha}$ represents the standard Riemann-Liouville fractional derivatives, $I_i \in C(\mathbb{R},\mathbb{R})(i = 1,2,\cdots,p)$, f is continuous at every point $(t,u) \in (t_i,t_{i+1}] \times \mathbb{R}$ for $i = 0,1,2,\cdots,p$. The assumptions imposed on f and I_i are similar to those used in [69].

We observed that in the above-mentioned work, the authors all require that the nonlinear term f is bounded and continuous, if the impulse functions I_k, J_k are bounded, it is easy to see that these conditions are very strongly restrictive and difficult to satisfy in applications. We observed that in the above-mentioned work, the authors all require that the nonlinear term f is bounded and continuous, if the impulse functions I_k, J_k are bounded, it is easy to see that these conditions are very strongly restrictive and difficult to satisfy in applications. Furthermore, there has been few papers discussed the existence of solutions of the periodic boundary value problems for impulsive fractional differential equations involving other fractional derivatives such as the impulsive Hadamard type fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}x(t) - \lambda x(t) = f(t,x(t)), t \in (t_i,t_{i+1}], i = 0,1,\cdots,m, \\ \Delta x(t_i) = I(t_i,x(t_i)), i = 1,2,\cdots,m, \end{cases} \tag{1.6}$$

where $D_{0^+}^{\alpha}$ is the so called Hadamard type fractional derivative of order $\alpha \in (0,1)$.

In this paper, we will study the existence of solutions of four classes of impulsive integral type boundary value problems of singular fractional differential systems. The first one is as follows:

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) - \lambda x(t) = p(t)f(t,x(t)), t \in (t_i,t_{i+1}], i = 0,1, \\ x(1) - \lim_{t\to 0} t^{1-\alpha}x(t) = \int_0^1 \phi(s)G(s,x(s))ds, \\ \lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t) - x(t_1) = I(t_1,x(t_1)), \end{cases} \tag{1.7}$$

where

(a) $0<\alpha<1$, $\lambda\in R$, ${}^{RL}D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α,

(b) $0=t_0<t_1<t_2=1$,

(c) $\phi:(0,1)\mapsto\mathbb{R}$ satisfy $\phi|_{(0,t_1)},\in L^1(0,t_1),\phi|_{(t_1,1)}\in L^1(t_1,1)$,

(d) $p:(0,1)\mapsto\mathbb{R}$ satisfy the growth conditions: there exist constants k,l with $k>-1$ and $\max\{-\alpha,-k-1\}<l\le 0$ such that $|p(t)|\le t^k(1-t)^l, t\in(0,1)$,

(e) f,G defined on $(0,1]\times\mathbb{R}$ are **impulsive II-Carathéodory functions**, $I:\{t_1\}\times\mathbb{R}\to\mathbb{R}$ is a **Discrete II-Carathéodory function**.

The second one is following

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t)-\lambda x(t)=p(t)f(t,x(t)), t\in(t_i,t_{i+1}], i=0,1,\\ x(1)-\lim\limits_{t\to 0^+}x(t)=\int_0^1\phi(s)G(s,x(s))ds,\\ \lim\limits_{t\to t_1^+}x(t)-x(t_1)=I(t_1,x(t_1)), \end{cases} \tag{1.8}$$

where

(f) $0<\alpha<1$, $\lambda\in\mathbb{R}$, ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order α, t_i satisfies (b), $\phi:(0,1)\mapsto\mathbb{R}$ satisfy (c), $p:(0,1)\mapsto\mathbb{R}$ satisfy that there exist constants k,l with $k>-1$, $l\le 0, l\le 0$ with $\alpha+l>0$, $\alpha+k+l>0$ such that $|p(t)|\le t^k(1-t)^l,\ t\in(0,1)$,

(g) f,G defined on $(0,1]\times\mathbb{R}$ are **impulsive I-Carathéodory functions**, $I:\{t_1\}\times\mathbb{R}\mapsto\mathbb{R}$ is a **Discrete I-Carathéodory function**.

We emphasize that much work on fractional boundary value problems involves either Riemann-Liouville or Caputo type fractional differential equations see [4–6, 11]. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [35], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. Recent studies can be seen in [12, 14, 15].

Thirdly we study the following impulsive integral type boundary value problems of singular fractional differential systems

$$\begin{cases} {}^{RLH}D_{1^+}^{\alpha}x(t)-\lambda x(t)=p(t)f(t,x(t)), t\in(t_i,t_{i+1}], i=0,1,\\ x(e)-\lim\limits_{t\to 1^+}(\log t)^{1-\alpha}x(t)=\int_1^e\phi(s)G(s,x(s))ds,\\ \lim\limits_{t\to t_1^+}\left(\log\frac{t}{t_1}\right)^{1-\alpha}x(t)-x(t_1)=I(t_1,x(t_1)), \end{cases} \tag{1.9}$$

where

(h) $0<\alpha<1$, $\lambda\in R$, ${}^{RLH}D_{1^+}^{\alpha}$ is the Hadamard fractional derivative of order α,

(i) $1=t_0<t_1<t_2=e,\phi\in L^1(1,e)$, $p:(1,e)\mapsto\mathbb{R}$ are continuous and satisfy the growth conditions: there exist constants k,l with $k>-1$ and $\max\{-\alpha,-k-1\}<l\le 0$ such that $|p(t)|\le(\log t)^k(1-\log t)^l,\ t\in(1,e)$,

(j) f,G defined on $(1,e]\times\mathbb{R}$ are **impulsive III-Carathéodory functions**, $I:\{t_1\}\times\mathbb{R}\mapsto\mathbb{R}$ is a **Discrete III-Carathéodory function**.

Finally we study the following impulsive integral type boundary value problems of singular fractional differential systems

$$\begin{cases} {}^{CH}D_{1^+}^{\alpha}x(t)-\lambda x(t)=p(t)f(t,x(t)), t\in(t_i,t_{i+1}], i=0,1,\\ x(e)-\lim\limits_{t\to 1^+}x(t)=\int_1^e\phi(s)G(s,x(s))ds,\\ \lim\limits_{t\to t_1^+}x(t)-x(t_1)=I(t_1,x(t_1)), \end{cases} \tag{1.10}$$

where

(k) $0<\alpha<1$, $\lambda\in\mathbb{R}$, ${}^{CH}D_{1^+}^{\alpha}$ is the Caputo type Hadamard fractional derivative of order α'

(l) $1=t_0<t_1<t_2=e,\phi\in L^1(1,e)$ and $p:(1,e)\mapsto\mathbb{R}$ satisfies that there exist constants k,l with $k>-1$ and $\max\{-\alpha,-k-\alpha\}<l\le 0$ such that $|p(t)|\le(\log t)^k(1-\log t)^l,\ t\in(1,e)$,

(m) f, G defined on $(1,e] \times \mathbb{R}$ are **impulsive I-Carathéodory functions**, $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ is a **Discrete I-Carathéodory function**.

A function $x : (0,1] \mapsto \mathbb{R}$ is called a solution of BVP(1.7) (or BVP(1.8)) if $x|_{(t_i,t_{i+1}]}(i = 0,1)$ is continuous, the limits below exist $\lim_{t \to t_i^+}(t-t_i)^{1-\alpha}x(t), i = 0,1$ (or $\lim_{t \to t_i^+} x(t)(i = 0,1)$ and x satisfies all equations in (1.7) (or (1.8)).

A function $x : (1,e] \mapsto \mathbb{R}$ is called a solution of BVP(1.9) (or BVP(1.10)) if $x|_{(t_i,t_{i+1}]}(i = 0,1)$ is continuous, the limits below exist $\lim_{t \to t_i^+}\left(\log \frac{t}{t_i}\right)^{1-\alpha} x(t), i = 0,1$ (or $\lim_{t \to t_i^+} x(t)(i = 0,1)$ and x satisfies all equations in (1.9) (or (1.10)).

To get solutions of a boundary value problem of fractional differential equations, we firstly define a Banach space X, then we transform the boundary value problem into a integral equation and define a nonlinear operator T on X by using the integral equation obtained, finally, we prove that T has fixed point in X. The fixed points are just solutions of the boundary value problem. Three difficulties occur in known papers: one is how to transform the boundary value problem into a integral equation; the other one is how to define and prove a Banach space and the completely continuous property of the nonlinear operator defined; the third one is to choose a suitable fixed point theorem and impose suitable growth conditions on functions to get the fixed points of the operator.

To the best of the authors knowledge, no one has studied the existence of solutions of BVP(1.i) $(i = 7,8,9,10)$. This paper fills this gap. Another purpose of this paper is to illustrate the similarity and difference of these three kinds of fractional differential equations. We obtain results on the existence of at least one solution for BVP(1.i) $(i = 7,8,9,10)$ respectively. Some examples are given to illustrate the efficiency of the main theorems. For simplicity we only consider the left-sided operators here. The right-sided operators can be treated similarly.

The remainder of this paper is as follows: in Section 2, we present related definitions; in Section 3 some preliminary results are given. In Sections 4, the main theorems and their proof are given. In Section 5, a mistake happened in cited paper is showed and a corrected expression of solutions is given.

2 Related definitions

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [41, 63, 66].

Let the Gamma function, Beta function and the classical Mittag-Leffler special function be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx, \quad \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad E_{\delta,\sigma}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\delta k+\sigma)}$$

respectively for , $\alpha > 0, p > 0, q > 0, \delta > 0, \sigma > 0$. We note that $E_{\delta,\delta}(x) > 0$ for all $x \in \mathbb{R}$ and $E_{\delta,\delta}(x)$ is strictly increasing in x. Then for $x > 0$ we have $E_{\delta,\delta}(-x) < E_{\delta,\delta}(0) = \frac{1}{\Gamma(\delta)} < E_{\delta,\delta}(x)$.

Definition 2.1. *[41]. Let $c \in \mathbb{R}$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (c,\infty) \mapsto \mathbb{R}$ is given by*

$$I_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} g(s)ds,$$

provided that the right-hand side exists.

Definition 2.2. *[41]. Let $c \in \mathbb{R}$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $g : (c,+\infty) \mapsto \mathbb{R}$ is given by*

$${}^{RL}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_c^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $\alpha < n < \alpha+1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.3. *[41]. Let $c \in \mathbb{R}$. The Caputo fractional derivative of order $\alpha > 0$ of a function $g : (c,+\infty) \mapsto \mathbb{R}$ is given by*

$${}^{C}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $\alpha < n < \alpha+1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.4. *[41]. Let $c > 0$. The Hadamard fractional integral of order $\alpha > 0$ of a function $g : [c, +\infty) \mapsto \mathbb{R}$ is given by*

$$^{H}I^{\alpha}_{c^+}g(t) = \frac{1}{\Gamma(\alpha)}\int_c^t \left(\log\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s},$$

provided that the right-hand side exists.

Definition 2.5. *[41]. Let $c > 0$. The Hadamard fractional derivative of order $\alpha > 0$ of a function $g : [c, +\infty) \mapsto \mathbb{R}$ is given by*

$$^{RLH}D^{\alpha}_{c^+}g(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_c^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} g(s)\frac{ds}{s},$$

where $\alpha < n < \alpha + 1$, i.e., $n = \lceil\alpha\rceil$, provided that the right-hand side exists.

Definition 2.6. *[38]. Let $c > 0$. The Caputo type Hadamard fractional derivative of order $\alpha > 0$ of a function $g : [c, +\infty) \mapsto \mathbb{R}$ is given by*

$$^{CH}D^{\alpha}_{c^+}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_c^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n g(s)\frac{ds}{s},$$

where $\alpha < n < \alpha + 1$, i.e., $n = \lceil\alpha\rceil$, provided that the right-hand side exists.

Definition 2.7. *We call $F : \bigcup_{i=0}^{1} (t_i, t_{i+1}) \times \mathbb{R} \mapsto \mathbb{R}$ an* **impulsive I-Carathéodory function** *if it satisfies*

(i) *$t \mapsto F(t, u)$ is measurable on $(t_i, t_{i+1})(i = 0, 1)$ for any $u \in \mathbb{R}$,*
(ii) *$u \mapsto F(t, u)$ are continuous on $\mathbb{R}$ for all $t \in (t_i, t_{i+1})(i = 0, 1)$,*
(iii) *for each $r > 0$ there exists $M_r > 0$ such that*

$$|F(t, u)| \leq M_r, t \in (t_i, t_{i+1}), |u| \leq r, (i = 0, 1).$$

Definition 2.8. *We call $F : \bigcup_{i=0}^{1} (t_i, t_{i+1}) \times \mathbb{R} \mapsto \mathbb{R}$ an* **impulsive II-Carathéodory function** *if it satisfies*

(i) *$t \mapsto F\left(t, (t - t_i)^{\alpha-1}u\right)$ is measurable on $(t_i, t_{i+1})(i = 0, 1)$ for any $u \in \mathbb{R}$,*
(ii) *$u \mapsto F\left(t, (t - t_i)^{\alpha-1}u\right)$ are continuous on $\mathbb{R}$ for all $t \in (t_i, t_{i+1})(i = 0, 1)$,*
(iii) *for each $r > 0$ there exists $M_r > 0$ such that*

$$\left|F\left(t, (t - t_i)^{\alpha-1}u\right)\right| \leq M_r, t \in (t_i, t_{i+1}), |u| \leq r, (i = 0, 1).$$

Definition 2.9. *We call $F : \bigcup_{i=0}^{1} (t_i, t_{i+1}) \times \mathbb{R} \mapsto \mathbb{R}$ an* **impulsive III-Carathéodory function** *if it satisfies*

(i) *$t \mapsto F\left(t, \left(\log\frac{t}{t_i}\right)^{\alpha-1} u\right)$ is measurable on $(t_i, t_{i+1})(i = 0, 1)$ for any $u \in \mathbb{R}$,*
(ii) *$u \mapsto F\left(t, \left(\log\frac{t}{t_i}\right)^{\alpha-1} u\right)$ are continuous on $\mathbb{R}$ for all $t \in (t_i, t_{i+1})(i = 0, 1)$,*
(iii) *for each $r > 0$ there exists $M_r > 0$ such that*

$$\left|F\left(t, \left(\log\frac{t}{t_i}\right)^{\alpha-1} u\right)\right| \leq M_r, t \in (t_i, t_{i+1}), |u| \leq r, (i = 0, 1).$$

Definition 2.10. *We call $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ an* **discrete I-Carathéodory function** *if it satisfies*

(i) *$u \mapsto I(t_1, u)$ are continuous on $\mathbb{R}$,*
(ii) *for each $r > 0$ there exists $M_r > 0$ such that $|I(t_1, u)| \leq M_r, |u| \leq r$.*

Definition 2.11. *We call $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ an* **discrete II-Carathéodory function** *if it satisfies*

(i) *$u \mapsto I\left(t_1, t_1^{\alpha-1}u\right)$ are continuous on $\mathbb{R}$,*
(ii) *for each $r > 0$ there exists $M_r > 0$ such that $\left|I\left(t_1, t_1^{\alpha-1}u\right)\right| \leq M_r, |u| \leq r$.*

Definition 2.12. *We call $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ an* **discrete III-Carathéodory function** *if it satisfies*

(i) *$u \mapsto I\left(t_1, (\log t_1)^{\alpha-1} u\right)$ are continuous on $\mathbb{R}$,*
(ii) *for each $r > 0$ there exists $M_r > 0$ such that $\left|I\left(t_1, (\log t_1)^{\alpha-1} u\right)\right| \leq M_r, |u| \leq r$.*

Definition 2.13. *[57].Let E and F be Banach spaces. A operator $T : E \mapsto F$ is called a completely continuous operator if T is continuous and maps any bounded set into relatively compact set.*

The following Banach spaces are used:

(i) *Let $a < b$ be constants. $C(a,b]$ denote the set of all continuous functions on $(a,b]$ with the limit $\lim_{t\to a^+} x(t)$ existing, and the norm $||x|| = \sup_{t\in(a,b]} |x(t)|$;*

(ii) *Let $a < b$ be constants. $C_{1-\alpha}(a,b]$ the set of all continuous functions on $(a,b]$ with the limit $\lim_{t\to a^+}(t-a)^{1-\alpha}x(t)$ existing, the norm $||x||_{1-\alpha} = \sup_{t\in(a,b]} (t-a)^{1-\alpha}|x(t)|$;*

(iii) *Let $0 < a < b$. $LC_{1-\alpha}(a,b]$ denote the set of all continuous functions on $(a,b]$ with the limit $\lim_{t\to a^+} \left(\log\frac{t}{a}\right)^{1-\alpha} x(t)$ existing, and the norm $||x|| = \sup_{t\in(a,b]} \left(\log\frac{t}{a}\right)^{1-\alpha} |x(t)|$.*

Let m be a positive integer and $\mathbb{N}_0 = \{0,1,2,\cdots,m\}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. The following Banach spaces are also used in this paper:

$$P_mC_{1-\alpha}(0,1] = \left\{x : (0,1] \mapsto \mathbb{R} : \ x|_{(t_i,t_{i+1}]} \in C_{1-\alpha}(t_i,t_{i+1}] : i \in \mathbb{N}_0 \right\}$$

with the norm

$$||x|| = ||x||_{P_mC_{1-\alpha}} = \max\left\{\sup_{t\in(t_i,t_{i+1}]} (t-t_i)^{1-\alpha}|x(t)| : i \in \mathbb{N}_0\right\},$$

$$P_mC(0,1] = \left\{x : (0,1] \mapsto \mathbb{R} : \ x|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}] : i \in \mathbb{N}_0 \right\}$$

with the norm

$$||x|| = ||x||_{P_mC(0,1]} = \max\left\{\sup_{t\in(t_i,t_{i+1}]} |x(t)| : i \in \mathbb{N}_0\right\}.$$

Let $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$. We also use the Banach spaces

$$LP_mC_{1-\alpha}(1,e] = \left\{x : (1,e] \mapsto \mathbb{R} : \begin{array}{c} x|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}], i \in \mathbb{N}_0, \\ \text{there exist the limits} \\ \lim_{t\to t_i^+} \left(\log\frac{t}{t_i}\right)^{1-\alpha} x(t), i \in \mathbb{N}_0 \end{array}\right\}$$

with the norm

$$||x|| = ||x||_{LP_mC_{1-\alpha}} = \max\left\{\sup_{t\in(t_i,t_{i+1}]} \left(\log\frac{t}{t_i}\right)^{1-\alpha} |x(t)|, i \in \mathbb{N}_0\right\},$$

$$P_mC(1,e] = \left\{x : (1,e] \mapsto \mathbb{R} : \ x|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}], i \in \mathbb{N}_0 \right\}$$

with the norm

$$||x|| = ||x||_{P_mC} = \max\left\{\sup_{t\in(t_i,t_{i+1}]} |x(t)|, i \in \mathbb{N}_0\right\}.$$

3 Some preliminary results

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.i)(i=7,8,9,10) respectively.

3.1 Basic theory for linear fractional differential equation

Lakshmikantham et al. [51–54] investigated the basic theory of initial value problems for fractional differential equations involving Riemann-Liouville differential operators of order $q \in (0,1)$. The existence and uniqueness of solutions of the following initial value problems of fractional differential equations were discussed under the assumption that $f \in C_r[0,1]$. We will establish existence and uniqueness results for these problems under more weaker assumptions see (A1)-(A4) in the sequel.

Let $\eta \in R$, $F, A : (0,1) \mapsto \mathbb{R}$ and $B, G : (1,e) \mapsto \mathbb{R}$ are continuous functions. We will consider the following four classes of initial value problems of non-homogeneous linear fractional differential equations:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha} x(t) = A(t)x(t) + F(t),\ t \in (0,1), \\ \lim\limits_{t \to 0^+} x(t) = \eta, \end{cases} \tag{3.1.1}$$

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha} x(t) = A(t)x(t) + F(t),\ t \in (0,1), \\ \lim\limits_{t \to 0^+} t^{1-\alpha} x(t) = \eta, \end{cases} \tag{3.1.2}$$

$$\begin{cases} {}^{RLH}D_{0^+}^{\alpha} x(t) = B(t)x(t) + G(t),\ t \in (1,e), \\ \lim\limits_{t \to 1^+} (\log t)^{1-\alpha}\ x(t) = \eta, \end{cases} \tag{3.1.3}$$

$$\begin{cases} {}^{CH}D_{0^+}^{\alpha} x(t) = B(t)x(t) + G(t),\ t \in (1,e), \\ \lim\limits_{t \to 1^+} x(t) = \eta, \end{cases} \tag{3.1.4}$$

To get solutions of (3.1.1), we need the following assumptions:

(A1) there exists constants $k_i > -1$, $l_i \le 0$ with $l_i > \{-\alpha, -\alpha - k_i\}$ $(i = 1,2)$, $M_A \ge 0$ and $M_F \ge 0$ such that $|A(t)| \le M_A t^{k_1}(1-t)^{l_1}$ and $|F(t)| \le M_F t^{k_2}(1-t)^{l_2}$ for all $t \in (0,1)$.

Choose Picard function sequence as

$$\phi_0(t) = \eta,\ t \in (0,1],$$

$$\phi_n(t) = \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_{n-1}(s) + F(s)]ds, t \in (0,1], n = 1,2,\cdots.$$

Claim 1. $\phi_n \in C(0,1]$. One sees $\phi_0 \in C(0,1]$. Then ϕ_1 is continuous on $(0,1]$, together with

$$\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_0(s) + F(s)]ds\right| \le ||x|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |A(s)|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |F(s)|ds$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [M_A|\eta| s^{k_1}(1-s)^{l_1} + M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le M_A|\eta| \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1} ds + M_F \int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2} ds$$

$$= M_A|\eta| t^{\alpha+k_1+l_1} \int_0^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1} dw + M_F t^{\alpha+k_2+l_2} \int_0^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)} w^{k_2} dw$$

$$= M_A|\eta| t^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{\alpha+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \to 0 \text{ as } t \to 0^+,$$

we see that $\lim\limits_{t \to 0^+} \phi_1(t)$ exists. So $\phi_1 \in C(0,1]$. By mathematical induction method, we can prove that $\phi_n \in C(0,1]$.

Claim 2. $\{\phi_n\}$ is convergent uniformly on $(0,1]$. In fact we have for $t \in (0,1]$ that

$$|\phi_1(t) - \phi_0(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_0(s) + F(s)]ds\right|$$

$$\le M_A|\eta| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1} ds + M_F \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(1-s)^{l_2} ds$$

$$\le M_A|\eta| \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1} ds + M_F \int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2} ds$$

$$= M_A|\eta| t^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{\alpha+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So

$$|\phi_2(t)-\phi_1(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_1(s)-\phi_0(s)]ds\right|$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_A s^{k_1}(1-s)^{l_1}\left(M_A|\eta|s^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}+M_F s^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\right)ds$$

$$\le |\eta|M_A^2\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+2k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}ds$$

$$+M_AM_F\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+k_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}ds$$

$$=|\eta|M_A^2 t^{2\alpha+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}$$

$$+M_AM_F t^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}.$$

Now suppose that

$$|\phi_j(t)-\phi_{j-1}(t)| \le |\eta|M_A^j t^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^{j-1}M_F t^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

We get that

$$|\phi_{j+1}(t)-\phi_j(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_j(s)-\phi_{j-1}(s)]ds\right|$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_A\left(|\eta|M_A^j s^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}\right.$$

$$\left.+M_A^{j-1}M_F s^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}\right)s^{k_1}(1-s)^{l_1}ds$$

$$\le \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}M_A\left(|\eta|M_A^j s^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}\right.$$

$$\left.+M_A^{j-1}M_F s^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}\right)s^{k_1}ds$$

$$\le |\eta|M_A^{j+1}t^{(j+1)\alpha+(j+1)k_1+(j+1)l_1}\prod_{i=0}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^jM_F t^{(j+1)\alpha+jk_1+k_2+jl_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

From the mathematical induction method, we get for every $n=1,2,\cdots$ that

$$|\phi_{n+1}(t)-\phi_n(t)| \le |\eta|M_A^{n+1}t^{(n+1)\alpha+(n+1)k_1+(n+1)l_1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^nM_F t^{(n+1)\alpha+nk_1+k_2+nl_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}$$

$$\le |\eta|M_A^{n+1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^nM_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}, t\in[0,1].$$

Consider

$$\sum_{n=1}^{+\infty} u_n = \sum_{n=1}^{+\infty} |\eta| M_A^{n+1} \prod_{i=0}^{n} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)},$$

$$\sum_{n=1}^{+\infty} v_n = \sum_{n=1}^{+\infty} M_A^n M_F \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{i=1}^{n} \frac{\mathbf{B}(\alpha+l_1, i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

One sees for sufficiently large n that

$$\frac{u_{n+1}}{u_n} = M_A \frac{\mathbf{B}(\alpha+l_1,(n+1)\alpha+(n+1)k_1+(n+1)l_1)}{\Gamma(\alpha)} = M_A \int_0^1 (1-x)^{\alpha+l_1-1} x^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} dx$$

$$\leq M_A \int_0^\delta (1-x)^{\alpha+l_1-1} x^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} dx + M_A \int_\delta^1 (1-x)^{\alpha+l_1-1} dx \text{ with } \delta \in (0,1)$$

$$\leq M_A \int_0^\delta (1-x)^{\alpha+l_1-1} dx \delta^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} + \frac{M_A}{\alpha+l_1} \delta^{\alpha+l_1}$$

$$\leq \frac{M_A}{\alpha+l_1} \delta^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} + \frac{M_A}{\alpha+l_1} \delta^{\alpha+l_1}.$$

For any $\epsilon > 0$, it is easy to see that there exists $\delta \in (0,1)$ such that $\frac{M_a}{\alpha+l_1}\delta^{\alpha+l_1} < \frac{\epsilon}{2}$. For this δ, there exists an integer $N > 0$ sufficiently large such that $\frac{M_a}{\alpha+l_1}\delta^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} < \frac{\epsilon}{2}$ for all $n > N$. So $0 < \frac{u_{n+1}}{u_n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $n > N$. It follows that $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 0$. Then $\sum_{n=1}^{+\infty} u_n$ is convergent. Similarly we get $\sum_{n=1}^{+\infty} v_n$ is convergent. Hence

$$\phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)] + \cdots, t \in [0,1]$$

is uniformly convergent. Then $\{\phi_n(t)\}$ is convergent uniformly on $(0,1]$.

Claim 3. $\phi(t) = \lim_{n\to+\infty} \phi_n(t)$ defined on $(0,1]$ is a unique continuous solution of the integral equation

$$x(t) = \eta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [A(s)x(s) + F(s)] ds, t \in (0,1]. \; l \tag{3.1.5}$$

Proof. By $\phi(t) = \lim_{n\to+\infty} \phi_n(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $[0,1]$ by definging $x(t)|_{t=0} = \lim_{t\to 0^+} x(t)$. From

$$\left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_{n-1}(s) + F(s)] ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_{m-1}(s) + F(s)] ds \right|$$

$$\leq M_A ||\phi_{n-1} - \phi_{m-1}|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1} (1-s)^{l_1} ds$$

$$\leq M_A ||\phi_{n-1} - \phi_{m-1}|| t^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}$$

$$\leq M_A ||\phi_{n-1} - \phi_{m-1}|| \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \to 0 \text{ as } m,n \to +\infty,$$

it follows that

$$\phi(t) = \lim_{n\to+\infty} \phi_n(t) = \lim_{n\to+\infty} \left[\eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_{n-1}(s) + F(s)] ds \right]$$

$$= \eta + \lim_{n\to+\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_{n-1}(s) + F(s)] ds$$

$$= \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[A(s) \lim_{n\to+\infty} \phi_{n-1}(s) + F(s) \right] ds$$

$$= \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi(s) + F(s)] ds.$$

Then ϕ is a continuous solution of (3.1.5) defined on $(0,1]$.

Suppose that ψ defined on $(0,1]$ is also a solution of (3.1.5). Then

$$\psi(t) = \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\psi(s) + F(s)] ds, t \in (0,1].$$

We need to prove that $\phi(t)\equiv\psi(t)$ on $[0,1]$. Then

$$|\psi(t)-\phi_0(t)|=\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)\psi(s)+F(s)|ds\right|$$

$$\leq|\eta|M_At^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}+M_Ft^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$|\psi(t)-\phi_1(t)|=\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s)-\phi_0(s)]ds\right|$$

$$\leq|\eta|M_A^2t^{2\alpha+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}$$

$$+M_AM_Ft^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}.$$

Now suppose that

$$|\psi(t)-\phi_{j-1}(t)|\leq|\eta|M_A^jt^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^{j-1}M_Ft^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

Then

$$|\psi(t)-\phi_j(t)|=\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s)-\phi_{j-1}(s)]ds\right|$$

$$\leq|\eta|M_A^{j+1}t^{(j+1)\alpha+(j+1)k_1+(j+1)l_1}\prod_{i=0}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^jM_Ft^{(j+1)\alpha+jk_1+k_2+jl_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

Hence

$$|\psi(t)-\phi_n(t)|\leq|\eta|M_A^{n+1}t^{(n+1)\alpha+(n+1)k_1+(n+1)l_1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^nM_Ft^{(n+1)\alpha+nk_1+k_2+nl_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}$$

$$\leq|\eta|M_A^{n+1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^nM_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}\text{ for all }n=1,2,\cdots.$$

Similarly we have

$$\lim_{n\to+\infty}|\eta|M_A^{n+1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}=0,$$

$$\lim_{n\to+\infty}M_A^nM_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}=0.$$

Then $\lim_{n\to+\infty}\phi_n(t)=\psi(t)$ uniformly on $(0,1]$. Then $\phi(t)\equiv\psi(t)$. Then (3.1.5) has a unique solution ϕ. The proof is complete.

Theorem 3.1. *Suppose that (A1) holds. Then x is a solution of IVP(3.1.1) if and only if x is a solution of the integral equation (3.1.5).*

Proof. Suppose that $x \in C(0,1]$ is a solution of IVP(3.1.1). Then $\lim_{t\to 0^+} x(t) = \eta$ and $||x|| = r < +\infty$, From (A1), we have

$$\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\right| \le ||x|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|F(s)|ds$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A r s^{k_1}(1-s)^{l_1} + M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le M_A r \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1} ds + M_F \int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2} ds$$

$$= M_A r t^{\alpha+k_1+l_1} \int_0^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1} dw + M_F t^{\alpha+k_2+l_2} \int_0^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)} w^{k_2} dw$$

$$= M_A r t^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{\alpha+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So $t \to \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is defined on $(0,1]$ and

$$\lim_{t\to 0^+} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A(s)x(s)ds = \lim_{t\to 0^+} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds = 0. \tag{3.1.6}$$

Furthermore, we have for $t_1, t_2 \in (0,1]$ with $t_1 < t_2$ that

$$\left|\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\right|$$

$$\le \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)x(s)+F(s)|ds + \int_0^{t_1} \frac{|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|}{\Gamma(\alpha)}|A(s)x(s)+F(s)|ds$$

$$\le M_A r\left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1}ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1}ds\right]$$

$$+M_F\left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(1-s)^{l_2}ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(1-s)^{l_2}ds\right]$$

$$\le M_A r\left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1}ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(t_2-s)^{l_1}ds\right]$$

$$+M_F\left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2}ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(t_2-s)^{l_2}ds\right]$$

$$= M_A r\left[t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1}dw + \int_0^{t_1} \frac{(t_1-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1}ds - \int_0^{t_1} \frac{(t_2-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1}ds\right]$$

$$+M_F\left[t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)} w^{k_2}dw + \int_0^{t_1} \frac{(t_1-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2}ds - \int_0^{t_1} \frac{(t_2-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2}ds\right]$$

$$= M_A r\left[t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1}dw\right.$$

$$\left.+t_1^{\alpha+k_1+l_1}\int_0^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1}dw - t_2^{\alpha+k_1+l_1}\int_0^{\frac{t_1}{t_2}} \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1}dw\right]$$

$$+M_F\left[t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)} w^{k_2}dw\right.$$

$$+t_1^{\alpha+k_2+l_2}\int_0^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw - t_2^{\alpha+k_2+l_2}\int_0^{\frac{t_1}{t_2}} \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\Bigg]$$

$$= M_A r\left[t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw\right.$$

$$\left.+|t_1^{\alpha+k_1+l_1} - t_2^{\alpha+k_1+l_1}|\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} - t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw\right]$$

$$+M_F\left[t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\right.$$

$$\left.+|t_1^{\alpha+k_2+l_2} - t_2^{\alpha+k_2+l_2}|\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} - t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\right] \to 0 \text{ as } t_1 \to t_2.$$

So $t \mapsto \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is continuous on $(0,1]$ by defining

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\Big|_{t=0} = \lim_{t\to 0^+}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds.$$

We have $I_{0^+}^{\alpha}\,{}^CD_{0^+}^{\alpha}x(t) = I_{0^+}^{\alpha}[A(t)x(t)+F(t)]$. So

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds = I_{0^+}^{\alpha}[A(t)x(t)+F(t)] = I_{0^+}^{\alpha}\,{}^CD_{0^+}^{\alpha}x(t)$$

$$= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^s (s-w)^{-\alpha}x'(w)dw\right)ds \text{ interchange the order of ingrals}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t\int_w^t (t-s)^{\alpha-1}(s-w)^{-\alpha}dsx'(w)dw \text{ use } \frac{s-w}{t-w} = u$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t\int_0^1 (1-u)^{\alpha-1}u^{-\alpha}dux'(w)dw \text{ by } \mathbf{B}(\alpha,1-\alpha) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)}$$

$$= \int_0^t x'(w)dw = x(t) - \lim_{t\to 0^+}x(t) = x(t) - \eta.$$

Then $x \in C(0,1]$ is a solution of (3.1.5).

On the other hand, if x is a solution of (3.1.5), together Cases 1,2 and 3, we have $x \in C(0,1]$ and $\lim_{t\to 0^+} x(t) = \eta$. So $x \in C(0,1]$. Furthermore, we have

$${}^CD_{0^+}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left(\eta + \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}[A(w)x(w)+F(w)]dw\right)' ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left(\int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}[A(w)x(w)+F(w)]dw\right)' ds$$

$$= \left[\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{1}{\Gamma(1-(1-\alpha))}\left(\int_0^s (s-w)^{-(1-\alpha)}[A(w)x(w)+F(w)]dw\right)' ds\right]'$$

$$= \frac{1}{\Gamma(2-\alpha)}\left[(t-s)^{1-\alpha}\frac{1}{\Gamma(1-(1-\alpha))}\int_0^s (s-w)^{-(1-\alpha)}[A(w)x(w)+F(w)]dw\Big|_0^t\right.$$

$$\left.+(1-\alpha)\int_0^t (t-s)^{-\alpha}\frac{1}{\Gamma(1-(1-\alpha))}\int_0^s (s-w)^{-(1-\alpha)}[A(w)x(w)+F(w)]dwds\right]'$$

$= \frac{1}{\Gamma(1-\alpha)}\left[\int_0^t (t-s)^{-\alpha}\frac{1}{\Gamma(1-(1-\alpha))}\int_0^s (s-w)^{-(1-\alpha)}[A(w)x(w)+F(w)]dwds\right]'$ by (3.1.6)

$= \frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\left[\int_0^t\int_w^t (t-s)^{-\alpha}(s-w)^{-(1-\alpha)}ds[A(w)x(w)+F(w)]dw\right]'$ by chenging the order of integrals

$= \frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\left[\int_0^t\int_0^1 (1-u)^{-\alpha}u^{\alpha-1}du[A(w)x(w)+F(w)]dw\right]'$ by $\frac{s-w}{t-w}=u$

$= \left[\int_0^t [A(w)x(w)+F(w)]dw\right]'$ by $\mathbf{B}(1-\alpha,\alpha)=\Gamma(1-\alpha)\Gamma(\alpha) = A(t)x(t)+F(t)$.

So $x \in C(0,1]$ is a solution of IVP(3.1.1). The proof is completed. □

Theorem 3.2. *Suppose that (A1) holds. Then (3.1.1) has a unique solution. If there exists constants $k_2 > -1$, $l_2 \le 0$ with $l_2 > \{-\alpha, -\alpha-k_2\}$, $M_F \ge 0$ such that $|F(t)| \le M_F t^{k_2}(1-t)^{l_2}$ for all $t \in (0,1)$, then the following special problem*

$$\begin{cases} {}^C D_{0^+}^{\mathbf{ff}} x(t) = \lambda x(t) + F(t),\ t \in (0,1], \\ \lim\limits_{t\to 0^+} x(t) = \eta \end{cases} \tag{3.1.7}$$

has a unique solution

$$x(t) = \eta E_{\alpha,1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds,\ t \in (0,1]. \tag{3.1.8}$$

Proof. From Claims 1, 2 and 3, Theorem 3.1 implies that (3.1.1) has a unique solution. From the assumption and $A(t) \equiv \lambda$, it is easy to see that (A1) holds with $k_1 = l_1 = 0$ and k_2, l_2 mentioned. Thus (3.1.7) has a unique solution. We get from the Picard function sequence that

$$\phi_n(t) = \eta + \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_{n-1}(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds$$

$$= \eta + \eta\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \lambda^2\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}\phi_{n-2}(w)dwds$$

$$+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}F(w)dwds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds$$

$$= \eta + \frac{\eta\lambda}{\Gamma(\alpha+1)}t^\alpha + \lambda^2\int_0^t\int_w^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}ds\phi_{n-2}(w)dw$$

$$+\lambda\int_0^t\int_w^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}dsF(w)dw + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds$$

$$= \eta + \frac{\eta\lambda}{\Gamma(\alpha+1)}t^\alpha + \lambda^2\int_0^t (t-w)^{2\alpha-1}\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\frac{u^{\alpha-1}}{\Gamma(\alpha)}du\phi_{n-2}(w)dw$$

$$+\lambda\int_0^t (t-w)^{2\alpha-1}\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\frac{u^{\alpha-1}}{\Gamma(\alpha)}duF(w)dw + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds$$

$$= \eta\left(1+\frac{\lambda t^\alpha}{\Gamma(\alpha+1)}\right) + \lambda^2\int_0^t \frac{(t-w)^{2\alpha-1}}{\Gamma(2\alpha)}\phi_{n-2}(w)dw + \int_0^t (t-s)^{\alpha-1}\left(\frac{\lambda(t-s)^\alpha}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha)}\right)F(s)ds$$

$$= \cdots\cdots\cdots$$

$$= \eta\sum_{j=0}^{n-1}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)} + \eta\lambda^n\int_0^t \frac{(t-w)^{n\alpha-1}}{\Gamma(n\alpha)}dw + \int_0^t (t-s)^{\alpha-1}\left(\sum_{j=0}^{n-1}\frac{\lambda^j(t-s)^{j\alpha}}{\Gamma((j+1)\alpha)}\right)F(s)ds$$

$$= \eta\sum_{j=0}^{n}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)} + \int_0^t (t-s)^{\alpha-1}\left(\sum_{j=0}^{n}\frac{\lambda^j(t-s)^{j\alpha}}{\Gamma((j+1)\alpha)}.\right)F(s)ds$$

$$\to \eta E_{\alpha,1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds.$$

Then we get (3.1.8). The proof is complete. □

To get solutions of (3.1.2), we need the following assumptions:

(A2) there exists constants $k_i > -\alpha$, $l_i \le 0$ with $l_1 > \{-\alpha, -\alpha - k_1\}$, $l_2 > \max\{-\alpha, -1 - k_2\}$, $M_A \ge 0$ and $M_F \ge 0$ such that $|A(t)| \le M_A t^{k_1}(1-t)^{l_1}$ and $|F(t)| \le M_F t^{k_2}(1-t)^{l_2}$ for all $t \in (0,1)$.

Choose Picard function sequence as

$$\phi_0(t) = \eta t^{\alpha-1},\ t \in (0,1],$$

$$\phi_n(t) = \eta t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds, t \in (0,1], n = 1,2,\cdots.$$

Claim 1. $\phi_n \in C_{1-\alpha}(0,1]$. Since $\phi_0 \in C_{1-\alpha}(0,1]$, then ϕ_1 is continuous on $(0,1]$, together with

$$t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s) + F(s)]ds\right|$$

$$= t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)s^{\alpha-1}s^{1-\alpha}\phi_0(s) + F(s)]ds\right|$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A|\eta|s^{\alpha-1}s^{k_1}(1-s)^{l_1} + M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le t^{1-\alpha}M_A|\eta|\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+k_1-1}ds + t^{1-\alpha}M_F\int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds$$

$$= M_A|\eta|t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1)}{\Gamma(\alpha)} + M_F t^{1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \to 0 \text{ as } t \to 0^+,$$

we see $\phi_1 \in C_{1-\alpha}(0,1]$. By mathematical induction method, we can prove that $\phi_n \in C_{1-\alpha}(0,1]$.

Claim 2. $\{t \to t^{1-\alpha}\phi_n(t)\}$ is convergent uniformly on $(0,1]$. In fact we have for $t \in (0,1]$ that

$$t^{1-\alpha}|\phi_1(t) - \phi_0(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s) + F(s)]ds\right|$$

$$\le t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A|\eta|s^{k_1}(1-s)^{l_1} + M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le |\eta|M_A t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}ds + M_F t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds$$

$$= |\eta|M_A t^{k_1+l_1+1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So

$$t^{1-\alpha}|\phi_2(t) - \phi_1(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_1(s) - \phi_0(s)]ds\right|$$

$$\le t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_A s^{k_1}(1-s)^{l_1}\left(|\eta|M_A s^{k_1+l_1+1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F s^{k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\right)ds$$

$$\le |\eta|M_A^2 t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{2k_1+l_1+1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}ds$$

$$+M_A M_F t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1+k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}ds$$

$$= |\eta|M_A^2 t^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_A M_F t^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}.$$

$$t^{1-\alpha}|\phi_3(t)-\phi_2(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_2(s)-\phi_1(s)]ds\right|$$

$$\leq t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_A s^{k_1}(1-s)^{l_1}\left(|\eta|M_A^2 s^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\right.$$

$$\left.+M_A M_F s^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2+k_1+k_2+l_2)}{\Gamma(\alpha)}\right)ds$$

$$\leq |\eta|M_A^3 t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{3k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2+2k_1+l_1)}{\Gamma(\alpha)}ds$$

$$+M_A^2 M_F t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{2k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}ds$$

$$=|\eta|M_A^3 t^{3k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}$$

$$+M_A^2 M_F t^{2k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}.$$

$$t^{1-\alpha}|\phi_4(t)-\phi_3(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_3(s)-\phi_2(s)]ds\right|$$

$$\leq t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_A s^{k_1}(1-s)^{l_1}\left(|\eta|M_A^3 s^{3k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}\right.$$

$$\left.+\ M_A^2 M_F s^{2k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}\right)ds$$

$$\leq |\eta|M_A^4 t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{4k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}ds$$

$$+M_A^3 M_F t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{3k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}ds$$

$$=|\eta|M_A^4 t^{4k_1+4l_1+4}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}ds\frac{\mathbf{B}(\alpha+l_1,4k_1+3l_1+4)}{\Gamma(\alpha)}$$

$$+M_A^3 M_F t^{3k_1+k_2+3l_1+l_2+4}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+k_2+2l_1+l_2+4)}{\Gamma(\alpha)}.$$

Similarly by the mathematical induction method, we get for every $n=1,2,\cdots$ that

$$t^{1-\alpha}|\phi_n(t)-\phi_{n-1}(t)| \leq |\eta|M_A^n t^{nk_1+nl_1+n}\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_A^{n-1}M_F t^{(n-1)k_1+k_2+(n-1)l_1+l_2+n}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

$$\leq |\eta|M_A^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_A^{n-1}M_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}, t\in[0,1].$$

Similarly we can prove that both

$$\sum_{n=1}^{+\infty}u_n = \sum_{n=1}^{+\infty}|\eta|M_A^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)},$$

$$\sum_{n=1}^{+\infty}v_n = \sum_{n=1}^{+\infty}M_A^{n-1}M_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

are convergent. Hence

$$t^{1-\alpha}\phi_0(t)+t^{1-\alpha}[\phi_1(t)-\phi_0(t)]+t^{1-\alpha}[\phi_2(t)-\phi_1(t)]+\cdots+t^{1-\alpha}[\phi_n(t)-\phi_{n-1}(t)]+\cdots, t\in[0,1]$$

is uniformly convergent. Then $\{t \to t^{1-\alpha}\phi_n(t)\}$ is convergent uniformly on $(0,1]$.

Claim 3. $\phi(t) = t^{\alpha-1}\lim_{n\to+\infty} t^{1-\alpha}\phi_n(t)$ defined on $(0,1]$ is a unique continuous solution of the integral equation

$$x(t) = \eta t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s) + F(s)]ds, t \in (0,1]. \tag{3.1.9}$$

Proof. By $\lim_{n\to+\infty} t^{1-\alpha}\phi_n(t) = t^{1-\alpha}\phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $(0,1]$. From

$$t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{m-1}(s) + F(s)]ds\right|$$

$$\leq M_A||\phi_{n-1} - \phi_{m-1}||t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}s^{\alpha-1}ds$$

$$\leq M_A||\phi_{n-1} - \phi_{m-1}||t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+k_1-1}ds$$

$$\leq M_A||\phi_{n-1} - \phi_{m-1}||t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1)}{\Gamma(\alpha)}$$

$$\leq M_A||\phi_{n-1} - \phi_{m-1}||\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1)}{\Gamma(\alpha)} \to 0 \text{ uniformly as } m,n \to +\infty,$$

we know that

$$\phi(t) = t^{\alpha-1}\lim_{n\to\infty} t^{1-\alpha}\phi_n(t) = \lim_{n\to+\infty}\left[\eta + t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds\right]$$

$$= \eta t^{\alpha-1} + t^{\alpha-1}\lim_{n\to+\infty} t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds$$

$$= \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[A(s)\phi(s) + F(s)\right]ds.$$

Then ϕ is a continuous solution of (3.1.9) defined on $(0,1]$.

Suppose that ψ defined on $(0,1]$ is also a solution of (3.1.9). Then

$$\psi(t) = \eta t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\psi(s) + F(s)]ds, t \in [0,1].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $(0,1]$. Then

$$t^{1-\alpha}|\psi(t) - \phi_0(t)| = t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)\psi(s) + F(s)|ds\right|$$

$$\leq |\eta|M_A t^{k_1+l_1+1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$t^{1-\alpha}|\psi(t) - \phi_1(t)| = t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s) - \phi_0(s)]ds\right|$$

$$\leq |\eta|M_A^2 t^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_A M_F t^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}.$$

By mathematical induction method, we can get that

$$t^{1-\alpha}|\psi(t) - \phi_n(t)| = t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s) - \phi_{n-1}(s)]ds\right|$$

$$\leq |\eta|M_A^n t^{nk_1+nl_1+n}\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_A^{n-1}M_F t^{(n-1)k_1+k_2+(n-1)l_1+l_2+n}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}.$$

Hence

$$t^{1-\alpha}|\psi(t)-\phi_n(t)| \le |\eta| M_A^n \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_A^{n-1}M_F \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}, \text{ for all } n=1,2,\cdots.$$

Similarly we have $\lim_{n\to+\infty} t^{1-\alpha}\phi_n(t) = t^{1-\alpha}\psi(t)$ uniformly on $(0,1]$. Then $\phi(t)\equiv\psi(t)$ on $(0,1]$. Then (3.1.9) has a unique solution ϕ. The proof is complete.

Theorem 3.3. *Suppose that (A2) holds. Then $x \in C_{1-\alpha}(0,1]$ is a solution of IVP(3.1.2) if and only if $x \in C_{1-\alpha}(0,1]$ is a solution of the integral equation (3.1.9).*

Proof. Suppose that $x \in C_{1-\alpha}(0,1]$ is a solution of IVP(3.1.2). Then $t \to t^{1-\alpha}x(t)$is continuous on $(0,1]$ by defining $t^{1-\alpha}x(t)|_{t=0} = \lim_{t\to0^+} t^{1-\alpha}x(t)$ and $||x|| = r < +\infty$. So by $\frac{w}{s}=u$, we get

$$\lim_{s\to0^+}\int_0^s (s-w)^{-\alpha}x(w)dw = \lim_{s\to0^+}\int_0^s (s-w)^{-\alpha}w^{\alpha-1}w^{1-\alpha}x(w)dw$$

$$= \lim_{s\to0^+}\xi^{1-\alpha}x(\xi)\int_0^s (s-w)^{-\alpha}w^{\alpha-1}dw \text{ by mean value theorem of integral, } \xi\in(0,s)$$

$$= \lim_{s\to0^+}\xi^{1-\alpha}x(\xi)\int_0^1 (1-u)^{-\alpha}u^{\alpha-1}du = \eta\mathbf{B}(1-\alpha,\alpha).$$

From (A2), we have

$$t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\right|$$

$$= t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)s^{\alpha-1}s^{1-\alpha}x(s)+F(s)]ds\right|$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A r s^{\alpha-1}s^{k_1}(1-s)^{l_1}+M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le t^{1-\alpha}M_A r\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+k_1-1}ds + t^{1-\alpha}M_F\int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds$$

$$= M_A r t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1)}{\Gamma(\alpha)} + M_F t^{1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So $t \to t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is defined on $(0,1]$ and

$$\lim_{t\to0^+} t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds = 0. \tag{3.1.10}$$

Furthermore, we have similarly to Theorem 3.1 that $t \to \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is continuous on $(0,1]$. So $t \to t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is continuous on $[0,1]$ by defining

$$t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\Big|_{t=0} = \lim_{t\to0^+} t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds. \tag{3.1.11}$$

We have $I_{0^+}^{\alpha}\,{}^{RL}D_{0^+}^{\alpha}x(t) = I_{0^+}^{\alpha}[A(t)x(t)+F(t)]$. So

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds = I_{0^+}^{\alpha}[A(t)x(t)+F(t)] = I_{0^+}^{\alpha}\,{}^{RL}D_{0^+}^{\alpha}x(t)$$

$$= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{1}{\Gamma(1-\alpha)}\left(\int_0^s (s-w)^{-\alpha}x(w)dw\right)'\right]ds$$

$$= \left[\int_0^t \frac{(t-s)^\alpha}{\Gamma(\alpha+1)} \frac{1}{\Gamma(1-\alpha)} \left(\int_0^s (s-w)^{-\alpha} x(w)dw\right)' ds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \left[(t-s)^\alpha \int_0^s (s-w)^{-\alpha} x(w)dw\Big|_0^t + \alpha \int_0^t (t-s)^{\alpha-1} \int_0^s (s-w)^{-\alpha} x(w)dwds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \left[-t^\alpha \lim_{s\to 0^+} \int_0^s (s-w)^{-\alpha} x(w)dw + \alpha \int_0^t (t-s)^{\alpha-1} \int_0^s (s-w)^{-\alpha} x(w)dwds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \left[\alpha \int_0^t \int_w^t (t-s)^{\alpha-1} (s-w)^{-\alpha} ds x(w)dw\right]' - \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} t^{\alpha-1} \lim_{s\to 0^+} \int_0^s (s-w)^{-\alpha} x(w)dw$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \left[\alpha \int_0^t \int_0^1 (1-u)^{\alpha-1} u^{-\alpha} du x(w)dw\right]' - \frac{t^{\alpha-1}}{\Gamma(1-\alpha)\Gamma(\alpha)} \lim_{s\to 0^+} \int_0^s (s-w)^{-\alpha} x(w)dw \text{ by } \frac{s-w}{t-w} = u$$

$$= \left[\int_0^t x(w)dw\right]' = x(t) - \frac{t^{\alpha-1}}{\Gamma(1-\alpha)\Gamma(\alpha)} \eta \mathbf{B}(1-\alpha, \alpha) = x(t) - \eta t^{\alpha-1}.$$

Then $x \in C_{1-\alpha}(0,1]$ is a solution of (3.1.9).

On the other hand, if $x \in C_{1-\alpha}(0,1]$ is a solution of (3.1.9), together with (3.1.10)-(3.1.11) implies $\lim_{t\to 0^+} t^{1-\alpha} x(t) = \eta$. Furthermore, we have

$${}^{RL}D_{0^+}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t (t-s)^{-\alpha} x(s)ds\right)$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t (t-s)^{-\alpha} \left(\eta s^{\alpha-1} + \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} [A(w)x(w) + F(w)]dw\right) ds\right)'$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\eta \int_0^t (t-s)^{-\alpha} s^{\alpha-1} ds\right)' + \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t (t-s)^{-\alpha} \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} [A(w)x(w) + F(w)]dwds\right)'$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\eta \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} du\right)' + \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t \int_w^t (t-s)^{-\alpha} \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} ds [A(w)x(w) + F(w)]dw\right)'$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t \int_0^1 (1-u)^{-\alpha} \frac{u^{\alpha-1}}{\Gamma(\alpha)} du [A(w)x(w) + F(w)]dw\right)' = A(t)x(t) + F(t).$$

So $x \in C_{1-\alpha}(0,1]$ is a solution of IVP(3.1.2). The proof is completed. □

Theorem 3.4. *Suppose that (A2) holds. Then (3.1.2) has a unique solution. If $A(t) \equiv \lambda$ and there exists constants $k_2 > -1$, $l_2 \leq 0$ with $l_2 > \{-\alpha, -1-k_1\}$ and $M_F \geq 0$ such that $|F(t)| \leq M_F t^{k_2}(1-t)^{l_2}$ for all $t \in (0,1)$, then following special problem*

$$\begin{cases} {}^{RL}D_{0^+}^{\mathbf{ff}} x(t) = \lambda x(t) + F(t), \ t \in (0,1], \\ \lim_{t\to 0^+} t^{1-\alpha} x(t) = \eta \end{cases} \tag{3.1.12}$$

has a unique solution

$$x(t) = \eta\Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) F(s)ds, \ t \in (0,1]. \tag{3.1.13}$$

Proof. From Claims 1, 2 and 3, (3.1.2) and Theorem 3.3 has a unique solution. From the assumption and $A(t) \equiv \lambda$, one sees that (A2) holds with $k_1 = l_1 = 0$ and k_2, l_2 mentioned. Thus (3.1.12) has a unique solution. We get from the Picard function sequence that

$$\phi_n(t) = \eta t^{\alpha-1} + \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_{n-1}(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds$$

$$= \eta t^{\alpha-1} + \eta\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{\alpha-1} ds + \lambda^2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} \phi_{n-2}(w)dwds$$

$$+\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} F(w)dwds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds$$

$$= \eta t^{\alpha-1} + \frac{\eta\lambda\Gamma(\alpha)t^{2\alpha-1}}{\Gamma(2\alpha)} + \lambda^2 \int_0^t \int_w^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} ds\phi_{n-2}(w)dw$$

$$+\lambda \int_0^t \int_w^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} dsF(w)dw + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds$$

$$= \cdots\cdots\cdots$$

$$= \eta\Gamma(\alpha)t^{\alpha-1} \sum_{j=0}^{n-1} \frac{\lambda^j t^{j\alpha}}{\Gamma((j+1)\alpha)} + \eta\lambda^n \int_0^t \frac{(t-w)^{n\alpha-1}}{\Gamma(n\alpha)} dw + \int_0^t (t-s)^{\alpha-1} \left(\sum_{j=0}^{n-1} \frac{\lambda^j (t-s)^{j\alpha}}{\Gamma((j+1)\alpha)} \right) F(s)ds$$

$$= \eta\Gamma(\alpha)t^{\alpha-1} \sum_{j=0}^{n} \frac{\lambda^j t^{j\alpha}}{\Gamma((j+1)\alpha)} + \int_0^t (t-s)^{\alpha-1} \left(\sum_{j=0}^{n} \frac{\lambda^j (t-s)^{j\alpha}}{\Gamma((j+1)\alpha)}. \right) F(s)ds$$

$$\to \eta\Gamma(\alpha)t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha) F(s)ds.$$

Then we get (3.1.13). The proof is complete. □

To get solutions of (3.1.3), we need the following assumptions:

(A3) there exists constants $k_i > -\alpha$, $l_i \le 0$ with $l_1 > \{-\alpha, -\alpha - k_1\}$, $l_2 > \max\{-\alpha, -1 + k_2\}$, $M_B \ge 0$ and $M_G \ge 0$ such that $|B(t)| \le M_B(\log t)^{k_1}(1-\log t)^{l_1}$ and $|G(t)| \le M_G(\log t)^{k_2}(1-\log t)^{l_2}$ for all $t \in (1,e)$.

Choose Picard function sequence as

$$\phi_0(t) = \eta(\log t)^{\alpha-1},\ t \in (1,e],$$

$$\phi_n(t) = \eta(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s}, t \in (1,e], n = 1, 2, \cdots.$$

Claim 1. $\phi_n \in LC_{1-\alpha}(1,e]$. In fact, $\phi_0 \in LC_{1-\alpha}(1,e]$ and

$$(\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_0(s) + G(s)] \frac{ds}{s} \right|$$

$$\le (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [M_B|\eta|(\log s)^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1} + M_G(\log s)^{k_2}(1-\log s)^{l_2}] \frac{ds}{s}$$

$$\le (\log t)^{1-\alpha} M_B|\eta| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{\alpha+k_1-1} \frac{ds}{s} + (\log t)^{1-\alpha} M_G \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_2-1} (\log s)^{k_2} \frac{ds}{s}$$

$$= M_B|\eta|(\log t)^{\alpha+k_1+l_1} \mathbf{B}(\alpha+l_1, k_1+\alpha) + M_G(\log t)^{1+k_1+l_1} \mathbf{B}(\alpha+l_2, k_2+1) \to 0 \text{ as } t \to 0^+,$$

we know that $t \to \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_0(s) + G(s)] \frac{ds}{s}$ is continuous on $(1,e]$ and $\lim_{t\to 0^+} (\log t)^{1-\alpha}\phi_1(t)$ exists. Then $\phi_1 \in LC_{1-\alpha}(1,e]$. By mathematical induction method, we can show $\phi_n \in LC_{1-\alpha}(1,e]$.

Claim 2. $\{t \to (\log t)^{1-\alpha}\phi_n(t)\}$ is convergent uniformly on $(1,e]$. In fact we have for $t \in (1,e]$ that

$$(\log t)^{1-\alpha}|\phi_1(t) - \phi_0(t)| = \frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_0(s) + G(s)] \frac{ds}{s} \right|$$

$$\le \frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [|\eta| M_B(\log s)^{k_1}(1-\log s)^{l_1} + M_G(\log s)^{k_2}(1-\log s)^{l_2}] \frac{ds}{s}$$

$$\le |\eta| M_B \frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{k_1} \frac{ds}{s}$$

$$+ M_G \frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{k_2}(1-\log s)^{l_2} \frac{ds}{s}$$

$$= |\eta| M_B(\log t)^{k_1+l_1+1} \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)} + M_G(\log t)^{k_2+l_2+1} \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)}.$$

So

$$(\log t)^{1-\alpha}|\phi_2(t) - \phi_1(t)| = \frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} B(s)[\phi_1(s) - \phi_0(s)] \frac{ds}{s} \right|$$

$$\le (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} M_B(\log s)^{k_1}(1-\log s)^{l_1} \left(|\eta| M_B(\log s)^{k_1+l_1+1} \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)} \right.$$

$$+M_G(\log s)^{k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\Big)\frac{ds}{s}$$

$$\leq|\eta|M_B^2(\log t)^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_BM_G(\log t)^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}.$$

$$(\log t)^{1-\alpha}|\phi_3(t)-\phi_2(t)|=\frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}B(s)[\phi_2(s)-\phi_1(s)]ds\right|$$

$$\leq\frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l_1-1}M_B(\log s)^{k_1}\Big(|\eta|M_B^2(\log s)^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_BM_G(\log s)^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\Big)\frac{ds}{s}$$

$$\leq|\eta|M_B^3(\log t)^{3k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}$$

$$+M_B^2M_G(\log t)^{2k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}.$$

$$(\log t)^{1-\alpha}|\phi_4(t)-\phi_3(t)|=\frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}B(s)[\phi_3(s)-\phi_2(s)]ds\right|$$

$$\leq\frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l_1-1}M_B(\log s)^{k_1}\Big(|\eta|M_B^3(\log s)^{3k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}$$

$$+M_B^2M_G(\log s)^{2k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}\Big)\frac{ds}{s}$$

$$\leq|\eta|M_B^4(\log t)^{4k_1+4l_1+4}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}ds\frac{\mathbf{B}(\alpha+l_1,4k_1+3l_1+4)}{\Gamma(\alpha)}$$

$$+M_B^3M_G(\log t)^{3k_1+k_2+3l_1+l_2+4}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+k_2+2l_1+l_2+4)}{\Gamma(\alpha)}.$$

Similarly by the mathematical induction method, we get for every $n=1,2,\cdots$ that

$$(\log t)^{1-\alpha}|\phi_n(t)-\phi_{n-1}(t)|\leq|\eta|M_B^n(\log t)^{nk_1+nl_1+n}\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G(\log t)^{(n-1)k_1+k_2+(n-1)l_1+l_2+n}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l_1,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

$$\leq|\eta|M_B^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)},t\in(1,e].$$

Similarly we can prove that both

$$\sum_{n=1}^{+\infty}u_n=\sum_{n=1}^{+\infty}|\eta|M_B^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)},$$

$$\sum_{n=1}^{+\infty}v_n=\sum_{n=1}^{+\infty}M_B^{n-1}M_G\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

are convergent. Hence

$$(\log t)^{1-\alpha}\phi_0(t)+(\log t)^{1-\alpha}[\phi_1(t)-\phi_0(t)]+(\log t)^{1-\alpha}[\phi_2(t)-\phi_1(t)]+\cdots+(\log t)^{1-\alpha}[\phi_n(t)-\phi_{n-1}(t)]+\cdots,$$

$t\in(1,e]$ is uniformly convergent. Then $\{t\to(\log t)^{1-\alpha}\phi_n(t)\}$ is convergent uniformly on $(1,e]$.

Claim 3. $\phi(t) = (\log t)^{\alpha-1} \lim_{n\to+\infty} (\log t)^{1-\alpha}\phi_n(t)$ defined on $(1,e]$ is a unique continuous solution of the integral equation

$$x(t) = \eta(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s}, t \in (1,e]. \tag{3.1.14}$$

Proof. By $\lim_{n\to+\infty} (\log t)^{1-\alpha}\phi_n(t) = (\log t)^{1-\alpha}\phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $(1,e]$. From

$$(\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [A(s)\phi_{n-1}(s) + F(s)] \frac{ds}{s} - \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{m-1}(s) + G(s)] \frac{ds}{s} \right|$$

$$\le M_B||\phi_{n-1} - \phi_{m-1}||(\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{k_1} (\log s)^{\alpha-1} \frac{ds}{s}$$

$$\le M_B||\phi_{n-1} - \phi_{m-1}||(\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{\alpha+k_1-1} \frac{ds}{s}$$

$$\le M_B||\phi_{n-1} - \phi_{m-1}||(\log t)^{\alpha+k_1+l_1} \mathbf{B}(\alpha + l_1, \alpha + k_1)$$

$$\le M_B||\phi_{n-1} - \phi_{m-1}||\mathbf{B}(\alpha + l_1, \alpha + k_1) \to 0 \text{ uniformly as } m,n \to +\infty,$$

we know that

$$\phi(t) = (\log t)^{\alpha-1} \lim_{n\to+\infty} (\log t)^{1-\alpha}\phi_n(t) = \lim_{n\to+\infty} \left[\eta + (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s}\right]$$

$$= \eta(\log t)^{\alpha-1} + (\log t)^{\alpha-1} \lim_{n\to+\infty} (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s}$$

$$= \eta(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi(s) + G(s)] \frac{ds}{s}.$$

Then ϕ is a continuous solution of (3.1.14) defined on $(1,e]$.

Suppose that ψ defined on $(1,e]$ is also a solution of (3.1.14). Then

$$\psi(t) = \eta(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\psi(s) + G(s)] \frac{ds}{s}, t \in (1,e].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $(1,e]$. Then

$$(\log t)^{1-\alpha}|\psi(t) - \phi_0(t)| = (\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |B(s)\psi(s) + G(s)| \frac{ds}{s} \right|$$

$$\le |\eta|M_B(\log t)^{k_1+l_1+1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_G(\log t)^{k_2+l_2+1} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$(\log t)^{1-\alpha}|\psi(t) - \phi_1(t)| = (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} B(s)[\psi(s) - \phi_0(s)] \frac{ds}{s} \right|$$

$$\le |\eta|M_B^2(\log t)^{2k_1+2l_1+2} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_BM_G(\log t)^{k_1+k_2+l_1+l_2+2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}.$$

By mathematical induction method, we can get that

$$(\log t)^{1-\alpha}|\psi(t) - \phi_n(t)| = (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} B(s)[\psi(s) - \phi_{n-1}(s)]ds \right|$$

$$\leq |\eta|M_B^n(\log t)^{nk_1+nl_1+n}\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G(\log t)^{(n-1)k_1+k_2+(n-1)l_1+l_2+n}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

$$\leq |\eta|M_B^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)},\text{ for all } n=1,2,\cdots.$$

Similarly we have $\lim_{n\to+\infty}(\log t)^{1-\alpha}\phi_n(t)=(\log t)^{1-\alpha}\psi(t)$ uniformly on $(1,e]$. Then $\phi(t)\equiv\psi(t)$ on $(1,e]$. Then (3.1.14) has a unique solution ϕ. The proof is complete.

Theorem 3.5. *Suppose that (A3) holds. Then $x\in LC_{1-\alpha}(1,e]$ is a solution of IVP(3.1.3) if and only if $x\in LC_{1-\alpha}(1,e]$ is a solution of the integral equation (3.1.14).*

Proof. Suppose that $x\in C_{1-\alpha}(0,1]$ is a solution of IVP(3.1.3). Then $t\to(\log t)^{1-\alpha}x(t)$is continuous on $(1,e]$ by defining $(\log t)^{1-\alpha}x(t)|_{t=1}=\lim_{t\to1^+}(\log t)^{1-\alpha}x(t)$ and $||x||=r<+\infty$. So

$$\lim_{s\to1^+}\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}=\lim_{s\to1^+}\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}(\log w)^{\alpha-1}(\log w)^{1-\alpha}x(w)\frac{dw}{w}$$

$$=\lim_{s\to1^+}(\log\xi)^{1-\alpha}x(\xi)\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}(\log w)^{\alpha-1}\frac{dw}{w}\text{ by mean value theorem of integral, }\xi\in(1,s)$$

$$=\lim_{s\to1^+}(\log\xi)^{1-\alpha}x(\xi)\int_0^1(1-u)^{-\alpha}u^{\alpha-1}du\text{ by }\frac{\log w}{\log s}=u$$

$$=\eta\mathbf{B}(1-\alpha,\alpha).$$

From (A3), we have

$$(\log t)^{1-\alpha}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\right|$$

$$\leq(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[M_Br(\log s)^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}+M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}$$

$$\leq(\log t)^{1-\alpha}M_Br\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l_1-1}(\log s)^{\alpha+k_1-1}\frac{ds}{s}+(\log t)^{1-\alpha}M_G\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l_2-1}(\log s)^{k_2}\frac{ds}{s}$$

$$=M_Br(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+\alpha)+M_G(\log t)^{1+k_1+l_1}\mathbf{B}(\alpha+l_2,k_2+1).$$

So $t\to(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is defined on $(1,e]$ and

$$\lim_{t\to1^+}(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}=0.\tag{3.1.15}$$

Furthermore, we have similarly to Theorem 3.1.1 that $t\to\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $(1,e]$. So $t\to(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $[1,e]$ by defining

$$(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\Big|_{t=1}=\lim_{t\to1^+}(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}.\tag{3.1.16}$$

We have ${}^{H}I_{1^+}^{\alpha}\,{}^{RLH}D_{1^+}^{\alpha}x(t)={}^{H}I_{1^+}^{\alpha}[B(t)x(t)+G(t)]$. So

$$\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}={}^{H}I_{1^+}^{\alpha}[B(t)x(t)+G(t)]={}^{H}I_{1^+}^{\alpha}\,{}^{RLH}D_{1^+}^{\alpha}x(t)$$

$$=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{\Gamma(1-\alpha)}s\left(\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\right)'\frac{ds}{s}$$

$$=\frac{1}{\Gamma(\alpha+1)}\frac{1}{\Gamma(1-\alpha)}t\left[\int_1^t\left(\log\frac{t}{s}\right)^{\alpha}\left(\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\right)'ds\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)}t\left[\left(\log\frac{t}{s}\right)^{\alpha}\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\Big|_1^t+\alpha\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\frac{ds}{s}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)}t\left[(\log t)^{\alpha}\lim_{s\to1^+}\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}+\alpha\int_1^t\int_w^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{w}\right)^{-\alpha}\frac{ds}{s}x(w)\frac{dw}{w}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)}t\left[(\log t)^{\alpha}\eta\mathbf{B}(1-\alpha,\alpha)+\alpha\int_1^t\mathbf{B}(\alpha,1-\alpha)x(w)\frac{dw}{w}\right]'$$

$$=x(t)-\eta(\log t)^{\alpha-1}.$$

Then $x\in LC_{1-\alpha}(1,e]$ is a solution of (3.1.14).

On the other hand, if x is a solution of (3.1.14), together with Cases 1, 2, 3 and (3.1.15) implies $\lim_{t\to1^+}(\log t)^{1-\alpha}x(t)=\eta$. Then $x\in LC_{1-\alpha}(1,e]$. Furthermore, we have by Definition 2.5 that

$${}^{RLH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}t\left(\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}x(s)\frac{ds}{s}\right)'$$

$$=\frac{1}{\Gamma(1-\alpha)}t\left[\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\eta(\log s)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\frac{s}{w}\right)^{\alpha-1}[A(w)x(w)+F(w)]\frac{dw}{w}\right)\frac{ds}{s}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}t\left[\eta\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}(\log s)^{\alpha-1}\frac{ds}{s}\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}t\left[\frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\frac{s}{w}\right)^{\alpha-1}[A(w)x(w)+F(w)]\frac{dw}{w}\frac{ds}{s}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}t\left[\eta\mathbf{B}(1-\alpha,\alpha)\right]'+\frac{1}{\Gamma(1-\alpha)}t\left[\frac{1}{\Gamma(\alpha)}\int_1^t\int_w^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{w}\right)^{\alpha-1}\frac{ds}{s}[A(w)x(w)+F(w)]\frac{dw}{w}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}t\left[\frac{1}{\Gamma(\alpha)}\int_1^t\mathbf{B}(1-\alpha,\alpha)[B(w)x(w)+G(w)]\frac{dw}{w}\right]'$$

$$=B(t)x(t)+G(t).$$

So $x\in LC_{1-\alpha}(1,e]$ is a solution of IVP(3.1.3). The proof is completed. □

Theorem 3.6. *Suppose that (A3) holds. Then (3.1.14) has a unique solution. If $B(t)\equiv\lambda$ and there exists constants $k_2>-1$, $l_2\le0$ with $l_2>\{-\alpha,-1-k_2\}$ and $M_G\ge0$ such that $|G(t)|\le M_G(\log t)^{k_2}(1-\log t)^{l_2}$ for all $t\in(1,e)$, then following special problem*

$$\begin{cases}{}^{RLH}D_{1^+}^{\mathbf{ff}}x(t)=\lambda x(t)+G(t),\ t\in(1,e],\\ \lim_{t\to0^+}(\log t)^{1-\alpha}x(t)=\eta\end{cases}\tag{3.1.17}$$

has a unique solution

$$x(t)=\eta\Gamma(\alpha)(\log t)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\log t)^{\alpha})+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s},\ t\in(1,e].\tag{3.1.18}$$

Proof. From Claims 1, 2 and 3, (3.1.14) has a unique solution. From the assumption and $B(t)\equiv\lambda$, one sees that (A3) holds with $k_1=l_1=0$ and k_2,l_2 mentioned in assumption. Thus (3.1.17) has a unique solution. We get from the Picard function sequence that

$$\phi_n(t) = \eta(\log t)^{\alpha-1} + \lambda\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\phi_{n-1}(s)ds + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

$$= \eta(\log t)^{\alpha-1} + \frac{\eta\lambda}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\frac{ds}{s} + \frac{\lambda^2}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\frac{s}{w}\right)^{\alpha-1}\phi_{n-2}(w)\frac{dw}{w}\frac{ds}{s}$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\left(\log\frac{s}{w}\right)^{\alpha-1}G(w)\frac{dw}{w}\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

$$= \eta(\log t)^{\alpha-1} + \frac{\eta\lambda(\log t)^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{\lambda^2}{\Gamma(\alpha)}\int_1^t\int_w^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\left(\log\frac{s}{w}\right)^{\alpha-1}\frac{ds}{s}\phi_{n-2}(w)\frac{dw}{w}$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_1^t\int_w^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{w}\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\frac{ds}{s}G(w)\frac{dw}{w} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

$$= \cdots\cdots\cdots$$

$$= \eta\Gamma(\alpha)(\log t)^{\alpha-1}\sum_{j=0}^{n-1}\frac{\lambda^j(\log t)^{j\alpha}}{\Gamma((j+1)\alpha)} + \frac{\eta\lambda^n}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{dw}{w} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\sum_{j=0}^{n-1}\frac{\lambda^j(t-s)^{j\alpha}}{\Gamma((j+1)\alpha)}\right)F(s)\frac{ds}{s}$$

$$= \eta\Gamma(\alpha)(\log t)^{\alpha-1}\sum_{j=0}^{n}\frac{\lambda^j(\log t)^{j\alpha}}{\Gamma((j+1)\alpha)} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\sum_{j=0}^{n}\frac{\lambda^j\left(\frac{t}{s}\right)^{j\alpha}}{\Gamma((j+1)\alpha)}\right)G(s)\frac{ds}{s}$$

$$\to \eta\Gamma(\alpha)(\log t)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\log t)^{\alpha}) + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s}.$$

Then we get (3.1.18). The proof is complete. □

To get solutions of (3.1.4), we need the following assumptions:

(A4) there exists constants $k_i > -1$, $l_i \le 0$ with $l_i > \{-\alpha, -\alpha - k_i\}$, $M_B \ge 0$ and $M_G \ge 0$ such that $|B(t)| \le M_B(\log t)^{k_1}(1-\log t)^{l_1}$ and $|G(t)| \le M_G(\log t)^{k_2}(1-\log t)^{l_2}$ for all $t \in (1,e)$.

Choose Picard function sequence as

$$\phi_0(t) = \eta,\ t \in (1,e],$$

$$\phi_n(t) = \eta + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)\phi_{n-1}(s) + G(s)]\frac{ds}{s}, t \in (1,e], n = 1,2,\cdots.$$

Claim 1. $\phi_n \in C(1,e]$. Since $\phi_0 \in C(1,e]$, then ϕ_1 is continuous on $(1,e]$ and

$$\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)\phi_0(s) + G(s)]\frac{ds}{s}\right|$$

$$\le \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}[M_B|\eta|(\log s)^{k_1}(1-\log s)^{l_1} + M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}$$

$$\le M_B|\eta|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}\frac{ds}{s}$$

$$+M_G\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_2}(1-\log s)^{l_2}\frac{ds}{s}$$

$$= M_B|\eta|(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1, k_1+1) + M_G(\log t)^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2, k_2+1) \to 0 \text{ as } t \to 0^+,$$

we get that $\lim_{t\to 1^+}\phi_1(s)$ exists. Then $\phi_1 \in C(1,e]$. By mathematical induction method, we see that $\phi_n \in C(1,e]$.

Claim 2. ϕ_n is convergent uniformly on $(1,e]$. In fact we have for $t \in (1,e]$ that

$$|\phi_1(t) - \phi_0(t)| = \left|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)\phi_0(s) + G(s)]\frac{ds}{s}\right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}[M_B|\eta|(\log s)^{k_1}(1-\log s)^{l_1} + M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}$$

$$\leq |\eta| M_B \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{k_1} \frac{ds}{s} + M_G \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_2-1} (\log s)^{k_2} \frac{ds}{s}$$

$$= |\eta| M_B (\log t)^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_G (\log t)^{\alpha+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So

$$|\phi_2(t)-\phi_1(t)| = \left|\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} B(s)[\phi_1(s)-\phi_0(s)] \frac{ds}{s}\right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [M_B (\log s)^{k_1} (1-\log s)^{l_1} \Big(|\eta| M_B (\log s)^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}$$

$$+M_G (\log s)^{\alpha+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\Big) \frac{ds}{s}$$

$$\leq \frac{1}{\Gamma(\alpha)} |\eta| M_B^2 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{\alpha+2k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$+M_B M_G \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{\alpha+k_1+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$= |\eta| M_B^2 (\log t)^{2k_1+2l_1+2} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}$$

$$+M_B M_G (\log t)^{2\alpha+k_1+k_2+l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)},$$

and

$$|\phi_3(t)-\phi_2(t)| = \left|\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} B(s)[\phi_2(s)-\phi_1(s)] \frac{ds}{s}\right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} M_B (\log s)^{k_1} (1-\log s)^{l_1} \Big(|\eta| M_B^2 (\log s)^{2k_1+2l_1+2} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}$$

$$+M_B M_G (\log s)^{2\alpha+k_1+k_2+l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\Big) \frac{ds}{s}$$

$$\leq |\eta| M_B^3 \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log) s^{2\alpha+3k_1+2l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$+M_B^2 M_G \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{2\alpha+2k_1+k_2+l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$= |\eta| M_B^3 (\log t)^{3\alpha+3k_1+3l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+3k_1+2l_1+1)}{\Gamma(\alpha)}$$

$$+M_B^2 M_G (\log t)^{3\alpha+2k_1+k_2+2l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+k_2+l_1+l_2+1)}{\Gamma(\alpha)}.$$

$$|\phi_4(t)-\phi_3(t)| = \left|\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} B(s)[\phi_3(s)-\phi_2(s)] \frac{ds}{s}\right|$$

$$\frac{|\eta| M_B^4}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{3\alpha+4k_1+3l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+3k_1+2l_1+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$+\frac{M_B^3 M_G}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{3\alpha+3k_1+k_2+2l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+k_2+l_1+l_2+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$\leq |\eta| M_B^4 (\log t)^{4\alpha+4k_1+4l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+3k_1+2l_1+1)}{\Gamma(\alpha)} \frac{ds}{s} \frac{\mathbf{B}(\alpha+l_1,3\alpha+4k_1+3l_1+1)}{\Gamma(\alpha)}$$

$$+M_B^3 M_G (\log t)^{4\alpha+3k_1+k_2+3l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+k_2+l_1+l_2+1)}{\Gamma(\alpha)} \times$$

$$\frac{\mathbf{B}(\alpha+l_1,3\alpha+3k_1+k_2+2l_1+l_2+1)}{\Gamma(\alpha)}.$$

Similarly by the mathematical induction method, we get for every $n = 1, 2, \cdots$ that

$$|\phi_n(t) - \phi_{n-1}(t)| \leq |\eta| M_B^n (\log t)^{n\alpha + nk_1 + nl_1} \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha + l_1, i\alpha + (i+1)k_1 + il_1 + 1}{\Gamma(\alpha)}$$

$$+ M_B^{n-1} M_G (\log t)^{n\alpha + (n-1)k_1 + k_2 + (n-1)l_1 + l_2} \frac{\mathbf{B}(\alpha + l_2, k_2 + 1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha + l, i\alpha + ik_1 + k_2 + (i-1)l_1 + l_2 + 1)}{\Gamma(\alpha)}$$

$$\leq |\eta| M_B^n \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha + l_1, i\alpha + (i+1)k_1 + il_1 + 1}{\Gamma(\alpha)}$$

$$+ M_B^{n-1} M_G \frac{\mathbf{B}(\alpha + l_2, k_2 + 1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha + l, i\alpha + ik_1 + k_2 + (i-1)l_1 + l_2 + 1)}{\Gamma(\alpha)}, t \in (1, e].$$

Similarly we can prove that both

$$\sum_{n=1}^{+\infty} u_n = \sum_{n=1}^{+\infty} |\eta| M_B^n \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha + l_1, i\alpha + (i+1)k_1 + il_1 + 1}{\Gamma(\alpha)},$$

and

$$\sum_{n=1}^{+\infty} v_n = \sum_{n=1}^{+\infty} M_B^{n-1} M_G \frac{\mathbf{B}(\alpha + l_2, k_2 + 1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha + l, i\alpha + ik_1 + k_2 + (i-1)l_1 + l_2 + 1)}{\Gamma(\alpha)}$$

are convergent. Hence

$$t^{1-\alpha}\phi_0(t) + t^{1-\alpha}[\phi_1(t) - \phi_0(t)] + t^{1-\alpha}[\phi_2(t) - \phi_1(t)] + \cdots + t^{1-\alpha}[\phi_n(t) - \phi_{n-1}(t)] + \cdots, t \in (1, e]$$

is uniformly convergent. Then $\{\phi_n(t)\}$ is convergent uniformly on $(1, e]$.

Claim 3. $\phi(t) = \lim_{n \to +\infty} \phi_n(t)$ defined on $(1, e]$ is a unique continuous solution of the integral equation

$$x(t) = \eta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} [B(s)x(s) + G(s)] \frac{ds}{s}. \tag{3.1.19}$$

Proof. By $\lim_{n \to +\infty} \phi_n(t) = \phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $(1, e]$. From

$$\left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s} - \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} [B(s)\phi_{m-1}(s) + G(s)] \frac{ds}{s} \right|$$

$$\leq M_B ||\phi_{n-1} - \phi_{m-1}|| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} (\log s)^{k_1} (1 - \log s)^{l_1} \frac{ds}{s}$$

$$\leq M_B ||\phi_{n-1} - \phi_{m-1}|| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha + l_1 - 1} (\log s)^{k_1} \frac{ds}{s}$$

$$\leq M_B ||\phi_{n-1} - \phi_{m-1}|| (\log t)^{\alpha + k_1 + l_1} \frac{\mathbf{B}(\alpha + l_1, k_1 + 1)}{\Gamma(\alpha)}$$

$$\leq M_B ||\phi_{n-1} - \phi_{m-1}|| \frac{\mathbf{B}(\alpha + l_1, k_1 + 1)}{\Gamma(\alpha)} \to 0 \text{ uniformly as } m, n \to +\infty,$$

we know that

$$\phi(t) = \lim_{n \to \infty} \phi_n(t) = \lim_{n \to +\infty} \left[\eta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s} \right]$$

$$= \eta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} [B(s)\phi(s) + G(s)] \frac{ds}{s}.$$

Then ϕ is a continuous solution of (3.1.19) defined on $(1, e]$.

Suppose that ψ defined on $(1, e]$ is also a solution of (3.19). Then

$$\psi(t) = \eta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} [B(s)\phi(s) + G(s)] \frac{ds}{s}, t \in (1, e].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $(0,1]$. Now we have

$$|\psi(t)-\phi_0(t)| = \frac{1}{\Gamma(\alpha)}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)\phi(s)+G(s)]\frac{ds}{s}\right|$$

$$\leq |\eta| M_B(\log t)^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_G(\log t)^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$|\psi(t)-\phi_1(t)| = \frac{1}{\Gamma(\alpha)}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}B(s)[\psi(s)-\phi_0(s)]\frac{ds}{s}\right|$$

$$\leq |\eta| M_B^2(\log t)^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}$$

$$+M_BM_G(\log t)^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}.$$

By mathematical induction method, we can get that

$$|\psi(t)-\phi_n(t)| = \frac{1}{\Gamma(\alpha)}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}B(s)[\psi(s)-\phi_{n-1}(s)]\frac{ds}{s}\right|$$

$$\leq |\eta| M_B^n(\log t)^{n\alpha+nk_1+nl_1}\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G(\log t)^{n\alpha+(n-1)k_1+k_2+(n-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}$$

$$\leq |\eta| M_B^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}, t\in(1,e].$$

Similarly we have $\lim_{n\to+\infty}\phi_n(t) = \psi(t)$ uniformly on $(1,e]$. Then $\phi(t)\equiv\psi(t)$ on $(1,e]$. Then (3.1.19) has a unique solution ϕ. The proof is complete.

Theorem 3.7. *Suppose that (A4) holds. Then $x \in C(1,e]$ is a solution of IVP(3.1.4) if and only if $x \in C(1,e]$ is a solution of the integral equation (3.1.19).*

Proof. Suppose that $x \in C(1,e]$ is a solution of IVP(3.1.4). Then $t \to x(t)$is continuous on $[0,1]$ by defining $x(t)|_{t=0} = \lim_{t\to 0^+} x(t)$ and $||x|| = r < +\infty$. One can see that

$$\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}\frac{ds}{s} \leq \int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l_1-1}(\log s)^{k_1}\frac{ds}{s} \text{ by } \frac{\log s}{\log t} = u$$

$$= (\log t)^{\alpha+k_1+l_1}\int_0^1(1-u)^{\alpha+l_1-1}u^{k_1}du \leq (\log t)^{\alpha+k_1+l_1}\int_0^1(1-u)^{\alpha+l_1-1}u^{k_1}du$$

$$= (\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1).$$

From (A4), we have for $t\in(1,e]$ that

$$\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\right|$$

$$\leq \int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[M_Br(\log s)^{k_1}(1-\log s)^{l_1}+M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}$$

$$\leq M_Br\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}\frac{ds}{s}$$

$$+M_G\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_2}(1-\log s)^{l_2}\frac{ds}{s}$$

$$= M_B r(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1)$$

$$+M_G(\log t)^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1).$$

So $t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is defined on $(1,e]$ and

$$\lim_{t\to 1^+}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}=0. \tag{3.1.20}$$

Furthermore, we have similarly to Theorem 3.1 that $t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $(0,1]$. So $t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $[0,1]$ by defining

$$\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\Big|_{t=1}=0. \tag{3.1.21}$$

One sees that

$$\int_w^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left(\log \frac{s}{w}\right)^{-\alpha}\frac{ds}{s} \text{ by } \frac{\log s-\log w}{\log t-\log w}=u$$

$$=\int_0^1 (1-u)^{\alpha-1}u^{-\alpha}du=\Gamma(1-\alpha)\Gamma(\alpha).$$

We have by Definition 2.6 and ${}^{H}I_{1^+}^{\alpha}\,{}^{CH}D_{1^+}^{\alpha}x(t)={}^{H}I_{1^+}^{\alpha}[B(t)x(t)+G(t)]$. So

$$\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[A(s)x(s)+F(s)]\frac{ds}{s}={}^{H}I_{1^+}^{\alpha}[B(t)x(t)+G(t)]$$

$$={}^{H}I_{1^+}^{\alpha}\,{}^{CH}D_{1^+}^{\alpha}x(t)$$

$$=\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left[\frac{1}{\Gamma(1-\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{-\alpha} wx'(w)\frac{dw}{w}\right]\frac{ds}{s}$$

$$=\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^s \left(\log \frac{s}{w}\right)^{-\alpha} x'(w)dw\right]\frac{ds}{s}$$

$$=\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha)}\int_1^t\int_w^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left(\log \frac{s}{w}\right)^{-\alpha}\frac{ds}{s}x'(w)dw$$

$$=\int_1^t x'(w)dw=x(t)-\lim_{t\to 1^+}x(t)=x(t)-\eta.$$

Then $x\in C(1,e]$ is a solution of (3.1.19).

On the other hand, if $x\in C(1,e]$ is a solution of (3.1.19), together with (3.1.20) implies $\lim\limits_{t\to 1^+}(t)=\eta$. Furthermore, we have that

$${}^{CH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha}sx'(s)\frac{ds}{s}$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha}\left(\eta+\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\right)'ds$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\right)'ds$$

$$=\frac{t}{\Gamma(2-\alpha)}\left(\int_1^t \left(\log \frac{t}{s}\right)^{1-\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\right)'ds\right)'$$

$$=\frac{t}{\Gamma(2-\alpha)}\Bigg[\left(\log \frac{t}{s}\right)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\Big|_1^t$$

$$+(1-\alpha)\frac{1}{s}\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}ds\Bigg]'$$

$$= \frac{t}{\Gamma(1-\alpha)} \frac{1}{\Gamma(\alpha)} \left[\frac{1}{s} \int_1^t \left(\log \frac{t}{s}\right)^{-\alpha} \int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1} [B(w)x(w) + G(w)] \frac{dw}{w} ds \right]'$$

$$= \frac{t}{\Gamma(1-\alpha)} \frac{1}{\Gamma(\alpha)} \left[\int_1^t \int_w^t \left(\log \frac{t}{s}\right)^{-\alpha} \left(\log \frac{s}{w}\right)^{\alpha-1} \frac{ds}{s} [B(w)x(w) + G(w)] \frac{dw}{w} \right]' \text{ by } \frac{\log s}{\log t} = u$$

$$= \frac{t}{\Gamma(1-\alpha)} \frac{1}{\Gamma(\alpha)} \left[\int_1^t \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} du [B(w)x(w) + G(w)] \frac{dw}{w} \right]' \text{ by } \mathbf{B}(1-\alpha,\alpha) = \Gamma(1-\alpha)\Gamma(\alpha)$$

$$= B(t)x(t) + G(t).$$

So $x \in C(1,e]$ is a solution of IVP(3.1.4). The proof is completed. □

Theorem 3.8. *Suppose that (A4) holds. Then (3.1.4) has a unique solution. If $B(t) \equiv \lambda$ and there exists constants $k_2 > -1$, $l_2 \le 0$ with $l_2 > \{-\alpha, -\alpha - k_2\}$ and $M_G \ge 0$ such that $|G(t)| \le M_G t^{k_2}(1-t)^{l_2}$ for all $t \in (1,e)$, then following special problem*

$$\begin{cases} {}^{CH}D_{0^+}^{\mathbf{ff}} x(t) = \lambda x(t) + G(t),\ t \in (1,e], \\ \lim\limits_{t \to 1^+} x(t) = \eta \end{cases} \tag{3.1.22}$$

has a unique solution

$$x(t) = \eta E_{\alpha,1}(\lambda (\log t)^\alpha) + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{s}\right)^\alpha\right) G(s) \frac{ds}{s},\ t \in (1,e]. \tag{3.1.23}$$

Proof. From Claims 1, 2 and 3, Theorem 3.7, (3.1.4) has a unique solution. From the assumption and $A(t) \equiv \lambda$, one sees that (A4) holds with $k_1 = l_1 = 0$ and k_2, l_2 mentioned. Thus (3.1.22) has a unique solution. We get from the Picard function sequence that

$$\phi_n(t) = \eta + \lambda \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_{n-1}(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} G(s) \frac{ds}{s}$$

$$= \eta + \eta\lambda \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} + \lambda^2 \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1} \phi_{n-2}(w) \frac{dw}{w} \frac{ds}{s}$$

$$+\lambda \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1} G(w) \frac{dw}{w} \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} G(s) \frac{ds}{s}$$

$$= \eta + \frac{\eta\lambda(\log t)^\alpha}{\Gamma(\alpha+1)} + \lambda^2 \frac{1}{\Gamma(\alpha)} \int_1^t \int_w^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{s}{w}\right)^{\alpha-1} \frac{ds}{s} \phi_{n-2}(w) \frac{dw}{w}$$

$$+\lambda \frac{1}{\Gamma(\alpha)} \int_0^t \int_w^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{s}{w}\right)^{\alpha-1} \frac{ds}{s} F(w) \frac{dw}{w} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} F(s) \frac{ds}{s}$$

$$= \eta \left(1 + \frac{\lambda(\log t)^\alpha}{\Gamma(\alpha+1)}\right) + \lambda^2 \frac{1}{\Gamma(2\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{2\alpha-1} \phi_{n-2}(w) \frac{dw}{w}$$

$$+\lambda \frac{1}{\Gamma(2\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{2\alpha-1} G(w) \frac{dw}{w} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} G(s) \frac{ds}{s}$$

$$= \cdots\cdots\cdots$$

$$\eta \sum_{j=0}^n \frac{\lambda^j (\log t)^{j\alpha}}{\Gamma(j\alpha+1)} + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\sum_{j=0}^n \frac{\lambda^j \left(\log \frac{t}{s}\right)^{j\alpha}}{\Gamma((j+1)\alpha)} \right) G(s) ds$$

$$\to \eta E_{\alpha,1}(\lambda(\log t)^\alpha) + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{s}\right)^\alpha\right) G(s) \frac{ds}{s}.$$

Then we get (3.1.23). The proof is complete. □

Theorem 3.9. *(Schaefer's fixed point theorem). Let E be a Banach spaces and $T : E \mapsto E$ be a completely continuous operator. If the set $E(T) = \{x = \theta(Tx) :$ for some $\theta \in [0,1], x \in E\}$ is bounded, then T has at least a fixed point in E.*

3.2 Exact piecewise continuous solutions of FDEs

In this section, we present exact piecewise continuous solutions of the following fractional differential equations, respectively

$$^{C}D_{0^+}^{\mathbf{ff}}x(t)=\lambda x(t)+F(t),\ t\in(t_i,t_{i+1}],i\in\mathbb{N}_0, \tag{3.2.1}$$

$$^{RL}D_{0^+}^{\mathbf{ff}}x(t)=\lambda x(t)+F(t),\ t\in(t_i,t_{i+1}],i\in\mathbb{N}_0, \tag{3.2.2}$$

$$^{RLH}D_{0^+}^{\mathbf{ff}}x(t)=\lambda x(t)+G(t),\ t\in(s_i,s_{i+1}],i\in\mathbb{N}_0, \tag{3.2.3}$$

and

$$^{CH}D_{0^+}^{\mathbf{ff}}x(t)=\lambda x(t)+G(t),\ t\in(s_i,s_{i+1}],i\in\mathbb{N}_0, \tag{3.2.4}$$

where $\lambda\in\mathbb{R}$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$ in (3.2.1) and (3.2.2) and $1=t_0<t_1<\cdots<t_m<t_{m+1}=e$ in (3.2.3) and (3.2.4). We say that $x:(0,1]\mapsto\mathbb{R}$ is a piecewise solution of (3.2.1) (or (3.2.2) if $x\in P_mC(0,1]$ (or $P_mC_{1-\alpha}(0,1]$ and satisfies (3.2.1) or (3.2.2). We say that $x(1,e]\mapsto\mathbb{R}$ is a piecewise continuous solutions of (3.2.3) (or (3.2.4)) if $x\in LP_mC_{1-\alpha}(1,e]$, (or $LP_mC(1,e]$) and x satisfies all equations in (3.2.3) (or (3.2.4)).

Theorem 3.10. *Suppose that F is continuous on $(0,1)$ and there exist constants $k>-1$ and $l\in(-\alpha,-\alpha-k,0]$ such that $|F(t)|\le t^k(1-t)^l$ for all $t\in(0,1)$. Then x is a piecewise solution of (3.2.1) if and only if x and there exists constants $c_i(i\in\mathbb{N}_0)\in\mathbb{R}$ such that*

$$x(t)=\sum_{v=0}^{j}c_v\mathbf{E}_{\alpha,1}(\lambda(t-t_v)^\alpha)+\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds,t\in(t_j,t_{j+1}],j\in\mathbb{N}_0. \tag{3.2.5}$$

Proof. Firstly, we have for $t\in(t_i,t_{i+1}]$ that

$$\begin{aligned}
&\left|\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds\right|\le\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)|F(s)|ds\\
&\le\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)s^k(1-s)^l ds\\
&=\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\int_0^t(t-s)^{\alpha-1}(t-s)^{\alpha j}s^k(1-s)^l ds\\
&\le\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\int_0^t(t-s)^{\alpha+l-1}(t-s)^{\alpha j}s^k ds\\
&=\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}t^{\alpha+\alpha j+k+l}\int_0^1(1-w)^{\alpha+\alpha j+l-1}w^k dw\\
&\le\sum_{j=0}^{+\infty}\frac{\lambda^j t^{\alpha j}}{\Gamma((j+1)\alpha)}t^{\alpha+k+l}\int_0^1(1-w)^{\alpha+l-1}w^k dw=t^{\alpha+k+l}\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)\mathbf{B}(\alpha+l,k+1).
\end{aligned}$$

Then $\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds$ is convergent and is continuous on $[0,1]$. If x is a piecewise continuous solution of (3.2.5), then we know that $x\in P_mC(0,1]$ and $\lim_{t\to t_i^+}x(t)(i\in\mathbb{N}_0)$ exist. Now we prove that x satisfies differential equation in (3.2.1). In fact, for $t\in(t_i,t_{i+1}](i\in\mathbb{N}_0)$, we have that

$$^{C}D_{0^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}x'(s)ds=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}x'(s)ds+\int_{t_i}^t(t-s)^{-\alpha}x'(s)ds\right]$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{\kappa=0}^{j}c_\kappa\mathbf{E}_{\alpha,1}(\lambda(s-t_\kappa)^\alpha)+\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)F(v)dv\right)'ds\right.$$

$$\left.+\int_{t_i}^{t}(t-s)^{-\alpha}\left(\sum_{\kappa=0}^{i}c_\kappa\mathbf{E}_{\alpha,1}(\lambda(s-t_\kappa)^\alpha)+\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)F(v)dv\right)'ds\right]$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left[\sum_{\kappa=0}^{j}c_\kappa\mathbf{E}_{\alpha,1}(\lambda(s-t_\kappa)^\alpha)\right]'ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\int_{t_i}^{t}(t-s)^{-\alpha}\left[\sum_{\kappa=0}^{i}c_\kappa\mathbf{E}_{\alpha,1}(\lambda(s-t_\kappa)^\alpha)\right]'ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}\left[\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)F(v)dv\right]'ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m(s-t_\kappa)^{m\alpha}}{\Gamma(m\alpha+1)}\right]'ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{\kappa=0}^{i}c_\kappa\int_{t_i}^{t}(t-s)^{-\alpha}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m(s-t_\kappa)^{m\alpha}}{\Gamma(m\alpha+1)}\right]'ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{m=0}^{+\infty}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_0^t(t-s)^{-\alpha}\left[\int_0^s(s-v)^{\alpha+m\alpha-1}F(v)dv\right]'ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa\sum_{m=1}^{+\infty}\frac{\lambda^m m\alpha}{\Gamma(m\alpha+1)}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}(s-t_\kappa)^{m\alpha-1}ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i}c_\kappa\int_{t_i}^{t}(t-s)^{-\alpha}(s-t_\kappa)^{m\alpha-1}ds+\sum_{m=0}^{+\infty}\lambda^m D_{0^+}^{\alpha}I_{0^+}^{\alpha(m+1)}F(t)$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{\lambda^m m\alpha}{\Gamma(m\alpha+1)}\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa(t-t_\kappa)^{m\alpha-\alpha}\int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}}(1-w)^{-\alpha}w^{m\alpha-1}dw$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha m-\alpha}\int_{\frac{t_i-y_\kappa}{t-t_\kappa}}^{1}(1-w)^{-\alpha}w^{m\alpha-1}dw+\sum_{m=0}^{+\infty}\lambda^m I_{0^+}^{\alpha m}F(t)$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{\lambda^m m\alpha}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{m\alpha-\alpha}\sum_{j=\kappa}^{i-1}\int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}}(1-w)^{-\alpha}w^{m\alpha-1}dw$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{\alpha m-\alpha}\int_{\frac{t_i-y_\kappa}{t-t_\kappa}}^{1}(1-w)^{-\alpha}w^{m\alpha-1}dw$$

$$+\frac{c_i}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\int_0^1(1-w)^{-\alpha}w^{m\alpha-1}dw+f(t)+\lambda\int_0^t\sum_{m=1}^{+\infty}(t-s)^{\alpha-1}\frac{\lambda^{m-1}(t-s)^{\alpha(m-1)}}{\Gamma(\alpha m)}F(s)ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{\lambda^m m\alpha}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{m\alpha-\alpha}\int_0^1(1-w)^{-\alpha}w^{m\alpha-1}dw$$

$$+\frac{c_i}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\int_0^1(1-w)^{-\alpha}w^{m\alpha-1}dw+f(t)+\lambda\int_0^t\sum_{m=1}^{+\infty}(t-s)^{\alpha-1}\frac{\lambda^{m-1}(t-s)^{\alpha(m-1)}}{\Gamma(\alpha m)}F(s)ds$$

$$= \lambda x(t)+F(t).$$

We have done that x satisfies (3.2.1) if x satisfies (3.2.5).

Now, we suppose that x is a solution of (3.2.1). We will prove that x satisfies (3.2.5) by mathematical induction method. Since x is continuous on $(t_i, t_{i+1}]$ and the limit $\lim\limits_{t\to t_i^+} x(t)(i \in N_0)$ exists, then $x \in P_mC(0,1]$. For $t \in (t_0, t_1]$, we know from Theorem 3.2 that there exists $c_0 \in \mathbb{R}$ such that

$$x(t) = c_0\mathbf{E}_{\alpha,1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds, t \in (t_0, t_1].$$

Then (3.2.5) holds for $j = 0$. We suppose that (3.2.5) holds for all $j = 0, 1, \cdots, i$. We derive the expression of x on $(t_{i+1}, t_{i+2}]$. Suppose that

$$x(t) = \Phi(t) + \sum_{j=0}^{i} c_j\mathbf{E}_{\alpha,1}(\lambda(t-t_j)^\alpha) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds, t \in (t_{i+1}, t_{i+2}]. \quad (3.2.6)$$

By ${}^CD_{0^+}^\alpha x(t) - \lambda x(t) = f(t), t \in (t_{i+1}, t_{i+2}]$, we get

$$F(t) + \lambda x(t) = {}^CD_{0^+}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\left(\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}x'(s)ds + \int_{t_{i+1}}^t (t-s)^{-\alpha}x'(s)ds\right)$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{v=0}^{j} c_v\mathbf{E}_{\alpha,1}(\lambda(s-t_v)^\alpha) + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)' ds\right.$$

$$\left.+ \int_{t_{i+1}}^t (t-s)^{-\alpha}\left(\Phi(s) + \sum_{v=0}^{i} c_v\mathbf{E}_{\alpha,1}(\lambda(s-t_v)^\alpha) + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)' ds\right]$$

$$= {}^CD_{t_{i+1}^+}^\alpha\Phi(t) + \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{v=0}^{j} c_v\mathbf{E}_{\alpha,1}(\lambda(s-t_v)^\alpha)\right)' ds\right.$$

$$\left.+ \int_{t_{i+1}}^t (t-s)^{-\alpha}\left(\sum_{v=0}^{i} c_v\mathbf{E}_{\alpha,1}(\lambda(s-t_v)^\alpha)\right)' ds + \int_0^t (t-s)^{-\alpha}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)' ds\right]$$

$$= {}^CD_{t_{i+1}^+}^\alpha\Phi(t) + \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\sum_{v=0}^{j} c_v\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota(s-t_v)^{\iota\alpha}}{\Gamma(\alpha\iota+1)}\right)' ds\right.$$

$$\left.+ \sum_{v=0}^{i} c_v\int_{t_{i+1}}^t (t-s)^{-\alpha}\left(\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota(s-t_v)^{\iota\alpha}}{\Gamma(\alpha\iota+1)}\right)' ds\right]$$

$$+ \frac{1}{\Gamma(2-\alpha)}\left[\int_0^t (t-s)^{1-\alpha}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)' ds\right]'$$

$$= {}^CD_{t_{i+1}^+}^\alpha\Phi(t) + \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\sum_{v=0}^{j} c_v\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{\iota=1}^{+\infty}\frac{(\iota\alpha)\lambda^\iota(s-t_v)^{\iota\alpha-1}}{\Gamma(\alpha\iota+1)}\right) ds\right.$$

$$\left.+ \sum_{v=0}^{i} c_v\int_{t_{i+1}}^t (t-s)^{-\alpha}\left(\sum_{\iota=1}^{+\infty}\frac{(\alpha\iota)\lambda^\iota(s-t_v)^{\iota\alpha-1}}{\Gamma(\alpha\iota+1)}\right) ds\right]$$

$$+ \frac{1}{\Gamma(2-\alpha)}\left[(t-s)^{1-\alpha}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)\Big|_0^t\right.$$

$$\left.+ (1-\alpha)\int_0^t (t-s)^{-\alpha}\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)duds\right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}} \Phi(t) + \frac{1}{\Gamma(1-\alpha)} \left[\sum_{j=0}^{i} \sum_{v=0}^{j} c_v \sum_{\iota=1}^{+\infty} \frac{\lambda^{\iota}}{\Gamma(\alpha\iota)} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha}(s-t_v)^{\iota\alpha-1} ds \right.$$

$$\left. + \sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty} \frac{\lambda^{\iota}}{\Gamma(\alpha\iota)} \int_{t_{i+1}}^{t} (t-s)^{-\alpha}(s-t_v)^{\iota\alpha-1} ds \right]$$

$$+ \frac{1}{\Gamma(2-\alpha)} \left[(1-\alpha) \sum_{\iota=0}^{+\infty} \frac{\lambda^{\iota}}{\Gamma(\alpha(j+1))} \int_0^t \int_u^t (t-s)^{-\alpha}(s-u)^{\alpha j+\alpha-1} ds F(u) du \right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}} \Phi(t) + \frac{1}{\Gamma(1-\alpha)} \left[\sum_{v=0}^{i} c_v \sum_{j=v}^{i} \sum_{\iota=1}^{+\infty} \frac{\lambda^{\iota}(t-tv)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)} \int_{\frac{t_j-t_v}{t-t_v}}^{\frac{t_{j+1}-t_v}{t-t_v}} (1-w)^{-\alpha} w^{\iota\alpha-1} dw \right.$$

$$\left. + \sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty} \frac{\lambda^{\iota}(t-t_v)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)} \int_{\frac{t_{i+1}-t_v}{t-t_v}}^{1} (1-w)^{-\alpha} w^{\iota\alpha-1} dw \right]$$

$$+ \frac{1}{\Gamma(2-\alpha)} \left[(1-\alpha) \sum_{\iota=0}^{+\infty} \frac{\lambda^{\iota}}{\Gamma(\alpha(j+1))} \int_0^t (t-u)^{\alpha j} \int_0^1 (1-w)^{-\alpha} w^{\alpha j+\alpha-1} dw F(u) du \right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}} \Phi(t) + \frac{1}{\Gamma(1-\alpha)} \left[\sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty} \frac{\lambda^{\iota}(t-tv)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)} \int_0^{\frac{t_{i+1}-t_v}{t-t_v}} (1-w)^{-\alpha} w^{\iota\alpha-1} dw \right.$$

$$\left. + \sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty} \frac{\lambda^{\iota}(t-t_v)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)} \int_{\frac{t_{i+1}-t_v}{t-t_v}}^{1} (1-w)^{-\alpha} w^{\iota\alpha-1} dw \right] + \left[\sum_{\iota=0}^{+\infty} \frac{\lambda^{\iota}}{\Gamma(\alpha j+1)} \int_0^t (t-u)^{\alpha j} F(u) du \right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}} \Phi(t) + \frac{1}{\Gamma(1-\alpha)} \left[\sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty} \frac{\lambda^{\iota}(t-tv)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)} \int_0^1 (1-w)^{-\alpha} w^{\iota\alpha-1} dw \right]$$

$$+ \left[\sum_{\iota=0}^{+\infty} \frac{\lambda^{\iota}}{\Gamma(\alpha j+1)} \int_0^t (t-u)^{\alpha j} F(u) du \right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}} \Phi(t) + \sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty} \frac{\lambda^{\iota}(t-tv)^{\alpha(\iota-1)}}{\Gamma(\alpha(\iota-1)+1)} + \sum_{\iota=1}^{+\infty} \frac{\lambda^{\iota}}{\Gamma(\alpha j)} \int_0^t (t-u)^{\alpha j-1} F(u) du$$

$$= F(t) + \lambda x(t) + {}^c D^{\alpha}_{t^+_{i+1}} \Phi(t) - \lambda\Phi(t).$$

It follows that ${}^C D^{\alpha}_{t^+_{i+1}} \Phi(t) - \lambda\Phi(t) = 0$ for all $t \in (t_{i+1}, t_{i+2}]$. By Theorem 3.2, we know that there exists $c_{i+1} \in R$ such that $\Phi(t) = c_{i+1}\mathbf{E}_{\alpha,1}(\lambda(t-t_{i+1})^{\alpha})$ for $t \in (t_{i+1}, t_{i+2}]$. Substituting Φ into (3.2.6), we get that (3.2.5) holds for $j = i+1$. Now suppose that (3.2.5) holds for all $j \in \mathbb{N}_0$. By the mathematical induction method, we know that x satisfies (3.2.5) and $x|_{(t_i,t_{i+1}]}$ is continuous and $\lim_{t\to t_i^+} x(t)$ exists. The proof is complete. □

Theorem 3.11. *Suppose that F is continuous on $(0,1)$ and there exist constants $k > -1$ and $l \in (-\alpha, -1-k, 0]$ such that $|F(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$. Then x is a solution of (3.2.2.) if and only if there exists constants $c_i(i \in \mathbb{N}_0) \in \mathbb{R}$ such that*

$$x(t) = \sum_{v=0}^{j} c_v (t-t_v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_v)^{\alpha}) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})F(s)ds, t \in (t_j, t_{j+1}], j \in \mathbb{N}_0. \quad (3.2.7)$$

Proof. For $t \in (t_j, t_{j+1}](j \in N_0)$, similarly to the beginning of the proof of Theorem 3.10 we know that

$$t^{1-\alpha} \left| \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})F(s)ds \right| \le \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})|F(s)|ds$$

$$\le t^{1-\alpha} \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})s^k(1-s)^l ds$$

$$= t^{1-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\mathbf{\Gamma}((j+1)\alpha)}\int_0^t (t-s)^{\alpha-1}(t-s)^{\alpha j}s^k(1-s)^l ds$$

$$\le t^{1-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\mathbf{\Gamma}((j+1)\alpha)}\int_0^t (t-s)^{\alpha+l-1}(t-s)^{\alpha j}s^k ds$$

$$= t^{1-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\mathbf{\Gamma}((j+1)\alpha)}t^{\alpha+\alpha j+k+l}\int_0^1 (1-w)^{\alpha+\alpha j+l-1}w^k dw$$

$$\le t^{1-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j t^{\alpha j}}{\mathbf{\Gamma}((j+1)\alpha)}t^{\alpha+k+l}\int_0^1 (1-w)^{\alpha+l-1}w^k dw = t^{1+k+l}\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)\mathbf{B}(\alpha+l,k+1).$$

So $t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds$ is convergent and is continuous on $[0,1]$.

If x is a solution of (3.2.7), we have $x\in P_mC_{1-\alpha}(0,1]$. It follows for $t\in(t_i,t_{i+1}]$ that

$$^{RL}D_{0^+}^\alpha x(t) = \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\int_0^t (t-s)^{-\alpha}x(s)ds\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{\kappa=0}^{j}c_\kappa(s-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^\alpha)\right.\right.$$

$$\left.\left.+\int_0^s (s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)f(v)dv\right)ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\int_{t_i}^t (t-s)^{-\alpha}\left(\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^\alpha)\right.\right.$$

$$\left.\left.+\int_0^s (s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)F(v)dv\right)ds\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}(s-t_\kappa)^{\alpha-1}\sum_{m=0}^{+\infty}\frac{\lambda^m(s-t_\kappa)^{\alpha m}}{\mathbf{\Gamma}(\alpha(m+1))}ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{\kappa=0}^{i}c_\kappa\int_{t_i}^t (t-s)^{-\alpha}(t-t_\kappa)^{\alpha-1}\sum_{m=0}^{+\infty}\frac{\lambda^m(s-t_\kappa)^{\alpha m}}{\mathbf{\Gamma}(\alpha(m+1))}ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\int_0^t (t-s)^{-\alpha}\int_0^s (s-v)^{\alpha-1}\sum_{m=0}^{+\infty}\frac{\lambda^m(s-v)^{\alpha m}}{\mathbf{\Gamma}(\alpha(m+1))}F(v)dvds\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m}{\mathbf{\Gamma}(\alpha(m+1))}\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}(s-t_\kappa)^{\alpha+\alpha m-1}ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{\kappa=0}^{i}c_\kappa\sum_{m=0}^{+\infty}\frac{\lambda^m}{\mathbf{\Gamma}(\alpha(m+1))}\int_{t_i}^t (t-s)^{-\alpha}(t-t_\kappa)^{\alpha+\alpha m-1}ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{m=0}^{+\infty}\lambda^m\int_0^t (t-s)^{-\alpha}\int_0^s \frac{(s-v)^{\alpha+\alpha m-1}}{\mathbf{\Gamma}(\alpha(m+1))}F(v)dvds\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m}{\mathbf{\Gamma}(\alpha(m+1))}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{\alpha m}\sum_{j=\kappa}^{i-1}\int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}}(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{m=0}^{+\infty}\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha m}\frac{\lambda^m}{\mathbf{\Gamma}(\alpha(m+1))}\int_{\frac{t_i-t_\kappa}{t-t_\kappa}}^1 (1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\lambda^m\int_0^t\int_v^t(t-s)^{-\alpha}\frac{(s-v)^{\alpha+\alpha m-1}}{\Gamma(\alpha(m+1))}dsF(v)dv\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{\alpha m}\int_0^{\frac{t_i-t_\kappa}{t-t_\kappa}}(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_{\frac{t_i-t_\kappa}{t-t_\kappa}}^1(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right.$$

$$\left.+\sum_{m=0}^{+\infty}c_i(t-t_i)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_0^1(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\lambda^m\int_0^t(t-v)^{\alpha m}\int_0^1(1-w)^{-\alpha}\frac{w^{\alpha+\alpha m-1}}{\Gamma(\alpha(m+1))}dwF(v)dv\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{\kappa=0}^{i-1}c_\kappa\sum_{m=0}^{+\infty}(t-t_\kappa)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_0^1(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[c_i\sum_{m=0}^{+\infty}(t-t_i)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_0^1(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\lambda^m\int_0^t(t-v)^{\alpha m}\int_0^1(1-w)^{-\alpha}\frac{w^{\alpha+\alpha m-1}}{\Gamma(\alpha(m+1))}dwF(v)dv\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{\kappa=0}^{i-1}c_\kappa\sum_{m=0}^{+\infty}(t-t_\kappa)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\frac{\Gamma(1-\alpha)\Gamma(\alpha(m+1))}{\Gamma(\alpha m)}\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[c_i\sum_{m=0}^{+\infty}(t-t_i)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\frac{\Gamma(1-\alpha)\Gamma(\alpha(m+1))}{\Gamma(\alpha m)}\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\frac{\Gamma(1-\alpha)\Gamma(\alpha(m+1))}{\Gamma(\alpha m)}\int_0^t(t-v)^{\alpha m}F(v)dv\right]'$$

$$=\sum_{\kappa=0}^{i-1}c_\kappa\sum_{m=1}^{+\infty}(t-t_\kappa)^{\alpha m-1}\frac{(\alpha m)\lambda^m}{\Gamma(\alpha m)}+c_i\sum_{m=1}^{+\infty}(t-t_i)^{\alpha m-1}\frac{(\alpha m)\lambda^m}{\Gamma(\alpha m)}$$

$$+\sum_{m=1}^{+\infty}\frac{(\alpha m)\lambda^m}{\Gamma(\alpha m)}\int_0^t(t-v)^{\alpha m-1}F(v)dv+F(t)$$

$$=F(t)+\lambda\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-t_\kappa)^\alpha)+\sum_{m=1}^{+\infty}\frac{(\alpha m)\lambda^m}{\Gamma(\alpha(m+1))}\int_0^t(t-v)^{\alpha m-1}F(v)dv$$

$$=F(t)+\lambda\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_\kappa)^\alpha)+\lambda\int_0^t(t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^\alpha)F(v)dv$$

$$=\lambda x(t)+F(t),t\in(t_i,t_{i+1}].$$

It follows that x is a solution of (3.2.2).

Now we prove that if x is a solution of (3.2.2), then x satisfies (3.2.7) and $x\in P_mC_{1-\alpha}(0,1]$ by mathematical induction method. By Theorem 3.4, we know that there exists a constant $c_0\in\mathbb{R}$ such that

$$x(t)=c_0t^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)+\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds,t\in(t_0,t_1].$$

Hence (3.2.7) holds for $j=0$. Assume that (3.2.7) holds for $j=0,1,2,\cdots,i\leq m$, we will prove that (3.2.7) holds for $j = i+1$. Suppose that

$$x(t) = \Phi(t) + \sum_{j=0}^{i} c_j(t-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_j)^{\alpha}) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s)ds, t \in (t_{i+1}, t_{i+2}].$$

Then for $t \in (t_{i+1}, t_{i+2}]$ we have

$$\begin{aligned}
F(t) + \lambda x(t) &= {}^{RL}D_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}x(s)ds + \int_{t_{i+1}}^{t}(t-s)^{-\alpha}x(s)ds\right]' \\
&= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{\kappa=0}^{j} c_\kappa(s-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^{\alpha})\right.\right. \\
&\quad \left. + \int_0^s (s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^{\alpha})F(v)dv\right) ds \\
&\quad + \int_{t_{i+1}}^{t}(t-s)^{-\alpha}\left(\Phi(s) + \sum_{\kappa=0}^{k} c_\kappa(t-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^{\alpha})\right. \\
&\quad \left.\left. + \int_0^s (s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^{\alpha})F(v)dv\right) ds\right]' \\
&= {}^{RL}D_{t_{i+1}^+}^{\alpha}\Phi(t) + \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\sum_{\kappa=0}^{j} c_\kappa(s-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^{\alpha})ds\right. \\
&\quad + \int_{t_{i+1}}^{t}(t-s)^{-\alpha}\sum_{\kappa=0}^{k} c_\kappa(t-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^{\alpha})ds \\
&\quad \left. + \int_0^t (t-s)^{-\alpha}\int_0^s (s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^{\alpha})F(v)dvds\right]'.
\end{aligned}$$

Similarly to the proof of Theorem 3.10 we can get that

$$F(t) + \lambda x(t) = {}^{RL}D_{0^+}^{\alpha} x(t) = F(t) + \lambda x(t) + {}^{RL}D_{t_{i+1}^+}^{\alpha}\Phi(t) - \lambda\Phi(t).$$

So ${}^{RL}D_{t_{i+1}^+}^{\alpha}\Phi(t) - \lambda\Phi(t) = 0$ on $(t_{i+1}, t_{i+2}]$. Then Theorem 3.4 implies that there exists a constant $c_{i+1} \in \mathbb{R}$ such that $\Phi(t) = c_{i+1}(t-t_{i+1})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_{i+1})^{\alpha})$ on $(t_{i+1}, t_{i+2}]$. Hence

$$x(t) = \sum_{j=0}^{i+1} c_j(t-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_j)^{\alpha}) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s)ds, t \in (t_{i+1}, t_{i+2}].$$

By mathematical induction method, we know that (3.2.7) holds for $j \in \mathbb{N}_0$. The proof is complete. □

Theorem 3.12. *Suppose that G is continuous on $(1,e)$ and there exist constants $k > -1$ and $l \in (-\alpha, -1-k, 0]$ such that $|G(t)| \leq (\log t)^k(1-\log t)^l$ for all $t \in (1,e)$. Then x is a solution of (3.2.3) if and only if there exists constants $c_i(i \in \mathbb{N}_0) \in \mathbb{R}$ such that*

$$\begin{aligned}
x(t) &= \sum_{v=0}^{j} c_v\Gamma(\alpha)\left(\log\frac{t}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{t_v}\right)^{\alpha}\right) \\
&\quad + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s}, t \in (t_j, t_{j+1}], j \in \mathbb{N}_0.
\end{aligned} \tag{3.2.8}$$

Proof. For $t \in (t_j, t_{j+1}](j \in \mathbb{N}_0)$, similarly to the beginning of the proof of Theorem 3.10 we know that

$$\begin{aligned}
&(\log t)^{1-\alpha}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s}\right| \\
&\leq (\log t)^{1-\alpha}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)(\log s)^k(1-\log s)^l\frac{ds}{s} \\
&\leq (\log t)^{1-\alpha}\sum_{\iota=0}^{+\infty}\frac{\lambda^{\iota}}{\Gamma(\alpha(\iota+1))}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha\iota+\alpha+l-1}(\log s)^k\frac{ds}{s} \text{ by } \frac{\log s}{\log t} = w
\end{aligned}$$

$$= (\log t)^{1-\alpha}\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha(\iota+1))}(\log t)^{\alpha\iota+\alpha+k+l}\int_0^1(1-w)^{\alpha\iota+\alpha+l-1}w^k dw$$

$$\leq (\log t)^{1+k+l}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)\int_0^1(1-w)^{\alpha+l-1}w^k dw$$

$$= (\log t)^{1+k+l}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)\mathbf{B}(\alpha+l,k+1).$$

So $\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)G(s)\frac{ds}{s}$ is convergent for all $t\in(1,e]$ and

$$\lim_{t\to1^+}(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)G(s)\frac{ds}{s}\text{ exists.}$$

If x is a solution of (3.2.8), we have $x\in LP_mC_{1-\alpha}(1,e]$. By using Definition 2.5, it follows for $t\in(t_i,t_{i+1}]$ that

$$^{RLH}D_{1^+}^\alpha x(t) = \frac{1}{\mathbf{\Gamma}(1-\alpha)}t\left[\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}x(s)\frac{ds}{s}\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}t\left[\sum_{j=0}^{i-1}\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{j}c_v\Gamma(\alpha)\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\right.\right.$$

$$\left.+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right)G(u)\frac{du}{u}\right)\frac{ds}{s}$$

$$+\int_{t_i}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{i}c_v\Gamma(\alpha)\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\right.$$

$$\left.\left.+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right)G(u)\frac{du}{u}\right)\frac{ds}{s}\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}t\left[\Gamma(\alpha)\sum_{j=0}^{i-1}\sum_{v=0}^{j}c_v\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\frac{ds}{s}\right.$$

$$+\Gamma(\alpha)\sum_{v=0}^{i}c_v\int_{t_i}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\frac{ds}{s}$$

$$\left.+\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right)G(u)\frac{du}{u}\frac{ds}{s}\right]'.$$

One sees that

$$\int_{t_i}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\frac{ds}{s}$$

$$= \sum_{\kappa=0}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha(\iota+1))}\int_{t_i}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{t_v}\right)^{\alpha\kappa+\alpha-1}\frac{ds}{s}\text{ by }\frac{\log s-\log t_v}{\log t-\log t_v}=w$$

$$= \sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa}\int_{\frac{\log t_i-\log t_v}{\log t-\log t_v}}^1(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dw$$

and

$$\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\frac{ds}{s}$$

$$= \sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa}\int_{\frac{\log t_i-\log t_v}{\log t-\log t_v}}^{\frac{\log t_{i+1}-\log t_v}{\log t-\log t_v}}(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dw.$$

Similarly

$$\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) \int_1^s \left(\log \frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{s}{u}\right)^{\alpha}\right) G(u)\frac{du}{u}\frac{ds}{s}$$

$$= \int_1^t \int_u^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) \left(\log \frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{s}{u}\right)^{\alpha}\right) \frac{ds}{s}G(u)\frac{du}{u}$$

$$= \sum_{\kappa=0}^{+\infty} \int_1^t \left(\log \frac{t}{u}\right)^{\alpha\kappa} \frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))} \int_0^1 (1-w)^{-\alpha} w^{\alpha\kappa+\alpha-1} dw G(u)\frac{du}{u}.$$

So

$${}^{RLH}D_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} t \left[\Gamma(\alpha) \sum_{v=0}^{i-1} c_v \sum_{\kappa=0}^{+\infty} \frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))} \left(\log \frac{t}{t_v}\right)^{\alpha\kappa} \sum_{j=v}^{i-1} \int_{\frac{\log t_j - \log t_v}{\log t - \log t_v}}^{\frac{\log t_{j+1} - \log t_v}{\log t - \log t_v}} (1-w)^{-\alpha} w^{\alpha\kappa+\alpha-1} dw\right.$$

$$+\Gamma(\alpha) \sum_{v=0}^{i} c_v \sum_{\kappa=0}^{+\infty} \frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))} \left(\log \frac{t}{t_v}\right)^{\alpha\kappa} \int_{\frac{\log t_i - \log t_v}{\log t - \log t_v}}^{1} (1-w)^{-\alpha} w^{\alpha\kappa+\alpha-1} dw$$

$$\left.+ \sum_{\kappa=0}^{+\infty} \int_1^t \left(\log \frac{t}{u}\right)^{\alpha\kappa} \frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))} \int_0^1 (1-w)^{-\alpha} w^{\alpha\kappa+\alpha-1} dw G(u)\frac{du}{u}\right]'$$

$$= t\left[\Gamma(\alpha) \sum_{v=0}^{i} c_v \sum_{\kappa=0}^{+\infty} \frac{\lambda^\kappa}{\Gamma(\alpha\kappa+1)} \left(\log \frac{t}{t_v}\right)^{\alpha\kappa} + \sum_{\kappa=0}^{+\infty} \int_1^t \left(\log \frac{t}{u}\right)^{\alpha\kappa} \frac{\lambda^\kappa}{\Gamma(\alpha\kappa+1)} G(u)\frac{du}{u}\right]'$$

$$= F(t) + t\left[\frac{\Gamma(\alpha)}{t} \sum_{v=0}^{i} c_v \sum_{\kappa=1}^{+\infty} \frac{\lambda^\kappa}{\Gamma(\alpha\kappa)} \left(\log \frac{t}{t_v}\right)^{\alpha\kappa-1} + \sum_{\kappa=1}^{+\infty} \int_1^t \left(\log \frac{t}{u}\right)^{\alpha\kappa-1} \frac{\lambda^\kappa}{\Gamma(\alpha\kappa)} G(u)\frac{du}{u}\right]$$

$$= \lambda x(t) + F(t), t \in (t_i, t_{i+1}].$$

It follows that x is a solution of (3.2.3).

Now we prove that if x is a solution of (3.2.3), then x satisfies (3.2.8) and $x \in LP_mC_{1-\alpha}(1,e]$ by mathematical induction method. By Theorem 3.6, we know that there exists a constant $c_0 \in \mathbb{R}$ such that

$$x(t) = c_0(\log t)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log t)^{\alpha}) + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\left(\lambda \left(\frac{t}{s}\right)^{\alpha}\right) F(s)ds, t \in (t_0, t_1].$$

Hence (3.2.8) holds for $j = 0$. Assume that (3.2.8) holds for $j = 0, 1, 2, \cdots, i \le m$, we will prove that (3.2.8) holds for $j = i+1$. Suppose that

$$x(t) = \Phi(t) + \sum_{v=0}^{i} c_v\Gamma(\alpha) \left(\log \frac{t}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{t_v}\right)^{\alpha}\right)$$

$$+ \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) G(s)\frac{ds}{s}, t \in (t_{i+1}, t_{i+2}].$$

Then for $t \in (t_{i+1}, t_{i+2}]$ we have

$$F(t) + \lambda x(t) = {}^{RLH}D_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \left[\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} \left(\log \frac{t}{s}\right)^{-\alpha} x(s)\frac{ds}{s} + \int_{t_{i+1}}^t \left(\log \frac{t}{s}\right)^{-\alpha} x(s)\frac{ds}{s}\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{i}c_v\Gamma(\alpha)\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right.\right.$$

$$+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\Big)\frac{ds}{s}$$

$$+\int_{t_{i+1}}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\Phi(s)+\sum_{v=0}^{i}c_v\Gamma(\alpha)\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right.$$

$$\left.\left.+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)\frac{ds}{s}\right]'$$

$$= {}^{RL}D^{\alpha}_{t_{i+1}^+}\Phi(t)+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{i}c_v\Gamma(\alpha)\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right.\right.$$

$$+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\Big)\frac{ds}{s}$$

$$+\int_{t_{i+1}}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{i}c_v\Gamma(\alpha)\left(\log\frac{s}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right.$$

$$\left.\left.+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)\frac{ds}{s}\right]'.$$

Similarly to above discussion we can get that

$$F(t)+\lambda x(t)={}^{RLH}D^{\alpha}_{1^+}x(t)=F(t)+\lambda x(t)+{}^{RLH}D^{\alpha}_{t_{i+1}^+}\Phi(t)-\lambda\Phi(t).$$

So ${}^{RLH}D^{\alpha}_{t_{i+1}^+}\Phi(t)-\lambda\Phi(t)=0$ on $(t_{i+1},t_{i+2}]$. Then Theorem 3.6 implies that there exists a constant $c_{i+1}\in\mathbb{R}$ such that $\Phi(t)=c_{i+1}\Gamma(\alpha)\left(\log\frac{t}{t_{i+1}}\right)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\left(\lambda\left(\frac{t}{t_{i+1}}\right)^{\alpha}\right)$ on $(t_{i+1},t_{i+2}]$. Hence

$$x(t)=\sum_{v=0}^{i+1}c_v\Gamma(\alpha)\left(\log\frac{t}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{t_v}\right)^{\alpha}\right)$$

$$+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s},t\in(t_{i+1},t_{i+2}].$$

By mathematical induction method, we know that (3.2.8) holds for $j\in\mathbb{N}_0$. The proof is complete. □

Theorem 3.13. *Suppose that G is continuous on $(1,e)$ and there exist constants $k>-1$ and $l\in(-\alpha,-\alpha+k,0]$ such that $|G(t)|\le(\log t)^k(1-\log t)^l$ for all $t\in(1,e)$. Then x is a piecewise solution of (3.2.4) if and only if x and there exists constants $c_i(i\in\mathbb{N}_0)\in\mathbb{R}$ such that*

$$x(t)=\sum_{v=0}^{j}c_vE_{\alpha,1}\left(\lambda\left(\log\frac{t}{t_v}\right)^{\alpha}\right)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s},t\in(t_j,t_{j+1}],j\in\mathbb{N}_0. \qquad (3.2.9)$$

Proof. For $t\in(t_j,t_{j+1}](j\in\mathbb{N}_0)$, similarly to the beginning of the proof of Theorem 3.12 we know that

$$\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s}\right|$$

$$\le(\log t)^{\alpha+k+l}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)\mathbf{B}(\alpha+l,k+1).$$

So $\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s}$ is convergent for all $t\in(1,e]$ and

$$\lim_{t\to1^+}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s}\text{ exists.}$$

If x is a solution of (3.2.9), we have $x\in LP_mC(1,e]$. By using Definition 2.6, it follows for $t\in(t_i,t_{i+1}]$ that

$$
\begin{aligned}
&{}^{CH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}sx'(s)\frac{ds}{s}\\
&=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{j}c_v\Gamma(\alpha)E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right.\right.\\
&+\left.\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\\
&+\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{i}c_v\Gamma(\alpha)E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right.\\
&+\left.\left.\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\right]\\
&=\frac{1}{\Gamma(1-\alpha)}t\left[\sum_{j=0}^{i-1}\sum_{v=0}^{j}c_v\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right)'ds\right.\\
&+\sum_{v=0}^{i}c_v\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\left(E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right)'ds\\
&+\left.\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\right].
\end{aligned}
$$

One sees that

$$
\begin{aligned}
&\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\left(E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right)'ds=\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{\kappa=0}^{+\infty}\frac{\lambda^{\kappa}}{\Gamma(\alpha\kappa+1)}\left(\log\frac{s}{t_v}\right)^{\kappa\alpha}\right)'ds\\
&=\sum_{\kappa=1}^{+\infty}\frac{(\kappa\alpha)\lambda^{\kappa}}{\Gamma(\alpha\kappa+1)}\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{t_v}\right)^{\kappa\alpha-1}ds=\sum_{\kappa=1}^{+\infty}\frac{(\kappa\alpha)\lambda^{\kappa}}{\Gamma(\alpha\kappa+1)}\left(\log\frac{t}{t_v}\right)^{\alpha(\kappa-1)}\int_{\frac{t_i-t_v}{t-t_v}}^{1}(1-w)^{-\alpha}w^{\kappa\alpha-1}dw
\end{aligned}
$$

and

$$
\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right)'ds=\sum_{\kappa=1}^{+\infty}\frac{(\alpha\kappa)\lambda^{\kappa}}{\Gamma(\alpha\kappa+1)}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa-1}\int_{\frac{\log t_i-\log t_v}{\log t-\log t_v}}^{\frac{\log t_{i+1}-\log t_v}{\log t-\log t_v}}(1-w)^{-\alpha}w^{\alpha\kappa-1}dw.
$$

Similarly

$$
\begin{aligned}
&\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\\
&=\frac{t}{1-\alpha}\left[\int_1^t\left(\log\frac{t}{s}\right)^{1-\alpha}\left(\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\right]'\\
&=\frac{t}{1-\alpha}\left[\left(\log\frac{t}{s}\right)^{1-\alpha}\left(\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)\Big|_1^t\right.\\
&\left.+(1-\alpha)\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\frac{ds}{s}\right]'\\
&=t\left[\sum_{\kappa=0}^{+\infty}\frac{\lambda^{\kappa}}{\Gamma(\alpha(\kappa+1))}\int_1^t\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{u}\right)^{\alpha\kappa+\alpha-1}\frac{ds}{s}G(u)\frac{du}{u}\right]'\\
&=t\left[\sum_{\kappa=0}^{+\infty}\frac{\lambda^{\kappa}}{\Gamma(\alpha(\kappa+1))}\int_1^t\left(\log\frac{t}{u}\right)^{\alpha\kappa}\int_0^1(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}\right]'\\
&=F(t)+t\sum_{\kappa=1}^{+\infty}\frac{(\alpha\kappa)\lambda^{\kappa}}{\Gamma(\alpha(\kappa+1))}\int_1^t\left(\log\frac{t}{u}\right)^{\alpha\kappa-1}\int_0^1(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}.
\end{aligned}
$$

So

$$^{CH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}t\left[\sum_{j=0}^{i-1}\sum_{v=0}^{j}c_v\sum_{\kappa=1}^{+\infty}\frac{(\alpha\kappa)\lambda^\kappa}{\Gamma(\alpha\kappa+1)}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa-1}\int_{\frac{\log t_j-\log t_v}{\log t-\log t_v}}^{\frac{\log t_{j+1}-\log t_v}{\log t-\log t_v}}(1-w)^{-\alpha}w^{\alpha\kappa-1}dw\right.$$

$$+\sum_{v=0}^{i}c_v\sum_{\kappa=1}^{+\infty}\frac{(\kappa\alpha)\lambda^\kappa}{\Gamma(\alpha\kappa+1)}\left(\log\frac{t}{t_v}\right)^{\alpha(\kappa-1)}\int_{\frac{t_i-t_v}{t-t_v}}^{1}(1-w)^{-\alpha}w^{\kappa\alpha-1}dw$$

$$+F(t)+t\sum_{\kappa=1}^{+\infty}\frac{(\alpha\kappa)\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\int_1^t\left(\log\frac{t}{u}\right)^{\alpha\kappa-1}\int_0^1(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}$$

$$=\lambda x(t)+F(t),t\in(t_i,t_{i+1}].$$

It follows that x is a solution of (3.2.4).

Now we prove that if x is a solution of (3.2.4), then x satisfies (3.2.9) and $x\in LP_mC(1,e]$ by mathematical induction method. By Theorem 3.8, we know that there exists a constant $c_0\in\mathbb{R}$ such that

$$x(t)=c_0\mathbf{E}_{\alpha,\alpha}(\lambda(\log t)^\alpha)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\left(\lambda\left(\frac{t}{s}\right)^\alpha\right)F(s)ds,t\in(t_0,t_1].$$

Hence (3.2.9) holds for $j=0$. Assume that (3.2.9) holds for $j=0,1,2,\cdots,i\le m$, we will prove that (3.2.9) holds for $j=i+1$. Suppose that

$$x(t)=\Phi(t)+\sum_{v=0}^{i}c_vE_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{t_v}\right)^\alpha\right)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)G(s)\frac{ds}{s},t\in(t_{i+1},t_{i+2}].$$

Then for $t\in(t_{i+1},t_{i+2}]$ we have

$$F(t)+\lambda x(t)={}^{CH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}\left(\log\frac{t}{s}\right)^{-\alpha}x'(s)ds+\int_{t_{i+1}}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}x'(s)ds\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{i}c_vE_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right)G(u)\frac{du}{u}\right)'ds\right.$$

$$\left.+\int_{t_{i+1}}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\Phi(s)+\sum_{v=0}^{i}c_vE_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right)G(u)\frac{du}{u}\right)'ds\right].$$

Similarly to above discussion we can get that

$$F(t)+\lambda x(t)={}^{CH}D_{1^+}^{\alpha}x(t)=F(t)+\lambda x(t)+{}^{CH}D_{t_{i+1}^+}^{\alpha}\Phi(t)-\lambda\Phi(t).$$

So ${}^{CH}D_{t_{i+1}^+}^{\alpha}\Phi(t)-\lambda\Phi(t)=0$ on $(t_{i+1},t_{i+2}]$. Then Theorem 3.8 implies that there exists a constant $c_{i+1}\in\mathbb{R}$ such that $\Phi(t)=c_{i+1}\mathbf{E}_{\alpha,\alpha}\left(\lambda\left(\frac{t}{t_{i+1}}\right)^\alpha\right)$ on $(t_{i+1},t_{i+2}]$. Hence

$$x(t)=\sum_{v=0}^{i+1}c_vE_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{t_v}\right)^\alpha\right)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)G(s)\frac{ds}{s},t\in(t_{i+1},t_{i+2}].$$

By mathematical induction method, we know that (3.2.9) holds for $j\in\mathbb{N}_0$. The proof is complete. □

3.3 Preliminary for BVP(1.7)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.7). For ease expression, denote

$$\delta_{\alpha,\lambda}(t,s)=(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha),$$

$$\Lambda=-1+\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)+\Gamma(\alpha)^2\delta_{\alpha,\lambda}(t_1,0)\delta_{\alpha,\lambda}(1,t_1).$$

Lemma 3.1. *Suppose that $\Lambda \neq 0$ and $\sigma : (0,1) \to R$ is continuous and satisfies that there exist numbers $k > -1$ and $\max\{-\alpha, -k-1\} < l \leq 0$ such that $|\sigma(t)| \leq t^k(1-t)^l$ for all $t \in (0,1)$. The x is a solutions of*

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha} x(t) - \lambda x(t) = \sigma(t), t \in (t_i, t_{i+1}], i = 0, 1, \\ x(1) - \lim\limits_{t\to 0} t^{1-\alpha} x(t) = a_0, \quad \lim\limits_{t \to t_1^+} (t - t_1)^{1-\alpha} x(t) - x(t_1) = I_0 \end{cases} \tag{3.3.1}$$

if and only if $x \in P_1C_{1-\alpha}(0,1]$ and

$$x(t) = \begin{cases} \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[a_0 - \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1) I_0 \; - \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds \\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1) \int_0^{t_1} \delta_{\alpha,\lambda}(t_1,s)\sigma(s)ds\Big] + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (0, t_1], \\ \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[a_0 - \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1) I_0 \; - \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds \\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1) \int_0^{t_1} \delta_{\alpha,\lambda}(t_1,s)\sigma(s)ds\Big] + \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_1)}{\Lambda}\Big[\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)a_0 \\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0) - 1) I_0 \; -\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0) \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds \\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0) - 1) \int_0^{t_1} \delta_{\alpha,\lambda}(t_1,s)\sigma(s)ds\Big] + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (t_1, 1]. \end{cases} \tag{3.3.2}$$

Proof. Let x be a solution of (3.3.1). By Theorem 3.11, we know that there exist numbers $A_0, A_1 \in \mathbb{R}$ such that

$$x(t) = A_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds, t \in (t_0, t_1] \tag{3.3.3}$$

and

$$\begin{aligned} x(t) = {} & A_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) + A_1\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(t-t_1)^{\alpha})(t-t_1)^{\alpha-1} \\ & + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds, t \in (t_1, t_2]. \end{aligned} \tag{3.3.4}$$

Note $E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}$. It follows from the boundary conditions and the impulse assumption in (3.3.1) that

$$\begin{aligned} & A_0\Gamma(\alpha)E_{\alpha,\alpha}(\lambda) + A_1\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1} \\ & + \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds - A_0 = a_0, \\ & A_1 - \Big[A_0\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t_1^{\alpha}) + \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha})\sigma(s)ds\Big] = I_0. \end{aligned}$$

Then

$$\begin{aligned} A_0 = {} & \frac{1}{\Lambda}\Big[a_0 - \Gamma(\alpha)(1-t_1)^{\alpha}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})I_0 - \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds \\ & -\Gamma(\alpha)(1-t_1)^{\alpha}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha}) \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha})\sigma(s)ds\Big], \\ A_1 = {} & \frac{1}{\Lambda}\Big[\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t_1^{\alpha})a_0 + (\Gamma(\alpha)E_{\alpha,\alpha}(\lambda) - 1)I_0 \\ & -\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t_1^{\alpha}) \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds \\ & +(\Gamma(\alpha)E_{\alpha,\alpha}(\lambda) - 1) \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha})\sigma(s)ds\Big]. \end{aligned} \tag{3.3.5}$$

Substituting A_0, A_1 into (3.3.3) and (3.3.4), we get (3.3.2) obviously.

On the other hand, if x satisfies (3.3.2), then $x|_{(0,t_1]}$ and $x|_{(t_1,1]}$ are continuous and the limits $\lim\limits_{t\to 0} t^{1-\alpha}x(t)$

and $\lim_{t\to t_1}(1-t_1)^{1-\alpha}x(t)$ exist. So $x\in P_1C_{1-\alpha}(0,1]$. Using (3.3.5), we rewrite x by

$$x(t)=\begin{cases} A_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\sigma(s)ds, t\in(0,t_1],\\ A_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)+A_1\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(t-t_1)^\alpha)(t-t_1)^{\alpha-1}\\ +\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\sigma(s)ds, t\in(t_1,1].\end{cases}$$

Since σ is continuous on $(0,1)$ and $|\sigma(t)|\le t^k(1-t)^l$, one can show easily that x is continuous on $(t_i,t_{i+1}](i=0,1)$ and using the method at the beginning of the proof of this lemma, we know that both the limits $\lim_{t\to0^+}t^{1-\alpha}x(t)$ and $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t)$ exist. So $x\in P_1C_{1-\alpha}(0,1]$. Furthermore, by direct computation, we have $x(1)-\lim_{t\to0}t^{1-\alpha}x(t)=a_0$, and $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t)-x(t_1)=I_0$. One have from Theorem 3.12 easily for $t\in(t_0,t_1]$ that $D_{0^+}^\alpha x(t)=\lambda x(t)+\sigma(t)$ and for $t\in(t_1,t_2]$ that

$${}^{RL}D_{0^+}^\alpha x(t)=\frac{1}{\Gamma(1-\alpha)}\left[\int_0^t(t-s)^{-\alpha}x(s)ds\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[\int_0^{t_1}(t-s)^{-\alpha}\left(A_0\Gamma(\alpha)s^{\alpha-1}E_{\alpha,\alpha}(\lambda s^\alpha)+\int_0^s(s-w)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-w)^\alpha)\sigma(w)dw\right)ds\right.$$

$$+\int_{t_1}^t(t-s)^{-\alpha}\left(A_0\Gamma(\alpha)s^{\alpha-1}E_{\alpha,\alpha}(\lambda s^\alpha)+A_1\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(s-t_1)^\alpha)(s-t_1)^{\alpha-1}\right.$$

$$\left.\left.+\int_0^s(s-w)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-w)^\alpha)\sigma(w)dw\right)ds\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[A_0\Gamma(\alpha)\int_0^t(t-s)^{-\alpha}s^{\alpha-1}E_{\alpha,\alpha}(\lambda s^\alpha)ds+A_1\Gamma(\alpha)\int_{t_1}^t(t-s)^{-\alpha}E_{\alpha,\alpha}(\lambda(s-t_1)^\alpha)(s-t_1)^{\alpha-1}ds\right.$$

$$\left.+\int_0^t(t-s)^{-\alpha}\int_0^s(s-w)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-w)^\alpha)\sigma(w)dwds\right]'.$$

One has by variable transformation $w=\frac{s-t_1}{t-t_1}$ and $\mathbf{B}(1-\alpha,j\alpha+\alpha)=\frac{\Gamma(1-\alpha)\Gamma(j\alpha+\alpha)}{\Gamma(j\alpha+1)}$ that

$$\int_{t_1}^t(t-s)^{-\alpha}E_{\alpha,\alpha}(\lambda(s-t_1)^\alpha)(s-t_1)^{\alpha-1}ds=\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}\int_{t_1}^t(t-s)^{-\alpha}(s-t_1)^{j\alpha+\alpha-1}ds$$

$$=\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}(t-t_1)^{j\alpha}\int_0^1(1-w)^{-\alpha}w^{j\alpha+\alpha-1}dw=\sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha).$$

Then

$${}^{RL}D_{0^+}^\alpha x(t)=\frac{1}{\Gamma(1-\alpha)}\left[A_0\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^jt^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)+A_1\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)\right.$$

$$\left.+\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}\int_0^t\int_w^t(t-s)^{-\alpha}(s-w)^{j\alpha+\alpha-1}ds\sigma(w)dw\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[A_0\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^jt^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)+A_1\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)\right.$$

$$+\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}\int_0^t(t-w)^{j\alpha}\int_0^1(1-u)^{-\alpha}u^{j\alpha+\alpha-1}du\sigma(w)dw\Bigg]' \text{ by } \frac{s-w}{t-w}=u$$

$$=\frac{1}{\Gamma(1-\alpha)}\Bigg[A_0\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)+A_1\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)$$

$$+\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\int_0^t(t-w)^{j\alpha}\sigma(w)dw\Gamma(1-\alpha)\Bigg]'$$

$$=\Bigg[A_0\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)}+A_1\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}+\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\int_0^t(t-w)^{j\alpha}\sigma(w)dw\Bigg]'$$

$$=\lambda x(t)+\sigma(t).$$

So x is a solution of (3.3.1). The proof is completed. □

For ease expression, denote for a function $H:(0,1)\times R\to R$ that $H_x(t)=H(t,x(t))$. Define the nonlinear operator T on $P_1C_{1-\alpha}(0,1]$ for $x\in P_1C_{1-\alpha}(0,1]$ by

$$(Tx)(t)=\begin{cases}\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[\int_0^1\phi(s)G_x(s)ds-\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_x(t_1)\ -\int_0^1\delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)f_x(s)ds\Big]+\int_0^t\delta_{\alpha,\lambda}(t,s)p(s)f_x(s))ds, t\in(0,t_1],\\ \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[\int_0^1\phi(s)G_x(s)ds-\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_x(t_1)\ -\int_0^1\delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)f_x(s)ds\Big]+\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_1)}{\Lambda}\Big[\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)\int_0^1\phi(s)G_x(s)ds\\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1)I_x(t_1)\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)\int_0^1\delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)f_x(s)ds\Big]+\int_0^t\delta_{\alpha,\lambda}(t,s)p(s)f_x(s)ds, t\in(t_1,1].\end{cases}$$

Lemma 3.2. *Suppose that (a)-(e) hold, $\Lambda\neq 0$, and f,G are* **impulsive II-Carathéodory functions***, I a* **discrete II-Carathódory functions***. Then $x\in P_1C_{1-\alpha}(0,]$ is a solution of BVP(1.7) if and only if $x\in P_1C_{1-\alpha}(0,1]$ is a fixed point of T, $T:P_1C_{1-\alpha}(0,1]\to P_1C_{1-\alpha}(0,1]$ is well defined and is completely continuous.*

Proof. **Step (i)** Prove that $T:P_mC_{1-\alpha}(0,1]\to P_mC_{1-\alpha}(0,1]$ is well defined.

It comes from the method in Theorem 3.12 that $Tx|_{(0,t_1]}$, $Tx|_{(t_1,1]}(i=0,1)$ are continuous and the limits $\lim_{t\to 0}t^{1-\alpha}(Tx)(t)$ and $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}(Tx)(t)$ exist. We see from Lemma 3.3.1 that $x\in P_1C_{1-\alpha}(0,1]$ is a solution of BVP(1.7) if and only if $x\in P_1C_{1-\alpha}(0,1]$ is a fixed point of T in $P_1C_{1-\alpha}(0,1]$.

Step (ii) We prove that T is continuous.

Let $x_n\in P_1C_{1-\alpha}(0,1]$ with $x_n\to x_0$ as $n\to+\infty$. We can show that $Tx_n\to Tx_0$ as $n\to+\infty$ by using the dominant convergence theorem. We refer the readers to the papers [65, 77, 81].

Step (iii) Prove that T is compact, i.e., prove that $T(\overline{\Omega})$ is relatively compact for every bounded subset $\Omega\subset P_1C_{1-\alpha}(0,1]$.

Let Ω be a bounded open nonempty subset of $P_1C_{1-\alpha}(0,1]$. We have

$$||x||=\max\left\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{1-\alpha}|x(t)|:i=0,1\right\}\le r<+\infty,(x,y)\in\overline{\Omega}. \tag{3.3.6}$$

Since f,G are **impulsive II-Carathéodory functions**, I ia a **discrete II-Carathéodory function**, then there exists constants $M_f, M_I, M_G \geq 0$ such that

$$\begin{aligned}
&|f(t,x(t)))| = \left|f\left(t,(t-t_i)^{\alpha-1}(t-t_i)^{1-\alpha}x(t)\right)\right| \leq M_f, t\in(t_i,t_{i+1}], i=0,1,\\
&|G(t,x(t))| \leq M_G, t\in(t_i,t_{i+1}], i=0,1,\\
&|I(t_1,x(t_1))| = |I(t_1,t^{\alpha-1}t_1^{1-\alpha}x(t_1))| \leq M_I.
\end{aligned} \tag{3.3.7}$$

This step is done by three sub-steps:

Sub-step (iii1) Prove that $T(\overline{\Omega})$ is uniformly bounded.

Using (3.3.2) and (3.3.7), we have for $t\in(0,t_1]$ that

$$\begin{aligned}
t^{1-\alpha}|(Tx)(t)| &\leq \tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{|\Lambda|}\left[||\phi||_1 M_G + \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)|M_I\right.\\
&+\int_0^1(1-s)^{\alpha-1}\sum_{v=0}^{+\infty}\tfrac{\lambda^v(1-s)^{\alpha v}}{\Gamma(\alpha(v+1))}M_f s^k(1-s)^l ds\\
&\left.+\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}(t_1-s)^{\alpha-1}\sum_{v=0}^{+\infty}\tfrac{\lambda^v(t_1-s)^{\alpha v}}{\Gamma(\alpha(v+1))}M_f s^k(1-s)^l ds\right]\\
&+M_f t^{1-\alpha}\sum_{v=0}^{+\infty}\tfrac{\lambda^v(t-s)^{\alpha v}}{\Gamma(\alpha(v+1))}\int_0^t(t-s)^{\alpha+l-1}s^k ds\\
&\leq \tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{|\Lambda|}\left[||\phi||_1 M_G + \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)|M_I\right.\\
&+M_f\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-s)^{\alpha v+\alpha+l-1}s^k ds\\
&\left.+\Gamma(\alpha)M_f\delta_{\alpha,\lambda}(1,t_1)\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^{t_1}(t_1-s)^{\alpha v+\alpha+l-1}s^k ds\right]\\
&+M_f t^{1-\alpha}\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^t(t-s)^{\alpha v+\alpha+l-1}s^k ds\\
&\leq \tfrac{\Gamma(\alpha)||\phi||_1 E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}M_G + \tfrac{\Gamma(\alpha)^2 E_{\alpha,\alpha}(|\lambda|)^2(1-t_1)^{\alpha-1}}{|\Lambda|}|M_I\\
&+\left(\tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|}+\tfrac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|}+E_{\alpha,\lambda}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1).
\end{aligned}$$

For $t\in(t_1,t_2]$, we have similarly that

$$\begin{aligned}
(t-t_1)^{1-\alpha}|(Tx)(t)| &\leq \tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\left[||\phi||_1 M_G + \Gamma(\alpha)(1-t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)M_I\right.\\
&+\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-s)^{\alpha v+\alpha+l-1}s^k ds M_f\\
&\left.+\Gamma(\alpha)(1-t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^{t_1}(t_1-s)^{\alpha v+\alpha+l-1}s^k ds M_f\right]\\
&+\tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(t-t_1)^\alpha)}{|\Lambda|}\left[\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)||\phi||_1 M_G\right.\\
&+(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)M_I\ +\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-s)^{\alpha v+\alpha+l-1}s^k ds M_f\\
&\left.+(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^{t_1}(t_1-s)^{\alpha v+\alpha+l-1}s^k ds M_f\right]
\end{aligned}$$

$$+M_f(t-t_1)^{1-\alpha}\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^t(t-s)^{\alpha v+\alpha+l-1}s^kds$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\Big[||\phi||_1M_G+\Gamma(\alpha)(1-t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)M_I$$

$$+E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)M_f$$

$$+\Gamma(\alpha)(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)^2\mathbf{B}(\alpha+l,k+1)M_f\Big]$$

$$+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\Big[\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)||\phi||_1M_G$$

$$+(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)M_I\ +\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)^2\mathbf{B}(\alpha+l,k+1)M_f$$

$$+(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)t_1^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)M_f\Big]$$

$$+M_f(t-t_1)^{1-\alpha}t^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)$$

$$\leq\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1t_1^{\alpha-1}}{|\Lambda|}\right)M_G$$

$$+\left(\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2(1-t_1)^{\alpha-1}}{|\Lambda|}+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\right)M_I$$

$$+\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3t_1^{\alpha-1}}{|\Lambda|}\right.$$

$$\left.+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)^2t_1^{\alpha+k+l}}{|\Lambda|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)M_f.$$

From above discussion, we get

$$\begin{aligned}||Tx||\leq&\left[\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1}{|\Lambda|t_1^{1-\alpha}}\right]M_G\\&+\left[\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|(1-t_1)^{1-\alpha}}+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\right]M_I\\&+\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3t_1^{\alpha-1}}{|\Lambda|}\right.\\&\left.+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)^2t_1^{\alpha+k+l}}{|\Lambda|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)M_f.\end{aligned}\tag{3.3.8}$$

From above discussion, $T(\overline{\Omega})$ is uniformly bounded.

Sub-step (iii2) Prove that $t\to(t-t_i)^{1-\alpha}T(\overline{\Omega})$ is equi-continuous on $(t_i,t_{i+1}](i=0,1)$. Let

$$(t-t_i)^{1-\alpha}\overline{(T_1(x,y))}(t)=\begin{cases}(t-t_i)^{1-\alpha}(T_1(x,y))(t),t\in(t_i,t_{i+1}],\\ \lim_{t\to t_i^+}(t-t_i)^{1-\alpha}(T_1(x,y))(t),t=t_i.\end{cases}$$

Then $t\to(t-t_i)^{1-\alpha}\overline{(Tx)}(t)$ is continuous on $[t_i,t_{i+1}]$. Let $s_2\leq s_1$ and $s_1,s_2\in[t_0,t_1]$. By AscoliCArzela theorem on the closed interval, We can prove that

$$\left|s_1^{1-\alpha}\overline{(Tx)}(s_1)-s_2^{1-\alpha}\overline{(Tx)}(s_2)\right|\to 0\text{ unoformly as }s_1\to s_2$$

and for $s_2\leq s_1$ and $s_1,s_2\in(t_1,t_2]$, we have

$$\left|(s_1-t_1)^{1-\alpha}\overline{(Tx)}(s_1)-(s_2-t_1)^{1-\alpha}\overline{(Tx)}(s_2)\right|\to 0\text{ unoformly as }s_1\to s_2.$$

Then $t \to (t-t_i)^{1-\alpha}T(\overline{\Omega})$ is equi-continuous on $(t_i, t_{i+1}](i=0,1)$. So $T(\overline{\Omega})$ is relatively compact. Then T is completely continuous. The proofs are completed. □

3.4 Preliminary for BVP(1.8)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.8).

Lemma 3.3. *Suppose that $E_{\alpha,1}(\lambda)-1 \neq 0$ and $\sigma:(0,1)\mapsto \mathbb{R}$ is continuous and satisfies that there exist numbers $k>-1$ and $l \le 0$ with $l \in (\max\{-\alpha,-\alpha-k\},0]$ such that $|\sigma(t)| \le t^k(1-t)^l$ for all $t\in(0,1)$. The x is a solutions of*

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) - \lambda x(t) = \sigma(t), t \in (t_i, t_{i+1}], i = 0,1, \\ x(1) - \lim_{t\to 0} x(t) = a_0, \quad \lim_{t\to t_1^+} x(t) - x(t_1) = I_0 \end{cases} \tag{3.4.1}$$

if and only if x and

$$x(t) = \begin{cases} \frac{E_{\alpha,1}(\lambda t^{\alpha})}{E_{\alpha,1}(\lambda)-1}\left[a_0 - E_{\alpha,1}(\lambda(1-t_1)^{\alpha})I_0 - \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds\right] \\ + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t\in(0,t_1], \\ \frac{E_{\alpha,1}(\lambda t^{\alpha})}{E_{\alpha,1}(\lambda)-1}\left[a_0 - E_{\alpha,1}(\lambda(1-t_1)^{\alpha})I_0 - \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds\right] \\ + E_{\alpha,1}(\lambda(t-t_1)^{\alpha})I_0 + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t\in(t_1,1]. \end{cases} \tag{3.4.2}$$

Proof. Let x be a solution of (3.4.1). We know by Theorem 3.10 that there exist numbers $A_0, A_1 \in \mathbb{R}$ such that

$$x(t) = A_0 E_{\alpha,1}(\lambda t^{\alpha}) + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t\in(t_0,t_1] \tag{3.4.3}$$

and

$$x(t) = A_0 E_{\alpha,1}(\lambda t^{\alpha}) + A_1 E_{\alpha,1}(\lambda(t-t_1)^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds, t\in(t_1,t_2]. \tag{3.4.4}$$

Note $E_{\alpha,1}(0)=1$. It follows from (3.4.3), (3.4.4), the boundary conditions and the impulse assumption in (3.4.1) that

$$A_0E_{\alpha,1}(\lambda) + A_1E_{\alpha,1}(\lambda(1-t_1)^{\alpha}) + \int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds - A_0 = a_0,$$

$$A_1 = I_0.$$

Then

$$A_0 = \frac{1}{E_{\alpha,1}(\lambda)-1}\left[a_0 - E_{\alpha,1}(\lambda(1-t_1)^{\alpha})I_0 - \int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds\right]. \tag{3.4.5}$$

Substituting A_0, A_1 into (3.4.3) and (3.4.4), we get (3.3.2) obviously.

On the other hand, if x satisfies (3.4.2), then both $x|_{(0,t_1]}$ and $x|_{(t_1,1]}$ are continuous and the limits $\lim_{t\to 0} x(t)$ and $\lim_{t\to t_1} x(t)$ exist. So $x\in P_1C(0,1]$. Using (3.4.5) and $A_1 = I_0$, we rewrite x by

$$x(t) = \begin{cases} A_0E_{\alpha,1}(\lambda t^{\alpha}) + \int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds, t\in(0,t_1], \\ A_0E_{\alpha,1}(\lambda t^{\alpha}) + A_1E_{\alpha,1}(\lambda(t-t_1)^{\alpha}) + \int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds, t\in(t_1,1]. \end{cases}$$

Since σ is continuous on $(0,1)$ and $|\sigma(t)| \le t^k(1-t)^l$, one can show easily that x is continuous on $(t_i,t_{i+1}](i=0,1)$ and using the method at the beginning of the proof of this lemma, we know that both the limits $\lim_{t\to 0^+} x(t)$ and $\lim_{t\to t_1^+} x(t)$ exist. So $x\in P_1C(0,1]$. Furthermore, by direct computation, we have $x(1) - \lim_{t\to 0} x(t) = a_0$, and

$\lim_{t\to t_1^+} x(t) - x(t_1) = I_0$. One have easily from Theorem 3.2.1 for $t \in (t_0, t_1]$ that ${}^C D_{0^+}^{\alpha} x(t) = \lambda x(t) + \sigma(t)$ and for $t \in (t_1, t_2]$ that

$$
\begin{aligned}
&{}^C D_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_1} (t-s)^{-\alpha} \left(A_0 E_{\alpha,1}(\lambda s^{\alpha}) + \int_0^s (s-w)^{\alpha-1} E_{\alpha,\alpha}(\lambda (s-w)^{\alpha}) \sigma(w) dw\right)' ds \\
&+ \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^t (t-s)^{-\alpha} \left(A_0 E_{\alpha,1}(\lambda s^{\alpha}) + A_1 E_{\alpha,1}(\lambda (s-t_1)^{\alpha}) + \int_0^s (s-w)^{\alpha-1} E_{\alpha,\alpha}(\lambda (s-w)^{\alpha}) \sigma(w) dw\right)' ds \\
&= \frac{A_0}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left(E_{\alpha,1}(\lambda s^{\alpha})\right)' ds + \frac{A_1}{\Gamma(1-\alpha)} \int_{t_1}^t (t-s)^{-\alpha} \left(E_{\alpha,1}(\lambda (s-t_1)^{\alpha})\right)' ds \\
&+ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left(\int_0^s (s-w)^{\alpha-1} E_{\alpha,\alpha}(\lambda (s-w)^{\alpha}) \sigma(w) dw\right)' ds \\
&= \frac{A_0}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left(\sum_{j=0}^{+\infty} \frac{\lambda^j}{\Gamma(j\alpha+1)} s^{j\alpha}\right)' ds + \frac{A_1}{\Gamma(1-\alpha)} \int_{t_1}^t (t-s)^{-\alpha} \left(\sum_{j=0}^{+\infty} \frac{\lambda^j}{\Gamma(j\alpha+1)} (s-t_1)^{j\alpha}\right)' ds \\
&+ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left(\int_0^s (s-w)^{\alpha-1} \sum_{j=0}^{+\infty} \frac{\lambda^j}{\Gamma(\alpha(j+1))} (s-w)^{j\alpha} \sigma(w) dw\right)' ds \\
&= \frac{A_0}{\Gamma(1-\alpha)} \sum_{j=1}^{+\infty} \frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)} \int_0^t (t-s)^{-\alpha} s^{j\alpha-1} ds + \frac{A_1}{\Gamma(1-\alpha)} \sum_{j=1}^{+\infty} \frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)} \int_{t_1}^t (t-s)^{-\alpha} (s-t_1)^{j\alpha-1} ds \\
&+ \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{+\infty} \frac{\lambda^j}{\Gamma(\alpha(j+1))} \int_0^t (t-s)^{-\alpha} \left(\int_0^s (s-w)^{\alpha-1} (s-w)^{j\alpha} \sigma(w) dw\right)' ds \\
&= \frac{A_0}{\Gamma(1-\alpha)} \sum_{j=1}^{+\infty} \frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)} t^{\alpha j-\alpha} \int_0^1 (1-w)^{-\alpha} w^{j\alpha-1} dw + \frac{A_1}{\Gamma(1-\alpha)} \sum_{j=1}^{+\infty} \frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)} (t-t_1)^{\alpha j-\alpha} \int_0^1 (1-u)^{-\alpha} u^{j\alpha-1} du \\
&+ \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{+\infty} \frac{\lambda^j}{\Gamma(\alpha(j+1))} \Gamma(j\alpha+\alpha) \int_0^t (t-s)^{-\alpha} \left(I_{0^+}^{\alpha j+\alpha} \sigma(s)\right)' ds
\end{aligned}
$$

One has by $\mathbf{B}(1-\alpha, j\alpha) = \frac{\Gamma(1-\alpha)\Gamma(j\alpha)}{\Gamma((j-1)\alpha+1)}$ that

$$
\begin{aligned}
{}^C D_{0^+}^{\alpha} x(t) &= \frac{A_0}{\Gamma(1-\alpha)} \sum_{j=1}^{+\infty} \frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)} t^{\alpha j-\alpha} \frac{\Gamma(1-\alpha)\Gamma(j\alpha)}{\Gamma((j-1)\alpha+1)} \\
&+ \frac{A_1}{\Gamma(1-\alpha)} \sum_{j=1}^{+\infty} \frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)} (t-t_1)^{\alpha j-\alpha} \frac{\Gamma(1-\alpha)\Gamma(j\alpha)}{\Gamma((j-1)\alpha+1)} + \sum_{j=0}^{+\infty} \lambda^j {}^C D_{0^+}^{\alpha} \left(I_{0^+}^{\alpha j+\alpha} \sigma(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda A_0 E_{\alpha,1}(\lambda t^{\alpha}) + \lambda A_1 E_{\alpha,1}(\lambda (t-t_1)^{\alpha}) + \sigma(t) + \sum_{j=1}^{+\infty} \lambda^j I_{0^+}^{\alpha j} \sigma(t) \\
&= \lambda A_0 E_{\alpha,1}(\lambda t^{\alpha}) + \lambda A_1 E_{\alpha,1}(\lambda (t-t_1)^{\alpha}) + \sigma(t) + \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) \sigma(s) ds \\
&= \lambda x(t) + \sigma(t).
\end{aligned}
$$

So x is a solution of (3.4.1). The proof is completed. □

Define the nonlinear operator Q on $P_1C(0,1]$ by Qx for $x \in P_1C(0,1]$ with

$$(Qx)(t) = \begin{cases} \frac{E_{\alpha,1}(\lambda t^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[\int_0^1 \phi(s)G_x(s)ds - E_{\alpha,1}(\lambda(1-t_1)^\alpha)I_x(t_1) - \int_0^1 \delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\right] \\ + \int_0^t \delta_{\alpha,\lambda}(t,s)p(s)f_x(s)ds, t \in (0,t_1], \\ \frac{E_{\alpha,1}(\lambda t^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[\int_0^1 \phi(s)G_x(s)ds - E_{\alpha,1}(\lambda(1-t_1)^\alpha)I_x(t_1) - \int_0^1 \delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\right] \\ + E_{\alpha,1}(\lambda(t-t_1)^\alpha)I_0 + \int_0^t \delta_{\alpha,\lambda}(t,s)p(s)f_x(s)ds, t \in (t_1,1]. \end{cases}$$

Lemma 3.4. *Suppose that (b), (c), (f)-(g) hold, $E_{\alpha,1}(\lambda) - 1 \neq 0$ and f, G are* **impulsive I-Carathéodory functions**, *I a* **discrete I-Carathódory function**. *Then $x \in P_1C(),1]$ is a solution of BVP(1.0.8) if and only if $x \in P_1C(0,1]$ is a fixed point of Q, $Q : P_1C(0,1] \to P_1C(0,1]$ is well defined and is completely continuous.*

Proof. The proof is similar to that of Lemma 3.2 and is omitted. □

3.5 Preliminary for BVP(1.9)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.9). For ease expression, denote

$$\varrho_{\alpha,\lambda}(t,s) = \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right),$$

$$\Lambda_1 = -1 + \Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1) + \Gamma(\alpha)^2\varrho_{\alpha,\lambda}(t_1,1)\varrho_{\alpha,\lambda}(e,1).$$

Lemma 3.5. *Suppose that $\Lambda_1 \neq 0$ and $\sigma : (0,1) \mapsto \mathbb{R}$ is continuous and satisfies that there exist numbers $k > -1$ and $l \leq 0$ with $l > \max\{-\alpha, -k-1\}$ such that $|\sigma(t)| \leq (\log t)^k (1-\log t)^l$ for all $t \in (1,e)$. The x is a solutions of*

$$\begin{cases} {}^{RLH}D_{1^+}^\alpha x(t) - \lambda x(t) = \sigma(t), t \in (t_i, t_{i+1}], i = 0,1, \\ x(e) - \lim\limits_{t\to 1^+} (\log t)^{1-\alpha} x(t) = a_0, \ \lim\limits_{t\to t_1^+}\left(\log\frac{t}{t_1}\right)^{1-\alpha} x(t) - x(t_1) = I_0 \end{cases} \tag{3.5.1}$$

if and only if $x \in LPC_{1-\alpha}(1,e]$ and

$$x(t) = \begin{cases} \frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)}{\Lambda_1}\left[a_0 - \Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_0 - \int_1^e \varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right. \\ \left. + \Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right] + \int_0^t \varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s}, t \in (1,t_1], \\ \frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)}{\Lambda_1}\left[a_0 - \Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_0 - \int_1^e \varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right. \\ \left. + \Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right] \\ + \frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,t_1)}{\Lambda_1}\left[\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)a_0 - (1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))I_0\right. \\ \left. - \Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)\int_1^e \varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s} + (1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right] \\ + \int_0^t \varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s}, t \in (1,e]. \end{cases} \tag{3.5.2}$$

Proof. Let x be a solution of (3.5.1). We know from Theorem 3.12 that there exist numbers $A_0, A_1 \in R$ such that

$$x(t) = A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1) + \int_1^t \varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s}, t \in (t_0,t_1] \tag{3.5.3}$$

and

$$x(t) = A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1) + A_1\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,t_1) + \int_1^t \varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s}, t \in (t_1,t_2]. \tag{3.5.4}$$

Note $E_{\alpha,\alpha}(0)=\frac{1}{\Gamma(\alpha)}$. It follows from (3.5.3), (3.5.4), the boundary conditions and the impulse assumption in (3.5.1) that

$$(-1+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\,A_0+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)A_1=a_0-\int_1^e\varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s},$$

$$-\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)A_0+A_1=I_0-\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}.$$

Then

$$A_0=\frac{1}{\Lambda_1}\left[a_0-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_0-\int_1^e\varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right],$$

$$A_1=\frac{1}{\Lambda_1}\left[\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)a_0-(1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\,I_0\right.\tag{3.5.5}$$

$$\left.-\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)\int_1^e\varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}+(1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right].$$

Substituting A_0, A_1 into (3.5.3) and (3.5.4), we get (3.5.2) obviously.

On the other hand, if x satisfies (3.5.2), then $x|_{(1,t_1]}$ and $x|_{(t_1,e]}$ are continuous and the limits $\lim\limits_{t\to1^+}(\log t)^{1-\alpha}x(t)$ and $\lim\limits_{t\to t_1^+}(\log t-\log t_1)^{1-\alpha}x(t)$ exist. So $x\in LP_1C_{1-\alpha}(1,e]$. Using (3.5.5), we rewrite x by

$$x(t)=\begin{cases}A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)+\int_1^t\varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s},\,t\in(0,t_1],\\[2mm] A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)+A_1\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,t_1)+\int_1^t\varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s},\,t\in(t_1,1].\end{cases}$$

Furthermore, by direct computation, we have $x(e)-\lim\limits_{t\to1^+}(\log t)^{1-\alpha}x(t)=a_0$, and $\lim\limits_{t\to t_1^+}(\log t-\log t_1)^{1-\alpha}x(t)-x(t_1)=I_0$. One have from Theorem 3.2.3 easily for $t\in(t_0,t_1]$ that ${}^{RLH}D_{0^+}^{\alpha}x(t)=\lambda x(t)+\sigma(t)$ for $t\in(t_0,t_1]$. For $t\in(t_1,t_2]$, we have by Definition 2.5 that

$${}^{RLH}D_{1^+}^{\alpha}x(t)=\frac{t}{\Gamma(1-\alpha)}\left(\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}x(s)\frac{ds}{s}\right)'$$

$$=\frac{t}{\Gamma(1-\alpha)}\left[\int_1^{t_1}\left(\log\frac{t}{s}\right)^{-\alpha}\left(A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(s,1)+\int_1^s\varrho_{\alpha,\lambda}(s,u)\sigma(u)\frac{du}{u}\right)\frac{ds}{s}\right.$$

$$\left.+\int_{t_1}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(s,1)+A_1\Gamma(\alpha)\varrho_{\alpha,\lambda}(s,t_1)+\int_1^s\varrho_{\alpha,\lambda}(s,u)\sigma(u)\frac{du}{u}\right)\frac{ds}{s}\right]'$$

$$=\frac{t}{\Gamma(1-\alpha)}\left[\Gamma(\alpha)A_0\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,1)\frac{ds}{s}+\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\int_1^s\varrho_{\alpha,\lambda}(s,u)\sigma(u)\frac{du}{u}\frac{ds}{s}\right.$$

$$\left.+\Gamma(\alpha)A_1\int_{t_1}^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,t_1)\frac{ds}{s}\right]'$$

$$=\frac{t}{\Gamma(1-\alpha)}\left[\Gamma(\alpha)A_0\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,1)\frac{ds}{s}+\int_1^t\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,u)\frac{ds}{s}\sigma(u)\frac{du}{u}\right.$$

$$\left.+\Gamma(\alpha)A_1\int_{t_1}^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,t_1)\frac{ds}{s}\right]'$$

One finds that

$$\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,u)\frac{ds}{s}=\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{u}\right)^{\alpha-1}\sum_{j=0}^{+\infty}\frac{\lambda^j(\log s-\log u)^{j\alpha}}{\Gamma((j+1)\alpha)}\frac{ds}{s}$$

$$=\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{u}\right)^{\alpha j+\alpha-1}\frac{ds}{s}\text{ by }\frac{logs-\log u}{\log t-\log u}=w$$

$$=\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\left(\log\frac{t}{u}\right)^{\alpha j}\int_0^1(1-w)^{-\alpha}w^{\alpha j+\alpha-1}dw$$

$$=\Gamma(1-\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{u}\right)^{\alpha j}\text{ by }\mathbf{B}(1-\alpha,\alpha(j+1))=\frac{\Gamma(1-\alpha)\Gamma(\alpha(j+1))}{\Gamma(\alpha j-1))}.$$

So

$$^{RLH}D_{1^+}^{\alpha}x(t)=\frac{t}{\Gamma(1-\alpha)}\left[\Gamma(\alpha)A_0\Gamma(1-\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}(\log t)^{\alpha j}\right.$$

$$\left.+\int_1^t\Gamma(1-\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{u}\right)^{\alpha j}\sigma(u)\frac{du}{u}+\Gamma(\alpha)A_1\Gamma(1-\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{t_1}\right)^{\alpha j}\right]'$$

$$=t\left[\frac{\Gamma(\alpha)A_0}{t}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}(\log t)^{\alpha j-1}+\int_1^t\sigma(u)\frac{du}{u}+\frac{1}{t}\int_1^t\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{u}\right)^{\alpha j-1}\sigma(u)\frac{du}{u}\right.$$

$$\left.+\frac{\Gamma(\alpha)A_1}{t}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{t_1}\right)^{\alpha j-1}\right]=\lambda x(t)+\sigma(t).$$

So x is a solution of (3.5.1). The proof is completed. □

Define the nonlinear operator R on $LP_1C_{1-\alpha}(0,1]$ for $x\in LP_1C_{1-\alpha}(0,1]$ by $(Rx)(t)$ by

$$(Rx)(t)=\begin{cases}\frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)}{\Lambda_1}\left[\int_0^1\phi(s)G_x(s)ds-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_x(t_1)-\int_1^e\varrho_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}\right.\\ \left.+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)p(s)f_x(s)\frac{ds}{s}\right]\\ \quad+\int_1^t\varrho_{\alpha,\lambda}(t,s)p(s)f_x(s)\frac{ds}{s},t\in(1,t_1],\\ \frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)}{\Lambda_1}\left[\int_0^1\phi(s)G_x(s)ds-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_x(t_1)-\int_1^e\varrho_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}\right.\\ \left.+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)p(s)f_x(s)\frac{ds}{s}\right]\\ +\frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,t_1)}{\Lambda_1}\left[\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)\int_0^1\phi(s)G_x(s)ds-(1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))I_x(t_1)\right.\\ \left.-\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)\int_1^e\varrho_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}+(1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)p(s)f_x(s)\frac{ds}{s}\right]\\ \quad+\int_1^t\varrho_{\alpha,\lambda}(t,s)p(s)f_x(s)\frac{ds}{s},t\in(1,e].\end{cases}$$

Lemma 3.6. *Suppose that (h), (i) and (j) hold,* $\Lambda_1\neq 0$*, and* f,G *are* **impulsive III-Carathéodory functions***,* I **a discrete III-Carathódory function***. Then* $R:LP_1C_{1-\alpha}(1,e]\to LP_1C_{1-\alpha}(1,e]$ *is well defined and is completely continuous.*

Proof. The proof is similar to that of the proof of Lemma 3.2 and is omitted. □

3.6 Preliminary for BVP(1.10)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.10).

Lemma 3.7. *Suppose that* $E_{\alpha,1}(\lambda)-1\neq 0$ *and* $\sigma:(0,1)\mapsto\mathbb{R}$ *is continuous and satisfies that there exist numbers* $k>-1$ *and* $l\leq 0$ *with* $l\in(\max\{-\alpha,-\alpha-k\},0]$ *such that* $|\sigma(t)|\leq(\log t)^k(1-\log t)^l$ *for all* $t\in(0,1)$*. The x is a solutions of*

$$\begin{cases}^{CH}D_{1^+}^{\alpha}x(t)-\lambda x(t)=\sigma(t),t\in(t_i,t_{i+1}],i=0,1,\\ x(e)-\lim_{t\to 1^+}x(t)=a_0,\quad\lim_{t\to t_1^+}x(t)-x(t_1)=I_0\end{cases}\tag{3.6.1}$$

if and only if $x \in PC(1,e]$ *and*

$$x(t)=\begin{cases}\frac{E_{\alpha,1}(\lambda(\log t)^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[a_0-E_{\alpha,1}(\lambda(1-\log t_1)^\alpha)I_0-\int_1^e\delta_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right]\\ \quad+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)\sigma(s)\frac{ds}{s},t\in(0,t_1],\\ \frac{E_{\alpha,1}(\lambda(\log t)^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[a_0-E_{\alpha,1}(\lambda(1-\log t_1)^\alpha)I_0-\int_1^e\delta_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right]\\ +E_{\alpha,1}\left(\lambda\left(\log\frac{t}{t_1}\right)^\alpha\right)I_0+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)\sigma(s)\frac{ds}{s},t\in(t_1,e].\end{cases}\quad(3.6.2)$$

Proof. Let x be a solution of (3.6.1). We know by Theorem 3.13 that there exist numbers $A_0,A_1\in\mathbb{R}$ such that

$$x(t)=A_0E_{\alpha,1}(\lambda(\log t)^\alpha)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)\sigma(s)\frac{ds}{s},t\in(t_0,t_1]\quad(3.6.3)$$

$$x(t)=A_0E_{\alpha,1}(\lambda(\log t)^\alpha)+A_1E_{\alpha,1}(\lambda(\log t-\log t_1)^\alpha)+\int_1^t\delta_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s},t\in(t_1,t_2].\quad(3.6.4)$$

Note $E_{\alpha,1}(0)=1$. It follows from (3.6.3), (3.6.4), the boundary conditions and the impulse assumption in (3.6.1) that

$$A_0E_{\alpha,1}(\lambda)+A_1E_{\alpha,1}(\lambda(1-\log t_1)^\alpha)+\int_1^e\delta_{\alpha,\lambda}(e,s)\sigma(s)ds-A_0=a_0,\ \ A_1=I_0.$$

Then

$$A_0=\frac{1}{E_{\alpha,1}(\lambda)-1}\left[a_0-E_{\alpha,1}(\lambda(1-\log t_1)^\alpha)I_0-\int_1^e\delta_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right].\quad(3.6.5)$$

Substituting A_0,A_1 into (3.6.3) and (3.6.4), we get (3.6.2) obviously.

On the other hand, if x satisfies (3.6.2), then $x|_{(1,t_1]}$ and $x|_{(t_1,1]}$ are continuous and the limits $\lim_{t\to1^+}x(t)$ and $\lim_{t\to t_1^+}x(t)$ exist. So $x\in P_1C(1,e]$. Using (3.6.5) and $A_1=I_0$, we rewrite x by

$$x(t)=\begin{cases}A_0E_{\alpha,1}(\lambda(\log t)^\alpha)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)\sigma(s)\frac{ds}{s},t\in(1,t_1],\\ A_0E_{\alpha,1}(\lambda(\log t)^\alpha)+A_1E_{\alpha,1}(\lambda(\log t-\log t_1)^\alpha)+\int_1^t\delta_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s},t\in(t_1,e].\end{cases}$$

One have from Theorem 3.2.4 easily for $t\in(t_0,t_1]$ that ${}^{CH}D_{1^+}^\alpha x(t)=\lambda x(t)+\sigma(t)$ and for $t\in(t_1,t_2]$ that

$$\begin{aligned}&{}^{CH}D_{1^+}^\alpha x(t)=\frac{1}{\Gamma(1-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}x'(s)\frac{ds}{s}\\ &=\frac{1}{\Gamma(1-\alpha)}\int_1^{t_1}\left(\log\frac{t}{s}\right)^{-\alpha}\left(A_0E_{\alpha,1}(\lambda(\log s)^\alpha)+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right)\sigma(u)\frac{du}{u}\right)'\frac{ds}{s}\\ &+\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(A_0E_{\alpha,1}(\lambda(\log s)^\alpha)+A_1E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_1}\right)^\alpha\right)+\int_1^s\delta_{\alpha,\lambda}(s,u)\sigma(u)\frac{du}{u}\right)'\frac{ds}{s}\\ &=\lambda x(t)+\sigma(t).\end{aligned}$$

So x is a solution of (3.6.1). The proof is completed. □

Define the nonlinear operator J on $LP_1C(1,e]$ by (Jx) by

$$(Jx)(t)=\begin{cases}\frac{E_{\alpha,1}(\lambda(\log t)^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[\int_0^1\phi(s)G_x(s)ds-E_{\alpha,1}(\lambda(1-\log t_1)^\alpha)I_x(t_1)-\int_1^e\delta_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}\right]\\ \quad+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)p(s)f_x(s)\frac{ds}{s},t\in(0,t_1],\\ \frac{E_{\alpha,1}(\lambda(\log t)^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[\int_0^1\phi(s)G_x(s)ds-E_{\alpha,1}(\lambda(1-\log t_1)^\alpha)I_x(t_1)-\int_1^e\delta_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}\right]\\ +E_{\alpha,1}\left(\lambda\left(\log\frac{t}{t_1}\right)^\alpha\right)I_x(t_1)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)p(s)f_x(s)\frac{ds}{s},t\in(t_1,e].\end{cases}$$

Lemma 3.8. *Suppose that (k), (l) and (m) hold,* $E_{\alpha,1}(\lambda)-1\neq0$*, and* f,G *are* **impulsive I-Carathéodory functions***,* I *a* **discrete I-Carathódory function***. Then* $R:LP_1C(1,e]\to LP_1C(1,e]$ *is well defined and is completely continuous.*

Proof. The proof is similar to that of the proof of Lemma3.2 and is omitted. □

4 Solvability of BVP(1.7)-BVP(1.10)

Now, we prove that main theorems in this paper by using the Schaefer's fixed point theorem [57].

(B1) there exists nonnegative a constant I_0, nondecreasing functions $b, B, \overline{B} : [0,+\infty) \mapsto \mathbb{R}$, bounded continuous functions $\phi, \psi : (0,1) \mapsto \mathbb{R}$ such that

$$|f(t,(t-t_i)^{\alpha-1}x)-\phi(t)| \le b(|x|), t\in(t_i,t_{i+1}], i=0,1, x\in\mathbb{R},$$

$$|G(t,(t-t_i)^{\alpha-1}x)-\psi(t)| \le B(|x|), t\in(t_i,t_{i+1}], i=0,1, x\in\mathbb{R},$$

$$|I(t_1,t_1^{\alpha-1}x)-I_0| \le \overline{B}(|x|), x\in\mathbb{R}.$$

Let

$$\Phi(t)=\begin{cases}\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[\int_0^1\phi(s)\psi(s)ds-\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_0\ -\int_0^1\delta_{\alpha,\lambda}(1,s)p(s)\phi(s)ds\\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)\phi(s)ds\Big]+\int_0^t\delta_{\alpha,\lambda}(t,s)p(s)\phi(s))ds, t\in(0,t_1],\\ \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[\int_0^1\phi(s)\psi(s)ds-\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_0\ -\int_0^1\delta_{\alpha,\lambda}(1,s)p(s)\phi(s)ds\\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)\phi(s)ds\Big]+\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_1)}{\Lambda}\Big[\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)\int_0^1\phi(s)\psi(s)ds\\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1)I_0\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)\int_0^1\delta_{\alpha,\lambda}(1,s)p(s)\phi(s)ds\\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)\phi(s)ds\Big]+\int_0^t\delta_{\alpha,\lambda}(t,s)p(s)\phi(s)ds, t\in(t_1,1].\end{cases}$$

Theorem 4.14. *Suppose that (a)-(e), (B1) hold, $\Lambda\neq 0$. Then BVP(1.7) has at least one solution if there exists a $r_0>0$ such that*

$$\frac{A_1B(r_0+||\Phi||)+A_2\overline{B}(r_0+||\Phi||)+A_3b(r_0+||\Phi||)}{r_0}<1, \tag{4.1}$$

where

$$A_1=\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1}{|\Lambda|t_1^{1-\alpha}},$$

$$A_2=\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|(1-t_1)^{1-\alpha}}+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|},$$

$$A_3=\Bigg(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3t_1^{\alpha-1}}{|\Lambda|}$$
$$+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)^2t_1^{\alpha+k+l}}{|\Lambda|}+E_{\alpha,\alpha}(|\lambda|)\Bigg)\mathbf{B}(\alpha+l,k+1).$$

Proof. From Lemma 3.1, Lemma 3.2, the definition of T, $x\in P_1C_{1-\alpha}(0,1]$ is a solution of BVP(1.7) if and only if $x\in P_1C_{1-\alpha}(0,1]$ is a fixed point of T in $P_1C_{1-\alpha}(0,1]$. Lemma 3.2 implies that T is a completely continuous operator.

For $r>0$, denote $\Omega=\{x\in P_1C_{1-\alpha}(0,1] : ||x-\Phi||\le r\}$. For $x\in\Omega$, we get $||x||\le||x-\Phi||+||\Phi||\le r+||\Phi||$ Then (B1) implies that

$$|f(t,x(t))-\phi(t)|=|f(t,(t-t_i)^{\alpha-1}(t-t_i)^{1-\alpha}x(t))-\phi(t)|\le b(|(t-t_i)^{1-\alpha}x(t)|)$$

$$\le b(||x||)\le b(r+||\Phi||), t\in(t_i,t_{i+1}], i=0,1,$$

$$|G(t,x(t))-\psi(t)|\le B(||x||)\le B(r+||\Phi||), t\in(t_i,t_{i+1}], i=0,1,$$

$$|I(t_1,x(t_1))-I_0|\le\overline{B}(||x||)\le\overline{B}(r+||\Phi||), t\in(0,1).$$

By the definition of T and the method used in **Step (iii1)** in the proof of Lemma 3.2, we have

$$||Tx|| \leq \left[\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda|} + \frac{\Gamma(\alpha)^2 E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1}{|\Lambda|t_1^{1-\alpha}}\right] B(r+||\Phi||)$$

$$+\left[\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|(1-t_1)^{1-\alpha}} + \frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\right]\overline{B}(r+||\Phi||)$$

$$+\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3t_1^{\alpha-1}}{|\Lambda|}\right.$$

$$\left.+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)^2t_1^{\alpha+k+l}}{|\Lambda|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(r+||\Phi||)$$

$$= A_1B(r+||\Phi||) + A_2\overline{B}(r+||\Phi||) + A_3b(r+||\Phi||).$$

From (4.1), there exists a constant $r_0 > 0$ such that $A_1B(r_0+||\Phi||) + A_2\overline{B}(r_0+||\Phi||) + A_3b(r_0+||\Phi||) < r_0$. Choose $\Omega = \{x \in P_1C_{1-\alpha}(0,1] : ||x-\Phi|| \leq r_0\}$. For $x \in \partial\Omega$, we see easily that $x =\neq \lambda(Tx)$ for all $\lambda \in [0,1]$. In fact, if $x = Tx$ for some $x \in \partial\Omega$ and $\lambda \in [0,1]$, then $r_0 = ||x|| = \lambda||Tx|| \leq ||Tx|| \leq A_1B(r_0+||\Phi||) + A_2\overline{B}(r_0+||\Phi||) + A_3b(r_0+||\Phi||) < r_0$, a contradiction. So Lemma 3.9 implies that T has at least one fixed point in Ω. Then BVP(1.7) has at least one solution. The proof is complete. □

Theorem 4.15. *Suppose that $\Lambda \neq 0$, $\theta \geq 0$, $|\psi(t)| \leq B_\psi$, $|\phi(t)| \leq b_\phi$ for all $t \in (0,1)$ and $b(x) = b_1x^\theta$, $B(x) = B_1x^\theta$ and $\overline{B}(x) = \overline{B}_1x^\theta$ in (B1). Then BVP(1.7) has at least one solution if one of the following item holds:*

(i) $\theta \in [0,1)$;

(ii) $\theta = 1$ *with* $A_1B + A_2\overline{B}r + A_3b < 1$;

(iii) $\theta > 1$ *with* $(A_1B_1 + A_2\overline{B}_1 + A_3b_1)(A_1B_\psi + A_2|I_0| + A_3b_\phi) \leq \frac{(\theta-1)^{\theta-1}}{\theta^\theta}$.

Proof. It is easy to see that $\Phi \in P_1C_{1-\alpha}(0,1]$. By using the method in **Step (iii2)** in the proof of Lemma 3.2, we get $||\Phi|| \leq A_1B_\psi + A_2|I_0| + A_3b_\phi$. By Theorem 4.1, we know that BVP(1.0.7) has at least one solution if there exists $r_0 > 0$ such that (4.1) holds.

When $\theta \in [0,1)$, we have

$$\inf_{r\in(0,+\infty)} \frac{A_1B(r+||\Phi||)+A_2\overline{B}(r+||\Phi||)+A_3b(r+||\Phi||)}{r}$$

$$= \inf_{r\in(0,+\infty)} \frac{A_1B_1[r+||\Phi||]^\theta+A_2\overline{B}_1[r+||\Phi||]^\theta+A_3b_1[r+||\Phi||]^\theta}{r} = 0 < 1.$$

Then Theorem 4.1 implies that BVP(1.7) has at least one solution.

When $\theta = 1$, we have from $A_1B + A_2\overline{B}r + A_3b < 1$ that

$$\inf_{r\in(0,+\infty)} \frac{A_1B[r+||\Phi||]+A_2\overline{B}[r+||\Phi||]+A_3b[r+||\Phi||]}{r} < 1.$$

Then there $r_0 > 0$ such that (4.1) holds. Then Theorem 4.1 implies that BVP(1.7) has at least one solution.

When $\theta > 1$, it is easy to see that $(A_1B_1 + A_2\overline{B}_1 + A_3b_1)(A_1B_\psi + A_2|I_0| + A_3b_\phi) \leq \frac{(\theta-1)^{\theta-1}}{\theta^\theta}$ implies that $(A_1B_1 + A_2\overline{B}_1 + A_3b_1)||\Phi|| \leq \frac{(\theta-1)^{\theta-1}}{\theta^\theta}$. Choose $r_0 = \frac{||\Phi||}{\theta-1}$. It is easy to check that

$$\frac{A_1B_1(r_0+||\Phi||)^\theta+A_2\overline{B}_1(r_0+||\Phi||)^\theta+A_3b_1(r_0+||\Phi||)^\theta}{r_0} \leq 1.$$

Then Theorem 4.1 implies that BVP(1.7) has at least one solution. The proof of Theorem 4.2 is complete. □

Remark 4.1. **(i)** *When $\alpha \in (0,1)$, $G(t,x) \equiv 0$, and replace $I(t_1,x)$ in (1.0.7) by $I(x)-x$, we see that BVP(1.7) becomes BVP(1.3). According Theorem 4.2, BVP(1.3) has at leat one solution if both f and I are bounded.*

(ii) *When $\alpha \in (0,1)$, one chooses $G(t,x) \equiv 0$, $f(t,x) = 1+t^2+(t-t_i)^{1-\alpha}x$ for $t \in (t_i,t_{i+1}](i=0,1)$, $0 = t_0 < t_1 = \frac{1}{2} < t_2 = 1$, and $I(t_1,x) \equiv 0$, then BVP(1.7) becomes BVP(1.3) with $I(x) = x$. According Theorem 4.2, BVP(1.3) has at least one solution. But The results in [69] can not be applied.*

Theorem 4.16. *Suppose that (b), (c), (f), (g) hold, $E_{\alpha,1}(\lambda)-1\neq 0$, and*

(B2) *there exist nondecreasing functions $b, B, \overline{B} : [0,+\infty)\mapsto \mathbb{R}$ such that*

$$|f(t,x)|\le b(|x|), t\in(0,1), x\in\mathbb{R},$$

$$|G(t,x)|\le B(|x|), t\in(0,1), x\in\mathbb{R},$$

$$|I(t_1,x)|\le \overline{B}(|x|), x\in\mathbb{R}.$$

Then BVP(1.8) has at least one solution if there exists $r_0>0$ such that

$$\begin{aligned}&\tfrac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(r_0)+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,1}(|\lambda|)\right)\overline{B}(r_0)\\&+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(r_0)<r_0.\end{aligned}\tag{4.2}$$

Proof. From Lemma 3.3, Lemma 3.4 and the definition of Q, $x\in P_1C(0,1]$ is a solution of BVP(1.8) if and only if $x\in P_1C(0,1]$ is a fixed point of Q. Lemma 3.4 implies that Q is a completely continuous operator. From (B2), we have for $x\in P_1C(0,1]$ that

$$|f(t,x(t))|\le b(|x(t)|)\le b(||x||), t\in(0,1),$$

$$|G(t,x(t))|\le B(||x||), t\in(0,1),$$

$$|I(t_1,x(t_1))|\le \overline{B}(||x||).$$

We consider the set $\Omega=\{x\in P_1C(0,1]: x=\lambda(Tx),\text{ for some }\lambda\in[0,1]\}$. For $x\in\Omega$, we have for $t\in(t_0,t_1]$ that

$$\begin{aligned}|(Qx)(t)|\le{}&\tfrac{E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}\Big[||\phi||_1B(||x||)+E_{\alpha,1}(|\lambda|)\overline{B}(||x||)\\&+\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-s)^{\alpha v+\alpha-1}s^k(1-s)^l ds\, b(||x||)\Big]\\&+\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^t(t-s)^{\alpha v+\alpha-1}s^k(1-s)^l ds\, b(||x||)\\&\le\tfrac{E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}\Big[||\phi||_1B(||x||)+E_{\alpha,1}(|\lambda|)\overline{B}(||x||)\\&+E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)\Big]+E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)\\&=\tfrac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||)+\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}\overline{B}(||x||)\\&+\left(\tfrac{E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}E_{\alpha,\alpha}(|\lambda|)+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).\end{aligned}$$

For $t\in(t_1,t_2]$, one has that

$$\begin{aligned}|(Qx)(t)|\le{}&\tfrac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||)+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,1}(|\lambda|)\right)\overline{B}(||x||)\\&+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).\end{aligned}$$

It follows that

$$\begin{aligned}||x||=\lambda||Tx||\le||Tx||\le{}&\tfrac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||)+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,1}(|\lambda|)\right)\overline{B}(||x||)\\&+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).\end{aligned}$$

From (4.2), we choose $\Omega = \{x \in P_1C(),1] : ||x|| \leq r_0\}$. For $x \in \Omega$, we get $x \neq \lambda(Tx)$ for any $\lambda \in [0,1]$ and $x \in \partial\Omega$.

As a consequence of Schaefer's fixed point theorem, we deduce that Q has a fixed point which is a solution of the problem BVP(1.8). The proof is completed. The proof of Theorem 4.3 is complete. □

Theorem 4.17. *Suppose that (h), (i), (j) hold, $\Lambda_1 \neq 0$, and*

(B3) *there exist nondecreasing functions $b, B, \overline{B} : [0,+\infty) \mapsto \mathbb{R}$ such that*

$$|f(t,t^{\alpha-1}x)| \leq b(|x|), t \in (t_i,t_{i+1}], i = 0,1, x \in \mathbb{R},$$

$$|G(t,t^{\alpha-1}x)| \leq B(|x|), t \in (t_i,t_{i1}], i = 0,1, x \in \mathbb{R},$$

$$|I(t_1,t_1^{\alpha-1}x)| \leq \overline{B}(|x|), x \in \mathbb{R}.$$

Then BVP(1.9) has at least one solution if there exists a constant $r_0 > 0$ such that

$$B_1B(r_0) + B_2\overline{B}(r_0) + B_3b(r_0) < r_0, \tag{4.3}$$

where

$$B_1 = \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda_1|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1(\log t_1)^{\alpha-1}}{|\Lambda_1|},$$

$$B_2 = \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|} + \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|}{|\Lambda_1|},$$

$$B_3 = \left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha+k+l}}{|\Lambda_1|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha-1}}{|\Lambda_1|}\right.$$
$$\left.+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|(\log t_1)^{\alpha+k+l}}{|\Lambda_1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1).$$

Proof. From Lemma 3.5, Lemma 3.6 and the definition of R, $x \in LP_1C_{1-\alpha}(1,e]$ is a solution of BVP(1.9) if and only if $x \in LP_1C_{1-\alpha}(1,e]$ is a fixed point of R. Lemma 3.6 implies that R is a completely continuous operator.

From (B3), we have for $x \in LP_1C_{1-\alpha}(1,e]$ that

$$|f(t,x(t))| = \left|f\left(t,\left(\log\tfrac{t}{s}\right)^{\alpha-1}\left(\log\tfrac{t}{s}\right)^{1-\alpha}x(t)\right)\right| \leq b\left(\left|\left(\log\tfrac{t}{s}\right)^{-\alpha}x(t)\right|\right) \leq b(||x||), t \in (t_i,t_{i+1}], i = 0,1,$$

$$|G(t,x(t))| \leq B(||x||), t \in (1,e),$$

$$|I(t_1,x(t_1))| \leq \overline{B}(||x||).$$

We consider the set $\Omega = \{x \in LP_1C_{1-\alpha}(0,1] : x = \lambda(Rx)$, for some $\lambda \in [0,1]\}$. For $x \in \Omega$, we have for $t \in (t_0,t_1]$ that

$$\left|(\log t)^{1-\alpha}(Rx)(t)\right| \leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}\left(\lambda(\log t)^{\alpha}\right)}{|\Lambda_1|}\Big[||\phi||_1B(||x||) + \Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\overline{B}(||x||)$$
$$+\int_1^e\left(\log\tfrac{e}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\tfrac{e}{s}\right)^{\alpha}\right)(\log s)^k(1-\log s)^l\tfrac{ds}{s}b(||x||)$$
$$+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t_1}{s}\right)^{\alpha}\right)(\log s)^k(1-\log s)^l\tfrac{ds}{s}b(||x||)\Big]$$
$$+(\log t)^{1-\alpha}\int_1^t\left(\log\tfrac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\tfrac{t}{s}\right)^{\alpha}\right)(\log s)^k(1-\log s)^l\tfrac{ds}{s}b(||x||)$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda_1|}\left[||\phi||_1 B(||x||)+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\overline{B}(||x||)\right.$$

$$+\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)$$

$$\left.+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)\right]$$

$$+(\log t)^{1-\alpha}\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda_1|}\left[||\phi||_1 B(||x||)+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\overline{B}(||x||)\right.$$

$$+\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-w)^{\alpha v+\alpha+l-1}w^k dw b(||x||)$$

$$\left.+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}(\log t_1)^{\alpha v+\alpha+k+l}\int_0^1(1-w)^{\alpha v+\alpha+l-1}w^k dw b(||x||)\right]$$

$$+(\log t)^{1-\alpha}\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}(\log t)^{\alpha v+\alpha+k+l}\int_0^1(1-w)^{\alpha v+\alpha+l-1}w^k dw b(||x||)$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda_1|}B(||x||)+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|}\overline{B}(||x||)$$

$$+\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha+k+l}}{|\Lambda_1|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).$$

For $t\in(t_1,t_2]$, one has that

$$\left|\left(\log\frac{t}{t_1}\right)^{1-\alpha}(Rx)(t)\right|\leq\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda_1|}\left[||\phi||_1B(||x||)+\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)\overline{B}(||x||)\right.$$

$$\left.+E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)+\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2(\log t_1)^{\alpha+k+l}\mathbf{B}(\alpha+l,k+1)\right]$$

$$+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda_1|}\left[\Gamma(\alpha)(\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)||\phi||_1B(||x||)+\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|\overline{B}(||x||)\right.$$

$$+\Gamma(\alpha)(\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)^2\mathbf{B}(\alpha+l,k+1)b(||x||)$$

$$\left.+\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|(\log t_1)^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)\right]$$

$$+\left(\log\frac{t}{t_1}\right)^{1-\alpha}(\log t)^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)$$

$$\leq\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda_1|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1(\log t_1)^{\alpha-1}}{|\Lambda_1|}\right)B(||x||)$$

$$+\left(\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|}+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|}{|\Lambda_1|}\right)\overline{B}(||x||)$$

$$+\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha+k+l}}{|\Lambda_1|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha-1}}{|\Lambda_1|}\right.$$

$$\left.+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|(\log t_1)^{\alpha+k+l}}{|\Lambda_1|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||)$$

$$=B_1B(||x||)+B_2\overline{B}(||x||)+B_3b(||x||).$$

It follows that
$$||x|| = \lambda||Rx|| \le ||Rx|| \le B_1B(||x||) + B_2\overline{B}(||x||) + B_3b(||x||).$$
From (4.3), we choose $\Omega = \{x \in LP_1C_{1-\alpha}(0,1] : ||x|| \le r_0\}$. For $x \in \partial\Omega$, we get $x \ne \lambda(Rx)$ for any $\lambda \in [0,1]$. In fact, if there exists $x \in \partial\Omega$ such that $x = \lambda(Rx)$ for some $\lambda \in [0,1]$. Then $r_0 = ||x|| = \lambda||Rx|| \le ||Rx|| \le B_1B(r_0) + B_2\overline{B}(r_0) + B_3b(r_0) < r_0$, a contradiction.

As a consequence of Schaefer's fixed point theorem, we deduce that R has a fixed point which is a solution of the problem BVP(1.9). The proof is completed. The proof of Theorem 4.4 is complete. □

Theorem 4.18. *Suppose that (k), (l), (m) hold, $E_{\alpha,1}(\lambda) - 1 \ne 0$, and*

(B4) *there exist nondecreasing functions $b, B, \overline{B} : [0,+\infty) \mapsto \mathbb{R}$ such that*
$$|f(t,x)| \le b(|x|), t \in (t_i, t_{i+1}], i = 0,1, x \in \mathbb{R},$$
$$|G(t,x)| \le B(|x|), t \in (t_i, t_{i1}], i = 0,1, x \in \mathbb{R},$$
$$|I(t_1,x)| \le \overline{B}(|x|), x \in \mathbb{R}.$$

Then BVP(1.10) has at least one solution if there exists a constant $r_0 > 0$ such that
$$\frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(r_0) + \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,1}(|\lambda|)\right)\overline{B}(r_0)$$
$$+ \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(r_0) < r_0. \tag{4.4}$$

Proof. From Lemma 3.7, Lemma 3.8 and the definition of J, $x \in LP_1C(1,e]$ is a solution of BVP(1.10) if and only if $x \in LP_1C(1,e]$ is a fixed point of R. Lemma 3.8 implies that J is a completely continuous operator.

From (B4), we have for $x \in LP_1C(1,e]$ that
$$|f(t,x(t))| \le b(|x(t)|) \le b(||x||), t \in (t_i, t_{i+1}], i = 0,1,$$
$$|G(t,x(t))| \le B(||x||), t \in (1,e),$$
$$|I(t_1,x(t_1))| \le \overline{B}(||x||).$$
We consider the set $\Omega = \{x \in LP_1C(0,1] : x = \lambda(Jx),$ for some $\lambda \in [0,1]\}$. For $x \in \Omega$, we have for $t \in (t_0,t_1]$ that
$$|(Jx)(t)| \le \frac{E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}\Big[||\phi||_1B(||x||) + E_{\alpha,1}(|\lambda|)\overline{B}(||x||)$$
$$+ \sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)\Big]$$
$$+ \sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)$$
$$\le \frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||) + \frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|}\overline{B}(||x||)$$
$$+ \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||)$$
For $t \in (t_1,t_2]$, one has that
$$|(Jx)(t)| \le \frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||) + \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,1}(|\lambda|)\right)\overline{B}(||x||)$$
$$+ \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).$$

It follows that

$$||x|| = \lambda||Rx|| \leq ||Rx|| \leq \frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||) + \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,1}(|\lambda|)\right)\overline{B}(||x||)$$

$$+ \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).$$

From (4.4), we choose $\Omega = \{x \in LP_1C(0,1] : ||x|| \leq r_0\}$. For $x \in \partial\Omega$, we get $x \neq \lambda(Jx)$ for any $\lambda \in [0,1]$. In fact, if there exists $x \in \partial\Omega$ such that $x = \lambda(Jx)$ for some $\lambda \in [0,1]$. Then

$$r_0 = ||x|| = \lambda||Jx|| \leq ||Jx|| \leq \frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(r_0) + \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,1}(|\lambda|)\right)\overline{B}(r_0)$$

$$+ \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(r_0) < r_0,$$

a contradiction.

As a consequence of Schaefer's fixed point theorem, we deduce that R has a fixed point which is a solution of the problem BVP(1.10). The proof is completed. The proof of Theorem 4.5 is complete. □

5 Applications

In [33, 78, 89], authors studied the existence and uniqueness of solutions of BVP(1.1). It was proved in [33] that if f is a jointly continuous function and there is a constant $\overline{\lambda} \in \left[0, 1-\frac{1}{p}\right]$ for some $p \in \left(1, \frac{1}{1-q}\right]$ and $L > 0$ such that $|f(t,x)| \leq L(1+|x|^{\overline{\lambda}}]$ for each $t \in [0,T]$ and $x \in R$. Then BVP(1.1) has at least one solution. It seems that the solvability of BVP(1.1) is not related to the impulse function I_k. For periodic boundary value problem, this is not true. For example, the following problem

$${}^CD_{0^+}^{\frac{1}{2}}x(t) = 1, t \in (t_i, t_{i+1}], i \in N_0,\ x(0) = x(T), \Delta x(t_i) = I_i > 0, i \in \mathbb{N}_0,$$

has no solution. In fact, from Theorem 3.10 and ${}^CD_{0^+}^{\frac{1}{2}}x(t) = 1, t \in (t_i, t_{i+1}]$, we get that there exist constants $c_0(i \in \mathbb{N}_0)$ such that $x(t) = \sum_{j=0}^{i} c_j + \frac{2t^{1/2}}{\Gamma(1/2)}$. By $x(0) = x(T)$, we know $c_0 = \sum_{j=0}^{m} c_j + \frac{2T^{1/2}}{\Gamma(1/2)}$. From $\Delta x(t_i) = I_i$, we see that $c_i = I_i$. Thus $\sum_{j=1}^{m} I_j + \frac{2T^{1/2}}{\Gamma(1/2)} = 0$, a contradiction.

Consider the following periodic boundary value problem of fractional differential equation

$$\begin{cases} {}^CD_{0^+}^{\alpha}x(t) = p(t)f(t,x(t)), t \in (t_i, t_{i+1}], i = 0, 1, \\ \Delta x(t_1) = b_1x(t_1) + I(t_1, x(t_1)), \\ x(0) = x(T) + \int_0^T \phi(s)G(s,x(s))ds, \end{cases} \tag{5.1}$$

where $t_0 = 0 < t_1 < t_2 = T$, $b_1 \in R$ with $b_1 \neq 0$, $\phi : (0,T) \mapsto \mathbb{R}$ with $\phi \in L^1(0,T)$, $p : (0,T) \mapsto \mathbb{R}$ is continuous and there exist numbers $k > -1, l \in (-\alpha, -\alpha-k, 0]$ such that $|p(t) \leq t^k(T-t)^l$ for all $t \in (0,T)$, $f : (0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is a II-Carathéodory function, $I : \{t_1,\}, \times\mathbb{R} \mapsto \mathbb{R}$ is a **discrete II-Carathéodory function**.

If x is a solution of (5.1), then by Theorem 3.1, we see that there exist constants c_0, c_1 such that

$$x(t) = \sum_{j=0}^{i} c_i + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t \in (t_i, t_{i+1}], i = 0, 1.$$

By $x(0) = x(T) + \int_0^T \phi(s)G(s,x(s))ds$, we get $c_0 = c_0 + c_1 + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds + \int_0^T \phi(s)G(s,x(s))ds$. So $c_1 = -\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds - \int_0^T \phi(s)G(s,x(s))ds$. By $\Delta x(t_1) = b_1x(t_1) + I_1(t_1,x(t_1))$, we get $c_1 = b_1\left(c_0 + \int_0^{t_1} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\right) + I(t_1,x(t_1))$. The $c_0 = -\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds - \frac{1}{b_1}I(t_1,x(t_1)) -$

$\frac{1}{b_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds - \frac{1}{b_1}\int_0^T \phi(s)G(s,x(s))ds$. So

$$x(t)=\begin{cases} -\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-\frac{1}{b_1}I(t_1,x(t_1))-\frac{1}{b_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\\ -\frac{1}{b_1}\int_0^T\phi(s)G(s,x(s))ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in(0,t_1],\\ -\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-\frac{1}{b_1}I(t_1,x(t_1))-\left(1+\frac{1}{b_1}\right)\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\\ -\left(1+\frac{1}{b_1}\right)\int_0^T\phi(s)G(s,x(s))ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in(t_1,T].\end{cases}$$

It is easy to show that if x satisfies above integral equation, then x is a solution of (5.1).

Define the operator $T: P_1C(0,T]\mapsto P_1C(0,T]$ by

$$(Tx)(t)=\begin{cases} -\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-\frac{1}{b_1}I(t_1,x(t_1))-\frac{1}{b_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\\ -\frac{1}{b_1}\int_0^T\phi(s)G(s,x(s))ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in(0,t_1],\\ -\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-\frac{1}{b_1}I(t_1,x(t_1))-\left(1+\frac{1}{b_1}\right)\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\\ -\left(1+\frac{1}{b_1}\right)\int_0^T\phi(s)G(s,x(s))ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in(t_1,T].\end{cases}$$

Theorem 5.19. *$T: P_1C(0,T]\mapsto P_1C(0,T]$ is well defined and is completely continuous. x is a solution of mentioned problem (5.1) if and only if x is a fixed point of T in $P_1C(0,T]$.*

Proof. It follows from Theorem 3.10 and the details are omitted. □

Theorem 5.20. *Suppose that there exist nondecreasing functions $b, B, \overline{B}: [0,+\infty)\mapsto \mathbb{R}$ such that*

$$|f(t,x)|\le b(|x|), t\in(0,1), x\in R,$$

$$|G(t,x)|\le B(|x|), t\in(0,1), x\in R,$$

$$|I(t_1,x)|\le \overline{B}(|x|), x\in R.$$

Then problem (5.1) has at least one solution if there exists $r_0>0$ such that

$$\left(1+\frac{||\phi||_1}{|b_1|}\right)B(r_0)+\frac{1}{|b_1|}\overline{B}(r_0) + \left(\frac{t_1^{\alpha+k+l}}{\Gamma(\alpha)}+\left(1+\frac{1}{|b_1|}\right)\frac{T^{\alpha+k+l}}{\Gamma(\alpha)}+\frac{T^{\alpha+k+l}}{\Gamma(\alpha)}\right)\mathbf{B}(\alpha+l,k+1)b(r_0)<r_0. \tag{5.2}$$

Proof. In fact, for $x\in P_1C(0,T]$, we have $|f(t,x(t))|\le b(||x||), |G(t,x(t))|\le B(|x||)$ and $|I(t_1,x(t_1))|\le \overline{B}(||x||)$ for all $t\in(0,T]$. Then

$$||Tx||\le\left(1+\frac{||\phi||_1}{|b_1|}\right)B(||x||)+\frac{1}{|b_1|}\overline{B}(||x||) + \left(\frac{t_1^{\alpha+k+l}}{\Gamma(\alpha)}+\left(1+\frac{1}{|b_1|}\right)\frac{T^{\alpha+k+l}}{\Gamma(\alpha)}+\frac{T^{\alpha+k+l}}{\Gamma(\alpha)}\right)\mathbf{B}(\alpha+l,k+1)b(||x||).$$

The remainder of the proof is similar to the proof of Theorem 4.3 and is omitted. □

Example 5.1. *Consider the following problem*

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t)-\lambda x(t)=t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}(t^2+\arctan x(t)), t\in(t_i,t_{i+1}], i=0,1,\\ x(1)=\lim_{t\to0^+}t^{1-\alpha}x(t),\\ \lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t)-x(t_1)=0,\end{cases} \tag{5.3}$$

where $\alpha \in (0,1)$, $0 = t_0 < \frac{1}{2} = t_1 < t_2 = 1$, $\lambda \in R$ are fixed constant. Corresponding to BVP(1.7), we have $\phi(t) \equiv 0$, $G(t,x) \equiv 0$ and $I(t_1,x) \equiv 0$.

It is easy to see that (a), (b) and (c) hold. $p(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}$ satisfies (d) with $k = -\frac{1}{4}, l = -\frac{1}{8}$. $f(t,x) = t^2 + \arctan x$. One sees that f, G, I satisfy (e). Choose $\phi(t) = t^2$, $\psi(t) = 0$ and $I_0 = 0$. Then (B1) holds with $B(x) = \overline{B}(x) = 0$ and $b(x) = \frac{\pi}{2}$. Thus by Theorem 4.1, we know BVP(5.3) has at least one solution since there exists a constant $r_0 > 0$ such that (4.1) holds obviously.

According to the results in [69], BVP(5.3) can not be solved since the nonlinearity $p(t)f(t,x) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}(t^2 + \arctan x)$ is unbounded and the impulse function $I(x) = x$ is also unbounded.

Example 5.2. *Consider the following BVP*

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) - \lambda x(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}(t^2 + \sqrt[3]{(t-t_i)^{1-\alpha}x(t)}), t \in (t_i, t_{i+1}], i = 0, 1, \\ x(1) - \lim\limits_{t \to 0^+} t^{1-\alpha}x(t) = \int_0^1 s^{-\frac{1}{2}}(1-s)^{-\frac{1}{6}}[s + \sqrt{(t-t_i)^{1-\alpha}|x(s)|}]ds, \\ \lim\limits_{t \to t_1^+} (t-t_1)^{1-\alpha}x(t) - x(t_1) = 8 + \sqrt[5]{t_1^{1-\alpha}x(t_1)}, \end{cases} \tag{5.4}$$

where $\alpha \in (0,1)$, $0 = t_0 < \frac{1}{2} = t_1 < t_2 = 1$, $\lambda \in R$ are fixed constant.

Corresponding to BVP(1.7), we have $\phi(t) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{6}}$, $G(t,x) = t + \sqrt{(t-t_i)^{1-\alpha}|x|}$ and $I(t_1,x) = 8 + \sqrt[5]{x}$. It is easy to see that (a), (b) and (c) hold.$p(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}$ satisfies (d) with $k = -\frac{1}{4}, l = -\frac{1}{8}$. $f(t,x) = t^2 + \sqrt[3]{(t-t_i)^{1-\alpha}x}$. One sees that f, G, I satisfy (e). Choose $\phi(t) = t^2$, $\psi(t) = t$ and $I_0 = 8$. Then (B1) holds with $B(x) = \sqrt{|x|}, \overline{B}(x) = \sqrt[5]{x}$ and $b(x) = \sqrt[3]{x}$. Thus by Theorem 4.1, we know BVP(5.4) has at least one solution since there exists a constant $r_0 > 0$ such that (4.1) holds obviously.

Example 5.3. *Consider the following BVP*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}\sqrt[3]{x(t)}, t \in (t_i, t_{i+1}], i = 0, 1, \\ x(0) - x(1) = \int_0^1 s^{-\frac{1}{2}}(1-s)^{-\frac{1}{6}}\sqrt[3]{x(s)}ds, \\ \Delta x(t_1) = x(t_1^+) - x(t_1) = x(t_1) + 8, \end{cases} \tag{5.5}$$

where $\alpha \in (0,1)$, $0 = t_0 < \frac{1}{2} = t_1 < t_2 = 1$ are fixed constant.

Corresponding to BVP(5.1), we have $\phi(t) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{6}}$, $G(t,x) = \sqrt[3]{x}$ and $I(t_1,x) = 8$. It is easy to see that (a), (b) and (c) hold.$p(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}$ satisfies (d) with $k = -\frac{1}{4}, l = -\frac{1}{8}$. $f(t,x) = \sqrt[3]{x}$. One sees that f, G, I satisfy (e). Then (B1) holds with $B(x) = \sqrt[3]{|x|}, \overline{B}(x) = 8$ and $b(x) = \sqrt[3]{x}$. Thus by Theorem 5.0.2, we know BVP(5.5) has at least one solution since there exists a constant $r_0 > 0$ such that

$$\left(1 + \frac{\mathbf{B}(1/2, 5/6)}{|b_1|}\right)\sqrt[3]{r_0} + \frac{8}{|b_1|} + \left(\frac{t_1^{\alpha+k+l}}{\Gamma(\alpha)} + \left(2 + \frac{1}{|b_1|}\right)\frac{1}{\Gamma(\alpha)}\right)\mathbf{B}(\alpha + l, k+1)\sqrt[3]{r_0} < r_0.$$

holds obviously.

Example 5.4. *Consider the following periodic boundary value problem*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = r(t), t \in (t_i, t_{i+1}], i \in N_0, \\ x(0) - x(T) = 0, \\ \Delta x(t_i) = x(t_i^+) - x(t_1) = b_i x(t_i), i \in N, \end{cases} \tag{5.6}$$

where $0 = t_0, < t_1 < \cdots < t_m < t_{m+1} = T$, $\alpha \in (0,1)$, $r : (0,T) \mapsto \mathbb{R}$ satisfies that there exist constants $k > -1$ and $l \in (-\alpha, -\alpha - k, 0]$ such that $|r(t)| \le t^k(T-t)^l$ for all $t \in (0,T)$, $b_i \in \mathbb{R}(i \in \mathbb{N})$. Then BVP(5.6) has a unique solution if and only if $\sum\limits_{i=1}^{m} b_i \prod\limits_{j=1}^{i-1}(1 + b_j) \ne 0$.

By Theorem 3.10, we see that there exist constants $c_i (i \in \mathbb{N}_0)$ such that

$$x(t) = \sum_{j=0}^{i} c_i + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds, t \in (t_i, t_{i+1}], i \in N_0.$$

By $x(0) = x(T)$, we get

$$c_0 = \sum_{j=0}^{m} c_i + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds.$$

So

$$\sum_{j=1}^{m} c_j = -\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds. \tag{5.7}$$

By $\Delta x(t_i) = b_i x(t_1)$, we get

$$c_i = b_i \left(\sum_{j=0}^{i-1} c_j + \int_0^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds \right).$$

Then

$$\frac{c_1}{b_1} = c_0 + \int_{t_0}^{t_1} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds,$$

$$\frac{c_i}{b_i} = \sum_{j=1}^{i-1} b_j \frac{c_j}{b_j} + c_0 + \int_0^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds, i \in N.$$

Thus

$$\frac{c_i}{b_i} = \prod_{j=1}^{i-1}(1+b_j)c_0 + \sum_{j=0}^{i} \prod_{v=j}^{i-1}(1+b_v) \int_{t_{j-1}}^{t_j} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds.$$

Substituting c_i into (5.7), we get

$$\sum_{i=1}^{m} b_i \left(\prod_{j=1}^{i-1}(1+b_j)c_0 + \sum_{j=0}^{i} \prod_{v=j}^{i-1}(1+b_v) \int_{t_{j-1}}^{t_j} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds \right) = -\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds.$$

It is easy to see that if (5.6) has a unique solution if and only if $\sum_{i=1}^{m} b_i \prod_{j=1}^{i-1}(1+b_j) \neq 0$.

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Some results for the Jacobi-Dunkl transform in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$

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Abstract

In this paper, using a generalized Jacobi-Dunkl translation operator, we prove an analog of Titchmarsh's theorem for functions satisfying the Jacobi-Dunkl Lipschitz condition in $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$, $\alpha \geq \beta \geq \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$.

Keywords: Jacobi-Dunkl operator, Jacobi-Dunkl transform, generalized Jacobi-Dunkl translation.

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1 Introduction

Titchmarsh's [[10], Theorem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

Theorem 1.1. *[10] Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following statements are equivalents:*

(a) $\|f(t+h) - f(t)\| = 0(h^\alpha), \quad as \quad h \to 0,$

(b) $\displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha}), \quad as \quad r \to \infty,$

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we prove in analog of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the Jacobi-Dunkl Lipschitz condition in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$. For this purpose, we use the generalized translation operator. Similar results have been established in the context of non compact rank one Riemannian symetric spaces [9].

In section 2 below, we recapitulate from [[1],[2],[3],[5]] some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$. Section 3 is devoted to the main result after defining the class $Lip(\delta, 2, \alpha, \beta)$ of functions in $L^2_{\alpha,\beta}(\mathbb{R})$ satisfying the Lipschitz condition correspondent to the generalized Jacobi-Dunkl translation.

2 Notation and Preliminaries

The Jacobi-Dunkl function with parameters (α, β), $\alpha \geq \beta \geq \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$, is defined by the formula:

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda}\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) & \text{if } \lambda \in \mathbb{C}\backslash\{0\}, \\ 1 & \text{if } \lambda = 0, \end{cases}$$

with $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$ and $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function given by:

$$\varphi_\mu^{\alpha,\beta}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}, \alpha + 1, -(\sinh x)^2\right),$$

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where F is the Gausse hypergeometric function (see [1],[6] and [7]).
$\psi_\lambda^{\alpha,\beta}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}\mathcal{U} = i\lambda\mathcal{U}, \quad \lambda \in \mathbb{C}, \\ \mathcal{U}(0) = 1, \end{cases}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta}\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \times \frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}].$$

The operator $\Lambda_{\alpha,\beta}$ is a particular case of the operator D given by

$$D\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + \frac{A'(x)}{A(x)} \times \left(\frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}\right),$$

where $A(x) = |x|^{2\alpha+1}B(x)$, and B a function of class C^∞ on $\mathbb{R}$, even and positive. The operator $\Lambda_{\alpha,\beta}$ corresponds to the function

$$A(x) = A_{\alpha,\beta}(x) = 2^\rho (\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

Using the relation

$$\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) = -\frac{\mu^2+\rho^2}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function $\psi_\lambda^{\alpha,\beta}$ can be written in the form below (see [2])

$$\psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i\frac{\lambda}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x), \quad x \in \mathbb{R},$$

where $\lambda^2 = \mu^2 + \rho^2, \quad \rho = \alpha + \beta + 1$.
Denote $L^2_{\alpha,\beta}(\mathbb{R}) = L^2_{\alpha,\beta}(\mathbb{R}, A_{\alpha,\beta}(t)dt)$ the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{L^2_{\alpha,\beta}(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(t)|^2 A_{\alpha,\beta}(t)dt\right)^{1/2} < +\infty.$$

Using the eigenfunctions $\psi_\lambda^{\alpha,\beta}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform of a function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ by:

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = \int_{\mathbb{R}} f(t)\psi_\lambda^{\alpha,\beta}(t)A_{\alpha,\beta}(t)dt, \quad \lambda \in \mathbb{R},$$

and the inversion formula

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}f(\lambda)\psi_{-\lambda}^{\alpha,\beta}(t)d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|}\mathbb{I}_{\mathbb{R}\backslash]-\rho,\rho[}(\lambda)d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}, \quad \mu \in \mathbb{C}\backslash(i\mathbb{N}),$$

and $\mathbb{I}_{\mathbb{R}\backslash]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\backslash]-\rho,\rho[$.
Denote $L^2_\sigma(\mathbb{R}) = L^2(\mathbb{R}, d\sigma(\lambda))$.
The Jacobi-Dunkl transform is a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2_\sigma(\mathbb{R})$, i.e.,

$$\|f\| := \|f\|_{L^2_{\alpha,\beta}(\mathbb{R})} = \|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2_\sigma(\mathbb{R})}. \tag{2.1}$$

The operator of Jacobi-Dunkl translation is defined by:

$$T_x f(y) = \int_{\mathbb{R}} f(z) d\nu_{x,y}^{\alpha,\beta}(z), \quad \forall x, y \in \mathbb{R}, \tag{2.2}$$

where $\nu_{x,y}^{\alpha,\beta}(z), \quad x, y \in \mathbb{R}$, are the signed measures given by

$$d\nu_{x,y}^{\alpha,\beta}(z) = \begin{cases} K_{\alpha,\beta}(x,y,z) A_{\alpha,\beta}(z) dz & \text{if } x, y \in \mathbb{R}^*, \\ \delta_x & \text{if } y = 0, \\ \delta_y & \text{if } x = 0, \end{cases}$$

here, δ_x is the Dirac measure at x. And,

$$\begin{aligned} K_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha} \mathbb{I}_{I_{x,y}} \times \int_0^\pi \rho_\theta(x,y,z) \\ &\times (g_\theta(x,y,z))_+^{\alpha-\beta-1} \sin^{2\beta}\theta d\theta, \\ I_{x,y} &= [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|], \\ \rho_\theta(x,y,z) &= 1 - \sigma_{x,y,z}^\theta + \sigma_{z,x,y}^\theta + \sigma_{z,y,x}^\theta, \end{aligned}$$

$$\forall z \in \mathbb{R}, \theta \in [0,\pi], \sigma_{x,y,z}^\theta = \begin{cases} \frac{\cosh x + \cosh y - \cosh z \cos\theta}{\sinh x \sinh y}, & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0, \end{cases}$$

$$g_\theta(x,y,z) = 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2\cosh x \cosh y \cosh z \cos\theta,$$

$$t_+ = \begin{cases} t, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

and,

$$M_{\alpha,\beta} = \begin{cases} \frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} & \text{if } \alpha > \beta, \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

In [2], we have

$$\mathcal{F}_{\alpha,\beta}(T_h f)(\lambda) = \psi_\lambda^{\alpha,\beta}(h) \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad \lambda, h \in \mathbb{R}. \tag{2.3}$$

For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind defined by:

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

Moreover, we see that

$$\lim_{z \to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0,$$

by consequence, there exists $C_1 > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Rightarrow |j_\alpha(z) - 1| \geq C_1 |z|^2. \tag{2.4}$$

Lemma 2.1. *The following inequalities are valids for Jacobi functions* $\varphi_\mu^{\alpha,\beta}(t)$:
(c) $|\varphi_\mu^{\alpha,\beta}(t)| \leq 1$,
(d) $|1 - \varphi_\mu^{\alpha,\beta}(t)| \leq t^2(\mu^2 + \rho^2)$.
Proof. *(See [8], Lemma 3.1, Lemma 3.2).*

Lemma 2.2. *Let* $\alpha \geq \beta \geq \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$. *Then for* $|\nu| \leq \rho$, *there exists a positive constant* C_2 *such that*

$$|1 - \varphi_{\mu+i\nu}^{\alpha,\beta}(t)| \geq C_2 |1 - j_\alpha(\mu t)|.$$

Proof. *(See [4], Lemma 9).*

3 Main results

In this section we introduce and prove an analog of Theorem 1.1. Firstly we have to define, for functions in $L^2_{\alpha,\beta}(\mathbb{R})$, the conditions of Cauchy-Lipschitz related to the Jacobi-Dunkl translation operator given in 2.2.

Definition 3.1. *Let $\delta \in (0,1)$. A function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ is said to be in the Jacobi-Dunkl-Lipschitz class, denoted by $Lip(\delta,2,\alpha,\beta)$, if*

$$\|N_h\Lambda^m_{\alpha,\beta}f\| = O(h^\delta), \quad as \quad h \to 0,$$

where $N_h = T_h + T_{-h} - 2I$, I is the unit operator in the space $L^2_{\alpha,\beta}(\mathbb{R})$ and $m = 0,1,2,....$

Lemma 3.3. *For $f \in L^2_{\alpha,\beta}(\mathbb{R})$, then*

$$\|N_h\Lambda^m_{\alpha,\beta}f\|^2 = 4\int_{\mathbb{R}} \lambda^{2m}|\varphi^{\alpha,\beta}_\mu(h) - 1|^2|\mathcal{F}_{\alpha,\beta}f(\lambda)|^2 d\sigma(\lambda).$$

Proof. Since $\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}f)(\lambda) = i\lambda\mathcal{F}_{\alpha,\beta}(f)(\lambda)$, we have

$$\mathcal{F}_{\alpha,\beta}(\Lambda^m_{\alpha,\beta}f)(\lambda) = i^m\lambda^m\mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad m = 0,1,2,.... \tag{3.5}$$

We us formulas 2.3 and 3.5, we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h\Lambda^m_{\alpha,\beta}f)(\lambda) = i^m(\psi^{\alpha,\beta}_\lambda(h) + \psi^{\alpha,\beta}_\lambda(-h) - 2)\lambda^m\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

Since

$$\psi^{\alpha,\beta}_\lambda(h) = \varphi^{\alpha,\beta}_\mu(h) + i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi^{\alpha+1,\beta+1}_\mu(h),$$

$$\psi^{\alpha,\beta}_\lambda(-h) = \varphi^{\alpha,\beta}_\mu(-h) - i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi^{\alpha+1,\beta+1}_\mu(-h),$$

and $\varphi^{\alpha,\beta}_\mu$ is even (see [2]), then

$$\mathcal{F}_{\alpha,\beta}(N_h\Lambda^m_{\alpha,\beta}f)(\lambda) = 2i^m(\varphi^{\alpha,\beta}_\mu(h) - 1)\lambda^m\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

Now by Parseval's identity (formula 2.1), we have the result.

Theorem 3.1. *Let $f \in L^2_{\alpha,\beta}(\mathbb{R})$. Then the following statements are equivalents:*

(i) $f \in Lip(\delta,2,\alpha,\beta)$,

(ii) $\displaystyle\int_{|\lambda|\geq r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}), \quad as \quad r \to \infty.$

Proof. $(i) \Rightarrow (ii)$. Assume that $f \in Lip(\delta,2,\alpha,\beta)$, then we have

$$\|N_h\Lambda^m_{\alpha,\beta}f\| = O(h^\delta), \quad as \quad h \to 0.$$

From Lemma 3.3, we have

$$\|N_h\Lambda^m_{\alpha,\beta}f\|^2 = 4\int_{\mathbb{R}} \lambda^{2m}|1 - \varphi^{\alpha,\beta}_\mu(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

By 2.4 and Lemma 2.2, we get:

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{2m}|1 - \varphi^{\alpha,\beta}_\mu(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \geq C_1^2C_2^2\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |\mu h|^4|\lambda^{2m}\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

From $\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}$ we have

$$\left(\frac{\eta}{2h}\right)^2 - \rho^2 \leq \mu^2 \leq \left(\frac{\eta}{h}\right)^2 - \rho^2$$

$$\Rightarrow \mu^2h^2 \geq \frac{\eta^2}{4} - \rho^2h^2.$$

Take $h \leq \frac{\eta}{3\rho}$, then we have $\mu^2 h^2 \geq C_3 = C_3(\eta)$.
So,

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \geq C_1^2C_2^2C_3^2 \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

There exists then a positive constant C such that

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C\int_{\mathbb{R}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq Ch^{2\delta}.$$

For all $0 < h < \frac{\eta}{3\rho}$, then we have

$$\int_{r\leq|\lambda|\leq 2r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq Cr^{-2\delta}, \quad r\to\infty.$$

Furthermore, we obtain

$$\begin{aligned}\int_{|\lambda|\geq r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \sum_{i=0}^{\infty}\int_{2^ir\leq|\lambda|\leq 2^{i+1}r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda)\\ &\leq C\sum_{i=0}^{\infty}(2^ir)^{-2\delta}\\ &\leq Cr^{-2\delta}.\end{aligned}$$

This proves that

$$\int_{|\lambda|\geq r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}), \quad as \quad r\to\infty.$$

$(ii) \Rightarrow (i)$. Suppose now that

$$\int_{|\lambda|\geq r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}), \quad as \quad r\to\infty,$$

and write

$$\begin{aligned}\int_{\mathbb{R}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_{|\lambda|<\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda)\\ &+ \int_{|\lambda|\geq\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).\end{aligned}$$

Using the inequality (c) of Lemma 2.1, we get

$$\int_{|\lambda|\geq\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq 4\int_{|\lambda|\geq\frac{1}{h}} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

Then

$$\int_{|\lambda|\geq\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(h^{2\delta}), \quad as \quad h\to 0. \tag{3.6}$$

Set

$$\phi(\lambda) = \int_{\lambda}^{\infty} x^{2m}|\mathcal{F}_{\alpha,\beta}(f)(x)|^2 d\sigma(x).$$

An integration by parts gives:

$$\begin{aligned}\int_0^x \lambda^2\lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_0^x -\lambda^2\phi'(\lambda)d\lambda\\ &= -x^2\phi(x) + 2\int_0^x \lambda\phi(\lambda)d\lambda\\ &\leq 2\int_0^x O(\lambda^{1-2\delta})d\lambda\\ &= O(x^{2-2\delta}).\end{aligned}$$

From Lemma 2.1, we get

$$\begin{aligned}\int_{|\lambda|\leq\frac{1}{h}}\lambda^{2m}|1-\varphi_{\mu}^{\alpha,\beta}(h)|^{2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^{2}d\sigma(\lambda) &\leq \int_{|\lambda|\leq\frac{1}{h}}\lambda^{2m}|1-\varphi_{\mu}^{\alpha,\beta}(h)||\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^{2}d\sigma(\lambda)\\ &\leq \int_{|\lambda|\leq\frac{1}{h}}\lambda^{2m}(\mu^{2}+\rho^{2})h^{2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^{2}d\sigma(\lambda)\\ &\leq h^{2}\int_{|\lambda|\leq\frac{1}{h}}\lambda^{2}\lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^{2}d\sigma(\lambda)\\ &= O(h^{2}h^{-2+2\delta}).\end{aligned}$$

Hence,

$$\int_{|\lambda|\leq\frac{1}{h}}\lambda^{2m}|1-\varphi_{\mu}^{\alpha,\beta}(h)|^{2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^{2}d\sigma(\lambda)=O(h^{2\delta}). \tag{3.7}$$

Finally, we conclude from 3.6 and 3.7 that

$$\begin{aligned}\int_{\mathbb{R}}\lambda^{2m}|1-\varphi_{\mu}^{\alpha,\beta}(h)|^{2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^{2}d\sigma(\lambda) &= \int_{|\lambda|<\frac{1}{h}}+\int_{|\lambda|\geq\frac{1}{h}}\\ &= O(h^{2\delta})+O(h^{2\delta})\\ &= O(h^{2\delta}).\end{aligned}$$

And this ends the proof.

Corollary 3.1. *Let $f\in L^{2}_{\alpha,\beta}(\mathbb{R})$, and let*

$$\|N_{h}\Lambda^{m}_{\alpha,\beta}f\|=O(h^{\delta}),\quad as\quad h\to 0,$$

Then

$$\int_{|\lambda|\geq r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^{2}d\sigma(\lambda)=O(r^{-2m-2\delta}),\quad as\quad r\to\infty.$$

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Stability of traveling fronts in a population model with nonlocal delay and advection

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Abstract

In this paper, we are concerned with the stability of traveling fronts in a population model with nonlocal delay and advection under the large initial perturbation (i.e. the initial perturbation around the traveling wave decays exponentially as $x \to -\infty$, but it can be arbitrarily large in other locations). The globally exponential stability of traveling fronts is established by the weighted-energy method combining with comparison principle, including even the slower waves whose wave speed are close to the critical speed.

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1 Introduction

In this paper, we consider a population model with nonlocal delay and advection

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} - B\frac{\partial u}{\partial x} + d(u(t,x)) = \varepsilon \int_{-\infty}^{+\infty} b(u(t-r, x-y+Br))g(y)dy, \quad t > 0, x \in R \tag{1.1}$$

with the initial condition

$$u(s,x) = u_0(s,x), \quad s \in [-r, 0],\ x \in R, \tag{1.2}$$

which describes the population growth of a single-species population dynamics with two age classes and a fixed maturation period living in a spatial transport field(see [5, 24, 26]). Here, $u(t,x)$ denotes the total mature population in time $t \geq 0$ and at location $x \in R$, $D > 0$ is the diffusion rate for the mature, $\varepsilon > 0$ is the impact of the death rate of the immature. $r > 0$ is the mature age of the species, which is the so-called time delay, $\alpha > 0$ represents the effect of the dispersal rate of the immature and satisfies $\alpha \leq rD$, $B \in R$ is the velocity of the spatial transport field and $g(y)$ is the heat kernel

$$g(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{y^2}{4\alpha}} \ \mathit{with} \ \int_{-\infty}^{+\infty} g(y)dy = 1.$$

Lastly, $d(u)$ and $b(u)$ denote the death and birth rates of the mature, respectively, and satisfy the following hypotheses:

(A_1) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $\varepsilon b'(0) > d'(0) \geq 0$, $d(u_+) = \varepsilon b(u_+)$, and $0 \leq \varepsilon b'(u_+) < d'(u_+)$;

(A_2) For $0 \leq u \leq u_+$, $d''(u) \geq 0$, $b''(u) \leq 0$, and $d(u), b(u) \in C^2[0, u_+]$.

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Particularly, the birth function $b(u)$ can be taken as the following three important types

$$b_1(u) = pue^{-au^q}, \ \ b_2(u) = \frac{pu}{1+au^q}, \ \ or\ b_3(u) = \begin{cases} pu(1-\frac{u^q}{K^q}) & 0 \le u \le K \\ 0, & u > K \end{cases} \quad a,p,q>0, \tag{1.3}$$

where $K > 0$ represents the carrying capacity and $p > 0$ is the effect of varying the birth rate. When $q = 1$, $b_1(u)$ is the so-called Nicholson's birth function, $b_2(u)$ is the Monod function and $b_3(u)$ is the nonlinear term of Logistic model.

On the other hand, when $B \neq 0$, by taking the death rate of the mature population as $d(u) = \delta u,\ \delta > 0$, (1.1) reduces to the following reaction-advection-diffusion model with nonlocal delayed effect in [5]

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} - B\frac{\partial u}{\partial x} + \delta u = \varepsilon \int_{-\infty}^{+\infty} b(u(t-r, x-y+Br))g(y)dy, \quad t>0,\ x\in R. \tag{1.4}$$

When $B = 0$, (1.1) can be reduced to all kinds of reaction-diffusion models with nonlocal delay, such as So, Wu and Zou's age-structured population model(where $d(u) = \delta u, \delta > 0$, see[19]), Nicholson's blowflies model(where $d(u) = \delta u,\ \delta > 0,\ b(u) = b_1(u)$ with $q = 1$, see [12, 13, 15]), Al-Omari and Gourley's age-structured population model(where $d(u) = \delta u^2$ and $\varepsilon b(u) = \tilde{p}e^{-\gamma r}u,\ \tilde{p} > 0,\ \gamma > 0$, see [1]), and so on.

Recent years, there have been extensively investigations on the stability of traveling waves for various reaction-diffusion equations with nonlocal delay and the issue of the stability of traveling waves become more interesting and important, please refer to [2–5, 8, 9, 11–18, 21, 23, 24] and references therein. One of the most effective methods for the stability of monostable waves is the weighted energy method used and developed by Mei(see [11–16]). For example, Mei[13, 15] established the exponential stability of monostable waves for the Nicholson's blowflies equations with nonlocal delay by the weighted energy method combining with the comparison principle. Motivated by Mei's idea, Wu's[24] established the exponential stability of the traveling wavefronts in monostable reaction-advection-diffusion equations with nonlocal delay, but it is only for the faster waves(i.e. the noncritical speed). More recently, by the weighted energy method combining with the comparison principle and the Green function technique, Mei[15, 16] proved the stability for the noncritical waves including the slower waves whose wave speed are close to the critical speed and even for the critical waves of (1.1) when $B = 0$. As a result, here we are interested in the stability of traveling waves of the reaction-advection-diffusion equation (1.1) for all faster waves including those slow waves by the weighted energy method combining with the comparison principle, which recovers and improves Wu[24] stability results. By constructing a non-piecewise weighted function concerning with the critical speed [20, 27, 28] which is different from Mei[14, 16], the difficulty caused by the nonlocality can be overcome, some energy estimates in the weighted L^2 space is first established, and the energy estimates in the H^1 space is thus built up. Finally, we prove the globally exponential stability of traveling fronts of (1.1). The stability of critical waves is our pursuit in another subsequent paper [10].

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and state our main result on the stability of traveling fronts. In Section 3, we prove the main result after establishing the boundedness of solutions , the comparison principle and some energy estimates.

2 Preliminaries and Main Result

Throughout this paper, $C > 0$ denotes a generic constant, $C_i > 0\ (i = 1, 2, ...)$ represents a specific constant. Let I be an interval. $L^2(I)$ is the space of the square integrable functions defined on I, and $H^k(I)(k \ge 0)$ is the Sobolev space of the L^2-functions $h(x)$ defined on the interval I whose derivatives $\frac{d^i}{dx^i}h(x)\ (i = 1, 2, ..., k)$ also belong to $L^2(I)$. $L^p_w(I)$ denotes the weighted L^p-space with a weight function $w(x) > 0$ and its norm is defined by

$$\|h\|_{L^p_w} = \left(\int_I w(x)|h(x)|^p dx\right)^{\frac{1}{p}},$$

$H^k_w(I)$ is the weighted Sobolev space with the norm given by

$$\|h\|_{H^k_w} = \left(\sum_{i=0}^{k}\int_I w(x)|\frac{d^i}{dx^i}h(x)|^2 dx\right)^{\frac{1}{2}}.$$

Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0,T];\mathcal{B})$ the space of the $\mathcal{B}-$valued continuous functions on $[0,T]$. $L^2([0,T];\mathcal{B})$ is the space of the $\mathcal{B}-$valued L^2-functions on $[0,T]$. The corresponding spaces of $\mathcal{B}-$valued functions on $[0,\infty)$ are defined similarly.

From (A_1), it is not difficult to verify that (1.1) has two constant equilibria $u_\pm$ and from (A_2) we know that u_- is unstable and u_+ is stable. A traveling wavefront of (1.1) connecting with u_- and u_+ is a monotone solution in the form of $u(t,x) = \phi(x+ct)$ with a speed c and satisfies the following differential equation

$$\begin{cases} (c-B)\phi'(\xi) - D\phi''(\xi) + d(\phi) = \varepsilon \int_{-\infty}^{+\infty} b(\phi(\xi - y + (B-c)r))g(y)dy, \\ \phi(\pm\infty) = u_\pm, \end{cases} \tag{2.5}$$

where $\xi = x + ct, \phi' = \frac{d\phi}{d\xi}$.

By using the monotone iteration technique as well as the upper and lower solution method as in [5–7, 21, 22], the existence of traveling wavefronts of (1.1) can be obtained in a similar way as follows.

Proposition 2.1. (Existence of traveling wavefronts) (see[21, 22, 24].) *Assume that (A_1)-(A_2) hold. Then there exist a minimum wave speed (is also called the critical wave speed) $\widetilde{c}_* = c_* + B$, where $c_* = c_*(\gamma,\alpha,\varepsilon,D,d'(0),b'(0)) \in (0, 2\sqrt{D(\varepsilon b'(0) - d'(0)})$ is the the critical wave speed of traveling wavefront of (1.1) when $B = 0$, and a corresponding number $\lambda_* = \lambda_*(\widetilde{c}_*) > 0$ satisfying*

$$\Delta(\lambda_*, c_* + B) = 0, \quad \frac{\partial}{\partial\lambda}\Delta(\lambda, c_* + B)|_{\lambda=\lambda_*} = 0, \tag{2.6}$$

where

$$\Delta(\lambda, c) = F_c(\lambda) - G_c(\lambda) = \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda(c-B)r} - [(c-B)\lambda - D\lambda^2 + d'(0)], \tag{2.7}$$

such that for all $c \geq c_* + B$, the traveling wavefront $\phi(x+ct)$ of (1.1) connecting with u_- and u_+ exists. Furthermore, $(\lambda_*, c_* + B)$ is the tangent point of $F_c(\lambda) = \varepsilon b'(0)e^{\alpha\lambda^2 - \lambda(c-B)r}$ and $G_c(\lambda) = (c-B)\lambda - D\lambda^2 + d'(0)$, i.e., for $c = c_* + B$, it holds that

$$\varepsilon b'(0) e^{\alpha\lambda_*^2 - \lambda_* c_* r} = c_*\lambda_* - D\lambda_*^2 + d'(0), \tag{2.8}$$

and for $c > c_* + B$, there exist two numbers depending on c: $\lambda_1 = \lambda_1(c) > 0$ and $\lambda_2 = \lambda_2(c) > 0$ as the solutions to the equation $F_c(\lambda_i) = G_c(\lambda_i)$, i.e.,

$$\varepsilon b'(0) e^{\alpha\lambda_i^2 - \lambda_i cr + \lambda_i Br} = c\lambda_i - B\lambda_i - D\lambda_i^2 + d'(0), \quad i = 1, 2 \tag{2.9}$$

such that

$$F_c(\lambda) < G_c(\lambda) \quad \text{for } \lambda_1 < \lambda < \lambda_2,$$

and particularly

$$F_c(\lambda_*) = G_c(\lambda_*) \quad \text{for } \lambda_1 < \lambda_* < \lambda_2.$$

Now, we define a weight function as

$$w(x) = e^{-2\lambda_* x}, \quad x \in R,$$

where $\lambda_* = \lambda_*(\widetilde{c}_*)$ is the positive constant determined in Proposition2.1 and it satisfies $\frac{c_*}{2D} = \frac{\widetilde{c}_* - B}{2D} < \lambda_* < \frac{\widetilde{c}_* - B}{D} = \frac{c_*}{D}$ as shown in[25]. Obviously, $w(x) \to +\infty$ as $x \to -\infty$ and $w(x) \to 0$ as $x \to +\infty$.

Next, we are going to state our main result about the stability of the traveling wavefront of (1.1) and (1.2).

Theorem 2.1. (Nonlinear Stability). *Let $d(u)$ and $b(u)$ satisfy (A_1)-(A_2). For a given traveling wavefront $\phi(x+ct)$ of (1.1) with $c > c_* + B$ and $\phi(\pm\infty) = u_\pm$, if c satisfies*

$$e^{\lambda_*^2 \alpha} < \frac{(c-B)\lambda_* - D\lambda_*^2 + d'(0)}{c_*\lambda_* - D\lambda_*^2 + d'(0)}, \tag{2.10}$$

the initial data holds $u_- \leq u_0(s,x) \leq u_+$ for $(s,x) \in [-r,0] \times R$, and the initial perturbation is $u_0(s,x) - \phi(x + cs) \in C([-r,0]; H_w^1(R))$, then the solution of (1.1) and (1.2) satisfies $u(t,x) - \phi(x+ct) \in C([0,\infty); H_w^1(R))$, and

$$u_- \leq u(t,x) \leq u_+, \quad \text{for } (t,x) \in R_+ \times R,$$

$$\|(u-\phi)(t)\|_{H^1_w(R)} \le Ce^{-\mu t}, \quad t \ge 0,$$

for some positive constant μ.

In particular, $u(t,x)$ *also converges asymptotically exponential to the wavefront* $\phi(x+ct)$ *in the* $L^\infty-$*norm:*

$$\sup_{x\in R} |u(t,x)-\phi(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0.$$

Remark 2.1. (i) When $\alpha \to 0$ in $g(y)$, by the property of the heat kernel, we have $\lim_{\alpha\to 0}\int_{-\infty}^{+\infty} b(u(t-r,x-y))g(y)dy = b(v(t-r,x))$, then the nonlocal equation (1.1) is reduced to a reaction-advection-diffusion equation with local delay. So the result of this paper also includes the stability of traveling fronts of (1.1) with local nonlinearity.

(ii) Obviously, the condition (2.10) is equivalent to $\alpha < \frac{1}{\lambda_*^2}\ln\frac{c\lambda_*-\lambda_*B-D\lambda_*^2+d'(0)}{c_*\lambda_*-D\lambda_*^2+d'(0)}$. Here α is independent of c, D and d, but both λ_* and c_* are related to α. For given α, we need the wave speed c to be large shown as in (2.10). Conversely, when c is sufficiently large, one can easily verify that $\frac{1}{\lambda_*^2}\ln\frac{c\lambda_*-\lambda_*B-D\lambda_*^2+d'(0)}{c_*\lambda_*-D\lambda_*^2+d'(0)}$ is sufficiently large. This ensures α is sufficiently large as well. When c is sufficiently close to the critical wave speed $\widetilde{c}_* = c_* + B$, then one can recognize that $\frac{1}{\lambda_*^2}\ln\frac{c\lambda_*-\lambda_*B-D\lambda_*^2+d'(0)}{c_*\lambda_*-D\lambda_*^2+d'(0)} \ll 1$, which means α needed to be sufficiently small. Thus, when α is small enough, we may obtain the stability for those slower waves with the speed $c \in (c_*+B, 2\sqrt{D(\varepsilon b'(0)-d'(0))}+B)$.

3 Proof of asymptotic stability

As shown in [11] , we can similarly prove the global existence and uniqueness of the solution for the initial value problem (1.1) and (1.2). In order to prove our stability result, we also need to prove the following boundedness and establish the comparison principle for (1.1) and (1.2), which can be proved similarly as shown in [9, 11, 12, 14, 15], here we omit them.

Lemma 3.1. (Boundedness). *Let the initial data satisfy*

$$u_- = 0 \le u_0(s,x) \le u_+, \textit{ for } (s,x) \in [-r,0]\times R. \tag{3.11}$$

Then the solution $u(t,x)$ *of the Cauchy problem* (1.1) *and* (1.2) *satisfies*

$$u_- \le u(t,x) \le u_+, \textit{ for } (t,x) \in [0,\infty)\times R. \tag{3.12}$$

Lemma 3.2. (Comparison principle). *Let* $\bar{u}(t,x)$ *and* $\underline{u}(t,x)$ *be the solutions of* (1.1) *and* (1.2) *with the initial data* $\bar{u}_0(s,x)$ *and* $\underline{u}_0(s,x)$*, respectively. If*

$$u_- \le \underline{u}_0(s,x) \le \bar{u}_0(s,x) \le u_+, \textit{ for } (s,x)\in[-r,0]\times R, \tag{3.13}$$

then

$$u_- \le \underline{u}(t,x) \le \bar{u}(t,x) \le u_+, \textit{ for } (t,x)\in[0,\infty)\times R. \tag{3.14}$$

In what follows, we are going to prove the main result, Theorem 2.1, by means of the weighted-energy method combining with comparison principle.

For given initial data $u_0(s,x)$ satisfying

$$u_- = 0 \le u_0(s,x) \le u_+, \text{ for } (s,x)\in[-r,0]\times R,$$

let

$$\begin{cases} U_0^+(s,x) = \max\{u_0(s,x),\phi(x+cs)\}, \text{ for } (s,x)\in[-r,0]\times R. \\ U_0^-(s,x) = \min\{u_0(s,x),\phi(x+cs)\}, \text{ for } (s,x)\in[-r,0]\times R. \end{cases} \tag{3.15}$$

so

$$u_- \le U_0^-(s,x) \le u_0(s,x) \le U_0^+(s,x) \le u_+, \text{ for } (s,x)\in[-r,0]\times R, \tag{3.16}$$

$$u_- \le U_0^-(s,x) \le \phi(x+cs) \le U_0^+(s,x) \le u_+, \text{ for } (s,x)\in[-r,0]\times R. \tag{3.17}$$

Define $U^+(t,x)$ and $U^-(t,x)$ as the corresponding solutions of Eq.(1.1) with respect to the initial data $U_0^+(s,x)$ and $U_0^-(s,x)$, respectively, i.e.,

$$\begin{cases} \frac{\partial U^\pm}{\partial t} - D\frac{\partial^2 U^\pm}{\partial x^2} - B\frac{\partial U^\pm}{\partial x} + d(U^\pm) = \varepsilon \int_{-\infty}^{+\infty} b(U^\pm(t-r,x-y+Br))g(y)dy, \quad t>0, x\in R, \\ U^\pm(s,x) = U_0^\pm(s,x), \quad (s,x)\in[-r,0]\times R. \end{cases} \tag{3.18}$$

By the comparison principle, it follows that

$$u_- \le U^-(t,x) \le u(t,x) \le U^+(t,x) \le u_+, \text{ for } (t,x)\in R_+\times R, \tag{3.19}$$

$$u_- \le U^-(t,x) \le \phi(x+ct) \le U^+(t,x) \le u_+, \text{ for } (t,x)\in R_+\times R. \tag{3.20}$$

In order to prove the stability of the traveling wavefronts presented in Theorem 2.1, we also need the following three steps as shown in [9, 14–16].

Step 1. The convergence of $U^+(t,x)$ to $\phi(x+ct)$.

Let $\xi := x+ct$ and

$$v(t,\xi) := U^+(t,x) - \phi(x+ct), \quad v_0(s,\xi) := U_0^+(s,x) - \phi(x+cs), \tag{3.21}$$

then by (3.17) and (3.20), we have

$$v(t,\xi)\ge 0 \text{ and } v_0(s,\xi)\ge 0. \tag{3.22}$$

From Eq.(1.1) and (A_1) , it can be verified that $v(t,\xi)$ defined in (3.21) satisfies (by linearizing it at 0)

$$\begin{cases} v_t + (c-B)v_\xi - Dv_{\xi\xi} + d'(0)v - \varepsilon b'(0)\int_{-\infty}^{+\infty} v(t-r,\xi-y+(B-c)r)g(y)dy \\ = -Q_1(t,\xi) + \varepsilon\int_{-\infty}^{+\infty} Q_2(t-r,\xi-y+(B-c)r)g(y)dy + [d'(0)-d'(\phi(\xi))]v \\ \ +\varepsilon\int_{-\infty}^{+\infty}[b'(\phi(\xi-y+(B-c)r)) - b'(0)]v(t-r,\xi-y+(B-c)r)g(y)dy \\ =: J_1(t,\xi)+J_2(t,\xi)+J_3(t,\xi)+J_4(t,\xi), \quad (t,\xi)\in R_+\times R, \end{cases} \tag{3.23}$$

with the initial data

$$v(s,\xi) = v_0(s,\xi), \quad (s,\xi)\in[-r,0]\times R,$$

where

$$Q_1(t,\xi) = d(\phi+v) - d(\phi) - d'(\phi)v. \tag{3.24}$$

with $\phi=\phi(\xi)$ *and* $v=v(t,\xi)$.

$$Q_2(t-r,\xi-y+(B-c)r) = b(\phi+v) - b(\phi) - b'(\phi)v. \tag{3.25}$$

with $\phi=\phi(\xi-y+(B-c)r)$ *and* $v=v(t-r,\xi-y+(B-c)r)$.

Multiplying (3.23) by $e^{2\mu t}w(\xi)v(t,\xi)$, we obtain

$$\begin{aligned} &\{\frac{1}{2}e^{2\mu t}wv^2\}_t + e^{2\mu t}\{\frac{c-B}{2}wv^2 - Dwvv_\xi\}_\xi + De^{2\mu t}wv_\xi^2 + De^{2\mu t}w'vv_\xi \\ &+\{-\frac{c-B}{2}\cdot\frac{w'}{w} + d'(0) - \mu\}e^{2\mu t}wv^2 \\ &-\varepsilon e^{2\mu t}w(\xi)v(t,\xi)b'(0)\int_{-\infty}^{+\infty} v(t-r,\xi-y+(B-c)r)g(y)dy \\ = \ & e^{2\mu t}w(\xi)v(t,\xi)[J_1(t,\xi)+J_2(t,\xi)+J_3(t,\xi)+J_4(t,\xi)]. \end{aligned} \tag{3.26}$$

By the Cauchy-Schwarz inequality $xy\le x^2+\frac{1}{4}y^2$, we have

$$|De^{2\mu t}w'vv_\xi| = De^{2\mu t}w|v_\xi\cdot\frac{w'}{w}v| \le De^{2\mu t}wv_\xi^2 + \frac{D}{4}e^{2\mu t}(\frac{w'}{w})^2wv^2,$$

then (3.26) is reduced to

$$
\begin{aligned}
&\{\frac{1}{2}e^{2\mu t}wv^2\}_t + e^{2\mu t}\{\frac{c-B}{2}wv^2 - Dwvv_\xi\}_\xi \\
&+\{-\frac{c-B}{2}\cdot\frac{w'}{w} + d'(0) - \mu - \frac{D}{4}(\frac{w'}{w})^2\}e^{2\mu t}wv^2 \\
&-\varepsilon e^{2\mu t}w(\xi)v(t,\xi)b'(0)\int_{-\infty}^{+\infty} v(t-r,\xi-y+(B-c)r)g(y)dy \\
\leq\ & e^{2\mu t}w(\xi)v(t,\xi)[J_1(t,\xi)+J_2(t,\xi)+J_3(t,\xi)+J_4(t,\xi)]. \qquad (3.27)
\end{aligned}
$$

Integrating (3.27) over $R\times[0,t]$ with respect to ξ and t, and noting the vanishing term at far fields,

$$\{\frac{c-B}{2}wv^2 - Dwvv_\xi\}|_{\xi=-\infty}^{\xi=+\infty} = 0,$$

because $\sqrt{w}v \in H^1(R)$, this implies, by the property of Sobolev space $H^1(R)$, that $(\sqrt{w}v)|_{\xi=\pm\infty} = 0$ and $(\sqrt{w}v_\xi)|_{\xi=\pm\infty} = 0$. Thus we further have

$$
\begin{aligned}
&e^{2\mu t}\|v(t)\|^2_{L^2_w} + \int_0^t\int_R\{(B-c)\cdot\frac{w'}{w} + 2d'(0) - 2\mu - \frac{D}{2}(\frac{w'}{w})^2\}e^{2\mu s}w(\xi)v^2(s,\xi)d\xi ds \\
&-2\varepsilon b'(0)\int_0^t\int_R\int_R e^{2\mu s}w(\xi)v(s,\xi)v(s-r,\xi+(B-c)r-y)g(y)dyd\xi ds \\
\leq\ & \|v_0(0)\|^2_{L^2_w} + 2\int_0^t\int_R e^{2\mu s}w(\xi)v(s,\xi)[J_1(t,\xi)+J_2(t,\xi)+J_3(t,\xi)+J_4(t,\xi)]d\xi ds. \\
& \qquad (3.28)
\end{aligned}
$$

We now turn to estimate the third term on the left-hand-side of (3.28). First of all, by changing variables : $\xi-y+(B-c)r\to\xi, s-r\to s, y\to y$, we have

$$
\begin{aligned}
& b'(0)\int_0^t\int_R\int_R e^{2\mu s}w(\xi)v^2(s-r,\xi+(B-c)r-y)g(y)dyd\xi ds \\
=\ & b'(0)\int_{-r}^{t-r}\int_R\int_R e^{2\mu(s+r)}w(\xi+y+(c-B)r)v^2(s,\xi)g(y)dyd\xi ds \\
=\ & e^{2\mu r}b'(0)\int_0^{t-r}\int_R e^{2\mu s}[\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v^2(s,\xi)d\xi ds \\
&+e^{2\mu r}b'(0)\int_{-r}^0\int_R e^{2\mu s}[\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v_0^2(s,\xi)d\xi ds \\
\leq\ & e^{2\mu r}b'(0)\int_0^t\int_R e^{2\mu s}[\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v^2(s,\xi)d\xi ds \\
&+e^{2\mu r}b'(0)\int_{-r}^0\int_R e^{2\mu s}[\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v_0^2(s,\xi)d\xi ds. \\
& \qquad (3.29)
\end{aligned}
$$

Again, using the Cauchy-Schwarz inequality we obtain

$$|v(s,\xi)v(s-r,\xi+(B-c)r-y)| \leq \frac{\eta}{2}v^2(s,\xi) + \frac{1}{2\eta}v^2(s-r,\xi+(B-c)r-y) \qquad (3.30)$$

for any positive constant η, which will be specified later, and using (3.29), we have

$$
\begin{aligned}
& 2\varepsilon b'(0)|\int_0^t \int_R \int_R e^{2\mu s} w(\xi) v(s,\xi) v(s-r,\xi+(B-c)r-y) g(y) dy d\xi ds| \\
\leq\ & \varepsilon b'(0)|\int_0^t \int_R \int_R e^{2\mu s} w(\xi)[\eta v^2(s,\xi) + \frac{1}{\eta} v^2(s-r,\xi+(B-c)r-y)] g(y) dy d\xi ds| \\
=\ & \varepsilon\eta b'(0) \int_0^t \int_R \int_R e^{2\mu s} w(\xi) v^2(s,\xi) g(y) dy d\xi ds \\
& + \frac{\varepsilon}{\eta} b'(0) \int_0^t \int_R \int_R e^{2\mu s} w(\xi) v^2(s-r,\xi+(B-c)r-y) g(y) dy d\xi ds \\
\leq\ & \varepsilon\eta b'(0) \int_0^t \int_R e^{2\mu s} w(\xi)[\int_R g(y) dy] v^2(s,\xi) d\xi ds \\
& + \frac{\varepsilon e^{2\mu r}}{\eta} b'(0) \int_0^t \int_R e^{2\mu s}[\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)} g(y) dy] w(\xi) v^2(s,\xi) d\xi ds \\
& + \frac{\varepsilon e^{2\mu r}}{\eta} b'(0) \int_{-r}^0 \int_R e^{2\mu s}[\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)} g(y) dy] w(\xi) v_0^2(s,\xi) d\xi ds. \qquad (3.31)
\end{aligned}
$$

Substituting (3.31) into (3.28) leads to

$$
\begin{aligned}
& e^{2\mu t}\|v(t)\|^2_{L^2_w} + \int_0^t \int_R e^{2\mu s} B_{\eta,\mu,w}(\xi) w(\xi) v^2(s,\xi) d\xi ds \\
\leq\ & \|v_0(0)\|^2_{L^2_w} + \frac{\varepsilon e^{2\mu r} b'(0)}{\eta} \int_{-r}^0 \int_R e^{2\mu s}[\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)} g(y) dy] w(\xi) v_0^2(s,\xi) d\xi ds \\
& + 2\int_0^t \int_R e^{2\mu s} w(\xi) v(s,\xi)[J_1(t,\xi)+J_2(t,\xi)+J_3(t,\xi)+J_4(t,\xi)] d\xi ds. \qquad (3.32)
\end{aligned}
$$

where

$$
B_{\eta,\mu,w}(\xi) := A_{\eta,w}(\xi) - 2\mu - \frac{\varepsilon}{\eta}(e^{2\mu r}-1) b'(0) \int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)} g(y) dy. \qquad (3.33)
$$

where

$$
\begin{aligned}
A_{\eta,w}(\xi) := & -(c-B)\cdot\frac{w'}{w} + 2d'(0) - \frac{D}{2}(\frac{w'}{w})^2 \\
& -\varepsilon\eta b'(0) \int_R g(y) dy \\
& -\frac{\varepsilon b'(0)}{\eta} \int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)} g(y) dy. \qquad (3.34)
\end{aligned}
$$

Using Taylor's formula for the nonlinearity $Q_1(t,\xi)$, $Q_2(t-r,\xi-y+(B-c)r)$, and noting $(A2)$, we have

$$Q_1(t,\xi) = d(\phi+v) - d(\phi) - d'(\phi)v = d''(\bar\phi_1)v^2 \geq 0.$$

$$Q_2(t-r,\xi-y+(B-c)r) = b(\phi+v) - b(\phi) - b'(\phi)v = b''(\bar\phi_2)v^2 \leq 0$$

for some $\bar\phi_1,\ \bar\phi_2 \in [\phi,\phi+v]$, namely,

$$J_1(t,\xi) \leq 0 \quad \text{and} \quad J_2(t,\xi) \leq 0.$$

Notice that $d'(u)$ is increasing and $b'(u)$ is decreasing from (A_2) and the fact $v(t,\xi) \geq 0$(see (3.22)), which implies

$$d'(0) - d'(\phi) \leq 0 \ \ and \ \ b'(\phi) - b'(0) \leq 0 \ \ for\ \phi \geq 0,$$

namely,

$$J_3(t,\xi) \leq 0 \quad \text{and} \quad J_4(t,\xi) \leq 0,$$

which leads that

$$2\int_0^t\int_R e^{2\mu s}w(\xi)v(s,\xi)[J_1(t,\xi)+J_2(t,\xi)+J_3(t,\xi)+J_4(t,\xi)]d\xi ds\le 0. \tag{3.35}$$

On the other hand, since $\frac{w(\xi+(c-B)r+y)}{w(\xi)}=e^{-2\lambda_*((c-B)r+y)}$, and using the fact

$$\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy=e^{4\alpha\lambda_*^2-2\lambda_*(c-B)r},$$

it follows that

$$\begin{aligned}
&\frac{\varepsilon e^{2\mu r}}{\eta}b'(0)\int_{-r}^0\int_R e^{2\mu s}[\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v_0^2(s,\xi)d\xi ds\\
=&\ \frac{\varepsilon b'(0)e^{2\mu r}}{\eta}\int_{-r}^0 e^{2\mu s}\int_R e^{4\alpha\lambda_*^2-2\lambda_*(c-B)r}w(\xi)v_0^2(s,\xi)d\xi ds\\
=&\ \frac{\varepsilon b'(0)}{\eta}e^{2\mu r+4\alpha\lambda_*^2-2\lambda_*(c-B)r}\int_{-r}^0\int_R e^{2\mu s}w(\xi)v_0^2(s,\xi)d\xi ds\\
\le&\ C\int_{-r}^0\|v_0(s)\|_{L_w^2}^2ds. \qquad (3.36)
\end{aligned}$$

Applying (3.35) and (3.36) in (3.32), we then obtain

$$\begin{aligned}
&e^{2\mu t}\|v(t)\|_{L_w^2}^2+\int_0^t\int_R e^{2\mu s}B_{\eta,\mu,w}(\xi)w(\xi)v^2(s,\xi)d\xi ds\\
\le&\ \|v_0(0)\|_{L_w^2}^2+C\int_{-r}^0\|v_0(s)\|_{L_w^2}^2ds. \qquad (3.37)
\end{aligned}$$

Next, we will prove $B_{\eta,\mu,w}(\xi)>0$ by selecting the numbers η,μ. For that purpose, we need the following lemma.

Lemma 3.3. *Let $\eta=e^{2\alpha\lambda_*^2-\lambda_*(c-B)r}$. Then*

$$A_{\eta,w}(\xi)\ge C_1>0,\ \ \xi\in R \tag{3.38}$$

for some positive constant C_1.

Proof. Notice that $\eta=e^{2\alpha\lambda_*^2-\lambda_*(c-B)r}$, $w(\xi)=e^{-2\lambda_*\xi}$, $\frac{w'(\xi)}{w(\xi)}=-2\lambda_*$ and $\int_R g(y)dy=1$. We may obtain

$$\begin{aligned}
A_{\eta,w}(\xi)\ =&\ -(c-B)\cdot\frac{w'}{w}+2d'(0)-\frac{D}{2}(\frac{w'}{w})^2-\varepsilon\eta b'(0)\int_R g(y)dy\\
&-\frac{\varepsilon b'(0)}{\eta}\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy\\
=&\ 2(c-B)\lambda_*+2d'(0)-2D\lambda_*^2-\varepsilon\eta b'(0)-\frac{\varepsilon b'(0)}{\eta}e^{4\alpha\lambda_*^2-2\lambda_*(c-B)r}\\
=&\ 2[(c-B)\lambda_*+d'(0)-D\lambda_*^2-\varepsilon\eta b'(0)]\\
=&\ 2[(c-B)\lambda_*+d'(0)-D\lambda_*^2-\varepsilon b'(0)e^{2\alpha\lambda_*^2-\lambda_*(c-B)r}]\\
=&\ 2[(c-B)\lambda_*+d'(0)-D\lambda_*^2-\varepsilon b'(0)e^{\alpha\lambda_*^2-\lambda_*(c-B)r}e^{\alpha\lambda_*^2}]\\
\ge&\ 2[(c-B)\lambda_*+d'(0)-D\lambda_*^2-\varepsilon b'(0)e^{\alpha\lambda_*^2-\lambda_*c_*r}e^{\alpha\lambda_*^2}]\\
=&\ 2[((c-B)\lambda_*+d'(0)-D\lambda_*^2)-(c_*\lambda_*+d'(0)-D\lambda_*^2)e^{\alpha\lambda_*^2}][by(2.8)]\\
=&\ 2(c_*\lambda_*+d'(0)-D\lambda_*^2)[\frac{(c-B)\lambda_*+d'(0)-D\lambda_*^2}{c_*\lambda_*+d'(0)-D\lambda_*^2}-e^{\alpha\lambda_*^2}]\\
=:&\ C_1>0.[by(2.10),\ see\ Remark\ 2.1\ (ii)).] \qquad (3.39)
\end{aligned}$$

This completes the proof. □

Lemma 3.4. *Let $\mu_1 > 0$ be the unique solution of the equation*

$$C_1 = 2\mu + \varepsilon\eta b'(0)(e^{2\mu r} - 1) \tag{3.40}$$

If $0 < \mu < \mu_1$, then

$$B_{\eta,\mu,w}(\xi) \geq C_2 > 0, \ \ \xi \in R. \tag{3.41}$$

Proof. Applying (3.39) to (3.33), it can be examined that

$$\begin{aligned} B_{\eta,\mu,w}(\xi) \ \ &\geq \ \ C_1 - 2\mu - \frac{\varepsilon b'(0)}{\eta}(e^{2\mu r} - 1)\int_R \frac{w(\xi + y + (c-B)r)}{w(\xi)} g(y)dy \\ &= \ \ C_1 - 2\mu - \frac{\varepsilon b'(0)}{\eta}(e^{2\mu r} - 1)e^{4\alpha\lambda_*^2 - 2\lambda_*(c-B)r} \\ &= \ \ C_1 - 2\mu - \varepsilon\eta b'(0)(e^{2\mu r} - 1) \\ &=: \ \ C_2 > 0, \ \textit{for } 0 < \mu < \mu_1. \end{aligned} \tag{3.42}$$

This completes the proof. □

Applying (3.41) to (3.37), and dropping $\int_0^t \int_R e^{2\mu s} B_{\eta,\mu,w}(\xi) w(\xi) v^2(s,\xi) d\xi ds$, we then immediately establishes the first basic energy estimate as follows which is crucial for our main stability result.

Lemma 3.5. *It holds that*

$$e^{2\mu t}\|v(t)\|^2_{L^2_w} \leq C\left(\|v_0(0)\|^2_{L^2_w} + \int_{-r}^0 \|v_0(s)\|^2_{L^2_w} ds\right), \quad t \geq 0. \tag{3.43}$$

Similarly, differentiating Eq. (3.23) with respect to ξ, and multiplying it by $e^{2\mu t} w(\xi) v_\xi(t,\xi)$, and integrating the resultant equation over $R \times [0,t]$ with respect to ξ and t, then by using (3.43) in Lemma 3.5, we can obtain the second energy estimate as follows.

Lemma 3.6. *It holds that*

$$e^{2\mu t}\|v_\xi(t)\|^2_{L^2_w} \leq C\left(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v_0(s)\|^2_{H^1_w} ds\right), \quad t \geq 0. \tag{3.44}$$

Therefore, (3.43) and (3.44) imply the following fact.

Lemma 3.7. *It holds that*

$$\|v(t)\|^2_{H^1_w} \leq Ce^{-2\mu t}\left(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v_0(s)\|^2_{H^1_w} ds\right), \quad t \geq 0. \tag{3.45}$$

Notice that $w(\xi) \to 0$ as $\xi \to \infty$, we can not conclude $H^1_w(R) \hookrightarrow C(R)$. However, for any interval $I = (-\infty, \xi_*]$ for some large $\xi_* \gg 1$, there is the Sobolev's embedding result $H^1_w(I) \hookrightarrow C(I)$, which can be combined with (3.45) and be given the following $L^\infty-$ estimate.

Lemma 3.8. *It holds that*

$$\sup_{\xi \in I} |v(t,\xi)| \leq Ce^{-\mu t}\left(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v_0(s)\|^2_{H^1_w} ds\right)^{\frac{1}{2}}, \quad t \geq 0, \tag{3.46}$$

for any interval $I = (-\infty, \xi_*]$ with some large $\xi_* \gg 1$.

However, we need the $L^\infty-$ convergence in (3.46) in the whole space $(-\infty, +\infty)$. Thus, we are going to prove the convergence at $\xi = +\infty$.

Lemma 3.9. *It holds that*

$$\lim_{\xi \to +\infty} |v(t,\xi)| \leq Ce^{-\mu_2 t}, \quad t \geq 0, \tag{3.47}$$

where $\mu_2 > 0$ and μ_2 satisfies $d'(u_+) - \mu_2 - \varepsilon b'(u_+)e^{\mu_2 r} > 0$.

Proof. From Eq.(1.1), it can be verified that $v(t,\xi)$ defined in (3.21) satisfies

$$\begin{aligned} & v_t + cv_\xi - Dv_{\xi\xi} - Bv_\xi + d'(\phi(\xi))v \\ & -\varepsilon \int_{-\infty}^{+\infty} b'(\phi(\xi - y + (B-c)r))v(t-r, \xi - y + (B-c)r)g(y)dy \\ = & -Q_1(t,\xi) + \varepsilon \int_{-\infty}^{+\infty} Q_2(t-r, \xi - y + (B-c)r))g(y)dy \\ = & J_1(t,\xi) + J_2(t,\xi), \quad (t,\xi) \in R_+ \times R. \end{aligned} \tag{3.48}$$

As shown in proving (3.35), $J_1(t,\xi) \le 0$ *and* $J_2(t,\xi) \le 0$, above equation is reduced to

$$\begin{aligned} & v_t + cv_\xi - Dv_{\xi\xi} - Bv_\xi + d'(\phi(\xi))v \\ & -\varepsilon \int_{-\infty}^{+\infty} b'(\phi(\xi - y + (B-c)r))v(t-r, \xi - y + (B-c)r)g(y)dy \le 0. \end{aligned} \tag{3.49}$$

Taking limits as $\xi \to +\infty$, and noting that $v_\xi(t,+\infty) = 0$, $v_{\xi\xi}(t,+\infty) = 0$ by the fact that the boundedness of $v(t,\xi)$ for all $\xi \in R$, and $\int_{-\infty}^{+\infty} g(y)dy = 1$, we obtain

$$\frac{d}{dt}v(t,\infty) + d'(u_+)v(t,\infty) - \varepsilon b'(u_+)v(t-r,\infty) \le 0. \tag{3.50}$$

Multiplying (3.50) by $e^{\mu_2 t}$(μ_2 is a positive constant to be specified later) and integrating it over $[0,t]$, we then have

$$\int_0^t e^{\mu_2 s}\frac{d}{ds}v(s,\infty)ds + d'(u_+)\int_0^t e^{\mu_2 s}v(s,\infty)ds - \varepsilon b'(u_+)\int_0^t e^{\mu_2 s}v(s-r,\infty)ds \le 0. \tag{3.51}$$

For the first term in (3.51), we get

$$\int_0^t e^{\mu_2 s}\frac{d}{ds}v(s,\infty)ds = e^{\mu_2 t}v(t,\infty) - v_0(0,\infty) - \mu_2\int_0^t e^{\mu_2 s}v(s,\infty)ds. \tag{3.52}$$

By the change of variable $s - r \to s$ for the third term in (3.51), we obtain

$$\begin{aligned} & \varepsilon b'(u_+)\int_0^t e^{\mu_2 s}v(s-r,\infty)ds \\ = & \varepsilon b'(u_+)\int_{-r}^{t-r} e^{\mu_2(s+r)}v(s,\infty)ds \\ \le & \varepsilon b'(u_+)e^{\mu_2 r}\int_{-r}^{t} e^{\mu_2 s}v(s,\infty)ds \\ = & \varepsilon b'(u_+)e^{\mu_2 r}[\int_{-r}^{0} e^{\mu_2 s}v_0(s,\infty)ds + e^{\mu_2 r}\int_0^t e^{\mu_2 s}v(s,\infty)ds] \\ = & \varepsilon b'(u_+)e^{\mu_2 r}\int_{-r}^{0} e^{\mu_2 s}v_0(s,\infty)ds + e^{\mu_2 r}\varepsilon b'(u_+)\int_0^t e^{\mu_2 s}v(s,\infty)ds. \end{aligned} \tag{3.53}$$

Substituting (3.52) and (3.53) into (3.51), we have

$$e^{\mu_2 t}v(t,\infty) + [d'(u_+) - \mu_2 - \varepsilon b'(u_+)e^{\mu_2 r}]\int_0^t v(s,\infty)e^{\mu_2 s}ds \le C_3, \tag{3.54}$$

where $C_3 = v_0(0,\infty) + \varepsilon b'(u_+)e^{\mu_2 r}\int_{-r}^0 v_0(s,\infty)e^{\mu_2 s}ds$. Moreover, by (A_1), we can obtain $d'(u_+) - \varepsilon b'(u_+) > 0$. Then, there exist $\mu_2 > 0$ such that $d'(u_+) - \mu_2 - \varepsilon b'(u_+)e^{\mu_2 r} > 0$, and therefore (3.54) yields

$$v(t,\infty) \le Ce^{-\mu_2 t}.$$

This completes the proof. □

Combining Lemma 3.8 and Lemma 3.9, and taking $0 < \mu < \min\{\mu_1, \mu_2\}$, we prove the $L^\infty-$ convergence in Theorem 2.1 for all $\xi \in R$, i.e.,

Lemma 3.10. *It holds that*

$$\sup_{x\in R}|U^+(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0, \tag{3.55}$$

where $0 < \mu < \min\{\mu_1, \mu_2\}$.

Step 2. The convergence of $U^-(t,x)$ to $\phi(x+ct)$.

Let $\xi := x + ct$ and

$$v(t,\xi) = \phi(x+ct) - U^-(t,x), \quad v_0(s,\xi) = \phi(x+cs) - U_0^-(s,x). \tag{3.56}$$

As shown in the process of Step 1, we can similarly prove the convergence of $U^-(t,x)$ to $\phi(x+ct)$, i.e.,

Lemma 3.11. *It holds that*

$$\sup_{x\in R} |U^-(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0. \tag{3.57}$$

for $0 < \mu < \min\{\mu_1, \mu_2\}$.

Step 3. The convergence of $u(t,x)$ to $\phi(x+ct)$.

In this step, we are going to prove Theorem 2.1, namely,

Lemma 3.12. *It holds that*

$$\sup_{x\in R} |u(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0,$$

for $0 < \mu < \min\{\mu_1, \mu_2\}$.

Proof. Since the initial data satisfy $U_0^-(s,x) \le u_0(s,x) \le U_0^+(s,x)$, by Lemma3.2, it can be proved that the corresponding solutions of (1.1) and (1.2) satisfy $U^-(t,x) \le u(t,x) \le U^+(t,x),\ (t,x) \in R_+ \times R$. Thanks to Lemma3.10, Lemma3.11, we have the following convergence results:

$$\sup_{x\in R} |U^\pm(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0.$$

Using the squeeze Theorem, we finally prove

$$\sup_{x\in R} |u(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0.$$

This completes the proof. □

As a final remark, we consider a reaction-advection-diffusion equation with nonlocal delay (see [21, 24])

$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u + B\frac{\partial u}{\partial x} + f(u(t,x), \tilde{g} * S(u)), & t > 0, x \in R, \\ u(s,x) = u_0(s,x), & s \in [-r, 0],\ x \in R, \end{cases} \tag{3.58}$$

which is a generalized version of the model (1.1) in the case where $S(u) = b(u)$, $f(u,v) = -d(u) + \varepsilon v$, $\tilde{g}(t,x) = J_\alpha(x + Bt)\delta(t-r)$, $r > 0$ is the time delay and $J_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}}$. Under appropriate assumptions, the exponential stability of noncritical traveling fronts of (3.58) can be obtained similarly as in this paper, including even the slower waves whose wave speed are close to the critical speed, which recovers and improves Wang and Wu's stability results [21, 24] for the noncritical waves. Particularly, the spreading speed and its coincidence with the minimal wave speed(i.e. the critical speed) can be established for (3.58) in the same way as Zhao's [20, 27, 28].

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On certain subclass of p - valent analytic functions associated with differintegral operator

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Abstract

In this paper, by making use of the fractional differintegral operator, we introduce a certain subclass of multivalent analytic functions. We study some properties such as inclusion relationship, integral preserving, convolution and some interesting results for multivalent starlikeness are proved.

Keywords: Multivalent function, subordination, superordination, hadamard product, differintegral operator, starlike function.

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1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a, m)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots$.

Let $\mathcal{A}_p$ be the class of functions analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \geq 1). \tag{1.1}$$

and let $\mathcal{A} = \mathcal{A}_1$.

For the functions $f(z)$ of the form (1.1) and $g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, b_{n+p}\, z^{n+p}.$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

In our present investigation, we shall also make use of the Guassian hypergeometric function

$$\begin{aligned} {}_2F_1(a, b; c; z) &= 1 + \frac{ab}{c}.\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}.\frac{z^2}{2!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!}, \qquad (a,\, b,\, c \in \mathbb{C}, \quad with \quad c \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}) \end{aligned} \tag{1.2}$$

where the Pochhammer symbol $(x)_k$ is defined, in terms of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & if\, k = 0 \\ x(x+1)(x+2)\ \ldots\ (x+k-1) & if\, k \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

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Definition 1.1. *Let $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$, then the generalized fractional integral operator $I_{0,z}^{\alpha,\beta,\gamma}$ of order α of a function $f(z)$ is defined by*

$$I_{0,z}^{\alpha,\beta,\gamma} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} \quad {}_2F_1\left(\alpha+\beta, \gamma; \alpha; 1-\frac{t}{z}\right) f(t)dt, \tag{1.3}$$

where the function $f(z)$ is analytic in a simply - connected region of the z - plane containing the origin and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$ provided further that

$$f(z) = O\left(|z|\right)^{\epsilon}, z \to 0 \quad for \quad \epsilon > max\left(0, \beta-\gamma\right) - 1. \tag{1.4}$$

Definition 1.2. *Let $0 \le \alpha < 1$ and $\beta, \gamma \in \mathbb{R}$, then the generalized fractional derivative operator $J_{0,z}^{\alpha,\beta,\gamma}$ of order α of a function $f(z)$ is defined by*

$$\begin{aligned} J_{0,z}^{\alpha,\beta,\gamma} f(z) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left[z^{\alpha-\beta} \int_0^z (z-t)^{-\alpha} \quad {}_2F_1\left(\beta-\alpha, 1-\gamma; 1-\alpha; 1-\frac{t}{z}\right) f(t)dt \right] \\ &= \frac{d^n}{dz^n} J_{0,z}^{\alpha-n,\beta,\gamma} f(z) \end{aligned} \tag{1.5}$$

where the function $f(z)$ is analytic in a simply - connected region of the z - plane containing the origin, with the order as given in (1.4)and multiplicity of $(z-t)^{\alpha}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 1.3. *For real number $\alpha\,(-\infty < \alpha < 1)$, $\beta\,(-\infty < \beta < 1)$ and a positive real number γ, the fractional operator $U_{0,z}^{\alpha,\beta,\gamma} : A_p \to A_p$ is defined in terms of $J_{0,z}^{\alpha,\beta,\gamma}$ and $I_{0,z}^{\alpha,\beta,\gamma}$ by*

$$U_{0,z}^{\alpha,\beta,\gamma} = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+p+\gamma-\beta)_n}{(1+p-\beta)_n (1+p+\gamma-\alpha)_n} a_{n+p} z^{n+p} \tag{1.6}$$

which for $f(z) \neq 0$ may be written as

$$U_{0,z}^{\alpha,\beta,\gamma} = \begin{cases} \frac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} J_{0,z}^{\alpha,\beta,\gamma} f(z); & 0 \le \alpha \le 1 \\ \frac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} I_{0,z}^{-\alpha,\beta,\gamma} f(z); & if\ -\infty \le \alpha < 0. \end{cases}$$

where $J_{0,z}^{\alpha,\beta,\gamma} f(z)$ and $I_{0,z}^{-\alpha,\beta,\gamma} f(z)$ are, respectively the fractional derivative of f of order α if $0 \le \alpha < 1$ and the fractional integral of f of order $-\alpha$ if $\infty \le \alpha < 0$.

Recently, using the operator $U_{0,z}^{\alpha,\beta,\gamma}$, Ahmed S. Galiz [1], introduce the linear operator $\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f : A_p \to A_p$ by

$$\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z) = z^p + \sum_{n=1}^{\infty} \left[\frac{p+l+\lambda n}{p+l}\right]^m \frac{(1+p)_n (1+p+\gamma-\beta)_n}{(1+p-\beta)_n (1+p+\gamma-\alpha)_n} a_{n+p} z^{n+p} \tag{1.7}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $l \ge 0$, $\lambda \ge 0$ and $p \in \mathbb{N}$.

The above operator generates several operators studied by many authors such as El - Ashwah and Aouf [4], Selvaraj and Karthikeyan [21], Dziok - Srivastava operator [6], Salagean [19], Goyal and Prajapat [7] and others.

From (1.7), we can easily verified that

$$\lambda z \left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)' = (p+l)\, \phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda} f(z) - [p(1-\lambda)+l]\, \phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z). \tag{1.8}$$

On differentiating (1.8), we get

$$\lambda z \left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'' = (p+l) \left(\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda} f(z)\right)' - [p+l+(1-p)\lambda] \left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'. \tag{1.9}$$

We note that the operator $\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}$ is a generalization of several familiar operators and we will show some of the interesting special cases:

(1). If $m = 0$, $\alpha = \lambda$, $\beta = \mu$ and $\gamma = \eta$ then the operator is reduced into the well - known fractional differintegral operator $I_p^{\lambda}(\mu, \eta)$ which was introduced and investigated by Goyal and Prajapat [7].

(2). If we take $m = 0$, $\alpha = \lambda$, $\beta = \mu$ and $\gamma = \eta = 0$ then the operator is reduced into the known fractional differintegral operator Ω_p^{λ}. It was studied by Patel and Mishra [17] and also in [18].

(3). $\phi_{-\alpha,0,\beta-1}^{0,l,\lambda} = \mathcal{Q}_{\beta,p}^{\alpha}$ $(\beta > -p)$, where $\mathcal{Q}_{\beta,p}^{\alpha}$ is the Liu - Owa operator (see in [11] and [3]). Also put $p = 1$, it is well known Jung - Kim - Srivastava operator [8].

(4). $\phi_{-1,0,\beta-1}^{0,l,\lambda} = \mathcal{J}_{\beta,p}$ $(\beta > -1)$, where $\mathcal{J}_{\beta,p}$ is the Bernardi integral operator (see [5]).

2 Definitions and Preliminaries

We denote by $\mathcal{P}$ the class of functions $\chi(z)$ given by

$$\chi(z) = 1 + c_1 z + c_2 z^2 + \cdots, \tag{2.10}$$

which are analytic in $\mathbb{U}$ and satisfy the following inequality $Re\{\chi(z)\} > 0$ for $z \in \mathbb{U}$.

Definition 2.4. *A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$ if it satisfies the following subordination condition;*

$$1 + \frac{1}{b}\left(\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} - 1\right) \prec \psi(z) \qquad (z \in \mathbb{U}; \psi \in \mathcal{P}) \tag{2.11}$$

where (and throughout this paper unless otherwise mentioned) the parameters p, γ, λ, b and β are constrained as follows:
$p \in \mathbb{N}, b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \gamma \in \mathbb{R}, \beta < p + 1, -\infty < \alpha < \gamma + p + 1$ *and* $\lambda \geq 0$.

For the sake of conveniance, we set

$$\mathcal{S}_{p,b}^{m,l,\lambda}\left(\alpha, \beta, \gamma; \frac{1 + Az}{1 + Bz}\right) = \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; A, B) \qquad (-1 \leq B < A \leq 1).$$

For $A = 1 - \frac{2\eta}{p}$, $B = -1$, we have

$$\mathcal{S}_{p,b}^{m,l,\lambda}\left(\alpha, \beta, \gamma; 1 - \frac{2\eta}{p}, B = -1\right) = \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \eta) \qquad (0 \leq \eta < 1).$$

In order to estabilish our main results, we shall require the following known lemmas:

Lemma 2.1. *[10] Let the function $\psi(z)$ be analytic and convex(univalent) in $\mathbb{U}$ with $\psi(0) = 1$. Suppose also that the function $\phi(z)$ given by*

$$\phi(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + \cdots \tag{2.12}$$

is analytic in $\mathbb{U}$. If

$$\phi(z) + \frac{z\phi'(z)}{\nu} \prec \psi(z) \quad (\mathcal{R}(\nu) > 0; \nu \neq 0; z \in \mathbb{U}), \tag{2.13}$$

then

$$\phi(z) \prec q(z) = \frac{\nu}{k} z^{-\frac{\nu}{k}} \int_0^z \psi(t) t^{\frac{\nu}{k} - 1} dt \prec \psi(z),$$

and $q(z)$ is the best dominant of (2.12).

Lemma 2.2. *[26] Let μ be a positive measure on the unit interval $[0, 1]$. Let $g(z, t)$ be a complex valued function defined on $\mathbb{U} \times [0, 1]$ such that $g(0, t)$ is analytic in $\mathbb{U}$ for each $t \in [0, 1]$ and such that $g(z, 0)$ is μ integrable on $[0, 1]$ for all $z \in \mathbb{U}$. In addition, suppose that $Re\{g(z, t)\} > 0$, $g(-r, t)$ is real and*

$$Re\left\{\frac{1}{g(z, t)}\right\} \geq \frac{1}{g(-r, t)} \qquad (|z| \leq r < 1; t \in [0, 1]).$$

If G is defined by $G(z) = \int_0^1 g(z, t) d\mu(t)$, then

$$Re\left\{\frac{1}{G(z)}\right\} \geq \frac{1}{G(-r)} \qquad (|z| \leq r < 1).$$

Lemma 2.3. *[15] Let φ be analytic in $\mathbb{U}$ with $\varphi(0) = 1$ and $\varphi(z) = 0$ for $0 < |z| < 1$ and let $A, B \in \mathbb{C}$ with $A \neq B$, $|B| \leq 1$.*

(i). Let $B \neq 0$ and $v \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ satisfy either $\left|\frac{v(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{v(A-B)}{B} + 1\right| \leq 1$. If φ satisfies

$$1 + \frac{z\varphi'(z)}{v\varphi(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{2.14}$$

then

$$\varphi(z) \prec (1 + Bz)^{v\left(\frac{A-B}{B}\right)},$$

and this is best dominant.

(ii). Let $B = 0$ and $v \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be such that $|vA| < \pi$. If φ satisfies (2.14), then

$$\varphi(z) \prec e^{vAz}$$

and this is the best dominant.

Lemma 2.4. *[12] Let $\kappa, \tau \in \mathbb{C}$. Suppose that ϕ is convex and univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $Re(\kappa\phi + \tau) > 0$. If the function g is analytic in $\mathbb{U}$ with $g(0) = 1$, then the subordination*

$$g(z) + \frac{zg'(z)}{\kappa g(z) + \tau} \prec \phi(z) \qquad (z \in \mathbb{U})$$

implies that

$$g(z) \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Lemma 2.5. *[20] Let the function g be analytic in $\mathbb{U}$ with $g(0) = 1$ and $Re\{g(z)\} > \frac{1}{2}$. Then, for any function F analytic in $\mathbb{U}$, $(g * F)(\mathbb{U})$ is contained in the convex hull of $F(\mathbb{U})$.*

Lemma 2.6. *[25] For real and complex numbers a, n and c $(c \notin \mathbb{Z}_0^-)$*

$$\int_0^1 t^{n-1}(1-t)^{c-n-1}(1-tz)^{-a}dt = \frac{\Gamma(n)\Gamma(c-n)}{\Gamma(c)}\, {}_2F_1(a, n; c; z) \quad (Re\{n\}, Re\{c\} > 0), \tag{2.15}$$

$${}_2F_1(a, n; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-n; c; \frac{z}{z-1}\right). \tag{2.16}$$

Motivated by the concept of Aouf et. al. [2], Huo Tang, Guan Tie Deng and Shu Hai Li [24] and Selvaraj et. al. [22], in this paper, we investigate some inclusion relations and other interesting properties for certain classes of p - valent functions involving an integral operator.

3 Inclusion Relationship

Theorem 3.1. *Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $l \geq 0$, $\lambda \geq 0$, $b = b_1 + ib_2 \neq 0$, $\tan\sigma = \frac{b_1}{b_2}$ and $\psi \in \mathcal{P}$ with $Im(\psi) < (Re(\psi) - 1)\cot\sigma$. Then*

$$\mathcal{S}_{p,b}^{m+1,l,\lambda}(\alpha, \beta, \gamma; \psi) \subset \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi) \tag{3.17}$$

Proof. Let $\mathcal{S}_{p,b}^{m+1,l,\lambda}(\alpha, \beta, \gamma; \psi)$ and suppose that

$$g(z) = 1 + \frac{1}{b}\left[\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} - 1\right], \qquad (z \in \mathbb{U}) \tag{3.18}$$

where g is analytic in $\mathbb{U}$ with $g(0) = 1$. In view of (1.8) and (3.18), we obtain

$$(p+l)\frac{\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda} f(z)}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} = \lambda bp\,(g(z) - 1) + (p+l). \tag{3.19}$$

Differentiating (3.19) both sides with respect to z, and using (3.18), we get

$$1 + \frac{1}{b}\left[\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} - 1\right] = g(z) + \frac{\lambda zg'(z)}{\lambda bp(g(z) - 1) + p + l}. \tag{3.20}$$

Since $Re(\lambda bp(\psi(z) - 1) + p + l) > 0$ for $Im(\psi) < (Re(\psi) - 1)\cot\sigma$ and where $\tan\sigma = \frac{b_1}{b_2}$, so applying Lemma 2.4 to (3.20), it follows that $g(z) \prec \psi(z)$, that is $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$. □

Taking $\psi(z) = \dfrac{1+Az}{1+Bz}$ in Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $l \geq 0$, $\lambda \geq 0$, $b = b_1 + ib_2 \neq 0$ and $-1 \leq B < A \leq 1$, then*

$$\mathcal{S}_{p,b}^{m+1,l,\lambda}(\alpha, \beta, \gamma; A, B) \subset \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; A, B).$$

Remark 3.1. *If we put $m = 0$, $\alpha = \lambda$, $\beta = \mu$ and $\gamma = \eta$, then this result is reduced into the class of functions $M_p^{\lambda}(\mu, \eta; \gamma; \phi)$ which is studied by [24].*

4 Convolution properties

Now, we derive certain convolution properties for the function class $\mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$.

Theorem 4.1. *Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$. Then*

$$f(z) = \left[z^p \cdot \exp\left(bp\int_0^z \frac{\psi(\omega(\xi))-1}{\xi}d\xi\right)\right] * \left(z^p + \sum_{n=1}^{\infty}\left(\frac{p+l+\lambda n}{p+l}\right)^{-m}\frac{(1+p-\beta)_n(1+p+\gamma-\alpha)_n}{(1+p)_n(1+p+\gamma-\beta)_n}z^{n+p}\right), \tag{4.21}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$.

Proof. Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$. From (2.11)

$$\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)} = [\psi(\omega(z))-1]\,bp + p \tag{4.22}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$. By virtue of (4.22), we can easily find that

$$\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)} - \frac{p}{z} = \frac{[\psi(\omega(z))-1]\,bp}{z} \tag{4.23}$$

Integrating (4.23), we get

$$\begin{aligned} \log\left(\frac{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)}{z^p}\right) &= bp\int_0^z \frac{[\psi(\omega(\xi))-1]\,bp}{\xi}d\xi \\ \Rightarrow \phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z) &= z^p \cdot \exp\left[bp\int_0^z \frac{[\psi(\omega(\xi))-1]}{\xi}d\xi\right] \end{aligned} \tag{4.24}$$

Then, from (1.7) and (4.24), we deduce that the required assertion of the Theorem 4.1. □

Corollary 4.2. *Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; A, B)$ with $-1 \leq B < A \leq 1$. Then*

$$f(z) = \left[z^p \cdot \exp\left(bp\int_0^z \frac{(A-B)(\omega(\xi))-1}{\xi}d\xi\right)\right] * \left(z^p + \sum_{n=1}^{\infty}\left(\frac{p+l+\lambda n}{p+l}\right)^{-m}\frac{(1+p-\beta)_n(1+p+\gamma-\alpha)_n}{(1+p)_n(1+p+\gamma-\beta)_n}z^{n+p}\right),$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$.

Theorem 4.2. *Let $f \in \mathcal{A}_p$ and $\psi \in \mathcal{P}$. Then $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$ if and only if*

$$\begin{aligned} &\frac{1}{z^p}\left\{f * \left(pz^p + \sum_{n=1}^{\infty}\left[\frac{p+l+\lambda n}{p+l}\right]^m \frac{(n+p)(1+p)_n(1+p+\gamma-\beta)_n}{(1+p-\beta)_n(1+p+\gamma-\alpha)_n}z^{n+p}\right.\right. \\ &\left.\left.-p\left[(b\psi(e^{i\theta})-1)+1\right] \times \left(z^p + \sum_{n=1}^{\infty}\left(\frac{p+l+\lambda n}{p+l}\right)^{-m}\frac{(1+p-\beta)_n(1+p+\gamma-\alpha)_n}{(1+p)_n(1+p+\gamma-\beta)_n}z^{n+p}\right)\right)\right\} \neq 0 \end{aligned} \tag{4.25}$$

$(z \in \mathbb{U}; 0 < \theta < 2\pi)$.

Proof. Suppose that $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$. We know that (2.11) holds true, which implies that

$$1 + \frac{1}{b}\left(\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)} - 1\right) \neq \psi(e^{i\theta}) \qquad (z \in \mathbb{U}; 0 < \theta < 2\pi). \tag{4.26}$$

One can easily verify that, from (4.26)

$$\frac{1}{z^p}\left\{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)' - p\left[(b\psi(e^{i\theta})-1)+1\right]\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right\} \neq 0 \qquad (z\in\mathbb{U}; 0<\theta<2\pi). \tag{4.27}$$

On the otherhand, we find from (1.7) that

$$z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)' = pz^p + \sum_{n=1}^{\infty}\left[\frac{p+l+\lambda n}{p+l}\right]^m \frac{(n+p)(1+p)_n(1+p+\gamma-\beta)_n}{(1+p-\beta)_n(1+p+\gamma-\alpha)_n} a_{n+p}z^{n+p} \tag{4.28}$$

Combining (1.7), (4.27) and (4.28), we can easily get the convolution property (4.25) asserted by Theorem 4.2. □

5 Some properties of the operator $\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}$

Now we discuss some properties of the operator $\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}$.

Theorem 5.1. *Let $\sigma > 0$, $\gamma\in\mathbb{R}$, $p\in\mathbb{N}\setminus\{1\}$, $-1\leq B<A\leq 1$ and the function $f\in\mathcal{A}_p$ satisfies the following subordination:*

$$(1-\sigma)\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}, \qquad (z\in\mathbb{U}). \tag{5.29}$$

Then

$$\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} \prec \psi(z) \prec \frac{1+Az}{1+Bz}, \tag{5.30}$$

where

$$\psi(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1} \quad {}_2F_1\left(1,1;\frac{p+l}{k\sigma\lambda}+1;\frac{Bz}{1+Bz}\right) & for B\neq 0,\\ 1+\frac{p+l}{k\sigma\lambda+p+l}Az & for\ B=0.\end{cases} \tag{5.31}$$

is the best dominant of (5.30). Furthermore,

$$f\in\mathcal{S}^{m,l,\lambda}_{p,b}(\alpha,\beta,\gamma;\delta) \tag{5.32}$$

where

$$\delta = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1} \quad {}_2F_1\left(1,1;\frac{p+l}{k\sigma\lambda}+1;\frac{B}{B-1}\right) & for B\neq 0,\\ 1+\frac{p+l}{k\sigma\lambda+p+l}A & for\ B=0.\end{cases}$$

The result is best possible.

Proof. Let

$$g(z) = \frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}}, \tag{5.33}$$

where g is of the form (2.12) and is analytic in $\mathbb{U}$. Differentiating (5.33) with respect to z and making use of (1.9), we get

$$\begin{aligned}(1-\sigma)\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} &= g(z)+\frac{\lambda\sigma z g'(z)}{p+l}\\ &\prec \frac{1+Az}{1+Bz}. \qquad (z\in\mathbb{U})\end{aligned}$$

Applying Lemma 2.1 and Lemma 2.6, we have

$$\begin{aligned}\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} &\prec \psi(z)\\ &= \frac{p+l}{k\sigma\lambda}z^{-\frac{p+l}{k\sigma\lambda}}\int_0^z t^{\frac{p+l}{k\sigma\lambda}-1}\left(\frac{1+At}{1+Bt}\right)dt\\ &= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1} \quad {}_2F_1\left(1,1;\frac{p+l}{k\sigma\lambda}+1;\frac{Bz}{1+Bz}\right) & for B\neq 0,\\ 1+\frac{p+l}{k\sigma\lambda+p+l}Az & for\ B=0.\end{cases}\end{aligned}$$

This proves the assertion (5.30) of Theorem 5.1. Next, inorder to prove the assertion (5.32), it suffices to prove that

$$\inf_{|z|<1} \{Re(\psi(z))\} = \psi(-1).$$

Indeed, we have

$$Re\frac{1+Az}{1+Bz} \geq \frac{1-Ar}{1-Br} \qquad (|z| = r < 1)$$

Setting

$$G(z,\zeta) = \frac{1+A\zeta z}{1+B\zeta z} \qquad \text{and} \qquad dv(\zeta) = \frac{p+l}{k\sigma\lambda}\zeta^{\frac{p+l}{k\sigma\lambda}-1}d\zeta \qquad (0 \leq \zeta \leq 1),$$

which is a positive measure on the closed interval [0, 1], we get

$$\psi(z) = \int_0^z G(z,\zeta)dv(\zeta).$$

Then

$$Re\{\psi(z)\} \geq \int_0^1 \frac{1-A\zeta r}{1-B\zeta r}dv(\zeta) = \psi(-r) \qquad (|z| = r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (5.32). Finally, the estimate (5.32) is best possible as ψ is the best dominant of (5.30). This completes the proof of the theorem. □

Theorem 5.2. *Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \eta) \quad (0 \leq \eta < 1)$, then*

$$Re\left\{(1-\sigma)\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda}f(z)\right)'}{pz^{p-1}}\right\} > \eta \qquad (|z| < R),$$

where

$$R = \left\{\frac{\sqrt{(p+l)^2 + (\sigma\lambda k)^2} - \sigma\lambda k}{(p+l)}\right\}^{\frac{1}{k}}. \tag{5.34}$$

The result is best possible.

Proof. Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \eta)$, then we write

$$\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{pz^{p-1}} = \eta + (1-\eta)u(z) \qquad (z \in \mathbb{U}) \tag{5.35}$$

where u is of the form (2.12) and is analytic in $\mathbb{U}$. Differentiating (5.35) with respect to z, we have

$$\frac{1}{1-\eta}\left\{(1-\sigma)\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda}f(z)\right)'}{pz^{p-1}} - \eta\right\} = u(z) + \frac{\sigma\lambda zu'(z)}{(p+l)} \tag{5.36}$$

Applying the following well-knowing estimate [9]:

$$\frac{|zu'(z)|}{Re\{u(z)\}} \leq \frac{2kr^k}{1-r^{2k}} \qquad (|z| = r < 1),$$

in (5.36), we have

$$\begin{aligned} &\frac{1}{1-\eta}Re\left\{(1-\sigma)\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda}f(z)\right)'}{pz^{p-1}} - \eta\right\} \\ &\geq Re\{u(z)\}\left(1 - \frac{2\sigma\lambda kr^k}{(p+l)(1-r^{2k})}\right), \end{aligned} \tag{5.37}$$

such that the right hand side of (5.37) is positive, if $r < R$, where R is given by (5.34). In order to show that the bound R is best possible, we consider the function $f \in \mathcal{A}_p$ defined by

$$\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{pz^{p-1}} = \eta + (1-\eta)\frac{1+z^k}{1-z^k} \qquad (0 \leq \eta < 1; z \in \mathbb{U}).$$

Note that

$$\frac{1}{1-\eta}\left\{(1-\sigma)\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda} f(z)\right)'}{pz^{p-1}} - \eta\right\} = \frac{(p+l)(1-z^{2k}) - 2\sigma\lambda k z^k}{(p+l)(1-z^{2k})} = 0, \tag{5.38}$$

for $z = Re^{\frac{i\pi}{k}}$. □

For a function $f \in \mathcal{A}_p$, the generalized Bernardi - Libera - Livingston integral operator $F_{c,p}$ is defined by

$$\begin{aligned} F_{c,p}f(z) &= \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt = \left(z^p + \sum_{n=1}^{\infty}\frac{c+p}{c+p+n}z^{n+p}\right) * f(z) \\ &= z^p\,{}_2F_1(1, c+p; c+p+1; z) * f(z) \qquad (c > -p; z \in \mathbb{U}) \end{aligned} \tag{5.39}$$

From (1.7) and (5.39), we have

$$z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} F_{c,p}(f(z))\right)' = (c+p)\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z) - c\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} F_{c,p}(f(z)) \tag{5.40}$$

Theorem 5.3. *Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; A, B)$ and $F_{c,p}$ be defined by (5.39). Then*

$$\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} F_{c,p}(f(z))\right)'}{pz^{p-1}} \prec \theta(z) \prec \frac{1+Az}{1+Bz}, \tag{5.41}$$

where

$$\theta(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1} \quad {}_2F_1\left(1,\, 1;\, \frac{p+c}{k} + 1;\, \frac{Bz}{1+Bz}\right) & for B \neq 0, \\ 1 + \frac{p+c}{k+p+c}Az & for\ B = 0. \end{cases} \tag{5.42}$$

is the best dominant of (5.41). Furthermore,

$$Re\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} F_{c,p}(f(z))\right)'}{pz^{p-1}} > \mu$$

where

$$\mu = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1} \quad {}_2F_1\left(1,\, 1;\, \frac{p+c}{k} + 1;\, \frac{B}{B-1}\right) & for B \neq 0, \\ 1 + \frac{p+c}{k+p+c}Az & for\ B = 0. \end{cases}$$

The result is best possible.

Proof. Let

$$K(z) = \frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} F_{c,p}(f(z))\right)'}{pz^{p-1}} \qquad (z \in \mathbb{U}). \tag{5.43}$$

where K is of the form (2.12) and is analytic in $\mathbb{U}$. Using (5.40) and (5.43) and differentiating the resulting equation with respect to z we have

$$\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} F_{c,p}(f(z))\right)'}{pz^{p-1}} = K(z) + \frac{zK'(z)}{p+c} \prec \frac{1+Az}{1+Bz}.$$

The remaining part of the proof is similar to that of Theorem 5.1 and so we omit it. □

Theorem 5.4. *Let $f, g \in \mathcal{A}_p$ satisfy the following inequality:*

$$Re\left(\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)}{z^p}\right) > 0 \qquad (z \in \mathbb{U}).$$

If

$$\left|\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)} - 1\right| < 1 \qquad (z \in \mathbb{U}).$$

then

$$Re\left(\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}\right) > 0 \qquad (|z| < R_1; z \in \mathbb{U}),$$

where

$$R_1 = \left(\frac{-3k + \sqrt{9k^2 + 4p(p+k)}}{2(p+k)}\right)^{\frac{1}{k}}. \tag{5.44}$$

Proof. Let

$$q(z) = \frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)} - 1 = c_k z^k + c_{k+1}z^{k+1} + \cdots, \tag{5.45}$$

where $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = 0$ and $|q(z)| \leq |z|^k$. Then, by applying the familiar Schwarz Lemma [14], we have $q(z) = z^k\chi(z)$, where χ is analytic in $\mathbb{U}$ and $\chi(z) \leq 1$.
From (5.45),

$$\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z) = \phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)\left[1 + z^k\chi(z)\right] \tag{5.46}$$

Differentiating (5.46) logarithmically w.r.t z, we have

$$\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)} = \frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)} + \frac{z^k\left[k\chi(z) + z\chi'(z)\right]}{1 + z^k\chi(z)}. \tag{5.47}$$

Letting

$$\omega(z) = \frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)}{z^p} \qquad (z \in \mathbb{U}),$$

where ω is in the form (2.12) is analytic in $\mathbb{U}$, $Re\{\omega(z)\} > 0$ and

$$\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)} = \frac{z\omega'(z)}{\omega(z)} + p$$

then we have

$$Re\left\{\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}\right\} \geq p - \left|\frac{z\omega'(z)}{\omega(z)}\right| - \left|\frac{z^k\left[k\chi(z) + z\chi'(z)\right]}{1 + z^k\chi(z)}\right| \tag{5.48}$$

Using the following known estimates [9],

$$\left|\frac{\omega'(z)}{\omega(z)}\right| \leq \frac{2kr^{k-1}}{1 - r^{2k}} \quad \text{and} \quad \left|\frac{k\chi(z) + z\chi'(z)}{1 + z^k\chi(z)}\right| \leq \frac{k}{1 - r^k} \qquad (|z| = r < 1),$$

in (5.48), we have

$$Re\left\{\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}\right\} \geq \frac{p - 3kr^k - (p+k)r^{2k}}{1 - r^{2k}} \tag{5.49}$$

which is certainly positive, provided that $r < R_1$, where R_1 is given by (5.44). □

Theorem 5.5. *Let $f \in \mathcal{S}^{m,l,\lambda}_{p,b}(\alpha, \beta, \gamma; A, B)$ and $g \in \mathcal{A}_p$ satisfy the following inequality:*

$$Re\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2} \qquad (z \in \mathbb{U}) \tag{5.50}$$

then $(f * g)(z) \in \mathcal{S}^{m,l,\lambda}_{p,b}(\alpha, \beta, \gamma; A, B)$.

Proof. We have

$$\frac{(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}(f * g)(z))'}{pz^{p-1}} = \frac{(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}(f)(z))'}{pz^{p-1}} * \frac{g(z)}{z^p} \qquad (z \in \mathbb{U}),$$

where g satisfies (5.50) and $\frac{1+Az}{1+Bz}$ is convex in $\mathbb{U}$. By using (5.30) and applying Lemma 2.5, we get the required assertion of this theorem. □

Theorem 5.6. *Let $\vartheta \in \mathbb{C}^*$ and $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. Suppose that*

$$\left|\frac{\vartheta(p+l)(A-B)}{\lambda B} - 1\right| \leq 1 \quad or \quad \left|\frac{\vartheta(p+l)(A-B)}{\lambda B} + 1\right| \leq 1, if B \neq 0,$$

$$\left|\frac{\vartheta(p+l)}{\lambda} A\right| \leq \pi, if B = 0.$$

If $f \in \mathcal{A}_p$ with $\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z) \neq 0$ for all $z \in \mathbb{U}^* = \mathbb{U} \setminus \{0\}$, then

$$\frac{\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)} \prec \frac{1+Az}{1+Bz},$$

implies $\left(\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\vartheta} \prec g_1(z)$. where

$$g_1(z) = \begin{cases} (1+Bz)^{\frac{\vartheta(p+l)(A-B)}{\lambda B}} & for B \neq 0, \\ e^{\frac{\vartheta(p+l)}{\lambda} Az} & for\ B = 0, \end{cases}$$

is the best dominant.

Proof. Let

$$\varphi(z) = \left(\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\vartheta} \qquad (z \in \mathbb{U}). \tag{5.51}$$

Then φ is analytic in $\mathbb{U}$, $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for all $\mathbb{U}$. Taking the logrithamic differentiation on both sides of (5.51) and using the identity (1.8), we obtain

$$1 + \frac{\lambda z\varphi'(z)}{\vartheta(p+l)\varphi(z)} = \frac{\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)} \prec \frac{1+Az}{1+Bz}.$$

Now the assertions of Theorem 5.6 follows from Lemma 2.3. □

Taking $B = -1$ and $A = 1 - 2\eta, 0 \leq \eta < 1$ in Theorem 5.6, we get the following corollary:

Corollary 5.3. *Let $\vartheta \in \mathbb{C}^*$ satisfies either*

$$\left|\frac{2\vartheta(p+l)(1-\eta)}{\lambda} - 1\right| \leq 1 \quad or \quad \left|\frac{2\vartheta(p+l)(1-\eta)}{\lambda} + 1\right| \leq 1.$$

If $f \in \mathcal{A}_p$ with $\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z) \neq 0$ for all $z \in \mathbb{U}^*$, then

$$Re\left\{\frac{\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)}\right\} \prec \frac{1+Az}{1+Bz}$$

implies

$$\left(\frac{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)}{z^p}\right)^{\vartheta} \prec g_1(z)$$

where

$$g_1(z) = (1-z)^{\frac{-2\vartheta(p+l)(1-\eta)}{\lambda}}$$

is the best dominant.

Theorem 5.7. *Let $\sigma > 0$, $\epsilon > 0$, $-1 \leq B < A \leq 1$ and the function $f \in \mathcal{A}_p$ satisfies the following subordination:*

$$(1-\sigma)\frac{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)}{z^p} + \sigma \frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}, \qquad (z \in \mathbb{U}). \tag{5.52}$$

Then

$$Re\left\{\frac{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)}{z^p}\right\}^{\frac{1}{\epsilon}} > \delta^{\frac{1}{\epsilon}}, \tag{5.53}$$

where

$$\delta = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1} \quad {}_2F_1\left(1, 1; \frac{p}{k\sigma} + 1; \frac{B}{B-1}\right) & forB \neq 0, \\ 1 + \frac{p}{k\sigma + p} A & for\, B = 0. \end{cases}$$

The result is best possible.

Proof. Let

$$G(z) = \frac{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)}{z^p}, \tag{5.54}$$

where G is of the form (2.12) and is anlytic in $\mathbb{U}$. Differentiating (5.54) with respect to z, we get

$$\begin{aligned}(1-\sigma)\frac{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)}{z^p} + \sigma \frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{pz^{p-1}} &= G(z) + \frac{\sigma z G'(z)}{p} \\ &\prec \frac{1+Az}{1+Bz}. \qquad (z \in \mathbb{U})\end{aligned}$$

Now, applying similar steps invoved in Theorem 5.1 and using the elementary inequality

$$Re\{\Omega^{\kappa}\} \geq Re\{\Omega\}^{\kappa} \qquad (Re\{\Omega\} > 0; \kappa \in \mathbb{N}),$$

we obtain the required result. □

Remark 5.2. *Taking $m = 0$ and the choices of α, β and γ , this subclass is reduced into the class $S_{p,n}^{\lambda}(A, B)$ which is studied by A. O. Mostafa and M.K.Aouf [13].*

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Total eccentricity index of some composite graphs

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Abstract

The total eccentricity index of a graph G is the sum of eccentricities of all the vertices of G. In this paper, we first derive some sharp upper and lower bounds of total eccentricity index of different subdivision graphs and then determine some explicit expression of the total eccentricity index of the double graph, extended double cover graph and some generalized thorn graphs.

Keywords: Eccentricity, graph invariant, total eccentricity index, composite graphs, graph operations.

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1 Introduction

Let $G = (V(G), E(G))$ be a connected graph with number of vertices $|V(G)| = n$ and number of edges $|E(G)| = m$. For any two vertices $u, v \in V(G)$, the distance between u and v, denoted by $d_G(u, v)$, is defined as the number of edges in the shortest path connecting u and v. The eccentricity of a vertex v, denoted by $\varepsilon_G(v)$, is the largest distance of v and any other vertex u of G. The degree of a vertex v is the number of vertices adjacent with the vertex v. A vertex v is called well-connected if $\deg_G(v) = n - 1$, *i.e.*, it is adjacent to any other vertex of G. A number of topological indices based on vertex eccentricity are already subject to various studies. The total eccentricity index of G is defined as $\zeta(G) = \sum_{v \in V(G)} \varepsilon_G(v)$. Similar to this index, Dankelmann et.al. [3] and Tang et al. [16] studied average eccentricity of graphs. Fathalikhani et al. in [12], studied total eccentricity of some graph operations. In [7], the present authors present total eccentricity of the generalized hierarchical product graphs. As usual, let $K_n, S_n, C_n, K_{m,n}$ denote the complete graph with n vertices, the star graph on $(n+1)$ vertices, the cycle on n vertices and the complete bipartite graph with $(m+n)$ vertices respectively. In this paper, we first find some sharp upper and lower bounds of total eccentricity index of different subdivision graphs and then determine some explicit expression of the total eccentricity index of the double graph, extended double cover graph and some generalized thorn graphs.

2 Total eccentricity index of subdivision graphs

In this section, we derive sharp upper and lower bounds of total eccentricity index of four types of graphs resulting from edge subdivisions, such as $S(G)$, $R(G)$, $Q(G)$ and $T(G)$. For different study of these subdivision graphs see [11, 17, 18]. For a given graph G, the line graph $L(G)$ is the graph whose vertices are the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting an additional vertex in each edge of G so that $|V(S(G))| = |V(G)| + |E(G)|$and $|E(S(G))| = 2|E(G)|$.

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Lemma 2.1. *[18] Let G be a connected graph. Then*
(i) For each $v \in V(G)$, $\varepsilon_{S(G)}(v) = 2\varepsilon_G(v)$,
(ii) For each $e \in E(G)$, $2\varepsilon_{L(G)}(e) \leq \varepsilon_{S(G)}(e) \leq 2\varepsilon_{L(G)}(e) + 1$.

Theorem 2.1. *Let G be a connected graph. Then*
(i) $\zeta(S(G)) \leq 2\zeta(L(G)) + 2\zeta(G) + |E(G)|$,
(ii) $\zeta(S(G)) \geq 2\zeta(L(G)) + 2\zeta(G)$.

Proof. From definition of $S(G)$, we have

$$\begin{aligned}\zeta(S(G)) &= \sum_{v\in V(S(G))} \varepsilon_{S(G)}(v) \\ &= \sum_{v\in V(G)} \varepsilon_{S(G)}(v) + \sum_{e\in E(G)} \varepsilon_{S(G)}(e) \\ &= \sum_{v\in V(G)} 2\varepsilon_G(v) + \sum_{e\in E(G)} \varepsilon_{S(G)}(e) \\ &= 2\zeta(G) + \sum_{e\in E(G)} \varepsilon_{S(G)}(e).\end{aligned}$$

Now using Lemma 2.1, the above sum $\sum_{e\in E(G)} \varepsilon_{S(G)}(e)$ is bounded above by $2\sum_{v\in V(G)} \varepsilon_{L(G)}(v) + |E(G)|$ and bounded below by $2\sum_{v\in V(G)} \varepsilon_{L(G)}(v)$. This completes the proof. □

As explained in [18], the equality attains in Theorem 2.1 when the eccentricities of all the vertices attained at the vertices of degree one (including trees). Therefore the following corollary follows.

Corollary 2.1. *Let G be a connected graph such that eccentricity of each vertex is attained just at a pendant vertex. Then $\zeta(S(G)) = 2\zeta(L(G)) + 2\zeta(G) + |E(G)|$.*

Example 2.1. *Let S_n and $P_n (n \geq 1)$ be star and path on n vertices respectively. Then we have, $\zeta(S_n) = 2n - 1$ and*

$$\zeta(P_n) = \begin{cases} \frac{3}{4}n^2 - \frac{1}{2}n, & \text{when } n \text{ is even} \\ \frac{3}{4}n^2 - \frac{1}{2}n - \frac{1}{4}, & \text{when } n \text{ is odd.} \end{cases}$$

(i) Since $L(S_n) = K_{1,n}$, the total eccentricity index of subdivision graph of star graph is given by

$$\zeta(S(S_n)) = 2\zeta(L(S_n)) + 2\zeta(S_n) + |E(S_n)| = 7n - 5.$$

(ii) Since $S(P_n) = P_{2n-1}, L(P_n) = P_{n-1}$, $|E(P_n)| = n - 1$ so the total eccentricity index of subdivision graph of path graph is given by

$$\zeta(S(P_n)) = 2\zeta(L(P_n)) + 2\zeta(P_n) + |E(P_n)| = 3n^2 - 4n + 1.$$

The triangle parallel graph of a graph G is denoted by $R(G)$ and is obtained by replacing each edge of G by a triangle, so that $|V(R(G))| = |V(G)| + |E(G)|$ and $|E(R(G))| = 3|E(G)|$.

Lemma 2.2. *[18] Let G be a connected graph. Then*
(i) For each $v \in V(G)$, $\varepsilon_G(v) \leq \varepsilon_{R(G)}(v) \leq \varepsilon_G(v) + 1$,
(ii) For each $e \in E(G)$, $\varepsilon_{R(G)}(e) = \varepsilon_{L(G)}(e) + 1$.

Theorem 2.2. *Let G be a connected graph. Then*
(i) $\zeta(R(G)) \leq \zeta(L(G)) + \zeta(G) + |V(G)| + |E(G)|$,
(ii) $\zeta(R(G)) \geq \zeta(L(G)) + \zeta(G) + |E(G)|$.

Proof. From definition of $R(G)$, we have

$$\zeta(R(G)) = \sum_{v\in V(R(G))} \varepsilon_{R(G)}(v) = \sum_{v\in V(G)} \varepsilon_{R(G)}(v) + \sum_{e\in E(G)} \varepsilon_{R(G)}(e).$$

Using Lemma 2.2, we have $\zeta(G) \leq \sum_{v\in V(G)} \varepsilon_{R(G)}(v) \leq \zeta(G) + |V(G)|$ and $\sum_{e\in E(G)} \varepsilon_{R(G)}(e) = \zeta(L(G)) + |E(G)|$. Combining these, the desired result follows. □

Like previous corollary, the equality attains in Theorem 2.2 when the eccentricities of all the vertices attained only at the pendent vertices. So the following corollary follows.

Corollary 2.2. *Let G be a connected graph such that eccentricity of each vertex is attained only at a pendant vertex. Then,*

$$\zeta(R(G)) = \zeta(L(G)) + \zeta(G) + |E(G)|.$$

Example 2.2. *Similarly using the last corollary the total eccentricity index of triangle parallel graph of star graph and path graphs are given respectively as*

(i) $\zeta(R(S_n)) = \zeta(L(S_n)) + \zeta(S_n) + |E(S_n)| = 4n - 3.$

(ii) $\zeta(R(P_n)) = \zeta(L(P_n)) + \zeta(P_n) + |E(P_n)| = \frac{3}{2}n(n-1).$

Another subdivision graph, the line superposition graph $Q(G)$ of a graph G is obtained by inserting a new vertex to each edge of G and then by joining each new vertex to the end vertices of the edge corresponding to it, so that $|V(Q(G))| = |V(G)| + |E(G)|$ and $|E(Q(G))| = 3|E(G)| + |E(L(G))|$.

Lemma 2.3. *[18] Let G be a connected graph. Then*

(i) For each $v \in V(G)$, $\varepsilon_{Q(G)}(v) = \varepsilon_G(v) + 1$,

(ii) For each $e \in E(G)$, $\varepsilon_{L(G)}(e) \leq \varepsilon_{Q(G)}(e) \leq \varepsilon_{L(G)}(e) + 1$.

Theorem 2.3. *Let G be a connected graph. Then*

(i) $\zeta(Q(G)) \leq \zeta(L(G)) + \zeta(G) + |V(G)| + |E(G)|$,

(ii) $\zeta(Q(G)) \geq \zeta(L(G)) + \zeta(G) + |V(G)|$.

Proof. Using definition of $Q(G)$, we have

$$\zeta(Q(G)) = \sum_{v \in V(Q(G))} \varepsilon_{Q(G)}(v) = \sum_{v \in V(G)} \varepsilon_{Q(G)}(v) + \sum_{e \in E(G)} \varepsilon_{Q(G)}(e)$$

From Lemma 2.3, we have $\sum_{v \in V(G)} \varepsilon_{Q(G)}(v) = \zeta(G) + |V(G)|$ and $\zeta(L(G)) \leq \sum_{e \in E(G)} \varepsilon_{Q(G)}(e) \leq \zeta(L(G)) + |E(G)|$. From where, the desired result follows. □

Similarly, we get the following corollary.

Corollary 2.3. *Let G be a connected graph such that eccentricity of each vertex is attained only at a pendant vertex. Then*

$$\zeta(Q(G)) = \zeta(L(G)) + \zeta(G) + |V(G)| + |E(G)|.$$

Example 2.3. *The total eccentricity of line superposition graph of star graph and path graph are given by*

(i) $\zeta(Q(S_n)) = \zeta(L(S_n)) + \zeta(S_n) + |V(S_n)| + |E(S_n)| = 5n - 3.$

(ii) $\zeta(R(P_n)) = \zeta(L(P_n)) + \zeta(P_n) + |E(P_n)| = \frac{3}{2}n(n-1).$

For the total graph $T(G)$ of a graph G, any two vertices being adjacent if and only if the corresponding elements of G are either adjacent or incident, so that $|V(T(G))| = |V(G)| + |E(G)|$ and $|E(T(G))| = 2|V(G)| + |E(L(G))|$.

Lemma 2.4. *[18] Let G be a connected graph. Then*

(i) For each $v \in V(G)$, $\varepsilon_G(v) \leq \varepsilon_{T(G)}(v) \leq \varepsilon_G(v) + 1$,

(ii) For each, $\varepsilon_{L(G)}(e) \leq \varepsilon_{T(G)}(e) \leq \varepsilon_{L(G)}(e) + 1$.

Theorem 2.4. *Let G be a connected graph. Then*

(i) $\zeta(T(G)) \leq \zeta(L(G)) + \zeta(G) + |V(G)| + |E(G)|$,

(ii) $\zeta(T(G)) \geq \zeta(L(G)) + \zeta(G)$.

Proof. From the construction of $R(G)$, we have

$$\zeta(T(G)) = \sum_{v \in V(T(G))} \varepsilon_{T(G)}(v) = \sum_{v \in V(G)} \varepsilon_{T(G)}(v) + \sum_{e \in E(G)} \varepsilon_{T(G)}(e).$$

Now similar to the previous theorem, using Lemma 2.4, we have $\zeta(G) \leq \sum_{v \in V(G)} \varepsilon_{T(G)}(v) \leq \zeta(G) + |V(G)|$ and $\zeta(L(G)) \leq \sum_{e \in E(G)} \varepsilon_{T(G)}(e) \leq \zeta(L(G)) + |E(G)|$. Combining, the desired result follows. □

Corollary 2.4. *Let T be a tree. Then*

$$\zeta(T(T)) = \zeta(L(T)) + \zeta(T) + |E(T)|.$$

Proof. Since for any tree T, $v \in V(G)$, $\varepsilon_{T(T)}(v) = \varepsilon_T(v)$ and, $\varepsilon_{T(G)}(e) = \varepsilon_T(e) + 1$. □

Example 2.4. *From the above corollary the total eccentricity of the total graph of star graph and path graph are given by*
(i) $\zeta(T(S_n)) = \zeta(L(S_n)) + \zeta(S_n) + |V(S_n)| + |E(S_n)| = 5n - 3.$
(ii) $\zeta(T(P_n)) = \zeta(L(P_n)) + \zeta(P_n) + |E(P_n)| = \frac{3}{2}n(n-1).$

3 Total eccentricity index of double graph and extended double cover

In this section, we derive total eccentricity index of double graph and extended double cover graph. The double graph of G denoted by G^*, constructed by making two copies of G and for each vertex $u_i \in V(G)$ there are two vertices x_i and $y_i$in $V(G^*)$, so that for any edge $u_iu_j \in E(G)$ there will be two edges x_iy_j and x_jy_i including the edges x_iy_i and x_jy_j in G^*. Different applications of double graph of a graph ware investigated in [2, 9, 10, 14].

Theorem 3.5. *The total eccentricity index of the double graph* G^* *is given by* $\zeta(G^*) = 2\zeta(G) + 2\|n-1\|_G$ *where,* $\|n-1\|_G$ *the number of vertices with eccentricity one i.e. of degree* $(n-1)$.

Proof. From the definition of double graph it is clear $\varepsilon_{G^*}(x_i) = \varepsilon_{G^*}(y_i) = \varepsilon_G(v_i)$, *when* $\varepsilon_G(v_i) \geq 2$ and $\varepsilon_{G^*}(x_i) = \varepsilon_{G^*}(y_i) = \varepsilon_G(v_i) + 1 = 2$, *when* $\varepsilon_G(v_i) = 1$. Thus the connective eccentric index of double graph G^* is

$$\begin{aligned}
\zeta(G^*) &= \sum_{i=1}^{n} \varepsilon_{G^*}(x_i) + \sum_{i=1}^{n} \varepsilon_{G^*}(y_i) = 2\left[\sum_{\varepsilon_G(v_i)\geq 2} \{\varepsilon_G(v_i) + 1\} + \sum_{\varepsilon_G(v_i)\geq 1} 2\right] \\
&= 2\left[\sum_{\varepsilon_G(v_i)\geq 2} \varepsilon_G(v_i) + \sum_{\varepsilon_G(v_i)=1} \{\varepsilon_G(v_i) + 1\}\right] \\
&= 2\left[\sum_{\varepsilon_G(v_i)\geq 2} \varepsilon_G(v_i) + \sum_{\varepsilon_G(v_i)=1} \varepsilon_G(v_i)\right] + 2\sum_{\varepsilon_G(v_i)=1} 1 \\
&= 2\zeta(G) + 2\|n-1\|_G.
\end{aligned}$$

where, $\|n-1\|_G$ the number of vertices with eccentricity one i.e. of degree $(n-1)$. □

From the above result the following corollary is obvious.

Corollary 3.5. *If G does not contain any well connected vertices then* $\zeta(G^*) = 2\zeta(G)$.

Example 3.5. *Let* G_{2n} *be the double graph of* P_n. *Then the total eccentricity index of* G_{2n} *is given by*

$$\zeta(G_{2n}) = \begin{cases} \frac{3}{2}n^2 - n & , if\ n\ is\ even \\ \frac{3}{2}n^2 - n - \frac{1}{2}, & if\ n\ is\ odd. \end{cases}$$

The extended double cover was introduced by Alon [1] in 1986. Let G be a simple connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. The extended double cover of G, denoted by G^{**}, is the bipartite graph with bipartion (P, Q) where $P = \{x_1, x_2, ..., x_n\}$ and $Q = \{y_1, y_2, ..., y_n\}$ in which x_i and y_i are adjacent if and only if $i = j$.

Theorem 3.6. *The total eccentricity index of the extended double cover* G^{**} *is given by* $\zeta(G^{**}) = 2\zeta(G) + 2n$.

Proof. If G is a graph with n vertices and m edges then from definition of extended double cover graph G^{**} consists of $2n$ vertices and $(n+2m)$ edges and $\varepsilon_{G^*}(x_i) = \varepsilon_{G^*}(y_i) = \varepsilon_G(v_i)+1$, for $i = 1,2,,n$. Thus the connective eccentric index of double graph G^{**} is

$$\begin{aligned}\zeta(G^{**}) &= \sum_{i=1}^{n}\varepsilon_{G^{**}}(x_i) + \sum_{i=1}^{n}\varepsilon_{G^{**}}(y_i)\\ &= 2\sum_{i=1}^{n}\{\varepsilon_G(v_i)+1\}\\ &= 2\zeta(G)+2n\end{aligned}$$

as desired. □

Example 3.6. *(i) Let H_{2n} be the extended double cover of P_n. Then the total eccentricity index of H_{2n} is given by*

$$\zeta(H_{2n}) = \begin{cases} \frac{3}{4}n^2+\frac{3}{2}n, & \text{when } n \text{ is even}\\ \frac{3}{4}n^2+\frac{3}{2}n-\frac{1}{4}, & \text{when } n \text{ is odd.}\end{cases}$$

(ii) Since the extended double cover of K_n is $K_{n,n}$, so from Theorem 3.6 it can be verified that

$$\zeta(K_n^{**}) = 2\zeta(K_n)+2n = 4n = \zeta(k_{n,n}).$$

4 Total eccentricity index of generalized thorn graphs

Now we determine total eccentricity index of two special types of graphs G_{K_p} and G_{P_m}(see [15]), named as generalized thorn graph as the thorny graph are special cases of these graphs. Let G_{K_p} be the graph obtained from G by attaching t complete graph of order p i.e. K_p at every vertex of G. Let the vertices of G are denoted by $v_1, v_2, ..., v_k$. Also the vertices attached to the vertex v_i are denoted by $v_{i1}^{(r)}, v_{i2}^{(r)}, ..., v_{im}^{(r)}$; $i = 1,2,...,k$; $r = 1,2,...,t$. Now we find the total eccentricity index of this graph G_{K_p}. Let the vertex v_i is identified with $v_{ip}^{(r)}$ $i = 1,2,...,k; r = 1,2,...,t$.

Theorem 4.7. *For any simple connected graph G the $\zeta(G_{K_p})$ and $\zeta(G)$ are related as*

$$\zeta(G_{K_p}) = (pt-t+1)\zeta(G) + k(2pt-2t+1)$$

(2) where G_{K_p} is the graph obtained from G by attaching t complete graphs at each vertex of G.

Proof. From the construction, the eccentricities of the vertices of G_{K_p} are given by

$$\varepsilon_{G_{K_p}}(v_i) = \varepsilon_G(v_i)+1, \text{for } i = 1,2,...,k;$$

$$\varepsilon_{G_{K_p}}(v_{ij}^{(r)}) = \varepsilon_G(v_i)+2, \text{for } i = 1,2,...,k; j = 1,2,...,m; r = 1,2,...,t.$$

Therefore the total eccentricity index of G_{K_p} is given by

$$\begin{aligned}\zeta(G_{K_p}) &= \sum_{i=1}^{k}\varepsilon_{G_{K_p}}(v_i) + \sum_{i=1}^{k}\sum_{j=1}^{p-1}\sum_{r=1}^{t}\varepsilon_{G_{K_p}}(v_{ij}^{(r)})\\ &= \sum_{i=1}^{k}\{\varepsilon_G(v_i)+1\} + \sum_{i=1}^{k}\sum_{j=1}^{p-1}\sum_{r=1}^{t}\{\varepsilon_G(v_i)+2\}\\ &= \sum_{i=1}^{k}\varepsilon_G(v_i)+k+t(p-1)\sum_{i=1}^{k}\{\varepsilon_G(v_i)+2\}\\ &= \{1+t(p-1)\}\zeta(G)+k+2kt(p-1)\end{aligned}$$

as desired. □

Any edge (u,v) of a graph G is called a thorn if and only if, either $\deg_G(u) = 1$ or $\deg_G(v) = 1$. The t-thorny graph G^t is obtained from G by attaching t thorns at each and every vertices of G. This type of graphs were introduced in [13] and for different eccentricity based topological indices of thorn graphs see [4–6, 8]. Since the thorn of a graph can be treated as K_2 , so it is easy to show that by substituting $p = 2$ in (2), we get the total eccentricity of the t-thorny graphs as follows, which already derived in [7].

Corollary 4.6. *[7] The total eccentricity index of the t-thorn graph G^t is computed as follows*

$$\zeta(G^t) = (t+1)\zeta(G) + n(2t+1).$$

Next, we construct another graph denoted by G_{P_m} by attaching t paths of order m (≥ 2) at each vertex v_i,$1 \leq i \leq p$ of G. The vertices of the r-th path attached at v_i are denoted by $v_{i1}^{(r)}, v_{i2}^{(r)}, ..., v_{im}^{(r)}$; $i = 1,2,...,p; r = 1,2,...,t$. Let the vertex $v_{i1}^{(r)}$ is identified with the i-th vertex v_i of G. Clearly the resulting graph G_{P_m} consists of $\{(m-1)t + p\}$ number of vertices.

Theorem 4.8. *For any simple connected graph G the $\zeta(G_{P_m})$ and $\zeta(G)$ are related as*

$$\zeta(G_{P_m}) = \{t(m-1)+1\}\,\zeta(G_{P_m}) + \frac{kt}{2}\left(3m^2 - 5m + 2\right) + k(m-1)$$

(3) where G_{P_m} is the graph obtained from G by attaching t paths each of length m at each vertex of G.

Proof. From the construction of G_{P_m}, eccentricities of the vertices are given by

$$\varepsilon_{G_{P_m}}(v_i) = \varepsilon_G(v_i) + (m-1), \text{for } i = 1,2,...,k;$$

$$\varepsilon_{G_{P_m}}(v_{ij}^{(r)}) = \varepsilon_G(v_i) + m + j - 2, \text{for } i = 1,2,...,k; j = 1,2,...,m; r = 1,2,...,t.$$

Therefore the total eccentricity index of G_{P_m} is given by

$$\begin{aligned}
\zeta(G_{P_m}) &= \sum_{i=1}^{k} \varepsilon_{G_{P_m}}(v_i) + \sum_{i=1}^{k}\sum_{j=2}^{m}\sum_{r=1}^{t} \varepsilon_{G_{P_m}}(v_{ij}^{(r)}) \\
&= \sum_{i=1}^{k} \{\varepsilon_G(v_i) + (m-1)\} + \sum_{i=1}^{k}\sum_{j=2}^{m}\sum_{r=1}^{t} \{\varepsilon_G(v_i) + m + j - 2\} \\
&= \sum_{i=1}^{k} \varepsilon_G(v_i) + k(m-1) + kt(m-1)(m-2) + t(m-1)\sum_{i=1}^{k} \varepsilon_G(v_i) + kt\sum_{j=2}^{m} j \\
&= \{t(m-1)+1\}\,\zeta(G_{P_m}) + k(m-1)\,\{t(m-2)+1\} + kt\left\{\frac{m(m+1)}{2} - 1\right\}
\end{aligned}$$

from where the desired result follows. □

Again, the thorn of the graph can be treated as P_2, so by substituting $m = 2$ in (3) we can obtain the Corollary 4.6.

5 Conclusion

In this paper, first we derive some sharp upper and lower bounds of total eccentricity index of different subdivision graphs and then apply those results to find total eccentricity index of some particular graphs. Then we determine some explicit expression of the total eccentricity index of the double graph, extended double cover graph and some generalized thorn graphs; from where we get the total eccentricity of the t-thorny graphs or t-fold bristled graph. For further study, total eccentricity index of some other graph operations can be computed.

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Qualitative behavior of rational difference equations of higher order

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Abstract

In this paper we study the behavior of the solution of the following rational difference equation

$$x_{n+1} = \frac{ax_{n-r}^2 + bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \quad n = 0, 1, ...,$$

where the parameters a, b, c and d are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, ..., x_{-1}$ and x_0 are posistive real numbers where $t = \max\{r, k, l\}$.

Keywords: stability, rational difference equation, global attractor, periodic solution.

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1 Introduction

In the past two decades, the study of Difference Equations has been growing continuously. This is largely due to the fact that difference equations appear as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. Moreover, difference equations also appear in the study of discretization schemes for nonlinear differential equations. The need for a discretization of nonlinear differential equations arises from the fundamental realization that nonlinear systems generally do not have analytic solutions expressible in terms of a finite representation of elementary functions. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole. Our objective in this paper is to investigate the global stability character, boundedness and the periodicity of solutions of the rational difference equation

$$x_{n+1} = \frac{ax_{n-r}^2 + bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \quad n = 0, 1, ..., \tag{1.1}$$

where the parameters a, b, c and d are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, ..., x_{-1}$ and x_0 are positive real numbers where $t = \max\{r, k, l\}$.

Recently there has a lot of interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations for example ([1], [2], [3], [4], [5], [6], [7], [8], [9]).

Many researchers studied qualitative behaviors of the solution of difference equations for example; in [5] Elabbasy et al studied the global stability character, boundedeness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}.$$

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Elabbasy et al. [4] analyzed the global stability, periodicity character and gave the solution of special cases of the difference equation

$$x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a.$$

Wang et al. [24] studied the global attractivity of equilibrium points and the asymptotic behavior of the solutions of the solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{\alpha + bx_{n-s} + cx_{n-t}}.$$

Saleh and Baha [17] investigated the behavior of nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}.$$

Yan, Li and Zhao [26] studied boundedeness, periodic character, invariant intervals and the global asymptotic stability of the all nonnegative solutions of the difference equation

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_n}.$$

See also ([10],[11],[12],[13],[14],[15],[16],[17],[18]). Other related results can be found in ([19],[20],[21],[22],[23],[24],[25],[27],[28],[29],[30],[31],[32]).Let us introduce some basic definitions and some theorems that we need sequel.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

Definition 1.1. *([13], [16])(Equilibrium Point) A point $\bar{x} \in I$ is called an equilibrium point of Eq.(1.2) if*

$$\bar{x} = F(\bar{x}, \bar{x}, ..., \bar{x}).$$

Definition 1.2. *([13], [16])The difference equation (1.2) is said to be persistence if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that*

$$m \leq x_0 \leq M \quad \text{for all } n \geq N.$$

Definition 1.3. *([13], [16]) Stability*

(a) The equilibrium point $\bar{x}$ of Eq.(1.2) is called stable (or locally stable) if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\| x_0 - \bar{x} \| < \delta$ implies $\| x_n - \bar{x} \| < \epsilon$ for $n \geq 0$. Otherwise the equilibrium $\bar{x}$ is called unstable.

(b) The equilibrium point $\bar{x}$ of Eq.(1.2) is called asymptotically stable (or locally asymptotically stable) if it stable and there exists $\gamma > 0$ such that $\| x_0 - \bar{x} \| < \gamma$ implies

$$\lim_{n\to\infty} \| x_n - \bar{x} \| = 0.$$

(c) The equilibrium point $\bar{x}$ of Eq.(1.2) is called globally asymptotically stable if it is asymptotically stable, and if every x_0,

$$\lim_{n\to\infty} \| x_n - \bar{x} \| = 0.$$

(d) The equilibrium point $\bar{x}$ of Eq.(1.2) is called globally asymptotically stable relative to a set $s \subset \mathbb{R}^{k+1}$ if it is asymptotically stable, and if for every $x_0 \in s$,

$$\lim_{n\to\infty} \| x_n - \bar{x} \| = 0.$$

(e) The equilibrium point $\bar{x}$ of Eq.(1.2) is said to be a global attractor with basin of attraction a set $s \subset \mathbb{R}^{k+1}$ if

$$\lim_{n\to\infty} x_n = \bar{x}$$

for every solution with $x_0 \in s$.

Theorem 1.1. ([13], [16])*Assume $p, q \in \mathbb{R}$. Then a necessary and sufficient for the asymptotic stability of the difference equation*

$$x_{n+2} + px_{n+1} + qx_n = 0, \quad n = 0, 1, ... \tag{1.3}$$

is that

$$| p | < 1 + q < 2.$$

Theorem 1.2. ([13], [16])*Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, ...\}$. Then*

$$| p | + | q | < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ... \tag{1.4}$$

Theorem 1.3. ([13], [16])*Assume $p_1, ..., p_k \in \mathbb{R}$ and $k \in \{1, 2, ...\}$. Then the difference equation*

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0$$

is asymptotically stable provided that

$$\sum_{i=1}^{k} | p_i | < 1$$

Remark 1.1. ([12], [13])*The Linear equation*

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0, \quad n = 0, 1, 2, ... \tag{1.5}$$

where $p_1, ..., p_m \in (0, \infty)$ and $k_1, ...k_m$ are positive integers, is asymptotically stable provided that

$$\sum_{i=1}^{m} k_i p_i < 1.$$

([12], [13])Periodicity (a) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if

$$x_{n+p} = x_n \quad \text{for } n \geq -k. \tag{1.6}$$

The theory of Full Limiting Sequences was indicated in [15]. The following theorem was given in [5].

Theorem 1.4. ([12], [13])*Let $F \in [I^{k+1}, I]$ for some interval I of real numbers and for some non-negative integer k, and consider the difference equation*

$$x_{n+1} = F(x_n, x_{n-1}..., x_{n-k}), \tag{1.7}$$

Let l_0 be a limit point of the sequence $\{x_n\}_{n=-k}^{\infty}$. Then the following statements are true.

(i) There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of Eq.(1.7), called a full limiting sequence of $\{x_n\}_{n=-k}^{\infty}$, such that $L_0 = l_0$, and such that for every $N \in \{..., -1, 0, 1, ...\}$ L_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$.

(ii) For every $i_0 \leq -k$, there exists a subsequence $\{x_i\}_{i=-0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that

$$L_N = \lim_{i\to\infty} x_{r_i+N} \quad \textit{for all} \quad N \geq i_0.$$

2 Local Stability of the Equilibrium Point

In this section we investigate the local stability character of the solutions of Eq.(1.1). Eq.(1.1) has an equilibrium points are given by

$$\begin{aligned} \overline{x} &= f(\overline{x},\overline{x}) \\ &= \frac{a\overline{x}^2 + b\overline{x}^3}{c\overline{x}^2 + d\overline{x}^3} \\ &= \frac{a + b\overline{x}}{c + d\overline{x}} \end{aligned}$$

Then Eq.(1.1) has an equilibrium points $\overline{x} = \frac{b-c\pm\sqrt{(b-c)^2+4ad}}{2d}$.

Let $f:(0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v,w) = \frac{au^2 + bvw^2}{cu^2 + dvw^2}. \tag{2.8}$$

Therefore it follows that

$$f_u(u,v,w) = \frac{2uvw^2(ad-bc)}{(cu^2+dvw^2)^2},$$

$$f_v(u,v,w) = -\frac{u^2w^2(ad-bc)}{(cu^2+dvw^2)^2},$$

$$f_w(u,v,w) = -\frac{2u^2vw(ad-bc)}{(cu^2+dvw^2)^2}$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x}) = \frac{2(ad-bc)}{(c+d\overline{x})^2} = -c_0,$$

$$f_v(\overline{x},\overline{x},\overline{x}) = -\frac{(ad-bc)}{(c+d\overline{x})^2} = -c_1,$$

$$f_w(\overline{x},\overline{x},\overline{x}) = -\frac{2(ad-bc)}{(c+d\overline{x})^2} = -c_2.$$

At $\overline{x} = \frac{b-c+\sqrt{(b-c)^2+4ad}}{2d}$, one has $(c+d\overline{x})^2 = \frac{1}{4}(b+c+\sqrt{(b-c)^2+4ad})$.

Thus

$$f_u(\overline{x},\overline{x},\overline{x}) = \frac{8(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} = -c_0,$$

$$f_v(\overline{x},\overline{x},\overline{x}) = -\frac{4(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} = -c_1,$$

$$f_w(\overline{x},\overline{x},\overline{x}) = -\frac{8(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} = -c_2.$$

Then the linearized equation of Eq.(1.1) about is $\overline{x}$ is

$$y_{n+1} + c_0y_{n-r} + c_1y_{n-l} + c_2y_{n-k} = 0 \tag{2.9}$$

Theorem 2.5. *Assume that*

$$20 \mid (ad-bc) \mid < b+c+\sqrt{(b-c)^2+4ad}.$$

Then the positive equilibrium point $\overline{x} = \frac{b-c+\sqrt{(b-c)^2+4ad}}{2d}$ *of Eq.(1.1) is locally asymptotically stable.*

Proof. It follows by Theorem 1.3 that, Eq.(1.1) is asymptotically stable if

$$\mid c_0 \mid + \mid c_1 \mid + \mid c_2 \mid < 1$$

$$\mid \frac{8(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} \mid + \mid \frac{4(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} \mid + \mid \frac{8(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} \mid < 1,$$

or

$$20 \mid (ad-bc) \mid < b+c+\sqrt{(b-c)^2+4ad}.$$

The proof is complete. □

3 Existence of Periodic Solutions

In this section we study the existence of prime period two solutions of Eq.(1.1).

Theorem 3.6. *(i) Let r, l, k odd, then Eq.(1.1) has a prime period two solution for all $a, b, c, d \in \mathbb{R}^+$.*
(ii) Let r, k even, l odd, then Eq.(1.1) has a prime period two solution for all $a, b, c, d \in \mathbb{R}^+$.

Proof. We will prove the theorem when Case (i) is true. The proof of Case (ii) is similar.
First suppose that there exists a prime period two solution

$$..., p, q, p, q, ...,$$

of Eq.(1.1).
We see from Eq.(1.1) that

$$p = \frac{ap^2+bp^3}{cp^2+dp^3} = \frac{a+bp}{c+dp},$$

and

$$q = \frac{aq^2+bq^3}{cq^2+dq^3} = \frac{a+bq}{c+dq}.$$

Then

$$cp + dp^2 = a + bp \tag{3.10}$$

and

$$cq + dq^2 = a + bq \tag{3.11}$$

Subtracting (3.10) from (3.11) gives

$$c(p-q) + d(p^2-q^2) = b(p-q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{b-c}{d}. \tag{3.12}$$

Also, since p and q are positive, $(b-c)$ should be positive.
Again, adding (3.10) and (3.11) yields

$$c(p+q) + d(p^2+q^2) = 2a + b(p+q). \tag{3.13}$$

It follows by (3.12), (3.13) and the relation

$$p^2+q^2 = (p+q)^2 - 2pq \quad \text{for all } \; p, q \in \mathbb{R},$$

that

$$pq = -\frac{a}{d}. \tag{3.14}$$

It is clear that p and q are two real distinct roots of quadratic equation given by:

$$dt^2 - (b-c)t - a = 0,$$

for all $a, b, c, d \in \mathbb{R}^+$.

Second suppose that $a, b, c, d \in \mathbb{R}^+$.We will show that Eq.(1.1) has prime period two solutions.
Assume that

$$p = \frac{(b-c) + \sqrt{(b-c)^2 + 4ad}}{2d},$$

and

$$q = \frac{(b-c) - \sqrt{(b-c)^2 + 4ad}}{2d}$$

Therefore p and q are distinct real numbers.
Set

$$x_{-t} = p, x_{-t+1} = q, \cdots, x_{-1} = p, x_0 = q.$$

We wish to show that

$$x_1 = x_{-1} = p \text{ and } x_2 = x_0 = q.$$

It follows from Eq.(1.1) that

$$x_1 = \frac{a + bp}{c + dp} = p$$

Similarly we see that

$$x_2 = q.$$

Then Eq.(1.1) has the prime period two solution

$$..., p, q, p, q, ...,$$

where p and q are distinct roots of a quadratic equation and the proof is complete. □

4 Global Attractor of the Equilibrium Point of Eq.(1.1)

In this section we investigate the global attractivity character of solutions of Eq.(1.1).

Lemma 4.1. *For any values of the quotient $\frac{a}{c}$ and $\frac{b}{d}$, the function $f(u, v, w)$ defined by Eq.(2.8) is monotone in each of its three arguments.*

Theorem 4.7. *The equilibrium point $\overline{x}$ of Eq.(1.1) is global attractor if one of the following statments hold:*
(i) $ad \geq bc$ and $4c(\frac{b}{d})^4 - 4a(\frac{b}{d})^3 > -(b+c)(\frac{a}{c})^4$.
(ii) $ad \leq bc$ and $5d(\frac{a}{c})^4 - 4b(\frac{a}{c})^3 > -a(\frac{b}{d})^2$.

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be solution of Eq.(1.1) and again let f be function defined by Eq.(2.8).
We will prove the theorem when Case (i) is true. The proof of Case (ii) is similar. In case of (i), when $ad \geq bc$, the function $f(u, v, w)$ is non-decreasing in u and non-increasing in v, w. Thus from Eq.(1.1), we see that

$$x_{n+1} = \frac{ax_{n-r}^2 + bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \leq x_{n+1} = \frac{ax_{n-r}^2 + b(0)}{cx_{n-r}^2 + d(0)} = \frac{a}{c}.$$

Then

$$x_n \leq \frac{a}{c} = H \text{ for all } n \geq 1. \tag{4.15}$$

$$\begin{aligned} x_{n+1} &= \frac{ax_{n-r}^2 + bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \geq x_{n+1} = \frac{a(0) + bx_{n-l}x_{n-k}^2}{c(0) + dx_{n-l}x_{n-k}^2} \\ &\geq \frac{b}{d} = h \text{ for all } n \geq 1. \end{aligned} \tag{4.16}$$

Then from Eq.(4.15) and Eq.(4.16), we see that

$$0 < h = \frac{b}{d} \leq x_n \leq \frac{a}{c} = H \quad \text{for all } n \geq 1.$$

let $\{x_n\}_{n=0}^{\infty}$ be solution of Eq.(1.1) with

$$I = \liminf_{n\to\infty} x_n \text{ and } S = \limsup_{n\to\infty}.$$

We want to show that $I = S$.

Now it follows from Eq.(1.1) that

$$I \geq f(I, S, S),$$

or

$$I \geq \frac{aI^2 + bS^3}{cI^2 + dS^3}.$$

and so

$$aI^2 + bS^3 - cI^3 \leq dIS^3. \tag{4.17}$$

Similarly, we see from Eq.(1.1) that

$$S \leq f(S, I, I),$$

or

$$S \leq \frac{aS^2 + bI^3}{cS^2 + dI^3},$$

and so

$$aS^2 + bI^3 - cS^3 \geq dSI^3. \tag{4.18}$$

Therefore it follows from Eq.(4.17) and Eq.(4.18) that

$$aI^4 + bI^2S^3 - cI^5 \leq dI^3S^3 \leq aS^4 + bI^3S^2 - cS^5$$

$$c(I^5 - S^5) + bI^2S^2(I - S) - a(I^4 - S^4) \geq 0,$$

if and only if

$$(I - S)[c(I^4 + I^3S + I^2S^2 + IS^3 + S^4) + bI^2S^2 - a(I + S)(I^2 + S^2)] \geq 0,$$

and so $I \geq S$ if

$$c(I^4 + I^3S + I^2S^2 + IS^3 + S^4) + bI^2S^2 - a(I + S)(I^2 + S^2) \geq 0. \tag{4.19}$$

Inequality (4.19) can be written as:

$$c(I^4 + I^3S + IS^3 + S^4) + (b + c)I^2S^2 - a(I + S)(I^2 + S^2) \geq 0.$$

To prove Inequality (4.19), let us consider

$$\tau = c(I^4 + I^3S + IS^3 + S^4) - a(I + S)(I^2 + S^2)$$

Then, one has

$$\begin{aligned} \tau &\geq 4c(\frac{b}{d})^4 - 4a(\frac{b}{d})^3 \\ &\geq -(b + c)(\frac{a}{c})^4 \\ &\geq -(b + c)I^2S^2, \end{aligned}$$

and so it follows that

$$I \geq S.$$

Therefore

$$I = S.$$

This complete the proof. □

5 Boundedness of Solutions of Eq.(1.1)

In this section we study the boundedness of solutions of Eq.(1.1)

Theorem 5.8. *Every solution of Eq.(1.1) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of Eq.(1.1). Then

$$\begin{aligned} x_{n+1} &= \frac{ax_{n-r}^2 + bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \\ & \frac{ax_{n-r}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} + \frac{bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \\ &\leq \frac{ax_{n-r}^2}{cx_{n-r}^2} + \frac{bx_{n-l}x_{n-k}^2}{dx_{n-l}x_{n-k}^2} \\ &= \frac{a}{c} + \frac{b}{d}. \end{aligned}$$

Thus $x_N \leq \frac{a}{c} + \frac{b}{d} = M$ for all $N \geq 1$.

Let there exists $m > 0$ such that $x_N \geq m$ for all $N \geq 1$. Taking $x_N = \frac{1}{y_N}$, then one has

$$\begin{aligned} y_{n+1} &= \frac{cy_{n-r}^2 + dy_{n-l}y_{n-k}^2}{ay_{n-r}^2 + by_{n-l}y_{n-k}^2} \\ & \frac{cy_{n-r}^2}{ay_{n-r}^2 + by_{n-l}y_{n-k}^2} + \frac{dy_{n-l}y_{n-k}^2}{ay_{n-r}^2 + by_{n-l}y_{n-k}^2} \\ &\leq \frac{cy_{n-r}^2}{ay_{n-r}^2} + \frac{dy_{n-l}y_{n-k}^2}{by_{n-l}y_{n-k}^2} \\ &= \frac{c}{a} + \frac{d}{b}. \end{aligned}$$

Thus $x_N = \frac{1}{y_N} \geq \frac{1}{h} = \frac{ab}{ad+bc} = m$ for all $N \geq 1$. Hence, $m \leq x_N \leq M$ for all $N \geq 1$. □

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On boundary value problems for fractional integro-differential equations in Banach spaces

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Abstract

This paper aims to study the existence and uniqueness of solutions of fractional integro-differential equations in Banach spaces by applying a new generalized singular type Gronwall's inequality, fixed point theorems and Hölder inequality. Example is provided to illustrate the main results.

Keywords: Fractional integro-differential equations, boundary value problems, existence and uniqueness, generalized singular type Gronwall's inequality, fixed point theorems.

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1 Introduction

This paper deals with the existence and uniqueness of solutions of boundary value problems (for short BVP) for fractional integro-differential equations given by

$$\begin{cases} {}^cD^{\alpha}x(t) = f\big(t, x(t), (Sx)(t)\big), t \in J = [0, T], \alpha \in (0, 1], \\ ax(0) + bx(T) = c, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative of order $\alpha, f : J \times X \times X \to X$ is a given function satisfying some assumptions that will be specified later and a, b, c are real numbers with $a + b \neq 0$ and S is a nonlinear integral operator given by $(Sx)(t) = \int_0^t k\big(t, s, x(s)\big)ds$, where $k \in C(J \times J \times X, X)$.

The ordinary differential equations is considered the basis of the fractional differential equations. In the last few decades, fractional order models are found to be more adequate than integer order models for some real world problems. For more details about fractional calculus and its applications we refer the reader to the monographs of Hilfer [6], Kilbas et al. [8], Miller and Ross [9], Podlubny [10], Samko et al. [11] and the references given therein. Recently, some fractional differential equations and optimal controls in Banach spaces were studied by Balachandran and Park [2], El-Borai [3], Henderson and Ouahab [4], Hernandez et al. [5], Wang et al. [14] and Wang et al. [15, 16]. Very recently, Karthikeyan and Trujillo [7] and Wang et al. [13] have extended the work in [1] from real line $\mathbb{R}$ to the abstract Banach space X by using more general assumptions on the nonlinear function f. Our attempt is to generalize the results proved in [1, 7, 13].

This paper is organized as follows. In Section 2, we set forth some preliminaries. Section 3 introduces a new generalized singular type Gronwall inequality to establish the estimate for priori bounds. In Section 4, we prove our main results by applying Banach contraction principle and Schaefer's fixed point theorem. Finally, in Section 5, application of the main results is exhibited.

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2 Preliminaries

Before proceeding to the statement of our main results, we set forth some preliminaries. Let the Banach space of all continuous functions from J into X with the supremum norm $\|x\|_\infty := \sup\{\|x(t)\| : t \in J\}$ is denoted by $C(J, X)$. For measurable functions $m : J \to \mathbb{R}$, define the norm $\|m\|_{L^p(J,\mathbb{R})} = \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}}, 1 \le p < \infty$, where $L^p(J, \mathbb{R})$ the Banach space of all Lebesgue measurable functions m with $\|m\|_{L^p(J,\mathbb{R})} < \infty$.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a suitable function h is defined by*

$$I^\alpha_{a+}h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

where $a \in \mathbb{R}$ and Γ is the Gamma function.

Definition 2.2. *For a suitable function h given on the interval $[a, b]$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ of h, is defined by*

$$(D^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Definition 2.3. *For a suitable function h given on the interval $[a, b]$, the Caputo fractional order derivative of order $\alpha > 0$ of h, is defined by*

$$({}^cD^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Lemma 2.1. *([8, 17]) Let $\alpha > 0$; then the differential equation ${}^cD^\alpha h(t) = 0$, has the following general solution $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where $n = [\alpha] + 1$.*

Lemma 2.2. *([8, 17]) Let $\alpha > 0$; then*

$$I^\alpha({}^cD^\alpha h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where $n = [\alpha] + 1$.

Definition 2.4. *A function $x \in C^1(J, X)$ is said to be a solution of the fractional BVP (1.1) if x satisfies the equation ${}^cD^\alpha x(t) = f\big(t, x(t), (Sx)(t)\big)$ a.e. on J, and the condition $ax(0) + bx(T) = c$.*

For the existence of solutions for the fractional BVP (1.1), we need the following auxiliary lemma.

Lemma 2.3. *Let $\overline{f} : J \to X$ be continuous. A function $x \in C(J, X)$ is solution of the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s)ds - \frac{1}{a+b}\Big[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \overline{f}(s)ds - c\Big], \tag{2.2}$$

if and only if x is a solution of the following fractional BVP

$$\begin{cases} {}^cD^\alpha x(t) = \overline{f}(t), t \in J = [0, T], \alpha \in (0, 1], \\ ax(0) + bx(T) = c. \end{cases} \tag{2.3}$$

Proof. Assume that x satisfies fractional BVP (2.3); then by using Lemma 2.2 and Def. 2.1, we get

$$x(t) + c_0 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s)ds,$$

where $c_0 \in \mathbb{R}$, that is:

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s) ds - c_0. \tag{2.4}$$

By applying boundary condition $ax(0) + bx(T) = c$, we have

$$c_0 = \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \overline{f}(s) ds - \frac{c}{a+b}.$$

Now, by substituting the value of c_0 in (2.4), we obtain

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \overline{f}(s) ds + \frac{c}{a+b}.$$

Conversely, it is clear that if x satisfies fractional integral equation (2.2), then fractional BVP (2.3) is also satisfied. □

As a consequence of lemma 2.3, we have the following result which is useful in what follows.

Lemma 2.4. *Let $f : J \times X \times X \to X$ be continuous function. Then, $x \in C(J, X)$ is a solution of the fractional integral equation*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big) ds \\ &\quad - \frac{1}{a+b} \Big[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big) ds - c \Big], \end{aligned} \tag{2.5}$$

if and only if x is solution of the fractional BVP (1.1).

Lemma 2.5. *(Bochner theorem) A measurable function $f : J \to X$ is Bochner integrable if $\|f\|$ is Lebesgue integrable.*

Lemma 2.6. *(Mazur theorem, [12]) Let X be a Banach space. If $U \subset X$ is relatively compact, then $conv(U)$ is relatively compact and $\overline{conv}(U)$ is compact.*

Lemma 2.7. *(Ascoli-Arzela theorem) Let $S = \{s(t)\}$ is a function family of continuous mappings $s : [a, b] \to X$. If S is uniformly bounded and equicontinuous, and for any $t^* \in [a, b]$, the set $\{s(t^*)\}$ is relatively compact, then, there exists a uniformly convergent function sequence $\{s_n(t)\} (n = 1, 2, \ldots, t \in [a, b])$ in S.*

Lemma 2.8. *(Schaefer's fixed point theorem) Let $F : X \to X$ be a completely continuous operator. If the set $E(F) = \{x \in X : x = \eta F x$ for some $\eta \in [0, 1]\}$ is bounded, then, F has fixed points.*

3 A generalized singular type Gronwall's inequality

Before dealing with the main reslts, we need to introduce a new generalized singular Gronwall type inequality with mixed type singular integral operator.

We, first, state a generalized Gronwall inequality from [15].

Lemma 3.9. *(Lemma 3.2, [15]) Let $x \in C(J, X)$ satisfies the following inequality:*

$$\begin{aligned} \|x(t)\| \le a &+ b \int_0^t \|x(\theta)\|^{\lambda_1} d\theta + c \int_0^T \|x(\theta)\|^{\lambda_2} d\theta + d \int_0^t \|x_\theta\|_B^{\lambda_3} d\theta \\ &+ e \int_0^T \|x_\theta\|_B^{\lambda_4} d\theta, t \in J, \end{aligned}$$

where $\lambda_1, \lambda_3 \in [0, 1], \lambda_2, \lambda_4 \in [0, 1), a, b, c, d, e \ge 0$ are constants and
$\|x_\theta\|_B = \sup_{0 \le s \le \theta} \|x(s)\|$. Then there exists a constant $L > 0$ such that

$$\|x(t)\| \le L.$$

Using the above generalized Gronwall inequality, we can obtain the following new generalized singular type Gronwall inequality.

Lemma 3.10. *Let $x \in C(J, X)$ satisfies the following inequality:*

$$\begin{aligned} \|x(t)\| &\le a + b\int_0^t (t-s)^{\alpha-1}\|x(s)\|^{\lambda}ds + c\int_0^T (T-s)^{\alpha-1}\|x(s)\|^{\lambda}ds \\ &\quad + d\int_0^t (t-s)^{\alpha-1}\|x_s\|_B^{\lambda}ds + e\int_0^T (T-s)^{\alpha-1}\|x_s\|_B^{\lambda}ds, \end{aligned} \tag{3.6}$$

where $\alpha \in (0,1], \lambda \in [0, 1-\frac{1}{p})$ for some $1 < p < \frac{1}{1-\alpha}, \|x_s\|_B = \sup_{0\le\tau\le s}\|x(\tau)\|$ and $a, b, c, d, e \ge 0$ are constants.Then, there exists a constant $L > 0$, such that

$$\|x(t)\| \le L.$$

Proof. Let

$$y(t) = \begin{cases} 1, & \|x(t)\| \le 1, \\ x(t), & \|x(t)\| > 1. \end{cases}$$

Using (3.6) and Hölder inequality, we get

$$\begin{aligned} \|x(t)\| &\le \|y(t)\| \\ &\le (a+1) + b\int_0^t (t-s)^{\alpha-1}\|y(s)\|^{\lambda}ds + c\int_0^T (T-s)^{\alpha-1}\|y(s)\|^{\lambda}ds \\ &\quad + d\int_0^t (t-s)^{\alpha-1}\|y_s\|_B^{\lambda}ds + e\int_0^T (T-s)^{\alpha-1}\|y_s\|_B^{\lambda}ds \\ &\le (a+1) + b\left(\int_0^t (t-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t \|y(s)\|^{\frac{\lambda p}{p-1}}ds\right)^{\frac{p-1}{p}} \\ &\quad + c\left(\int_0^T (T-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}}\left(\int_0^T \|y(s)\|^{\frac{\lambda p}{p-1}}ds\right)^{\frac{p-1}{p}} \\ &\quad + d\|y_s\|_B^{\lambda}\int_0^t (t-s)^{\alpha-1}ds + e\|y_s\|_B^{\lambda}\int_0^T (T-s)^{\alpha-1}ds \\ &\le (a+1) + b\left(\frac{t^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{\frac{1}{p}}\int_0^t \|y(s)\|^{\frac{\lambda p}{p-1}}ds \\ &\quad + c\left(\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{\frac{1}{p}}\int_0^T \|y(s)\|^{\frac{\lambda p}{p-1}}ds \\ &\quad + d\|y_s\|_B^{\lambda}\frac{t^{\alpha}}{\alpha} + e\|y_s\|_B^{\lambda}\frac{T^{\alpha}}{(\alpha)} \\ &\le (a+1) + d\|y_s\|_B^{\lambda}\frac{T^{\alpha}}{\alpha} + e\|y_s\|_B^{\lambda}\frac{T^{\alpha}}{(\alpha)} \\ &\quad + b\left(\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{\frac{1}{p}}\int_0^t \|y(s)\|^{\frac{\lambda p}{p-1}}ds \\ &\quad + c\left(\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{\frac{1}{p}}\int_0^T \|y(s)\|^{\frac{\lambda p}{p-1}}ds, \end{aligned}$$

where $0 < \frac{\lambda p}{p-1} < 1$.
Hence, by lemma 3.9 there exists a constant $L > 0$, such that $\|x(t)\| \le L$.

□

4 Main results

For convenience, we list hypotheses that will be used in our further discussion.

• (H1) The function $f : J \times X \times X \to X$ is measurable with respect to t on J and is continuous with respect to x on X.
• (H2) There exists a constant $\alpha_1 \in (0, \alpha)$ and real-valued functions $m_1(t), m_2(t) \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R})$, such that

$$\|f(t, x(t), (Sx)(t)) - f(t, y(t), (Sy)(t))\| \leq m_1(t)\big(\|x(t) - y(t)\| + \|Sx(t) - Sy(t)\|\big),$$
$$\|k(t, s, x(s)) - k(t, s, y(s))\| \leq m_2(t)\|x(s) - y(s)\|,$$

for each $s \in [0, t]$, $t \in J$ and all $x, y \in X$.
• (H3) There exists a constant $\alpha_2 \in (0, \alpha)$ and real-valued function $h(t) \in L^{\frac{1}{\alpha_2}}(J, \mathbb{R})$, such that $\|f(t, x(t), (Sx)(t))\| \leq h(t)$, for each $t \in J$, and all $x \in X$.

For brevity, let $M = \|m_1 + m_1 m_2 T\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R})}$ and $H = \|h\|_{L^{\frac{1}{\alpha_2}}(J,\mathbb{R})}$.
• (H4) There exist constants $\lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{1-\alpha}$ and $N_f, N_k > 0$, such that

$$\|f(t, x(t), (Sx)(t))\| \leq N_f\big(1 + \|x(t)\|^{\lambda} + \|(Sx)(t)\|\big),$$
$$\|k(t, s, x(s))\| \leq N_k\big(1 + \|x(s)\|^{\lambda}\big),$$

for each $s \in [0, t]$, $t \in J$ and all $x \in X$.
• (H5) For every $t \in J$, the set
$K_1 = \big\{(t - s)^{\alpha-1} f(s, x(s), (Sx)(s)) : x \in C(J, X), s \in [0, t]\big\}$ is relatively compact.

Now, we are in position to deal with our main results.

Theorem 4.1. *Assume that (H1)-(H3) hold. If*

$$\Omega_{\alpha,T} = \frac{M}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left(1 + \frac{|b|}{|a+b|}\right) < 1. \tag{4.7}$$

Then, the fractional BVP (1.1) has a unique solution on J.

Proof. By making use of hypothesis (H3) and Hölder inequality, for each $t \in J$, we have

$$\begin{aligned}
\int_0^t \|(t-s)^{\alpha-1} f(s, x(s), (Sx)(s))\| ds &\leq \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| ds \\
&\leq \int_0^t (t-s)^{\alpha-1} h(s) ds \\
&\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\right)^{1-\alpha_2} \left(\int_o^t (h(s))^{\frac{1}{\alpha_2}} ds\right)^{\alpha_2} \\
&\leq H \left(\left[\frac{-(t-s)^{\frac{\alpha-1}{1-\alpha_2}+1}}{\frac{\alpha-1}{1-\alpha_2}+1}\right]_0^t\right)^{1-\alpha_2} \leq H \frac{T^{\alpha-\alpha_2}}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}.
\end{aligned}$$

Thus, $\|(t - s)^{\alpha-1} f(s, x(s), (Sx)(s))\|$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in J$ and $x \in C(J, X)$. Then, $(t - s)^{\alpha-1} f(s, x(s), (Sx)(s))$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in J$ due to lemma 2.5.

Hence, the fractional BVP (1.1) is equivalent to the following fractional integral equation

$$\begin{aligned}
x(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds \\
&- \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds - c\right],
\end{aligned} \tag{4.8}$$

Now, let $B_r = \{x \in C(J, X) : \|x\|_\infty \leq r\}$, where

$$r \geq \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|b|}{|a+b|} \times \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|c|}{|a+b|}. \tag{4.9}$$

Define the operator F on B_r as follows:

$$(F(x))(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big)ds - \frac{1}{a+b}\Big[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big)ds - c\Big], t \in J. \tag{4.10}$$

Clearly, the solution of the fractional BVP (1.1) is the fixed point of the operator F on B_r. We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $F(x) \in B_r$ for every $x \in B_r$.
For every $x \in B_r$ and $\delta > 0$, by (H3) and Hölder inequality, we have

$$\begin{aligned}
&\|(F(x))(t+\delta) - (F(x))(t)\| \\
&\le \Big\| \frac{1}{\Gamma(\alpha)}\int_0^{t+\delta} (t+\delta-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big)ds \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big)ds \Big\| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t \Big[(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}\Big] \|f\big(s, x(s), (Sx)(s)\big)\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^{t+\delta} (t+\delta-s)^{\alpha-1}\|f\big(s, x(s), (Sx)(s)\big)\|ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}]h(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^{t+\delta} (t+\delta-s)^{\alpha-1}h(s)ds \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} \\
&\quad - \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_t^{t+\delta} (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_t^{t+\delta} (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} \\
&\le \frac{H}{\Gamma(\alpha)}\left(\frac{-\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} + \frac{(t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2} - \frac{H}{\Gamma(\alpha)}\left(\frac{t^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2} + \frac{H}{\Gamma(\alpha)}\left(\frac{\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}.
\end{aligned}$$

It is obvious that the right-hand side of the above inequality tends to zero as $\delta \to 0$. Therefore, F is continuous on J, that is, $F(x) \in C(J, X)$. Moreover, for $x \in B_r$ and all $t \in J$, by using (4.9), we have

$$\begin{aligned}
\|(F(x))(t)\| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f\big(s, x(s), (Sx)(s)\big)\|ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|f\big(s, x(s), (Sx)(s)\big)\|ds + \frac{|c|}{|a+b|} \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds + \frac{|c|}{|a+b|} \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} \\
&\quad + \frac{|b|}{(|a+b|)\Gamma(\alpha)}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^T (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} + \frac{|c|}{|a+b|}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|b|}{|a+b|} \times \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|c|}{|a+b|} \\
&\leq r.
\end{aligned}
$$

Thus, $\|F(x)\|_\infty \leq r$ and we conclude that for all $x \in B_r$, $F(x) \in B_r$, that is, $F : B_r \to B_r$.

Step 2. F is contraction mapping on B_r.
For $x, y \in B_r$ and any $t \in J$, by using (4.7), (H2) and Hölder inequality, we have

$$
\begin{aligned}
&\|(F(x))(t) - (F(y))(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f\big(s, x(s), (Sx)(s)\big) - f\big(s, y(s), (Sy)(s)\big) \right\| ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left\| f\big(s, x(s), (Sx)(s)\big) - f\big(s, y(s), (Sy)(s)\big) \right\| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) \big(\|x(s) - y(s)\| + \|(Sx)(s) - (Sy)(s)\| \big) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m_1(s) \big(\|x(s) - y(s)\| + \|(Sx)(s) - (Sy)(s)\| \big) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) \\
&\quad \times \left(\|x(s) - y(s)\| + \int_0^s \left\| k\big(s,\tau,x(\tau)\big) - k\big(s,\tau,y(\tau)\big) \right\| d\tau \right) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m_1(s) \\
&\quad \times \left(\|x(s) - y(s)\| + \int_0^s \left\| k\big(s,\tau,x(\tau)\big) - k\big(s,\tau,y(\tau)\big) \right\| d\tau \right) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) \\
&\quad \times \left(\|x(s) - y(s)\| + \int_0^s m_2(s) \|x(\tau) - y(\tau)\| d\tau \right) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m_1(s) \\
&\quad \times \left(\|x(s) - y(s)\| + \int_0^s m_2(s) \|x(\tau) - y(\tau)\| d\tau \right) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) \big(\|x-y\|_\infty + m_2(s) T \|x-y\|_\infty \big) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m_1(s) \big(\|x-y\|_\infty + m_2(s) T \|x-y\|_\infty \big) ds \\
&\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \left(\int_0^t \big(m_1(s) + m_1(s) m_2(s) T\big)^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \\
&\quad \times \left(\int_0^T \big(m_1(s) + m_1(s) m_2(s) T\big)^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\
&\leq \frac{M\|x-y\|_\infty}{\Gamma(\alpha)} \left(\frac{t^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}} \right)^{1-\alpha_1} + \frac{M|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)} \left(\frac{T^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}} \right)^{1-\alpha_1} \\
&\leq \left[\frac{M}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left(1 + \frac{|b|}{|a+b|} \right) \right] \|x-y\|_\infty. \\
&= \Omega_{\alpha,T} \|x-y\|_\infty.
\end{aligned}
$$

Thus, we have

$$\|F(x) - F(y)\|_\infty \leq \Omega_{\alpha,T}\|x - y\|_\infty.$$

Since $\Omega_{\alpha,T} < 1$, F is contraction. By Banach contraction principle, we can deduce that F has a unique fixed point which is the unique solution of the fractional BVP (1.1). □

Our second main result is based on the well known Schaefer's fixed point theorem.

Theorem 4.2. *Assume that(H1), (H4) and (H5) hold. Then the fractional BVP (1.1) has at least one solution on J.*

Proof. Transform the fractional BVP (1.1) into a fixed point problem. Consider the operator $F : C(J, X) \to C(J, X)$ defined as (4.10). It is obvious that F is well defined due to (H1), Hölder inequality and the lemma 2.5.

For the sake of convenience, we subdivide the proof into several steps.
Step 1. F is continuous operator.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in C(J,X). Then for each $t \in J$, we have

$$\begin{aligned}
&\|(F(x_n))(t) - (F(x))(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|f\big(s, x_n(s), (Sx_n)(s)\big) - f\big(s, x(s), (Sx)(s)\big)\right\| ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left\|f\big(s, x_n(s), (Sx_n)(s)\big) - f\big(s, x(s), (Sx)(s)\big)\right\| ds \\
&\leq \|f(., x_n(.), (Sx_n)(.)) - f(., x(.), (Sx)(.))\|_\infty \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \\
&\quad + \left\|f\big(., x_n(.), (Sx_n)(.)\big) - f\big(., x(.), (Sx)(.)\big)\right\|_\infty \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} ds \\
&\leq \left\|f\big(., x_n(.), (Sx_n)(.)\big) - f\big(., x(.), (Sx)(.)\big)\right\|_\infty \frac{t^\alpha}{\alpha\Gamma(\alpha)} \\
&\quad + \left\|f\big(., x_n(.), (Sx_n)(.)\big) - f\big(., x(.), (Sx)(.)\big)\right\|_\infty \frac{|b|}{|a+b|\Gamma(\alpha)}\frac{T^\alpha}{\alpha} \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right)\left\|f\big(., x_n(.), (Sx_n)(.)\big) - f\big(., x(.), (Sx)(.)\big)\right\|_\infty.
\end{aligned}$$

Taking supremum, we get

$$\begin{aligned}
&\|Fx_n - Fx\|_\infty \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right)\left\|f\big(., x_n(.), (Sx_n)(.)\big) - f\big(., x(.), (Sx)(.)\big)\right\|_\infty,
\end{aligned}$$

since f is continuous, we have

$$\|Fx_n - Fx\|_\infty \to 0 \text{ as } n \to \infty.$$

Therefore, F is continuous operator.

Step 2. F maps bounded sets into bounded sets in $C(J, X)$.
Indeed, it is enough to show that for any $\eta^* > 0$, there exists a $l > 0$ such that for each

$x \in B_{\eta^*} = \{x \in C(J, X) : \|x\|_\infty \leq \eta^*\}$, we have $\|Fx\|_\infty \leq l$.
For each $t \in J$, by (H4), we get

$$\begin{aligned}
\|(F(x))(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| ds + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f (1 + \|x(s)\|^\lambda + \|(Sx)(s)\|) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} N_f (1 + \|x(s)\|^\lambda + \|(Sx)(s)\|) ds + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \left(1 + \|x(s)\|^\lambda + \int_0^s \|k(s, \tau, x(\tau))\| d\tau\right) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \\
&\quad \times N_f \left(1 + \|x(s)\|^\lambda + \int_0^s \|k(s, \tau, x(\tau))\| d\tau\right) ds + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k (1 + \|x(\tau)\|^\lambda) d\tau\right) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \\
&\quad \times N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k (1 + \|x(\tau)\|^\lambda) d\tau\right) ds + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f (1 + \|x\|_\infty^\lambda + N_k (1 + \|x\|_\infty^\lambda) T) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \\
&\quad \times N_f (1 + \|x\|_\infty^\lambda + N_k (1 + \|x\|_\infty^\lambda) T) ds + \frac{|c|}{|a+b|} \\
&\leq \frac{N_f (1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&\quad + \frac{N_f (1 + (\eta^*)^\lambda)(1 + N_k T)|b|}{\Gamma(\alpha)|a+b|} \int_0^T (T-s)^{\alpha-1} ds + \frac{|c|}{|a+b|} \\
&\leq \frac{N_f (1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} \\
&\quad + \frac{N_f (1 + (\eta^*)^\lambda)(1 + N_k T)|b|}{\Gamma(\alpha)|a+b|} \frac{T^\alpha}{\alpha} + \frac{|c|}{|a+b|} \\
&\leq \left(1 + \frac{|b|}{|a+b|}\right) \frac{T^\alpha}{\Gamma(\alpha+1)} N_f (1 + (\eta^*)^\lambda)(1 + N_k T) + \frac{|c|}{|a+b|} \\
&\leq l,
\end{aligned}$$

where

$$l := \left(1 + \frac{|b|}{|a+b|}\right) \frac{T^\alpha}{\Gamma(\alpha+1)} N_f (1 + (\eta^*)^\lambda)(1 + N_k T) + \frac{|c|}{|a+b|}.$$

Thus, we have
$\|(F(x))(t)\| \leq l$ and hence $\|Fx\|_\infty \leq l$.

Step 3. F maps bounded sets into equicontinuous sets of $C(J, X)$.
Let $0 \leq t_1 \leq t_2 \leq T, x \in B_{\eta^*}$. Using (H4), again we have

$$
\begin{aligned}
&\|(F(x))(t_2)-(F(x))(t_1)\| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\|f(s,x(s),(Sx)(s))\|\,ds \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|f(s,x(s),(Sx)(s))\|\,ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]N_f\big(1+\|x(s)\|^{\lambda}+\|(Sx)(s)\|\big)ds \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}N_f\big(1+\|x(s)\|^{\lambda}+\|(Sx)(s)\|\big)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]N_f\big(1+\|x\|_\infty^{\lambda}+N_k(1+\|x\|_\infty^{\lambda})T\big)ds \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}N_f\big(1+\|x\|_\infty^{\lambda}+N_k(1+\|x\|_\infty^{\lambda})T\big)ds \\
&\le \frac{N_f(1+(\eta^*)^{\lambda})(1+N_kT)}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds \\
&\quad+\frac{N_f(1+(\eta^*)^{\lambda})(1+N_kT)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\le \frac{N_f(1+(\eta^*)^{\lambda})(1+N_kT)}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha})
\end{aligned}
$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero and since x is an arbitrary in B_{η^*}, F is equicontinuous.

Now, let $\{x_n\}, n=1,2,\dots$ be a sequence on B_{η^*}, and

$$(Fx_n)(t)=(F_1x_n)(t)+(F_2x_n)(T), t\in J,$$

where

$$
\begin{aligned}
(F_1x_n)(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_n(s),(Sx_n)(s))ds, t\in J,\\
(F_2x_n)(T)&=-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s,x_n(s),(Sx_n)(s))ds+\frac{c}{a+b}.
\end{aligned}
$$

In view of hypothesis (H5) and lemma 2.6, the set $\overline{conv}K_1$ is compact. For any $t^*\in J$,

$$
\begin{aligned}
(F_1x_n)(t^*)&=\frac{1}{\Gamma(\alpha)}\int_0^{t^*}(t^*-s)^{\alpha-1}f(s,x_n(s),(Sx_n)(s))ds\\
&=\frac{1}{\Gamma(\alpha)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}\left(t^*-\frac{it^*}{k}\right)^{\alpha-1}f\left(\frac{it^*}{k},x_n(\frac{it^*}{k}),(Sx_n)(\frac{it^*}{k})\right)\\
&=\frac{t^*}{\Gamma(\alpha)}\zeta_n,
\end{aligned}
$$

where

$$\zeta_n=\lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}\left(t^*-\frac{it^*}{k}\right)^{\alpha-1}f\left(\frac{it^*}{k},x_n(\frac{it^*}{k}),(Sx_n)(\frac{it^*}{k})\right).$$

Now, we have $\{(F_1x_n)(t)\}$ is a function family of continuous mappings $F_1x_n : J\to X$, which is uniformly bounded and equicontinuous. As $\overline{conv}K_1$ is convex and compact, we know $\zeta_n\in\overline{conv}K_1$. Hence, for any

$t^* \in J = [0,T]$, the set $\{(F_1x_n)(t^*)\}$, is relatively compact. Therefore by lemma 2.7, every $\{(F_1x_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_1x_{n_k})(t)\}, k = 1,2,\dots,$ on J. Thus, $\{F_1x : x \in B_{\eta^*}\}$ is relatively compact. Similarly, one can obtain $\{(F_2x_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_2x_{n_k})(t)\}, k = 1,2,\dots,$ on J. Thus, $\{F_2x : x \in B_{\eta^*}\}$ is relatively compact. As a result, the set $\{Fx : x \in B_{\eta^*}\}$ is relatively compact.

As a consequence of steps 1-3, we can conclude that F is continuous and completely continuous.

Step 4. A priori bounds.
Now it remains to show that the set

$$E(F) = \{x \in C(J,X) : x = \eta Fx \text{ for some } \eta \in [0,1]\},$$

is bounded
Let $x \in E(F)$, then $x = \eta Fx$ for some $\eta \in [0,1]$. Thus, for each $t \in J$, we have

$$\begin{aligned} x(t) = \eta \bigg(&\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big) ds \\ &- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big) ds + \frac{c}{a+b} \bigg). \end{aligned}$$

Using (H4), for each $t \in J$, we have

$$\begin{aligned} \|x(t)\| &\le \|(F(x))(t)\| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k(1+\|x(\tau)\|^\lambda) d\tau\right) ds \\ &\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \\ &\quad \times N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k(1+\|x(\tau)\|^\lambda) d\tau\right) ds + \frac{|c|}{|a+b|} \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f\big(1 + \|x(s)\|^\lambda + N_k(1+\|x_s\|_B^\lambda)T\big) ds \\ &\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \\ &\quad \times N_f\big(1 + \|x(s)\|^\lambda + N_k(1+\|x_s\|_B^\lambda)T\big) ds + \frac{|c|}{|a+b|} \\ &\le \frac{N_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{N_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|^\lambda ds \\ &\quad + \frac{N_f N_k T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{N_f N_k T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s\|_B^\lambda ds \\ &\quad + \frac{|b|N_f}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \\ &\quad + \frac{|b|N_f}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|x(s)\|^\lambda ds \\ &\quad + \frac{|b|N_f N_k T}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \\ &\quad + \frac{|b|N_f N_k T}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|x_s\|_B^\lambda ds + \frac{|c|}{|a+b|} \\ &\le \frac{N_f T^\alpha}{\Gamma(\alpha+1)} + \frac{N_f N_k T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{|b|N_f T^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{|b|N_f N_k T^{\alpha+1}}{|a+b|\Gamma(\alpha+1)} + \frac{|c|}{|a+b|} \\ &\quad + \frac{N_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|^\lambda ds + \frac{|b|N_f}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|x(s)\|^\lambda ds \end{aligned}$$

$$+\frac{N_f N_k T}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x_s\|_B^\lambda ds + \frac{|b|N_f N_k T}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|x_s\|_B^\lambda ds.$$

By lemma 3.10, there exists a $N > 0$ such that $\|x(t)\| \le N, t \in J$.
Thus for every $t \in J$, we have $\|x\|_\infty \le N$. This show that the set $E(F)$ is bounded.
As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point that is solution of fractional BVP (1.1). □

5 Examples

In this section, we give one example to illustrate the usefulness of our main results.

Example 5.1.

$$\begin{cases} {}^cD^{\frac{1}{2}}x(t) = \frac{e^{-\sigma t}}{1+e^t}\left(\frac{|x(t)|}{1+|x(t)|} + \int_0^t \frac{(s+|x(s)|)}{(2+t)^2(1+|x(s)|)}ds\right), t \in J_1, \alpha \in (0,1], \\ x(0)+x(T) = 0, \end{cases} \tag{5.11}$$

where $\sigma > 0$ is constant.

Take $X_1 = [0,\infty), J_1 = [0,1]$ and so $T = 1$.

Set

$$f_1(t,x(t),(Sx)(t)) = \frac{e^{-\sigma t}}{1+e^t}\left(\frac{|x(t)|}{1+|x(t)|} + (Sx)(t)\right), \quad k_1(t,s,x(s)) = \frac{s+|x(s)|}{(2+t)^2(1+|x(s)|)}.$$

Let $x_1, x_2 \in C(J_1, X_1)$ and $t \in [0,1]$, we have

$$\begin{aligned} |k_1(t,s,x_1(s)) - k_1(t,s,x_2(s))| &\le \frac{1}{(2+t)^2}\left|\frac{(1-s)\left(|x_1(s)|-|x_2(s)|\right)}{(1+|x_1(s)|)(1+|x_2(s)|)}\right| \\ &\le \frac{1}{(2+t)^2}\left|x_1(s)-x_2(s)\right| \\ &\le \frac{1}{4}\left|x_1(s)-x_2(s)\right|, \end{aligned}$$

and

$$\begin{aligned} &|f_1(t,x_1(t),(Sx_1)(t)) - f_1(t,x_2(t),(Sx_2)(t))| \\ &\le \frac{e^{-\sigma t}}{1+e^t}\left(\left|\frac{|x_1(t)|}{1+|x_1(t)|} - \frac{|x_2(t)|}{1+|x_2(t)|}\right| + \left|(Sx_1)(t)-(Sx_2)(t)\right|\right) \\ &\le \frac{e^{-\sigma t}}{2}\left(\left|\frac{|x_1(t)|-|x_2(t)|}{(1+|x_1(t)|)(1+|x_2(t)|)}\right| + \left|(Sx_1)(t)-(Sx_2)(t)\right|\right) \\ &\le \frac{e^{-\sigma t}}{2}\left(\left|x_1(t)-x_2(t)\right| + \left|(Sx_1)(t)-(Sx_2)(t)\right|\right). \end{aligned}$$

Also, for all $x \in C(J_1, X_1)$ and each $t \in J_1$, we have

$$\begin{aligned} |f_1(t,x(t),(Sx)(t))| &\le \frac{e^{-\sigma t}}{1+e^t}\left(\left|\frac{|x(t)|}{1+|x(t)|}\right| + \int_0^t \left|\frac{s+|x(s)|}{(2+t)^2(1+|x(s)|)}\right| ds\right) \\ &\le \frac{e^{-\sigma t}}{2}\left(1+\frac{1}{4}\right) \le \left(\frac{5}{4}\right)\frac{e^{-\sigma t}}{2}. \end{aligned}$$

For $t \in J_1, \beta \in (0, \frac{1}{2})$, we have
$m_1(t) = \frac{e^{-\sigma t}}{2} \in L^{\frac{1}{\beta}}(J_1, \mathbb{R}), m_2(t) = \frac{1}{4} \in L^{\frac{1}{\beta}}(J_1, \mathbb{R}), h(t) = \left(\frac{5}{4}\right) \frac{e^{-\sigma t}}{2} \in L^{\frac{1}{\beta}}(J_1, \mathbb{R})$ and $M = \| \left(\frac{5}{4}\right) \frac{e^{-\sigma t}}{2} \|_{L^{\frac{1}{\beta}}(J_1, \mathbb{R})}$.
Choosing some $\sigma > 0$ large enough and $\beta = \frac{1}{4} \in (0, \frac{1}{2})$, one can arrive at the following inequality

$$\Omega_{\frac{1}{2},\frac{1}{4}} = \frac{M}{\Gamma(\frac{1}{2})} \frac{1}{(\frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{1}{4}})^{1-\frac{1}{4}}} \left(1 + \frac{1}{2}\right) < 1.$$

All the assumptions in Theorem 4.1 are satisfied, and therefore, the fractional BVP 5.11 has a unique solution on J_1.

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Orthogonal stability of the new generalized quadratic functional equation

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Abstract

In this paper, the authors investigate the Hyers - Ulam - Rassias stability and J. M. Rassias mixed type product- sum of powers of norms stability of a orthogonally generalized quadratic functional equation of the form

$$f(nx+y)+f(nx-y)=n[f(x+y)+f(x-y)]+2n(n-1)f(x)-2(n-1)f(y),$$

where $f : A \to B$ be a mapping from a orthogonality normed space A into a Banach Space B, $\perp$ is orthogonality in the sense of Ratz with $x \perp y$ for all $x, y \in A$.

Keywords: : Hyers - Ulam - Rassias stability, J. M. Rassias mixed type product - sum of powers of norms stability,Example, Orthogonally quadratic functional equation, Orthogonality space, Quadratic mapping.

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1 Introduction

The stability problem of functional equations originated from the following question of Ulam[19]: Under what condition does there exist an additive mapping near an approximately additive mapping? In 1941, Hyers [8] gave a partial armative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy dierence.

The idea of generalized Hyers-Ulam stability is extended to various functional equations like additive equations, Jensen's equations, Hosszu's equations, homogeneous equations, logarithmic equations, exponential equations, multiplicative equations, trigonometric and gamma functional equations .

It is easy to see that the quadratic function $f(x) = kx^2$ is a solution of each of the following functional equations

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x), \tag{1.2}$$

$$f(x-y-z)+f(x)+f(y)+f(z)=f(x-y)+f(y+z)+f(z-x), \tag{1.3}$$

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(x-y-z)=4f(x)+4f(y)+4f(z). \tag{1.4}$$

So it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x(see [1, 9]). The bi-additive function B is given by

$$B(x,y)=\frac{1}{4}[f(x+y)-f(x-y)]. \tag{1.5}$$

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A Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was first treated by F. Skof for functions $f : A \to B$ where A is a normed space and B is a Banach space (see [17]). Cholewa [2] noticed that the theorem of Skof is still true if relevent domain A is replaced by abelian group. Czerwik [3] proved the Hyers-Ulam-Rassias stability of the equation (1.1).

In 1982-1984, J.M. Rassias [12, 13] proved the following theorem in which he generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms.

Theorem 1.1. *[12, 13] Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon \|x\|^p \|y\|^q$$

for all $x, y \in E$, where ϵ and p, q are constants with $\epsilon > 0$ and $r = p + q \neq 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{\epsilon}{2 - 2^r} \|x\|^r$$

for all $x \in E$. If, in addition, for every $x \in E$, $f(tx)$ is continuous in real t for each fixed x, then L is linear.

The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability. Later, J.M.Rassias [15] discused the stability of quadratic functional equation

$$f(mx+y) + f(mx-y) = 2f(x+y) + 2f(x-y) + 2(m^2-2)f(x) - 2f(y)$$

for any arbitrary but fixed real constant m with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ using the mixed powers of norms.

Now we present the results connected with functional equation in orthogonal space. The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), x \perp y \tag{1.6}$$

in which $\perp$ is an abstract orthogonally was first investigated by S. Gudder and D. Strawther . R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1.6) in [7]. The orthogonally quadratic functional equation (1.1) was first investigated by F. Vajzovic [20] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. This result was then generalized by H. Drljevic [4], M. Fochi [5], M.Moslehian [10, 11] and G. Szabo [18].

Definition 1.1. *A vector space X is called an orthogonality vector space if there is a relation $x \perp y$ on X such that*
(i) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
(ii) independence: if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;
(iii) homogeneity: if $x \perp y$, then $ax \perp by$ for all $a, b \in \mathbb{R}$;
(iv) the Thalesian property: if P is a two-dimensional subspace of X ; then
(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;
(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.
Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector x, y define $x \perp y$ iff x, y are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all x, $y \in X$.

Definition 1.2. *The pair $(x, \perp)$ is called an orthogonality space. It becomes orthogonality normed space when the orthogonality space is equipped with a norm.*

Definition 1.3. *Let X be an orthogonality space and Y be a real Banach space. A mapping $f : X \to Y$ is called orthogonally quadratic if it satisfies the so called orthogonally Euler-Lagrange (or Jordan - von Neumann) quadratic functional equation*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.7}$$

for all $x, y \in X$ with $x \perp y$.

In this paper, we obtain the general solution of new quadratic functional equation

$$f(nx+y)+f(nx-y)=n[f(x+y)+f(x-y)]+2n(n-1)f(x)-2(n-1)f(y) \tag{1.8}$$

and study the Hyers - Ulam - Rassias stability and J. M. Rassias mixed type product-sum of powers of norms stability in the concept of orthogonality.

Definition 1.4. *A mapping $f : A \to B$ is called orthogonal quadratic if it satisfies the quadratic functional equation (1.8) for all $x, y \in A$ with $x \perp y$ where A be an orthogonality space and B be a real Banach space.*

Through out this paper, let $(A, \perp)$ denote an orthogonality normed space with norm $\|\cdot\|_A$ and $(B, \|\cdot\|_B)$ is a Banach space. We define

$$\begin{aligned} Df(x,y) &= f(nx+y)+f(nx-y) \\ &\quad -n[f(x+y)+f(x-y)]-2n(n-1)f(x)+2(n-1)f(y). \end{aligned} \tag{1.9}$$

for all $x, y \in A$ with $x \perp y$.

Now we proceed to find the general solution of the functional equation (1.8).

2 The General Solution of the Functional Equation (1.8)

In this section, we obtain the general solution of the functional equation (1.8). Through out this section, let X and Y be real vector spaces.

Theorem 2.2. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation*

$$\begin{aligned} f(nx+y)+f(nx-y) &= n[f(x+y) \\ &\quad +f(x-y)]+2n(n-1)f(x)-2(n-1)f(y) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ if and only if it satisfies the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2.2}$$

for all $x, y \in X$.

Proof. Suppose a function $f : X \to Y$ satisfies (2.1). Putting $x = y = 0$ in (2.1), we get $f(0) = 0$. Let $x = 0$ and $y = 0$ in (2.1), we obtain $f(-y) = f(y)$ and $f(nx) = n^2 f(x)$, respectively. Setting $(x, y) = (x, x+y)$ in (2.1), we obtain

$$f((n+1)x+y)+f((n-1)x-y)=n[f(2x+y)+f(-y)]+2f(nx)-2nf(x) \tag{2.3}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.3) and adding the resultant with (2.3), we obtain

$$\begin{aligned} &f((n+1)x+y)+f((n+1)x-y)+f((n-1)x+y)+f((n-1)x-y) \\ &= n[f(2x+y)+f(2x-y)]+2n[f(x+y)+f(x-y)]+2[f(x+y)+f(x-y)] \\ &+2nf(y)+4f(nx)-4nf(x) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Setting $n = n+1, n = n-1$ and $n = 2$ respectively in (2.1), we obtain the following equations

$$\begin{aligned} &f((n+1)x+y)+f((n+1)x-y) \\ &= (n+1)[f(x+y)+f(x-y)]+2n^2f(x)+2nf(x)-2nf(y) \end{aligned} \tag{2.5}$$

$$\begin{aligned} f((n-1)x+y)+f((n-1)x-y) &= (n-1)[f(x+y)+f(x-y)] \\ &+2n^2f(x)-6nf(x)+4f(x)-2nf(y)+4f(y) \end{aligned} \tag{2.6}$$

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+4f(x)-2f(y) \tag{2.7}$$

for all $x, y \in X$. Substitute (2.5), (2.6) and (2.7) in (2.4), we arrive (2.2).

Conversely, assume f satisfies the functional equation (2.2). Letting (x, y) by $(0, 0)$ in (2.2), we get $f(0) = 0$. Putting $x = 0$ in (2.2), we obtain $f(-y) = f(y)$ for all $y \in X$. Thus f is an even function. Substituting (x, y) by (x, x) and $(x, 2x)$ in (2.2), we get

$$f(2x) = 4f(x), f(3x) = 9f(x) \tag{2.8}$$

respectively for all $x \in X$. Setting $(x, y) = (nx + y, nx - y)$ in (2.2), we obtain

$$f(nx + y) + f(nx - y) = 2n^2 f(x) + 2f(y) \tag{2.9}$$

for all $x, y \in X$. Multiplying (2.2) by n and subtracting the resultant from (2.9), we arrive (2.1). □

3 Hyers - Ulam - Rassias Stability of (1.8)

In this section, we present the Hyers - Ulam - Rassias stability of the functional equation (1.8) involving sum of powers of norms.

Theorem 3.3. *Let μ and $s(s < 2)$ be non-negative real numbers. Let $f : A \to B$ be a mapping fulfilling*

$$\|D\, f\,(x, y)\|_B \leq \mu \left\{\|x\|_A^s + \|y\|_A^s\right\} \tag{3.1}$$

for all $x, y \in A$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q : A \to B$ such that

$$\|f(y) - Q(y)\|_B \leq \frac{\mu}{2(n^2 - n^s)} \|x\|_A^s \tag{3.2}$$

for all $x \in A$. The function $Q(x)$ is defined by

$$Q(y) = \lim_{k\to\infty} \frac{f(n^k x)}{(n^2)^k} \tag{3.3}$$

for all $x \in A$.

Proof. Replacing (x, y) by $(0, 0)$ in (3.1) we get $f(0) = 0$. Setting (x, y) by $(x, 0)$ in (3.1), we obtain

$$\left\|f(nx) - n^2 f(x)\right\|_B \leq \frac{\mu}{2}(\|x\|_A^s) \tag{3.4}$$

for all $x \in A$. Since $x \perp 0$, we have

$$\left\|\frac{f(nx)}{n^2} - f(x)\right\|_B \leq \frac{\mu}{2n^2} \|x\|_A^s \tag{3.5}$$

for all $x \in A$. Now replacing x by nx and dividing by n^2 in (3.5) and summing resulting inequality with (3.5), we arrive

$$\left\|\frac{f(n^2 x)}{(n^2)^2} - f(x)\right\|_B \leq \frac{\mu}{2n^2} \left\{1 + \frac{n^s}{n^2}\right\} \|x\|_A^s \tag{3.6}$$

for all $x \in A$. In general, using induction on a positive integer n we obtain that

$$\begin{aligned}\left\|\frac{f(n^k x)}{(n^2)^k} - f(x)\right\|_B &\leq \frac{\mu}{2n^2} \sum_{t=0}^{k-1} \frac{n^{st}}{(n^2)^t} \|x\|_A^s \\ &\leq \frac{\mu}{2n^2} \sum_{t=0}^{\infty} \frac{n^{st}}{(n^2)^t} \|x\|_A^s\end{aligned} \tag{3.7}$$

for all $x \in A$. In order to prove the convergence of the sequence $\{f(n^k x)/(n^2)^k\}$ replace x by $n^m x$ and divide by $(n^2)^m$ in (3.7), for any $k, m > 0$, we obtain

$$\begin{aligned}\left\|\frac{f\left(n^k n^m x\right)}{(n^2)^{(k+m)}} - \frac{f(n^m x)}{(n^2)^m}\right\|_B &= \frac{1}{(n^2)^m} \left\|\frac{f\left(n^k n^m x\right)}{(n^2)^k} - f\left(n^m x\right)\right\|_B \\ &\leq \frac{1}{(n^2)^m} \frac{\mu}{2n^2} \sum_{t=0}^{k-1} \frac{n^{st}}{(n^2)^t} \|n^m x\|_A^s \\ &\leq \frac{\mu}{2n^2} \sum_{t=0}^{\infty} \frac{1}{n^{(2-s)(t+m)}} \|x\|_A^s .\end{aligned} \tag{3.8}$$

As $s < 2$, the right hand side of (3.8) tends to 0 as $m \to \infty$ for all $x \in A$. Thus $\{f(n^k x)/(n^2)^k\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $Q : A \to B$ such that

$$Q(x) = \lim_{k\to\infty} \frac{f(n^k x)}{(n^2)^k} \quad \forall x \in A.$$

Letting $k \to \infty$ in (3.7), we arrive the formula (3.2) for all $x \in A$. To prove Q satisfies (1.8), replace (x, y) by $(n^k x, n^k y)$ in (3.1) and divide by $(n^2)^k$ then it follows that

$$\begin{aligned}\frac{1}{(n^2)^k}\|f(n^k(nx+y)) + f(n^k(nx-y)) - n[f(n^k(x+y)) - f(n^k(x-y))]\\ - 2n(n-1)f(n^k x) - 2(n-1)f(n^k y)\|_B \le \frac{\mu}{(n^2)^k}\left\{\left\|n^k x\right\|_A^s + \left\|n^k y\right\|_A^s\right\}.\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned}\|Q(nx+y) + Q(nx-y) - n[Q(x+y) - Q(x-y)]\\ - 2n(n-1)Q(x) + 2(n-1)Q(y)\|_B \le 0.\end{aligned}$$

which gives

$$Q(nx+y) + Q(nx-y) = n[Q(x+y) - Q(x-y)] + 2n(n-1)Q(x) - 2(n-1)Q(y)$$

by taking limit as $k \to \infty$ in (3.7), we obtain

$$\|f(x) - Q(x)\|_B \le \frac{\mu}{2(n^2 - n^s)}\|x\|_A^s \tag{3.9}$$

for all $x, y \in A$ with $x \perp y$. Therefore $Q : A \to B$ is an orthogonally quadratic mapping which satisfies (1.8). To prove the uniqueness: Let Q' be another orthogonally quadratic mapping satisfying (1.8) and the inequality (3.2). Then

$$\begin{aligned}\|Q(x) - Q'(x)\|_B &= \frac{1}{(n^2)^k}\left\|Q\left(n^k x\right) - Q'\left(n^k x\right)\right\|_B\\ &\le \frac{1}{(n^2)^k}\left(\|Q(k^t x) - f(k^t x)\|_B + \left\|f\left(n^k x\right) - Q'\left(n^k x\right)\right\|_B\right)\\ &\le \frac{\mu}{n^2 - n^s}\frac{1}{n^{k(2-s)}}\|x\|_A^s\\ &\to 0 \text{ as } k \to \infty\end{aligned}$$

for all $x \in A$. Therefore Q is unique. This completes the proof of the theorem. □

Theorem 3.4. *Let μ and $s(s > 2)$ be nonnegative real numbers. Let $f : A \to B$ be a mapping satisfying (3.1) for all $x, y \in A$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q : A \to B$ such that*

$$\|f(x) - Q(x)\|_B \le \frac{\mu}{2(n^s - n^2)}\|x\|_A^s \tag{3.10}$$

for all $x \in A$. The function $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty}(n^2)^n f\left(\frac{x}{n^k}\right) \tag{3.11}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{2}$ in (3.4), the rest of the proof is similer to that of Theorem 3.1. □

4 J.M. Rassias Mixed Type Product - Sum of Powers of Norms Stability of (1.8)

In this section, we discuss the J.M. Rassias mixed type product - sum of powers of norms stability of the functional equation (1.8).

Theorem 4.5. *Let $f : A \to B$ be a mapping satisfying the inequality*

$$\|D f(x,y)\|_B \leq \mu \left\{ \|x\|_A^{2s} + \|y\|_A^{2s} + \|x\|_A^{s} \|y\|_A^{s} \right\} \tag{4.1}$$

for all $x, y \in A$ where μ and s are constants with, $\mu, s > 0$ and $s < 1$. Then the limit

$$Q(x) = \lim_{k\to\infty} \frac{f\left(n^k x\right)}{(n^2)^k} \tag{4.2}$$

exists for all $x \in A$ and $Q : A \to B$ is the unique quadratic mapping such that

$$\|f(x) - Q(x)\|_B \leq \frac{\mu}{2(n^2 - n^{2s})} \|x\|_A^{ns} \tag{4.3}$$

for all $x \in A$.

Proof. Letting (x, y) by $(0, 0)$ in (4.1), we get $f(0) = 0$. Again substituting (x, y) by $(x, 0)$ in (4.1), we obtain

$$\left\| \frac{f(nx)}{n^2} - f(x) \right\|_B \leq \frac{\mu}{2n^2} \|x\|_A^{ns} \tag{4.4}$$

for all $x \in A$. Now replacing x by nx and dividing by n^2 in (4.4) and summing resulting inequality with (4.4), we arrive

$$\left\| \frac{f(n^2 x)}{(n^2)^2} - f(x) \right\|_B \leq \frac{\mu}{2n^2} \left\{ 1 + \frac{n^{2s}}{n^2} \right\} \|x\|_A^{2s} \tag{4.5}$$

for all $x \in A$. Using induction on a positive integer k, we obtain that

$$\begin{aligned} \left\| \frac{f\left(n^k x\right)}{(n^2)^k} - f(x) \right\|_B &\leq \frac{\mu}{2n^2} \sum_{t=0}^{k-1} \left(\frac{n^{2s}}{n^2} \right)^t \|x\|_A^{2s} \\ &\leq \frac{\mu}{2n^2} \sum_{t=0}^{\infty} \left(\frac{n^{2s}}{n^2} \right)^t \|x\|_A^{2s} \end{aligned} \tag{4.6}$$

for all $x \in A$. In order to prove the convergence of the sequence $\{f(n^k x)/4^k\}$ replace x by $n^m x$ and divide by $(n^2)^m$ in (4.6), for any $k, m > 0$, we obtain

$$\begin{aligned} \left\| \frac{f\left(n^k n^m x\right)}{(n^2)^{(k+m)}} - \frac{f(n^m x)}{(n^2)^m} \right\|_B &= \frac{1}{(n^2)^m} \left\| \frac{f\left(n^k n^m x\right)}{(n^2)^k} - f(n^m x) \right\|_B \\ &\leq \frac{1}{(n^2)^m} \frac{\mu}{2n^2} \sum_{t=0}^{k-1} \left(\frac{n^{2s}}{n^2} \right)^t \|n^m x\|_A^{2s} \\ &\leq \frac{\mu}{2n^2} \sum_{t=0}^{\infty} \frac{1}{n^{(n-2s)(t+m)}} \|x\|_A^{2s} \end{aligned} \tag{4.7}$$

As $s < 1$, the right hand side of (4.7) tends to 0 as $m \to \infty$ for all $x \in A$. Thus $\{f(n^k x)/(n^2)^k\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $Q : A \to B$ such that

$$Q(x) = \lim_{k\to\infty} \frac{f(n^k x)}{(n^2)^k} \quad \forall x \in A.$$

Letting $n \to \infty$ in (4.6), we arrive the formula (4.2) for all $x \in A$. To show that Q is unique and it satisfies (1.8), the rest of the proof is similar to that of theorem 3.1 □

Theorem 4.6. *Let $f : A \to B$ be a mapping satisfying the inequality (4.1) for all $x, y \in A$ where μ and s are constants with, $\mu, s > 0$ and $s > 2$. Then the limit*

$$Q(x) = \lim_{k\to\infty} (n^2)^k f\left(\frac{x}{n^k}\right) \tag{4.8}$$

exists for all $x \in A$ and $Q : A \to B$ is the unique quadratic mapping such that

$$\|f(x) - Q(x)\|_B \leq \frac{\mu}{2(n^{2s} - n^2)} \|x\|_A^{2s} \tag{4.9}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{3}$ in (4.4), the proof is similar to that of Theorem 4.1. □

Now we will provide an example to illustrate that the functional equation (1.8) is not stable for $s = 2$.

Example 4.1. *Let $\varphi : X \to X$ be a function defined by*

$$\phi(x) = \begin{cases} \mu \|x\|^2 & , \|x\| < 1 \\ \mu & otherwise \end{cases} \tag{4.10}$$

where $\mu > 0$ is a constant and we define a function $f : X \to Y$ by

$$f(x) = \sum_{m=0}^{\infty} \frac{\varphi(n^m x)}{(n^2)^m} \tag{4.11}$$

for all $x \in X$. Then f satisfies the functional inequality

$$\|D(f(x, y))\| \leq \frac{2n^2}{(n-1)} \mu (\|x\|^2 + \|y\|^2) \tag{4.12}$$

for all $x, y \in X$. Then there exist any quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \leq \eta \|x\|^2 \tag{4.13}$$

for $x \in X$.

Proof. From the equation (4.10) and (4.11) , we obtain

$$f(x) \leq \sum_{m=0}^{\infty} \frac{\phi(n^m x)}{n^{2m}} = \sum_{k=0}^{\infty} \frac{\mu}{n^{2m}} \leq \mu(\frac{n^2}{n^2 - 1}) \tag{4.14}$$

for all $x \in X$. Therefore we see that f is bounded. We are going to prove that f satisfies (4.12).

If $(\|x\|^2 + \|y\|^2) \geq 1$ then the left hand side of (4.12) is less than

$$\frac{2n^2}{(n-1)}$$

. Now we suppose that $0 \leq \|x\|^2 + \|y\|^2 \leq 1$. Then there exist a positive integer k such that

$$\frac{1}{n^{2k-1}} \leq \|x\|^2 + \|y\|^2 < \frac{1}{n^{2k}} \tag{4.15}$$

for all $x \in X$. so that

$$n^{2k}\|x\|^2 < 1, n^{2k}\|y\|^2 < 1$$

and consequently, $n^{k-1}\|x\| < 1$, $n^{k-1}\|y\| < 1$, $n^{k-1}\|x + y\| < 1$, $n^{k-1}\|x - y\| < 1$, $n^{k-1}\|nx + y\| < 1$, $n^k\|nx - y\| < 1$ for all $m \in 0, 1, 2, ..., k - 1$

$$n^{k-1}\|x\| < 1, n^{k-1}\|y\| < 1, n^{k-1}(\|x + y\|) < 1,$$
$$n^{k-1}(\|x - y\|) < 1, n^{k-1}(\|nx + y\|) < 1, n^{k-1}(\|nx - y\|) < 1.$$

for all $x \in \{0, 1, 2, ...k-1\}$.

$$\|D(f(x,y))\| \le \sum_{m=k}^{\infty} \frac{2n(n+1)}{n^{2m}}\ \mu$$
$$\le \frac{2n(n+1)}{n^{2m}} \left(\frac{n^2}{n^2-1}\right) \mu$$
$$\le \frac{2n^2}{n-1}\, \mu \left(\|x\|^2 + \|y\|^2\right)$$

Thus f satisfies the inequality (4.12) Let us consider the an orthogonally quadratic mapping satisfying $Q : X \to Y$ and a constant $\eta > 0$ such that

$$\|f(x) - Q(x)\| \le \eta\|x\|^2$$

for all $x \in X$.Since f is bounded,Q is also bounded on any open interval containing the origin zero. we have

$$Q(x) = c\|x\|^2$$

for all $x \in X$ and C is constant.Thus we obtain

$$\left\|f(x) - c\|x\|^2\right\| \le \eta\|x\|^2$$
$$\|f(x)\| \le (\|c\| + \eta)\,\|x\|^2 \tag{4.16}$$

for all $x \in X$. But we can choose a positive integer

$$p, p\mu > \eta + |c|$$

. If $x \in \left(0, \frac{1}{n^{p-1}}\right)$, then $n^m x \in (0,1)$ for all $m = 0, 1, \ldots, p-1$. For this x, we get $f(x) = \sum_{m=0}^{\infty} \frac{\phi(n^m x)}{n^{2m}} \ge \sum_{m=0}^{\infty} \frac{\mu(n^{2m})\|x\|^2}{n^{2m}} = p\mu\|x\|^2 > (\eta + |c|)\,\|x\|^2$ which contradicts (4.16). Therefore the functional equation (1.8) is not stable in sense of Ulam, Hyers and Rassias if $s = 2$, assumed in the inequality (4.16). □

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Some new results on intuitionistic fuzzy H-ideal in BCI-algebra

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Abstract

Intuitionistic fuzzy set, involving membership, non-membership and hesitancy consideration present mathematically a very general structure. Because of these considerations it is possible to define several operations / compositions of these sets. In the existing literature ten different operations on such sets are defined. These ten operations on intuitionistic fuzzy sets bear interesting properties. In this paper we have identified and proved several of these properties, particularly those involving the operation $A \to B$ defined as standard intuitionistic fuzzy implicational with other operations.

Keywords: Intuitionistic fuzzy sets, equality, intuitionistic fuzzy implication, operation intuitionistic fuzzy H-ideal.

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1 Introduction

Intuitionistic fuzzy sets (IFS) as a generalization of fuzzy sets [7], was introduced by atanassov [1, 2], it assigns to each element degrees of membership, non-membership hesitancy. Some results x, with intuitionistic fuzzy sets based on operations(denoted by $\cup, \cap, \oplus, \otimes, \odot, \Theta, *, @, \#, \$$) have been established in [1, 2, 3, 4, 5 and 6]. The paper is organized as follows: In section 2 some basic definitions related to intuitionistic fuzzy set theory are presented. In section 3 results associated with standard intuitionistic fuzzy implication are proved.

2 Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)(((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)(x * x = 0)$,

(IV) $(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y)$.

We can define a partial order $'\leq'$ on X by $x \leq y$ if and only if $x * y = 0$. Any BCI-algebra X has the following properties:

(a1) $(\forall x \in X)(x * 0 = x)$.

(a2) $(\forall x, y, z \in X)((x * y) * z = (x * z) * y)$.

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(a3) $(\forall x, y, z \in X)(x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.

A mapping $\mu : X \rightarrow [0,1]$, where X is an arbitrary nonempty set, is called a fuzzy set in X. For any fuzzy set μ in X and any $t \in [0,1]$ we define two sets $U(\mu;t) = \{x \in X | \mu(x) \geq t\}$ and $L(\mu;t) = \{x \in X | \mu(x) \leq t\}$, which are called an upper and lower t-level cut of μ and can be used to the characterization of μ. As an important generalization of the notion of fuzzy sets in X, Atanassov[1,2] introduced the concept of an intuitionstic fuzzy set (IFS for short) defined on a nonempty set X as objects having the form $A = \{< x, \mu_A(x), \lambda_A(x) > | x \in X\}$, where the functions μ_A: $X \rightarrow [0,1]$ and $\lambda_A : X \rightarrow [0,1]$ denote the degree of membership (namely μ_A(x)) and the degree of non-membership (namely λ_A(x)) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. Such defined objects are studied by many authors (see for Example two journals: 1. Fuzzy Sets and Systems and 2. Notes on Intuitionistic Fuzzy Sets) and have many interesting applications not only in mathematics (see Chapter 5 in the book [3]). For the sake of simplicity, we shall use the symbol $A = \{X, \mu_A, \lambda_A\}$ for the intuitionistic fuzzy set $A = \{< x, \mu_A(x), \lambda_A(x) >: x \in X\}$.

A nonempty subset A of a BCI-algebra X is called an ideal of X if it satisfies:

(11) $0 \in A$,

(12) $(\forall x, y \in X)(\forall y \in A)(x * y \in A \Rightarrow x \in A)$.

A nonempty subset A of a BCI-algebra X is called H-ideal of X if it satisfies (11) and (12) $(\forall x, y \in X)(\forall z \in A)((x * (y * z)), y \in A \Rightarrow x * z \in A)$

Definition:2.1

An IFS $A = \{< x, \mu_A(x), \lambda_A(x) >: x \in X\}$ in a BCI-algebra X is called an intuitionistic fuzzy ideal of X if it satisfies:$(\forall x \in X)(\mu_A(0) \geq \mu_A(x), \lambda_A(0) \leq \lambda_A(x))$ and
$(\forall x, y \in X)(\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\})$,
$(\forall x, y \in X)(\lambda_A(x) \leq \max\{\lambda_A(x * y), \lambda_A(y)\}$.

3 Intuitionistic fuzzy H-ideals

In what follows, let X denotes a BCI-algebra unless otherwise specified. We first consider the intuitionistic fuzzification of the notion of H-ideals in a BCI-algebra as follows.

Definition:3.1. $[1,2]$:

An intuitionistic fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, x_3, x_4, \cdots x_n\}$ is given by $A = \{< \mu_A(x), \lambda_A(x) >: x \in X\}$, Where $\mu_A : X \rightarrow [0,1]$ and $\lambda_A : X \rightarrow [0,1]$ such that $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$. The number $\mu_A(x)$ and $\lambda_A(x)$ denote the degree of membership and non-membership of $x \in X$ to A, respectively. For each IFS A in X, if $\pi_A(x) = 1 - \mu_A(x) - \lambda_A(x), x \in X$, then $\pi_A(x)$ represent the degree of hesitance of x to A, $\Pi_A(x)$ is called Intuitionistic index. Obviously, when $\pi_A(x) = 0$, i.e, $\lambda_A(x) = 1 - \mu_A(x)$ for each x in X, then the IFS set A becomes fuzzy set. Thus, fuzzy sets are the special cases of IFSs. For studying sets, there is need to consider relations and operations, which in the study of Intuitionistic fuzzy sets are defined as follows.

Definition:3.2

An IFS A = $\{< x, \mu_A(x), \lambda_A(x) >: x \in X\}$ in X is called an intuitionistic fuzzy H-ideal of X if it satisfies $(\forall x \in X)(\mu_A(0) \geq \mu_A(x), \lambda_A(0) \leq \lambda_A(x))$ and
$(\forall x, y, z \in X)(\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\})$,
$(\forall x, y, z \in X)(\lambda_A(y * x) \leq \max\{\lambda_A(x * (y * z)), \lambda_A(y)\})$

Definition:3.3 Set Operators on Intuitionistic Fuzzy Set

Let IFSs(X) denotes the family of all IFSs (X) on the universe, $_B(x)$: $x \in X\}$

1. $A \cup B = \{< X, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\} >: x \in X\}$
2. $A \cap B = \{< X, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\} >: x \in X\}$

3. $A^C = \{< X, \mu_A(x), \lambda_A(x) >: x \in X\}$

4. $A@B = \{< X, \frac{1}{2}[\mu_A(x) + \mu_B(x)], \frac{1}{2}[\lambda_A(x) + \lambda_B(x)] >: x \in X\}$

5. $A\$B = \left\{ < X, \sqrt{\mu_A(x)\mu_B(x)}, \sqrt{\lambda_A(x)\lambda_B(x)} >: x \in X \right\}$

6. $A\#B = \left\{ < X, \frac{2\mu_A(x).\mu_B(x)}{\mu_A(x)+\mu_B(x)}, \frac{2\lambda_A(x)\lambda_B(x)}{\lambda_A(x)+\lambda_B(x)} >: x \in X \right\}$

7. $A * B = \left\{ < X, \frac{\mu_A(x)+\mu_B(x)}{2[\mu_A(x).\mu_B(x)+1]}, \frac{\lambda_A(x)+\lambda_B(x)}{2[\lambda_A(x).\lambda_B(x)+1]} >: x \in X \right\}$

8. $A \oplus B = \left\{ < X, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \lambda_A(x).\lambda_B(x) >: x \in X \right\}$

9. $A \otimes B = \{< X, \mu_A(x) + \mu_B(x), \lambda_A(x) + \lambda_B(x) - \lambda_A(x)\lambda_B(x) >: x \in X\}$

10. Hamacher Union Function
$\mu_{A\cup B}(x) = \frac{\mu_A(x)+\mu_B(x)-(2-\gamma)\mu_A(x)\mu_B(x)}{1-(1-\gamma)\mu_A(x).\mu_B(x)}$, Where $\gamma \geq 0$

11. Hamacher Intersection Function
$\mu_{A\cap B}(x) = \frac{\mu_A(x).\mu_B(x)}{\gamma+(1-\gamma)[\mu_A(x)+\mu_B(x)-\mu_A(x).\mu_B(x)]}$, Where $\gamma \geq 0$

12. Bounded Difference: $\mu_{A\ominus B}(x) = Max\{0, \mu_A(x) - \mu_B(x) : x \in X\}$

13. Bounded Product: $\mu_{A\odot B}(x) = Max\{0, \mu_A(x) + \mu_B(x) - 1 : x \in X\}$

14. Bounded Sum: $\mu_{A\oplus B}(x) = Min\{1, \mu_A(x) + \mu_B(x) : x \in X\}$

15. Simple Disjunctive Sum:
$\mu_{A\otimes B}(x) = Max\{Min\{\mu_A(x), 1 - \mu_B(x)\}, Min\{1 - \mu_A(x), \mu_B(x) : x \in X\}$

16. Disjoint Sum: $\mu_{A\Delta B}(x) = |\mu_A(x) - \mu_B(x)|$

Example 3.1. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X by A

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X.Obviously, the union $(A \cup B)$ is not an intuitionistic fuzzy H- ideal of X,

Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3 * 0) \geq \min\{\mu_A(3 * (4 * 0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3 * 4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\cup B}(3) \geq Min\{\mu_{A\cup B}(2), \mu_{A\cup B}(4)\}$
$\Rightarrow Max\{0.43, 0.51\} \geq Min\{Max\{0.43, 0.57\}, Max\{0.66, 0.51\}\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.66\}$
$\Rightarrow 0.51 \not\geq 0.57$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x * z) * (y * z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x * z) * (y * z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A\cup B}(3) \leq Max\{\lambda_{A\cup B}(2), \lambda_{A\cup B}(4)\}$
$\Rightarrow Min\{0.54, 0.45\} \leq Max\{Min\{0.54, 0.41\}, Min\{0.33, 0.45\}\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.33\}$
$\Rightarrow 0.45 \not\leq 0.41$

Example 3.2. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X.

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.

We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, the intersection $(A \cap B)$ is an intuitionistic fuzzy H- ideal of X,

Since $x = 3, y = 4, z = 0$.

$\Rightarrow \mu_A(x*z) \geq min\{\mu_A(x*(y*z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3*0) \geq min\{\mu_A(3*(4*0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq min\{\mu_A(3*4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\cap B}(3) \geq Min\{\mu_{A\cap B}(2), \mu_{A\cap B}(4)\}$
$\Rightarrow Max\{0.43, 0.51\} \geq Min\{Min\{0.43, 0.57\}, Min\{0.66, 0.51\}\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.51\}$
$\Rightarrow 0.43 \geq 0.43$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\Rightarrow \lambda_B(x) \leq Max\{\lambda_B((x*z)*(y*z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A\cap B}(3) \leq Max\{\lambda_{A\cap B}(2), \lambda_{A\cap B}(4)\}$
$\Rightarrow Max\{0.54, 0.45\} \leq Max\{Max\{0.55, 0.51\}, Max\{0.33, 0.45\}\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.45\}$
$\Rightarrow 0.54 \leq 0.54$

Example 3.3. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X by A

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, then $A \oplus B$ is not an intuitionistic fuzzy P- ideal of X,
Since $x = 3$, $y = 4$, $z = 0$.
$\Rightarrow \mu_A(x*z) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3*0) \geq \min\{\mu_A(3*(4*0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3*4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$

$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\oplus B}(3) \geq Min\{\mu_{A\oplus B}(2), \mu_{A\oplus B}(4)\}$
$\Rightarrow \mu_A(3) + \mu_B(3) - \mu_A(3).\mu_B(3) \geq Min\{\mu_A(2) + \mu_B(2) - \mu_A(2).\mu_B(2), \mu_A(4) + \mu_B(4) - \mu_A(4).\mu_B(4)\}$
$\Rightarrow 0.7207 \geq Min\{0.7549, 0.8334\}$
$\Rightarrow 0.7207 \ngeq 0.7549$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x * z) * (y * z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x * z) * (y * z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A\oplus B}(3) \leq Max\{\lambda_{A\oplus B}(2), \lambda_{A\oplus B}(4)\}$
$\Rightarrow \lambda_A(3)\lambda_B(3) \leq Max\{\lambda_A(2)\lambda_B(2), \lambda_A(4)\lambda_B(4)\}$
$\Rightarrow 0.243 \leq max\{0.2214, .1485\}$
$0.243 \nleq 0.2214$

Example 3.4. *Let* $X = \{0, 1, 2, 3, 4\}$ *be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, then $A \otimes B$ is not an intuitionistic fuzzy H- ideal of X,
Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3 * 0) \geq \min\{\mu_A(3 * (4 * 0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3 * 4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\otimes B}(3) \geq Min\{\mu_{A\otimes B}(2), \mu_{A\otimes B}(4)\}$

$\Rightarrow \mu_A(3)\mu_B(3) \geq Min\{\mu_A(2)\mu_B(2), \mu_A(4), \mu_B(4)\}$
$\Rightarrow 0.2193 \geq Min\{0.2551, 0.3366\}$
$\Rightarrow 0.2193 \ngeq 0.2451$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x*z)*(y*z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A\otimes B}(3) \leq Max\{\lambda_{A\otimes B}(2), \lambda_{A\otimes B}(4)\}$
$\Rightarrow \lambda_A(3) + \lambda_B(3) - \lambda_A(3)\lambda_B(3) \leq Max\{\lambda_A(2) + \lambda_B(2) - \lambda_A(2)\lambda_B(2), \lambda_A(4) + \lambda_B(4) - \lambda_A(4)\lambda_B(4)\}$
$\Rightarrow 0.747 \leq Max\{0.7286, 0.6315\}$
$\Rightarrow 0.747 \nleq 0.6315$

Example 3.5. *Let* $X = \{0,1,2,3,4\}$ *be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X.

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, the hamacher union is not an intuitionistic fuzzy H- ideal of X,

Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x*z) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3*0) \geq \min\{\mu_A(3*(4*0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3*4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\cup B}(3) \geq Min\{\mu_{A\cup B}(2), \mu_{A\cup B}(4)\}$ and $\gamma = 0.46$
$\Rightarrow 0.6839 \geq Min\{0.7183, 0.7972\}$
$\Rightarrow 0.6839 \ngeq 0.7183$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$

$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x*z)*(y*z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A\cup B}(3) \leq Max\{\lambda_{A\cup B}(2), \lambda_{A\cup B}(4)\}$ and $\gamma = 0.46$
$\Rightarrow 0.7095 \leq Max\{0.692, 0.6000\}$
$\Rightarrow 0.7095 \nleq 0.6925$

Example 3.6. *Let $X = \{0,1,2,3,4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X.

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, the hamacher intersection is an intuitionistic fuzzy H-ideal of X,

Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x*z) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3*0) \geq \min\{\mu_A(3*(4*0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3*4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\cap B}(3) \geq Min\{\mu_{A\cap B}(2), \mu_{A\cap B}(4)\}$ and $\gamma = 0.46$
$\Rightarrow 0.2574 \geq Min\{0.2816, 0.3697\}$
$\Rightarrow 0.2574 \ngeq 0.2816$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x*z)*(y*z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$

Now $\lambda_{A\cap B}(3) \leq Max\{\lambda_{A\cap B}(2), \lambda_{A\cap B}(4)\}$ and $\gamma = 0.46$
$\Rightarrow 0.1845 \leq Max\{0.2585, 0.1845\}$
$\Rightarrow 0.1845 \leq 0.2585$

Example 3.7. *Let $X = \{0,1,2,3,4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, the bounded difference is not an intuitionistic fuzzy H- ideal of X,

Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3 * 0) \geq \min\{\mu_A(3 * (4 * 0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3 * 4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\Theta B}(3) \geq Min\{\mu_{A\Theta B}(2), \mu_{A\Theta B}(4)\}$
$\Rightarrow 0 \geq Min\{0, 0.15\}$
$\Rightarrow 0 \geq 0$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x * z) * (y * z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x * z) * (y * z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A\Theta B}(3) \leq Max\{\lambda_{A\Theta B}(2), \lambda_{A\Theta B}(4)\}$ and $\gamma = 0.46$
$\Rightarrow 0.45 \leq Max\{0.13, 0\}$
$\Rightarrow 0.45 \nleq 0.13$

Example 3.8. *Let $X = \{0,1,2,3,4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, the simple disjunctive sum is an intuitionistic fuzzy H- ideal of X,

Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3 * 0) \geq \min\{\mu_A(3 * (4 * 0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3 * 4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\oplus B}(3) \geq Min\{\mu_{A\oplus B}(2), \mu_{A\oplus B}(4)\}$
$Max\{Min\{0.43, 0.49\}, Min\{0.57, 0.51\}\} \geq$
$Min\{Max\{Min\{0.43, 0.43\}, Min\{0.57, 0.57\}, \{Max\{Min\{0.66, 0.49\}, Min\{0.34, 0.51\}\}$
$\Rightarrow Max\{0.43, 0.51\} \geq Min\{Max\{0.43, 0.57\}, Max\{0.49, 0.34\}\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.49\}$
$\Rightarrow 0.51 \geq 0.49$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x * z) * (y * z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x * z) * (y * z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A\oplus B}(3) \leq Max\{\lambda_{A\oplus B}(2), \lambda_{A\oplus B}(4)\}$
$Max\{Min\{0.54, 0.55\}, Min\{0.46, 0.45\}\} \leq$
$Max\{Max\{Min\{0.54, 0.59\}, Min\{0.46, 0.41\}, \{Max\{Min\{0.33, 0.55\}, Min\{0.67, 0.45\}\}$
$\Rightarrow Max\{0.54, 0.45\} \leq Max\{Max\{0.54, 0.41\}, Max\{0.33, 0.45\}\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.45\}$
$\Rightarrow 0.54 \leq 0.54$

Example 3.9. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, the bounded product is an intuitionistic fuzzy H- ideal of X,

Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3 * 0) \geq \min\{\mu_A(3 * (4 * 0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3 * 4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A \odot B}(3) \geq Min\{\mu_{A \odot B}(2), \mu_{A \odot B}(4)\}$
$\Rightarrow Max\{0, -0.06\} \geq Min\{Max\{0, 0\}, Max\{0, 0.17\}\}$
$\Rightarrow 0 \geq Min\{0, 0.17\}$
$\Rightarrow 0 \geq 0$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x * z) * (y * z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x * z) * (y * z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A \odot B}(3) \leq Max\{\lambda_{A \odot B}(2), \lambda_{A \odot B}(4)\}$
$\Rightarrow Max\{0, -0.01\} \leq Max\{Max\{0, -0.05\}, Max\{0, -0.22\}\}$
$\Rightarrow 0 \leq Max\{0, 0\}$
$\Rightarrow 0 \leq 0$

Example 3.10. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, the bounded sum is not an intuitionistic fuzzy H- ideal of X,

Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3 * 0) \geq \min\{\mu_A(3 * (4 * 0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3 * 4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\oplus B}(3) \geq Min\{\mu_{A\oplus B}(2), \mu_{A\oplus B}(4)\}$
$\Rightarrow Min\{1, 0.94\} \geq Min\{Min\{1, 1\}, Min\{1, 1.17\}\}$
$\Rightarrow 0.94 \geq Min\{1, 1\}$
$\Rightarrow 0.94 \not\geq 1$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x * z) * (y * z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x * z) * (y * z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A\oplus B}(3) \leq Max\{\lambda_{A\oplus B}(2), \lambda_{A\oplus B}(4)\}$
$\Rightarrow Min\{1, 0.99\} \leq Max\{Min\{1, 0.95\}, Min\{1, 0.78\}\}$
$\Rightarrow .99 \leq Max\{.95, 0.78\}$
$\Rightarrow .99 \not\leq .95$

Example 3.11. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, then $A@B$ is not an intuitionistic fuzzy H- ideal of X,
Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3 * 0) \geq \min\{\mu_A(3 * (4 * 0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3 * 4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A@B}(3) \geq Min\{\mu_{A@B}(2), \mu_{A@B}(4)\}$
$\Rightarrow 0.47 \geq Min\{0.5, 0.585\}$
$\Rightarrow 0.47 \not\geq 0.5$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x * z) * (y * z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x * z) * (y * z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A@B}(3) \leq Max\{\lambda_{A@B}(2), \lambda_{A@B}(4)\}$
$\Rightarrow 0.495 \leq Max\{0.475, 0.393\}$
$\Rightarrow 0.495 \not\leq 0.475$

Example 3.12. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:*

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, then $A\$B$ is not an intuitionistic fuzzy H- ideal of X,

Since $x = 3, y = 4, z = 0$.

$\Rightarrow \mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3 * 0) \geq \min\{\mu_A(3 * (4 * 0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3 * 4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\$B}(3) \geq Min\{\mu_{A\$B}(2), \mu_{A\$B}(4)\}$
$\Rightarrow 0.4682 \geq Min\{0.4950, 0.5801\}$
$\Rightarrow 0.4682 \not\geq 0.4950$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x * z) * (y * z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x * z) * (y * z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \not\leq 0.45$
Now $\lambda_{A\$B}(3) \leq Max\{\lambda_{A\$B}(2), \lambda_{A\$B}(4)\}$
$\Rightarrow 0.4929 \leq Max\{0.4705, 0.3853\}$
$\Rightarrow 0.4929 \not\leq 0.4705$

Example 3.13. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.

We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, then $A\#B$ is not an intuitionistic fuzzy H- ideal of X,

Since $x = 3, y = 4, z = 0$.

$\Rightarrow \mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3 * 0) \geq \min\{\mu_A(3 * (4 * 0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3 * 4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$

$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A\#B}(3) \geq Min\{\mu_{A\#B}(2), \mu_{A\#B}(4)\}$
$\Rightarrow 0.4665 \geq Min\{0.4902, 0.5753\}$
$\Rightarrow 0.4665 \ngeq 0.4902$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x*z)*(y*z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A\#B}(3) \leq Max\{\lambda_{A\#B}(2), \lambda_{A\#B}(4)\}$
$\Rightarrow 0.4909 \leq Max\{0.4661, 0.3807\}$
$\Rightarrow 0.4909 \nleq 0.4661$

Example 3.14. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

We define an intuitionistic fuzzy set $A = < X, \mu_A, \lambda_A >$ in X

X	0	1	2	3	4
μ_A	.77	.66	.43	.43	.66
λ_A	.22	.33	.54	.54	.33

By routine calculation, we know that $A = < X, \mu_A, \lambda_A >$ is an intuitionistic fuzzy H-ideal of X.
We define an intuitionistic fuzzy set $B = < X, \mu_B, \lambda_B >$ in X .

X	0	1	2	3	4
μ_B	.78	.51	.57	.51	.51
λ_B	.21	.45	.41	.45	.45

By routine calculation, we know that $B = < X, \mu_B, \lambda_B >$ is an intuitionistic fuzzy H-ideal of X. Then A and B are intuitionistic fuzzy H-ideal Of X. Obviously, then $A * B$ is not an intuitionistic fuzzy H- ideal of X,
Since $x = 3, y = 4, z = 0$.
$\Rightarrow \mu_A(x*z) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\}$
$\Rightarrow \mu_A(3*0) \geq \min\{\mu_A(3*(4*0)), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(3*4), \mu_A(4)\}$
$\Rightarrow \mu_A(3) \geq \min\{\mu_A(2), \mu_A(4)\}$
$\Rightarrow 0.43 \geq Min\{0.43, 0.66\}$
$\Rightarrow 0.43 \geq 0.43$
And $\mu_B(3) \geq \min\{\mu_B(2), \mu_B(4)\}$
$\Rightarrow 0.51 \geq Min\{0.57, 0.51\}$
$\Rightarrow 0.51 \geq 0.51$
Now $\mu_{A*B}(3) \geq Min\{\mu_{A*B}(2), \mu_{A*B}(4)\}$
$\Rightarrow 0.3854 \geq Min\{0.4015, 0.4376\}$

$\Rightarrow 0.3854 \not\geq 0.4015$
$\Rightarrow \lambda_A(x) \leq Max\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\}$
$\Rightarrow \lambda_A(3) \leq Max\{\lambda_A(2), \lambda_A(4)\}$
$\Rightarrow 0.54 \leq Max\{0.54, 0.33\}$
$\Rightarrow 0.54 \leq 0.54$
And $\lambda_B(x) \leq Max\{\lambda_B((x*z)*(y*z)), \lambda_B(y)\}$
$\Rightarrow \lambda_B(3) \leq Max\{\lambda_B(2), \lambda_B(4)\}$
$\Rightarrow 0.45 \leq Max\{0.41, 0.45\}$
$\Rightarrow 0.45 \leq 0.45$
Now $\lambda_{A*B}(3) \leq Max\{\lambda_{A*B}(2), \lambda_{A*B}(4)\}$
$\Rightarrow 0.3982 \leq Max\{0.3888, 0.3395\}$
$\Rightarrow 0.3982 \nleq 0.3888.$

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Some multiple series identities and their hypergeometric forms

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Abstract

In this paper, we obtain solutions of some multiple series identities involving bounded multiple sequences. We also derive hypergeometric forms of these identities involving Kampé de Fériet double hypergeometric function, Srivastava's triple hypergeometric function.

Keywords: Pochhammer Symbol; Bounded Sequences; Multiple Series Identities; Srivastava's triple double Hypergeometric Function ; Kampé de Fériet double Hypergeometric Function.

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1 Introduction

Pochhammer's Symbol
The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k) = (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k = 1,2,3,\cdots \\ 1 & ; \text{ if } k = 0 \\ k! & ; \text{ if } b = 1,\ k = 1,2,3,\cdots \end{cases} \tag{1.1}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

Generalized Gaussian Hypergeometric Function[17,p.42(1)]
Generalized ordinary hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{array}{lcc} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

or

$${}_AF_B\left[\begin{array}{lcc} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lcc} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.2}$$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.
If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z(real or complex).
If $A = B+1$, then series ${}_AF_B$ is convergent when $|z| < 1$.

$$((a_A))_{2m} = 4^{Am}\left(\left(\frac{a_A}{2}\right)\right)_m \left(\left(\frac{1+a_A}{2}\right)\right)_m$$

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$$((a_A))_{1+2m} = 4^{Am}\Big(\Big(\frac{1+a_A}{2}\Big)\Big)_m\Big(\Big(\frac{2+a_A}{2}\Big)\Big)_m \prod_{i=1}^{A}(a_i)$$

where $m = 0, 1, 2, 3, \ldots$

Kampé de Fériet's General Double Hypergeometric Function[17,p.63(16); see also 16]

In 1921, Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 and their confluent forms $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ were unified and generalized by Kampé de Fériet.

We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H.M.Srivastava and R.Panda:

$$F^{A:B;D}_{E:G;H}\left[\begin{array}{ll}(a_A):(b_B);(d_D) & ; \\ & x,\, y \\ (e_E):(g_G);(h_H) & ;\end{array}\right] = \sum_{m,n=0}^{\infty}\frac{((a_A))_{m+n}\,((b_B))_m\,((d_D))_n\,x^m\,y^n}{((e_E))_{m+n}\,((g_G))_m\,((h_H))_n\,m!\,n!} \tag{1.3}$$

where for convergence

(i) $A+B < E+G+1,\ A+D < E+H+1\ ;|x| < \infty,\ |y| < \infty$, or

(ii) $A+B = E+G+1,\ A+D = E+H+1$, and

$$\begin{cases} |x|^{\frac{1}{(A-E)}} + |y|^{\frac{1}{(A-E)}} < 1 & ,\text{if } E < A \\ \max\ \{|x|, |y|\} < 1 & ,\text{if } E \geq A \end{cases}$$

Srivastava's Triple Hypergeometric Function[17,p.69(39,40)]

In 1967, H. M. Srivastava defined a general triple hypergeometric function $F^{(3)}$ in the following form

$$F^{(3)}\left[\begin{array}{ll}(a_A) :: (b_B); (d_D); (e_E) : (g_G); (h_H); (l_L); & \\ & x,\, y,\, z \\ (m_M) :: (n_N); (p_P); (q_Q) : (r_R); (s_S); (t_T); & \end{array}\right]$$

$$= \sum_{i,j,k=0}^{\infty}\frac{((a_A))_{i+j+k}\ ((b_B))_{i+j}\ ((d_D))_{j+k}\ ((e_E))_{k+i}\ ((g_G))_i\ ((h_H))_j\ ((l_L))_k\ x^i\ y^j\ z^k}{((m_M))_{i+j+k}\ ((n_N))_{i+j}\ ((p_P))_{j+k}\ ((q_Q))_{k+i}\ ((r_R))_i\ ((s_S))_j\ ((t_T))_k\ i!\ j!\ k!} \tag{1.4}$$

Some Series Identities

We recall the following identities which are potentially useful in the series rearrangement techniques.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n-1}\Psi(m,\ n,\ r) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(0,\ n+r+1,\ r) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+r+1,\ n,\ r) +$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+1,\ n+r+1,\ r+m+1) \tag{1.5}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n}\Psi(m,\ n,\ r) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+r,\ n,\ r) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m,\ n+r+1,\ r+m+1) \tag{1.6}$$

where $\{\Psi(m,n,r)\}_{m,n,r=0}^{\infty}$ are suitably bounded double and triple sequences of essentially arbitrary(real or complex) parameters respectively.

Some Useful Indefinite Integrals

When $m = 0, 1, 2, 3, \cdots,$ then

$$\int \sinh^{2m}\theta\, \mathrm{d}\theta = \left\{\frac{-(\frac{1}{2})_m(-1)^m\ \sinh\theta\cosh\theta}{(1)_m}\sum_{r=0}^{m-1}\frac{(1)_r(-1)^r\ \sinh^{2r}\theta}{(\frac{3}{2})_r}\right\} + \left\{\frac{\theta\,(-1)^m\,(\frac{1}{2})_m}{(1)_m}\right\} + Constant \tag{1.7}$$

$$\int \cosh^{2m}\theta\, \mathrm{d}\theta = \left\{ \frac{(\frac{1}{2})_m \sinh\theta\cosh\theta}{(1)_m} \sum_{r=0}^{m-1} \frac{(1)_r \cosh^{2r}\theta}{(\frac{3}{2})_r} \right\} + \left\{ \frac{\theta\,(\frac{1}{2})_m}{(1)_m} \right\} + Constant \tag{1.8}$$

$$\int \sinh^{2m+1}\theta\, \mathrm{d}\theta = \frac{(1)_m (-1)^m \cosh\theta}{(\frac{3}{2})_m} \sum_{r=0}^{m} \frac{(\frac{1}{2})_r (-1)^r \sinh^{2r}\theta}{(1)_r} + Constant \tag{1.9}$$

$$\int \cosh^{2m+1}\theta\, \mathrm{d}\theta = \frac{(1)_m \sinh\theta}{(\frac{3}{2})_m} \sum_{r=0}^{m} \frac{(\frac{1}{2})_r \cosh^{2r}\theta}{(1)_r} + Constant \tag{1.10}$$

Above formulas (1.7)-(1.10) can be verified for $m = 0, 1, 2, 3, \ldots$ and it is the convention that the empty sum $\sum_{r=0}^{-1} F(r)$ is treated as zero.

2 A family of multiple-series identities

Theorem 1. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \left(\int_0^{\gamma} \sinh^{2m+2n}\theta\, \mathrm{d}\theta \right) \frac{y^m z^n}{m!\, n!}$$

$$= \gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \frac{(\frac{1}{2})_{m+n} (-y)^m (-z)^n}{(m+n)!\, m!\, n!} + \frac{z \sinh\gamma\cosh\gamma}{2} \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{0,n+r+1} \frac{(\frac{3}{2})_{n+r} (1)_r (-z)^n (z\sinh^2\gamma)^r}{(2)_{n+r} (2)_{n+r} (\frac{3}{2})_r} +$$

$$+ \frac{y \sinh\gamma\cosh\gamma}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+r+1,n} \frac{(\frac{3}{2})_{m+n+r} (1)_r (-y)^m (-z)^n (y\sinh^2\gamma)^r}{(2)_{m+n+r} (2)_{m+r} (\frac{3}{2})_r (1)_n} +$$

$$+ \frac{y z \sinh^3\gamma\cosh\gamma}{4} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+1,n+r+1} \frac{(\frac{5}{2})_{m+n+r} (2)_{m+r} (y\sinh^2\gamma)^m (-z)^n (z\sinh^2\gamma)^r}{(3)_{m+n+r} (\frac{5}{2})_{m+r} (2)_{n+r} (2)_m} \tag{2.1}$$

provided that each of the series involved is absolutely convergent.

Theorem 2. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \left(\int_0^{\gamma} \cosh^{2m+2n}\theta\, \mathrm{d}\theta \right) \frac{y^m z^n}{m!\, n!}$$

$$= \gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \frac{(\frac{1}{2})_{m+n} y^m z^n}{(m+n)!\, m!\, n!} + \frac{z \sinh\gamma\cosh\gamma}{2} \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{0,n+r+1} \frac{(\frac{3}{2})_{n+r} (1)_r z^n (z\cosh^2\gamma)^r}{(2)_{n+r} (2)_{n+r} (\frac{3}{2})_r} +$$

$$+ \frac{y \sinh\gamma\cosh\gamma}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+r+1,n} \frac{(\frac{3}{2})_{m+n+r} (1)_r y^m z^n (y\cosh^2\gamma)^r}{(2)_{m+n+r} (2)_{m+r} (\frac{3}{2})_r (1)_n} +$$

$$+ \frac{y z \sinh\gamma\cosh^3\gamma}{4} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+1,n+r+1} \frac{(\frac{5}{2})_{m+n+r} (2)_{m+r} (y\cosh^2\gamma)^m z^n (z\cosh^2\gamma)^r}{(3)_{m+n+r} (\frac{5}{2})_{m+r} (2)_{n+r} (2)_m} \tag{2.2}$$

provided that each of the series involved is absolutely convergent.

Theorem 3. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \left(\int_0^{\gamma} \sinh^{2m+2n+1}\theta\, \mathrm{d}\theta \right) \frac{y^m z^n}{m!\, n!}$$

$$= \cosh\gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+r,n} \frac{(1)_{m+n+r} (\frac{1}{2})_r (-y)^m (-z)^n (y\sinh^2\gamma)^r}{(\frac{3}{2})_{m+n+r} (1)_{m+r}\, n!\, r!} +$$

$$+ \frac{z \sinh^2\gamma\cosh\gamma}{3} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m,n+r+1} \frac{(2)_{m+n+r} (\frac{3}{2})_{m+r} (y\sinh^2\gamma)^m (-z)^n (z\sinh^2\gamma)^r}{(\frac{5}{2})_{m+n+r} (2)_{n+r} (2)_{m+r}\, m!} -$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(1)_{m+n}\,(-y)^m\,(-z)^n}{(\frac{3}{2})_{m+n}\;m!\,n!} \tag{2.3}$$

provided that each of the series involved is absolutely convergent.

Theorem 4. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\left(\int_0^{\gamma}\cosh^{2m+2n+1}\theta\,\mathrm{d}\theta\right)\frac{y^m\,z^n}{m!\;n!}$$

$$=\sinh\gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r,n}\frac{(1)_{m+n+r}\,(\frac{1}{2})_r\,y^m\,z^n(y\cosh^2\gamma)^r}{(\frac{3}{2})_{m+n+r}\,(1)_{m+r}\,n!\,r!}+$$

$$+\frac{z\sinh\gamma\cosh^2\gamma}{3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m,n+r+1}\frac{(2)_{m+n+r}\,(\frac{3}{2})_{m+r}\,(y\cosh^2\gamma)^m\,z^n\,(z\cosh^2\gamma)^r}{(\frac{5}{2})_{m+n+r}\,(2)_{n+r}\,(2)_{m+r}\,m!} \tag{2.4}$$

provided that each of the series involved is absolutely convergent.

3 Derivations

Suppose left hand side of (2.1) is denoted by "T" and using the integral (1.7), then we get

$$T=-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n-1}\Lambda_{m,n}\frac{\sinh\gamma\cosh\gamma\,(\frac{1}{2})_{m+n}\,(1)_r\,(-1)^{m+n}\,(-1)^r\,y^m\,z^n(\sinh^2\gamma)^r}{(1)_{m+n}(1)_m(1)_n(\frac{3}{2})_r}+$$

$$+\gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(\frac{1}{2})_{m+n}\,(-1)^{m+n}\,y^m\,z^n}{(1)_{m+n}\;m!\,n!}$$

$$=-\sinh\gamma\cosh\gamma\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{0,n+r+1}\frac{(\frac{1}{2})_{n+r+1}\,(1)_r\,(-1)^{n+r+1}\,(-1)^r\;z^{n+r+1}(\sinh^2\gamma)^r}{(1)_{n+r+1}\,(1)_{n+r+1}(\frac{3}{2})_r}-$$

$$-\sinh\gamma\cosh\gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r+1,n}\frac{(\frac{1}{2})_{m+n+r+1}\,(1)_r\,(-1)^{m+n+r+1}\,(-1)^r\,y^{m+r+1}\,z^n(\sinh^2\gamma)^r}{(1)_{m+n+r+1}(1)_{m+r+1}(1)_n(\frac{3}{2})_r}-$$

$$-\sinh\gamma\cosh\gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+1,n+r+1}\frac{(\frac{1}{2})_{m+n+r+2}\,(1)_{m+r+1}\,(-1)^{m+n+r+2}\,(-1)^{m+r+1}}{(1)_{m+n+r+2}(1)_{m+1}(1)_{n+r+1}}\times$$

$$\times\frac{y^{m+1}\,z^{n+r+1}(\sinh^2\gamma)^{m+r+1}}{(\frac{3}{2})_{m+r+1}}+\gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(\frac{1}{2})_{m+n}\,(-1)^{m+n}\,y^m\,z^n}{(1)_{m+n}\;m!\,n!}$$

$$=\frac{z\sinh\gamma\cosh\gamma}{2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{0,n+r+1}\frac{(\frac{3}{2})_{n+r}\,(1)_r\,(-z)^n(z\sinh^2\gamma)^r}{(2)_{n+r}\,(2)_{n+r}(\frac{3}{2})_r}+$$

$$+\frac{y\sinh\gamma\cosh\gamma}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r+1,n}\frac{(\frac{3}{2})_{m+n+r}\,(1)_r\,(-y)^m\,(-z)^n(y\sinh^2\gamma)^r}{(2)_{m+n+r}(2)_{m+r}(1)_n(\frac{3}{2})_r}+$$

$$+\frac{yz\sinh^3\gamma\cosh\gamma}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+1,n+r+1}\frac{(\frac{5}{2})_{m+n+r}\,(2)_{m+r}\,(y\sinh^2\gamma)^m\,(-z)^n(z\sinh^2\gamma)^r}{(3)_{m+n+r}(2)_m(2)_{n+r}(\frac{5}{2})_{m+r}}\times$$

$$+\gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(\frac{1}{2})_{m+n}\,(-y)^m\,(-z)^n}{(1)_{m+n}\;m!\,n!}$$

which is the right hand side of (2.1).

Similarly we can derive (2.2) to (2.4) by means of series identities (1.5) and (1.6).

4 Hypergeometric generalizations of integrals and their solutions

Setting $\Lambda_{m,n} = \frac{((a_A))_{m+n}((d_D))_m((g_G))_n}{((b_B))_{m+n}((e_E))_m((h_H))_n}$, in theorems (2.1) to (2.4), using some algebraic properties of Pochhammer symbol and multiple power series in hypergeometric notations given by (1.2) to (1.4), we get the analytical solutions of following integrals.

$$\int_0^\gamma F^{A:D;G}_{B:E;H}\left[\begin{array}{c} (a_j)_{j=1}^A : (d_j)_{j=1}^D ; (g_j)_{j=1}^G \;; \\ (b_j)_{j=1}^B : (e_j)_{j=1}^E ; (h_j)_{j=1}^H \;; \end{array} y\sinh^2\theta, z\sinh^2\theta\right] \mathrm{d}\theta$$

$$= \gamma\, F^{A+1:D;G}_{B+1:E;H}\left[\begin{array}{c} (a_j)_{j=1}^A, \frac{1}{2} : (d_j)_{j=1}^D ; (g_j)_{j=1}^G \;; \\ (b_j)_{j=1}^B, 1 : (e_j)_{j=1}^E ; (h_j)_{j=1}^H \;; \end{array} -y, -z\right] + \frac{z\sinh\gamma\cosh\gamma \prod_{i=1}^A (a_i) \prod_{i=1}^G (g_i)}{2\prod_{i=1}^B (b_i) \prod_{i=1}^H (h_i)} \times$$

$$\times\, F^{A+G+1:1;2}_{B+H+2:0;1}\left[\begin{array}{c} \frac{3}{2}, (1+a_j)_{j=1}^A, (1+g_j)_{j=1}^G : 1 ; 1, 1 \;; \\ 2, 2, (1+b_j)_{j=1}^B, (1+h_j)_{j=1}^H : — ; \frac{3}{2} \;; \end{array} -z, z\sinh^2\gamma\right] +$$

$$+ \frac{y\sinh\gamma\cosh\gamma \prod_{i=1}^A (a_i) \prod_{i=1}^D (d_i)}{2\prod_{i=1}^B (b_i) \prod_{i=1}^E (e_i)} \times$$

$$\times\, F^{(3)}\left[\begin{array}{c} \frac{3}{2}, (1+a_j)_{j=1}^A :: — ; — ; (1+d_j)_{j=1}^D : 1 ; (g_j)_{j=1}^G ; 1, 1 \;; \\ 2, (1+b_j)_{j=1}^B :: — ; — ; (1+e_j)_{j=1}^E, 2 : — ; (h_j)_{j=1}^H ; \frac{3}{2} \;; \end{array} -y, -z, y\sinh^2\gamma\right] +$$

$$+ \frac{yz\sinh^3\gamma\cosh\gamma \prod_{i=1}^A (a_i) \prod_{i=1}^A (1+a_i) \prod_{i=1}^D (d_i) \prod_{i=1}^G (g_i)}{4\prod_{i=1}^B (b_i) \prod_{i=1}^B (1+b_i) \prod_{i=1}^E (e_i) \prod_{i=1}^H (h_i)} \times$$

$$\times F^{(3)}\left[\begin{array}{c} \frac{5}{2}, (2+a_j)_{j=1}^A :: — ; (1+g_j)_{j=1}^G ; 2 : 1, (1+d_j)_{j=1}^D ; 1 ; 1 \;; \\ 3, (2+b_j)_{j=1}^B :: — ; (1+h_j)_{j=1}^H, 2 ; \frac{5}{2} : 2, (1+e_j)_{j=1}^E ; — ; — \;; \end{array} y\sinh^2\gamma, -z, z\sinh^2\gamma\right] \tag{4.1}$$

$$\int_0^\gamma F^{A:D;G}_{B:E;H}\left[\begin{array}{c} (a_j)_{j=1}^A : (d_j)_{j=1}^D ; (g_j)_{j=1}^G \;; \\ (b_j)_{j=1}^B : (e_j)_{j=1}^E ; (h_j)_{j=1}^H \;; \end{array} y\cosh^2\theta, z\cosh^2\theta\right] \mathrm{d}\theta$$

$$= \gamma\, F^{A+1:D;G}_{B+1:E;H}\left[\begin{array}{c} (a_j)_{j=1}^A, \frac{1}{2} : (d_j)_{j=1}^D ; (g_j)_{j=1}^G \;; \\ (b_j)_{j=1}^B, 1 : (e_j)_{j=1}^E ; (h_j)_{j=1}^H \;; \end{array} y, z\right] + \frac{z\sinh\gamma\cosh\gamma \prod_{i=1}^A (a_i) \prod_{i=1}^G (g_i)}{2\prod_{i=1}^B (b_i) \prod_{i=1}^H (h_i)} \times$$

$$\times\, F^{A+G+1:1;2}_{B+H+2:0;1}\left[\begin{array}{c} \frac{3}{2}, (1+a_j)_{j=1}^A, (1+g_j)_{j=1}^G : 1 ; 1, 1 \;; \\ 2, 2, (1+b_j)_{j=1}^B, (1+h_j)_{j=1}^H : — ; \frac{3}{2} \;; \end{array} z, z\cosh^2\gamma\right] +$$

$$+ \frac{y\sinh\gamma\cosh\gamma \prod_{i=1}^A (a_i) \prod_{i=1}^D (d_i)}{2\prod_{i=1}^B (b_i) \prod_{i=1}^E (e_i)} \times$$

$$\times\, F^{(3)}\left[\begin{array}{l} \frac{3}{2},(1+a_j)_{j=1}^{A}::-;-;(1+d_j)_{j=1}^{D}\ :1;(g_j)_{j=1}^{G};1,1\ ; \\ 2,\,(1+b_j)_{j=1}^{B}::-;-;(1+e_j)_{j=1}^{E},2:-;(h_j)_{j=1}^{H};\frac{3}{2}\ ; \end{array} y,\,z,\,y\cosh^2\gamma\right]+$$

$$+\frac{y\,z\,\sinh\gamma\,\cosh^3\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{A}(1+a_i)\,\prod_{i=1}^{D}(d_i)\,\prod_{i=1}^{G}(g_i)}{4\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{B}(1+b_i)\,\prod_{i=1}^{E}(e_i)\,\prod_{i=1}^{H}(h_i)}\times$$

$$\times F^{(3)}\left[\begin{array}{l} \frac{5}{2},(2+a_j)_{j=1}^{A}::-;(1+g_j)_{j=1}^{G}\ ;2:1,(1+d_j)_{j=1}^{D};1;1\ ; \\ 3,\,(2+b_j)_{j=1}^{B}::-;(1+h_j)_{j=1}^{H},2;\frac{5}{2}:2,(1+e_j)_{j=1}^{E};-;-\ ; \end{array} y\cosh^2\gamma,\,z,\,z\cosh^2\gamma\right] \tag{4.2}$$

$$\int_0^{\gamma}\ \sinh\theta\, F_{B:E;H}^{A:D;G}\left[\begin{array}{l} (a_j)_{j=1}^{A}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G}\ ; \\ (b_j)_{j=1}^{B}:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H}\ ; \end{array} y\sinh^2\theta,\,z\sinh^2\theta\right]\mathrm{d}\theta$$

$$=\ \cosh\gamma\, F^{(3)}\left[\begin{array}{l} 1,\,(a_j)_{j=1}^{A}::-;-;(d_j)_{j=1}^{D}\ :1;(g_j)_{j=1}^{G};\ \frac{1}{2}\ ; \\ \frac{3}{2},\,(b_j)_{j=1}^{B}::-;-;1,\,(e_j)_{j=1}^{E}:-;(h_j)_{j=1}^{H};-\ ; \end{array} -y,\,-z,\,y\sinh^2\gamma\right]+$$

$$+\frac{z\,\sinh^2\gamma\,\cosh\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{G}(g_i)}{3\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{H}(h_i)}\times$$

$$\times F^{(3)}\left[\begin{array}{l} 2,\,(1+a_j)_{j=1}^{A}::-;(1+g_j)_{j=1}^{G}\ ;\frac{3}{2}:(d_j)_{j=1}^{D};\ 1\ ;\ 1\ ; \\ \frac{5}{2},\,(1+b_j)_{j=1}^{B}::-;2,\,(1+h_j)_{j=1}^{H};2:(e_j)_{j=1}^{E};-;-\ ; \end{array} y\sinh^2\gamma,\,-z,\,z\sinh^2\gamma\right]+$$

$$+F_{B+1:E;H}^{A+1:D;G}\left[\begin{array}{l} 1,\,(a_j)_{j=1}^{A}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G}\ ; \\ \frac{3}{2},\,(b_j)_{j=1}^{B}:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H}\ ; \end{array} -y,\,-z\right] \tag{4.3}$$

$$\int_0^{\gamma}\ \cosh\theta\, F_{B:E;H}^{A:D;G}\left[\begin{array}{l} (a_j)_{j=1}^{A}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G}\ ; \\ (b_j)_{j=1}^{B}:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H}\ ; \end{array} y\cosh^2\theta,\,z\cosh^2\theta\right]\mathrm{d}\theta$$

$$=\ \sinh\gamma\, F^{(3)}\left[\begin{array}{l} 1,\,(a_j)_{j=1}^{A}::-;-;(d_j)_{j=1}^{D}\ :1;(g_j)_{j=1}^{G};\ \frac{1}{2}\ ; \\ \frac{3}{2},\,(b_j)_{j=1}^{B}::-;-;1,\,(e_j)_{j=1}^{E}:-;(h_j)_{j=1}^{H};-\ ; \end{array} y,\,z,\,y\cosh^2\gamma\right]+$$

$$+\frac{z\,\sinh\gamma\,\cosh^2\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{G}(g_i)}{3\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{H}(h_i)}\times$$

$$\times F^{(3)}\left[\begin{array}{l} 2,\,(1+a_j)_{j=1}^{A}::-;(1+g_j)_{j=1}^{G}\ ;\frac{3}{2}:(d_j)_{j=1}^{D};\ 1\ ;\ 1\ ; \\ \frac{5}{2},\,(1+b_j)_{j=1}^{B}::-;2,\,(1+h_j)_{j=1}^{H};2:(e_j)_{j=1}^{E};-;-\ ; \end{array} y\cosh^2\gamma,\,z,\,z\cosh^2\gamma\right] \tag{4.4}$$

provided that each of the series as well as associated integrals involved are convergent.

These solutions are not found in Ramanujan's notebooks[11-13], Five notebooks of B. C. Berndt[5-9], Three volumes of R. P. Agarwal[1-3] and other literature[4;10;14;15] on special functions.

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Further results on nonsplit dom strong domination number

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Abstract

A subset S of V is called a dom strong dominating set if for every vertex $v \in V - S$, there exists $u_1, u_2 \in S$ such that $u_1v, u_2v \in E(G)$ and $d(u_1) \geq d(v)$. The minimum cardinality of a dom strong dominating set is called the dom strong domination number and is denoted by $\gamma_{ds}(G)$. A dom strong dominating set S is said to be a non split dom strong dominating set if the induced subgraph $\langle V - S \rangle$ is connected. The minimum cardinality of a non split dom strong dominating set is called the non split dom strong domination number of a graph and is denoted by $\gamma_{nsds}(G)$. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper, we find an upper bound for the sum of nonsplit dom strong domination number and connectivity of a graph and characterise the corresponding extremal graphs.

Keywords: Nonsplit dom strong domination number and connectivity.

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1 Introduction

In this paper we consider simple and undirected graphs. The sets V and E are the vertex set and the edge set of the graph G respectively. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. The degree of a vertex u in G is the number of edges incident with u and is denoted by $d(u)$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For terminology we refer to Chartrand and Lesniak [1].

A vertex dominates itself and its neighbors. A set $S \subset V$ is a dominating set of G if every vertex of G is dominated by some vertex in S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A subset S of V is called a dom strong dominating set if for every $v \in V - S$, there exists $u_1, u_2 \in S$ such that $u_1v, u_2v \in E(G)$ and $(u_1) \geq (v)$. The minimum cardinality of a dom strong dominating set is called the dom strong domination number and is denoted by $\gamma_{ds}(G)$. The nonsplit dom strong domination number was introduced by G.Mahadevan et.al. [5]. A dom strong dominating set S is said to be a non split dom strong dominating set(NSDSD-set) if the induced subgraph $\langle V - S \rangle$ is connected. The minimum cardinality of a non split dom strong dominating set is called the non split dom strong domination number of a graph and is denoted by $\gamma_{nsds}(G)$.

Several authors have studied the problem of obtaining an upper bound for the sum of a dominating parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. In [6], the authors found an upper bound for the sum of the domination number and connectivity of graphs and charecterized the corresponding extremal graphs. Motivated by the above, we find an upper bound for sum

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of the nonsplit dom strong domination number and connectivity of graphs and characterize the corresponding extremal graphs. Also we characterize the graphs with this sum is greater than or equal to $2n-4$.

2 Preliminaries

Theorem 2.1. [1] *For any graph G, $\kappa(G) \leq \delta(G)$.*

Theorem 2.2. [5] $2 \leq \gamma_{nsds}(G) \leq n$.

Theorem 2.3. [5] *For any connected graph G, $\gamma_{nsds}(G) = n$ if and only if G is a star.*

Notation 1. *$C_n(P_k)$ is the graph by attaching the end vertices of P_k path graph to any one vertex of the cycle graph. $K_n(P_k)$ is the graph by attaching the end vertices of P_k path graph to any one vertex of the complete graph. $C_n(P_k, P_m, 0, \ldots)$ is the graph by attaching an end vertex of P_k path graph to any one of the vertex C_n and attaching the end vertices of P_m path graph to another vertex of C_n. $K_n(P_k, P_m, 0, \ldots)$ is the graph by attaching the end vertex of P_k path graph to any one of the vertex K_n and attaching the end vertices of P_m path graph to another vertex of K_n. $K_n(u(P_n, P_m), 0, \ldots)$ is the graph obtained from K_n by attaching the end vertices of P_n and P_m path graph to $u \in V(G)$ which is one of the vertex in K_n graph. $K_n(P_n, P_m, \ldots)$ is the graph obtained from K_n by attaching the every end vertices of P_n, $P_m, \ldots$ of path graph to every vertex in K_n graph. $K_n(mP_k)$ is the graph obtained from K_n by attaching the m times P_k path graph to any one vertex of K_n.*

3 Main Results

Theorem 3.1. *For any connected graph G, $\gamma_{nsds}(G) + \kappa(G) \leq 2n-1$ and the equality holds if and only if G is a complete graph of order 2.*

Proof. $\gamma_{nsds}(G) + \kappa(G) \leq n + \delta \leq n + n - 1 = 2n-1$.
Let $\gamma_{nsds}(G) + \kappa(G) = 2n-1$. Then $\gamma_{nsds}(G) = n$ and $\kappa(G) = n-1$ which gives G is a star as well as a complete graph. Hence $G \cong K_2$. The converse is obvious. □

Theorem 3.2. *For any connected graph G, $\gamma_{nsds}(G) + \kappa(G) = 2n-2$ if and only if $G \cong K_3$ or $K_{1,2}$*

Proof. Let $\gamma_{nsds}(G) + \kappa(G) = 2n-2$. Then there are two cases to consider.
(i)$\gamma_{nsds}(G) = n$ and $\kappa(G) = n-2$ (ii)$\gamma_{nsds}(G) = n-1$ and $\kappa(G) = n-1$
Case 1.$\gamma_{nsds}(G) = n$ and $\kappa(G) = n-2$
Since $\gamma_{nsds}(G) = n, G$ is a star and hence $\kappa(G) = 1$ which gives $n = 3$. Thus $G \cong K_{1,2}$.

Case 2.$\gamma_{nsds}(G) = n-1$ and $\kappa(G) = n-1$
Since $k(G) = n-1, G$ is a complete graph, this gives $\gamma_{nsds}(G) = 2$. Then $n = 3$ and hence $G \cong K_3$. The converse is obvious.

□

Theorem 3.3. *For any connected graph G, $\gamma_{nsds}(G) + \kappa(G) = 2n-3$ if and only if $G \cong K_4$ or $K_4 - e$ or $K_{1,3}$ or C_4*

Proof. Let $\gamma_{nsds}(G) + \kappa(G) = 2n-3$. Then there are three cases to consider.
(i)$\gamma_{nsds}(G) = n$ and $\kappa(G) = n-3$ (ii)$\gamma_{nsds}(G) = n-1$ and $\kappa(G) = n-2$ (iii)$\gamma_{nsds}(G) = n-2$ and $\kappa(G) = n-1$

Case 1.$\gamma_{nsds}(G) = n$ and $\kappa(G) = n-3$
Since $\gamma_{nsds}(G) = n$ we have G is a star and hence $\kappa(G) = 1$ which gives $n = 4$. Thus $G \cong K_{1,3}$.

Case 2.$\gamma_{nsds}(G) = n-1$ and $\kappa(G) = n-2$
Since $\kappa(G) = n-2$ we have $n-2 \leq \delta(G)$. If $\delta = n-1$ then $G \cong K_n$, which is a contradiction. Hence $\delta(G) = n-2$. Then $G \cong K_n - Q$ where Q is a matching in K_n. Then $\gamma_{nsds}(G) \leq 3$. If $\gamma_{nsds}(G) = 3$ then $n = 4$ and hence G is isomorphic to either C_4 *or* $K_4 - e$. If $\gamma_{nsds} = 2$ then $n = 3$ and hence $G \cong K_{1,2}$, which is a contradiction.

Case 3.$\gamma_{nsds}(G) = n-2$ and $\kappa(G) = n-1$

Since $\kappa(G) = n-1$, G is a complete graph. Since $\gamma_{nsds}(K_n) = 2$ we have $n = 4$. Hence $G \cong K_4$. The converse is obvious.

□

Theorem 3.4. *For any connected graph G,$\gamma_{nsds}(G) + \kappa(G) = 2n-4$ if and only if $G \cong K_5$ or $K_{1,4}$ or P_4 or $K_3(1, 0, 0)$ or $C_5 + e$ or $K_5 - Q$ where Q is the maximum matching in K_5 or the graph obtained from $K_{2,3}$ by joining the vertices of degree three by an edge.*

Proof. Let $\gamma_{nsds}(G) + \kappa(G) = 2n-4$. Then there are four cases to consider.
(i)$\gamma_{nsds}(G) = n$ and $\kappa(G) = n-4$ (ii)$\gamma_{nsds}(G) = n-1$ and $\kappa(G) = n-3$ (iii)$\gamma_{nsds}(G) = n-2$ and $\kappa(G) = n-2$ (iv)$\gamma_{nsds}(G) = n-3$ and $\kappa(G) = n-1$

Case 1.$\gamma_{nsds}(G) = n$ and $\kappa(G) = n-4$

Since $\gamma_{nsds}(G) = n$ we have G is a star and hence $\kappa(G) = 1$ which gives $n = 5$. Thus $G \cong K_{1,4}$.

Case 2.$\gamma_{nsds}(G) = n-1$ and $\kappa(G) = n-3$

Since $\kappa(G) = n-3$ we have $n-3 \leq \delta$. If $\delta = n-1$ then $G \cong K_n$ which is a contradiction. If $\delta = n-2$ then $G \cong K_n - Q$ where Q is a matching in K_n. Then $\gamma_{nsds}(G) \leq 3$. If $\gamma_{nsds}(G) = 3$ then $n = 4$. Hence $G \cong K_4 - e$ or C_4. For these two graphs $\kappa(G) \neq n-3$ which is a contradiction. Hence $\delta(G) = n-3$. Let $S = \{u_1, u_2, \ldots, u_{n-3}\}$ be the minimum vertex cut of G and let $V - S = \{y_1, y_2, y_3\}$.

Subcase 2.1. $< V - S >= \bar{K_3}$

Let us assume $< S >$ be connected. If $| S |= 1$ then G is isomorphic to $K_{1,3}$ which is a contradiction. If $| S |= 2$ then G is isomorphic to the graph obtained from $K_{2,3}$ by joining the vertices of degree 3 by an edge. Let $| S |\geq 3$ and let $d(u_2) \geq d(u_i)$, i = 1 or 3. Then $\{u_1, u_2, y_1, y_2\}$ is a NSDSD-set of G. Hence $\gamma_{nsds} \leq 4$. Then $n \leq 5$ which is a contradiction.

Subcase 2.2. $< V - S >= K_1 \cup K_2$

Let $y_1y_2 \in E(G)$. Then y_3 is adjacent to all the vertices of S and y_1, y_2 are not adjacent to at most one vertex of S. Let us assume $d(y_1) = n-2$. Suppose $\langle S \rangle$ is disconnected. Then $| S |\leq 3$. If $| S |= 2$ then G is isomorphic to $C_5 + e$. If $| S |= 3$ then we get the graphs with $\gamma_{nsds} \leq 4$ which is a contradiction. Suppose $\langle S \rangle$ is connected. If $d(u_i) \leq n-2$ for all i and $| S |\geq 2$ then $\{y_1, y_2, y_3\}$ is a NSDSD-set of G which is a contradiction. If $| S |= 1$ then G is isomorphic to P_4. Let $u_1 \in S$ such that $d(u_1) = n-1$. If $| S |\geq 3$ then $V - \{y_1, y_2\}$ is a NSDSD-set of G which is a contradiction. If $| S |= 1$ then G is isomorphic to $C_3(1, 0, 0)$. If $| S |= 2$ then we obtain the graphs with $\gamma_{nsds}(G) \neq n-1$.

Suppose $d(y_1) = d(y_2) = n-3$. If $| S |\geq 3$ then $V - \{y_1, y_2\}$ is a NSDSD-set of G which is a contradiction. If $| S |= 2$ then we obtain the graphs with $\gamma_{nsds}(G) + \kappa(G) \neq 2n-4$

Case 3.$\gamma_{nsds}(G) = n-2$ and $\kappa(G) = n-2$

If $\kappa(G) = n-2$, then $n-2 \leq \delta(G)$. If $\delta(G) = n-1$ then $G \cong K_n$, which is a contradiction and we have $\delta(G) = n-2$. Then $G \cong K_n - Q$ where Q is the matching in K_n. Then $\gamma_{nsds}(G) \leq 3$. If $\gamma_{nsds}(G) = 3$ then $n = 5$. Hence G is $K_5 - Q$.If $| Q |= 1$ then $G \cong K_5 - Q$ where Q is a matching in K_5 with $| Q |= 2$. If $\gamma_{nsds}(G) = 2$ then $n = 4$. Thus $G \cong K_4 - e$ *or* C_4

Case 4.$\gamma_{nsds}(G) = n-3$ and $\kappa(G) = n-1$

If $\kappa(G) = n-1$, then $G \cong K_n$ on n vertices. But $\gamma_{nsds}(G) = 2$ then $n = 5$ and hence $G \cong K_5$. The converse is obvious.

□

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Spectrum of fuzzy prime filters of a 0 - distributive lattice

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Abstract

Stone's topology on the set of fuzzy prime filters of a bounded 0 - distributive lattice is introduced and many properties of this space of fuzzy prime filters are furnished.

Keywords: fuzzy sublattice, fuzzy filter, fuzzy prime filter, 0-distributive lattice.

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1 Introduction

For topological concepts which have now become commonplace the reader is referred to [5] and for lattice theoretic concepts the reader is referred to [4]. Venkatanarasimhan [15] has studied the Stone's space of prime filters for a pseudocomplemented lattice. The concept of a 0-distributive lattice introduced by Varlet [13] is a generalization of a distributive lattice and a pseudo-complemented lattice. A 0-distributive lattice is a lattice L with 0 in which for all $a, b, c \in L$, $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$. In [9], [2] authors have studied Stone's topology on set of prime filters of a 0 - distributive lattice. Such a study of prime spectrum plays an important role in the field of lattice theory.

Fuzzy set theory introduced by Zadeh [16] is generalization of classical set theory. After the inception of the notion fuzzy sets, Rosenfield started the pioneering work in the domain of fuzzification of algebraic objects viz fuzzy groups [11]. Many researchers have applied this concept to mathematical branches such as semi-group, ring, semi-ring, field, near ring, lattice etc. In particular while fuzzifying the notions in lattice theory, Bo et al [3] and Swami et al [12] have laid down the foundation for fuzzy ideals, fuzzy filters of a lattice. In [10], we have introduced and studied spectrum of L - fuzzy prime ideals of a bounded distributive lattice. In this paper our aim is to introduce Stone's topology τ on the set of fuzzy prime filters Σ of a bounded 0 - distributive lattice L and study many properties of the space $Fspec\,(L) = (\Sigma, \tau)$. Mainly we prove $Fspec\,(L)$ is compact and it contains a subspace homeomorphic with the spectrum of L which is dense in it. If L and L' are isomorphic bounded 0 - distributive lattices, $Fspec\,(L)$ and $Fspec\,(L')$ are homeomorphic.

2 Preliminaries

In this article we collect basic definitions and results which are used in subsequent sections.

Let $L = \langle L, \wedge, \vee \rangle$ be a bounded lattice.

Definition 2.1. *A fuzzy subset of L is a map of L into $\langle [0,1], \wedge, \vee \rangle$, where $\alpha \wedge \beta = min(\alpha, \beta)$ and $\alpha \vee \beta = max(\alpha, \beta)$ for all $\alpha, \beta \in [0,1]$. Let μ be a fuzzy subset of L. For $\alpha \in [0,1]$, the set $\mu_\alpha = \{x \in L : \mu(x) \geq \alpha\}$ is called α - cut (or α - level set) of μ.*

Definition 2.2. *A fuzzy subset μ of L is said to be a fuzzy sublattice of L if for all $x, y \in L$, $\mu\,(x \wedge y) \wedge \mu\,(x \vee y) \geq \mu\,(x) \wedge \mu(y)$.*

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Definition 2.3. *A monotonic fuzzy sublattice is a fuzzy filter of L. Here μ is monotonic means $\mu(x) \leq \mu(y)$ whenever $x \leq y$ in L.*

Definition 2.4. *The smallest fuzzy filter containing fuzzy subset μ of L is called fuzzy filter generated by μ and is denoted by $\langle\mu\rangle$. Here by a fuzzy subset σ contains a fuzzy subset μ we mean $\mu(a) \leq \sigma(a), \forall a \in L$ and will be denoted by $\mu \subseteq \sigma$.*

Definition 2.5. *A proper fuzzy filter of L is a non-constant fuzzy filter of L.*

Definition 2.6. *A proper fuzzy filter of L is said to be a fuzzy prime filter of L if for any $x, y \in L$, $\mu(x \vee y) \leq \mu(x) \vee \mu(y)$*

Result 2.1. *A fuzzy subset μ of L is a fuzzy filter of L if and only if $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$.*

Result 2.2. *A fuzzy subset μ of L is a fuzzy prime filter of L if and only if μ is a homomorphism from $\langle L, \wedge, \vee\rangle$ into $\langle [0,1], \wedge, \vee\rangle$.*

Result 2.3. *A fuzzy subset μ of L is a fuzzy filter of L if and only if each level set μ_α is a filter of L, $\forall\, \alpha \in [0,1]$ such that $\mu_\alpha \neq \emptyset$.*

Result 2.4. *If μ is a fuzzy subset of L, then*

$$\langle\mu\rangle = \bigcap \{\sigma \mid \sigma \text{ is a fuzzy filter of } L,\ \mu \subseteq \sigma\}.$$

Also $\chi_{[S)} = \langle\chi_S\rangle$, where $S \subseteq L$.

Result 2.5. *A non-constant fuzzy filter μ of L is a fuzzy prime filter of L if and only if each level set μ_α is a prime filter of L; $\forall\, \alpha \in [0,1]$ such that μ_α is a proper filter of L.*

Result 2.6. *A non-empty subset P of L is a prime filter of L if and only if χ_P is a fuzzy prime filter of L.*

Result 2.7. *([3]) Let L and L' be two lattices and $f : L \to L'$ an onto homomorphism. Then*
(i) If μ is a fuzzy sublattice (ideal, filter) of L then $f(\mu)$ is a fuzzy sublattice (ideal, filter) of L' where $f(\mu)$ is defined as

$$f(\mu)(y) = sup\{\mu(x) : f(x) = y, x \in L\} \text{ for all } y \in L';$$

(ii) If ν is a fuzzy sublattice (ideal, prime ideal, filter, prime filter) of L' then $f^{-1}(\nu)$ is a fuzzy sublattice (ideal, prime ideal, filter, prime filter) of L where $f^{-1}(\nu)$ is defined as

$$f^{-1}(\nu)(x) = \nu(f(x)) \text{ for all } x \in L.$$

Result 2.8. *Let L, L' be two bounded lattices. Let $f : L \to L'$ be a lattice isomorphism. If μ is a fuzzy prime filter of L, then $f(\mu)$ is a fuzzy prime filter of L' and $f^{-1}(f(\mu)) = \mu$.*

3 Spectrum of fuzzy prime filters

Now onwards L will denote a bounded 0 - distributive lattice. Let Σ denote the set of fuzzy prime filters of L. For each $\mu \in \Sigma$, we assume that $\mu(1) = 1$. For a fuzzy subset θ of L define $V(\theta) = \{\mu \in \Sigma \mid \theta \subseteq \mu\}$. If $\theta = \chi_{\{a\}}$ then we denote $V(\theta)$ by $V(a)$.

At the outset we prove some properties of $V\ (.)$.

Theorem 3.1. *Let θ and σ be fuzzy subsets of L.*

1. *If $\theta \subseteq \sigma$, then $V(\sigma) \subseteq V(\theta)$.*
2. *$V(\sigma) \cup V(\theta) \subseteq V(\sigma \cap \theta)$.*
3. *$V(\theta) = V(\langle\theta\rangle)$*
4. *$V(0) = \emptyset$ and $V(1) = \Sigma$.*

Proof. Proof of (1) follows by definition of $V(.)$.

(2) We have $\sigma \cap \theta \subseteq \sigma$ and $\sigma \cap \theta \subseteq \theta$. By using (1) we get $V(\sigma) \subseteq V(\sigma \cap \theta)$ and $V(\theta) \subseteq V(\sigma \cap \theta)$. Hence $V(\sigma) \cup V(\theta) \subseteq V(\sigma \cap \theta)$.

(3) As $\theta \subseteq \langle \theta \rangle$, again by (1) we get $V(\langle \theta \rangle) \subseteq V(\theta)$. Let $\mu \in V(\theta)$ then $\theta \subseteq \mu$. Therefore $\bigcap \{\sigma \in \Sigma : \theta \subseteq \sigma\} \subseteq \mu$ that is $\langle \theta \rangle \subseteq \mu$. Thus $\mu \in V(\langle \theta \rangle)$ proving that $V(\theta) \subseteq V(\langle \theta \rangle)$. Combining both inclusions we get $V(\theta) = V(\langle \theta \rangle)$.

(4) $V(0) = \{\mu \in \Sigma : \chi_{\{0\}} \subseteq \mu\} = \{\mu \in \Sigma : \mu(0) = 1\}$. As $\mu \in \Sigma$ and our assumption that $\mu(1) = 1$ it follows $V(0) = \emptyset$. Again by assumption that $\mu(1) = 1\ \forall \mu \in \Sigma$ we get $V(1) = \{\mu \in \Sigma : \chi_{\{1\}} \subseteq \mu\} = \{\mu \in \Sigma : \mu(1) = 1\} = \Sigma$. □

Remark 3.1. *Let $F(L)$ be the set of all fuzzy subsets of L and $\wp(\Sigma)$ be the power set of Σ. Then $V(.)$ defines a function $V : F(L) \longrightarrow \wp(\Sigma)$ such that $V(\theta) = \{\mu \in \Sigma \mid \theta \subseteq \mu\}$. Clearly it is not an injective map as $V(\theta) = V(\langle \theta \rangle)$.*

Theorem 3.2. *Let I and J be filters of L. Then $V(\chi_I) \cup V(\chi_J) = V(\chi_{I \cap J})$.*

Proof. By Theorem 3.1 (2) we have $V(\chi_I) \cup V(\chi_J) \subseteq V(\chi_I \cap \chi_J) = V(\chi_{I\cap J})$. Let $\mu \in V(\chi_{I\cap J})$. Then $\chi_{I\cap J} \subseteq \mu$ implies $\mu(x) = 1$ for all $x \in I \cap J$. If $\chi_I \nsubseteq \mu$ and $\chi_J \nsubseteq \mu$, then there exist $x \in I$ and $y \in J$ such that $\mu(x) \neq 1$ and $\mu(y) \neq 1$. But I and J being filters we have $x \vee y \in I \cap J$ so that $\mu(x \vee y) = 1$. As μ is a fuzzy prime filter of L, by Result 2.2, $\mu(x \vee y) = \mu(x) \vee \mu(y) = 1$ that is $\mu(x) = 1$ or $\mu(y) = 1$; which contradicts to the choice of x and y. Hence either $\chi_I \subseteq \mu$ or $\chi_J \subseteq \mu$. Therefore $\mu \in V(\chi_I)$ or $\mu \in V(\chi_J)$ and consequently $\mu \in V(\chi_I) \cup V(\chi_J)$. Thus $V(\chi_I \cap \chi_J) \subseteq V(\chi_I) \cup V(\chi_J)$ and the result follows. □

Theorem 3.3. *If $\{\theta_i \mid i \in \Lambda\}$ (Λ is any indexing set) is a family of fuzzy subsets of L, then $V(\bigcup \{\theta_i \mid i \in \Lambda\}) = \bigcap \{V(\theta_i) \mid i \in \Lambda\}$.*

Proof. We have

$$\begin{aligned} \mu \in V\left(\bigcup \{\theta_i \mid i \in \Lambda\}\right) &\iff \bigcup \{\theta_i \mid i \in \Lambda\} \subseteq \mu \\ &\iff \theta_i \subseteq \mu \quad \forall\, i \in \Lambda \\ &\iff \mu \in V(\theta_i) \quad \forall\, i \in \Lambda \\ &\iff \mu \in \bigcap \{V(\theta_i) \mid i \in \Lambda\}. \end{aligned}$$

This sows that $V(\bigcup \{\theta_i \mid i \in \Lambda\}) = \bigcap \{V(\theta_i) \mid i \in \Lambda\}$. □

Remark 3.2. *Unlike in a crisp case $\{V(\theta) \mid \theta \text{ is a fuzzy subset of } L\}$ does not offer a system of closed sets for a topology on the set Σ though $V(\bigcup \{\theta_i | i \in \Lambda\}) = \bigcap \{V(\theta_i) | i \in \Lambda\}$ (where Λ is any indexing set) holds. This happens as $V(\sigma) \cup V(\theta) \neq V(\sigma \cap \theta)$ for some fuzzy subsets θ and σ of L.*

Theorem 3.4. *1. $V(a) \cup V(b) = V(a \vee b)$; $\forall\, a, b \in L$*

2. $V(\chi_E) = \bigcap \{V(a) \mid a \in E\}$; $\forall\, E \subseteq L$.

Proof. (1) Let $\mu \in V(a) \cup V(b)$. Then $\mu \in V(a)$ or $\mu \in V(b)$. If $\mu \in V(a)$, then $\mu(a) = 1$. As μ is a fuzzy filter, $\mu(a \vee b) \geq \mu(a) = 1$ that is $\mu(a \vee b) = 1$. But then $\mu \in V(a \vee b)$ so that $V(a) \subseteq V(a \vee b)$. Similarly if $\mu \in V(b)$, then $\mu \in V(a \vee b)$. Therefore $V(b) \subseteq V(a \vee b)$. Hence $V(a) \cup V(b) \subseteq V(a \vee b)$. Let $\mu \in V(a \vee b)$ then $\mu(a \vee b) = 1$. μ being a fuzzy prime filter, by Result 2.2, $\mu(a \vee b) = \mu(a) \vee \mu(b)$. Thus either $\mu(a) = 1$ or $\mu(b) = 1$ that is $\mu \in V(a)$ or $\mu \in V(b)$ so that $\mu \in V(a) \cup V(b)$. Thus $V(a \vee b) \subseteq V(a) \cup V(b)$. Combining both the inclusions, (1) follows
(2) As $E = \bigcup \{a \mid a \in E\}$, we have $\chi_E = \bigcup \left\{\chi_{\{a\}} \mid a \in E\right\}$. Therefore $V(\chi_E) = V\left(\bigcup \left\{\chi_{\{a\}} \mid a \in E\right\}\right) = \bigcap \{V(a) \mid a \in E\}$ (By Theorem 3.3). □

Theorem 3.5. *Let $\mathfrak{B} = \{X(a) \mid a \in L\}$ where $X(a) = X\left(\chi_{\{a\}}\right) = \Sigma \setminus V(a)$. Then $\mathfrak{B}$ constitutes a base for the open sets of some topology on Σ.*

Proof. By Theorem 3.1, (4), we have $X(0) = \Sigma \setminus V(0) = \Sigma \setminus \emptyset = \Sigma$. Therefore

$$\Sigma = \bigcup \{X(a) \mid a \in L\} \qquad (1)$$

Let $a, b \in L$. Then

$$\begin{aligned} \mu \in X(a) \cap X(b) &\iff \mu \in (\Sigma \setminus V(a)) \cap (\Sigma \setminus (V(b))) \\ &\iff \mu \in \Sigma \setminus (V(a) \cup V(b)) \\ &\iff \mu \in \Sigma \setminus V(a \vee b) \ \text{(by Theorem 3.4 (1))} \\ &\iff \mu \in X(a \vee b) \end{aligned}$$

Thus

$$X(a) \cap X(b) = X(a \vee b) \qquad (2)$$

From (1) and (2) it follows that $\mathfrak{B}$ forms a base for a topology on Σ. □

Let τ denote the topology with base $\mathfrak{B}$ on Σ. The topological space $\langle \Sigma, \tau \rangle$ is called fuzzy prime spectrum of L and is denoted by $Fspec(L)$.

Theorem 3.6. *If L is a chain or a finite lattice, then $\mathfrak{B} = \tau$.*

Proof. Any open set O in $Fspec(L)$ is expressed as $O = \bigcup\{X(a) \mid a \in A \subseteq L\}$. By assumption, $[A) = [t)$ for some $t \in A$. Hence

$$\begin{aligned} O = \bigcup \{X(a) \mid a \in A \subseteq L\} &= \Sigma \setminus \bigcap \{V(a) \mid a \in A\} \\ &= \Sigma \setminus V(\chi_A) \ \text{(by Theorem 3.4 (2))} \\ &= \Sigma \setminus V\left(\chi_{[A)}\right) \ \text{(by Theorem 3.1 (3))} \\ &= \Sigma \setminus V\left(\chi_{[t)}\right) \\ &= \Sigma \setminus V(t) \\ &= X(t) \end{aligned}$$

Thus any open set $O = X(t)$ for some $t \in L$ imply $\tau \subseteq \mathfrak{B}$. But always we have $\mathfrak{B} \subseteq \tau$. Hence $\mathfrak{B} = \tau$. □

Theorem 3.7. *The space $Fspec(L)$ is a compact space.*

Proof. Consider an open cover $\{X(a) \mid a \in A \subseteq L\}$ of Σ consisting of basic open sets. Therefore

$$\begin{aligned} \Sigma &= \bigcup \{X(a) \mid a \in A\} \\ &= \bigcup \{\Sigma \setminus V(a) \mid a \in A\} \\ &= \Sigma \setminus \bigcap \{V(a) \mid a \in A\} \\ &= \Sigma \setminus V(\chi_A) \quad \text{(by Theorem 3.4, (2))} \\ &= \Sigma \setminus V\left(\chi_{[A)}\right) \text{(by Theorem 3.1, (3))} \end{aligned}$$

Therefore $V\left(\chi_{[A)}\right) = \emptyset$ (1)

If $[A) \subset L$, then as every proper filter of a 0 - distributive lattice is contained in a prime filter (see [14]) there exists a prime filter P of L containing $[A)$. Hence $\chi_{[A)} \subseteq \chi_P$ and χ_P is a fuzzy prime filter of L (by Result 2.6). Thus $\chi_P \in V\left(\chi_{[A)}\right) = \emptyset$ (by (1)); a contradiction. This proves that $[A) = L$. This results into $0 \in [A)$ and consequently $0 = a_1 \wedge a_2 \wedge \ldots \wedge a_n$; n is finite, $a_i \in A \quad \forall\, i = 1, 2, \ldots, n$. Let $S = \{a_1, a_2, \ldots, a_n\} \subseteq A$. Then by using (1) and the fact that $[A) = L = [S)$ we have $V(\chi_S) = V\left(\chi_{[S)}\right) = V(\chi_L) = V\left(\chi_{[A)}\right) = \emptyset$
Therefore

$$\begin{aligned} \bigcup \{X(a_i) \mid i = 1, 2, \ldots, n \text{ and } a_i \in S\} &= \bigcup \{\Sigma \setminus V(a_i) \mid i = 1, 2, \ldots, n \text{ and } a_i \in S\} \\ &= \Sigma \setminus \bigcap \{V(a_i) \mid i = 1, 2, \ldots, n \text{ and } a_i \in S\} \\ &= \Sigma \setminus V(\chi_S) \\ &= \Sigma \setminus \emptyset = \Sigma \end{aligned}$$

This shows that $\{X(a_i) \mid a_i \in S,\ i = 1, 2, \ldots, n\}$ is a finite subcover of the basic open cover $\{X(a) \mid a \in A\}$ of Σ. Hence $Fspec(L)$ is compact. □

Theorem 3.8. *A subset $\mathcal{F}$ of Σ is closed in $Fspec(L)$ if and only if there exists $F \subseteq L$ such that $\mathcal{F} = V(\chi_F)$*

Proof. Let $\mathcal{F} = V(\chi_F)$ for some subset F of L. Then $\mathcal{F} = V(\chi_F) = \bigcap\{V(a) \mid a \in F\}$ (by Theorem 3.4, (2)). As $V(a)$ is a closed set in $Fspec(L)$, we get $\mathcal{F}$ is closed in $Fspec(L)$.
Conversely suppose $\mathcal{F}$ is a closed set in $Fspec(L)$. Then $\Sigma \setminus \mathcal{F}$ is open in $Fspec(L)$ and therefore we have

$$\begin{aligned} \Sigma \setminus \mathcal{F} &= \bigcup\{X(a) \mid a \in F\} \text{ for some } F \subseteq L \\ &= \bigcup\{\Sigma \setminus V(a) \mid a \in F\} \\ &= \Sigma \setminus \bigcap\{V(a) \mid a \in F\} \\ &= \Sigma \setminus V(\chi_F) \quad \text{(by Theorem 3.4, (2))} \end{aligned}$$

Thus $\mathcal{F} = V(\chi_F)$ for some $F \subseteq L$. This completes the proof.

□

Theorem 3.9. *Let $\wp$ denote the set of all prime filters of L. Then the set $\mathcal{H} = \{\chi_P \mid P \in \wp\}$ is dense in $Fspec(L)$.*

Proof. By Result 2.6, $\mathcal{H} \subseteq \Sigma$. Let $\mu \in \Sigma \setminus \mathcal{H}$. Let $X(a)$ be a basic open subset of $Fspec(L)$ containing μ. Then the 1 - level subset $\mu_1 = \{x \in L \mid \mu(x) = 1\}$ is proper and hence a prime filter of L (By Result 2.5). $\mu \in X(a) = \Sigma \backslash V(a)$ imply $\mu(a) \neq 1$ so that $a \notin \mu_1$ that is $\chi_{\mu_1}(a) = 0 \neq 1$. This gives $\chi_{\mu_1} \in X(a)$. As μ_1 is a prime filter of L, $\chi_{\mu_1} \in \mathcal{H}$. Thus any $X(a)$ containing μ contains a point of $\mathcal{H}$. Thus every member of $\Sigma \backslash \mathcal{H}$ is a limit point of $\mathcal{H}$. Hence $\overline{\mathcal{H}} = \Sigma$ which proves the result. □

Remark 3.3. *The subspace $\mathcal{H}$ of $Fspec(L)$ is homeomorphic with the prime spectrum $\wp$ (the set equipped with Zariski topology) under the homeomorphism $f : \wp \to \mathcal{H}$ defined by $f(P) = \chi_P\,; \forall P \in \wp$.*

Theorem 3.10. *For any subset $\mathcal{U}$ of Σ, the closure of $\mathcal{U}$ i.e. $\overline{\mathcal{U}} = V(\chi_F)$, where $F = \bigcap\{\mu_1 \mid \mu \in \mathcal{U}\}$.*

Proof. Let $\sigma \in \mathcal{U}$. Also if $\chi_F(x) = 1$ then $x \in F$ and by definition of F we have $x \in \sigma_1$ (since $\sigma \in \mathcal{U}$). But then $\sigma(x) = 1$ which gives $\chi_F \subseteq \sigma$. Thus $\sigma \in V(\chi_F)$. This proves that $\mathcal{U} \subseteq V(\chi_F)$. As $V(\chi_F)$ is a closed set we get $\overline{\mathcal{U}} \subseteq V(\chi_F)$
Now let $\mu \in V(\chi_F)$. If $\mu \in \mathcal{U}$, then $\mu \in \overline{\mathcal{U}}$ and we get $V(\chi_F) \subseteq \overline{\mathcal{U}}$. Otherwise suppose $X(a)$ be a basic open set containing μ. Then $\mu \notin V(a)$ so that $\mu(a) \neq 1$. As $\mu \in V(\chi_F)$ we get $\chi_F(a) \neq 1$. Thus $a \notin F = \bigcap\{\sigma_1 \mid \sigma \in \mathcal{U}\}$ imply $\sigma(a) \neq 1$ for some $\sigma \in \mathcal{U}$. Therefore $\chi_{\{a\}} \nsubseteq \sigma$ and consequently $\sigma \notin V\left(\chi_{\{a\}}\right) = V(a)$ that is $\sigma \in X(a)$. Thus any basic open set $X(a)$ containing μ contains a point σ of $\mathcal{U}$. Therefore μ is a limit point of $\mathcal{U}$. Thus $\mu \in \overline{\mathcal{U}}$. Hence $V(\chi_F) \subseteq \overline{\mathcal{U}}$. Combining both the inclusions the result follows. □

Corollary 3.1. *For any $\mu \in \Sigma$, $\overline{\{\mu\}} = V(\chi_{\mu_1})$ and for $\mu, \theta \in \Sigma$, $\overline{\{\mu\}} = \overline{\{\theta\}}$ if and only if $\mu_1 = \theta_1$.*

Proof. $\overline{\{\mu\}} = V(\chi_{\mu_1})$ follows by Theorem 3.10.
Suppose $\overline{\{\mu\}} = \overline{\{\theta\}}$. Then $V(\chi_{\mu_1}) = V(\chi_{\theta_1})$ (by Theorem 3.10).
Now

$$\begin{aligned} a \in \mu_1 &\Rightarrow \chi_{\mu_1}(a) = 1 \\ &\Rightarrow \sigma(a) = 1\,; \forall \sigma \in V(\chi_{\mu_1}) \\ &\Rightarrow \sigma(a) = 1 \ \forall \sigma \in V(\chi_{\theta_1}) \ \ldots \left(\text{since } V(\chi_{\mu_1}) = V(\chi_{\theta_1})\right) \\ &\Rightarrow \chi_{\theta_1}(a) = 1 \ \ldots \left(\text{as } \chi_{\theta_1} \in V(\chi_{\theta_1})\right) \\ &\Rightarrow a \in \theta_1 \end{aligned}$$

This shows that $\mu_1 \subseteq \theta_1$. Similarly we can prove $\theta_1 \subseteq \mu_1$. Therefore $\mu_1 = \theta_1$.
Conversely suppose $\mu_1 = \theta_1$. Then $\chi_{\mu_1} = \chi_{\theta_1} \Longrightarrow V(\chi_{\mu_1}) = V(\chi_{\theta_1}) \Longrightarrow \overline{\{\mu\}} = \overline{\{\theta\}}$ (by Theorem 3.10). □

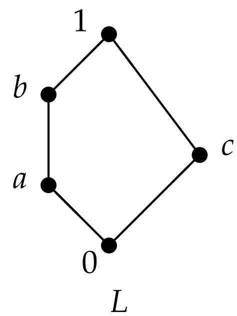

Figure 1:

This result suggests that for a 0 - distributive lattice L the space $Fspec(L)$ need not be T_0 - space. To verify this consider the 0 - distributive lattice L as depicted by the Hasse diagram in Figure 1.

Define $\mu = \{(0,0.2),(a,0.4),(b,0.8),(c,1),(1,1)\}$ and $\theta = \{(0,0.3),(a,0.5),(b,0.7),(c,1),(1,1)\}$. Then μ and θ are fuzzy prime filters of L i.e. $\mu, \theta \in Fspec(L)$. Clearly, $\mu \neq \theta$. But $\mu_1 = \theta_1 = \{c, 1\}$. Hence by Corollary 3.1 we have $\overline{\{\mu\}} = \overline{\{\theta\}}$. This shows that $Fspec(L)$ is not a T_0 - space.

Theorem 3.11. *Let L and L' be bounded 0 - distributive lattices and let $f : L \to L'$ be a lattice homomorphism. For each $\mu' \in Fspec(L')$ define $f^* : Fspec(L') \to Fspec(L)$ by $f^*(\mu') = f^{-1}(\mu')$. Then*
(i) f^* is a continuous mapping.
(ii) If f is surjective, then f^* is injective.

Proof. By Result 2.7, f^* is well defined map.
(i) For any basic closed set $V(a)$ in $Fspec(L)$ where $a \in L$ we have

$$\begin{aligned} f^{*-1}(V(a)) &= \{\mu' \in Fspec(L') \mid f^*(\mu') \in V(a)\} \\ &= \{\mu' \in Fspec(L') \mid [f^*(\mu')](a) = 1\} \\ &= \left\{\mu' \in Fspec(L') \mid \left[f^{-1}(\mu')\right](a) = 1\right\} \\ &= \{\mu' \in Fspec(L') \mid \mu'(f(a)) = 1\} \\ &= \left\{\mu' \in Fspec(L') \mid \chi_{\{f(a)\}} \subseteq \mu'\right\} \\ &= V\left(\chi_{\{f(a)\}}\right) = V(f(a)) \end{aligned}$$

which is a closed set in $Fspec(L')$. Thus inverse image under f^* of a basic closed set in $Fspec(L)$ is a closed set in $Fspec(L')$. Hence f^* is continuous.
(ii) Let f be surjective and $\mu', \theta' \in Fspec(L')$ such that $f^*(\mu') = f^*(\theta')$. Then

$$\begin{aligned} (f^*(\mu'))(x) = (f^*(\theta'))(x)\ ;\ \forall\, x \in L &\Rightarrow \left(f^{-1}(\mu')\right)(x) = \left(f^{-1}(\theta')\right)(x)\ ;\ \forall\, x \in L \\ &\Rightarrow \mu'(f(x)) = \theta'(f(x)) \quad \forall\, x \in L \\ &\Rightarrow \mu' = \theta' \end{aligned}$$

This proves f^* is an injective map. □

Theorem 3.12. *If $f : L \to L'$ is an isomorphism, then $Fspec(L)$ is homeomorphic to $Fspec(L')$.*

Proof. By hypothesis, the functions $f^* : Fspec(L') \longrightarrow Fspec(L)$ defined by $f^*(\mu') = f^{-1}(\mu')$ and $g^* : Fspec(L) \longrightarrow Fspec(L')$ defined by $g^*(\mu) = f(\mu)$, are well defined and inverses of each other (see Result 2.7 and Result 2.8). Further, analogous to the proof in theorem 3.17, it can be proved that they are continuous. □

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Periodic boundary value problem for the graph differential equation and the matrix differential equation

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Abstract

A network can be represented by graph which is isomorphic to its adjacency matrix. Thus the analysis of networks involving rate of change with respect to time reduces to the study of graph differential equations and its associated matrix differential equations. In this paper we develop monotone iterative technique for graph differential equations and its associated matrix differential equations using Periodic boundary value problem.

Keywords: Graph differential equation, Matrix differential equation, coupled lower and upper solutions, monotone iterative technique.

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1 Introduction

It is a well established fact that a graph represents interconnections in a physical or a biological system and a graph that varies with time models physical phenomena that are time dependent. Such systems naturally arise in all disciplines of knowledge including social sciences. In [1,2] the concept of the derivative of a graph was considered and graph differential equations(GDEs) were introduced. In [2] a solution of the graph differential equations was obtained by developing monotone iterative technique [3] for initial value problems(IVPs) of the corresponding matrix differential equations(MDEs). In [4] the concept of a pseudo graph was introduced and a graph was used to study prey predator problem. The idea of using MDEs to study the corresponding GDEs is justified by the existence of an isomorphism between graphs with n-vertices and $n \times n$ matrices.

In this paper, we obtain a solution for the periodic boundary value problem(PBVP) of a GDE by studying the corresponding PBVP of the MDE. Using the approach used in [5] we develop a sequence of monotone iterates which are solutions of a sequence of IVPs of linear matrix differential equations. We develop different types of iterative sequences as in [6,7] using the iterative scheme given in [8] to obtain the monotone sequences. Next, we show that the sequence converges to a solution of the PBVP and that the corresponding graph function is a solution for the considered PBVP for a GDE.

2 Preliminaries

In this section, we give certain definitions,notations,results and preliminary facts related to GDEs that are required to study the main results in the problem.

Definition 2.1. Pseudo simple graph: A simple graph having loops is called as a pseudo simple graph.

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Let $v_1, v_2, ...v_N$ be N vertices,where N is any positive integer. Let D_N be the set of all weighted directed pseudo simple graphs D=(V, E). Then $(D_N, +, .)$ is a linear space w.r.t the operations + and . defined in [1,2].

Let the set of all corresponding adjacency matrices be E_N. Then $(E_N, +, .)$ is a matrix linear space where '+' denotes matrix addition and '.' denotes scalar multiplication. With this basic structure defined, the comparison theorems, existence and uniqueness results of a solution of a MDE and the corresponding GDE follow as in [2].

Definition 2.2. Continuous and differentiable matrix function:
(1) A matrix function $E : I \to \mathbb{R}^{n\times n}$ defined by $E(t) = (e_{ij}(t))_{N\times N}$ is said to be continuous if and only if each entry $e_{ij}(t)$ is continuous for all $i, j = 1, 2, \ldots, N$ where $e_{ij} : I \to \mathbb{R}$.
(2) A continuous matrix function E(t) is said to be differentiable if and only if each entry $e_{ij}(t)$ is differentiable for all $i, j = 1, 2, \ldots, N$. The derivative of E(t) (if exists) is denoted by E' and is given by $E'(t) = (e'_{ij})_{N\times N}$.

Definition 2.3. Continuous and differentiable graph function: Let $D : I \to D_N$ be a graph function and $E : I \to \mathbb{R}^{n\times n}$ be its associated adjacency matrix function. Then
(1) D(t) is said to be continuous if and only if E(t) is continuous.
(2) D(t) is said to be differentiable if and only if E(t) is differentiable.

If for any graph function D the corresponding adjacency matrix function is differentiable then we say that D is differentiable and the derivative of D (if exists) is denote by D'.

Consider the initial value problem

$$D' = G(t, D), \quad D(t_0) = D_0 \tag{2.1}$$

Let E, E_0 be adjacency matrices corresponding to any graph D and the initial graph D_0.

Then the MDE is given by

$$E' = F(t, E), \quad E(t_0) = E_0 \tag{2.2}$$

where F(t,E) is the adjacency matrix function corresponding to G(t,D).

Definition 2.4. Any continuous differentiable matrix function E(t) is said to be a solution of (2.2), if and only if it satisfies (2.2).

Definition 2.5. By a solution of GDE (2.1), we mean the graph function D(t) corresponding to the matrix solution E(t) of the MDE (2.2).

In order to obtain a solution of (2.1), we use the corresponding adjacency MDE. As there exists an isomorphism between graphs and matrices, the graph function corresponding to the solution obtained for the MDE will be a solution of the corresponding GDE.

Definition 2.6. Let $\{E_n\}$ be a sequence of matrices and E be a matrix. Then E_n converges to E if and only if given $\epsilon > 0$ there exist n $\geq$ N such that $||E_n - E|| \leq$ ffl for all n $\geq$ N. This means $e_{n_{ij}} \to e_{ij}$ for all $1 \leq i, j \leq N$.

Definition 2.7. Consider two matrices A and B of order N. We say that $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i, j = 1, 2, \ldots, N$.

Definition 2.8.

Let D_1 and D_2 be two graphs. Let $e^1_{ij}(t)$ be the weight of the edge joining the vertex v_j to v_i, $i, j = 1, 2, ..., n$ in D_1 and $e^2_{ij}(t)$ be the weight of the edge joining the vertex v_j to v_i, $i, j = 1, 2, ..., n$ in D_2 then we say that $D_1 \leq D_2$ if and only if $e^1_{ij} \leq e^2_{ij}$.

Theorem 2.1.

If $\{U_n(t)\} \in C^1[I, \mathbb{R}^{n\times n}]$ is a sequence of equicontinuous and equibounded multimappings defined on an interval I, then we can extract a subsequence that converges uniformly to a continuous multimapping $U(t)$ on I.

With the necessary preliminaries in place, we proceed to the next section to develop the main results.

3 Main results

In this section, we obtain a solution for the PBVP for GDE through the solution of the PBVP for MDE. Consider the PBVP for GDE given by

$$D' = G_1(t,D) + G_2(t,D), \quad D(0) = D(T), \tag{3.3}$$

where $G_1, G_2 \in C[I \times D_N, D_N]$ and I=[0,T].

The graphs G_1 and G_2 generate two matrix mappings $F_1, F_2 \in C^1[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]$ such that the weight of the edge joining the vertex v_k to the vertex v_l, $k,l = 1,2,...,n$ is given by $e^i_{lk}(t)$, for $i = 1,2$ that is, $(e^i_{lk}(t))_{n\times n}$ is the weight matrix corresponding to the graph G_i and is denoted by $F_i = (e^i_{lk}(t))_{n\times n}$. Further, let $E(t) = (e_{lk}(t))$ be any arbitrary matrix function corresponding to any arbitrary graph function D(t) and then consider the following MDE corresponding to GDE (3.3) as

$$E' = F_1(t,E) + F_2(t,E), \quad E(0) = E(T), \tag{3.4}$$

where $F_1, F_2 \in C[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]$.

We next proceed to gives several possible notions of lower and upper solutions relative to MDE (3.4).

Definition 3.9. *Let $V_0, W_0 \in C^1[I, \mathbb{R}^{n\times n}]$. Then V_0, W_0 are said to be*
(a) natural lower and upper solutions of (3.4) if

$$\left.\begin{array}{r} V_0' \le F_1(t,V_0) + F_2(t,V_0), \;\; V_0(0) \le V_0(T). \\ W_0' \ge F_1(t,W_0) + F_2(t,W_0), \;\; W_0(0) \ge W_0(T), \;\; t \in I; \end{array}\right\} \tag{3.5}$$

(b) coupled lower and upper solutions of Type I of (3.4) if

$$\left.\begin{array}{r} V_0' \le F_1(t,V_0) + F_2(t,W_0), \;\; V_0(0) \le V_0(T), \\ W_0' \ge F_1(t,W_0) + F_2(t,V_0), \;\; W_0(0) \ge W_0(T), \;\; t \in I; \end{array}\right\} \tag{3.6}$$

(c) coupled lower and upper solutions of Type II of (3.4) if

$$\left.\begin{array}{r} V_0' \le F_1(t,W_0) + F_2(t,V_0), \;\; V_0(0) \le V_0(T), \\ W_0' \ge F_1(t,V_0) + F_2(t,W_0), \;\; W_0(0) \ge W_0(T), \;\; t \in I; \end{array}\right\} \tag{3.7}$$

(d) coupled lower and upper solutions of Type III of (3.4) if

$$\left.\begin{array}{r} V_0' \le F_1(t,W_0) + F_2(t,W_0), \;\; V_0(0) \le V_0(T), \\ W_0' \ge F_1(t,V_0) + F_2(t,V_0), \;\; W_0(0) \ge W_0(T), \;\; t \in I. \end{array}\right\} \tag{3.8}$$

We observe that whenever $V_0(t) \le W_0(t)$, $t \in I$, if $F_1(t,E)$ is nondecreasing in E for each $t \in I$ and $F_2(t,E)$ is nonincreasing in E for each $t \in I$, the lower and upper solutions defined by (3.5) and (3.8) reduce to (3.7) and consequently, it is sufficient to investigate the cases (3.6) and (3.7).

We prove the following lemma that deals with developing the MIT for the equation (3.4).

Lemma 3.1. *Let $P \in C^1[I, \mathbb{R}^{n\times n}]$ such that $P'(t) \le 0$ and $P(0) \le 0$ then $P(t) \le 0$*

Proof. Consider the linear matrix differential equation

$$P'(t) = M(t)P + H(t), \quad P(0) = 0$$

Whose unique solution is given by

$$P(t) = e^{M(t-t_0)}P(0) + \int_{t_0}^{t} e^{M(t-s)}H(s)ds$$

Then by hypothesis, we get, $P(t) \le 0$

Theorem 3.2. *Assume that*
(A_1) $V_0, W_0 \in C^1[I, \mathbb{R}^{n\times n}]$ are coupled lower and upper solutions of Type I relative to (3.4) with $V_0(t) \leq W_0(t)$, $t \in I$;
(A_2) $F_1, F_2 \in C[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]$, $F_1(t,E)$ is nondecreasing in E for each $t \in I$ and $F_2(t,E)$ is nonincreasing in E for each $t \in I$;
(A_3) F_1 and F_2 map bounded sets into bounded sets in $\mathbb{R}^{n\times n}$.

Then there exist monotone sequences $\{V_n\}$,$\{W_n\}$ in $\mathbb{R}^{n\times n}$ such that $V_n \to \rho, W_n \to R$ where (ρ, R) are the coupled minimal and maximal solutions of (3.4), that is, they satisfy

$$\rho' = F_1(t,\rho) + F_2(t,R), \quad \rho(0) = \rho(T),$$
$$R' = F_1(t,R) + F_2(t,\rho), \quad R(0) = R(T).$$

Proof. For each $n \geq 0$, consider the initial value problems

$$V'_{n+1} = F_1(t,V_n) + F_2(t,W_n), \quad V_{n+1}(0) = V_n(T), \tag{3.9}$$

$$W'_{n+1} = F_1(t,W_n) + F_2(t,V_n), \quad W_{n+1}(0) = W_n(T), \tag{3.10}$$

where $V(0) \leq W(0)$.

Our aim is to prove that the solutions of (3.9) and (3.10) satisfy the relation,

$$V_0 \leq V_1 \leq ... \leq V_n \leq W_n \leq ... \leq W_1 \leq W_0, \quad t \in I. \tag{3.11}$$

Since V_o is the coupled lower solution of Type I of (3.4), we have, using the fact $V_0 \leq W_0$ and the nondecreasing character of F,

$$V'_0 \leq F_1(t,V_0) + F_2(t,W_0).$$

Also from (3.9) we get for $n = 0$,

$$V'_1 = F_1(t,V_0) + F_2(t,W_0), \tag{3.12}$$

$$V_1(0) = V_0(T). \tag{3.13}$$

Clearly equations (3.12), (3.13) have a unique solution denoted by $V_1(t), t \in I$. Now we show that $V_0 \leq V_1$ on I. Set $P(t) = V_0(t) - V_1(t)$,

$$P'(t) \leq F_1(t,V_0) + F_2(t,W_0) - F_1(t,V_0) - F_2(t,V_0) \leq 0$$

and $P(0) = V_0(0) - V_1(0) \leq V_0(T) - V_0(T) \leq 0$. Then by Lemma 3.2, we get $P(t) \leq 0$. Thus $V_0 \leq V_1$ on I. A similar argument shows that $W_1 \leq W_0$ on I. For the purpose of showing $V_1 \leq W_1$, consider (3.12), (3.13) along with

$$W'_1 = F_1(t,W_0) + F_2(t,V_0), \tag{3.14}$$

$$W_1(0) = W_0(T). \tag{3.15}$$

Put $P(t) = V_1(t) - W_1(t)$, *then* $P'(t) = F_1(t,V_0) + F_2(t,W_0) - F_1(t,W_0) - F_2(t,V_0)$. Since $V_0 \leq W_0$ on I, using the monotone nature of F_1 and F_2 gives
$P'(t) \leq 0$ and $P(0) = V_1(0) - W_1(0) = V_0(T) - W_0(T) \leq 0$. Therefore by Lemma 3.2, we get $P(t) \leq 0$. Thus $V_1 \leq W_1$ on I. Hence

$$V_0 \leq V_1 \leq W_1 \leq W_0, \text{ on } I. \tag{3.16}$$

Assume that for $j \geq 1$,

$$V_{j-1} \leq V_j \leq W_j \leq W_{j-1}, \text{ on } I. \tag{3.17}$$

Then we will show that

$$V_j \leq V_{j+1} \leq W_{j+1} \leq W_j, \text{ on } I. \tag{3.18}$$

To do this consider,

$$V'_j = F_1(t,V_{j-1}) + F_2(t,W_{j-1}), \tag{3.19}$$

$$V_j(0) = V_{j-1}(T), \tag{3.20}$$

$$V'_{j+1} = F_1(t, V_j) + F_2(t, W_j), \tag{3.21}$$

$$V_{j+1}(0) = V_j(T). \tag{3.22}$$

Now we show that $V_j(t) \leq V_{j+1}(t)$ on I. Consider $P(t) = V_j(t) - V_{j+1}(t)$. Then $P'(t) = F_1(t, V_{j-1}) + F_2(t, W_{j-1}) - F_1(t, V_j) - F_2(t, W_j)$, Now using the fact that $V_{j-1} \leq V_j$, $W_j \leq W_{j-1}$, and the monotone nature of F_1 and F_2, we obtain $P'(t) \leq 0$ and $P(0) = V_j(0) - V_{j+1}(0) = V_{j-1}(T) - V_j(T) \leq 0$. Again by using Lemma 3.2, we get $P(t) \leq 0$. Thus $V_j \leq V_{j+1}$ on I. Similarly we get $W_{j+1} \leq W_j$ on I.

Next we show that $V_{j+1} \leq W_{j+1}$ $t \in I$. We have from (3.9), (3.10)

$$V'_{j+1} = F_1(t, V_j) + F_2(t, W_j), \quad V_{j+1}(0) = V_j(T), \tag{3.23}$$

$$W'_{j+1} = F_1(t, W_j) + F_2(t, V_j), \quad W_{j+1}(0) = W_j(T). \tag{3.24}$$

Set $P(t) = V_{j+1}(t) - W_{j+1}(t)$, then $P'(t) = F_1(t, V_j) + F_2(t, W_j) - F_1(t, W_j) - F_2(t, V_j)$, Arguing as earlier we conclude $P'(t) \leq 0$ and $P(0) = V_{j+1}(0) - W_{j+1}(0) = V_j(T) - W_j(T) \leq 0$. By Lemma 3.2, we get $P(t) \leq 0$. Thus $V_{j+1} \leq W_{j+1}$ on I.
Hence (3.18) follows and consequently, by induction (3.18) is valid for all n. Clearly the sequences $\{V_n\}$,$\{W_n\}$ are uniformly bounded on I.
To show that these sequences are equicontinuous, consider for any $s \geq t$, where $t, s \in I$,

$$\begin{aligned}
|V_n(t) - V_n(s)| = |V_n(0) + \int_0^t (F_1(\xi, V_{n-1}(\xi)) + F_2(\xi, W_{n-1}(\xi)))d\xi \\
- V_n(0) - \int_0^s (F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)))d\xi| \\
= |\int_0^t (F_1(\xi, V_{n-1}(\xi)) + F_2(\xi, W_{n-1}(\xi)))d\xi \\
- \int_0^s (F_1(\xi, V_{n-1}(\xi)) + F_2(\xi, W_{n-1}(\xi)))d\xi| \\
\leq |\int_s^t (F_1(\xi, V_{n-1}(\xi)) + F_2(\xi, W_{n-1}(\xi)))d\xi| \\
\leq M|t - s|.
\end{aligned}$$

Here we utilized the properties of integral, together with the fact F_1 and F_2 are bounded and $\{V_n\}$,$\{W_n\}$ are uniformly bounded. Hence $\{V_n(t)\}$ is equicontinuous on I. The corresponding Ascoli's Theorem, Theorem 2.9, gives a subsequence $\{V_{n_k}\}$ which converges uniformly to $\rho(t) \in \mathbb{R}^{n\times n}, t \in I$, and since $\{V_n(t)\}$ is nondecreasing sequence, the entire sequence $\{V_n(t)\}$ converges uniformly to $\rho(t)$ on I.

Similar arguments apply to the sequence $\{W_n(t)\}$ and we obtain $W_n(t) \to R(t)$ uniformly on I. It therefore follows, using the integral representation of (3.9), (3.10) that $\rho(t)$, R(t) satisfy

$$\rho'(t) = F_1(t, \rho(t)) + F_2(t, R(t)), \quad \rho(0) = \rho(T);$$

$$R'(t) = F_1(t, R(t)) + F_2(t, \rho(t)), \quad R(0) = R(T).$$

and that

$$V_0 \leq \rho \leq R \leq W_0, \ \ t \in I.$$

Next we claim that (ρ, R) are coupled minimal and maximal solutions of (3.4), that is, if U(t) is any solution of (3.4) such that

$$V_0 \leq U \leq W_0, \ \ on\ I, \tag{3.25}$$

then

$$V_0 \leq \rho \leq U \leq R \leq W_0, \ \ on\ I. \tag{3.26}$$

Suppose that for some n,

$$V_n \leq U \leq W_n \ \ \ on\ I. \tag{3.27}$$

Then we have using the monotone nature of F_1, F_2 and (3.4)

$$U' = F_1(t,U) + F_2(t,U) \geq F_1(t,V_n) + F_2(t,W_n), \quad U(0) = U(T).$$

$$V'_{n+1} = F_1(t,V_n) + F_2(t,W_n), \quad V_{n+1}(0) = V_n(T).$$

Now we show that $V_{n+1} \leq U$ on I. Set $P(t) = V_{n+1}(t) - U(t)$, $P'(t) = F_1(t,V_n) + F_2(t,W_n) - F_1(t,V_n) - F_2(t,W_n) \leq 0$ and $P(0) = V_{n+1}(0) - U(0) = V_n(T) - U(T) \leq 0$. Then by Lemma 3.2, we get $P(t) \leq 0$. Thus $V_{n+1} \leq U$ on I. Similarly $W_{n+1} \geq U$ on I.
Hence by induction (3.27) is true for all $n \geq 1$. Now taking the limit as $n \to \infty$, we get (3.26), proving the claim. The proof is complete.

In the following theorem we use a different set of iterative scheme to form the existence a result for the MDE (3.4).

Theorem 3.3. *Let the hypothesis of Theorem 3.4 hold with $V_0 \leq W_0$ on I. Then the iterative scheme given by*

$$V'_{n+1} = F_1(t,W_n) + F_2(t,V_n), \tag{3.28}$$

$$V_{n+1}(0) = W_n(T). \tag{3.29}$$

and

$$W'_{n+1}(t) = F_1(t,V_n) + F_2(t,W_n), \tag{3.30}$$

$$W_{n+1}(0) = V_n(T). \tag{3.31}$$

yield alternating sequences $\{V_{2n}, W_{2n+1}\}$ converging to ρ and $\{W_{2n}, V_{2n+1}\}$ converging to R uniformly on I such that the relation

$$V_0 \leq W_1 \leq ... \leq V_{2n} \leq W_{2n+1} \leq U \leq V_{2n+1} \leq W_{2n} \leq ... \leq V_1 \leq W_0. \tag{3.32}$$

holds on I. Further ρ and R are coupled minimal and maximal solutions of Type II for the MDE (3.4).

Proof.

Clearly the IVPs (3.28), (3.29), (3.30) and (3.31) have unique solutions for each $n = 0,1,2,...$ denoted by V_{n+1}(t) and W_{n+1}(t) respectively. Setting $n = 0$ in the iterative scheme we obtain that V_1 and W_1 are solutions of the IVPs for MDEs given by

$$V'_1 = F_1(t,W_0) + F_2(t,V_0), \tag{3.33}$$

$$V_1(0) = W_0(T), \tag{3.34}$$

and

$$W'_1(t) = F_1(t,V_0) + F_2(t,W_0), \tag{3.35}$$

$$W_1(0) = V_0(T). \tag{3.36}$$

First we show that $V_0 \leq W_1$ on I. Set $P(t) = V_0(t) - W_1(t)$, then $P'(t) \leq 0$, due to the fact that $V_0 \leq W_0$ and F_1 and F_2 are monotone in E. Also $P(0) \leq 0$. Then by Lemma 3.2, we get $P(t) \leq 0$. Thus $V_0 \leq W_1$ on I. A similar argument shows that $V_1 \leq W_0$, $W_1 \leq V_1$ on I. Thus, $V_0 \leq W_1 \leq V_1 \leq W_0$ on I.
We now proceed to prove that $V_0 \leq W_1 \leq V_2 \leq W_3 \leq V_3 \leq W_2 \leq V_1 \leq W_0$ on I. To do this, set $n = 1$ in (3.28), (3.29) then

$$V'_2 = F_1(t,W_1) + F_2(t,V_1), \quad V_2(0) = W_1(T). \tag{3.37}$$

Now we show that $W_1 \leq V_2$ on J. Put $P(t) = W_1(t) - V_2(t)$, then by using $V_0 \leq W_0$ and monotone nature of F_1 and F_2, we arrive at $P'(t) \leq 0$ and also $P(0) \leq 0$. Hence by Lemma 3.2, we get $P(t) \leq 0$. Thus $W_1 \leq V_2$ on I. Working in a similar fashion we shows that $W_2 \leq V_1$ on I.
To prove $V_2 \leq W_3$, set $n = 1$ in (3.28), (3.29) and $n = 2$ in (3.30), (3.31) then

$$V'_2 = F_1(t,W_1) + F_2(t,V_1),$$

$$V_2(0) = W_1(T).$$

and

$$W'_3 = F_1(t,V_2) + F_2(t,W_2),$$

$$W_2(0) = V_2(T).$$

We now proceed to prove that $V_2 \le W_3$ on I. Consider $P(t) = V_2(t) - W_3(t)$. Since $W_1 \le V_2$, $W_2 \le V_1$ on I and using the monotone nature of F_1 and F_2, gives $P'(t) \le 0$ and also we get $P(0) \le 0$. Then by Lemma 3.2, we have $P(t) \le 0$. Thus $V_2 \le W_3$ on I. Working in a similar fashion we shows that $W_2 \le V_1$ on I. Working as earlier, it can be easily shown that $W_3 \le V_3$ on I.
Now assume that the relation (3.32) holds for some integer $n = k$ such that

$$W_{2k-1} \le V_{2k} \le W_{2k+1} \le U \le V_{2k+1} \le W_{2k} \le V_{2k-1}. \tag{3.38}$$

To apply mathematical induction we need to prove that

$$W_{2k+1} \le V_{2k+2} \le W_{2k+3} \le U \le V_{2k+3} \le W_{2k+2} \le V_{2k+1} \;\; on \; I. \tag{3.39}$$

For this, set $n = 2k+1$ in (3.28), (3.29) and $n = 2k$ in (3.30), (3.31). Then,

$$V'_{2k+2} = F_1(t, W_{2k+1}) + F_2(t, V_{2k+1}), \quad V_{2k+2}(0) = W_{2k+1}(T). \tag{3.40}$$

and

$$W'_{2k+1} = F_1(t, V_{2k}) + F_2(t, W_{2k}), \quad W_{2k+1}(0) = V_{2k}(T). \tag{3.41}$$

Further, we show that $V_{2k+2} \le W_{2k+3}$ on I. Set $P(t) = V_{2k+2}(t) - W_{2k+3}(t)$. Then $P'(t) \le 0$, due to the fact that $W_1 \le V_2, W_2 \le V_1$ and monotone nature of F_1 and F_2. Also $P(0) \le 0$. Applying by Lemma 3.2, we get $P(t) \le 0$. Thus $V_2 \le W_3$ on I. Similarly, $V_{2k+3} \le W_{2k+2}$, $W_{2k+2} \le V_{2k+1}$ all hold on I.
Thus we are in a position to apply mathematical induction and claim that the relation (3.32) holds. Working as in Theorem 3.3, we can show that the sequences $\{V_{2n}\}, \{V_{2n+1}\}, \{W_{2n}\}, \{W_{2n+1}\}$ are equicontinuous and uniformly bounded. Thus from Theorem 2.9, which is the Arzela-Ascoli Theorem, we conclude that they are uniformly convergent and that $V_{2n} \to \rho, W_{2n+1} \to \rho$ and $W_{2n} \to R$ and $V_{2n+1} \to R$ as $n \to \infty$.

The proof is complete if we show that ρ and R are coupled minimal and maximal solutions of the MDE (3.4). This follows by considering the corresponding Hukuhara integral and using the properties of uniform continuity of F_1 and F_2 and uniform convergence of the sequences $\{V_{2n}\}, \{W_{2n+1}\}$ and $\{V_{2n+1}\}, \{W_{2n}\}$. As the details are routine, we omit them and the proof of the theorem is complete.

Now in order to extend our results to GDEs, we define the various notions of lower and upper solutions of (3.3) and use the results obtained earlier to obtain solutions of the PBVP (3.3).

Definition 3.10. *Let $X_0, Y_0 \in C^1[I, D_N]$ be graph functions then we say that X_0 and Y_0 are*
(a) natural lower and upper solutions of (3.3) if

$$\left.\begin{array}{l} X'_0 \le G_1(t, X_0) + G_2(t, X_0), \;\; X_0(0) \le X_0(T). \\ Y'_0 \ge G_1(t, Y_0) + G_2(t, Y_0), \; Y_0(0) \ge Y_0(T), \;\; t \in I; \end{array}\right\} \tag{3.42}$$

(b) coupled lower and upper solutions of Type I of (3.3) if

$$\left.\begin{array}{l} X'_0 \le G_1(t, X_0) + G_2(t, Y_0), \;\; X_0(0) \le X_0(T), \\ Y'_0 \ge G_1(t, Y_0) + G_2(t, X_0), \;\; Y_0(0) \ge Y_0(T), \;\; t \in I; \end{array}\right\} \tag{3.43}$$

(c) coupled lower and upper solutions of Type II of (3.3) if

$$\left.\begin{array}{l} X'_0 \le G_1(t, Y_0) + G_2(t, X_0), \;\; X_0(0) \le X_0(T), \\ Y'_0 \ge G_1(t, X_0) + G_2(t, Y_0), \;\; Y_0(0) \ge Y_0(T), \;\; t \in I; \end{array}\right\} \tag{3.44}$$

(d) coupled lower and upper solutions of Type III of (3.3) if

$$\left.\begin{array}{l} X'_0 \le G_1(t, Y_0) + G_2(t, Y_0), \;\; X_0(0) \le X_0(T), \\ Y'_0 \ge G_1(t, X_0) + G_2(t, X_0), \;\; Y_0(0) \ge Y_0(T), \;\; t \in I. \end{array}\right\} \tag{3.45}$$

Theorem 3.4. *Assume that*
(A_1) $X_0, Y_0 \in C^1[I, D_N]$ are coupled lower and upper solutions of Type I relative to (3.3) with $X_0(t) \le Y_0(t)$, $t \in I$;

(A_2) $G_1, G_2 \in C^1[I \times D_N, D_N]$, $G_1(t, E)$ is nondecreasing in D for each $t \in I$ and $G_2(t, D)$ is nonincreasing in D for

each $t \in I$;

(A_3) G_1 and G_2 map bounded sets into bounded sets in D_N.

Then there exists solutions $G_\rho(t)$ and $G_R(t)$ where (G_ρ, G_R) are the coupled minimal and maximal solutions of (3.3), that is, they satisfy

$$G'_\rho(t) = G_1(t, G_\rho) + G_2(t, G_R), \quad G_\rho(0) = G_\rho(T),$$
$$G'_R(t) = G_1(t, G_R) + G_2(t, G_\rho), \quad G_R(0) = G_R(T),$$

Proof.

Consider the given graph differential equation (3.3). and its corresponding MDE is (3.4).
By hypothesis we have that X_0, Y_0 are coupled lower and upper solutions of type I of GDE (3.3). with $X_0(t) \leq Y_0(t), t \in I$
By using the isomorphism that exists between graphs and matrices we note that corresponding to $X_0, Y_0 \in C[I, D_N]$ there exists $V_0, W_0 \in C^1[I, \mathbb{R}^{n\times n}]$ such that V_0 *and* W_0 are coupled lower and upper solutions of MDE (3.4) with $V_0(t) \leq W_0(t), t \in I$. Hence hypothesis (A_1) of Theorem 3.3 holds.
Now here $G_1(t, D) \in C[I \times D_N, D_N]$ is nondecreasing in D and there exists $F_1(t, E) \in C[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]$ which is nondecreasing in E and similarly $F_2(t, E)$ is nonincreasing in E for each t. Therefore the hypothesis $(A_2), (A_3)$ of Theorem 3.3 holds. Clearly F_1 and F_2 map bounded sets into bounded sets. Since G_1 *and* G_2 map bounded sets into bounded sets.

Thus we conclude that there exists $\rho(t)$, $R(t) \in C[I,\ \mathbb{R}^{n\times n}]$ such that $(\rho(t),\ R(t))$ are the coupled minimal and maximal solution of the MDE (3.4). By using isomorphism between are obtained that there exist $G_\rho(t), G_R(t) \in C^1[I,\ D_N]$ such that $(G_\rho(t), G_R(t))$ are the coupled minimal and maximal solution of GDE (3.3).

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Double quadrilateral snakes on k-odd sequential harmonious labeling of graphs

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Abstract

The objective of this paper is to investigate some k-odd sequential harmonious labeling of graphs. In particular, we show that k-odd sequential harmonious labeling of double quadrilateral snakes ($2Q_x$-snakes) for each $x \geq 1$. We also prove that, $2mQ_x$-snakes are k-odd sequential harmonious labeling of graphs for each $m, x \geq 1$. Finally, we present some examples and verified to illustrate proposed theories.

Keywords: Labeling, Harmonious, k-odd sequential harmonious, Double quadrilateral snake.

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1 Introduction

All the graphs in this paper are finite,simple and undirected.The symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G.

The cardinality of the vertex set is called the order of G. The cardinality of the edge set is called the size of G. A graph with p vertices and q edges is called a (p, q) graph.

Definition : 1.1

If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.

Definition : 1.2

A graph G is said to be harmonious if there exist an injection $f : V(G) = \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f^+ : E(G)$ =$\{0, 1, 2, \cdots, 2q-1\}$ defined by $f^+(uv)$ = $(f(u) + f(v))(\mod q-1)$ is a bijection and f is said to be harmonious labeling of G.

Definition : 1.3

An odd sequential harmonious labeling if there exist an injection f from the vertex set V to $\{0, 1, 2, \cdots, 2q-1\}$ such that the induced mapping f^+ from the edge set E to $\{1, 3, 5, \cdots, 2q-1\}$ defined by

$$f^+(uv) = \begin{cases} f(u) + f(v), & \text{if } f(u) + f(v) \text{ is even} \\ f(u) + f(v) + 1, & \text{if } f(u) + f(v) \text{ is odd are distinct.} \end{cases}$$

A graph G is said to be an odd sequential harmonious graph if it admits an odd sequential harmonious labeling.

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Definition : 1.4

For any integer $k \geq 1$, A labeling is an k-odd sequential harmonious labeling if there exist an injection f from the vertex set V to $\{k-1, k, k+1, \cdots, k+2q-2\}$ such that the induced mapping f^+ from the edge set E to $\{2k-1, 2k+1, 2k+3, \cdots, 2k+2q-3\}$ defined by

$$f^+(uv) = \begin{cases} f(u)+f(v), & \text{if } f(u)+f(v) \text{ is even} \\ f(u)+f(v)+1, & \text{if } f(u)+f(v) \text{ is odd are distinct.} \end{cases}$$

A graph G is said to be an k-odd sequential harmonious graph if it admits an k-odd sequential harmonious labeling.

In this paper, we investigate Double Quadrilateral Snakes on k-odd sequential harmonious labeling of graphs. Throughout this paper, k denote any positive integer ≥ 1. For brevity, we use k-OSHL for k-odd sequential harmonious labeling.

2 Main Results

Definition : 2.1

The Quadrilateral snake Q_x is obtained from the $v_1, v_2, v_3 \ldots, v_n$ by joining v_i and v_{i+1} to new vertices u_{2i-1} and u_{2i}. That is, every edge of a path is replaced by a cycle C_4.

Definition : 2.2

Let Q_x be the Quadrilateral snake is obtained from the path $v_1, v_2, v_3, \cdots, v_n$. Then the double quadrilateral snake $D(Q_x)$ is obtained from Q_x by adding new vertices $w_1, w_2, w_3, \cdots, w_{2n-2}$ and new edges $v_i w_{2i-2}$ for $2 \leq i \leq n$ and $w_{2i-1}\ w_{2i}$, $v_i\ w_{2i-1}$ for $1 \leq i \leq n-1$.

Theorem : 2.3

Double quadrilateral snake is a k-odd sequential harmonious graph for each $x \geq 1$.

Proof. Let $2Q_x$ - snake be a double quadrilateral snake.

Let the vertices of $2Q_x$ be $\{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq 2n-2\} \cup \{w_i : 1 \leq i \leq 2n-2\}$.

The edges of $2Q_x$ be $\{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_{2i-1} u_{2i} : 1 \leq i \leq n-1\} \cup \{w_{2i-1} w_{2i} : 1 \leq i \leq n-1\} \cup \{v_i u_{2i-1} : 1 \leq i \leq n-1\} \cup \{v_i u_{2i-2} : 2 \leq i \leq n\} \cup \{v_i w_{2i-1} : 1 \leq i \leq n-1\} \cup \{v_i w_{2i-2} : 2 \leq i \leq n\}$, which are denoted in Fig.2.3(a).

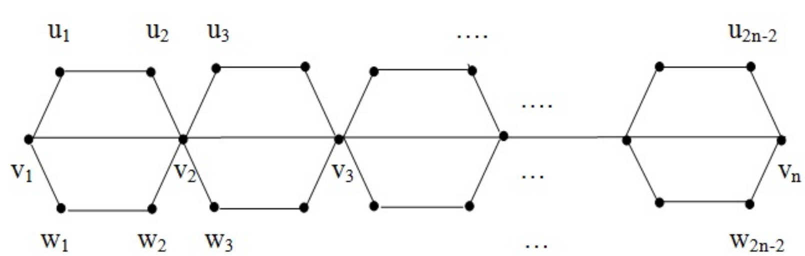

Fig.2.3(a) : D(Qx) with ordinary labeling

We first, label the vertices of $D(Q_x)$ as follows,

Define $f : V(DQ_x) \mathring{\{} k-1, k, k+1, \cdots, k+2q-2\}$ by

$f(v_i) = 3i - 1 + k \qquad 1 \leq i \leq n$

$f(u_i) = 3i - 2 + k \qquad 1 \leq i \leq 2n-2$

$f(w_i) = 3i - k + 1 \qquad 1 \leq i \leq 2n-2$

Then the induced edge labels are

$$\left.\begin{aligned} f^+(v_iv_{i+1}) =& \quad 2i+2k+1 \\ f^+(u_{2i-1}u_{2i}) =& \quad 4i-2k+1 \end{aligned}\right\} 1 \le i \le n-1 \begin{cases} f^+(w_{2i-1}w_{2i}) = 6i-2k+1 \\ f^+(v_iu_{2i-1}) = 6i+2k-1 \end{cases}$$

$f^+(v_iu_{2i-2}) = 6i+2k+1 \qquad 2 \le i \le n$
$f^+(v_iw_{2i-1}) = 6i+2k+3 \qquad 1 \le i \le n-1$
$f^+(v_iw_{2i-2}) = 6i+2k-3 \qquad 2 \le i \le n$

Clearly, the edge labels are odd and distinct, $f^+(E) = \{2k-1, 2k+1, 2k+3, \cdots, 2k+2q-3\}$. Hence, the graph $D(Q_x)$ is a k-odd sequential harmonious graph. □

EXAMPLE : 2-OSHL of $D(Q_3)$ is shown in Fig.2.3(b).

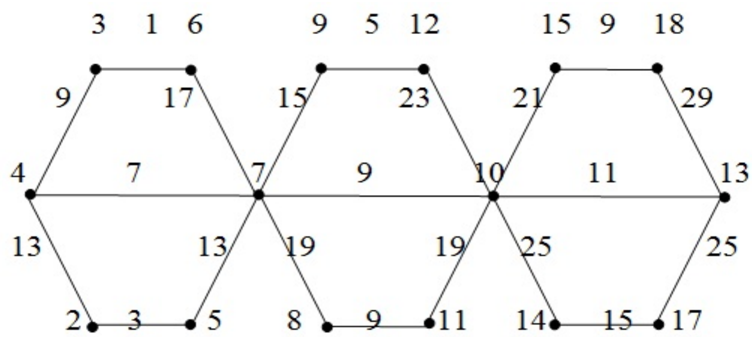

Fig.2.3(b) : 2-OSHL of 2(Q₃)

Theorem : 2.4

Double m-quadrilateral snake is a k-odd sequential harmonious graph for each $m, x \ge 1$.

Proof. Let $2mQ_x$ -snake be a double m-quadrilateral snake.

Let the vertices of $2mQ_x$ be $\{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le 2n-2\} \cup \{u_i^1 : 1 \le i \le 2n-2\} \cup \{u_i^2 : 1 \le i \le 2n-2\} \cup \{w_i : 1 \le i \le 2n-2\} \cup \{w_i^1 : 1 \le i \le 2n-2\} \cup \{w_i^2 : 1 \le i \le 2n-2\}$.

The edges of $2mQ_x$ be $\{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{u_{2i-1}u_{2i} : 1 \le i \le n-1\} \cup \{u_{2i-1}^1u_{2i}^1 : 1 \le i \le n-1\} \cup \{u_{2i-1}^2u_{2i}^2 : 1 \le i \le n-1\} \cup \{w_{2i-1}w_{2i} : 1 \le i \le n-1\} \cup \{w_{2i-1}^1w_{2i}^1 : 1 \le i \le n-1\} \cup \{w_{2i-1}^2w_{2i}^2 : 1 \le i \le n-1\} \cup \{v_iu_{2i-1} : 1 \le i \le n-1\} \cup \{v_i^1u_{2i-1}^1 : 1 \le i \le n-1\} \cup \{v_i^2u_{2i-1}^2 : 1 \le i \le n-1\} \cup \{v_iu_{2i-2} : 2 \le i \le n\} \cup \{v_i^1u_{2i-2}^1 : 2 \le i \le n\} \cup \{v_i^2u_{2i-2}^2 : 2 \le i \le n\} \cup \{v_iw_{2i-1} : 1 \le i \le n-1\}\{v_i^1w_{2i-1}^1 : 1 \le i \le n-1\}\{v_i^2w_{2i-1}^2 : 1 \le i \le n-1\} \cup \{v_iw_{2i-2} : 2 \le i \le n\} \cup \{v_i^1w_{2i-2}^1 : 2 \le i \le n\} \cup \{v_i^2w_{2i-2}^2 : 2 \le i \le n\}$, which are denoted in Fig.2.4(a).

We first, label the vertices of $2m(Q_x)$ as follows,

Define $f : V(2mQ_x)\mathring{\{}k-1, k, k+1, \cdots, k+2q-2\}$ by

$f(v_i) = 4i-6+k \qquad 1 \le i \le n$

$$\left.\begin{aligned} f(u_i) =& \quad 4i-4+k \\ f(u_i^1) =& \quad 4i+4+k \\ f(u_i^2) =& \quad 4i-2-k \end{aligned}\right\} 1 \le i \le 2n-2 \begin{cases} f(w_i) = 4i+k-1 \\ f(w_i^1) = 4i+k+1 \\ f(w_i^2) = 4i-k+1 \end{cases}$$

Then the induced edge labels are

$f^+(v_iv_{i+1}) = 8i+2k+3 \qquad 1 \le i \le n-1$

$$\left.\begin{aligned} f^+(u_{2i-1}u_{2i}) &= 8i+4k-1 \\ f^+(u_{2i-1}^1u_{2i}^1) &= 8i-4k+1 \\ f^+(u_{2i-1}^2u_{2i}^2) &= 8i-4k+3 \end{aligned}\right\} 1 \le i \le n-1 \begin{cases} f^+(w_{2i-1}w_{2i}) = 6i-2k-1 \\ f^+(w_{2i-1}^1w_{2i}^1) = 6i-2k+3 \\ f^+(w_{2i-1}^2w_{2i}^2) = 8i-2k+3 \end{cases}$$

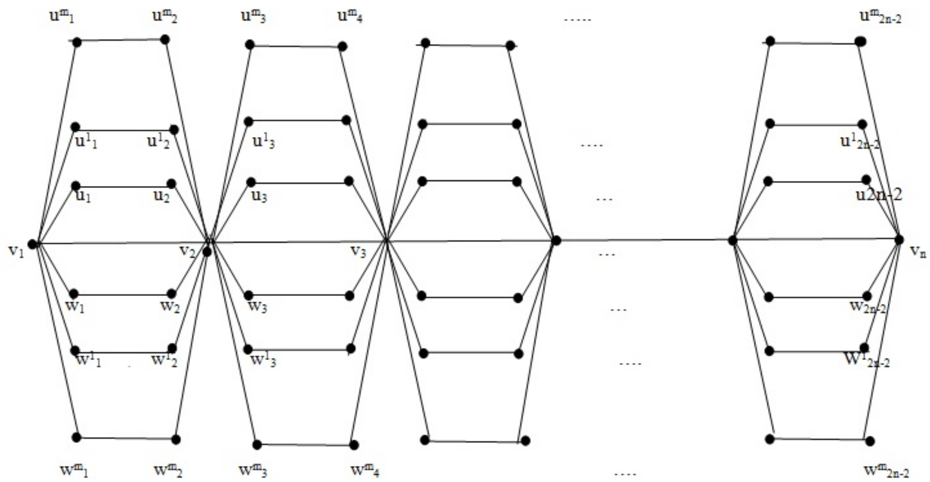

Fig.2.4(a) : 2m(Q$_x$) with ordinary labeling

$$\left.\begin{aligned} f^+(v_iu_{2i-1}) &= 8i+4k-3\\ f^+(v_iu^1_{2i-1}) &= 2i+2k+3\\ f^+(v_iu^2_{2i-1}) &= 2i+2k-3 \end{aligned}\right\}\ 1\le i\le n-1 \left\{\begin{aligned} f^+(v_iw_{2i-1}) &= 8i+2k-3\\ f^+(v_iw^1_{2i-1}) &= 8i-2k-3\\ f^+(v_iw^2_{2i-1}) &= 10i-2k+1 \end{aligned}\right.$$

$$\left.\begin{aligned} f^+(v_iu_{2i-2}) &= 2i+2k-1\\ f^+(v_iu^1_{2i-2}) &= 4i+2k+1\\ f^+(v_iu^2_{2i-2}) &= 4i+2k-1 \end{aligned}\right\}\ 2\le i\le n \left\{\begin{aligned} f^+(v_iw_{2i-2}) &= 10i-2k-1\\ f^+(v_iw^1_{2i-2}) &= 10i+2k+1\\ f^+(v_iw^2_{2i-2}) &= 10i+2k-1 \end{aligned}\right.$$

Clearly, the edge labels are odd and distinct, $f^+(E) = \{2k-1, 2k+1, 2k+3, \cdots, 2k+2q-3\}$. Hence, the graph $2m(Q_x)$ is a k-odd sequential harmonious graph. □

EXAMPLE :

2-OSHL of $6(Q_3)$ is shown in Fig.2.4(b)

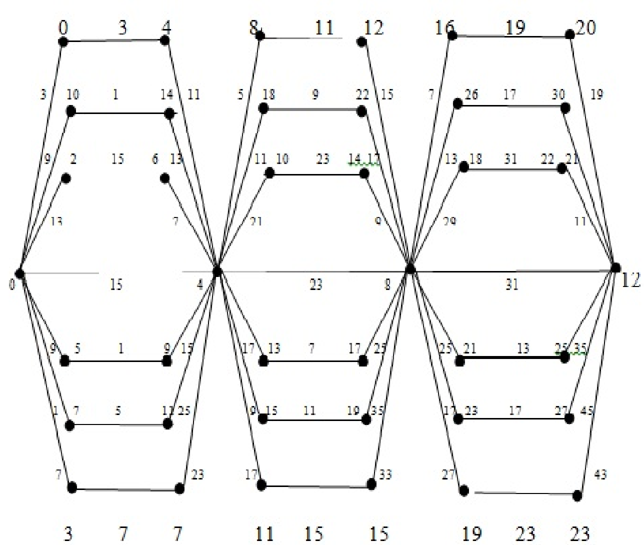

Fig.2.4(b) : 2-OSHL of 6(Q$_3$)

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