ISSN 2319-3786
VOLUME 4, ISSUE 1, JANUARY 2016

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Malaya Journal of Matematik

an international journal of mathematical sciences

UNIVERSITY PRESS
5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),
Coimbatore- 641114, Tamil Nadu, India.
www.malayajournal.org

The Malaya Journal of Matematik is published quarterly in single volume annually and four issues constitute one volume appearing in the months of January, April, July and October.

Subscription

The subscription fee is as follows:

USD 350.00 For USA and Canada

Euro 190.00 For rest of the world

Rs. 4000.00 In India. (For Indian Institutions in India only)

Prices are inclusive of handling and postage; and issues will be delivered by Registered Air-Mail for subscribers outside India.

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Contact No. : +91-9585408402

E-mail : info@mkdpress.com; editorinchief@malayajournal.org; publishingeditor@malayajournal.org

Website : https://mkdpress.com/index.php/index/index

Vol. 4 No. 01 (2016): Malaya Journal of Matematik (MJM)

Malaya J. Mat. 4(1)(2016) 1-7

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

THE DOM-CHROMATIC NUMBER OF A GRAPH

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Abstract

For a given χ-coloring of a graph $G = (V, E)$. A dominating set $S \subseteq V(G)$ is said to be *dom-coloring set* if it contains at least one vertex from each color class of G. The dom-chromatic number $\gamma_{dc}(G)$ is the minimum cardinality taken over all dom-coloring sets of G. In this paper, we initiate a study on $\gamma_{dc}(G)$ and its exact values for some classes of graphs have been established. Also its relationship with other graph theoretic parameters are investigated.

Keywords: Graph; Chromatic number; Domination number; Dom-chromatic number.

2010 MSC: 05C15, 05C70.

1 Introduction

All the graphs $G = (V, E)$ considered here are simple, finite and undirected, where $|V| = p$ denotes number of vertices and $|E| = q$ denotes number of edges of G. In general we use $\langle X \rangle$ to denote subgraph induced by the set of vertices X and $N(v)$ and $N[v]$ denote open and closed neighborhood of a vertex v, respectively. Let $deg(v)$ be the degree of vertex v and usual $\delta(G)$ the minimum degree and $\Delta(G)$ the maximum degree of a graph G. A subgraph H of a graph G is called a component of G, if H is maximally connected sub graph of G. Any undefined term in this paper may be found in Harary [5].

A coloring of a graph G is an assignment of colors to its vertices. So, that no two adjacent vertices have the same color. The set of all vertices with any one color is independent and is called a *color class*. An k-coloring of a graph G uses k-colors. The *chromatic number* $\chi(G)$ is defined as the minimum k for which G has an k-coloring. For complete review on theory of coloring we refer [8] and [10].

A set D of vertices in a graph G is a *dominating set* if every vertex in $V - D$ is adjacent to some vertex in D. *The domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A minimum dominating set of a graph G is called a γ*-set* of G. For more information on domination and its related parameters, we refer [1], [6], [7] and [14].

Analogously, we initiate the study on domination and coloring theory in terms of dom-chromatic number as follows: For a given χ-coloring of G, a dominating set $S \subseteq V(G)$ is said to be *dom-coloring set* if it contains at least one vertex from each color class of G. The *dom-chromatic number* $\gamma_{dc}(G)$ is the minimum cardinality taken over all dom-coloring sets of a graph G.

2 Bounds and characterization

First, we begin with couple of observations.

Observation 1. *In a graph G with χ-coloring, not all dominating sets are dom-coloring sets.*

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For example, consider a complete graph on five vertices say v_1, v_2, v_3, v_4, v_5. The dominating set is v_1, which is not dom-coloring set. The set $S = \{v_1, v_2, v_3, v_4, v_5\}$ is the dom-coloring set.

Observation 2. *Let G be a nontrivial graph with $n \geq 2$ components. Then*

$$\gamma_{dc}(G) \leq \gamma_{dc}(G_1) + \gamma_{dc}(G_2) + \cdots + \gamma_{dc}(G_n).$$

Theorem 2.1. *For any graph G,*

$$max\{\gamma(G), \chi(G)\} \leq \gamma_{dc}(G) \leq \gamma(G) + \chi(G) - 1.$$

The bounds are sharp.

Proof. Since every dom-coloring set is a dominating set of a graph G and hence $\gamma_{dc}(G) \geq \gamma(G)$. Then a dom-coloring set contains at least one vertex from each color class, we have $\gamma_{dc}(G) \geq \chi(G)$. Thus the lower bound follows.

Since every minimum dominating set contains at least one vertex with any color classes say c_1. Clearly, $S = D \cup T$ is a dom-coloring set with $|S| = \gamma(G)\chi(G) - 1$, where T consists of $\chi(G) - 1$ vertices with distinct colors, distinct from c_1 used in D. Hence the upper bound.

The lower bound attains for P_p, $p \geq 2$ vertices and upper bound attains for K_p or $\overline{K}_p$. □

To prove our next result we make use of the following definition.

Definition 2.1. *In a graph G, the minimum dominating is said to be optimized dominating set if it contains maximum number of vertices with distinct colors, where maximum is taken over all minimum dominating set. The optimized dominating set denoted by D_μ, where μ is the number of colors used in the optimized dominating set.*

For illustrative example of optimized dominating set D_μ. We consider a cycle C_4 with vertices in the form of $\{v_1 v_2 v_3 v_4 v_1\}$. The set of all minimum dominating sets are $D_1 = \{v_1, v_2\}$, $D_2 = \{v_2, v_3\}$, $D_3 = \{v_3, v_4\}$, $D_4 = \{v_4, v_1\}$, $D_5 = \{v_1, v_3\}$ and $D_6 = \{v_2, v_4\}$. Among all above said the minimum dominating set we can take any of D_1, D_2, D_3 and D_4 as D_μ. Since these contains vertices with two colors. D_5 and D_6 can not be taken as D_μ. Since these contains vertices with only one color.
Note: In any nontrivial graph G, $\mu \geq 1$.

Theorem 2.2. *For any graph G,*

$$\gamma_{dc}(G) \leq \gamma(G) + \chi(G) - \mu.$$

Proof. Let G be any nontrivial graph with optimized dominating set D_μ. We claim $S = D_\mu \cup T$ is a dom-coloring set, where T is the set of vertices with all distinct colors which are not used in D_μ. Since $D_\mu \subseteq S$, clearly S is a dominating set and also S contains at least one vertex from each color class. Hence, S is a dom-coloring set. The number of vertices in S is given by $|S| = |D_\mu| + \chi(G) - \mu$. Hence the result follows. □

Theorem 2.3. *For a graph G, $\gamma_{dc}(G) = 1$ if and only if $G = K_1$.*

Theorem 2.4. *For any graph G, $\gamma_{dc}(G) = p$ if and only if $G \cong K_p$ or $\overline{K}_p$.*

Proof. Suppose $\gamma_{dc}(G) = p$. On the contrary, if $G \neq K_p$ or $\overline{K}_p$, then the following cases arise.
Case 1. G is a connected graph.
In G, there exist at least two non adjacent vertices say u, v and each with degree at least one which receive same color. Hence, the set $V(G) - \{u\}$ is the dominating set which contain at least one vertex from each color class. Hence, $\gamma_{dc}(G) < p$, a contradiction.
Case 2. G is a disconnected graph.
Suppose G contains n components, say $G_1, G_2, \ldots, G_n$ and let G_j be the component which uses maximum number of colors in the χ-coloring of G. The set $V(G_j) - \{u\} \cup S$ is the dominating set, where u is any vertex in G_j and S is the union of all components other than G_j. There fore, $\gamma_{dc}(G) < p$, a contradiction. Hence $G = K_p$ or $\overline{K}_p$.
Conversely, suppose $G = K_p$ then $\chi(G) = p$. Hence $\gamma_{dc}(G) = p$. Now suppose $G = \overline{K}_p$ then $\gamma(G) = p$. Thus $\gamma_{dc}(G) = p$. □

Theorem 2.5. *Let G be a connected graph of order at least three with $\delta(G) \geq 2$. Then $\gamma_{dc}(G) = p-1$ if and only if G is a noncomplete graph containing K_{p-1} as its induced subgraph.*

Proof. Let G be a connected graph of order at least three, $\delta(G) \geq 2$ and $\gamma_{dc}(G) = p-1$. On the contrary suppose G contains no K_{p-1} as its induced subgraph then there exist at least four vertices v_1, v_2, v_3 and v_4 such that the edges $e_1 = (v_1v_2)$ and $e_2 = (v_3v_4)$ does not belongs to G. Hence by assigning the same color, say, c_1 to the vertices v_1, v_2 and by assigning the same color say c_2 to the vertices v_3, v_4, we get a $k \leq p-2$ coloring of G. The set $V(G) - \{v_1, v_3\}$ is a dominating set containing at least one vertex from each color class. Hence $\gamma_{dc}(G) \leq p-2$, which is a contradiction.
Conversely, suppose G is a non complete graph containing K_{p-1} as its induced subgraph. The graph G require exactly $p-1$ colors. Thus the set $V(K_{p-1})$ is a minimum dom-coloring set. Hence the proof. □

Theorem 2.6. *Let G be a connected graph with $p \geq 4$ vertices. If $\delta(G) \geq 2$ satisfying the following conditions:*
(i) G contains K_{p-2} as its induced subgraph.
(ii) G contains four vertices v_1, v_2, v_3 and v_4 such that the edges $e_1 = (v_1v_2)$ and $e_2 = (v_3v_4)$ does not belongs to G, then $\gamma_{dc}(G) = p-2$.

Proof. Let G be a connected graph of order at least four vertices with $\delta(G) \geq 2$. Then by condition (i) the vertices of K_{p-2} require $p-2$ colors and dominate other two vertices say v_1 and v_3 and by condition (ii) the colors given to v_1 and v_3 are already used in $V(K_{p-2})$. Thus, $V(K_{p-2})$ is the minimum dom-coloring set with cardinality $p-2$ vertices. Hence the result follows. □

Definition 2.2. *A dominator coloring of a graph G is a proper coloring of G in which every vertex dominates every vertex of at least one color class. The minimum number of colors required for a dominator coloring of G is called the dominator chromatic number of G and is denoted by $\chi_d(G)$.*

To prove our next result we make use of the following result.

Theorem 2.7. *[4] Let G be a connected graph. Then*

$$max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G).$$

The bounds are sharp.

Theorem 2.8. *Let G be a connected graph with $\chi(G) = \chi_d(G)$. Then*

$$\gamma_{dc}(G) = \chi_d(G).$$

Proof. Suppose $\chi(G) = \chi_d(G)$, then by Theorem 2.7, we have for each color class of G, let x_i be a vertex in the class i, where $1 \leq i \leq \chi_d(G)$. We show that the set $S = \{x_i : 1 \leq i \leq \chi_d(G)\}$ is a dominating set. Let $v \in V(G)$. Then v dominates a color class i, for some i $(1 \leq i \leq \chi(G))$. Clearly, D is also a dom-coloring set, which can not be minimized further as it contains exactly one vertex from each color class. Hence the result follows. □

3 Bipartite graph

Theorem 3.9. *For any bipartite graph, with $p \geq 2$ vertices,*

$$2 \leq \gamma_{dc}(G) \leq \lceil \frac{p}{3} \rceil.$$

Further, lower bound exists if in each partite set there exists a vertex with degree equal to cardinality of the other set and upper bound exist if the graph is isomorphic with path P_p, $p \geq 2$ vertices.

Observation 3. *If G is isomorphic to P_2 or P_3, then $\gamma_{dc}(G) \neq \gamma(G)$.*

Theorem 3.10. *For any path P_p with $p \geq 4$ vertices,*

$$\gamma_{dc}(P_p) = \gamma(P_p).$$

Proof. Let a path P_p, $p \geq 4$ be labeled as $1, 2, 3, \ldots, p$. First we prove $\gamma_{dc}(P_p) = \gamma(P_p)$ for $p \geq 6$. We know that $\gamma(P_p) = \lceil \frac{p}{3} \rceil$ for $p \geq 1$, and hence it is true for $p \geq 6$ is also. Now we show the existence of dom-coloring set with cardinality equal to $\lceil \frac{p}{3} \rceil$. Here two cases arise.

Case 1. p is a multiple of 3.

The set $D = \{3m - 1/1 \leq m \leq \frac{p}{3}\}$ is the minimum dominating set which contains at least one vertex from each color class. Hence, $\gamma_{dc}(P_p) = \gamma(P_p)$.

Case 2. p is not a multiple of 3.

Take a largest subpath $P^{'}$ of order multiple of 3, starting from the first vertex of P_p. Form minimum dominating set $D^{'}$ of this path $P^{'}$ as defined in the above case. The set $D = D^{'} \cup \{v\}$ is the minimum dominating set of P_p with cardinality $\lceil \frac{p}{3} \rceil$, where v is the any vertex of P_p not in $P^{'}$. The set D contains at least one vertex from each color class. Hence, $\gamma_{dc}(P_p) = \gamma(P_p)$.

Now we prove the result is true for $p = 4, 5$.

For $p = 4$, $D = \{v_2, v_3\}$ is the minimum dominating set of P_4 which is also dom-coloring set. Then $\gamma_{dc}(P_4) = \gamma(P_4)$.

For $p = 5$, $D = \{v_2, v_5\}$ is the minimum dominating set of P_5 which is also dom-coloring set. Then, $\gamma_{dc}(P_5) = \gamma(P_5)$. □

Theorem 3.11. *For any cycle C_{2p} with $p \geq 2$ vertices,*

$$\gamma_{dc}(C_{2p}) = \gamma(C_p).$$

Theorem 3.12. *For any complete bipartite graph $K_{m,n}$ with $2 \leq m \leq n$ vertices,*

$$\gamma_{dc}(K_{m,n}) = \gamma(K_{m,n}).$$

Proof. By taking one vertex from each partite set. We get a minimum dominating set of $K_{m,n}$ with $2 \leq m \leq n$ vertices, which is also dom-coloring set of $K_{m,n}$. Thus the result follows. □

4 Splitting graph

Definition 4.3. *The splitting graph $S^{'}(G)$ of a graph G is obtained by adding a new vertex $v^{'}$ corresponding to each vertex v of G such that $N(v) = N(v^{'})$, where $N(v)$ and $N(v^{'})$ are the neighborhood sets of v and $v^{'}$ respectively in $S^{'}(G)$.*

Theorem 4.13. *For any non trivial graph G,*

$$\gamma_{dc}(S^{'}(G)) \leq 2\gamma_{dc}(G).$$

Proof. Let G be any graph with $\chi(G)$-coloring and $D = \{v_1, v_2, \ldots, v_{\gamma_{dc}(G)}\}$ be the minimum dom-coloring set of G. The splitting graph $S^{'}(G)$ can also be colored with $\chi(G)$-colors by assigning each $v^{'}$, as that of its the same color corresponding copy in G. In $S^{'}(G)$, D dominates all the vertices except possibly the copies $v_1^{'}, v_2^{'}, \ldots, v_{\gamma_{dc}(G)}^{'}$ of the vertices in D. Hence, $D \cup \{v_1^{'}, v_2^{'}, \ldots, v_{\gamma_{dc}(G)}^{'}\}$ dominates all the vertices of $S^{'}(G)$. Also $D \cup \{v_1^{'}, v_2^{'}, \ldots, v_{\gamma_{dc}(G)}^{'}\}$ contains at least one vertex from each color class as D contains so. Hence, $\gamma_{dc}(S^{'}(G)) \leq |D \cup \{v_1^{'}, v_2^{'}, \ldots, v_{\gamma_{dc}(G)}^{'}\}| = 2\gamma_{dc}(G)$. □

Observation 4. *If G is isomorphic to C_3 or C_4 or C_5 then, $\gamma_{dc}(S^{'}(G)) = \gamma_{dc}(G)$.*

Theorem 4.14. *For any cycle C_p, $p \geq 6$ vertices,*

$$\gamma_{dc}(S^{'}(C_p)) = \begin{cases} 2\gamma_{dc}(C_p), & \text{if } p = 3n+3,\ n \geq 1 \\ 2\gamma_{dc}(C_p) - 1, & \text{if } p = 3n+5,\ n \geq 1 \\ 2\gamma_{dc}(C_p) - 2, & \text{if } p = 3n+4,\ n \geq 1 \end{cases}$$

Proof. Let C_p be labeled as $v_1, v_2, \ldots, v_p$. Here three cases arise.

Case 1. $p = 3n + 3$, $n \geq 1$.

Subcase 1.1. p is even.

Here the cycle C_p is bi-colorable and hence $S^{'}(C_p)$ is also bi-colorable. The set $D = \{v_{3m-2}/1 \leq m \leq n+1\}$ is the minimum dominating set of C_p. Clearly the set D contains at least one vertex from each color class, D dominates all the vertices of $S^{'}(C_p)$ except $D^{'} = \{v^{'}_{3m-2}/1 \leq m \leq n+1\}$. Since there exist no common neighbor ($v \in V(C_p)$ of any two of the vertices in $D^{'}$, to dominate the vertices of $D^{'}$ we must include all the vertices of $D^{'}$ into the dominating set. That is $D_s = D \cup D^{'}$, hence D_s is the minimum dominating set of $S^{'}(C_p)$ containing at least one vertex from each color class. Hence $\gamma_{dc}(S^{'}(C_p)) = 2\gamma_{dc}(C_p)$.

Subcase 1.2. p is odd.

Here the cycle C_p is 3-colorable and hence $S^{'}(C_p)$ is also 3-colorable. Let the first vertex v_1 be colored with color 3, then the set $D_s = D \cup D^{'}$ is the minimum dominating set of $S^{'}(C_p)$ containing at least one vertex from each color class, where D and $D^{'}$ are the sets as defined in the above subcase. Hence $\gamma_{dc}(S^{'}(C_p)) = 2\gamma_{dc}(C_p)$.

Case 2. $p = 3n+5, n \geq 1$.

Subcase 2.1. p is even.

Here the cycle C_p is bi-colorable and hence $S^{'}(C_p)$ is also bi-colorable. The set $D = \{v_{3m-2}/1 \leq m \leq n+2\}$ is the minimum dominating set of C_p. Clearly, the set D contains at least one vertex from each color class, D dominates all the vertices of $S^{'}(C_p)$ except $D^{'} = \{v^{'}_{3m-2}/1 \leq m \leq n+2\}$. There exists a common neighbor v_p of the vertices $v^{'}_{p-1}$ and $v^{'}_1$, hence we take v_p into dominating set and there exist no common neighbor of other vertices of $D^{'}$. We must include the vertices of $D^{'}$ except $v^{'}_1$ and $v^{'}_{p-1}$ into the dominating set, That is $D_s = D \cup \{v_p\} \cup D^{'} \backslash \{v^{'}_1, v^{'}_{p-1}\}$ is the minimum dominating set of $S^{'}(C_p)$ containing at least one vertex from each color class. Hence, $\gamma_{dc}(S^{'}(C_p)) = \gamma_{dc}(C_p) + 1 + \gamma_{dc}(C_p) - 2 = 2\gamma_{dc}(C_p) - 1$.

Subcase 2.2. p is odd.

Here the cycle C_p is 3-colorable and hence $S^{'}(C_p)$ is also 3-colorable. Let the first vertex v_1 be colored with color 3, then the set $D_s = D \cup \{v_p\} \cup D^{'} \backslash \{v^{'}_1, v^{'}_{p-1}\}$ is the minimum dominating set of $S^{'}(C_p)$ containing at least one vertex from each color class, where D and $D^{'}$ are the sets as defined in the above subcase. Hence, $\gamma_{dc}(S^{'}(C_p)) = 2\gamma_{dc}(C_p) - 1$.

Case 3. $p = 3n+4, n \geq 1$.

Subcase 3.1. p is even.

Here the cycle C_p is bi-colorable and hence $S^{'}(C_p)$ is also bi-colorable. The set $D = \{v_{3m-2}/1 \leq m \leq n+2\}$ is the minimum dominating set of C_p. Clearly the set D contains at least one vertex from each color class. The set D dominates all the vertices of $S^{'}(C_p)$ except $D^{'} \backslash \{v^{'}_1, v^{'}_p\} = \{v^{'}_{3m-2}/1 \leq m \leq n+2\} \backslash \{v^{'}_1, v^{'}_p\} = \{v^{'}_{3m-2}/2 \leq m \leq n+1\}$. Since there exist no common neighbor of any of the vertices in $D^{'} \backslash \{v^{'}_1, v^{'}_p\}$, we must include the vertices of $D^{'} \backslash \{v^{'}_1, v^{'}_p\} = \{v^{'}_{3m-2}/2 \leq m \leq n+1\}$ in to the dominating set. i.e., $D_s = D \cup D^{'} \backslash \{v^{'}_1, v^{'}_p\}$ is the minimum dominating set of $S^{'}(C_p)$ containing at least one vertex from each color class. Hence, $\gamma_{dc}(S^{'}(C_p)) = \gamma_{dc}(C_p) + \gamma_{dc}(C_p) - 2 = 2\gamma_{dc}(C_p) - 2$.

Subcase 3.2. p is odd.

Here the cycle C_p is 3-colorable and hence $S^{'}(C_p)$ is also 3-colorable. Let the first vertex v_1 be colored with color 3, then the set $D_s = D \cup D^{'} \backslash \{v^{'}_1, v^{'}_p\}$ is the minimum dominating set of $S^{'}(C_p)$ containing at least one vertex from each color class, where D and $D^{'}$ are the sets as defined in the above subcase 3.1. Hence, $\gamma_{dc}(S^{'}(C_p)) = 2\gamma_{dc}(C_p) - 2$. □

5 Mycielski's graph

Definition 5.4. *From a simple graph G, Mycielski's construction produces a simple graph $\mu(G)$ containing G. Begining with G having vertex set $V = \{v_1, v_2, \ldots, v_p\}$, add vertices $U = \{u_1, u_2, \ldots, u_p\}$ and one more vertex w. Add edges to make u_i adjacent to all of $N_G(v_i)$, and finally let $N(w) = U$.*

To prove our next result we make use of the following results.

Theorem 5.15. *[9] If G is any graph, then*

$$\chi(\mu(G)) = \chi(G) + 1.$$

Theorem 5.16. *[3] If G is any graph, then*

$$\gamma(\mu(G)) = \chi(G) + 1.$$

Theorem 5.17. *For any nontrivial graph G,*

$$\gamma_{dc}(\mu(G)) = \gamma_{dc}(G) + 1.$$

Proof. If T is a dom-coloring set of G then in $\mu(G)$, $T \cup \{w\}$ is the dom-coloring of set. Since w dominates the vertices of U and the vertices in U receives the same color as that of their respective preimages in V. Thus,

$$\gamma_{dc}(\mu(G)) \leq \gamma_{dc}(G) + 1.$$

Let D be a γ_{dc}-set of $\mu(G)$. Clearly, D contains w. Then $D' = D - \{w\}$ dominates V, since $N_{\mu(G)}(v_i) = N_G(v_i) \cup B$, where B is the set of all mirror images of neighbors of v_i. The set D' contains at least one vertex from each color class except the color used for w. Let $D'_{(G)}$ consists of those vertices v_i where either $v_i \in D'$ or $u_i \in D'$. Thus $D'_{(G)}$ is dom-coloring set. Hence

$$\gamma_{dc}(G) \leq |D'_{(G)}| \leq |D'| = \gamma_{dc}(\mu(G)) - 1$$

$$\gamma_{dc}(G) + 1 = \gamma_{dc}(\mu(G)).$$

By virtue of the above facts, we have

$$\gamma_{dc}(\mu(G)) = \gamma_{dc}(G) + 1.$$

□

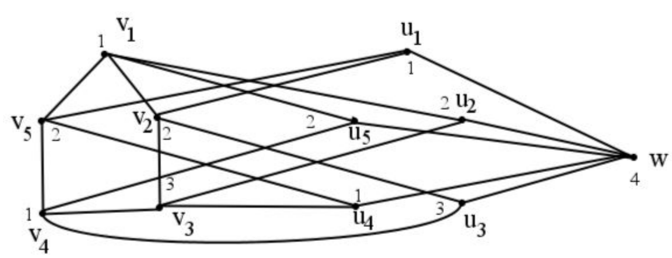

Figure 1: Mycielski's graph of $\mu(C_5)$

For example, consider $\mu(C_5)$ as shown in Figure 1. The minimum dom-coloring set in C_5 is $\{v_1, v_3, v_5\}$ and the minimum dom-coloring set in $\mu(C_5)$ is $\{v_1, v_3, v_5, w\}$. Hence $\gamma_{dc}(C_5) = 3$ and $\gamma_{dc}(\mu(C_5)) = 4$.

6 Acknowledgment

Thanks are due to Prof. N. D. Soner for his help and valuable suggestions in the preparation of this paper.

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Received: November 20, 2015; *Accepted*: January 01, 2016

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 8-18

Malaya Journal of Matematik

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an international journal of mathematical sciences with computer applications...

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ISSN : 2319-3786

Approximating solutions of nonlinear second order ordinary differential equations via Dhage iteration principle

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Abstract

In this paper the authors prove algorithms for the existence as well as approximation of the solutions for an initial and a periodic boundary value problem of nonlinear second order ordinary differential equations. The main results rely on the Dhage iteration principle embodied in a recent hybrid fixed point theorem of Dhage (2013) in the partially ordered normed linear spaces and the numerical solution of the considered equations is obtained under weaker mixed partial continuity and partial Lipschitz conditions. Our hypotheses and results are also illustrated by some numerical examples.

Keywords: Approximating solutions, Dhage iteration principle, hybrid fixed point theorem, initial value problems, periodic boundary value problems.

2010 MSC: 45G10, 45G99.

1 Introduction

The study of nonlinear differential equations via successive approximations has been a topic of great interest since long time. It is Picard who first devised a constructive method for the initial value problems of nonlinear ordinary differential equations in terms of a sequence of successive approximations converging to a unique solution of the related differential equations. The method is commonly known as Picard's iteration method in nonlinear analysis and frequently used for nonlinear equations in the literature. It employs the Lipschitz condition of the nonlinearities together with a certain restriction on Lipschitz constant. The Picard's method is further abstracted to metric spaces by Banach which thereby made it possible to relax the condition on Lipschitz constant. Many attempts have been made in the literature to weaken the Lipschitz condition. Nieto and Lopéz [13] weakened Lipschitz condition to partial Lipschitz condition guaranteeing the conclusion of the Picard's method under certain additional conditions. But in any circumstances the hypothesis of Lipschitz condition is unavoidable to guarantee the conclusion of Picard's method. Very recently, the author in [1–4] proved some abstract hybrid fixed point theorems in the setting of a partially ordered metric spaces and to some nonlinear differential and integral equations without using any kind of geometric condition and still the conclusion of Picard's method holds. See Dhage [5–7] and Dhage and Dhage [8–10] and the references therein. However, in this case the order relation and the metric are required to satisfy certain compatibility condition. See [8], [11] and the references therein. In this paper, we use this hybrid fixed point theorem in the study of initial and boundary value problems of nonlinear second order ordinary differential equations and prove a stronger conclusion of Picard method.

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The paper will be organized as follows. In Section 2 we give some preliminaries and key fixed point theorem that will be used in subsequent part of the paper. In Section 3 we discuss the existence result for initial value problems and in Section 4 we discuss the existence result for periodic boundary value problems of ordinary differential equations.

2 Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of a partially ordered normed linear space appear in Dhage [2] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [12], Nieto and Lopéz [13] and the references therein.

We need the following definitions in the sequel.

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **monotone nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called* **monotone nonincreasing** *if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called* **monotonic** *or simply* **monotone** *if it is either monotone nondecreasing or monotone nonincreasing on E.*

The following terminologies may be found in any book on nonlinear analysis and applications.

Definition 2.2. *An operator $\mathcal{T}$ on a normed linear space E into itself is called* **compact** *if $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called* **totally bounded** *if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is continuous and totally bounded, then it is called* **completely continuous** *on E.*

Definition 2.3 (Dhage [2]). *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.4 (Dhage [1, 2]). *A non-empty subset S of the partially ordered Banach space E is called* **partially bounded** *if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called **partially bounded** if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called* **uniformly partially bounded** *if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant.*

Definition 2.5 (Dhage [1, 2]). *A non-empty subset S of the partially ordered Banach space E is called* **partially compact** *if every chain C in S is compact. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Remark 2.1. *Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.*

Definition 2.6 (Dhage [1]). *The order relation $\preceq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called* **Janhavi** *if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a* **Janhavi metric** *or* **Janhavi Banach space***.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

The Dhage iteration principle or method (in short DIP or DIM) developed in Dhage [1–4] may be described as " ***the sequence of successive approximations of a nonlinear equation beginning with a lower or an upper solution as its first or initial approximation converges monotonically to the solution*** " forms a basic and powerful tool in the study of nonlinear differential and integral equations. See Dhage and Dhage [8, 9] and the references therein. The following applicable hybrid fixed point theorem of Dhage [3] containing the DIP is used as a key tool for the work of this paper.

Theorem 2.1 (Dhage [2]). *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible. Let $\mathcal{T} : E \to E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, then the operator equation $\mathcal{T}x = x$ has a solution x^* in E and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Remark 2.2. *The conclusion of Theorem 2.1 also remains true if we replace the compatibility of E with respect to the order relation $\leq$ and the norm $\|\cdot\|$ by the weaker condition of the compatibility of every compact chain C in E (cf. [3, 8]). The later condition holds if $\leq$ and $\|\cdot\|$ are compatible in every partially compact subset of E.*

Remark 2.3. *The regularity of E in above Theorem 2.1 may be replaced with a stronger continuity condition of the operator $\mathcal{T}$ on E which is a result proved in Dhage [2].*

3 Initial Value Problems

Given a closed and bounded interval $J = [t_0, t_0 + a]$ of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$ and $a > 0$, consider the initial value problem (in short IVP) of second order ordinary nonlinear hybrid differential equation,

$$\left.\begin{aligned} x''(t) &= f(t, x(t)),\ t \in J, \\ x(t_0) &= \alpha_0\,,\ x'(t_0) = \alpha_1, \end{aligned}\right\} \tag{3.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

By a *solution* of the IVP (3.1) we mean a function $x \in C^2(J, \mathbb{R})$ that satisfies equation (3.1), where $C^2(J, \mathbb{R})$ is the space of twice continuously differentiable real-valued functions defined on J.

The IVP (3.1) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. In the present paper it is proved that the existence of the solutions may be proved under weaker partially continuity and partially compactness type conditions.

The equivalent integral form of the IVP (3.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.2}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.3}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. The following lemma follows immediately form Arzellá-Ascoli theorem.

Lemma 3.1. *Let $(C(J, \mathbb{R}), \leq, \|\cdot\|)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.2) and (3.3) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(J, \mathbb{R})$.*

Proof. The proof of the lemma appears in Dhage and Dhage [8] and Dhage [5], but since it is not well-known, we give the details of the proof for the sake of completeness. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots \tag{ND}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent. By monotone characterization, the sequence $\{x_n(t)\}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in J$. This shows that the sequence $\{x_n(t)\}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.7. *A function $u \in C^2(J,\mathbb{R})$ is said to be a lower solution of the IVP (3.1) if it satisfies*

$$\left.\begin{aligned} u''(t) \leq &\quad f(t,u(t)),\ t \in J,\\ u(t_0) \leq &\quad \alpha_0, \quad u'(t_0) \leq \alpha_1, \end{aligned}\right\} \tag{$*$}$$

Similarly, an upper solution v to the IVP (3.1) is defined on J, by reversing the above inequalities.

We consider the following set of assumptions in what follows:

(H_1) There exists a constant $K > 0$ such that $|f(t,x)| \leq K$ for all $t \in J$ and $x \in \mathbb{R}$.

(H_2) The mapping $x \mapsto f(t,x)$ is monotone nondecreasing for each $t \in J$.

(H_3) The IVP (3.1) has a lower solution $u \in C^2(J,\mathbb{R})$.

Lemma 3.2. *For a given integrable function $h : J \to \mathbb{R}$, a function $u \in C^2(J,\mathbb{R})$ is a solution of the IVP*

$$\left.\begin{aligned} x''(t) = h(t),\ t \in J,\\ x(t_0) = \alpha_0, \quad x'(t_0) = \alpha_1, \end{aligned}\right\} \tag{3.4}$$

if and only if it is a solution of the nonlinear integral equation,

$$x(t) = \alpha_0 + x_1(t-t_0) + \int_{t_0}^{t} (t-s)h(s)\,ds,\ t \in J. \tag{3.5}$$

Theorem 3.2. *Assume that hypotheses (H_1) through (H_3) hold. Then the IVP (3.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = x(t) = \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t} (t-s)f(s,x_n(s))\,ds, \tag{3.6}$$

for all $t \in \mathbb{R}$, where $x_0 = u$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then by Lemma 3.1, every compact chain in E is compatible with respect to the norm $\|\cdot\|$ and order relation $\leq$. Define the operator $\mathcal{T}$ by

$$\mathcal{T}x(t) = \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t} (t-s)f(s,x(s))\,ds,\ t \in J. \tag{3.7}$$

From the continuity of the integral, it follows that $\mathcal{T}$ defines the map $\mathcal{T} : E \to E$. Now by Lemma 3.2, the IVP (3.1) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t),\ t \in J. \tag{3.8}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: *$\mathcal{T}$ is nondecreasing on E.*

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (H$_2$), we obtain

$$\begin{aligned} \mathcal{T}x(t) &= \alpha_0 + \alpha_1(t-t_0) + \textstyle\int_{t_0}^t (t-s) f(s, x(s))\, ds \\ &\leq \alpha_0 + \alpha_1(t-t_0) + \textstyle\int_{t_0}^t (t-s) f(s, y(s))\, ds \\ &= \mathcal{T}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is nondecreasing operator on E into E.

Step II: *$\mathcal{T}$ is partially continuous on E.*

Let $\{x_n\}$ be a sequence in a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \left[\alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^t (t-s) f(s, x_n(s))\, ds\right] \\ &= \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^t (t-s) \left[\lim_{n\to\infty} f(s, x_n(s))\right] ds \\ &= \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^t (t-s) f(s, x(s))\, ds \\ &= \mathcal{T}x(t), \end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{T}x_n\}$ converges to $\mathcal{T}x$ pointwise on J.

Next, we will show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned} |\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| &\leq |\alpha_1|\, |t_2 - t_1| + \left|\int_{t_0}^{t_2} (t_2 - s) f(s, x_n(s))\, ds - \int_{t_0}^{t_1} (t_1 - s) f(s, x_n(s))\, ds\right| \\ &\leq |\alpha_1|\, |t_2 - t_1| + \left|\int_{t_0}^{t_1} (t_2 - t_1) f(s, x_n(s))\, ds\right| + \left|\int_{t_1}^{t_2} (t_2 - s) f(s, x_n(s))\, ds\right| \\ &\leq |\alpha_1|\, |t_2 - t_1| + K \int_{t_0}^{t_0+a} |t_2 - t_1|\; ds + K \int_{t_1}^{t_2} |t_2 - s|\, ds \\ &\leq \big(|\alpha_1| + 2a\, K\big)\, |t_2 - t_1| \\ &\to 0 \quad as \quad t_1 \to t_2, \end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \to \mathcal{T}x$ is uniformly and hence $\mathcal{T}$ is partially continuous on E.

Step III: *$\mathcal{T}$ is partially compact on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned} |\mathcal{T}x(t)| &\leq |\alpha_0| + |\alpha_1|(t-t_0) + \left|\int_{t_0}^t (t-s) f(s, x(s))\, ds\right| \\ &\leq |\alpha_0| + |\alpha_1| a + \int_{t_0}^{t_0+a} |t-s|\, |f(s, x(s))|\, ds \\ &\leq |\alpha_0| + |\alpha_1| a + a^2 K \\ &= r, \end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{T}x\| \leq r$ for all $x \in C$. Hence $\mathcal{T}$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| &\leq |\alpha_1|\,|t_2 - t_1| + \left|\int_{t_0}^{t_2}(t_2 - s)f(s,x(s))\,ds - \int_{t_0}^{t_1}(t_1 - s)f(s,x(s))\,ds\right| \\ &\leq |\alpha_1|\,|t_2 - t_1| + \left|\int_{t_0}^{t_1}(t_2 - t_1)f(s,x(s))\,ds\right| + \left|\int_{t_1}^{t_2}(t_2 - s)f(s,x(s))\,ds\right| \\ &\leq |\alpha_1|\,|t_2 - t_1| + K\int_{t_0}^{t_0+a}|t_2 - t_1|\,ds + K\int_{t_1}^{t_2}|t_2 - s|\,ds \\ &\leq \big(|\alpha_1| + 2a\,K\big)\,|t_2 - t_1| \\ &\to 0 \quad as \quad t_1 \to t_2,\end{aligned}$$

uniformly for all $x \in C$. Hence $\mathcal{T}(C)$ is compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \leq \mathcal{T}u$.*

Since the hypothesis (H_3) holds, u is a lower solution of (3.1) defined on J. Then

$$u''(t) \leq f(t,u(t)), \tag{3.9}$$

satisfying,

$$u(t_0) \leq \alpha_0, \quad u'(t_0) \leq \alpha_1, \tag{3.10}$$

for all $t \in J$.

Integrating (3.9) from t_0 to t, we obtain

$$u'(t) \leq \alpha_1 + \int_{t_0}^{t} f(s,u(s))\,ds. \tag{3.11}$$

Again integrating (3.11) from t_0 to t,

$$u(t) \leq \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t}(t - s)f(s,u(s))\,ds = \mathcal{T}u(t)$$

for all $t \in J$. This show that u is a lower solution of the operator equation $x = \mathcal{T}x$.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 in view of Remark 2.2 and we apply to conclude that the operator equation $\mathcal{T}x = x$ has a solution. Consequently the integral equation and the IVP (3.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.1) converges monotonically to x^*. This completes the proof. □

Remark 3.4. *The conclusion of Theorem 3.2 also remains true if we replace the hypothesis (H_3) with the following one:*

(H_3') The IVP (3.1) has an upper solution $v \in C^2(J,\mathbb{R})$.

Example 3.1. *Given a closed and bounded interval $J = [0,1]$, consider the IVP,*

$$\left.\begin{aligned} x''(t) &= \tanh x(t),\ t \in J, \\ x(0) &= 0,\ x'(0) = 1. \end{aligned}\right\} \tag{3.12}$$

Here, $f(t,x) = \tanh x$. Clearly, the functions f is continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $K = 1$. Moreover, the function $f(t,x) = \tanh x$ is nonincreasing in x for each $t \in J$ and so the hypothesis (H_2) is satisfied.

Finally, the IVP (3.1) has a lower solution $u(t) = t - \dfrac{t^2}{2}$ defined on J. Thus all hypotheses of Theorem 3.2 are satisfied. Hence we apply Theorem 3.2 and conclude that the IVP (3.12) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = t + \int_0^t (t - s)\tanh x_n(s)\,ds, \tag{3.13}$$

for all $t \in J$, where $x_0 = u$, converges monotonically to x^*.

Remark 3.5. *In view of Remark 3.4, the existence of the solutions x^* of the IVP (3.12) may be obtained under the assumption of an upper solution $v(t) = \dfrac{t^2}{2} + t$ defined on J and the sequence $\{x_n\}$ defined by*

$$x_{n+1}(t) = t + \int_0^t (t-s)\tanh x_n(s)\,ds, \tag{3.14}$$

for all $t \in J$, where $x_0 = v$, converges monotonically to x^*.

4 Periodic Boundary Value Problems

Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$ consider the periodic boundary value problem (in short PBVP) of second order ordinary nonlinear differential equation

$$\left.\begin{aligned} x''(t) &= f(t, x(t)),\ t \in J, \\ x(0) &= x(T),\ x'(0) = x'(T), \end{aligned}\right\} \tag{4.15}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

By a solution of the PBVP (4.15) we mean a function $x \in C^2(J, \mathbb{R})$ that satisfies equation (4.15).

The PBVP (4.15) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. In the present paper it is proved that the existence as well as algorithm of the solutions may be proved for periodic boundary value problems of nonlinear second order ordinary differential equations under weaker partially continuity and partially compactness type conditions.

We need the following definition in what follows.

Definition 4.8. *A function $u \in C^2(J, \mathbb{R})$ is said to be a lower solution of the of PBVP (4.15) if it satisfies*

$$\left.\begin{aligned} u''(t) &\le f(t, u(t)),\ t \in J, \\ u(0) &\le u(T),\ u'(0) \le u'(T). \end{aligned}\right\} \tag{**}$$

Similarly, an upper solution v to the PBVP (4.15) is defined on J, by reversing the above inequalities.

(H_4) The PBVP (4.15) has a lower solution $u \in C^2(J, \mathbb{R})$.

Consider the PBVP

$$\left.\begin{aligned} x''(t) + h(t)x(t) &= \tilde{f}(t, x(t)),\ t \in J, \\ x(0) &= x(T),\ x'(0) = x'(T), \end{aligned}\right\} \tag{4.16}$$

where $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ and

$$\tilde{f}(t, x) = f(t, x) + h(t)x. \tag{4.17}$$

Remark 4.6. *A function $u \in C^2(J, \mathbb{R})$ is a lower solution of the PBVP (4.15) if and only if it is a solution of the PBVP (4.16) defined on J. A similar assertion is also true for an upper solution. A function $u \in C^2(J, \mathbb{R})$ is a solution of the PBVP (4.15) if and only if it is a lower as well as an upper solution of (4.15) defined on J.*

The following useful lemma may be found in Torres [14].

Lemma 4.3. *If h is a continuous function then for every $\sigma \in C[0, T]$, the linear periodic boundary value problem*

$$\left.\begin{aligned} x''(t) + h(t)x(t) &= \sigma(t),\quad t \in J, \\ x(0) &= x(T),\quad x'(0) = x'(T), \end{aligned}\right\} \tag{4.18}$$

has a unique solution expressed by

$$x(t) = \int_0^T G_h(t, s)\sigma(s)\,ds \tag{4.19}$$

where $G_h \in C([0, T] \times [0, T], \mathbb{R})$ is the Green function of the linear periodic boundary value problem (4.18), which satisfies the positivity $G_h(t, s) > 0$, for every $(t, s) \in [0, T] \times [0, T]$.

The Green's function G_h is continuous on $J \times J$ and therefore, the number

$$M_h := \max\{\,|G_h(t,s)| \,:\, t,s \in [0,T]\,\}$$

exists for all $h \in L^1(J,\mathbb{R}^+)$. In particular, if $h = 1$, then for the sake of convenience we write $G_1(t,s) = G(t,s)$ and $M_1 = M$.

By an application of above Lemma 4.3 we obtain

Lemma 4.4. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a given continuous function. Then a function $u \in C(J,\mathbb{R})$ is a solution of the PBVP (4.15) if and only if it is a solution of the nonlinear integral equation*

$$x(t) = \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds \tag{4.20}$$

for all $t \in J$.

Theorem 4.3. *Assume that hypotheses (H$_1$), (H$_2$) and (H$_4$) hold. Then the PBVP (4.15) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = \int_0^T G(t,s)\tilde{f}(s,x_n(s))\,ds \tag{4.21}$$

for all $t \in J$, where $x_0 = u$ converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then by Lemma 3.1, every compact chain in E is compatible with respect to the norm $\|\cdot\|$ and order relation $\leq$. Define the operator $\mathcal{T}$ on E by

$$\mathcal{T}x(t) = \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds, \quad t \in J. \tag{4.22}$$

From the continuity of the integral, it follows that $\mathcal{T}$ defines the map $\mathcal{T} : E \to E$. Now by Lemma 4.4, the PBVP (4.15) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t), \quad t \in J. \tag{4.23}$$

We shall show that the operators $\mathcal{T}$ satisfies all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: *$\mathcal{T}$ is monotone nondecreasing on E.*

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (H$_1$), we obtain

$$\begin{aligned}\mathcal{T}x(t) &= \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds \\ &\leq \int_0^T G(t,s)\tilde{f}(s,y(s))\,ds \\ &= \mathcal{T}y(t),\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is nondecreasing operator on E into E.

Step II: *$\mathcal{T}$ is partially continuous on E.*

Let $\{x_n\}$ be a sequence in a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \int_0^T G(t,s)\tilde{f}(s,x_n(s))\,ds \\ &= \int_0^T G(t,s)\Big[\lim_{n\to\infty} \tilde{f}(s,x_n(s))\Big]\,ds \\ &= \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds \\ &= \mathcal{T}x(t),\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}x_n$ converges to $\mathcal{T}x$ pointwise on J.

Next, we will show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned} |\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| &= \left| \int_0^T G(t_2,s)\tilde{f}(s,x_n(s))\,ds - \int_0^T G(t_1,s)\tilde{f}(s,x_n(s))\,ds \right| \\ &= \left| \int_0^T |G(t_2,s) - G(t_1,s)||\tilde{f}(s,x_n(s))|\,ds \right| \\ &\leq K \int_0^T |G(t_1,s) - G(t_2,s)|\,ds \\ &\to 0 \quad as \quad t_2 - t_1 \to 0 \end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \to \mathcal{T}x$ is uniformly and hence $\mathcal{T}$ is partially continuous on E.

Step III: *$\mathcal{T}$ is partially compact on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned} |\mathcal{T}x(t)| &= \left| \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds \right| \\ &\leq \int_0^T G(t,s)|\tilde{f}(s,x(s))|\,ds \\ &\leq \int_0^T MK\,ds \\ &\leq MKT = r_1, \end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{T}x\| \leq r_1$ for all $x \in C$. Hence $\mathcal{T}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned} |\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| &= \left| \int_0^T [G(t_2,s) - G(t_1,s)]\tilde{f}(s,x(s))\,ds \right| \\ &\leq \int_0^T |G(t_2,s) - G(t_1,s)||\tilde{f}(s,x(s))|\,ds \\ &\leq \int_0^T |G(t_2,s) - G(t_1,s)|\,K\,ds \\ &\to 0 \quad as \quad t_1 \to t_2 \end{aligned}$$

uniformly for all $x \in C$. Hence $\mathcal{T}(C)$ is compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \leq \mathcal{T}u$.*

By hypothesis (H_4), the PBVP (4.15) has a lower solution u. Then we have

$$\left.\begin{aligned} &u''(t) \leq f(t,u(t)),\ t \in J, \\ &u(0) \leq u(T)\,,\ u'(0) \leq u'(T). \end{aligned}\right\} \tag{4.24}$$

Integrating (4.24) twice which together with the definition of the operator $\mathcal{T}$ implies that $u(t) \leq \mathcal{T}u(t)$ for all $t \in J$. See lemma 4.5.1 of Heikkilä and Lakshmikantham [12] and references therein. Consequently, u is a lower solution to the operator equation $x = \mathcal{T}x$.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 with $x_0 = u$ and we apply it to conclude that the operator equation $\mathcal{T}x = x$ has a solution. Consequently the integral equation and the PBVP (4.15) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (4.15) converges monotonically to x^*. This completes the proof. □

Remark 4.7. *The conclusion of Theorem 4.3 also remains true if we replace the hypothesis* (H_4) *with the following one:*

(H_4') *The PBVP* (4.15) *has an upper solution* $v \in C^2(J, \mathbb{R})$.

Example 4.2. *Given a closed and bounded interval* $J = [0, 1]$ *in* $\mathbb{R}$, *consider the PBVP,*

$$\left.\begin{aligned} x''(t) = f(t, x(t)) - x(t),\ t \in J, \\ x(0) = x(1)\,,\ x'(0) = x'(1), \end{aligned}\right\} \tag{4.25}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ *is defined as*

$$f(t, x) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 + \dfrac{tx}{1+x}, & \text{if } x > 0. \end{cases}$$

Here, $\tilde{f}(t, x(t)) = f(t, x(t))$. Clearly, the functions f is continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $K = 1$.

Again, $\tilde{f}$ is nondecreasing on $J \times \mathbb{R}$ and thus hypothesis (H_2) holds. Finally, the PBVP (4.25) has a lower solution $u(t) = \int_0^T G(t, s)\, ds$ defined on J. Thus all hypotheses of Theorem 4.3 are satisfied. Hence we apply Theorem 4.3 and conclude that the PBVP (4.15) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = \int_0^T G(t, s)\tilde{f}(s, x_n(s))\, ds, \tag{4.26}$$

for all $t \in J$, where $x_0 = u$, converges monotonically to x^*.

Remark 4.8. *In view of Remark 4.7, the existence of the solutions* x^* *of the PBVP* (4.15) *may be obtained under the assumption of an upper solution* $v(t) = 2\int_0^T G(t, s)\, ds$ *defined on* J *and the sequence* $\{x_n\}$ *defined by*

$$x_{n+1}(t) = \int_0^T G(t, s)\tilde{f}(s, x_n(s))\, ds, \tag{4.27}$$

for all $t \in J$, *where* $x_0 = v$, *converges monotonically to* x^*.

5 Conclusion

From the foregoing discussion it is clear that unlike Schauder fixed point principle, the proofs of Theorems 3.2 and 4.3 do not invoke the construction of a non-empty, closed, convex and bounded subset of the Banach space of navigation which is mapped into itself by the operators related to the given differential equations. The convexity hypothesis is altogether omitted from the discussion and still we have proved the existence of the solutions for the differential equations considered in this paper with stronger conclusion. Similarly, unlike the use of Banach fixed point theorem, Theorems 3.2 and 4.3 do not make any use of any type of Lipschitz condition on the nonlinearity involved in the differential equations (3.1) and (4.15), but even then the algorithms for the solutions of the differential equations (3.1) and (4.15) are proved in terms of the successive iteration scheme. The nature of the convergence of the algorithms is not geometrical and so we are not able to obtain the rate of convergence of the algorithms to the solutions of the related problems. However, in a way we have been able to prove the existence results for the IVP (3.1) and PBVP (4.15) under much weaker conditions with a stronger conclusion of the monotone convergence of successive approximations to the solutions than those proved in the existing literature on nonlinear differential equations.

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Received: March 25, 2015; *Accepted*: August 23, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 19–24

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Interval-Valued Fuzzy Ideals in Ternary Semirings

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Abstract

In this paper we introduce the notion of interval-valued fuzzy ternary subsemirings and interval-valued fuzzy ideals in ternary semirings and investigate some of the properties. Also the homomorphism image and inverse image are investigated.

Keywords: Interval-valued fuzzy ternary subsemirings, interval-valued fuzzy ideal.

2010 MSC: 08A72, 16Y60, 20M12, 17A40.

1 Introduction

The notion of ternary algebraic system was introduced by Lehmer [12] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [13] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [1] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings [1–7, 10]. The theory of fuzzy sets was first studied by Zadeh [15] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Interval-valued fuzzy sets were introduced independently by Zadeh [16], Grattan-Guiness [8], Jahn [9], sambuc [14] in the same year 1975 as a generalization of fuzzy set. In the field of application, the success of the use of fuzzy set theory depends on the choice of the membership function that we take. However there are application in which experts do not have precise knowledge of the function that should be taken. In the cases, it is appropriate to represent the membership degree of each element to the fuzzy set by means of an interval. From these considerations arises the extension of fuzzy sets called theory of Interval-valued fuzzy set (IVFS) that is, fuzzy sets such that the membership degree of each element of the fuzzy set is given by a closed subinterval of the interval $[0,1]$. Thus an interval-valued fuzzy set is defined by an interval-valued membership function. It is important to note that not only vagueness (lack of sharp class boundaries), but also a feature of uncertainty (lack of information) can be addressed intuitively by interval valued fuzzy set. Kavikumar et al.[11] studied fuzzy ideals in ternary semirings. In this paper we introduce the notion of interval-valued fuzzy ternary subsemirings and interval-valued fuzzy ideals in ternary semirings and investigate some of the properties. Also the homomorphism image and inverse image are investigated.

2 Preliminaries

In this section, we first give some basic definitions of the theory of ternary semirings which will be used in this paper.

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Definition 2.1. *A nonempty set S together with a binary operation called, addition $+$ and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S, +)$ is a commutative semigroup satisfying the following conditions:*
(i) $(abc)de = a(bcd)e = ab(cde)$,
(ii) $(a + b)cd = acd + bcd$,
(iii) $a(b + c)d = abd + acd$
and (iv) $ab(c + d) = abc + abd$ for all $a, b, c, d, e \in S$.

Throughout this paper S denotes a ternary semiring with zero.

Definition 2.2. *Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then 0 is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.*

Definition 2.3. *An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.*

Definition 2.4. *An additive subsemigroup I of S is called a left [resp. right, lateral] ideal of S if $s_1s_2i \in I$ [resp. $is_1s_2 \in I, s_1is_2 \in I$] for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S, then I is called an ideal of S.*

It is obvious that every ideal of a ternary semiring with zero contains the zero element.

Definition 2.5. *Let S_1 and S_2 be ternary semirings. A mapping $f : S_1 \to S_2$ is said to be a homomorphism if $f(x + y) = f(x) + f(y)$ and $f(xyz) = f(x)f(y)f(z)$ for all $x, y, z \in S_1$.*

Let $f : S_1 \to S_2$ be an onto homomorphism of ternary semirings. Note that if I is an ideal of S_1, then $f(I)$ is an ideal of S_2. If S_1 and S_2 be ternary semirings with zero 0, then $f(0) = 0$.

Definition 2.6. *An interval number on $[0, 1]$, denoted by $\widetilde{a}$, is defined as the closed sub interval of $[0, 1]$, where $\widetilde{a} = [a^-, a^+]$ satisfying $0 \leq a^- \leq a^+ \leq 1$.*

The set of all interval numbers is denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$.

Definition 2.7. *Let $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ be two interval numbers in $D[0, 1]$. Then*
i) $\widetilde{a} \leq \widetilde{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$
ii) $\widetilde{a} = \widetilde{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$
iii) $inf\widetilde{a_i} = [\bigwedge_{i\in I} a_i{}^-, \bigwedge_{i\in I} a_i{}^+]$, $sup\widetilde{a_i} = [\bigvee_{i\in I} a_i{}^-, \bigvee_{i\in I} a_i{}^+]$ for interval numbers $\widetilde{a_i} = [a_i{}^-, a_i{}^+] \in D[0, 1], i \in I$.

Definition 2.8. *Let $\{\widetilde{a_i}\}, i = 1, 2, \cdots, n$ for some $n \in Z^+$ be a finite number of interval numbers, where $\widetilde{a_i} = [a_i{}^-, a_i{}^+]$. Then we define $Max^i\{\widetilde{a_i}\} = [\ max\{a_i{}^-\},\ max\{a_i{}^+\}\ ]$ and $Min^i\{\widetilde{a_i}\} = [\ min\{a_i{}^-\},\ min\{a_i{}^+\}\ ]$.*

In this paper we assume that any two interval numbers in $D[0, 1]$ are comparable. i.e. for any two interval numbers $\widetilde{a}$ and $\widetilde{b}$ in $D[0, 1]$, we have either $\widetilde{a} \leq \widetilde{b}$ or $\widetilde{a} > \widetilde{b}$. It is clear that $(D[0, 1], \leq, \vee, \wedge)$ is a complete lattice with $\widetilde{0} = [0, 0]$ as the least element and $\widetilde{1} = [1, 1]$ as the greatest element.
Let X be a non-empty set. A map $\widetilde{\mu} : X \to D[0, 1]$ is called an interval-valued fuzzy subset of X.
Note: We can write $\widetilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$ for all $x \in X$, for any interval-valued fuzzy subset $\widetilde{\mu}$ of a non empty set X, where μ^- and μ^+ are some fuzzy subsets of X.

Definition 2.9. *Let $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ be two interval-valued fuzzy subsets of a non-empty set X. Then $\widetilde{\mu}_1$ is said to be a subset of $\widetilde{\mu}_2$, denoted by $\widetilde{\mu}_1 \subseteq \widetilde{\mu}_2$ if $\widetilde{\mu}_1(x) \leq \widetilde{\mu}_2(x)$, i.e., $\mu_1^-(x) \leq \mu_2^-(x)$ and $\mu_1^+(x) \leq \mu_2^+(x)$, for all $x \in X$ where $\widetilde{\mu}_1(x) = [\mu_1^-(x), \mu_1^+(x)]$ and $\widetilde{\mu}_2(x) = [\mu_2^-(x), \mu_2^+(x)]$.*

Definition 2.10. *Let $\widetilde{\mu}$ be an interval-valued fuzzy subset of a non-empty set X and $[\alpha, \beta] \in D[0, 1]$. Then the level subset of $\widetilde{\mu}$, denoted by $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$, is defined as $\overline{U}(\widetilde{\mu}, [\alpha, \beta]) = \{x \in X : \widetilde{\mu}(x) \geq [\alpha, \beta]\}$.*

If we consider two interval numbers $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ such that $[\alpha_1, \beta_1] > [\alpha_2, \beta_2]$, then we have $[\alpha_1, \beta_1] \geq [\alpha_2, \beta_2]$ and $[\alpha_1, \beta_1] \neq [\alpha_2, \beta_2]$. In this case, we find that $\overline{U}(\widetilde{\mu}, [\alpha_1, \beta_1]) \subseteq \overline{U}(\widetilde{\mu}, [\alpha_2, \beta_2])$, since for any $x \in \overline{U}(\widetilde{\mu}, [\alpha_1, \beta_1]) \Rightarrow \widetilde{\mu}(x) \geq [\alpha_1, \beta_1] \geq [\alpha_2, \beta_2] \Rightarrow x \in \overline{U}(\widetilde{\mu}, [\alpha_2, \beta_2])$.

Definition 2.11. *Let I be any subset of a ternary semiring S. The interval-valued characteristic function of I denoted by $\widetilde{\chi}_I$ is defined as*

$$\widetilde{\chi}_I = \begin{cases} \widetilde{1} & if \;\; x \in I \\ \widetilde{0} & otherwise \end{cases}$$

Definition 2.12. *Let $\widetilde{\mu}_1$,$\widetilde{\mu}_2$ and $\widetilde{\mu}_3$ be any three interval-valued fuzzy subsets of a ternary semiring S. Then $\widetilde{\mu}_1 \cap \widetilde{\mu}_2$, $\widetilde{\mu}_1 \cup \widetilde{\mu}_2$, $\widetilde{\mu}_1 + \widetilde{\mu}_2$, $\widetilde{\mu}_1 \circ \widetilde{\mu}_2 \circ \widetilde{\mu}_3$ are interval-valued fuzzy subsets of S defined by for all $x \in S$*
$(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x) = Min^i\{\widetilde{\mu}_1(x),\ \widetilde{\mu}_2(x)\}$
$(\widetilde{\mu}_1 \cup \widetilde{\mu}_2)(x) = Max^i\{\widetilde{\mu}_1(x),\ \widetilde{\mu}_2(x)\}$

$$(\widetilde{\mu}_1 + \widetilde{\mu}_2)(x) = \begin{cases} sup\{Min^i\{\widetilde{\mu}_1(y), \widetilde{\mu}_2(z)\}\} & if \;\; x = y + z \\ \widetilde{0} & otherwise \end{cases}$$

$$(\widetilde{\mu}_1 \circ \widetilde{\mu}_2 \circ \widetilde{\mu}_3)(x) = \begin{cases} sup\{Min^i\{\widetilde{\mu}_1(u), \widetilde{\mu}_2(v), \widetilde{\mu}_3(w)\}\} & if \;\; x = uvw, \\ \widetilde{0} & otherwise \end{cases}$$

3 Interval-Valued Fuzzy ideals

Definition 3.13. *Let $\widetilde{\mu}$ be an interval-valued fuzzy subset of S. Then $\widetilde{\mu}$ is called an interval-valued fuzzy ternary subsemiring of S if*
1. $\widetilde{\mu}(x + y) \geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$
2. $\widetilde{\mu}(xyz) \geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y), \widetilde{\mu}(z)\}$ for all $x, y, z \in S$.

Theorem 3.1. *Let $\widetilde{\mu}$ be an interval-valued fuzzy subset of a ternary semiring S. Then $\widetilde{\mu}$ is an interval-valued fuzzy ternary subsemiring of S if and only if $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$ is a ternary subsemiring of S for all $[\alpha, \beta] \in Im\widetilde{\mu}$.*

Proof: Let $\widetilde{\mu}$ be an interval-valued fuzzy ternary subsemiring of S and $[\alpha, \beta]$ be an arbitrary element in $Im\widetilde{\mu}$. Let $x, y, z \in \overline{U}(\widetilde{\mu}, [\alpha, \beta])$ then $\widetilde{\mu}(x) \geq [\alpha, \beta]$, $\widetilde{\mu}(y) \geq [\alpha, \beta]$, $\widetilde{\mu}(z) \geq [\alpha, \beta]$. Now $\widetilde{\mu}(x + y) \geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} \geq [\alpha, \beta]$. Thus $\widetilde{\mu}(x + y) \geq [\alpha, \beta]$. Hence $x + y \in \overline{U}(\widetilde{\mu}, [\alpha, \beta])$. Similarly $xyz \in \overline{U}(\widetilde{\mu}, [\alpha, \beta])$. Hence $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$ is a ternary subsemiring of S.
Conversely, let $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$ be a ternary subsemiring of S for all $[\alpha, \beta] \in Im\widetilde{\mu}$. If there exist $x, y, z \in S$ such that $\widetilde{\mu}(x + y) < [\alpha, \beta] = Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$ then $[\alpha, \beta] \in Im\widetilde{\mu}$ and $x, y \in \overline{U}(\widetilde{\mu}, [\alpha, \beta])$ with $x + y \notin \overline{U}(\widetilde{\mu}, [\alpha, \beta])$ this contradicts to that $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$ is a ternary subsemiring. Hence $\widetilde{\mu}(x + y) \geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$. Similarly we have $\widetilde{\mu}(xyz) \geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y), \widetilde{\mu}(z)\}$. Therefore $\widetilde{\mu}$ is an interval-valued fuzzy ternary subsemiring of S.

Example 3.1. *Consider the set of integer modulo 6, non-positive integer $Z_6^- = \{0, -1, -2, -3, -4, -5\}$ with the addition modulo 6 and ternary multiplication modulo 6. Then $(Z_6^-, \oplus_6, \odot_6)$ is a ternary semiring. Let an interval-valued fuzzy subset $\widetilde{\mu} : Z_6^- \to D[0, 1]$ be defined by $\widetilde{\mu}(0) = [0.7, 0.9]$, $\widetilde{\mu}(-1) = [0.1, 0.2]$, $\widetilde{\mu}(-2) = [0.7, 0.9]$, $\widetilde{\mu}(-3) = [0.1, 0.2]$, $\widetilde{\mu}(-4) = [0.7, 0.9]$ and $\widetilde{\mu}(-5) = [0.1, 0.2]$. Then $\widetilde{\mu}$ is an interval-valued fuzzy ternary subsemiring of S.*

Definition 3.14. *Let $\widetilde{\mu}$ be an interval-valued fuzzy subset of a ternary semiring S. $\widetilde{\mu}$ is called an interval-valued fuzzy right (resp. left, lateral) ideal of S if*
1. $\widetilde{\mu}(x + y) \geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$
2. $\widetilde{\mu}(xyz) \geq \widetilde{\mu}(x)$ (resp. $\widetilde{\mu}(xyz) \geq \widetilde{\mu}(z)$, $\widetilde{\mu}(xyz) \geq \widetilde{\mu}(y)$) for all $x, y, z \in S$.

Definition 3.15. *Let $\widetilde{\mu}$ be an interval-valued fuzzy subset of a ternary semiring S. $\widetilde{\mu}$ is called an interval-valued fuzzy ideal of S if*
1. $\widetilde{\mu}(x + y) \geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$
2. $\widetilde{\mu}(xyz) \geq Max^i\{\widetilde{\mu}(x), \widetilde{\mu}(y), \widetilde{\mu}(z)\}$ for all $x, y, z \in S$.

Theorem 3.2. *Let $\widetilde{\mu}$ be an interval-valued fuzzy subset of a ternary semiring S. Then $\widetilde{\mu}$ is an interval-valued fuzzy right (resp. left, lateral) ideal of S if and only if $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$ is a right (resp. left, lateral) ideal of S for all $[\alpha, \beta] \in Im\widetilde{\mu}$.*

Proof: Let $\widetilde{\mu}$ be an interval-valued fuzzy right ideal of S and $[\alpha, \beta] \in Im\widetilde{\mu}$. By Theorem 3.1, $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$ is a ternary subsemiring. Let $x \in \overline{U}(\widetilde{\mu}, [\alpha, \beta])$ and $y, z \in S$. Then $\widetilde{\mu}(xyz) \geq \widetilde{\mu}(x) \geq [\alpha, \beta]$. Thus $\widetilde{\mu}(xyz) \geq [\alpha, \beta]$, then $xyz \in \overline{U}(\widetilde{\mu}, [\alpha, \beta])$. Hence $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$ is a right ideal of S. Conversely, let $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$ be a right ideal for all $[\alpha, \beta] \in Im\widetilde{\mu}$. By Theorem 3.1, $\widetilde{\mu}$ is an interval-valued fuzzy ternary subsemiring. If there exist $x, y, z \in S$ such that $\widetilde{\mu}(xyz) < [\alpha, \beta] = \widetilde{\mu}(x)$, then $x \in \overline{U}(\widetilde{\mu}, [\alpha, \beta])$ and $y, z \in S$ with $xyz \notin \overline{U}(\widetilde{\mu}, [\alpha, \beta])$. This contradicts that $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$ is a right ideal. Hence $\widetilde{\mu}(xyz) \geq \widetilde{\mu}(x)$. Therefore $\widetilde{\mu}$ is an interval-valued fuzzy right ideal of S.

Example 3.2. *Let S be a ternary semiring consists of non-positive integers with usual addition and ternary multiplication. Let*

$$\widetilde{\mu}(x) = \begin{cases} [0.9, 1] & if \;\; x = 0 \\ [0.5, 0.8] & if \;\; x \text{ is even} \\ [0.2, 0.3] & if \;\; x \text{ is odd} \end{cases}$$

Then $\widetilde{\mu}$ is an interval-valued fuzzy right ideal of S.

Theorem 3.3. *$\widetilde{\chi}_I$ is an interval-valued fuzzy ideal of S if and only if I is an ideal of S.*

Proof: Let $\widetilde{\chi}_I$ be an interval-valued fuzzy ideal of a ternary semiring S. Let $x, y \in I$. Then $\widetilde{\chi}_I(x + y) \geq Min^i\{\widetilde{\chi}_I(x), \widetilde{\chi}_I(y)\} = \widetilde{1}$ which implies $\widetilde{\chi}_I(x + y) = \widetilde{1}$. Thus $x + y \in I$. Similarly if $x \in I$; $y, z \in S$ we have $xyz,\ yzx,\ yxz \in I$. Therefore I is an ideal of S.
Conversely, if there exist $x, y \in S$ such that $\widetilde{\chi}_I(x + y) < \widetilde{1} = Min^i\{\widetilde{\chi}_I(x), \widetilde{\chi}_I(y)\}$ then $\widetilde{\chi}_I(x + y) = \widetilde{0},\ \widetilde{\chi}_I(x) = \widetilde{1},\ \widetilde{\chi}_I(y) = \widetilde{1}$. Thus $x, y \in I$ and $x + y \notin I$ which is a contradiction. Thus $\widetilde{\chi}_I(x + y) \geq Min^i\{\widetilde{\chi}_I(x), \widetilde{\chi}_I(y)\}$. Similarly we prove that $\widetilde{\chi}_I(xyz) \geq Max^i\{\widetilde{\chi}_I(x), \widetilde{\chi}_I(y), \widetilde{\chi}_I(z)\}$. Therefore $\widetilde{\chi}_I$ is an interval-valued fuzzy ideal of S.

Theorem 3.4. *Let $f : S_1 \to S_2$ be an epimorphism of ternary semirings. Let $\widetilde{\mu}$ be an interval-valued fuzzy subset of S_2. Then $\widetilde{\mu}$ is an interval-valued fuzzy ideal of S_2 if and only if $f^{-1}(\widetilde{\mu})$ is an interval-valued fuzzy ideal of S_1 where $[f^{-1}(\widetilde{\mu})](x) = \widetilde{\mu}(f(x))$ for all $x \in S_1$.*

Proof: Let $\widetilde{\mu}$ be an interval-valued fuzzy ideal of S_2. Let $x_1, x_2, x_3 \in S_1$. Now $f^{-1}(\widetilde{\mu})(x_1 + x_2) = \widetilde{\mu}(f(x_1 + x_2)) = \widetilde{\mu}(f(x_1) + f(x_2))$
$\geq Min^i\{\widetilde{\mu}(f(x_1)), \widetilde{\mu}(f(x_2))\} = Min^i\{f^{-1}(\widetilde{\mu})(x_1), f^{-1}(\widetilde{\mu})(x_2)\}$.
Now $f^{-1}(\widetilde{\mu})(x_1x_2x_3) = \widetilde{\mu}(f(x_1x_2x_3)) = \widetilde{\mu}(f(x_1)f(x_2)f(x_3))$
$\geq Max^i\{\widetilde{\mu}(f(x_1)), \widetilde{\mu}(f(x_2)), \widetilde{\mu}(f(x_3))\}$
$= Max^i\{f^{-1}(\widetilde{\mu})(x_1), f^{-1}(\widetilde{\mu})(x_2), f^{-1}(\widetilde{\mu})(x_3)\}$. Thus $f^{-1}(\widetilde{\mu})$ is an interval-valued fuzzy ideal of S_1. Conversely let $f^{-1}(\widetilde{\mu})$ is an interval-valued fuzzy ideal of S_1. Let $y_1, y_2, y_3 \in S_2$ such that $f(x_1) = y_1$, $f(x_2) = y_2$ and $f(x_3) = y_3$ where $x_1, x_2, x_3 \in S_1$.
$\widetilde{\mu}(y_1 + y_2) = \widetilde{\mu}(f(x_1) + f(x_2)) = \widetilde{\mu}(f(x_1 + x_2)) = f^{-1}(\widetilde{\mu})(x_1 + x_2)$
$\geq Min^i\{f^{-1}(\widetilde{\mu})(x_1), f^{-1}(\widetilde{\mu})(x_2)\} = Min^i\{\widetilde{\mu}(f(x_1)), \widetilde{\mu}(f(x_2))\} = Min^i\{\widetilde{\mu}(y_1), \widetilde{\mu}(y_2)\}$. Similarly we have $\widetilde{\mu}(y_1y_2y_3) \geq Max^i\{\widetilde{\mu}(y_1), \widetilde{\mu}(y_2), \widetilde{\mu}(y_3)\}$. Thus $\widetilde{\mu}$ is an interval-valued fuzzy ideal of S_2.

Theorem 3.5. *Let $f : S_1 \to S_2$ be an epimorphism of ternary semirings and let $\widetilde{\mu}$ be an interval-valued fuzzy subset of S_1. If $\widetilde{\mu}$ is an interval-valued fuzzy ideal of S_1 then $f(\widetilde{\mu})$ is an interval-valued fuzzy ideal of S_2 where $f(\widetilde{\mu})(y) = sup\{\widetilde{\mu}(x)/f(x) = y\}$.*

Proof: Let us assume that $\widetilde{\mu}$ is an interval-valued fuzzy ideal of S_1. Let $y_1, y_2, y_3 \in S_2$ then there exist $x_1, x_2, x_3 \in S_1$ such that $f(x_1) = y_1,\ f(x_2) = y_2$ and $f(x_3) = y_3$. Now $f(\widetilde{\mu})(y_1 + y_2) = \sup\limits_{x \in S_1}\{\widetilde{\mu}(x)/f(x) = y_1 + y_2\}$
$\geq \sup\limits_{x_1, x_2 \in S_1}\{\widetilde{\mu}(x_1 + x_2)/f(x_1 + x_2) = y_1 + y_2\}$
$= \sup\limits_{x_1, x_2 \in S_1}\{\widetilde{\mu}(x_1 + x_2)/f(x_1) + f(x_2) = y_1 + y_2\}$
$\geq \sup\{Min^i\{\widetilde{\mu}(x_1), \widetilde{\mu}(x_2)\}/f(x_1) = y_1,\ f(x_2) = y_2\}$
$= Min^i\{\sup\{\widetilde{\mu}(x_1)/f(x_1) = y_1\}, \sup\{\widetilde{\mu}(x_2)/f(x_2) = y_2\}\}$
$= Min^i\{f(\widetilde{\mu})(y_1), f(\widetilde{\mu})(y_2)\}$.
Now $f(\widetilde{\mu})(y_1y_2y_3) \geq \sup\{\widetilde{\mu}(x_1x_2x_3)/f(x_1x_2x_3) = y_1y_2y_3\}$

$= \sup\{\widetilde{\mu}(x_1x_2x_3)/f(x_1)f(x_2)f(x_3) = y_1y_2y_3\}$
$\geq \sup\{Max^i\{\widetilde{\mu}(x_1), \widetilde{\mu}(x_2), \widetilde{\mu}(x_3)\}/f(x_1) = y_1,\ f(x_2) = y_2,\ f(x_3) = y_3\}$
$= Max^i\{\sup\{\widetilde{\mu}(x_1)/f(x_1) = y_1\}, \sup\{\widetilde{\mu}(x_2)/f(x_2) = y_2\},$
$\quad \sup\{\widetilde{\mu}(x_3)/f(x_3) = y_3\}\}$
$= Max^i\{f(\widetilde{\mu})(y_1), f(\widetilde{\mu})(y_2), f(\widetilde{\mu})(y_3)\}.$
Thus $f(\widetilde{\mu})$ is an interval-valued fuzzy ideal of S_2. The following example shows that converse of the above theorem need not be true.

Example 3.3. *Let Z_0^- and Z_6^- be the ternary semirings of negative integers and negative integer modulo 6 respectively. The mapping f defined by $f : Z_0^- \to Z_6^-$, $f(x) = k$ (mod 6) where $k \equiv x$(mod 6), $-5 \leq k \leq 0$ is a homomorphism and f is onto. Let*

$$\widetilde{\mu}(x) = \begin{cases} [0.7, 0.8] & if\ \ x = -18 \\ [0.4, 0.5] & if\ \ x \in \langle -9 \rangle\ \ and\ \ x \neq -18 \\ [0.6, 0.7] & if\ \ x \in \langle -3 \rangle\ \ and\ \ x \notin \langle -9 \rangle \\ [0.1, 0.2] & otherwise \end{cases}$$

Then

$$f(\widetilde{\mu})(x) = \begin{cases} [0.7, 0.8] & if\ \ x = 0 \\ [0.6, 0.7] & if\ \ x = -3 \\ [0.1, 0.2] & otherwise \end{cases}$$

Clearly $f(\widetilde{\mu})$ is an interval-valued fuzzy ideal of Z_6^- but $\widetilde{\mu}$ is not an interval-valued fuzzy ideal of Z_0^-, since $\widetilde{\mu}(-18 + (-18)) < Min^i\{\widetilde{\mu}(-18), \widetilde{\mu}(-18)\}$.

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Received: October 19, 2015; *Accepted*: January 05, 2016

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 25-36

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

On Optional Deterministic Server Vacations in a Batch Arrival Queueing System with a Single Server Providing First Essential Service Followed by One of the Two Types of Additional Optional Service

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Abstract

We analyze a batch arrival queue with a single server providing first essential service (FES) followed by one of the two types of additional optional service (AOS). After completion of the FES, a customer has the option to leave the system or to choose one of the two types of AOS and as soon as a customer leaves (either after the FES or after completing one of the chosen AOS, the server may take a vacation or may continue staying in the system. The vacation times are assumed to be deterministic and the server vacations are based on Bernoulli schedules under a single vacation policy. We obtain explicit queue size distribution at a random epoch under the steady state. In addition, some important performance measures such as the steady state expected queue size and the expected waiting time of a customer at a random epoch are obtained. Further, some interesting particular cases are also discussed.

Keywords: Batch arrivals, compound Poisson process, first essential service (FES), additional optional service (AOS), deterministic server vacation, queue size distribution.

AMS subject classification: Primary 60 K 25, Secondary 60J15, 90B10.

1 Introduction

Server vacations are a common phenomenon in many real life queueing situations. In recent years, research on queueing systems with server vacations has acquired great importance. Queueing systems with a wide variety of vacation policies have been studied by a large number of authors. Many researchers including Borthakur and Choudhury [1], Choudhury [2, 3], Madan and Choudhury [15], Gaver [5], Kielson and Servi [6], Lee and Srinivan [8], Rosenberg and Yechialli [18] and Tegham [22] have studied queues with Bernoulli schedule vacations or modified Bernoulli schedule vacations. Among the authors who studied queueing systems with vacation policies other than the Bernoulli type vacations, we mention Shanthikumar [19] who studied generalized vacations, Takagi [20] and, Madan and Abu-Rub [12–14] who studied vacations based exhaustive service and Madan [9] considered a priority queueing system with exhaustive service in which the server cannot take a vacation till all priority units present in the system are served. Recently, Krishnamoorthy and Sreeniwan [7] and Tao et al [21] studied queueing system with working vacations wherein they assume that the server is on vacation but keeps working in the system at a lower arte. Further, most of the above-mentined authors assumed that the server takes a single vacation and some, e.g. Choudhury and Madan [4] studied a queueing system in which server may take multiple vacations. Further, majority of authors who studied vacation queues assume that the server takes a vacation of random length. However, in many real life situations, the server may take a break or a vacation of fixed lenghth as it happens in a factory, a bank, a railway station and a post office etc.. In order to minimize uncertainty of availability of a server, a fixed length vacation is more realistic in many queueing situations. Madan [10, 16] studied queueing systems with deterministic vacations. Further, Madan [11] introduced the idea of a second optional service in a single server queue without server vacations. Very recently Madan [17] cosidered a queueing system which provides two stages of general service followed by a third stage optional service with deterministic server vacations. In the present paper, we study a queueing sytem with a single server providing the first essentail service followed by one of the two types of additional optional service. This system allows deterministic server vacations. We generalize results obtained by Madan [10], Madan [11] and Madan [16].

2 The Mathematical Model

Customers (units) arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i\, dt$ $(i = 1, 2, 3, \cdots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t,\ t + dt]$, where $0 \le c_i \le 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. There is a single server which provides first essential service (FES) to every customer. We assume that

the first essential service (FES) time random variable S_E follows a general probability law with the distribution function $B(S_E)$ and the probability density function $b(S_E)$ with the k-th moment $E(S_E^k)$, $(k = 1, 2, 3, \ldots)$.

Let $h(x)$ be the conditional probability of completion of FES during the interval $(x,\ x + dx]$, given that the elapsed service time is x, so that

$$h(x) = \frac{h(x)}{1 - H(x)}\,, \tag{2.1}$$

and therefore,

$$h(S_E) = h(S_E) \exp\left[- \int_0^s h(x)\, dx \right]. \tag{2.2}$$

After completion of his FES, a customer may opt to take one of the two kinds of additional optional service (AOS) with probability α or else may leave the system with probability $1 - \alpha$. Each customer opting for the AOS has the option to choose type 1, AOS (1) with probability θ_1 or type 2, AOS (2) with probability θ_2, where $\theta_1 + \theta_2 = 1$. We assume that the service time random variable S_j of type j, AOS (j) follows a general probability law with $B_j(S_j)$ as the distribution function, $b_j(S_j)$ as the probability density function and $E(S_j^k)$ as the k-th moment $(k = 1, 2, 3, \cdots)$ of the service time, $j = 1, 2$.

Let $\mu_j(x)$ be the conditional probability of completion of AOS (j), $j = 1, 2$, during the interval $(x,\ x + dx]$, given that the elapsed service time is x, so that

$$\mu_j(x) = \frac{b_j(x)}{1 - B_j(x)}\,,\ \ j = 1, 2, \tag{2.3}$$

and therefore,

$$b_j(s_j) = \mu_j(s_j) \left[- \int_0^s \mu_j(x)\, dx \right],\ \ j = 1, 2, \tag{2.4}$$

Next, we assume that as soon as the number of services required by a customer i.e. either FES alone or FES followed by one of the two types of AOS are complete, the server may decide to take a vacation for a constant duration d with probability δ or else with probability $1 - \delta$ may continue to be in the system, either providing service to the next customer, if any or else remains idle and waits for the next batch of customers to arrive.

We further assume that whenever the server takes a vacation, it is always a single vacation. In other words, on completion of a vacation, the server must be back to the system even if there is no customer present in the system.

Finally, it is assumed that the inter-arrival times of the customers, the service times of each kind of service and vacation times of the server, all these stochastic processes involved in the system are independent of each other.

3 Definitions and Notations

Assuming that the steady state exists, let $P_n^E(x)$ denote the steady state probability that there are n (≥ 0) customers in the queue excluding one customer in FES and the elapsed service time of this customer is x. Accordingly, $P_n^E = \int_0^\infty P_n^E(x)\,dx$ denotes the corresponding steady state probability irrespective of the elapsed service time x. Next, we define $P_{n,j}(x)$ to be the steady state probability that that there are n (≥ 0) customers in the queue excluding one customer in AOS (j) with elapsed service time x. Accordingly, $P_{n,j} = \int_0^\infty P_{n,j}(x)\,dx$ is the steady state probability that that there are n (≥ 0) customers in the queue excluding one customer in AOS (j), irrespective of the elapsed service time x. Next, we define V_n as a steady state probability that there are n (≥ 0) customers in the queue and the server is on vacation. Finally, let Q denote the steady state probability that the system is empty, i.e., there is no customer either in queue or in service and the server is idle but available in the system. We further assume that K_r is the probability of r arrivals during the vacation period and therefore, ,

$$K_r = \frac{-\exp(\lambda d)(\lambda d)^r}{r!}, r = 0, 1, 2, \dots . \tag{3.1}$$

In addition, we define the following probability generating functions (PGFs):

$$P^E(x,z) = \sum_{n=0}^{\infty} P_n^E(x) z^n, \quad P^E(z) = \sum_{n=0}^{\infty} P^E z^n, \tag{3.2}$$

$$P_j(x,z) = \sum_{n=0}^{\infty} P_{n,j}(x) z^n, \quad P_j(z) = \sum_{n=0}^{\infty} P_{n,j} z^n, j = 1, 2, \tag{3.3}$$

$$K(z) = \sum_{n=0}^{\infty} K_n z^n = \sum_{n=0}^{\infty} \frac{\exp(-\lambda d)(\lambda d)^n}{n!} z^n = \exp\left[-\lambda d(1-z)\right], |z| \leq 1, \tag{3.4}$$

$$C(z) = \sum_{n=0}^{\infty} c_n z^n, |z| \leq 1. \tag{3.5}$$

Following the usual probability arguments, we obtain the following steady equations for our model.

4 Steady State Equations Governing the System

$$\frac{d}{dx} P_n^E(x) + (\lambda + h(x)) P_n^E(x) = \lambda \sum_{i=1}^{n} P_n^E(x), n \geq 1, \tag{4.1}$$

$$\frac{d}{dx} P_0^E(x) + (\lambda + h(x)) P_0^E(x) = 0, \tag{4.2}$$

$$\frac{d}{dx} P_{n,1}(x) + (\lambda + \mu_1(x)) P_{n,1}(x) = \lambda \sum_{i=1}^{n} P_{n-i,1}(x), n \geq 1, \tag{4.3}$$

$$\frac{d}{dx} P_{0,1}(x) + (\lambda + \mu_1(x)) P_{0,1}(x) = 0, \tag{4.4}$$

$$\frac{d}{dx} P_{n,2}(x) + (\lambda + \mu_2(x)) P_{n,2}(x) = \lambda \sum_{i=1}^{n} P_{n-i,2}(x), n \geq 1, \tag{4.5}$$

$$\frac{d}{dx} P_{0,2}(x) + (\lambda + \mu_2(x)) P_{0,2}(x) = 0, \tag{4.6}$$

$$V_n = \delta(1-\alpha) \int_0^\infty P_n^E(x) h(x)\,dx + \delta \sum_{j=1}^{2} \int_0^\infty P_{n,j}(x) \mu_j(x)\,dx, n \geq 0, \tag{4.7}$$

$$\lambda Q = (1-\delta)(1-\alpha) \int_0^\infty P_0^E(x) h(x)\,dx + (1-\delta) \sum_{j=1}^{2} \int_0^\infty P_{0,j}(x) \mu_j(x)\,dx + V_0 K_0. \tag{4.8}$$

We will solve the above equations subject to the following boundary conditions:

$$P_n^E(0) = (1-\delta)(1-\alpha)\int_0^\infty P_{n+1}^E(x)h(x)\,dx + (1-\delta)\sum_{j=1}^{2}\int_0^\infty P_{n+1,j}(x)\mu_j(x)\,dx + V_0K_{n+1} \tag{4.9}$$

$$+V_1K_n + \cdots + V_{n+1}K_0 + \lambda c_{n+1}Q\,, n \geq 1\,, \tag{4.10}$$

$$P_0^E(0) = (1-\delta)\sum_{j=1}^{2}\int_0^\infty P_{1,j}(x)\mu_j(x)\,dx + V_0K_1 + V_1K_0 + \lambda c_1 Q\,, n \tag{4.11}$$

$$P_{n,1}(0) = \alpha\theta_1\int_0^\infty P_n^E(x)h(x)\,dx\,, n \geq 0\,, \tag{4.12}$$

$$P_{n,2}(0) = \alpha\theta_2\int_0^\infty P_n^E(x)h(x)\,dx\,, n \geq 0\,. \tag{4.13}$$

5 Steady State Solution in Terms of Probability Generating Functions

We multiply both sides of equation (4.1) by suitable powers of z, add equation (4.2) in the result and use (3.2) and on simplifying we obtain,

$$\frac{d}{dz}P^E(x,z) + (\lambda - \lambda C(z) + h(x))P^E(x,z) = 0\,. \tag{5.1}$$

Similar operations on equations (4.3) and (4.4); (4.5) and (4.6); and (4.7) yield

$$\frac{d}{dz}P_1(x,z) + (\lambda - \lambda C(z) + \mu_1(x))P_1(x,z) = 0\,, \tag{5.2}$$

$$\frac{d}{dz}P_2(x,z) + (\lambda - \lambda C(z) + \mu_2(x))P_2(x,z) = 0\,, \tag{5.3}$$

$$V(z) = \delta(1-\alpha)\int_0^\infty P^E(x,z)h(x)\,dx + \delta\sum_{j=1}^{2}\int_0^\infty P_j(x,z)\mu_j(x)\,dx\,. \tag{5.4}$$

(5.5)

Yet again we use a similar operation on (4.9) and (4.10), use (4.8) and simplify. Thus we obtain

$$zP^E(0,z) = (1-\delta)(1-\alpha)\int_0^\infty P^E(x,z)h(x)\,dx + (1-\delta)\sum_{j=1}^{2}\int_0^\infty P_j(x,z)\mu_j(x)\,dx \tag{5.6}$$

$$+V(z)K(z) + \lambda(C(z)-1)Q\,. \tag{5.7}$$

Finally, with the similar operations of (4.11) and (4.12), we obtain,

$$P_1(0,z) = \alpha\theta_1\int_0^\infty P^E(x,z)h(x)\,dx\,, \tag{5.8}$$

$$P_2(0,z) = \alpha\theta_2\int_0^\infty P^E(x,z)h(x)\,dx\,. \tag{5.9}$$

Next, we integrate equations (5.1), (5.2) and (5.3) between the limits 0 and x and obtain

$$P^E(x,z) = P^E(0,z)\exp\left[-(\lambda-\lambda C(z))x - \int_0^\infty h(t)\,dt\right]\,, \tag{5.10}$$

$$P_1(x,z) = P_1(0,z)\exp\left[-(\lambda-\lambda C(z))x - \int_0^\infty \mu_1(t)\,dt\right]\,, \tag{5.11}$$

$$P_2(x,z) = P_2(0,z)\exp\left[-(\lambda-\lambda C(z))x - \int_0^\infty \mu_2(t)\,dt\right]\,. \tag{5.12}$$

where $P^E(0,z)$, $P_1(0,z)$ and $P_2(0,z)$ have been obtained above in (5.5), (5.6) and (5.7) respectively.

We again integrate (5.8), (5.9) and (5.10) with respect to x and obtain

$$P^E(z) = P^E(0,z)\frac{1-\overline{H}(\lambda-\lambda C(z))}{\lambda-\lambda C(z)}, \tag{5.13}$$

$$P_1(z) = P_1(0,z)\frac{1-\overline{B_1}(\lambda-\lambda C(z))}{\lambda-\lambda C(z)}, \tag{5.14}$$

$$P_2(z) = P_1(0,z)\frac{1-\overline{B_2}(\lambda-\lambda C(z))}{\lambda-\lambda C(z)}, \tag{5.15}$$

where $\overline{H}(\lambda-\lambda C(z)) = \int\limits_0^\infty \exp\left[-(\lambda-\lambda C(z))x\right]\, dH(x)$ is the Laplace-Stieltjes transform of S_E, the service time of FES and $\overline{B_j}(\lambda-\lambda C(z)) = \int\limits_0^\infty \exp\left[-(\lambda-\lambda C(z))x\right]\, dB_j(x)$ is the Laplace-Stieltjes transform of S_j, the service time of AOS (j), $j=1,2$.

Now, we multiply equations (5.11), (5.12) and (5.13) by $h(x)$, $\mu_1(x)$ and $\mu_2(x)$ respectively and integrate them with respect to x and use (2.2) and (2.4). Thus we obtain

$$\int_0^\infty P^E(x,z)h(x)\,dx = P^E(0,z)\overline{H}(\lambda-\lambda C(z)), \tag{5.16}$$

$$\int_0^\infty P_1(x,z)\mu_1(x)\,dx = P_1(0,z)\overline{B_1}(\lambda-\lambda C(z)), \tag{5.17}$$

$$\int_0^\infty P_2(x,z)\mu_2(x)\,dx = P_2(0,z)\overline{B_2}(\lambda-\lambda C(z)). \tag{5.18}$$

We further use (5.14), (5.15) and (5.16) into equations (5.4), (5.5), (5.6) and (5.7), simplify and get

$$V(z) = \delta(1-\alpha)P^E(0,z)\overline{H}(\lambda-\lambda C(z)) + P_1(0,z)\overline{B_1}(\lambda-\lambda C(z)) + P_2(0,z)\overline{B_2}(\lambda-\lambda C(z)), \tag{5.19}$$

$$zP^E(0,z) = (1-\delta)(1-\alpha)P^E(0,z)\overline{H}(\lambda-\lambda C(z)) + (1-\delta)P_1(0,z)\overline{B_1}(\lambda-\lambda C(z)) \tag{5.20}$$

$$+(1-\delta)P_2(0,z)\overline{B_2}(\lambda-\lambda C(z)) + V(z)K(z) + \lambda(C(z)-1)Q, \tag{5.21}$$

$$P_1(0,z) = \alpha\theta_1 P^E(0,z)\overline{H}(\lambda-\lambda C(z)), \tag{5.22}$$

$$P_2(0,z) = \alpha\theta_2 P^E(0,z)\overline{H}(\lambda-\lambda C(z)), \tag{5.23}$$

Now, solving (5.17), (5.18), (5.19) and (5.20), utilizing (5.11), (5.12) and (5.13) and simplifying, we obtain

$$V(z) = \frac{\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{5.24}$$

$$P^E(z) = \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{5.25}$$

$$P_1(z) = \frac{\alpha\theta_1\overline{H}(\lambda-\lambda C(z))\left(\overline{B_1}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{5.26}$$

$$P_2(z) = \frac{\alpha\theta_2\overline{H}(\lambda-\lambda C(z))\left(\overline{B_2}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{5.27}$$

where

$$\begin{aligned}\Psi(z) &= (1-\alpha)(1-\delta)+(1-\delta)\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+(1-\delta)\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\\ &\quad+\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]K(z).\end{aligned}$$

Now, in order to determine the only unknown Q, we proceed as follows:

$$\begin{aligned}V(1) &= \lim_{z\to 1} V(z)\\ &= \lim_{z\to 1}\frac{\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}\\ &= \frac{\delta\lambda E(I)Q}{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]},\end{aligned} \tag{5.28}$$

where $E(I)$ is the average batch size, $E(S_E)$, $E(S_1)$ and $E(S_2)$ are the average service time of FES, AOS (1) and AOS(2), respectively.

$$\begin{aligned} P^E(1) &= \lim_{z\to 1} P^E(z) = \lim_{z\to 1} \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))} \\ &= \frac{\lambda E(I)E(S_E)Q}{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]}, \end{aligned} \tag{5.29}$$

$$\begin{aligned} P_1(1) &= \lim_{z\to 1} P_1(z) = \lim_{z\to 1} \frac{\alpha\theta_1\overline{H}(\lambda-\lambda C(z))\left(\overline{B_1}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))} \\ &= \frac{\alpha\theta_1\lambda E(I)E(S_1)Q}{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]}, \end{aligned} \tag{5.30}$$

$$\begin{aligned} P_2(1) &= \lim_{z\to 1} P_2(z) = \lim_{z\to 1} \frac{\alpha\theta_2\overline{H}(\lambda-\lambda C(z))\left(\overline{B_2}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))} \\ &= \frac{\alpha\theta_2\lambda E(I)E(S_2)Q}{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]}, \end{aligned} \tag{5.31}$$

Next, we use the results found in (4.37), (4.38), (4.39) and (4.40) in the normalizing condition:

$$Q+V(1)+P^E(1)+P_1(1)+P_2(1)=1\,. \tag{5.32}$$

On simplifying, (4.41) yields

$$Q=\frac{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]}{1+\delta\lambda E(I)-\delta\lambda d}\,. \tag{5.33}$$

The result (4.42) gives the probability that the server is idle and the stability condition which emerges from this equation is given by

$$\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]<1\,. \tag{5.34}$$

Now, we define ρ, the utilization factor of the system as the proportion of time the server is providing any kind of service and using results (4.38), (4.39) and (4.40) and simplifying, we get

$$\rho=P^E(1)+P_1(1)+P_2(1)=\frac{\lambda E(I)\left[E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right]}{1+\delta\lambda E(I)-\delta\lambda d}\,. \tag{5.35}$$

6 Steady State Average Queue Length and Average Waiting Time

Let $P_Q(z)$ be the steady state probability generating function for the number of customers in the queue so that adding (4.37), (4.38), (4.39) and (4.40) we get

$$P_Q(z)=V(z)+P^E(z)+P_1(z)+P_2(z)=\frac{N(z)}{D(z)}\,. \tag{6.1}$$

Next, we define L_q to be the steady state average number of customers in the queue. Then $L_Q=\frac{d}{dz}P_Q(z)\,|_{z=1}$. However, since $P_Q(z)=0/0$ at $z=1$, we use double differentiation and obtain

$$L_Q=\lim_{z\to 1}\frac{d}{dz}P_Q(z)=\lim_{z\to 1}\frac{D'(z)N''(z)-N'(z)D''(z)}{2\left(D'(z)\right)^2}=\frac{D'(1)N''(1)-N'(1)D''(1)}{2\left(D'(1)\right)^2}, \tag{6.2}$$

where primes mean derivatives with respect to z and after a lot of algebra and simplification, we obtain

$$\begin{aligned} L_Q =& \frac{(\lambda E(I))^2\left[E(S_E^2)+\alpha\theta_1 E(S_1^2)+\alpha\theta_2 E(S_2^2)+2(E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2))\right]}{2\left\{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]\right\}} \\ &+\frac{2\delta\lambda^2 dE(I)\left[E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right]+\delta\lambda^2 d}{2\left\{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]\right\}} \\ &+\frac{\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]E(I(I-1))}{2\left\{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\right\}+\delta\lambda d\right]\right\}}, \end{aligned} \tag{6.3}$$

where $E(S_E^2)$, $E(S_1^2)$ and $E(S_2^2)$ are the second moments of the FES, AOS (1) and AOS (2) service times respectively and $E(I(I-1))$ is the second factorial moment of the batch size.

Note that using L_q obtained in (6.3) into Littles formulae, we can obtain the following:
The steady state average number of customers in the system is

$$L = L_q + \rho\,. \tag{6.4}$$

where ρ is given by (5.32). Further, the steady state average waiting time in the queue is

$$W_q = \frac{L_q}{\lambda}\,. \tag{6.5}$$

The steady state average waiting time in the system is

$$W = \frac{L}{\lambda}\,. \tag{6.6}$$

7 Particular Cases

Case 1: We assume single Poisson arrivals with FES, AOS (1) and AOS (2) all having exponential distribution

In this case we have $E(I) = 1$, $E(S_E) = 1/h$, $E(S_E^2) = 2/h^2$, $E(S_1) = 1/\mu_1$, $E(S_1^2) = 2/\mu_1^2$, $E(S_2) = 1/\mu_2$, $E(S_2^2) = 2/\mu_2^2$, and $E(I(I-1)) = 0$. Furthermore,

$$\overline{H}\,[\,\lambda - \lambda C(z)\,] = \frac{h}{h+\lambda-\lambda C(z)}\,,\ \overline{B_1}\,[\,\lambda - \lambda C(z)\,] = \frac{\mu_1}{\mu_1+\lambda-\lambda C(z)} \text{ and } \overline{B_2}\,[\,\lambda - \lambda C(z)\,] = \frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}\,.$$

Consequently,

$$\frac{1-\overline{H}\,[\,\lambda-\lambda C(z)\,]}{\lambda-\lambda C(z)} = \frac{1}{h+\lambda-\lambda C(z)}\,,\quad \frac{1-\overline{B_1}\,[\,\lambda-\lambda C(z)\,]}{\lambda-\lambda C(z)} = \frac{1}{\mu_1+\lambda-\lambda C(z)}\,,$$

and

$$\frac{1-\overline{B_2}\,[\,\lambda-\lambda C(z)\,]}{\lambda-\lambda C(z)} = \frac{1}{\mu_2+\lambda-\lambda C(z)}\,.$$

Substituting these values in the main results, we obtain:

$$V(z) = \frac{\delta\left[\,(1-\alpha)+\alpha\theta_1\left(\frac{\mu_1}{\mu_1+\lambda-\lambda C(z)}\right)+\alpha\theta_2\left(\frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}\right)\right]\lambda\,[\,C(z)-1\,]\left(\frac{h}{h+\lambda-\lambda C(z)}\right)Q}{z-\Psi(z)\left(\frac{h}{h+\lambda-\lambda C(z)}\right)}\,, \tag{7.1}$$

$$P^E(z) = \frac{\left(\frac{h}{h+\lambda-\lambda C(z)}-1\right)Q}{z-\Psi(z)\left(\frac{h}{h+\lambda-\lambda C(z)}\right)}\,, \tag{7.2}$$

$$P_1(z) = \frac{\alpha\theta_1\left(\frac{h}{h+\lambda-\lambda C(z)}\right)\left(\frac{\mu_1}{\mu_1+\lambda-\lambda C(z)}-1\right)Q}{z-\Psi(z)\left(\frac{h}{h+\lambda-\lambda C(z)}\right)}\,, \tag{7.3}$$

$$P_2(z) = \frac{\alpha\theta_2\left(\frac{h}{h+\lambda-\lambda C(z)}\right)\left(\frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}-1\right)Q}{z-\Psi(z)\left(\frac{h}{h+\lambda-\lambda C(z)}\right)}\,, \tag{7.4}$$

where, in this case,

$$\begin{aligned}\Psi(z) &= (1-\alpha)(1-\delta)+(1-\delta)\alpha\theta_1\left(\frac{\mu_1}{\mu_1+\lambda-\lambda C(z)}\right)+(1-\delta)\alpha\theta_2\left(\frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}\right)\\ &\quad+\delta\left[\,(1-\alpha)+\alpha\theta_1\left(\frac{\mu_1}{\mu_1+\lambda-\lambda C(z)}\right)+\alpha\theta_2\left(\frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}\right)\right]K(z)\,.\end{aligned}$$

and

$$Q = \frac{1-\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}+\delta d\right]}{1+\delta\lambda-\delta\lambda d}. \tag{7.5}$$

Further,

$$V(1) = \frac{\delta\lambda}{1+\delta-\delta\lambda d}, \tag{7.6}$$

$$P^E(1) = \frac{\lambda}{h(1+\delta-\delta\lambda d)}, \tag{7.7}$$

$$P_1(1) = \frac{\alpha\theta_1\lambda}{\mu_1(1+\delta-\delta\lambda d)}, \tag{7.8}$$

$$P_2(1) = \frac{\alpha\theta_2\lambda}{\mu_2(1+\delta-\delta\lambda d)}, \tag{7.9}$$

$$\begin{aligned} L_Q = {} & \frac{\lambda^2\left[\frac{2}{h^2}+\frac{2\alpha\theta_1}{\mu_1^2}+\frac{2\alpha\theta_2}{\mu_2^2}+2\left(\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}\right)\right]}{2-2\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}+\delta d\right]} + \frac{2\delta\lambda^2 d\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}\right]+\delta\lambda^2 d}{2-2\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}+\delta d\right]} \\ & + \frac{\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}\right]}{2-2\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}+\delta d\right]}. \end{aligned} \tag{7.10}$$

Next, the steady state average number of customers in the system,

$$L = L_q + \rho, \tag{7.11}$$

where

$$\rho = \frac{\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}\right]}{1+\delta\lambda-\delta\lambda d}. \tag{7.12}$$

Further, the steady state average waiting time in the queue is

$$W_q = \frac{L_q}{\lambda}. \tag{7.13}$$

The steady state average waiting time in the system is

$$W = \frac{L}{\lambda}. \tag{7.14}$$

Case 2: The first essential service is compulsorily followed by one of AOS (1) or AOS (2)

The results corresponding to this particular case can be obtained by putting $\alpha = 1$ in the main results.

$$V(z) = \frac{\delta\left[\theta_1\overline{B_1}(\lambda-\lambda C(z))+\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.15}$$

$$P^E(z) = \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.16}$$

$$P_1(z) = \frac{\theta_1\overline{H}(\lambda-\lambda C(z))\left(\overline{B_1}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.17}$$

$$P_2(z) = \frac{\theta_2\overline{H}(\lambda-\lambda C(z))\left(\overline{B_2}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.18}$$

where

$$\Psi(z) = (1-\delta)\theta_1\overline{B_1}(\lambda-\lambda C(z)) + (1-\delta)\theta_2\overline{B_2}(\lambda-\lambda C(z)) + \delta\left[\theta_1\overline{B_1}(\lambda-\lambda C(z)) + \theta_2\overline{B_2}(\lambda-\lambda C(z))\right]K(z),$$

$$Q = \frac{1-[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]}{1+\delta\lambda E(I)-\delta\lambda d}, \tag{7.19}$$

$$\rho = \frac{\lambda E(I)[E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)]}{1+\delta\lambda E(I)-\delta\lambda d}, \tag{7.20}$$

$$L_Q = \frac{(\lambda E(I))^2\left[E(S_E^2)+\theta_1 E(S_1^2)+\theta_2 E(S_2^2)+2(E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2))\right]}{2\{1-[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]\}} + \frac{2\delta\lambda^2 dE(I)[E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)]+\delta\lambda^2 d}{2\{1-[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]\}} + \frac{[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]E(I(I-1))}{2\{1-[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]\}}, \tag{7.21}$$

Case 3: No AOS(1): $M^X/(G_E,G_2)/D/1$ **Queue**

In this case we put $\theta_1 = 0$ in the main results (5.20) to (5.23), (5.30), (5.31) and (6.2) to obtain

$$V(z) = \frac{\delta\left[(1-\alpha)+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]\lambda[C(z)-1]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.22}$$

$$P^E(z) = \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.23}$$

$$P_1(z) = 0, \tag{7.24}$$

$$P_2(z) = \frac{\alpha\theta_2\overline{H}(\lambda-\lambda C(z))\left(\overline{B_2}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.25}$$

where

$$\Psi(z) = (1-\alpha)(1-\delta)+(1-\delta)\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))+\delta\left[(1-\alpha)+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]K(z),$$

$$Q = \frac{1-[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]}{1+\delta\lambda E(I)-\delta\lambda d},\quad \lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d<1, \tag{7.26}$$

$$L_Q = \frac{(\lambda E(I))^2\left[E(S_E^2)+\alpha\theta_2 E(S_2^2)+2(E(S_E)+\alpha\theta_2 E(S_2))\right]}{2\{1-[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]\}} + \frac{2\delta\lambda^2 dE(I)[E(S_E)+\alpha\theta_2 E(S_2)]+\delta\lambda^2 d}{2\{1-[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]\}} + \frac{[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]E(I(I-1))}{2\{1-[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]\}}. \tag{7.27}$$

Case 4: No AOS (2): $M^X/(G_E, G_1)/D/1$ **Queue**

In this case we put $\theta_2 = 0$ in the main results (5.20) to (5.23), (5.30), (5.31) and (6.2) to obtain

$$V(z) = \frac{\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))\right]\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.28}$$

$$P^E(z) = \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.29}$$

$$P_1(z) = \frac{\alpha\theta_1\overline{H}(\lambda-\lambda C(z))\left(\overline{B_1}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.30}$$

$$P_2(z) = 0, \tag{7.31}$$

where

$$\Psi(z) = (1-\alpha)(1-\delta)+(1-\delta)\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))\right]K(z),$$

$$Q = \frac{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]}{1+\delta\lambda E(I)-\delta\lambda d},\ \ \lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d<1, \tag{7.32}$$

$$\begin{aligned} L_Q = {} & \frac{(\lambda E(I))^2\left[E(S_E^2)+\alpha\theta_1E(S_1^2)+2(E(S_E)+\alpha\theta_1E(S_1))\right]}{2\left\{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]\right\}} \\ & +\frac{2\delta\lambda^2dE(I)\left[E(S_E)+\alpha\theta_2E(S_1)\right]+\delta\lambda^2d}{2\left\{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]\right\}} \\ & +\frac{\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]E(I(I-1))}{2\left\{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]\right\}}. \end{aligned} \tag{7.33}$$

Case 5: None of the AOS: $M^X/G_E/D/1$ **Queue**

In this case we put $\alpha = \theta_1 = \theta_2 = 0$ in the main results (5.20) to (5.23), (5.30), (5.31) and (6.2) to obtain

$$V(z) = \frac{\delta\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\left\{1-\delta+\delta K(z)\right\}\overline{H}(\lambda-\lambda C(z))}, \tag{7.34}$$

$$P^E(z) = \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\left\{1-\delta+\delta K(z)\right\}\overline{H}(\lambda-\lambda C(z))}, \tag{7.35}$$

$$P_1(z) = 0, \tag{7.36}$$

$$P_2(z) = 0, \tag{7.37}$$

where

$$Q = \frac{1-\lambda E(I)E(S_E)-\delta\lambda d}{1+\delta\lambda E(I)-\delta\lambda d},\ \ \lambda E(I)E(S_E)+\delta\lambda d<1, \tag{7.38}$$

$$\begin{aligned} L_Q = {} & \frac{(\lambda E(I))^2\left[E(S_E^2)+2E(S_E)\right]}{2\left\{1-\left[\lambda E(I)E(S_E)+\delta\lambda d\right]\right\}}+\frac{2\delta\lambda^2dE(I)E(S_E)+\delta\lambda^2d}{2\left\{1-\left[\lambda E(I)E(S_E)+\delta\lambda d\right]\right\}} \\ & +\frac{\left[\lambda E(I)E(S_E)+\delta\lambda d\right]E(I(I-1))}{2\left\{1-\left[\lambda E(I)E(S_E)+\delta\lambda d\right]\right\}}. \end{aligned} \tag{7.39}$$

Case 6: No Server Vacations: $M^X/G_E/1$ Queue

In this case we put $\delta = 0$ and $d = 0$. Consequently $K(z) = 1$ in the results of case 4 and obtain

$$V(z) = 0\,, \tag{7.40}$$

$$P^E(z) = \frac{\left(1 - \overline{H}(\lambda - \lambda C(z))\right) Q}{z - \overline{H}(\lambda - \lambda C(z))}\,, \tag{7.41}$$

$$P_1(z) = 0\,, \tag{7.42}$$

$$P_2(z) = 0\,, \tag{7.43}$$

$$Q = 1 - \lambda E(I)E(S_E)\,, \quad \lambda E(I)E(S_E) < 1\,, \tag{7.44}$$

$$L_Q = \frac{(\lambda E(I))^2 \left[E(S_E^2) + 2E(S_E) \right] + \lambda E(I)E(S_E)E(I(I-1))}{2[1 - \lambda E(I)E(S_E)]}\,. \tag{7.45}$$

The results of this case are known results of the ordinary $M^X/G/1$ queue.

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Received: June 26, 2015; *Accepted*: August 23, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 37-41

Malaya Journal of Matematik
MJM
an international journal of mathematical sciences with computer applications...
www.malayajournal.org
ISSN : 2319-3786

Certain coefficient inequalities for p-valent functions

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Abstract

In the present paper, applying lemmas due to Nunokawa et al. [3] and Jack's lemma we obtain some coefficient inequalities for certain subclass of p-valent functions.

Keywords: Analytic, univalent, p-valent, starlike and convex functions, Jack's lemma.

2010 MSC: 30C45.

1 Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad p \in \mathbb{N} := \{1, 2, 3, \ldots\}, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Put $\mathcal{A}_1 = \mathcal{A}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in Δ is denoted by $\mathcal{S}$. A function $f \in \mathcal{S}$ is called starlike (with respect to 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0, 1]$. A function $f \in \mathcal{S}$ that maps Δ onto a convex domain, denoted by $f \in \mathcal{K}$, is called a convex function. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order $0 \le \gamma < 1$ if it satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, \qquad z \in \Delta.$$

We denote by $\mathcal{S}^*(\gamma)$ the subclass of $\mathcal{A}$ consisting of all starlike functions of order γ in Δ. Furthermore, let $\mathcal{M}(\beta)$ be the class of functions $f(z) \in \mathcal{A}$ which satisfy

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \qquad z \in \Delta.$$

for some real number β with $\beta > 1$. The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi, Ganigi and Sarangi [6].

Further, let $\mathcal{P}(\gamma, p)$ denote the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ which satisfy

$$\mathfrak{Re}\left\{\frac{f(z)}{z^p}\right\} > \gamma, \qquad z \in \Delta,$$

for some real $0 \le \gamma < p$. The class $\mathcal{P}(1/2, 1) \equiv \mathcal{P}(1/2)$ was studied by Obradović et al. in [5]. We remark that $\mathcal{K} \subset \mathcal{P}(1/2)$.

Nunokawa, Cho, Kwon and Sokół [3] obtained the following results.

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Lemma 1.1. *Let $B(z)$ and $C(z)$ be analytic in Δ with*

$$|\Im\mathfrak{m}\{C(z)\}| < \Re\mathfrak{e}\{B(z)\}.$$

If $p(z)$ is analytic in Δ with $p(0) = 1$, and if

$$|\arg\{B(z)zp'(z) + C(z)p(z)\}| < \pi/2 + t(z),$$

where

$$t(z) = \begin{cases} \arg\{C(z) + iB(z)\} & \text{when } \arg\{C(z) + iB(z)\} \in [0, \pi/2] \\ \arg\{C(z) + iB(z)\} - \pi/2 & \text{when } \arg\{C(z) + iB(z)\} \in (\pi/2, \pi], \end{cases}$$

then we have

$$\Re\mathfrak{e}\{p(z)\} > 0, \qquad z \in \Delta.$$

Lemma 1.2. *Let $B(z)$ and $C(z)$ be analytic in Δ with*

$$\Re\mathfrak{e}\left\{\frac{C(z)}{B(z)}\right\} \geq -1, \qquad z \in \Delta.$$

If $p(z)$ is analytic in Δ with $p(0) = 0$, and if

$$|B(z)zp'(z) + C(z)p(z)| < |B(z) + C(z)|, \qquad z \in \Delta, \tag{1.2}$$

then we have

$$|p(z)| < 1, \qquad z \in \Delta.$$

In this paper, applying the Lemma 1.1, Lemma 1.2 and Jack's Lemma, we obtain coefficient conditions for some certain subclasses of p-valent functions.

2 Main results

Our first result is contained in the following:

Theorem 2.1. *Assume that $f \in \mathcal{A}_p$. If*

$$\left|\arg\left\{z^{1-p}f'(z) - (p-1)\frac{f(z)}{z^p} - \frac{\gamma}{p}\right\}\right| < \frac{3\pi}{4}, \qquad z \in \Delta, \tag{2.3}$$

then

$$\Re\mathfrak{e}\left\{\frac{f(z)}{z^p}\right\} > \frac{\gamma}{p}, \qquad z \in \Delta, \tag{2.4}$$

where $0 \leq \gamma < p$, that is $f \in \mathcal{P}(\gamma/p, p)$.

Proof. Let $f(z) \neq 0$ for $z \neq 0$ and let $p(z)$ be defined by

$$\left(1 - \frac{\gamma}{p}\right)p(z) + \frac{\gamma}{p} = \frac{f(z)}{z^p}, \qquad z \in \Delta, \tag{2.5}$$

where $0 \leq \gamma < p$. Then $p(z)$ is analytic in Δ, $p(0) = 1$ and

$$\left(1 - \frac{\gamma}{p}\right)p(z) + \left(1 - \frac{\gamma}{p}\right)zp'(z) = z^{1-p}f'(z) - (p-1)\frac{f(z)}{z^p} - \frac{\gamma}{p}.$$

If we put $B(z) = C(z) = 1 - \frac{\gamma}{p}$, from (2.3) and applying Lemma 1.1 we obtain (2.4) immediately. □

If we take $p = 1$ in Theorem 2.1, then it becomes the result from [4] of the following form:

Corollary 2.1. *Let $f \in \mathcal{A}$. If*

$$|\arg\{f'(z) - \gamma\}| < \frac{3\pi}{4}, \qquad z \in \Delta,$$

then

$$\Re\mathfrak{e}\left\{\frac{f(z)}{z}\right\} > \gamma, \qquad z \in \Delta.$$

Theorem 2.2. *Assume that $f \in \mathcal{A}_p$. If*

$$\left| z^{1-p} f'(z) - (p-1)\frac{f(z)}{z^p} - \frac{\gamma}{p} \right| < 2\left(1 - \frac{\gamma}{p}\right), \qquad z \in \Delta, \tag{2.6}$$

then $f \in \mathcal{P}(\gamma/p, p)$, where $0 \le \gamma < p$.

Proof. For $0 \le \gamma < p$, let $p(z)$ be defined by (2.5). Then from (2.6) and applying Lemma 1.2, we can obtain the result. □

Putting $p = 1$, in Theorem 2.2, we have:

Corollary 2.2. *Let $f \in \mathcal{A}$. If*

$$|f'(z) - \gamma| < 2(1-\gamma), \qquad z \in \Delta,$$

then

$$\mathfrak{Re}\left\{ \frac{f(z)}{z} \right\} > \gamma, \qquad z \in \Delta.$$

Putting $\gamma = 1/2$, in Corollary 2.2, we have:

Corollary 2.3. *Let $f \in \mathcal{A}$. If*

$$\left| f'(z) - \frac{1}{2} \right| < 1, \qquad z \in \Delta,$$

then $f \in \mathcal{P}(1/2)$.

The following Lemma (popularly known *Jack's lemma* (see [1])) will be required on our present investigation.

Lemma 2.3. *Let the (nonconstant) function $w(z)$ be analytic in Δ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a the point $z_0 \in \Delta$, then*

$$z_0 w'(z_0) = c w(z_0),$$

where c is a real number and $c \ge 1$.

Theorem 2.3. *Assume that $f(z)/z^p \ne \gamma$ and that $f \in \mathcal{A}_p$ satisfies the inequality*

$$\mathfrak{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > p + \frac{\gamma - 1}{2(\gamma+1)}, \qquad z \in \Delta, \tag{2.7}$$

then $f \in \mathcal{P}(\frac{1+\gamma}{2}, p)$, where $0 \le \gamma < p$.

Proof. Define the function $w(z)$ by

$$\frac{f(z)}{z^p} = \frac{1 + \gamma w(z)}{1 + w(z)}, \qquad (w(z) \ne -1, |z| < 1), \tag{2.8}$$

where $0 \le \gamma < p$. Because $f(z)/z^p \ne \gamma$, then $w(z)$ is analytic in Δ and $w(0) = 0$. From (2.7), some computation yields

$$\frac{zf'(z)}{f(z)} = p + \frac{\gamma z w'(z)}{1 + \gamma w(z)} - \frac{z w'(z)}{1 + w(z)}. \tag{2.9}$$

Suppose there exists a point $z_0 \in \Delta$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|.$$

Applying Lemma 2.3, then we have

$$z_0 w'(z_0) = c w(z_0) \quad (c \ge 1,\, w(z_0) = e^{i\theta}, \theta \in \mathbb{R}). \tag{2.10}$$

Thus, by using (2.9) and (2.10), it follows that

$$\begin{aligned}\mathfrak{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} &= p+\mathfrak{Re}\left\{\frac{c\gamma e^{i\theta}}{1+\gamma e^{i\theta}}\right\}-\mathfrak{Re}\left\{\frac{\gamma e^{i\theta}}{1+e^{i\theta}}\right\} \\ &= p+\frac{c\gamma(\gamma+\cos\theta)}{1+\gamma^2+2\gamma\cos\theta}-\frac{c}{2} \\ &\leq \frac{(2p+1)\gamma+(2p-1)}{2(1+\gamma)},\end{aligned}$$

which contradicts the hypothesis (2.7). It follows that $|w(z)|<1$, that is,

$$\left|\frac{(f(z)/z^p)-1}{\gamma-(f(z)/z^p)}\right|<1, \qquad (z\in\Delta,\ 0\leq\gamma<p).$$

This evidently completes the proof of Theorem 2.3. □

If we take $\gamma=0$ and $p=1$, in Theorem 2.3, we get:

Corollary 2.4. *Let $f\in\mathcal{A}$. If*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\frac{1}{2}, \qquad z\in\Delta,$$

then $f\in\mathcal{P}(1/2)$, that is $\mathcal{S}^*(1/2)\subset\mathcal{P}(1/2)$.

Theorem 2.4. *Assume that $f\in\mathcal{A}_p$ satisfies the inequality*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<p+\frac{1-\gamma}{2-\gamma}, \qquad z\in\Delta, \tag{2.11}$$

then

$$\left|\frac{f(z)}{z^p}-1\right|<|1-\gamma|, \qquad z\in\Delta,$$

where $0\leq\gamma<p$.

Proof. Let us $f(z)/z^p\neq\gamma$. Consider the function $w(z)$ defined by

$$\frac{f(z)}{z^p}=1+(1-\gamma)w(z), \qquad |z|<1, \tag{2.12}$$

where $0\leq\gamma<p$. Then $w(z)$ is analytic in Δ and $w(0)=0$. From (2.12), some computation yields

$$\frac{zf'(z)}{f(z)}=p+\frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)}. \tag{2.13}$$

Suppose there exists a point $z_0\in\Delta$ such that

$$|w(z_0)|=1 \quad \text{and} \quad |w(z)|<1 \quad \text{when} \quad |z|<|z_0|.$$

Applying Lemma 2.3, then we have

$$z_0 w'(z_0)=cw(z_0) \quad (c\geq 1,\ w(z_0)=e^{i\theta},\theta\in\mathbb{R}). \tag{2.14}$$

Thus, by using (2.13) and (2.14), it follows that

$$\begin{aligned}\mathfrak{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} &= p+\mathfrak{Re}\left\{\frac{c(1-\gamma)e^{i\theta}}{1+(1-\gamma)e^{i\theta}}\right\} \\ &= p+\frac{c(1-\gamma)(1-\gamma+\cos\theta)}{1+(1-\gamma)^2+2(1-\gamma)\cos\theta} \\ &\geq \frac{(2p+1)-\gamma(p+1)}{2-\gamma},\end{aligned}$$

which contradicts the hypothesis (2.11). It follows that $|w(z)|<1$, that is,

$$\left|\frac{f(z)}{z^p}-1\right|<|1-\gamma|, \qquad (z\in\Delta, 0\leq\gamma<p).$$

This evidently completes the proof of Theorem 2.4. □

Corollary 2.5. *Assume that $f \in \mathcal{A}$. If f satisfies the inequalities*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{1-\gamma}{2-\gamma}, \qquad z \in \Delta,$$

then

$$\mathfrak{Re}\left\{\frac{f(z)}{z}\right\} > \gamma, \qquad z \in \Delta,\ 0 \leq \gamma < 1.$$

Taking $\gamma = 1/2$ in Corollary 2.5, we have:

Corollary 2.6. *Assume that $f \in \mathcal{A}$. If f satisfies the inequalities*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{4}{3}, \qquad z \in \Delta,$$

then

$$f \in \mathcal{P}(1/2), \qquad z \in \Delta,$$

that is, $\mathcal{M}(4/3) \subset \mathcal{P}(1/2)$.

Combining Corollary 2.4 and 2.6, we have:

Corollary 2.7. *Assume that $f \in \mathcal{A}$. If f satisfies the following two-sided inequality*

$$\frac{1}{2} < \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{4}{3}, \qquad z \in \Delta,$$

then

$$f \in \mathcal{P}(1/2), \qquad z \in \Delta,$$

that is, $\mathcal{S}(1/2, 3/4) \subset \mathcal{P}(1/2)$, where the class $\mathcal{S}(\alpha, \beta)$, $\alpha < 1$ and $\beta > 1$, was recently considered by K. Kuroki and S. Owa in [2].

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Received: June 28, 2015; *Accepted*: August 23, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 42–52

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

n-power Quasi-isometry and n-power Normal Composition Operators on L^2-spaces

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Abstract

In this paper, we give the characterizations of n-power Quasi-isometry and n-power normal composition operators. Further, we also discuss the characterization of the n-power Quasi-isometry composite multiplication operator.

Keywords: n-power quasi-isometry operator, n-power normal operator, composite multiplication operator.

2010 MSC: 47B20, 47B33, 47B38.

1 Introduction

Let $B(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H. An operator A is an n- power quasi-isometry if $A^{n-1}A^{*2}A^2 = A^*AA^{n-1}$ for all $n \in Z^+$[5]. The operator A is normal if $A^*A = AA^*$ and n-power normal if $A^nA^* = A^*A^n$ for all $n \geq 2$[3]. We denote the class of n-power normal operators and n-power quasi-isometry operators by $[nN]$ and $[nQI]$ respectively. The class of normal operators $\subset$ class$[nN]$. Also A is an n- power normal operator if and only if A^n is normal[3].

Let $(X, \sum, \lambda)$ be a sigma-finite measure space and let T be a measurable transformation from X to itself. If T is a measurable transformation then T^n is also a measurable transformation. Further, if T is non-singular then λT^{-1} is absolutely continuous with respect to λ and it follows that $\lambda(T^{-1})^n$ is absolutely continuous with respaect to λ. The Radon-Nikodym derivative of $\lambda(T^{-1})^n$ with respect to λ is denoted by h_n.

Associated with each transformation T is a conditional expectation operator
$E(f|T^{-1}(\sum)) = E(f)$ is defined for each non-negative function $f \in L^p$ $(1 \leq p < \infty)$ and is uniquely determined by the following conditions:
(i)$E(f)$ is $T^{-1}(\sum)$ measurable.
(ii) If B is any $T^{-1}(\sum)$ measurable set for which $\int_B f d\lambda$ converges then we have $\int_B f d\lambda = \int_B E(f)d\lambda$.

The conditional expectation operator E has the following properties:
(i) $E(f.(g \circ T)) = E(f).(g \circ T)$
(ii) E is monotonically increasing. (i.e) if $f \leq g$ a.e then $E(f) \leq E(g)$ a.e.
(iii) $E(1) = 1$.

When E is defined on a possible infinite σ-finite measure space it behaves similarly to expectations on standard probability spaces. As an operator on $L^2(\lambda)$, E is the projection operator onto the closure of the range of C_T.

Let π be an essentially bounded function. The multiplication operator M_π on $L^2(\lambda)$ induced by π, is given by, $M_\pi f = \pi.f$ for $f \in L^2(\lambda)$.

A composition operator C_T on $L^2(\lambda)$ is a bounded linear operator given by composition with a map $T : X \to X$ as, $C_T f = f \circ T$ for all $f \in L^2(\lambda)$ and C_T^* is given by $C_T^* f = hE(f) \circ T^{-1}$ for all $f \in L^2(\lambda)$. A weighted

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composition operator W is a linear transformation acting on a set of complex valued $\sum$- measurable functions f of the form, $Wf = wC_Tf$ where w is a complex valued $\sum$- measurable function. When $w = 1$ we say W is a composition operator.

The adjoint W^* is defined as, $W^*f = hE(wf) \circ T^{-1}$ for $f \in L^2(\lambda)$. Also, $w_n = w.(w \circ T).(w \circ T^2).....(w \circ T^{n-1})$. For $f \in L^2(\lambda)$, $W^n f = w_n f \circ T^n$, $W^{*^n} f = h_n E(w_n.f) \circ T^{-n}$.

A composite multiplication operator is a linear transformation acting on a set of complex valued $\sum$ measurable functions f of the form $M_{u,T}(f) = C_T M_u(f) = (uf) \circ T = u \circ T.f \circ T$ where u is a complex valued, $\sum$ measurable function. In case, $u = 1$ almost everywhere, $M_{u,T}$ becomes a composition operator. The adjoint of $M_{u,T}$ is given by $M^*_{u,T} f = uh.E(f) \circ T^{-1}$.

Various properties of composition operators and weighted composition operators on L^2 spaces have been analyzed by many authors. In particular, spectra of composition operators and their generalized Alugthe transformations as weighted composition operators are characterized in[4]. In this paper we study the characterisations of the $[nQI]$ and $[nN]$ class of composition operators. The characterisations of class$[nN]$ operators A are evaluated mainly by the aid of the normality of A^n. In[7], the characterisations of n-power normal and n-power quasinormal composite multiplication operators are studied. We study the characterisations of quasi-isometry and n-power quasi-isometry composite multiplication operators.

2 Characterization of the class$[nQI]$ composition operators

The following Lemmas of [2] and [8] play an important role in the following Theorems:

Lemma 2.1. *[2, 8] Let P be the projection of $L^2(\lambda)$ onto $\overline{R(C_T)}$, where $\overline{R(C_T)}$ denotes the closure of the range of C_T. Then,*

(i) $C^*_T C_T f = hf$ *and* $C_T C^*_T f = h \circ TPf \ \ \forall f \in L^2(\lambda)$.

(ii) $\overline{R(C_T)} = \{f \in L^2(\lambda) : f \text{ is } T^{-1}(\sum) \text{ measurable}\}$.

(iii) If f is $T^{-1}(\sum)$ measurable, and g and fg belong to $L^2(\lambda)$, then $P(fg) = fP(g)$ (f need not be in $L^2(\lambda)$). Also, for $k \in \mathbb{N}$

(iv) $(C^*_T C_T)^k f = h^k f$.

(v) $(C_T C^*_T)^k f = (h \circ T)^k Pf$.

(vi) E is the identity operator on $L^2(\lambda)$ if and only if $T^{-1}(\sum) = \sum$.

The following Theorem gives the characterization of n- power quasi-isometry operators.

Theorem 2.1. *Let $C_T \in B(L^2(\lambda))$. Then C_T is in the class$[nQI]$ if and only if $h \circ T^{n-1}.E(h) \circ T^{n-2} = h$.*

Proof.

$$\begin{aligned}
C_T \in [nQI] \quad &\Leftrightarrow \quad C^{n-1}_T C^{*^2}_T C^2_T f = C^*_T C^n_T f, \text{ where } C^*_T f = h.E(f) \circ T^{-1}. \\
&\Leftrightarrow \quad C^{n-1}_T C^{*^2}_T (f \circ T^2) = C^*_T (f \circ T^n) \\
&\Leftrightarrow \quad C^{n-1}_T C^*_T [h.E(f \circ T^2) \circ T^{-1}] = h.E(f \circ T^n) \circ T^{-1} \\
&\Leftrightarrow \quad C^{n-1}_T C^*_T [h.f \circ T] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C^{n-1}_T [h.E(h.f \circ T) \circ T^{-1}] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C^{n-1}_T [h.E(h) \circ T^{-1}.f] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C^{n-2}_T [h.E(h) \circ T^{-1}.f] \circ T = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C^{n-2}_T [h \circ T.E(h).f \circ T] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C^{n-3}_T [h \circ T^2.E(h) \circ T.f \circ T^2] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C^{n-4}_T [h \circ T^3.E(h) \circ T^2.f \circ T^3] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad h \circ T^{n-1}.E(h) \circ T^{n-2}.f \circ T^{n-1} = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad h \circ T^{n-1}.E(h) \circ T^{n-2} = h
\end{aligned}$$

□

Theorem 2.2. *Let $C_T \in B(L^2(\lambda))$. Then C_T^* is in the class$[nQI]$ if and only if $h.E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)} = h \circ T.E[h].E[h] \circ T^{-1}...E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)}$.*

Proof.

$$C_T^* \in [nQI] \Leftrightarrow C_T^{*^{n-1}} C_T^2 C_T^{*^2} f = C_T C_T^{*^n} f. \tag{2.1}$$

Now,

$$\begin{aligned}
C_T^{*^{n-1}} C_T^2 C_T^{*^2} f &= C_T^{*^{n-1}} C_T^2 h.E(h) \circ T^{-1}.E(f) \circ T^{-2} \\
&= C_T^{*^{n-1}} h \circ T^2.E(h) \circ T.E(f) \\
&= C_T^{*^{n-2}} h.E[h \circ T^2.E(h) \circ T.E(f)] \circ T^{-1} \\
&= C_T^{*^{n-2}} h.h \circ T.E[h].E(f) \circ T^{-1} \\
&= C_T^{*^{n-3}} h.E[h] \circ T^{-1}.h.E[h] \circ T^{-1}.E(f) \circ T^{-2} \\
&= C_T^{*^{n-4}} h.E[h] \circ T^{-1}.E[h] \circ T^{-2}.E[h] \circ T^{-1}.E[h] \circ T^{-2}.E(f) \circ T^{-3} \\
&= C_T^{*^{n-5}} h.E[h] \circ T^{-1}.E[h] \circ T^{-2}.E[h] \circ T^{-3}.E[h] \circ T^{-2}.E[h] \circ T^{-3}.E(f) \circ T^{-4} \\
&= h.E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}. \\
&\quad E(f) \circ T^{-(n-1)}.
\end{aligned}$$

And

$$\begin{aligned}
C_T C_T^{*^n} f &= C_T C_T^{*^{(n-1)}} h.E(f) \circ T^{-1} \\
&= C_T C_T^{*^{(n-2)}} h.E[h.E(f) \circ T^{-1}] \circ T^{-1} \\
&= C_T C_T^{*^{(n-2)}} h.E[h] \circ T^{-1}.E(f) \circ T^{-2} \\
&= C_T C_T^{*^{(n-3)}} h.E[h] \circ T^{-1}.E[h] \circ T^{-2}.E(f) \circ T^{-3} \\
&= C_T h.E[h] \circ T^{-1}.E[h] \circ T^{-2}...E[h] \circ T^{-(n-1)}.E(f) \circ T^{-n} \\
&= h \circ T.E[h].E[h] \circ T^{-1}...E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)}.
\end{aligned}$$

Now (2.1) becomes $C_T^* \in [nQI] \Leftrightarrow h.E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)} = h \circ T.E[h].E[h] \circ T^{-1}...E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)}$. □

Example 2.1. *Let $X = \mathbb{N}$, the set of all natural numbers and λ be a counting measure on it. $T : \mathbb{N} \to \mathbb{N}$ is defined by $T(k) = k+1, k \in \mathbb{N}$. Since $T^{n-1}(k) = k_1$where $k_1 \in \mathbb{N}$, $h_2 \circ T^{n-1}(k) = 1$ and $h(k) = 1$, C_T is of class$[nQI]$. Here C_T is the unilateral shift operator on l^2.*

3 Weighted composition operators of class$[nQI]$

Theorem 3.3. *Let W be a weighted composition operator then $W \in [nQI]$ if and only if $w_{n-1}.h \circ T^{n-1}.E(w) \circ T^{n-2}.E(h) \circ T^{n-2}.E(w.w_2) \circ T^{n-3}.f \circ T^{n-1} = h.E[w.w_n] \circ T^{-1}.f \circ T^{n-1}$.*

Proof.

$$W \in [nQI] \Leftrightarrow W^{n-1} W^{*^2} W^2 f = W^* W^n f. \tag{3.2}$$

Consider

$$\begin{aligned}
W^{n-1}W^{*^2}W^2f &= W^{n-1}W^{*^2}W[wf\circ T]\\
&= W^{n-1}W^{*^2}w[w.f\circ T]\circ T\\
&= W^{n-1}W^{*^2}w_2.f\circ T^2\\
&= W^{n-1}W^*h.E[w.w_2.f\circ T^2]\circ T^{-1}\\
&= W^{n-1}W^*h.E[w.w_2]\circ T^{-1}.f\circ T\\
&= W^{n-1}h.E(w)\circ T^{-1}.E(h)\circ T^{-1}.E(w.w_2)\circ T^{-2}.f\\
&= W^{n-2}w.h\circ T.E(w).E(h).E(w.w_2)\circ T^{-1}.f\circ T\\
&= W^{n-3}w.w\circ T.h\circ T^2.E(w)\circ T.E(h)\circ T.E(w.w_2).f\circ T^2\\
&= W^{n-4}w.w\circ T.w\circ T^2.h\circ T^3.E(w)\circ T^2.E(h)\circ T^2.\\
&\quad E(w.w_2)\circ T.f\circ T^3\\
&= w.w\circ T.w\circ T^2...w\circ T^{n-2}.h\circ T^{n-1}.E(w)\circ T^{n-2}.E(h)\circ T^{n-2}.\\
&\quad E(w.w_2)\circ T^{n-3}.f\circ T^{n-1}\\
&= w_{n-1}.h\circ T^{n-1}.E(w)\circ T^{n-2}.E(h)\circ T^{n-2}.\\
&\quad E(w.w_2)\circ T^{n-3}.f\circ T^{n-1}
\end{aligned}$$

And

$$\begin{aligned}
W^*W^nf &= W^*W^{n-1}w.f\circ T\\
&= W^*W^{n-2}w.[w.f\circ T]\circ T\\
&= W^*W^{n-2}w.w\circ T.f\circ T^2\\
&= W^*w.w\circ T...w\circ T^{n-1}.f\circ T^n\\
&= h.E[w.w.w\circ T...w\circ T^{n-1}.f\circ T^n]\circ T^{-1}\\
&= h.E[w.w_n]\circ T^{-1}.f\circ T^{n-1}.
\end{aligned}$$

Now (3.2) becomes
$W\in[nQI]\Leftrightarrow w_{n-1}.h\circ T^{n-1}.E(w)\circ T^{n-2}.E(h)\circ T^{n-2}.E(w.w_2)\circ T^{n-3}.f\circ T^{n-1}=h.E[w.w_n]\circ T^{-1}.f\circ T^{n-1}.$ □

Theorem 3.4. *Let W be a weighted composition operator then $W^*\in[nQI]$ if and only if* $h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E[w]\circ T^{-3}.E[h]\circ T^{-3}....E[w]\circ T^{n-2}.E[h]\circ T^{n-2}.E[w.w_2]\circ T^{-(n-1)}.h\circ T^{-(n-3)}.E[w]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}=w.h\circ T.E[w].E[h].E[w]\circ T^{-1}.E[h]\circ T^{-1}....E[w]\circ T^{-(n-2)}.$
$E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}.$

Proof.

$$W^*\in[nQI]\Leftrightarrow W^{*^{n-1}}W^2W^{*^2}f=WW^{*^n}f. \tag{3.3}$$

$$\begin{aligned}
W^{*^{n-1}}W^2W^{*^2}f &= W^{*^{n-1}}W^2h.E(w)\circ T^{-1}.E(h)\circ T^{-1}.E(wf)\circ T^{-2}\\
&= W^{*^{n-1}}W.w.h\circ T.E[w].E[h].E(wf)\circ T^{-1}\\
&= W^{*^{n-1}}w_2.h\circ T^2.E[w]\circ T.E[h]\circ T.E(wf)\\
&= W^{*^{n-2}}h.E[w.w_2.h\circ T^2.E[w]\circ T.E[h]\circ T.E(wf)]\circ T^{-1}\\
&= W^{*^{n-2}}h.E[w.w_2]\circ T^{-1}.h\circ T.E[w].E[h].E(wf)\circ T^{-1}\\
&= W^{*^{n-3}}h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w.w_2]\circ T^{-2}.h.\\
&\quad E[w]\circ T^{-1}.E[h]\circ T^{-1}.E(wf)\circ T^{-2}\\
&= W^{*^{n-4}}h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E[w.w_2]\circ T^{-3}.\\
&\quad h\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E(wf)\circ T^{-3}\\
&= h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E[w]\circ T^{-3}.E[h]\circ T^{-3}....\\
&\quad E[w]\circ T^{n-2}.E[h]\circ T^{n-2}.E[w.w_2]\circ T^{-(n-1)}.h\circ T^{-(n-3)}.E[w]\circ T^{-(n-2)}.\\
&\quad E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}.
\end{aligned}$$

And,

$$\begin{aligned}
WW^{*^n}f &= WW^{*^{n-1}}h.E(wf)\circ T^{-1}\\
&= WW^{*^{n-2}}h.E[w.h.E(wf)\circ T^{-1}]\circ T^{-1}\\
&= WW^{*^{n-2}}h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E(wf)\circ T^{-2}\\
&= WW^{*^{n-3}}h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E(wf)\circ T^{-3}\\
&= Wh.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}\\
&\quad ....E[w]\circ T^{-(n-1)}.E[h]\circ T^{-(n-1)}.E(wf)\circ T^{-n}\\
&= w.h\circ T.E[w].E[h].E[w]\circ T^{-1}.E[h]\circ T^{-1}\\
&\quad ....E[w]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}.
\end{aligned}$$

Now (3.3) becomes $W^* \in [nQI] \Leftrightarrow h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E[w]\circ T^{-3}.E[h]\circ T^{-3}....E[w]\circ T^{n-2}.E[h]\circ T^{n-2}.E[w.w_2]\circ T^{-(n-1)}.h\circ T^{-(n-3)}.E[w]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)} = w.h\circ T.E[w].E[h].E[w]\circ T^{-1}.E[h]\circ T^{-1}....E[w]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}$. □

4 Charecterisations of class $[nN]$ composition operators

In this section we discuss the characterization of n-power normal composition operators on L^2- spaces.

Lemma 4.2. *[1] Let α and β be non-negative functions with $S = support\ \ \alpha$. Then the following are equivalent:*

(i) For every $f \in L^2(X, \sum, \lambda)$, $\int_X \alpha |f|^2 d\lambda \geq \int_X |E(\beta f|F)|^2 d\lambda$, where F is a sub-sigma algebra of $\sum$.

(ii) Support $\beta \subset S$ and $E(\frac{\beta^2}{\alpha}\chi_S|F) \leq 1a.e.$

Theorem 4.5. *$C_T \in B(L^2(\lambda))$ is of class$[nN]$ if and only if $h_n > 0$ and $E(\frac{1}{h_n}) = \frac{1}{h_n\circ T^n}$.*

Proof.

$$\begin{aligned}
\left\langle C_T^{*^n}C_T^nf, f\right\rangle &= \langle h_nf, f\rangle = \int_X h_n |f|^2 d\lambda\\
\left\langle C_T^nC_T^{*^n}f, f\right\rangle &= \langle h_n\circ T^nEf, f\rangle = \int_X h_n\circ T^nE(f)\overline{f}d\lambda = \int_X \left|E(h_n^{\frac{1}{2}}\circ T^nf)\right|^2 d\lambda.
\end{aligned}$$

Let $S = support\ \ h_n$. By Lemma 4.2, C_T is of class$[nN]$ if and only if $support\ \ h_n^{\frac{1}{2}}\circ T^n \subset\ \ support\ \ h_n$ and $E\left(\frac{\chi_S h_n\circ T^n}{h_n}\right) \leq 1$.

As $h_n\circ T^n > 0$, the condition involving supports is true if and only if $h_n > 0$ (so that $\chi_S = 1$). The inequality is then equivalent to $E(\frac{1}{h_n}) = \frac{1}{h_n\circ T^n}$ since $h_n\circ T$ is $T^{-1}(\sum)$ measurable. □

Theorem 4.6. *(i)* C_T *is of class*$[nN]$ $\Leftrightarrow \left\|h^{\frac{1}{2}}h_{n-1}^{\frac{1}{2}}f\right\|^2 = \left\|h_{n-1}^{\frac{1}{2}}h^{\frac{1}{2}}\circ TPf\right\|^2 \Leftrightarrow \left\|h_n^{\frac{1}{2}}f\right\|^2 = \left\|(h_n\circ T^n)^{\frac{1}{2}}Pf\right\|^2$.

(ii) $\left\|h^{\frac{1}{2}}h_{n-1}^{\frac{1}{2}}Pf\right\|^2 = \left\|h_{n-1}^{\frac{1}{2}}h^{\frac{1}{2}}\circ TPf\right\|^2$.

Proof.

$$\begin{aligned}
0 &= \left\langle C_T^*C_T^n f, f\circ T^{n-1}\right\rangle - \left\langle C_T^nC_T^* f, f\circ T^{n-1}\right\rangle\\
&= \left\langle hC_T^{n-1} f, f\circ T^{n-1}\right\rangle - \left\langle C_T^{n-1}h\circ TPf, f\circ T^{n-1}\right\rangle\\
&= \left\langle hf\circ T^{n-1}, f\circ T^{n-1}\right\rangle - \left\langle h\circ TPf\circ T^{n-1}, f\circ T^{n-1}\right\rangle\\
&= \left\langle hf\circ T^{n-1}, f\circ T^{n-1}\right\rangle - \left\langle Ph\circ Tf\circ T^{n-1}, f\circ T^{n-1}\right\rangle\\
&= \int h\,|f|^2\circ T^{n-1}d\lambda - \int Ph\circ T\,|f|^2\circ T^{n-1}d\lambda\\
&= \int h\,|f|^2\,d\lambda\circ T^{n-1} - \int Ph\circ T\,|f|^2\,d\lambda\circ T^{n-1}\\
&= \langle hh_{n-1}f, f\rangle - \langle Ph\circ Th_{n-1}f, f\rangle\\
&= \langle hh_{n-1}f, f\rangle - \langle h\circ Th_{n-1}Pf, f\rangle
\end{aligned}$$

Since C_T and C_T^n commutes $\forall n\in\mathbb{N}, h, h_n$ also commutes $\forall n\in\mathbb{N}$.

$$\left\|h^{\frac{1}{2}}h_{n-1}^{\frac{1}{2}}f\right\|^2 = \left\|h_{n-1}^{\frac{1}{2}}h^{\frac{1}{2}}\circ TPf\right\|^2$$

Also,

$$\left\langle (C_T^{*^n}C_T^n - C_T^nC_T^{*^n})f, f\right\rangle = \langle (h_n - h_n\circ T^nP)f, f\rangle$$

And hence it follows that $\left\|h_n^{\frac{1}{2}}f\right\|^2 = \left\|(h_n\circ T^n)^{\frac{1}{2}}Pf\right\|^2$.
(ii) follows directly from (i)

□

Theorem 4.7. *Let* C_T *be an n-power normal composition operator on* $L^2(\lambda)$ *then for all* $m>n, f\in L^2$ *and* $i=m-n$ *we have*

$$\left\langle C_T^mC_T^{*^m}f, f\right\rangle = \left\langle Ph_i\circ T^ih_nf, f\right\rangle. \tag{4.4}$$

Proof. For $m=n+1$, we have

$$\begin{aligned}
\left\langle C_T^{n+1}C_T^{*^{n+1}}f, f\right\rangle &= \left\langle C_TC_T^*C_T^nC_T^{*^n}f, f\right\rangle\\
&= \langle h\circ TPh_nf, f\rangle\\
&= \langle Ph\circ Th_nf, f\rangle
\end{aligned}$$

Suppose (4.4) holds for $m=n+1, n+2, ...n+k, i=1,2,...k$ and all $f\in L^2$. Then

$$\begin{aligned}
\left\langle C_T^{n+k+1}C_T^{*^{n+k+1}}f, f\right\rangle &= \left\langle C_T^{k+1}C_T^{*^{k+1}}C_T^nC_T^{*^n}f, f\right\rangle\\
&= \left\langle h_{k+1}\circ T^{k+1}Ph_nf, f\right\rangle\\
&= \left\langle Ph_{k+1}\circ T^{k+1}h_nf, f\right\rangle
\end{aligned}$$

And hence (4.4) follows by induction. □

Theorem 4.8. *Let C_T be an n-power normal composition operator on $L^2(\lambda)$ then for all $m > n, f \in L^2$ and $i = m - n$ we have*

$$\left\langle C_T^{*^m} C_T^m f, f \right\rangle = \langle h_i h_n f, f \rangle . \tag{4.5}$$

Proof. For $m = n + 1$, we have

$$\begin{aligned} \left\langle C_T^{*^{n+1}} C_T^{n+1} f, f \right\rangle &= \left\langle C_T^* C_T C_T^n C_T^{*^n} f, f \right\rangle \\ &= \langle h h_n f, f \rangle \\ &= \langle h h_n f, f \rangle \end{aligned}$$

Suppose (4.5) holds for $m = n+1, n+2, ...n+k, i = 1, 2, ...k$ and all $f \in L^2$. Then

$$\begin{aligned} \left\langle C_T^{*^{n+k+1}} C_T^{n+k+1} f, f \right\rangle &= \left\langle C_T^{*^{k+1}} C_T^{k+1} C_T^n C_T^{*^n} f, f \right\rangle \\ &= \langle h_{k+1} h_n f, f \rangle \\ &= \langle h_{k+1} h_n f, f \rangle \end{aligned}$$

And hence (4.5) follows by induction. □

Theorem 4.9. *Let C_{T_1} and C_{T_2} be n-power normal composition operators on $L^2(\lambda)$ then for all $m > n, p > n$, such that m and p are multiples of n,$C_{T_1}^m C_{T_2}^p$ is normal.*

Proof. On applying Theorem 4.8 in the subsequent equations the assertion is proved. Denote $C_{T_1}^{*^k} C_{T_1}^k$ by $M_{h(1)k}$ and $C_{T_2}^{*^k} C_{T_2}^k$ by $M_{h(2)k}$ respectively.

$$\begin{aligned} \left\langle (C_{T_1}^m C_{T_2}^p)^* (C_{T_1}^m C_{T_2}^p) f, f \right\rangle &= \left\langle C_{T_2}^{*^p} C_{T_1}^{*^m} C_{T_1}^m C_{T_2}^p f, f \right\rangle \\ &= \left\langle h_{(1)m-n} h_{(1)n} C_{T_2}^p f, C_{T_2}^p f \right\rangle \\ &= \left\langle h_{(1)m-n} h_{(1)n} C_{T_2}^{*^p} C_{T_2}^p f, f \right\rangle \\ &= \left\langle h_{(1)m-n} h_{(1)n} h_{(2)p-n} h_{(2)n} f, f \right\rangle \\ \left\langle (C_{T_1}^m C_{T_2}^p)(C_{T_1}^m C_{T_2}^p)^* f, f \right\rangle &= \left\langle C_{T_2}^p C_{T_2}^{*^p} C_{T_1}^{*^m} f, C_{T_1}^{*^m} f \right\rangle \\ &= \left\langle h_{(2)p-n} h_{(2)n} C_{T_1}^{*^m} C_{T_1}^m f, f \right\rangle \\ &= \left\langle h_{(1)m-n} h_{(1)n} h_{(2)p-n} h_{(2)n} f, f \right\rangle \end{aligned}$$

From the above equalities it follows that $C_{T_1}^m C_{T_2}^p$ is normal. □

Corollary 4.1. *Let C_{T_1} and C_{T_2} be n-power normal composition operators on $L^2(\lambda)$ then for all $m > n, p > n$, such that m and p are multiples of n,$(C_{T_1}^m C_{T_2}^p)^q, \ \forall q \in \mathbb{N}$ is normal.*
In particular, for $q = n$, $C_{T_1}^m C_{T_2}^p$ is of class$[nN]$.

Proof. From Theorem 4.9 it is obvious that $(C_{T_1}^m C_{T_2}^p)^q, \ \forall q \in \mathbb{N}$ is normal. And for $q = n$, it follows from the normality of class$[nN]$ operators that $C_{T_1}^m C_{T_2}^p$ is of class$[nN]$ □

Now we establish a close relationship between $\sigma(C_T)$ and $E(h)$ where $E(h)$ denotes the essential range of the Radon-Nikodym derivative h.

Theorem 4.10. *Let C_T be an n-power normal composition operator on $L^2(\lambda)$ then*
$\sigma(C_T) \subset \left\{ \alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \text{ and } |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}} \right\}$

Proof. C_T is in class$[nN]$ implies C_T^n is normal and hence by Spectral mapping Theorem, for normal operators, $\sigma(C_T^{*^n} C_T^n) = \left\{ |\alpha|^2 : \alpha \in \sigma(C_T^n) \right\}$.
But $C_T^{*^n} C_T^n = M_{h_n}$.

Therefore $\sigma(M_{h_n}) = \left\{|\alpha|^2 : \alpha \in \sigma(C_T^n)\right\}$. Because $\sigma(M_{h_n}) = E(h_n)$. We have,
$E(h_n) = \left\{|\alpha|^2 : \alpha \in \sigma(C_T^n)\right\} = \left\{|\alpha|^2 : \alpha \in \sigma(C_T)^n\right\}$.
Thus $\sigma(C_T)^n \subset \left\{\alpha : \alpha \in \mathbb{C} \text{ and } |\alpha|^2 \in E(h_n)\right\}$, which implies
$\sigma(C_T) \subset \left\{\alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \text{ and } |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}}\right\}$ □

Theorem 4.11. *Let C_T be an n-power normal composition operator on $L^2(\lambda)$ such that 1 does not belong to $E(h_n)^{\frac{1}{n}}$. Then $\sigma(C_T) = \left\{\alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \text{ and } |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}}\right\}$ and hence $\sigma(C_T)$ has cyclic symmetry.*

Proof. From the proof of Theorem 4.10, we have, $E(h_n)^{\frac{1}{n}} = \left\{|\alpha|^{\frac{2}{n}} : \alpha^{\frac{1}{n}} \in \sigma(C_T)\right\}$. Thus for every $m \in E(h_n)^{\frac{1}{n}}$ there is an $\alpha^{\frac{1}{n}} \in \sigma(C_T)$ such that $|\alpha|^{\frac{2}{n}} = m$. If $\alpha \in \sigma(C_T)$ and $|\alpha| \neq 1$, then every β such that $|\alpha| = |\beta|$ is in $\sigma(C_T)$[6]. Since by assumption, $1 \notin E(h_n)^{\frac{1}{n}}$ there is no $\alpha^{\frac{1}{n}} \in \sigma(C_T)$ such that $|\alpha|^{\frac{2}{n}} = 1$. Hence $\left\{\alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \text{ and } |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}}\right\} \subset \sigma(C_T)$. The opposite inclusion follows from Theorem 4.10 and hence $\sigma(C_T) = \left\{\alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \text{ and } |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}}\right\}$. □

5 Quasi-isometry and n-power Quasi-isometry Composite multiplication operators

In this section we give a characterization of quasi-isometry and n- power quasi-isometry composite multiplication operators.

Theorem 5.12. *Let $M_{u,T}$ on $L^2(\lambda)$ be a composite mulitiplication operator, then for $\lambda \geq 0$, $M_{u,T}$ is a quasi-isometry if and only if $u.h.E[uh] \circ T^{-1}.E[u \circ Tu \circ T^2] \circ T^{-2}.f = u^2.h.f$.*

Proof.

$$
\begin{aligned}
M_{u,T} \text{ is a quasi-isometry} \quad &\Leftrightarrow \quad M_{u,T}^{*2} M_{u,T}^2 f = M_{u,T}^* M_{u,T} f \\
&\Leftrightarrow \quad M_{u,T}^{*2} M_{u,T} u \circ T.f \circ T = M_{u,T}^*.u \circ T.f \circ T \\
&\Leftrightarrow \quad M_{u,T}^{*2}[u \circ T(u \circ T.f \circ T) \circ T] = u.h.E[u \circ T.f \circ T] \circ T^{-1} \\
&\Leftrightarrow \quad M_{u,T}^{*2}[u \circ T.u \circ T^2.f \circ T^2] = u.h.u \circ T \circ T^{-1}.f \circ T \circ T^{-1} \\
&\Leftrightarrow \quad M_{u,T}^* u.h.E[u \circ T.u \circ T^2.f \circ T^2] \circ T^{-1} = u.h.u.f \\
&\Leftrightarrow \quad M_{u,T}^* u.h.E[u \circ T.u \circ T^2] \circ T^{-1}.f \circ T = u^2.h.f \\
&\Leftrightarrow \quad u.h.E[u.h.E[u \circ T.u \circ T^2] \circ T^{-1}.f \circ T] \circ T^{-1} = u^2.h.f \\
&\Leftrightarrow \quad u.h.E[uh] \circ T^{-1}.E[u \circ T.u \circ T^2] \circ T^{-2}.f = u^2.h.f.
\end{aligned}
$$

□

Corollary 5.2. *$C_T \in B(L^2(\lambda))$ is quasi-isometry if and only if $h.E[h] \circ T^{-1}.f = h.f$.*

Proof. The proof is obtained by putting $u = 1$ in Theorem 5.12. □

Theorem 5.13. *Let $M_{u,T}$ on $L^2(\lambda)$ be a composite mulitiplication operator, then for $\lambda \geq 0$, $M_{u,T}^*$ is a quasi-isometry if and only if $u \circ T.u \circ T^2.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[f] = u \circ T.u \circ T.h \circ T.E[f]$*

Proof.

$$
\begin{aligned}
M^*_{u,T} \text{ is a quasi-isometry} &\Leftrightarrow M^2_{u,T}M^{*^2}_{u,T}f = M_{u,T}M^*_{u,T}f \\
&\Leftrightarrow M^2_{u,T}M^*_{u,T}u.h.E[f]\circ T^{-1} = M_{u,T}u.h.E[f]\circ T^{-1} \\
&\Leftrightarrow M^2_{u,T}u.h.E[u.h.E[f]\circ T^{-1}]\circ T^{-1} = u\circ T.[u.h.E[f]\circ T^{-1}]\circ T \\
&\Leftrightarrow M^2_{u,T}u.h.E[u.h]\circ T^{-1}.E[f]\circ T^{-2} = u\circ T.u\circ T.h\circ T.E[f] \\
&\Leftrightarrow M_{u,T}u\circ T.[u.h.E[u.h]\circ T^{-1}.E[f]\circ T^{-2}]\circ T \\
&\quad = u\circ T.u\circ T.h\circ T.E[f] \\
&\Leftrightarrow M_{u,T}u\circ T.u\circ T.h\circ T.E[u.h].E[f]\circ T^{-1} \\
&\quad = u\circ T.u\circ T.h\circ T.E[f] \\
&\Leftrightarrow u\circ T[u\circ Tu\circ T.h\circ T.E[u.h].E[f]\circ T^{-1}]\circ T \\
&\quad = u\circ T.u\circ T.h\circ T.E[f] \\
&\Leftrightarrow u\circ T.u\circ T^2.u\circ T^2.h\circ T^2.E[u.h]\circ T.E[f] \\
&\quad = u\circ T.u\circ T.h\circ T.E[f].
\end{aligned}
$$

□

Theorem 5.14. *Let $M_{u,T}$ on $L^2(\lambda)$ be a composite mulitiplication operator, then for $\lambda \geq 0$, $M_{u,T}$ is an n-power quasi-isometry operator if and only if $u\circ T.u\circ T^2.u\circ T^3....u\circ T^{n-1}.u\circ T^{n-1}.h\circ T^{n-1}.E[u.h]\circ T^{n-2}E[u\circ T.u\circ T^2]\circ T^{n-3}.f\circ T^{n-1} = u.h.E[u\circ T.u\circ T^2.u\circ T^3...u\circ T^n]\circ T^{-1}.f\circ T^{n-1}$.*

Proof.

$$
M_{u,T} \text{ is } n\text{-power quasi-isometry} \Leftrightarrow M^{n-1}_{u,T}M^{*^2}_{u,T}M^2_{u,T}f = M^*_{u,T}M_{u,T}M^{n-1}_{u,T}f \tag{5.6}
$$

Now,

$$
\begin{aligned}
M^{n-1}_{u,T}M^{*^2}_{u,T}M^2_{u,T}f &= M^{n-1}_{u,T}u.h.E[u.h]\circ T^{-1}.E[u\circ T.u\circ T^2]\circ T^{-2}.f \\
&= M^{n-2}_{u,T}u\circ T[u.h.E[u.h]\circ T^{-1}.E[u\circ T.u\circ T^2]\circ T^{-2}.f]\circ T \\
&= M^{n-2}_{u,T}u\circ T.u\circ T.h\circ T.E[u.h].E[u\circ T.u\circ T^2]\circ T^{-1}.f\circ T \\
&= M^{n-3}_{u,T}u\circ T[u\circ T.u\circ T.h\circ T.E[u.h].E[u\circ T.u\circ T^2]\circ T^{-1}.f\circ T]\circ T \\
&= M^{n-3}_{u,T}u\circ T.u\circ T^2.u\circ T^2.h\circ T^2.E[u.h]\circ T.E[u\circ T.u\circ T^2].f\circ T^2.
\end{aligned}
$$

Continuing in a similar manner, we arrive at the following expression,

$$
\begin{aligned}
M^{n-1}_{u,T}M^{*^2}_{u,T}M^2_{u,T}f = \;& u\circ T.u\circ T^2.u\circ T^3....u\circ T^{n-1}.u\circ T^{n-1} \\
& h\circ T^{n-1}.E[u.h]\circ T^{n-2}.E[u\circ T.u\circ T^2]\circ T^{n-3}. \\
& f\circ T^{n-1}.
\end{aligned}
$$

$$
\begin{aligned}
M^*_{u,T}M^n_{u,T}f &= M^*_{u,T}M^{n-1}_{u,T}u\circ T.f\circ T \\
&= M^*_{u,T}M^{n-2}_{u,T}u\circ T[u\circ T.f\circ T]\circ T \\
&= M^*_{u,T}M^{n-2}_{u,T}u\circ T.u\circ T^2.f\circ T^2 \\
&= . \\
&= . \\
&= . \\
&= M^*_{u,T}u\circ T.u\circ T^2.u\circ T^3...u\circ T^n.f\circ T^n \\
&= u.hE[u\circ T.u\circ T^2.u\circ T^3...u\circ T^n.f\circ T^n]\circ T^{-1} \\
&= u.hE[u\circ T.u\circ T^2.u\circ T^3...u\circ T^n]\circ T^{-1}.f\circ T^{n-1}.
\end{aligned}
$$

Hence equation (5.6) becomes,
$M_{u,T}$ is n- power quasi-isometry $\Leftrightarrow u \circ T.u \circ T^2.u \circ T^3....u \circ T^{n-1}.u \circ T^{n-1}$
$h \circ T^{n-1}.E[u.h] \circ T^{n-2}.E[u \circ T.u \circ T^2] \circ T^{n-3}.f \circ T^{n-1} = u.h.E[u \circ T.u \circ T^2.u \circ T^3...u \circ T^n] \circ T^{-1}.f \circ T^{n-1}$. □

Corollary 5.3. *$C_T \in B(L^2(\lambda))$ is n-power quasi-isometry if and only if $h \circ T^{n-1}.E[h] \circ T^{n-2}.f \circ T^{n-1} = h.f \circ T^{n-1}$.*

Proof. The proof is obtained by putting $u = 1$ in Theorem 5.14. □

Theorem 5.15. *Let $M_{u,T}$ on $L^2(\lambda)$ be a composite mulitiplication operator, then for $\lambda \geq 0$, $M^*_{u,T}$ is an n-power quasi-isometry operator if and only if $u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}....E[u.h] \circ T^{-(n-2)}E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-(n-1)}.$ $E[h] \circ T^{-(n-3)}.E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)} = u \circ T.u \circ T.h \circ T.E[u.h].E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}....E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}$.*

Proof.

$$M^*_{u,T} \text{ is } n\text{-power quasi-isometry} \Leftrightarrow M^{*^{n-1}}_{u,T} M^2_{u,T} M^{*^2}_{u,T} f = M_{u,T} M^*_{u,T} M^{*^{n-1}}_{u,T} f \tag{5.7}$$

Now,

$$\begin{aligned}
M^{*^{n-1}}_{u,T} M^2_{u,T} M^{*^2}_{u,T} f &= M^{*^{n-1}}_{u,T}[u \circ T.u \circ T^2.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[f]] \\
&= M^{*^{n-2}}_{u,T} u.h.E[u \circ T.u \circ T^2.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[f]] \circ T^{-1} \\
&= M^{*^{n-2}}_{u,T} u.h.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-1}.h \circ T.E[u.h].E[f] \circ T^{-1} \\
&= M^{*^{n-3}}_{u,T} u.h.E[u.h.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-1}. \\
&\quad h \circ T.E[u.h].E[f] \circ T^{-1}] \circ T^{-1} \\
&= M^{*^{n-3}}_{u,T} u.h.E[u.h] \circ T^{-1}.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-2}. \\
&\quad h.E[u.h] \circ T^{-1}.E[f] \circ T^{-2} \\
&= . \\
&= . \\
&= . \\
&= u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}....E[u.h] \circ T^{-(n-2)} \\
&\quad E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-(n-1)}.E[h] \circ T^{-(n-3)} \\
&\quad E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}.
\end{aligned}$$

And,

$$\begin{aligned}
M_{u,T} M^{*^n}_{u,T} &= M_{u,T} M^{*^{n-1}}_{u,T} u.h.E[f] \circ T^{-1} \\
&= M_{u,T} M^{*^{n-2}}_{u,T} u.h.E[u.h.E[f] \circ T^{-1}] \circ T^{-1} \\
&= M_{u,T} M^{*^{n-2}}_{u,T} u.h.E[u.h] \circ T^{-1}.E[f] \circ T^{-2} \\
&= . \\
&= . \\
&= . \\
&= M_{u,T} u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}.E[u.h] \circ T^{-3}.... \\
&\quad E[u.h] \circ T^{-(n-1)}.E[f] \circ T^{-n} \\
&= u \circ T[u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}.E[u.h] \circ T^{-3}.... \\
&\quad E[u.h] \circ T^{-(n-1)}.E[f] \circ T^{-n}] \circ T \\
&= u \circ T.u \circ T.h \circ T.E[u.h].E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}.... \\
&\quad E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}.
\end{aligned}$$

Hence equation (5.7) becomes,
$M^*_{u,T}$ is n-power quasi-isometry $\Leftrightarrow$ $u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}....E[u.h] \circ T^{-(n-2)}.E[u \circ T.u \circ T^2.u \circ T^2] \circ$

$T^{-(n-1)}.E[h] \circ T^{-(n-3)}.E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)} = u \circ T.u \circ T.h \circ T.E[u.h].E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}$ $....E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}$. □

Corollary 5.4. *$C_T^* \in B(L^2(\lambda))$ is n-power quasi-isometry if and only if $h.E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.$ $E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)} = h \circ T.E[h].E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}$.*

Proof. The proof is obtained by putting $u = 1$ in Theorem 5.15. □

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Received: September 2, 2015; *Accepted*: January 23, 2016

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 53–58

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Minimal and Maximal Soft Open Sets

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Abstract

In this paper, we introduce new types of minimal and maximal sets via soft topological spaces namely minimal and maximal soft open sets and their complements. These sets are depended on the soft open sets. Many interested result are presented to reveal some properties of these new sets.

Keywords: Soft set, soft topology, minimal soft open, maximal soft open.

2010 MSC: 34G20.

1 Introduction

The phenomena of uncertainty can be emerged in many fields such as economy, social and medical sciences, engineering and so on. To deal with such uncertainties many mathematical tools have been introduced such as probability, fuzzy sets, rough sets and etc. However, these tools have their own limitations. In fact the limitations here always associated with the inadequacy of the parametrization tools. Molodtsov [1] initiated another efficient tool, soft set theory, which is more flexible to deal with uncertainty and to treat some limitation obstacles that other tools suffered to handle them. The theory of the soft set has been being investigating intensively and various applications of this theory have been done in many different fields.

Shabir and Naz [2] introduced the concept of the soft topological space. Heavily investigations were followed to this new kind of topological space and many generalizations depending on the generalizations of soft open and closed sets were introduced as well.

On the other hand, the notation of maximal open sets and minimal open sets were introduced by F. Nakaoka and N. Oda in [3] and [4] . Many generalizations of these concepts have been introduced depending on the various generalizations of the concept of open set. In this paper we introduced the concepts of maximal and minimal soft open sets.

2 Preliminaries

Definition 2.1. *[1] Let E be a set of parameters and A be a subset of E, a soft set F_A on the universe set U is denoted by the set of ordered pairs:*

$$F_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}$$

where $f_A : E \longrightarrow P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.
f_A is called an approximate function of the soft set F_A. The value of f_A may be arbitrary, some of them may be empty, or may have nonempty intersection.

Remark 2.1. *The set of all soft sets over U will be denoted by $S(U)$.*

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Example 2.1. *Suppose that there are eight cars in the universe* $U = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ *and let* $E = \{x_1, x_2, x_3, x_4, x_5\}$ *is the set of decision parameters such that* $x_1 = new$,$x_2 = expensive$, $x_3 = high - tech$, $x_4 = model$, $x_5 =$ *interor desgin . Consider the map* $f_A \equiv cars(atributes)$, *so* $f_A(x_3)$ *means "cars(high-tech)". Thus the functional value of* $f_A(x_3)$ *is the set* $\{c \in U : c$ *is a high-tech* $\}$. *Now let* $A = \{x_2, x_3, x_5\}$ *and* $f_A(x_2) = \{c_2, c_6\}$, $f_A(x_3) = \{c_1, c_3, c_4\}$ *and* $f_A(x_5) = \{c_1, c_7, c_8\}$. *Then the soft set* $F_A = \{(x_2, \{c_2, c_6\}), (x_3, \{c_1, c_3, c_4\}), (x_5, \{c_1, c_7, c_8\})\}$.

Definition 2.2. [5] *Let* $F_A \in S(U)$, *if* $f_A(x) = \emptyset$ *for all* $x \in E$, *then* F_A *is called an empty set, and denoted by* F_Φ.

Example 2.2. *Let* $U = \{u_1, u_2, u_3, u_4\}$ *and* $E = \{x_1, x_2, x_3\}$, *then* $F_\Phi = \{(x_1, \emptyset), (x_2, \emptyset), (x_3, \emptyset)\}$

Definition 2.3. [5] *Let* $F_A \in S(U)$, *if* $f_A(x) = U$ *for all* $x \in A$, *then* F_A *is called an A-univers soft set and is denoted by* $F_{\tilde{A}}$.
If $A = E$, *then* $F_{\tilde{E}}$ *is called a universe soft set.*

Example 2.3. *Let* $U = \{u_1, u_2, u_3, \}$, $E = \{x_1, x_2, x_3\}$ *and* $A = \{x_1, x_2\}$, *then* $F_{\tilde{A}} = \{(x_1, U), (x_2, U)\}$ *and* $F_{\tilde{E}} = \{(x_1, U), (x_2, U), (x_3, U)\}$.

Definition 2.4. [5] *Let* $F_A, F_B \in S(U)$. *Then* F_A *is a soft subset of* F_B *if* $f_A(x) \subseteq f_B(x)$ *for all* $x \in E$ *and is denotd by* $F_A \tilde{\subseteq} F_B$.
If $F_A \neq F_B$, *then* F_A *is a proper soft subset of* F_B *and is denoted by* $F_A \tilde{\subset} F_B$.

Example 2.4. *Let* $U = \{u_1, u_2, u_3, u_4, u_5\}$, $E = \{x_1, x_2, x_3, x_4, x_5\}$, $A = \{x_1, x_4\}$, $B = \{x_4\}$, $F_A = \{(x_1, \{u_1, u_5\}), (x_4, \{u_2, u_3, u_4\})\}$ *and* $F_B = \{(x_4, \{u_2, u_3\})\}$. *It is clear that* $F_B \tilde{\subset} F_A$.

Definition 2.5. [5] *Let* $F_A, F_B \in S(U)$. *Then* F_A *and* F_B *are soft equal if* $f_A(x) = f_B(x)$ *for all* $x \in E$ *and is denoted by* $F_A = F_B$.

Definition 2.6. [5] *Let* $F_A, F_B \in S(U)$. *Then the soft union of* F_A *and* F_B *(denoted by* $F_A \tilde{\bigcup} F_B$*) is defined by the following:* $F_A \tilde{\bigcup} F_B = f_A(x) \bigcup f_B(x)$ *for all* $x \in E$.

Example 2.5. *Let* $U = \{u_1, u_2, u_3\}$, $E = \{x_1, x_2, x_3\}$, $A = \{x_1, x_2\}$, $B = \{x_2, x_3\}$, $F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\}$ *and* $F_B = \{(x_2, \{u_1, u_3\}), (x_3, \{u_3\})\}$. *Then* $F_A \tilde{\bigcup} F_B = \{(x_1, \{u_1, u_2\}), (x_2, U), (x_3, \{u_3\})\}$

Definition 2.7. [5] *Let* $F_A, F_B \in S(U)$. *Then the soft intersection of* F_A *and* F_B *(denoted by* $F_A \tilde{\bigcap} F_B$*) is defined by the following:* $F_A \tilde{\bigcap} F_B = f_A(x) \bigcap f_B(x)$ *for all* $x \in E$.

Example 2.6. *Let* $U = \{u_1, u_2, u_3, u_4\}$, $E = \{x_1, x_2, x_3, x_4\}$, $A = \{x_1, x_4\}$, $B = \{x_1\}$, $F_A = \{(x_1, \{u_1\}), (x_4, \{u_2, u_3, u_4\})\}$ *and* $F_B = \{(x_1, \{u_1, u_3\})\}$. *So* $F_A \tilde{\bigcap} F_B = \{(x_1, \{u_1\}\}$.

Definition 2.8. [5] *Let* $F_A \in S(U)$. *Then the soft complement of* F_A *(denoted by* $F_A^{\tilde{c}}$*) is defined by the approximate function:* $F_A^{\tilde{c}} = f_A^c(x)$, *where* $f_A^c(x) = U - f_A(x)$ *for all* $x \in A$.

Example 2.7. *Let* $U = \{u_1, u_2, u_3\}$, $E = \{x_1, x_2, x_3\}$, $A = \{x_1, x_3\}$ *and* $F_A = \{(x_1, \{u_2\}), (x_3, \{u_1, u_3\})\}$, *then* $F_A^{\tilde{c}} = \{(x_1, \{u_1, u_3\}), (x_3, \{u_2\})\}$.

Definition 2.9. [6] *Let* $F_A \in S(U)$. $\alpha = (x, \{u\})$ *is a nonempty soft element of* F_A , *denoted by* $\alpha \tilde{\in} F_A$ *if* $x \in E$ *and* $u \in f_A(x)$.

Remark 2.2. *The pair* $(x, \emptyset)$, *where* $x \in E$, *is called the empty soft element of* F_A.

Example 2.8. *Let* $U = \{u_1, u_2, u_3\}$, $E = \{x_1, x_2, x_3\}$ *and* $A = \{x_2, x_3\}$ *and let* $F_A = \{(x_2, \{u_2, u_3\}), (x_3, \{u_1, u_2\})\}$, *then the following are nonempty elements in* F_A:
$\alpha_1 = (x_2, \{u_2\}) \tilde{\in} F_A$; *since* $u_2 \in f_A(x_2) = \{u_2, u_3\}$
$\alpha_2 = (x_2, \{u_3\}) \tilde{\in} F_A$; *since* $u_3 \in f_A(x_2) = \{u_2, u_3\}$
$\alpha_3 = (x_3, \{u_1\}) \tilde{\in} F_A$; *since* $u_1 \in f_A(x_3) = \{u_1, u_2\}$
$\alpha_4 = (x_3, \{u_2\}) \tilde{\in} F_A$; *since* $u_2 \in f_A(x_3) = \{u_1, u_2\}$

Definition 2.10. [2] *Let* $F_E \in S(U)$. *A soft topology on* F_E, *denoted by* $\tilde{\tau}$, *is a collection of soft subsets of* F_E *satisfing the following properties*

1. $F_{\Phi}, F_E \in \tilde{\tau}$.

2. If $\{F_{E_i} \tilde{\subseteq} F_E : i \in I \subseteq \mathbb{N}\} \subset \tilde{\tau}$, then $\tilde{\bigcup}_{i \in I} F_{E_i} \in \tilde{\tau}$.

3. If $\{F_{E_i} \tilde{\subseteq} F_E : 1 \leq i \leq n, n \in \mathbb{N}.\} \subset \tilde{\tau}$, then $\bigcap_{i=1}^{\tilde{n}} F_{E_i} \in \tilde{\tau}$.

Then $\tilde{\tau}$ is called a soft topology and the pair $(F_E, \tilde{\tau})$ is called a soft topological space.

Example 2.9. *Let $U = \{u_1, u_2, u_3\}$, $E = \{x_1, x_2, x_3\}$, $A = \{x_1, x_2\}$, then $(F_A, \tilde{\tau}) = \{F_{\phi}, F_{A_1}, F_{A_2}, F_{A_3}, F_A\}$ is a soft topological space,*
where $F_{A_1} = \{(x_1, \{u_2\})\}$,$F_{A_2} = \{(x_1, \{u_2\}), (x_2, \{u_2\})\}$,$F_{A_3} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\}$.

Definition 2.11. *Let $(F_A, \tilde{\tau})$ be a soft topological space and $\alpha \tilde{\in} F_A$. If there is a soft open set F_B such that $\alpha \tilde{\in} F_B$, then F_B is called a soft open neighbourhood (or soft neighbourhood) of α.*

Definition 2.12. *[2] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then F_B is said to be a soft closed if $F_B^{\tilde{c}}$ is a soft open.*

Definition 2.13. *[2] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then the soft closure of F_B is the intersection of all soft closed set that contain F_B and it is denoted by $\tilde{\overline{F}}_B$.*

3 Minimal and maximal soft open sets

Definition 3.14. *A proper nonempty soft open subset F_K of a soft topological space $(F_A, \tilde{\tau})$ is said to be minimal soft open set if any soft open set which is contained in F_K is F_{Φ} or F_K.*

Definition 3.15. *A proper nonempty soft open subset F_K of a soft topological space $(F_A, \tilde{\tau})$ is said to be maximal soft open set if any soft open set which contains F_K is F_A or F_K.*

Example 3.10. *Let $U = \{u_1, u_2, u_3, u_4\}$, $E = \{x_1, x_2, x_3, x_4\}$, $A = \{x_1, x_2, x_3\}$, and let $(F_A, \tilde{\tau}) = \{F_{\phi}, F_{A_1}, F_{A_2}, F_{A_3}, F_A\}$ be a soft topological space, where $F_{A_1} = \{(x_1, \{u_1, u_3\}), (x_2, \{u_2, u_4\})\}$, $F_{A_2} = \{(x_2, \{u_2\})\}$, $F_{A_3} = \{(x_1, \{u_1, u_2, u_3\}), (x_2, U), (x_3, \{u_2\})\}$. Then F_{A_2} is a minimal soft open set and F_{A_3} is a maximal soft open set.*

Proposition 3.1. *Let F_K and F_H be soft open subsets of a soft topological space $(F_A, \tilde{\tau})$, if F_K is minimal soft open then $F_K \tilde{\bigcap} F_H = F_{\phi}$ or $F_K \tilde{\subseteq} F_H$.*

Proof. Suppose that $F_K \tilde{\bigcap} F_H \neq F_{\phi}$, so $F_K \tilde{\bigcap} F_H \tilde{\subseteq} F_H$. But F_H is minimal soft open, hence $F_K \tilde{\bigcap} F_H = F_K$. Therefore $F_K \tilde{\subseteq} F_H$. □

Proposition 3.2. *Let F_K and F_H be minimal soft open subsets of a soft topological space $(F_A, \tilde{\tau})$,then $F_K \tilde{\bigcap} F_H = F_{\phi}$ or $F_K = F_H$.*

Proof. Suppose that $F_K \tilde{\bigcap} F_H \neq F_{\phi}$, so $F_K \tilde{\bigcap} F_H \tilde{\subseteq} F_H$. But F_H is minimal soft open, hence $F_K \tilde{\bigcap} F_H = F_H$. Therefore $F_K \tilde{\subseteq} F_H$.
By using the same argument,we get $F_H \tilde{\subseteq} F_K$. Therefore $F_K = F_H$. □

Proposition 3.3. *Let F_H be a minimal soft open set. If $\alpha \tilde{\in} F_H$, then $F_H \tilde{\subset} F_K$ for any soft open neighbourhood F_K of α.*

Proof. Let F_K be a soft open neighbourhood of α and suppose $F_H \tilde{\not\subset} F_K$, then $F_H \tilde{\bigcap} F_K \neq F_{\phi}$ and it is proper soft open subset of F_H. So we get a contradiction of being F_H is minimal. □

Proposition 3.4. *Let F_K be a nonempty finite soft open subset of a soft topological space $(F_A, \tilde{\tau})$. Then there exists at least one finite minimal soft open set F_H such that $F_H \tilde{\subset} F_K$.*

Proof. Let F_K be a nonempty finite soft open set. If F_K is minimal then set $F_K = F_H$. Otherwise, there exists a soft open set F_{K_1} such that $F_\emptyset \neq F_{K_1} \tilde{\subset} F_K$. So if F_{K_1} is minimal then set $F_H = F_{K_1}$. Otherwise, there exists F_{K_2} such that $F_\emptyset \neq F_{K_2} \tilde{\subset} F_{K_1} \tilde{\subset} F_K$. Now, if F_{K_2} is minimal then set $F_H = F_{K_2}$. Otherwise there exists a finite open soft set F_{K_3} such that $F_\emptyset \neq F_{K_3} \tilde{\subset} F_{K_2} \tilde{\subset} F_{K_1} \tilde{\subset} F_K$. Indeed, since F_K is finite, so if we continue this process we will reach to a final soft open set, say F_{K_n} for some $n \in \mathbb{N}$, which is of course minimal such that $F_\emptyset \neq F_{K_n} \tilde{\subset} F_{K_{n-1}} \tilde{\subset} ... \tilde{\subset} F_{K_2} \tilde{\subset} F_{K_1} \tilde{\subset} F_K$. Set $F_H = F_{K_n}$ as required. □

Proposition 3.5. *Let F_K and F_H be soft open subsets of a soft topological space $(F_A, \tilde{\tau})$. If F_K is maximal soft open, then $F_K \tilde{\bigcup} F_H = F_A$ or $F_H \tilde{\subseteq} F_K$.*

Proof. Suppose that $F_K \tilde{\bigcup} F_H \neq F_A$, so $F_K \tilde{\subseteq} F_K \tilde{\bigcup} F_H$. But F_K is maximal soft open, hence $F_K = F_K \tilde{\bigcup} F_H$. Therefore $F_H \tilde{\subseteq} F_K$. □

Proposition 3.6. *Let F_K and F_H be maximal soft open sets of a soft topological space $(F_A, \tilde{\tau})$,then $F_K \tilde{\bigcup} F_H = F_A$ or $F_K = F_H$.*

Proof. Suppose that $F_K \tilde{\bigcup} F_H \neq F_A$, so $F_H \tilde{\subseteq} F_K \tilde{\bigcup} F_H$ and $F_K \tilde{\subseteq} F_K \tilde{\bigcup} F_H$. But F_H is maximal soft open, hence $F_K \tilde{\bigcup} F_H = F_H$. Therefore $F_K \tilde{\subseteq} F_H$.
Using the same argument we get $F_H \tilde{\subseteq} F_K$. Therefore $F_K = F_H$. □

Proposition 3.7. *Let F_M be a proper nonempty cofinite soft open set of a soft topological space $(F_A, \tilde{\tau})$. Then there exists at least one cofinite maximal soft open set F_N such that $F_M \tilde{\subset} F_N$.*

Proof. Let F_M be a proper nonempty cofinite soft open set. If F_M is maximal then set $F_M = F_N$. Otherwise, there exists a proper soft open set F_{N_1} such that $F_M \tilde{\subset} F_{N_1}$. So if F_{N_1} is maximal then set $F_N = F_{N_1}$. Otherwise, there exists a proper soft open set F_{N_2} such that $F_M \tilde{\subset} F_{N_1} \tilde{\subset} F_{N_2}$. Now, if F_{N_2} is maximal then set $F_M = F_{N_2}$. In fact, since F_M is cofinite, so if we continue this process we will reach to a cofinal soft open set, say F_{N_n} for some $n \in \mathbb{N}$, which is of course maximal such that $F_M \tilde{\subset} F_{N_1} \tilde{\subset} F_{N_2} \tilde{\subset} ... \tilde{\subset} F_{N_n} \neq F_A$. Set $F_N = F_{N_n}$ as required. □

Proposition 3.8. *Let F_K be a maximal soft open subset of a soft topological space $(F_A, \tilde{\tau})$ and $\alpha \tilde{\notin} F_K$. Then $F_K^{\tilde{c}} \tilde{\subseteq} F_H$ for any soft open set F_H containing α.*

Proof. Since $\alpha \tilde{\notin} F_K$, then for any F_H containing α, so we have $F_H \tilde{\not\subseteq} F_K$. Hence by using proposition 3.5, we get $F_K \tilde{\bigcup} F_H = F_A$ and this means $F_K^{\tilde{c}} \tilde{\subseteq} F_H$. □

Proposition 3.9. *Let F_K be a maximal soft open subset of a soft topological space $(F_A, \tilde{\tau})$. Then either the following holds:*

1. *For each $\alpha \tilde{\in} F_K^{\tilde{c}}$ and soft open set F_H containing α, we have $F_H = F_A$.*
2. *There exists a soft open set F_H such that $F_K^{\tilde{c}} \tilde{\subseteq} F_H$ and $F_H \tilde{\subset} F_A$.*

Proof. Suppose (1) does not hold, so there exists $\alpha \tilde{\in} F_K^{\tilde{c}}$ and a soft open set F_H containing α such that $F_H \tilde{\subset} F_A$. So by proposition 3.8 we have that $F_K^{\tilde{c}} \tilde{\subseteq} F_H$. □

Proposition 3.10. *Let F_K be a maximal soft open subset of a soft topological space $(F_A, \tilde{\tau})$. Then either the following holds:*

1. *For each $\alpha \tilde{\in} F_K^{\tilde{c}}$ and each soft open neighbourhood set F_H containing α, we have $F_K^{\tilde{c}} \tilde{\subset} F_H$.*
2. *There exists a proper soft open set F_H such that $F_K^{\tilde{c}} = F_H$*

Proof. Suppose that(2) does not hold, so by proposition 3.8, we get $F_K^{\tilde{c}} \tilde{\subset} F_H$ for each $\alpha \tilde{\in} F_K^{\tilde{c}}$ and each soft open neighbourhood F_H of α. □

Proposition 3.11. *Let F_K be a maximal soft open subset of a soft topological space $(F_A, \tilde{\tau})$. Then $\tilde{\overline{F}}_K = F_A$ or $\tilde{\overline{F}}_K = F_K$*

Proof. Let F_K be a maximal soft open set. By using proposition 3.10, so we have only two cases:

1. From the first condition of proposition 3.10, let $\alpha \tilde{\in} F_K^{\tilde{c}}$ and each soft open neighbourhood F_H of α, then $F_K^{\tilde{c}} \tilde{\subset} F_H$. So $F_K \tilde{\bigcap} F_H \neq F_\phi$. i.e. $\alpha \tilde{\in} \tilde{\overline{F}}_K$. Thus $F_K^{\tilde{c}} \tilde{\subset} \tilde{\overline{F}}_K$. But $F_A = F_K \bigcup F_K^{\tilde{c}} \tilde{\subset} F_K \bigcup \tilde{\overline{F}}_K = \tilde{\overline{F}}_K \tilde{\subset} F_A$. Consequently we get $\tilde{\overline{F}}_K = F_A$.

2. From the second condition of proposition 3.10, there exits a soft open set F_H such that $F_K^{\tilde{c}} = F_H \neq F_A$, so $F_K^{\tilde{c}}$ is soft open set and thus F_K is soft closed, i.e. $\tilde{\overline{F}}_K = F_K$.

□

Definition 3.16. *A proper nonempty soft closed subset F_C of a soft topological space $(F_A, \tilde{\tau})$ is said to be minimal soft closed set if any soft closed set which is contained in F_C is F_Φ or F_C.*

Definition 3.17. *A proper nonempty soft closed subset F_C of a soft topological space $(F_A, \tilde{\tau})$ is said to be maximal soft closed set if any soft closed set which contains F_C is F_A or F_C.*

Example 3.11. *Consider example 2.9, then $F_B = \{(x_1, \{u_1, u_3\})\}$ is a minimal soft closed set and $F_C = \{(x_1, \{u_1, u_3\}), (x_2, \{u_1, u_3\})\}$ is a maximal soft closed set.*

Proposition 3.12. *Let F_C and F_D be soft closed sets of a soft topological space $(F_A, \tilde{\tau})$.*

1. *If F_C is minimal, then $F_C \tilde{\bigcap} F_D = F_\Phi$ or $F_C \tilde{\subseteq} F_D$.*

2. *If F_C and F_D are minimal, then $F_C \tilde{\bigcap} F_D = F_\Phi$ or $F_C = F_D$.*

3. *If F_C is maximal, then $F_C \tilde{\bigcup} F_D = F_A$ or $F_D \tilde{\subseteq} F_C$.*

4. *If F_C and F_D are maximal, then $F_C \tilde{\bigcup} F_D = F_A$ or $F_C = F_D$.*

Proof. (1) Suppose that $F_C \tilde{\bigcap} F_D \neq F_\phi$, so $F_C \tilde{\bigcap} F_D \tilde{\subseteq} F_C$. But F_C is minimal soft closed, hence $F_C \tilde{\bigcap} F_D = F_C$. Therefore $F_C \tilde{\subseteq} F_D$.
(2) Suppose that $F_C \tilde{\bigcap} F_D \neq F_\phi$, so $F_C \tilde{\bigcap} F_D \tilde{\subset} F_C$. But F_D is minimal soft closed, hence $F_C \tilde{\bigcap} F_D = F_D$. Therefore $F_D \tilde{\subseteq} F_C$. But from (1) we have $F_C \tilde{\subseteq} F_D$. Therefore $F_C = F_D$.
(3) Suppose that $F_D \tilde{\bigcup} F_C \neq F_A$, so $F_C \tilde{\subseteq} F_C \tilde{\bigcup} F_D$. But F_C is maximal soft closed, hence $F_C = F_C \tilde{\bigcup} F_D$. Therefore $F_D \tilde{\subseteq} F_C$.
(4) Suppose that $F_C \tilde{\bigcup} F_D \neq F_A$, so $F_D \tilde{\subseteq} F_C \tilde{\bigcup} F_D$ and $F_C \tilde{\subseteq} F_C \tilde{\bigcup} F_D$. But F_D is maximal soft closed, hence $F_C \tilde{\bigcup} F_D = F_D$. Therefore $F_C \tilde{\subseteq} F_D$.
Using the same argument, we get $F_D \tilde{\subseteq} F_C$. Therefore $F_C = F_D$. □

Proposition 3.13. *Let $(F_A, \tilde{\tau})$ be a soft topological space. If F_K is a proper maximal soft open subset of F_A, then $F_K^{\tilde{c}}$ is a minimal soft closed set.*

Proof. Suppose $F_K^{\tilde{c}}$ is not minimal soft closed set, so there exists a soft closed set F_C such that $F_\Phi \neq F_C \tilde{\subset} F_K^{\tilde{c}}$. Hence $F_K \tilde{\subset} F_C^{\tilde{c}} \tilde{\subset} F_A$. This means that F_K is not maximal which is contradicting of being F_K is maximal. □

Proposition 3.14. *Let $(F_A, \tilde{\tau})$ be a soft topological space. If F_K be a proper minimal soft open subset of F_A then $F_K^{\tilde{c}}$ is a maximal soft closed set.*

Proof. Suppose $F_K^{\tilde{c}}$ is not maximal soft closed set, so there exists a soft closed set F_C such that $F_K^{\tilde{c}} \tilde{\subset} F_C \tilde{\subset} F_A$. Hence $F_\Phi \neq F_C^{\tilde{c}} \tilde{\subset} F_K$. This means that F_K is not minimal which is contradicting of being F_K is minimal. □

Proposition 3.15. *Let F_C and $\{F_{D_\alpha} : \lambda \in \Lambda\}$be minimal soft closed subsets of a soft topological space $(F_A, \tilde{\tau})$. If $F_C \neq F_{D_\lambda}$ for each λ, then $(\tilde{\bigcup}_{\lambda \in \Lambda} F_{D_\lambda}) \tilde{\bigcap} F_C = F_\Phi$.*

Proof. Suppose that $(\tilde{\bigcup}_{\lambda \in \Lambda} F_{D_\lambda}) \tilde{\bigcap} F_C \neq F_\Phi$, so there exist some $\lambda_o \in \Lambda$ such that $F_{D_{\lambda_o}} \tilde{\bigcap} F_C \neq F_\Phi$. But F_C and $F_{D_{\lambda_o}}$ are minimal so by 3.12 we get $F_C = F_{D_{\lambda_o}}$ which is a contradiction. □

4 Acknowledgement

The author would like to thank the Iraqi ministry of higher education and scientific research for its support.

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Received:July 15, 2015 ; *Accepted*: October 12, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 59-64

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

A Study On Linear and Non linear Schrodinger Equations by Reduced Differential Transform Method

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Abstract

In this paper, reduced differential transform method (RDTM) is used to obtain the exact solution of nonlinear Schrodinger equation. Compared to other existing analytical/numerical methods, RDTM is more efficient and easy to apply.

Keywords: non linear Schrodinger equations, reduced differential transform, reduced differential inverse transform, analytic solution.

2010 MSC: 35J10, 35F25.

1 Introduction

In this paper we consider the one dimensional linear Schrodinger equation of the form

$$iu_t + u_{xx} = 0, \qquad u(x,0) = f(x), \qquad x \in R, t \geq 0 \tag{1.1}$$

and the nonlinear Schrodinger equation of the form

$$iu_t + u_{xx} + v|u|^2 u = 0, \qquad u(x,0) = f(x), \qquad x \in R, t \geq 0, \tag{1.2}$$

where v is a real constant, $u(x,t)$ is a complex function and $i = \sqrt{-1}$. Such equations has been widely used for studying nonlinear waves in fluid-filled viscoelastic tubes, solitary waves in piezoelectric semiconductors, nonlinear optical waves, hydrodynamics and plasma waves.

Recently, an enormous work has been carried out to the search for efficient analytical / numerical methods for finding the solution of nonlinear Schrodinger equations. For example, Sadhigi and Ganji [6] investigated the linear and nonlinear one dimensional Schrodinger equations using adomian decomposition method and homotopy perturbation method. Wang [4] proposed the finite difference method to obtain the numerical solution of the nonlinear Schrodinger equation. Biazar and Gazvini [5] applied homotopy perturbation method to obtain the solution of cubic Schrodinger equation. Khuri [9] used adomian decomposition method to solve the nonlinear Schrodinger equation. Wazwaz [2] presented the exact solution of the linear and nonlinear one dimensional Schrodinger equation obtained by means of variation iteration method. Borhanifar and Reza Abazari [7] proposed differential transform method to obtain the exact solution of Schrodinger equation.

The aim of this work is to propose employ reduced differential transform method (RDTM) on nonlinear Schrodinger equations. The reduced DTM was first envisioned by Keskin [13] and successfully applied to many nonlinear partial differential equations. Also, Keskin and Oturnac [11, 12] applied this method to obtain the analytical solutions of generalized KdV equations. RDTM has been widely used by many researchers [10] and [14–17] successfully for different nonlinear physical systems such as higher dimensional Burger

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equations, Burgers-Huxley equations, Newell-Whitehead-Segel equation and generalized Hirota-Satsuma coupled KdV equation. By comparing with adomian decomposition method, reduced diferential transform can apply directly to solve the problem without using adomian polynomials. The solution procedure is very simple and easy in RDTM and the computational time is also less than that of traditional differential transform method. Another advantage of RDTM is that it does not require auxiliary parameter to control the convergence region, which was used in homotopy analysis method.

This paper is sketched as follows: Section 2 describes to show, how to use the reduced differential transform method (RDTM) will be presented and we show how to use the RDTM to approximate the solution. In section 3, we apply this method to some linear and nonlinear Schrodinger equations. The results of numerical experiments are also presented in this section.

2 Basic Idea of Reduced Differential Transform

With reference to the articles [10–17], the basic definitions of reduced differential transform are introduced as follows:

Consider a function $u(x,t)$ of two variables and assume that it can be represented as a product of two single variable functions, i.e., $u(x,t) = f(x)g(t)$. On the basis of the properties of the one dimensional differential transform, the function $u(x,t)$ can be represented as

$$u(x,t) = \sum_{h=0}^{\infty} F(h)x^h \sum_{k=0}^{\infty} G(k)t^k = \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} U(h,k)x^h t^k$$

where $U(h,k) = F(h)G(k)$ is called the spectrum of $u(x,t)$.

The basic definitions and properties of reduced differential transform method(RDTM) are introduced below.

The reduced differential transform of $u(x,t)$ at $t=0$ is defined as

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} \tag{2.3}$$

Where $u(x,t)$ is the given function and $U_k(x)$ is the transformed function.

The reduced differential inverse transform of $U_k(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k \tag{2.4}$$

and from (1.1) and (1.2), we have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} t^k \tag{2.5}$$

In this work, the lower case $u(x,t)$ represents the original function while upper case $U_k(x)$ represents the transformed function. On the basis of the definions (2.3) and (2.4), we have the following results:

Theorem 2.1. *If* $w(x,t) = u(x,t) + v(x,t)$ *then* $W_k(x) = U_k(x) + V_k(x)$

Theorem 2.2. *If* $w(x,t) = \alpha u(x,t)$ *then* $W_k(x) = \alpha U_k(x)$

Theorem 2.3. *If* $w(x,t) = \alpha\frac{\partial^n u(x,t)}{\partial t^n}$ *then* $W_k(x) = \alpha\frac{(k+n)!}{k!}U_{k+n}(x)$

Theorem 2.4. *If* $w(x,t) = x^m t^n$ *then* $W_k(x) = x^m\delta(k-n) = \begin{cases} x^m & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}$

Theorem 2.5. *If* $w(x,t) = \alpha\frac{\partial^n u(x,t)}{\partial x^n}$ *then* $W_k(x) = \alpha\frac{\partial^n U_k(x)}{\partial x^n}$

Theorem 2.6. *If* $w(x,t) = u(x,t)v(x,t)$ *then* $W_k(x) = \sum_{k_1=0}^{k} U_{k_1}(x)V_{k-k_1}(x)$

Theorem 2.7. *If* $w(x,t) = x^m t^n u(x,t)$ *then* $W_k(x) = x^m U_{k-n}(x)$

Theorem 2.8. *If* $w(x,t) = t^n u(x,t)$ *then* $W_k(x) = U_{k-n}(x)$

3 Numerical Examples

Example 3.1. *We first consider the one dimensional linear Schrodinger equation*

$$iu_t + u_{xx} = 0, \qquad x \in R, t \geq 0 \tag{3.6}$$

Subject to the initial condition

$$u(x,0) = 1 + 2\cosh ax, \quad \textit{where a is constant} \tag{3.7}$$

The transformed version of eqn. (3.6) is

$$i(k+1)U_{k+1}(x) = -\frac{\partial^2}{\partial x^2}U_k(x) \tag{3.8}$$

The transformed version of eqn. (3.7) is

$$U_0(x) = 1 + 2\cosh ax \tag{3.9}$$

Using the recurrence equation (3.8), with transformed initial condition (3.9), for $k = 0,1,2,3,4$, the first few components of $U_k(x)$ are obtained as follows:

$$U_1(x) = -ia^2 coshax$$
$$U_2(x) = -\frac{a^4}{2}\cosh ax$$
$$U_3(x) = i\frac{a^6}{6}\cosh ax$$
$$U_4(x) = \frac{a^8}{24}\cosh ax$$
$$U_5(x) = -i\frac{a^{10}}{120}\cosh ax$$

and so on. Finally substituting all values of $U_k(x)$ in to Eq. (2.4), we obtain the series solution as follows:

$$u(x,t) = 1 + \cosh ax\left(1 - \frac{(ia^2t)}{1!} + \frac{(ia^2t)^2}{2!} - \frac{(ia^2t)^3}{3!} + \frac{(ia^2t)^4}{4!} - \frac{(ia^2t)^5}{5!} + \cdots\right)$$

Consequently the series in the closed form

$$u(x,t) = 1 + \cosh axe^{-a^2it} \tag{3.10}$$

which is exactly the same as the results obtained by DTM [3], HPM [1, 6], HAM [8] and ADM [6] by setting $a = 2$ in the eqn. (3.10).

Example 3.2. *Consider the one dimensional nonlinear Schrodinger equation*

$$iu_t + u_{xx} + m|u|^2u = 0, \quad x \in R, t \geq 0 \tag{3.11}$$

Subject to the initial condition

$$u(x,0) = e^{inx} \tag{3.12}$$

Where m is constant.

The transformed version of eqn. (3.11) is

$$i(k+1)U_{k+1}(x) = -\frac{\partial^2}{\partial x^2}U_k(x) - m\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\overline{U}_{k_1}(x)U_{k_2-k_1}(x)U_{k-k_2}(x) \tag{3.13}$$

The transformed version of initial condition (3.12) is

$$U_0(x) = e^{inx} \tag{3.14}$$

Similar to the previous problem, the first few components of $U_k(x)$ are obtained as follows:

$$\begin{aligned}
U_1(x) &= i(m-n^2)e^{inx}\\
U_2(x) &= -\frac{1}{2}(m-n^2)^2e^{inx}\\
U_3(x) &= -\frac{i}{6}(m-n^2)^3e^{inx}\\
U_4(x) &= \frac{1}{24}(m-n^2)^4e^{inx}\\
U_5(x) &= \frac{i}{120}(m-n^2)^5e^{inx}
\end{aligned}$$

and so on. Substituting all these values in Eq. (2.4) yields

$$u(x,t) = e^{inx} + i(m-n^2)e^{inx}t + \frac{i^2}{2!}(m-n^2)^2e^{inx}t^2 + \frac{i^3}{3!}(m-n^2)^3e^{inx}t^3 + \frac{i^4}{4!}(m-n^2)^4t^4 + \cdots$$

Consequently the solution in closed form is

$$u(x,t) = e^{i(nx+(m-n^2)t)} \tag{3.15}$$

which is exactly the same as the result obtained by HPM [6], VIM [2], DTM [3] with $m = 2$, $n = 1$ and ADM [6] with $m = -2$, $n = 1$ in the eqn. (3.15).

Example 3.3. *Finally we consider the following non-linear Schrodinger equation*

$$iu_t = mu_{xx} + u\cos^2 x + |u|^2u \quad x \in R, t \geq 0 \tag{3.16}$$

Subject to the initial condition

$$u(x,0) = \sin ax \tag{3.17}$$

where m and a are constants.

The transformed version of (3.16) gives the following recursive formula:

$$i(k+1)U_{k+1}(x) = m\frac{\partial^2}{\partial x^2}U_k(x) + U_k(x)\cos^2 x + \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\overline{U}_{k_1}(x)U_{k_2-k_1}(x)U_{k-k_2}(x) \tag{3.18}$$

Where $|u| = \overline{u}u$ and $\overline{u}$ is the conjugate of u. The transformed version of initial condition (3.17) is

$$U_0(x) = \sin ax \tag{3.19}$$

Substituting the eqn. (3.19) in to the eqn. (3.18), we obtained the first few components of $U_k(x)$ as

$$\begin{aligned}
U_1(x) &= i(ma^2-1)\sin ax\\
U_2(x) &= -\frac{1}{2}(ma^2-1)^2\sin ax\\
U_3(x) &= -\frac{i}{6}(ma^2-1)^3\sin ax\\
U_4(x) &= \frac{1}{24}(ma^2-1)^4\sin ax\\
U_5(x) &= \frac{i}{120}(ma^2-1)^5\sin ax
\end{aligned}$$

and so on and finally substituting all these values in the eqn. (2.4) yields

$$u(x,t) = \sin ax + i(ma^2-1)t\sin ax + \frac{i^2}{2!}(ma^2-1)^2t^2\sin ax + \frac{i^3}{3!}(ma^2-1)^3t^3 sinax + \cdots$$

Consequently, the solution in closed form is

$$u(x,t) = \sin ax e^{i(ma^2-1)t} \tag{3.20}$$

which is exactly the same as the results obtained by DTM [3, 7], HPM [5], HAM [8] and FDM [4] by setting $m = -1/2$ and $a = 1$ in the eqn. (3.20).

4 Conclusion

In this paper, the reduced differential transform method was applied to obtain the exact solution of linear and nonlinear Schrodinger equations. The results of test examples confirmed that, RDTM is efficient and powerful tool in finding the analytical solution of linear and nonlinear partial differential equations. This method is better than numerical methods since it is free from round of error and does not require large computer memory. Comparing with other existing methods it does not require linearization, perturbation or discretization.

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Received: November 11, 2015; *Accepted*: January 17, 2016

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 65–78

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Oscillation of First Order Neutral Difference Equations

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Abstract

In this paper, we consider a class of first order neutral difference equations of the form

$$\Delta[r(n)(x(n)+p(n)x(n-\tau))]+q(n)x(n-\sigma)=0, \quad n\geq n_0. \qquad (*)$$

Some sufficient conditions for the oscillation of all solutions of (*) are established. Our result extend and improve some of the previous results in the literature. Some examples are considered to illustrate our results.

Keywords: Oscillation, nonoscillation, neutral, difference equations.

2010 MSC: 39A10, 39A12.

1 Introduction

During the past few decades, neutral difference equations have been studied extensively and the oscillatory theory for these equations is well developed; see [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] and the references cited therein. A survey of the most significant efforts in this theory can be found in the excellent monographs of Györi and Ladas [6] and Agarwal [1], [2].

Consider the first order neutral difference equations of the form

$$\Delta[r(n)(x(n)+p(n)x(n-\tau))]+q(n)x(n-\sigma)=0, \quad n\geq n_0 \tag{1.1}$$

where $\{p(n)\}$ is a sequence of real numbers, $\{r(n)\}$ and $\{q(n)\}$ are sequences of positive real numbers, τ and σ are positive integers, and Δ is the forward difference operator given by $\Delta x(n)=x(n+1)-x(n)$.

Let us choose a positive integer $n^*>\max\{\tau,\sigma\}$. By a solution of (1.1) on $N(n_0)=\{n_0,n_0+1,...\}$, we mean a real sequence $\{x(n)\}$ which is defined on $n\geq n_0-n^*$ and which satisfies (1.1) for $n\in N(n_0)$. A solution $\{x(n)\}$ of (1.1) on $N(n_0)$ is said to be oscillatory if for every positive integer $N_0>n_0$ there exists $n\geq N_0$ such that $x(n)x(n+1)\leq 0$, otherwise $\{x(n)\}$ is said to be non-oscillatory.

There are numerous numbers of oscillation criteria obtained for oscillation of all solutions of (1.1). In particular, Murugesan and Suganthi [9] investigated the oscillation behavior of (1.1) and obtained some new oscillation results under the condition

$$\sum_{n=n_0}^{\infty} q(n)=\infty. \tag{1.2}$$

For oscillation of (1.1) when $r(n)\equiv 1$ and $p(n)$ is equal to a constant, we refer the readers to the papers by Lalli [7] and the references cited therein. For further oscillation results on the oscillating behavior of solutions of (1.1), when $r(n)\equiv 1$, we refer the reader to the monographs by Agarwal [1], [2] as well as the papers of

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Ying Gao and Zhang [15], Murugesan and Suganthi [9], Chen et. al [3], Tang et. al [13] and the references cited therein.

Define the sequences $\{z(n)\}$ and $\{w(n)\}$ as follows:

$$z(n) = x(n) + p(n)x(n-\tau), \tag{1.3}$$

$$w(n) = z(n) + p(n)z(n-\tau). \tag{1.4}$$

If $\{x(n)\}$ is an eventually positive solution of the equation

$$\Delta[x(n) + px(n-\tau)] + q(n)x(n-\sigma) = 0, \tag{1.5}$$

where p is a real constant then $\{z(n)\}$ and $\{w(n)\}$ are also solutions of (1.5).

In the sequel, unless otherwise specified, when we write a functional inequality, we assume that it holds for all sufficiently large n.

2 Some Useful Lemmas

In the proof of our main results, we need the following Lemmas. The Lemma 2.2 and 2.3 are discrete analogues of the Lemma 1.5.1 and 1.5.3 respectively in [6].

Lemma 2.1. [9] *Assume that (1.2) holds. Let $\{x(n)\}$ be an eventually positive solution of equation (1.1). Then the following statements are true.*

(i) if $p(n) \leq -1$ then $z(n) < 0$;

(ii) if $-1 \leq p(n) \leq 0$ and $\{r(n)\}$ is a decreasing sequence of positive real numbers, then $z(n) > 0$ and $\lim_{n\to\infty} z(n) = 0$.

Lemma 2.2. *Let $\{f(n)\}$ and $\{g(n)\}$ be sequence of real numbers such that*

$$f(n) = g(n) + \mu g(n-c); \quad n \geq n_0 + \max\{0, c\},$$

where $\mu \in R$, $\mu \neq 1$ and c is a positive integer. Assume that $\lim_{n\to\infty} f(n) = l \in R$ exists and $\liminf_{n\to\infty} g(n) = a \in R$. Then $l = (1+p)a$.

Lemma 2.3. *Let $0 \leq \lambda < 1$, c be a positive integer and $n_0 \in N$ and $\{x(n)\}$ be a sequence of positive real numbers and assume that for every $\in > 0$ there exists a $n_\in \geq n_0$ such that*

$$x(n) \leq (\lambda + \in)x(n-c) + \in \quad for \quad n \geq n_\in.$$

Then

$$\lim_{n\to\infty} x(n) = 0.$$

Lemma 2.4. *Assume that (1.2) holds, p is a real number with $p \neq 1$ and $\{q(n)\}$ is a τ-periodic sequence of positive real numbers. Let $\{x(n)\}$ be an eventually positive solution of (1.5).*

Then

(a) $\{z(n)\}$ is decreasing sequence and either

$$\lim_{n\to\infty} z(n) = -\infty; \tag{2.6}$$

or

$$\lim_{n\to\infty} z(n) = 0. \tag{2.7}$$

(b) The following statements are equivalent:

(i) (2.6) holds;

(ii) $p < -1$;

(iii) $\lim_{n\to\infty} x(n) = \infty$;

(iv) $w(n) > 0, \Delta w(n) > 0$.

(c) The following statements are equivalent:

(i) (2.7) holds;

(ii) $p > -1$;

(iii) $\lim_{n\to\infty} x(n) = 0$;

(iv) $w(n) > 0, \Delta w(n) < 0$.

Proof. (a) we have

$$\Delta z(n) = -q(n)x(n-\sigma) < 0 \tag{2.8}$$

and so $\{z(n)\}$ is strictly decreasing sequence. If (2.6) is not true, then there exists $l \in R$ such that $\lim_{n\to\infty} z(n) = l$. By summing (2.8) from n_1 to ∞, with n_1 sufficiently large, we find

$$l - z(n) = -\sum_{s=n_1}^{\infty} q(s)x(s-\sigma). \tag{2.9}$$

In view of (1.2) this implies that $\liminf_{n\to\infty} x(n) = 0$ and so by Lemma 2.2, $l = (1+p)0 = 0$. The proof of (a) is complete.

Now we turn to the proofs of (b) and (c). First assume that (2.6) holds. Then p must be negative and $\{x(n)\}$ is unbounded. Therefore there exists a $n^* \geq n_0$ such that $z(n^*) < 0$ and

$$x(n^*) \geq \max_{s\leq n^*} x(s) > 0.$$

Then

$$0 > z(n^*) = x(n^*) + px(n^*-\tau) \geq x(n^*)(1+p)$$

which implies that $p < -1$. Also

$$z(n) = x(n) + px(n-\tau) > px(n-\tau)$$

and (2.6) implies that $\lim_{n\to\infty} x(n) = \infty$. Now assume that (2.7) holds.

If $p \geq 0$, then from (1.3) it follows that $\lim_{n\to\infty} x(n) = 0$. Next assume that $p \in (-1, 0)$. Then by Lemma 2.3, $\lim_{n\to\infty} x(n) = 0$.

Finally if $p \leq -1$, then $x(n) > -px(n-\tau) \geq x(n-\tau)$ which shows that $\{x(n)\}$ is bounded from below by a positive constant, say m. Then (2.9) yields.

$$l - z(n_1) + m\sum_{s=n_1}^{\infty} q(s) \leq 0,$$

which is a contradiction. Therefore, if (2.7) holds $p > -1$. Observe that under the hypothesis (2.6), we have

$$\Delta w(n) = -q(n)z(n-\sigma) > 0. \tag{2.10}$$

If (2.6) holds, then

$$\lim_{n\to\infty} w(n) = \infty. \tag{2.11}$$

From (2.10) and (2.11) we have $w(n) > 0$ eventually. By a similar proof, under the hypothesis (2.7), we have $\Delta w(n) < 0$ and $w(n) > 0$. On the basis of the above discussions, the proof of (b) and (c) are now obvious. □

Lemma 2.5. *Assume that*

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\sigma-1} q(s) > 0. \tag{2.12}$$

If $\{x(n)\}$ is an eventually positive solution of the delay difference equation

$$\Delta x(n) + q(n)x(n-\sigma) = 0, \quad n \geq n_0, \tag{2.13}$$

then

$$\liminf_{n\to\infty} \frac{x(n-\sigma)}{x(n)} < \infty. \tag{2.14}$$

Proof. In view of the assumption there exists a constant $d > 0$ and a sequence $\{n_k\}$ of positive integers such that $n_k \to \infty$ as $k \to \infty$ and

$$\sum_{s=n_k}^{n_k+\sigma-1} q(s) \geq d; \quad k = 1,2,3,....$$

Then there exists a $\xi_k \in \{n_k, n_k+1, ..., n_k+\sigma\}$ for each k such that

$$\sum_{s=n_k}^{\xi_k} q(s) \geq \frac{d}{2} \quad and \quad \sum_{s=\xi_k}^{n_k+\sigma} q(s) \geq \frac{d}{2}. \tag{2.15}$$

Summing the equation (2.13) from n_k to ξ_k and ξ_k to $n_k+\sigma$, we find

$$x(\xi_k+1) - x(n_k) + \sum_{s=n_k}^{\xi_k} q(s)x(s-\sigma) = 0 \tag{2.16}$$

and

$$x(n_k+\sigma+1) - x(\xi_k) + \sum_{s=\xi_k}^{n_k+\sigma} q(s)x(s-\sigma) = 0. \tag{2.17}$$

By omitting the first terms in (2.16) and (2.17) and by using the decreasing nature of $\{x(n)\}$ and (2.15), we find

$$-x(n_k) + \frac{d}{2}x(\xi_k-\sigma) \leq 0$$

and

$$-x(\xi_k) + \frac{d}{2}x(n_k) \leq 0$$

(or)

$$\frac{x(\xi_k-\sigma)}{x(\xi_k)} \leq \left(\frac{2}{d}\right)^2.$$

This completes the proof. □

Lemma 2.6. *If the equation (2.13) has an eventually positive solution, then one has eventually that*

$$\sum_{s=n+1}^{n+\sigma} q(s) \leq 1. \tag{2.18}$$

Proof. Let $\{x(n)\}$ be an eventually positive solution of (2.13). On the contrary, assume that

$$\sum_{s=n+1}^{n+\sigma} q(s) > 1, \tag{2.19}$$

eventually. Summing the equation (2.13) from $n+1$ to $n+\sigma$ and using the decreasing nature of $\{x(n)\}$, we have

$$x(n+\sigma+1) - x(n+1) + \sum_{s=n+1}^{n+\sigma} q(s)x(s-\sigma) \leq 0,$$

or

$$x(n+\sigma+1) - x(n) + x(n)\sum_{s=n+1}^{n+\sigma} q(s) \leq 0,$$

or

$$x(n+\sigma+1) + x(n)\left(\sum_{s=n+1}^{n+\sigma} q(s) - 1\right) \leq 0,$$

eventually.

This is a contradiction and the proof is complete. □

Lemma 2.7. [6] *The delay difference inequality*

$$\Delta x(n) + q(n)x(n-\sigma) \leq 0 \tag{2.20}$$

has an eventually positive solution if and only if the delay difference equation

$$\Delta y(n) + q(n)y(n-\sigma) = 0 \tag{2.21}$$

has an eventually positive solution.

Lemma 2.8. [6] *The advanced difference inequality*

$$\Delta x(n) - q(n)x(n+\sigma) \leq 0 \tag{2.22}$$

has an eventually negative solution if and only if the advanced difference equation

$$\Delta y(n) - q(n)y(n+\sigma) = 0 \tag{2.23}$$

has an eventually negative solution.

Lemma 2.9. *Assume that*

$$\limsup_{n\to\infty} \sum_{s=n-\sigma+1}^{n-1} q(s) > 0. \tag{2.24}$$

If $\{x(n)\}$ is an eventually negative solution of the advanced difference equation

$$\Delta x(n) - q(n)x(n+\sigma) = 0, \quad n \geq n_0, \tag{2.25}$$

then

$$\liminf_{n\to\infty} \frac{x(n+\sigma)}{x(n+1)} < \infty.$$

Proof. In view of the assumption there exists a constant $d > 0$ and a sequence $\{n_k\}$ of positive integers such that $n_k \to \infty$ as $k \to \infty$ and

$$\sum_{s=n_k-\sigma+1}^{n_k-1} q(s) \geq d, \quad k = 1, 2, 3....$$

Then there exists $\xi_k \in \{n_k - \sigma, n_k - \sigma + 1, ..., n_k\}$ for all k such that

$$\sum_{s=n_k-\sigma+1}^{\xi_k} q(s) \geq \frac{d}{2} \quad \text{and} \quad \sum_{s=\xi_k}^{n_k} q(s) \geq \frac{d}{2}. \tag{2.26}$$

Summing the equation (2.25) from $n_k - \sigma$ to ξ_k and ξ_k to n_k, we find

$$x(\xi_k + 1) - x(n_k - \sigma) - \sum_{s=n_k-\sigma+1}^{\xi_k} q(s)x(s+\sigma) = 0 \tag{2.27}$$

and

$$x(\xi_k + 1) - x(\xi_k) - \sum_{s=\xi_k}^{n_k} q(s)x(s+\sigma) = 0. \tag{2.28}$$

By omitting the second terms in (2.27) and (2.16) and by using the decreasing nature of $\{x(n)\}$ and (2.26), we find

$$x(\xi_k + 1) - x(n_k + 1)\frac{d}{2} \leq 0$$

and

$$x(n_k + 1) - x(\xi_k + \sigma)\frac{d}{2} \leq 0$$

or

$$\frac{x(\xi_k + \sigma)}{x(\xi_k + 1)} \leq \left(\frac{2}{d}\right)^2.$$

This completes the proof. □

Lemma 2.10. *If the equation (2.25) has an eventually negative solution, than one has eventually that*

$$\sum_{s=n-\sigma+1}^{n-1} q(s) \le 1. \tag{2.29}$$

Proof. Let $\{x(n)\}$ be an eventually negative solution of (2.25). On the contrary, let us assume that

$$\sum_{s=n-\sigma+1}^{n-1} q(s) > 1, \tag{2.30}$$

eventually. Summing the equation (2.25) from $n-\sigma+1$ to n and using the decreasing nature of $\{x(n)\}$, we have

$$x(n+1) - x(n-\sigma+1) - \sum_{s=n-\sigma+1}^{n} q(s)x(s+\sigma) = 0$$

or

$$x(n+1) - x(n-\sigma+1) - x(n+1) \sum_{s=n-\sigma+1}^{n} q(s) \le 0$$

or

$$-x(n-\sigma+1) + x(n+1)\left(1 - \sum_{s=n-\sigma+1}^{n} q(s)\right) \le 0.$$

This is a contradiction and the proof is complete. □

3 Main Results

Theorem 3.1. *Assume that (1.2) hold with $-1 \le p(n) \le 0$ and $\{r(n)\}$ is a decreasing sequence positive real numbers. Suppose that*

$$\sum_{n=n_0}^{\infty} \left[\frac{q(n)}{r(n-\sigma)} \ln \left(e \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma} \right) \right] = \infty. \tag{3.31}$$

Then every solution of (1.1) is oscillatory.

Proof. Assume, for the sake of a contradiction, that (1.1) has an eventually positive solution $\{x(n)\}$. Then there exists an integers $n_1 \ge n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$ and $x(n-\sigma > 0$ for $n \ge n_1$.

Set $z(n)$ to be defined as in (1.3). Then by Lemma 2.1 (ii), it follows that

$$z(n) > 0, \quad \textit{eventually}. \tag{3.32}$$

As $x(n) > z(n)$, it follows from (1.1) that

$$\Delta(r(n)z(n)) + q(n)z(n-\sigma) \le 0. \tag{3.33}$$

Dividing the last inequality by $r(n) > 0$, we obtain

$$\Delta z(n) + \frac{\Delta r(n)}{r(n)} z(n+1) + \frac{q(n)}{r(n)} z(n-\sigma) \le 0. \tag{3.34}$$

Let

$$z(n) = \frac{y(n)}{r(n)}. \tag{3.35}$$

This implies that $y(n) > 0$. Substituting in (3.34) yields

$$\Delta y(n) + \frac{q(n)}{r(n-\sigma)} y(n-\sigma) \le 0, \quad n \ge n_0 \tag{3.36}$$

So by Lemma 2.5, we have that the delay difference equation

$$\Delta y(n) + \frac{q(n)}{r(n-\sigma)} y(n-\sigma) = 0, \quad n \ge n_0 \tag{3.37}$$

has an eventually positive solution as well. Let

$$\lambda(n) = -\frac{\Delta y(n)}{y(n)} \tag{3.38}$$

Then $\{\lambda(n)\}$ satisfies

$$\lambda(n) \geq \bar{Q}(n) \exp\left\{\sum_{s=n-\sigma}^{n-1} \lambda(s)\right\}, \tag{3.39}$$

where

$$\bar{Q}(n) = \frac{q(n)}{r(n-\sigma)}. \tag{3.40}$$

Let

$$R(n) = \sum_{s=n+1}^{n+\sigma} \bar{Q}(s). \tag{3.41}$$

Therefore

$$\lambda(n) \geq \bar{Q}(n) \exp\left\{\frac{1}{R(n)} R(n) \sum_{s=n-\sigma}^{n-1} \lambda(s)\right\}. \tag{3.42}$$

Applying the inequality

$$e^{ax} \geq x + \frac{\ln(ea)}{a}, \quad \forall \quad x, a > 0, \tag{3.43}$$

to (3.42), we have

$$\lambda(n) \geq \bar{Q}(n) \left\{\frac{1}{R(n)} \sum_{s=n-\sigma}^{n-1} \lambda(s) + \frac{\ln(e(R(n))}{R(n)}\right\}, \tag{3.44}$$

or

$$\lambda(n) \left(\sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right) - \bar{Q}(n) \sum_{s=n-\sigma}^{n-1} \lambda(s) \geq \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.45}$$

Then, for $M > N$, we have

$$\sum_{n=N}^{M-1} \lambda(n) \left(\sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right) - \sum_{n=N}^{M-1} \bar{Q}(n) \sum_{s=n-\sigma}^{n-1} \lambda(s) \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.46}$$

By interchanging the order of summation, we get

$$\sum_{n=N}^{M-1} \bar{Q}(n) \sum_{s=n-\sigma}^{n-1} \lambda(s) \geq \sum_{s=N}^{M-\sigma-1} \lambda(n) \left(\sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.47}$$

From (3.45) and (3.47), we find that

$$\sum_{n=M-\sigma}^{M-1} \lambda(n) \left(\sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right) \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.48}$$

By Lemma 2.4, we have

$$\sum_{s=n+1}^{n-\sigma} \bar{Q}(s) \leq 1 \quad eventually. \tag{3.49}$$

Therefore, from (3.48) and (3.49), we get

$$\sum_{n=M-\sigma}^{M-1} \lambda(n) \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right), \tag{3.50}$$

$$\sum_{n=M-\sigma}^{M-1} \left(1 - \frac{y(n+1)}{y(n)}\right) \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.51}$$

Using the inequality

$$\log x < x - 1, \quad for \quad 0 < x < 1$$

in (3.51), we get

$$\ln \frac{y(M-\sigma)}{y(M)} \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln \left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s) \right) \tag{3.52}$$

which implies by condition (3.31) that

$$\lim_{n\to\infty} \frac{y(n-\sigma)}{y(n)} = \infty. \tag{3.53}$$

On the other hand, (3.31) implies that there exists a sequence $\{n_k\}$ of positive integers with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{s=n_k+1}^{n_k+\sigma} \frac{q(s)}{r(s-\sigma)} \geq \frac{1}{e} \quad \text{for} \quad \text{all} \quad k. \tag{3.54}$$

Hence by Lemma 2.3, we obtain

$$\liminf_{n\to\infty} \frac{y(n-\sigma)}{y(n)} < \infty \tag{3.55}$$

This contradicts (3.53) and completes the proof. □

Theorem 3.2. *Assume that (1.2) hold with $p(n) \equiv p > -1$, $r(n) \equiv r > 0$ and $\sigma > \tau$. Assume further that $\{q(n)\}$ is a τ -periodic and*

$$\sum_{n=n_0}^{\infty} \left[\frac{q(n)}{r(1+p)} \ln \left(e \sum_{s=n+1}^{n+\sigma-\tau} \frac{q(s)}{r(1+p)} \right) \right] = \infty. \tag{3.56}$$

Then every solution of (1.1) is oscillatory.

Proof. Assume the contrary. Without loss of generality we may assume that $\{x(n)\}$ be an eventually positive solution of (1.1). Then there exists an integers $n_1 \geq n_0$ such that

$$x(n) > 0, \quad x(n-\tau) > 0 \quad \text{and} \quad x(n-\sigma) > 0 \quad \text{for} \quad \text{all} \quad n \geq n_1. \tag{3.57}$$

Let $z(n)$ and $w(n)$ be defined as in (1.3) and (1.4). It is easily seen, by direct substituting, that $\{z(n)\}$ and $\{w(n)\}$ are also solutions of (1.1) when p and r are constants, that is

$$r\Delta z(n) + pr\Delta z(n-\tau) + q(n)z(n-\sigma) = 0, \tag{3.58}$$

$$r\Delta w(n) + pr\Delta w(n-\tau) + q(n)w(n-\sigma) = 0. \tag{3.59}$$

By Lemma 2.2, we have that $\{z(n)\}$ is decreasing and $w(n) > 0$. Also we have indeed that

$$\Delta w(n) = \frac{-1}{r} q(n)z(n-\sigma) \geq \frac{-1}{r} q(n) \geq (n-\sigma-\tau)$$

$$= \frac{-1}{r} q(n-\tau)z(n-\sigma-\tau) = \Delta w(n-\tau).$$

Then

$$\Delta w(n) \geq \Delta w(n-\tau) \tag{3.60}$$

Using (3.60) in (3.59) implies that

$$r(1+p)\Delta w(n-\tau) + q(n)w(n-\sigma) \leq 0. \tag{3.61}$$

As $p > -1$, we have $1 + p > 0$. Then

$$\Delta w(n-\tau) + \frac{q(n)}{r(1+p)} w(n-\sigma) \leq 0. \tag{3.62}$$

In view of the τ -periodicity of $q(n)$, (3.62) implies that

$$\Delta w(n) + \frac{q(n)}{r(1+p)} w(n-(\sigma-\tau)) \leq 0. \tag{3.63}$$

As $\{w(n)\}$ is a positive solution, so by Lemma 2.5, the delay difference equation

$$\Delta w(n) + \frac{q(n)}{r(1+p)} w(n-(\sigma-\tau)) = 0 \tag{3.64}$$

has an eventually positive solution as well. Let

$$\lambda(n) = -\frac{\Delta w(n)}{w(n)}. \tag{3.65}$$

Then $\lambda(n)$ satisfies

$$\lambda(n) \geq \bar{Q}_1(n) \exp\left\{\sum_{s=n-\sigma+\tau}^{n-1} \lambda(s)\right\}, \tag{3.66}$$

where

$$\bar{Q}_1(x) = \frac{q(n)}{r(1+p)}.$$

Let

$$R_1(n) = \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s). \tag{3.67}$$

Therefore,

$$\lambda(n) \geq \bar{Q}_1(n) \exp\left\{\frac{1}{R_1(n)} R_1(n) \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s)\right\}. \tag{3.68}$$

Applying the inequality (3.43) to (3.68), we have

$$\lambda(n) \geq \bar{Q}_1(n) \left\{\frac{1}{R_1(n)} \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s) + \frac{\ln(eR_1(n))}{R_1(n)}\right\}, \tag{3.69}$$

or

$$\lambda(n) \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s) - \bar{Q}_1(n) \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s) \geq \bar{Q}_1(n) \left\{\ln\left(e \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s)\right)\right\}.$$

Then, for $M > N$, we have

$$\sum_{n=N}^{M-1} \lambda(n) \left(\sum_{s=n+1}^{n+\sigma-1} \bar{Q}_1(s)\right) - \sum_{n=N}^{M-1} \bar{Q}_1(n) \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s)$$
$$\geq \sum_{n=N}^{M-1} \bar{Q}_1(n) \left(\ln\left(e \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s)\right)\right). \tag{3.70}$$

By interchanging the order of summation, we get

$$\sum_{n=N}^{M-1} \bar{Q}_1(n) \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s) \geq \sum_{n=N}^{M-\sigma+\tau-1} \lambda(n) \sum_{s=n+1}^{n+\sigma-\tau} Q_1(s). \tag{3.71}$$

From (3.70) and (3.71), we find that

$$\sum_{n=M-\sigma+\tau}^{M-1} \lambda(n) \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s) \geq \sum_{n=N}^{M-1} \bar{Q}_1(n) \left(\ln\left(e \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s)\right)\right). \tag{3.72}$$

By Lemma 2.4, we have

$$\sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s) \leq 1 \quad \text{eventually}. \tag{3.73}$$

Using (3.73) in (3.72), we get

$$\sum_{n=M-\sigma+\tau}^{M-1} \lambda(n) \geq \sum_{n=N}^{M-1} Q_1(n) \left(\ln\left(e \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s)\right)\right), \tag{3.74}$$

or

$$\sum_{n=M-\sigma+\tau}^{M-1}\left(1-\frac{y(n+1)}{y(n)}\right) \geq \sum_{n=N}^{M-1} Q_1(n)\left(\ln\left(e\sum_{s=n+1}^{n+\sigma-\tau}\bar{Q}_1(s)\right)\right). \tag{3.75}$$

Using the inequality $\ln x < x-1$ for $0<x<1$ in (3.75), we get

$$\ln\left(\frac{y(M-\sigma+\tau)}{y(M)}\right) \geq \sum_{n=N}^{M-1}\bar{Q}_1(n)\left(\ln\left(e\sum_{s=n+1}^{n+\sigma-\tau}\bar{Q}_1(s)\right)\right), \tag{3.76}$$

which implies by condition (3.56) that

$$\lim_{n\to\infty}\frac{y(n-\sigma+\tau)}{y(n)} = \infty. \tag{3.77}$$

On the other hand, (3.56) implies that there exists a sequence $\{n_k\}$ of positive integers with $n_k\to\infty$ as $k\to\infty$ such that

$$\sum_{s=n_k+1}^{n_k+\sigma-\tau}\frac{q(s)}{r(1+p)} \geq \frac{1}{e} \quad \text{for} \quad \text{all} \quad \text{k}. \tag{3.78}$$

Hence by Lemma 2.3, we obtain

$$\liminf_{n\to\infty}\frac{y(n-\sigma+\tau)}{y(n)} < \infty.$$

This contradicts (3.77) and completes the proof. □

Theorem 3.3. *Assume that (1.2) hold with $p(n)\leq -1$ and $\tau-\sigma>1$. Assume further that*

$$\sum_{n=n_0}^{\infty}\left[\frac{-q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)}\ln\left(e\sum_{s=n-\tau+\sigma+1}^{n-1}\frac{-q(s)}{p(s+\tau-\sigma)r(s+\tau-\sigma)}\right)\right] = \infty. \tag{3.79}$$

Then every solution of (1.1) is oscillatory.

Proof. Assume that (1.1) has a nonoscillatory solution. Without loss of generality we may assume that $\{x(n)\}$ is an eventually positive solution of (1.1). Then there exists an integer $n_1\geq n_0$ such that $x(n)>0$, $x(n-\tau)>0$ and $x(n-\sigma)>0$ for $n\geq n_1$.

Set $z(n)$ to be defined as in (1.3). Then by Lemma 2.1, it follows that

$$z(n) < 0 \quad \text{eventually}.$$

As $z(n) > p(n)x(n-\tau)$, it follows from (1.1) that

$$\Delta(r(n)z(n)) + \frac{q(n)}{p(n+\tau-\sigma)}z(n+\tau-\sigma) \leq 0. \tag{3.80}$$

Dividing the last inequality by $r(n)>0$, we obtain

$$\Delta z(n) + \frac{\Delta r(n)}{r(n)}z(n+1) + \frac{q(n)}{p(n+\tau-\sigma)r(n)}z(n+\tau-\sigma) \leq 0. \tag{3.81}$$

Let

$$y(n) = r(n)z(n). \tag{3.82}$$

This implies that $y(n)>0$. Substituting in (3.82) yields

$$\Delta y(n) + \frac{q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)}y(n+\tau-\sigma) \leq 0, \quad n\geq n_0. \tag{3.83}$$

So by Lemma 2.8, we have that the advanced difference equation

$$\Delta y(n) + \frac{q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)}y(n+\tau-\sigma) = 0, \quad n\geq n_0 \tag{3.84}$$

has an eventually negative solution as well as.

Let

$$\lambda(n)=\frac{\Delta y(n)}{y(n+1)}. \tag{3.85}$$

Then $\{\lambda(n)\}$ is positive sequence. Furthermore, $\{\lambda(n)\}$ satisfies

$$\lambda(n)\geq \bar{Q}_2(n)\exp\left(\sum_{s=n+1}^{n+\tau-\sigma-1}\lambda(s)\right), \tag{3.86}$$

where

$$\bar{Q}_2(n)=\frac{-q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)}>0. \tag{3.87}$$

Let

$$R_2(n)=\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s). \tag{3.88}$$

Therefore

$$\lambda(n)\geq \bar{Q}_2(n)\exp\left\{\frac{1}{R_2(n)}R_2(n)\sum_{s=n+1}^{n+\tau-\sigma-1}\lambda(s)\right\}. \tag{3.89}$$

Applying the inequality (3.43) to (3.89), we have

$$\lambda(n)\geq \bar{Q}_2(n)\left\{\frac{1}{R_2(n)}\sum_{s=n+1}^{n+\tau-\sigma-1}\lambda(s)+\frac{\ln(eR_2(n))}{R_2(n)}\right\}, \tag{3.90}$$

or

$$\lambda(n)\sum_{s=n-\tau+\sigma}^{n-1}\bar{Q}_2(s)-\bar{Q}_2(n)\sum_{s=n+1}^{n+\tau-\sigma-1}\lambda(s)\geq \bar{Q}_2(n)\ln\left(e\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s)\right).$$

Then for $M>N$, we have

$$\sum_{n=N+1}^{M}\lambda(n)\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s)-\sum_{n=N+1}^{M}\bar{Q}_2(n)\sum_{s=n+1}^{n+\tau-\sigma-1}\lambda(s)$$
$$\geq \sum_{n=N+1}^{M}\bar{Q}_2(n)\ln\left(e\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s)\right). \tag{3.91}$$

By interchanging the order of summation we get

$$\sum_{n=N+1}^{M}\bar{Q}_2(n)\sum_{s=n+1}^{n+\tau-\sigma-1}\lambda(s)\geq \sum_{n=N+\tau-\sigma}^{M}\lambda(n)\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s). \tag{3.92}$$

From (3.91) and (3.92), we find that

$$\sum_{n=N+1}^{n+\tau-\sigma-1}\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s)\geq \sum_{n=N}^{M-1}\bar{Q}_2(n)\ln\left(e\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s)\right). \tag{3.93}$$

However, using Lemma 2.10, it follows that

$$\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s)\leq 1 \quad \text{eventually.} \tag{3.94}$$

Therefore from (3.94) in (3.94), we get

$$\sum_{n=N+1}^{N+\tau-\sigma-1}\lambda(n)\geq \sum_{n=N}^{M-1}\bar{Q}_2(n)\ln\left(e\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s)\right),$$

or

$$\sum_{n=N+1}^{N+\tau-\sigma-1}\left(1-\frac{y(n)}{y(n+1)}\right)\geq \sum_{n=N}^{M-1}\bar{Q}_2(n)\ln\left(e\sum_{s=n-\tau+\sigma+1}^{n-1}\bar{Q}_2(s)\right). \tag{3.95}$$

Using the inequality $\ln x < x - 1$, for $0 < x < 1$ in (3.95), we get

$$\log \frac{y(N+\tau-\sigma)}{y(N+1)} \geq \sum_{n=N}^{M-1} \bar{Q}_2(n) \ln \left(e \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s) \right)$$

which implies by condition (3.79) that

$$\lim_{n\to\infty} \frac{y(n+\tau-\sigma)}{y(n+1)} = \infty. \tag{3.96}$$

On the other hand, (3.79) implies that there exists a sequence $\{n_k\}$ of positive integers with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{s=n_k-\tau+\sigma+1}^{n_k-1} \frac{-q(s)}{p(s+\tau-\sigma)r(s+\tau-\sigma)} \geq \frac{1}{e}, \quad \text{for all k}. \tag{3.97}$$

Hence by Lemma 2.9, we obtain

$$\liminf_{n\to\infty} \frac{y(n+\tau-\sigma)}{y(n+1)} < \infty. \tag{3.98}$$

This contradicts (3.96) and completes the proof. □

4 Some Examples

Example 4.1. *Consider the equation*

$$\Delta \left[\frac{1}{n+2} \left(x(n) - \frac{n+1}{n+2} x(n-2) \right) \right] + \frac{1}{n+1} x(n-1) = 0, \quad n = 0, 1, 2, ..., \tag{4.99}$$

where

$$r(n) = \frac{1}{n+1}, \quad q(n) = \frac{1}{n+1}, \quad p(n) = -\frac{n+1}{n+2}, \tau = 2 \quad \text{and} \quad \sigma = 1.$$

Observe that

$$\sum_{n=0}^{\infty} q(n) = \infty.$$

Also

$$\sum_{n=0}^{\infty} \left[\frac{q(n)}{r(n-\sigma)} \ln \left(e \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} \right) \right] = \sum_{n=0}^{\infty} \ln(e\sigma) = \infty.$$

All conditions of the Theorem 3.1 are satisfied. Then all solutions of (4.99) oscillate.

Example 4.2. *Consider the equation*

$$\Delta \left[2 \left(x(n) - \frac{1}{2} x(n-2) \right) \right] + (2 + (-1)^n) x(n-3) = 0, \quad n = 0, 1, 2, ..., \tag{4.100}$$

where

$$-1 \leq p(n) = \frac{1}{2}, \tau = 2, \sigma = 3, r(n) = 2 \quad \text{and} \quad q(n) = 2 + (-1)^n.$$

Observe that

$$\sum_{n=0}^{\infty} \left[\frac{q(n)}{r(1+p)} \ln \left(e \sum_{s=n+1}^{n+\sigma-\tau} \frac{q(s)}{r(1+p)} \right) \right] = \sum_{n=0}^{\infty} [(2+(-1)^n) \ln(e(2-(-1)^n))] = \infty.$$

Then all conditions of Theorem 3.2 are satisfied and therefore all solution of (4.100) oscillate.

Example 4.3. *Consider the difference equation*

$$\Delta\left[\frac{1}{2^n}(x(n)-2^n x(n-3))\right]+e^n x(n-1)=0, \quad n=0,1,2,..., \tag{4.101}$$

where

$$-1 \geq p(n)=-2^n, r(n)=\frac{1}{2^n}, q(n)=e^n, \tau=3 \quad \sigma=1.$$

Observe that

$$\sum_{n=0}^{\infty}\left[\frac{-q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)}\ln\left(e\sum_{s=n-\tau+\sigma+1}^{n-1}\frac{-q(s)}{p(s+\tau-\sigma)r(s+\tau-\sigma)}\right)\right]$$
$$=\sum_{n=0}^{\infty} ne^n=\infty.$$

Then all conditions of Theorem 3.3 are satisfied and therefore all solutions of (4.101) oscillate.

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Received: December 22, 2015; *Accepted*: January 12, 2016

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 79–83

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

A Note on Some Modular Equations Using Theta Functions

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Abstract

In this paper, we have given simple proof to the modular equations using theta function identities.

Keywords: Theta functions, Modular equations.

2010 MSC: 05C15, 05C70.

1 Introduction

We begin this section by introducing the standard notation

$$(a;q)_0 := 1$$

and

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

We now give Ramanujans denition of his general theta-function. For $|ab|<1$, define

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

We introduce theta-functions that play major roles. In [4, Entry 22, P. 36], they are defined by, for $q=e^{2iz}$,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2{}_\infty (q^2;q^2)_\infty = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

After Ramanujan,we define

$$\chi(q) := (-q;q^2)_\infty,$$

Ramanujan's modular equations involve quotients of the function $f(-q)$ at certain arguments. For example [5, P. 206], let

$$P := \frac{f(-q)}{q^{1/6}f(-q^5)} \qquad \text{and} \qquad Q := \frac{f(-q^2)}{q^{1/3}f(-q^{10})},$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{1.1}$$

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These modular equations are also called Schläfli type. Since the publication of [5] several authors, including N.D. Baruah [1], [2], M. S. M. Naika [8], [9] and K. R. Vasuki [10], [11], have found additional modular equations of the type (1.1). After that several new $P-Q$ eta function identities have been discovered and employed them in finding the explicit evaluation of continued fractions, class invariants and ratio of theta functions by many mathematicians.

There are many definitions of a modular equation in the literature. We now give the definition of a modular equation as given by Ramanujan. A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;z):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n \qquad |z|<1,$$

denotes an ordinary hypergeometric function with

$$(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}.$$

Then, we say that β is of n^{th} degree over α and call the ratio

$$m:=\frac{z_1}{z_n},$$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

In Section 2 of this paper, we recall some theta function identities and in Section 3, we prove few modular equations using the same. Recently N. D. Baruah and N. Saikia [3] have obtained few modular equations for the explicit evaluations of Ramanujan's theta functions with two parameters. We are obtaining these modular equations using theta functions. The purpose of this paper is to provide direct proofs of some of P-Q eta function identities. Our proofs use nothing more than theta function identities. However our proofs are much more elementary and can be extended to prove other modular equations.

2 Preliminary Results

For convenience, we denote $f(-q^n)$ by f_n for a positive integer n. It is easy to see that

$$\begin{aligned}\varphi(-q) &= \frac{f_1^2}{f_2}, \qquad \psi(q)=\frac{f_2^2}{f_1}, \qquad \varphi(q)=\frac{f_2^5}{f_1^2f_4^2}, \qquad \psi(-q)=\frac{f_1f_4}{f_2},\\ \chi(q) &= \frac{f_2^2}{f_1f_4}, \qquad \chi(-q)=\frac{f_1}{f_2} \qquad \text{and} \qquad f(q)=\frac{f_2^3}{f_1f_4}.\end{aligned} \tag{2.1}$$

We have

$$\varphi(q)\psi(q^2) = \psi^2(q), \tag{2.2}$$

$$\varphi^2(q)+\varphi^2(-q) = 2\varphi^2(q^2), \tag{2.3}$$

$$\varphi^2(q)-\varphi^2(-q) = 8q\psi^2(q^4), \tag{2.4}$$

$$\varphi^2(q)-\varphi^2(q^5) = 4q\chi(q)\chi(q^5)\psi^2(-q^5), \tag{2.5}$$

$$\varphi^2(q)-5\varphi^2(q^5) = -4\frac{f_2^2\chi(q^5)}{\chi(q)}, \tag{2.6}$$

$$\psi^2(-q)+q\psi^2(-q^5) = \frac{\varphi^2(q^5)}{\chi(q)\chi(q^5)}, \tag{2.7}$$

$$\psi^2(-q)+5q\psi^2(-q^5) = f_1^2\frac{\chi(q)}{\chi(q^5)}. \tag{2.8}$$

The identities (2.2) - (2.4) are due to Ramanujan and for a proof see [4]. Again the identities (2.5) - (2.8) are due to Ramanujan [4], S. -Y. Kang [7], has given the proof of (2.5) - (2.8) by employing theta function identities. Recently S. Bhargava, K. R. Vasuki and Rajanna [6] deduced (2.5) - (2.8) from Ramanujan's ${}_1\psi_1$ summation formula.

3 Main Results

Theorem 3.1. *If*

$$P := \frac{\psi(q)}{q^{1/8}\psi(q^2)} \qquad \text{and} \qquad Q := \frac{\psi(q^2)}{q^{1/4}\psi(q^4)}$$

then

$$P^2 - \left(\frac{2}{PQ}\right)^2 - \left(\frac{Q}{P}\right)^2 = 0.$$

Proof. Consider

$$\begin{aligned} Q^2 + \frac{4}{Q^2} &= \frac{\psi^4(q^2) + 4q\psi^4(q^4)}{q^{1/2}\psi^2(q^2)\psi^2(q^4)} \\ &= \frac{1}{q^{1/2}\psi^2(q^2)}\left\{\frac{\psi^4(q^2)}{\psi^2(q^4)} + 4q\psi^2(q^2)\right\} \\ &= \frac{1}{q^{1/2}\psi^2(q^2)}\left\{\varphi^2(q^2) + 4q\psi^2(q^2)\right\} \\ &= \frac{\varphi^2(q)}{q^{1/2}\psi^2(q^2)} = P^4. \end{aligned}$$

where we used (2.2) after changing q to q^2, (2.3), (2.4) consecutively. Further on employing (2.2) again we have the result. □

Theorem 3.2. *If*

$$P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \qquad \text{and} \qquad Q := \frac{\varphi(q)}{\varphi(q^5)}$$

then

$$PQ - \frac{5}{PQ} = \frac{P}{Q} - \frac{Q}{P}.$$

Proof. From the definition of P and Q, it is easy to see that

$$PQ = \frac{\psi(-q)\varphi(q)}{q^{1/2}\psi(-q^5)\varphi(q^5)} \qquad \text{and} \qquad \frac{P}{Q} = \frac{\psi(-q)\varphi(q^5)}{q^{1/2}\varphi(q)\psi(-q^5)}.$$

Thus in order to prove the result, it suffices to prove the following identity:

$$\psi^2(-q)\left\{\varphi^2(q) - \varphi^2(q^5)\right\} = q\psi^2(q^5)\left\{5\varphi^2(q^5) - \varphi^2(q)\right\}.$$

Using (2.5) and (2.6), we have

$$\frac{\psi^2(-q)}{q\psi^2(-q^5)} - \frac{f_2^2}{q\chi^2(q)\psi^2(-q^5)} = 0.$$

Further, on using (2.1) we have the result. □

Theorem 3.3. *If*

$$P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \qquad \text{and} \qquad Q := \frac{\varphi(-q^2)}{\varphi(q^5)}$$

then

$$PQ + \frac{Q}{P} = \frac{5}{PQ} + \frac{P}{Q}.$$

Proof. Consider

$$\begin{aligned}\frac{P^2+5}{P^2+1} &= \frac{\psi^2(-q)+5q\psi^2(-q^5)}{\psi^2(-q)+q\psi^2(-q^5)}\\ &= f_1^2\frac{\chi^2(q)}{\varphi^2(q^5)}\\ &= \frac{\varphi^2(-q^2)}{\varphi^2(q^5)} = Q^2.\end{aligned}$$

where we employed (2.7), (2.8) and then (2.1). Hence the result. □

Theorem 3.4. *If*

$$P := \frac{\varphi(q)}{\varphi(q^5)} \qquad \text{and} \qquad Q := \frac{\varphi(-q)}{\varphi(-q^5)}$$

then

$$PQ + \frac{5}{PQ} - 4 = \frac{Q}{P} + \frac{P}{Q}.$$

Proof. Consider

$$\begin{aligned}\frac{5P-P^3}{Q^3-5Q} &= \frac{\varphi(q)\varphi^3(-q^3)\left\{5\varphi^2(q^5)-\varphi^2(q)\right\}}{\varphi(-q)\varphi^3(q^5)\left\{\varphi^2(-q)-5\varphi^2(-q^5)\right\}}\\ &= -\frac{\varphi(q)\varphi^2(-q^3)\chi(-q)\chi(q^5)}{\varphi(-q)\varphi^3(q^5)\chi(q)\chi(-q^5)}\\ &= \frac{P^2-1}{Q^2-1}.\end{aligned}$$

Where we used (2.6) twice and (2.1). Now on factorizing and on dividing throughout by PQ we have the result. □

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Received: May 2, 2015; *Accepted*: September 16, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 84–92

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

ORTHOGONAL STABILITY OF THE NEW GENERALIZED QUADRATIC FUNCTIONAL EQUATION

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Abstract

In this paper, the authors investigate the Hyers - Ulam - Rassias stability and J. M. Rassias mixed type product- sum of powers of norms stability of a orthogonally generalized quadratic functional equation of the form

$$f(nx+y)+f(nx-y)=n[f(x+y)+f(x-y)]+2n(n-1)f(x)-2(n-1)f(y).$$

Where $f : A \to B$ be a mapping from a orthogonality normed space A into a Banach Space B, $\perp$ is orthogonality in the sense of Ratz with $x \perp y$ for all $x, y \in A$.

Keywords: : Hyers - Ulam - Rassias stability, J. M. Rassias mixed type product - sum of powers of norms stability,Example, Orthogonally quadratic functional equation, Orthogonality space, Quadratic mapping.

2010 MSC: : 39B55, 39B52, 39B82,46H25

1 Introduction

The stability problem of functional equations originated from the following question of Ulam[19]: Under what condition does there exist an additive mapping near an approximately additive mapping? In 1941, Hyers [8] gave a partial armative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy dierence.

The idea of generalized Hyers-Ulam stability is extended to various functional equations like additive equations, Jensen's equations, Hosszu's equations, homogeneous equations, logarithmic equations, exponential equations, multiplicative equations, trigonometric and gamma functional equations .

It is easy to see that the quadratic function $f(x) = kx^2$ is a solution of each of the following functional equations

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x), \tag{1.2}$$

$$f(x-y-z)+f(x)+f(y)+f(z)=f(x-y)+f(y+z)+f(z-x), \tag{1.3}$$

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(x-y-z)=4f(x)+4f(y)+4f(z). \tag{1.4}$$

So it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x(see [1, 9]). The bi-additive function B is given by

$$B(x,y)=\frac{1}{4}[f(x+y)-f(x-y)]. \tag{1.5}$$

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Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was first treated by F. Skof for functions $f : A \to B$ where A is a normed space and B is a Banach space (see [17]). Cholewa [2] noticed that the theorem of Skof is still true if relevent domain A is replaced by abelian group. Czerwik [3] proved the Hyers-Ulam-Rassias stability of the equation (1.1).

In 1982-1984, J.M. Rassias [12, 13] proved the following theorem in which he generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms.

Theorem 1.1. *[12, 13] Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \|x\|^p \|y\|^q$$

for all $x, y \in E$, where ϵ and p, q are constants with $\epsilon > 0$ and $r = p + q \neq 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2 - 2^r} \|x\|^r$$

for all $x \in E$. If, in addition, for every $x \in E$, $f(tx)$ is continuous in real t for each fixed x, then L is linear.

The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability. Later, J.M.Rassias [15] discused the stability of quadratic functional equation

$$f(mx+y) + f(mx-y) = 2f(x+y) + 2f(x-y) + 2(m^2-2)f(x) - 2f(y)$$

for any arbitrary but fixed real constant m with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ using the mixed powers of norms.

Now we present the results connected with functional equation in orthogonal space. The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), x \perp y \tag{1.6}$$

in which $\perp$ is an abstract orthogonally was first investigated by S. Gudder and D. Strawther . R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1.6) in [7]. The orthogonally quadratic functional equation (1.1) was first investigated by F. Vajzovic [20] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. This result was then generalized by H. Drljevic [4], M. Fochi [5], M.Moslehian [10, 11] and G. Szabo [18].

Definition 1.1. *A vector space X is called an orthogonality vector space if there is a relation $x \perp y$ on X such that*
(i) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
(ii) independence: if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;
(iii) homogeneity: if $x \perp y$, then $ax \perp by$ for all $a, b \in \mathbb{R}$;
(iv) the Thalesian property: if P is a two-dimensional subspace of X ; then
(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;
(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.
Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector x, y define $x \perp y$ iff x, y are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all x, $y \in X$.

Definition 1.2. *The pair $(x, \perp)$ is called an orthogonality space. It becomes orthogonality normed space when the orthogonality space is equipped with a norm.*

Definition 1.3. *Let X be an orthogonality space and Y be a real Banach space. A mapping $f : X \to Y$ is called orthogonally quadratic if it satisfies the so called orthogonally Euler-Lagrange (or Jordan - von Neumann) quadratic functional equation*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.7}$$

for all $x, y \in X$ with $x \perp y$.

In this paper, we obtain the general solution of new quadratic functional equation

$$f(nx+y)+f(nx-y)=n[f(x+y)+f(x-y)]+2n(n-1)f(x)-2(n-1)f(y) \tag{1.8}$$

and study the Hyers - Ulam - Rassias stability and J. M. Rassias mixed type product-sum of powers of norms stability in the concept of orthogonality.

Definition 1.4. *A mapping $f : A \to B$ is called orthogonal quadratic if it satisfies the quadratic functional equation (1.8) for all $x, y \in A$ with $x \perp y$ where A be an orthogonality space and B be a real Banach space.*

Through out this paper, let $(A, \perp)$ denote an orthogonality normed space with norm $\|\cdot\|_A$ and $(B, \|\cdot\|_B)$ is a Banach space. We define

$$\begin{aligned} Df(x,y) &= f(nx+y)+f(nx-y) \\ &- n[f(x+y)+f(x-y)]-2n(n-1)f(x)+2(n-1)f(y). \end{aligned} \tag{1.9}$$

for all $x, y \in A$ with $x \perp y$.

Now we proceed to find the general solution of the functional equation (1.8).

2 The General Solution of the Functional Equation (1.8)

In this section, we obtain the general solution of the functional equation (1.8). Through out this section, let X and Y be real vector spaces.

Theorem 2.2. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation*

$$\begin{aligned} f(nx+y)+f(nx-y) &= n[f(x+y) \\ &+ f(x-y)]+2n(n-1)f(x)-2(n-1)f(y) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ if and only if it satisfies the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2.2}$$

for all $x, y \in X$.

Proof. Suppose a function $f : X \to Y$ satisfies (2.1). Putting $x = y = 0$ in (2.1), we get $f(0) = 0$. Let $x = 0$ and $y = 0$ in (2.1), we obtain $f(-y) = f(y)$ and $f(nx) = n^2 f(x)$, respectively. Setting $(x, y) = (x, x + y)$ in (2.1), we obtain

$$f((n+1)x+y)+f((n-1)x-y)=n[f(2x+y)+f(-y)]+2f(nx)-2nf(x) \tag{2.3}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.3) and adding the resultant with (2.3), we obtain

$$\begin{aligned} &f((n+1)x+y)+f((n+1)x-y)+f((n-1)x+y)+f((n-1)x-y) \\ &= n[f(2x+y)+f(2x-y)]+2n[f(x+y)+f(x-y)]+2[f(x+y)+f(x-y)] \\ &+ 2nf(y)+4f(nx)-4nf(x) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Setting $n = n + 1, n = n - 1$ and $n = 2$ respectively in (2.1), we obtain the following equations

$$\begin{aligned} &f((n+1)x+y)+f((n+1)x-y) \\ &= (n+1)[f(x+y)+f(x-y)]+2n^2f(x)+2nf(x)-2nf(y) \end{aligned} \tag{2.5}$$

$$\begin{aligned} f((n-1)x+y)+f((n-1)x-y) &= (n-1)[f(x+y)+f(x-y)] \\ &+ 2n^2f(x)-6nf(x)+4f(x)-2nf(y)+4f(y) \end{aligned} \tag{2.6}$$

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+4f(x)-2f(y) \tag{2.7}$$

for all $x, y \in X$. Substitute (2.5), (2.6) and (2.7) in (2.4), we arrive (2.2).

Conversely, assume f satisfies the functional equation (2.2). Letting (x,y) by $(0,0)$ in (2.2), we get $f(0)=0$. Putting $x=0$ in (2.2), we obtain $f(-y)=f(y)$ for all $y\in X$. Thus f is an even function. Substituting (x,y) by (x,x) and $(x,2x)$ in (2.2), we get

$$f(2x)=4f(x), f(3x)=9f(x) \tag{2.8}$$

respectively for all $x\in X$. Setting $(x,y)=(nx+y,nx-y)$ in (2.2), we obtain

$$f(nx+y)+f(nx-y)=2n^2f(x)+2f(y) \tag{2.9}$$

for all $x,y\in X$. Multiplying (2.2) by n and subtracting the resultant from (2.9), we arrive (2.1). □

3 Hyers - Ulam - Rassias Stability of (1.8)

In this section, we present the Hyers - Ulam - Rassias stability of the functional equation (1.8) involving sum of powers of norms.

Theorem 3.3. *Let μ and $s(s<2)$ be non-negative real numbers. Let $f:A\to B$ be a mapping fulfilling*

$$\|Df(x,y)\|_B\le\mu\left\{\|x\|_A^s+\|y\|_A^s\right\} \tag{3.1}$$

for all $x,y\in A$ with $x\perp y$. Then there exists a unique orthogonally quadratic mapping $Q:A\to B$ such that

$$\|f(x)-Q(x)\|_B\le\frac{\mu}{2(n^2-n^s)}\|x\|_A^s \tag{3.2}$$

for all $x\in A$. The function $Q(x)$ is defined by

$$Q(x)=\lim_{k\to\infty}\frac{f(n^kx)}{(n^2)^k} \tag{3.3}$$

for all $x\in A$.

Proof. Replacing (x,y) by $(0,0)$ in (3.1) we get $f(0)=0$. Setting (x,y) by $(x,0)$ in (3.1), we obtain

$$\left\|f(nx)-n^2f(x)\right\|_B\le\frac{\mu}{2}(\|x\|_A^s) \tag{3.4}$$

for all $x\in A$. Since $x\perp 0$, we have

$$\left\|\frac{f(nx)}{n^2}-f(x)\right\|_B\le\frac{\mu}{2n^2}\|x\|_A^s \tag{3.5}$$

for all $x\in A$. Now replacing x by nx and dividing by n^2 in (3.5) and summing resulting inequality with (3.5), we arrive

$$\left\|\frac{f(n^2x)}{(n^2)^2}-f(x)\right\|_B\le\frac{\mu}{2n^2}\left\{1+\frac{n^s}{n^2}\right\}\|x\|_A^s \tag{3.6}$$

for all $x\in A$. In general, using induction on a positive integer n we obtain that

$$\begin{aligned}\left\|\frac{f(n^kx)}{(n^2)^k}-f(x)\right\|_B &\le\frac{\mu}{2n^2}\sum_{t=0}^{k-1}\frac{n^{st}}{(n^2)^t}\|x\|_A^s\\ &\le\frac{\mu}{2n^2}\sum_{t=0}^{\infty}\frac{n^{st}}{(n^2)^t}\|x\|_A^s\end{aligned} \tag{3.7}$$

for all $x\in A$. In order to prove the convergence of the sequence $\{f(n^kx)/(n^2)^k\}$ replace x by n^mx and divide by $(n^2)^m$ in (3.7), for any $k,m>0$, we obtain

$$\begin{aligned}\left\|\frac{f\left(n^kn^mx\right)}{(n^2)^{(k+m)}}-\frac{f(n^mx)}{(n^2)^m}\right\|_B &=\frac{1}{(n^2)^m}\left\|\frac{f\left(n^kn^mx\right)}{(n^2)^k}-f(n^mx)\right\|_B\\ &\le\frac{1}{(n^2)^m}\frac{\mu}{2n^2}\sum_{t=0}^{k-1}\frac{n^{st}}{(n^2)^t}\|n^mx\|_A^s\\ &\le\frac{\mu}{2n^2}\sum_{t=0}^{\infty}\frac{1}{n^{(2-s)(t+m)}}\|x\|_A^s.\end{aligned} \tag{3.8}$$

As $s < 2$, the right hand side of (3.8) tends to 0 as $m \to \infty$ for all $x \in A$. Thus $\{f(n^k x)/(n^2)^k\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $Q : A \to B$ such that

$$Q(x) = \lim_{k\to\infty} \frac{f(n^k x)}{(n^2)^k} \quad \forall x \in A.$$

Letting $k \to \infty$ in (3.7), we arrive the formula (3.2) for all $x \in A$. To prove Q satisfies (1.8), replace (x, y) by $(n^k x, n^k y)$ in (3.1) and divide by $(n^2)^k$ then it follows that

$$\frac{1}{(n^2)^k}\|f(n^k(nx+y)) + f(n^k(nx-y)) - n[f(n^k(x+y)) - f(n^k(x-y))] \\ - 2n(n-1)f(n^k x) - 2(n-1)f(n^k y)\|_B \le \frac{\mu}{(n^2)^k}\left\{\left\|n^k x\right\|_A^s + \left\|n^k y\right\|_A^s\right\}.$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$\|Q(nx+y) + Q(nx-y) - n[Q(x+y) - Q(x-y)] \\ - 2n(n-1)Q(x) + 2(n-1)Q(y)\|_B \le 0,$$

which gives

$$Q(nx+y) + Q(nx-y) = n[Q(x+y) - Q(x-y)] + 2n(n-1)Q(x) - 2(n-1)Q(y)$$

by taking limit as $k \to \infty$ in (3.7), we obtain

$$\| f(x) - Q(x)\|_B \le \frac{\mu}{2(n^2 - n^s)} \|x\|_A^s \tag{3.9}$$

for all $x, y \in A$ with $x \perp y$. Therefore $Q : A \to B$ is an orthogonally quadratic mapping which satisfies (1.8). To prove the uniqueness: Let Q' be another orthogonally quadratic mapping satisfying (1.8) and the inequality (3.2). Then

$$\begin{aligned}\|Q(x) - Q'(x)\|_B &= \frac{1}{(n^2)^k}\left\|Q\left(n^k x\right) - Q'\left(n^k x\right)\right\|_B \\ &\le \frac{1}{(n^2)^k}\left(\|Q(k^t x) - f(k^t x)\|_B + \left\|f\left(n^k x\right) - Q'\left(n^k x\right)\right\|_B\right) \\ &\le \frac{\mu}{n^2 - n^s}\frac{1}{n^{k(2-s)}}\|x\|_A^s \\ &\to 0 \text{ as } k \to \infty\end{aligned}$$

for all $x \in A$. Therefore Q is unique. This completes the proof of the theorem. □

Theorem 3.4. *Let μ and $s (s > 2)$ be nonnegative real numbers. Let $f : A \to B$ be a mapping satisfying (3.1) for all $x, y \in A$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q : A \to B$ such that*

$$\|f(x) - Q(x)\|_B \le \frac{\mu}{2(n^s - n^2)} \|x\|_A^s \tag{3.10}$$

for all $x \in A$. The function $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} (n^2)^n f\left(\frac{x}{n^k}\right) \tag{3.11}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{2}$ in (3.4), the rest of the proof is similar to that of Theorem 3.1. □

4 J.M. Rassias Mixed Type Product - Sum of Powers of Norms Stability of (1.8)

In this section, we discuss the J.M. Rassias mixed type product - sum of powers of norms stability of the functional equation (1.8).

Theorem 4.5. *Let $f : A \to B$ be a mapping satisfying the inequality*

$$\|D f(x,y)\|_B \leq \mu \left\{ \|x\|_A^{2s} + \|y\|_A^{2s} + \|x\|_A^{s} \|y\|_A^{s} \right\} \tag{4.1}$$

for all $x, y \in A$ where μ and s are constants with, $\mu, s > 0$ and $s < 1$. Then the limit

$$Q(x) = \lim_{k\to\infty} \frac{f\left(n^k x\right)}{(n^2)^k} \tag{4.2}$$

exists for all $x \in A$ and $Q : A \to B$ is the unique quadratic mapping such that

$$\|f(x) - Q(x)\|_B \leq \frac{\mu}{2(n^2 - n^{2s})} \|x\|_A^{ns} \tag{4.3}$$

for all $x \in A$.

Proof. Letting (x, y) by $(0, 0)$ in (4.1), we get $f(0) = 0$. Again substituting (x, y) by $(x, 0)$ in (4.1), we obtain

$$\left\| \frac{f(nx)}{n^2} - f(x) \right\|_B \leq \frac{\mu}{2n^2} \|x\|_A^{ns} \tag{4.4}$$

for all $x \in A$. Now replacing x by nx and dividing by n^2 in (4.4) and summing resulting inequality with (4.4), we arrive

$$\left\| \frac{f(n^2 x)}{(n^2)^2} - f(x) \right\|_B \leq \frac{\mu}{2n^2} \left\{ 1 + \frac{n^{2s}}{n^2} \right\} \|x\|_A^{2s} \tag{4.5}$$

for all $x \in A$. Using induction on a positive integer k, we obtain that

$$\begin{aligned} \left\| \frac{f\left(n^k x\right)}{(n^2)^k} - f(x) \right\|_B &\leq \frac{\mu}{2n^2} \sum_{t=0}^{k-1} \left(\frac{n^{2s}}{n^2} \right)^t \|x\|_A^{2s} \\ &\leq \frac{\mu}{2n^2} \sum_{t=0}^{\infty} \left(\frac{n^{2s}}{n^2} \right)^t \|x\|_A^{2s} \end{aligned} \tag{4.6}$$

for all $x \in A$. In order to prove the convergence of the sequence $\{f(n^k x)/4^k\}$ replace x by $n^m x$ and divide by $(n^2)^m$ in (4.6), for any $k, m > 0$, we obtain

$$\begin{aligned} \left\| \frac{f\left(n^k n^m x\right)}{(n^2)^{(k+m)}} - \frac{f(n^m x)}{(n^2)^m} \right\|_B &= \frac{1}{(n^2)^m} \left\| \frac{f\left(n^k n^m x\right)}{(n^2)^k} - f(n^m x) \right\|_B \\ &\leq \frac{1}{(n^2)^m} \frac{\mu}{2n^2} \sum_{t=0}^{k-1} \left(\frac{n^{2s}}{n^2} \right)^t \|n^m x\|_A^{2s} \\ &\leq \frac{\mu}{2n^2} \sum_{t=0}^{\infty} \frac{1}{n^{(n-2s)(t+m)}} \|x\|_A^{2s} \end{aligned} \tag{4.7}$$

As $s < 1$, the right hand side of (4.7) tends to 0 as $m \to \infty$ for all $x \in A$. Thus $\{f(n^k x)/(n^2)^k\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $Q : A \to B$ such that

$$Q(x) = \lim_{k\to\infty} \frac{f(n^k x)}{(n^2)^k} \quad \forall x \in A.$$

Letting $n \to \infty$ in (4.6), we arrive the formula (4.2) for all $x \in A$. To show that Q is unique and it satisfies (1.8), the rest of the proof is similar to that of theorem 3.1 □

Theorem 4.6. *Let $f : A \to B$ be a mapping satisfying the inequality (4.1) for all $x, y \in A$ where μ and s are constants with, $\mu, s > 0$ and $s > 2$. Then the limit*

$$Q(x) = \lim_{k\to\infty} (n^2)^k f\left(\frac{x}{n^k}\right) \tag{4.8}$$

exists for all $x \in A$ and $Q : A \to B$ is the unique quadratic mapping such that

$$\|f(x) - Q(x)\|_B \leq \frac{\mu}{2(n^{2s} - n^2)} \|x\|_A^{2s} \tag{4.9}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{3}$ in (4.4), the proof is similar to that of Theorem 4.5. □

Now we will provide an example to illustrate that the functional equation (1.8) is not stable for $s = 2$.

Example 4.1. *Let $\varphi : X \to X$ be a function defined by*

$$\phi(x) = \begin{cases} \mu \|x\|^2 & , \|x\| < 1 \\ \mu & otherwise \end{cases} \tag{4.10}$$

where $\mu > 0$ is a constant and we define a function $f : X \to Y$ by

$$f(x) = \sum_{m=0}^{\infty} \frac{\varphi(n^m x)}{(n^2)^m} \tag{4.11}$$

for all $x \in X$. Then f satisfies the functional inequality

$$\|D(f(x,y))\| \leq \frac{2n^2}{(n-1)} \mu (\|x\|^2 + \|y\|^2) \tag{4.12}$$

for all $x, y \in X$. Then there exist any quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \leq \eta \|x\|^2 \tag{4.13}$$

for $x \in X$.

Proof. From the equation (4.10) and (4.11) , we obtain

$$f(x) \leq \sum_{m=0}^{\infty} \frac{\phi(n^m x)}{n^{2m}} = \sum_{k=0}^{\infty} \frac{\mu}{n^{2m}} \leq \mu(\frac{n^2}{n^2-1}) \tag{4.14}$$

for all $x \in X$. Therefore we see that f is bounded. We are going to prove that f satisfies (4.12).

If $(\|x\|^2 + \|y\|^2) \geq 1$ then the left hand side of (4.12) is less than

$$\frac{2n^2}{(n-1)}$$

. Now we suppose that $0 \leq \|x\|^2 + \|y\|^2 \leq 1$. Then there exist a positive integer k such that

$$\frac{1}{n^{2k-1}} \leq \|x\|^2 + \|y\|^2 < \frac{1}{n^{2k}} \tag{4.15}$$

for all $x \in X$. so that

$$n^{2k}\|x\|^2 < 1, n^{2k}\|y\|^2 < 1$$

and consequently, $n^{k-1}\|x\| < 1$, $n^{k-1}\|y\| < 1$, $n^{k-1}\|x+y\| < 1$, $n^{k-1}\|x-y\| < 1$, $n^{k-1}\|nx+y\| < 1$, $n^k\|nx-y\| < 1$ for all $m \in 0, 1, 2, ..., k-1$

$$\begin{aligned} &n^{k-1}\|x\| < 1, n^{k-1}\|y\| < 1, n^{k-1}(\|x+y\|) < 1, \\ &\qquad n^{k-1}(\|x-y\|) < 1, n^{k-1}(\|nx+y\|) < 1, n^{k-1}(\|nx-y\|) < 1. \end{aligned}$$

for all $x \in \{0, 1, 2, ...k-1\}$.

$$\|D(f(x,y))\| \leq \sum_{m=k}^{\infty} \frac{2n(n+1)}{n^{2m}} \mu$$
$$\leq \frac{2n(n+1)}{n^{2m}} \left(\frac{n^2}{n^2-1}\right) \mu$$
$$\leq \frac{2n^2}{n-1} \mu \left(\|x\|^2 + \|y\|^2\right)$$

Thus f satisfies the inequality (4.12) Let us consider the an orthogonally quadratic mapping satisfying $Q : X \to Y$ and a constant $\eta > 0$ such that

$$\|f(x) - Q(x)\| \leq \eta \|x\|^2$$

for all $x \in X$.Since f is bounded,Q is also bounded on any open interval containing the origin zero. we have

$$Q(x) = c\|x\|^2$$

for all $x \in X$ and c is constant.Thus we obtain

$$\left\|f(x) - c\|x\|^2\right\| \leq \eta \|x\|^2$$
$$\|f(x)\| \leq (\|c\| + \eta) \|x\|^2 \tag{4.16}$$

for all $x \in X$. But we can choose a positive integer

$$p, p\mu > \eta + |c|$$

. If $x \in \left(0, \frac{1}{n^{p-1}}\right)$, then $n^m x \in (0,1)$ for all $m = 0, 1, \ldots, p-1$. For this x, we get $f(x) = \sum_{m=0}^{\infty} \frac{\phi(n^m x)}{n^{2m}} \geq \sum_{m=0}^{\infty} \frac{\mu(n^{2m})\|x\|^2}{n^{2m}} = p\mu \|x\|^2 > (\eta + |c|) \|x\|^2$ which contradicts (4.16). Therefore the functional equation (1.8) is not stable in sense of Ulam, Hyers and Rassias if $s = 2$, assumed in the inequality (4.16). □

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Received: October 10, 2015; *Accepted*: January 23, 2016

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 93–99

Malaya Journal of Matematik
www.malayajournal.org

MJM an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Some integral inequalities of fractional quantum type

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Abstract

In this work, some of the most important fractional integral inequalities involving the Riemann Liouville are extended to quantum calculus on the specific time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n,\ n \text{ a nonnegative integer}\} \cup \{0\}$, where $t_0 \in \mathbb{R}$ and $0 < q < 1$.

Keywords: q-integral inequalities, Riemann-Liouville Fractional Integral.

2010 MSC: 26A33, 26D10, 26D10.

1 Introduction

2 Introduction

The literature on fractional inequalities is now vast, and fractional inequalities are important in studying the existence, uniqueness, and other properties of fractional differential equations. Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville and Caputo derivative, for more details see [7–13] and references cited therein.

The study of the fractional q-integral inequalities play a fundamental role in the theory of differential equations and fractional differential equations. In the past several years, integral inequalities have been studied extensively by several researchers in the quantum, for more details we may refer to [1–6] and the references therein.

In this work, we have used some new Riemann-Liouville integral inequalities and we have obtained some new fractional q-integral inequalities on the specific time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n,\ n \text{ a nonnegative integer}\} \cup \{0\}$, where $t_0 \in \mathbb{R}$ and $0 < q < 1$. Our results are extension of [7].

3 Preliminaries

In this section, we give some necessary definitions and properties which will be used in the next section of this paper. For more details, we may refer to [1–5].

Definition 3.1. *[1–3] The specific time scale $\mathbb{T}_{t_0}$ is defined as*

$$\mathbb{T}_{t_0} = \{t : t = t_0 q^n,\ n \text{ is a nonnegative integer}\} \cup \{0\}, \quad 0 < q < 1. \tag{3.1}$$

If there is no confusion concerning t_0, we will denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$.

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Definition 3.2. *The q-factorial function is defined in the following way*

$$(t-s)^{\underline{(n)}} = (t-s)(t-qs)\,(t-q^s)\cdots(t-q^n s)\,, \quad \textit{when } n \in \mathbb{N},$$
$$(t-s)^{\underline{(n)}} = t^n \prod_{k=0}^{\infty} \frac{1-(s/t)q^k}{1-(s/t)q^{n+k}}, \quad \textit{when } n \notin \mathbb{N}. \tag{3.2}$$

Definition 3.3. *The q-derivative of the q-factorial function with respect to t is*

$$\nabla_q (t-s)^{\underline{(n)}} = \frac{1-q^n}{1-q}(t-s)^{\underline{(n-1)}}, \tag{3.3}$$

and the q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)^{\underline{(n)}} = -\frac{1-q^n}{1-q}(t-qs)^{\underline{(n-1)}}. \tag{3.4}$$

Definition 3.4. *The q-exponential function is defined as*

$$e_q(t) = \prod_{k=0}^{\infty}\left(1-q^k t\right), \quad e_q(0) = 1. \tag{3.5}$$

Definition 3.5. *The q-Gamma function is defined by*

$$\Gamma_q(v) = \frac{1}{1-q}\int_0^1 \left(\frac{t}{1-q}\right)^{v-1} e_q(qt)\nabla t, \quad v \in \mathbb{R}^+. \tag{3.6}$$

Remark 3.1. *Observe that*

$$\Gamma_q(v+1) = [v]_q\,\Gamma_q(v), \quad v \in \mathbb{R}^+ \textit{ and } [v]_q = \frac{1-q^v}{1-q}.$$

Definition 3.6. *The fractional q-integral is defined as*

$$\nabla_q^{-v} f(t) = \frac{1}{\Gamma_q(v)}\int_0^t (t-qs)^{\underline{(v-1)}} f(s)\nabla s. \tag{3.7}$$

Remark 3.2. *For $f(t) = 1$, the above definition gives*

$$\nabla_q^{-v}(1) = \frac{1}{\Gamma_q(v)}\frac{q-1}{q^v-1}t^{\underline{v}} = \frac{1}{\Gamma_q(v+1)}t^{\underline{v}}.$$

4 Main Results

In this section, we will state our main results and give their proofs. We begin with the following lemmas:

Lemma 4.1. *Let f, g be two positive functions on $\mathbb{T}_{t_0}$. Then for all $t > 0, v > 0$, we have*

$$\nabla_r^{-v}\left[\frac{(f(t))^p}{(g(t))^{\frac{p}{q}}}\right] \geq \frac{(\nabla_r^{-v} f(t))^p}{\left(\nabla_r^{-v} g(t)\right)^{\frac{p}{q}}}, \tag{4.8}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let ϕ and ψ tow functions, then using the fractional Hölder inequality we can write

$$\nabla_r^{-v}|\phi(t)\psi(t)| \leq \left(\nabla_r^{-v}|\phi(t)|^p\right)^{\frac{1}{p}}\left(\nabla_r^{-v}|\psi(t)|^q\right)^{\frac{1}{q}}, \quad t > 0, \tag{4.9}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Putting $\phi(t) = \frac{f(t)}{(g(t))^{\frac{1}{q}}}$ and $\psi(t) = (g(t))^{\frac{1}{q}}$ in Eq. (4.9), we get

$$\nabla_r^{-v} f(t) = \nabla_r^{-v}\left(\frac{f(t)}{(g(t))^{\frac{1}{q}}}(g(t))^{\frac{1}{q}}\right) \leq \left[\nabla_r^{-v}\left(\frac{(f(t))^p}{(g(t))^{\frac{p}{q}}}\right)\right]^{\frac{1}{p}}\left[\nabla_r^{-v} g(t)\right]^{\frac{1}{q}}$$

and the proof is complete. □

Lemma 4.2. *Let f, g be two positive functions on $\mathbb{T}_{t_0}$, such that $\nabla_r^{-v} f^p(t) < \infty$, $\nabla_r^{-v} g^q(t) < \infty$, and $t > 0$. If*

$$0 < m \leq \frac{f(s)}{g(s)} \leq M < \infty, \quad s \in [0, t]. \tag{4.10}$$

Then for any $v > 0, p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left[\nabla_r^{-v} f(t)\right]^{\frac{1}{p}} \left[\nabla_r^{-v} g(t)\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \nabla_r^{-v} \left[(f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}}\right]. \tag{4.11}$$

Proof. Since $\frac{f(s)}{g(s)} \leq M$ for all $s \in [0, t], t > 0$, then we have

$$[g(s)]^{\frac{1}{q}} \geq M^{-\frac{1}{q}} [f(s)]^{\frac{1}{q}}. \tag{4.12}$$

Multiplying both side of (4.12) by $[f(s)]^{\frac{1}{p}}$, we have

$$[f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \geq M^{-\frac{1}{q}} f(s). \tag{4.13}$$

Multiplying both side of (4.13) by $\frac{(t-rs)^{\overline{(v-1)}}}{\Gamma_r(v)}$, we have

$$\frac{(t-rs)^{\overline{(v-1)}}}{\Gamma_r(v)} [f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \geq M^{-\frac{1}{q}} \frac{(t-rs)^{\overline{(v-1)}}}{\Gamma_r(v)} f(s). \tag{4.14}$$

Integrating both sides of (4.14) with respect to s on $(0, t)$, we obtain

$$\frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{\overline{(v-1)}} [f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \nabla s \geq M^{-\frac{1}{q}} \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{\overline{(v-1)}} f(s) \nabla s,$$

or equivalently,

$$\nabla_r^{-v} \left[(f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}}\right] \geq M^{-\frac{1}{q}} \nabla_r^{-v} f(t).$$

This leads to

$$\left(\nabla_r^{-v} \left[(f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}}\right]\right)^{\frac{1}{p}} \geq M^{-\frac{1}{pq}} \left(\nabla_r^{-v} f(t)\right)^{\frac{1}{p}}. \tag{4.15}$$

But then, since $m\, g(s) \leq f(s)$ for all $s \in [0, t], t > 0$, so we have

$$[f(s)]^{\frac{1}{p}} \geq m^{\frac{1}{p}} [g(s)]^{\frac{1}{p}}. \tag{4.16}$$

Multiplying both side of (4.16) by $[g(s)]^{\frac{1}{q}}$, we have

$$[f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \geq m^{\frac{1}{p}} g(s). \tag{4.17}$$

Multiplying both side of (4.17) by $\frac{(t-rs)^{\overline{(v-1)}}}{\Gamma_r(v)}$ and integrating the resulting identity with respect to s on $(0, t)$, we obtain

$$\frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{\overline{(v-1)}} [f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \nabla s \geq m^{\frac{1}{p}} \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{\overline{(v-1)}} g(s) \nabla s,$$

or equivalently,

$$\nabla_r^{-v} \left[(f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}}\right] \geq m^{\frac{1}{p}} \nabla_r^{-v} g(t).$$

Hence, we can write

$$\left(\nabla_r^{-v} \left[(f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}}\right]\right)^{\frac{1}{p}} \geq m^{\frac{1}{pq}} \left(\nabla_r^{-v} g(t)\right)^{\frac{1}{p}}. \tag{4.18}$$

Combining the inequalities (4.15) and (4.18), we obtain the inequality (4.11). □

Lemma 4.3. *Let f, g be two positive functions on $\mathbb{T}_{t_0}$, such that $\nabla_r^{-v} f^p(t) < \infty$, $\nabla_r^{-v} g^q(t) < \infty$, and $t > 0$. If*

$$0 < m \leq \frac{f(s)}{g(s)} \leq M < \infty, \quad s \in [0, t].$$

Then for any $v > 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left[\nabla_r^{-v} f^p(t)\right]^{\frac{1}{p}} \left[\nabla_r^{-v} g^q(t)\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \nabla_r^{-v} \left[f(t)\, g(t)\right]. \tag{4.19}$$

Proof. Replacing $f(s)$ and $g(s)$ by $(f(s))^p$ and $(g(s))^q$, $s \in [0, t]$, $t > 0$, respectively, in Lemma 4.2, we obtain the required inequality (4.19). □

Theorem 4.1. *Let f be a positive function on $\mathbb{T}_{t_0}$, such that for all $t > 0$, $v > 0$ and $p > 1$,*

$$\nabla_r^{-v} f(t) \geq \left(\frac{t^{\underline{v}}}{\Gamma_r(v+1)}\right)^{p-1}. \tag{4.20}$$

Then, we have

$$\nabla_r^{-v} [f(t)]^p \geq \left[\nabla_r^{-v} f(t)\right]^{p-1}. \tag{4.21}$$

Proof. In view of Lemma 4.1, we can write

$$\nabla_r^{-v} [f(t)]^p = \nabla_r^{-v} \left(\frac{[f(t)]^p}{1^{p-1}}\right) \geq \frac{[\nabla_r^{-v} f(t)]^p}{\left[\nabla_r^{-v} 1\right]^{p-1}} = \left(\frac{\Gamma_r(v+1)}{t^{\underline{v}}}\right)^{p-1} \left[\nabla_r^{-v} f(t)\right]^p. \tag{4.22}$$

But from the condition (4.20), we have

$$\left(\frac{\Gamma_r(v+1)}{t^{\underline{v}}}\right)^{p-1} \geq \left[\nabla_r^{-v} f(t)\right]^{-1}. \tag{4.23}$$

Combining (4.22) and (4.23), we obtain (4.21). □

Theorem 4.2. *Let $v > 0$, $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let f be a positive functions on $\mathbb{T}_{t_0}$, such that $\nabla_r^{-v} f^p(t) < \infty$, $t > 0$. If*

$$0 < m \leq f^p(s) \leq M < \infty, \quad s \in [0, t]. \tag{4.24}$$

Then, we have

$$\left[\nabla_r^{-v} f^p(t)\right]^{\frac{1}{p}} \leq \left(\frac{M}{m}\right)^{\frac{2}{pq}} \left(\frac{t^{\underline{v}}}{\Gamma_r(v+1)}\right)^{-\frac{p+1}{q}} \left[\nabla_r^{-v} \left(f^{\frac{1}{p}}(t)\right)\right]^p. \tag{4.25}$$

Proof. Putting $g(s) = 1$ into lemma 4.3, we can write

$$\left[\nabla_r^{-v} f^p(t)\right]^{\frac{1}{p}} \left[\nabla_r^{-v} 1^q(t)\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \nabla_r^{-v} \left[f(t) \times 1\right],$$

or equivalently,

$$\left[\nabla_r^{-v} f^p(t)\right]^{\frac{1}{p}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\frac{t^{\underline{v}}}{\Gamma_r(v+1)}\right)^{-\frac{1}{q}} \nabla_r^{-v} f(t). \tag{4.26}$$

Now, put $g(s) = 1$ into lemma 4.2 to write

$$\left[\nabla_r^{-v} f(t)\right]^{\frac{1}{p}} \left[\nabla_r^{-v}(1)\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \nabla_r^{-v} \left[f^{\frac{1}{p}(t)}\right],$$

or equivalently,

$$\left[\nabla_r^{-v} f(t)\right]^{\frac{1}{p}} \leq \left(\frac{M}{m}\right)^{\frac{1}{p^2 q}} \left(\frac{t^{\underline{v}}}{\Gamma_r(v+1)}\right)^{-\frac{1}{q}} \nabla_r^{-v} \left[f^{\frac{1}{p}}(t)\right].$$

Obviously,

$$\left[\nabla_r^{-v} f(t)\right]^{\frac{1}{p}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\frac{t^{\underline{v}}}{\Gamma_r(v+1)}\right)^{-\frac{p}{q}} \left[\nabla_r^{-v} \left(f^{\frac{1}{p}}(t)\right)\right]^{p} \tag{4.27}$$

Combining (4.26) and (4.27), we obtain the inequality (4.25). □

Theorem 4.3. *Let f, g be two nonnegative functions on $\mathbb{T}_{t_0}$, such that g is non-decreasing. If*

$$\nabla_r^{-v} f(t) \geq \nabla_r^{-v} g(t), \quad t > 0. \tag{4.28}$$

Then for any $v > 0, \gamma > 0, \delta > 0$ and $\gamma - \delta \geq 1$, we have

$$\nabla_r^{-v}\left[f^{\gamma-\delta}(t)\right] \leq \nabla_r^{-v}\left[f^{\gamma}(t) g^{-\delta}(t)\right]. \tag{4.29}$$

Proof. Using the arithmetic-geometric inequality, for $\gamma > 0, \delta > 0$, we can write

$$\frac{\gamma}{\gamma-\delta} f^{\gamma-\delta}(s) - \frac{\delta}{\gamma-\delta} g^{\gamma-\delta}(s) \leq f^{\gamma}(s) g^{-\delta}(s), \quad s \in [0,t], t > 0. \tag{4.30}$$

Multiplying both side of (4.30) by $\frac{(t-rs)^{\underline{(v-1)}}}{\Gamma_r(v)}$, we have

$$\begin{aligned} &\frac{(t-rs)^{\underline{(v-1)}}}{\Gamma_r(v)} \frac{\gamma}{\gamma-\delta} f^{\gamma-\delta}(s) - \frac{(t-rs)^{\underline{(v-1)}}}{\Gamma_r(v)} \frac{\delta}{\gamma-\delta} g^{\gamma-\delta}(s) \\ &\leq \frac{(t-rs)^{\underline{(v-1)}}}{\Gamma_r(v)} f^{\gamma}(s) g^{-\delta}(s). \end{aligned} \tag{4.31}$$

Integrating both side of (4.31) with respect to s on $[0,t]$, we get

$$\begin{aligned} &\frac{\gamma}{\gamma-\delta} \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{\underline{(v-1)}} f^{\gamma-\delta}(s) \nabla s - \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{\underline{(v-1)}} g^{\gamma-\delta}(s) \nabla s \\ &\leq \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{\underline{(v-1)}} f^{\gamma}(s) g^{-\delta}(s) \nabla s. \end{aligned}$$

Obviously,

$$\frac{\gamma}{\gamma-\delta} \nabla_r^{-v}\left[f^{\gamma-\delta}(t)\right] - \frac{\delta}{\gamma-\delta} \nabla_r^{-v}\left[g^{\gamma-\delta}(t)\right] \leq \nabla_r^{-v}\left[f^{\gamma}(t) g^{-\delta}(t)\right].$$

This leads to

$$\frac{\gamma}{\gamma-\delta} \nabla_r^{-v}\left[f^{\gamma-\delta}(t)\right] \leq \nabla_r^{-v}\left[f^{\gamma}(t) g^{-\delta}(t)\right] + \frac{\delta}{\gamma-\delta} \nabla_r^{-v}\left[g^{\gamma-\delta}(t)\right].$$

This ends the proof. □

Theorem 4.4. *Let $v > 0$ and f, g be two positive functions on $\mathbb{T}_{t_0}$, such that f is non-decreasing and g is non-increasing. Then for any $t > 0, \gamma > 0, \delta > 0$, we have*

$$\nabla_r^{-v}\left[f^{\gamma}(t) g^{\delta}(t)\right] \leq \frac{\Gamma_r(v+1)}{t^{\underline{v}}} \nabla_r^{-v}\left[f^{\gamma}(t)\right] \nabla_r^{-v}\left[g^{\delta}(t)\right]. \tag{4.32}$$

Proof. For any $t > 0, \gamma > 0, \delta > 0$, we have

$$\left(f^{\gamma}(s) - f^{\gamma}(\rho)\right)\left(g^{\delta}(s) - g^{\delta}(\rho)\right) \geq 0, \quad s, \rho \in [0,t].$$

This may be written as

$$f^{\gamma}(s)g^{\delta}(\rho)+f^{\gamma}(\rho)g^{\delta}(s)\geq f^{\gamma}(\rho)g^{\delta}(\rho)+f^{\gamma}(s)g^{\delta}(s).$$

So,

$$\begin{aligned}&\nabla_r^{-v}\left[f^{\gamma}(t)g^{\delta}(t)\right]+\frac{t^{\underline{v}}}{\Gamma_r(v+1)}f^{\gamma}(\rho)g^{\delta}(\rho)\\&\leq g^{\delta}(\rho)\nabla_r^{-v}\left[f^{\gamma}(t)\right]+f^{\gamma}(\rho)\nabla_r^{-v}\left[g^{\delta}(t)\right].\end{aligned}\tag{4.33}$$

Multiplying both side of (4.33) by $\frac{(t-r\rho)^{\underline{(v-1)}}}{\Gamma_r(v)}$, $\rho\in(0,t)$, we get

$$\begin{aligned}&\frac{(t-r\rho)^{\underline{(v-1)}}}{\Gamma_r(v)}\nabla_r^{-v}\left[f^{\gamma}(t)g^{\delta}(t)\right]+\frac{t^{\underline{v}}}{\Gamma_r(v+1)}\frac{(t-r\rho)^{\underline{(v-1)}}}{\Gamma_r(v)}f^{\gamma}(\rho)g^{\delta}(\rho)\\&\leq\frac{(t-r\rho)^{\underline{(v-1)}}}{\Gamma_r(v)}g^{\delta}(\rho)\nabla_r^{-v}\left[f^{\gamma}(t)\right]+\frac{(t-r\rho)^{\underline{(v-1)}}}{\Gamma_r(v)}f^{\gamma}(\rho)\nabla_r^{-v}\left[g^{\delta}(t)\right].\end{aligned}\tag{4.34}$$

Integrating both side of (4.34) with respect to ρ on $[0,t]$, we obtain

$$\begin{aligned}&\frac{t^{\underline{v}}}{\Gamma_r(v+1)}\nabla_r^{-v}\left[f^{\gamma}(t)g^{\delta}(t)\right]+\frac{t^{\underline{v}}}{\Gamma_r(v+1)}\nabla_r^{-v}\left[f^{\gamma}(t)g^{\delta}(t)\right]\\&\leq\nabla_r^{-v}\left[g^{\delta}(t)\right]\nabla_r^{-v}\left[f^{\gamma}(t)\right]+\nabla_r^{-v}\left[f^{\gamma}(t)\right]\nabla_r^{-v}\left[g^{\delta}(t)\right],\end{aligned}$$

which implies (4.32). □

Theorem 4.5. *Let $v>0$ and f,g be two positive functions on $\mathbb{T}_{t_0}$, such that f is non-decreasing and g is non-increasing. Then for any $t>0,\gamma>0,\delta>0$, we have*

$$\begin{aligned}&\frac{t^{\underline{v}}}{\Gamma_r(v+1)}\nabla_r^{-v}\left[f^{\gamma}(t)g^{\delta}(t)\right]+\frac{t^{\underline{\omega}}}{\Gamma_r(\omega+1)}\nabla_r^{-\omega}\left[f^{\gamma}(t)g^{\delta}(t)\right]\\&\leq\nabla_r^{-v}\left[f^{\gamma}(t)\right]\nabla_r^{-\omega}\left[g^{\delta}(t)\right]+\nabla_r^{-\omega}\left[f^{\gamma}(t)\right]\nabla_r^{-v}\left[g^{\delta}(t)\right].\end{aligned}\tag{4.35}$$

Proof. Multiplying both side of (4.33) by $\frac{(t-r\rho)^{\underline{(\omega-1)}}}{\Gamma_r(\omega)}$, $\rho\in(0,t)$, we get

$$\begin{aligned}&\frac{(t-r\rho)^{\underline{(\omega-1)}}}{\Gamma_r(\omega)}\nabla_r^{-v}\left[f^{\gamma}(t)g^{\delta}(t)\right]+\frac{t^{\underline{\omega}}}{\Gamma_r(\omega+1)}\frac{(t-r\rho)^{\underline{(\omega-1)}}}{\Gamma_r(\omega)}f^{\gamma}(\rho)g^{\delta}(\rho)\\&\leq\frac{(t-r\rho)^{\underline{(\omega-1)}}}{\Gamma_r(\omega)}g^{\delta}(\rho)\nabla_r^{-v}\left[f^{\gamma}(t)\right]+\frac{(t-r\rho)^{\underline{(\omega-1)}}}{\Gamma_r(\omega)}f^{\gamma}(\rho)\nabla_r^{-v}\left[g^{\delta}(t)\right].\end{aligned}\tag{4.36}$$

Integrating both side of (4.36) with respect to ρ on $[0,t]$, we obtain

$$\begin{aligned}&\nabla_r^{-v}\left[f^{\gamma}(t)g^{\delta}(t)\right]\int_0^1\frac{(t-r\rho)^{\underline{(\omega-1)}}}{\Gamma_r(\omega)}\nabla\rho+\frac{t^{\underline{\omega}}}{\Gamma_r(\omega+1)}\int_0^1\frac{(t-r\rho)^{\underline{(\omega-1)}}}{\Gamma_r(\omega)}f^{\gamma}(\rho)g^{\delta}(\rho)\nabla\rho\\&\leq\nabla_r^{-v}\left[f^{\gamma}(t)\right]\int_0^1\frac{(t-r\rho)^{\underline{(\omega-1)}}}{\Gamma_r(\omega)}g^{\delta}(\rho)\nabla\rho+\nabla_r^{-v}\left[g^{\delta}(t)\right]\int_0^1\frac{(t-r\rho)^{\underline{(\omega-1)}}}{\Gamma_r(\omega)}f^{\gamma}(\rho)\nabla\rho,\end{aligned}$$

and this ends the proof. □

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Received: October 10, 2015; *Accepted*: January 27, 2016

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 100–110

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Some new integral inequalities for k-fractional integrals

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Abstract

The aim of the present paper is to investigate some new integral inequalities for k-fractional integrals. Moreover, special cases of the integral inequalities in this paper have been obtained by Tariboon *et.al.* in [22].

Keywords: k-Reimann-Liouville calculus; fractional integral inequalities; Grüss inequality.

2010 MSC: 26A33, 26D10, 26D15.

1 Introduction and Preliminaries

Integration with weight functions is used in countless mathematical problems such as approximation theory, spectral analysis, statistical analysis and the theory of distributions. Grüss developed an integral inequality [11, p. 236] in 1935. During the last few years, many researchers focused their attention on the study and generalizations of the Grüss inequality [7–9, 14, 18]. The integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals is known in the literature as the Grüss inequality. The Grüss inequality is as follows:

Theorem 1.1. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions such that $\varphi < f(x) < \Phi$ and $\psi < g(x) < \Psi$ for all $x \in [a, b]$, where $\varphi, \Phi, \psi, \Psi$ are constants. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \frac{1}{b-a} \int_a^b g(x) \, dx \right| \leq \frac{1}{4} (\Phi - \varphi)(\Psi - \psi),$$

where the constant $\frac{1}{4}$ is sharp. (see, [11, p. 236])

Fractional calculus and its widely applications have recently been paid more and more attention. For more recent development on fractional calculus, we refer the reader to [1–4, 10, 16, 19, 20, 24]. There are several known forms of the fractional integrals which have been studied extensively for their applications [5, 13, 15, 21, 23].

The first is the Riemann-Liouville fractional integral of order $\alpha \geq 0$ for a continuous function f on $[a, b]$ which is defined by

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \; \alpha \geq 0, \; a \leq t \leq b.$$

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This integral is motivated by the well known Cauchy formula:

$$\int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n)dt_n = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt;\ n \in \mathbb{N}.$$

The second is the Hadamard fractional integral introduced by Hadamard [12]. It is given by:

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t}, \alpha > 0,\ x > a > 0.$$

The Hadamard integral is based on the generalization of the integral

$$\int_a^x \frac{dt_1}{t_1} \int_a^{t_1} \frac{dt_2}{t_2} \cdots \int_a^{t_{n-1}} \frac{f(t_n)}{t_n} dt_n = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{n-1} f(t)\frac{dt}{t}$$

for $n \in \mathbb{N}$.

Recently, in [6], Diaz and Pariguan have defined new functions called k-gamma and k-beta functions and the Pochhammer k-symbol, that is respectively generalization of the classical gamma and beta functions and the classical Pochhammer symbol:

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n\,(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ (k>0)\,,$$

where $(x)_{n,k}$ is the Pochhammer k-symbol for factorial function defined by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+nk),\ \ k \in \mathbb{R},\ n \in \mathbb{N}.$$

It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, explicitly given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt,\ x > 0.$$

Clearly,

$$\Gamma(x) = \lim_{k\longrightarrow 1} \Gamma_k(x),\ \Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \text{ and } \Gamma_k(x+k) = x\Gamma_k(x).$$

Later, under the above definitions, in [17], Mubeen and Habibullah have introduced the k-fractional integral of the Riemann-Liouville type as follows:

$$_kJ_a^\alpha f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt;\ \alpha > 0,\ x > a. \tag{1.1}$$

Note that when $k \to 1$, then it reduces to the classical Riemann-liouville fractional integral.

Recently in [22] some new fractional integral inequalities of Grüss type were proved, by replacing the constants appeared as bounds of the functions f and g, by four integrable functions. In this paper we extend the results of [22] to k-fractional integral inequalities of Grüss type.

2 Main results

Throughout of this paper, we denote the Riemann-Liouville fractional integral of order α of a function f which have limit zero by $_kJ_0^\alpha f(t) = {}_kJ^\alpha f(t)$

Theorem 2.1. *Let f be an integrable function on $[0,\infty)$. Assume that:*

(H_1) *There exist two integrable functions φ_1, φ_2 on $[0,\infty)$ such that*

$$\varphi_1(t) \le f(t) \le \varphi_2(t) \quad \text{for all} \quad t \in [0,\infty).$$

Then, for $t > 0$, $\alpha, \beta > 0$, $k > 0$, we have

$$_kJ^\beta \varphi_1(t)_kJ^\alpha f(t) + {}_kJ^\alpha \varphi_2(t)_kJ^\beta f(t) \ge {}_kJ^\alpha \varphi_2(t)_kJ^\beta \varphi_1(t) + {}_kJ^\alpha f(t)_kJ^\beta f(t). \tag{2.2}$$

Proof. From (H_1), for all $\tau \geq 0, \rho \geq 0$, we have

$$(\varphi_2(\tau) - f(\tau))\,(f(\rho) - \varphi_1(\rho)) \geq 0.$$

Therefore

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \tag{2.3}$$

Multiplying both sides of (2.3) by $(t-\tau)^{\frac{\alpha}{k}-1}/k\Gamma_k(\alpha)$, $\tau \in (0,t)$, we get

$$f(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau) + \varphi_1(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau) \geq \varphi_1(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau) + f(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau). \tag{2.4}$$

Integrating both sides of (2.4) with respect to τ on $(0,t)$, we obtain

$$\begin{aligned} f(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau)d\tau + \varphi_1(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau)d\tau \\ \geq \varphi_1(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau)d\tau + f(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau)d\tau, \end{aligned}$$

which yields

$$f(\rho)\,{}_kJ^\alpha\varphi_2(t) + \varphi_1(\rho)\,{}_kJ^\alpha f(t) \geq \varphi_1(\rho)\,{}_kJ^\alpha\varphi_2(t) + f(\rho)\,{}_kJ^\alpha f(t). \tag{2.5}$$

Multiplying both sides of (2.5) by $(t-\rho)^{\frac{\beta}{k}-1}/k\Gamma_k(\beta)$, $\rho \in (0,t)$, we have

$$\begin{aligned} J^\alpha\varphi_2(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}f(\rho) + J^\alpha f(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\varphi_1(\rho) \\ \geq J^\alpha\varphi_2(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\varphi_1(\rho) + J^\alpha f(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}f(\rho). \end{aligned} \tag{2.6}$$

Integrating both sides of (2.6) with respect to ρ on $(0,t)$, we get

$$\begin{aligned} {}_kJ^\alpha\varphi_2(t)\int_0^t \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}f(\rho)d\rho + J^\alpha f(t)\int_0^t \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\varphi_1(\rho)d\rho \\ \geq {}_kJ^\alpha\varphi_2(t)\int_0^t \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\varphi_1(\rho)d\rho + {}_kJ^\alpha f(t)\int_0^t \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}f(\rho)d\rho. \end{aligned}$$

Hence, we deduce inequality (2.2) as requested. This completes the proof. □

As special cases, we give the following results for the Theorem 2.1.

Remark 2.1. *If we take $k=1$ in the Theorem 2.1, we obtain the Theorem 2 in [22].*

Corollary 2.1. *If we take $\alpha=\beta$ in the Theorem 2.1, we obtain*

$${}_kJ^\alpha\,(\varphi_1+\varphi_2)\,(t)\,{}_kJ^\alpha f(t) \geq {}_kJ^\alpha\varphi_1(t)\,{}_kJ^\alpha\varphi_2(t) + \left({}_kJ^\alpha f(t)\right)^2.$$

Corollary 2.2. *Let f be an integrable function on $[0,\infty)$ satisfying $m \leq f(t) \leq M$, for all $t \in [0,\infty)$ and $m, M \in \mathbb{R}$. Then for $t>0$ and $\alpha,\beta>0$, we have*

$$m\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^\alpha f(t) + M\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\beta f(t) \geq mM\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + {}_kJ^\alpha f(t)\,{}_kJ^\beta f(t).$$

Corollary 2.3. *If we take $\alpha=\beta$ in the Corollary 2.2, we obtain*

$$(m+M)\,\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha f(t) \geq mM\frac{t^{\frac{2\alpha}{k}}}{(\Gamma_k(\alpha+k))^2} + \left({}_kJ^\alpha f(t)\right)^2.$$

Theorem 2.2. *Let f be an integrable function on $[0,\infty)$ and constants $\theta_1,\theta_2>0$ satisfying $1/\theta_1+1/\theta_2=1$. Suppose that (H_1) holds. Then, for $t>0$, $\alpha,\beta>0$ and $k>0$, we have*

$$\begin{aligned}\frac{1}{\theta_1}\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}\left((\varphi_2-f)^{\theta_1}\right)(t)+\frac{1}{\theta_2}\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}\left((f-\varphi_1)^{\theta_2}\right)(t)\\+{}_kJ^{\alpha}\varphi_2(t){}_kJ^{\beta}\varphi_1(t)+{}_kJ^{\alpha}f(t){}_kJ^{\beta}f(t)\\\geq{}_kJ^{\alpha}\varphi_2(t){}_kJ^{\beta}f(t)+{}_kJ^{\alpha}f(t){}_kJ^{\beta}\varphi_1(t).\end{aligned}\tag{2.7}$$

Proof. According to the well-known Young's inequality

$$\frac{1}{\theta_1}x^{\theta_1}+\frac{1}{\theta_2}y^{\theta_2}\geq xy,\qquad\forall x,y\geq0,\;\;\theta_1,\theta_2>0,\;\;\frac{1}{\theta_1}+\frac{1}{\theta_2}=1,$$

setting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho)$, $\tau,\rho>0$, we have

$$\frac{1}{\theta_1}(\varphi_2(\tau)-f(\tau))^{\theta_1}+\frac{1}{\theta_2}(f(\rho)-\varphi_1(\rho))^{\theta_2}\geq(\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho)).\tag{2.8}$$

Multiplying both sides of (2.8) by $\dfrac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$, $\tau,\rho\in(0,t)$, we get

$$\begin{aligned}&\frac{1}{\theta_1}\frac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}(\varphi_2(\tau)-f(\tau))^{\theta_1}+\frac{1}{\theta_2}\frac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}(f(\rho)-\varphi_2(\rho))^{\theta_2}\\\geq\;&\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}(\varphi_2(\tau)-f(\tau))\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}(f(\rho)-\varphi_1(\rho)).\end{aligned}$$

Integrating the above inequality with respect to τ and ρ from 0 to t, we have

$$\frac{1}{\theta_1}{}_kJ^{\beta}(1)(t){}_kJ^{\alpha}(\varphi_2-f)^{\theta_1}(t)+\frac{1}{\theta_2}{}_kJ^{\alpha}(1)(t){}_kJ^{\beta}(f-\varphi_1)^{\theta_2}(t)\geq{}_kJ^{\alpha}(\varphi_2-f)(t){}_kJ^{\beta}(f-\varphi_1)(t),$$

which implies (2.7). □

Corollary 2.4. *Let f be an integrable function on $[0,\infty)$ satisfying $m\leq f(t)\leq M$, for all $t\in[0,\infty)$ and $m,M\in\mathbb{R}$. Then for $t>0$, $\alpha,\beta>0$ and $k>0$, we have*

$$\begin{aligned}&(m+M)^2\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}f^2(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}f^2(t)+2{}_kJ^{\alpha}f(t){}_kJ^{\beta}f(t)\\\geq\;&2(m+M)\left(\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}f(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}f(t)\right).\end{aligned}$$

Theorem 2.3. *Let f be an integrable function on $[0,\infty)$ and constants $\theta_1,\theta_2>0$ satisfying $\theta_1+\theta_2=1$. In addition, suppose that (H_1) holds. Then, for $t>0$, $\alpha,\beta>0$ and $k>0$, we have*

$$\begin{aligned}&\theta_1\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}\varphi_2(t)+\theta_2\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}f(t)\\&\quad\geq{}_kJ^{\alpha}(\varphi_2-f)^{\theta_1}(t){}_kJ^{\beta}(f-\varphi_1)^{\theta_2}(t)+\theta_1\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}f(t)\\&\quad+\theta_2\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}\varphi_1(t).\end{aligned}\tag{2.9}$$

Proof. From the well-known Weighted AM-GM inequality

$$\theta_1x+\theta_2y\geq x^{\theta_1}y^{\theta_2},\qquad\forall x,y\geq0,\;\;\theta_1,\theta_2>0,\;\;\theta_1+\theta_2=1,$$

and setting $x = \varphi_2(\tau) - f(\tau)$ and $y = f(\rho) - \varphi_1(\rho)$, $\tau, \rho > 0$, we have

$$\theta_1(\varphi_2(\tau) - f(\tau)) + \theta_2(f(\rho) - \varphi_1(\rho)) \geq (\varphi_2(\tau) - f(\tau))^{\theta_1}(f(\rho) - \varphi_1(\rho))^{\theta_2}. \tag{2.10}$$

Multiplying both sides of (2.10) by $\dfrac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$, $\tau, \rho \in (0,t)$, we get

$$\begin{aligned}\theta_1\frac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}(\varphi_2(\tau)-f(\tau)) + \theta_2\frac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}(f(\rho)-\varphi_2(\rho)) \\ \geq \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}(\varphi_2(\tau)-f(\tau))^{\theta_1}\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}(f(\rho)-\varphi_1(\rho))^{\theta_2}.\end{aligned}$$

Integrating the above inequality with respect to τ and ρ from 0 to t, we have

$$\begin{aligned}\theta_{1k}J^{\beta}(1)(t)_kJ^{\alpha}(\varphi_2 - f)(t) + \theta_{2k}J^{\alpha}(1)(t)_kJ^{\beta}(f-\varphi_1)(t) \\ \geq {}_kJ^{\alpha}(\varphi_2 - f)^{\theta_1}(t)_kJ^{\beta}(f-\varphi_1)^{\theta_2}(t).\end{aligned}$$

Therefore, we deduce inequality (2.9). □

Corollary 2.5. *Let f be an integrable function on $[0,\infty)$ satisfying $m \leq f(t) \leq M$, for all $t \in [0,\infty)$ and $m, M \in \mathbb{R}$. Then for $t > 0$, $\alpha, \beta > 0$ and $k > 0$, we have*

$$\begin{aligned}M\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}f(t) \quad &\geq \quad m\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + \frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}f(t) \\ &+2{}_kJ^{\alpha}(M-f)^{\frac{1}{2}}(t)_kJ^{\beta}(f-m)^{\frac{1}{2}}(t).\end{aligned}$$

Theorem 2.4. *Let f and g be two integrable functions on $[0,\infty)$. Suppose that (H_1) holds and moreover we assume that:*

(H_2) *There exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that*

$$\psi_1(t) \leq g(t) \leq \psi_2(t) \quad \textit{for all} \quad t \in [0,\infty).$$

Then, for $t > 0$, α, $\beta > 0$, $k > 0$ the following inequalities hold:

(a) ${}_kJ^{\beta}\psi_1(t)\,{}_kJ^{\alpha}f(t) + {}_kJ^{\alpha}\varphi_2(t)\,{}_kJ^{\beta}g(t) \geq {}_kJ^{\beta}\psi_1(t)\,{}_kJ^{\alpha}\varphi_2(t) + {}_kJ^{\alpha}f(t)\,{}_kJ^{\beta}g(t)$.

(b) ${}_kJ^{\beta}\varphi_1(t)\,{}_kJ^{\alpha}g(t) + {}_kJ^{\alpha}\psi_2(t)\,{}_kJ^{\beta}f(t) \geq {}_kJ^{\beta}\varphi_1(t)\,{}_kJ^{\alpha}\psi_2(t) + {}_kJ^{\beta}f(t)\,{}_kJ^{\alpha}g(t)$.

(c) ${}_kJ^{\alpha}\varphi_2(t)\,{}_kJ^{\beta}\psi_2(t) + {}_kJ^{\alpha}f(t)\,{}_kJ^{\beta}g(t) \geq {}_kJ^{\alpha}\varphi_2(t)\,{}_kJ^{\beta}g(t) + {}_kJ^{\beta}\psi_2(t)\,{}_kJ^{\alpha}f(t)$.

(d) ${}_kJ^{\alpha}\varphi_1(t)J^{\beta}\psi_1(t) + {}_kJ^{\alpha}f(t)\,{}_kJ^{\beta}g(t) \geq {}_kJ^{\alpha}\varphi_1(t)\,{}_kJ^{\beta}g(t) + {}_kJ^{\beta}\psi_1(t)\,{}_kJ^{\alpha}f(t)$.

Proof. To prove (a), from (H_1) and (H_2), we have for $t \in [0,\infty)$ that

$$(\varphi_2(\tau) - f(\tau))\,(g(\rho) - \psi_1(\rho)) \geq 0.$$

Therefore

$$\varphi_2(\tau)g(\rho) + \psi_1(\rho)f(\tau) \geq \psi_1(\rho)\varphi_2(\tau) + f(\tau)g(\rho). \tag{2.11}$$

Multiplying both sides of (2.11) by $(t-\tau)^{\frac{\alpha}{k}-1}/k\Gamma_k(\alpha)$, $\tau \in (0,t)$, we get

$$\begin{aligned}g(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau) + \psi_1(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau) \\ \geq \psi_1(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau) + g(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau).\end{aligned} \tag{2.12}$$

Integrating both sides of (2.12) with respect to τ on $(0,t)$, we obtain

$$g(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau)d\tau+\psi_1(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau)d\tau$$
$$\geq \psi_1(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{k\Gamma(\alpha)}\varphi_2(\tau)d\tau+g(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{k\Gamma(\alpha)}f(\tau)d\tau.$$

Then we have

$$g(\rho)_kJ^\alpha\varphi_2(t)+\psi_1(\rho)_kJ^\alpha f(t)\geq \psi_1(\rho)_kJ^\alpha\varphi_2(t)+g(\rho)_kJ^\alpha f(t). \tag{2.13}$$

Multiplying both sides of (2.13) by $(t-\rho)^{\frac{\beta}{k}-1}/k\Gamma_k(\beta)$, $\rho\in(0,t)$, we have

$$\begin{aligned}&{}_kJ^\alpha\varphi_2(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}g(\rho)+{}_kJ^\alpha f(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\psi_1(\rho)\\ &\geq {}_kJ^\alpha\varphi_2(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\psi_1(\rho)+{}_kJ^\alpha f(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}g(\rho).\end{aligned} \tag{2.14}$$

Integrating both sides of (2.14) with respect to ρ on $(0,t)$, we get the desired inequality (a).

To prove (b)-(d), we use the following inequalities
(b) $(\psi_2(\tau)-g(\tau))(f(\rho)-\varphi_1(\rho))\geq 0.$
(c) $(\varphi_2(\tau)-f(\tau))(g(\rho)-\psi_2(\rho))\leq 0.$
(d) $(\varphi_1(\tau)-f(\tau))(g(\rho)-\psi_1(\rho))\leq 0.$ □

Remark 2.2. *If we take $k=1$ in the Theorem 2.4, we obtain the Theorem 5 in [22].*

As a special case of Theorem 2.4, we have the following Corollary.

Corollary 2.6. *Let f and g be two integrable functions on $[0,\infty)$. Assume that:*

(H_3) *There exist real constants m, M, n, N such that*

$$m\leq f(t)\leq M \quad \textit{and} \quad n\leq g(t)\leq N \quad \textit{for all} \quad t\in[0,\infty).$$

Then, for $t>0$, α, $\beta>0$, $k>0$ we have

(a_1) $$\frac{nt^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^\alpha f(t)+\frac{Mt^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\beta g(t)\geq \frac{nMt^{\frac{\alpha+\beta}{K}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+{}_kJ^\alpha f(t)_kJ^\beta g(t).$$

(b_1) $$\frac{mt^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^\alpha g(t)+\frac{Nt^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\beta f(t)\geq \frac{mNt^{\frac{\alpha+\beta}{k}}}{\Gamma_K(\alpha+k)\Gamma_k(\beta+1)}+{}_kJ^\beta f(t)_kJ^\alpha g(t).$$

(c_1) $$\frac{MNt^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+{}_kJ^\alpha f(t)_kJ^\beta g(t)\geq \frac{Mt^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\beta g(t)+\frac{Nt^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^\alpha f(t).$$

(d_1) $$\frac{mnt^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+{}_kJ^\alpha f(t)_kJ^\beta g(t)\geq \frac{mt^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\beta g(t)+\frac{nt^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^\alpha f(t).$$

Theorem 2.5. *Let f and g be two positive integrable functions on $[0,\infty)$. Assume that (H_1) and (H_2) hold with*

(H_4) *$\varphi_1(t)>0$ and $\psi_1(t)>0$ for all $t\in[0,\infty)$.*

Then, for $t>0$, α, $\beta>0$ and $k>0$ the following inequalities holds

$$\frac{{}_kJ^\alpha\varphi_1\varphi_2(t)_kJ^\beta\psi_1\psi_2(t)_kJ^\alpha f^2(t)_kJ^\beta g^2(t)}{\left({}_kJ^\alpha\varphi_1 f(t)_kJ^\beta\psi_1 g(t)+{}_kJ^\alpha\varphi_2 f(t)_kJ^\beta\psi_2 g(t)\right)^2}\leq\frac{1}{4}. \tag{2.15}$$

Proof. To prove (2.15), using the conditions (H_1)-(H_3), we obtain

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\rho)} - \frac{f(\tau)}{g(\rho)}\right) \geq 0,$$

and

$$\left(\frac{f(\tau)}{g(\rho)} - \frac{\varphi_1(\tau)}{\psi_2(\rho)}\right) \geq 0,$$

which imply that

$$\left(\frac{\varphi_1(\tau)}{\psi_2(\rho)} + \frac{\varphi_2(\tau)}{\psi_1(\rho)}\right)\frac{f(\tau)}{g(\rho)} \geq \frac{f^2(\tau)}{g^2(\rho)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\rho)\psi_2(\rho)}. \tag{2.16}$$

Multiplying both sides of (2.16) by $\psi_1(\rho)\psi_2(\rho)g^2(\rho)$, we have

$$\varphi_1(\tau)f(\tau)\psi_1(\rho)g(\rho) + \varphi_2(\tau)f(\tau)\psi_2(\rho)g(\rho) \geq \psi_1(\rho)\psi_2(\rho)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\rho). \tag{2.17}$$

Multiplying both sides of (2.17) by $\dfrac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$, and double integrating with respect to τ and ρ from 0 to t, we have

$${}_kJ^\alpha\varphi_1 f(t)\,{}_kJ^\beta\psi_1 g(t) + {}_kJ^\alpha\varphi_2 f(t)\,{}_kJ^\beta\psi_2 g(t) \geq {}_kJ^\alpha f^2(t)\,{}_kJ^\beta\psi_1\psi_2(t) + {}_kJ^\alpha\varphi_1\varphi_2(t)\,{}_kJ^\beta g^2(t).$$

Applying the AM-GM inequality, we get

$${}_kJ^\alpha\varphi_1 f(t)\,{}_kJ^\beta\psi_1 g(t) + {}_kJ^\alpha\varphi_2 f(t)\,{}_kJ^\beta\psi_2 g(t) \geq 2\sqrt{{}_kJ^\alpha f^2(t)\,{}_kJ^\beta\psi_1\psi_2(t)\,{}_kJ^\alpha\varphi_1\varphi_2(t)\,{}_kJ^\beta g^2(t)},$$

which leads to the desired inequality in (2.15). The proof is completed. □

As a special case of Theorem 2.5, we get the following result:

Corollary 2.7. *Let f and g be two positive integrable functions on $[0,\infty)$ satisfying (H_3) with $m,n > 0$. Then for $t > 0$, $\alpha,\beta > 0$ and $k > 0$, we have*

$$\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}\frac{{}_kJ^\alpha f^2(t)\,{}_kJ^\beta g^2(t)}{\left({}_kJ^\alpha f(t)\,{}_kJ^\beta g(t)\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}}\right)^2.$$

Lemma 2.1. *Let f be an integrable function on $[0,\infty)$ and φ_1, φ_2 are two integrable functions on $[0,\infty)$. Assume that the condition (H_1) holds. Then, for $t > 0$, $\alpha > 0$, $k > 0$, we have*

$$\begin{aligned}
\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha f^2(t) - \left({}_kJ^\alpha f(t)\right)^2 \;=\; & \left({}_kJ^\alpha\varphi_2(t) - {}_kJ^\alpha f(t)\right)\left({}_kJ^\alpha f(t) - {}_kJ^\alpha\varphi_1(t)\right)\\
& -\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\left((\varphi_2(t) - f(t))(f(t) - \varphi_1(t))\right)\\
& +\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\varphi_1 f(t) - {}_kJ^\alpha\varphi_1(t)\,{}_kJ^\alpha f(t)\\
& +\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\varphi_2 f(t) - {}_kJ^\alpha\varphi_2(t)\,{}_kJ^\alpha f(t)\\
& + {}_kJ^\alpha\varphi_1(t)\,{}_kJ^\alpha\varphi_2(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\varphi_1\varphi_2(t).
\end{aligned} \tag{2.18}$$

Proof. For any $\tau > 0$ and $\rho > 0$, we have

$$\begin{aligned}
&(\varphi_2(\rho) - f(\rho))(f(\tau) - \varphi_1(\tau)) + (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho))\\
&+(\varphi_2(\tau) - f(\tau))(f(\tau) - \varphi_1(\tau)) - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho))\\
&= f^2(\tau) + f^2(\rho) - 2f(\tau)f(\rho) + \varphi_2(\rho)f(\tau) + \varphi_1(\tau)f(\rho) - \varphi_1(\tau)\varphi_2(\rho)\\
&+\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) - \varphi_1(\rho)\varphi_2(\tau) - \varphi_2(\tau)f(\tau) + \varphi_1(\tau)\varphi_2(\tau)\\
&\varphi_1(\tau)f(\tau) - \varphi_2(\rho)f(\rho) + \varphi_1(\rho)\varphi_2(\rho) - \varphi_1(\rho)f(\rho).
\end{aligned} \tag{2.19}$$

Multiplying (2.19) by $(t-\tau)^{\frac{\alpha}{k}-1}/k\Gamma_k(\alpha)$, $\tau\in(0,t)$, $t>0$ and integrating the resulting identity with respect to τ from 0 to t, we get

$$\begin{aligned}
&(\varphi_2(\rho)-f(\rho))\left({}_kJ^\alpha f(t)-{}_kJ^\alpha\varphi_1(t)\right)+\left({}_kJ^\alpha\varphi_2(t)-{}_kJ^\alpha f(t)\right)(f(\rho)-\varphi_1(\rho))\\
&+{}_kJ^\alpha\left((\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\right)-(\varphi_2(\rho)-f(\rho))(f(\rho)-\varphi_1(\rho))\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&=\ {}_kJ^\alpha f^2(t)+f^2(\rho)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}-2f(\rho)_kJ^\alpha f(t)+\varphi_2(\rho)_kJ^\alpha f(t)+f(\rho)_kJ^\alpha\varphi_1(t)\\
&+\varphi_2(\rho)_kJ^\alpha\varphi_1(t)+f(\rho)_kJ^\alpha\varphi_2(t)+\varphi_1(\rho)_kJ^\alpha f(t)-\varphi_1(\rho)_kJ^\alpha\varphi_2(t)-{}_kJ^\alpha\varphi_2 f(t)\\
&+{}_kJ^\alpha\varphi_1\varphi_2(t)-{}_kJ^\alpha\varphi_1 f(t)-\varphi_2(\rho)f(\rho)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}+\varphi_1(\rho)\varphi_2(\rho)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&+\varphi_1(\rho)f(\rho)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}.
\end{aligned}\tag{2.20}$$

Multiplying (2.20) by $(t-\rho)^{\frac{\alpha}{k}-1}/k\Gamma_k(\alpha)$, $\rho\in(0,t)$, $t>0$ and integrating the resulting identity with respect to ρ from 0 to t, we have

$$\begin{aligned}
&\left({}_kJ^\alpha\varphi_2(t)-{}_kJ^\alpha f(t)\right)\left({}_kJ^\alpha f(t)-{}_kJ^\alpha\varphi_1(t)\right)\\
&+\left({}_kJ^\alpha\varphi_2(t)-{}_kJ^\alpha f(t)\right)\left({}_kJ^\alpha f(t)-{}_kJ^\alpha\varphi_1(t)\right)\\
&{}_kJ^\alpha\left((\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\right)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&{}_kJ^\alpha\left((\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\right)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&=\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha f^2(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha f^2(t)-2{}_kJ^\alpha f(t)_kJ^\alpha f(t)+{}_kJ^\alpha\varphi_2(t)_kJ^\alpha f(t)\\
&+{}_kJ^\alpha\varphi_1(t)_kJ^\alpha f(t)-{}_kJ^\alpha\varphi_1(t)_kJ^\alpha\varphi_2(t)+{}_kJ^\alpha\varphi_2(t)_kJ^\alpha f(t)+{}_kJ^\alpha\varphi_1(t)_kJ^\alpha f(t)\\
&+{}_kJ^\alpha\varphi_1(t)_kJ^\alpha\varphi_2(t)-\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\varphi_2 f(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\varphi_1\varphi_2(t)\\
&+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^\alpha\varphi_1 f(t)-\frac{t^{\alpha}}{\Gamma_k(\alpha+1)}\,{}_kJ^\alpha\varphi_2 f(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\varphi_1\varphi_2(t)\\
&+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^\alpha\varphi_1 f(t),
\end{aligned}\tag{2.21}$$

which implies (2.18). □

Remark 2.3. *If we take $k=1$ in the Lemma 2.1, we obtain the Lemma 7 in [22].*

If $\varphi_1(t)\equiv m$ and $\varphi_2(t)\equiv M$, $m,M\in\mathbb{R}$, for all $t\in[0,\infty)$, then inequality (2.18) reduces to the following corollary.

Corollary 2.8. *Let f be an integrable function on $[0,\infty)$ satisfying $m\le f(t)\le M$, for all $t\in[0,\infty)$. The for all $t>0$, $\alpha>0$ we have*

$$\begin{aligned}
\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha f^2(t)-\left({}_kJ^\alpha f(t)\right)^2 &=\left(M\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}-{}_kJ^\alpha f(t)\right)\left({}_kJ^\alpha f(t)-m\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right)\\
&\quad-\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\left((M-f(t))(f(t)-m)\right).
\end{aligned}\tag{2.22}$$

Remark 2.4. *If we take $k=1$ in the Corollary 2.8, we obtain the Corollary 8 in [22].*

Theorem 2.6. *Let f and g be two integrable functions on $[0,\infty)$ and φ_1, φ_2, ψ_1 and ψ_2 are four integrable functions on $[0,\infty)$ satisfying the conditions (H_1) and (H_2) on $[0,\infty)$. Then for all $t>0$, $\alpha>0$,$k>0$, we have*

$$\left|\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha fg(t)-{}_kJ^\alpha f(t)\,{}_kJ^\alpha g(t)\right|\le\sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)},\tag{2.23}$$

where $T(u,v,w)$ is defined by

$$\begin{aligned} T(u,v,w) &= ({}_kJ^{\alpha}w(t) - {}_kJ^{\alpha}u(t))({}_kJ^{\alpha}u(t) - {}_kJ^{\alpha}v(t)) + \tfrac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}vu(t) - {}_kJ^{\alpha}v(t)\, {}_kJ^{\alpha}u(t) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} J^{\alpha}wu(t) - {}_kJ^{\alpha}w(t)\, {}_kJ^{\alpha}u(t) + {}_kJ^{\alpha}v(t)\, {}_kJ^{\alpha}w(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}vw(t). \end{aligned} \tag{2.24}$$

Proof. Let f and g be two integrable functions defined on $[0,\infty)$ satisfying (H_1) and (H_2). Define

$$H(\tau,\rho) := (f(\tau) - f(\rho))(g(\tau) - g(\rho)), \quad \tau,\rho \in (0,t),\ t > 0.$$

Multiplying both sides of (2.24) by $(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1}/k^2\Gamma_k^2(\alpha)$, $\tau,\rho \in (0,t)$ and integrating the resulting identity with respect to τ and ρ from 0 to t, we can state that

$$\frac{1}{k^2\Gamma_k^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1} H(\tau,\rho)d\tau d\rho = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}fg(t) - {}_kJ^{\alpha}f(t)\, {}_kJ^{\alpha}g(t). \tag{2.25}$$

Applying the Cauchy-Schwarz inequality to (2.25), we have

$$\begin{aligned} &\left(\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}fg(t) - {}_kJ^{\alpha}f(t)\, {}_kJ^{\alpha}g(t)\right)^2 \\ &\le \left(\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}f^2(t) - ({}_kJ^{\alpha}f(t))^2\right)\left(\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}g^2(t) - ({}_kJ^{\alpha}g(t))^2\right). \end{aligned} \tag{2.26}$$

Since $(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \ge 0$ and $(\psi_2(t) - g(t))(g(t) - \psi_1(t)) \ge 0$ for $t \in [0,\infty)$, we have

$$\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\left((\varphi_2(t) - f(t))(f(t) - \varphi_1(t))\right) \ge 0$$

and

$$\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\left((\psi_2(t) - g(t))(g(t) - \psi_1(t))\right) \ge 0.$$

Thus, from Lemma 2.1, we get

$$\begin{aligned} \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}f^2(t) - ({}_kJ^{\alpha}f(t))^2 &\le ({}_kJ^{\alpha}\varphi_2(t) - {}_kJ^{\alpha}f(t))({}_kJ^{\alpha}f(t) - {}_kJ^{\alpha}\varphi_1(t)) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\varphi_1 f(t) - {}_kJ^{\alpha}\varphi_1(t)\, {}_kJ^{\alpha}f(t) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\varphi_2 f(t) - {}_kJ^{\alpha}\varphi_2(t)\, {}_kJ^{\alpha}f(t) \\ &+ {}_kJ^{\alpha}\varphi_1(t) J^{\alpha}\varphi_2(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\varphi_1\varphi_2(t) \\ &= T(f,\varphi_1,\varphi_2), \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}g^2(t) - ({}_kJ^{\alpha}g(t))^2 &\le ({}_kJ^{\alpha}\psi_2(t) - {}_kJ^{\alpha}g(t))({}_kJ^{\alpha}g(t) - {}_kJ^{\alpha}\psi_1(t)) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\psi_1 g(t) - {}_kJ^{\alpha}\psi_1(t)\, {}_kJ^{\alpha}g(t) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\psi_2 g(t) - {}_kJ^{\alpha}\psi_2(t)\, {}_kJ^{\alpha}g(t) \\ &+ {}_kJ^{\alpha}\psi_1(t)\, {}_kJ^{\alpha}\psi_2(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\psi_1\psi_2(t) \\ &= T(g,\psi_1,\psi_2). \end{aligned} \tag{2.28}$$

From (2.26), (2.27) and (2.28), we obtain (2.23). □

Remark 2.5. *If we take $k = 1$ in the Theorem 2.6, we obtain the Theorem 9 in [22].*

Remark 2.6. *If $T(f,\varphi_1,\varphi_2) = T(f,m,M)$ and $T(g,\psi_1,\psi_2) = T(g,p,P)$, $m,M,p,P \in \mathbb{R}$ then inequality (2.23) reduces to*

$$\left| \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \,{}_kJ^{\alpha} fg(t) - {}_kJ^{\alpha} f(t)\, {}_kJ^{\alpha} g(t) \right| \leq \left(\frac{t^{\frac{\alpha}{k}}}{2\Gamma_k(\alpha+k)} \right)^2 (M-m)(P-p). \tag{2.29}$$

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Received: September 15, 2015; *Accepted*: January 13, 2016

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 111–118

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Two fluid Axially Symmetric Cosmological Models in f(R,T) Theory of Gravitation

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Abstract

In this paper we have investigated two fluid axially symmetric cosmological models in $f(R,T)$ theory of gravitation. To get the deterministic model, we have assumed a supplementary condition $H_3 = kH_1$, where H_1 and H_3 are Hubble parameters and k is constant. Two-fluid model in $f(R,T)$ theory of gravitation, one fluid represents the matter content of the universe and another fluid is chosen to model the cosmic microwave background radiation. Some geometric aspects of the model are also discussed.

Keywords: Two fluid, Axially symmetric, $f(R,T)$ theory.

2010 MSC: 05C15, 05C70.

1 Introduction

In recent years, several modified gravity theories like f(R) gravity, f(G) gravity, f(T) gravity and so on investigated by many workers. Nojiri and Odintsov [1]-[2] have proposed Noteworthy amongst them are f (R) theory of gravity and a general scheme for the modified f(R) gravity reconstruction from any realistic FRW cosmology . The presence of a late time cosmic acceleration of the universe in f(R) gravity studied by Carroll et al.[3]. FRW models in f(R) gravity evaluated by Paul et al.[4]. A physically viable f(R) gravity model, which showed the unification of early time inflation and late time acceleration, has developed by Shamir [5]. Ali Shojai and Fatimah Shojai [6] have discussed some new exact static spherically symmetric interior solutions of metric f(R) gravitational theories.

f(R,T) modified theory of gravity developed by Harko et al [7] , where T denotes the trace of the energy momentum tensor and R is the curvature scalar. A spatially homogeneous Bianchi type-III cosmological model in the presence of a perfect fluid source in f(R,T) theory with negative constant deceleration parameter investigated by Reddy et al [8]. Rao and Neelima [9]-[10] have presented Bianchi type-VIo universes and perfect fluid Einstein-Rosen in f(R,T) gravity. A new class of Bianchi cosmological models in f(R,T) gravity evaluated by Chaubey and Shukla [11]. Reddy and Kumar [12] have obtained by LRS Bianchi type -II Universe in f(R,T) theory of gravity. FRW viscous fluid cosmological model in f(R,T) gravity derived by Naidu et al.[13]. Shri Ram et al. [14] have examined anisotropic cosmological models in f(R,T) theory of gravitation. Pawar and Solanke [15] have investigated the physical behavior of LRS Bianchi type I cosmological model in f(R,T) theory of gravity. Dark energy cosmological models in f(R,T) theory of gravity studied by Pawar and Agrawal [16].

Cosmological models with two fluids have derived by McIntosh [17]. Bianchi type VI0 model with two fluid source developed by Coley and Dunn [18]. Pant and Oli [19] have discussed two- fluid Bianchi type II cosmological models. Oli [20] has constructed Biachi type-I two fluid cosmological models with a variable G

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and Λ. Verma [21] has examined qualitative analysis of two fluids FRW cosmological models. Bainchi type IX two fluids cosmological models in General Relativity have obtained by Pawar and Dagwal [22]. Two-fluid cosmological model in Bianchi type V space time without variable G and Λ studied by Singh et al.[23]. Two fluid cosmological models in Kaluza-Klein space time examined by Samanta [24]. Venkateswarlu [25] has presented Kaluza-Klein mesonic cosmological model with two-fluid source. Singh et al.[26] have obtained two-fluid cosmological model of Bianchi type-V with negative constant deceleration parameter. Recently, two fluids tilted cosmological model in General Relativity and Axially Bianchi type-I mesonic cosmological models with two fluid sources in Lyra Geometry presented by Pawar and Dagwal [27]-[28].
Axially symmetric cosmological models with string dust cloud source developed by Bhattacharaya and Karade [29]. Reddy and Rao [30] have constructed axially symmetric Bianchi type-I cosmological model. Axially symmetric Bianchi type-I cosmological model with negative constant deceleration parameter presented by Reddy et al.[31]. Axially symmetric perfect fluid cosmological models in Brans-Dicke scalar tensor theory of gravitation derived by Rao et al.[32]. Axially symmetric space-time with strange quark matter attached to string cloud in bimetric theory studied by Sahoo[33].Axially Symmetric Bianchi Type-I Bulk-Viscous Cosmological Models with Time-Dependent Λ and q investigated by Nawsad Ali [34].

2 Model and Field Equations

We consider metric in the form

$$ds^2 = dt^2 - A^2[d\chi^2 + \alpha^2(\chi)d\phi^2] - B^2dz^2, \tag{2.1}$$

where A and B are functions of t alone and α is function of χ.
The Einsteins field equation in $f(R,T)$ theory of gravity for the function given by

$$f(R,T) = R + 2f(T) \tag{2.2}$$

as

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij} + 2f'T_{ij} + [2pf'(T) + f(T)]g_{ij}, \tag{2.3}$$

The energy momentum tensor for perfect fluids given by

$$T_{ij} = T_{ij}^{(m)} + T_{ij}^{(r)}, \tag{2.4}$$

where $T_{ij}^{(m)}$ is energy-momentum tensor for the matter field with density ρ_m ,pressure p_m and four-velocity $u_1^{(m)} = (0,0,0,1)$ with $g^{ij}u_i^{(m)}u_j^{(m)} = 1$. $T_{ij}^{(r)}$ is energy-momentum tensor for the radiation field with density ρ_r , pressure $p_r = \frac{1}{3}\rho_r$ and four-velocity $u_1^{(r)} = (0,0,0,1)$ with $g^{ij}u_i^{(r)}u_j^{(r)} = 1$
Thus,

$$T_{ij}^{(m)} = (\rho_m + p_m)u_i^{(m)}u_j^{(m)} - p_m g_{ij}, \tag{2.5}$$

$$T_{ij}^{(r)} = \frac{4}{3}\rho_r u_i^{(r)}u_j^{(r)} - \frac{1}{3}\rho_r g_{ij}, \tag{2.6}$$

and the prime denotes differentiation with respect to the argument.
We choose the function $f(T)$ as the trace of the stress energy tensor of the matter so that

$$f(T) = \lambda T, \tag{2.7}$$

where λ is an arbitrary constant.
The field equation (3) for metric (1) reduce to

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} = (1+3\lambda)p_m + (\frac{\rho_r}{3} - \lambda\rho_m), \tag{2.8}$$

$$2\frac{A_{44}}{A} + (\frac{A_4}{A})^2 - \frac{\alpha_{11}}{A^2\alpha} = (1+3\lambda)p_m + (\frac{\rho_r}{3} - \lambda\rho_m), \tag{2.9}$$

$$(\frac{A_4}{A})^2 + 2\frac{A_4B_4}{AB} - \frac{\alpha_{11}}{A^2\alpha} = -(1+3\lambda)\rho_m - (1+\frac{8\lambda}{3})\rho_r + \rho_m\lambda, \tag{2.10}$$

equation of state

$$p_m = (\gamma - 1)\rho_m \qquad 1 \leq \gamma \leq 2 \tag{2.11}$$

Here the index 4 after a field variable denotes the differentiation with respect to time t.
The function dependence of the metric together with Equation (9) and (10) imply,

$$\frac{\alpha_{11}}{\alpha} = m^2, \tag{2.12}$$

m is constant
If $m = 0$, then $\alpha(\chi) = c_1\chi + c_2, \chi > 0$, where c_1 and c_2 are integration constant. Without loss of generality, by taking $c_1 = 1$ and $c_2 = 0$ we get $\alpha(\chi) = \chi$.
With the help of equation (12), the set of equation (8)-(11) reduces to

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} = (1+3\lambda)p_m + (\frac{\rho_r}{3} - \lambda\rho_m), \tag{2.13}$$

$$2\frac{A_{44}}{A} + (\frac{A_4}{A})^2 = (1+3\lambda)p_m + (\frac{\rho_r}{3} - \lambda\rho_m), \tag{2.14}$$

$$(\frac{A_4}{A})^2 + 2\frac{A_4B_4}{AB} = -(1+3\lambda)\rho_m - (1+\frac{8\lambda}{3})\rho_r + \rho_m\lambda, \tag{2.15}$$

equation of state

$$p_m = (\gamma - 1)\rho_m \qquad 1 \leq \gamma \leq 2 \tag{2.16}$$

Solving equation (13) and (14) we get

$$\frac{A_{44}}{A} + (\frac{A_4}{A})^2 - \frac{B_{44}}{B} - \frac{A_4B_4}{AB} = 0. \tag{2.17}$$

The directional Hubble parameters in the direction of χ, ϕ and z

$$H_1 = H_2 = \frac{A_4}{A}, \quad and \quad H_3 = \frac{B_4}{B}. \tag{2.18}$$

From equation (17) and (18)

$$(H_1)_4 + 2(H_1)^2 - (H_3)_4 - (H_3)^2 - H_1H_3 = 0. \tag{2.19}$$

The shear scalar is proportional to the expansion scalar which envisages a linear relationship between Hubble parameter H_1 and H_3 .

$$H_3 = kH_1, \tag{2.20}$$

where k is constant.
Using equation (19) and (20) we get

$$(H_1)_4 + \frac{k^2+k-2}{k-1}(H_1)^2 = 0. \tag{2.21}$$

Integrate equation (21) we get

$$H_1 = \frac{1}{T} \quad and \quad H_3 = \frac{k}{T}, \tag{2.22}$$

where $T = (Mt - a)$, $M = \frac{k^2+k-2}{k-1}$ and a is integration constant.
From equation (22)

$$A = b_1T^{(\frac{1}{M})}, \quad B = b_2T^{(\frac{k}{M})} \tag{2.23}$$

where b_1 and b_2 are integration constant.
Equation (23) can be rewritten as

$$A = T^{(\frac{1}{M})}, \quad B = T^{(\frac{k}{M})} \tag{2.24}$$

where $b_1 = b_2 = 1$ without loss of generality.
Hence the line element (1) reduced to

$$ds^2 = \frac{dT^2}{M^2} - T^{\frac{2}{M}}[d\chi^2 + \alpha^2(\chi)d\phi^2] - T^{\frac{2k}{M}}dz^2, \tag{2.25}$$

3 Some Physical and Geometrical Property

Conservation law separated for radiation and matter

$$(\gamma - 2 - 3\gamma)\rho_{m4} + [2H_1 + H_3](2\gamma - 3 - \lambda + 3\lambda\gamma)\rho_m = 0, \tag{3.26}$$

$$(1 + \frac{8\lambda}{3})\rho_{r4} + 4[2H_1 + H_3](\frac{1+2\lambda}{3})\rho_r = 0. \tag{3.27}$$

From equation (22), (26) and (27) we get

$$\rho_m = NT^{\frac{-[2+k](2\gamma-3-\lambda+3\lambda\gamma)}{\gamma-2-3\lambda}}, \tag{3.28}$$

$$\rho_r = N_1 T^{\frac{-[2+k](1+2\lambda)}{3+8\lambda}}. \tag{3.29}$$

where N and N_1 are integration constant.
The density parameter as

$$\Omega_m = \frac{N}{3(2+k)^2} T^{\frac{-[2+k](2\gamma-3-\lambda+3\lambda\gamma)}{\gamma-2-3\lambda}+2}, \tag{3.30}$$

$$,\Omega_r = \frac{N_1}{3(2+k)^2} T^{\frac{-[2+k](1+2\lambda)}{3+8\lambda}+2}. \tag{3.31}$$

Total density parameter as

$$\Omega = \frac{1}{3(2+k)^2}[NT^{\frac{-[2+k](2\gamma-3-\lambda+3\lambda\gamma)}{\gamma-2-3\lambda}+2}+, N_1 T^{\frac{-[2+k](1+2\lambda)}{3+8\lambda}+2}]. \tag{3.32}$$

The scalar expansion and shear scalar are

$$\theta = \frac{2+k}{T}, \tag{3.33}$$

$$\sigma^2 = \frac{2(1-k)^2}{3T^2}. \tag{3.34}$$

The deceleration parameter as

$$q = \frac{m(6+k) - 4(1+k)}{2(1+k)^2}. \tag{3.35}$$

The spatial volume and the rate of expansion H_i in the direction of x, y, z-axis are

$$V = T^{\frac{2+k}{M}},$$

$$H_1 = H_2 = \frac{1}{T}, \quad H_3 = \frac{k}{T}. \tag{3.36}$$

Initially the density parameter for matter Ω_m , the density parameter for radiation Ω_r and total density parameter Ω are infinite. For large value of T the density parameter for matter Ω_m , the density parameter for radiation Ω_r and total the density parameter for Ω are vanish. When $T = 0$, the scalar expansion and shear scalar are infinity but at $T = \infty$, the scalar expansion and shear scalar are zero. The deceleration parameter is constant and Spatial Volume is constant at $M = \infty$ .The shear scalar are zero at $k = 1$. The rate of expansion H_i in the direction of x, y, z-axis is vanishing at $T = \infty$ but initially the rate of expansion H_i in the direction of x, y, z-axis is infinite. At $\lambda = -\frac{3}{8}$, the density parameter for radiation Ω_r is zero. since $\lim_{T\to\infty}(\frac{\sigma}{\theta}) \neq 0$ the models not approach isotropy for large value of T. In dust Universe $(\gamma = 1)$, the density parameter for matter Ω_m is infinite at $\lambda = -\frac{1}{3}$ .For zeldovich Universe $(\gamma = 2)$, when $\lambda \to 0$, the density parameter for matter Ω_m is infinite. The density parameter for matter Ω_m is infinite at $\lambda \to -\frac{2}{9}$ in radiation Universe$(\gamma = \frac{4}{3})$.

3.1 Case-I $k = 2$

The line element (1) reduced to

$$ds^2 = \frac{dT^2}{4} - T^{\frac{1}{2}}[d\chi^2 + \alpha^2(\chi)d\phi^2] - Tdz^2, \tag{3.37}$$

where $T = (4t - a_1)$, a_1 is integration constant.
The energy density of matter ρ_m and energy density of radiation ρ_r

$$\rho_m = N_3 T^{\frac{-4(2\gamma-3-\lambda+3\lambda\gamma)}{\gamma-2-3\lambda}}, \tag{3.38}$$

$$\rho_r = N_4 T^{\frac{-4(1+2\lambda)}{3+8\lambda}}. \tag{3.39}$$

where N_3 and N_4 are integration constant.
The density parameter as

$$\Omega_m = \frac{N_3}{48} T^{\frac{6\gamma-8+2\lambda+12\lambda\gamma}{2+3\lambda-\gamma}}, \tag{3.40}$$

$$,\Omega_r = \frac{N_4}{48} T^{\frac{2(1+4\lambda)}{3+8\lambda}}. \tag{3.41}$$

Total density parameter as

$$\Omega = \frac{1}{48}[N_3 T^{\frac{6\gamma-8+2\lambda+12\lambda\gamma}{2+3\lambda-\gamma}} + N_4 T^{\frac{2(1+4\lambda)}{3+8\lambda}}]. \tag{3.42}$$

The scalar expansion and shear scalar are

$$\theta = \frac{4}{T}, \tag{3.43}$$

$$\sigma^2 = \frac{2}{3T^2}. \tag{3.44}$$

The deceleration parameter as

$$q = \frac{10}{9}. \tag{3.45}$$

The spatial volume and the rate of expansion H_i in the direction of x, y, z-axis are

$$V = T,$$

$$H_1 = H_2 = \frac{1}{T}, \quad H_3 = \frac{2}{T}. \tag{3.46}$$

When $T = 0$, the density parameter for matter $\Omega_m = 0$, the density parameter for radiation $\Omega_r = 0$ and total the density parameter $\Omega = 0$ but $T = \infty$, the density parameter for matter $\Omega_m = \infty$, the density parameter for radiation $\Omega_r = \infty$ and total the density parameter $\Omega = \infty$. The scalar expansion and shear scalar are infinity at $T = 0$ .For large value of T the scalar expansion and shear scalar are zero. The deceleration parameter is constant. The rate of expansion H_i in the direction of x, y, z-axis is vanishing at $T = \infty$ but initially the rate of expansion H_i in the direction of x, y, z-axis is infinite. since $\lim_{T\to\infty}(\frac{\sigma}{\theta}) \neq 0$ the models not approach isotropy for large value of T . At $\lambda = -\frac{3}{8}$, the density parameter for radiation Ω_r is infinite but at $\lambda = -\frac{1}{4}$ the density parameter for radiation Ω_r is constant. In dust Universe $(\gamma = 1)$, the density parameter for matter Ω_m is infinite at $\lambda = -\frac{1}{3}$ but at $\lambda = \frac{2}{14}$, the density parameter for matter is constant. For zeldovich Universe $(\gamma = 2)$, when $\lambda \to 0$, the density parameter for matter Ω_m is infinite and $\lambda \to -\frac{4}{26}$, the density parameter for matter Ω_m is constant. The density parameter for matter Ω_m is infinite at $\lambda \to -\frac{2}{9}$ and the density parameter for matter Ω_m is constant at $\lambda \to 0$ in radiation Universe $(\gamma = \frac{4}{3})$.

3.2 Case-I $k = -2$

The line element (1) reduced to

$$ds^2 = dt^2 - e^{2(l_1 - \frac{t}{a_2})}[d\chi^2 + \alpha^2(\chi)d\phi^2] - e^{2(l_2 + \frac{2t}{a_2})}dz^2, \tag{3.47}$$

where a_2 is integration constant.
The energy density of matter ρ_m and energy density of radiation ρ_r

$$\begin{aligned} \rho_m &= N_5 \\ \rho_r &= N_6. \end{aligned} \tag{3.48}$$

where N_5 and N_6 are integration constant.
The density parameter as

$$\begin{aligned} \Omega_m &= \infty, \\ ,\Omega_r &= \infty. \end{aligned} \tag{3.49}$$

Total density parameter as

$$\Omega = \infty. \tag{3.50}$$

The rate of expansion H_i in the direction of x, y, z-axis, the scalar expansion and shear scalar are

$$\begin{aligned} H_1 = H_2 = -\frac{1}{a}, \quad H_3 &= \frac{2}{a} \\ \theta &= 0, \\ \sigma^2 &= \frac{6}{T^2}. \end{aligned} \tag{3.51}$$

The deceleration parameter as

$$q = 2. \tag{3.52}$$

The spatial volume as

$$V = 1. \tag{3.53}$$

The models are non expanding and shearing universe .The energy density of matter ρ_m and energy density of radiation ρ_r, deceleration parameter, Spatial Volume, The rate of expansion H_i in the direction of x, y, z-axis are constant therefore no effect of T. The shear scalar are infinity at $T = 0$. When $T = \infty$ shear scalar are zero. Since, $\lim_{T\to\infty}(\frac{\sigma}{\theta}) = 0$ the models approach isotropy for large value of T.

4 conclusion:

The models are expanding and shearing universe but in case-II the models is non expanding. Initially the density parameter for matter Ω_m , the density parameter for radiation Ω_r and total the density parameter Ω are infinite. For large value of T the density parameter for matter Ω_m , the density parameter for radiation Ω_r and total the density parameter Ω are vanish. In dust Universe $(\gamma = 1)$, the density parameter for matter Ω_m is infinite at $\lambda = -\frac{1}{3}$. For zeldovich Universe $(\gamma = 2)$, when $\lambda \to 0$, the density parameter for matter Ω_m is infinite. The density parameter for matter Ω_m is infinite at $\lambda = -\frac{2}{9}$ in radiation Universe $(\gamma = \frac{4}{3})$. At$\lambda = -\frac{3}{8}$, the density parameter for radiation Ω_r is zero.
For case-I When $T = 0$, the density parameter for matter $\Omega_m = 0$, the density parameter for radiation $\Omega_r = 0$ and total the density parameter $\Omega = 0$ but $T = \infty$, the density parameter for matter $\Omega_m = \infty$, the density parameter for radiation $\Omega_r = \infty$ and total the density parameter $\Omega = \infty$. At $\lambda = -\frac{3}{8}$, the density parameter for radiation Ω_r is infinite but at $\lambda = -\frac{1}{4}$ the density parameter for radiation Ω_r is constant. In dust Universe

$(\gamma = 1)$, the density parameter for matter Ω_m is infinite at $\lambda = -\frac{1}{3}$ but at $\lambda = \frac{2}{14}$, the density parameter for matter is constant. For zeldovich Universe $(\gamma = 2)$, when $\lambda \to 0$, the density parameter for matter Ω_m is infinite and $\lambda \to -\frac{4}{26}$, the density parameter for matter Ω_m is constant. The density parameter for matter Ω_m is infinite at $\lambda \to -\frac{2}{9}$ and the density parameter for matter Ω_m is constant at $\lambda \to 0$ in radiation Universe $(\gamma = \frac{4}{3})$.All density parameter are infinite in case-II. The scalar expansion and shear scalar are infinity at $T = 0$ .For large value of T the scalar expansion and shear scalar are zero. The rate of expansion H_i in the direction of x, y, z-axis is vanishing at $T = \infty$ but initially the rate of H_i expansion in the direction of x, y, z-axis is infinite. In case-II the rate of expansion H_i in the direction of x, y, z-axis is constant. The deceleration parameter is constant for all case. Since $\lim_{T\to\infty}(\frac{\sigma}{\theta}) \neq 0$ the models not approach isotropy for large value of T. But in case-II,$\lim_{T\to\infty}(\frac{\sigma}{\theta}) = 0$ the models approach isotropy for large value of T.

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Received: November 20, 2015; *Accepted*: January 01, 2016

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 119-125

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Lyapunov Approach for Stability of Integro-Differential Equations with Non Instantaneous Impulse Effect

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Abstract

In this paper an integro-differential system of equations, with fixed moments of non instantaneous impulse effects is considered. Sufficient conditions for stability and asymptotic stability of this system have been worked out. The investigations are carried out by means of piecewise continuous functions, analogous to Lyapunov functions and by means of the theory of differential inequalities for such functions. A new comparison lemma, connecting the solution of the given impulsive integro-differential system to the solution of a scalar impulsive differential system is also established.

Keywords: Impulsive integro-differential systems, non instantaneous impulses, Lyapunov stability, asymptotic stability, Lyapunov function.

2010 MSC: 34C20; 34D20; 34A37; 34K4; 92D25.

1 Introduction

The literature on abstract impulsive differential equations considers basically the problems on existence and qualitative properties of solutions of equations of the type

$$x' = f(t, x), \quad t \neq t_i, \tag{1.1}$$

$$\Delta x = I_i(x), \quad t = t_i, \tag{1.2}$$

where, $i \in N, t \in R^+, I_i(x) = x(t_i^+) - x(t_i^-)$, $x \in R^n$, $f : R^+ \times R^n \to R^n$ and $0 = t_0 < t_1 < t_2 < t_3 < \cdots < \infty$. Here I_i:$R^n \to R^n$ is a sequence of instantaneous impulse operators and have been used to describe abrupt changes such as shocks, harvesting , natural disasters etc.

It seems that the above instantaneous impulsive differential equations models can not characterize the dynamics of evolution process completely in pharmacotherapy. For example as in [1], consider the hemodynamical equilibrium of a person. In the case of decompensation (e.g. high or low levels of glucose), one can prescribe some intravenous drugs(insulin) and the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes. In fact the above situation has fallen in new impulsive action which starts abruptly and stays active on a finite time interval. Thus we have to use a new model to describe such an evolution process.

To the best of our knowledge, Hernandez and O'Regan [1] in 2013, initially offered to study a new class of abstract impulsive differential equations with non instantaneous impulses in a PC_α-normed Banach space.Then Pierre and Rolnik[2] continued the work in a PC_α- normed Banach space and developed the results in [1].UlamHyers stability and Lyapunov stability of this type of non instantaneous differential

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systems were established recently in [6,8] and [4] respectively.Existence of solutions for integer/fractional differential and integro-differential equations with non instantaneous impulses was presented in[3, 5,7].

Motivated by the above stated work on non instantaneous impulsive differential systems, in this paper, we consider the following, new model of impulsive integro -differential equations to describe an evolution process, in which an impulse action starts at an arbitrary fixed point and keeps active on a finite time interval and establish sufficient conditions its stability and asymptotic stability

$$x'(t) = f(t, x(t), Lx), \quad t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, 3, \cdots, m, \tag{1.3}$$

$$x(t) = g_i(t, x(t), Mx), \quad t \in (t_i, s_i], \quad i = 0, 1, 2, 3, \cdots, m, \tag{1.4}$$

where $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 \leq \cdots s_{m-1} \leq t_m \leq s_m \leq t_{m+1} = T$ are pre fixed numbers, $f : J \times R^n \times R^n \to R^n$, where $J = [0, T]$ is continuous and g_i:$[t_i, s_i] \times R^n \times R^n \to R^n$ is continuous for all $i = 1, 2, \cdots, m$, which are called non instantaneous impulses.

$Lx = \int_{t_0}^{t} K(s, x(s))ds$, K: $J \times R^n \to R^n$ and $Mx = \int_{t_0}^{t} I(s, x(s))ds$, I: $J \times R^n \to R^n$ are continuous functions. As far as existence of solution of system (1.3)-(1.4), is concerned, we refer [3,7]. Assume that we can measure the state of the process at any time to get a function $x(\cdot)$ as a solution of (1.3)-(1.4). To ensure the existence of trivial solution of the system (1.3)-(1.4), let us assume that $f(t, 0, 0) = 0$, $K(t, 0) = 0$, $g_i(t, 0, 0) = 0$, $I(t, 0)$.

The novelty of our paper is to establish stability and asymptotic stability of solutions of integro-differential system of equations with non instantaneous impulses. A new comparison lemma for this non instantaneous impulsive systems is proved and by using this, the study of the solution of impulsive integro-differential system is replaced by the study of the solutions of a scalar Impulsive integro-differential system as done in [12,15] .

In section 2, some preliminaries notes and definitions are given. In section 3, a new comparison lemma, connecting the solution of the given impulsive integro-differential system to the solution of a scalar impulsive integro-differential system is worked out. This lemma plays an important role in establishing the main results of the paper. Sufficient conditions for stability and asymptotic stability of impulsive integro-differential system of equations with non instantaneous fixed time impulse effect, are established by using the lemma.

2 PRELIMINARIES

Let $C(J, R^n)$ be the Banach space of all continuous function from J into R^n with the norm $\|x\| = Max\{\|x_1\|_C, \|x_2\|_C, \|x_3\|_C, \cdots, \|x_n\|_C\}$ for $x \in C(J, R^n)$, where $\|x_k\|_C = \sup|x_k(t)|$. Also we use the Banach space $PC(J, R^n) = \{x : J \to R^n : x \in C((t_k, t_{k+1}], R^n) : \text{k} = 0, 1, 2, \cdots, m\}$ and for $k = 1, 2, \cdots, m$ there exists $x(t_k^-)$ and $x(t_k^+)$ such that $x(t_k^-) = x(t_k^+)$ with the norm $\|x\|_{PC} = \max\{\|x_1\|_{PC}, \|x_2\|_{PC}, \|x_3\|_{PC}, \cdots, \|x_n\|_{PC}\}$. Denote $PC^1(J, R^n) = \{x \in PC(J, R^n) : x' \in PC(J, R^n)\}$. Set $\|x\|_{PC^1} = \max\{\|x\|_{PC}, ||x'||_{PC}$. Clearly $PC^1(J, R^n)$ endowed with the norm $|| \cdot ||_{PC^1}$ is also a Banach space. If $x, y \in R^n$, $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ by $x \leq y$ we mean that $x_i \leq y_i$ $\forall i = 1, 2, \cdots, n$.

Referring [3,5,7] , a function $x \in PC^1(J, R^n)$ is called classical solution of the impulsive Cauchy problem

$$\begin{aligned} x'(t) &= f(t, x(t), Lx), \quad t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \cdots, m, \\ x(t) &= g_i(t, x(t), Mx), \quad t \in (t_i, s_i], \quad i = 0, 1, 2, \cdots, m, \\ x(0) &= x_0. \end{aligned}$$

If satisifes $x(0) = x_0$, $x(t) = g_i(t, x(t), Mx)$, $t \in (t_i, s_i]$, $i = 1, 2, \cdots, m$. And

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} f\left(s, x(s), \int_{t_0}^{t} K(\tau, x(\tau))d\tau\right)ds, \quad t \in (0, t_1], \\ x(t) &= g_i\left(s_i, x(s_i), \int_{t_0}^{t} I(\tau, x(\tau))d\tau\right) + \int_{t_0}^{t} f\left(s, x(s), \int_{t_0}^{t} K(\tau, x(\tau))d\tau\right), \quad t \in (s_i, t_{i+1}], \\ &\quad i = 1, 2, \cdots, m. \end{aligned}$$

Let us introduce the intervals : $G_i = (s_i, t_{i+1}] : i = 1, 2, \cdots, m$ with $G = \cup_{i=0}^{m} G_i$ and $H_i = (t_i, s_i] : i = 0, 1, 2, \cdots, m$ with $H = \cup_{i=0}^{m} H_i$.

Definition 2.1. *[10, 14] A function V: $J \times R^n \to R^+$ is said to belong to class V_0 if* *(i)* V *is continuous in* $G_i \cup H_i$, $i = 0, 1, 2, \cdots, m$.

(ii) V *is locally Lipschitz continuous in its second argument on each of* G_i, $i = 0, 1, 2, \cdots, m$.

(iii) $V(t+0, g_i(t,x)) \leq V(t,x)$ *for each* $x \in H_i$, $i = 0, 1, 2, \cdots, m$.

(iv) *For* $i = 1, 2, \cdots, m$ *in* , $V(t_i - 0, x) = V(t_i, x)$ *and* $V(t_i + 0, x) = \lim_{t \to t_i + 0} V(t_i, x)$.

Further for $t \in G_i$ *and* $x \in PC(J, R^n)$,*we define the following derivative* ,

$$D^+_{(2)}V(t,x) = \lim_{h \to 0} \sup \frac{1}{h}\left[V\left(t+h, x + hf\left(t, x(t), \int_{t_0}^{t} K(s, x(s))ds\right)\right) - V(t,x)\right]$$

Note that if $x(t)$ *is a solution of the system (1.3)-(1.4), then* $D^+_{(2)}V(t,x) = V'_{(2)}(t,x)$. *We shall now use the following classes of functions:*

$\mathcal{K} = \{a \in C[J, R^+] : a(\cdot)$ *is monotonically increasing and* $a(0) = 0\}$.

$C\mathcal{K} = \{a \in C[J \times R^+, R^+] : a(t, \cdot) \in \mathcal{K}$ *for each* $t \in J\}$.

Together with system (1.3)-(1.4), we consider the following scalar impulsive differential system of equations:

$$u'(t) = g(t, u(t), Pu), \quad t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \cdots, m, \tag{2.1}$$

$$u(t) = f_i(t, u(t), Qu), \quad t \in (t_i, s_i], \quad i = 0, 1, 2, \cdots, m, \tag{2.2}$$

$$x(t_0^+) = u_0 \geq 0, \tag{2.3}$$

where g: $J \times R^+ \times R^+ \to R^+$, f_i: $[t_i, s_i] \times R^+ \times R^+ \to R^+$, $Pu = \int_{t_0}^{t} K_1(\tau, u(\tau))d\tau$, K_1: $J \times R^+ \to R^+$ *and* $Qu = \int_{t_0}^{t} I_1(\tau, u(\tau))d\tau$, I_1: $J \times R^+ \to R^+$. *Let* $S(\rho) = \{(t,x) \in J \times R^n : ||x|| < \rho, \rho > 0\}$. *We shall say that the conditions (A) are satisfied if the following hold:*

(A1) $f \in PC(S(\rho) \times R^n, R^n)$.

(A2) $K \in C(S(\rho), R^n)$.

(A3) $g(t, 0, 0) = 0$ *for* $t \in J$.

(A4) $\psi_k \in C[R^+, R^+]$ *is non decreasing function with* $\psi_k(0) = 0$ *and* $\psi_k(r_{k-1}(t_k; t_{k-1}, u^+_{k-1})) = u^+_k$, *where* $r_k(t, t_k, u^+_k)$ *is the maximal solution of the system (2.1)-(2.3), if it occurs in* $(t_k, t_{k+1}] = H_k \cup G_k$, $k = 0, 1, 2, \cdots, m$.

(A5) *Let* $||g_i(t, x(t), Mx)|| < \rho$ *for each* $t \in H_i$, $i = 0, 1, 2, \cdots, m$.

Definition 2.2. *[14] The system (1.3)-(1.4) is said to be stable, if for each* $\epsilon > 0$, $\exists$ *a* $\delta = \delta(t_0, \epsilon) > 0$ *such that for any solution* $x(t) = x(t, t_0, x_0)$ *of (1.3)-(1.4), the inequality* $||x_0|| \leq \delta \Rightarrow ||x(t)|| < \epsilon$, $\forall t \geq t_0$.

Definition 2.3. *[14] The system (1.3)-(1.4) is said to be attractive, if for each* $\epsilon > 0$, $\exists$ *two numbers* $\delta = \delta(t_0) > 0$ *and* $\Gamma = \Gamma(t_0, \epsilon) > 0$ *such that for any solution* $x(t) = x(t, t_0, x_0)$ *of (1.3)-(1.4), the inequality* $||x_0|| \leq \delta \Rightarrow ||x(t)|| < \epsilon$, $\forall t \geq t_0 + \Gamma$.

Definition 2.4. *[14] The system (1.3)-(1.4) is said to be asymptotically stable if it is stable and attractive.*

Definition 2.5. *[14] A function* $V \in V_0$*is said to be :*

(i) *positive definite if there exists a* $\delta > 0$ *and a function* $a \in \mathcal{K}$ *such that* $||x|| < \delta \Rightarrow a(||x||) \leq V(t,x)$.

(ii) *weakly decrescent if there exists a* $\delta > 0$ *and a function* $b \in C\mathcal{K}$ *such that* $||x|| < \delta \Rightarrow V(t,x) \leq b(t, ||x||)$.

3 MAIN RESULTS

Before establishing the main results of the paper, we will prove the following lemma:

Lemma 3.1. *Let the following conditions be fulfilled:*

1. *Conditions (A1)-(A4) hold.*

2. *Let $V \in S(\rho) \to R^+$ and $V \in V_0$. Assume that*

   (i) $D^+_{(2)}V(t, x(t)) \leq g(t, V(t, x(t))) : t \in G_i$.

   (ii) $V(t_k + 0, x(t_k + 0)) \leq \psi_k(V(t_k, x(t_k))) : k = 1, 2, \cdots, m$.

   (iii) $V(s, x(s)) \leq V(t, x(t))$ *:for* $t, s \in H_i$ *such that* $0 \leq t \leq s$.

3. *The solution* $x(t) = x(t; t_0, x_0)$ *of system (1.3)-(1.4) is such that* $(t, x(t + 0, t_0, x_0)) \in S(\rho)$ *for* $t \in J$.

4. *Let* $r(t, t_0, u_0)$ *, the maximal solution of (2.1)-(2.3) satisfying* $u_0 \geq V(t_0 + 0, x_0)$*exists on* J *.*

Then

$$V(t, x(t; t_0, x_0)) \leq r(t; t_0, u_0) : t \in J. \tag{3.1}$$

Proof. The maximal solution $r(t, t_0, u_0)$ of the system (2.1)-(2.3) can be defined as follows:

$$r(t, t_0, u_0) = \begin{cases} r_0(t, t_0, u_0^+), & t_0 < t \leq t_1 \\ r_1(t, t_1, u_1^+), & t_1 < t \leq t_2 \\ \cdots \\ \cdots \\ r_m(t, t_m, u_m^+), & t_m < t \leq t_{m+1}, \end{cases}$$

where $r_i(t, t_1, u_i^+)$ is the maximal solution of the system (2.1)-(2.3), in $(t_i, t_{i+1}] = H_i \cup G_i$, for which $\psi_i(r_{i-1}(t_i; t_{i-1}, u_{i-1}^+)) = u_i^+$, $i = 1, 2, \cdots, m$ and $u_0^+ = u_0$. We claim (3.1) by considering the following three cases;

Case 1: For $t \in (s_i, t_{i+1}) = G_i - \{t_{i+1}\}$ let us say $m(t) = V(t, x(t; t_0, x_0))$ so that for small $h > 0$ we have

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, x(t+h)) - V(t, x(t)) \\ &= V(t+h, x(t+h)) - V\left(t+h, x + hf\left(t, x(t), \int_{t_0}^t K(s, x(s))ds\right)\right) \\ &\quad + V\left(t+h, x + hf\left(t, x(t), \int_{t_0}^t K(s, x(s))ds\right)\right) - V(t, x(t)). \end{aligned}$$

As in Definition 2.1, $V(t, x(t))$ is locally Lipschitzian in x for $t \in G$, using assumption 2(i) of the statement of lemma, we arrive at $D^+_{(2)}m(t) \leq g(t, m(t)) : t \in G_i - \{t_{i+1}\}$. Then by theorem 3.1.1 in [13] we observe that

$$m(t) \leq r(t), i.e. V(t, x(t; t_0, x_0)) \leq r(t; t_0, u_0) : t \in G_i - \{t_{i+1}\}. \tag{3.2}$$

Case 2: For $t \in H_i = (t_i, s_i]$,Using condition 2(iii) of the lemma and the fact that $u_0 \geq V(t_0 + 0, x_0)$, it is clear that

$$V(t, x(t; t_0, x_0)) \leq V(t_0 + 0, x_0) \leq u_0 \leq r(t; t_0, u_0). \tag{3.3}$$

Case 3: For $t \in \{t_1, t_2, \cdots, t_{m+1}\}$, i.e. the moments of impulses .w.l.o.g. let us assume that $t = t_k$. Then by using assumption 2(ii) of the lemma, we see

$$V(t_i + 0, x(t_i + 0; t_0, x_0)) \leq \psi_i(V(t_i, x(t_i; t_0, x_0))) \leq \psi_i(r_i(t_i, t_{i-1}, u_{i-1}^+)) = u_i^+.$$

Again using this condition $V(t, x(t; t_i + 0, x_0)) \leq u_i^+$ in place of $u_0 \geq V(t_0 + 0, x_0)$ and as done in case 1,we get

$$V(t, x(t; t_0, x_0)) \leq r(t; t_0, u_0) \text{for each} = 1, 2, \cdots, m. \tag{3.4}$$

Thus from (3.2), (3.3) and (3.4), we conclude that $V(t, x(t; t_0, x_0)) \leq r(t; t_0, u_0) : t \in J$ and hence (3.1) is established. □

Theorem 3.1. *Let the following conditions hold:*

1. *Conditions (A) are satisfied.*
2. *Let $V \in S(\rho) \to R^+$ and $V \in V_0$ such that it is positive definite and weakly decrescent.*
3. *Assume that for $t > t_0 \geq 0$*
   - *(i)* $D^+_{(2)}V(t,x(t)) \leq g(t,V(t,x(t))) : t \in G_i$,
   - *(ii)* $V(t_k+0,x(t_k+0)) \leq \psi_k(V(t_k,x(t_k)))$: $k = 1,2,\cdots,m$
   - *(iii)* $V(s,x(s)) \leq V(t,x(t))$: *for* $t,s \in H_i$ *such that* $0 \leq t \leq s$

   Then,

   - *(a) if the zero solution of the system (2.1)-(2.3) is stable then the system (1.3)-(1.4) will also be stable.*
   - *(b) if the zero solution of the system (2.1)-(2.3) is asymptotically stable then the system (1.3)-(1.4) will also be asymptotically stable.*

Proof. **proof of (a):**
Since $V \in V_0$ is positively definite in $S(\rho)$, there exists a function $a \in \mathcal{K}$ and a such that $\delta_1 > 0, 0 < \delta_1 \leq \rho||x|| < \delta_1$ implies

$$a(||x||) \leq V(t,x). \tag{3.5}$$

Since $V \in V_0$ is weakly decrescent, there exists a function $a \in C\mathcal{K}$ and a $\delta_2 > 0$ such that $||x|| < \delta_2$ implies

$$V(t+0,x) \leq b(t+0,||x||). \tag{3.6}$$

Now as assertion in the statement of theorem, zero solution of the system (2.1)-(2.3) is stable, by definition of stability that there exists a $\delta_3(t_0,\epsilon) > 0$ such that $||u_0|| < \delta_3$ implies $\big|\big|u(t;t_0,u_0)\big|\big| < a(\epsilon)$ for $t \geq t_0$ and in particular $\big|\big|r(t;t_0,u_0)\big|\big| < a(\epsilon)$ for $t \geq t_0$, where $r(t;t_0,u_0)$ is the maximal solution of (2.1)-(2.3) for which

$$r(t_0+0;t_0,u_0) = u_0. \tag{3.7}$$

Choose $\delta_4 = \delta_4(t_0,\epsilon)$ satisfying $b(t_0+0,\delta_4) < \delta_3$ and let $\delta = \delta(t_0,\epsilon) = \min\{\delta_3,\delta_4\}$. Then for $||x_0|| < \delta$, we have $V(t_0+0,x_0) \leq b(t_0+0,||x||) \leq (t_0+0,\delta) < \delta_3$. Therefore we see that

$$V(t_0+0,x_0) < \delta_3, \text{ for } ||x|| < \delta. \tag{3.8}$$

Thus $\big|\big|r(t;t_0,V(t_0+0,x_0))\big|\big| < a(\epsilon)$ for $t \geq t_0$. Let $x(t) = x(t;t_0,x_0)$ be a solution of system (1.3)-(1.4). Then to prove that zero solution of system (1.3)-(1.4) will be stable,we claim that for above mentioned $\delta = \delta(t_0,\epsilon) > 0$, $||x|| < \delta$ implies that for every $\epsilon > 0$ we have

$$||x(t)|| < \epsilon, \text{ for } t \geq t_0. \tag{3.9}$$

If possible let this be false. Then there exists some $t^* > t_0$ such that $t_k < t^* \leq t_{k+1}$ for some k satisfying $\|x(t^*)\| \geq \epsilon$ and

$$\|x(t)\| < \epsilon, \text{ for } t_0 < t \leq t_k. \tag{3.10}$$

Again for $\|x(t_k^+)\| = \|g_i(t,x(t),Mx)\| < \rho$ and $\|x(t_k)\| < \epsilon$ from(3.9) . Hence we can find a t^0 such that $t_k < t^0 \leq t^*$ and $\epsilon \leq \|x(t^0)\| < \rho$ with

$$\|x(t)\| < \rho, \text{ for } t_0 \leq t \leq t^0. \tag{3.11}$$

Set $m(t) = V(t,x(t;t_0,x_0))$ for $t_0 \leq t \leq t^0$. We note here that $m(t_0^+) = V(t_0^+,x(t;t_0,x_0)) = u_0$ and all the conditions of lemma 3.1, are fulfilled in the interval $[t_0,t^0]$. Therefore, applying the lemma for the system (2.1)-(2.3) of integro-differntial equations in the interval $[t_0,t^0]$, instead of interval $J = [0,T]$, we have the following inequality:

$$\begin{aligned} m(t) &= V(t,x(t;t_0,x_0)) \leq r(t;t_0,u_0) \\ &= r\big(t;t_0,V(t_0^+,x_0)\big) : t \in \big[t_0,t^0\big], \end{aligned} \tag{3.12}$$

where $r(t;t_0,u_0)$ is the maximal solution of (2.1)-(2.3) in $[t_0,t^0]$. We then have, by using(3.10), (3.5), (3.12), (3.11) and (3.7)

$$a(\epsilon) \leq \left(\left\| x(t^0) \right\| \right) \leq V\left(t^0, x(t^0)\right) \leq r(t;t_0,V(t_0^+,x_0)) < a(\epsilon)$$

which is a contradiction. Therefore (3.9) holds true and hence for every given $\epsilon > 0$, there exists a $\delta = \delta(t_0,\epsilon)$, such that $||x_0|| < \delta$ implies $||x(t)|| < \epsilon$ for $t \geq t_0$. Thus the zero solution of the system (1.3)-(1.4) is stable.
proof of (b): It is given that the zero solution of the system (2.1)-(2.3) is asymptotically stable, which means it is stable and attractive. As proved in part (a), stability of system (2.1)-(2.3) implies that the system (1.3)-(1.4) is stable. Therefore, by definition,there exists a $\delta_{01} = \delta_{01}(t_0,\epsilon) > 0$, such that $||x_0|| < \delta_{01}$ implies $||x(t)|| < \epsilon$ for $t \geq t_0$. In particular if we take $\epsilon = \rho$ we have, $||x(t)|| < \rho$ for $||x_0|| < \delta_{01}$, $t \geq t_0$.

Again as zero solution of system (2.1)-(2.3) is attractive, by definition 2.3, for each $\epsilon > 0$, $\exists$ two numbers $\delta_{02} = \delta_{02}(t_0) > 0$ and $\Gamma = \Gamma(t_0,\epsilon)$ such that

$$||u_0|| \leq \delta_{02} \Rightarrow \left\| r(t,t_0,u_0) \right\| < a(\epsilon), t \geq t_0 + \Gamma. \tag{3.13}$$

Choose $\delta_{03} = \delta_{03}(t_0) > 0$ such that

$$\delta_{03} < \delta_{02}, \text{ with } b(t_0,\delta_{03}) < \delta_{02}. \tag{3.14}$$

Then from (3.6) and (3.14), we get,

$$V(t_0+0,x_0) \leq b(t_0+0,||x_0||) \leq b(t_0+0,\delta_{03}) < \delta_{02}.$$

Therefore, $\left\| r(t;t_0,V(t_0+0,x_0)) \right\| < a(\epsilon)$ for $t \geq t_0+\Gamma$. Set $\delta_0 = \delta_0(t_0) = \min\{\delta_{01},\delta_{02},\delta_{03}\}$. Now let $||x_0|| < \delta_0$ and apply lemma, it follows that if $x(t) = x(t;t_0,x_0)$ is a solution of the system(1.3)-(1.4), then

$$V(t,x(t;t_0+x_0)) \leq r(t;t_0,V(t_0+0,x_0)) : t \geq t_0+\Gamma. \tag{3.15}$$

Now, to prove that system (1.3)-(1.4) is attractive , we claim that for above mentioned $\delta_0 = \delta_0(t_0) > 0$, $||x_0|| < \delta$ implies that for every $\epsilon > 0$, we have $\left\| x(t) \right\| < \epsilon$ for $t \geq t_0+\Gamma$. If possible let it be false. Then as done in the proof of part(a), by using (3.13), (3.14) and (3.15) we will arrive at contradiction and hence the system (1.3)-(1.4) is attractive. Thus the system (1.3)-(1.4) is asymptotically stable. □

4 CONCLUSION

A variety of results concerning Lyapunov stability , eventual stability and practical stability for the impulsive differential systems of type (1.1)-(1.2) with instantaneous impulses(fixed time and variable time impulses), are established in literature by using Lyapunov functions along with comparison theorems ([9,10,11,14] and the references there in). in this paper, we established stability and asymptotic stability for a new impulsive integro-differential system in which, impulses are non instantaneous and that to the best of our knowledge, are proved for the first time. A new lemma by which the study of the solution of impulsive integro-differential system is replaced by the study of the solutions of a scalar Impulsive integro-differential system is also proved. The desired results are obtained by using Lyapunov functions and comparison differential inequalities.

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Received: September 11, 2015; *Accepted*: October 27, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 126–135

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Analytical Solution of Non-Integer Extra-Ordinary Differential Equation Via Adomian Decomposition Method

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Abstract

In the present paper, we obtain the analytical solution of the linear extraordinary fractional equations with constant coefficients by Adomian decomposition method under nonhomgeneous initial value condition, this method is a powerful method which consider the approximate solution as an infinite series usually converges to the exact solution.

Keywords: Extraordinary Fractional differential equation, Adomian decomposition method.

2010 MSC: 34A08, 74H10.

1 Introduction

Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [39]. Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry and biological sciences [43]. There have been many excellent books and monographs available on this field [20, 34, 41, 43, 46, 50]. In [34], the authors gave the most recent and up-to-date developments on fractional differential and fractional integro-differential equations with applications involving many different potentially useful operators of fractional calculus. In a recent work by Jaimini et.al. [33] the authors have given the corresponding Leibnitz rule for fractional calculus. For the history of fractional calculus, interested reader may see the recent review paper by Machado et. al. [38].

Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The fractional calculus has allowed the operations of integration and differentiation to any fractional order. The order may take on any real or imaginary value. Recently theory of fractional differential equations attracted many scientists and mathematicians to work on [16, 28, 29, 43–45, 51]. For the existence of solutions for fractional differential equations, one can see [9, 15, 17–19, 21–25, 30–32, 35, 36, 53] and references therein. The results have been obtained by using fixed point theorems like Picard's, Schauder fixed-point theorem and Banach contraction mapping principle. About the development of existence theorems for fractional functional differential equations, many contribution exists [7, 8, 13, 16, 23, 37, 54]. Many applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivative of fractional order. The results of several studies clearly stated that the fractional derivatives seem to arise generally and

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universally from important mathematical reasons. Recently, interesting attempts have been made to give the physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives proposed in [27, 29, 44, 45].

Ahmed et. al. [10] considered the fractional order predator-prey model and the fractional order rabies model. They have shown the existence and uniqueness of solutions of the model system and also studied the stability of equilibrium points. The motivation behind fractional order system are discussed in [10]. Lakshmikantham and Vatsala in [35, 36] and Lakshmikantham in [37] defined and proved the existence of the solution of fractional initial value problems.

Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the Adomian Decomposition Method[3, 4, 6]. This method is much more simpler in computation and quicker in convergence than any other method available in the open literature.

The application of the fractional differential equation in physical problems is available in the book of Bracewell [12]. Recently, the solution of the fractional differential equation has been obtained through the Adomian Decomposition Method by the researchers in [11, 26].

The purpose of this paper is to develop further the applicability of the Decomposition Method to solve extraordinary differential equations of fractional order.

For the sake of convenience, we first of all give definitions of fractional integral and fractional derivative introduced by Riemann-Liouville

Definition 1.1. (*Fractional integral*) ([11])

Let $q > 0$ denote a real number. Assuming $f(x)$ to be a function of class $C^{(n)}$ (the class of function with continuous nth derivative), the fractional integral of a function f of order $-q$ is given by

$$\frac{d^{-q}f(x)}{dx^{-q}} = \frac{1}{\Gamma(q)} \int_0^x \frac{f(t)dt}{(x-t)^{1-q}}, \tag{1.1}$$

Definition 1.2. (*Fractional derivative*) [11]

Let $q > 0$ denote a real number and n the smallest integer exceeding q such that $n - q > 0$ $(n = 0$ if $q < 0)$. Assuming $f(x)$ to be a function of class $C^{(n)}$ (the class of function with continuous nth derivative), the fractional derivative of a function f of order q is given by

$$\frac{d^q f(x)}{dx^q} = \frac{d^n}{dx^n}\left(\frac{d^{-(n-q)}f(x)}{dx^{-(n-q)}}\right) = \frac{1}{\Gamma(n-q)}\frac{d^n}{dx^n}\int_0^x \frac{f(t)dt}{(x-t)^{1-n+q}} \tag{1.2}$$

Definition 1.3. (*Mittag-Leffler function*) [2, 43]

A two-parameter function of the Mittag-Leffler type is defined by the series expansion:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad (\alpha > 0, \beta > 0), \tag{1.3}$$

In particular

$$\begin{aligned} E_{\frac{1}{2},1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\frac{k}{2}+1\right)} = \exp\left(z^2\right)(1 + erf(z)) \\ &= \exp\left(z^2\right) erf(-z) \end{aligned}$$

$$\begin{aligned} E_{\frac{3}{2},2}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\frac{3k}{2}+2\right)} = \exp\left(z^2\right)(1 + erf(z)) \\ &= \exp\left(z^2\right) erf(-z). \end{aligned}$$

2 The Decomposition Method

We consider an equation in the form

$$Lu + Ru + Nu = g, \tag{2.4}$$

where L is an easily or trivially invertible linear operator, R is the remaining linear part, and N represents a nonlinear operator.

The general solution of the given equation is decomposed into the sum

$$u = \sum_{n=0}^{\infty} u_n, \tag{2.5}$$

where u_0 is the solution of the linear part.

Our approach will be to write any nonlinear term in terms of the Adomian A_n polynomials. Its has been derived by Adomian that $Nu = \sum_{n=0}^{\infty} A_n$,where the A_n are special polynomials obtained for the particular nonlinearity $Nu = f(u)$ and generated by Adomian [3-5]. These A_n polynomials depends, of course, on the particular nonlinearity.

The A_n are given as

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= u_1 \left(\frac{d}{du_0}\right) f(u_0), \\ A_2 &= u_2 \left(\frac{d}{du_0}\right) f(u_0) + \left(\frac{u_1^2}{2!}\right)\left(\frac{d^2}{du_0^2}\right) f(u_0), \\ A_3 &= u_3 \left(\frac{d}{du_0}\right) f(u_0) + u_1 u_2 \left(\frac{d^2}{du_0^2}\right) f(u_0) \\ &\quad + \left(\frac{u_1^3}{3!}\right)\left(\frac{d^3}{du_0^3}\right) f(u_0), \\ &\quad \dots\dots\dots\dots \end{aligned} \tag{2.6}$$

and can be found the formula (for $n \geq 1$)

$$A_n = \sum_{\nu=1}^{n} c(\nu, n) f^{(\nu)}(u_0), \tag{2.7}$$

where the $c(\nu, n)$ are products (or sums of products) of ν components of u whose subscripts sum to n, divided by the factorial of the number of repeated subscripts [5].

Therefore, the general solution becomes

$$\begin{aligned} u &= u_0 - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1}Nu \\ &= u_0 - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n, \end{aligned} \tag{2.8}$$

where $u_0 = \phi + L^{-1}g$ and $L\phi = 0$.

To identify the terms in ${}_{n=1}^{\infty} u_n$, it has been derived by Adomian that

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \qquad n \geq 0. \tag{2.9}$$

From (2.9) , we can write $u_1 = -L^{-1}Ru_0 - L^{-1}A_0$. Thus u_1 can be calculated in terms of the known u_0. Now,

$$\begin{aligned} u_2 &= -L^{-1}Ru_1 - L^{-1}A_1, \\ u_3 &= -L^{-1}Ru_2 - L^{-1}A_2, \end{aligned} \tag{2.10}$$

and so on.

Hence all the terms of u are now calculated and the general solution is obtained as

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.11}$$

Recently, the Adomian Decomposition Method was reviewed and a mathematical model of Adomian polynomials was introduced in [1].

3 Solution of An Extraordinary Fractional Differential Equation

A relationship involving one or more derivatives of an unknown function f with respect to its independent variable x is known as an ordinary differential equation. A similair relationships involving at least one differintegral of noninteger order may be termed as an extraordinary differential equation. Such an equation is solved when an explicit expression for f is exhibited. As with ordinary differential equations, the solutions of extraordinary differential equations often involve integrals and contain arbitrary constants as discussed in [42]. These types of equations are also known as fractional differential equations. The application of extraordinary differential equation is now available in many physical and technical areas [43]. It can be mentioned here that the simplified fractional order differential equation appearing in applied problems is of the form

$$\begin{aligned} D^m y(t) + \lambda D^\alpha y(t) &= ty(t), m = 1,2 \text{ and } 0 < \alpha < 1 \\ y(0) &= k_0, y^{'}(0) = k_1. \end{aligned} \tag{3.12}$$

where k_0 and k_1 are constant

Appling $D^{-m}$to both sides of (3.12), we obtain

$$y(t) + = D^{-m} ty(t) - \lambda D^{\alpha - m} y(t) \tag{3.13}$$

According to the above procedure of solving the fractional differential equations and using Adomian decomposition method, Let

$$y_0(t) = \sum_{i=0}^{1} k_i t^i \tag{3.14}$$

$$\begin{aligned} y_1(t) &= D^{-m} t y_0(t) - \lambda D^{\alpha-m} y_0(t) \\ &= -\left(\lambda - D^{-\alpha} t\right) D^{\alpha-m} y_0(t) \end{aligned} \tag{3.15}$$

$$\begin{aligned} y_2(t) &= D^{-m} t y_1(t) - \lambda D^{\alpha-m} y_1(t) \\ &= (-1)^2 \left(\lambda - D^{-\alpha} t\right)^2 D^{2(\alpha-m)} y_0(t) \end{aligned} \tag{3.16}$$

$$\begin{aligned} y_3(t) &= D^{-m} t y_2(t) - \lambda D^{\alpha-m} y_2(t) \\ &= (-1)^3 \left(\lambda - D^{-\alpha} t\right)^3 D^{3(\alpha-m)} y_0(t) \end{aligned} \tag{3.17}$$

$$\begin{aligned} y_n(t) &= D^{-m} t y_{n-1}(t) - \lambda D^{\alpha-m} y_{n-1}(t) \\ &= (-1)^n \left(\lambda - D^{-\alpha} t\right)^n D^{n(\alpha-m)} y_0(t) \end{aligned} \tag{3.18}$$

Adding all terms,we obtain the solution of the equation by Adomian decomposition method as follows

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{3.19}$$

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} (-1)^n \left(\lambda - D^{-\alpha} t\right)^n D^{n(\alpha-m)} y_0(t) \\ &= \sum_{i=0}^{1} k_i \sum_{n=0}^{\infty} (-1)^n \left(\lambda - D^{-\alpha} t\right)^n D^{n(\alpha-m)} t^i \end{aligned} \tag{3.20}$$

4 Applications and Results

Our objective in this section is to apply the Adomian Decomposition Method to solve some fractional differential equations in physics. This technique can be carried out to solve the problem defined by the problem (3.12). The method is easily applicable via Matlab 7.9.3 program. So we can give the following example.

Example 4.1. *Consider the following fractional differential equation*

$$\begin{aligned} Dy(t) + D^{\frac{1}{2}}y(t) &= ty(t), \\ y(0) &= 1, \end{aligned} \tag{4.21}$$

In the light of the Adomian Decomposition Method, we assume $y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots$ to be the solution of (4.21) where

$$y_0(t) = 1$$

$$\begin{aligned} y_1(t) &= D^{-1}ty_0(t) - D^{-\frac{1}{2}}y_0(t) \\ &= -\left(1 - D^{-\frac{1}{2}}t\right) D^{-\frac{1}{2}}y_0(t) \\ &= -\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right) \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}t^{\frac{1}{2}} \\ &= \frac{16}{3\pi}t^2 - \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}. \end{aligned}$$

$$\begin{aligned} y_2(t) &= D^{-1}ty_1(t) - D^{-\frac{1}{2}}y_1(t) \\ &= (-1)^2\left(1 - D^{-\frac{1}{2}}t\right)^2 D^{2(-\frac{1}{2})}y_0(t) \\ &= (-1)^2\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^2 \frac{\Gamma(1)}{\Gamma(2)}t \\ &= \frac{32}{9\pi}t^4 - \frac{8}{3\sqrt{\pi}}t^{\frac{5}{2}} + \frac{1}{2}t. \end{aligned}$$

$$\begin{aligned} y_3(t) &= D^{-1}ty_2(t) - D^{-\frac{1}{2}}y_2(t) \\ &= (-1)^3\left(1 - D^{-\frac{1}{2}}t\right)^3 D^{3(-\frac{1}{2})}y_0(t) \\ &= (-1)^3\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^3 \frac{\Gamma(1)}{\Gamma\left(\frac{5}{2}\right)}t^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} y_n(t) &= D^{-1}ty_{n-1}(t) - D^{-\frac{1}{2}}y_{n-1}(t) \\ &= (-1)^n\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^n D^{n(-\frac{1}{2})}y_0(t) \\ &= (-1)^n\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^n \frac{\Gamma(1)}{\Gamma\left(\frac{n}{2}+1\right)}t^{\frac{n}{2}} \\ &= E_{\frac{1}{2},1}\left(\frac{8}{3\sqrt{\pi}}t^2 - \sqrt{t}\right) \\ &= \exp\left(\frac{8}{3\sqrt{\pi}}t^2 - \sqrt{t}\right)^2 erf\left(\sqrt{t} - \frac{8}{3\sqrt{\pi}}t^2\right). \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n(t) = \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right)^n \frac{1}{\Gamma\left(\frac{n}{2}+1\right)} t^{\frac{n}{2}} \\ &= \sum_{n=0}^{\infty} \frac{\left(-\left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right) t^{\frac{1}{2}}\right)^n}{\Gamma\left(\frac{n}{2}+1\right)} \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{8}{3\sqrt{\pi}} t^2 - \sqrt{t}\right)^n}{\Gamma\left(\frac{n}{2}+1\right)} \\ &= E_{\frac{1}{2},1}\left(\frac{8}{3\sqrt{\pi}} t^2 - \sqrt{t}\right) \\ &= \exp\left(\frac{8}{3\sqrt{\pi}} t^2 - \sqrt{t}\right)^2 erf\left(\sqrt{t} - \frac{8}{3\sqrt{\pi}} t^2\right). \end{aligned}$$

Example 4.2. *Consider the fractional differential equation*

$$\begin{aligned} D^2 y(t) + D^{\frac{1}{2}} y(t) &= t y(t), \\ y(0) &= 0, y'(0) = 1. \end{aligned} \tag{4.22}$$

In the light of the Adomian decomposition method, we assume $y(t) = y_0(t) + y_1(t) + y_2(t) + ...$ to be the solution of (4.22) where

$$y_0(t) = t$$

$$\begin{aligned} y_1(t) &= D^{-2} t y_0(t) - D^{-\frac{3}{2}} y_0(t) \\ &= -\left(1 - D^{-\frac{1}{2}} t\right) D^{-\frac{3}{2}} y_0(t) \\ &= -\left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right) \frac{\Gamma(2)}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}} \end{aligned}$$

$$\begin{aligned} y_2(t) &= D^{-2} t y_1(t) - D^{-\frac{3}{2}} y_1(t) \\ &= (-1)^2 \left(1 - D^{-\frac{1}{2}} t\right)^2 D^{2\left(-\frac{3}{2}\right)} y_0(t) \\ &= (-1)^2 \left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right)^2 \frac{\Gamma(2)}{\Gamma(5)} t^4 \end{aligned}$$

$$\begin{aligned} y_3(t) &= D^{-2} t y_2(t) - D^{-\frac{3}{2}} y_2(t) \\ &= (-1)^3 \left(1 - D^{-\frac{1}{2}} t\right)^3 D^{3\left(-\frac{3}{2}\right)} y_0(t) \\ &= (-1)^3 \left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right)^3 \frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} t^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} y_n(t) &= D^{-2} t y_{n-1}(t) - D^{-\frac{3}{2}} y_{n-1}(t) \\ &= (-1)^n \left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right)^n \frac{\Gamma(2)}{\Gamma\left(\frac{3n}{2}+2\right)} t^{\frac{3n}{2}+1} \\ &= t.E_{\frac{3}{2},2}\left(\frac{8}{3\sqrt{\pi}} t^3 - t^{\frac{3}{2}}\right) \\ &= t.\exp\left(\frac{8}{3\sqrt{\pi}} t^3 - t^{\frac{3}{2}}\right)^2 erf\left(t^{\frac{3}{2}} - \frac{8}{3\sqrt{\pi}} t^3\right). \end{aligned}$$

Therefore, the solution is

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} y_n(t) \\
&= \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right)^n \frac{\Gamma(2)}{\Gamma\left(\frac{3n}{2}+1\right)} t^{\frac{3n}{2}+1} \\
&= t.\sum_{n=0}^{\infty} \frac{\left(-\left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right) t^{\frac{3}{2}}\right)^n}{\Gamma\left(\frac{3n}{2}+2\right)} \\
&= t.\sum_{n=0}^{\infty} \frac{\left(\frac{8}{3\sqrt{\pi}} t^3 - t^{\frac{3}{2}}\right)^n}{\Gamma\left(\frac{3n}{2}+2\right)} \\
&= t.E_{\frac{3}{2},2}\left(\frac{8}{3\sqrt{\pi}} t^3 - t^{\frac{3}{2}}\right) \\
&= t.\exp\left(\frac{8}{3\sqrt{\pi}} t^3 - t^{\frac{3}{2}}\right)^2 erf\left(t^{\frac{3}{2}} - \frac{8}{3\sqrt{\pi}} t^3\right).
\end{aligned}$$

Conclusion

In this paper, Adomian's Method is effectively implemented to determine the (approximate) analytic solution of the fractional-order extraordinary differential equation. Such a solution is expressed in the form of a series with easily computable components. An illustrative example is presented.

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Received: July 10, 2015; *Accepted*: December 23, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 136–154

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

A note on inventory model for perishable items with trapezoidal type market demand and time-varying holding cost under partial backlogging

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Abstract

In this paper, we studied an inventory model for perishable items with time dependent trapezoidal type demand. Shortages are allowed and partially backlogged with a constant rate. Holding cost is assumed to be linearly dependent with time. The rate of deterioration of the items dependent on both time and life of the products. The numerical solution of the model is obtained. Sensitivity analysis is performed to show the effect of changes in the parameter on the optimum solution.

Keywords: Trapezoidal type demand function, Partial backlogging, Time dependent deterioration rate, Time-varying holding cost.

2010 MSC: 90B05.

1 Introduction

An inventory system with time dependent deteriorating items is one of considerable attention in the recent years. In daily situations such as failure of electric bulb, batteries as they age, expiry of drugs, evaporation of volatile liquids, are common problem to all of us, so, we should not neglect the effect of deterioration on the replenishment policies. In fact, the stock level of inventory is continuously depleting because of the combined effects of its demand and deterioration. In the last few years, many attention has been given to the inventory control system involve with deteriorating items. Ghare and Schrader [1] developed an inventory model by taking into account the effect of deterioration of items in storage. In their model, they introduced a constant deterioration rate, while the demand rate was also taken to be constant. Afterward, Covert and Philip [2] and Tadikamalla [3] extended Ghare and Schrader's work by introducing variable rates of deterioration. Then, immediately Shah [29] provided a further generalization of all these models by considering shortages and using a general distribution for the deterioration rate.

All the above inventory models are based on static environment where the demand is assumed to be constant and steady over a finite replenishment cycle. However, in the real business market scenario, demand should not be constant which is increasing with time during the market growth phase. Then, after reaching its peak, the demand becomes stable for a finite time period. Thereafter, the demand starts decreasing with time. For example, we can easily think that some kind of winter season winter products. In the beginning of the winter season, about October or November, the sale increases up to the month of December and the sale reaches its climatic and maintain this climate sales situation until the end of the winter season. This type of market demand may be approximated by a ramp type demand function. The inventory control with ramp type demand rate first time proposed by Hill [5], they introduced the inventory models for increasing demand followed by a constant demand. Thereafter, Hariga [6] developed an

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inventory lot sizing model for deteriorating items with general continuous time varying demand over a finite planning horizon under three replenishment policies and considered deterioration rate is a constant fraction of the on hand inventory, shortages are allowed and completely back-ordered. Subsequently, several authors discussed inventory models with ramp type demand rates from various aspects. Here, (please see the table-1) we listed some authors those who have used ramp type demand function to study inventory systems in different environments.

Apart from the above discussion, we may think, when some seasonal goods are coming to the market, the demand rate of such type of items may increase with time up to the certain time and then reaches a peak, the demand becomes stable for a finite time period, and finally the market demand rate gradually decreases to a constant or zero. We hope such type of demand is more realistic to construct some EOQ model. This type of demand is named as trapezoidal type demand. Cheng and Wang [27] first introduced trapezoidal type demand. They extended Hill's [5] ramp type demand rate to trapezoidal type demand. Subsequently, several authors discussed inventory models with trapezoidal type demand rates from different angles. Here we listed (please see the table-II) some authors those who have considered a trapezoidal type demand function to formulate some EOQ models in different domain.

All articles given in table-2, shows that, all the researchers studied economic order quantity model by considering the trapezoidal type demand, deterioration (constant/linear/Weibull), shortages (allowed/not allowed), backlogging (partial/complete), and constant holding costs. However, always constant holding cost may not help to develop a better approximate EOQ model in real life scenario, perhaps. So, holding cost may not be constant over time always, as there is a change in time value of money and change in the price index.

Hence, the motivation behind this article is, to prepare a more general inventory model, which includes; (a) Trapezoidal type demand, which is piecewise linear continuous function with time (b) Shortages are allowed with partially backlogged, and backlogging rate is constant (c) Deterioration depends on both time and life on an item, which is reflected more realistic than constant. (d) Linear increasing holding cost with time.

2 Notations and assumptions

The model is based on following assumptions and notations:

1. The demand rate $D(t)$ is assumed to be a trapezoidal type function, which is piecewise linear continuous with time , defined as follows;
$$D(t) = \begin{cases} a_1 + b_1 t & \text{if } 0 \le t \le \mu_1 \\ D_0 & \text{if } \mu_1 \le t \le \mu_2 \\ a_2 - b_2 t & \text{if } \mu_2 \le t \le T \le \frac{a_2}{b_2} \end{cases}$$
where μ_1 is the point in time axis, when demand reaches peak position and maintain constant, and μ_2 is the point in time axis, when demand start decreases.
2. The replenishment rate is infinite, thus replenishment is instantaneous, i. e. lead time is zero.
3. T is the length of each ordering cycle.
4. $I(t)$ is level of inventory at time t, $0 \le t \le T$.
5. $S = I(0)$ is the maximum inventory level for the ordering cycle.
6. $\theta(t) = \frac{1}{1+R-t}$ is the deteriorating rate of inventory items, where R is the maximum life time of item.
7. t_1 is the time when the inventory level reaches zero due to both demand and deterioration.
8. Shortage is allowed and partially backlogged.
9. β is the backlogging rate; $0 \le \beta \le 1$, if β is 1 or 0, then shortage is completely backlogged or lost.
10. $H(t) = h + \alpha t$ is the holding cost, where $\alpha > 0$, $h > 0$.
11. c_1 is the constant shortage cost per unit per unit time.

12. c is the constant purchasing cost per unit.
13. L is the constant lost sale cost per unit.
14. A is the fixed ordering cost per order.
15. $C_1(t_1)$ is the total average cost per unit (when $0 \leq t_1 \leq \mu_1$).
16. $C_2(t_1)$ is the total average cost per unit (when $\mu_1 \leq t_1 \leq \mu_2$).
17. $C_3(t_1)$ is the total average cost per unit (when $\mu_2 \leq t_1 < T$).
18. t_1^* is the optimal time, when the inventory level reaches zero.

3 Formulation of mathematical model and its solutions

Here, we consider the time dependent deteriorating inventory model with trapezoidal type demand rate. Inventory level attains maximum at $t = 0$, when replenishment occurs. From $t = 0$ to $t = t_1$, the level of inventory reduces due to both demand and deterioration. At t_1 the inventory level reaches zero, then shortage starts occurring during the time interval (t_1, T), and all the demand during the shortage period (t_1, T) is partially backlogged with constant backlogging rate (β), $(0 \leq \beta \leq 1)$. The total number of backlogged items is replaced by the next replenishment. The rate of change of the inventory during the positive stock period $(0, t_1)$ and shortage period (t_1, T) is described by the following differential equations:

$$\frac{dI(t)}{dt} = -\theta(t)I(t) - D(t), \;\; 0 < t < t_1 \tag{3.1}$$

and

$$\frac{dI(t)}{dt} = -\beta D(t), \;\; t_1 < t < T. \tag{3.2}$$

with boundary condition $I(t_1) = 0$.
As per the nature of the demand function, our work can be completed through three cases, because, the shortage of inventory may occur during $(0, \mu_1]$, or $[\mu_1, \mu_2]$, or $[\mu_2, T)$. Hence, to make a complete study of the inventory model, we should take care about all three cases. These three cases are given as follows.

3.1 Case-I $(0 < t \leq \mu_1)$

Due to demand and deterioration, the inventory level gradually decreases during the time interval $(0, t_1]$ and finally falls to zero at time $t = t_1$, i. e. shortage starts during $(0, \mu_1]$. Hence equations (3.1) and (3.2) reduce to

$$\frac{dI(t)}{dt} = -\frac{I(t)}{1+R-t} - (a_1 + b_1 t), \;\; 0 < t < t_1, \tag{3.3}$$

$$\frac{dI(t)}{dt} = -(a_1 + b_1 t)\beta, \;\; t_1 < t < \mu_1, \tag{3.4}$$

$$\frac{dI(t)}{dt} = -D_0\beta, \;\; \mu_1 < t < \mu_2, \tag{3.5}$$

and

$$\frac{dI(t)}{dt} = -(a_2 - b_2 t)\beta, \;\; \mu_2 < t < T. \tag{3.6}$$

Solving the above differential equations (3.3-3.6) with the condition $I(t_1) = 0$ and continuity property of $I(t)$, we get

$$I(t) = (1+R-t)\left[a_1 \ln\left(\frac{1+R-t}{1+R-t_1}\right) + b_1(1+R)\ln\left(\frac{1+R-t}{1+R-t_1}\right) + b_1(t-t_1)\right], \; 0 \leq t \leq t_1, \tag{3.7}$$

$$I(t) = \beta a_1(t_1 - t) + \beta\frac{b_1}{2}(t_1^2 - t^2), \; t_1 \leq t \leq \mu_1, \tag{3.8}$$

$$I(t) = -D_0\beta t + a_1\beta t_1 + \beta\frac{b_1}{2}(t_1^2 + \mu_1^2), \; \mu_1 \leq t \leq \mu_2, \tag{3.9}$$

and

$$I(t) = \beta a_1 t_1 - \beta a_2 t + \beta\frac{b_2}{2}(t^2 + \mu_2^2) + \beta\frac{b_1}{2}(t_1^2 + \mu_1^2),\ \mu_2 \leq t \leq T. \tag{3.10}$$

The beginning inventory level can be obtained as

$$S = I(0) = (1+R)\left[\ln\left(\frac{1+R}{1+R-t_1}\right)(a_1 + b_1(1+R)) - b_1 t_1\right]. \tag{3.11}$$

Inventory is available in the system during the time period $(0, t_1)$. So, the cost for holding inventory in stock is computed for time period $(0, t_1)$ only.
Holding cost is as follows:

$$\begin{aligned}
HC &= \int_0^{t_1} H(t)I(t)dt \\
&= \int_0^{t_1} (h+\alpha t)(1+R-t)\left[\ln\left(\frac{1+R-t}{1+R-t_1}\right)(a_1 + b_1(1+R)) + b_1(t-t_1)\right]dt \\
&= (a_1 + b_1(1+R))(t_1 - (1+R))\left[t_1 + (1+R)\ln\left(\frac{1+R-t_1}{1+R}\right) + \frac{(R\alpha+\alpha-h)}{2}\left[(1+R)t_1 + \frac{t_1^2}{2}\right.\right. \\
&+ \left.\left.(1+R)^2\ln\left(\frac{1+R-t_1}{1+R}\right)\right] - \alpha\left[(1+R)^2 t_1 + (1+R)\frac{t_1^2}{2} + \frac{t_1^3}{3} + (1+R)^3\ln\left(\frac{1+R-t_1}{1+R}\right)\right]\right] \\
&+ b_1\left(\frac{\alpha}{12}t_1^4 - \frac{(R\alpha+\alpha-h)}{6}t_1^3 - \frac{h(1+R)}{2}t_1^2\right).
\end{aligned} \tag{3.12}$$

Shortage due to stock out is accumulated in the system during the time period (t_1, T). The optimum level of shortage is occur at $t = T$, hence, the total shortage cost during the above mentioned time period is as follows:

$$\begin{aligned}
SC &= c_1\int_{t_1}^{T} -I(t)dt \\
&= c_1\left[-\int_{t_1}^{\mu_1} I(t)dt - \int_{\mu_1}^{\mu_2} I(t)dt - \int_{\mu_2}^{T} I(t)dt\right] \\
&= -c_1\left[\int_{t_1}^{\mu_1}\left(\beta a_1(t_1 - t) + \beta\frac{b_1}{2}(t_1^2 - t^2)\right)dt + \int_{\mu_1}^{\mu_2}\left(-D_0\beta t + \beta a_1 t_1 + \beta\frac{b_1}{2}(t_1^1 + \mu_1^2)\right)dt\right. \\
&+ \left.\int_{\mu_2}^{T}\left(\beta a_1 t_1 - \beta a_2 t + \beta\frac{b_2}{2}(t^2 + \mu_2^2) + \beta\frac{b_1}{2}(t_1^2 + \mu_1^2)\right)dt\right] \\
&= c_1\left[\beta\frac{a_1}{2}(t_1 - \mu_1)(t_1 + \mu_1 - 2T) + \beta\frac{b_1}{6}(2t_1^3 - 2\mu_1^3 + 3T\mu_1^2 - 3Tt_1^2) + \beta\frac{a_2}{2}(\mu_2^2 - T^2)\right. \\
&+ \left.\beta\frac{b_2}{6}(3T\mu_2^2 - T^3 - 2\mu_2^3) + \beta\frac{D_0}{2}(\mu_1 - \mu_2)(\mu_1 + \mu_2 - 2T)\right].
\end{aligned} \tag{3.13}$$

Due to stock out during the time period (t_1, T), shortage is accumulated, but not all customers are willing to wait for the next lot size to arrive. Hence, this results in some loss of sale which accounts to loss in profit.
Lost sale cost is calculated as follows:

$$\begin{aligned}
LSC &= L\int_{t_1}^{T}(1-\beta)D(t)dt \\
&= L(1-\beta)\left[\int_{t_1}^{\mu_1} D(t)dt + \int_{\mu_1}^{\mu_2} D(t)dt + \int_{\mu_2}^{T} D(t)dt\right] \\
&= L(1-\beta)\left[a_1(\mu_1 - t_1) + \frac{b_1}{2}(\mu_1^2 - t_1^2) + D_0(\mu_2 - \mu_1) + a_2(T - \mu_2) - \frac{b_2}{2}(T^2 - \mu_2^2)\right].
\end{aligned} \tag{3.14}$$

Purchase cost is as follows:

$$\begin{aligned}
PC &= c\left[I(0) + \int_{t_1}^{T}\beta D(t)dt\right] \\
&= c(1+R)\left[a_1\ln\left(\frac{1+R}{1+R-t_1}\right) + b_1(1+R)\ln\left(\frac{1+R}{1+R-t_1}\right) - b_1 t_1\right] + c\beta\left[a_1(\mu_1 - t_1)\right. \\
&+ \left.\frac{b_1}{2}(\mu_1^2 - t_1^2) + D_0(\mu_2 - \mu_1) + a_2(T - \mu_2) - \frac{b_2}{2}(T^2 - \mu_2^2)\right].
\end{aligned} \tag{3.15}$$

The total average cost is given by

$$\begin{aligned} C_1(t_1) &= \frac{1}{T}\Big[A+PC+HC+SC+SLC\Big] \\ &= \frac{1}{T}\Big[A+c(1+R)\Big[a_1\ln\left(\frac{1+R}{1+R-t_1}\right)+b_1(1+R)\ln\left(\frac{1+R}{1+R-t_1}\right)-b_1t_1\Big] \\ &+ c\beta\Big[a_1(\mu_1-t_1)+\frac{b_1}{2}(\mu_1^2-t_1^2)+D_0(\mu_2-\mu_1)+a_2(T-\mu_2)-\frac{b_2}{2}(T^2-\mu_2^2)\Big] \\ &+ L(1-\beta)\Big[a_1(\mu_1-t_1)+\frac{b_1}{2}(\mu_1^2-t_1^2)+D_0(\mu_2-\mu_1)+a_2(T-\mu_2)-\frac{b_2}{2}(T^2-\mu_2^2)\Big] \\ &+ c_1\beta\frac{a_1}{2}(t_1-\mu_1)(t_1-\mu_1-2T)+c_1\frac{\beta b_1}{6}(2t_1^3-2\mu_2^3+3T\mu_1^2-3Tt_1^2) \\ &+ c_1\beta\frac{a_2}{2}(\mu_2-T)^2+c_1\beta\frac{b_2}{6}(3T\mu_2^2-T^3-2\mu_2^3)+c_1\beta\frac{D_0}{2}(\mu_1-\mu_2)(\mu_1+\mu_2-2T) \\ &+ (a_1+b_1(1+R))(t_1-(1+R))\Big[t_1+(1+R)\ln\left(\frac{1+R-t_1}{1+R}\right)+\frac{(R\alpha+\alpha-h)}{2}\Big[(1+R)t_1+\frac{t_1^2}{2} \\ &+ (1+R)^2\ln\left(\frac{1+R-t_1}{1+R}\right)\Big]-\alpha\left((1+R)^2t_1+(1+R)\frac{t_1^2}{2}+\frac{t_1^3}{3}+(1+R)^3\ln\left(\frac{1+R-t_1}{1+R}\right)\right)\Big] \\ &+ b_1\left(\frac{\alpha}{12}t_1^4-\frac{(R\alpha+\alpha-h)}{6}t_1^3-\frac{h(1+R)}{2}t_1^2\right)\Big]. \end{aligned} \tag{3.16}$$

Equation (3.16) is highly non linear in nature with t_1. We can find the optimum values of t_1 for minimum average cost $C_1(t_1)$ from the solutions of the following equations by the help of Mathematica 10,

$$\frac{dC_1(t_1)}{dt_1}=0. \tag{3.17}$$

3.2 Case-II (for $t_1 \in [\mu_1, \mu_2]$)

The differential equations governing the inventory model can be expressed as follows:

$$\frac{dI(t)}{dt}=-\frac{I(t)}{1+R-t}-(a_1+b_1t),\ \ 0<t<\mu_1, \tag{3.18}$$

$$\frac{dI(t)}{dt}=-\frac{I(t)}{1+R-t}-D_0,\ \ \mu_1<t<t_1, \tag{3.19}$$

$$\frac{dI(t)}{dt}=-D_0\beta,\ \ t_1<t<\mu_2 \tag{3.20}$$

and

$$\frac{dI(t)}{dt}=-\beta(a_2-b_2t),\ \ \mu_2<t<T. \tag{3.21}$$

Solving the above differential equations (3.18-3.21) with the help of $I(t_1)=0$ and continuity property of $I(t)$, we obtain

$$\begin{aligned} I(t) &= (1+R-t)\Big[a_1\ln\left(\frac{1+R-t}{1+R-t_1}\right)+b_1(1+R)\ln\left(\frac{1+R-t}{1+R-\mu_1}\right) \\ &+ b_1\mu_1\ln\left(\frac{1+R-\mu_1}{1+R-t_1}\right)+b_1(t-\mu_1)\Big],\ \ 0\le t\le\mu_1, \end{aligned} \tag{3.22}$$

$$I(t)=(1+R-t)D_0\ln\left(\frac{1+R-t}{1+R-t_1}\right),\ \ \mu_1\le t\le t_1, \tag{3.23}$$

$$I(t)=\beta D_0(t_1-t),\ \ t_1\le t\le\mu_2 \tag{3.24}$$

and

$$I(t)=-\beta a_2(t-t_1)-\beta b_2\mu_2t_1+\frac{\beta b_2}{2}(t^2+\mu_2^2),\ \ \mu_2\le t\le T. \tag{3.25}$$

The beginning inventory level can be obtained as

$$\begin{aligned} S &= I(0) \\ &= (1+R)\left[a_1 \ln\left(\frac{1+R}{1+R-t_1}\right) + b_1(1+R) + b_1(1+R)\ln\left(\frac{1+R}{1+R-t_1}\right)\right. \\ &+ \left. b_1\mu_1 \ln\left(\frac{1+R-\mu_1}{1+R-t_1}\right) - b_1\mu_1\right]. \end{aligned} \tag{3.26}$$

The total cost per ordering cycle is consists by following five different costs, these are as follows:

1. Ordering cost.

$$OC = A. \tag{3.27}$$

2. Holding cost

$$\begin{aligned} HC &= \int_0^{\mu_1} (h+\alpha t)(1+R-t)\left[a_1 \ln\left(\frac{1+R-t}{1+R-t_1}\right) + b_1(1+R)\ln\left(\frac{1+R-t}{1+R-\mu_1}\right)\right. \\ &+ \left. b_1\mu_1 \ln\left(\frac{1+R-\mu_1}{1+R-t_1}\right) + b_1(t-\mu_1)\right]dt + \int_{\mu_1}^{t_1} (h+\alpha t) b_0 \ln\left(\frac{1+R-t}{1+R-t_1}\right) dt \\ &= \left[a_1 \ln\left(\frac{1+R-\mu_1}{1+R-t_1}\right) + b_1\mu_1 \ln\left(\frac{1+R-\mu_1}{1+R-t_1}\right) - b_1\mu_1\right]\left[h(1+R)\mu_1 + (R\alpha+\alpha-h)\frac{\mu_1^2}{2} - \frac{\alpha\mu_1^3}{3}\right] \\ &+ \left[ah(1+R)(t_1-(1+R)) + b_1 h(1+R)^2(\mu_1-(1+R))\right]\left[\mu_1 + (1+R)\ln\left(\frac{1+R-\mu_1}{1+R}\right)\right] \\ &+ \left[(1+R)\mu_1 + \frac{\mu_1^2}{2} + (1+R)^2 \ln\left(\frac{1+R-\mu_1}{1+R}\right)\right]\left[a_1(t_1-(1+R))\left(\frac{R\alpha+\alpha-h}{2}\right)\right. \\ &+ \left. 3(1+R)(\mu_1-(1+R))\left(\frac{R\alpha+\alpha-h}{2}\right)\right] - \left[(1+R)^2\mu_1 + \left(\frac{1+R}{2}\right)\mu_1^2 + \frac{\mu_1^3}{3}\right] \\ &\times \left[\frac{\alpha}{3}a_1(t_1-(1+R)) + \frac{\alpha}{3}b_1(1+R)(\mu_1-(1+R))\right] + b_1 h(1+R)\frac{\mu_1^2}{2} + (R\alpha+\alpha-h)b_1\frac{\mu_1^3}{3} \\ &- \frac{\alpha b_1 \mu_1^4}{4} + D_0(t_1-(1+R))h\left[(t_1-\mu_1) + (1+R)\ln\left(\frac{1+R-t_1}{1+R-\mu_1}\right)\right] \\ &+ (t_1-(1+R))\frac{\alpha D_0}{2}\left[(1+R)(t_1-\mu_1) + \frac{t_1^2-\mu_1^2}{2} + (1+R)^2\ln\left(\frac{1+R-t_1}{1+R-\mu_1}\right)\right]. \end{aligned} \tag{3.28}$$

3. Purchase cost

$$\begin{aligned} PC &= c\left[(1+R)\left(a_1 \ln\left(\frac{1+R}{1+R-t_1}\right) + b_1(1+R)\ln\left(\frac{1+R}{1+R-t_1}\right) + b_1\mu_1\ln\left(\frac{1+R-\mu_1}{1+R-t_1}\right) - b_1\mu_1\right)\right. \\ &+ \left. \beta D_0(\mu_2-t_1) + \beta a_2(T-\mu_2) - \frac{\beta b_2}{2}(T^2-\mu_2^2)\right]. \end{aligned} \tag{3.29}$$

4. Shortage cost

$$\begin{aligned} SC &= \frac{c_1\beta D_0}{2}(t_1-\mu_2)^2 + \frac{c_1\beta a_2}{2}\left[(T-t_1^2) - (\mu_2-t_1^2)\right] \\ &+ c_1\beta b_2\mu_2 t_1(T-\mu_2) + \frac{c_1\beta b_2}{2}\left[\frac{T^3-\mu_2^3}{3} + \mu_2^2(T-\mu_2)\right]. \end{aligned} \tag{3.30}$$

5. Lost sale cost

$$LSC = L(1-\beta)\left[D_0(\mu_2-t_1) + a_2(T-\mu_2) - \frac{b_2}{2}(T^2-\mu_2^2)\right]. \tag{3.31}$$

The total average cost is given by

$$
\begin{aligned}
C_2(t_1) &= \frac{1}{T}\Big[OC+HC+PC+SC+LSC\Big]\\
&= \frac{1}{T}\Bigg[A+c\Bigg[(1+R)\left(a_1\ln\left(\frac{1+R}{1+R-t_1}\right)+b_1(1+R)\ln\left(\frac{1+R}{1+R-t_1}\right)+b_1\mu_1\ln\left(\frac{1+R-\mu_1}{1+R-t_1}\right)-b_1\mu_1\right)\\
&+ \beta D_0(\mu_2-t_1)+\beta a_2(T-\mu_2)-\frac{\beta b_2}{2}(T^2-\mu_2^2)\Bigg]\\
&+ L(1-\beta)\left[D_0(\mu_2-t_1)+a_2(T-\mu_2)-\frac{b_2}{2}(T^2-\mu_2^2)\right]\\
&+ \frac{c_1\beta D_0}{2}(t_1-\mu_2)^2+\frac{c_1\beta a_2}{2}\left[(T-t_1^2)-(\mu_2-t_1^2)\right]\\
&+ c_1\beta b_2\mu_2 t_1(T-\mu_2)+\frac{c_1\beta b_2}{2}\left[\frac{T^3-\mu_2^3}{3}+\mu_2^2(T-\mu_2)\right]\\
&+ \left[a_1\ln\left(\frac{1+R-\mu_1}{1+R-t_1}\right)+b_1\mu_1\ln\left(\frac{1+R-\mu_1}{1+R-t_1}\right)-b_1\mu_1\right]\left[h(1+R)\mu_1+(R\alpha+\alpha-h)\frac{\mu_1^2}{2}-\frac{\alpha\mu_1^3}{3}\right]\\
&+ \left[ah(1+R)(t_1-(1+R))+b_1h(1+R)^2(\mu_1-(1+R))\right]\left[\mu_1+(1+R)\ln\left(\frac{1+R-\mu_1}{1+R}\right)\right]\\
&+ \left[(1+R)\mu_1+\frac{\mu_1^2}{2}+(1+R)^2\ln\left(\frac{1+R-\mu_1}{1+R}\right)\right]\left[a_1(t_1-(1+R))\left(\frac{R\alpha+\alpha-h}{2}\right)\right.\\
&+ \left.3(1+R)(\mu_1-(1+R))\left(\frac{R\alpha+\alpha-h}{2}\right)\right]-\left[(1+R)^2\mu_1+\left(\frac{1+R}{2}\right)\mu_1^2+\frac{\mu_1^3}{3}\right]\\
&\times \left[\frac{\alpha}{3}a_1(t_1-(1+R))+\frac{\alpha}{3}b_1(1+R)(\mu_1-(1+R))\right]+b_1h(1+R)\frac{\mu_1^2}{2}+(R\alpha+\alpha-h)b_1\frac{\mu_1^3}{3}\\
&- \frac{\alpha b_1\mu_1^4}{4}+D_0(t_1-(1+R))h\left[(t_1-\mu_1)+(1+R)\ln\left(\frac{1+R-t_1}{1+R-\mu_1}\right)\right]\\
&+ (t_1-(1+R))\frac{\alpha D_0}{2}\left[(1+R)(t_1-\mu_1)+\frac{t_1^2-\mu_1^2}{2}+(1+R)^2\ln\left(\frac{1+R-t_1}{1+R-\mu_1}\right)\right]. \qquad (3.32)
\end{aligned}
$$

Equation (3.32) is highly non linear in nature with t_1. We can find the optimum values of t_1 for minimum average cost $C_2(t_1)$ from the solutions of the following equations by the help of Mathematica 10,

$$\frac{dC_2(t_1)}{dt_1}=0. \qquad (3.33)$$

3.3 Case-III ($t_1 \in [\mu_2, T]$)

The differential equation governing the inventory model can be expressed as follows:

$$\frac{dI(t)}{dt}=-\frac{I(t)}{1+R-t}-(a_1+b_1t),\;\; 0<t<\mu_1, \qquad (3.34)$$

$$\frac{dI(t)}{dt}=-\frac{I(t)}{1+R-t}-D_0,\;\; \mu_1\le t<\mu_2, \qquad (3.35)$$

$$\frac{dI(t)}{dt}=-\frac{I(t)}{1+R-t}-(a_2-b_2t),\;\; \mu_2\le t<t_1 \qquad (3.36)$$

and

$$\frac{dI(t)}{dt}=-\beta(a_2-b_2t),\;\; t_1<t<T. \qquad (3.37)$$

Solving the above differential equations (3.34-3.37) with the help of $I(t_1)=0$ and continuity property of $I(t)$, we obtain

$$
\begin{aligned}
I(t) &= (1+R-t)\Big[a_1\ln(1+R-t)+b_1(1+R)\ln(1+R-t)+b_1t+(b_2(1+R)-a_2)\ln(1+R-t_1)\\
&- b_2R\ln(1+R-\mu_2)-b_2(t_1-\mu_2)-b_1R\ln(1+R-\mu_1)-b_1\mu_1\Big],\;\; 0\le t<\mu_1, \qquad (3.38)
\end{aligned}
$$

$$\begin{aligned} I(t) &= (1+R-t)\Big[D_0\ln(1+R-t)+(b_2(1+R)-a_2)\ln(1+R-t_1)\\ &- b_2R\ln(1+R-\mu_2)+b_2(t_1-\mu_2)\Big],\ \ \mu_1\le t\le \mu_2, \end{aligned} \tag{3.39}$$

$$I(t)=(1+R-t)\left[a_2\ln\left(\frac{1+R-t}{1+R-t_1}\right)-b_2(1+R)\ln\left(\frac{1+R-t}{1+R-t_1}\right)+b_2(t_1-t),\ \mu_2\le t\le t_1\right. \tag{3.40}$$

and

$$I(t)=\beta a_2(t_1-t)+\frac{\beta b_2}{2}(t^2-t_1^2),\ \ t_1<t<T. \tag{3.41}$$

In this case the begging inventory level can be obtained as

$$\begin{aligned} S &= I(0)\\ &= (1+R)\Big[a_1\ln(1+R)+b_1(1+R)\ln(1+R)+(b_2(1+R)-a_2)\ln(1+R-t_1)\\ &- b_2R\ln(1+R-\mu_2)-b_2(t_1-\mu_2)-b_1R\ln(1+R-\mu_1)-b_1\mu_1\Big]. \end{aligned} \tag{3.42}$$

The total cost per order cycle is consist by following different costs, these are as follows:

1. Ordering cost

$$OC=A. \tag{3.43}$$

2. Holding cost

$$
\begin{aligned}
HC &= \ln(1+R-\mu_1)\left[h(1+R)\mu_1+(R\alpha+\alpha-h)\frac{\mu_1^2}{2}-\frac{2\mu_1^3}{3}\right](a_1+b_1(1+R))-(a_1+b_1(1+R))h(1+R)\\
&\times \left[\mu_1+(1+R)\ln\left(\frac{1+R-\mu_1}{1+R}\right)\right]-(a_1+b_1(1+R))\left(\frac{R\alpha+\alpha-h}{2}\right)\left[(1+R)\mu_1\right.\\
&+ \left.\frac{\mu_1^2}{2}+(1+R)^2\ln\left(\frac{1+R-\mu_1}{1+R}\right)\right]+(a_1+b_1(1+R))\frac{\alpha}{3}\left[(1+R)^2\mu_1+(1+R)\frac{\mu_1^2}{2}\right.\\
&+ \left.\frac{\mu_1^3}{3}+(1+R)^3\ln\left(\frac{1+R-\mu_1}{1+R}\right)\right]+h(1+R)b_1\frac{\mu_1^2}{2}+(R\alpha+\alpha-h)\frac{b_1\mu_1^3}{3}-\frac{\alpha b_1\mu_1^4}{4}\\
&+ \left[(b_1(1+R)-a_2)\ln(1+R-t_1)-b_2R\ln(1+R-\mu_2)-b_2(t_1-\mu_2)-b_1R\ln(1+R-\mu_1)\right.\\
&- \left.b_1\mu_1\right]\left[h(1+R)\mu_1+(R\alpha+\alpha-h)\frac{\mu_1^2}{2}-\frac{\alpha\mu_1^3}{3}\right]+D_0\ln\left(\frac{1+R-\mu_2}{1+R-\mu_1}\right)\left[h(1+R)(\mu_2-\mu_1)\right.\\
&+ \left.(R\alpha+\alpha-h)\frac{\mu_2^2-\mu_1^2}{2}-\frac{\alpha(\mu_2^3-\mu_1^3)}{3}\right]+D_0h(1+R)\left[\mu_2-\mu_1+(1+R)\ln\left(\frac{1+R-\mu_2}{1+R-\mu_1}\right)\right]\\
&+ D_0\left(\frac{R\alpha+\alpha-h}{2}\right)\left[(1+R)(\mu_2-\mu_1)+\frac{\mu_2^2-\mu_1^2}{2}+(1+R)^2\ln\left(\frac{1+R-\mu_2}{1+R\mu_1}\right)\right]\\
&- \frac{\alpha}{3}D_0\left[(1+R)^2(\mu_2-\mu_1)+\frac{1+R}{2}(\mu_2^2-\mu_1^2)+\frac{\mu_2^3-\mu_1^3}{3}\right]+\left[(b_2(1+R)-a_2)\ln(1+R-t_1)\right.\\
&- \left.b_2R\ln(1+R-\mu_2)+b_2(t_1-\mu_2)\right]\left[h(1+R)(\mu_2-\mu_1)+\frac{R\alpha+\alpha-h}{2}(\mu_2^2-\mu_1^2)-\frac{\alpha}{3}(\mu_2^3-\mu_1^3)\right]\\
&+ (a_2-b_2(1+R))(1+R-t_1)\frac{\alpha}{3}\left[(1+R)^2(t_1-\mu_2)+\frac{1+R}{2}(t_1^2-\mu_2^2)+(1+R)^3\ln\left(\frac{1+R-t_1}{1+R-\mu_2}\right)\right]\\
&- (a_2-b_2(1+R))\ln\left(\frac{1+R-\mu_2}{1+R-t_1}\right)\left[h(1+R)\mu_2+(R\alpha+\alpha-h)\frac{\mu_2^2}{2}-\frac{\alpha\mu_2^3}{3}\right]\\
&- (a_2-b_2(1+R))(1+R-t_1)h(1+R)\left[t_1-\mu_2+(1+R)\ln\left(\frac{1+R-t_1}{1+R-\mu_2}\right)\right]\\
&- (a_2-b_2(1+R))(1+R-t_1)\frac{R\alpha+\alpha-h}{2}\left[(1+R)(t_1-\mu_2)+\frac{t_1^2-\mu_2^2}{2}+(1+R)^2\ln\left(\frac{1+R-t_1}{1+R-\mu_2}\right)\right]\\
&+ h(1+R)b_2t_1(t_1-\mu_2)+(R\alpha+\alpha-h)b_2t_1\frac{t_1^2-\mu_2^2}{2}-\alpha b_2t_1\frac{t_1^3-\mu_2^3}{3}\\
&+ h(1+R)b_2\frac{t_1^2-\mu_2^2}{2}+(R\alpha+\alpha-h)b_2\frac{t_1^3-\mu_2^3}{3}-\alpha b_2\frac{t_1^4-\mu_2^4}{4}. \qquad (3.44)
\end{aligned}
$$

3. Purchase cost

$$
\begin{aligned}
PC &= c\left[(1+R)\left[a_1\ln(1+R)+b_1(1+R)\ln(1+R)+(b_2(1+R)-a_2)\ln(1+R-t_1)+\beta a_2(T-t_1)\right.\right.\\
&- \left.\left.b_2R\ln(1+R-\mu_2)-b_2(t_1-\mu_2)-b_1R\ln(1+R-\mu_1)-b_1\mu_1\right]-\beta\frac{b_2}{2}(T^2-t_1^2)\right]. \qquad (3.45)
\end{aligned}
$$

4. Shortage cost

$$SC = c_1\left[\beta a_2t_1(t_1-T)+\beta a_2\frac{T^2-t_1^2}{2}-\frac{\beta b_2}{6}(T^3-t_1^3)+\frac{\beta b_2}{2}t_1^2(T-t_1)\right]. \qquad (3.46)$$

5. Lost sale cost

$$LSC = L(1-\beta)\left[a_2(T-t_1)-\frac{b_2}{2}(T^2-t_1^2)\right]. \qquad (3.47)$$

The total average cost is given by

$$
\begin{aligned}
C_3(t_1) &= \frac{1}{T}\Big[A + \ln(1+R-\mu_1)\Big[h(1+R)\mu_1 + (R\alpha+\alpha-h)\frac{\mu_1^2}{2} - \frac{2\mu_1^3}{3}\Big](a_1+b_1(1+R)) - (a_1+b_1(1+R)) \\
&\times h(1+R)\mathit{bigg}[\mu_1 + (1+R)\ln\left(\frac{1+R-\mu_1}{1+R}\right)\Big] - (a_1+b_1(1+R))\left(\frac{R\alpha+\alpha-h}{2}\right)\Big[(1+R)\mu_1 \\
&+ \frac{\mu_1^2}{2} + (1+R)^2\ln\left(\frac{1+R-\mu_1}{1+R}\right)\Big] + (a_1+b_1(1+R))\frac{\alpha}{3}\Big[(1+R)^2\mu_1 + (1+R)\frac{\mu_1^2}{2} \\
&+ \frac{\mu_1^3}{3} + (1+R)^3\ln\left(\frac{1+R-\mu_1}{1+R}\right)\Big] + h(1+R)b_1\frac{\mu_1^2}{2} + (R\alpha+\alpha-h)\frac{b_1\mu_1^3}{3} - \frac{\alpha b_1\mu_1^4}{4} \\
&+ \Big[(b_1(1+R)-a_2)\ln(1+R-t_1) - b_2R\ln(1+R-\mu_2) - b_2(t_1-\mu_2) - b_1R\ln(1+R-\mu_1) \\
&- b_1\mu_1\Big]\Big[h(1+R)\mu_1 + (R\alpha+\alpha-h)\frac{\mu_1^2}{2} - \frac{\alpha\mu_1^3}{3}\Big] + D_0\ln\left(\frac{1+R-\mu_2}{1+R-\mu_1}\right)\Big[h(1+R)(\mu_2-\mu_1) \\
&+ (R\alpha+\alpha-h)\frac{\mu_2^2-\mu_1^2}{2} - \frac{\alpha(\mu_2^3-\mu_1^3)}{3}\Big] + D_0h(1+R)\Big[\mu_2-\mu_1+(1+R)\ln\left(\frac{1+R-\mu_2}{1+R-\mu_1}\right)\Big] \\
&+ D_0\left(\frac{R\alpha+\alpha-h}{2}\right)\Big[(1+R)(\mu_2-\mu_1) + \frac{\mu_2^2-\mu_1^2}{2} + (1+R)^2\ln\left(\frac{1+R-\mu_2}{1+R\mu_1}\right)\Big] \\
&- \frac{\alpha}{3}D_0\Big[(1+R)^2(\mu_2-\mu_1) + \frac{1+R}{2}(\mu_2^2-\mu_1^2) + \frac{\mu_2^3-\mu_1^3}{3}\Big] + \Big[(b_2(1+R)-a_2)\ln(1+R-t_1) \\
&- b_2R\ln(1+R-\mu_2) + b_2(t_1-\mu_2)\Big]\Big[h(1+R)(\mu_2-\mu_1) + \frac{R\alpha+\alpha-h}{2}(\mu_2^2-\mu_1^2) - \frac{\alpha}{3}(\mu_2^3-\mu_1^3)\Big] \\
&+ (a_2-b_2(1+R))(1+R-t_1)\frac{\alpha}{3}\Big[(1+R)^2(t_1-\mu_2) + \frac{1+R}{2}(t_1^2-\mu_2^2) + (1+R)^3\ln\left(\frac{1+R-t_1}{1+R-\mu_2}\right)\Big] \\
&- (a_2-b_2(1+R))\ln\left(\frac{1+R-\mu_2}{1+R-t_1}\right)\Big[h(1+R)\mu_2 + (R\alpha+\alpha-h)\frac{\mu_2^2}{2} - \frac{\alpha\mu_2^3}{3}\Big] \\
&- (a_2-b_2(1+R))(1+R-t_1)h(1+R)\Big[t_1-\mu_2+(1+R)\ln\left(\frac{1+R-t_1}{1+R-\mu_2}\right)\Big] \\
&- (a_2-b_2(1+R))(1+R-t_1)\frac{R\alpha+\alpha-h}{2}\Big[(1+R)(t_1-\mu_2) + \frac{t_1^2-\mu_2^2}{2} + (1+R)^2\ln\left(\frac{1+R-t_1}{1+R-\mu_2}\right)\Big] \\
&+ h(1+R)b_2t_1(t_1-\mu_2) + (R\alpha+\alpha-h)b_2t_1\frac{t_1^2-\mu_2^2}{2} - \alpha b_2t_1\frac{t_1^3-\mu_2^3}{3} \\
&+ h(1+R)b_2\frac{t_1^2-\mu_2^2}{2} + (R\alpha+\alpha-h)b_2\frac{t_1^3-\mu_2^3}{3} - \alpha b_2\frac{t_1^4-\mu_2^4}{4} \\
&+ c\Big[(1+R)\Big[a_1\ln(1+R) + b_1(1+R)\ln(1+R) + (b_2(1+R)-a_2)\ln(1+R-t_1) + \beta a_2(T-t_1) \\
&- b_2R\ln(1+R-\mu_2) - b_2(t_1-\mu_2) - b_1R\ln(1+R-\mu_1) - b_1\mu_1\Big] - \beta\frac{b_2}{2}(T^2-t_1^2)\Big] \\
&+ c_1\Big[\beta a_2t_1(t_1-T) + \beta a_2\frac{T^2-t_1^2}{2} - \frac{\beta b_2}{6}(T^3-t_1^3) + \frac{\beta b_2}{2}t_1^2(T-t_1)\Big] \\
&+ L(1-\beta)\Big[a_2(T-t_1) - \frac{b_2}{2}(T^2-t_1^2)\Big]\Big]. \qquad (3.48)
\end{aligned}
$$

Equation (48) is highly non linear in nature with t_1. We can find the optimum values of t_1 for minimum average cost $C_3(t_1)$ from the solutions of the following equations by the help of Mathematica 10,

$$\frac{dC_3(t_1)}{dt_1} = 0. \qquad (3.49)$$

4 Numerical example and sensitivity analysis

In this section, we use Mathematica 10 to get a numerical solutions and sensitivity analysis of model for different parameters.

Example 4.1. *The parameter values is given as follows:*
$A = \$350$ *per order,* $T = 12$ *weeks,* $\mu_1 = 4$ *weeks,* $\mu_2 = 10$ *weeks,* $a_1 = 150$ *unit,* $b_1 = 50$ *unit,* $a_2 = 450$ *unit,* $b_2 = 10$ *unit,* $c = \$75$ *per unit,* $R = 13$ *weeks,* $L = \$7$ *per unit,* $\alpha = 0.2$ *unit,* $h = 5$ *unit,* $c_1 = \$5$ *per unit,* $\beta = 0.4$, $t_1^* = 3.20578$ *weeks, minimum cost* $C_1(t_1^*) = \$90137.07715$.

Example 4.2. *The parameter values is given as follows:*
$A = \$350$ *per order,* $T = 12$ *weeks,* $\mu_1 = 2.5$ *weeks,* $\mu_2 = 9$ *weeks,* $a_1 = 150$ *unit,* $b_1 = 50$ *unit,* $a_2 = 450$ *unit,* $b_2 = 10$ *unit,* $c = \$75$ *per unit,* $R = 13$ *weeks,* $L = \$7$ *per unit,* $\alpha = 0.2$ *unit,* $h = 5$ *unit,* $c_1 = \$5$ *per unit,* $\beta = 0.4$, $t_1^* = 4.901507$ *weeks, minimum cost* $C_2(t_1^*) = \$62503.3333$.

Example 4.3. *The parameter values is given as follows:*
$A = \$350$ *per order,* $T = 12$ *weeks,* $\mu_1 = 2$ *weeks,* $\mu_2 = 4.5$ *weeks,* $a_1 = 150$ *unit,* $b_1 = 40$ *unit,* $a_2 = 450$ *unit,* $b_2 = 10$ *unit,* $c = \$75$ *per unit,* $R = 13$ *weeks,* $L = \$7$ *per unit,* $\alpha = 0.2$ *unit,* $h = 5$ *unit,* $c_1 = \$5$ *per unit,* $\beta = 0.4$, $t_1^* = 8.051037$ *weeks, minimum cost* $C_3(t_1^*) = \$11873.09731$.

Based on the above numerical examples (1, 2 & 3), the performed sensitivity analysis by $\pm 50\%$ and $\pm 25\%$ changing one parameter at a time and keeping the remaining parameters at their original values. The table (3, 4 & 5) summarize the results of sensitivity analysis. Based on the results of sensitivity analysis from table (3, 4 & 5) the following observations can be done.

From table-3, the total average cost decreases with decrease in the parameters α, β, h, c_1, L, c, a_1, a_2, b_1, b_2 and A, however, the total average cost increases with decrease in the value of R. The total average cost is highly sensitive to changes in R, c_1, c, a_1, a_2, and b_1. It is less sensitive to changes in α, β, h, L, b_2 and A. The time duration of shortage is increases with decrease in the parameters α, β, h, a_2, b_1, A and R, however, the time duration of shortage is decreases with decrease in the parameters c_1, L, c, a_1 and b_2.

From table-4, the total average cost gradually decreases with decrease in the parameters α, β, h, c_1, L, c, a_1, a_2, b_1, b_2 and A, however, the total average cost increases with decrease in the value of R. The total average cost is less sensitive to changes in all parameters. The time duration of shortage is decreases with decrease in the parameters α, whereas, the time duration of shortage is increases with decrease in the parameters c_1, L, c, a_1, b_2.β, h, a_2, b_1, A and R.

From table-5, the total average cost gradually decreases with decrease in the parameters α, β, h, c_1, L, c, a_1, a_2, b_1, b_2 and A, however, the total average cost increases with decrease in the value of the parameter R. The total average cost is less sensitive to changes in all parameters, except R. The model is highly sensitive to change the parameter R. The time duration of shortage is increases with decrease the value of all parameters.

5 Conclusions

In this paper, we developed an economic order quantity model when the market demand is followed by trapezoidal type function. Shortages are allowed and partially backlogged. Time varying holding cost, deterioration rate of the items dependent on both time and life of items are considered. Numerical examples are carried out to illustrate the model and the solution procedure. Subsequently, sensitivity analysis is carried out with respect to all key parameters to observe interesting managerial insights. Finally, in this paper, from numerical example and sensitivity analysis, we conclude that the model without shortage is more profitable than the model allow with shortage. This is happening because of the deterioration rate is not only depend on time but also depend on life of the product. If the life of product is more than the length of replenishment cycle, then the model without shortage is profitable. In this paper, we have taken life of the products is more than the length of replenishment cycle. Therefore, our paper is conclude that the model without shortage is more profitable than shortage.

6 Acknowledgment

The authors would like to convey their sincere thanks to the reviewer and editor.

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Table 1: List of authors those who have used ramp type demand

Authors	Objective	Contribution	Remarks
Hill (1995)	Finding EOQ model	First time introduced ramp type demand	Constant holding cost
Mandal and Pal (1998)	Finding EOQ model	Deterministic and probabilistic demand are discussed and validate their model with numerical techniques	Constant holding cost
Wu et al. (1999)	Finding EOQ model	Weibull distribution, numerical techniques, sensitivity analysis	Constant holding cost
Wu and Ouyang (2000)	Finding EOQ model	Ramp type demand function, shortages are allowed	Constant holding cost
Wu (2001)	Finding EOQ model	Weibull distribution deterioration, variable backlogging rate and dependent on waiting time for the next replenishment.	Constant holding cost
Giri et al. (2003)	Finding EOQ model	Deterioration rate of items are follow Weibull distribution, shortages are allowed and fully backlogged	Constant holding cost
Deng (2005)	Finding EOQ model	Weibull distribution, sensitivity analysis and numerical techniques	Constant holding cost
Chen et al. (2006)	Finding EOQ model	Time dependent deterioration, shortages are allowed, sensitivity analysis and numerical examples.	Constant holding cost
Manna and Chadhuri (2006)	Finding EOQ model	Linear time dependent deterioration and numerical techniques, model with no shortage and with shortages are discussed	Constant holding cost
Deng et al. (2007)	Finding EOQ model	Point out some questionable results of Mandal and Pal (1998) and Wu and Ouyang (2000)	Constant holding cost
Panda et al. (2008)	Finding EOQ model	Constant deterioration, shortages are not allowed	Constnat holding cost
Wu et al. (2008)	Finding EOQ model	Model allows shortage, completely backlogged and constant deterioration	Constant holding cost
Skouri et al. (2009)	Finding EOQ model	Weibull distribution, partial backlogging, model starting with no shortages and with shortages	Constant holding cost
Panda et al. (2009)	Finding EOQ model	Uniqueness and existence of the solution is established	Constnat holding cost
Mahata and Goswami (2009)	Finding EOQ model	Fuzzy cost coefficients, fuzzy replenishment fuzzy multi-objective mathematical programming with the triangular fuzzy number	Constant holding cost
Panda and Saha (2010)	Finding EOQ model	Shortages are not allowed, uniform deterioration rate and sensitivity analysis.	Constant holding cost
Roy and Choudhuri (2011)	Finding EOQ model	Shortages and without shortages are allowed, numerical solutions provided, finite time horizon	Constnat holding cost
Agarwal and Banerjee (2011)	Finding EOQ model	Shortages are allowed and partially backlogged, an algorithm is developed	Constant holding cost
Tripathy and Mishra (2011)	Finding EOQ model	Weibull distribution, sensitivity analysis	Constant holding cost
Goyal et al. (2013)	Finding EOQ model	Genetic algorithm is implemented, finite time horizon	Constant holding cost
Sanni and Chukwa (2013)	Finding EOQ model	Three parameter Weibull distribution, shortages are allowed and complete backlog	Constant holding cost

Table 2: List of authors those who have used trapezoidal type demand

Authors	Objective	Contribution	Remarks
Cheng and Wang (2009)	Finding EOQ model	First time trapezoidal type demand introduced, shortages are allowed and completely backlogged	Constant holding cost, constant deterioration rate
Cheng et al. (2011)	Finding EOQ model	Trapezoidal type demand, shortages are allowed, partial backlogging	Constant holding cost
Saha et al. (2011)	Finding EOQ model	Price discount mathematical modelling, numerical techniques	Constant holding cost
Cheng (2012)	Finding EOQ model	Trapezoidal type demand supply chain management	Constant holding cost, constant deterioration rate
Uthayakumar and Rameswari (2012)	Finding EOQ model	Euler-Lagrang method used, shortages are not allowed	Constant holding cost, constant deterioration rate
Chuang et al. (2013)	Finding EOQ model	Trapezoidal type demand function, with and without shortage, partially backlogged	Constant holding cost, constant deterioration rate
Singh and Pattnayak (2013)	Finding EOQ model	Deterioration rate is Weibull distribution, trapezoidal type demand	Constant holding cost
Zhao (2014)	Finding EOQ model	Trapezoidal type demand, Weibull distribution deterioration rate, numerical techniques	Constant holding cost
Lin et al. (2014)	Finding EOQ model	Deterioration and partial backlogging	Constant holding cost
Dem et al. (2014)	Finding EOQ model	Trapezoidal type demand, numerical techniques	Constant holding cost, constant deterioration rate
Debata et al. (2015)	Finding EOQ model	Quadratic trapezoidal type demand, shortages are allowed	Constant holding cost, constant deterioration rate

Table 3: Sensitivity analysis with respect to the different parameters of the EOQ model for Case-I

Parameter	Change(%)	t_1^*	$C_1(t_1^*)$
α	+50	3.27051	93750.20153
	+25	3.25062	93043.01798
	-25	3.25051	92097.52768
	-50	3.20105	92001.00796
β	+50	3.20031	92508.01392
	+25	3.05780	92001.79231
	-25	3.00921	92001.00701
	-50	2.97013	91909.27592
h	+50	3.25079	92707.09376
	+25	3.20701	92701.12093
	-25	3.19230	92435.01783
	-50	3.10273	92013.87054
c_1	+50	2.97301	93079.01253
	+25	3.00791	93005.73458
	-25	3.15240	92739.00157
	-50	3.19076	92543.07532
L	+50	3.10520	92805.70153
	+25	3.20513	92510.15304
	-25	3.22075	92502.07352
	-50	3.22071	92401.03731
c	+50	3.00253	93517.05943
	+25	3.21075	92759.13709
	-25	3.20148	92050.00195
	-50	3.22057	91907.25079
a_1	+50	2.50713	94725.01536
	+25	2.81346	93079.17254
	-25	3.70193	90137.02549
	-50	3.84076	90731.02716
a_2	+50	3.80173	94053.01732
	+25	3.35162	92750.30457
	-25	2.90125	92080.17062
	-50	2.80157	91652.90173
b_1	+50	3.07523	92503.00171
	+25	2.90536	92275.36072
	-25	2.81603	92032.07480
	-50	———	——-
b_2	+50	2.70532	92901.05736
	+25	3.01257	92712.63527
	-25	3.30764	92204.72065
	-50	3.51480	92201.30531
A	+50	3.22053	92545.30512
	+25	3.21001	92544.30125
	-25	3.20732	92498.07326
	-50	3.01560	92440.69075
R	+50	3.60523	91705.86301
	+25	3.46798	92052.01983
	-25	2.90763	94925.01267
	-50	2.53075	97098.92576

Table 4: Sensitivity analysis with respect to the different parameters of the EOQ model for Case-II

Parameter	Change(%)	t_1^*	$C_2(t_1^*)$
α	+50	4.70127	62015.02357
	+25	4.81013	61973.16743
	-25	4.90072	61862.05732
	-50	4.99053	61802.50643
β	+50	4.95073	61989.20579
	+25	4.90879	61907.57832
	-25	4.90705	61896.42793
	-50	4.89235	61850.06925
h	+50	4.91205	61975.05769
	+25	4.90753	61960.17368
	-25	4.90601	61890.25376
	-50	4.90076	61875.03425
c_1	+50	4.98075	62901.75321
	+25	4.94362	61957.01893
	-25	4.90201	61032.70685
	-50	4.87057	61015.81342
L	+50	5.01379	63079.16983
	+25	4.90739	62582.75032
	-25	4.89074	61037.06517
	-50	4.80153	60979.35792
c	+50	5.10572	62932.05132
	+25	4.98374	61875.23691
	-25	4.90725	61013.71528
	-50	4.70792	60978.06493
a_1	+50	4.96053	61890.05271
	+25	4.94712	61867.79825
	-25	4.90358	61805.08543
	-50	4.89532	61801.73105
a_2	+50	4.98053	62904.50732
	+25	4.80148	61852.73201
	-25	4.67592	61801.05937
	-50	5.60985	61790.75301
b_1	+50	4.95780	61980.25710
	+25	4.92045	61875.05923
	-25	4.90752	61701.17682
	-50	4.89271	61692.87052
b_2	+50	4.97530	61979.01532
	+25	4.93125	61900.33251
	-25	4.90048	61897.01572
	-50	4.90101	61865.79858
A	+50	4.95667	61954.85791
	+25	4.93514	61901.03725
	-25	4.80532	61895.17523
	-50	4.80254	61890.53201
R	+50	5.07521	60073.00125
	+25	4.99079	62573.27918
	-25	4.30253	65379.82460
	-50	3.90792	69024.31026

Table 5: Sensitivity analysis with respect to the different parameters of the EOQ model for Case-III

Parameter	Change(%)	t_1^*	$C_3(t_1^*)$
α	+50	8.70531	12251.07963
	+25	8.31982	11083.70342
	-25	7.90871	10735.04251
	-50	7.89253	10071.96521
β	+50	8.40215	13781.07193
	+25	8.17082	11057.80134
	-25	7.95026	10042.71692
	-50	7.90631	9075.06741
h	+50	8.50921	12705.31206
	+25	8.19052	12069.61283
	-25	8.01304	10317.00791
	-50	7.97302	9419.10742
c_1	+50	8.60931	13924.80349
	+25	8.26310	11739.15723
	-25	8.01972	10705.90618
	-50	7.96052	9107.69042
L	+50	8.20981	15921.09275
	+25	8.19804	12074.83210
	-25	8.00541	10182.98024
	-50	7.89071	9471.07315
c	+50	8.30781	18076.42197
	+25	8.09561	14071.00814
	-25	7.94861	11806.70581
	-50	7.90671	10174.60832
a_1	+50	8.10893	11075.91372
	+25	8.07052	11026.11732
	-25	8.01246	10642.90742
	-50	8.01004	10173.63410
a_2	+50	8.19672	10798.38013
	+25	8.10729	10201.07300
	-25	8.01492	9842.09215
	-50	8.00710	9062.00921
b_1	+50	8.20573	12801.13780
	+25	8.07819	12017.49001
	-25	8.04162	10038.29410
	-50	7.96502	9056.19042
b_2	+50	8.19063	10961.80562
	+25	8.05721	10208.83041
	-25	8.00300	9072.90521
	-50	7.98402	9004.00461
A	+50	8.20963	13042.07521
	+25	8.07491	11562.06531
	-25	8.01962	10281.64081
	-50	7.92081	8042.90513
R	+50	9.04810	19302.98032
	+25	8.70831	10571.64812
	-25	8.08941	10049.64192
	-50	7.49802	9834.94210

Received: October 29, 2015; *Accepted*: January 17, 2016

UNIVERSITY PRESS

Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 155-168

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Sobolev type fractional stochastic integro-differential evolution

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Abstract

In this paper, we prove the existence of $\alpha-$mild solutions for a class of fractional stochastic integro-differential evolution equations of sobolev type with fractional sobolev stochastic nonlocal conditions in a real separable Hilbert space. To establish our main results, we use the Banach contraction mapping principle, fractional calculus, stochastic analysis and an analytic semigroup of linear operators. An example is given to illustrate the feasibility of our abstract result..

Keywords: Fractional stochastic evolution equations, Fixed point technique, fractional stochastic nonlocal condition.

2010 MSC: 26A33, 46E39, 34K50.

1 Introduction

Let (Ω, Γ, P) be a complete probability space equipped with a normal filtration $\Gamma_t, t \in J$ satisfying the usual conditions (i.e., right continuous and Γ_0 containing all P-null sets). We consider three real separable spaces X, Y and E, and Q-Wiener process on (Ω, Γ_b, P) with the linear bounded covariance operator Q such that $trQ < \infty$. We assume that there exist complete orthonormal systems $\{e_{1,n}\}_{n\geq 1}, \{e_{2,n}\}_{n\geq 1}$ in E, bounded sequences of non-negative real numbers $\{\lambda_{1,n}\}, \{\lambda_{2,n}\}$ such that $Qe_{1,n} = \lambda_{1,n}e_{1,n}, Qe_{2,n} = \lambda_{2,n}e_{2,n}, n = 1, 2, ...,$ and sequences $\{\beta_{1,n}\}_{n\geq 1}, \{\beta_{2,n}\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle w_1(t), e_1\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_{1,n}}\, \langle e_{1,n}, e_1\rangle\, \beta_{1,n}(t), \quad e_1 \in E,\ t \in J,$$

$$\langle w_2(t), e_2\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_{2,n}}\, \langle e_{2,n}, e_2\rangle\, \beta_{2,n}(t), \quad e_2 \in E,\ t \in J,$$

and $\Gamma_t = \Gamma_t^{w_1,w_2}$, where $\Gamma_t^{w_1,w_2}$ is the sigma algebra generated by $\{(w_1(s), w_2(s)) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{1/2}E; X)$ be the space of all Hilbert–Schmidt operators from $Q^{1/2}$ E to X with the inner product $\langle \psi, \pi\rangle\, L_2^0 = tr[\psi Q \pi^*]$.

In this paper we consider the following Sobolev type fractional stochastic integro-differential evolution equations with fractional sobolev stochastic nonlocal conditions of the form

$${}^{C}D_t^{q}\left[Lx\left(t\right)\right] = Ax\left(t\right) + \sigma_1\left(t, x\left(t\right), Hx\left(t\right)\right)\frac{dw_1\left(t\right)}{dt}, \quad t \in J, \tag{1.1}$$

$${}^{L}D_t^{1-q}\left[Mx(0)\right] = \sigma_2(t, x(t))\frac{dw_2(t)}{dt}. \tag{1.2}$$

where ${}^{C}D_t^{q}$ and ${}^{L}D_t^{1-q}$ are the Caputo and Riemann–Liouville fractional derivatives with $0 < q \leq 1$, the state $x\left(\cdot\right)$ takes its values in the Hilbert space X.

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The operators $A : D(A) \subset X \to X$, $L : D(L) \subset X \to X$ and $M : D(M) \subset X \to X$ are closed linear operators in X, σ_1 and σ_2 are given functions to be specified later, $J = [0, b]$, $b > 0$ is a constant. The term $Hx(t)$ is given by

$$Hx(t) = \int_0^t K(t,s)\, x(s)\, ds,$$

where $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \left\{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le b\right\}$.

During the past three decades, fractional differential equations and their applications have gained a lot of importance, mainly because this field has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering ([2], [3], [10], [11], [23], [24]) . Recently, there has been a significant development in the existence results for boundary value problems of nonlinear fractional differential equations ([1], [7]).

The problem with nonlocal condition, which is a generalization of the problem of classical condition, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski see ([4], [5], [6]). Since it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems, differential equations with nonlocal conditions have been studied by many authors and some basic results on nonlocal problems have been obtained, see ([12], [17], [22]) and the references therein for more comments and citations, A. Debbouche, D. Baleanu and R. P. Agarwal [13] proved the existence of mild and strong solutions for fractional nonlocal nonlinear integro-differential equations of Sobolev type using Schauder fixed point theorem, Gelfand-Shilov principles combined with semigroup theory. A. Debbouche and J.J. Nieto [11] studied the existence and uniqueness of mild solutions for a class of Sobolev type fractional nonlocal abstract evolution equations with nonlocal conditions and optimal multi-controls in Banach spaces by using fractional calculus, semigroup theory, a singular version of Gronwall inequality and Leray–Schauder fixed point theorem.

Stochastic differential and integro-differential equations have attracted great interest due to its applications in various fields of science and engineering. There are many interesting results on the existence, uniqueness and asymptotic stability of solutions to stochastic differential equations, see ([8], [14], [18], [19], [28], [29], [30], [35], [36]) and the references therein. In particular, fractional stochastic differential equations have also been studied by several authors, see ([9], [31], [32]).

More recently, El-Borai [16] studied the existence of mild solutions for a class of semilinear stochastic fractional differential equations by using Leray-Schauder fixed point theorem. Cui and Yan [9] studied the existence of mild solutions for a class of fractional neutral stochastic integro-differential equationswith infinite delay in Hilbert spaces by means of Sadovskii's fixed point theorem. Sakthivel et al. [31] investigated the existence and asymptotic stability in pth moment of mild solutions to a class of nonlinear fractional neutral stochastic differential equations with infinite delays in Hilbert spaces by using semigroup theory and fixed point technique. The existence of mild solutions for impulsive fractional stochastic differential equations with infinite delay has also been established in [32].

For our best knowledge, there is no work reported on the existence of α-mild solutions for Sobolev type fractional stochastic integro-differential evolution equations with fractional sobolev stochastic nonlocal conditions in fractional power space X_α.

Motivated by the above works, we introduce here a new nonlocal fractional stochastic condition of Sobolev type, and we prove the existence of α-mild solutions for the problem (1.1)-(1.2) by using a new strategy which relies on the compactness of the operator semigroup generated by $T = AL^{-1}$, Schauder fixed point theorem and approximating techniques. The rest of this paper is organized as follows. In Section 2 we present some essential facts in fractional calculus, semigroup theory, stochastic analysis that will be used to obtain our main results. In Section 3, we state and prove existence results on α-mild solutions for Sobolev type fractional stochastic system (1.1)-(1.2). Finally, in Section 4, as an example, a fractional stochastic parabolic partial differential equation with a sobolev type fractional stochastic nonlocal condition is considered.

2 Preliminaries

Throughout this paper, $(X, ||\cdot||)$ is a separable Hilbert space.

The operators $A : D(A) \subset X \to X$, $L : D(L) \subset X \to X$ and $M : D(M) \subset X \to X$ satisfy the following conditions:

$(A1)$ $L,$ A and M are closed linear operators.

$(A2)$ $D(M) \subset D(L) \subset D(A)$ and L and M are bijective.

$(A3)$ $L^{-1} : X \to D(L) \subset X$ and $M^{-1} : X \to D(M) \subset X$ are linear, bounded, and compact operators.

From $(A3)$, we deduce that L^{-1} is bounded operators. Note $(A3)$ also implies that L is closed since the fact: L^{-1} is closed and injective, then its inverse is also closed. It comes from $(A1) - (A3)$ and the closed graph theorem, we obtain the boundedness of the linear operator $AL^{-1} : X \to X$. Consequently, $-AL^{-1}$ generates a semigroup $\left\{S(t) = e^{AL^{-1}t},\ t \geq 0\right\}$. We suppose that $K_0 = \sup_{t\geq 0} ||S(t)|| < \infty$, and for short, we denote by $C_1 = \left|\left|L^{-1}\right|\right|, C_2 = \left|\left|M^{-1}\right|\right|$ and $T = AL^{-1}$.

$(A4)$ The resolvent $R(\lambda, T)$ is compact for some $\lambda \in \rho(T)$, the resolvent set of T.

Without loss of generality, we assume that $0 \in \rho(T)$. Then it is possible to define the fractional power T^α for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(T^\alpha)$ with inverse $T^{-\alpha}$. Hereafter, we denote by X_α the Banach space $D(T^\alpha)$ normed with $\|x\|_\alpha$.

Lemma 2.1 (See [26]). *Let T be the infinitesimal generator of an analytic semigroup $S(t)$. If $0 \in \rho(T)$, then*

(a) *$D(T^\alpha)$ is a Hilbert space with the norm $||x||_\alpha = ||T^\alpha x||$ for $x \in D(T^\alpha)$.*

(b) *$S(t) : X \to D(T^\alpha)$ for each $t > 0$ and $\alpha \geq 0$.*

(c) *$S(t)T^\alpha x = T^\alpha S(t)x$ for each $x \in D(T^\alpha)$ and $t \geq 0$.*

(d) *If $0 < \alpha \leq \beta \leq 1$, then $D(T^\beta) \hookrightarrow D(T^\alpha)$.*

(e) *For every $t > 0,$ $T^\alpha S(t)$ is bounded on X and there exist $K_\alpha > 0$ and $\delta > 0$ such that*

$$||T^\alpha S(t)|| \leq \frac{K_\alpha}{t^\alpha} e^{-\delta t} \leq \frac{K_\alpha}{t^\alpha}.$$

(vi) *$T^{-\alpha}$ is a bounded linear operator in X with $D(T^\alpha) = Im\left(T^{-\alpha}\right)$.*

Definition 2.1 *The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b], R^+)$ is given by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

where Γ is the gamma function. If $a = 0$, we can write $I^\alpha f(t) = (g_\alpha * f)(t)$, where

$$g_\alpha(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and as usual, $*$ denotes the convolution of functions. Moreover, $\lim\limits_{\alpha \to 0} g_\alpha(t) = \delta(t)$, with δ the delta Dirac function.

Definition 2.2 *The Riemann–Liouville derivative of order $n-1 < \alpha < n,$ $n \in N$, for a function $f \in C([0, +\infty))$ is given by*

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \ \ t > 0.$$

Definition 2.3 *The Caputo derivative of order $n-1 < \alpha < n,$ $n \in N$, for a function $f \in C([0, +\infty))$ is given by*

$$^C D^\alpha f(t) =^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \ \ t > 0.$$

Remark 2.1 *The following properties hold. Let $n-1 < \alpha < n,$ $n \in N$*

(i) *If $f(t) \in C^n([0, \infty))$, then*

$$^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(s), \ \ t > 0.$$

(*ii*) *The Caputo derivative of a constant is equal to zero.*

(*iii*) *The Riemann–Liouville derivative of a constant function is given by*

$$^{L}D_{a^+}^{\alpha}C = \frac{C}{\Gamma(1-\alpha)}(x-a)^{-\alpha}.$$

If f is an abstract function with values in X, then integrals which appear in Definitions 2.1-2.3 are taken in Bochner's sense.

According to previous definitions, it is suitable to rewrite problem (1.1)-(1.2) as the equivalent integral equation

$$\begin{aligned} x(t) &= x(0) + \tfrac{1}{\Gamma(q)}\int_0^t L^{-1}(t-s)^{q-1}Ax(s)\,ds \\ &+ \tfrac{1}{\Gamma(q)}\int_0^t L^{-1}(t-s)^{q-1}\sigma_1(s,x(s),Hx(s))dw_1(s) \end{aligned} \tag{2.1}$$

Remark 2.2 *We note that:*

(a) For the nonlocal condition, the function $x(0)$ is dependent on t.

(b) $^{L}D_t^{1-q}[Mx(0)]$ is well defined, i.e., if $q=1$ and M is the identity, then (1.2) reduces to the usual nonlocal condition.

(c) The function $x(0)$ takes the form

$$M^{-1}x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}M^{-1}\sigma_2(s,x(s))dw_2(s),$$

where $Mx(0)|_{t=0} = x_0$.

(d) The explicit and implicit integrals given in (2.1) exist (taken in Bochner's sense).

Let $L^2(\Gamma_b, X_\alpha)$ be the Banach space of all Γ_b-measurable square integrable random variables with values in the Hilbert space X_α. Let $E(\cdot)$ denotes the expectation with respect to the measure P. An important subspace of $L^2(\Gamma_b, X_\alpha)$ is given by $L_2^0(\Gamma_b, X_\alpha) = \{x \in L^2(\Gamma_b, X_\alpha) : x \text{ is } \Gamma_0 - \text{measurable}\}$.

Let $C(J, L^2(\Gamma, X_\alpha))$ be the Hilbert space of continuous maps from J into $L^2(\Gamma, X_\alpha)$ satisfying $\sup_{t\in J} E\,||x(t)||_\alpha^2 < \infty$. Let $\mathcal{H}_2(J, X_\alpha)$ is a closed subspace of $C(J, L^2(\Gamma, X_\alpha))$ consisting of measurable and Γ_t-adapted X_α-valued process $x \in C(J, L^2(\Gamma, X_\alpha))$ endowed with the norm $||x||_{\mathcal{H}_2} = (\sup_{t\in J} E\,||x(t)||_\alpha^2)^{1/2}$.

Then it easy to check that $(\mathcal{H}_2, \|\cdot\|_{\mathcal{H}_2})$ is a Hilbert space. For any constant $\tau > 0$, let $B_\tau = \left\{x \in \mathcal{H}_2 : \|x\|_{\mathcal{H}_2} \le \tau\right\}$, clearly that B_τ is a bounded closed conves set in $\mathcal{H}_2$.

Definition 2.4 *By the α-mild solution of the problem (1.1)-(1.2), we mean that the Γ_t-adapted stochastic process $x \in H_2$ which satisfies*

1. $x(0) \in L^2(\Gamma, X_\alpha)$, *where* $x(0) = M^{-1}x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}M^{-1}\sigma_2(s,x(s))dw_2(s)$ *and* $Mx(0)|_{t=0} = x_0$;

2. $x(t) \in X_\alpha$ *has cádlág paths on* $t \in J$ *almost surely and for each* $t \in J$, $x(t)$ *satisfies the integral equation*

$$\begin{aligned} x(t) &= \mathcal{S}_q(t)M^{-1}\left[x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s,x(s))dw_2(s)\right] 2.2 \\ &+ \int_0^t (t-s)^{q-1}L^{-1}\mathcal{T}_q(t-s)\sigma_1(s,x(s),Hx(s))dw_1(s) \end{aligned} \tag{2.1}$$

where $S_q(t)x = \int_0^{+\infty} h_q(s)S(t^q s)\,x ds$ *and* $T_q(t)x = q\int_0^{+\infty} sh_q(s)S(t^q s)\,x ds$.

Here, $S(t)$ is a C_0-semigroup generated by a linear operator $T = AL^{-1} : X \to X$, h_q is a probability density function defined on $(0,\infty)$, that is $h_q(s) \ge 0$, $s \in (0,\infty)$ and $\int_0^\infty h_q(s)ds = 1$.

The following lemma follows from the results in ([15], [16], [21], [33], [34]) and will be used throughout this paper.

Lemma 2.2. *The operators $S_q(t)$ and $T_q(t)$ have the following properties:*

(1) *For any fixed $t \ge 0$, $S_q(t)$ and $T_q(t)$ are linear and bounded operators in X_α,*

$$\text{i.e.for any } x \in X_\alpha, \quad ||\mathcal{S}_q(t)x|| \le K_0\,||x||_\alpha, \qquad ||\mathcal{T}_q(t)x|| \le \frac{qK_0}{\Gamma(1+q)}\,||x||_\alpha.$$

(2) *The operators* $\{S_q(t) : t \geq 0\}$ *and* $\{T_q(t) : t \geq 0\}$ *are strongly continuous.*

(3) *For every* $t > 0$, $S_q(t)$ *and* $T_q(t)$ *are compact operators in* X*, and hence they are norm-continuous.*

(4) *For every* $t > 0$*, the restriction of* $S_q(t)$ *to* X_α *and the restriction of* $T_q(t)$ *to* X_α *are compact operators in* X_α*.*

(5) *The restriction of* $S_q(t)$ *to* X_α *and the restriction of* $T_q(t)$ *to* X_α *are continuous in* $(0, +\infty)$ *by the operator norm* $\|\cdot\|_\alpha$*.*

(6) *For any* $x \in X$ *and* $t \in J$, $\left|\left|T^\alpha \mathcal{T}_q(t)x\right|\right| \leq A_\alpha t^{-\alpha q} \|x\|$ *, where* $A_\alpha = \frac{qK_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}$*.*

3 Main results

In this section, we give the existence of α-mild solutions for the problem (1.1)-(1.2). We impose the following assumptions on the data of the problem (1.1)-(1.2).

$(H1)$ The functions $\sigma_1 : J \times X_\alpha \times X_\alpha \to L_2^0$ satisfies the Carathéodory type conditions, i.e. $\sigma_1(\cdot, x, Hx)$ is strongly measurable for all $x \in X_\alpha$, and $\sigma_1(t,\cdot,\cdot)$ is continuous for a.e. $t \in J$.

$(H2)$ For some $\tau > 0$, there exist constants $q_1 \in [\frac{1}{2}, q(1-\alpha))$, $\rho_1 > 0$ and functions $\varphi_\tau \in L^{\frac{1}{2q_1-1}}(J, \mathbb{R}^+)$ such that for a.e. $t \in J$,

$$\sup_{\|x\|^2_{\mathcal{H}_2} \leq \tau} E\left\|\sigma_1(t, x, Hx)\right\|^2 \leq \varphi_\tau(t) \quad \text{and} \quad \liminf_{\tau \to +\infty} \frac{\|\varphi_\tau\|_{L^{\frac{1}{2q_1-1}}[0,b]}}{\tau} = \rho_1 < +\infty.$$

$(H3)$ The nonlocal function $\sigma_2 : J \times X_\alpha \to L_2^0$ is continuous, and there exist a constant $\rho_2 > 0$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for some $\tau > 0$ and all $x \in B_\tau$,

$$E\left\|\sigma_2(t, x)\right\|_\alpha^2 \leq \Phi(\tau) \quad \text{and} \quad \liminf_{\tau \to +\infty} \frac{\Phi(\tau)}{\tau} = \rho_2 < +\infty.$$

$(H4)$ There exists a constant $\delta \in (0, b)$ such that $\sigma_2(t, x) = \sigma_2(t, y)$ for any $x, y \in \mathcal{H}_2$ with $x(t) = y(t)$, $t \in [\delta, b]$.

Theorem 3.1 *If the assumptions (H1)–(H4) are satisfied, then the problem (1.1)-(1.2) has at least one* α*-mild solution provided that*

$$\begin{aligned}&\left\{\left(\tfrac{C_2K_0}{\Gamma(1-q)}\right)^2 Tr\,(Q)\, \tfrac{1}{2q-1} b^{2q-1}\rho_2\right.\\ &\left.+ (C_1A_\alpha)^2\, Tr\,(Q)\, b^{2(q-q_1-\alpha q)} \left(\tfrac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1} \rho_1\right\} < \tfrac{1}{2}.\end{aligned} \tag{3.1}$$

Proof Let $\{\delta_n : n \in \mathbb{N}\}$ be a decreasing sequence in $(0, b)$ such that $\lim_{n\to\infty}\delta_n = 0$. We first prove the following problem

$${}^{C}D_t^q\left[Lx(t)\right] = Ax(t) + \sigma_1(t, x(t), Hx(t))\frac{dw_1(t)}{dt}, \quad t \in J, \tag{3.2}$$

$$x(0) = S(\delta_n)\left(M^{-1}x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q} M^{-1}\sigma_2(s, x(s))dw_2(s)\right) \tag{3.3}$$

has at least one α-mild solution $x_n \in \mathcal{H}_2$. To this end, for fixed $n \in \mathbb{N}$, we define an operator $\Psi_n : \mathcal{H}_2 \to \mathcal{H}_2$ by

$$\begin{aligned}(\Psi_n x)(t) \;&=\; \mathcal{S}_q(t)S(\delta_n)M^{-1}\left[x_0 + \tfrac{1}{\Gamma(1-q)}\textstyle\int_0^t (t-s)^{-q}\sigma_2(s, x(s))dw_2(s)\right]\\ &+\; \textstyle\int_0^t (t-s)^{q-1} L^{-1}\mathcal{T}_q(t-s)\sigma_1(s, x(s), Hx(s))dw_1(s), \quad t \in J.\end{aligned} \tag{3.4}$$

By direct calculation, we know that Ψ_n is well defined. From Definition 2.4, it is easy to see that the α-mild solution of the problem (3.2)-(3.3) is equivalent to the fixed point of the operator Ψ_n.

In what follows, we prove that there exists a positive constant R, such that $\Psi_n(B_R) \subset B_R$.

If this is not true, then for each $\tau > 0$, there would exist $x_\tau \in B_\tau$ and $t_\tau \in J$ such that $E\left\|(\Psi_n x_\tau)(t_\tau)\right\|_\alpha^2 > \tau$. It follows from Lemma 2.2 (1) and (6), the assumption (H2) and Hölder inequality that

$$\begin{aligned}
\tau &< E\left\|(\Psi_n x_\tau)(t_\tau)\right\|_\alpha^2 \\
&\leq 2E\left\|\mathcal{S}_q(t_\tau)S(\delta_n)M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)}\int_0^{t_\tau}(t_\tau-s)^{-q}\sigma_2(s,x_\tau(s))dw_2(s)\right]\right\|_\alpha^2 \\
&\quad +2E\left\|\int_0^{t_\tau}(t_\tau-s)^{q-1}L^{-1}\mathcal{T}_q(t_\tau-s)\sigma_1(s,x_\tau(s),Hx_\tau(s))dw_1(s)\right\|_\alpha^2 \\
&\leq 2\left\|M^{-1}\right\|^2\left\|\mathcal{S}_q(t_\tau)\right\|^2 Tr(Q)\left(\frac{1}{\Gamma(1-q)}\right)^2 \\
&\quad \times\int_0^{t_\tau}(t_\tau-s)^{-2q}E\left\|\sigma_2(s,x_\tau(s)\right\|^2 ds3.5 \\
&\quad +2\left\|L^{-1}\right\|^2 Tr(Q)\int_0^{t_\tau}(t_\tau-s)^{2(q-1)}E\left\|T^\alpha\mathcal{T}_q(t_\tau-s)\sigma_1(s,x_\tau(s),Hx_\tau(s))\right\|^2 ds \\
&\leq 2C_2^2K_0^2 Tr(Q)\left(\frac{1}{\Gamma(1-q)}\right)^2\Phi(\tau)\left(\int_0^{t_\tau}(t_\tau-s)^{-2q}ds\right) \\
&\quad +2C_1^2 Tr(Q)A_\alpha^2\left(\int_0^{t_\tau}(t_\tau-s)^{\frac{2q-2-2\alpha q}{2-2q_1}}ds\right)^{2-2q_1}\times\left(\int_0^{t_\tau}\varphi_{\tau_1}^{\frac{1}{2q_1-1}}(s)\,ds\right)^{2q_1-1} \\
&\leq 2C_2^2K_0^2 Tr(Q)\left(\frac{1}{\Gamma(1-q)}\right)^2\Phi(\tau)\frac{1}{-2q+1}b^{-2q+1} \\
&\quad +2C_1^2 Tr(Q)A_\alpha^2 b^{2(q-q_1-\alpha q)}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1}\left\|\varphi_{\tau_1}\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}.
\end{aligned} \tag{3.2}$$

Dividing both side of (3.5) by τ, then taking the lower limit as $\tau \to +\infty$, we get

$$\begin{aligned}
&\Big\{2C_2^2K_0^2 Tr(Q)\left(\tfrac{1}{\Gamma(1-q)}\right)^2\tfrac{1}{-2q+1}b^{-2q+1}\rho_2 \\
&+2C_1^2 Tr(Q)A_\alpha^2 b^{2(q-q_1-\alpha q)}\left(\tfrac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1}\rho_1\Big\} \geq 1.
\end{aligned}$$

which contradicts (3.1).

Next, we prove that Ψ_n is continuous in B_R. To this end, let $\{x_m\}_{m=1}^{\infty} \subset B_R$ be a sequence such that $\lim_{m\to\infty} x_m = x$ in B_R. By the Carathéodory continuity of σ_1 and σ_2, we have

$$\lim_{m\to\infty}\sigma_1(s,x_m(s),Hx_m(s)) = \sigma_1(s,x(s),Hx(s)), \quad a.e.\ s\in J. \tag{3.6}$$

$$\lim_{m\to\infty}\sigma_2(s,x_m(s)) = \sigma_2(s,x(s)), \quad a.e.\ s\in J. \tag{3.7}$$

From the assumption (H2), we get that for each $t \in J$,

$$\begin{aligned}
&(t-s)^{2(q-1-\alpha q)}E\left\|\sigma_1(s,x_m(s),Hx_m(s))-\sigma_1(s,x(s),Hx(s))\right\|^2 \\
&\leq (t-s)^{2(q-1-\alpha q)}\Big(2E\left\|\sigma_1(s,x_m(s),Hx_m(s))\right\|^2 \\
&+2E\left\|\sigma_1(s,x_m(s),Hx_m(s))\right\|^2\Big) \leq 4(t-s)^{2(q-1-\alpha q)}\varphi_R(s).
\end{aligned} \tag{3.8}$$

From the assumption (H3), we get that for each $t \in J$,

$$\begin{aligned}
&(t-s)^{-2q}E\left\|\sigma_2(s,x_m(s))-\sigma_2(s,x(s))\right\|^2 \\
&\leq 4(t-s)^{-2q}\varphi_R(s).
\end{aligned} \tag{3.9}$$

Using the fact that the functions $s \to 4(t-s)^{2(q-1-\alpha q)}\varphi_R(s)$ and $s \to 4(t-s)^{-2q}\varphi_R(s)$ are Lebesgue integrables for $s \in [0,t]$, $t \in J$, by Lemma 2.2 (1) and (6), (3.6), (3.7), (3.8), (3.9) and the Lebesgue dominated convergence theorem, we get that

$$
\begin{aligned}
& E\left\|\left(\Psi_{n} x_{m}\right)(t)-\left(\Psi_{n} x\right)(t)\right\|_{\alpha}^{2} 3.10 \qquad (3.3)\\
\leq\ & 2\left\|M^{-1}\right\|^{2}\left\|\mathcal{S}_{q}(t)\right\|^{2} Tr(Q)\left(\frac{1}{\Gamma(1-q)}\right)^{2} \int_{0}^{t}(t-s)^{-2q} \\
& \times E\left\|\sigma_{2}(s, x_{m}(s))-\sigma_{2}(s, x(s)\right\|^{2} ds \\
& +2\left\|L^{-1}\right\|^{2} Tr(Q) \int_{0}^{t}(t-s)^{2(q-1)} \\
& \times E\left\|T^{\alpha} \mathcal{T}_{q}(t-s)\left(\sigma_{1}(s, x_{m}(s), H x_{m}(s))-\sigma_{1}(s, x(s), H x(s))\right)\right\|^{2} ds \\
\leq\ & 2 C_{2}^{2} K_{0}^{2} Tr(Q)\left(\frac{1}{\Gamma(1-q)}\right)^{2} \frac{1}{-2q+1} t^{-2q+1} \\
& \times E\left\|\sigma_{2}(s, x_{m}(s))-\sigma_{2}(s, x(s))\right\|^{2} \\
& +C_{1}^{2} Tr(Q) A_{\alpha}^{2} \int_{0}^{t}(t-s)^{2(q-1-\alpha q)} \\
& \times E\left\|\sigma_{1}(s, x_{m}(s), H x_{m}(s))-\sigma_{1}(s, x(s), H x(s))\right\|^{2} ds \\
\rightarrow\ & 0 \quad \text{as} \quad m \rightarrow \infty.
\end{aligned}
$$

Therefore, by (3.10) we know that

$$
E\left\|\left(\Psi_{n} x_{m}\right)-\left(\Psi_{n} x\right)\right\|_{H_{2}}^{2}=\sup_{t \in J} E\left\|\left(\Psi_{n} x_{m}\right)(t)-\left(\Psi_{n} x\right)(t)\right\|_{\alpha}^{2} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty.
$$

which means that Ψ_n is continuous in B_R.

Now, we demonstrate that $\Psi_n : B_R \to B_R$ is a compact operator. We first prove that $\{(\Psi_n x)(t) : x \in B_R\}$ is relatively compact in X_α for all $t \in J$. For $t = 0$, since the compactness of $S(t)$ for every $t > 0$ implies that the restriction of $S(t)$ to X_α is compact semigroup in X_α, for $\forall n \in \mathbb{N}$ we can deduce, by the assumption (H3), that $\{(\Psi_n x)(0) : x \in B_R\}$ is relatively compact in X_α. For $0 < t \leq b$, $\epsilon \in (0, t)$, arbitrary $\delta > 0$ and $x \in B_R$, we define the operator $\Psi_n^{\epsilon,\delta}$ by

$$
\begin{aligned}
\left(\Psi_{n}^{\epsilon, \delta} x\right)(t) =\ & \mathcal{S}_{q}(t) S\left(\delta_{n}\right) M^{-1}\left[x_{0}+\frac{1}{\Gamma(1-q)} \int_{0}^{t}(t-s)^{-q} \sigma_{2}(s, x(s)) d w_{2}(s)\right] 3.11 \qquad (3.4)\\
& +\int_{0}^{t-\epsilon} \int_{\delta}^{\infty} q \tau(t-s)^{q-1} L^{-1} h_{q}(\tau)\left(S(t-s)^{q} \tau\right) \sigma_{1}(s, x(s), H x(s)) d \tau d w_{1}(s) \\
=\ & \mathcal{S}_{q}(t) S\left(\delta_{n}\right) M^{-1}\left[x_{0}+\frac{1}{\Gamma(1-q)} \int_{0}^{t}(t-s)^{-q} \sigma_{2}(s, x(s)) d w_{2}(s)\right] \\
& +L^{-1} S\left(\epsilon^{q} \delta\right) \int_{0}^{t-\epsilon} \int_{\delta}^{\infty} q \tau(t-s)^{q-1} h_{q}(\tau)\left(S(t-s)^{q} \tau-\epsilon^{q} \delta\right) \sigma_{1}(s, x(s), H x(s)) d \tau d w_{1}(s)
\end{aligned}
$$

Since the restriction of $S(\epsilon^q \delta)$ $(\epsilon^q \delta > 0)$ to X_α is compact semigroup in X_α, by Lemma 2.2 (4) we know that the set $\left\{\left(\Psi_n^{\epsilon,\delta} x\right)(t) : x \in B_R\right\}$ is relatively compact in X_α for $\forall \epsilon \in (0, t)$ and $\forall \delta > 0$. Moreover, for every

$x \in B_R$, by assumption (H2), Lemma 2.2 (6) and Hölder inequality we know that

$$\begin{aligned}
& E\left\|\left(\Psi_n x\right)(t)-\left(\Psi_n^{\epsilon,\delta} x\right)(t)\right\|_\alpha^2 \\
\leq\ & 2E\left\|\int_0^t \int_0^\delta q\tau(t-s)^{q-1} L^{-1} h_q(\tau)\left(S(t-s)^q \tau\right) \sigma_1(s, x(s), Hx(s)) d\tau dw_1(s)\right\|_\alpha^2 \\
& +2E\left\|\int_{t-\epsilon}^t \int_\delta^\infty q\tau(t-s)^{q-1} L^{-1} h_q(\tau)\left(S(t-s)^q \tau\right) \sigma_1(s, x(s), Hx(s)) d\tau dw_1(s)\right\|_\alpha^2 \\
\leq\ & 2\left\|L^{-1}\right\|^2 Tr(Q) \int_0^t (t-s)^{2(q-1)}\left\|\int_0^\delta q\tau h_q(\tau) T^\alpha\left(S(t-s)^q \tau\right) d\tau\right\|^2 \\
& \times E\left\|\sigma_1(s, x(s), Hx(s))\right\|^2 ds \\
& +2\left\|L^{-1}\right\|^2 Tr(Q) \int_{t-\epsilon}^t (t-s)^{2(q-1)}\left\|\int_\delta^\infty q\tau h_q(\tau) T^\alpha\left(S(t-s)^q \tau\right) d\tau\right\|^2 \\
& \times E\left\|\sigma_1(s, x(s), Hx(s))\right\|^2 ds \\
\leq\ & 2C_1^2 Tr(Q) M_\alpha^2 \left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1} \\
& \times t^{2(q-q_1-\alpha q)}\left(\int_0^\delta q\tau^{1-\alpha} h_q(\tau) d\tau\right)^2 \\
& +2C_1^2 Tr(Q) A_\alpha^2 \left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1} \epsilon^{2(q-q_1-\alpha q)}.
\end{aligned}$$

Therefore, letting $\delta, \epsilon \rightarrow 0$, we see that there are relatively compact sets arbitrarily close to the set $\{(\Psi_n x)(t) : x \in B_R\}$ in X_α for $0 < t \leq b$. Hence, the set $\{(\Psi_n x)(t) : x \in B_R\}$ is also relatively compact in X_α for $0 < t \leq b$. And since $\{(\Psi_n x)(t) : x \in B_R\}$ is relatively compact in X_α, we have the relatively compactness of $\{(\Psi_n x)(t) : x \in B_R\}$ in X_α for all $t \in J$.

Next, we prove that $\Psi_n(B_R)$ is equicontinuous. For $t = 0$, since $S(\delta_n)$ is a compact operator for $\forall n \in \mathbb{N}$, we know that the functions $\{(\Psi_n x)(t) : x \in B_R\}$ are equicontinuous at $t = 0$. For any $x \in B_R$ and $0 < t_1 < t_2 \leq b$, we get that

$$\begin{aligned}
& E\left\|\left(\Psi_n x\right)(t_2)-\left(\Psi_n x\right)(t_1)\right\|_\alpha^2 \\
\leq\ & 4E\left\|\mathcal{S}_q(t_2) S(\delta_n) M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)} \int_0^{t_2}(t-s)^{-q} \sigma_2(s, x(s)) dw_2(s)\right]\right. \\
& \left.-\mathcal{S}_q(t_1) S(\delta_n) M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)} \int_0^{t_1}(t-s)^{-q} \sigma_2(s, x(s)) dw_2(s)\right]\right\|_\alpha^2 \\
& +4E\left\|\int_{t_1}^{t_2}(t_2-s)^{q-1} L^{-1} \mathcal{T}_q(t_2-s) \sigma_1(s, x(s), Hx(s)) dw_1(s)\right\|_\alpha^2 \\
& +4E\left\|\int_0^{t_1}\left((t_2-s)^{q-1}-(t_1-s)^{q-1}\right) L^{-1} \mathcal{T}_q(t_2-s) \sigma_1(s, x(s), Hx(s)) dw_1(s)\right\|_\alpha^2 \\
& +4E\left\|\int_0^{t_1}(t_1-s)^{q-1} L^{-1}\left(\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right) \sigma_1(s, x(s), Hx(s)) dw_1(s)\right\|_\alpha^2 \\
:\ & =I_1+I_2+I_3+I_4.
\end{aligned}$$

where

$$\begin{aligned}
I_1 =\ & 4E\left\|\mathcal{S}_q(t_2) S(\delta_n) M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)} \int_0^{t_2}(t-s)^{-q} \sigma_2(s, x(s)) dw_2(s)\right]\right. \\
& \left.-\mathcal{S}_q(t_1) S(\delta_n) M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)} \int_0^{t_1}(t-s)^{-q} \sigma_2(s, x(s)) dw_2(s)\right]\right\|_\alpha^2,
\end{aligned}$$

$$I_2 = 4E\left\|\int_{t_1}^{t_2}(t_2-s)^{q-1} L^{-1} \mathcal{T}_q(t_2-s) \sigma_1(s, x(s), Hx(s)) dw_1(s)\right\|_\alpha^2,$$

$$I_3 = 4E \left\| \int_0^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) L^{-1} \mathcal{T}_q(t_2 - s) \sigma_1(s, x(s), Hx(s)) dw_1(s) \right\|_\alpha^2,$$

$$I_4 = 4E \left\| \int_0^{t_1} (t_1 - s)^{q-1} L^{-1} \left(\mathcal{T}_q(t_2 - s) - \mathcal{T}_q(t_1 - s) \right) \sigma_1(s, x(s), Hx(s)) dw_1(s) \right\|_\alpha^2.$$

Therefore, we only need to check $I_i \to 0$ independently of $x \in B_R$ when $t_2 - t_1 \to 0$, $i = 1, 2, \cdots, 4$.
For I_1, by Lemma 2.2 (1) and (5) and the assumption (H3), we know that

$$\begin{aligned}
I_1 \quad \leq \quad & 4E \left\| \left(\mathcal{S}_q(t_2) - \mathcal{S}_q(t_1) \right) S(\delta_n) M^{-1} \left[\frac{1}{\Gamma(1-q)} \int_0^{t_1} (t_1 - s)^{-q} \sigma_2(s, x(s)) dw_2(s) \right] \right\|_\alpha^2 \\
& + 4E \left\| \mathcal{S}_q(t_2) S(\delta_n) M^{-1} \left[\frac{1}{\Gamma(1-q)} \int_0^{t_1} \left((t_2 - s)^{-q} - (t_1 - s)^{-q} \right) \sigma_2(s, x(s)) dw_2(s) \right] \right\|_\alpha^2 \\
& + 4E \left\| \mathcal{S}_q(t_2) S(\delta_n) M^{-1} \left[\frac{1}{\Gamma(1-q)} \int_{t_1}^{t_2} (t_2 - s)^{-q} \sigma_2(s, x(s)) dw_2(s) \right] \right\|_\alpha^2 \\
\leq \quad & 4 \left\| M^{-1} \right\|^2 K_0^2 Tr(Q) \left(\frac{1}{\Gamma(1-q)} \right)^2 \frac{t_1^{-2q+1}}{-2q+1} \Phi(\tau) E \left\| \mathcal{S}_q(t_2) - \mathcal{S}_q(t_1) \right\|_\alpha^2 \\
& + 4 \left\| M^{-1} \right\|^2 K_0^2 Tr(Q) \left\| \mathcal{S}_q(t_2) \right\|^2 \Phi(\tau) \left(\int_0^{t_1} \left((t_2 - s)^{-q} - (t_1 - s)^{-q} \right)^2 ds \right) \\
& + 4 \left\| M^{-1} \right\|^2 K_0^2 Tr(Q) \left\| \mathcal{S}_q(t_2) \right\|^2 \left(\frac{1}{\Gamma(1-q)} \right)^2 \frac{(t_2 - t_1)^{-2q+1}}{-2q+1} \Phi(\tau) \\
\longrightarrow \quad & 0 \quad \text{as} \quad t_2 - t_1 \longrightarrow 0.
\end{aligned}$$

For I_2, by the assumption (H2), Lemma 2.2 (6) and Hölder inequality, we have

$$\begin{aligned}
I_2 \quad \leq \quad & 4 \left\| L^{-1} \right\|^2 Tr(Q_1) \int_{t_1}^{t_2} (t_2 - s)^{q-1} \left\| T^\alpha \mathcal{T}_q(t_2 - s) \right\|^2 E \left\| \sigma_1(s, x(s), H_1 x(s)) \right\|^2 ds \\
\leq \quad & 4C_1^2 Tr(Q_1) A_\alpha^2 \int_{t_1}^{t_2} (t - s)^{2q--2-2\alpha q} \varphi_R(s) ds \\
& 4C_1^2 Tr(Q_1) A_\alpha^2 \left\| \varphi_R \right\|_{L^{\frac{1}{2q_1 - 1}}[0,b]} \left(\frac{1 - q_1}{q - q_1 - \alpha q} \right)^{2 - 2q_1} (t_2 - t_1)^{2(q - q_1 - \alpha q)} \\
\longrightarrow \quad & 0 \quad \text{as} \quad t_2 - t_1 \longrightarrow 0.
\end{aligned}$$

For I_3, by the assumption (H2), Lemma 2.2 (6) and Hölder inequality, we get that

$$\begin{aligned}
I_3 \quad \leq \quad & 4 \left\| L^{-1} \right\|^2 Tr(Q) \int_0^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \left\| T^\alpha \mathcal{T}_q(t_2 - s) \right\|^2 E \left\| \sigma_1(s, x(s), Hx(s)) \right\|^2 ds \\
\leq \quad & 4C_1^2 Tr(Q) A_\alpha^2 \int_0^{t_1} \left((t_2 - s)^{2q--2-2\alpha} - (t_1 - s)^{2q--2-2\alpha} \right) \varphi_R(s) ds \\
\leq \quad & 4C_1^2 Tr(Q) A_\alpha^2 \left(\int_0^{t_1} \left((t_2 - s)^{\frac{2q--2-2\alpha}{2-2q_1}} - (t_1 - s)^{\frac{2q--2-2\alpha}{2-2q_1}} \right) ds \right)^{2-2q_1} \times \left\| \varphi_R \right\|_{L^{\frac{1}{2q_1 - 1}}[0,t_1]} \\
\leq \quad & 4C_1^2 Tr(Q) A_\alpha^2 \left\| \varphi_R \right\|_{L^{\frac{1}{2q_1 - 1}}[0,b]} \left(\frac{1 - q_1}{q - q_1 - \alpha q} \right)^{2 - 2q_1} (t_2 - t_1)^{2(q - q_1 - \alpha q)} \\
\longrightarrow \quad & 0 \quad \text{as} \quad t_2 - t_1 \longrightarrow 0.
\end{aligned}$$

For $\epsilon > 0$ small enough, by Lemma 2.3 (5) and (6), the assumption (H2) and Hölder inequality, we know that

$$\begin{aligned}
I_4 \quad \leq \quad & \sup_{s\in[0,t_1-\epsilon]} \left\|\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right\|_\alpha^2 \cdot 8\left\|L^{-1}\right\|^2 Tr\,(Q)\int_0^{t_1-\epsilon}(t_1-s)^{2(q-1)} \\
& \times E\left\|\sigma_1(s,x(s),Hx\,(s))\right\|^2 ds \\
& +8\left\|L^{-1}\right\|^2 Tr\,(Q)\int_{t_1-\epsilon}^{t_1}(t_1-s)^{2(q-1)}\left\|T^\alpha\left(\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right)\right\|^2 \\
& \times E\left\|\sigma_1(s,x(s),Hx\,(s))\right\|^2 ds \\
\leq \quad & \sup_{s\in[0,t_1-\epsilon]} \left\|\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right\|_\alpha^2 \cdot 8C_1^2 Tr\,(Q)\int_0^{t_1-\epsilon}(t_1-s)^{2(q-1)}\varphi_R\,(s)\,ds \\
& +16C_1^2 Tr\,(Q)\int_{t_1-\epsilon}^{t_1}(t_1-s)^{2(q-1)}\left((t_2-s)^{-2\alpha q}-(t_1-s)^{-2\alpha q}\right)\varphi_R\,(s)\,ds \\
\leq \quad & \sup_{s\in[0,t_1-\epsilon]} \left\|\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right\|_\alpha^2 \cdot 8C_1^2 Tr\,(Q)\left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]} \\
& \times\left(\frac{1-q_1}{q-q_1}\right)^{2-2q_1}\left(t_1^{\frac{q-q_1}{1-q_1}}-\epsilon^{\frac{q-q_1}{1-q_1}}\right)^{2-2q_1} \\
& +32C_1^2 Tr\,(Q)\,A_\alpha^2\left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1}\epsilon^{2(q-q_1-\alpha q)} \\
\longrightarrow \quad & 0 \quad\text{as}\quad t_2-t_1\longrightarrow 0 \quad\text{and}\quad \epsilon\longrightarrow 0.
\end{aligned}$$

As a result, $E\left\|(\Psi_n x)(t_2)-(\Psi_n x)(t_1)\right\|_\alpha^2$ tends to zero independently of $x\in B_R$ as $t_2-t_1\to 0$, which means that $\Psi_n : B_R\to B_R$ is equicontinuous. Hence by the Arzela-Ascoli theorem one has that $\Psi_n : B_R\to B_R$ is a compact operator. Therefore, by Schauder fixed point theorem we obtain that for each $n\in\mathbb{N}$, Ψ_n has at least one fixed point $x_n\in B_R$ which is in turn a α-mild solution of the problem (3.2)-(3.3). Furthermore, for any $t\in J$, $x_n(t)$ is given by

$$\begin{aligned}
x_n\,(t) \quad = \quad & \mathcal{S}_q(t)S\,(\delta_n)\,M^{-1}\left[x_0+\tfrac{1}{\Gamma(1-q)}\int_0^t(t-s)^{-q}\sigma_2(s,x_n(s))dw_2(s)\right] \\
+ \quad & \int_0^t(t-s)^{q-1}L^{-1}\mathcal{T}_q(t-s)\sigma_1(s,x_n(s),Hx_n\,(s))dw_1(s).
\end{aligned} \tag{3.12}$$

Finally, we show that the set $\{x_n : n\in\mathbb{N}\}\subset B_R$ is precompact in $\mathcal{H}_2$. Denote by

$$x_n^1\,(t) \quad = \quad \mathcal{S}_q(t)S\,(\delta_n)\,M^{-1}\left[x_0+\tfrac{1}{\Gamma(1-q)}\int_0^t(t-s)^{-q}\sigma_2(s,x_n(s))dw_2(s)\right],\quad t\in J$$

and

$$x_n^2\,(t) \quad = \quad \int_0^t(t-s)^{q-1}L^{-1}\mathcal{T}_q(t-s)\sigma_1(s,x_n(s),Hx_n\,(s))dw_1(s),\quad t\in J.$$

Therefore, it is sufficient to show that the sets $\{x_n^1 : n\in\mathbb{N}\}$ and $\{x_n^2 : n\in\mathbb{N}\}$ are precompact in $\mathcal{H}_2$.

Let $\xi\in(0,\delta)$ be fixed, where δ is the constant in (H4). By taking the method similar to the proof of the compactness of the operator Ψ_n, we can see that the sets $\{x_n^1 : n\in\mathbb{N}\}\big|_{[\xi,b]}$ and $\{x_n^2 : n\in\mathbb{N}\}$ are precompact in $C([\xi,b],L^2(\Gamma,X_\alpha))$ and $C([0,b],L^2(\Gamma,X_\alpha))$, respectively. In particular, the set $\{x_n^2 : n\in\mathbb{N}\}\big|_{[\xi,b]}$ is precompact in $C([\xi,b],L^2(\Gamma,X_\alpha))$. Therefore, we have proved that the set $\{x_n : n\in\mathbb{N}\}\big|_{[\xi,b]}$ is precompact in $C([\xi,b],L^2(\Gamma,X_\alpha))$.

Without loss of generality, we let

$$x_n\longrightarrow x \quad\text{in}\quad C([\xi,b],L^2(\Gamma,X_\alpha)) \quad\text{as}\quad n\longrightarrow\infty.$$

Denote by

$$x_n^\delta\,(t)=\begin{cases} x_n\,(\delta)\,,\ t\in[0,\delta]\,, \\ x_n\,(t)\,,\ t\in[\delta,b]\,. \end{cases},\quad x^\delta\,(t)=\begin{cases} x\,(\delta)\,,\ t\in[0,\delta]\,, \\ x\,(t)\,,\ t\in[\delta,b]\,. \end{cases}$$

Then we have

$$x_n^\delta\to x^\delta \quad\text{in}\quad C([0,b],L^2(\Gamma,X_\alpha)) \quad\text{as}\quad n\longrightarrow\infty.$$

Therefore, to prove that the set $\{x_n : n \in \mathbb{N}\}$ is precompact in $C([0,b], L^2(\Gamma, X_\alpha))$, we only need to prove that the set $\{x_n^1 : n \in \mathbb{N}\}\big|_{[0,\xi]}$ is precompact in $C([0,\xi], L^2(\Gamma, X_\alpha))$. By the strong continuity of the semigroup $S(t)(t \geq 0)$ and the assumptions (H3) and (H4), we have

$$\begin{aligned}
& E\left\|S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right)\right. \\
& \left.-M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x(s))dw_2(s)\right)\right\|_\alpha^2 \\
\leq\ & 2\left\|M^{-1}\right\|^2\left(\frac{1}{\Gamma(1-q)}\right)^2 \\
& \times E\left\|S(\delta_n)\int_0^t (t-s)^{-q}T^\alpha\left(\sigma_2(s, x_n^\delta(s)) - -\sigma_2(s, x^\delta(s)\right)dw_2(s)\right\|^2 \\
& +2\left\|M^{-1}\right\|^2\left(\frac{1}{\Gamma(1-q)}\right)^2 \times E\left\|S(\delta_n)\int_0^t (t-s)^{-q}T^\alpha\sigma_2(s, x^\delta(s))dw_2(s)\right. \\
& \left.-\int_0^t (t-s)^{-q}T^\alpha\sigma_2(s, x^\delta(s))dw_2(s)\right\|^2 \\
\leq\ & 2C_2^2\left(\frac{1}{\Gamma(1-q)}\right)^2 K_0^2 E\left\|\int_0^t (t-s)^{-q}\sigma_2(s, x_n^\delta(s))dw_2(s)\right. \\
& \left.-\int_0^t (t-s)^{-q}\sigma_2(s, x^\delta(s), H_2x^\delta(s))dw_2(s)\right\|_\alpha^2 \\
& +2C_2^2\left(\frac{1}{\Gamma(1-q)}\right)^2 \times E\left\|(S(\delta_n) - I)\int_0^t (t-s)^{-q}T^\alpha\sigma_2(s, x^\delta(s))dw_2(s)\right\|^2 \\
\longrightarrow\ & 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

which means that the set $\left\{S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right) : n \in \mathbb{N}\right\}$ is precompact in X_α. By the continuity of the operator $\mathcal{S}_q(t)(t \geq 0)$, we know that the set $\left\{\mathcal{S}_q(t)S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right) : n \in \mathbb{N}\right\}$ is precompact in X_α for $t \in [0,\xi]$. By Lemma 2.2 (2) and the assumption (H3), we know that for every $n \in \mathbb{N}$ and $t_1, t_2 \in [0,\xi]$ with $t_1 < t_2$,

$$\begin{aligned}
& E\left\|\mathcal{S}_q(t_2)S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^{t_2} (t_2-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right)\right. \\
& \left.-\mathcal{S}_q(t_1)S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^{t_1} (t_1-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right)\right\|_\alpha^2 \\
\leq\ & 3E\left\|\left(\mathcal{S}_q(t_2) - \mathcal{S}_q(t_1)\right)S(\delta_n) M^{-1}\left[\frac{1}{\Gamma(1-q)}\int_0^{t_1} (t_1-s)^{-q}T^\alpha\sigma_2(s, x_n(s))dw_2(s)\right]\right\|_\alpha^2 \\
& +3E\left\|\mathcal{S}_q(t_2)S(\delta_n) M^{-1}\left[\frac{1}{\Gamma(1-q)}\int_0^{t_1}\left((t_2-s)^{-q} - (t_1-s)^{-q}\right)T^\alpha\sigma_2(s, x_n(s))dw_2(s)\right]\right\|_\alpha^2 \\
& +3E\left\|\mathcal{S}_q(t_2)S(\delta_n) M^{-1}\left[\frac{1}{\Gamma(1-q)}\int_{t_1}^{t_2} (t_2-s)^{-q}T^\alpha\sigma_2(s, x_n(s))dw_2(s)\right]\right\|_\alpha^2 \\
\longrightarrow\ & 0 \text{ as } t_2 - t_1 \longrightarrow 0.
\end{aligned}$$

This means that the set $\{x_n^1(t) : n \in \mathbb{N}\}$ is equicontinuous for $t \in [0,\xi]$. Therefore, applying Arzela-Ascoli theorem again one obtains that the set $\{x_n^1 : n \in \mathbb{N}\}\big|_{[0,\xi]}$ is precompact in $C([0,\xi], L^2(\Gamma, X_\alpha))$.

Therefore,we have proved that the set $\{x_n : n \in \mathbb{N}\}$ is precompact in $C([0,b], L^2(\Gamma, X_\alpha))$.

Hence, without losing the generality, we may suppose that

$$x_n \to x^* \text{ in } C([0,b], L^2(\Gamma, X_\alpha)) \text{ as } n \longrightarrow \infty.$$

Taking limits in (3.12) one has

$$\begin{aligned}
x^*(t) &= \mathcal{S}_q(t)S(\delta_n) M^{-1}\left[x_0 + \tfrac{1}{\Gamma(1-q)}\textstyle\int_0^t (t-s)^{-q}\sigma_2(s, x^*(s))dw_2(s)\right] \\
&+ \textstyle\int_0^t (t-s)^{q-1} L^{-1}\mathcal{T}_q(t-s)\sigma_1(s, x^*(s), Hx^*(s))dw_1(s).
\end{aligned}$$

for all $t \in J$, which means that $x^* \in C([0,b], L^2(\Gamma, X_\alpha))$ is a α-mild solution of the problem (1.1)-(1.2) and the proof of Theorem 3.1 is completed.

4 An example

n this section, we present an example, which do not aim at generality but indicate how our abstract result can be applied to concrete problem. Let $N \geq 1$ be an integer, $U \subset \mathbb{R}^N$ be a bounded domain, whose boundary $\partial\Omega$ is an $(N-1)$-dimensional $C^{2+\mu}$-manifold for some $0 < \mu < 1$. We consider the nonlocal problem of Sobolev type fractional stochastic parabolic partial differential equation of the form

$$ {}^C D_t^{\frac{2}{3}} \left[x(z,t) - x_{zz}(z,t)\right] - \frac{\partial^2}{\partial z^2} x(z,t) = \frac{\sin\left(z,t,x(z,t), \int_0^t K(t,s)\, x(z,s)\, ds\right) d\hat{w}_1(t)}{e^t dt}, \; z \in U, \; t \in J, \tag{4.1} $$

$$ x(z,0) = \frac{\partial^2}{\partial z^2} \left[x_0(z) + \frac{1}{\Gamma\left(\frac{1}{3}\right)} \sum_{k=1}^{m} c_k \int_0^t (t-s)^{-\frac{2}{3}} x(z,t_k)\, \hat{w}_2(s) \right], \; z \in U \tag{4.2} $$

where ${}^C D_t^q$ is the Caputo fractional derivative of order $q \in (0,1)$, $0 < t_1 < ... < t_m < b$ and c_k are positive constants, $k = 1, ..., m$; the functions $x(t)(z) = x(z,t)$, $\sigma_1(t, x(t), Hx(t))(z) = \frac{\sin\left(z,t,x(z,t), \int_0^t K(t,s)x(z,s)ds\right)}{e^t}$ and $\sigma_2(t, x(t))(z) = \sum_{k=1}^m c_k x(z, t_k)$; $\hat{w}_1(t)$ and $\hat{w}_2(t)$ are two sided and standard one dimensional Brownian motions defined on the filtered probability space (Ω, Γ, P), $J = [0,b]$, $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \left\{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq b\right\}$.

Let $X = L^2(U)$, define the operators $L : D(L) \subset X \to X$, $A : D(A) = H^2(U) \cap H_0^1(U) \subset X \to X$ and $M : D(M) \subset X \to X$ by $Lx = x - x''$, $Ax = -x''$ and $M^{-1}x = x''$ where the domains $D(L)$, $D(A)$ and $D(M)$ are given by

$$ \{x \in X : x, x' \text{ are absolutely continuous}, \; x'' \in X, \, x|_{\partial U} = 0\}. $$

It is easy to see that L^{-1} is compact, bounded with $\|L^{-1}\| \leq 1$ and $T = AL^{-1}$ generates the above strongly continuous semigroup $S(t)$ on $L^2(U)$ with $\|S(t)\| \leq e^{-t} \leq 1$. Therefore, with the above choices, the system (4.1)-(4.2) can be written as an abstract formulation of (1.1)-(1.2).

From the definitions of σ_1 and σ_2, it is easy to verify that $\sigma_1 : X_0 \times X_0 \to L_2^0$ and $\sigma_2 : X_0 \to L_2^0$ whenever $x \in C(J, L^2(\Gamma, X_0))$. Moreover, we see that the assumptions (H1)–(H4) and the condition (3.1) hold with

$$ q_1 = \frac{1}{2}, \quad \varphi_\tau(t) = \frac{|U|}{e^t}, \quad \Phi(\tau) = m\,|U| \sum_{k=1}^{m} c_k^2 \tau^{\frac{2}{3}}, \quad \rho_1 = \rho_2 = 0, \quad \delta = t_1. $$

Therefore, by Theorem 3.1, we have the following result.

Theorem 4.1 *The nonlocal problem of Sobolev type fractional stochastic parabolic partial differential equation has at least one* 0*-mild solution.*

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Received: February 20, 2015; *Accepted*: August 20, 2015

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 169–177

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On the Stability of $\alpha-$Cauchy-Jensen type functional equation in Banach Algebras

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Abstract

Using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the following $\alpha-$Cauchy-Jensen functional equation:

$$f(\frac{x+y}{\alpha}+z)+f(\frac{x-y}{\alpha}+z)=\frac{2}{\alpha}f(x)+2f(z),$$

where $\alpha \in \mathbb{N}_{\geq 2}$.

Keywords: Cauchy-Jensen type functional equation, fixed point, homomorphism in Banach algebra, generalized Hyers-Ulam stability.

2010 MSC: 39A10, 39B52, 47H10, 39B82.

1 Introduction

The study of stability problems for functional equations is related to a question of S. M. Ulam [25] concerning the stability of group homomorphisms.

Let $(G,.)$ be a group and let $(H,.,d)$ be a metric group with the metric d. Given $\delta > 0$, does there exist $\epsilon > 0$ such that if a mapping $h : G \to H$ satisfies the inequality

$$d(h(xy), h(x)h(y)) \leq \delta$$

for all $x, y \in G$, then there is a homomorphism $a : G \to H$ with

$$d(h(x), a(x)) \leq \epsilon$$

for all $x \in G$?

In 1941, Hyers [7] considered the case of approximately additive mappings $f : E \to F$, where E and F are Banach spaces and f satisfies Hyers inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$ and $\epsilon > 0$. He proved that then there exists a unique additive mapping $T : E \to F$ satisfying

$$\|f(x) - T(x)\| \leq \epsilon$$

for all $x \in E$.

In 1950, T. Aoki [1] was the second author to study this problem for additive mappings.

In 1978, Th. M. Rassias [18] generalized the result of Hyers by considering the stability problem for unbounded Cauchy differences. This phenomenon of stability introduced by Th. M. Rassias [18] is called the Hyers-Ulam-Rassias stability.

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Theorem 1.1 ([18, Th. M. Rassias]). *Let $f : E \to F$ be a mapping from a real normed vector space E into a Banach space F satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \tag{1.1}$$

for all $x, y \in E$, where θ and p are constants with $\theta > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p, \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.3) for $x \neq 0$. Also, if the function $t \to f(tx)$ from $\mathbb{R}$ into F is continuous for each fixed $x \in E$, then T is linear.

The above inequality (1.3) has produced a lot of influence on the development of what we now call the Hyers-Ulam-Rassias stability of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Rassias [20], following the spirit of the innovative approach of Rassias [18] for the unbounded Cauchy difference, proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p.\|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [21] for a number of other new results).

Theorem 1.2 ([19, 20]). *Let E be a real normed linear space and F a real complete normed linear space. Assume that $f : E \to F$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R}\backslash\{1\}$ such that f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta.\|x\|^{p/2}.\|y\|^{p/2},$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \leq \frac{\theta}{|2^p - 2|}\|x\|^p,$$

for all $x \in E$. If, in addition, $f : E \to F$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}-$linear.

Găvruta [6] generalized Rassias' result. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 3, 8–16, 22–24]).

We now recall one of fundamental results of fixed point theory. For the proof, we refer to [5] : Let X a set, a function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

1. $d(x, y) = 0$ if and only if $x = y$.

2. $d(x, y) = d(y, x)$ for all $x, y \in X$.

3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.3 ([5, The alternative of fixed point]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strict contractive mapping with a Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integer n or there exists a positive integer n_0 such that

1. *$d(J^n x, J^{n+1} x) < \infty \; \forall n \geq n_0$;*

2. *the sequence $J^n x$ converge to a fixed y^* for J;*

3. *y^* is the unique fixed point of J in the set $Y = \{y \in X, d(J^{n_0} x, y) < \infty\}$;*

4. *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$.*

In [17], Park proved the generalized Hyers-Ulam stability of the Cauchy-Jensen functional equation

$$f(\frac{x+y}{2}+z)+f(\frac{x-y}{2}+z)=f(x)+2f(z) \tag{1.3}$$

in Banach Algebras, by using the fixed theorem.

In the present paper, using the fixed point method, we prove the Hyers-Ulam-Rassias stability of homomorphisms in real Banach algebras for the α-Cauchy-Jensen functional equation

$$f(\frac{x+y}{\alpha}+z)+f(\frac{x-y}{\alpha}+z)=\frac{2}{\alpha}f(x)+2f(z), \tag{1.4}$$

where $\alpha \in \mathbb{N}_{\geq 2}$.

2 Stability of homomorphisms in real Banach algebras

Throughout this section, assume that $\mathbb{N}$ the set of all positive integers, $\mathbb{N}_{\geq 2}=\mathbb{N}\backslash\{0,1\}$, A is a real Banach algebra with norm $\|.\|_A$ and that B is a real Banach algebra with norm $\|.\|_B$.

For a given mapping $f: A \to B$,we define

$$D_\alpha f(x,y,z)=f(\frac{x+y}{\alpha}+z)+f(\frac{x-y}{\alpha}+z)-\frac{2}{\alpha}f(x)-2f(z),$$

for all $x,y,z \in A$ and $\alpha \in \mathbb{N}_{\geq 2}$.

Lemma 2.1. *If a mapping $f: A \to B$ satisfies (1.4), then f is a Jensen type additive-additive mapping.*

Proof. Letting $x=y=z=0$ in (1.4), we get $f(0)=0$. Setting $y=z=0$ in (1.4), we obtain

$$2f(\frac{x}{\alpha})=\frac{2}{\alpha}f(x), \tag{2.5}$$

for all $x \in A$. Replacing y by 0 in (1.4) and by (2.5), we get

$$2f(\frac{x}{\alpha}+z)=\frac{2}{\alpha}f(x)+2f(z)=2f(\frac{x}{\alpha})+2f(z),$$

for all $x,z \in A$. Then

$$f(t+z)=f(t)+f(z), \tag{2.6}$$

for all $t,z \in A$, with $t=x/\alpha$. This implies that f is an additive mapping.
Now, we substitute $t=\frac{u+v}{2}$ and $z=\frac{u-v}{2}$ in (2.6), we obtain

$$f(\frac{u+v}{2})+f(\frac{u-v}{2})=f(u), \tag{2.7}$$

for all $u,v \in A$. Therefore f is a Jensen type additive mapping. □

Using the fixed point method, we establish the Hyers-Ulam-Rassias stability of homomorphisms in real Banach algebras for the functional equation $D_\alpha f(x,y,z)=0$.

Theorem 2.4. *Let $f: A \to B$ be a mapping for which there exists a function $\varphi: A^3 \to [0,\infty)$ such that*

$$\|D_\alpha f(x,y,z)\|_B \leq \varphi(x,y,z); \tag{2.8}$$

$$\|f(xy)-f(x)f(y)\|_B \leq \varphi(x,y,0); \tag{2.9}$$

$$\sum_{n\geq 0} a^{-n}\varphi(a^n x, a^n y, a^n z) < \infty \tag{2.10}$$

and

$$\varphi(ax,ax,ax) \leq aL\varphi(x,x,x) \tag{2.11}$$

for all $x,y,z \in A$, with $L<1$ and $a=1+2/\alpha$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $h: A \to B$ such that

$$\|f(x)-h(x)\|_B \leq \frac{1}{a-aL}\varphi(x,x,x), \tag{2.12}$$

for all $x \in A$.

Proof. Letting $x = y = z$ in (2.8), we get

$$\left\| \frac{f(ax)}{a} - f(x) \right\|_B \leq \frac{1}{a}\varphi(x,x,x); \tag{2.13}$$

for all $x \in A$, with $a = 1 + 2/\alpha$.

Consider the set

$$S := \{g : A \to B\}$$

and introduce the generalized metric on S:

$$d(g,k) = \inf\{C \in \mathbb{R}_+ : \|g(x) - k(x)\| \leq C\varphi(x,x,x), \forall x \in A\},$$

with the convention $\inf \emptyset := +\infty$. As in [4], one can prove that the generalized metric space (S,d) is complete.

Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{a}g(ax) \tag{2.14}$$

for all $x \in A$. Hence $d(Jf,f) \leq \frac{1}{a} < \infty$.

First, we start by proving that J is strictly contractive on the (S,d). To this end, let $f,g \in S$ be given. Without loss of generality, we may suppose that $d(g,k)$ is finite. In this case, let $C \in \mathbb{R}_+$ be an arbitrary constant such that

$$\|g(x) - k(x)\| \leq C\varphi(x,x,x)$$

for all $x \in A$. Then

$$\begin{aligned} \|Jg(x) - Jk(x)\|_B &= \|\frac{g(ax)}{a} - \frac{k(ax)}{a}\|_B \\ &\leq \frac{1}{a}\varphi(ax,ax,ax) \quad (2.15) \\ &\leq CL\varphi(x,x,x) \quad (2.16) \end{aligned}$$

for every $x \in A$, i.e, $d(Jg,Jk) \leq CL$. This implies that

$$d(Jg,Jk) \leq Ld(f,g)$$

for all $g,k \in S$. As $L < 1$, then operator J is strictly contractive.

By Theorem 1.3, there exists a mapping $h : A \to B$ satisfying the following

(1) h is a fixed point of J, that is,

$$h(ax) = ah(x)$$

for all $x \in A$. The mapping h is a unique fixed point of J in the set

$$S^* = \{g \in S : d(g,k) < \infty\}.$$

(2) $\lim_{n\to\infty} d(J^n f,h) = 0$. This implies the equality

$$\lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^n} = h(x) \tag{2.17}$$

for all $x \in A$.

(3) $d(f,h) \leq \frac{1}{1-L}d(Jf,f)$, which implies the inequality

$$d(f,g) \leq \frac{1}{a - aL}.$$

This implies that the inequality (2.12) holds.

It follows from (2.8), (2.10) and (2.17) that

$$\|D_\alpha h(x,y,z)\|_B = \lim_{n\to\infty}\frac{1}{a^n}\|D_\alpha f(a^nx,a^ny,a^nz)\|_B \quad (2.18)$$

$$\leq \lim_{n\to\infty}\frac{1}{a^n}\varphi(a^nx,a^ny,a^nz)=0 \quad (2.19)$$

for all $x,y,z\in A$ and $n\in\mathbb{N}$. So

$$h(\frac{x+y}{\alpha}+z)+h(\frac{x-y}{\alpha}+z)=\frac{2}{\alpha}h(x)+2h(z),$$

for all $x,y,z\in A$. By Lemma 2.1, the mapping $h: A\to B$ is Cauchy additive.

By the same reasoning as the proof of theorem of [18], the mapping $h: A\to B$ is $\mathbb{R}$-linear.

It follow from (2.9) that

$$\begin{aligned}\|h(xy)-h(x)h(y)\| &= \lim_{n\to\infty}\frac{1}{a^{2n}}\|f(a^nxa^ny)-f(a^nx)f(a^ny)\|_B\\ &\leq \lim_{n\to\infty}\frac{1}{a^{2n}}\varphi(a^nx,a^ny,0)\\ &\leq \lim_{n\to\infty}\frac{1}{a^n}\varphi(a^nx,a^ny,0)=0\end{aligned}$$

for all $x,y\in A$. So $h(xy)=h(x)h(y)$ for all $x,y\in A$. Thus $h: A\to B$ is a homomorphism satisfying (2.12), as desired. □

Corollary 2.1 ([17]). *Let $f: A\to B$ be a mapping for which there exists a function $\varphi: A^3\to[0,\infty)$ such that*

$$\|D_2f(x,y,z)\|_B\leq\varphi(x,y,z);$$

$$\|f(xy)-f(x)f(y)\|_B\leq\varphi(x,y,0);$$

$$\sum_{n\geq0}2^{-n}\varphi(2^nx,2^ny,2^nz)<\infty$$

and

$$\varphi(2x,2x,2x)\leq 2L\varphi(x,x,x)$$

for all $x,y,z\in A$, with $L<1$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$, then there exists a unique homomorphism $h: A\to B$ such that

$$\|f(x)-h(x)\|_B\leq\frac{1}{2-2L}\varphi(x,x,x),$$

for all $x\in A$.

Corollary 2.2. *Let $p<1$ and δ be nonnegative real numbers and let $f: A\to B$ be a mapping satisfying*

$$\|D_\alpha f(x,y,z)\|_B\leq\delta(\|x\|_A^p+\|y\|_A^p+\|z\|_A^p);$$

and

$$\|f(xy)-f(x)f(y)\|_B\leq\delta(\|x\|_A^p+\|y\|_A^p);$$

for all $x,y,z\in A$, with $a=1+2/\alpha$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$, then there exists a unique homomorphism $h: A\to B$ such that

$$\|f(x)-h(x)\|_B\leq\frac{3\delta}{a-a^p}\|x\|_A^p,$$

for all $x\in A$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x,y,z):=\delta(\|x\|_A^p+\|y\|_A^p+\|z\|_A^p)$$

for all $x,y,z\in A$. Then, $L=a^{p-1}$ (with $a=1+2/\alpha$) and we get the desired result. □

Theorem 2.5. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0, \infty)$ such that*

$$\|D_\alpha f(x,y,z)\|_B \leq \varphi(x,y,z); \tag{2.20}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \varphi(x,y,0); \tag{2.21}$$

$$\sum_{n\geq 0} a^{2n}\varphi(a^{-n}x, a^{-n}y, a^{-n}z) < \infty \tag{2.22}$$

and

$$a\varphi(x,x,x) \leq L\varphi(ax,ax,ax) \tag{2.23}$$

for all $x,y,z \in A$, with $L < 1$ and $a = 1 + 2/\alpha$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{L}{a - aL}\varphi(x,x,x), \tag{2.24}$$

for all $x \in A$.

Proof. We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := ag(\frac{x}{a})$$

for all $x \in A$ with $a = 1 + 2/\alpha$.

It follows from (2.13) that

$$\|f(x) - af(x/a)\|_B \leq \varphi(x/a, x/a, x/a) \leq \frac{L}{a}\varphi(x,x,x) \tag{2.25}$$

for all $x \in A$. Hence $d(f, Jf) \leq \frac{L}{a}$.

Let $g, k \in S$ and $C \in \mathbb{R}_+$ be an arbitrary constant such that

$$\|g(x) - k(x)\| \leq C\varphi(x,x,x)$$

for all $x \in A$. Then

$$\begin{aligned}\|Jg(x) - Jk(x)\|_B &= \|ag(\frac{x}{a}) - ak(\frac{x}{a})\|_B \\ &\leq aC\varphi(\frac{x}{a}, \frac{x}{a}, \frac{x}{a}) \\ &\leq CL\varphi(x,x,x)\end{aligned} \tag{2.26}$$

for all $x \in A$, i.e, $d(Jg, Jk) \leq CL$. We hence conclude that

$$d(Jg, Jk) \leq Ld(g,k)$$

for all $g, k \in S$. As $L < 1$, then operator J is strictly contractive.

By Theorem 1.3, there exists a mapping $h : A \to B$ satisfying the following

(1) h is a fixed point of J such that $\lim_{n\to\infty} d(J^n f, h) = 0$. This implies the equality

$$h(x) = \lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} a^n f(\frac{x}{a^n}) \tag{2.27}$$

and $ah(\frac{x}{a}) = h(x)$ for all $x \in A$. The mapping h is a unique fixed point of J in the set

$$S^* = \{g \in S : d(f,g) < \infty\}.$$

(2) $d(f,h) \leq \frac{1}{1-L}d(Jf, f)$, which implies the inequality

$$d(f,g) \leq \frac{L}{a - aL}.$$

This implies that the inequality (2.24) holds.

It follows from (2.20), (2.22) and (2.27) that

$$\|D_\alpha h(x,y,z)\|_B = \lim_{n\to\infty} a^n \left\| D_\alpha f(\frac{1}{a^n}x, \frac{1}{a^n}y, \frac{1}{a^n}z)\right\|_B$$

$$\leq \lim_{n\to\infty} a^n \varphi(\frac{1}{a^n}x, \frac{1}{a^n}y, \frac{1}{a^n}z) \quad (2.28)$$

$$\leq \lim_{n\to\infty} a^{2n} \varphi(\frac{1}{a^n}x, \frac{1}{a^n}y, \frac{1}{a^n}z) = 0 \quad (2.29)$$

for all $x, y, z \in A$ and $n \in \mathbb{N}$. So

$$h(\frac{x+y}{\alpha}+z) + h(\frac{x-y}{\alpha}+z) = \frac{2}{\alpha}h(x) + 2h(z),$$

for all $x, y, z \in A$. By Lemma 2.1, the mapping $h : A \to B$ is Cauchy additive.

By the same reasoning as the proof of theorem of [18], the mapping $h : A \to B$ is $\mathbb{R}$-linear.

It follow from (2.21) that

$$\|h(xy) - h(x)h(y)\| = \lim_{n\to\infty} a^{2n}\|f(\frac{xy}{a^{2n}}) - f(\frac{x}{a^n})f(\frac{y}{a^n})\|_B$$

$$\leq \lim_{n\to\infty} a^{2n}\varphi(a^{-n}x, a^{-n}y, 0) = 0$$

for all $x, y \in A$. So $h(xy) = h(x)h(y)$ for all $x, y \in A$.

Thus $h : A \to B$ is a homomorphism satisfying (2.24), as desired. □

Corollary 2.3 ([17])**.** *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ such that*

$$\|D_2 f(x,y,z)\|_B \leq \varphi(x,y,z);$$

$$\|f(xy) - f(x)f(y)\|_B \leq \varphi(x,y,0);$$

$$\sum_{n\geq 0} 2^{2n}\varphi(2^{-n}x, 2^{-n}y, 2^{-n}z) < \infty$$

and

$$2\varphi(x,x,x) \leq L\varphi(2x,2x,2x)$$

for all $x, y, z \in A$, with $L < 1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{L}{2-2L}\varphi(x,x,x),$$

for all $x \in A$.

Corollary 2.4. *Let $p > 1$ and δ be nonnegative real numbers and let $f : A \to B$ be a mapping satisfying*

$$\|D_\alpha f(x,y,z)\|_B \leq \delta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p);$$

and

$$\|f(xy) - f(x)f(y)\|_B \leq \delta(\|x\|_A^p + \|y\|_A^p);$$

for all $x, y, z \in A$, with $a = 1 + 2/\alpha$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{3\delta}{a^p - a}\|x\|_A^p,$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x,y,z) := \delta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$$

for all $x, y, z \in A$. Then, $L = a^{1-p}$ (with $a = 1 + 2/\alpha$) and we get the desired result. □

3 Acknowledgment

The authors would like to thank the referee for his valuable comments and suggestions.

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Received: October 10, 2015; *Accepted*: January 23, 2016

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Website: http://www.malayajournal.org/

Malaya J. Mat. 4(1)(2016) 178–185

Malaya Journal of Matematik
www.malayajournal.org

MJM an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Compactons solutions for the fractional nonlinear dispersive K(2,2) equations by the homotopy perturbation method

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Abstract

In this paper, the homotopy perturbation is successively used to obtain approximate analytical solutions of the nonlinear dispersive $K(2,2)$ equation with time and space derivative. Comparison between the numerical and the exact solutions revealed that HPM is an alternative analytical method for solving fractional differential equations.

Keywords: Caputo fractional derivative, homotopy perturbation method, K(2,2) equation, fractional differential equations.

2010 MSC: 35R11, 35Q53, 35A15, 47H15, 65P05.

1 Introduction

Fractional differential systems have recently been proved to be useful in physics, engineering and control processing in various fields of sciences such as viscoelasticity, diffusion, control, relaxation processes. Many contributions have been made to both the theory and applications of the fractional differential equations during the past decades ([7],[15],[18]). As in general, there exists no method that yields an exact solution, approximate solutions are then derived using the linearization ([7],[15]) or Adomian decomposition method (ADM) [1] . The variational iteration method (VIM) was first proposed by He [11] for solving non-linear problems and it is found to be an effective way to approximate the solutions of the fractional differential equations, both linear and nonlinear [14]. Momani and Odibat [17] and Yıldırım [20] applied the homotopy perturbation method (HPM) to fractional differential equations and revealed that HPM is an alternative analytical method for solving fractional differential equations.
Our concern in this work is to consider the numerical solution of the nonlinear dispersive $K(2,2)$ equation with time and space fractional derivatives of the form

$$^{c}D_t^{\alpha}u + (2u + 6u_{xx})^{c}\, D_x^{\beta}u + 2uu_{xxx} = 0, \;\; 0 < \alpha, \beta \leq 1, \tag{1.1}$$

with the initial condition

$$u(x,0) = g(x). \tag{1.2}$$

When $\alpha = \beta = 1$, this equation turns to the classical $K(2,2)$ equation

$$u_t + (u^2)_x + (u^2)_{xxx} = 0, \tag{1.3}$$

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developed in [19] for describing the compacton solution; i.e., a compact wave that preserves its shape after the interaction with another compact wave.
For the case $g(x) = (\frac{4}{3})\cos^2(\frac{x}{4})$, Eq. (1.3) has an exact solution $u(x,t) = (\frac{4}{3})\cos^2(\frac{x-t}{4})$ and it is developed in [8] for describing the compacton solution [14]. This example is solved numerically by the HPM method and by the variational homotopy method in [2]. It wil be used for comparing the exact and the numerical approximation in section 3.1.
We will extend the application of the HPM method in order to derive analytical approximate solutions to nonlinear time and space fractional $K(2,2)$ equations (1.1). Precisely, we use the new homotopy described in [17] for handling an iterative formula easy-to-use for computation. Observing the numerical results, and comparing with the exact solution, the proposed method reveals to be very close to the exact solution and consequently, an efficient way to solve the nonlinear fractional $K(2,2)$ equation (1.1)-(1.2). This method can take the advantages of the conventional perturbation method while eliminating its restrictions. HPM has been applied by many authors ([3],[4],[6],[9],[20]) and used for many types of linear and non-linear equations in science and engineering. This is the raison why we try to use it in this work.

2 Basic definitions

There are several definitions of a fractional derivative of order $\alpha > 0$ (see [7], [15], [16], [18]). The most commonly used definitions are the Riemann–Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper .

Definition 2.1. *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C\,[0,\infty)$, and it is said to be in the space C_μ^m, if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.*

Definition 2.2. *The left sided Riemann–Liouville fractional integral of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as*

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad x > 0, \alpha > 0, \tag{2.4}$$

$$I^0 f(x) = f(x).$$

Definition 2.3. *Let $f \in C_{-1}^m$, $m \in \mathbb{N}$. Then the (left sided) Caputo fractional derivative of f is defined as*

$$^cD^\alpha f(x) = I^{m-\alpha} D^m f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, & m-1 < \alpha < m, \\ \frac{d^m f(t)}{dt^m}, & \alpha = m. \end{cases} \tag{2.5}$$

According to (2.5), we can obtain

$$^cD^\alpha C = 0, \;\; \text{C is constant}$$

and

$$^cD^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \beta > \alpha - 1, \\ 0, & \beta \leq \alpha - 1. \end{cases} \tag{2.6}$$

Remark 2.1. *In this paper, we consider equation (1.1) with time-and space-fractional derivative in the Caputo sens. When $\alpha \in \mathbb{R}^+$, the time fractional derivative is defined as*

$$^cD_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (x-t)^{m-\alpha-1} \frac{\partial^m u(x,t)}{\partial \tau^m}(\tau) d\tau, m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, \quad \alpha = m. \end{cases} \tag{2.7}$$

The form of the space fractional derivative is similar to the above and we just omit it here.

3 The Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following non-linear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3.8}$$

with the following boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \tag{3.9}$$

where A is a general differential operator, $f(r)$ is a known analytic function, B is a boundary operator, is the unknown function, and Γ is the boundary of the domain Ω. The operator A can be generally divided into two operators, L and N, where L is a linear, and N a nonlinear operator. Equation (3.8) can be, therefore, written as follows

$$L(u) + N(u) - f(r) = 0. \tag{3.10}$$

Using the homotopy technique, we construct a homotopy $v(r, p) : \Omega \times [0, 1] \longrightarrow \mathbb{R}$, which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{3.11}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{3.12}$$

where $p \in [0, 1]$ is an embedding parameter, and $u_0$is the initial approximation of equation (3.8) which satisfies the boundary conditions. Clearly, from Eq. (3.11) and (3.12) we will have

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{3.13}$$
$$H(v, 1) = A(v) - f(r) = 0. \tag{3.14}$$

The changing process of p from zero to unity is just that of $v(r, p)$ changing from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. If the embedding parameter p $(0 \leq p \leq 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of equation (3.11) or (3.12) can be given as a power series in p

$$v = v_0 + pv_1 + p^2 v_2 + \cdots . \tag{3.15}$$

Setting $p = 1$, results in the approximate solution of Eq. (3.8)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{3.16}$$

The convergence of the series (3.16) has been proved in ([5],[6]).

3.1 New modification of the HPM

Momani and al. [17] introduce an algorithm to handle in a realistic and efficient way the nonlinear PDEs of fractional order. They consider the nonlinear partial differential equations with time fractional derivative of the form

$$\begin{cases} {}^cD_t^\alpha u(x, t) = f(u, u_x, u_{xx}) = L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + h(x, t), t > 0 \\ u^k(x, 0) = g_k(x), \quad k = 0, 1, 2, ...m - 1, \end{cases} \tag{3.17}$$

where L is a linear operator, N is a nonlinear operator which also might include other fractional derivatives of order less than α. The function h is considered to be a known analytic function and ${}^cD^\alpha$, $m - 1 < \alpha \leq m$, is the Caputo fractional derivative of order α.
In view of the homotopy technique, we can construct the following homotopy

$$\frac{\partial u^m}{\partial t^m} - L(u, u_x, u_{xx}) - h(x, t) = p[\frac{\partial u^m}{\partial t^m} + N(u, u_x, u_{xx}) -^c D_t^\alpha u], \tag{3.18}$$

or

$$\frac{\partial u^m}{\partial t^m} - h(x, t) = p[\frac{\partial u^m}{\partial t^m} + L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) -^c D_t^\alpha u], \tag{3.19}$$

where $p \in [0,1]$. The homotopy parameter p always changes from zero to unity. In case $p = 0$, Eq. (3.18) becomes the linearized equation

$$\frac{\partial u^m}{\partial t^m} = L(u, u_x, u_{xx}) + h(x,t), \tag{3.20}$$

or in the second form, Eq. (3.19) becomes the linearized equation

$$\frac{\partial u^m}{\partial t^m} = h(x,t). \tag{3.21}$$

When it is one, Eq. (3.18) or Eq. (3.19) turns out to be the original fractional differential equation (3.17). The basic assumption is that the solution of Eq. (3.18) or Eq. (3.19) can be written as a power series in p

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 \cdots . \tag{3.22}$$

Finally, we approximate the solution by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{3.23}$$

4 HPM method for the fractional K(2,2) equations

In this section, we apply the New modification of the HPM 3.1 for solving the K(2,2) equation with time-fractional derivative and we use the classical HPM to obtain analytical solution for K(2,2) equation with space-fractional derivative.

4.1 Numerical solutions of time-fractional K(2,2) equation

If one fixes $\beta = 1$ and considers the following form of the time-fractional $K(2,2)$ equation

$$^{c}D_t^{\alpha} u + (2\,u + 6\,u_{xx})\,u_x + 2\,uu_{xxx} = 0, \quad 0 < \alpha \leq 1, \tag{4.24}$$

with the initial condition

$$u(x,0) = (\frac{4}{3})\cos^2(\frac{x}{4}), \tag{4.25}$$

the exact solution of (4.24)-(4.25) for the special case $\alpha = 1$ is

$$u(x,t) = (\frac{4}{3})\cos^2(\frac{x-t}{4}). \tag{4.26}$$

In view of Eq. (3.19), the homotopy for Eq. (4.24) can be constructed as

$$\frac{\partial u}{\partial t} = p[\frac{\partial u}{\partial t} - (2\,u + 6\,u_{xx})\,u_x - 2\,uu_{xxx} - {}^{c}D_t^{\alpha} u]. \tag{4.27}$$

Substituting (3.22) into (4.27) and equating the terms with identical powers of p, one obtains the following set of linear partial differential equations

$$\begin{aligned}
&p^0: \tfrac{\partial u_0}{\partial t} = 0,\\
&p^1: \tfrac{\partial u_1}{\partial t} = \tfrac{\partial u_0}{\partial t} - (2u_0 + 6u_{0xx})\,u_{0x} - 2u_0 u_{0xxx} - {}^{c}D_t^{\alpha} u_0,\\
&p^2: \tfrac{\partial u_2}{\partial t} = \tfrac{\partial u_1}{\partial t} - (2u_0 + 6u_{0xx})\,u_{1x} - (2u_1 + 6u_{1xx})\,u_{0x} - 2u_0 u_{1xxx} - 2u_1 u_{0xxx} - {}^{c}D_t^{\alpha} u_1,\\
&p^3: \tfrac{\partial u_3}{\partial t} = \tfrac{\partial u_2}{\partial t} - (2u_0 + 6u_{0xx})\,u_{2x} - (2u_1 + 6u_{1xx})\,u_{1x} - (2u_2 + 6u_{2xx})\,u_{0x} - 2(u_0 u_{2xxx} + 2u_2 u_{0xxx} + 2u_1 u_{1xxx}) - {}^{c}D_t^{\alpha} u_2,\\
&\vdots
\end{aligned} \tag{4.28}$$

with the following conditions

$$u_0(x,0) = (\frac{4}{3})\cos^2(\frac{x}{4}), \tag{4.29}$$

$$u_i(x,0) = 0 \quad for \quad i = 1,2,.... \tag{4.30}$$

Selecting the initial value $u(x,0) = (\frac{4}{3})\cos^2(\frac{x}{4})$ for $u_0(x,t)$ and using equations (4.28) one obtains the following successive approximations

$$\begin{aligned}
&u_0(x,t) = \tfrac{4}{3}\cos^2(\tfrac{x}{4}),\\
&u_1(x,t) = \tfrac{1}{3}\sin(\tfrac{x}{2})t,\\
&u_2(x,t) = \tfrac{1}{3}\sin(\tfrac{x}{2})t - \tfrac{1}{12}\cos(\tfrac{x}{2})t^2 - \tfrac{1}{3}\sin(\tfrac{x}{2})\tfrac{t^{2-\alpha}}{\Gamma(3-\alpha)},\\
&u_3(x,t) = \tfrac{1}{3}\sin(\tfrac{x}{2})t - \tfrac{1}{6}\cos(\tfrac{x}{2})t^2 - \tfrac{1}{72}\sin(\tfrac{x}{2})t^3 - \tfrac{2}{3}\sin(\tfrac{x}{2})\tfrac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \tfrac{1}{3}\sin(\tfrac{x}{2})\tfrac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \tfrac{1}{3}\cos(\tfrac{x}{2})\tfrac{t^{3-\alpha}}{\Gamma(4-\alpha)},\\
&\vdots
\end{aligned} \tag{4.31}$$

The first four terms of the decomposition series solution for Eq. (4.24) is given as

$$u(x,t) = \frac{4}{3}\cos^2(\frac{x}{4}) + \sin(\frac{x}{2})t - \frac{1}{4}\cos(\frac{x}{2})t^2 - \frac{1}{72}\sin(\frac{x}{2})t^3 - \sin(\frac{x}{2})\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{1}{3}\sin(\frac{x}{2})\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{1}{3}\cos(\frac{x}{2})\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} \tag{4.32}$$

Case 1: substituting $\alpha = 1$ into (4.32), we obtain the following (four-terms) approximation of the IVP (4.24)-(4.25).

$$u(x,t) = (\frac{4}{3})\cos^2(\frac{x}{4}) + \frac{2}{3}(\cos(\frac{x}{4})\sin(\frac{x}{4}))t - \frac{1}{12}(-1+\cos^2(\frac{x}{4}))t^2 - \frac{1}{36}(\cos(\frac{x}{4})\sin(\frac{x}{4}))t^3. \tag{4.33}$$

Note that this result is the same obtained by the variational homotopy perturbation method (VHPM) in [2].

On the other hand, an expansion of the exact solution (4.26) in Taylor series over $t = 0$ to order 3 gives:

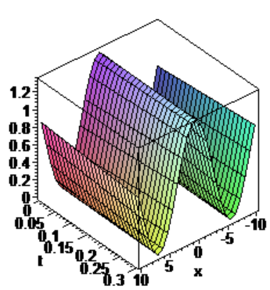
$$u(x,t) = (\frac{4}{3})\cos^2(\frac{x-t}{4}) = (4/3)\cos^2(\frac{x}{4}) + \frac{2}{3}(\cos(\frac{x}{4})\sin(\frac{x}{4}))t - \frac{1}{12}(-1+\cos^2(\frac{x}{4}))t^2 - \frac{1}{36}(\cos(\frac{x}{4})\sin(\frac{x}{4}))t^3 + O(t^4).$$

This confirms our result.

Case 2: substituting $\alpha = \frac{1}{2}$ into (4.32), one obtains

$$u(x,t) = (\frac{4}{3})\cos^2(\frac{x}{4}) + \sin(\frac{x}{2})t - \frac{4}{3\sqrt{\pi}}\sin(\frac{x}{2})t^{\frac{3}{2}} - \frac{1}{12}\cos(\frac{x}{2})t^2 + \frac{8}{45\sqrt{\pi}}\cos(\frac{x}{2})t^{\frac{5}{2}} - \frac{1}{72}\sin(\frac{x}{2})t^3.$$

In the same manner, the rest of components can be obtained using the iteration formula (4.37) and the Maple package.



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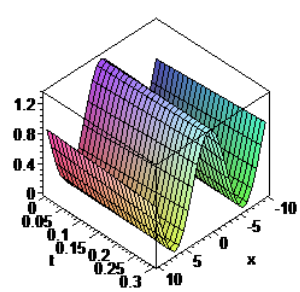


Figure 1: (Left): Exact solution for Eq. (4.24) with the initial condition (4.25); (Right): Approximative solution of Eq. (4.24) by HPM method for $\alpha = 1$ with four terms.

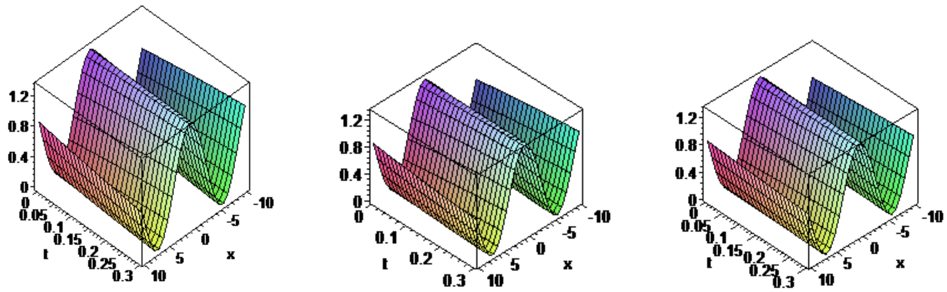
solution.png 

Figure 2: Series solution of Eq. (4.24) by HPM method with four terms for $\alpha = 1/4$ (left), $\alpha = 1/2$ (middle), and $\alpha = 0.9$ (right).

In Fig. 1, one has represented the graph of the exact solution (Left) of the initial value problem (4.24)-(4.25) and its series approximation (Right) with four terms and $\alpha = 1$. As it is shown, there is a similarity between the exact and the approximate solution by HPM method. In addition, it is clear that if one wants more accuracy, it is sufficient to increase the order of p.

The Fig. 2, represents the series solution obtained by the HPM method with four terms for respectively, $\alpha = 1/4$, $\alpha = 1/2$, and $\alpha = 0.9$.

4.2 Numerical solutions of space-fractional K(2,2) equation

We next consider the following space-fractional $K(2,2)$ equation with initial condition

$$u_t + (2u + 6u_{xx})^c\, D_x^{\beta} u + 2uu_{xxx} = 0, \quad 0 < \beta \leq 1, \tag{4.34}$$

with initial condition

$$u(x,0) = x^2. \tag{4.35}$$

This initial condition is taken as polynomial to avoid heavy calculations of fractional differentiation.

According to the HPM, we construct the following homotopy

$$u_t - v_{0t} + p[(2\,u + 6u_{xx})^c\, D_x^{\beta} u + 2uu_{xxx} + v_{0t}] = 0, \quad 0 < \beta \leq 1, \tag{4.36}$$

where $p \in [0,1]$ and $v_0 = u(x,0) = x^2$.

In view of the HPM, substituting equation (3.22) into equation (4.36) and equating the coefficients of like powers of p, we get the following set of differential equations

$$\begin{aligned} p^0 &: \tfrac{\partial u_0}{\partial t} = v_{0t}, \quad u_0(x,0) = x^2, \\ p^1 &: \tfrac{\partial u_1}{\partial t} = -(2u_0 + 6u_{0xx})^c\, D_x^{\beta} u_0 - 2u_0 u_{0xxx} - v_{0t}, \\ p^2 &: \tfrac{\partial u_2}{\partial t} = -(2u_0 + 6u_{0xx})^c\, D_x^{\beta} u_1 - (2u_1 + 6u_{1xx})^c\, D_x^{\beta} u_0 - 2(u_0 u_{1xxx} + u_1 u_{0xxx}), \\ &\vdots \end{aligned} \tag{4.37}$$

with the following conditions

$$u(x,0) = x^2, \tag{4.38}$$

$$u_i(x,0) = 0 \quad for \quad i = 1,2,.... \tag{4.39}$$

Using the initial conditions (4.38) and solving the above equations (4.37) yields

$$\begin{aligned} u_0\,(x,t) &= x^2, \\ u_1\,(x,t) &= (a_1 x^{2-\beta} + a_2 x^{4-\beta})t, \\ u_2\,(x,t) &= (a_3 x^{1-\beta} + a_4 x^{2-2\beta} + a_5 x^{4-2\beta} + a_6 x^{3-\beta} + a_7 x^{6-4\beta})\tfrac{t^2}{2}, \\ &\vdots \end{aligned} \tag{4.40}$$

where
$a_1 = \frac{-24}{\Gamma(3-\beta)}$, $a_2 = \frac{-4}{\Gamma(3-\beta)}$, $a_3 = 2(2-\beta)(1-\beta)(\beta)a_1$, $a_4 = \frac{288}{\Gamma(3-\beta)} + 18(2-\beta)(1-\beta)a_2^2$, $a_5 = \frac{48}{\Gamma(3-2\beta)} - 2\frac{\Gamma(5-\beta)}{\Gamma(5-2\beta)}a_1 + [6+3(4-\beta)(3-\beta)]a_2^2$, $a_6 = -2(4-\beta)(3-\beta)(2-\beta)a_2$, $a_7 = -2\frac{\Gamma(5-\beta)}{\Gamma(5-2\beta)}a_2 + a_2^2$.
Setting $p=1$ and adding the iteratives terms (4.40), yields the following general approximate solution

$$u(x,t) = x^2 + (a_1x^{2-\beta} + a_2x^{4-\beta})t + (a_3x^{1-\beta} + a_4x^{2-2\beta} + a_5x^{4-2\beta} + a_6x^{3-\beta} + a_7x^{6-4\beta})\frac{t^2}{2}. \tag{4.41}$$

While substituting $\beta = 1$ into (4.41), one obtains

$$u(x,t) = x^2 + (-24x - 4x^3)t + (288 + 664x^2)\frac{t^2}{2}, \tag{4.42}$$

and in the same manner, for $\beta = \frac{1}{2}$, it gives the following solution

$$u(x,t) = x^2 + \left(\frac{-32}{\sqrt{\pi}}x^{\frac{3}{2}} + \frac{-16}{3\sqrt{\pi}}x^{\frac{7}{2}}\right)t + \left(\begin{array}{c} \frac{-24}{\sqrt{\pi}}x^{\frac{1}{2}} + 384\frac{\sqrt{\pi}+1}{\pi}x \\ +\frac{140}{\sqrt{\pi}}x^{\frac{5}{2}} + \left(118 + \frac{2752}{3\pi}\right)x^3 \\ +\left(\frac{35}{3} + \frac{256}{9\pi}\right)x^4 \end{array}\right)\frac{t^2}{2}.$$

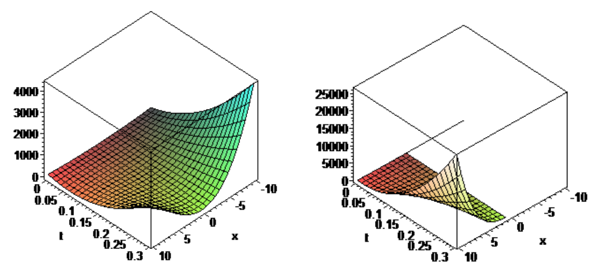


Figure 3: Series solution of Eq. (4.34-4.35) by HPM method with four terms for $\beta = 1$ (left) and $\beta = 1/2$ (right).

In Fig. 3, one has represented the graphs of the series solutions of Eq. (4.34-4.35) for $\beta = 1$ (left) and $\beta = 1/2$ (right).

5 Conclusion

In this work, homotopy perturbation method has been used for solving $K(2,2)$ equation with time and space fractional derivative. The final results obtained from HPM and compared with the exact solution shown that there is a similarity between the exact and the approximate solutions. In addition, it is obvious that; considering more power of p lead us to the more accurate results. This is the raison why one can say that HPM is an alternative analytical method for solving the general nonlinear dispersive $K(m,n)$ equation.

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Received: April 09, 2015; *Accepted*: August 23, 2015

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