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Reliability Measure of an Integrated H/W and S/W System with Redundancy and Preventive maintenance

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Abstract

The purpose of the study is to evaluate the reliability measures of an integrated h/w and s/w system with the concepts of redundancy and preventive maintenance. A stochastic model is developed considering two-identical units of the system- one unit is initially operative and other is in cold standby. In each unit h/w and s/w work together and may fail independently from normal mode. There is a single server who visits the system immediately to h/w repair and s/w up-gradation. The preventive maintenance of the system (unit) is conducted by the server after a maximum operation time. The failure time of h/w and s/w follows negative exponential distribution while the distributions of preventive maintenance, h/w repair and s/w up-gradation times are taken as arbitrary. The semi-Markov process and regenerative point technique is adopted to derive expressions for various measures of system effectiveness. The behaviour of some important reliability measures has been observed graphically giving particular values to various costs and parameters.

Keywords: Integrated h/w and s/w System, Redundancy, H/w Repair, S/w Up-gradation, Preventive Maintenance, Maximum Operation Time and Reliability Measures.

2010 MSC: 90B25, 60K10.

1 Introduction

Now a day's integrated h/w and s/w systems are of growing importance because of their use in almost all academic, business and industrial sectors. The continued operation and ageing of these systems gradually reduce their performance, reliability and safety. Therefore, a major challenge to the engineers and researchers is to develop such systems which can produce failure free services to the users with least cost. The method of redundancy has been used in many industrial systems not only to attain better reliability but also to reduce the frequency of failure up to a desired extent. Goel and Sharma[1] and Singh[2] discussed stochastically the two unit standby system under different repair policies of the server. But the technique of redundancy has not been used much more in case of integrated h/w and s/w systems. A few researchers including Malik and Anand[3] obtained reliability measures for a computer system by taking a redundant unit in cold standby. Further, it is proved that preventive maintenance can slow the deterioration process of operating system and restore them in a younger age or state. Thus, the method of preventive maintenance can be used to improve the performance of these systems. Recently, Malik and Kumar[4] investigated a reliability model for a computer system conducting preventive maintenance after a maximum operation time.

To strengthen the existing literature, here reliability measures for an integrated h/w and s/w system are obtained by introducing the concepts of redundancy and preventive maintenance. A stochastic model is developed considering two-identical units of the system- one unit is initially operative and other is in cold standby. In each unit h/w and s/w work together and may fail independently from normal mode. There

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is a single server who visits the system immediately to h/w repair and s/w up-gradation. The preventive maintenance of the system (unit) is conducted by the server after a maximum operation time. The failure time of h/w and s/w follows negative exponential distribution while the distributions of preventive maintenance, h/w repair and s/w up-gradation times are taken as arbitrary. The semi-Markov process and regenerative point technique is adopted to derive expressions for various measures of system effectiveness such as mean time to system failure, availability, busy period of the server due to preventive maintenance, busy period of the server due to h/w repair, busy period of the server due to software up-gradation, expected number of software up-gradations and expected number of visits of the server. The behaviour of some important reliability measures has been observed graphically giving particular values to various costs and parameters.

Notations

N_0	:	The unit is operative and in normal mode
C_s	:	The unit is cold standby
a/b	:	Probability that the system has hardware/software failure
λ^1/λ^2	:	Constant failure rate of hardware/software
α_0	:	Constant rate of Maximum Operation Time
Pm/PM	:	The unit is under preventive Maintenance/under preventive maintenance continuously from previous state
WPm/WPM	:	The unit is waiting for preventive Maintenance/waiting for preventive maintenance continuously from previous state
$HFur/HFUR$	:	The hardware is failed and is under repair/under repair continuously from previous state
$HFwr/HFWR$	:	The hardware is failed and is waiting for repair/waiting for repair continuously from previous state
$SFurp/SFURP$	:	The software is failed and is under up-gradation/under up-gradation continuously from previous state
$SFwrp/SFWRP$	:	The software is failed and is waiting for up-gradation/waiting for up-gradation continuously from previous state
$h(t)/H(t)$	:	pdf/cdf of software up-gradation time
$g(t)/G(t)$	:	pdf/cdf of repair time of the hardware
$f(t)/F(t)$	:	pdf/cdf of the time for preventive maintenance of the unit
$q^{ij}(t)/Q^{ij}(t)$	:	pdf/cdf of passage time from regenerative state i to a regenerative state j or to a failed state j without visiting any other regenerative state in $(0,t]$
pdf/cdf	:	Probability density function/ Cumulative density function
$q^{ij\cdot kr}(t)/Q^{ij\cdot kr}(t)$	:	pdf/cdf of direct transition time from regenerative state i to a regenerative state j or to a failed state j visiting state k, r once in $(0,t]$
$\mu_i(t)$	:	Probability that the system up initially in state $S_i \in E$ is up at time t without visiting to any regenerative state
$W_i(t)$	:	Probability that the server is busy in the state S_i up to time 't' without making any transition to any other regenerative state or returning to the same state via one or more non-regenerative states.
m_{ij}	:	Contribution to mean sojourn time (μ_i) in state S_i when system transit directly to state S_j so that $\mu_i = \sum_j m_{ij}$ and $m^{ij} = \int tdQ_{ij}(t) = -q_{ij}^{*'}(0)$?
$S/©$	:	Symbol for Laplace-Stieltjes convolution/Laplace convolution

Transition Probabilities and Mean Sojourn Times

Simple probabilistic considerations yield the following expressions for the non-zero elements

$$p_{ij} = Q_{ij}(\infty) = \int q_{ij}(t)dt \tag{1.1}$$

as

$$\left.\begin{aligned}
p^{01} &= \frac{\int_0 \alpha}{A}, & p^{02} &= \frac{\int_1 a\lambda}{A}, \\
p^{03} &= \frac{\int_2 b\lambda}{A}, & p^{10} &= f^*(A), \\
p^{1.10} &= \frac{\int_1 a\lambda}{A}[1-f^*(A)] = p^{12.10}, & p^{1.12} &= \frac{\int_2 b\lambda}{A}[1-f^*(A)] = p^{13.12}, \\
p^{1.4} &= \frac{\int_0 \alpha}{A}[1-f^*(A)] = p^{11.4}, & p^{20} &= g^*(A), \\
p^{29} &= \frac{\int_0 \alpha}{A}[1-g^*(A)] = p^{21.9}, & p^{27} &= \frac{\int_2 b\lambda}{A}[1-g^*(A)] = p^{23.7}, \\
p^{28} &= \frac{\int_1 a\lambda}{A}[1-g^*(A)] = p^{22.8}, & p^{30} &= h^*(A), \\
p^{35} &= \frac{\int_1 a\lambda}{A}[1-h^*(A)] = p^{32.5}, & p^{3.11} &= \frac{\int_0 \alpha}{A}[1-h^*(A)] = p^{3,1.11}, \\
p^{41} &= f^*(s), & p^{36} &= \frac{\int_2 b\lambda}{A}[1-h^*(A)] = p^{33.6}, \\
p^{52} &= h^*(s), & p^{63} &= h^*(s), \\
p^{73} &= g^*(s) = p^{82} = p^{91}, & p^{10.2} &= f^*(s), \\
p^{11.1} &= h^*(s), & p^{12.3} &= f^*(s)
\end{aligned}\right\} \tag{1.2}$$

where $A = \int_1 a\lambda + \int_2 b\lambda + \int_0 \alpha$.

It can be easily verified that

$$\left.\begin{aligned}
p^{01} + p^{02} + p^{03} &= p^{10} + p^{14} + p^{1.10} + p^{1.12} = p^{20} + p^{27} + p^{29} + p^{28} \\
&= p^{30} + p^{35} + p^{3.11} + p^{36} = p^{41} = p^{52} = p^{63} = p^{73} \\
&= p^{82} = p^{91} = p^{10.2} = p^{11.1} = p^{12.3} \\
&= p^{10} + p^{12.10} + p^{11.4} + p^{13.12} = p^{20} + p^{21.9} + p^{22,8} + p^{23.7} \\
&= p^{30} + p^{31.11} + p^{32.5} + p^{33.6} = 1
\end{aligned}\right\} \tag{1.3}$$

The mean sojourn times (μ_i) is the state S_i are μ

$$\left.\begin{aligned}
\mu_0 &= \frac{1}{\int_1 a\lambda + \int_2 b\lambda + \int_0 \alpha}, & \mu_1 &= \frac{1}{\int_1 a\lambda + \int_2 b\lambda + \int_0 \alpha + \alpha}, \\
\mu_2 &= \frac{1}{\int_1 a\lambda + \int_2 b\lambda + \int_0 \alpha + \theta}, & \mu_3 &= \frac{1}{\int_1 a\lambda + \int_2 b\lambda + \int_0 \alpha + \int \beta}, \\
\mu_1' &= \frac{1}{\alpha}, & \mu_2' &= \frac{1}{\theta}, \quad \mu_3' = \frac{1}{\beta}
\end{aligned}\right\} \tag{1.4}$$

The states S_0, S_1, S_2 and S_3 are regenerative states while S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11} and S_{12}, are non-regenerative states. Thus $E = \{S_0, S_1, S_2, S_3\}$. The possible transition between states along with transition rates for the model is shown in Fig. **??**.

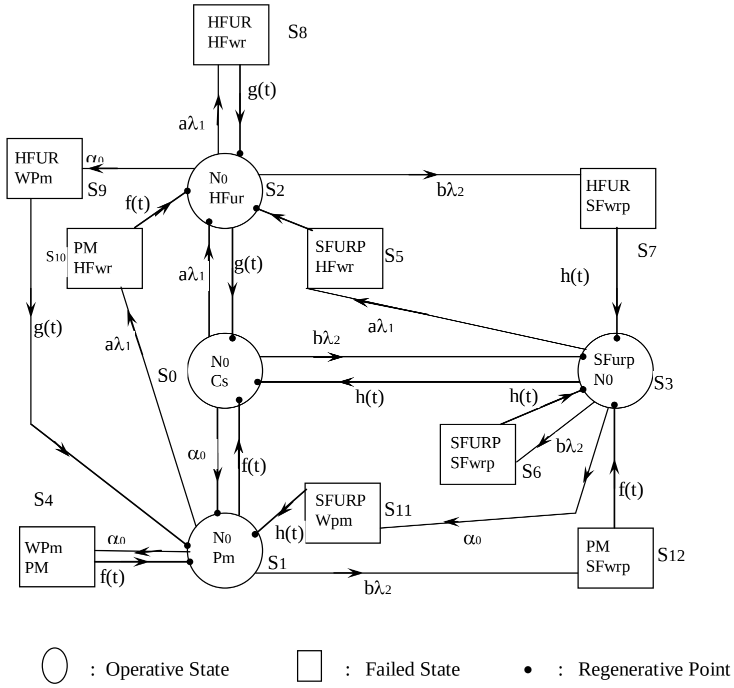

Figure 1:

Reliability and Mean Time to System Failure (MTSF)

Let $\phi_i(t)$ be the cdf of first passage time from the regenerative state i to a failed state. Regarding the failed state as absorbing state, we have the following recursive relations for $\phi_i(t)$:

$$\varphi_i(t) = \sum_j Q_{i,j}(t) + ®\varphi_j(t) + \sum_k Q_{i,j}(t), \tag{1.5}$$

where j is an un-failed regenerative state to which the given regenerative state i can transit and k is a failed state to which the state i can transit directly. Taking LST of above relation (1.5) and solving for $\tilde{\phi}_0(s)$. We have

$$R^*(s) = \frac{1 - \tilde{\phi}_0(s)}{s} \tag{1.6}$$

The reliability of the system model can be obtained by taking Laplace inverse transform of (1.6).

The mean time to system failure (MTSF) is given by

$$MTSF = \lim_{s \to 0} \frac{1 - \widetilde{\phi}_0(s)}{s} = \frac{N_1}{D_1} \tag{1.7}$$

where $N^1 = \mu_0 + p_{01}\mu_1 + p_{02}\mu_2 + +p_{03}\mu_3$ and $D^1 = 1 - p_{01}p_{10} - p_{02}p_{20} - p_{03}p_{30}$.

2 Steady State Availability

Let $A_i(t)$ be the probability that the system is in up-state at instant 't' given that the system entered regenerative state i at $t = 0$. The recursive relations for $A_i(t)$ are given as

$$A_i(t) = M_i(t) + \sum_j a_{ij}^{(n)}(t) © A_j(t) \tag{2.8}$$

where j is any successive regenerative state to which the regenerative state i can transit through n transitions. $M_i(t)$ is the probability that the system is up initially in state $S_i \in E$ up at time t without visiting to any other regenerative state, we have is

$$M_0(t) = e^{-(a\lambda_1+b\lambda_2+\alpha_0)^t}, \quad ,M_1(t) = e^{-(a\lambda_1+b\lambda_2+\alpha_0)^t}\overline{F(t)},$$
$$M_2(t) = e^{-(a\lambda_1+b\lambda_2+\alpha_0)^t}\overline{G(t)}, \quad ,M_3(t) = e^{-(a\lambda_1+b\lambda_2+\alpha_0)^t}\overline{H(t)}, \tag{2.9}$$

Taking LT of above relations (2.8) and solving for $A_0^*(s)$. The steady state availability is given by

$$A_0(\infty) = \lim_{s\to 0} sA_0^*(s) = \frac{N_2}{D_2}, \tag{2.10}$$

where

$$\begin{aligned} N_2 = \mu_0[&(1-p_{11.4})\{(1-p_{22.8})(1-p_{33.6}) - p_{23.7}p_{32.5}\} - p_{12.10}\{(1-p_{33.6})p_{21.9} + p_{31.11}p_{23.7}\} \\ &- p_{13.12}\{p_{21.9}p_{32.5} + (1-p_{22.8})p_{33.6}\}] + \mu_1[p_{01}\{(1-p_{22.8})(1-p_{33.6}) - p_{23.7}p_{32.5}\} \\ &+ p_{02}\{(1-p_{33.6})p_{21.9} + p_{31.11}p_{23.7}\} + p_{03}\{p_{21.9}p_{32.5} + (1-p_{22.8})p_{33.6}\}] \\ &+ \mu_2[p_{01}\{p_{12.10}(1-p_{33.6}) + p_{13.12}p_{32.5}\} + p_{02}\{(1-p_{33.6})(1-p_{11.4}) - p_{31.11}p_{13.12}\} \\ &+ p_{03}\{p_{31.11}p_{12.10} + (1-p_{11.4})p_{32.5}\}] + \mu_3[p_{01}\{p_{12.10}p_{23.7} + p_{13.12}(1-p_{22.8})\} \\ &+ p_{02}\{(1-p_{11.4})p_{23.7} + p_{21.9}p_{13.12}\} + p_{03}\{(1-p_{22.8})(1-p_{11.14}) - p_{21.9}p_{12.10}\}] \end{aligned}$$

and

$$\begin{aligned} D_2 = \mu_0[&(1-p_{11.4})\{(1-p_{22.8})(1-p_{33.6}) - p_{23.7}p_{32.5}\} - p_{12.10}\{(1-p_{33.6})p_{21.9} + p_{31.11}p_{23.7}\} \\ &- p_{13.12}\{p_{21.9}p_{32.5} + (1-p_{22.8})p_{33.6}\}] + [p_{01}\{(1-p_{22.8})(1-p_{33.6}) - p_{23.7}p_{32.5}\} \\ &+ p_{02}\{(1-p_{33.6})p_{21.9} + p_{31.11}p_{23.7}\} + p_{03}\{p_{21.9}p_{32.5} + (1-p_{22.8})p_{33.6}\}] \\ &+ [p_{01}\{p_{12.10}(1-p_{33.6}) + p_{13.12}p_{32.5}\} + p_{02}\{(1-p_{33.6})(1-p_{11.4}) - p_{31.11}p_{13.12}\} \\ &+ p_{03}\{p_{31.11}p_{12.10} + (1-p_{11.4})p_{32.5}\}] + [p_{01}\{p_{12.10}p_{23.7} + p_{13.12}(1-p_{22.8})\} \\ &+ p_{02}\{(1-p_{11.4})p_{23.7} + p_{21.9}p_{13.12}\} + p_{03}\{(1-p_{22.8})(1-p_{11.14}) - p_{21.9}p_{12.10}\}] \end{aligned}$$

Busy Period Analysis for Server

Let $B_i^P(t)$, $B_i^R(t)$ and $B_i^S(t)$ be the probability that the server is busy in preventive maintenance, hardware repair and software up-gradation of the system (unit) at an instant 't' given that the system entered state i at $t = 0$. The recursive relations for $B_i^P(t)$, $B_i^R(t)$ and $B_i^S(t)$ are as follows:

$$B_i^P(t) = W_i(t) + \sum_j q_{ij}^{(n)}(t) \copyright B_J^p(t) \tag{2.11}$$

$$B_i^R(t) = W_i(t) + \sum_j q_{ij}^{(n)}(t) \copyright B_J^R(t) \tag{2.12}$$

$$B_i^S(t) = W_i(t) + \sum_j q_{ij}^{(n)}(t) \copyright B_J^S(t) \tag{2.13}$$

Where j is any successive regenerative state to which the regenerative state i can transit through n transitions. $W_i(t)$ be the probability that the server is busy in state S_i due to PM, h/w repair and s/w up-gradation of the system up to time t without making any transition to any other regenerative state or returning to the same via one or more non-regenerative states. Taking LT of above relations (2.11) to (2.13) and solving for $B_0^{*P}(s)$, $B_0^{*R}(s)$ and $B_0^{*S}(s)$. The time for which server is busy due to preventive maintenance, h/w repair and s/w up-gradation respectively is given by

$$B_0^p = \lim_{s\to 0} sB_0^{*P}(s) = \frac{N_3^p}{D_2},\; B_0^R = \lim_{s\to 0} sB_0^{*R}(s) = \frac{N_3^R}{D_2} \text{ and } B_0^S = \lim_{s\to 0} sB_0^{*S}(s) = \frac{N_3^S}{D_2},$$

where

$$N_3^P = W_1^*(0)[p_{01}\{(1-p_{22.8})(1-p_{33.6})-p_{23.7}p_{32.5}\}+p_{02}\{(1-p_{33.6})p_{21.9}+p_{31.11}p_{23.7}\} + p_{03}\{p_{21.9}p_{32.5}+(1-p_{22.8})p_{33.6}\}] \quad (2.14)$$

$$N_3^P = W_2^*(0)[p_{01}\{p_{12.10}(1-p_{33.6})+p_{13.12}p_{32.5}\}+p_{02}\{(1-p_{33.6})(1-p_{11.4})-p_{31.11}p_{13.12}\} + p_{03}\{p_{31.11}p_{12.10}+(1-p_{11.4})p_{32.5}\}] \quad (2.15)$$

$$N_3^S = W_2^*(0)[p_{01}\{p_{12.10}p_{23.7}+p_{13.12}(1-p_{22.8})\}+p_{02}\{(1-p_{11.4})p_{23.7}+p_{21.9}p_{13.12}\} + p_{03}\{(1-p_{22.8})(1-p_{11.14})-p_{21.9}p_{12.10}\}] \quad (2.16)$$

Expected Number of S/w Up-gradations

Let $R_i^S(t)$ be the expected number of software up-gradations by the server in $(0,t]$ given that the system entered the regenerative state i at $t=0$. The recursive relations for $R_i^S(t)$ are given as

$$R_i^S(t) = \sum_j Q_{i,j}^{(n)}(t)\circledR[\delta_j + R_j^S(t)]\,. \quad (2.17)$$

Where j is any regenerative state to which the given regenerative state i transits and $\delta_j = 1$, if j is the regenerative state where the server does job afresh, otherwise $\delta_j = 0$.

Taking LST of relations (2.17) and solving for $\tilde{R}_0^S(s)$. The expected numbers of s/w up-gradations per unit time are given by

$$R_0^S(\infty) = \lim_{s\to 0}\tilde{R}_0^S = \frac{N_4^S}{D_2}\,. \quad (2.18)$$

Where D_2 is already mentioned.

$$N_4^S = [p_{01}\{p_{12.10}p_{23.7}+p_{13.12}(1-p_{22.8})\}+p_{02}\{(1-p_{11.4})p_{23.7}+p_{21.9}p_{13.12}\} + p_{03}\{(1-p_{22.8})(1-p_{11.14})-p_{21.9}p_{12.10}\}]$$

Expected Number of Visits by the Server

Let $N_i(t)$ be the expected number of visits by the server in $(0,t]$ given that the system entered the regenerative state i at $t=0$. The recursive relations for $N_i(t)$ are given as

$$N_i(t) = \sum_j Q_{i,j}^{(n)}(t)\circledR[\delta_j + N_j(t)] \quad (2.19)$$

Where j is any regenerative state to which the given regenerative state i transits and $\delta_j = 1$, if j is the regenerative state where the server does job afresh, otherwise $\delta_j = 0$. Taking LST of relation (2.19) and solving for $\tilde{N}_0(s)$. The expected number of visit per unit time by the $\tilde{N}_0(s)$ server are given by

$$N_0(\infty) = \lim_{s\to 0} s\tilde{N}_0(s) = \frac{N_2}{D_2}, \quad (2.20)$$

where

$$N_5 = [(1-p_{11.4})\{(1-p_{22.8})(1-p_{33.6})-p_{23.7}p_{32.5}\}-p_{12.10}\{(1-p_{33.6})p_{21.9}+p_{31.11}p_{23.7}\} - p_{13.12}\{p_{21.9}p_{32.5}+(1-p_{22.8})p_{33.6}\}]$$

Profit Analysis

The profit incurred to the system model in steady state can be obtained as

$$P = K_0A_0 - K_1B_0^P - K_2B_0^R - K_3B_0^S - K_4B_0^S - K_5N_0 \quad (2.21)$$

K_0 = Revenue per unit up-time of the system
K_1 = Cost per unit time for which server is busy due preventive maintenance
K_2 = Cost per unit time for which server is busy due to hardware failure
K_3 = Cost per unit time for which server is busy due to software up-gradation
K_4 = Cost per unit time s/w up-gradation
K_5 = Cost per unit time visit by the server

3 Conclusion

By considering a particular case $g(t) = \theta e^{-\theta t}$, $h(t) = \beta e^{-\beta t}$ and $f(t) = \alpha e^{-\alpha t}$, the numerical results some reliability measures are obtained for the system under study. The graphs for mean time to system failure (MTSF), availability and profit are drawn with respect to preventive maintenance (α) rate for fixed values of parameters as shown respectively in Figures 4, **??** and **??**. It is revealed that MTSF, Availability and profit increase with the increase of PM rate (α) and h/w repair rate (θ). But the value of these measures decrease with the increase of maximum operation time (α_0). Thus finally it is concluded that a system in which chances of h/w failure are high can be made reliable and economical to use

(i) By taking one more unit in cold standby.

(ii) By conducting PM of the system after a specific period of time.

(iii) By increasing h/w repair rate in case preventive maintenance of the system is not conducted after a maximum operation time.

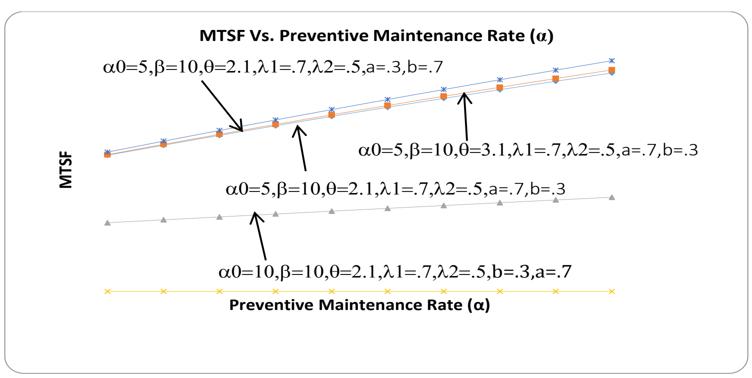

Figure 2:

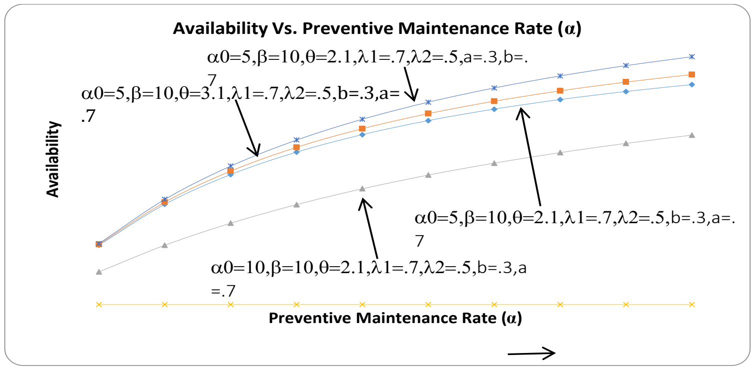

Figure 3:

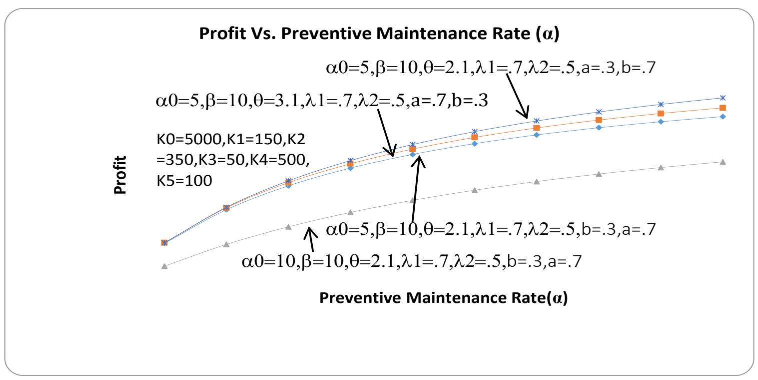

Figure 4:

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Existence and approximate solutions for hybrid fractional integro-differential equations

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Abstract

In this paper we prove existence and approximations of the solutions for initial value problems of nonlinear hybrid fractional differential equations, using the operator theoretic technique in a partially ordered metric space proved by Dhage.

Keywords: Fractional differential equation; Fixed point theorem; Dhage iteration method; Existence and uniqueness theorems.

2010 MSC: 534A08, 47H07, 47H10.

1 Introduction

In this paper we prove existence and approximations of the solutions for initial value problems of nonlinear hybrid fractional differential equations. Consider the following initial value problem of fractional differential equations,

$$\begin{cases} {}^{c}D^{\alpha}\left(\dfrac{x(t) - I^{\beta}h(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)), \;\; t \in J := [0,T], \\ x(0) = 0, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α, $0 < \alpha < 1$, I^{β} is the Riemann-Liouville fractional integral of order β, and $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g, h : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Fractional differential equations have aroused great interest, which is caused by both the intensive development of the theory of fractional calculus and the applications of physics, mechanics and chemistry engineering [22, 23]. For some recent development on the topic see [1–9] and the references cited therein. For some recent results on hybrid fractional differential equations we refer to [7], [10], [20], [24], [25] and the references cited therein.

The origin of the problem (1.1) lies in the initial value problems of first order quadratic differential equations with ordinary derivative wherein only existence of the solutions is proved using classical hybrid fixed point theorem of Dhage [11]. The problem (1.1) considered here is general in the sense that it includes the following three well-known classes of initial value problems of fractional differential equations.

Case I: Let $f(t,x) = 1$ and $h(t,x) = 0$ for all $t \in J$ and $x \in \mathbb{R}$. Then the problem (1.1) reduces to standard

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initial value problem of fractional differential equation

$$\begin{cases} {}^cD^{\alpha}x(t) = g(t, x(t)), \ \ t \in J := [0, T], \\ x(0) = 0. \end{cases} \tag{1.2}$$

Case II: If $h(t, x) = 0$ for all $t \in J$ and $x \in \mathbb{R}$ in (1.1), we obtain the following quadratic fractional differential equation,

$$\begin{cases} {}^cD^{\alpha}\left(\dfrac{x(t)}{f(t, x(t))}\right) = g(t, x(t)), \ \ t \in J := [0, T], \\ x(0) = 0. \end{cases} \tag{1.3}$$

Case III: If $f(t, x) = 1$ for all $t \in J$ and $x \in \mathbb{R}$ in (1.1), we obtain the following interesting fractional differential equation,

$$\begin{cases} {}^cD^{\alpha}\left[x(t) - I^{\beta}h(t, x(t))\right] = g(t, x(t)), \ \ t \in J := [0, T], \\ x(0) = 0. \end{cases} \tag{1.4}$$

Therefore, the main result of this paper also includes the existence as well as approximations of solutions of above mentioned initial value problems of fractional differential equations as special cases. Again our approach here in this paper is different than that employed in the related paper of Dhage [11].

In the present paper we prove the existence and approximations of the solutions of problem (1.1) under weaker partially compactness and partially Lipschitz type conditions via Dhage's iteration method [14]. Very recently, Dhage's iteration method has been applied in [14-16, 18,19] to nonlinear ordinary differential equations for proving the existence and algorithms of the solutions.

We recall the basic definitions of fractional calculus [22, 23] which are useful in what follows.

Definition 1.1. *The fractional integral of order q with the lower limit zero for a function f is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > 0, \quad q > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(q) = \int_0^{\infty} t^{q-1}e^{-t}dt$.

Definition 1.2. *The Riemann-Liouville fractional derivative of order $q > 0$, $n - 1 < q < n$, $n \in \mathbb{N}$, is defined as*

$$D^q_{0+} f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 1.3. *The Caputo derivative of order q for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$${}^cD^q f(t) = D^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n - 1 < q < n.$$

Remark 1.1. *If $f(t) \in C^n[0, \infty)$, then*

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t), \ t > 0, \ n - 1 < q < n.$$

Lemma 1.1. *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

In view of Lemma 1.1, it follows that

$$I^q\, {}^cD^q x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1}, \tag{1.5}$$

for some $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

The rest of the paper will be organized as follows. In Section 2 we give some preliminaries and key fixed point theorems that will be used in subsequent part of the paper. In Section 3 we discuss the main existence and approximation result for initial value problems of fractional differential equations (1.1). An illustrative example is also discussed.

2 Auxiliary Results

Unless otherwise mentioned, throughout this paper we let E denote a partially ordered real normed linear space with the order relation $\preceq$ and the norm $\|\cdot\|$ in which addition and scalar multiplication by positive real numbers are preserved by $\preceq$. A few details on such partially ordered normed linear spaces appear in Dhage [12] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all elements of C are comparable. We say that E is *regular* if for any nondecreasing (resp. nonincreasing) sequence $\{x_n\}$ in E such that $x_n \to x^*$ as $n \to \infty$, we have that $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [21] and the references therein.

We need the following definitions in the sequel.

Definition 2.4. *A mapping $\mathcal{B} : E \to E$ is called* **isotone** *or* **nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{B}x \preceq \mathcal{B}y$ for all $x, y \in E$.*

Definition 2.5 (Dhage [12]). *A mapping $\mathcal{B} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{B}x - \mathcal{B}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{B}$ called a partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{B}$ is a partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.6. *A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. A mapping $\mathcal{B} : E \to E$ is called* **partially bounded** *if $\mathcal{B}(C)$ is bounded for every chain C in E. $\mathcal{B}$ is called* **uniformly partially bounded** *if all chains $\mathcal{B}(C)$ in E are bounded by a unique constant. $\mathcal{B}$ is called* **bounded** *if $\mathcal{B}(E)$ is a bounded subset of E.*

Definition 2.7. *A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is compact. A mapping $\mathcal{B} : E \to E$ is called* **partially compact** *if $\mathcal{B}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{B}$ is called* **uniformly partially compact** *if $\mathcal{B}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{B}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{B}(C)$ is a relatively compact subset of E. If $\mathcal{B}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Definition 2.8 (Dhage [12]). *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* **compatible** *if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the whole sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called* **Janhavi** *if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a* **Janhavi metric** *or* **Janhavi Banach space***.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.9 (Dhage [12]). *A upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called* ***partially nonlinear $\mathcal{D}$-Lipschitz*** *if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x-y\|) \tag{2.6}$$

for all comparable elements $x, y \in E$. If $\psi(r) = kr$, $k > 0$, then $\mathcal{T}$ is called a partially Lipschitz with a Lipschitz constant k. Furthermore, if $\psi(r) < r$, $r > 0$, $\mathcal{T}$ is called a partially nonlinear $\mathcal{D}$-contraction on E.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}. \tag{2.7}$$

The elements of the set $\mathcal{K}$ are called the positive vectors in E. Then following lemma is immediate.

Lemma 2.2 (Dhage [12]). *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.10. *An operator $\mathcal{B} : E \to E$ is said to be positive if the range $R(\mathcal{B})$ of $\mathcal{B}$ is such that $R(\mathcal{B}) \subseteq \mathcal{K}$.*

The Dhage iteration method is embodied in the following hybrid fixed point theorem proved in Dhage [13] which are useful tools in what follows. A few other such hybrid fixed point theorems appear in Dhage [12, 13].

Theorem 2.1 (Dhage [14]). *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain C of E. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ and $\mathcal{C} : E \to E$ be three nondecreasing operators such that*

(a) $\mathcal{A}$ and $\mathcal{C}$ are partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ respectively.

(b) $\mathcal{B}$ is partially continuous and uniformly partially compact,

(c) $M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C \text{ is a chain in} E\}$, and

(d) there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0\mathcal{B}x_0 + \mathcal{C}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$.

Then the operator equation $\mathcal{A}x\mathcal{B}x + \mathcal{C}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\mathcal{B}x_n + \mathcal{C}x_n$, $n = 0, 1, \ldots$ converges monotonically to x^*.

Remark 2.2. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E is held if every partially compact subset S of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact is used to prove the desired characterization of the positive solution of the problem (1.1) on J.

3 Main Existence Result

The equivalent integral form of the problem (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.8}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.9}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and a lattice so that every pair of elements of E has a lower and an upper bound in it. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzellá-Ascoli theorem.

Lemma 3.3. *Let* $(C(J,\mathbb{R}),\leq,\|\cdot\|)$ *be a partially ordered Banach space with the norm* $\|\cdot\|$ *and the order relation* $\leq$ *defined by (3.8) and (3.9) respectively. Then* $\|\cdot\|$ *and* $\leq$ *are compatible in every partially compact subset of* $C(J,\mathbb{R})$*.*

Proof. The proof of the lemma is given in Dhage and Dhage [?]. Since the proof is not well-known, we give the details of proof. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t)\leq x_2(t)\leq\cdots\leq x_n(t)\leq\cdots,$$

for each $t\in J$.

Suppose that a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in\mathbb{N}}$ of the monotone real sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t\in J$. This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result, $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.11. *A function* $u\in C^1(J,\mathbb{R})$ *is said to be a lower solution of the problem (1.1) if*

$$\left.\begin{aligned} {}^cD^\alpha\left(\frac{u(t)-I^\beta h(t,u(t))}{f(t,u(t))}\right)&\leq g(t,u(t)),\ t\in J,\\ u(0)&\leq 0.\end{aligned}\right\}\qquad(*)$$

Similarly, an upper solution $v\in C^1(J,\mathbb{R})$ *to the problem (1.1) is defined on* J*, by the above inequalities with reverse sign.*

We consider the following set of assumptions in what follows:

(A_1) There exists a constant $M_f>0$ such that $0<f(t,x)\leq M_f$ for all $t\in J$ and $x\in\mathbb{R}$.

(A_2) There exists a $\mathcal{D}$-function ϕ such that

$$0\leq f(t,x)-f(t,y)\leq\phi(x-y)$$

for all $t\in J$ and $x,y\in\mathbb{R}$, $x\geq y$.

(A_3) There exists a constant $M_h>0$ such that $0\leq h(t,x)\leq M_h$ for all $t\in J$ and $x\in\mathbb{R}$.

(A_4) There exists a $\mathcal{D}$-function ω such that

$$0\leq h(t,x)-h(t,y)\leq\omega(x-y)$$

for all $t\in J$ and $x,y\in\mathbb{R}$, $x\geq y$.

(A_5) The function $g(t,x)$ is monotone nondecreasing in x for each $t\in J$.

(A_6) There exists a constant $M_g>0$ such that $0<g(t,x)\leq M_g$ for all $t\in J$ and $x\in\mathbb{R}$.

(A_7) The problem (1.1) has a lower solution $u\in C^1(J,\mathbb{R})$.

The following lemma is useful in what follows and may be found in Kilbas *et.al.* [22] and Podlubny [23].

Lemma 3.4. *Suppose that* $0<\alpha<1$ *and functions* f,g,h *satisfy problem (1.1). Then the unique solution of the hybrid fractional integro-differential problem (1.1) is given by*

$$x(t)=\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}h(s,x(s))ds+f(t,x(t))\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s))ds,\quad t\in J.\qquad(3.10)$$

Proof. By Lemma 1.1 we have

$$\frac{x(t) - I^{\beta}h(t, x(t))}{f(t, x(t))} = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s))ds + c_0,$$

where $c_0 \in \mathbb{R}$. Since $x(0) = 0, f(0,0) \neq 0$, it follows $c_0 = 0$. Thus (3.10) holds. This completes the proof. □

Theorem 3.2. *Assume that the hypotheses (A_1)-(A_7) hold. If*

$$M_g \frac{T^{\alpha}}{\Gamma(\alpha+1)} \phi(r) + \frac{T^{\beta}}{\Gamma(\beta+1)} \omega(r) < r,$$

then the problem (1.1) has a solution x^* defined on J and the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by

$$x_{n+1}(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, x_n(s))ds + f(t, x_n(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x_n(s))ds, \tag{3.11}$$

for all $t \in \mathbb{R}$, where $x_1 = u$, converges monotonically to x^*.

Proof. By Lemma 3.4, the problem (1.1) is equivalent to the nonlinear integral equation

$$x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, x(s))ds + f(t, x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s))ds, \ t \in J. \tag{3.12}$$

Set $E = C(J, \mathbb{R})$. Then, from Lemma 3.3 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ on E by

$$\mathcal{A}x(t) = f(t, x(t)), \ t \in J, \tag{3.13}$$

$$\mathcal{B}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s))\, ds, \ t \in J, \tag{3.14}$$

and

$$\mathcal{C}x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s, x(s))\, ds, \ t \in J. \tag{3.15}$$

From the continuity of the integrals, it follows that $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ define the maps $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ and $\mathcal{C} : E \to E$. Then, the problem (1.1) is equivalent to the operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t) + \mathcal{C}x(t) = x(t), \ \ t \in J. \tag{3.16}$$

We shall show that the operators $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: *$\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are nondecreasing operators on E.*

Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (A_2), we obtain

$$\mathcal{A}x(t) = f(t, x(t)) \geq f(t, y(t)) = \mathcal{A}y(t),$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E. Similarly, we have by (A_5),

$$\begin{aligned} \mathcal{B}x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s))\, ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, y(s))\, ds \\ &= \mathcal{B}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}$ is nondecreasing operator on E into itself. The proof that $\mathcal{C}$ is nondecreasing operator on E into itself is similar.

Step II: *$\mathcal{A}$ and $\mathcal{C}$ are partially bounded and partially $\mathcal{D}$-contraction on E.*

Let $x \in E$ be arbitrary. Then by (A_1),

$$|\mathcal{A}x(t)| \leq |f(t,x(t))| \leq M_f,$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\| \leq M_f$ and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then,

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |f(t,x(t)) - f(t,y(t))| \leq \phi(|x(t) - y(t)|) \leq \phi(\|x - y\|).$$

Then, $\|\mathcal{A}x - \mathcal{A}y\| \leq \phi(\|x - y\|)$ for all $x, y \in E$ with $x \geq y$ and hence $\mathcal{A}$ is a partially $\mathcal{D}$-Lipschitz on E with $\mathcal{D}$-functions $\phi(r)$, which further implies that $\mathcal{A}$ is a partially continuous on E.

Also we have

$$\begin{aligned}
|\mathcal{C}x(t)| &\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |h(s,x(s))| ds \\
&\leq M_h \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \\
&\leq M_h \frac{t^\beta}{\Gamma(\beta+1)} \\
&\leq M_h \frac{T^\beta}{\Gamma(\beta+1)},
\end{aligned}$$

which means that $\mathcal{C}$ is bounded and further partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then,

$$\begin{aligned}
|\mathcal{C}x(t) - \mathcal{C}y(t)| &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |h(s,x(s)) - h(s,y(s))| ds \\
&\leq \frac{T^\beta}{\Gamma(\beta+1)} \omega(\|x - y\|).
\end{aligned}$$

Hence $\mathcal{C}$ is a partially $\mathcal{D}$-Lipschitz on E with $\mathcal{D}$-functions $\dfrac{T^\beta}{\Gamma(\beta+1)}\omega(r)$, which further implies that $\mathcal{C}$ is a partially continuous on E.

Step III: *$\mathcal{B}$ is a partially continuous operator on E.*

Let $\{x_n\}$ be a sequence of points of a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,x_n(s))\, ds \right] \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\lim_{n\to\infty} g(s,x_n(s))\, ds \right] ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,x(s))\, ds \\
&= \mathcal{B}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{B}x_n\}$ converges to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|\, |g(s,x_n(s))|\, ds \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_1-s)^{\alpha-1} |g(s,x_n(s))|\, ds \right| \\
&\leq \frac{M_g}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha).
\end{aligned}$$

Consequently,

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniformly and hence $\mathcal{B}$ is a partially continuous on E.

Step IV: *$\mathcal{B}$ is a partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned} |\mathcal{B}x(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s, x(s))| \, ds \\ &\leq M_g \frac{T^\alpha}{\Gamma(\alpha+1)} \\ &= r, \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{B}x\| \leq r$ for all $x \in C$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then,

$$\begin{aligned} |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}| \, |g(s, x(s))| \, ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_1-s)^{\alpha-1} |g(s, x(s))| \, ds \right| \\ &\leq \frac{M_g}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha). \end{aligned}$$

Thus we have that

$$|\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1$$

uniformly for all $x \in C$. This shows that $\mathcal{B}(C)$ is an equicontinuous set in E. Hence $\mathcal{B}(C)$ is compact subset of E and consequently $\mathcal{B}$ is a partially compact operator on E into itself.

Step V: $\mathcal{D}$-functions ϕ and ω satisfy the growth condition $M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r,\ r > 0,$.

We have

$$M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) = M_g \frac{T^\alpha}{\Gamma(\alpha+1)} \phi(r) + \frac{T^\beta}{\Gamma(\beta+1)} \omega(r) < r,$$

by assumption.

Step VI: *u satisfies the operator inequality $u \leq \mathcal{A}u\mathcal{B}u + \mathcal{C}u$.*

Since the hypothesis (A_6) holds, u is a lower solution of (1.1) defined on J. Then,

$${}^cD^\alpha \left(\frac{u(t) - I^\beta h(t, u(t))}{f(t, u(t))} \right) \leq g(t, u(t)), \tag{3.17}$$

satisfying,

$$u(0) \leq 0, \tag{3.18}$$

for all $t \in J$.

Integrating (3.17) from 0 to t, we obtain

$$u(t) \leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, u(s)) ds + f(t, u(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, u(s)) ds, \tag{3.19}$$

for all $t \in J$. This show that u is a lower solution of the operator inequality $u \leq \mathcal{A}u\mathcal{B}u + \mathcal{C}u$.

Thus, the operators $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.1 in view of Remark 2.9 and we apply it to conclude that the operator equation $\mathcal{A}x\mathcal{B}x + \mathcal{C}x = x$ has a solution defined on J. Consequently the integral equation and the problem (1.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (3.11) converges monotonically to x^*. This completes the proof. □

Example 3.1. *Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the initial value problem of quadratic fractional nonlinear integro-differential equation,*

$$\begin{cases} {}^cD^{1/2}\left[\dfrac{x(t) - I^{3/2}(\arctan x(t))}{f(t,x(t))}\right] = \dfrac{2+\tanh x(t)}{12}, \quad t \in J := [0,1], \\ x(0) = 0, \end{cases} \tag{3.20}$$

where ${}^cD^{1/2}$ denotes the Caputo fractional derivative of order $1/2$, and $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is a function defined by

$$f(t,x) = \begin{cases} 1, & \text{if} \quad x \leq 0, \\ 1 + \dfrac{x}{1+x}, & \text{if} \quad x > 0. \end{cases}$$

If we take $h(t,x) = \arctan x$ and $g(t,x) = \dfrac{2+\tanh x}{12}$ for $t \in J$ and $x \in \mathbb{R}$, then it is easy to check that the conditions of Theorem 3.2 are satisfied with the lower solution u defined by $u(t) = -\dfrac{4t^{3/2}}{3\sqrt{\pi}} + \dfrac{t^{1/2}}{6\sqrt{\pi}}, t \in J$. Therefore, the problem (3.20) has a solution defined on $[0,1]$.

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Edge pair sum labeling of butterfly graph with shell order

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Abstract

The concept of an edge pair sum labeling was introduced in [3]. Let $G(p,q)$ be a graph. An injective map $f : E(G) \to \{\pm 1, \pm 2, \cdots, \pm q\}$is said to be an edge pair sum labeling if the induced vertex function $f^* : V(G) \to Z - \{0\}$ defined by $f^*(v) = \sum_{e \epsilon E_v} f(e)$ is one- one where E_v denotes the set of edges in G that are incident with a vertex v and $f^*(V(G))$ is either of the form $\left\{\pm k_1, \pm k_2, \cdots, \pm k_{\frac{p}{2}}\right\}$ or$\left\{\pm k_1, \pm k_2, \cdots, \pm k_{\frac{p-1}{2}}\right\}$ $\bigcup \left\{\pm k_{\frac{p+1}{2}}\right\}$ according as p is even or odd. A graph with an edge pair sum labeling is called an edge pair sum graph. In this paper we prove that the shell graph and butterfly graph with shell order are edge pair sum graphs.

Keywords: Edge pair sum labeling, edge pair sum graph, shell graph, butterfly graph.

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1 Introduction

Throughout this paper we consider finite, simple and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. G is also called a (p,q) graph. We follow the basic notations and terminologies of graph theory as in [2]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and for a dynamic survey of various graph labeling problems with extensive bibliography we refer to Gallian [1]. Ponraj et.al introduced the concept of pair sum labeling in [13]. An injective map $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling of a graph $G(p,q)$ if the induced edge function $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\left\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\right\}$ or $\left\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\right\} \cup \left\{\pm k_{\frac{q+1}{2}}\right\}$ according as q is even or odd. Analogous to pair sum labeling we define a new labeling called edge pair sum labeling [3] and we [4-12] establish that the path, cycle, star graph, $P_m \bigcup K_{1,n}$, $C_n \bigcup K_m^c$ if n is even, triangular snake, star graph, bistar, complete bipartite graphs, $k_{1,n}$, jelly fish, Y-tree, theta graph, spider graphs, ladder graph, $WT(n : k)$, subdivision of spokes in wheels, N quadrilateral graph, wheel graph, double triangular snake, flower graph, one point union of cycles, the perfect binary tree, shadow graph, total graph and P_n^2 are edge pair sum graphs. In this paper we prove that the shell graph and butterfly graph with shell order are edge pair sum graphs.

We use the following definitions in the subsequent sequel.

Definition 1.1. *A shell S_n is the graph obtained by taking (n-3)concurrent cords in a cycle C_n. The vertex at which all the chords are concurrent is called the apex. The shell is also called fan f_{n-1}.*

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Definition 1.2. *A multiple shell is a collection of edge disjoint shells that have their apex in common. Hence a double shell consists of two edge disjoint shells with a common apex.*

Definition 1.3. *A bow graph is a double shell in which each shell has any order.*

Definition 1.4. *A butterfly graph is a bow graph with exactly two pendant edges at the apex.*

2 Main results

Theorem 2.1. *The shell graph S_n is an edge pair sum graph.*

Proof. Let G be a shell graph.
Define $V(G) = \{u, v_i : 1 \leq i \leq (n-1)\}$ and
$E(G) = \{e_i^{'} = uv_i : 1 \leq i \leq (n-1); e_i = v_iv_{i+1} : 1 \leq i \leq (n-2)\}$ are the vertices and edges of the graph G.
Define an edge pair sum labeling $f : E(G) \to \{\pm1, \pm2, \pm3, ..., \pm(2n-3)\}$ by considering two cases.
Case(i). n is even and $n \geq 6$.
Define $f(e_{\frac{n-2}{2}}) = 2$, $f(e_{\frac{n}{2}}) = 1$,
for $1 \leq i \leq \frac{n-4}{2}$ $f(e_i) = -(n+1-2i)$, $f(e_{\frac{n}{2}+i}) = (3+2i)$,
$f(e_i^{'}) = -(2n-3-2i)$ and $f(e^{'}_{\frac{n+2}{2}+i}) = (n-1+2i)$,
$f(e^{'}_{\frac{n-2}{2}}) = -4 = -f(e^{'}_{\frac{n}{2}})$ and $f(e_{\frac{n+2}{2}}) = -3$.
For each edge label f, the induced vertex label f^* is calculated as follows:
$f^*(u) = -3 = -f^*(v_{\frac{n+2}{2}})$, $f^*(v_{\frac{n-2}{2}}) = -7 = -f^*(v_{\frac{n}{2}})$, $f^*(v_1) = -(3n-6) = -f^*(v_{n-1})$,
for $1 \leq i \leq (\frac{n-6}{2})$ $f^*(v_{1+i}) = (-4n+5+6i)$ and $f^*(v_{\frac{n+2}{2}+i}) = (n+7+6i)$.
$f^*(V(G)) = \{\pm3, \pm7, \pm(3n-6), \pm(n+13), \pm(n+19), \pm(n+25), ..., \pm(4n-11)\}$.
Hence f is an edge pair sum labeling.
The example for the edge pair sum graph labeling of S_6 is shown in Figure 1.

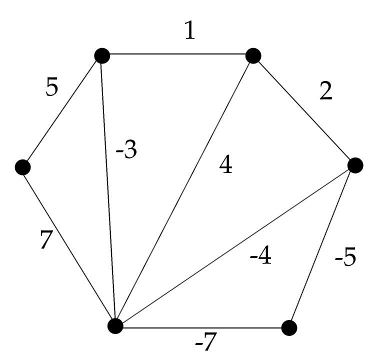


Figure 1

Case(ii). n is odd and $n \geq 6$.
Subcase (a). $n = 3$.
If $n = 3$ the shell graph becomes C_3 and proved that C_3 admit edge pair sum labeling [3].
Subcase (b). $n = 5$.
Define $f(e_1) = -2$, $f(e_2) = -1$, $f(e_3) = 3$, $f(e_1^{'}) = -7 = -f(e_3^{'})$, $f(e_2^{'}) = 4$ and $f(e_4^{'}) = -5$.
For each edge label f, the induced vertex label f^* is calculated as follows:
$f^*(u) = -1 = -f^*(v_2)$, $f^*(v_1) = -9 = -f^*(v_3)$ and $f^*(v_4) = -2$.
Then $f^*(V(G)) = \{\pm1, \pm9\} \bigcup \{-2\}$. Hence f is an edge pair sum labeling.
Subcase (c). $n \geq 7$.
Define $f(e_{\frac{n-1}{2}}) = -1$, $f(e_{\frac{n-3}{2}}) = -2$, $f(e_{\frac{n+1}{2}}) = 3$,
for $1 \leq i \leq \frac{n-5}{2}$ $f(e_i) = (n-2i)$, $f(e_{\frac{n+1}{2}+i}) = -(3+2i)$, $f(e_i^{'}) = (2n-3-2i)$ and $f(e^{'}_{\frac{n+3}{2}+i}) = -(n+2i)$,
$f(e^{'}_{\frac{n-1}{2}}) = -6 = -f(e^{'}_{\frac{n-3}{2}})$, $f(e^{'}_{\frac{n+1}{2}}) = 4$ and $f(e^{'}_{\frac{n+3}{2}}) = -10$.

For each edge label f, the induced vertex label f^* is calculated as follows:
$f^*(u) = -6 = -f^*(v_{\frac{n+1}{2}})$, $f^*(v_{\frac{n+3}{2}}) = -12$, $f^*(v_{\frac{n-3}{2}}) = 9 = -f^*(v_{\frac{n-1}{2}})$, $f^*(v_1) = (3n-7) = -f^*(v_{n-1})$,
for $1 \leq i \leq \frac{n-7}{2}$ $f^*(v_{1+i}) = (4n-7-6i)$ and $f^*(v_{\frac{n+3}{2}+i}) = -(n+8+6i)$. From the above vertex labeling we get $f^*(V(G)) = \{\pm 6, \pm 9, \pm(3n-7), \pm(n+14), \pm(n+20), \pm(n+26), ..., \pm(4n-13)\} \bigcup \{-12\}$.
Hence f is an edge pair sum labeling.
The examples for the edge pair sum graph labeling of S_5 and S_7 are shown in Figures 2 and 3 respectively.

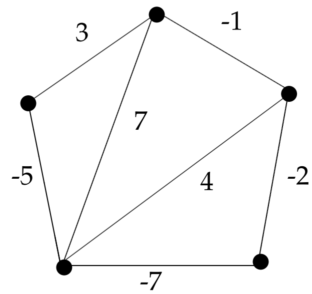


Figure 2

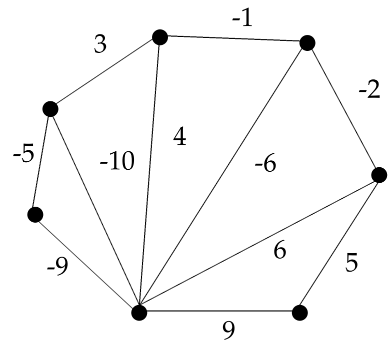


Figure 3

□

Theorem 2.2. *The butterfly graph with shell order m and m (order excludes the apex) is an edge pair sum graph.*

Proof. Let G be a butterfly graph with shells of order m and m excluding the apex.
Define $V(G) = \{w_0, w_1, w_2, v_i, u_i : 1 \leq i \leq m\}$ and
$E(G) = \{e_1' = w_0w_1, e_2' = w_0w_2, e_i = w_0u_i$ and $e_{2m-1+i} = w_0v_i : 1 \leq i \leq m, e_{m+i} = u_iu_{i+1}$
and $e_{3m-1+i} = v_iv_{i+1} : 1 \leq i \leq (m-1)\}$ are the vertices and edges of the graph G.
Define an edge labeling $f : E(G) \to \{\pm 1, \pm 2, \pm 3, ..., \pm 4m\}$.
Define $f(e_1') = 1$, $f(e_2') = -2$,
for $1 \leq i \leq m$ $f(e_i) = (2+2i) = -f(e_{2m-1+i})$ and
for $1 \leq i \leq (m-1)$ $f(e_{m+i}) = (2i+1) = -f(e_{3m-1+i})$.
For each edge label f, the induced vertex label f^* is calculated as follows:
$f^*(w_0) = -1 = -f^*(w_1)$, $f^*(w_2) = -2$, $f^*(u_1) = 7 = -f^*(v_1)$,
for $1 \leq i \leq (m-2)$ $f^*(u_{1+i}) = (8+6i) = -f^*(v_{1+i})$ and
$f^*(u_m) = (4m+1) = -f^*(v_m)$.
Then we get $f^*(V(G)) = \{\pm 1, \pm 7, \pm(4m+1), \pm 14, \pm 20, \pm 26, ..., \pm(6m-4)\} \bigcup \{-2\}$.
Hence f is an edge pair sum labeling.
The example for the edge pair sum graph labeling of graph G with $m = 5$ is shown in Figure 4.

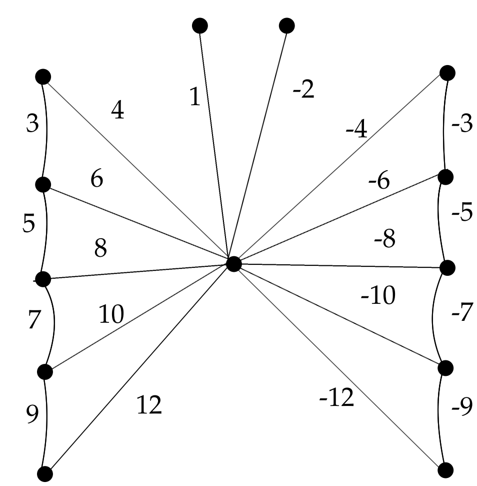


Figure 4

□

Theorem 2.3. *The butterfly graph with shell order m and 2m (order excludes the apex) is an edge pair sum graph for m is even.*

Proof. Let G be a butterfly graph with shell order m and $2m$ excluding the apex.
Define $V(G) = \{w_0, w_1, w_2, v_i : 1 \le i \le m, u_i : 1 \le i \le 2m\}$ and
$E(G) = \{e_1^{'} = w_0w_1, e_2^{'} = w_0w_2, e_i = w_0u_i : 1 \le i \le 2m, e_{2m+i} = u_iu_{1+i} : 1 \le i \le (2m-1),$
$e_{4m-1+i} = w_0v_i : 1 \le i \le m, e_{5m-1+i} = v_iv_{i+1} : 1 \le i \le (m-1)\}$ are the vertices and edges of the graph G.
Define an edge labeling $f : E(G) \to \{\pm 1, \pm 2, \pm 3, ..., \pm 6m\}$ by considering two cases.
Case(i). m = 2.
Define $f(e_1) = 3 = -f(e_4), f(e_2) = 5 = -f(e_3), f(e_5) = 2 = -f(e_7), f(e_6) = 1,$
$f(e_8) = 7 = -f(e_9), f(e_{10}) = 9, f(e_1^{'}) = -8$ and $f(e_2^{'}) = 6$.
For each edge label f, the induced vertex label f^* is calculated as follows:
$f^*(w_1) = -8 = -f^*(u_2), f^*(w_2) = 6 = -f^*(u_3), f^*(w_0) = -2 = -f^*(v_2),$
$f^*(u_1) = 5 = -f^*(u_4)$ and $f^*(v_1) = 16$.
Therefore we get $f^*(V(G)) = \{\pm 2, \pm 5, \pm 6, \pm 8\} \bigcup \{16\}$. Hence f is an edge pair sum labeling.
Case(ii). $m \ge 4$.
Define $f(e_1^{'}) = -(5m-2), f(e_2^{'}) = (5m-4), f(e_{3m}) = 1,$
for $1 \le i \le m$ $f(e_i) = (2i+1)$ and $f(e_{m+i}) = -(2m+3-2i),$
for $1 \le i \le (m-1)$ $f(e_{2m+i}) = (m-2+2i)$ and $f(e_{3m+i}) = -(3m-2-2i),$
for $1 \le i \le \frac{m}{2}$ $f(e_{4m-1+i}) = (2m+1+2i)$ and $f(e_{\frac{9m}{2}-1+i}) = -(3m+3-2i),$
for $1 \le i \le \frac{m}{2}-1$ $f(e_{5m-1+i}) = 2i$ and $f(e_{\frac{11m}{2}-1+i}) = -(m-2i)$ and $f(e_{\frac{11m}{2}-1}) = (4m+1)$.
For each edge label f, the induced vertex label f^* is calculated as follows:
$f^*(w_1) = -(5m-2) = -f^*(u_m), f^*(w_2) = (5m-4) = -f^*(u_{m+1}), f^*(w_0) = -2 = -f^*(v_{\frac{m}{2}+1}),$
$f^*(u_1) = (m+3) = -f^*(u_{2m}),$
for $1 \le i \le (m-2)$ $f^*(u_{1+i}) = (2m+1+6i)$ and $f^*(u_{m+1+i}) = -(8m-5-6i),$
for $1 \le i \le \frac{m-4}{2}$ $f^*(v_{1+i}) = (2m+5+6i)$ and $f^*(v_{\frac{m}{2}+1+i}) = -(5m-1-6i),$
$f^*(v_1) = (2m+5) = -f^*(v_m)$ and $f^*(v_{\frac{m}{2}}) = 8m$.
Then $f^*(V(G)) = \{\pm 2, \pm(2m+5), \pm(5m-2), \pm(5m-4), \pm(m+3), \pm(2m+7), \pm(2m+13), \pm(2m+19), ..., \pm(8m-11), \pm(2m+11), \pm(2m+17), \pm(2m+23), ..., \pm(5m-7)\} \bigcup \{8m\}$.
Hence f is an edge pair sum labeling.
The example for the edge pair sum graph labeling of graph G with $m = 4$ is shown in Figure 5.

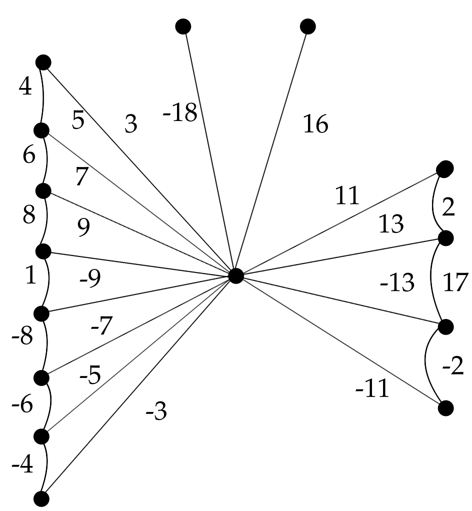
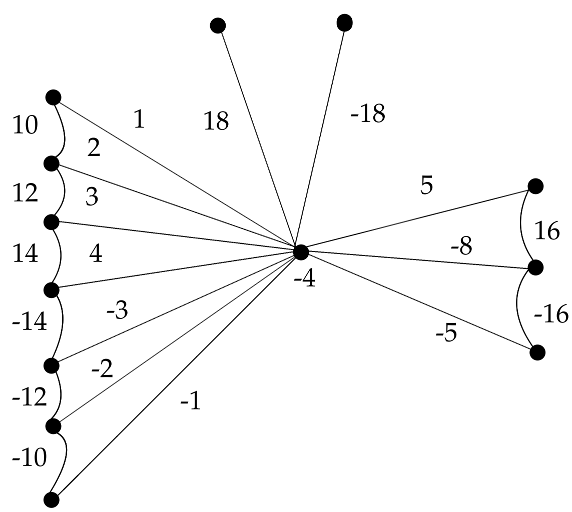


Figure 5

□

Theorem 2.4. *The butterfly graph with shell order m and (2m+1) (order excludes the apex) is an edge pair sum graph for m is odd.*

Proof. Let G be a butterfly graph with shells of order m and $2m+1$ excluding the apex.
Define $V(G) = \{w_0, w_1, w_2, v_i : 1 \le i \le m, u_i : 1 \le i \le (2m+1)\}$ and
$E(G) = \{e_1^{'} = w_0w_1, e_2^{'} = w_0w_2, e_i = w_0u_i : 1 \le i \le (2m+1), e_{2m+1+i} = u_iu_{1+i} : 1 \le i \le 2m,$
$e_{4m+1+i} = w_0v_i : 1 \le i \le m, e_{5m+1+i} = v_iv_{i+1} : 1 \le i \le (m-1)\}$ are the vertices and edges of the graph G.
Define an edge labeling $f : E(G) \to \{\pm1, \pm2, \pm3, ..., \pm(6m+2)\}$.
Define $f(e_1^{'}) = (5m+3) = -f(e_2^{'})$,
for $1 \le i \le (m+1)$ $f(e_i) = i$,
for $1 \le i \le m$ $f(e_{m+1+i}) = -(m+1-i), f(e_{2m+1+i}) = 2(m+1+i)$ and $f(e_{3m+1+i}) = -(4m+4-2i)$,
for $1 \le i \le \frac{m-1}{2}$ $f(e_{4m+1+i}) = (m+1+i)$ $f(e_{\frac{9m+3}{2}+i}) = -(\frac{3m+3}{2} - i)$,
$f(e_{5m+1+i}) = 2(2m+1+i)$ and $f(e_{\frac{11m+1}{2}+i}) = -(5m+3-2i)$ and
$f(e_{\frac{9m+3}{2}}) = -2(m+1)$.
For each edge label f, the induced vertex label f^* is calculated as follows:
$f^*(w_1) = (5m+3) = -f^*(w_2), f^*(w_0) = -(m+1) = -f^*(u_{m+1}), f^*(u_1) = (2m+5) = -f^*(u_{2m+1})$,
for $1 \le i \le (m-1)$ $f^*(u_{1+i}) = (4m+7+5i)$ and $f^*(u_{m+1+i}) = -(9m+7-5i)$,
for $1 \le i \le \frac{m-3}{2}$ $f^*(v_{1+i}) = (9m+8+5i)$ and $f^*(v_{\frac{m+1}{2}+i}) = -(\frac{23m+11}{2} - 5i)$,
$f^*(v_1) = (5m+6) = -f^*(v_m)$ and $f^*(v_{\frac{m+1}{2}}) = -2(m+1)$.
Then $f^*(V(G)) = \{\pm(5m+3), \pm(m+1), \pm(2m+5), \pm(5m+6), \pm(4m+12), \pm(4m+17), \pm(4m+22), ..., \pm(9m+2), \pm(9m+13), \pm(9m+18), \pm(9m+23), ..., \pm(\frac{23m+1}{2})\} \bigcup \{-2(m+1)\}$.
Hence f is an edge pair sum labeling.
The example for the edge pair sum graph labeling of graph G with $m = 3$ is shown in Figure 6.



□

Figure 6

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Quasistatic contact problem between thermo-electroelastic bodies with long-term memory and adhesion

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Abstract

We study of a quasistatic frictional contact problem between two thermo-electroelastic bodies with adhesion. The temperature of the materials caused by elastic deformations. The contact is modelled with a version of normal compliance condition and the associated Coulomb's law of friction in which the adhesion of contact surfaces is taken into account. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic equalities, differential equations and fixed point arguments.

Keywords: thermo-electroelastic materials, Adhesion, Coulomb's law of friction, Normal compliance, Fixed point.

2010 MSC: 49J40, 74H25.

1 Introduction

The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [1, 4, 9, 12] and recently in the monographs [7, 8]. The novelty in all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β, it describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. Following [2], the bonding field satisfies the restriction $0 \leq \beta \leq 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. The aim of this paper is to study the quasistatic contact in thermo-electroelastic materials. For this, we use an thermo-electroelastic constitutive law with long-term memory given by

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \theta^\ell\big) + \int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \theta^\ell(s)\big)\, ds - (\mathcal{E}^\ell)^* E^\ell(\varphi^\ell), \tag{1.1}$$

where $\boldsymbol{u}^\ell$ the displacement field, $\boldsymbol{\sigma}^\ell$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)$ represent the stress and the linearized strain tensor, respectively, θ^ℓ represents the absolute temperature and α^ℓ represents the damage field. Here $\mathcal{Q}^\ell$ is the relaxation operator, and $\mathcal{A}^\ell$ represents the thermo-elasticity operator with damage. $E(\varphi^\ell) = -\nabla\varphi^\ell$ is the electric field, $\mathcal{E}^\ell$ represents the third order piezoelectric tensor, $(\mathcal{E}^\ell)^*$ is its transposition. In this paper we study a quasistatic Coulomb's frictional contact problem between two thermo-electroelastic bodies with long-term memory. The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled with a surface variable, the bonding field. We derive a variational formulation of the problem

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and prove the existence of a unique weak solution. The paper is organized as follows. In section 2 we describe the mathematical models for the frictional contact problem between two thermo-electroelastic bodies with long-term memory. The contact is modelled with normal compliance and adhesion. We introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. We prove in section 3 the existence and uniqueness of the solution, where it is carried out in several steps and is based on a classical existence and uniqueness result on parabolic equalities, differential equations and fixed point arguments.

2 Problem statement and variational formulation

Let us consider two thermo-electroelastic bodies with long-term memory, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2, 3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, on one hand, and on two measurable parts Γ_a^ℓ and Γ_b^ℓ, on the other hand, such that $meas\Gamma_1^\ell > 0$, $meas\Gamma_a^\ell > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The Ω^ℓ body is submitted to $\boldsymbol{f}_0^\ell$ forces and volume electric charges of density q_0^ℓ. The bodies are assumed to be clamped on $\Gamma_1^\ell \times (0, T)$. The surface tractions $\boldsymbol{f}_2^\ell$ act on $\Gamma_2^\ell \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\ell \times (0, T)$ and a surface electric charge of density q_2^ℓ is prescribed on $\Gamma_b^\ell \times (0, T)$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The bodies is in adhesive contact over the surface Γ_3. The mechanical problem may be formulated as follows.

Problem P. For $\ell = 1, 2$, find a displacement field $\boldsymbol{u}^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{S}^d$, an electric potential field $\varphi^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}$, a temperature $\theta^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}$, a bonding field $\beta : \Gamma_3 \times (0, T) \longrightarrow \mathbb{R}$ and a electric displacement field $\boldsymbol{D}^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}^d$ such that

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \theta^\ell\big) + \int_0^t \mathcal{Q}^\ell\big(t - s, \varepsilon(\boldsymbol{u}^\ell(s)), \theta^\ell(s)\big)\, ds - (\mathcal{E}^\ell)^* E^\ell(\varphi^\ell), \quad \text{in } \Omega^\ell \times (0, T), \tag{2.2}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) + \mathcal{G}^\ell\big(E^\ell(\varphi^\ell)\big), \quad \text{in } \Omega^\ell \times (0, T), \tag{2.3}$$

$$\dot{\theta}^\ell - \kappa_0^\ell \Delta\theta^\ell = \Theta^\ell\big(\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \theta^\ell\big) + \rho^\ell \quad \text{in } \Omega^\ell \times (0, T), \tag{2.4}$$

$$\operatorname{Div} \boldsymbol{\sigma}^\ell + \boldsymbol{f}_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0, T), \tag{2.5}$$

$$\operatorname{div} \boldsymbol{D}^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0, T), \tag{2.6}$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0, T), \tag{2.7}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = \boldsymbol{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0, T), \tag{2.8}$$

$$\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \text{ where } \sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu \beta^2 R_\nu([u_\nu]) \quad \text{on } \Gamma_3 \times (0, T), \tag{2.9}$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\| \le \mu p_\nu([u_\nu]), \\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\| < \mu p_\nu([u_\nu]) \Rightarrow [\boldsymbol{u}_\tau] = 0, \quad \text{on } \Gamma_3 \times (0, T), \\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\| = \mu p_\nu([u_\nu]) \Rightarrow \exists \lambda \ge 0 \\ \qquad \text{such that } \boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]) = -\lambda [\boldsymbol{u}_\tau] \end{cases} \tag{2.10}$$

$$\dot{\beta} = -\Big(\beta\big(\gamma_\nu (R_\nu([u_\nu]))^2 + \gamma_\tau |\boldsymbol{R}_\tau([\boldsymbol{u}_\tau])|^2\big) - \varepsilon_a\Big)_+ \quad \text{on } \Gamma_3 \times (0, T), \tag{2.11}$$

$$\varphi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0, T), \tag{2.12}$$

$$\boldsymbol{D}^\ell . \boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0, T), \tag{2.13}$$

$$\kappa_0^\ell \frac{\partial^\ell \theta^\ell}{\partial \nu^\ell} + \lambda_0^\ell \theta^\ell = 0 \quad \text{on } \Gamma^\ell \times (0, T), \tag{2.14}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \ \theta^\ell(0) = \theta_0^\ell \quad \text{in } \Omega^\ell, \tag{2.15}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.16}$$

Here and below $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$, whereas "." and $\|.\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively; ν^ℓ is the unit outer normal vector on Γ^ℓ, and $r_+ = max\{r, 0\}$ denotes the positive part of r, equations (2.2) and (2.3) represent the thermo-electroelastic constitutive law with long term-memory. Equation (2.4) represents the energy conservation where Θ^ℓ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and ρ^ℓ is a given volume heat source. Equations (2.5) and (2.6) are the equilibrium equations for the stress and electric-displacement fields, respectively. Next, the equations (2.7) and (2.8) represent the displacement and traction boundary condition, respectively. Condition (2.9) represents the normal compliance conditions with adhesion where γ_ν is a given adhesion coefficient, p_ν is a given positive function which will be described below and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the displacements in normal direction, in this condition the interpenetrability between two bodies, that is $[u_\nu]$ can be positive on Γ_3.

$$R_\nu(s) = \begin{cases} L & \text{if} \quad s < -L, \\ -s & \text{if} \quad -L \le s \le 0, \\ 0 & \text{if} \quad s > 0. \end{cases} \qquad \boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \le L, \\ L\frac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L. \end{cases} \tag{2.17}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see, e.g., [9]). Condition (2.10) are a non local Coulomb's friction law conditions coupled with adhesive, where $[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ stands for the jump of the displacements in tangential direction. Next, the equation (2.11) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [1], see also [12, 14] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (2.11), $\dot\beta \le 0$. (2.12) and (2.13) represent the electric boundary conditions. The relation (2.14) represent a Fourier boundary condition for the temperature on Γ^ℓ. Finally the functions $\boldsymbol{u}_0, \theta_0$ and β_0 in (2.15)-(2.16) are the initial data.

We now proceed to obtain a variational formulation of Problem P. For this purpose, we introduce additional notation and assumptions on the problem data. Here and in what follows the indices i and j run between 1 and d, the summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. Let $H^\ell = L^2(\Omega^\ell)^d$, $H_1^\ell = H^1(\Omega^\ell)^d$, $\mathcal{H}^\ell = L^2(\Omega^\ell)_s^{d\times d}$, $\mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \in \mathcal{H}^\ell;\ \mathrm{div}\boldsymbol{\tau}^\ell \in H^\ell\}$. The spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell dx,\ (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell dx + \int_{\Omega^\ell} \nabla\boldsymbol{u}^\ell.\nabla\boldsymbol{v}^\ell dx,$$
$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx + \int_{\Omega^\ell} \mathrm{div}\,\boldsymbol{\sigma}^\ell.\,\mathrm{Div}\,\boldsymbol{\tau}^\ell dx$$

and the associated norms $\|.\|_{H^\ell}$, $\|.\|_{H_1^\ell}$, $\|.\|_{\mathcal{H}^\ell}$, and $\|.\|_{\mathcal{H}_1^\ell}$ respectively.
We introduce for the bonding field the set

$$\mathcal{Z} = \left\{\varsigma \in L^\infty(0,T;L^2(\Gamma_3));\ 0 \le \varsigma(t) \le 1\ \ \forall t \in [0,T],\ \text{a.e. on}\ \Gamma_3\right\},$$

and for the displacement field we need the closed subspace of H_1^ℓ defined by

$$V^\ell = \left\{\boldsymbol{v}^\ell \in H_1^\ell;\ \boldsymbol{v}^\ell = 0 \text{ on } \Gamma_1^\ell\right\}.$$

Since $meas\Gamma_1^\ell > 0$, the following Korn's inequality holds (see [7]) :

$$\|\varepsilon(\boldsymbol{v}^\ell)\|_{\mathcal{H}^\ell} \ge c_K \|\boldsymbol{v}^\ell\|_{H_1^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell. \tag{2.18}$$

Over the space V^ℓ we consider the inner product given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{V^\ell} = (\varepsilon(\boldsymbol{u}^\ell), \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in V^\ell, \tag{2.19}$$

and let $\|.\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (2.18) that the norms $\|.\|_{H_1^\ell}$ and $\|.\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell, \|.\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (2.19), there exists a constant $c_0 > 0$, depending only on Ω^ℓ, Γ_1^ℓ and Γ_3 such that

$$\|\boldsymbol{v}^\ell\|_{L^2(\Gamma_3)^d} \le c_0 \|\boldsymbol{v}^\ell\|_{V^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell. \tag{2.20}$$

We also introduce the spaces

$$E_0^\ell = L^2(\Omega^\ell), \quad E_1^\ell = H^1(\Omega^\ell), \quad W^\ell = \left\{\psi^\ell \in E_1^\ell;\ \psi^\ell = 0 \text{ on } \Gamma_a^\ell\right\},$$
$$\mathcal{W}^\ell = \left\{\boldsymbol{D}^\ell = (D_i^\ell);\ D_i^\ell \in L^2(\Omega^\ell),\ \operatorname{div}\boldsymbol{D}^\ell \in L^2(\Omega^\ell)\right\}.$$

Since $meas\Gamma_a^\ell > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi^\ell\|_{W^\ell} \geq c_F\|\psi^\ell\|_{H^1(\Omega^\ell)} \quad \forall\psi^\ell \in W^\ell, \tag{2.21}$$

where $c_F > 0$ is a constant which depends only on Ω^ℓ, Γ_a^ℓ. Over the space W^ℓ, we consider the inner product given by

$$(\varphi^\ell, \psi^\ell)_{W^\ell} = \int_{\Omega^\ell} \nabla\varphi^\ell.\nabla\psi^\ell dx \tag{2.22}$$

and let $\|.\|_{W^\ell}$ be the associated norm. It follows from (2.21) that $\|.\|_{H^1(\Omega^\ell)}$ and $\|.\|_{W^\ell}$ are equivalent norms on W^ℓ and therefore $(W^\ell, \|.\|_{W^\ell})$ is areal Hilbert space. The space $\mathcal{W}^\ell$ is real Hilbert space with the inner product

$$(\boldsymbol{D}^\ell, \boldsymbol{\Phi}^\ell)_{\mathcal{W}^\ell} = \int_{\Omega^\ell} \boldsymbol{D}^\ell.\boldsymbol{\Phi}^\ell dx + \int_{\Omega^\ell} \operatorname{div}\boldsymbol{D}^\ell.\operatorname{div}\boldsymbol{\Phi}^\ell dx,$$

where $\operatorname{div}\boldsymbol{D}^\ell = (\boldsymbol{D}_{i,i}^\ell)$, and the associated norm $\|.\|_{\mathcal{W}^\ell}$.

In order to simplify the notations, we define the product spaces

$$\boldsymbol{V} = V^1 \times V^2,\ H = H^1 \times H^2,\ H_1 = H_1^1 \times H_1^2,\ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2,\ \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2,$$
$$E_0 = E_0^1 \times E_0^2,\ E_1 = E_1^1 \times E_1^2,\ W = W^1 \times W^2,\ \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2.$$

The spaces $\boldsymbol{V}$, E_1, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(.,.)_{\boldsymbol{V}}$, $(.,.)_{E_1}$, $(.,.)_W$ and $(.,.)_{\mathcal{W}}$.

In the study of the Problem **P**, we consider the following assumptions:

The *thermo-elasticity operator* $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{A}^\ell} > 0 \text{ such that } : \forall\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, r_1, r_2 \in \mathbb{R}, \\ \qquad |\mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1, r_1) - \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2, r_2)| \leq L_{\mathcal{A}^\ell}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + \\ \qquad |r_1 - r_2|), \quad \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}, r) \text{ is measurable in } \Omega^\ell, \ \ \forall\boldsymbol{\xi} \in \mathbb{S}^d, r \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{0}, 0) \text{ belongs to } \mathcal{H}^\ell. \end{cases} \tag{2.23}$$

The *relaxation function* $\mathcal{Q}^\ell : \Omega^\ell \times (0, T) \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{Q}^\ell} > 0 \text{ such that } : \forall\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, r_1, r_2 \in \mathbb{R}, \\ \qquad |\mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}_1, r_1) - \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}_2, r_2)| \leq L_{\mathcal{Q}^\ell}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + \\ \qquad |r_1 - r_2|), \text{ for all } t \in (0, T), \quad \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}, r) \text{ is measurable in } \Omega^\ell, \\ \qquad \text{for any } t \in (0, T),\ \boldsymbol{\xi} \in \mathbb{S}^d,\ \ r \in \mathbb{R}. \\ \text{(c) The mapping } t \mapsto \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}, r) \text{ is continuous in } (0, T), \\ \qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d,\ \ r \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{0}, 0) \text{ belongs to } \mathcal{H}^\ell, \forall t \in (0, T). \end{cases} \tag{2.24}$$

The *energy function* $\Theta^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\Theta^\ell} > 0 \text{ such that } : \forall\,\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \alpha_1, \alpha_2 \in \mathbb{R}, \\ \quad |\Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\xi}_1, \alpha_1) - \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\xi}_2, \alpha_2)| \leq L_{\Theta^\ell}(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + \\ \quad |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\alpha_1 - \alpha_2|),\ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha) \text{ is measurable on } \Omega^\ell, \\ \quad \text{for any } \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}, \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \Theta^\ell(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0) \text{ belongs to } L^2(\Omega^\ell), \\ \text{(d) } \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha) \text{ is bounded for all } \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d, \alpha \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{2.25}$$

The *piezoelectric tensor* $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

$$\begin{cases} \text{(a) } \mathcal{E}^\ell(\boldsymbol{x},\tau) = (e^\ell_{ijk}(\boldsymbol{x})\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell),\ 1 \le i,j,k \le d. \end{cases} \tag{2.26}$$

The *electric permittivity operator* $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$, satisfies:

$$\begin{cases} \text{(a) } \mathcal{G}^\ell(\boldsymbol{x},\mathbf{E}) = (b^\ell_{ij}(\boldsymbol{x})E_j),\ b^\ell_{ij} = b^\ell_{ji},\ b^\ell_{ij} \in L^\infty(\Omega^\ell), \quad 1 \le i,j \le d. \\ \text{(b) There exists } m_{\mathcal{G}^\ell} > 0 \text{ such that :} \\ \quad \mathcal{G}^\ell\mathbf{E}.\mathbf{E} \ge m_{\mathcal{G}^\ell}|\mathbf{E}|^2,\ \forall \mathbf{E} \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{2.27}$$

The *normal compliance function* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) There exists } L_\nu > 0 \text{ such that :} \forall r_1, r_2 \in \mathbb{R}, \\ \quad |p_\nu(\boldsymbol{x},r_1) - p_\nu(\boldsymbol{x},r_2)| \le L_\nu|r_1 - r_2|,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } (p_\nu(\boldsymbol{x},r_1) - p_\nu(\boldsymbol{x},r_2))(r_1 - r_2) \ge 0, \quad \forall r_1, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\ \forall r \in \mathbb{R}. \\ \text{(d) } p_\nu(\boldsymbol{x},r) = 0,\ \text{ for all } r \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{2.28}$$

The *forces, tractions* have the regularity

$$\begin{gathered} \mathbf{f}^\ell_0 \in C(0,T;L^2(\Omega^\ell)^d),\ \ \mathbf{f}^\ell_2 \in C(0,T;L^2(\Gamma^\ell_2)^d), \\ q^\ell_0 \in C(0,T;L^2(\Omega^\ell)),\ \ q^\ell_2 \in C(0,T;L^2(\Gamma^\ell_b)),\ \ \rho^\ell \in C(0,T;L^2(\Omega^\ell)), \end{gathered} \tag{2.29}$$

The adhesion coefficients γ_ν, γ_τ and ε_a satisfy the conditions

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3),\ \ \varepsilon_a \in L^2(\Gamma_3),\ \gamma_\nu, \gamma_\tau, \varepsilon_a \ge 0,\ \text{a.e. on } \Gamma_3. \tag{2.30}$$

The energy coefficient κ^ℓ_0 and the microcrack diffusion coefficient κ^ℓ satisfies :

$$\kappa^\ell_0 > 0, \quad \kappa^\ell > 0. \tag{2.31}$$

Finally, the friction coefficient and the initial data satisfy:

$$\begin{gathered} \mu \in L^\infty(\Gamma_3), \quad \mu(x) \ge 0 \ \text{ a.e. on } \Gamma_3, \\ \boldsymbol{u}^\ell_0 \in \boldsymbol{V}^\ell,\ \theta^\ell_0 \in E^\ell_1,\ \beta_0 \in L^2(\Gamma_3),\ 0 \le \beta_0 \le 1,\ \text{a.e. on } \Gamma_3. \end{gathered} \tag{2.32}$$

We define the mappings $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2) : [0,T] \to \boldsymbol{V}$, $q = (q^1, q^2) : [0,T] \to W$, by

$$(\mathbf{f}(t), \boldsymbol{v})_V = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}^\ell_0(t)\boldsymbol{v}^\ell\,dx + \sum_{\ell=1}^{2} \int_{\Gamma^\ell_2} \mathbf{f}^\ell_2(t)\boldsymbol{v}^\ell\,da, \tag{2.33}$$

$$(q(t), \zeta)_W = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q^\ell_0(t)\zeta^\ell\,dx - \sum_{\ell=1}^{2} \int_{\Gamma^\ell_b} q^\ell_2(t)\zeta^\ell\,da \tag{2.34}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$, $\zeta \in W$ and $t \in [0,T]$, and note that conditions (2.29) imply that

$$\mathbf{f} \in C(0,T;\boldsymbol{V}),\ \ q \in C(0,T;W). \tag{2.35}$$

We introduce the following continuous functional $a_0 : E_1 \times E_1 \to \mathbb{R}$ by

$$a_0(\zeta,\xi) = \sum_{\ell=1}^{2} \kappa^\ell_0 \int_{\Omega^\ell} \nabla\zeta^\ell.\nabla\xi^\ell dx + \sum_{\ell=1}^{2} \lambda^\ell_0 \int_{\Gamma^\ell} \zeta^\ell\xi^\ell da. \tag{2.36}$$

Next, we define the four mappings $j_{ad} : L^2(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, $j_{\nu c} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $j_{fr} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, respectively, by

$$j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \Big(-\gamma_\nu\beta^2 R_\nu([u_\nu])[v_\nu] + \gamma_\tau\beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]).[v_\tau] \Big)\,da, \tag{2.37}$$

$$j_{\nu c}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu]\,da, \tag{2.38}$$

$$j_{fr}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \mu p_\nu([u_\nu])\big\|[v_\tau]\big\|\,da. \tag{2.39}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.2)–(2.16).

Problem PV. Find a displacement field $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2) : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} = (\sigma^1, \sigma^2) : [0,T] \to \mathcal{H}$, an electric potential field $\varphi = (\varphi^1, \varphi^2) : [0,T] \to W$, a temperature $\theta = (\theta^1, \theta^2) : [0,T] \to E_1$, a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ and a electric displacement field $\boldsymbol{D} = (\boldsymbol{D}^1, \boldsymbol{D}^2) : [0,T] \to \mathcal{W}$ such that, for a.e. $t \in (0,T)$,

$$\sigma^\ell = \mathcal{A}^\ell(\varepsilon(\boldsymbol{u}^\ell), \theta^\ell) + \int_0^t \mathcal{Q}^\ell(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \theta^\ell(s))\, ds - (\mathcal{E}^\ell)^* E^\ell(\varphi^\ell), \tag{2.40}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) + \mathcal{G}^\ell(E^\ell(\varphi^\ell)), \tag{2.41}$$

$$\begin{aligned} &\sum_{\ell=1}^{2} (\sigma^\ell, \varepsilon(\boldsymbol{v}^\ell) - \varepsilon(\boldsymbol{u}^\ell(t)))_{\mathcal{H}^\ell} + j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + j_{fr}(\boldsymbol{u}(t), \boldsymbol{v}) \\ &- j_{fr}(\boldsymbol{u}(t), \boldsymbol{u}(t)) + j_{vc}(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \geq (\mathbf{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_{\boldsymbol{V}}, \ \forall \boldsymbol{v} \in \boldsymbol{V}, \end{aligned} \tag{2.42}$$

$$\begin{aligned} &\forall \xi \in E_1, \quad \sum_{\ell=1}^{2} (\dot\theta^\ell(t) - \rho^\ell(t), \xi^\ell)_{L^2(\Omega^\ell)} + a_0(\theta(t), \xi) = \\ &\sum_{\ell=1}^{2} \Big(\Theta^\ell(\sigma^\ell(t), \varepsilon(\boldsymbol{u}^\ell(t)), \theta^\ell(t)), \xi^\ell\Big)_{L^2(\Omega^\ell)}, \end{aligned} \tag{2.43}$$

$$\sum_{\ell=1}^{2} \Big(\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell(t)) + \mathcal{G}^\ell(E^\ell(\varphi^\ell(t))),\ \nabla\phi^\ell\Big)_{H^\ell} = (-q(t), \phi)_W, \ \forall \phi \in W, \tag{2.44}$$

$$\dot\beta(t) = -\Big(\beta(t)\big(\gamma_\nu(R_\nu([u_\nu(t)]))^2 + \gamma_\tau |\boldsymbol{R}_\tau([\boldsymbol{u}_\tau(t)])|^2\big) - \varepsilon_a\Big)_+, \tag{2.45}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \theta(0) = \theta_0, \ \ \beta(0) = \beta_0. \tag{2.46}$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field, a temperature, a bonding field and a electric displacement field. The existence of the unique solution of Problem **PV** is stated and proved in the next section.

Remark 2.1. *We note that, in Problem **P** and in Problem **PV**, we do not need to impose explicitly the restriction* $0 \leq \beta \leq 1$. *Indeed, equation (2.45) guarantees that* $\beta(x,t) \leq \beta_0(x)$ *and, therefore, assumption (2.32) shows that* $\beta(x,t) \leq 1$ *for* $t \geq 0$, *a.e.* $x \in \Gamma_3$. *On the other hand, if* $\beta(x,t_0) = 0$ *at time* t_0, *then it follows from (2.45) that* $\dot\beta(x,t) = 0$ *for all* $t \geq t_0$ *and therefore,* $\beta(x,t) = 0$ *for all* $t \geq t_0$, *a.e.* $x \in \Gamma_3$. *We conclude that* $0 \leq \beta(x,t) \leq 1$ *for all* $t \in [0,T]$, *a.e.* $x \in \Gamma_3$.

First, we note that the functional j_{ad} and j_{vc} are linear with respect to the last argument and, therefore,

$$\begin{aligned} j_{ad}(\beta, \boldsymbol{u}, -\boldsymbol{v}) &= -j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}), \\ j_{vc}(\boldsymbol{u}, -\boldsymbol{v}) &= -j_{vc}(\boldsymbol{u}, \boldsymbol{v}). \end{aligned} \tag{2.47}$$

Next, using (2.38) and (2.28.b) imply

$$j_{vc}(\boldsymbol{u}_1, \boldsymbol{v}_2) - j_{vc}(\boldsymbol{u}_1, \boldsymbol{v}_1) + j_{vc}(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_{vc}(\boldsymbol{u}_2, \boldsymbol{v}_2) \leq 0. \tag{2.48}$$

Similar manipulations, based on the Lipschitz continuity of operators R_ν, $\boldsymbol{R}_\tau$ show that

$$|j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{v}) - j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{v})| \leq c\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{V}} \|\boldsymbol{v}\|_{\boldsymbol{V}}. \tag{2.49}$$

Next, using (2.39), (2.28)(a), keeping in mind (2.20), we obtain

$$\begin{aligned} &j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_2) - j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_1) + j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_2) \\ &\leq c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{V}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{V}}. \end{aligned} \tag{2.50}$$

3 Main Results

The main results are stated by the following theorems.

Theorem 3.1. *Assume that (2.23)–(2.32) hold. Then, there exists $\mu_0 > 0$ depending only on Ω^ℓ, Γ_1^ℓ, Γ_2^ℓ, Γ_3, p_ν, p_τ, and $\mathcal{A}^\ell$, $\ell = 1,2$ such that, if $\|\mu\| < \mu_0$, then Problem **PV** has a unique solution $\{\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \theta, \beta, \boldsymbol{D}\}$. Moreover, the solution satisfies*

$$\boldsymbol{u} \in C(0,T;\boldsymbol{V}), \tag{3.51}$$
$$\varphi \in C(0,T;W), \tag{3.52}$$
$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}, \tag{3.53}$$
$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1), \tag{3.54}$$
$$\theta \in L^2(0,T;E_1) \cap H^1(0,T;E_0), \tag{3.55}$$
$$\boldsymbol{D} \in W^{1,\infty}(0,T;\mathcal{W}). \tag{3.56}$$

The proof of Theorem 3.1 is carried out in several steps and is based on the following abstract result for variational inequalities.

Let X be a real Hilbert space, and consider the Problem of finding $\boldsymbol{u} \in X$ such that :

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_X + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}(t)) \geq (f, \boldsymbol{v} - \boldsymbol{u})_X \ \ \forall \boldsymbol{v} \in X. \tag{3.57}$$

To study problem (3.57) we need the following assumptions: The operator $A : X \to X$ is Lipschitz continuous and strongly monotone, *i.e.*,

$$\begin{cases} \text{(a) There exists } L_A > 0 \text{ such that} \\ \qquad \|A\boldsymbol{u}_1 - A\boldsymbol{u}_2\|_X \leq L_A \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in X, \\ \text{(b) There exists } m_A > 0 \text{ such that} \\ \qquad (A\boldsymbol{u}_1 - A\boldsymbol{u}_2,\ \boldsymbol{u}_1 - \boldsymbol{u}_2)_X \geq m_A \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in X. \end{cases} \tag{3.58}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) } j(\boldsymbol{u}, .) \text{ is convex and l.S.C. on } X \text{ for all } u \in X. \\ \text{(b) There exists } m_j > 0 \text{ such that} \\ \qquad j(\boldsymbol{u}_1, \boldsymbol{v}_2) - j(\boldsymbol{u}_1, \boldsymbol{v}_1) + j(\boldsymbol{u}_2, \boldsymbol{v}_1) - j(\boldsymbol{u}_2, \boldsymbol{v}_2) \\ \qquad\quad \leq m_j \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_X \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in X. \end{cases} \tag{3.59}$$

Finally, we assume that

$$f \in X. \tag{3.60}$$

The following existence, uniqueness result and regularity was proved in [13, p.51].

Theorem 3.2. *Let (3.57)–(3.60) hold, and $m_j < m_A$. Then:*

1. *There exists a unique solution $\boldsymbol{u} \in X$ of Problem (3.57).*
2. *If, moreover, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of (3.57) corresponding to the data $f_1, f_2 \in X$, then there exists $c > 0$ such that*

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X \leq c\|f_1 - f_2\|_X. \tag{3.61}$$

We turn now to the proof of Theorem 3.1 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments. To this end, we assume in what follows that (2.23)–(2.32) hold, and we consider that C is a generic positive constant which depends on Ω^ℓ, Γ_1^ℓ, Γ_1^ℓ, Γ_3, p_ν, p_τ, $\mathcal{A}^\ell$, $\mathcal{G}^\ell$, $\mathcal{Q}^\ell$, $\mathcal{E}^\ell$, γ_ν, γ_τ, Θ^ℓ, ϕ^ℓ, κ_0^ℓ, κ^ℓ, and T with $\ell = 1,2$. but does not depend on t nor of the rest of input data, and whose value may change from place to place.

In the first step. Let $\lambda \in C(0,T;E_0)$ and consider the auxiliary problem.

Problem PV$_\lambda$. Find $\theta_\lambda : [0,T] \to E_0$, such that

$$\textstyle\sum_{\ell=1}^2 (\dot{\theta}_\lambda^\ell(t) - \lambda^\ell(t) - \rho^\ell(t), \xi^\ell)_{L^2(\Omega^\ell)} + a_0(\theta_\lambda^\ell(t), \xi) = 0,\ \forall \xi \in E_0, \tag{3.62}$$
$$\theta_\lambda(0) = \theta_0. \tag{3.63}$$

Lemma 3.1. *There exists a unique solution θ_λ to the auxiliary problem $\boldsymbol{PV}_\lambda$ satisfying (3.55).*

Proof. Furthermore, by an application of the Poincaré-Friedrichs inequality, we can find a constant $c_0 > 0$ such that

$$\int_{\Omega^\ell} |\nabla\xi|^2 dx + \frac{\lambda_0^\ell}{\kappa_0^\ell}\int_{\Gamma^\ell} |\xi|^2 da \geq c_0 \int_{\Omega^\ell} |\xi|^2 dx, \quad \forall \xi \in E_1^\ell,\ \ell = 1,2.$$

Thus, we obtain

$$a_0(\xi,\xi) \geq c_1 \|\xi\|^2_{E_1}, \quad \forall \xi \in E_1,$$

where $c_1 = \kappa_0 \min(1, c_0)/2$, which implies that a_0 is E_1-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (3.62) has a unique solution θ_λ satisfying $\theta_\lambda(0) = \theta_0$ and the regularity (3.55). □

In the second step. Let $(\lambda, \eta) \in C(0, T; E_0 \times \boldsymbol{V})$, we use the θ_λ obtained in Lemma 3.1 and consider the auxiliary problem.

Problem PV$_{(\lambda,\eta)}$. Find $\boldsymbol{u}_{\lambda\eta} : [0, T] \to \boldsymbol{V}$, $\varphi_{\lambda\eta} : [0, T] \to W$, and $\beta_{\lambda\eta} : [0, T] \to L^2(\Gamma_3)$ such that

$$\begin{aligned}&\sum_{\ell=1}^{2}\Big(\mathcal{A}^\ell(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_{\lambda\eta}),\theta^\ell_\lambda),\ \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_{\lambda\eta}(t))\Big)_{\mathcal{H}^\ell}\\&+j_{vc}(\boldsymbol{u}_{\lambda\eta}(t),\boldsymbol{v}-\boldsymbol{u}_{\lambda\eta}(t))+j_{fr}(\boldsymbol{u}_{\lambda\eta}(t),\boldsymbol{v})-j_{fr}(\boldsymbol{u}_{\lambda\eta}(t),\boldsymbol{u}_{\lambda\eta}(t))\\&+(\eta(t),\boldsymbol{v}-\boldsymbol{u}_{\lambda\eta}(t))_{\boldsymbol{V}}\geq(\mathbf{f}(t),\boldsymbol{v}-\boldsymbol{u}_{\lambda\eta}(t))_{\boldsymbol{V}},\ \ \forall\boldsymbol{v}\in\boldsymbol{V},\end{aligned} \tag{3.64}$$

$$\textstyle\sum_{\ell=1}^2 \left(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell_{\lambda\eta}(t)) + \mathcal{G}^\ell E^\ell(\varphi^\ell_{\lambda\eta}(t)), \nabla\phi^\ell\right)_{H^\ell} = (-q(t), \phi)_W, \quad \forall \phi \in W, \tag{3.65}$$

$$\dot\beta_{\lambda\eta}(t) = -\Big(\beta_{\lambda\eta}(t)\big(\gamma_\nu(R_\nu([u_{\lambda\eta}\cdot(t)]))^2 + \gamma_\tau\left|\boldsymbol{R}_\tau([\boldsymbol{u}_{\lambda\eta\boldsymbol{o}}(t)])\right|^2\big) - \varepsilon_a\Big)_+, \tag{3.66}$$

$$\boldsymbol{u}_{\lambda\eta}(0) = \boldsymbol{u}_0, \ \ \beta_{\lambda\eta}(0) = \beta_0. \tag{3.67}$$

We have the following result

Lemma 3.2. *(1) There exists $\mu_0 > 0$ depending only on Ω^ℓ, Γ_1^ℓ, Γ_2^ℓ, Γ_3, p_ν, p_τ, and $\mathcal{A}^\ell$, $\ell = 1,2$ such that, if $\|\mu\| < \mu_0$, then Problem $\boldsymbol{PV}_{(\lambda,\eta)}$ has a unique solution $\{\boldsymbol{u}_{\lambda\eta}, \varphi_{\lambda\eta}, \beta_{\lambda\eta}\}$ which satisfies the regularity (3.51)–(3.53).*

(2) If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of (3.64) and (3.67) corresponding to the data (λ_1, η_1), $(\lambda_2, \eta_2) \in C(0, T; E_0 \times \boldsymbol{V})$, then there exists $c > 0$ such that, for $t \in [0, T]$,

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}} \leq c\|\eta_1(t) - \eta_2(t)\|_{\boldsymbol{V}}. \tag{3.68}$$

Proof. We apply Theorem 3.2 where $X = \boldsymbol{V}$, with the inner product $(\cdot,\cdot)_{\boldsymbol{V}}$ and the associated norm $\|.\|_{\boldsymbol{V}}$. Let $t \in [0, T]$. We use the Riesz representation theorem to define the operator $A : \boldsymbol{V} \to \boldsymbol{V}$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}} = \sum_{\ell=1}^{2} \left(\mathcal{A}^\ell(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \theta^\ell_\lambda),\ \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell}, \tag{3.69}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$, and define $\mathbf{f}_\eta \in X$ and the function $j : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$\mathbf{f}_\eta = \mathbf{f}(t) - \boldsymbol{\eta}(t), \tag{3.70}$$

$$j(\boldsymbol{u}, \boldsymbol{v}) = j_{vc}(\boldsymbol{u}, \boldsymbol{v}) + j_{fr}(\boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.71}$$

Assumptions (2.23) imply that the operators A satisfy conditions (3.58).
It follows from (2.28), (2.32), (2.38) and (2.39) that the functional j, (3.71), satisfies condition (3.59)(a). We use again (2.48), (2.50) and (3.71) to find

$$\begin{aligned}&j(\boldsymbol{u}_1, \boldsymbol{v}_2) - j(\boldsymbol{u}_1, \boldsymbol{v}_1) + j(\boldsymbol{u}_2, \boldsymbol{v}_1) - j(\boldsymbol{u}_2, \boldsymbol{v}_2)\\&\quad\leq c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{V}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{V}} \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{V},\end{aligned} \tag{3.72}$$

Using now (3.69)–(3.72) we find that (3.64) and (3.68) is a direct consequence of Theorem 3.2. Let now $t_1, t_2 \in [0,T]$, an argument based on (2.23), (2.49) and (2.50) shows that

$$\|\boldsymbol{u}_{\lambda\eta}(t_1) - \boldsymbol{u}_{\lambda\eta}(t_2)\|_V \le c\big(\|\lambda(t_1) - \lambda(t_2)\|_{E_0} + \|\eta(t_1) - \eta(t_2)\|_V + \|\mathbf{f}(t_1) - \mathbf{f}(t_2)\|_V\big). \tag{3.73}$$

Keeping in mind that $\mathbf{f} \in C(0,T;\boldsymbol{V})$ and recall that $(\lambda,\eta) \in C(0,T;E_0 \times \boldsymbol{V})$, it follows now from (3.73) that the mapping $\boldsymbol{u}_{\lambda\eta}$ satisfies the regularity (3.51).

Let us consider the form $G : W \times W \to \mathbb{R}$,

$$G(\varphi,\phi) = \sum_{\ell=1}^{2} (\mathcal{G}^\ell \nabla\varphi^\ell, \nabla\phi^\ell)_{H^\ell} \quad \forall \varphi,\phi \in W. \tag{3.74}$$

We use (2.21), (2.22), (2.27) and (3.74) to show that the form G is bilinear continuous, symmetric and coercive on W, moreover using (2.34) and the Riesz representation Theorem we may define an element $w_{\lambda\eta} : [0,T] \to W$ such that

$$(w_{\lambda\eta}(t),\phi)_W = (q(t),\phi)_W + \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell_{\lambda\eta}(t)), \nabla\phi^\ell)_{H^\ell} \quad \forall \phi \in W, t \in (0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_{\lambda\eta}(t) \in W$ such that

$$G(\varphi_{\lambda\eta}(t),\phi) = (w_{\lambda\eta}(t),\phi)_W \quad \forall \phi \in W. \tag{3.75}$$

It follows from (3.75) that $\varphi_{\lambda\eta}$ is a solution of the equation (3.65). Let $t_1, t_2 \in [0,T]$, it follows from (3.65) that

$$\|\varphi_{\lambda\eta}(t_1) - \varphi_{\lambda\eta}(t_2)\|_W \le C\big(\|\boldsymbol{u}_{\lambda\eta}(t_1) - \boldsymbol{u}_{\lambda\eta}(t_2)\|_V + \|q(t_1) - q(t_2)\|_W\big). \tag{3.76}$$

Now, from (2.29), (3.76) and $\boldsymbol{u}_{\lambda\eta} \in C(0,T;\boldsymbol{V})$, we obtain that $\varphi_{\lambda\eta} \in C(0,T;W)$.

On the other hand, we consider the mapping $H_{\lambda\eta} : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$,

$$H_{\lambda\eta}(t,\beta) = -\Big(\beta\big[\gamma_\nu(R_\nu([u_{\lambda\eta}(t)]))^2 + \gamma_\tau \Big|\boldsymbol{R}_\tau([\boldsymbol{u}_{\lambda\eta}(t)])\Big|^2\big] - \varepsilon_a\Big)_+,$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that $H_{\lambda\eta}$ is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \to H_{\lambda\eta}(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy-Lipschitz theorem (see [12, p.48], we deduce that there exists a unique function $\beta_{\lambda\eta} \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution of the equation (3.66). Also, the arguments used in Remark 2.1 show that $0 \le \beta_{\lambda\eta}(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Z}$, we find that $\beta_{\lambda\eta} \in \mathcal{Z}$. This completes the proof . □

In the third step, let us consider the element

$$\Lambda(\eta,\lambda)(t) = \big(\Lambda^1(\eta,\lambda)(t),\ \Lambda^2(\eta,\lambda)(t)\big) \in \boldsymbol{V} \times E_0, \tag{3.77}$$

defined by the equations

$$\begin{aligned}(\Lambda^1(\eta,\lambda)(t),\boldsymbol{v})_V = &-\sum_{\ell=1}^{2} \big((\mathcal{E}^\ell)^* E^\ell(\varphi^\ell_{\lambda\eta}),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell} + j_{ad}(\beta_{\lambda\eta}(t),\boldsymbol{u}_{\lambda\eta}(t),\boldsymbol{v}) \\ &+ \sum_{\ell=1}^{2} \Big(\int_0^t \mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_{\lambda\eta}(s)),\theta^\ell_\lambda(s)\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\Big)_{\mathcal{H}^\ell}, \quad \forall \boldsymbol{v} \in \boldsymbol{V},\end{aligned} \tag{3.78}$$

$$\Lambda^2(\eta,\lambda) = \Big(\Theta^1\big(\boldsymbol{\sigma}^1_{\lambda\eta},\boldsymbol{\varepsilon}(\boldsymbol{u}^1_{\lambda\eta}),\theta^1_\lambda\big),\ \Theta^2\big(\boldsymbol{\sigma}^2_{\lambda\eta},\boldsymbol{\varepsilon}(\boldsymbol{u}^2_{\lambda\eta}),\theta^2_\lambda\big)\Big), \tag{3.79}$$

where the mapping $\boldsymbol{\sigma}^\ell_{\lambda\eta}$ is given by

$$\boldsymbol{\sigma}^\ell_{\lambda\eta} = \mathcal{A}^\ell(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_{\lambda\eta}),\theta^\ell_\lambda) + \int_0^t \mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_{\lambda\eta}(s)),\theta^\ell_\lambda(s)\big)ds - (\mathcal{E}^\ell)^* E^\ell(\varphi^\ell_{\lambda\eta}). \tag{3.80}$$

Lemma 3.3. *The mapping* Λ *has a fixed point* $(\eta^*, \lambda^*) \in C(0,T;\boldsymbol{V} \times E_0)$.

Proof. Let (η_1, λ_1), $(\eta_2, \lambda_2) \in C(0,T;\boldsymbol{V} \times E_0)$ and denote by θ_i, $\boldsymbol{u}_i$, φ_i, β_i and $\boldsymbol{\sigma}_i$, the functions obtained in Lemmas 3.1, 3.2 and the relation (3.80), for $(\eta, \lambda) = (\eta_i, \lambda_i)$, $i = 1, 2$. Let $t \in [0, T]$. We use (2.26), (2.37) and the definition of R_ν, $\boldsymbol{R}_\tau$, we have

$$\|\Lambda^1(\eta_1,\lambda_1)(t) - \Lambda^1(\eta_2,\lambda_2)(t)\|_{\boldsymbol{V}}^2 \le \sum_{\ell=1}^{2} \|(\mathcal{E}^\ell)^*\nabla\varphi_1^\ell(t) - (\mathcal{E}^\ell)^*\nabla\varphi_2^\ell(t)\|_{\mathcal{H}^\ell}^2 +$$
$$\sum_{\ell=1}^{2}\int_0^t \left\|\mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}_1^\ell(s)), \theta_1^\ell(s)\big) - \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}_2^\ell(s)), \theta_2^\ell(s)\big)\right\|_{\mathcal{H}^\ell}^2 ds$$
$$+C\|\beta_1^2(t)R_\nu([u_{1\nu}(t)]) - \beta_2^2(t)R_\nu([u_{2\nu}(t)])\|_{L^2(\Gamma_3)}^2$$
$$+C\|\beta_1^2(t)\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(t)]) - \beta_2^2(t)\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(t)])\|_{L^2(\Gamma_3)}^2.$$

Therefore,

$$\|\Lambda^1(\eta_1,\lambda_1)(t) - \Lambda^1(\eta_2,\lambda_2)(t)\|_{\boldsymbol{V}}^2 \le C\Big(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\, ds +$$
$$\int_0^t \|\theta_1(s) - \theta_2(s))\|_{E_0}^2\, ds + \|\varphi_1(t) - \varphi_2(t)\|_W^2 + \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2\Big). \tag{3.81}$$

By similar arguments, from (3.79), (3.80) and (2.25) it follows that

$$\|\Lambda^2(\eta_1,\lambda_1)(t) - \Lambda^2(\eta_2,\lambda_2)(t)\|_{E_0}^2 \le C\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}}^2$$
$$+\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\, ds + \|\theta_1(t) - \theta_2(t))\|_{E_0}^2 +$$
$$\int_0^t \|\theta_1(s) - \theta_2(s))\|_{E_0}^2\, ds + \|\varphi_1(t) - \varphi_2(t)\|_W^2\Big). \tag{3.82}$$

It follows now from (3.81) and (3.82) that

$$\|\Lambda(\eta_1,\lambda_1)(t) - \Lambda(\eta_2,\lambda_2)(t)\|_{\boldsymbol{V}\times E_0}^2 \le C\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}}^2$$
$$+\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\, ds + \|\theta_1(t) - \theta_2(t))\|_{E_0}^2 + \int_0^t \|\theta_1(s) - \theta_2(s))\|_{E_0}^2\, ds$$
$$+\|\varphi_1(t) - \varphi_2(t)\|_W^2 + \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2\Big). \tag{3.83}$$

Also, from the Cauchy problem (3.66) we can write

$$\beta_i(t) = \beta_0 - \int_0^t \Big(\beta_i(s)\big(\gamma_\nu(R_\nu([u_{i\nu}(s)]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{i\tau}(s)])|^2\,\big) - \varepsilon_a\Big)_+ ds$$

and then

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \le C\int_0^t \|\beta_1(s)R_\nu([u_{1\nu}(s)])^2 - \beta_2(s)R_\nu([u_{2\nu}(s)])^2\|_{L^2(\Gamma_3)} ds$$
$$+ C\int_0^t \left\|\beta_1(s)\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(s)])|^2 - \beta_2(s)\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(s)])|^2\right\|_{L^2(\Gamma_3)} ds.$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we get

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \le C\Big(\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} ds + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{L^2(\Gamma_3)^d} ds\Big). \tag{3.84}$$

Next, we apply Gronwall's inequality and from the Sobolev trace theorem we obtain

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2 \le C\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{\boldsymbol{V}}^2 ds. \tag{3.85}$$

We use now (3.65), (2.21), (2.26) and (2.27) to find

$$\|\varphi_1(t)-\varphi_2(t)\|_W^2 \le C\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2. \tag{3.86}$$

From (3.62) we deduce that

$$(\dot\theta_1-\dot\theta_2,\theta_1-\theta_2)_{E_0}+a_0(\theta_1-\theta_2,\theta_1-\theta_2)+\big(\lambda_1-\lambda_2,\theta_1-\theta_2\big)_{E_0}=0.$$

We integrate this equality with respect to time, using the initial conditions $\theta_1(0)=\theta_2(0)=\theta_0$ and inequality $a_0(\theta_1-\theta_2,\theta_1-\theta_2)\ge 0$, to find

$$\frac{1}{2}\|\theta_1(t)-\theta_2(t)\|_{E_0}^2 \le \int_0^t \big(\lambda_1(s)-\lambda_2(s),\theta_1(s)-\theta_2(s)\big)_{E_0}\,ds,$$

which implies that

$$\|\theta_1(t)-\theta_2(t)\|_{E_0}^2 \le \int_0^t \|\lambda_1(s)-\lambda_2(s)\|_{E_0}^2\,ds+\int_0^t \|\theta_1(s)-\theta_2(s)\|_{E_0}^2\,ds.$$

This inequality combined with Gronwall's inequality leads to

$$\|\theta_1(t)-\theta_2(t)\|_{E_0}^2 \le C\int_0^t \|\lambda_1(s)-\lambda_2(s)\|_{E_0}^2\,ds \quad \forall t\in[0,T]. \tag{3.87}$$

We substitute (3.68), (3.85)-(3.86) in (3.83) to obtain

$$\|\Lambda(\eta_1,\lambda_1)(t)-\Lambda(\eta_2,\lambda_2)(t)\|_{\boldsymbol{V}\times E_0}^2 \le C\int_0^t \|(\eta_1,\lambda_1)(s)-(\eta_2,\lambda_2)(s)\|_{\boldsymbol{V}\times E_0}^2\,ds.$$

Reiterating this inequality m times we obtain

$$\|\Lambda^m(\eta_1,\lambda_1)-\Lambda^m(\eta_2,\lambda_2)\|_{C(0,T;\boldsymbol{V}\times E_0)}^2 \le \frac{C^mT^m}{m!}\|(\eta_1,\lambda_1)-(\eta_2,\lambda_2)\|_{C(0,T;\boldsymbol{V}\times E_0)}^2.$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\boldsymbol{V}\times E_0)$, and so Λ has a unique fixed point. □

Let $(\eta^*,\lambda^*)\in C(0,T;\boldsymbol{V}\times E_0)$, be the fixed point of Λ, and denote

$$\boldsymbol{u}_*=\boldsymbol{u}_{\lambda^*\eta^*},\quad \varphi_*=\varphi_{\lambda^*\eta^*},\quad \beta_*=\beta_{\lambda^*\eta^*},\quad \theta_*=\theta_{\lambda^*}, \tag{3.88}$$

$$\boldsymbol{\sigma}_*^\ell=\mathcal{A}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell),\theta_*^\ell\big)+\int_0^t \mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}_*^\ell(s)),\theta_*^\ell(s)\big)\,ds-(\mathcal{E}^\ell)^*E^\ell(\varphi_*^\ell), \tag{3.89}$$

$$\boldsymbol{D}_*^\ell=\mathcal{E}^\ell\varepsilon(\boldsymbol{u}_*^\ell)+\mathcal{G}^\ell\big(E^\ell(\varphi_*^\ell)\big). \tag{3.90}$$

We use : $\Lambda^1(\eta^*,\lambda^*)=\eta^*$ and $\Lambda^2(\eta^*,\lambda^*)=\lambda^*$, it follows:

$$\begin{aligned}(\eta^*(t),\boldsymbol{v})_{\boldsymbol{V}}=&-\sum_{\ell=1}^2\big((\mathcal{E}^\ell)^*E^\ell(\varphi_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}+j_{ad}(\beta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v})\\ &+\sum_{\ell=1}^2\left(\int_0^t \mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}_*^\ell(s)),\theta_*^\ell(s)\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell},\ \forall\boldsymbol{v}\in\boldsymbol{V},\end{aligned} \tag{3.91}$$

$$\lambda_*^\ell(t)=\Theta^\ell\big(\boldsymbol{\sigma}_*^\ell(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell(t)),\theta_*^\ell(t)\big),\ \ \ell=1,2. \tag{3.92}$$

Existence. We prove $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\theta_*,\beta_*,\boldsymbol{D}_*\}$ satisfies (2.40)–(2.46) and the regularites (3.51)–(3.56). Indeed, we write (3.64) for $(\eta,\lambda)=(\eta^*,\lambda^*)$ and use (3.88) to find

$$\begin{aligned}&\sum_{\ell=1}^2\big(\mathcal{A}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell),\theta_*^\ell\big),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell(t))\big)_{\mathcal{H}^\ell}+j_{vc}(\boldsymbol{u}_*(t),\boldsymbol{v}-\boldsymbol{u}_*(t))+j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{v})\\ &\quad -j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{u}_*(t))+(\eta^*(t),\boldsymbol{v}-\boldsymbol{u}_*(t))_{\boldsymbol{V}}\ge(\mathbf{f}(t),\boldsymbol{v}-\boldsymbol{u}_*(t))_{\boldsymbol{V}},\ \forall\boldsymbol{v}\in\boldsymbol{V}.\end{aligned} \tag{3.93}$$

Substitute (3.91) in (3.93) to obtain

$$\begin{aligned}
&\sum_{\ell=1}^{2}(\mathcal{A}^{\ell}(\boldsymbol{\varepsilon}(\boldsymbol{u}_*^{\ell}),\theta_*^{\ell}),\ \varepsilon(\boldsymbol{v}^{\ell})-\boldsymbol{\varepsilon}(\boldsymbol{u}_*^{\ell}(t)))_{\mathcal{H}^{\ell}}\\
&+\sum_{\ell=1}^{2}\left(\int_0^t \mathcal{Q}^{\ell}(t-s,\varepsilon(\boldsymbol{u}_*^{\ell}(s)),\theta_*^{\ell}(s),)\,ds,\ \varepsilon(\boldsymbol{v}^{\ell})-\boldsymbol{\varepsilon}(\boldsymbol{u}_*^{\ell}(t))\right)_{\mathcal{H}^{\ell}}\\
&+j_{ad}(\beta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v}-\boldsymbol{u}_*(t))+j_{vc}(\boldsymbol{u}_*(t),\boldsymbol{v}-\boldsymbol{u}_*(t))+j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{v})\\
&-j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{u}_*(t))-\sum_{\ell=1}^{2}((\mathcal{E}^{\ell})^*E^{\ell}(\varphi_*^{\ell}(t)),\ \varepsilon(\boldsymbol{v}^{\ell})-\boldsymbol{\varepsilon}(\boldsymbol{u}_*^{\ell}(t)))_{\mathcal{H}^{\ell}}\\
&\geq(\mathbf{f}(t),\boldsymbol{v}-\boldsymbol{u}_*(t))_{\boldsymbol{V}}\quad\forall\boldsymbol{v}\in\boldsymbol{V}\quad a.e.\ t\in[0,T],
\end{aligned}\tag{3.94}$$

and we substitute (3.92) in (3.62) to have

$$\sum_{\ell=1}^{2}(\dot{\theta}_*^{\ell}(t),\xi^{\ell})_{L^2(\Omega^{\ell})}+a_0(\theta_*^{\ell}(t),\xi)=\sum_{\ell=1}^{2}(\lambda_*^{\ell}(t)+\rho^{\ell}(t),\xi^{\ell})_{L^2(\Omega^{\ell})},\tag{3.95}$$

for all $\xi\in E_0$, a.e. $t\in(0,T)$.

We write now (3.66) for $(\eta,\lambda)=(\eta^*,\lambda^*)$ and use (3.88) to see that

$$\sum_{\ell=1}^{2}(\mathcal{G}^{\ell}E^{\ell}(\varphi_*^{\ell}(t)),\nabla\phi^{\ell})_{H^{\ell}}+\sum_{\ell=1}^{2}(\mathcal{E}^{\ell}\varepsilon(\boldsymbol{u}_*^{\ell}(t)),\nabla\phi^{\ell})_{H^{\ell}}=-(q(t),\phi)_W,\tag{3.96}$$

for all $\phi\in W$, a.e. $t\in(0,T)$. Additionally, we use $\boldsymbol{u}_{\lambda^*\mu^*\eta^*}$ in (3.66) and (3.88) to find

$$\dot{\beta}_*(t)=-\Big(\beta_*(t)\big(\gamma_\nu(R_\nu([u_{*\nu}(t)]))^2+\gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{*\tau}(t)])|^2\big)-\varepsilon_a\Big)_+,\tag{3.97}$$

a.e. $t\in[0,T]$. The relations (3.93)–(3.97), allow us to conclude now that $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\theta_*,\beta_*,\boldsymbol{D}_*\}$ satisfies (2.40)–(2.45). Next, (2.46) the regularity (3.51)–(3.53) and (3.55) follow from Lemmas 3.1 and 3.2. Since $\boldsymbol{u}_*$, φ_* and θ_* satisfies (3.51), (3.52) and (3.55), respectively, It follows from (3.89) that

$$\boldsymbol{\sigma}_*\in C(0,T;\mathcal{H}).\tag{3.98}$$

For $\ell=1,2$, we choose $\boldsymbol{v}=\boldsymbol{u}\pm\phi$ in (3.94), with $\phi=(\phi^1,\phi^2)$, $\phi^{\ell}\in D(\Omega^{\ell})^d$ and $\phi^{3-\ell}=0$, to obtain

$$\mathrm{Div}\,\boldsymbol{\sigma}_*^{\ell}(t)=-\boldsymbol{f}_0^{\ell}(t)\quad\forall t\in[0,T],\ \ \ell=1,2,\tag{3.99}$$

where $D(\Omega^{\ell})$ is the space of infinitely differentiable real functions with a compact support in Ω^{ℓ}. The regularity (3.54) follows from (2.29), (3.98) and (3.99). Let now $t_1,t_2\in[0,T]$, from (2.21), (2.26), (2.27) and (3.90), we conclude that there exists a positive constant $C>0$ verifying

$$\|\boldsymbol{D}_*(t_1)-\boldsymbol{D}_*(t_2)\|_H\leq C\left(\|\varphi_*(t_1)-\varphi_*(t_2)\|_W+\|\boldsymbol{u}_*(t_1)-\boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\right).$$

The regularity of $\boldsymbol{u}_*$ and φ_* given by (3.51) and (3.52) implies

$$\boldsymbol{D}_*\in C(0,T;H).\tag{3.100}$$

For $\ell=1,2$, we choose $\phi=(\phi^1,\phi^2)$ with $\phi^{\ell}\in D(\Omega^{\ell})^d$ and $\phi^{3-\ell}=0$ in (3.96) and using (2.34) we find

$$\mathrm{div}\,\boldsymbol{D}_*^{\ell}(t)=q_0^{\ell}(t)\quad\forall t\in[0,T],\ \ \ell=1,2.\tag{3.101}$$

Property (3.56) follows from (2.29), (3.100) and (3.101).

Finally we conclude that the weak solution $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\theta_*,\beta_*,\boldsymbol{D}_*\}$ of the problem **PV** has the regularity (3.51)–(3.56), which concludes the existence part of Theorem 3.1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator $\Lambda(.,.)$ defined by (3.78)-(3.79) and the unique solvability of the Problems PV_λ, and $\mathrm{PV}_{(\lambda,\eta)}$. □

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Hyers-Ulam-Rassias stability of nth order linear ordinary differential equations with initial conditions

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Abstract

In this paper, we investigate the stability of nth order linear ordinary differential non-homogeneous equation with initial conditions in the Hyers-Ulam-Rassias sense.

Keywords: Differential equation, Differential inequality, Hyers-Ulam-Rassias stability, Initial Value Problem, Integral Equation.

2010 MSC: 34A30, 34A40, 45D05.

1 Introduction

In 1940, S.M. Ulam while he was giving talk at Wisconsin University, he proposed the following problem: Under what conditions does there exist an additive mapping near an approximately additive mapping? for details see [18]. A year later, D.H. Hyers in [4] gave an answer to the problem of Ulam for additive functions defined on Banach spaces. Let E_1 and E_2 be two real Banach spaces and $f : X_1 \to X_2$ be a mapping. If there exist an $\epsilon \geq 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X_1$, then there exist a unique additive mapping $g : X_1 \to X_2$ with the property

$$\|f(x) - g(x)\| \leq \epsilon,$$

$\forall x \in X_1$. A generalized solution to Ulam's problem for approximately linear mappings was proved by Th.M. Rassias in 1978 [13]. He considered a mapping $f : E_1 \to E_2$ such that $t \to f(tx)$ is continuous in t for each fixed x. Assume that there exists $\theta \geq 0$ and $0 \geq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \left(\|x\|^p + \|y\|^p\right)$$

for any $x, y \in E_1$. After Hyers result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers result in various directions, see ([2], [5]).

Soon-Mo Jung [17], investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients. Miura et al [11], proved the Hyers-Ulam stability of the first-order linear differential equations of the form $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function, while Jung [14], proved the Hyers-Ulam stability of differential equations of the form $\varphi(t)y'(t) = y(t)$. Furthermore, the result of Hyers-Ulam stability for first-order linear differential equations has been generalized in ([15], [16], [19]).

A. Javadian, E. Sorouri, G.H. Kim and M. Eshaghi Gordji [6], investigated generalized Hyers-Ulam stability

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of the second order linear differential equations of the form $y'' + P(x)y' + q(x)y = f(x)$ with some conditions. Maher Nazmi Qorawani [10], investigated Hyers-Ulam stability of second order linear differential equations of the form $z'' + p(x)z' + (q(x) - \alpha(x))\, z = 0$ and nonlinear differential equations of the form $z'' + p(x)z' + q(x)z = h(x)\,|z|^{\beta}\, e^{(\frac{\beta-1}{2})\int p(x)dx} sgnz$ with initial conditions. Li and Yan [8], investigated the Hyers-Ulam Stability of nonhomogeneous second order Linear Differential Equations of the form $y'' + p(x)y' + q(x)y + r(x) = 0$ under some special conditions. Pasc Gavruta, Jung, Li [3], investigated the Hyers-Ulam stability for second order linear differential equations with boundary conditions of the form $y'' + \beta(x)y(x) = 0$. Jinghao Huang, Qusuay H. Alqifiary, and Yongjin Li [7], proved the generalized superstability of nth order linear differential equations with initial conditions of the form $y^{(n)}(x) + \beta(x)y(x) = 0$. Recently, M.I. Modebei, O.O. Olaiya, I. Otaide [12], investigated generalized Hyers-Ulam stability of second order linear ordinary differential equation $y'' + \beta(x)y = f(x)$ with initial condition.

In this paper, we investigate the Hyers-Ulam-Rassias Stability of nth order linear ordinary differential equations with initial condtions

$$y^{(n)} + \beta(x)y(x) = f(x)$$
$$y(a) = y'(a) = y''(a) = ... = y^{(n-1)}(a) = 0,$$

where $y \in C^n[a,b], \beta \in C[a,b]$ and $f : [a,b] \to \mathbb{R}$ continuous.

Let $(X, \|.\|)$ be a real or complex Banach space with $a, b \in \mathbb{R}$ where $-\infty < a < b < \infty$, ϵ be a positive real number. Let $y : (a,b) \to X$ be a continuouus function. We consider the following differential equation

$$y^{(n)}(t) = \sum_{k=0}^{n-1} P_k y^{(k)}(t), \quad t \in I \tag{1.1}$$

and the following differential inequality

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} P_k y^{(k)}(t) \right| \leq \epsilon, \quad t \in I \tag{1.2}$$

and

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} P_k y^{(k)}(t) \right| \leq \varphi(t), \quad t \in I \tag{1.3}$$

Definition 1.1. *The equation (1.1) is said to have the Hyers-Ulam stability for any $\epsilon > 0$, there exist a real number $K > 0$ such that for each approximate solution $y \in C^n(I, X)$ of (1.2) there exist a solution $y_0 \in C^n(I, X)$ of (1.1) with*

$$|y - y_0| \leq K\epsilon \quad \forall t \in I. \tag{1.4}$$

Definition 1.2. *The equation (1.1) is said to have the Hyers-Ulam-Rassias stability if there exist $\theta_\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$, such that for each approximate solution $y \in C^n(I, X)$ of (1.3) there exist a solution $y_0 \in C^n(I, X)$ of (1.1) with*

$$|y - y_0| \leq \theta_\varphi(t) \quad \forall t \in I. \tag{1.5}$$

Definition 1.3. *The equation $y^{(n)}(x) + \beta(x)y(x) = 0$ has the Hyers-Ulam stability with initial conditions $y(a) = y'(a) = ... = y^{(n-1)}(a) = 0$, if there exists a positive constant K with the following property: For every $\epsilon > 0, y \in C^n[a,b]$, if*

$$\left| y^{(n)}(x) + \beta(x)y(x) \right| \leq \epsilon, \tag{1.6}$$

and $y(a) = y'(a) = ... = y^{(n-1)}(a) = 0$, then there exists some $z \in C^n[a,b]$ satisfying $Z^{(n)} + \beta(x)z(x) = 0$ and $z(a) = z'(a) = ... = z^{(n-1)}(a) = 0$, such that

$$|y(x) - z(x)| \leq K\epsilon.$$

We need the following Lemma to prove our main results.

Lemma 1.1. ***(Generalized Replacement Lemma)*** *Suppose that $g : [a,b] \to \mathbb{R}$ is a continuous. Then*

$$\int_a^{s_{n-1}} \int_a^{s_{n-2}} ... \int_a^{s_2} \int_a^{s_1} \int_a^{x} g(s) ds ds_1 ds_2 ... ds_{n-1} = \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} g(s) ds, \forall x \in [a,b]$$

The details of the proof we can see [1].

Theorem 1.1. *If* $\max|\beta(x)| < \dfrac{n!}{(b-a)^n}$ *Then*

$$y^{(n)}(x) + \beta(x)y(x) = 0 \tag{1.7}$$

has the Hyers-Ulam stability with initial conditions

$$y(a) = y'(a) = ... = y^{(n-1)}(a) = 0 \tag{1.8}$$

where $y \in C^n[a,b], \beta \in C[a,b]$ *and* $f : [a,b] \to \mathbb{R}$ *continuous.*

Proof. For every $\epsilon > 0$, By using the Taylor formula, we have

$$y(x) = y(a) + y'(x-a) + ... + \frac{y^{(n)}(\xi)}{n!}(x-a)^n.$$

Thus

$$\begin{aligned} |y(x)| &= \left| \frac{y^{(n)}(\xi)}{n!}(x-a)^n \right| \\ &\le \max\left|y^{(n)}(x)\right| \frac{(b-a)^n}{n!} \qquad \forall x \in [a,b], \end{aligned}$$

then

$$\max|y(x)| \le \frac{(b-a)^n}{n!}\left[\max\left|y^{(n)}(x) - \beta(x)y(x) + \beta(x)y(x)\right|\right]$$

Now using (1.7), we obtain

$$\begin{aligned} \max|y(x)| &\le \frac{(b-a)^n}{n!}\left[\max\left|y^{(n)}(x) - \beta(x)y(x)\right| + \max|\beta(x)|\max|y(x)|\right] \\ &\le \frac{(b-a)^n}{n!}\epsilon + \frac{(b-a)^n}{n!}\max|\beta(x)|\max|y(x)|. \end{aligned}$$

Let $\eta = ((b-a)^n \max|\beta(x)|)/n!, \quad K = (b-a)^n/(n!(1-\eta))$. It is easy to see that $z_0(x) = 0$ is a solution of $y^{(n)}(x) - \beta(x)y = 0$ with initial conditions (1.8).

$$|y - z_0| \le K\epsilon.$$

Hence (1.7) has the Hyers-Ulam-Rassias stability with initial conditions (1.8). □

2 Main Result

In this section, we shall prove the Generalized Hyers-Ulam-Rassias Stability of the IVP

$$y^{(n)} + \beta(x)y(x) = f(x) \tag{2.9}$$

$$y(a) = y'(a) = y''(a) = ... = y^{(n-1)}(a) = 0, \tag{2.10}$$

where $y \in C^n[a,b], \beta \in C[a,b]$ and $f : [a,b] \to \mathbb{R}$ continuous.

Theorem 2.2. *Suppose* $|\beta(x)| < M$ *where* $M = \dfrac{n!}{(b-a)^n}$, $\varphi : [a,b] \to [0,\infty)$ *in an increasing function. The equation* (2.9) *has the Hyers-Ulam-Rassias stability if for* $\theta_\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and for each approximate solution* $y \in C^n[a,b]$ *of* (2.9) *satisfying*

$$\left|y^{(n)} - \beta(x)y(x) - f(x)\right| \le \varphi(x) \tag{2.11}$$

there exist a solution $z_0 \in C^n[a,b]$ *of* (2.9) *with condition* (2.10) *such that*

$$|y(x) - z_0(x)| \le \theta_\varphi(x). \tag{2.12}$$

Proof. From (2.11) we have that

$$-\varphi(x) \leq y^{(n)} - \beta(x)y(x) - f(x) \leq \varphi(x).$$

Integrating from a to x, and applying condition (2.10) we have

$$-\int_a^x \varphi(s)ds \leq y^{(n-1)}(x) - \int_a^x \beta(s)y(s)ds - \int_a^x f(s)ds \leq \int_a^x \varphi(s)ds.$$

On further integration and also applying condition (2.10) we have

$$-\int_a^{s_1}\int_a^x \varphi(s)dsds_1 \leq y^{(n-2)}(x) - \int_a^{s_1}\int_a^x \beta(s)y(s)dsds_1 - \int_a^{s_1}\int_a^x f(s)dsds_1 \leq \int_a^{s_1}\int_a^x \varphi(s)dsds_1.$$

Continuing the process finally we can get,

$$\begin{aligned}
-\int_a^{s_{n-1}}\int_a^{s_{n-2}}\dots\int_a^{s_2}\int_a^{s_1}\int_a^x \varphi(s)dsds_1\dots ds_{n-1}& \\
\leq y(x) - \int_a^{s_{n-1}}\int_a^{s_{n-2}}\dots\int_a^{s_2}\int_a^{s_1}\int_a^x \beta(s)y(s)dsds_1\dots ds_{n-1}& \\
-\int_a^{s_{n-1}}\int_a^{s_{n-2}}\dots\int_a^{s_2}\int_a^{s_1}\int_a^x f(s)dsds_1\dots ds_{n-1}& \\
\leq \int_a^{s_{n-1}}\int_a^{s_{n-2}}\dots\int_a^{s_2}\int_a^{s_1}\int_a^x \varphi(s)dsds_1\dots ds_{n-1}.&
\end{aligned}$$

Now applying Lemma (1.1), we obtain

$$-\int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds \leq y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\beta(s)y(s)ds - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}f(s)ds \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds.$$

Hence we have

$$\left|y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)y(s)ds + f(s)ds\right)\right| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds. \tag{2.13}$$

If we choose $z_0(x)$ such that it solves equation (2.9) with (2.10) such that

$$z_0(x) = \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)z_0(s)ds + f(s)ds\right),$$

thus we estimate

$$\begin{aligned}
|y(x) - z_0(x)| &\leq \left|y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)y(s)ds + f(s)ds\right)\right| \\
&+ \int_a^x \left|\frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)y(s)ds + f(s)ds\right) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)z_0(s)ds + f(s)\right)\right| ds
\end{aligned}$$

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds + \int_a^x \left|\frac{(x-s)^{n-1}}{(n-1)!}\beta(s)[y(s) - z_0(s)]\right| ds.$$

Now applying (2.13) and Theorem 1.1, we get

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds + |\beta(s)|\int_a^x \frac{(x-s)^{n-1}}{(n-1)!}|y(s) - z_0(s)|\,ds$$

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds + M\int_a^x \frac{(x-s)^{n-1}}{(n-1)!}|y(s) - z_0(s)|\,ds.$$

Applying Gronwall's inequality, we have

$$|y(x) - z_0(x)| \le \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds \exp\left\{M \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}ds\right\}$$
$$|y(x) - z_0(x)| \le \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds \exp\left\{M \left[\frac{(x-a)^n}{n!}\right]\right\}$$
$$|y(x) - z_0(x)| \le c\int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds$$

with

$$c = \exp\left\{\left[\frac{x-a}{b-a}\right]^n\right\}$$

and the proof is completed.

Remark: Note that as $x \to b$, then the above system considered is Hyers-Ulam stable. □

Conclusion

We obtained the Hyers-Ulam-Rassias stability of nth order linear ordinary differential nonhomogeneous equation with initial conditions. Hyers-Ulam-Rassias stability guarantees that there is a close exact solution of the system.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Hermite-Hadamard Type Inequalities for $(n, m, h_1, h_2, \varphi)$ −Convex Functions Via Fractional Integrals

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Abstract

In this paper, we obtain new generalizations for Hermite-Hadamard inequality by using Riemann-Liouville fractional integral and new type convex functions.

Keywords: Integral inequalities, Riemann-Liouville Fractional integral, Hermite-Hadamard Inequality, $(n, m, h_1, h_2, \varphi)$ −Convex Functions

2010 MSC: 26D15, 26A51, 26A33, 26A42.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

is known in the literature as Hermite-Hadamard inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f.

It is well known that the Hermite-Hadamard's inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [3, 5, 12, 13, 15, 16, 18, 20]) and the references there in.

Definition 1.1. ([9]) *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on I if inequality*

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b), \tag{1.2}$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

It is remarkable that Sarikaya et al. [11] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha > 0$.

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Definition 1.2. *([7]) Let $s \in (0,1]$. A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if*

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b), \tag{1.4}$$

holds for all $a, b \in I$ and $t \in [0,1]$. This class of $s-$convex functions is usually denoted by K_s^2.

Definition 1.3. *([10]) Let $(0,1) \subseteq J \subseteq \mathbb{R}$, $I \subseteq \mathbb{R}$ be an interval, and $h : I \to \mathbb{R}_0$ is said to be $h-$convex if the inequality*

$$f(ta + (1-t)b) \leq h(t) f(a) + h(1-t) f(b). \tag{1.5}$$

Definition 1.4. *([1, 8, 17]) Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integral $J_{a+}^{\alpha} f(x)$ and $J_{b-}^{\alpha} f(x)$ of order $\alpha \geq 0$ are defined by*

$$J_{a^+}^{\alpha} [f(x)] = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \quad x > a \tag{1.6}$$

and

$$J_{b^-}^{\alpha} [f(x)] = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt \quad x < b \tag{1.7}$$

respectively. Where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$ is Gamma function and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

We give the following properties:

$$J^{\alpha} J^{\beta} [f(t)] = J^{\alpha+\beta} [f(t)], \quad \alpha \geq 0, \beta \geq 0, \tag{1.8}$$

$$J^{\alpha} J^{\beta} [f(t)] = J^{\beta} J^{\alpha} [f(t)], \quad \alpha \geq 0, \beta \geq 0. \tag{1.9}$$

Definition 1.5. *([2]) A function f is said to be in the $L_p(a,b)$ space if*

$$L_p(a,b) = \left\{ f : \|f\|_{L_p} = \left(\int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty \right\}, \tag{1.10}$$

and for the case $p = \infty$

$$||f||_{\infty} = ess \sup_{a \leq t \leq b} |f(t)|. \tag{1.11}$$

Our goal in this paper is to state and prove the Hermite-Hadamard type inequality for convex functions. In order to achieve our goal, we give an important identity and then we prove some integral inequalities by using this identity.

In order to established main results, we first give following generalized definition.

In paper ([6]), (α, β, a, b) $-$convex functions are defined as solutions f of the functional inequality

$$f(\alpha(t) x + \beta(t) y) \leq a(t) f(x) + b(t) f(y)$$

where $0 \neq T \subseteq [0,1]$ and $\alpha, \beta, a, b : T \to \mathbb{R}$ are given functions. We introduce a definition of $(n, m, h_1, h_2, \varphi)$ $-$convex functions.

Definition 1.6. *Let $\varphi : [a,b] \subset \mathbb{R} \to [a,b]$. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$, $h_1, h_2 : [0,1] \to \mathbb{R}_0$, $m, n \in (0,1]$. Then f is said to be $(n, m, h_1, h_2, \varphi)$ $-$convex if the inequality*

$$f(nt\varphi(a) + m(1-t)\varphi(b)) \leq nh_1(t) f(\varphi(a)) + mh_2(t) f(\varphi(b)). \tag{1.12}$$

holds for all $a, b \in I$ and $t \in [0,1]$. If the inequality (1.12) reverses, then f is said to be $(n, m, h_1, h_2, \varphi)$ $-$concave on I.

Taking $\varphi(x) = x$, $h_1(t) = t$, $h_2(t) = 1-t$ and $m = n = 1$ in Definition 1.6, we obtain Definition 1.1,

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b).$$

Taking $\varphi(x) = x$, $h_1(t) = t$ and $h_2(t) = 1-t$ in Definition 1.6, we obtain $(n,m)-$convex functions in ([19]),

$$f(nta + m(1-t)b) \leq ntf(a) + m(1-t)f(b).$$

Taking $\varphi(x) = x$, $h_1(t) = t^{\beta}$ and $h_2(t) = 1 - t^{\alpha}$ in Definition 1.6, we obtain $(\beta, \alpha, n, m)-$convex functions in ([4]),

$$f(nta + m(1-t)b) \leq nt^{\beta} f(a) + m(1-t^{\alpha}) f(b).$$

The following Lemma will be used to established our main results:

Lemma 1.1. *([14]) Let $f : I \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$, then the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b)) + J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right] \\ &= \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(t\varphi(a) + (1-t)\varphi(b))dt. \end{aligned} \tag{1.13}$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \int_0^1 (1-t)^{\alpha} f'(t\varphi(a) + (1-t)\varphi(b))dt + \int_0^1 (-t^{\alpha}) f'(t\varphi(a) + (1-t)\varphi(b))dt \\ &= I_1 + I_2 \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^{\alpha} f'(t\varphi(a) + (1-t)\varphi(b))dt \\ &= (1-t)^{\alpha} \frac{f(t\varphi(a) + (1-t)\varphi(b))}{\varphi(a)-\varphi(b)}\bigg|_0^1 + \frac{\alpha}{\varphi(a)-\varphi(b)}\int_0^1 (1-t)^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{\varphi(b)-\varphi(a)}\int_0^1 (1-t)^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{(\varphi(b)-\varphi(a))^{\alpha+1}}\int_{\varphi(a)}^{\varphi(b)} (u-\varphi(a))^{\alpha-1} f(u)du \\ &= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}} J^{\alpha}_{\varphi(b)-}f(\varphi(a)), \end{aligned}$$

and similarly,

$$\begin{aligned} I_2 &= \int_0^1 (-t^{\alpha}) f'(t\varphi(a) + (1-t)\varphi(b))dt \\ &= (-t^{\alpha}) \frac{f(t\varphi(a) + (1-t)\varphi(b))}{\varphi(a)-\varphi(b)}\bigg|_0^1 + \frac{\alpha}{\varphi(a)-\varphi(b)}\int_0^1 t^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{\varphi(b)-\varphi(a)}\int_0^1 t^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{(\varphi(b)-\varphi(a))^{\alpha+1}}\int_{\varphi(a)}^{\varphi(b)} (\varphi(a)-u)^{\alpha-1} f(u)du \\ &= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}} J^{\alpha}_{\varphi(a)+}f(\varphi(b)). \end{aligned}$$

Thus can write,

$$I = I_1 + I_2 = \frac{f(\varphi(a)) + f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b)) + J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]$$

Multiplying the both sides by $\frac{\varphi(b)-\varphi(a)}{2}$, we obtain lemma which completes the proof. □

2 Main results

Theorem 2.1. *Let I be an interval $a,b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°. If $f' \in L_1([\varphi(a), \varphi(b)])$ for $\varphi(a), \varphi(b) \in I$, $n, m \in (0,1]$ and $h_1, h_2 : [0,1] \to \mathbb{R}_0$, then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b)) + J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\frac{2\left(1-2^{-\alpha}\right)}{\alpha+1}\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right| ||h_1||_{\infty} + m\left|f'\left(\frac{\varphi(b)}{m}\right)\right| ||h_2||_{\infty}\right). \tag{2.14}$$

Proof. From Lemma 1.1 and $(n, m, h_1, h_2, \varphi)$ −convexity of $|f'|$, we obtain

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b)) + J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right| \left|f'(t\varphi(a)+(1-t)\varphi(b))\right| dt$$

$$= \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right| \left|f'(nt\frac{\varphi(a)}{n} + m(1-t)\frac{\varphi(b)}{m})\right| dt$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right| \left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right| ||h_1||_{\infty} + m\left|f'\left(\frac{\varphi(b)}{m}\right)\right| ||h_2||_{\infty}\right) dt$$

$$= \frac{\varphi(b)-\varphi(a)}{2}\left\{\int_0^{1/2} \left[(1-t)^{\alpha} - t^{\alpha}\right] \left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right| ||h_1||_{\infty} + m\left|f'\left(\frac{\varphi(b)}{m}\right)\right| ||h_2||_{\infty}\right) dt\right.$$

$$\left.+ \int_{1/2}^1 \left[t^{\alpha} - (1-t)^{\alpha}\right] \left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right| ||h_1||_{\infty} + m\left|f'\left(\frac{\varphi(b)}{m}\right)\right| ||h_2||_{\infty}\right) dt\right\},$$

where

$$\int_0^{1/2} \left[(1-t)^{\alpha} - t^{\alpha}\right] dt = \int_{1/2}^1 \left[t^{\alpha} - (1-t)^{\alpha}\right] dt = \frac{1-2^{-\alpha}}{\alpha+1},$$

which completes the proof. □

Corollary 2.1. *Under the assumptions of Theorem 2.1 with $h_1(t) = h(t)$, $h_2(t) = h(1-t)$, , then the following inequality holds*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b)) + J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\frac{2-2^{1-\alpha}}{\alpha+1}||h_1||_{\infty}\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right| + m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\right).$$

Furthermore, if $n = m = 1$, then

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b)) + J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\frac{2-2^{1-\alpha}}{\alpha+1}||h_1||_{\infty}\left(\left|f'(\varphi(a))\right| + \left|f'(\varphi(b))\right|\right).$$

Corollary 2.2. *Under the assumptions of Corollary 2.1 with $h_1(t) = h(t) = t^s$, $n = m = 1$, then the following inequality holds*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\frac{2-2^{1-\alpha}}{\alpha+1}\frac{1}{s+1}\left(|f'(\varphi(a))|+|f'(\varphi(b))|\right).$$

Specially, $\alpha = s = n = m = 1$, then

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{(\varphi(b)-\varphi(a))}\int_{\varphi(a)}^{\varphi(b)} f(x)dx\right|$$
$$\leq \frac{\varphi(b)-\varphi(a)}{4}\left(|f'(\varphi(a))|+|f'(\varphi(b))|\right)$$

Theorem 2.2. *Let I be an interval $a, b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°. If $f' \in L_1([\varphi(a), \varphi(b)])$ for $\varphi(a), \varphi(b) \in I$, $n, m \in (0, 1]$ and $h_1, h_2 : [0, 1] \to \mathbb{R}_0$, then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right|+m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}. \tag{2.15}$$

Proof. From Lemma 1.1, Hölder inequality, and the $(n, m, h_1, h_2, \varphi)$ −convexity of $|f'|^q$, we obtain

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(t\varphi(a)+(1-t)\varphi(b))\right|dt$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(nt\frac{\varphi(a)}{n}+m(1-t)\frac{\varphi(b)}{m})\right|dt$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left[n\,|h_1(t)|\left|f'(\frac{\varphi(a)}{n})\right|+m\,|h_2(t)|\left|f'(\frac{\varphi(b)}{m})\right|\right]dt$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|^p dt\right)^{1/p}\left(\int_0^1 n^q\,|h_1(t)|^q\left|f'(\frac{\varphi(a)}{n})\right|^q dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|^p dt\right)^{1/p}\left(\int_0^1 m^q\,|h_2(t)|^q\left|f'(\frac{\varphi(b)}{m})\right|^q dt\right)^{1/q}\right\}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|^p dt\right)^{1/p}\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right| + m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left(\int_0^{1/2}\left[(1-t)^\alpha - t^\alpha\right]^p dt + \int_{1/2}^1 \left[t^\alpha-(1-t)^\alpha\right]^p dt\right)^{1/p}\right.$$

$$\left.\times\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right| + m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left(\int_0^{1/2}\left[(1-t)^{\alpha p} - t^{\alpha p}\right] dt + \int_{1/2}^1 \left[t^{\alpha p}-(1-t)^{\alpha p}\right] dt\right)^{1/p}\right.$$

$$\left.\times\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right| + m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right| + m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}$$

where

$$\int_0^{1/2}\left[(1-t)^{\alpha p} - t^{\alpha p}\right] dt = \int_{1/2}^1 \left[t^{\alpha p}-(1-t)^{\alpha p}\right] dt = \frac{1}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right),$$

which completes the proof. □

Corollary 2.3. *Under the assumptions of Theorem* 2.2 *with* $h_1(t) = h(t)$, $h_2(t) = h(1-t)$, *, then the following inequality holds*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\,||h_1||_q\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(n\left|f'(\frac{\varphi(a)}{n})\right| + m\left|f'(\frac{\varphi(b)}{m})\right|\right).$$

Furthermore, if $n = m = 1$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\,||h_1||_q\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))| + |f'(\varphi(b))|\right).$$

Corollary 2.4. *Under the assumptions of Corollary* 2.3 *with* $h_1(t) = h(t) = t^s$, $n = m = 1$ *, then the following inequality holds*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{sq+1}\right)^{1/q}\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))| + |f'(\varphi(b))|\right);$$

Specially, $\alpha = s = n = m = 1$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{1}{(\varphi(b)-\varphi(a))}\int_{\varphi(a)}^{\varphi(b)} f(x)dx\right|$$

$$\leq \frac{(\varphi(b)-\varphi(a))}{2}\left(\frac{1}{q+1}\right)^{1/q}\left\{\frac{2}{p+1}\left(1-\frac{1}{2^{p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))| + |f'(\varphi(b))|\right).$$

Corollary 2.5. *Under the assumptions of Corollary* 2.4 *with* $h_1(t) = h(t) = t,\ n = m = 1$ *and* $\varphi(x) = x,$ *, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right|$$

$$\leq \frac{b-a}{2}\left(\frac{1}{s+1}\right)^{1/q}\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(|f'(a)|+|f'(b)|\right);$$

Specially, $\alpha = s = n = m = 1$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)}{2}\left(\frac{1}{q+1}\right)^{1/q}\left\{\frac{2}{p+1}\left(1-\frac{1}{2^{p}}\right)\right\}^{1/p}\left(|f'(a)|+|f'(b)|\right).$$

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Intersection graph of subgroups of some non-abelian groups

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Abstract

The intersection graph of subgroups of a group G is a graph whose vertex set is the set of all proper subgroups of G and two distinct vertices are adjacent if and only if their intersection is non-trivial. In this paper, we obtain the clique number and degree of vertices of intersection graph of subgroups of dihedral group, quaternion group and quasi-dihedral group.

Keywords: Intersection graph, subgroups, non-abelian groups.

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1 Introduction

There are several graphs associated with algebraic structures to investigate some specific properties of algebraic structures. Among them the intersection graphs have its own importance, which have been studied in the literature over the past fifty years. In 1964, Bosak [1] initiated the study of the intersection graphs of semigroups. Later, Csákány and Pollák [5] defined the intersection graph of subgroups of finite group. Followed by this, Zelinca investigated the intersection graph of subgroups of a finite abelian group [7]. In the recent years, several interesting properties of the intersection graphs of subgroups groups have been obtained in the literature, see for instance [2], [4], [5], [6] and the references therein.

Let G be a group. *The intersection graph of subgroups of* G, denoted by $\mathscr{I}(G)$, is a graph with all the proper subgroups of G as its vertices and two distinct vertices in $\mathscr{I}(G)$ are adjacent if and only if the corresponding subgroups have a non-trivial intersection in G.

Let G be a simple graph. The *degree* of a vertex v in G, denoted by $deg_G(v)$ is the number of vertices to which v is adjacent. A *clique* of G is a complete subgraph of G. The *clique number* of G is the is the cardinality of a largest clique in G and it is denoted by $\omega(G)$.

For a positive integer n, $\tau(n)$ denotes the number of positive divisor of n; $\sigma(n)$ denotes the sum of all the positive divisors of n.

The aim of this paper is to find the clique number and degree of vertices of the intersection graph of subgroups of dihedral group, quaternion group and quasi-dihedral group.

We will use the following result of Chakrabarty *et al.* in the subsequent section.

Theorem 1.1. *([3]) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. If H is a proper subgroup of $\mathbb{Z}_n$ with $|H| = p_{i_1}^{\beta_{i_1}} p_{i_2}^{\beta_{i_2}} \dots p_{i_r}^{\beta_{i_r}}$, then $\deg_{\mathscr{I}(\mathbb{Z}_n)}(H) = \tau(n) - \prod_{j \notin \{i_1, i_2, \dots i_r\}} (\alpha_j + 1) - 3$.*

2 Properties of $\mathscr{I}(D_n)$, $\mathscr{I}(Q_n)$, $\mathscr{I}(QD_{2^\alpha})$

First, we start with the dihedral group. The dihedral group of order $2n$ $(n \geq 3)$ is defined by

$$D_n = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle.$$

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The subgroups of D_n are listed below:

(i) cyclic groups $H_0^r := \langle a^{\frac{n}{r}} \rangle$ of order r, where r is a divisor of n;

(ii) cyclic groups $H_i^1 := \langle a^i b \rangle$ of order 2, where $i = 1, 2, \ldots, n$;

(iii) dihedral groups $H_i^r := \langle a^{\frac{n}{r}}, a^i b \rangle$ of order $2r$, where r is a divisor of n, $r \neq 1, n$ and $i = 1, 2, \ldots, \frac{n}{r}$.

The number of subgroups of D_n listed in (i), (ii), (iii) are $\tau(n) - 1$, n, $\sigma(n) - n - 1$ respectively and so the total number of proper subgroups of D_n is $\tau(n) + \sigma(n) - 2$.

Theorem 2.2. *Let $n \geq 3$ be an integer with $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$, and let $r = p_{i_1}^{\beta_{i_1}} p_{i_2}^{\beta_{i_2}} \ldots p_{i_r}^{\beta_{i_r}}$ be a divisor of n. Then*

(1) $\deg_{\mathscr{I}(D_n)}(H_0^r) = \tau(n) + \sigma(n) - n - 3 - \prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (\alpha_j + 1) - \sum_{\substack{d|s \\ d \neq 1}} \frac{n}{d}$, *where* $s = \frac{n}{p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \ldots p_{i_k}^{\alpha_{i_k}}}$;

(2) *for each $r \neq 1, n$ and $i = 1, 2, \ldots, \frac{n}{r}$,* $\deg_{\mathscr{I}(D_n)}(H_i^r) = \tau(n) + \sigma(n) - n - 2 - \prod_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1)$;

(3) *for each $i = 1, 2, \ldots, n$,* $\deg_{\mathscr{I}(D_n)}(H_i^1) = \tau(n) - 2$.

Proof. (1): First we count the number of subgroups listed in (i) to which H_0^r is adjacent in $\mathscr{I}(D_n)$. Here $\langle a \rangle \cong \mathbb{Z}_n$, so by Theorem 1.1, H_0^r is adjacent with $\tau(n) - \prod_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1) - 2$ subgroups of $\mathbb{Z}_n$ including $\mathbb{Z}_n$. Clearly H_0^r is not adjacent with all the n subgroups of D_n listed in (ii). Finally, we count the number of subgroups listed in (iii) to which H_0^r is adjacent. For every divisor $d \neq 1$ of $s = \frac{n}{p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \ldots p_{i_k}^{\alpha_{i_k}}}$, H_0^r is not adjacent with H_i^d, $i = 1, 2, \ldots, \frac{n}{d}$; H_0^r is adjacent with each of the remaining proper subgroups of D_n listed in (iii). The total number of such subgroups is $\sigma(n) - n - 1 - \sum_{\substack{d|s \\ d \neq 1}} \frac{n}{d}$. Summing up all these values gives the degree of H_0^r.
(2): For each $r \neq 1, n$ and $i = 1, 2, \ldots, \frac{n}{r}$, H_0^r is the maximal cyclic subgroup of H_i^r and so the number of subgroups listed in (i) to which H_i^r is adjacent is the same as the number of subgroups listed in (i) to which H_0^r is adjacent including H_0^r. The number of such subgroup is $\tau(n) - \prod_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1) - 1$. Among the subgroups of D_n listed in (ii), H_i^r has exactly r subgroups as its subgroups and so H_i^r is adjacent with only these subgroups in the list. Finally, we count the number of subgroups listed in (iii) to which H_i^r is adjacent. For every divisor l of r, H_i^r is intersect with H_i^l; for every divisor d of s, $(d, r) = 1$ and so by chinese remainder theorem there exist an integer, let it be t such that H_t^1 is a subgroup of both H_i^r and H_i^d. So H_i^r adjacent with all the subgroups of D_n listed in (iii). The total number of such subgroups is $\sigma(n) - n - 1$. The degree of is just the sum of these three values.
(3): For each $i = 1, 2, \ldots, n$, the order of H_i^1 is 2. The number of subgroups of D_n contains H_i^1 is $\tau(n) - 2$ and H_i^1 is not intersect with remaining proper subgroups of D_n, since order of H_i^1 is prime. This completes the proof. □

Theorem 2.3. *For $n \geq 3$, $\omega(\mathscr{I}(D_n)) = \sigma(n) - n - 1 + \prod_{i=1}^{k} \alpha_i$.*

Proof. Take $\mathcal{A} := \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1 := \{H_i^r \mid r \mid n, r \neq 1, n, i = 1, 2, \ldots, \frac{n}{r}\}$ and $\mathcal{C}_2 := \bigcup \{\langle a^{\frac{n}{r}} \rangle \mid r \mid n, r \neq 1$ with r has every prime divisor of n as a factor$\}$. Clearly $\mathcal{A}$ is a maximal clique and $|\mathcal{A}| = |\mathcal{C}_1| + |\mathcal{C}_2| = (\sigma(n) - n - 1) + \prod_{i=1}^{k} \alpha_i$. Let $\mathcal{B}$ be another clique different from $\mathcal{A}$. Then $\mathcal{B}$ should contains either $\langle a^{\frac{n}{r}} \rangle$, for some $r \mid n$, $r \neq 1$ and r does not contains all the prime divisors of n or $\langle a^i b \rangle$, for some $i = 1, 2, \ldots, n$. If $\mathcal{B}$ contains the subgroup $\langle a^{\frac{n}{r}} \rangle$, for some $r \mid n$, $r \neq 1$ and r does not contains all the prime divisors of n, then let p_j be the prime divisor of n which is not a divisor of r. Here G has at least two subgroups of order $2p_j$ and so we cannot take the subgroups of order $2p_j$ in $\mathcal{B}$. It follows that $|\mathcal{B}| < |\mathcal{A}|$. If $\mathcal{B}$ contains the subgroup $\langle a^i b \rangle$, for some $i = 1, 2, \ldots, n$, then $\langle a^i b \rangle$ adjacent with $\tau(n) - 2$ and so we cannot take $\sigma(n) - \tau(n) + 1$ subgroups in $\mathcal{B}$. It follows that $|\mathcal{B}| < |\mathcal{A}|$. This completes the proof. □

Next, we consider the quaternion group. For any integer $n > 1$, the quaternion group of order $4n$, is defined by

$$Q_n = \langle a, b | a^{2n} = b^4 = 1, b^2 = a^n, ab = ba^{-1} \rangle.$$

The subgroups of Q_n are listed below:

(i) cyclic groups $H_{0,r} := \langle a^{\frac{2n}{r}} \rangle$, of order r, where r is a divisor of $2n$;

(ii) cyclic groups $H_{i,1} := \langle a^i b \rangle$ of order 4, where $i = 1, \ldots, n$;

(iii) quaternion groups $H_{i,r} := \langle a^{\frac{n}{r}}, a^i b \rangle$ of order $4r$, where r is a divisor of n, $i = 1, \ldots, \frac{n}{r}$.

The number of subgroups of Q_n listed in (i), (ii), (iii) are $\tau(2n) - 1$, n, $\sigma(n) - n - 1$ and so the total number of proper subgroups of Q_n is $\tau(2n) + \sigma(n) - 2$.

Theorem 2.4. *Let $n > 1$ be an integer with $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$, and let $r = p_{i_1}^{\beta_{i_1}} p_{i_2}^{\beta_{i_2}} \ldots p_{i_r}^{\beta_{i_r}}$ be a divisor of n.*

(1) *If r is even, then $\deg_{\mathscr{I}(Q_n)}(H_{0,r}) = \tau(2n) + \sigma(n) - 3 - \prod\limits_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1)$;*

(2) *If r is odd, then $\deg_{\mathscr{I}(Q_n)}(H_{0,r}) = \tau(2n) + \sigma(n) - n - 3 - \prod\limits_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1) - \sum\limits_{\substack{d|s \\ d \neq 1}} \frac{n}{d}$, where $s = \frac{n}{p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \ldots p_{i_k}^{\alpha_{i_k}}}$;*

(3) *For each $i = 1, \ldots, \frac{n}{r}$, $\deg_{\mathscr{I}(Q_n)}(H_{i,r}) = \tau(2n) + \sigma(n) - 3 - \prod\limits_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1)$;*

(4) *For each $i = 1, \ldots, n$, $\deg_{\mathscr{I}(Q_n)}(H_{i,1}) = \tau(2n) + \sigma(n) - 3 - \prod\limits_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1)$, where α_j's are powers of odd prime factors of n.*

Proof. (1)-(2): First we count the number of subgroups listed in (i) to which $H_{0,r}$ is adjacent. Here $\langle a \rangle \cong \mathbb{Z}_{2n}$, by Theorem 1.1, $H_{0,r}$ adjacent with $\tau(2n) - \prod\limits_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1) - 2$ subgroups of $\mathbb{Z}_{2n}$ including $\mathbb{Z}_{2n}$. Now we consider the following two cases:

Case a: r is even. Here Q_n has an unique subgroup of order 2 and so every subgroup of even order in Q_n are adjacent with each other, so $H_{0,r}$ is adjacent with $\sigma(n) - 1$ subgroups of Q_n excluding Q_n listed in (ii), (iii). This completes the proof of part (1).

Case b: r is odd. Clearly $H_{0,r}$ is not adjacent with all the n subgroups of Q_n listed in (ii), since order of $H_{1,r}$ is 4. Finally we count the number of subgroups listed in (iii) to which $H_{0,r}$ is adjacent. For every divisor $d \neq 1$ of $s = \frac{n}{p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \ldots p_{i_k}^{\alpha_{i_k}}}$, $H_{0,r}$ is not adjacent with $H_{i,d}$, $i = 1, 2, \ldots, \frac{n}{d}$; $H_{0,r}$ is adjacent with remaining proper subgroups of Q_n listed in (iii). The total number of such subgroups is $\sigma(n) - n - 1 - \sum\limits_{\substack{d|s \\ d \neq 1}} \frac{n}{d}$. This completes the proof of part (2).

(3): For each $i = 1, \ldots, \frac{n}{r}$, $H_{0,r}$ is the maximal cyclic subgroup of $H_{i,r}$ and so the number of subgroups listed in (i) to which $H_{i,r}$ is adjacent is the same as the number of subgroups listed in (i) to which $H_{0,r}$ is adjacent including $H_{0,r}$. The number of such subgroups is $\tau(n) - \prod\limits_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1) - 1$. Also Q_n has a unique subgroup of order 2 and so $H_{i,r}$ is adjacent with all the subgroups listed in (ii), (iii), since order of subgroups of Q_n listed in (ii), (iii) is even. The total number of such subgroups is $\sigma(n) - 1$. This completes the proof of part (3).

(4): Since Q_n has an unique subgroup of order 2, so $H_{i,1}$ is adjacent with all the subgroups listed in (ii), (iii). Also $H_{i,1}$ is adjacent all the even order subgroups of Q_n listed in (i). But $H_{i,1}$ is not adjacent with an odd order subgroups of Q_n listed in (i). The number of such subgroups is $\tau(2n) + \sigma(n) - \prod\limits_{j \notin \{i_1, i_2, \ldots i_r\}} (\alpha_j + 1) - 2$, where α_j's are powers of odd prime factors of n. This completes the proof of part (4). ☐

Theorem 2.5. *Let $n > 1$ be an integer and $2n = 2^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\omega(\mathscr{I}(Q_n)) = \sigma(n) + \alpha_1 \prod\limits_{i=2}^{k} (\alpha_i + 1) - 1$.*

Proof. Let $\mathcal{A}$ be the set of all even order subgroups of Q_n. Then $|\mathcal{A}| = \sigma(n) + \alpha_1 \prod_{i=2}^{k}(\alpha_i + 1) - 1$ and $\mathcal{A}$ is a maximal clique in $\mathscr{I}(Q_n)$. Let $\mathcal{B}$ be another clique different from $\mathcal{A}$. Then $\mathcal{B}$ should contains $\langle a^{\frac{2n}{r}} \rangle$, for some an odd divisor r of n, $r \neq 1$. Then $\mathcal{B}$ cannot contain the subgroups of order 4. It follows that $|\mathcal{B}| < |\mathcal{A}|$. This completes the proof. □

Finally, we consider the quasi-dihedral group. For any positive integer $\alpha > 3$, the quasi-dihedral group of order 2^α, is defined by

$$QD_{2^\alpha} = \langle a, b \mid a^{2^{\alpha-1}} = b^2 = 1, bab^{-1} = a^{2^{\alpha-2}-1} \rangle.$$

The proper subgroups of QD_{2^α} are listed below:

(i) cyclic groups $H_0^r = \langle a^{\frac{2^{\alpha-1}}{r}} \rangle$, where r is a divisor of $2^{\alpha-1}$, $r \neq 1$;

(ii) the dihedral group $H_1^{2^{\alpha-2}} = \langle a^2, b \rangle \cong D_{2^{\alpha-2}}$ and the dihedral subgroups H_i^r of $H_1^{2^{\alpha-2}}$, where r is a divisor of $2^{\alpha-2}$, $r \neq 2^{\alpha-2}$, $i \in \{1, 2, \ldots, \frac{2^{\alpha-2}}{r}\}$;

(iii) the quaternion group $H_{2,2^{\alpha-3}} = \langle a^2, ab \rangle \cong Q_{2^{\alpha-3}}$ and the quaternion subgroups $H_{i,r}$ of $H_{2,2^{\alpha-3}}$, where r is a divisor of $2^{\alpha-3}$, $r \neq 2^{\alpha-3}$, $i \in \{1, 2, \ldots, \frac{2^{\alpha-3}}{r}\}$.

The number of subgroups of QD_{2^α} listed in (i), (ii), (iii) are $\tau(2^{\alpha-1}) - 1$, $2^{\alpha-1} - 1$, $2^{\alpha-2} - 1$ and so the total number of proper subgroups of QD_{2^α} is $\alpha + 3(2^{\alpha-2} - 1)$.

Theorem 2.6. *If $\alpha \geq 4$, then*

(1) *for each divisor r of $2^{\alpha-1}$, $r \neq 1$, $\deg_{\mathscr{I}(QD_{2^\alpha})}(H_0^r) = \alpha + 2^{\alpha-1} - 4$;*

(2) *for each divisor r of $2^{\alpha-2}$, $r \neq 1$, $i = 1, 2, \ldots, \frac{2^{\alpha-2}}{r}$, $\deg_{\mathscr{I}(QD_{2^\alpha})}(H_i^r) = \alpha + 2^{\alpha-1} + r - 4$;*

(3) *for each divisor r of $2^{\alpha-3}$, $r \neq 1$, $i = 1, 2, \ldots, \frac{2^{\alpha-2}}{r}$, $\deg_{\mathscr{I}(QD_{2^\alpha})}(H_{i,r}) = \alpha + 2^{\alpha-1} - 4$;*

(4) *for $i = 2, 2^2, \ldots, 2^{\alpha-2}$, $\deg_{\mathscr{I}(QD_{2^\alpha})}(H_i^1) = \alpha - 2$;*

(5) *for $i = 1, 3, \ldots, 2^{\alpha-3}$, $\deg_{\mathscr{I}(QD_{2^\alpha})}(H_{i,1}) = \alpha + 2^{\alpha-1} - 4$.*

Proof. The only maximal subgroups of QD_2^α are $H_0^{2^{\alpha-1}}$, the dihedral subgroup $H_1^{2^{\alpha-2}}$ and quaternion subgroup $H_{2,2^{\alpha-3}}$. Here H_0^2 is a subgroup of all the subgroup of QD_{2^α} other than H_i^1, $i = 2, 2^2, \ldots, 2^{\alpha-2}$; also no subgroups listed in (i), (iii) are adjacent with H_i^1, $i = 2, 2^2, \ldots, 2^{\alpha-2}$. It follows that $\deg_{\mathscr{I}(QD_{2^\alpha})}(H_0^r) = \alpha + 3(2^{\alpha-2} - 1) - 2^{\alpha-2} - 1 = \alpha + 2^{\alpha-1} - 4$. Proofs of parts (3) and (5) are similar to the above.

Next, we count the number of subgroups of QD_{2^α} to which H_i^r is adjacent. By the above argument H_i^r is adjacent with all the subgroups listed in (i), (iii) and the dihedral subgroups of $H_1^{2^{\alpha-2}}$; also H_i^r has r subgroups of order 2 as its subgroups and so H_i^r adjacent with these subgroups, so $\deg_{\mathscr{I}(QD_{2^\alpha})}(H_i^r) = \alpha + 3(2^{\alpha-2} - 1) - (2^{\alpha-2} - r) - 1 = \alpha + 2^{\alpha-1} + r - 4$.

Finally, we count the number of subgroups of QD_{2^α} to which H_i^1 is adjacent. Note that $\deg_{\mathscr{I}(QD_{2^\alpha})}(H_i^1) = \deg_{\mathscr{I}(D_n)}(H_i^1) + 1$, since order of H_i^1 is 2, and it is not a subgroup of any subgroups of $H_{i,2^{\alpha-3}}$; $H_1^{2^{\alpha-2}}$ is also a vertex of $\mathscr{I}(QD_{2^\alpha})$. So by Theorem 2.2(3), we have $\deg_{\mathscr{I}(QD_{2^\alpha})}(H_i^1) = \tau(2^{\alpha-2}) - 1 = \alpha - 2$. Hence the proof. □

Theorem 2.7. *For $\alpha \geq 3$, $\omega(\mathscr{I}(QD_{2^\alpha})) = \alpha + 2^{\alpha-1} - 3$.*

Proof. Let $\mathcal{A}$ be the set of all subgroups of QD_{2^α} other than $\langle ba^i \rangle$, $i = 2, 2^2, \ldots, 2^{\alpha-2}$. Clearly $\mathcal{A}$ is a maximal clique in $\mathscr{I}(QD_{2^\alpha})$ and $|\mathcal{A}| = \alpha + 3(2^{\alpha-2} - 1) - 2^{\alpha-2} = \alpha + 2^{\alpha-1} - 3$. Let $\mathcal{B}$ be another clique in $\mathscr{I}(QD_{2^\alpha})$. Then $\mathcal{B}$ contains exactly one subgroup of the form $\langle ba^i \rangle$, $i = 2, 2^2, \ldots, 2^{\alpha-2}$. It follow that $|\mathcal{B}| < |\mathcal{A}|$, since for one cyclic subgroup in $\mathcal{B}$, we take more than one quaternion subgroups in $\mathcal{A}$. This completes the proof. □

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Results on Weighted Stepanov-like Pseudo Periodic Functions and Applications

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Abstract

In this article, we first established some new results on composition of weighted Stepanov-like pseudo periodic function of class r under a uniform continuity condition with respect to L^p norm. And then, we proved the existence and uniqueness of weighted pseudo periodic solutions to a semi-linear functional differential equation with finite delay under Stepanov-like nonlinear term.

Keywords: Weighted Stepanov-like pseudo periodic function of class r, L^p norm, differential equation with delay.

2010 MSC: 34G20.

1 Introduction

The periodic, almost periodic and almost automorphic solutions to differential equations can profoundly characterize the asymptotic behavior of the corresponding dynamic systems, which have gained great attention by many scholars [10, 17, 18]. Particularly, the concept of weighted pseudo periodicity was introduced in [3]; Xia presented the notion of weighted Stepanov-like pseudo periodicity in [20]. In order to investigate differential equations with delay, Xia in [21] further studied new types of functions so called weighted pseudo periodic of class r, weighted Stepanov-like pseudo periodic of class r, respectively. For more details on this topic, we refer to [1, 2, 4, 5, 7, 11–15, 19, 25, 26] and references therein.

The main purpose of present paper is to make a further investigation on the composition results for weighted Stepanov-like pseudo periodic function of class r. Considering the space of weighted Stepanov-like pseudo periodic function of class r with an integral norm coming from L^p norm, we first prove a new composition theorem for weighted Stepanov-like pseudo periodic function of class r under a uniform continuity condition with respect to the L^p norm suggested by [16]. And then, we apply the obtained results to prove the existence and uniqueness of weighted pseudo periodic solution to the following semi-linear delay differential equation with a weighted Stepanov-like pseudo periodic nonlinear term

$$u'(t) = Au(t) + f(t, u_t), \ \ t \in \mathbb{R}, \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$, $u_t \in \mathfrak{B}$ is defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r, 0]$, r is a no-negative constant, f and $\mathfrak{B}$ are specified in the later.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Section 3, we establish some new results on composition of weighted Stepanov-like pseudo periodic function of class r under a L^p norm uniform continuity condition. In Section 4, we prove the existence of pseudo periodic mild solutions to the existence and uniqueness of weighted pseudo periodic solutions to the equation (1.1) under Stepanov-like nonlinear term. An example is also given to illustrate the main results.

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2 Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces and $\mathbb{N}$, $\mathbb{R}$ stand for sets of natural numbers and real numbers, respectively. To facilitate the discussion below, we further introduce the following notations:

· $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$): The Banach spaces of bounded continuous function from $\mathbb{R}$ to $\mathbb{X}$ (respectively, from $\mathbb{R}\times\mathbb{Y}$ to $\mathbb{X}$) with the sup norm.

· $L^p(\mathbb{R}, \mathbb{X})$: The space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable function $f : \mathbb{R} \to \mathbb{X}$ such that $\|f\| \in L^p(\mathbb{R}, \mathbb{R})$.

· $L^p_{loc}(\mathbb{R}, \mathbb{X})$: The space of all classes of equivalence of measurable function $f : \mathbb{R} \to \mathbb{X}$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, \mathbb{X})$.

· $\mathfrak{B}$: The space $C([-r, 0], \mathbb{X})$ endowed with the sup norm $\|\psi\|_{\mathfrak{B}}$ on $[-r, 0]$.

Definition 2.1 ([21]). A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be anti-periodic if there exists a $\omega \in \mathbb{R}\setminus\{0\}$ with the property that $f(t+\omega) = -f(t)$ for all $t \in \mathbb{R}$. The least positive ω with this property is called the anti-periodic of f. The collection of such functions is denoted by $P_{\omega ap}(\mathbb{R}, \mathbb{X})$.

Definition 2.2 ([21]). A function $f \in C(\mathbb{R}, \mathbb{X})$ is called to be periodic if there exists a $\omega \in \mathbb{R}\setminus\{0\}$ such that $f(t+\omega) = f(t)$ for all $t \in \mathbb{R}$. The least positive ω with this property is called the periodic of f. The collection of those ω periodic functions is denoted by $P_\omega(\mathbb{R}, \mathbb{X})$.

Let $\mathbb{U}$ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $T > 0$ and for each $\rho \in \mathbb{U}$, we set $\mu(T, \rho) := \int_{-T}^{T} \rho(t)dt$. Thus the spaces of weights $\mathbb{U}_\infty$ and $\mathbb{U}_B$ are defined by

$$\mathbb{U}_\infty := \left\{\rho \in \mathbb{U} : \lim_{T\to\infty} \mu(T, \rho) = \infty\right\},$$

$$\mathbb{U}_B := \left\{\rho \in \mathbb{U}_\infty : \rho \text{ is bounded and } \inf_{t\in\mathbb{R}} \rho(t) > 0\right\}.$$

For a given $\rho_1, \rho_2 \in \mathbb{U}_\infty$, we define respectively

$$V(T, f, r, \rho_1, \rho_2) = \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\|\right) \rho_2 dt;$$

$$WPP_0(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2) := \left\{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \|f(t)\|\rho_2(t)dt = 0\right\};$$

$$WPP_0(\mathbb{R}, \mathbb{X}, , r, \rho_1, \rho_2) := \left\{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\|\right) \cdot\rho_2(t)dt = 0\right\};$$

$$WPP_0(\mathbb{R}\times\mathbb{Y}, \mathbb{X}, r, \rho_1, \rho_2) := \left\{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\|\right) \cdot\rho_2(t)dt = 0 \text{ and uniformly in } u \in \mathbb{Y}\right\}$$
$$= \left\{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} V(T, f, r, \rho_1, \rho_2) = 0 \text{ uniformly in } u \in \mathbb{Y}\right\}.$$

Definition 2.3 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called weighted pseudo anti-periodic for $\omega \in \mathbb{R}\setminus\{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_{\omega ap}(\mathbb{R}, \mathbb{X})$ and $\varphi \in WPP_0(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$. Denote by $WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$ the set of such function.

Definition 2.4 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called weighted pseudo periodic for $\omega \in \mathbb{R}\setminus\{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_\omega(\mathbb{R}, \mathbb{X})$ and $\varphi \in WPP_0(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$. Denote by $WPP_\omega(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$ the set of such function.

Definition 2.5 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called weighted pseudo anti-periodic of class r for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_{\omega ap}(\mathbb{R}, \mathbb{X})$ and $\varphi \in WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$. Denote by $WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ the set of such function.

Definition 2.6 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called weighted pseudo periodic of class r for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_\omega(\mathbb{R}, \mathbb{X})$ and $\varphi \in WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$. Denote by $WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ the set of such function.

Remark 2.1 ([21]). If $r = 0$, then the weighted pseudo anti-periodic function of class r reduces to the weighted pseudo anti-periodic function, the weighted pseudo periodic function of class r reduces to the weighted pseudo periodic function. That is, $WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, 0, \rho_1, \rho_2) = WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$, and $WPP_\omega(\mathbb{R}, \mathbb{X}, 0, \rho_1, \rho_2) = WPP_\omega(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$.

Let $\|\cdot\|_p$ denote the norm of space $L^p(0, 1; \mathbb{X})$ for $p \in [1, \infty)$, we give the following definitions.

Definition 2.7 ([6, 17]). Let $p \in [1, \infty)$. The space $BS^p(\mathbb{R}, \mathbb{X})$ of all Stepanov-like bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0, 1; \mathbb{X}))$, where f^b is the Bochner transform of f defined by $f^b(t, s) := f(t + s), t \in \mathbb{R}, s \in [0, 1]$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p d\tau \right)^{1/p} = \sup_{t \in \mathbb{R}} \|f(t + \cdot)\|_p.$$

For $\rho_1, \rho_2 \in U_\infty$, we list the following weighted ergodic space in $BS^p(\mathbb{R}, \mathbb{X})$:

$$R(T, f, r, \rho_1, \rho_2) = \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \left(\sup_{\theta \in [t-r, t]} \left(\int_\theta^{\theta+1} \|f_2(s)\|^p \cdot ds \right)^{\frac{1}{p}} \right) dt,$$

$$\begin{aligned} S^p WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2) := \Bigg\{ & f \in BS^p(\mathbb{R}, \mathbb{X}) : \lim_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \\ & \left(\sup_{\theta \in [t-r, t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) dt = 0 \Bigg\} \\ = \Bigg\{ & f \in BS^p(\mathbb{R}, \mathbb{X}) : \lim_{T \to \infty} R(T, f, r, \rho_1, \rho_2) = 0 \Bigg\}, \end{aligned}$$

$$\begin{aligned} S^p WPP_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, r, \rho_1, \rho_2) := \Bigg\{ & f \in BS^p(\mathbb{R}, \mathbb{X}) : \lim_{T \to \infty} R(T, f(\cdot, u), r, \rho_1, \rho_2) \\ & = 0 \text{ uniformly in } u \in \mathbb{Y} \Bigg\}. \end{aligned}$$

Definition 2.8 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo anti-periodic of class r (or S^p-weighted pseudo anti-periodic of class r) if there exist $\phi \in S^p WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ such that $g = f - \phi$ satisfied $g(t + \omega) + g(t) = 0$ a.e $t \in \mathbb{R}$. Denote by $S^p WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ the collection of such function.

Definition 2.9 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo periodic of class r (or S^p-weighted pseudo periodic of class r) if there exist $\phi \in S^p WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ such that $g = f - \phi$ satisfied $g(t + \omega) - g(t) = 0$ a.e $t \in \mathbb{R}$. Denote by $S^p WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ the collection of such function.

3 Results on composition theorem

The main aim of this section is to establish some new results on composition of weighted Stepanov-like pseudo periodic function of class r. We first list the following "uniform continuity condition" with respect to the L^p norm for a function h: $\mathbb{R} \times \mathbb{X} \to \mathbb{X}$ with $h(\cdot, u) \in L^p_{Loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{X}$, which was initially adopted in [16]:

(A1) For any $\varepsilon > 0$, there exists $\sigma > 0$ such that $x, y \in L^p(0,1,\mathbb{X})$ and $\|x-y\|_p < \sigma$ imply that

$$\|h(t+\cdot, x(\cdot)) - h(t+\cdot, y(\cdot))\|_p < \varepsilon, \quad t \in \mathbb{R}.$$

Definition 3.10 ([21]). Let $\rho_1, \rho_2 \in U_\infty$. The function ρ_1 is said to be equivalent to ρ_2(i.e. $\rho_1 \sim \rho_2$) if $\frac{\rho_1}{\rho_2} \in U_B$.

From arguments in [21], we know that the notation "$\sim$" is a binary equivalence relation on U_∞. For a given weight $\rho \in U_\infty$, its equivalence class can be denoted by $cl(\rho) = \{\varrho \in U_\infty : \rho \sim \varrho\}$. It is clear that $U_\infty = \bigcup_{\rho \in U_\infty} cl(\rho)$.

Let $\rho \in U_\infty$, $\tau \in \mathbb{R}$ be given, and define ρ^τ by $\rho^\tau(t) = \rho(t+\tau)$ for $t \in \mathbb{R}$. Denote [22]

$$U_T = \{\rho \in U_\infty : \rho \sim \rho^\tau \text{ for each } \tau \in \mathbb{R}\}.$$

In view of [21], we know the conclusion that for $\rho_1, \rho_2 \in U_T$ and $\inf_{T>0} \frac{\mu(T,\rho_1)}{\mu(T,\rho_2)} = \delta_0 > 0$, the space $(WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2), \|\cdot\|_\infty)$ is a Banach space with sup norm.

Using similar ideas as in [8, 9, 23, 24], one can easily show the following result.

Lemma 3.1. If $\rho_1, \rho_2 \in U_T$ and $\inf_{T>0} \frac{\mu(T,\rho_1)}{\mu(T,\rho_2)} = \delta_0 > 0$, then the decomposition of weighted Stepanov-like pseudo periodic function of class r is unique.

Lemma 3.2. Let $f \in BS^p(\mathbb{R}, \mathbb{X})$, $\rho_1, \rho_2 \in U_\infty$, $\sup_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} < \infty$, then $f \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$ if and only if for every $\varepsilon > 0$,

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,f)} \rho_2(t) \cdot dt = 0,$$

where $M(T,\varepsilon,f) = \left\{ t \in [-T,T] : \sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \geq \varepsilon \right\}$.

Proof. Sufficiency: From the statement of lemma it is clear that $\|f\|_{S^p} < \infty$ and for any $\varepsilon > 0$, there exist $T_0 > 0$ such that $T > T_0$,

$$\frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,f)} \rho_2(t) dt < \frac{\varepsilon}{\mathscr{M}+1}, \ \mathscr{M} = \|f\|_{S^p}.$$

Thus

$$\begin{aligned}
&\frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&:= \frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,f)} \left(\sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&+ \frac{1}{\mu(T,\rho_1)} \int_{[-T,T]\setminus M(T,\varepsilon,f)} \left(\sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&\leq \frac{\mathscr{M}}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,f)} \rho_2(t) dt + \frac{\varepsilon}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) dt \\
&\leq \frac{\mathscr{M}\varepsilon}{\mathscr{M}+1} + \frac{\varepsilon}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) dt \\
&\leq \varepsilon + \sup_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} \varepsilon.
\end{aligned}$$

So

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt = 0,$$

that is $f \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$.

Necessity: Suppose on the contrary that exist $\varepsilon_0 > 0$, such that

$$\frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon_0,f)} \rho_2(t) dt$$

does not converge to 0 as $T \to \infty$, then there exist $\delta > 0$ such that for each n,

$$\frac{1}{\mu(T,\rho_1)} \int_{M(T_n,\varepsilon_0,f)} \rho_2(t)dt \geq \delta, \text{ for some } T_n \geq n.$$

Then

$$\begin{aligned}
&\frac{1}{\mu(T_n,\rho_1)} \int_{-T_n}^{T_n} \left(\sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&\geq \frac{1}{\mu(T_n,\rho_1)} \int_{M(T_n,\varepsilon_0,f)} \left(\sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&\geq \frac{\varepsilon_0}{\mu(T_n,\rho_1)} \int_{M(T_n,\varepsilon_0,f)} \rho_2(t)dt \\
&\geq \varepsilon_0\delta,
\end{aligned}$$

which contradicts the fact that $f \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$, and the proof is complete. □

Lemma 3.3 ([16]). Let h be the function in (A1), and x: $\mathbb{R} \to \mathbb{X}$ with $\overline{x(\mathbb{R})}$ compact. For $\varepsilon > 0$, there exist a finite set $\{x_k\}_{k=1}^m \subset \overline{x(\mathbb{R})}$ such that

$$\|h(t+\cdot, x(t+\cdot))\|_p < \varepsilon + m \sup_{1\leq k\leq m} \|h(t+\cdot, x_k\|_p, \quad t \in \mathbb{R}.$$

Next, we establish main composition results for weighted Stepanov-like pseudo periodic function.

Theorem 3.1. Assume that $\rho_1, \rho_2 \in U_\infty, r \geq 0$, $f = g + \phi \in S^pWPP_\omega(\mathbb{R}\times\mathbb{X}, \mathbb{X}, r, \rho_1, \rho_2)$, $h = h_1 + h_2 \in S^pWPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ with $\overline{h_1(\mathbb{R})}$ compact, $g(t+\omega) - g(t) = 0$, $h_1(t+\omega) - h_1(t) = 0$. Assume g satisfies (A1), ϕ satisfies (A1) and $\{f(\cdot, z) : z \in \mathbb{K}\}$ is bounded in $S^pWPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ for any bounded $\mathbb{K} \subset \mathbb{X}$, then $f(\cdot, h(\cdot)) \in S^pWPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$.

Proof. Let $H_1(t) = g(t, h_1(t)), H_2(t) = f(t, h(t)) - f(t, h_1(t)), H_3(t) = \phi(t, h_1(t)), t \in \mathbb{R}$. Then

$$f(t,h(t)) = g(t,h_1(t)) + f(t,h(t)) - f(t,h_1(t)) + \phi(t,h_1(t)) = H_1(t) + H_2(t) + H_3(t).$$

Since

$$g(t+\omega, h_1(t+\omega)) = g(t, h_1(t+\omega)) = g(t, h_1(t)),$$

we have

$$H_1(t+\omega) - H_1(t) = 0.$$

Thus we need only to prove $H_2, H_3 \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$.

It is easy to see that $H_2 \in BS^p(\mathbb{R}, \mathbb{X})$ since $\{f(\cdot, z) : z \in \mathbb{K}\}$ is bounded in $S^pWPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ for any bounded $\mathbb{K} \subset \mathbb{X}$. Noticing that f satisfies (A1) since g and ϕ satisfy (A1), for $\varepsilon > 0$, let $\sigma > 0$ be given by (A1), then

$$\|H_2(t+\cdot)\|_p = \|f(t+\cdot, h(\cdot)) - f(t+\cdot, h_1(\cdot))\|_p < \varepsilon, \text{ for } \|h_2(t+\cdot)\| < \sigma.$$

This implies that $M_{T,\varepsilon}(H_2) \subset M_{T,\sigma}(h_2)$ by the notation defined in Lemma 3.2. Meanwhile since $h_2 \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$, $\sup_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} < \infty$ by Lemma 3.2,

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{M(T,\sigma,h_2)} \rho_2(t) \cdot dt = 0.$$

Thus

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,H_2)} \rho_2(t) \cdot dt = 0,$$

which shows that $H_2 \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$.

For $\varepsilon > 0$, let σ be given by (A1) with ϕ, in view of Lemma 3.3, there is a finite set $\{x_k\}_{k=1}^m \subset \overline{h_1(\mathbb{R})}$ such that for $t \in \mathbb{R}$,

$$\|\phi(t+\cdot, h_1(t+\cdot))\|_p < \varepsilon + m \sup_{1\leq k\leq m} \|\phi(t+\cdot, x_k\|_p, \quad t \in \mathbb{R},$$

so

$$\|\phi(t,h_1(t)\|_p < \varepsilon + m \sup_{1\le k\le m} \|\phi(t,x_k\|_p, \quad t\in\mathbb{R}.$$

Since $\phi(\cdot,x)\in S^pWPP_0(\mathbb{R},L^p(0,1,\mathbb{X}),r,\rho_1,\rho_2)$, for each $x\in\mathbb{X}$, there is $T>T_0$, $1\le K\le m$,

$$\begin{aligned} R\left(T,\phi(\cdot,x_k),r,\rho_1,\rho_2\right) &= \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_\theta^{\theta+1}\|\phi(s,x_k)\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)\cdot dt \\ &< \frac{\varepsilon}{m}. \end{aligned}$$

Then for $T>T_0$,

$$\begin{aligned} R\left(T,H_3,r,\rho_1,\rho_2\right) &:= \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_\theta^{\theta+1}\|\phi(s,h_1(\theta))\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)\cdot dt \\ &\le \varepsilon + m\sup_{1\le k\le m}\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_\theta^{\theta+1}\|\phi(s,x_k)\|^p ds\right)^{\frac{1}{p}}\right) \\ &\rho_2(t)\cdot dt \\ &= \varepsilon + mR\left(T,\phi(\cdot,x_k),r,\rho_1,\rho_2\right) \\ &= \varepsilon + m\cdot\frac{\varepsilon}{m} \\ &= 2\varepsilon. \end{aligned}$$

This yields that

$$\lim_{T\to\infty} R(T,H_3,r,\rho_1,\rho_2) = 0.$$

That is $H_3\in S^pWPP_0(\mathbb{R},L^P(0,1,\mathbb{X}),r,\rho_1,\rho_2)$. The proof is complete. □

According to Theorem 3.1, we can obtain the following corollaries.

Corollary 3.1. Assume that $\rho_1,\rho_2\in U_\infty, r\ge 0$, $f=g+\phi\in S^pWPP_{\omega ap}(\mathbb{R}\times\mathbb{X},\mathbb{X},r,\rho_1,\rho_2)$, $h=h_1+h_2\in S^pWPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$ with $\overline{h_1(\mathbb{R})}$ compact. $g(t+\omega)+g(t)=0$, $h_1(t+\omega)+h_1(t)=0$. Assume g satisfies (A1), ϕ satisfies (A1) and $\{f(\cdot,z): z\in\mathbb{K}\}$ is bounded in $S^pWPP_{\omega ap}(\mathbb{R}\times\mathbb{X},\mathbb{X},r,\rho_1,\rho_2)$ for any bounded $\mathbb{K}\subset\mathbb{X}$, then $f(\cdot,h(\cdot))\in S^pWPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$.

Corollary 3.2. Assume that $\rho_1,\rho_2\in U_\infty, r\ge 0$, $f=g+\phi\in S^pWPP_\omega(\mathbb{R}\times\mathbb{X},\mathbb{X},r,\rho_1,\rho_2)$, $h\in WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$. Assume g satisfies (A1), ϕ satisfies (A1) and $\{f(\cdot,z): z\in\mathbb{K}\}$ is bounded in $S^pWPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$ for any bounded $\mathbb{K}\subset\mathbb{X}$, then $f(\cdot,h(\cdot))\in S^pWPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$.

Corollary 3.3. Assume that $\rho_1,\rho_2\in U_\infty, r\ge 0$, $f=g+\phi\in S^pWPP_{\omega ap}(\mathbb{R}\times\mathbb{X},\mathbb{X},r,\rho_1,\rho_2)$, $h\in WPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$. Assume g satisfies (A1), ϕ satisfies (A1) and $\{f(\cdot,z): z\in\mathbb{K}\}$ is bounded in $S^pWPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$ for any bounded $\mathbb{K}\subset\mathbb{X}$, then $f(\cdot,h(\cdot))\in S^pWPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$.

4 Weighted pseudo periodic mild solution

In this section, we deal with weighted pseudo periodic mild solutions to the problem (1.1). We list the following basic assumptions:

(A2) $\rho_1,\rho_2\in U_T$, $\inf_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)}>0$ and $\sup_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)}<\infty$.

(A3) The operator $T(t)$ generated by A is exponentially stable, that is, there exist constants M, $c>0$ such that $\|T(t)\|\le Me^{-ct}$ for $t\ge 0$.

(A4) $f\in S^pWPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, and there exist a positive constant L_f such that for $\psi_i\in\mathfrak{B}$, $i=1,2$, $\|f(t,\psi_1)-f(t,\psi_2)\|_p\le L_f\|\psi_1-\psi_2\|_{\mathfrak{B}}$.

Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote

$$\alpha_0 = M\left(\frac{e^{qc-1}}{qc}\right)^{\frac{1}{q}}, \quad \alpha = \alpha_0 \sum_{k=1}^{\infty} e^{-ck}.$$

Under the condition (A3), we give the following definition.

Definition 4.11. A function u: $\mathbb{R} \to \mathbb{X}$ is said to be a mild solution to the problem (1.1) if

$$u(t) = \int_{-\infty}^{t} T(t-s) f(s, u_s) ds,$$

for all $t \in \mathbb{R}$.

Lemma 4.4. Let $\rho_1, \rho_2 \in U_T$,$u \in S^p WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$, then u_t belongs to $S^p WPP_\omega(\mathbb{R}, \mathfrak{B}, r, \rho_1, \rho_2)$.

Proof. Suppose that $u = \alpha + \beta$, where $\alpha \in P_\omega(\mathbb{R}, \mathbb{X})$ and $\beta \in S^p WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$, then $u_t = \alpha_t + \beta_t$ and $\alpha_t \in P_\omega(\mathbb{R}, \mathfrak{B}, r, \rho_1, \rho_2)$. On the other hand, for $T > 0$, we see that

$$\frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \sup_{\tau \in [-r,0]} \|\beta(s+\tau)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt$$

$$\leq \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-2r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt$$

$$\leq \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-2r,t-r]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right.$$

$$\left. + \sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt$$

$$\leq \frac{1}{\mu(T,\rho_1)} \int_{-T-r}^{T-r} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t+\tau) dt$$

$$+ \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt$$

$$\leq \frac{\mu(T+r,\rho_1)}{\mu(T,\rho_1)} \cdot \frac{1}{\mu(T+r,\rho_1)}$$

$$\int_{-T-r}^{T-r} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) \frac{\rho_2(t+r)}{\rho_2(t)} dt$$

$$+ \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt.$$

The fact $\rho_1, \rho_2 \in U_T$ implies that there exists $\eta > 0$ such that

$$\frac{\rho_1(t+r)}{\rho_1(t)} \leq \eta, \quad \frac{\rho_1(t-r)}{\rho_1(t)} \leq \eta, \quad \frac{\rho_2(t-r)}{\rho_2(t)} \leq \eta.$$

For $T > r$,

$$\begin{aligned}
\mu(T+r,\rho_1) &= \int_{-T-r}^{T-r} \rho_1(t) dt + \int_{T-r}^{T+r} \rho_1(t) dt \\
&\leq \int_{-T-r}^{T-r} \rho_1(t) dt + \int_{-T+r}^{T+r} \rho_1(t) dt \\
&= \int_{-T}^{T} \rho_1(t-r) dt + \int_{-T}^{T} \rho_1(t+r) dt \\
&\leq 2\eta \mu(T,\rho_1),
\end{aligned}$$

then

$$\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_{\theta}^{\theta+1}\sup_{\tau\in[-r,0]}\|\beta(s+\tau)\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)dt$$
$$\leq \frac{2\eta^2}{\mu(T+r,\rho_1)}\int_{-T-r}^{T+r}\left(\sup_{\theta\in[t-r,t]}\left(\int_{\theta}^{\theta+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)dt$$
$$+\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_{\theta}^{\theta+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)dt.$$

Note that $\beta\in S^pWPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, $\rho_1,\rho_2\in U_T$, then $\beta_t\in S^pWPP_0(\mathbb{R},\mathfrak{B},r,\rho_1,\rho_2)$. Therefore $u_t\in S^pWPP_\omega(\mathbb{R},\mathfrak{B},r,\rho_1,\rho_2)$. □

From the proof of Lemma 4.4, we can easily deduce the following corollary.

Corollary 4.4. ([21, Lemma 2.14]) Let $\rho_1,\rho_2\in U_T$,$u\in WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, then u_t belong to $WPP_\omega(\mathbb{R},\mathfrak{B},r,\rho_1,\rho_2)$.

Lemma 4.5 ([21]). Let $\phi_n\to\phi$ uniformly on $\mathbb{R}$ where each $\phi_n\in WPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$,$\rho_1,\rho_2\in U_\infty$, if $\sup_{T>0}\frac{\mu(T,\rho_1)}{\mu(T,\rho_2)}<\infty$, then $\phi\in WPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$.

Lemma 4.6. Assume that (A2)–(A3) hold, if $\phi\in S^pWPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, then

$$(\Gamma\phi)(t)=\int_{-\infty}^{t}T(t-s)\phi(s)ds\in WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2).$$

Proof. By $\phi\in S^pWPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, we let $\phi(s)=\phi_1(s)+\phi_2(s)$, where $\phi_2\in S^pWPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$ and $\phi_1(t+\omega)-\phi_1(t)=0$ a.e. $t\in\mathbb{R}$, then

$$(\Gamma\phi)(t)=\int_{-\infty}^{t}T(t-s)\phi_1(s)ds+\int_{-\infty}^{t}T(t-s)\phi_2(s)ds=(\Gamma_1\phi_1)(t)+(\Gamma_2\phi_2)(t).$$

First, we show that $\Gamma_2\phi_2\in WPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$. Consider the integrals

$$X_n(t)=\int_{t-n}^{t-n+1}T(t-s)\phi_2(s)ds.$$

Fix $n\in\mathbb{N}$ and $t\in\mathbb{R}$, we have

$$\|X_n(t+h)-X_n(t)\|\leq\int_{n-1}^{n}\|T(s)(\phi_2(t+h-s)-\phi_2(t-s))\|ds$$
$$\leq M\int_{t-n}^{t-n+1}\|\phi_2(s+h)-\phi_2(s)\|ds$$
$$\leq M\left(\int_{t-n}^{t-n+1}\|\phi_2(s+h)-\phi_2(s)\|^p ds\right)^{\frac{1}{p}}.$$

In view of $\phi_2\in L^p_{loc}(\mathbb{R},\mathbb{X})$, we get

$$\lim_{h\to 0}\int_{t-n}^{t-n+1}\|\phi_2(s+h)-\phi_2(s)\|^p ds=0,$$

which yields $\lim_{h\to 0}\|X_n(t+h)-X_n(t)\|$. This means that $X_n(t)$ is continuous.

By Hölder's inequality, one has

$$\begin{aligned}
\|X_n(t)\| &\leq \int_{n-1}^{n} \|T(s)\phi_2(t-s)\|ds \\
&\leq \int_{n-1}^{n} Me^{-cs}\|\phi_2(t-s)\|ds \\
&\leq Me^{-c(n-1)} \int_{n-1}^{n} \|\phi_2(t-s)\|ds \\
&\leq Me^{-c(n-1)} \int_{t-n}^{t-n+1} \|\phi_2(s)\|ds \\
&\leq Me^{-c(n-1)} \left(\int_{t-n}^{t-n+1} \|\phi_2(s)\|^p ds \right)^{\frac{1}{p}} \\
&\leq Me^{-c(n-1)}\|\phi_2\|_{S^p}.
\end{aligned}$$

Since

$$\sum_{n=1}^{\infty} Me^{-c(n-1)}\|\phi_2\|_{S^p} \leq \frac{M}{1-e^{-\delta}}\|\phi_2\|_{S^p} < +\infty,$$

it follows that $\sum_{n=1}^{\infty} X_n(t)$ converges uniformly on $\mathbb{R}$. Let $X(t) = \sum_{n=1}^{\infty} X_n(t)$ for $t \in \mathbb{R}$. Then

$$X(t) = (\Gamma_2\phi_2)(t) = \int_{-\infty}^{t} T(t-s)\phi_2(s)ds, \ t \in \mathbb{R}\,.$$

It is obvious that $X(t) \in BC(\mathbb{R}, \mathbb{X})$. So, we only need to show that

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) \left(\sup_{\theta\in[t-r,t)} \|X(\theta)\| \right) dt = 0. \tag{4.2}$$

In fact, by Hölder's inequality,

$$\begin{aligned}
\|X_n(t)\| &\leq \int_{n-1}^{n} Me^{-cs}\|\phi_2(t-s)\|ds \\
&\leq \widetilde{M} \int_{t-n}^{t-n+1} \|\phi_2(s)\|ds \\
&\leq \widetilde{M} \left(\int_{t-n}^{t-n+1} \|\phi_2(s)\|^p ds \right)^{\frac{1}{p}},
\end{aligned}$$

for some constant $\widetilde{M} > 0$, then

$$\begin{aligned}
&\frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) \left(\sup_{\theta\in[t-r,t)} \|X_n(\theta)\| \right) dt \\
&\leq \frac{\widetilde{M}}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) \left(\sup_{\theta\in[t-r,t)} \left(\int_{\theta-n}^{\theta-n+1} \|\phi_2(s)\|^p ds \right)^{\frac{1}{p}} \right) dt,
\end{aligned}$$

and hence $X_n \in WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ since $\phi_2 \in S^pWPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$. By Lemma 4.5, the equation (4.2) holds, whence $\Gamma_2\phi_2 \in WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$.

From $\phi_1(t+\omega) - \phi_1(t) = 0$ a.e. $t \in \mathbb{R}$, one has

$$(\Gamma_1\phi_1)(t+\omega) = \int_{-\infty}^{t+\omega} T(t+\omega-s)\phi_1(s)ds = (\Gamma_1\phi_1)(t), \text{ a.e. } t \in \mathbb{R}\,.$$

Hence $\Gamma\phi \in WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$. This completes of the proof. □

Theorem 4.2. Let conditions (A2)–(A4) hold, then the problem (1.1) has a unique weighted pseudo periodic mild solution if $\alpha L_f < 1$.

Proof. Define $\mathcal{F}: WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2) \to WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$ as $(\mathcal{F}u)(t) = \int_{-\infty}^{t} T(t-s)f(s,u_s)ds$.

If $u \in WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, by Corollary 4.4 and Corollary 3.2, $f(s,u_s) \in S^pWPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$. Owing to Lemma 4.6, it is not difficult to see that $\mathcal{F}(WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)) \subseteq WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$.

For any $u,v \in WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, we have

$$\begin{aligned}
\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\| &= \left\|\int_{-\infty}^{t} T(t-s)(f(s,u_s)-f(s,v_s))ds\right\| \\
&= \left\|\int_{0}^{\infty} T(s)\left(f(t-s,u_{t-s})-f(t-s,v_{t-s})\right)ds\right\| \\
&\le M\sum_{k=1}^{\infty}\left(\int_{k-1}^{k} e^{-cqs}ds\right)^{\frac{1}{q}}\left(\int_{k-1}^{k}\|f(s,u_s)-f(s,v_s)\|^p ds\right)^{\frac{1}{p}} \\
&= \alpha_0\sum_{k=1}^{\infty} e^{-ck}\|f(t+k-2+\cdot,u_{t+k-2+\cdot})-f(t+k-2+\cdot, \\
&\quad v_{t+k-2+\cdot})\|_p \\
&= \alpha\|f(t+k-2+\cdot,u_{t+k-2+\cdot})-f(t+k-2+\cdot,v_{t+k-2+\cdot})\|_p \\
&\le \alpha L_f\|u_{t+k-2+\cdot}-v_{t+k-2+\cdot}\|_{\mathfrak{B}} \\
&\le \alpha L_f\|u(t+k-2+\cdot)-v(t+k-2+\cdot)\| \\
&= \alpha L_f\|u-v\|,
\end{aligned}$$

then $\mathcal{F}$ is a contraction since $\alpha L_f < 1$. By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, which is the unique WPP_ω mild solution to the problem (1.1). □

Corollary 4.5. Assume that conditions (A2), (A3) and the following condition (A4′) are satisfied:

(A4′) $f \in S^pWPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, and there exist a positive constant L_f such that for $\psi_i \in \mathfrak{B}$, $i=1,2$, $\|f(t,\psi_1)-f(t,\psi_2)\|_p \le L_f\|\psi_1-\psi_2\|_{\mathfrak{B}}$.

Then the problem (1.1) admits a unique weighted pseudo anti-periodic mild solution provided that $\alpha L_f < 1$.

Example 4.1. Consider the partial differential equation which was inspired by [21]

$$\begin{aligned}
\frac{\partial}{\partial t}u(t,\xi) = \frac{\partial^2}{\partial \xi^2}u(t,\xi)+a_0(t)u(t,\xi)+\int_{-r}^{0} a_1(s)u(t+s,\xi),\quad (t,\xi)\in\mathbb{R}, \\
u(t,0)=u(t,\pi)=0,
\end{aligned} \tag{4.3}$$

where $a_0 \in WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, $\rho_1 = e^t$, $\rho_2 = 1+t^2$.

Let $X = \left(L^2([0,\pi],\mathbb{R}),\|\cdot\|_{L^2}\right)$ and define the operator A on $\mathbb{X}$. By $Au = u''$ with

$$D(A) = \left\{u\in\mathbb{X} : u''\in\mathbb{X}, u(0)=u(\pi)=0\right\}.$$

It is well-known that A is the infinitesimal generator of C_0-semigroup $(T(t))_{t>0}$ on $\mathbb{X}$ such that $\|T(t)\| \le e^{-t}$ for every $t \ge 0$.

Define the function $f:\mathbb{R}\times\mathfrak{B}\to\mathbb{X}$ by

$$f(t,\psi)(\xi) = a_0(t)\psi(0,\xi)+\int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

then the equation (4.3) can be rewritten as an abstract system in the form (1.1), where $u(t)=u(t,\cdot)$. Moreover, we can show

$$\|f(t,\cdot)\| \le \|a_0\| + \sqrt{r\left(\int_{-r}^{0} a^2(s)ds\right)},\quad t\in\mathbb{R}.$$

In view of Theorem 4.2, the equation (4.3) has a unique weighted pseudo periodic mild solution whenever $\|a_0\| + \sqrt{r\left(\int_{-r}^{0} a^2(s)ds\right)} < 1$.

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Harvesting Model for Fishery Resource with Reserve Area and Modified Effort Function

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Abstract

The aim of this paper is to study the dynamics of fishery resource system in an aquatic region consisting of two zones. One zone is free for fishing and other is restricted for any kind of fishing. In the proposed harvesting model, a modified effort function E is considered, which depends on the density effect of fish population. The criteria for local stability, global stability and instability are established for the proposed system. The theoretical results obtained are illustrated with numerical simulations in the last section.

Keywords: Fishery resource, Fishery effort, Stability, Harvesting.

2010 MSC: 92D25.

1 Introduction

Renewable resources like fishery, forestry and oil exploration are important sources of food and materials which play an important role for the survival and growth of biological population. The continuous and unplanned harvesting/exploration of these resources may lead to the extinction of the resources and it will further effect the survival of biological population. Therefore, its conservation and management are important fields to be analyzed through research. During the last few decades, there has been a considerable interest in modeling the dynamics of fishery resource systems [1, 2, 5–10].
Chaudhuri [5] in 1979 explained the problem of combined harvesting of two competing fish species and showed that the open-access fishery may possess a bionomic equilibrium which drives one species to extinction. Biological and economic interpretations of the results associated with the optimal equilibrium solution are discussed. Kitabatake [8] in 1982 studied a dynamical model for fishery resources with predator-prey relationships based on observational data for Lake Kasumigaura, Japan. He showed that the extensive use of diesel-powered trawling, which enables the large-scale catch of prey species in comparison with the traditional method of sailing trawling, may lead to the extinction of predator as well as prey species.

Mesterton-Gibbons [9] in 1996 described a technique to get the optimal harvesting policy for a Lotka-Volterra ecosystem of two interdependent populations, when the harvest rate is proportional to harvesting effort. The author explained that if two species coexist in the absence of harvesting, one species may be driven to extinction, if it is more catchable than the other. Fan and Wang [7] in 1998 examined the exploitation of single population modeled by time-dependent Logistic equation with periodic coefficients. Pradhan and Chaudhuri [10] in 1999 explained the dynamic reaction model of a fishery consisting of two competing species, each of which obeys the logistic law of growth. The authors use capital theoretic approach to formulate the dynamical system consisting of the growth equations of the two-species and fishing effort. Then they studied the existence of its steady states and their stability using eigen value analysis. Dubey et. al. [6] in 2002 explained that both the equilibrium density of fish population as well as the maximum sustainable yield

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increase as resource biomass density increase. The authors use Pontryagin's maximum Principle to discuss the optimal harvesting policy. Dubey et. al. [1] in 2003 proposed and analysed a mathematical model to study the dynamics of fishery resource system in an aquatic environment consisting of two zones: a free fishing zone and a reserve zone where fishing is not allowed. Here the authors obtained biological and bionomic equilibria of the system and showed that, even if, fishery is exploited continuously in the unreserved zone, fish population can be maintained at an appropriate equilibrium level in the habitat. Dubey and Patra [2] in 2013 proposed a dynamical model and analyzed the effect of the population on the resource biomass by taking into account the crowding effect. An appropriate Hamiltonian function is contructed for the discussion of optimal harvesting of resource which is utilized by the population using Pontryagin's Maximum Principal. The aim of this paper is to study the dynamics of fishery resource system in an aquatic region consisting of two zones. One zone is free for fishing and other is restricted for any kind of fishing. In the proposed harvesting model, a modified effort function E is considered which depends on the density effect of fish population. The criteria for local stability, global stability and instability are established for the proposed system. The theoretical results obtained are illustrated with the help of numerical simulations in the last section.

2 The Model

Consider a fishery resource system consisting of two zones: a free fishery zone and a reserve zone where fishing is not allowed. Let $x(t)$ be the biomass density of fish population inside the unreserved zone and $y(t)$ be the biomass density of same fish population inside the reserved zone at time t. If the fish population of reserved and unreserved zones are allowed to migrate within the zones, Dubey et. al. [1] in 2003 proposed and analyzed a mathematical model to study the dynamics of fishery resource system which is governed by following autonomous system of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= rx(1-\frac{x}{K}) - \sigma_1 x + \sigma_2 y - qEx,\\ \frac{dy}{dt} &= sy(1-\frac{y}{L}) + \sigma_1 x - \sigma_2 y,\end{aligned} \tag{2.1}$$

where $x(0) > 0, y(0) > 0$. In this model r and s are the intrinsic growth rates of fish subpopulation inside the unreserved and reserved areas respectively; K and L are the carrying capacities of fish species in the unreserved and reserved areas respectively; σ_1 is migration rate of fishes from unreserved area to reserved area and σ_2 is migration rate of fishes from reserved area to unreserved area; q is catchibality coefficient of fish species and E is the total effort applied for harvesting the fish species in the unreserved zone.
In this case the fishing effort E is simply taken as a function of t i.e. $E = E(t)$, which do not address the inverse effect of fish abundance on fishing effort. That is, it do not address the fact that higher the density of fishes, lesser the amount of effort needed to catch unit harvest. In order to overcome this deficiency, Idels and Wang [11] proposed a modified effort function which is a function of t as well as x and is given by

$$E(t,x) = \alpha(t) - \beta(t)\frac{1}{x}\frac{dx}{dt}, \tag{2.2}$$

where $\alpha \geq 0, \beta \geq 0$ are continuous functions of t. Incorporating (2.2), we get the following modified version of the model (2.1) as:

$$\begin{aligned}\frac{dx}{dt} &= \frac{r}{1-q\beta}x(1-\frac{x}{K}) - \frac{\sigma_1 x}{1-q\beta} + \frac{\sigma_2 y}{1-q\beta} - \frac{q\alpha x}{1-q\beta},\\ \frac{dy}{dt} &= sy(1-\frac{y}{L}) + \sigma_1 x - \sigma_2 y.\end{aligned} \tag{2.3}$$

The parameters $r, s, q, \sigma_1, \sigma_2, K$ and L are all assumed to be positive constants. It is to be noted that, if there is no migration of fish population from reserved area to unreserved area (i.e. $\sigma_2 = 0$) and $\frac{1}{1-q\beta}(r - \sigma_1 - q\alpha) < 0$, then $\dot{x} < 0$. Similarly, if there is no migration of fish population from unreserved area to reserved area (i.e. $\sigma_1 = 0$) and $s - \sigma_2 < 0$, then $\dot{y} < 0$. Hence throughout our analysis we assume that

$$1 - q\beta > 0,\ r - \sigma_1 - q\alpha > 0 \text{ and } s - \sigma_2 > 0. \tag{2.4}$$

Lemma 2.1. All the solutions of system (2.3) which initiate in $\Re_2^+$ are uniformly bounded.
Proof: We define a function

$$\omega(t) = x(t) + y(t). \tag{2.5}$$

The time derivative of (2.5) along the solution of system (2.3) is

$$\begin{aligned}\frac{d\omega}{dt} + \eta\omega &= \frac{dx}{dt} + \frac{dy}{dt} + \eta x + \eta y \\ &\leq \frac{K}{4r(1-q\beta)}(r + (1-q\beta)\eta - q\alpha - q\sigma_1\beta)^2 + \frac{L}{4s}(s + \eta + \frac{q\beta\sigma_2}{1-q\beta})^2 \\ &= \mu.\end{aligned}$$

Applying a theory of differential inequality (Birkhoff and Rota, [3] in 1982), we obtain

$$0 < \omega(x(t), y(t)) \leq \frac{\mu}{\eta}(1 - e^{-\eta t}) + \omega(x(0), y(0))e^{-\eta t},$$

and for $t \to \infty$, $0 < \omega \leq \frac{\mu}{\eta}$. This proves the lemma.

3 Dynamical Behavior of the System

The dynamical behavior of a system is studied at equilibrium points and equilibrium points of model (2.3) are obtained by solving $\dot{x} = \dot{y} = 0$. There are two feasible equilibrium points for the system (2.3), namely (i) $E_0 = (0,0)$, which is trivial equilibrium point and (ii) $E^* = (x^*, y^*)$, which is endemic equilibrium state, where x^*, y^* are positive solutions of following algebraic equations:

$$\sigma_2 y = q\alpha x + \sigma_1 x - rx + \frac{r}{K}x^2, \tag{3.6}$$

$$\sigma_1 x = \sigma_2 y - sy(1 - \frac{y}{L}). \tag{3.7}$$

Substituting the value of x from equation (3.7) into equation (3.6), we get a cubic equation in y as $ay^3 + by^2 + cy + d = 0$, where

$$\begin{aligned} a &= \frac{rs^2}{KL^2\sigma_1^2}, \\ b &= -2\frac{rs(s-\sigma_2)}{KL\sigma_1^2}, \\ c &= \frac{r}{K\sigma_1}(s-\sigma_2)^2 - \frac{s}{\sigma_1 L}(r - \sigma_1 - q\alpha), \\ d &= \frac{1}{\sigma_1}(r - \sigma_1 - q\alpha)(s - \sigma_2) - \sigma_2. \end{aligned}$$

The above equation has a unique positive solution $y = y^*$, if the following inequalities hold:

$$\frac{r}{K\sigma_1}(s-\sigma_2)^2 < \frac{s}{\sigma_1 L}(r - \sigma_1 - q\alpha), \tag{3.8}$$

$$(r - \sigma_1 - q\alpha)(s - \sigma_2) < \sigma_1\sigma_2. \tag{3.9}$$

After getting the value of y^*, the value of x^* can be easily computed from (3.7). It may be noted that for x^* to be positive, we must have

$$\sigma_2 + \frac{sy^*}{L} > s.$$

We now investigate the dynamical behaviour of system (2.3) at equilibrium points. The general variational matrix corresponding to the system (2.3) is given by

$$W = \begin{bmatrix} \frac{r}{1-q\beta} - \frac{2rx}{(1-q\beta)k} - \frac{\sigma_1}{1-q\beta} - \frac{q\alpha}{1-q\beta} & \frac{\sigma_2}{1-q\beta} \\ \sigma_1 & s - \frac{2sy}{L} - \sigma_2 \end{bmatrix}.$$

At $E_0(0,0)$, keeping in view equation (2.4), we note that all the eigen values of variational matrix are positive. Therefore, the trivial equilibrium E_0 is unstable. At $E^*(x^*, y^*)$, using the Routh-Hurwitz criteria, it is easy to check that all eigenvalues of the variational matrix corresponding to E^* have negative real parts, and hence E^* is locally asymptotic stable in XY plane. This implies that we can find a small circle with center E^* such that any solution $(x(t), y(t))$ of system (2.3), which is inside the circle at some time $t = t_0$, will remain inside the circle for all $t \geq t_0$ and will tend to (x^*, y^*) as $t \to \infty$.

Theorem 3.1. The nontrivial equilibrium E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive quadrant.
Proof: Consider the following positive definite function about E^*:

$$V = (x - x^* - x^* ln\frac{x}{x^*}) + \frac{y^*\sigma_2}{(1-q\beta)x^*\sigma_1}(y - y^* - y^* ln\frac{y}{y^*}). \tag{3.10}$$

Differentiating V with respect to time t along the solutions of model (2.3) and after some algebraic manipulation, we get

$$\frac{dV}{dt} = -\frac{r}{(1-q\beta)K}(x-x^*)^2 - \frac{s}{L}\frac{y^*\sigma_2}{(1-q\beta)x^*\sigma_1}(y-y^*)^2 - \frac{\sigma_2}{(1-q\beta)xx^*y}(x^*y - y^*x)^2 < 0. \tag{3.11}$$

This shows that $\frac{dV}{dt}$ is negative definite and hence by Liapunov's theorem on stability [14], it follows that the positive equilibrium E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive quadrant.
The above theorem implies that in an aquatic environment consisting of two zones, if one zone is reserved where fishing is not allowed and fish population are harvested only outside the reserved zone, then in both the reserved and unreserved zones fish species settle down to their respective equilibrium levels, whose magnitude depends upon the intrinsic growth rates of fish species, their migration coefficients and carrying capacities. This implies that fish populations may be sustained at an appropriate equilibrium level even after continuous harvesting of fish populations in unreserved zone.

Theorem 3.2. System (2.3) cannot have any limit cycle in the interior of positive quadrant.
Proof: Let

$$\begin{aligned}
H(x,y) &= \frac{1}{xy}, \\
h_1(x,y) &= \frac{rx}{1-q\beta}(1-\frac{x}{K}) - \frac{\sigma_1 x}{1-q\beta} + \frac{\sigma_2 y}{1-q\beta} - \frac{q\alpha x}{1-q\beta}, \\
h_2(x,y) &= sy(1-\frac{y}{L}) + \sigma_1 x - \sigma_2 y.
\end{aligned}$$

Clearly $H(x,y) > 0$ in the interior of the positive quadrant of XY plane. Then we have,

$$\delta(x,y) = \frac{\partial}{\partial x}(Hh_1) + \frac{\partial}{\partial y}(Hh_2) = -\frac{1}{(1-q\beta)y}\left[\frac{r}{K} + \frac{\sigma_2 y}{x^2}\right] - \frac{1}{x}\left[\frac{s}{L} + \frac{\sigma_2 x}{y^2}\right] < 0.$$

This shows that $\delta(x,y)$ does not change sign and is not identically zero in the positive quadrant of XY-plane. Then by Bendixson-Dulac criterion, the system (2.3) has no closed trajectory and hence there is no periodic solution in the interior of the positive quadrant of XY-plane.

4 Numerical Simulation

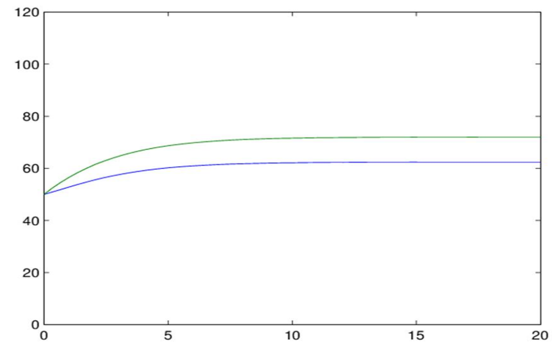
In order to investigate the dynamics of the model (2.3) with the help of computer simulations, we choose the following set of values of parameters (other set of parameter may also exist):

$$r = 0.9,\ K = 100,\ \sigma_1 = 0.3,\ q = 0.01,\ \beta = 1,\ \sigma_2 = 0.4,\ \alpha = 50,\ s = 0.5,\ L = 100 \tag{4.12}$$

with initial conditions $x(0) = 50$ and $y(0) = 50$. For this set of parameters, the conditions 3.8–3.9 for the existence of the interior equilibrium are satisfied. This shows that the endemic equilibrium point $E^*(x^*, y^*)$ exists and is given by

$$x^* = 62.3954$$
$$y^* = 71.9978.$$

Thus the interior equilibrium point $E^*(x^*, y^*)$ is locally asymptotically stable and Theorem 3.1 implies that non-trivial equilibrium E^* is globally asymptotically stable with respect to all solutions initiating in the positive quadrant. The behavior of x and y with respect to time t is plotted in Figure 1 for the set of values of parameters chosen in (4.12). From figure 1, we observe that the density of both populations settle down at their equilibrium levels, if the initial populations chosen are greater then 35. Now, we choose another set of



lower line – x population upper line – y population

Figure 1: Plot of x and y verses t for the values of parameters given in (12).

values of parameters as follows:

$$r = 0.9,\ K = 100,\ \sigma_1 = 0.1,\ q = 0.01,\ \beta = 1,\ \sigma_2 = 0.4,\ \alpha = 50,\ s = 0.5,\ L = 100 \tag{4.13}$$

with same initial values.

For the set of values of parameters given in (4.13), the behavior of x and y with respect to time t is plotted in Figure 2. It may be noted that the positive equilibrium $E^*(x^*, y^*)$ exists and it given by

$$x^* = 65.5747, y^* = 47.5698.$$

It may be checked that in this case (for the values of parameters given in (4.13), the conditions (3.8) and (3.9)

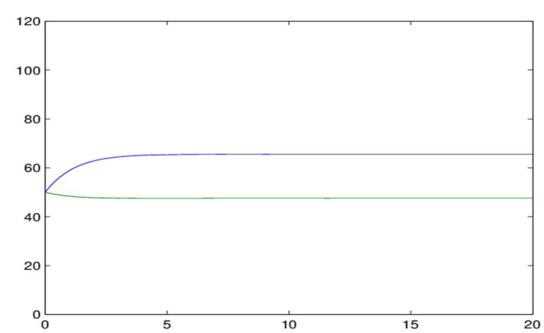


Figure 2: Plot of x and y verses t for the values of parameters given in (4.13)
lower line – x population upper line – y population

are satisfied. Then Theorem 3.1 shows that $E^*(x^*, y^*)$ is locally as well as globally asymptotically stable in the interior of the first quadrant. It may be noted that if σ_1 (migration rate from unreserved area to reserved area) decreases then the population inside reserved area will settle down to lower equilibrium point as we get in Figure 1 and Figure 2. In Figure 3, we have plotted the behavior of x and y with different initial values for set of parameters given in (4.12). Figure 3 shows that all the trajectories starting from different initial points converge to the point $E^*(62.3954, 71.9978)$. This shows that $E^*(62.3954, 71.9978)$ is globally asymptotically stable. This justifies our result in the theorem 3.1 that the system (2.3) is globally asymptotically stable with respect to all the solutions initiating in the interior of the positive quadrant. It is to be noted here that α and β

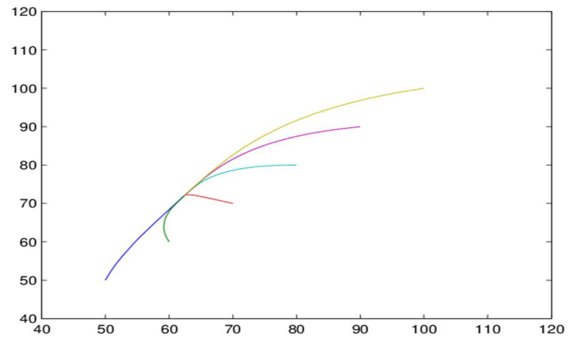
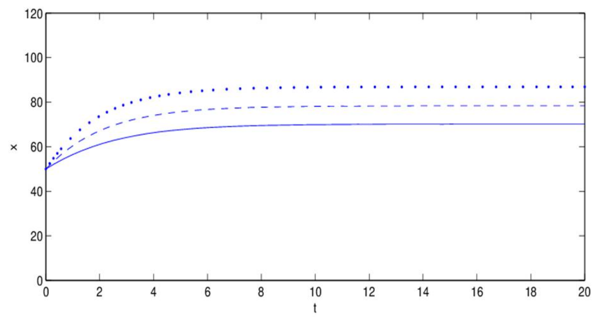


Figure 3: Global stability of $E^*(x^*, y^*)$.



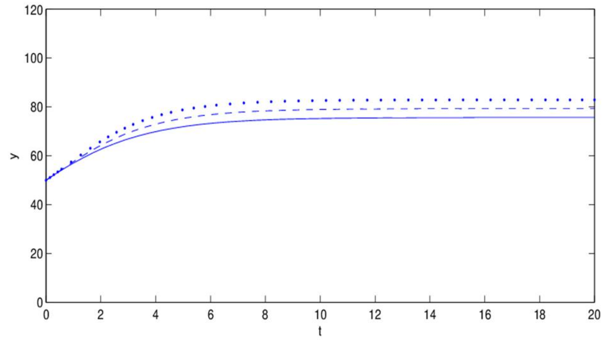
$\alpha = 40$
$\alpha = 30$
$\alpha = 20$

Figure 4: Plot of x verses time t for different values of α, the other values of parameters being same as given in (4.12).

are important parameters governing the dynamics of the system. Therefore, we have plotted the behavior of x and y with respect to time for different values of α in Figure 4 and Figure 5.
From Figure 4, we note that as the value of α decreases, the value of x increases. Similarly, the figure 5 implies that the value of y increases as the value of α decreases.

5 Conclusion

In this paper, a mathematical model representing harvesting of fishery resources with reserve area and modified effort function has been proposed. It has been assumed that the aquatic ecosystem consists of two zones: one free fishing zone and other reserved zone where fishing is strictly prohibited. It has been assumed that fish populations are growing logistically inside and outside the reserved zone and they migrate from reserved zone to unreserved zone and vice verse. Using stability theory of ordinary differential equation, it has been proved that the interior equilibrium exists under certain condition and it is globally asymptotically stable. It has been shown that the system under consideration does not have any limit cycle. It has been further found that if a reserved zone is created in an open-access fishery region where fishing is not allowed and harvesting of fish population is permitted only outside the reserved zone, the fish populations settle down at the respective equilibrium levels inside as well as outside the reserved zone.



$\alpha = 40$
$\alpha = 30$
$\alpha = 20$

Figure 5: Plot of y verses time t for different values of α, the other values of parameters being same as given in (4.12).

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Inverse Fourier Transform for Bi-Complex Variables

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Abstract

In this paper we examine the existence of bicomplexified inverse Fourier transform as an extension of its complexified inverse version within the region of convergence of bicomplex Fourier transform. In this paper we use the idempotent representation of bicomplex-valued functions as projections on the auxiliary complex spaces of the components of bicomplex numbers along two orthogonal,idempotent hyperbolic directions.

Keywords: Bicomplex numbers, Fourier transform, Inverse Fourier transform.

2010 MSC: 42A38, 44A10.

1 Introduction

In 1892, in search for special algebras, Corrado Segre [11] published a paper in which he treated an infinite family of algebras whose elements are commutative generalization of complex numbers called bicomplex numbers, tricomplex numbers,.....etc. Segre [11] defined a bicomplex number ξ as follows:

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,$$

where x_1, x_2, x_3, x_4 are real numbers, $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. The set of bicomplex numbers, complex numbers and real numbers are respectively denoted by $\mathbb{C}_2$, $\mathbb{C}_1$ and $\mathbb{C}_0$. Thus

$$\mathbb{C}_2 = \{\xi : \xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 : \ x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$$

$$\text{i.e., } \mathbb{C}_2 = \{\xi = z_1 + i_2 z_2 : \ z_1 (= x_1 + i_1 x_2), z_2 (= x_3 + i_1 x_4) \in \mathbb{C}_1\}.$$

There are two non trivial elements $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ in $\mathbb{C}_2$ with the properties $e_1^2 = e_1, e_2^2 = e_2, e_1 \cdot e_2 = e_2 \cdot e_1 = 0$ and $e_1 + e_2 = 1$ which means that e_1 and e_2 are idempotents alternatively called orthogonal idempotents. By the help of the idempotent elements e_1 and e_2, any bicomplex number

$$\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 = (a_0 + i_1 a_1) + i_2 (a_2 + i_1 a_3) = z_1 + i_2 z_2$$

where $a_0, a_1, a_2, a_3 \in \mathbb{C}_0$,, $z_1 (= a_0 + i_1 a_1)$ and $z_2 (= a_2 + i_1 a_3) \in \mathbb{C}_1$ can be expressed as

$$\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$$

where $\xi_1 (= z_1 - i_1 z_2) \in \mathbb{C}_1$ and $\xi_2 (= z_1 + i_1 z_2) \in \mathbb{C}_1$.

2 Fourier Transform

Let $f(t)$ be a real valued continuous function in $(-\infty, \infty)$ which satisfies the estimates

$$|f(t)| \leq C_1 \exp(-\alpha t), t \geq 0, \alpha > 0$$
$$\text{and } |f(t)| \leq C_2 \exp(-\beta t), t \leq 0, \beta > 0. \tag{1}$$

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Then the bicomplex Fourier transform [2] of $f(t)$ can be defined as

$$\widehat{f}(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} \exp(i_1 \omega t) f(t) dt, \omega \in \mathbb{C}_2.$$

The Fourier transform $\widehat{f}(\omega)$ exists and holomorphic for all $\omega \in \Omega$ where

$$\Omega = \{\omega = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \in \mathbb{C}_2 : -\infty < a_0, a_3 < \infty,$$

$$-\alpha + |a_2| < a_1 < \beta - |a_2| \; and \; 0 \leq |a_2| < \frac{\alpha + \beta}{2}\}$$

is the region of absolute convergence of $\widehat{f}(\omega)$ (see the figure 1 in the appendix).

2.1 Complex version of Fourier inverse transform.

We start with the complex version of Fourier inverse transform and in this connection we consider a continuous function $f(t)$ for $-\infty < t < \infty$ satisfying the estimates (1) possessing the Fourier transform $\widehat{f_1}$ in complex variable ω_1=$x_1 + i_1 x_2$ i.e.,

$$\begin{aligned} \widehat{f_1}(\omega_1) &= \int_{-\infty}^{\infty} \exp(i_1 \omega_1 t) f(t) dt \\ &= \int_{-\infty}^{\infty} \exp(i_1 x_1 t)\{\exp(-x_2 t) f(t)\} dt = \phi(x_1, x_2). \end{aligned}$$

In fact, one may identify $\phi(x_1, x_2)$ as the Fourier transform of $g(t) = \exp(-x_2 t) f(t)$ in usual complex exponential form [1, 6].

Towards this end, we assume that $f(t)$ is continuous and $f'(t)$ is piecewise continuous on the whole real line. Then $\widehat{f_1}(\omega_1)$ converges absolutely for $-\alpha < x_2 < \beta$ and

$$|\widehat{f_1}(\omega_1)\,| < \infty$$

which implies that

$$\begin{aligned} &\int_{-\infty}^{\infty} |\exp(i_1 \omega_1 t) f(t)| dt \\ &= \int_{-\infty}^{\infty} |\exp(i_1 x_1) g(t)| dt \\ &= \int_{-\infty}^{\infty} |g(t)| dt < \infty. \end{aligned}$$

The later condition shows $g(t)$ is absolutely integrable. Then by the Fourier inverse transform in complex exponential form [1, 6],

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i_1 x_1 t) \phi(x_1, x_2) dx_1$$

which implies that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(x_2 t) \exp(-i_1 x_1 t) \phi(x_1, x_2) dx_1.$$

Now if we integrate along a horizontal line then x_2 is constant and so for complex variable ω_1=$x_1 + i_1 x_2$ (which implies $d\omega_1 = dx_1$), the above inversion formula can be extended upto complex Fourier inverse transform

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-i_1(x_1 + i_1 x_2)t\} \phi(x_1, x_2) dx_1 \\ &= \frac{1}{2\pi} \int_{-\infty + i_1 x_2}^{\infty + i_1 x_2} \exp(-i_1 \omega_1 t) \widehat{f_1}(\omega_1) d\omega_1 \\ &= \frac{1}{2\pi} \lim_{x_1 \longrightarrow \infty} \int_{-x_1 + i_1 x_2}^{x_1 + i_1 x_2} \exp(-i_1 \omega_1 t) \widehat{f_1}(\omega_1) d\omega_1. \end{aligned} \tag{2}$$

Here the integration is to be performed along a horizontal line in complex ω_1-plane employing contour integration method.

We first consider the case $Im(\omega_1) = x_2 \geq 0$.We observe that $\widehat{f}_1(\omega_1)$ is continuous for $x_2 \geq 0$ and in particular it is holomorphic (and so it has no singularities) for $0 \leq x_2 < \beta$. We now introduce a contour Γ_R consisting of the segment $[-R, R]$ and a semicircle C_R of radius $|\omega_1| = R > \beta$ with centre at the origin. Then all possible singularities (if exists) of $\widehat{f}_1(\omega_1)$ must lie in the region above the horizontal line $x_2 = \beta$. At this stage we now consider the following two cases:

CaseI: We assume that $\widehat{f}_1(\omega_1)$ is holomorphic in $x_2 > \beta$ except for having a finite number of poles $\omega_1^{(k)}$ for $k = 1, 2, ...n$ therein (See Figure 2 in Appendix). By taking $R \to \infty$, we can guarantee that all these poles lie inside the contour Γ_R. Since $exp(-i_1\omega_1 t)$ never vanishes then the status of these poles $\omega_1^{(k)}$ of $\widehat{f}_1(\omega_1)$ is not affected by multiplication of it with $exp(-i_1\omega_1 t)$.Then by Cauchy's residue theorem,

$$\lim_{R\longrightarrow\infty}\int_{\Gamma_R}\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1$$
$$= 2\pi i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\}. \tag{3}$$

Furthermore as $x_2 \geq 0$, we can get $|\exp(-i_1\omega_1 t)| \leq 1$ for $\omega_1 \in C_R$ only when $t \leq 0$. In particular for $t < 0$,

$$M(R) = \max_{\omega_1\in C_R} |\widehat{f}_1(\omega_1)| = \max_{\omega_1\in C_R} |\int_{-\infty}^{0}\exp(i_1\omega_1 t)f(t)dt|$$
$$\leq C_2 \max_{\omega_1\in C_R} |\int_{-\infty}^{0}\exp\{(\beta + i_1\omega_1)t\}dt| = C_2 \max_{\omega_1\in C_R} |\frac{1}{\beta + i_1\omega_1}|$$
$$\leq C_2 \max_{\omega_1\in C_R} \frac{1}{\beta + |i_1||\omega_1|}$$

where we use the estimate 1. Now for $\omega_1| = R \to \infty$, we obtain that $M(R) \to 0$. Thus the conditions for Jordan's lemma [10] are met and so employing it we get that

$$\lim_{R\longrightarrow\infty}\int_{C_R}\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1 = 0. \tag{4}$$

Finally as,

$$\lim_{R\longrightarrow\infty}\int_{\Gamma_R}\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1$$
$$= \int_{C_R}\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1 + \int_{-R+i_1x_2}^{R+i_1x_2}\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1$$

then for $R \to \infty$, on using (3) and (4) we obtain that

$$\int_{-\infty+i_1x_2}^{\infty+i_1x_2}\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1$$
$$= 2\pi i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0$$

and so

$$f(t) = i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0.$$

Case II: Suppose $\widehat{f}_1(\omega_1)$ has infinitely many poles $\omega_1^{(k)}$ for $k = 1, 2, ...n$ in $x_2 > \beta$ and we arrange them in such a way that $\omega_1^{(1)} \leq |\omega_1^{(2)}| \leq |\omega_1^{(3)}|$..... where $\omega_1^{(k)}| \to \infty$ as $k \to \infty$. We then consider a sequence of contours Γ_k consisting of the segments $[-x_1^{(k)} + i_1x_2, x_1^{(k)} + i_1x_2]$ and the semicircles C_k of radii $r_k = \omega_1^{(k)}| > \beta$ enclosing the first k poles $\omega_1^{(1)}, \omega_1^{(2)}, \omega_1^{(3)}, ...\omega_1^{(k)}$ (See Figure 3 in Appendix).Then by Cauchy's residue theorem we get that

$$2\pi i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\}$$
$$= \int_{\Gamma_R} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1$$
$$= \int_{C_R} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1$$
$$+ \int_{-x_1^{(k)}+i_1x_2}^{x_1^{(k)}+i_1x_2} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1. \quad (5)$$

Now for $t<0$, in the procedure similar to Case I, employing Jordan lemma here also we may deduce that

$$\lim_{|\omega_1^{(k)}|\longrightarrow\infty} \int_{C_R} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1 = 0.$$

Hence in the limit $|\omega_1^{(k)}| \longrightarrow \infty$ (which implies that $|x_1^{(k)}| \longrightarrow \infty$) , (5) leads to

$$\int_{-\infty+i_1x_2}^{\infty+i_1x_2} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1$$
$$= 2\pi i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t<0$$

and as its consequence

$$f(t) = i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t<0.$$

Thus for $x_2 \geq 0$, whatever the number of poles is finite or infinite, from the above two cases we obtain the complex version of Fourier inverse transform as

$$f(t) = i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t<0. \quad (6)$$

We now consider the Case Im **(**ω_1**)** = $x_2 \leq 0$. The complex valued function $\widehat{f}_1(\omega_1)$ is continuous for $x_2 \leq 0$ and holomorphic in $-\alpha < x_2 \leq 0$. Introducing a contour $\Gamma'_{R'}$ consisting of the segment $[-R', R']$ and a semicircle $C'_{R'}$ of radius $\omega_1| = R' > \alpha$ with centre at the origin, we see that all possible singularities (if exists) of $\widehat{f}_1(\omega_1)$ must lie in the region below the horizontal line $x_2 = -\alpha$. If $\overline{\omega}_1^{(k)}$ for $k = 1, 2...$ are the poles in $x_2 < \alpha$, whatever the number of poles are finite or not for $R' \to \infty$, in similar to the previous consideration for $x_2 \geq 0$ we see that for $t>0$ the conditions for Jordan lemma are met and so

$$f(t) = -i_1 \sum_{Im(\omega_1^{(k)})<0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \overline{\omega}_1^{(k)}\} \text{ for } t>0. \quad (7)$$

We then assign the value of $f(t)$ at $t=0$ fulfilling the requirement of continuity of it in $-\infty < t < \infty$. This completes our procedure in complex ω_1 plane.

Similarly in $\omega_2 (= y_1 + i_1 y_2)$ plane the complex version of Fourier inverse transform of $\widehat{f}_2(\omega_2)$ will be

$$f(t) = \frac{1}{2\pi} \lim_{y_1\longrightarrow\infty} \int_{-y_1+i_1y_2}^{y_1+i_1y_2} \exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2)d\omega_2 \quad (8)$$

where the integration is to be performed along the horizontal line in ω_2 plane. Employing the contour integration method, we can obtain that

$$f(t) = i_1 \sum_{Im(\omega_2^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t<0$$
$$= -i_1 \sum_{Im(\omega_2^{(k)})<0} \text{Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t>0 \quad (9)$$

and the value of $f(t)$ at $t=0$ can be assigned fulfilling the requirement of continuity of it in $-\infty < t < \infty$.

2.2 Bicomplex version of Fourier inverse transform.

Suppose $\widehat{f}(\omega)$ is the bicomplex Fourier transform of the real valued continuous function $f(t)$ for $-\infty < t < \infty$ where $\omega = \omega_1 e_1 + \omega_2 e_2$ and $\widehat{f}(\omega) = \widehat{f_1}(\omega_1)e_1 + \widehat{f_2}(\omega_2)e_2$ in their idempotent representations. Here the symbols $\omega_1, \omega_2, \widehat{f_1}$ and $\widehat{f_2}$ have their same representation as defined in section 2.1. Then $\widehat{f}(\omega)$ is holomorphic in

$$\Omega = \{\omega = (x_1 + i_1 x_2)e_1 + (y_1 + i_1 y_2)e_2 \in \mathbb{C}_2 \\ : -\alpha < x_2, y_2 < \beta, -\infty < x_1, y_1 < \infty\}. \tag{10}$$

Now using complex inversions 2 and 8, we obtain that

$$\begin{aligned} f(t) &= f(t)e_1 + f(t)e_2 \\ &= [\frac{1}{2\pi}\int_{D_1} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1]e_1 + [\frac{1}{2\pi}\int_{D_2} \exp(-i_1\omega_2 t)\widehat{f_2}(\omega_2)d\omega_2]e_2 \\ &= \frac{1}{2\pi}\int_D \exp\{-i_1(\omega_1 e_1 + \omega_2 e_2)t\}\{\widehat{f_1}(\omega_1)e_1 + \widehat{f_2}(\omega_2)e_2\}d(\omega_1 e_1 + \omega_2 e_2) \\ &= \frac{1}{2\pi}\int_D \exp\{-i_1(\omega t)\widehat{f}(\omega)d\omega \end{aligned} \tag{11}$$

where

$$D_1 = \{\omega = x_1 + i_1 x_2 \in \mathbb{C}(i_1) : -\infty < x_1 < \infty, -\alpha < x_2 < \beta\},$$
$$D_2 = \{\omega = y_1 + i_1 y_2 \in \mathbb{C}(i_1) : -\infty < y_1 < \infty, -\alpha < y_2 < \beta\}$$

and D be such that $D_1 = P_1(D), D_2 = P_2(D)$. The integration in D_1 and D_2 are to be performed along the lines parallel to x_1 -axis in ω_1 plane and y_1-axis in ω_2 plane respectively inside the respective strips $-\alpha < x_2 < \beta$ and $-\alpha < y_2 < \beta$. As a result,

$$D = \{\omega \in \mathbb{C}_2 : \omega = \omega_1 e_1 + \omega_2 e_2 = (x_1 + i_1 x_2)e_1 + (y_1 + i_1 y_2)e_2\} \tag{12}$$

where $-\infty < x_1, y_1 < \infty, -\alpha < x_2, y_2 < \beta$. In four-component form D can be alternatively expressed as

$$D = \{\omega \in \mathbb{C}_2 : \frac{x_1 + y_1}{2} + i_1\frac{x_2 + y_2}{2} + i_2\frac{y_2 - x_2}{2} + i_1 i_2\frac{x_1 - y_1}{2}, \\ -\infty < x_1, y_1 < \infty, -\alpha < x_2, y_2 < \beta\}.$$

Conversely, if the integration in D is performed then the integrations in mutually complementary projections of D namely D_1 and D_2 are to be performed along the lines parallel to x_1-axis in ω_1 plane and y_1-axis in ω_2 plane respectively inside the strips $-\alpha < x_2, y_2 < \beta$ by using the contour integration technique. So using 2 and 8, we obtain that

$$\begin{aligned} &\frac{1}{2\pi}\int_D \exp\{-i_1(\omega t)\widehat{f}(\omega)d\omega \\ &= \frac{1}{2\pi}\int_D \exp\{-i_1(\omega_1 e_1 + \omega_2 e_2)t\}\{\widehat{f_1}(\omega_1)e_1 + \widehat{f_2}(\omega_2)e_2\}d(\omega_1 e_1 + \omega_2 e_2) \\ &= [\frac{1}{2\pi}\int_{D_1} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1]e_1 + [\frac{1}{2\pi}\int_{D_2} \exp(-i_1\omega_2 t)\widehat{f_2}(\omega_2)d\omega_2]e_2 \\ &= [\frac{1}{2\pi}\int_{-\infty+i_1x_2}^{\infty+i_1x_2} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1]e_1 + [\frac{1}{2\pi}\int_{-\infty+i_1y_2}^{\infty+i_1y_2} \exp(-i_1\omega_2 t)\widehat{f_2}(\omega_2)d\omega_2]e_2 \\ &= f(t)e_1 + f(t)e_2 = f(t) \end{aligned}$$

which guarantees the existence of Fourier inverse transform for bicomplex-valued functions.
In the following, we define the bicomplex version of Fourier inverse transform method.

Definition 1. *Let $\widehat{f}(\omega)$ be the bicomplex Fourier transform of a real valued continuous function $f(t)$ for $-\infty < t < \infty$ which is holomorphic in 12.The Fourier inverse transform of $\widehat{f}(\omega)$ is defined as*

$$f(t) = \frac{1}{2\pi}\int_D \exp\{-i_1(\omega t)\widehat{f}(\omega)d\omega$$

where D is given by 12. On using 6,7 and 9 this inversion method amounts to

$$f(t) = i_1 e_1 \sum_{Im(\omega_2^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\}$$
$$+ i_1 e_2 \sum_{Im(\omega_2^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t < 0 \tag{13}$$

and

$$f(t) = -i_1 e_1 \sum_{Im(\omega_1^{(k)})<0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \overline{\omega}_1^{(k)}\}$$
$$- i_1 e_2 \sum_{Im(\omega_2^{(k)})<0} \text{Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t > 0. \tag{14}$$

We assign the value of $f(t)$ at $t = 0$ fulfilling the requirement of continuity of it in the whole real line ($-\infty < t < \infty$).

The following examples will make our notion clear:

Example 2. *1. If* $\widehat{f}(\omega) = \frac{2a}{a^2+\omega^2}$ *for* $a>0$ *then*

$$\widehat{f}_1(\omega_1) = \frac{2a}{a^2+\omega_1^2},$$
$$\widehat{f}_2(\omega_2) = \frac{2a}{a^2+\omega_2^2}$$

and in each of ω_1 and ω_2 planes the poles are simple at $i_1 a$ and $i_1 a$. Now employing 13 and 14, for $t < 0$ we obtain that

$$f(t) = i_1 e_1 \text{ Res }\{\exp(-i_1\omega_1 t)\frac{2a}{a^2+\omega_1^2} : \omega_1 = i_1 a\}$$
$$+ i_1 e_2 \text{ Res }\{\exp(-i_1\omega_2 t)\frac{2a}{a^2+\omega_2^2} : \omega_2 = i_1 a\}$$
$$= i_1 e_1\{-i_1 \exp(at)\} + i_1 e_2\{-i_1 \exp(at)\} = \exp(-a|t|)$$

and for $t > 0$,

$$f(t) = -i_1 e_1 \text{ Res }\{\exp(-i_1\omega_1 t)\frac{2a}{a^2+\omega_1^2} : \omega_1 = i_1 a\}$$
$$- i_1 e_2 \text{ Res }\{\exp(-i_1\omega_2 t)\frac{2a}{a^2+\omega_2^2} : \omega_2 = i_1 a\}$$
$$= -i_1 e_1\{i_1 \exp(-at)\} - i_1 e_2\{i_1 \exp(at)\} = \exp(-a|t|).$$

Now for the continuity of t in the real line, we find $f(0) = 1$. Thus the Fourier inverse transform of $\widehat{f}(\omega)$ is $f(t) = \exp(-a|t|)$.

Example 3. *2. If*

$$\widehat{f}(\omega) = \frac{1}{2}[\frac{1}{\omega+\omega_0+\frac{i_1}{T}} - \frac{1}{\omega-\omega_0+\frac{i_1}{T}}] \textit{ for } T, \omega_0 > 0$$

then in each of ω_1 and ω_2 plane the poles are at $(\omega_0 - \frac{i_1}{T})$ and $(-\omega_0 - \frac{i_1}{T})$. For both the poles the imaginary components are negative and so the poles are in lower half of both the planes. In otherwords, no poles exist in upper half of ω_1 or ω_2 planes and as its consequence $f(t) = 0$ for $t < 0$. Now at $t > 0$,

$$f(t) = -i_1 e_1 \text{ Res }\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = -\omega_0 - \frac{i_1}{T}\}$$
$$-i_1 e_1 \text{ Res }\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_0 - \frac{i_1}{T}\}$$
$$-i_1 e_2 \text{ Res }\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = -\omega_0 - \frac{i_1}{T}\}$$
$$-i_1 e_2 \text{ Res }\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_0 - \frac{i_1}{T}\}$$

$$= -i_1 e_1 \frac{1}{2} \exp(-\frac{t}{T}) \exp(i_1 \omega_0 t) + i_1 e_1 \frac{1}{2} \exp(-\frac{t}{T}) \exp(-i_1 \omega_0 t)$$
$$- i_1 e_2 \frac{1}{2} \exp(-\frac{t}{T}) \exp(i_1 \omega_0 t) + i_1 e_2 \frac{1}{2} \exp(-\frac{t}{T}) \exp(-i_1 \omega_0 t)$$
$$= -i_1 \frac{1}{2} \exp(-\frac{t}{T}) \exp(i_1 \omega_0 t) + i_1 \frac{1}{2} \exp(-\frac{t}{T}) \exp(-i_1 \omega_0 t)$$
$$= \exp(-\frac{t}{T}) \sin(\omega_0 t).$$

Finally, the continuity of $f(t)$ in the whole real line implies that $f(0) = 0$.

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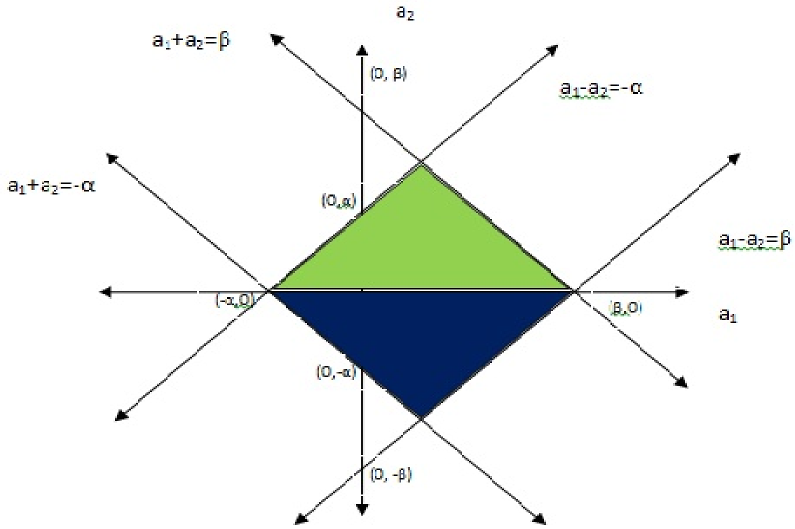
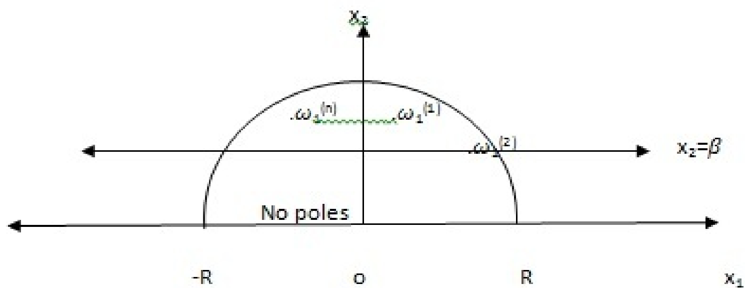
Appendix



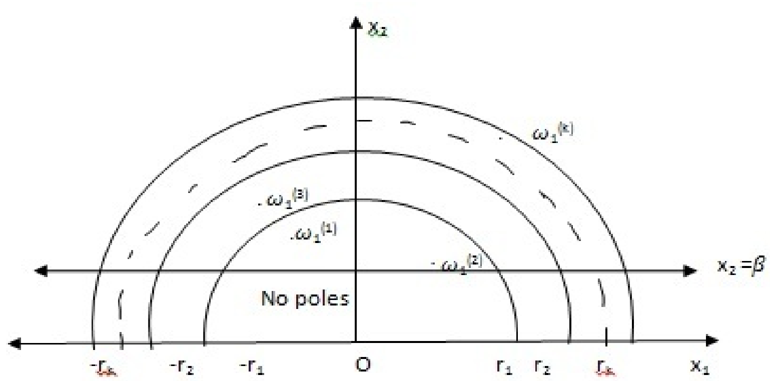
$a_1 - a_2$ plane in the concerned region of convergence

Fig-1



Upper half of ω_1 plane

Fig-1



Upper half of ω_1 plane

Fig-2

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Intuitionistic filter

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Abstract

The aim of this paper is to introduce a intuitionistic filter and study some of its properties.

Keywords: Intuitionistic set, Intuitionistic filter.

2010 MSC: 05C25, 05C10, 20F16.

1 Introduction and Preliminaries

In the philosophy of mathematics intuitionism is an approach where mathematics considered to be purely the result of the constructive mental activity of human rather than the discovery of fundamental principles claimed to exist in an objective reality.Intuitionistic sets and Intuitionistic points are introduced by D.Coker [3] in 1996. This concept is originated from the study of zadeh [7] who introduced Intuitionistic fuzzy set in the year 1965. This concept is the discrete form of Intuitionistic fuzzy set and it is also one of several ways of introducing vagueness in mathematical objects. After coker introduced Intuitionistic set and Intuitionistic topology several papers were published in intuitionistic fuzzy topology. It is known that filters are used to define convergence and hence limits. In this paper, we defined filters based on intuitionistic sets and derived various properties of intuitionistic filter.

Definition 1.1. *[3] : Let X be a nonempty fixed set. An intuitionistic set (IS for short) A is an object having the form* $A = \prec X, A^1, A^2 \succ$ *where* A^1 *and* A^2 *are subsets of X satisfying* $A^1 \cap A^2 = \phi$*. The set* A^1 *is called the set of members of A, while* A^2 *is called the set of non members of A.*

Definition 1.2. *[3] : Let X be a nonempty set. Let* $A = \prec X, A^1, A^2 \succ$ *and* $B = \prec X, B^1, B^2 \succ$ *be an intuitionistic sets on X and let* $\{A_i : i \in J\}$ *be an arbitrary family of IS's in X, where* $A^i = \prec X, {A_i}^1, {A_i}^2 \succ$ *.Then*

(1) $A \subseteq B$ *iff* $A^1 \subseteq B^1$ *and* $B^2 \subseteq A^2$.

(2) $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$.

(3) $\cup A_i = \prec X, \cup A_i^1, \cap {A_i}^2 \succ$.

(4) $\cap A_i = \prec X, \cap {A_i}^1, \cup {A_i}^2 \succ$

(5) $\tilde{X} = \prec X, X, \phi \succ$

(6) $\tilde{\phi} = \prec X, \phi, X \succ$.

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2 Intuitionistic filter

In this chapter we introduced intutionistic filters and study some of its basic properties.

Definition 2.3. *An intuitionistic filter ($\mathcal{I_F}$ for short)on a nonempty set X is a family of IS's in X satisfying the following axioms :*

($\mathcal{I_F}_1$) : $\tilde{\phi} \notin \mathcal{I_F}$

($\mathcal{I_F}_2$) : If $F \in \mathcal{I_F}$ and $H \supset F$, then $H \in \mathcal{I_F}$.

($\mathcal{I_F}_3$) : If $F \in \mathcal{I_F}$ and $H \in \mathcal{I_F}$, then $F \cap H \in \mathcal{I_F}$.

In this case the pair (X, $\mathcal{I_F}$) is called an intuitionistic filter.

Example 2.1. *Let $X = \{a, b\}$ and consider the family $\mathcal{I_F} = \{\tilde{X}, A_1, A_2\}$ where $A_1 = \prec X,\{a\},\{b\}\succ$, $A_2 = \prec X,\{a\},\phi\succ$.Then (X, $\mathcal{I_F}$) is an intuitionistic filter on X .*

Example 2.2. *Let $X = \{a, b, c\}$ and consider the family $\mathcal{I_F} = \{\tilde{X}, A_1, A_2\}$ where $A_1 = \prec X,\{a, b\},\phi \succ$, $A_2 = \prec X,\{b, c\},\phi\succ$. It is not an intuitionistic filter on X as $\prec X, \{a, b\}, \phi \succ \cap \prec X, \{b, c\}, \phi \succ = \prec X, \{b\}, \phi \succ$ which does not belong to $\mathcal{I_F}$ and hence axiom ($\mathcal{I_F}_3$) is not satisfied.*

Result 2.1. *Let $\{F_i : i \in J\}$ be a family of intuitionistic filters on X .Then $\cap_{i\in J} F_i$ is an intuitionistic filter on X .*

Proof. Let $F_i = \{\prec X, {F_i}^1, {F_i}^2 \succ: i \in J\}$ be any nonempty collection of intuitionistic filters on X.

Let $\mathcal{F} = \cap \{F_i : i \in J\}$.

To prove that $\mathcal{F}$ is an intuitionistic filter on X.

Since each F_i is an intuitionistic filter on X , $\prec$ X, X,$\phi \succ \in F_i$ for all i.

Hence $\prec$ X, X,$\phi \succ \in \cap F_i$.

$\Rightarrow \prec$ X, X,$\phi \succ \in \mathcal{F}$.

Therefore $\mathcal{F}$ is nonempty.

(i) Since $\prec$ X, ϕ, X $\succ \notin F_i$ for all i $\in$ J.

Therefore $\prec$ X, ϕ, X $\succ \notin \cap_{i\in J} F_i = \mathcal{F}$.

(ii) Let $\prec$ X, A^1,$A^2 \succ \in F_i$ for all i and $\prec$ X, B^1, $B^2 \succ \supset \prec$ X, A^1, $A^2 \succ$.

Since each F_i is an intuitionistic filter on X.

$\Rightarrow \prec$ X, B^1, $B^2 \succ \in F_i$ for all i .

$\Rightarrow \prec$ X, B^1, $B^2 \succ \in \cap_{i\in J} F_i =\mathcal{F}$.

(iii) Let A = $\prec$ X, A^1,$A^2 \succ \in \mathcal{F}$ and B = $\prec$ X, B^1,$B^2 \succ \in \mathcal{F}$.

$\Rightarrow \prec$ X, A^1,$A^2 \succ \in F_i$ for all i and $\prec$ X, B^1, $B^2 \succ \in F_i$ for all i.

As each F_i is an intuitionistic filter on X.

Therefore by Axiom ($\mathcal{I_F}_3$) $\prec$ X,$A^1 \cap B^1$, $A^2 \cup B^2 \succ \in F_i$ for all i .

Hence A $\cap$ B $\in \mathcal{F}$.

Therefore $\cap_{i\in J} F_i$ is an intuitionistic filter on X . □

Corollary 2.1. *Union of intuitionistic filters need not be an intuitionistic filter and it is Justified by the following example.*

Example 2.3. *Let $X = \{a, b\}$. $\mathcal{I_F}_1 = \{\prec X, \phi, \{a\} \succ, \prec X, \{a\}, \phi \succ, \prec X, \{b\}, \phi \succ, \prec X, \{b\}, \{a\} \succ, \prec X, X, \phi \succ, \prec X, \phi, \phi \succ\}$ and $\mathcal{I_F}_2 = \{\prec X, \phi, \{b\} \succ, \prec X, \{a\}, \phi \succ, \prec X, \{b\}, \phi \succ, \prec X, \{a\}, \{b\} \succ, \prec X, X, \phi \succ, \prec X, \phi, \phi \succ\}$. $\mathcal{I_F}_1 \cup \mathcal{I_F}_2 = \{\prec X, \phi, \{a\} \succ, \prec X, \phi, \{b\} \succ, \prec X, \{a\}, \phi \succ, \prec X, \{a\}, \{b\} \succ, \prec X, X, \phi \succ, \prec X, \phi, \phi \succ, \prec X, \{b\}, \phi \succ, \prec X, \{b\}, \{a\} \succ\}$ is not a intuitionistic filter as it does not satisfy the Axiom IF3.*

Definition 2.4. *A family $F_i = \{\prec X, {F_i}^1, {F_i}^2 \succ: i \in J\}$ of intuitionistic sets in X satisfies the finite intersection property(FIP for short)if every finite subfamily $\{F_i : i = 1, 2, ..n\}$ of $F_i = \{\prec X, {F_i}^1, {F_i}^2 \succ: i \in J\}$ satisfies the condition $\cap_{i=1}^{n} F_i \neq \tilde{\phi}$*

Theorem 2.1. *Let X be a nonempty set. Let $C = \{\prec X, {K_i}^1, {K_i}^2 \succ: i = 1, 2, 3, ...n\}$ be a nonempty family of intuitionistic sets of X.Then there exists a intuitionistic filter on X containing C iff C has finite intersection property.*

Proof. Suppose that $C = \{\prec X, {K_i}^1, {K_i}^2 \succ: i = 1, 2, 3, ...n\}$ has finite intersection property.

Let G = $\{B : B$ is the intersection of finite subfamily of C$\}$

As C has finite intersection property , it follows from definition, $\prec X, \phi, X \succ \notin$ G.

Consider the collection $\mathcal{I_F} = \{A_i = \prec X, {A_i}^1, {A_i}^2 \succ: A_i$ contains a member of G $\}$.
By the construction of $\mathcal{I_F}$,we have $\prec$ X, $\cap {K_i}^1, \cup {K_i}^2 \succ \subseteq \prec$ X, ${A_i}^1, {A_i}^2 \succ$
$\Rightarrow \cap {K_i}^1 \subseteq {A_i}^1$ and ${A_i}^2 \subseteq \cup {K_i}^2$.
So $\cap {K_i}^1 \subseteq {K_i}^1 \subseteq {A_i}^1$ and ${A_i}^2 \subseteq {K_i}^2 \subseteq \cup {K_i}^2$,
That is ${K_i}^1 \subseteq {A_i}^1$and ${A_i}^2 \subseteq {K_i}^2$.
Hence $\prec$ X, ${K_i}^1, {K_i}^2 \succ \subseteq \prec$ X, ${A_i}^1, {A_i}^2 \succ$
Therefore C $\subseteq \mathcal{I_F}$.
To prove that $\mathcal{I_F}$ is an intuitionistic filter on X.
Axiom ($\mathcal{I_{F1}}$) : By the construction of $\mathcal{I_F}$, we have $\prec$X, $\cap {K_i}^1, \cup {K_i}^2 \succ \subseteq \prec$ X, ${A_i}^1$,${A_i}^2 \succ$
$\Rightarrow \cap {K_i}^1 \subseteq {A_i}^1$ and ${A_i}^2 \subseteq \cup {K_i}^2$ and $\prec$ X ,ϕ, X $\succ \notin$ G. (by the finite intersection property)
Hence $\prec$ X,ϕ, X $\succ \notin \mathcal{I_F}$.
Axiom ($\mathcal{I_{F2}}$) : Let $\prec$ X, ${A_1}^1, {A_1}^2 \succ \in \mathcal{I_F}$.
$\Rightarrow \prec$ X, ${A_1}^1$,${A_1}^2 \succ \supseteq \prec$X, $\cap {K_1}^1, \cup {K_1}^2 \succ$.
If $\prec$ X, ${A_2}^1, {A_2}^2 \succ \supseteq \prec$ X, ${A_1}^1, {A_1}^2$ then $\prec$X ,${A_2}^1$,${A_2}^2 \succ \supseteq \prec$ X, $\cap {K_2}^1, \cup {K_2}^2 \succ$
Therefore $\prec X, {A_2}^1, {A_2}^2 \succ \in \mathcal{I_F}$.
Axiom ($\mathcal{I_{F3}}$) : Let $\prec X, {A_1}^1, {A_1}^2 \succ \in \mathcal{I_F}$ and $\prec X, {A_2}^1, {A_2}^2 \succ \in \mathcal{I_F}$
To prove that $\prec$ X ,${A_1}^1 \cap {A_2}^1, {A_1}^2 \cup {A_2}^2 \succ \in \mathcal{I_F}$.
Since $\prec$ X, ${A_1}^1, {A_1}^2 \succ \in \mathcal{I_F}$ and $\prec X, {A_2}^1, {A_2}^2 \succ \in \mathcal{I_F}$.
So that both $\prec$ X, ${A_1}^1, {A_1}^2 \succ$ and $\prec$X ,${A_2}^1, {A_2}^2 \succ$ contains some members of G say
$\prec$ X, ${A_1}^1$,${A_1}^2 \succ \supseteq \prec$ X, $\cap {K_1}^1, \cup {K_1}^2 \succ$, $\prec$ X ,${A_2}^1, {A_2}^2 \supseteq \prec$ X, $\cap {K_2}^1, \cup K_2^2 \succ$ Where $\prec$ X, $\cap {K_1}^1, \cup {K_1}^2 \succ$ and $\prec$ X, $\cap {K_2}^1, \cup {K_2}^2 \succ \in$ G.
Since $\prec$X, $\cap {K_1}^1, \cup {K_1}^2 \succ$ and $\prec$ X, $\cap {K_2}^1, \cup {K_2}^2 \succ \in$ C.
$\Rightarrow \prec$ X, $\cap {K_1}^1, \cup {K_1}^2 \succ \cap \prec$ X, $\cap {K_2}^1, \cup {K_2}^2 \succ \in$ C
$\Rightarrow \prec$ X, $(\cap {K_1}^1) \cap (\cap {K_2}^1)$, $(\cup {K_1}^2) \cup (\cup {K_2}^2) \succ \in$ C,
But $\prec$ X, ${A_1}^1$,${A_1}^2 \succ \supseteq \prec$X, $\cap {K_1}^1, \cup {K_1}^2 \succ$ and $\prec$ X ,${A_2}^1, {A_2}^2 \supseteq \prec$X, $\cap {K_2}^1, \cup {K_2}^2 \succ$
$\Rightarrow \prec$ X, ${A_1}^1$,${A_1}^2 \succ \cap \prec$ X ,A_2^1,$A_2^2 \succ \supseteq \prec$X, $\cap K_1^1, \cup K_1^2 \succ \cap \prec$X, $\cap K_2^1, \cup {K_2}^2 \succ$
$\Rightarrow \prec$X ,${A_1}^1 \cap {A_2}^1, {A_1}^2 \cup A_2^2 \succ \supseteq \prec$ X, $(\cap {K_1}^1) \cap (\cap {K_2}^1)$, $(\cup {K_1}^2) \cup (\cup {K_2}^2) \succ$
$\Rightarrow \prec$ X, $(\cap {K_1}^1) \cap (\cap {K_2}^1)$, $(\cup {K_1}^2) \cup (\cup {K_2}^2) \succ \subseteq \prec$ X ,$A_1^1 \cap A_2^1$, $A_1^2 \cup A_2^2 \succ$.
$\Rightarrow$ $(\cap {K_1}^1) \cap (\cap {K_2}^1) \subseteq {A_1}^1 \cap A_2^1$ and ${A_1}^2 \cup A_2^2 \subseteq (\cup {K_1}^2) \cup (\cup {K_2}^2) \succ$
Thus $\prec$X ,${A_1}^1 \cap A_2^1$, ${A_1}^2 \cup A_2^2 \succ \in \mathcal{I_F}$.
Therefore $\mathcal{I_F}$ is an intuitionistic filter on X containing C.
Conversely, Let $\mathcal{I_F}$ be an intuitionistic filter on X containing C .
Then $\mathcal{I_F} \supseteq$ C $\supseteq$ G.
Now $\mathcal{I_F}$ being an intuitionistic filter on X , $\prec$ X ,ϕ, X $\succ \notin \mathcal{I_F}$.
So $\prec$ X,ϕ, X $\succ \notin$ G.
Again $\prec$ X, $\cap_{i=1}{}^n {K_i}^1$, $\cup {K_i}^2 \succ \neq \prec$ X ,ϕ, X $\succ$
Therefore C must have finite intersection property. □

Remark 2.1. *The intuitionistic filter $\mathcal{I_F}$ as defined in Theorem 2.1, is said to be generated by C and C is said to be a sub base of $\mathcal{I_F}$.*
By Theorem 2.1, we have C is a sub base for $\mathcal{I_F} \Leftrightarrow$ C has F.I.P.
Also the intuitionistic filter $\mathcal{I_F}$ obtained above is the coarsest intuitionistic filter which contains C.
Because, if $\mathcal{I_{F1}}$ is any other intuitionistic filter containing C ,then $\mathcal{I_{F1}}$ must contain all finite intersections of members of C and their supersets.
Hence $\mathcal{I_{F1}} \supset \mathcal{I_F}$.
This implies $\mathcal{I_F}$ is coarsest of all intuitionistic filters on X which contains C.

Theorem 2.2. *Let $\mathcal{I_F}$ be an intuitionistic filter on a nonempty set X and A = $\prec$ X, A^1, $A^2 \succ$ be an intuitionistic set in X.Then there exists a intuitionistic filter $\mathcal{I_{F1}}$ finer than $\mathcal{I_F}$ such that $\prec$ X, A^1, $A^2 \succ \in \mathcal{I_{F1}}$ if and only if $\prec$ X, A^1, $A^2 \succ \cap \prec$ X, G^1, $G^2 \succ \neq \tilde{\phi}$ for every G = $\prec$ X, G^1, $G^2 \succ \in \mathcal{I_F}$.*

Proof. Let A $\cap$ G $\neq \tilde{\phi}$ for every G $\in \mathcal{I_F}$.
Let C = $\{A \cap G : G \in \mathcal{I_F}\}$.
We need to show that C has F.I.P.
Let $\{A \cap (G_i = \prec X, {G_i}^1, {G_i}^2 \succ) : i = 1, 2, 3, ...n\}$ be a collection of finite number of members of C .

Then $\cap \{A \cap (G_i = \prec X, {G_i}^1, {G_i}^2 \succ) : i = 1,2,3......n\}$ = A$\cap$ $\{\cap G_i : i = 1,2,3,....n\}$.
But by Axiom IF3 $\cap \{G_i : i = 1,2,3....n\} \in \mathcal{I}_{\mathcal{F}}$.
Therefore $\cap \{A \cap G_i : i = 1,2,3....n\}$ = A $\cap$ G where G =$\cap \{G_i : i = 1,2,3....n\} \neq \tilde{\phi}$ by hypothesis .
Thus C has finite intersection property and hence by Theorem 2.1, there exists an intuitionistic filter say $\mathcal{I}_{\mathcal{F}1}$ on X which contains C.
Let G be any member of $\mathcal{I}_{\mathcal{F}}$ so that A $\cap$ G $\neq \tilde{\phi}$ is a member of C.
Also as shown above $\mathcal{I}_{\mathcal{F}1}$ is an intuitionistic filter on X which contains C.
Hence A $\cap$ G is also a member of $\mathcal{I}_{\mathcal{F}1}$.
But G $\supset$ A $\cap$ G $\in \mathcal{I}_{\mathcal{F}1}$.
Therefore by Axiom ($\mathcal{I}_{\mathcal{F}2}$), G $\in \mathcal{I}_{\mathcal{F}1}$.
Since G $\in \mathcal{I}_{\mathcal{F}} \Rightarrow$ G $\in \mathcal{I}_{\mathcal{F}1}$.
Therefore $\mathcal{I}_{\mathcal{F}} \subset \mathcal{I}_{\mathcal{F}1}$.
i.e $\mathcal{I}_{\mathcal{F}1}$ is finer than $\mathcal{I}_{\mathcal{F}}$.
Conversely, let $\mathcal{I}_{\mathcal{F}1}$ be an intuitionistic filter on X and A $\in \mathcal{I}_{\mathcal{F}1}$ and $\mathcal{I}_{\mathcal{F}} \subset \mathcal{I}_{\mathcal{F}1}$.
Let G be any arbitrary member of $\mathcal{I}_{\mathcal{F}}$.
Since $\mathcal{I}_{\mathcal{F}} \subset \mathcal{I}_{\mathcal{F}1}$, we have G $\in \mathcal{I}_{\mathcal{F}1}$.
Also it is given that A $\in \mathcal{I}_{\mathcal{F}1}$.
Hence A $\cap$ G $\in \mathcal{I}_{\mathcal{F}1}$.
Further A $\cap$ G $\neq \tilde{\phi}$. □

3 Supremum and infinimum of intuitionistic set of intuitionistic filters

Definition 3.5. *Let $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ be a nonempty collection of intuitionistic filters on a nonempty set X such that $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ = $\{\mathcal{I}_{\mathcal{F}i} = \prec X, F_i^1, F_i^2 \succ$ and $\mathcal{I}_{\mathcal{F}i}$ is an intuitionistic filter on X$\}$. Then $\mathcal{I}_{\mathcal{F}i}$ is said to be the supremum of $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ if and only if*
(a) $\mathcal{I}_{\mathcal{F}i}$ is finer than every other intuitionistic filter in $\mathcal{M}_{\mathcal{I}\mathcal{F}}$.
(b) If ${I_{F_i}}^1$ is any other intuitionistic filter on X, which is finer than every other intuionistic filter in $\mathcal{M}_{\mathcal{I}\mathcal{F}}$, then $\mathcal{I}_{\mathcal{F}i}$ is coarser than ${I_{F_i}}^1$.

Definition 3.6. *: Let $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ be a nonempty collection of intuitionistic filters on a nonempty set X such that $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ = $\{\mathcal{I}_{\mathcal{F}i} = \prec X, F_i^1, F_i^2 \succ$ and $\mathcal{I}_{\mathcal{F}i}$ is an intuitionistic filter onX$\}$. Then $\mathcal{I}_{\mathcal{F}i}$ is said to be the infinimum of $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ if and only if*
(a) $\mathcal{I}_{\mathcal{F}i}$ is coarser than every other intuitionistic filter in $\mathcal{M}_{\mathcal{I}\mathcal{F}}$.
(b) If $\mathcal{I}_{\mathcal{F}_i}^1$ is any other intuitionistic filter on X, which is coarser than every other intuionistic filter in $\mathcal{M}_{\mathcal{I}\mathcal{F}}$, then $\mathcal{I}_{\mathcal{F}i}$ is finer than $\mathcal{I}_{\mathcal{F}_i}^1$.

Remark 3.2. *: If $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ is any nonempty class of intuitionistic filters on X, then infinimum of $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ always exists because we know that there is at least one intuitionistic filter $\{\prec X, X, \phi \succ\}$ on X which is coarser than every member of$\mathcal{M}_{\mathcal{I}\mathcal{F}}$. Also supremum of$\mathcal{M}_{\mathcal{I}\mathcal{F}}$ may or may not exist as will be clear from example given below.*
Let X = $\{a, b, c\}$ on which we have the following intuitionistic filters. $\mathcal{I}_{\mathcal{F}1}$ = $\{\prec X, X, \phi \succ\}$,
$\mathcal{I}_{\mathcal{F}2}$ = $\{\prec X, X, \phi \succ, \prec X, \{a, b\}, \phi \succ\}$, $\mathcal{I}_{\mathcal{F}3}$ = $\{\prec X, X, \phi \succ, \prec X, \{b, c\}, \phi \succ\}$,
$\mathcal{I}_{\mathcal{F}4}$ = $\{\prec X, X, \phi \succ, \prec X, \{a, c\}, \phi \succ\}$,
$\mathcal{I}_{\mathcal{F}5}$ = $\{\prec X, X, \phi \succ, \prec X, \{a\}, \phi \succ, \prec X, \{a, b\}, \phi \succ, \prec X, \{a, c\}, \phi \succ\}$ and
$\mathcal{I}_{\mathcal{F}6}$ = $\{\prec X, X, \phi \succ, \prec X, \{b\}, \phi \succ, \prec X, \{a, b\}, \phi \succ, \prec X, \{b, c\}, \phi \succ\}$.
Let $\mathcal{M}_{1\mathcal{I}\mathcal{F}}$ = $\{\mathcal{I}_{\mathcal{F}1}, \mathcal{I}_{\mathcal{F}2}, \mathcal{I}_{\mathcal{F}3}, \mathcal{I}_{\mathcal{F}4}\}$ clearly $\mathcal{I}_{\mathcal{F}1}$ is the infinimum of $\mathcal{M}_{1\mathcal{I}\mathcal{F}}$, as it is the only intuitionistic filter on X which is coarser than every member of$\mathcal{M}_{1\mathcal{I}\mathcal{F}}$. But on the other hand $\mathcal{M}_{1\mathcal{I}\mathcal{F}}$ has no supremum as there is no intuitionistic filter in $\mathcal{M}_{1\mathcal{I}\mathcal{F}}$ which is finer than each member of $\mathcal{M}_{1\mathcal{I}\mathcal{F}}$.
Again let $\mathcal{M}_{2\mathcal{I}\mathcal{F}}$ = $\{\mathcal{I}_{\mathcal{F}2}, \mathcal{I}_{\mathcal{F}3}, \mathcal{I}_{\mathcal{F}6}\}$.
Clearly $\mathcal{I}_{\mathcal{F}6}$ is the finest of all intuitionistic filters in $\mathcal{M}_{2\mathcal{I}\mathcal{F}}$ and it is coarsest of all intuitionistic filters on X which are finer than every member of $\mathcal{M}_{2\mathcal{I}\mathcal{F}}$.Therefore $\mathcal{I}_{\mathcal{F}6}$ is supremum of $\mathcal{M}_{2\mathcal{I}\mathcal{F}}$ and it is a member of $\mathcal{M}_{2\mathcal{I}\mathcal{F}}$.

Theorem 3.3. *: Let $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ = $\{F_i : i \in J\}$ be a nonempty collection of intuitionistic filters on a nonempty set X. Then $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ has a supremum if and only if the collection of all Intuitionistic subsets of X in the union of members of $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ has the finite intersection property.*

Proof. Let $\mathcal{I}_{\mathcal{F}} = \cup\{F_i : i \in J\}$ have the finite intersection property.
By Remark 2.1, there exists the coarsest intuitionistic filter on X containing $\mathcal{I}_{\mathcal{F}}$ and let that intuitionistic filter as $\mathcal{I}_{\mathcal{F}1}$.
But $\mathcal{I}_{\mathcal{F}1}$ is finer than every member of $\mathcal{M}_{\mathcal{I}\mathcal{F}}$.
Thus $\mathcal{I}_{\mathcal{F}1}$ is the coarsest intuitionistic filter on X which is finer than every member of $\mathcal{M}_{\mathcal{I}\mathcal{F}}$.
Hence by Definition 3.5, $\mathcal{I}_{\mathcal{F}1}$ is a supremum of $\mathcal{M}_{\mathcal{I}\mathcal{F}}$.
Conversely, Let $\mathcal{M}_{\mathcal{I}\mathcal{F}}$ has a supremum say $\prec X, F^1, F^2 \succ$.
By Definition 3.5, $\prec X, F^1, F^2 \succ$ is the coarsest of all intuitionistic filters on X which are finer than every member of $\mathcal{M}_{\mathcal{I}\mathcal{F}}$.
That is $\mathcal{I}_{\mathcal{F}}$ is the coarsest of all intuitionistic filters on X such that $\mathcal{I}_{\mathcal{F}} \supset \cup\{F_i : i \in J\}$.
Therefore $\cup\{F_i : i \in J\}$ must have finite intersection property. □

4 Intuitionistic filter base

Definition 4.7. *Let X be a any nonempty set. An intuitionistic filter base($\mathcal{I}_{\mathcal{F}\mathcal{B}}$ for short) on X is a nonempty family $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ of intuitionistic subsets of X satisfying the following axioms :*
(a) $\prec X, \phi, X \succ \notin \mathcal{I}_{\mathcal{F}\mathcal{B}}$.
(b) If $A \in \mathcal{I}_{\mathcal{F}\mathcal{B}}$ and $B \in \mathcal{I}_{\mathcal{F}\mathcal{B}}$, then there exists $C \in \mathcal{I}_{\mathcal{F}\mathcal{B}}$ such that $A \cap B \supset C$ or $C \subset A \cap B$

Example 4.4. *Let $X = \{a, b, c, d\}$. Then $\{\prec X, \{a\}, \phi \succ, \prec X, \{a, b\}, \phi \succ, \prec X, \{a, c\}, \phi \succ, \prec X, \{a, b, c\}, \{d\} \succ, \prec X, \{a, b, d\}, \phi \succ\}$ is an intuitionistic filter base in X.*

Remark 4.3. *: $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ has finite intersection property.*

Remark 4.4. *Every intuitionistic filter is an intuitionistic filter base.*

Theorem 4.4. *: Let $\mathcal{I}_{\mathcal{F}\mathcal{B}} = \{\prec X, {G_i}^1, {G_i}^2 \succ : i \in J\}$ be a family of intuitionistic subsets of a set X.Then $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ is an intuitionistic filter base on X if and only if the family $\mathcal{I}_{\mathcal{F}}$ consisting of all those intuitionistic subsets of X which contain a member of $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ is an intuitionistic filter on X.*

Proof. By definition of $\mathcal{I}_{\mathcal{F}}$, each member of $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ is also a member of $\mathcal{I}_{\mathcal{F}}$.
So that $\mathcal{I}_{\mathcal{F}\mathcal{B}} \subseteq \mathcal{I}_{\mathcal{F}}$ and as $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ is an intutionistic filter base i.e $\prec X, \phi, X \succ \notin \mathcal{I}_{\mathcal{F}\mathcal{B}}$.
Therefore $\prec X, \phi, X \succ \notin \mathcal{I}_{\mathcal{F}}$.
Let $\mathcal{I}_{\mathcal{F}} = \{\prec X, {F_i}^1, {F_i}^2 \succ : i \in J\}$ be an intuitionistic filter on X.
We need to show that $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ is an intutionistic filter base on X.
By Axiom ($\mathcal{I}_{\mathcal{F}1}$), we have $\prec X, \phi, X \succ \notin \mathcal{I}_{\mathcal{F}}$ and $\mathcal{I}_{\mathcal{F}\mathcal{B}} \subset \mathcal{I}_{\mathcal{F}}$.
Hence $\prec X, \phi, X \succ \notin \mathcal{I}_{\mathcal{F}\mathcal{B}}$.
Thus condition (a) for $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ is satisfied.
Now let $\prec X, {F_1}^1, {F_1}^2 \succ$ and $\prec X, {F_2}^1, {F_2}^2 \succ \in \mathcal{I}_{\mathcal{F}\mathcal{B}}$
then as $\{\prec X, {G_i}^1, {G_i}^2 \succ : i \in J\} \subset \{\prec X, {F_i}^1, {F_i}^2 \succ : i \in J\}$
It follows that $\prec X, {F_1}^1, {F_1}^2 \succ$ and $\prec X, {F_2}^1, {F_2}^2 \succ \in \mathcal{I}_{\mathcal{F}} \Rightarrow \prec X, {F_1}^1 \cap {F_2}^1, {F_1}^2 \cup {F_2}^2 \succ \in \mathcal{I}_{\mathcal{F}}$ by Axiom IF3 and hence by the definition of $\mathcal{I}_{\mathcal{F}}$, there exist a $\prec X, G^1, G^2 \succ \in \mathcal{I}_{\mathcal{F}\mathcal{B}}$
such that $\prec X, G^1, G^2 \succ \subset \prec X, {F_1}^1 \cap {F_2}^1, {F_1}^2 \cup {F_2}^2 \succ$
Thus corresponding to $\prec X, {F_1}^1, {F_1}^2 \succ$ and $\prec X, {F_2}^1, {F_2}^2 \succ \in \mathcal{I}_{\mathcal{F}\mathcal{B}}$
there exists a $\prec X, G^1, G^2 \succ \in \mathcal{I}_{\mathcal{F}\mathcal{B}}$
such that $\prec X, G^1, G^2 \succ \subset \prec X, {F_1}^1 \cap {F_2}^1, {F_1}^2 \cup {F_2}^2 \succ$.
Hence condition (b) for $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ to be an intuitionistic filter base is also satisfied.
Conversely, Let $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ be an intutionistic filter base on X.
We need to show that $\mathcal{I}_{\mathcal{F}} = \{\prec X, {F_i}^1, {F_i}^2 \succ\}$ is an intuitionistic filter on X.
By condition (a) of intuitionistic filter base $\prec X, \phi, X \succ \notin \mathcal{I}_{\mathcal{F}\mathcal{B}}$.
Hence $\prec X, \phi, X \succ \notin \mathcal{I}_{\mathcal{F}}$ as $\mathcal{I}_{\mathcal{F}}$ is the collection of all those intuitionistic subsets of X which contains a member of $\mathcal{I}_{\mathcal{F}\mathcal{B}}$.
Again let $\prec X, A^1, A^2 \succ \in \mathcal{I}_{\mathcal{F}}$ and $\prec X, B^1, B^2 \succ \supset \prec X, A^1, A^2 \succ$.
Then by definition of $\mathcal{I}_{\mathcal{F}}$, A contains a member of $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ say $\prec X, G^1, G^2 \succ$.

Therefore $\prec X, G^1, G^2 \succ \subset \prec X, A^1, A^2 \succ$ and $\prec X, A^1, A^2 \succ \subset \prec X, B^1, B^2 \succ$.
Hence $\prec X, G^1, G^2 \succ \subset \prec X, B^1, B^2 \succ$ and $\prec X, B^1, B^2 \succ \in \mathcal{I}_{\mathcal{F}}$.
Hence Axiom $(\mathcal{I}_{\mathcal{F}2})$ is satisfied.
let $\prec X, {F_1}^1, {F_1}^2 \succ$ and $\prec X, {F_2}^1, {F_2}^2 \succ \in \mathcal{I}_{\mathcal{F}}$ so that there exist members $\prec X, {G_1}^1, {G_1}^2 \succ \in \mathcal{I}_{\mathcal{F}\mathcal{B}}$ and $\prec X, {G_2}^1, {G_2}^2 \succ \in$ IFB such that $\prec X, {G_1}^1, {G_1}^2 \succ \subset \prec X, {F_1}^1, {F_1}^2 \succ$ and $\prec X, {G_2}^1, {G_2}^2 \succ \subset \prec X, {F_2}^1, {F_2}^2 \succ$.
Hence $\prec X, {G_1}^1 \cap {G_2}^1, {G_1}^2 \cup {G_2}^2 \succ \subset \prec X, {F_1}^1 \cap {F_2}^1, {F_1}^2 \cup {F_2}^2 \succ$.
Since $\prec X, {G_1}^1, {G_1}^2 \succ$ and $\prec X, {G_2}^1, {G_2}^2 \succ \in \mathcal{I}_{\mathcal{F}\mathcal{B}}$ and $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ is a intutionistic filter base on X, so by condition (b) of intutionistic filter base $\prec X, {G_1}^1, {G_1}^2 \succ \cap \prec X, {G_2}^1, {G_2}^2 \succ =$ $\prec X, {G_1}^1 \cap {G_2}^1, {G_1}^2 \cup {G_2}^2 \succ = \prec X, G^1, G^2 \succ$ say also belongs to $\mathcal{I}_{\mathcal{F}\mathcal{B}}$.
Hence $\prec X, G^1, G^2 \succ \subset \prec X, {F_1}^1 \cap {F_2}^1, {F_1}^2 \cup {F_2}^2 \succ$ or $\prec X, {F_1}^1 \cap {F_2}^1, {F_1}^2 \cup {F_2}^2 \succ$ contains a member of $\mathcal{I}_{\mathcal{F}\mathcal{B}}$.
So that $\prec X, {F_1}^1 \cap {F_2}^1, {F_1}^2 \cup {F_2}^2 \succ \in \mathcal{I}_{\mathcal{F}}$ whenever $\prec X, {F_1}^1, {F_1}^2 \succ$ and $\prec X, {F_2}^1, {F_2}^2 \succ \in \mathcal{I}_{\mathcal{F}}$.
Thus Axiom $(\mathcal{I}_{\mathcal{F}3})$ is satisfied.
Hence $\mathcal{I}_{\mathcal{F}}$ is an intutionistic filter on X and is known as the intuitionistic filter generated by the intuitionistic filter base $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ and $\mathcal{I}_{\mathcal{F}\mathcal{B}}$ is a subfamily of $\mathcal{I}_{\mathcal{F}}$. □

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New Riemann-Liouville generalizations for some inequalities of Hardy type

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Abstract

In this paper, we present new generalizations for some integral results related to Hardy inequalities. For our results, some recent results of Hardy type and other interesting inequalities of integer order of integration can be deduced as some special cases.

Keywords: Integral inequalities, Riemann-Liouville integral, Hardy ineqality.

2010 MSC: 26D15, 26D10.

1 Introduction

The classical integral inequality of Hardy is the following [4] :

$$\int_0^\infty x^{-p}\left(\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{1.1}$$

where $p > 1$, $x > 0$, f is a nonnegative integrable function on $[0,\infty[$. The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. During the past decades, the inequality has been developed and applied in almost unbelievable ways. This inequality plays an important role in analysis and its applications, see [1–3, 6–11, 13, 15, 18] and the references therein.

Let us recall some results that have motivated our work and have been reported in the previous literature. We begin by [12], where N. Levinson established the following generalization:

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_a^b f^p(t)dt, \tag{1.2}$$

where $f > 0$ on $[a,b] \subset [0,\infty[$, $p > 1$, and $F(x) = \int_0^x f(t)dt$.

Then, W.T. Sulaiman [17] presented the following similar Hardy inequality:

$$p\int_a^b \left(\frac{F(x)}{x}\right)^p dx \le (b-a)^p \int_a^b \left(\frac{f(x)}{x}\right)^p dx - \int_a^b \left(1-\frac{a}{x}\right)^p f^p(x)dx. \tag{1.3}$$

Recently, B. Sroysang [16] presented the following generalized result

$$p\int_a^b \frac{F^p(x)}{x^q}dx \le (b-a)^p \int_a^b \frac{f^p(x)}{x^q}dx - \int_a^b \frac{(x-a)^p}{x^q} f^p(x)dx. \tag{1.4}$$

The important integral results presented very recently in the paper of S. Wu et al. [14] is another motivation for us. For our results, some inequalities of this reference can be deduced as some special cases. We also generalise some results obtained by the authors of [16, 17].

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2 Preliminaries

In this section, we present some preliminaries that will be used to prove the main results [5].

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a,b]$ is defined as*

$$J_a^\alpha [f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ \alpha > 0,\ a < t \leq b, \tag{2.5}$$

where $\Gamma(\alpha) := \int\limits_0^\infty e^{-u} u^{\alpha-1} du.$

For $\alpha > 0$, $\beta > 0$, we have the following properties:

$$J_a^\alpha J_a^\beta [f(t)] = J_a^{\alpha+\beta} [f(t)] \tag{2.6}$$

and

$$J_a^\alpha J_a^\beta [f(t)] = J_a^\beta J_a^\alpha [f(t)]. \tag{2.7}$$

For the expression (2.5), when $f(t) = (t-a)^\mu$, we get:

$$J_a^\alpha (t-a)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-a)^{\mu+\alpha}, t \in [a,b]. \tag{2.8}$$

For $t = b$, we put

$$J_a^\alpha [f(b)] = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{2.9}$$

3 Main Results

Throughout the paper, all functions are assumed to be positive and all the integrals appear in the inequalities are exist and finite.

Theorem 3.1. *Let η be a nonnegative real number, and let $f > 0$ and $g > 0$ on $[a,b] \subseteq [0,\infty[$. If $\frac{x-a+\eta}{g(x)}$ is non-increasing, then for all $p > 1$, $\alpha > 0$, the fractional inequality*

$$\begin{aligned} & \int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)} \right)^p dx \\ \leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{(\alpha(p-1)-p+\frac{1}{p})\Gamma^{p-1}(1-\frac{1}{p}+\alpha)} \\ & \times \left[(b-a)^{\alpha(p-1)+\frac{1}{p}-p} \left(J_a^\alpha \left[\left(\frac{f(b)}{g(b)} (b-a+\eta) \right)^p (b-a)^{1-\frac{1}{p}} \right] \right) \right. \\ & \left. - J_a^\alpha \left[\left(\frac{f(b)}{g(b)} (b-a+\eta) \right)^p (b-a)^{1+\alpha(p-1)-p} \right] \right] \end{aligned} \tag{3.10}$$

is valid.

Proof. We have:

$$\begin{aligned} & \int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)} \right)^p dx \\ =\ & \int_a^b g^{-p}(x) \left(\int_a^x \frac{1}{\Gamma(\alpha)} (x-t)^{\alpha-1} f(t) (t-a)^{\frac{p-1}{p^2}} (t-a)^{\frac{1-p}{p^2}} dt \right)^p dx. \end{aligned} \tag{3.11}$$

Thanks to the fractional Hölder inequality [4], we obtain:

$$\begin{aligned} & \int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)} \right)^p dx \leq \frac{1}{\Gamma^p(\alpha)} \int_a^b g^{-p}(x) \\ & \times \left[\int_a^x \left((x-t)^{\alpha-1} f^p(t) (t-a)^{\frac{p-1}{p}} dt \right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{\alpha-1} (t-a)^{\left(\frac{1-p}{p^2}\right)\left(\frac{p}{p-1}\right)} dt \right)^{1-\frac{1}{p}} \right]^p dx. \end{aligned} \tag{3.12}$$

It yields then that

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{1}{\Gamma^p(\alpha)}\int_a^b g^{-p}(x)\int_a^x \left((x-t)^{\alpha-1}f^p(t)(t-a)^{\frac{p-1}{p}}dt\right) \\
&\times\left(\int_a^x (x-t)^{\alpha-1}(t-a)^{\frac{-1}{p}}dt\right)^{p-1} dx \qquad (3.13)\\
=\ & \frac{1}{\Gamma(\alpha)}\int_a^b g^{-p}(x)\int_a^x \left((x-t)^{\alpha-1}f^p(t)(t-a)^{\frac{p-1}{p}}dt\right)\left(J_a^\alpha (x-a)^{\frac{-1}{p}}\right)^{p-1} dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)}\int_a^b g^{-p}(x)(x-a)^{\left(\alpha-\frac{1}{p}\right)(p-1)} \qquad (3.14)\\
&\times\int_a^x \left((x-t)^{\alpha-1}f^p(t)(t-a)^{\frac{p-1}{p}}dt\right)dx.
\end{aligned}$$

This is to say that

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)}\int_a^b \left(\frac{x-a}{g(x)}\right)^p (x-a)^{\alpha(p-1)-1-p+\frac{1}{p}} \\
&\times\int_a^x \left((x-t)^{\alpha-1}f^p(t)(t-a)^{\frac{p-1}{p}}dt\right)dx \qquad (3.15)\\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)}\int_a^b \left(\frac{x-a+\eta}{g(x)}\right)^p (x-a)^{\alpha(p-1)-1-p+\frac{1}{p}} \\
&\times\int_a^x \left((x-t)^{\alpha-1}f^p(t)(t-a)^{\frac{p-1}{p}}dt\right)dx.
\end{aligned}$$

Since $\frac{x-a+\eta}{g(x)}$ is non-increasing and with the change of integration order, then we can write

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)}\int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\alpha-1}f^p(t)(t-a)^{\frac{p-1}{p}} \\
&\times\left(\int_t^b (x-a)^{\alpha(p-1)-1-p+\frac{1}{p}}dx\right)dt. \qquad (3.16)
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma^{p-1}(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)(\alpha(p-1)-p+\frac{1}{p})} \\
&\times\int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\alpha-1}f^p(t)(t-a)^{\frac{p-1}{p}} \qquad (3.17)\\
&\times\left((b-a)^{\alpha(p-1)-p+\frac{1}{p}}-(t-a)^{\alpha(p-1)-p+\frac{1}{p}}\right)dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\; & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{(\alpha(p-1)-p+\frac{1}{p})\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)} \\
&\times \Bigg[(b-a)^{\alpha(p-1)-p+\frac{1}{p}} \int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\alpha-1} f^p(t)(t-a)^{\frac{p-1}{p}} \\
&\quad - \int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\alpha-1} f^p(t)(t-a)^{\alpha(p-1)-p+1}\Bigg].
\end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\; & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{(\alpha(p-1)-p+\frac{1}{p})\Gamma^{p-1}(1-\frac{1}{p}+\alpha)} \\
&\times \Bigg[(b-a)^{\alpha(p-1)-p+\frac{1}{p}} J_a^\alpha \left[\left(\frac{f(b)}{g(b)}(b-a+\eta)\right)^p (b-a)^{\frac{p-1}{p}}\right] \\
&\quad - J_a^\alpha \left[\left(\frac{f(b)}{g(b)}(b-a+\eta)\right)^p (b-a)^{1+\alpha(p-1)-p}\right]\Bigg].
\end{aligned} \tag{3.19}$$

□

Remark 3.1. *Putting $\alpha = 1$ in Theorem 3.1, we obtain Theorem 3.1 of [14].*

Corollary 3.1. *Let f be a nonnegative function on $[a,b]$, $a > 0$ and $0 < \eta < a$. Then for all $p > 1, \alpha > 0$, we have*

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{x-a+\eta}\right)^p dx \\
\leq\; & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{(\alpha(p-1)-p+\frac{1}{p})\Gamma^{p-1}(1-\frac{1}{p}+\alpha)} \\
&\left[(b-a)^{\alpha(p-1)+\frac{1}{p}-p} J_a^\alpha \left(f^p(b)(b-a)^{1-\frac{1}{p}}\right) - J_a^\alpha \left(f^p(b)(b-a)^{1+\alpha(p-1)-p}\right)\right].
\end{aligned} \tag{3.20}$$

Remark 3.2. *Putting $\alpha = 1$ in Corollary 3.1, we obtain*

$$\begin{aligned}
&\int_a^b (x-a+\eta)^{-p} \left(\int_a^x f(t)dt\right)^p dx \\
\leq\; & \left(\frac{p}{p-1}\right)^p \left[\int_a^b f^p(t)\left(1-\frac{(t-a)^{1-\frac{1}{p}}}{(b-a)^{1-\frac{1}{p}}}\right) dt\right] \\
\leq\; & \left(\frac{p}{p-1}\right)^p \int_a^b f^p(t)dt.
\end{aligned} \tag{3.21}$$

Moreover, it is clear that for $0 < \eta < a$, we have

$$\int_a^b \left(\frac{J_a^\alpha f(x)}{x}\right)^p dx \leq \int_a^b \left(\frac{J_a^\alpha f(x)}{x-a+\eta}\right)^p dx. \tag{3.22}$$

Hence, the inequality (3.21) implies Levinson inequality (1.2).

We prove also the following theorem.

Theorem 3.2. *Let $f>0$ and $g>0$ on $[a,b]\subseteq[0,\infty[$ such that g is non decreasing, then for all $p>1$, $q>0$, $\alpha>0$, we have*

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ &\frac{1}{(\alpha p-\alpha+1)\Gamma^{p-1}(\alpha+1)}\\ &\times\left[(b-a)^{\alpha p-\alpha+1}J_a^\alpha\left(\frac{f^p(b)}{g^q(b)}\right)-J_a^\alpha\left(\frac{f^p(b)}{g^q(b)}(b-a)^{\alpha p-\alpha+1}\right)\right].\end{aligned} \tag{3.23}$$

Proof. We have:

$$\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx=\int_a^b g^{-q}(x)\left(\int_a^x \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}f(t)dt\right)^p dx, \tag{3.24}$$

and then,

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ &\int_a^b g^{-q}(x)\left[(J_a^\alpha f^p(x))^{\frac{1}{p}}(J_a^\alpha 1)^{1-\frac{1}{p}}\right]^p dx.\end{aligned} \tag{3.25}$$

Therefore,

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ &\int_a^b g^{-q}(x)\left[\left(\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f^p(t)dt\right)\left(\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}dt\right)^{p-1}\right]dx.\end{aligned} \tag{3.26}$$

Hence, we can write

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ &\frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)}\int_a^b g^{-q}(x)\left(\int_a^x(x-t)^{\alpha-1}f^p(t)dt\right)(x-a)^{\alpha(p-1)}dx.\end{aligned} \tag{3.27}$$

Since g is non decreasing and with the change of integration order, we obtain

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ &\frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)}\int_a^b g^{-q}(t)(b-t)^{\alpha-1}f^p(t)dt\int_t^b(x-a)^{\alpha(p-1)}dx.\end{aligned} \tag{3.28}$$

Therefore,

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ &\frac{1}{(\alpha p-\alpha+1)\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)}\int_a^b g^{-q}(t)(b-t)^{\alpha-1}f^p(t)\\ &\times\left[(b-a)^{\alpha p-\alpha+1}-(t-a)^{\alpha p-\alpha+1}\right]dt.\end{aligned} \tag{3.29}$$

This implies that

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ &\frac{1}{(\alpha p-\alpha+1)\Gamma^{p-1}(\alpha+1)}\\ &\times\left((b-a)^{\alpha p-\alpha+1}J_a^\alpha\left(\frac{f^p(b)}{g^q(b)}\right)-J_a^\alpha\left[\left(\frac{f^p(b)}{g^q(b)}\right)(b-a)^{\alpha p-\alpha+1}\right]\right).\end{aligned} \tag{3.30}$$

□

Remark 3.3. (i) : *Applying Theorem 3.2 for* $\alpha = 1$, *we obtain the first part of Theorem 3.5 in [14].*
(ii) : *Taking* $\alpha = 1$, $g(x) = x$ *in Theorem 3.2, we obtain Sroysang inequality (1.4).*
(iii) : *Taking* $\alpha = 1$, $g(x) = x$, $p = q$ *in Theorem 3.2, we obtain Sulaiman inequality (1.3).*

The third main result is given by the following theorem.

Theorem 3.3. *Let* $f \geq 0$ *and* $g > 0$ *on* $[a,b] \subseteq [0,\infty[$ *such that* g *is non decreasing. Then, for all* $0 < p < 1$, $q > 0$, $\alpha > 0$, *we have*

$$\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx \geq \frac{g^{-q}(b)}{(\alpha p - \alpha + 1)\Gamma^{p-1}(\alpha+1)} \left[\frac{(-1)^{\alpha p - \alpha + 1}}{\Gamma(\alpha)} \Gamma(\alpha p + 1) J_b^{\alpha p + 1} f^p(a) - (b-a)^{\alpha p - \alpha + 1} J_b^\alpha f^p(a) \right] \tag{3.31}$$

Proof. Using the weighted reverse Hölder inequality, we obtain

$$\begin{aligned} \int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx &\geq \frac{1}{\Gamma^p(\alpha)} \int_a^b g^{-q}(x) \left[\left(\int_a^x (x-t)^{\alpha-1} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{\alpha-1} dt \right)^{1-\frac{1}{p}} \right]^p dx \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b g^{-q}(x) \left[\left(\int_a^x (x-t)^{\alpha-1} f^p(t) dt \right) (J_a^\alpha 1)^{p-1} \right] dx. \end{aligned} \tag{3.32}$$

Therefore, we have

$$\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx \geq \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)} \int_a^b g^{-q}(x)(x-a)^{\alpha(p-1)} \left(\int_a^x (x-t)^{\alpha-1} f^p(t) dt \right) dx. \tag{3.33}$$

Thanks to the non decreasing of g and with the change of the order of integration, it yields that

$$\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx \geq \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)} \int_a^b g^{-q}(b)(x-a)^{\alpha(p-1)} \left(\int_a^x (x-t)^{\alpha-1} f^p(t) dt \right) dx. \tag{3.34}$$

Consequently,

$$\begin{aligned} \int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx &\geq \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)} \int_b^a g^{-q}(b)(a-t)^{\alpha-1} f^p(t) \left(\int_b^t (x-a)^{\alpha(p-1)} dx \right) dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)(\alpha p - \alpha + 1)} \int_b^a g^{-q}(b)(a-t)^{\alpha-1} f^p(t) \\ &\quad \times \left[(t-a)^{\alpha p - \alpha + 1} - (b-a)^{\alpha p - \alpha + 1} \right] dt. \end{aligned} \tag{3.35}$$

Therefore,

$$\begin{aligned} \int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx &\geq \frac{1}{(\alpha p - \alpha + 1)\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)} \\ &\quad \left[(b-a)^{\alpha p - \alpha + 1} \int_a^b (a-t)^{\alpha-1} g^{-q}(b) f^p(t) dt \right. \\ &\quad \left. - \int_a^b (a-t)^{\alpha-1} g^{-q}(b) f^p(t) (t-a)^{\alpha p - \alpha + 1} dt \right], \end{aligned} \tag{3.36}$$

and then,

$$\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx$$
$$\geq \frac{g^{-q}(b)}{(\alpha p-\alpha+1)\Gamma^{p-1}(\alpha+1)}\left[\frac{(-1)^{\alpha p-\alpha+1}}{\Gamma(\alpha)}\Gamma(\alpha p+1)J_b^{\alpha p+1}f^p(a)-(b-a)^{\alpha p-\alpha+1}J_b^\alpha f^p(a)\right].$$

This ends the proof. □

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Energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks

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Abstract

We consider the nonlinear wave equation in a bounded domain with a time varying delay term in the weakly nonlinear internal feedback

$$\left(|u_t|^{\gamma-2}u_t\right)_t - \Delta_x u - \int_0^t g(t-s)\Delta u(s)ds + \mu_1\psi(u_t(x,t)) + \mu_2\psi(u_t(x,t-\tau(t))) = 0,$$

we study the asymptotic behavior of solutions in using the Lyapunov functional , we extend and improve the previous result due to [30] ,

Keywords: Energy decay; viscoelastic term ; time varying delay term.

2010 MSC: 05C25, 05C10, 20F16.

1 Introduction

In this paper we investigate the decay properties of solutions for the initial boundary value problem of a nonlinear wave equation of the form

$$\begin{cases} \left(|u_t|^{\gamma-2}u_t\right)_t - \Delta_x u - \int_0^t g(t-s)\Delta u(s)ds + \mu_1\psi(u_t(x,t)) + \mu_2\psi(u_t(x,t-\tau(t))) = 0, & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0, & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), & \text{in } \Omega\times]0,\tau(0)[, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $\tau(t) > 0$ is a time varying delay, μ_1 and μ_2 are positive real numbers, and the initial data (u_0, u_1, f_0) belong to a suitable function space. In absence of delay ($\mu_2 = 0$), the problem of existence and energy decay have been extensively studied by several authors (see [3], [5], [6], [9], [12], [13], [17], [23]) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay). The decay rate of the energy (when t goes to infinity) depends on the function σ and on the function H which represents the growth at the origin of ψ.
Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see [25]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. Time delays so often arise in many physical, chemical, biological and economical phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1], [26], [28] and the references therein. In [7], the authors showed that a small delay in a boundary control could turn such well-behave hyperbolic system into a wild one and

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therefore, delay becomes a source of instability. However, sometimes it also can improve the performance of the systems (see [26]).
To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [18], [19], [27]). For instance in [18] the authors studied the wave equation with linear internal damping term with constant delay (ψ linear, $\tau(t) = const$ in the problem (1.1)). They determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (1.1) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [18], is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay are acting in the boundary. We also recall the result by Xu, Yung and Li [27], where the authors proved a result similar to the one in [18] for the one-space dimension by adopting the spectral analysis approach.
The case of time varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman [22] in one-space dimension and in the linear case (ψ linear in problem (1.1) and proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1-d}\mu_1,$$

where the constant d satisfies

$$\tau'(t) \leq d < 1, \quad \forall t > 0.$$

In [21] Nicaise, Pignotti and Valein extended the above result to higher-space dimension and established an exponential decay.
Our purpose in this paper is to give an energy decay estimate of the solution to problem (1.1) for a weakly nonlinear damping and in the presence of a time varying delay term.
In this article, we use some technique from (see [3]), [30]) and [31]).to give energy decay estimates of solutions to the problem (1.1) for a nonlinear damping and a time varying delay term. To prove decay estimates, we use a suitable energy and Lyapunov functionals and some properties of convex functions. These arguments of convexity were introduced and developed by Lasiecka et al. [4], and [13], and used by Liu and Zuazua [15], Eller et al [8].

2 Preliminaries and main results

In order to state and prove our results, we need some assumptions, as well as, some lemmas.
First assume the following hypotheses
(H1) $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l < 1,$$

and there exists a non-increasing differentiable function : $\xi : R_+ \to R_+$ such that $g'(t) \leq -\xi(t)g(t)$.
(H2) $\psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of the class $C(\mathbb{R})$ such that there exist ϵ_1, c_1, $c_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0, \infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon_1]$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon_1]$), such that

$$c_1|s| \leq |\psi(s)| \leq c_2|s| \quad \text{if } |s| \geq \epsilon_1, \tag{2.2}$$

$$s^2 + \psi^2(s) \leq H^{-1}(s\psi(s)) \quad \text{if } |s| \leq \epsilon_1. \tag{2.3}$$

$\psi : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|\psi'(s)| \leq c_3 \tag{2.4}$$

$$\alpha_1\, s\psi(s) \leq G(s) \leq \alpha_2\, s\psi(s), \tag{2.5}$$

where

$$G(s) = \int_0^s \psi(r)\, dr,$$

with l satisfying

$$\gamma - 1 \leq \frac{n+2}{n-2}, \quad if\ n > 2,$$
$$\gamma - 1 < \infty, \qquad if\ n \leq 2.$$

(H3) τ is a function such that

$$\tau \in W^{2,\infty}([0,T]), \forall T > 0, \tag{2.6}$$
$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \forall t > 0, \tag{2.7}$$
$$\tau'(t) \leq d < 1, \quad \forall t > 0,, \tag{2.8}$$

where τ_0 and τ_1 are two positive constants.
(H4) The weight of dissipation and the delay satisfy:

$$\mu_2 < \frac{\alpha_1(1-d)}{\alpha_2(1-\alpha_1 d)}\mu_1. \tag{2.9}$$

We now state some Lemmas needed later.

Lemma 2.1 (Sobolev-Poincaré's inequality)**.** *Let q be a number with $2 \leq q < +\infty$ $(n = 1,2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$. Then there exists a constant $c_* = c_*(\Omega, q)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2 \quad for \quad u \in H_0^1(\Omega).$$

Lemma 2.2. [30]. *For any $g \in C^1(R_+)$ and $\varphi \in H^1(0,T)$, we have*

$$\int_0^t \int_\Omega g(t-s)\varphi(x,s)\varphi_t(x,t)dxds = -\frac{1}{2}\frac{d}{dt}\left((go\varphi)(t) + \int_0^t g(s)ds\|\varphi\|_2^2\right) - g(t)\|\varphi\|_2^2 + (g'o\varphi)(t),$$

where

$$(go\varphi)(t) = \int_0^t g(t-s)\int_\Omega |\varphi(x,s) - \varphi(x,t)|^2 dxds,$$

and

$$\|\varphi\|_2^2 = \int_\Omega |\varphi(x,s)|^2 dx.$$

Lemma 2.3. [30]. *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(x,t) - u(x,s))ds\right)^2 dx \leq (1-l)c_s^2(go\nabla u)(t), \tag{2.10}$$

where

$$(go\nabla u)(t) = \int_0^t g(t-s)\int_\Omega |u(x,s) - u(x,t)|^2 dxds,$$

and c_s^2 is the poincaré constant and l is given in $(H1)$.

We introduce, as in [18], the new variable

$$z(x,\rho,t) = u_t(x, t-\tau(t)\rho), \quad x \in \Omega,\ \rho \in (0,1), \quad t > 0. \tag{2.11}$$

Then, we have

$$\tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, \text{ in } \Omega \times (0,1) \times (0,+\infty). \tag{2.12}$$

Therefore, problem (1.1) is equivalent to:

$$\begin{cases} \left(|u_t|^{\gamma-2}u_t\right)_t - \Delta_x u - \int_0^t g(t-s)\Delta u(s)ds + \mu_1\psi(u_t(x,t)) + \mu_2\psi(u_t(x,t-\tau(t))) = 0, & in\ \Omega\times]0,+\infty[, \\ \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, & in\ \Omega\times]0,1[\times]0,+\infty[, \\ u(x,t) = 0, & on\ \partial\Omega\times]0,+\infty[\\ z(x,0,t) = u_t(x,t), & on\ \Omega\times]0,+\infty[, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & in\ \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau(0)), & in\ \Omega\times]0,1[. \end{cases} \tag{2.13}$$

where ξ satisfies

$$\frac{\mu_2(1-\alpha_1)}{\alpha_1(1-d)} < \xi < \frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}. \tag{2.14}$$

We define the energy associated to the solution of the problem (2.13) by:

$$E(t) = \frac{\gamma-1}{\gamma}\|u_t(t)\|_\gamma^\gamma + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \xi(t)\tau(t)\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx, \tag{2.15}$$

Lemma 2.4. *Let (u,z) be a solution of the problem (2.13). Then, the energy functional defined by (2.15) satisfies*

$$\begin{aligned} E'(t) \le & -(\mu_1 - \xi(t)\alpha_2 - \mu_2\alpha_2)\int_\Omega u_t\psi(u_t)\,dx \\ & -\left(\xi(t)(1-\tau'(t))\alpha_1 - \mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)\psi(z(x,1,t))\,dx \\ & + \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ & \le 0. \end{aligned} \tag{2.16}$$

Proof. Multiplying the first equation in (2.13) by u_t, integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_\gamma^\gamma + \|\nabla u\|_2^2\right) + \mu_1\int_\Omega u_t\psi(u_t)\,dx + \mu_2\int_\Omega \psi(z(x,1,t))u_t(x,t)\,dx = 0. \tag{2.17}$$

We multiply the second equation in (2.13) by $\xi(t)\psi(z)$ and integrate over $\Omega\times(0,1)$, to obtain:

$$\xi(t)\tau(t)\int_\Omega\int_0^1 z_t\psi(z(x,\rho,t))\,d\rho\,dx = -\xi(t)\int_\Omega\int_0^1 (1-\tau'(t)\rho)\frac{\partial}{\partial\rho}G(z(x,\rho,t))\,d\rho\,dx. \tag{2.18}$$

Consequently,

$$\begin{aligned} & \frac{d}{dt}\left(\xi(t)\tau(t)\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx\right) \\ & = -\xi(t)\int_0^1\int_\Omega \frac{\partial}{\partial\rho}((1-\tau'(t)\rho)G(z(x,\rho,t)))\,d\rho\,dx + \xi_t(t)\tau(t)\int_0^1\int_\Omega G(z(x,\rho,t)))\,dxd\rho. \\ & = \xi(t)\int_\Omega (G(z(x,0,t)) - G(z(x,1,t)))\,dx + \xi(t)\tau'(t)\int_\Omega G(z(x,1,t)))\,dx \\ & + \xi_t(t)\tau(t)\int_0^1\int_\Omega G(z(x,\rho,t)))\,dxd\rho. \end{aligned} \tag{2.19}$$

From (2.17), (2.18), lemma 2.2 we get

$$\begin{aligned} E'(t) \le & -(\mu_1 - \xi(t)\alpha_2)\int_\Omega u_t\psi(u_t)\,dx - \xi(t)(1-\tau'(t))\int_\Omega G(z(x,1,t))\,dx \\ & -\mu_2\int_\Omega u_t(t)\psi(z(x,1,t))\,dx + \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \end{aligned} \tag{2.20}$$

Let us denote by G^* the conjugate function of the convex function G, i.e., $G^*(s) = \sup_{t\in\mathbb{R}_+}(st - G(t))$. Then G^* is the Legendre transform of G, which is given by (see Arnold [2], p. 61-62)

$$G^*(s) = s(G_2')^{-1}(s) - G[(G')^{-1}(s)], \quad \forall s \ge 0 \tag{2.21}$$

and satisfies the following inequality

$$st \le G^*(s) + G(t), \quad \forall s,t \ge 0. \tag{2.22}$$

Then, from the definition of G_2, we get

$$G^*(s) = s\psi^{-1}(s) - G(\psi^{-1}(s)).$$

Hence

$$\begin{aligned} G^*(\psi(z(x,1,t))) &= z(x,1,t)\psi(z(x,1,t)) - G(z(x,1,t)) \\ &\le (1-\alpha_1)z(x,1,t)\psi(z(x,1,t)). \end{aligned} \tag{2.23}$$

Making use of (2.19) and (2.22), we have

$$\begin{aligned} E'(t) \leq & -(\mu_1 - \xi(t)\alpha_2)\int_\Omega u_t\psi(u_t)\,dx - \xi(t)(1-\tau'(t))\int_\Omega G(z(x,1,t))\,dx \\ & + \mu_2\int_\Omega \left(G(u_t) + G^*(\psi(z(x,1,t)))\right)\,dx \\ & + \frac{1}{2}(g' o \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \end{aligned} \tag{2.24}$$

From (2.5) and (2.22), we obtain

$$\begin{aligned} E'(t) \leq & -(\mu_1 - \xi(t)\alpha_2 - \mu_2\alpha_2)\int_\Omega u_t\psi(u_t)\,dx \\ & - \left(\xi(t)(1-\tau'(t))\alpha_1 - \mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)\psi(z(x,1,t))\,dx \\ & + \frac{1}{2}(g' o \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \end{aligned}$$

Then, by using (2.8) and (2.14), our conclusion follows. □

3 Asymptotic Behavior

In this section we prove the energy decay result by constructing a suitable Lyapunov functionnal.
We denote by c various positive constants which may be different at different occurrences.
Now we define the following functional

$$L(t) = ME(t) + \epsilon\phi(t) + \epsilon\varphi(t) + \epsilon I(t), \tag{3.25}$$

where

$$\phi(t) = \int_\Omega u|u_t|^{\gamma-2}u_t dx, \tag{3.26}$$

$$\varphi(t) = -\int_\Omega |u_t|^{\gamma-2}u_t\int_0^t g(t-s)(u(t)-u(s))dsdx, \tag{3.27}$$

and

$$I(t) = \int_\Omega\int_0^1 e^{-2\rho\tau(t)}G(z(x,\rho,t))d\rho dx. \tag{3.28}$$

We need also the following lemma

Lemma 3.1. . *Let (u,z) be a solution of problem (2.13), then there exists two positive constants* λ_1,λ_2 *such that*

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t), \quad t \geq 0, \tag{3.29}$$

for M sufficiently large .

Proof. Thank's to the Holder and Young's inequalities, lemma 2.1 , we have

$$\begin{aligned} \int_\Omega u|u_t|^{\gamma-2}u_t dx & \leq C_\epsilon\int_\Omega |u|^\gamma dx + \epsilon\int_\Omega |u_t|^l dx \\ & \leq C_\epsilon\|\nabla u\|_2^\gamma + \epsilon\|u_t\|_\gamma^\gamma \\ & \leq C_\epsilon E^{\frac{\gamma}{2}}(t) + c\epsilon E(t) \\ & \leq C_\epsilon E^{\frac{\gamma-2}{2}}(0)E(t) + c\epsilon E(t), \end{aligned} \tag{3.30}$$

$$\begin{aligned} \int_\Omega u|u_t|^{\gamma-2}u_t dx & \geq -C_\epsilon\int_\Omega |u|^\gamma dx - \epsilon\int_\Omega |u_t|^\gamma dx \\ & \geq -C_\epsilon\|\nabla u\|_2^\gamma - \epsilon\|u_t\|_\gamma^\gamma \\ & \geq -C_\epsilon E^{\frac{\gamma}{2}}(t) - c\epsilon E(t) \\ & \geq -C_\epsilon E^{\frac{\gamma-2}{2}}(0)E(t) - c\epsilon E(t), \end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\varphi(t) &= \left|-\int_\Omega |u_t|^{\gamma-2}u_t \int_0^t g(t-s)(u(t)-u(s))dsdx\right| \\ &\leq \frac{1}{2}\|u_t\|_\gamma^\gamma + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 dx \\ &\leq \frac{1}{2}\left(\|u_t\|_\gamma^\gamma + (1-l)c_s^2\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2 ds\right) \\ &\leq \frac{1}{2}\left(\|u_t\|_\gamma^\gamma + (1-l)c_s^2(go\nabla u)(t)\right),\end{aligned} \tag{3.32}$$

it follows from (3.28) that $\forall c > 0$

$$\begin{aligned}|I(t)| &= \left|\int_\Omega\int_0^1 e^{-2\rho\tau(t)}G(z(x,\rho,s))dkdx\right| \\ &\leq c\int_\Omega\int_0^1 G(z(x,\rho,s))d\rho dx.\end{aligned} \tag{3.33}$$

Hence, combining (3.30)-(3.33). This yields

$$\begin{aligned}|L(t)-ME(t)| &= \epsilon\phi(t)+\varphi(t)+\epsilon I(t) \\ &\leq C_\epsilon E^{\frac{\gamma-2}{2}}(0)E(t) + c\epsilon E(t)\epsilon\|u_t\|_\gamma^\gamma + \epsilon(1-l)c_s^2(go\nabla u)(t) \\ &+ c\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx.\end{aligned} \tag{3.34}$$

Where

$$|L(t)-ME(t)| \leq c_5E(t), \tag{3.35}$$

where $c_5 = max(c_1,c_2,c_3,c_4)$. Thus, from the definition of E(t) and selecting M sufficiently large,

$$\beta_2E(t) \leq L(t) \leq \beta_1E(t). \tag{3.36}$$

Where $\beta_1 = (M-\epsilon c_5)$, $\beta_2 = (M+\epsilon c_5)$. This completes the proof. □

Lemma 3.2. *Let (u,z) be the solution of (2.13). Then it holds*

$$\begin{aligned}\frac{d}{dt}\phi(t) &\leq \left(\frac{(1+l)(1-l)^2+(\mu_1+\mu_2)\alpha c_s^2}{2}-1\right)\|\nabla u\|_2^2 + \frac{(1-l)}{2}(go\nabla u)(t) \\ &+ \frac{\mu_2}{4\alpha}\|\psi(z(x,1,t))\|_2^2 + \|u_t\|_\gamma^\gamma + \frac{\mu_1}{4\alpha}\|\psi(u_t)\|_2^2.\end{aligned} \tag{3.37}$$

Proof. We take the derivative of $\phi(t)$. It follows from (3.26) that

$$\frac{d}{dt}\phi(t) = \int_\Omega (|u_t|^{\gamma-2}u_t)_t u dx + \|u_t\|_\gamma^\gamma, \tag{3.38}$$

using the problem (2.13), then we have

$$\begin{aligned}\frac{d}{dt}\phi(t) &= \|u_t\|_\gamma^\gamma - \|\nabla u\|_2^2 + \int_\Omega\int_0^t g(t-s)\nabla u(s)\nabla u(t)dsdx \\ &-\mu_2\int_\Omega \psi(z(x,1,t))u(t)dx - \mu_1\int_\Omega \psi(u_t)u(t)dx,\end{aligned} \tag{3.39}$$

we estimate the third term in the right hand side of (3.39) as follows

$$\begin{aligned}&\left|\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx\right| \\ &\leq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)ds\right)^2 dx \\ &\leq \frac{1+(1+\lambda)(1-l)^2}{2}\|\nabla u\|_2^2 + \frac{(1+\frac{1}{\lambda})(1-l)}{2}(go\nabla u)(t),\end{aligned} \tag{3.40}$$

for the forth and fifth term in (3.39), Holder and Young's to get

$$\left|\int_\Omega \psi(u_t)udx\right| \le \alpha c_s^2\|\nabla u\|_2^2 + \frac{1}{4\alpha}\|\psi(u_t)\|_2^2, \tag{3.41}$$

and

$$\left|\int_\Omega \psi(z(x,1,t))udx\right| \le \alpha c_s^2\|\nabla u\|_2^2 + \frac{1}{4\alpha}\|\psi(z(x,1,t))\|_2^2. \tag{3.42}$$

Let $\lambda = \frac{l}{1-l}$ in (3.40) and using (3.41), (3.42), then (3.40) becomes

$$\begin{aligned} \frac{d}{dt}\phi(t) &\le \epsilon\left(\frac{(1+l)(1-l)^2+(\mu_1+\mu_2)\alpha c_s^2}{2} - 1\right)\|\nabla u\|_2^2 + \epsilon\frac{(1-l)}{2}(go\nabla u)(t) \\ &+ \epsilon\frac{\mu_2}{4\alpha}\|\psi(z(x,1,t))\|_2^2 + \|u_t\|_\gamma^\gamma + \frac{\epsilon\mu_1}{4\alpha}\|\psi(u_t)\|_2^2. \end{aligned} \tag{3.43}$$

This completes the proof. □

Lemma 3.3. . *Let (u,z) be the solution of (2.13). Then $\varphi(t)$ satisfies*

$$\begin{aligned} \varphi'(t) &\le \alpha\left(1+2(1-l)^2\right)\|\nabla u\|_2^2 - (g_0-\alpha)\|u_t\|_\gamma^\gamma \\ &+ \mu_1\|\psi(u_t)\|_2^2 + \frac{g(0)c_s^2}{4\alpha}(-g'o\nabla u)(t) \\ &+ \frac{\mu_2}{4\alpha}(1-l)(2(\alpha+1)+c_s^2)(go\nabla u)(t) \\ &+ \frac{1}{4\alpha}c_s^2(1-l)^2\mu_2\int_\Omega \psi^2(z(x,1,t))dx. \end{aligned} \tag{3.44}$$

Proof. Now Taking the derivatives of $\varphi(t)$, using the problem (2.13), we obtain

$$\begin{aligned} \varphi'(t) &= -\int_\Omega (|u_t|^{\gamma-2}u_t)_t\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &- \int_\Omega |u_t|^{\gamma-2}u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega u_t^\gamma dx \\ &= \int_\Omega \nabla u(t)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx \\ &- \int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx \\ &+ \int_\Omega \mu_1\psi(u(t))\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &+ \int_\Omega \mu_2\psi(z(x,1,t))\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &- \int_\Omega |u_t|^{\gamma-2}u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega u_t^\gamma dx. \end{aligned} \tag{3.45}$$

Next we will estimate the right hand side of (3.45), using Holder, Young's inequalities and $(H1)$ to have

$$\begin{aligned} &\int_\Omega \nabla u\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx \\ &\le \alpha\|\nabla u\|_2^2 + \frac{(1-l)}{4\alpha}(go\nabla u)(t), \end{aligned} \tag{3.46}$$

and

$$
\begin{aligned}
&-\int_\Omega \left(\int_0^t g(t-s)\nabla u_s ds\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\\
&\leq \alpha\int_\Omega \left(\int_0^t g(t-s)(|\nabla u(t)|_2^2 ds\right)^2 dx\\
&+\frac{1}{\alpha}\int_\Omega \left|\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right|^2 dx\\
&\leq \alpha\int_\Omega \left(\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|)ds\right)^2 dx\\
&+\frac{1}{\alpha}\left(\int_0^t g(t-s)ds\right)\int_\Omega\int_0^t g(t-s)(|\nabla u(t)|-\nabla u(s)|^2 dsdx\\
&\leq \alpha\left(\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2 dx\\
&+2\alpha(1-l)^2|\nabla u(t)|_2^2+\frac{1}{4\alpha}(1-l)(go\nabla u)(t)\\
&\leq 2\alpha(1-l)^2\|\nabla u(t)\|_2^2+\left(2\alpha+\frac{1}{4\alpha}\right)(1-l)(go\nabla u)(t),
\end{aligned}
\tag{3.47}
$$

where g is positive, continuous and $g(0)>0$, for any t_0, we have

$$
\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0, \quad \forall t\geq t_0,
\tag{3.48}
$$

then we use (3.48) to get

$$
\begin{aligned}
&\int_\Omega |u_t|^{\gamma-2}u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g(s)ds\right)\int_\Omega u_t^\gamma dx\\
&\leq \alpha\|u_t\|_\gamma^\gamma+\frac{g(0)c_s^2}{4\alpha}(-g'o\nabla u)(t)-g_0\|u_t\|_\gamma^\gamma,
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
&\left|-\int_\Omega \mu_1\psi(u_t)\int_0^t g(t-s)(u(t)-u(s))dsdx\right|\\
&\leq \mu_1\|\psi(u_t)\|_2^2+\frac{\mu_1(1-l)c_s^2}{4\alpha}(go\nabla u)(t),
\end{aligned}
\tag{3.50}
$$

and

$$
\begin{aligned}
&\left|-\int_\Omega \mu_2\psi(z(x,1,t))\int_0^t g(t-s)(u(t)-u(s))dsdx\right|\\
&\leq \mu_2\int_\Omega \psi^2(z(x,1,t))dx+\frac{\mu_2(1-l)c_s^2}{4\alpha}(go\nabla u)(t).
\end{aligned}
\tag{3.51}
$$

A substitution of (3.49)-(3.51) into (3.47) yields

$$
\begin{aligned}
\varphi'(t)\leq{}& \alpha\left(1+2(1-l)^2\right)\|\nabla u\|_2^2-(g_0-\alpha)\|u_t\|_\gamma^\gamma\\
&+\mu_1\|\psi(u_t)\|_2^2+\frac{g(0)c_s^2}{4\alpha}(-g'o\nabla u)(t)\\
&+\frac{\mu_2}{4\alpha}(1-l)(2(\alpha+1)+c_s^2)(go\nabla u)(t)\\
&+\frac{1}{4\alpha}c_s^2(1-l)^2\mu_2\int_\Omega \psi^2(z(x,1,t))dx.
\end{aligned}
\tag{3.52}
$$

□

Lemma 3.4. *. The functional defined by (3.28) can be estimated by*

$$
\frac{d}{dt}I(t)\leq -2I(t)-\frac{c\xi(t)}{2\tau_1}\int_\Omega G(z_1(x,1,t))dx+\frac{\xi(t)}{\tau_0}\|\psi(u_t)\|_2^2
\tag{3.53}
$$

where τ_0, τ_2 are some positive constant.

Proof. Differentiating (3.28) with respect to t and using the second equation in (2.13), we have

$$
\begin{aligned}
\frac{I(t)}{dt} &\leq \frac{d}{dt}\left[\xi(t)e^{-\rho\tau(t)}\int_\Omega\int_0^1 z^2(x,\rho,t))d\rho dx\right] \\
&= \left[\xi'(t)e^{-\tau(t)\rho}\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx - \xi(t)\rho e^{-\tau(t)\rho}\tau'(t)\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx\right] \\
&+ \frac{1}{\tau(t)}e^{-\tau(t)\rho}\tau(t)\xi(t)\int_\Omega\int_0^1 \frac{d}{dt}G(z(x,\rho,t))d\rho dx \\
&= \left[\xi'(t)e^{-\tau(t)\rho}\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx - \xi(t)\rho e^{-\tau(t)\rho}\tau'(t)\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx\right] \\
&+ \frac{1}{\tau(t)}e^{-\tau(t)\rho}\xi(t)\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}(1-\tau'(t)\rho)G(z(x,\rho,t))d\rho dx \\
&\leq -\xi(t)\rho e^{-\tau(t)\rho}\tau'(t)\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx + \xi(t)\frac{\beta}{\tau(t)}\int_\Omega G(z(x,1,t))dx \\
&+ \frac{1}{\tau(t)}\left[\xi(t)\int_\Omega [G(z(x,0,t))dx - G(z(x,1,t))]dx\right] \\
&\leq -2c\xi(t)I(t) - \frac{c\xi(t)}{2\tau_1}\int_\Omega G(z(x,1,t))dx + \frac{\xi(t)}{\tau_0}\|\psi(u_t)\|_2^2,
\end{aligned}
\tag{3.54}
$$

□

Theorem 3.1. . *Let* $(H1)-(H4)$ *hold. And* $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Omega\times(0,1))$ *be given. Then the solution of the problem (2.13) is global and bounded in time. Furthermore, we have the following decay estimates:*

$$E(t) \leq \omega_1 H_1^{-1}(\omega_2 t + \omega_3), \quad \forall t > 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}ds. \tag{3.55}$$

Proof. . First, we prove $T=\infty$, it is sufficient to show that $l\|\nabla u\|_2^2$ is bounded independently of t. We have from (2.15)

$$E(0) \geq E(t) \geq l\|\nabla u\|_2^2.$$

Then the energy is uniformly bounded.
Hence we conclude from lemma 3.2, lemma 3.3 and 3.4 that

$$
\begin{aligned}
\frac{dL(t)}{dt} &\leq \left\{\frac{M}{2} - \frac{\epsilon g(0)c_s^2}{4\alpha}\right\}(g' o\nabla u)(t) + \left\{\epsilon\left(\frac{\mu_2(1-l)}{4\alpha}\left(2(\alpha+1)+c_s^2+\frac{1-l}{2}\right)\right)\right\}(go\nabla u)(t) \\
&- \left\{\epsilon\left((1-\alpha-(1-l)^2(2+(1+l))) - (\mu_1+\mu_2)\alpha c_s^2\right)\right\}\|\nabla u\|_2^2 - \epsilon(g_0-\alpha-1)\|u_t\|_\gamma^\gamma \\
&- Mc_1\int_\Omega u_t\psi(u_t)dx - Mc_2\int_\Omega z(x,1,t)\psi(z(x,1,t))dx + \frac{\epsilon\mu_1}{4\alpha}\|\psi(u_t)\|_2^2 + \frac{\xi(t)}{\tau_0}\|\|\psi(u_t)\|_2^2 \\
&+ \epsilon\left\{\frac{\mu_2 c_s^2(1-l)^2}{4\alpha} + \frac{\mu_2}{4\alpha}\right\}\|\psi(z(x,1,t))\|_2^2 - 2\epsilon\int_\Omega\int_0^1 e^{-2\rho\tau(t)}G(z(x,\rho,t))d\rho dx \\
&- \epsilon\frac{e^{-2\tau(t)}}{\tau_1}\int_\Omega G(z(x,1,t))dx + \frac{\epsilon}{\tau_0}\int_\Omega G(z(x,0,t))dx - \frac{M}{2}g(t)\|\nabla u(t)\|_2^2.
\end{aligned}
\tag{3.56}
$$

Choosing carefully ϵ sufficiently small and M sufficiently large and put

$$\left\{\frac{M}{2} - \frac{\epsilon g(0)c_s^2}{4\alpha}\right\} = \eta_0 > 0,$$

$$\left\{\frac{\mu_2(1-l)}{2\alpha}\left(2(\alpha+1)+c_s^2+\frac{1-l}{2}\right)\right\} = \eta_1 > 0,$$

$$\left\{(1-\alpha-(1-l)^2(2+(1+l))) - (\mu_1+\mu_2)\alpha c_s^2\right\} = \eta_2 > 0,$$

$$\{g_0 - \alpha - 1\} = \eta_3 > 0,$$

then (3.56) takes the form

$$\frac{dL(t)}{dt} \leq -\theta\epsilon E(t) + \epsilon\frac{\eta_1}{2}(go\nabla u)(t) + \epsilon c\|\psi(u_t)\|_2^2, \tag{3.57}$$

where θ is positive constant, setting

$$\lambda_1 = \frac{\theta\epsilon}{\beta_2},\ \lambda_2 = \frac{\eta_1\epsilon}{2},\ \lambda_3 = \epsilon c,$$

the last inequality becomes

$$\frac{dL(t)}{dt} \leq -\lambda_1 E(t) + \lambda_2(go\nabla u)(t) + \lambda_3\|\psi(u_t)\|_2^2, \tag{3.58}$$

multiplying (3.58) by $\xi(t)$ we get

$$\begin{aligned}\xi(t)\frac{dL(t)}{dt} &\leq -\lambda_1\xi(t)E(t) + \lambda_2\xi(t)(go\nabla u)(t) + \lambda_3\xi(t)\|\psi(u_t)\|_2^2 \\ &\leq -\lambda_1\xi(t)E(t) - \lambda_2\xi(t)(g^{'}o\nabla u)(t) + \lambda_3\xi(t)\|\psi(u_t)\|_2^2 \\ &\leq -\lambda_1\xi(t)E(t) - cE^{'}(t) + \lambda_3\xi(t)\|\psi(u_t)\|_2^2,\end{aligned} \tag{3.59}$$

we consider the following partition on Γ_1

$$\Omega_{11} = \{x \in \Omega; |u_t| \geq \epsilon^{'}\}, \quad \Omega_{12} = \{x \in \Omega; |u_t| \leq \epsilon^{'}\},$$

then it is clear that $F = L(t) + c\xi(t)E(t)$ is equivalent to E(t), then

$$F^{'}(t) \leq -\lambda_1\xi(t)E(t) + \lambda_3\xi(t)\|\psi(u_t)\|_2^2 \quad \forall t \geq t_0, \tag{3.60}$$

from (2.2) and (2.3), it follows that

$$\int_{\Omega_{12}} |\psi(u_t)|^2 dx \leq \mu_1 \int_{\Omega_{12}} u_t\|\psi(u_t)\|_2^2 dx \leq -\mu_1 E^{'}(t). \tag{3.61}$$

case 1: H is linear then, according to $(H1)$

$$c_1^{'}|s| \leq |\psi(s)| \leq c_2^{'}|s|, \quad \forall s,$$

and so

$$\psi^2(s) \leq c_2^{'}s\psi(s), \quad \forall s.$$

H is linear on $[0,\epsilon^{'}]$. In this case one can easily check that there exists $\mu_1^{'} > 0$, such that $|\psi(s)| \leq \mu_1^{'}|s|$ for all $|s| \leq \epsilon^{'}$, and thus

$$\int_{\Omega_{11}} \|\psi(u_t)\|^2 dx \leq \mu_1^{'} \int_{\Omega_{11}} u_t\psi(u_t)dx \leq -\mu_1^{'}E^{'}(t), \tag{3.62}$$

using (3.61), (3.62) and the fact that $\xi^{'}(t) \leq 0$, it is clearly that $\vartheta = L(t)\xi(t) + c(\mu_1 + \mu_1^{'})E$ equivalent to E(t) then, from (3.60) produces

$$E(t) \leq ce^{-c\int_0^t \xi(s)ds} = H_1^{-1}\left(\int_0^t \xi(s)ds\right). \tag{3.63}$$

case2 : $H^{'}(0) = 0$ and $H^{''} > 0$ on $[0,\epsilon^{'}]$ since H is convex and increasing H^{-1} is concave and increasing by Jensen's inequality

$$\begin{aligned}\int_{\Omega_{12}} |\psi(u_t)|^2 dx &\leq \int_{\Omega_{12}} H^{-1}\left(u_t\psi(u_t)dx\right) \\ &\leq |\Omega_{12}|H^{-1}\left(\frac{1}{|\Omega_{12}|}u_t\psi(u_t)dx\right) \\ &\leq cH^{-1}(-c^{'}E^{'}(t)),\end{aligned} \tag{3.64}$$

then using (2.1), (3.62) and (3.64) we get

$$\begin{aligned}\int_{\Omega}|\psi(u_t)|^2dx &= \int_{\Omega_{11}}|\psi(u_t)|^2dx+\int_{\Omega_{12}}|\psi(u_t)|^2dx\\ &\le \int_{\Omega_{12}}H^{-1}u_t\psi(u_t)dx+\int_{\Omega_{12}}u_t\psi(u_t)dx\\ &\le |\Omega_{12}|H^{-1}\left(\frac{1}{|\Omega_{12}|}u_t\psi(u_t)dx\right)+\int_{\Omega_{12}}u_t\psi(u_t)dx\\ &\le cH^{-1}(-c^{'}E^{'}(t))-c\xi(t)\mu_1^{'}E^{'}(t),\end{aligned} \tag{3.65}$$

it is clearly $F=L(t)+c\mu_1E(t)$ equivalent to E(t) therefore (3.65) becomes

$$F^{'}(t)\le\lambda_1\xi(t)E(t)+cH^{-1}(-c^{'}E^{'}(t)),\quad \forall t\ge t_0. \tag{3.66}$$

Let us denote by H^* the conjugate function of the convex function H, i.e.,

$$H^*=\sup_{t\in R_+}(st-H(t)). \tag{3.67}$$

Then H^* is the Legendre transform of H which satisfies the following inequality

$$st\le H^*+H(t),\quad \forall s,t\ge 0, \tag{3.68}$$

and

$$H^*=s(H^{'})^{-1}(s)-H[(H^{'})^{-1}(s)],\forall s\ge 0, \tag{3.69}$$

the relation (3.69) and the fact that $H^{'}(0)=0$ and $(H^{'})^{-1}$, H are increasing function yield

$$H^*(s)\le s(H^{'})^{-1}(s),\quad \forall s\ge 0, \tag{3.70}$$

using the fact that $E^{'}\le 0$, $H^{'}\ge 0$, $H^{''}\ge 0$ we derive $\epsilon_0>0$ small enough we find that the functional F_1 defined by

$$F_1(t)=H^{'}(\epsilon_0E(t))F(t)+c_3E(t), \tag{3.71}$$

satisfies, for some $\nu_1,\nu_2>0$

$$\nu_1F_1(t)\le E(t)\le\nu_2F_1(t), \tag{3.72}$$

taking the derivative of (3.71)

$$\begin{aligned}F_1^{'}(t) &= \epsilon_0E^{'}(t)H^{''}(\epsilon_0E(t))(H^{'}(\epsilon_0E(t))F(t)+c_3E(t))+H^{'}(\epsilon_0E(t))(L^{'}(t)+c\mu_1E^{'}(t))+c_3E^{'}(t)\\ &\le -\lambda_1\xi(t)E(t)H^{'}(\epsilon_0E(t))+\hat{c}_3H^{'}(\epsilon_0E(t))H^{-1}(-c^{'}E^{'}(t))+\hat{c}_3c^{'}E^{'}(t)\\ &\le -\lambda_1\xi(t)E(t)H^{'}(\epsilon_0E(t))+\hat{c}_3H^*(H^{'}(\epsilon_0E(t)))-\hat{c}_3\xi(t)E^{'}(t)+c_3E^{'}(t)\\ &\le -\lambda_1\xi(t)E(t)H^{'}(\epsilon_0E(t))+\epsilon_0\hat{c}_3\xi(t)E(t)(H^{'}(\epsilon_0E(t)))-\hat{c}_3\xi(t)E^{'}(t)+c_3E^{'}(t)\\ &\le -c\xi(t)H_2E(t),\end{aligned} \tag{3.73}$$

where $H_2(t)=tH^{'}(\epsilon_0t)$ we can observe from lemma 3.1 that L(t) is equivalent to E(t). So, $F_1(t)$ is also equivalent to E(t). By the fact that H_2 is increasing we obtain

$$F_1^{'}(t)\le-\hat{c}\xi(t)H_2F_1(t),\forall t\ge 0. \tag{3.74}$$

Noting that $H_1^{'}=\frac{-1}{H_2}$, we infer from (3.74)

$$[F_1(t)H_1(F_1(t))]^{'}\ge\hat{c}\xi(t),\forall t\ge 0. \tag{3.75}$$

A simple integration over (0,t) yields

$$H_1(F_1(t))\ge\hat{c}\int_0^t\xi(s)ds+H_1(F_1(0)), \tag{3.76}$$

exploiting the fact that H_1^{-1} is decreasing , we infer

$$F_1(t) \leq H_1^{-1}\left(\hat{c}\int_0^t \xi(s)ds + H_1(F_1(0))\right), \tag{3.77}$$

the equivalence of L, F_1 and E, yields the estimate

$$E(t) \leq H_1^{-1}\left(\hat{c}\int_0^t \xi(s)ds + H_1(F_1(0))\right). \tag{3.78}$$

Which completes the proof. □

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On the Spectral Expansion Formula for a Class of Dirac Operators

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Abstract

This paper deals with a problem for the canonical Dirac differential equations system with piecewise continuous coefficient and spectral parameter dependent in boundary conditions. The resolvent operator is constructed. The completeness theorem for eigenvector functions is proved. The spectral expansion formula with respect to eigenvector functions is obtained and Parseval equality is given.

Keywords: Dirac operator, completeness theorem, expansion formula.

2010 MSC: 34L10, 34L40.

1 Introduction

Consider the following boundary value problem generated by Dirac differential equations system

$$By' + \Omega(x)y = \lambda\rho(x)y, \quad 0 < x < \pi \tag{1.1}$$

with boundary conditions

$$\begin{aligned} U_1(y) &:= b_1 y_2(0) + b_2 y_1(0) - \lambda\left(b_3 y_2(0) + b_4 y_1(0)\right) = 0, \\ U_2(y) &:= c_1 y_2(\pi) + c_2 y_1(\pi) + \lambda\left(c_3 y_2(\pi) + c_4 y_1(\pi)\right) = 0, \end{aligned} \tag{1.2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$p(x)$, $q(x)$ are real measurable functions, $p(x) \in L_2(0,\pi)$, $q(x) \in L_2(0,\pi)$, λ is a spectral parameter,

$$\rho(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha, & a < x \le \pi, \end{cases}$$

and $1 \ne \alpha > 0$. Let us define $k_1 = b_1 b_4 - b_2 b_3 > 0$, $k_2 = c_1 c_4 - c_2 c_3 > 0$.

In the finite interval, the spectral properties of Dirac operators by different aspects are examined by many authors, for example [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and etc. In the case of $\rho(x) \ne 1$, the solution of Dirac system was investigated in [5], in this work the representation of this solution has not operator transformation. The asymptotic formulas of eigenvalues and eigenfunctions of Dirac operator with discontinuous coefficient $\rho(x)$ were studied in [1]. Numerical computation of eigenvalues of Dirac system was worked in [11], [12]. Moreover, the theory of Dirac operators was comprehensively given in [6], [10].

This paper is organized as follows: in section 2, the operator formulation of the boundary value problem (1.1),(1.2) and the asymptotic formula of eigenvalues of the problem (1.1),(1.2) are given. In section 3, we prove completeness theorem of eigenfunctions. The expansion formula with respect to eigenfunctions and Parseval equality are obtained.

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2 Preliminaries

An inner product in Hilbert space $H_\rho = L_{2,\rho}(0,\pi;\mathbb{C}^2) \oplus \mathbb{C}^2$ is given by

$$\langle Y, Z\rangle = \int_0^\pi \left\{ y_1(x)\overline{z_1(x)} + y_2(x)\overline{z_2(x)} \right\} \rho(x)\,dx + \frac{1}{k_1} y_3\overline{z_3} + \frac{1}{k_2} y_4\overline{z_4}, \tag{2.3}$$

where

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3 \\ y_4 \end{pmatrix} \in H_\rho, \quad Z = \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3 \\ z_4 \end{pmatrix} \in H_\rho.$$

Let us define the operator L:

$$L(Y) := \begin{pmatrix} l(y) \\ b_1 y_2(0) + b_2 y_1(0) \\ -(c_1 y_2(\pi) + c_2 y_1(\pi)) \end{pmatrix}$$

with domain

$$D(L) := \Big\{ Y \mid Y = (y_1(x), y_2(x), y_3, y_4)^T \in H_\rho,\ y_1(x), y_2(x) \in AC[0,\pi],$$
$$y_3 = b_3 y_2(0) + b_4 y_1(0),\ y_4 = c_3 y_2(\pi) + c_2 y_1(\pi), l(y) \in L_{2,\rho}(0,\pi;\mathbb{C}^2) \Big\}$$

where

$$l(y) = \frac{1}{\rho(x)} \left\{ By' + \Omega(x) y \right\}.$$

Consequently, the boundary value problem (1.1),(1.2) is equivalent to the operator equation $LY = \lambda Y$.

Lemma 2.1. *The following properties for the operator L are valid:*

- *The eigenvector functions corresponding to different eigenvalues are orthogonal,*
- *The eigenvalues are real valued.*

Let $\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$ and $\psi(x,\lambda) = \begin{pmatrix} \psi_1(x,\lambda) \\ \psi_2(x,\lambda) \end{pmatrix}$ be solutions of the system (1.1) satisfying the initial conditions

$$\varphi(0,\lambda) = \begin{pmatrix} \lambda b_3 - b_1 \\ b_2 - \lambda b_4 \end{pmatrix}, \quad \psi(\pi,\lambda) = \begin{pmatrix} -c_1 - \lambda c_3 \\ c_2 + \lambda c_4 \end{pmatrix}.$$

The characteristic function of the problem (1.1),(1.2) is defined by

$$\Delta(\lambda) = W[\varphi(x,\lambda), \psi(x,\lambda)] = \varphi_2(x,\lambda)\psi_1(x,\lambda) - \varphi_1(x,\lambda)\psi_2(x,\lambda), \tag{2.4}$$

where $W[\varphi(x,\lambda), \psi(x,\lambda)]$ is Wronskian of the vector solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$. The Wronskian does not depend on x. It follows from (2.4) that

$$\Delta(\lambda) = b_2\psi_1(0,\lambda) + b_1\psi_2(0,\lambda) - \lambda\left(b_4\psi_1(0,\lambda) + b_3\psi_2(0,\lambda)\right) = U_1(\psi)$$

or

$$\Delta(\lambda) = -c_1\varphi_2(\pi,\lambda) - c_2\varphi_1(\pi,\lambda) - \lambda\left(c_3\varphi_2(\pi,\lambda) + c_4\varphi_1(\pi,\lambda)\right) = -U_2(\varphi).$$

Moreover, the zeros λ_n of characteristic function coincide with the eigenvalues of the boundary value problem (1.1),(1.2). The function $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions and there exist a sequence β_n such that

$$\psi(x,\lambda_n) = \beta_n \varphi(x,\lambda_n), \quad \beta_n \neq 0. \tag{2.5}$$

Definition 2.1. *Norming constants of the boundary value problem (1.1), (1.2) are defined as follows:*

$$\alpha_n := \int_0^\pi \left\{ \varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n) \right\} \rho(x)dx + \frac{1}{k_1}\left[b_3\varphi_2(0,\lambda_n) + b_4\varphi_1(0,\lambda_n)\right]^2 +$$
$$+ \frac{1}{k_2}\left[c_3\varphi_2(\pi,\lambda_n) + c_4\varphi_1(\pi,\lambda_n)\right]^2. \tag{2.6}$$

Lemma 2.2. *[7] The following relation is valid:*

$$\alpha_n \beta_n = \dot{\Delta}(\lambda_n), \tag{2.7}$$

where $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$.

Now, as a different from other works, the system (1.1) has $\rho(x)$ discontinuous coefficient. This coefficient influences the form of the solution of the equation (1.1). Therefore, the solution of the equation (1.1) has the integral representation (not operator transformation) as follows (detail in [5]): Assume that

$$\int_0^{\pi} \|\Omega(x)\| \, dx < +\infty$$

is satisfied for Euclidean norm of matrix function $\Omega(x)$. Then the integral representation of the solution of equation (1.1) satisfying the initial condition $E(0,\lambda) = I$, (I is unite matrix) can be represented

$$E(x,\lambda) = e^{-\lambda B \mu(x)} + \int_{-\mu(x)}^{\mu(x)} K(x,t) e^{-\lambda B t} dt,$$

where

$$\mu(x) = \begin{cases} x, & 0 \le x \le a, \\ \alpha x - \alpha a + a, & a < x \le \pi, \end{cases}$$

and for a kernel $K(x,t)$ the inequality

$$\int_{-\mu(x)}^{\mu(x)} \|K(x,t)\| \, dt \le e^{\sigma(x)} - 1,$$

$$\sigma(x) = \int_0^x \|\Omega(s)\| \, ds$$

holds.

Using this integral representation, the following lemma is proved:

Lemma 2.3. *[7] The solution* $\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$ *has the following integral representation*

$$\begin{aligned} \varphi_1(x,\lambda) &= (\lambda b_3 - b_1) \cos \lambda \mu(x) + (\lambda b_4 - b_2) \sin \lambda \mu(x) + \\ &+ (\lambda b_3 - b_1) \int_0^{\mu(x)} \left[\tilde{A}_{11}(x,t) \cos \lambda t + \approx{A}_{12}(x,t) \sin \lambda t \right] dt + \\ &+ (\lambda b_4 - b_2) \int_0^{\mu(x)} \left[\approx{A}_{11}(x,t) \sin \lambda t - \tilde{A}_{12}(x,t) \cos \lambda t \right] dt, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \varphi_2(x,\lambda) &= (\lambda b_3 - b_1) \sin \lambda \mu(x) + (b_2 - \lambda b_4) \cos \lambda \mu(x) + \\ &+ (\lambda b_3 - b_1) \int_0^{\mu(x)} \left[\tilde{A}_{21}(x,t) \cos \lambda t + \approx{A}_{22}(x,t) \sin \lambda t \right] dt + \\ &+ (\lambda b_4 - b_2) \int_0^{\mu(x)} \left[\approx{A}_{21}(x,t) \sin \lambda t - \tilde{A}_{22}(x,t) \cos \lambda t \right] dt, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &\tilde{A}_{1j}(x,t) = K_{1j}(x,-t) + K_{1j}(x,t), \\ &\approx{A}_{1j}(x,t) = K_{1j}(x,t) - K_{1j}(x,-t), \\ &\tilde{A}_{2j}(x,t) = K_{2j}(x,-t) + K_{2j}(x,t), \\ &\approx{A}_{2j}(x,t) = K_{2j}(x,t) - K_{2j}(x,-t), \end{aligned}$$

and $\tilde{A}_{1j}(x,.) \in L_2(0,\pi)$, $\approx{A}_{1j}(x,.) \in L_2(0,\pi)$, $\tilde{A}_{2j}(x,.) \in L_2(0,\pi)$, $\approx{A}_{2j}(x,.) \in L_2(0,\pi)$, $j = 1,2$.

Moreover, using (2.8) and (2.9), as $|\lambda| \to \infty$ uniformly in $x \in [0, \pi]$, the following asymptotic formulas hold:

$$\varphi_1(x, \lambda) = \lambda \left(b_3 \cos \lambda \mu(x) + b_4 \sin \lambda \mu(x)\right) + O\left(e^{|Im\lambda| \mu(x)}\right), \tag{2.10}$$

$$\varphi_2(x, \lambda) = \lambda \left(b_3 \sin \lambda \mu(x) - b_4 \cos \lambda \mu(x)\right) + O\left(e^{|Im\lambda| \mu(x)}\right). \tag{2.11}$$

Lemma 2.4. *[7] The eigenvalues $\lambda_n, (n \in \mathbb{Z})$ of the boundary value problem (1.1),(1.2) are in the form*

$$\lambda_n = \tilde{\lambda}_n + \epsilon_n,$$

where

$$\tilde{\lambda}_n = \left[n + \frac{1}{\pi} \arctan\left(\frac{c_3 b_4 - c_4 b_3}{b_3 c_3 + c_4 b_4}\right)\right] \frac{\pi}{\mu(\pi)}$$

and $\{\epsilon_n\} \in l_2$. Moreover, the eigenvalues are simple.

3 Completeness Theorem

Firstly, we construct the resolvent operator and then we prove the completeness theorem of the eigenfunctions of the problem (1.1), (1.2). The expansion formula respect to eigenfunctions is obtained and Parseval equality is given.

Lemma 3.5. *If λ is not a spectrum point of operator L, then the resolvent operator exists and has the following form*

$$y(x, \lambda) = \int_0^\pi R_\lambda(x, t) f(t) \rho(t) dt + \frac{f_4}{\Delta(\lambda)} \varphi(x, \lambda) + \frac{f_3}{\Delta(\lambda)} \psi(x, \lambda), \tag{3.12}$$

where

$$R_\lambda(x, t) = -\frac{1}{\Delta(\lambda)} \begin{cases} \psi(x, \lambda) \widetilde{\varphi}(t, \lambda), & t \le x, \\ \varphi(x, \lambda) \widetilde{\psi}(t, \lambda), & t \ge x, \end{cases} \tag{3.13}$$

here $\widetilde{\varphi}(t, \lambda)$ denotes the transposed vector function of $\varphi(t, \lambda)$.

Proof. Let $F(x) = \begin{pmatrix} f(x) \\ f_3 \\ f_4 \end{pmatrix} \in D(L)$, $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$. To construct the resolvent operator of L, we solve the following problem

$$By' + \Omega(x) y = \lambda \rho(x) y + \rho(x) f(x) \tag{3.14}$$

$$\begin{gathered} b_1 y_2(0) + b_2 y_1(0) - \lambda \left(b_3 y_2(0) + b_4 y_1(0)\right) = f_3, \\ c_1 y_2(\pi) + c_2 y_1(\pi) + \lambda \left(c_3 y_2(\pi) + c_4 y_1(\pi)\right) = -f_4. \end{gathered} \tag{3.15}$$

By applying the method of variation of parameters, we want to find the solution of problem (1.1), (1.2) which has a form

$$y(x, \lambda) = c_1(x, \lambda) \varphi(x, \lambda) + C_2(x, \lambda) \psi(x, \lambda). \tag{3.16}$$

Then, we get the equations system

$$c_1'(x, \lambda) \tilde{\psi}(x, \lambda) B \varphi(x, \lambda) = \tilde{\psi}(x, \lambda) \rho(x) f(x),$$

$$c_2'(x, \lambda) \tilde{\varphi}(x, \lambda) B \psi(x, \lambda) = \tilde{\varphi}(x, \lambda) \rho(x) f(x).$$

Using this system, we have

$$c_1(x, \lambda) = c_1(\pi, \lambda) - \frac{1}{\Delta(\lambda)} \int_x^\pi \tilde{\psi}(t, \lambda) f(t) \rho(t) dt, \tag{3.17}$$

$$c_2(x, \lambda) = c_2(0, \lambda) - \frac{1}{\Delta(\lambda)} \int_0^x \tilde{\varphi}(t, \lambda) f(t) \rho(t) dt. \tag{3.18}$$

Substituting the expression (3.17) and (3.18) into (3.16), we find

$$y(x,\lambda) = c_1(\pi,\lambda)\varphi(x,\lambda) + c_2(0,\lambda)\psi(x,\lambda) + \int_0^\pi R_\lambda(x,\lambda)f(t)\rho(t)dt,$$

where

$$R_\lambda(x,t) = -\frac{1}{\Delta(\lambda)}\begin{cases} \psi(x,\lambda)\widetilde{\varphi}(t,\lambda), & t \le x, \\ \varphi(x,\lambda)\widetilde{\psi}(t,\lambda), & t \ge x. \end{cases}$$

Taking the boundary conditions (3.15), we have

$$c_1(\pi,\lambda) = \frac{f_4}{\Delta(\lambda)}, \qquad c_2(0,\lambda) = \frac{f_3}{\Delta(\lambda)}.$$

Consequently,

$$y(x,\lambda) = \int_0^\pi R_\lambda(x,t)f(t)\rho(t)dt + \frac{f_4}{\Delta(\lambda)}\varphi(x,\lambda) + \frac{f_3}{\Delta(\lambda)}\psi(x,\lambda)$$

is obtained. □

Theorem 3.1. *The system of the eigenfunctions* $\{\varphi(x,\lambda_n)\}$, $(n \in \mathbb{Z})$ *of boundary value problem (1.1),(1.2) is complete in* $L_{2,\rho}(0,\pi;\mathbb{C}^2) \oplus \mathbb{C}^2$.

Proof. Taking into account (2.5) and (2.7) that $\psi(x,\lambda_n) = \frac{\dot{\Delta}(\lambda_n)}{\alpha_n}\varphi(x,\lambda_n)$. Using (3.12), (3.13) and this equality, we get

$$\underset{\lambda=\lambda_n}{Res}\ y(x,\lambda) = -\frac{1}{\alpha_n}\varphi(x,\lambda_n)\left\{\int_0^\pi \widetilde{\varphi}(x,\lambda_n)f(t)\rho(t)\,dt - \frac{f_4}{\beta_n} - f_3\right\}. \tag{3.19}$$

Let $F(x) \in L_{2,\rho}(0,\pi;\mathbb{C}^2) \oplus \mathbb{C}^2$ be such that

$$\langle F(x), \varphi(x,\lambda_n)\rangle = \int_0^\pi \widetilde{\varphi}(t,\lambda_n)f(t)\rho(t)dt + \frac{1}{k_1}f_3\left[b_3\varphi_2(0,\lambda_n) + b_4\varphi_1(0,\lambda_n)\right] +$$

$$+\frac{1}{k_2}f_4\left[c_3\varphi_2(\pi,\lambda_n) + c_4\varphi_1(\pi,\lambda_n)\right] = 0.$$

It follows from the boundary conditions (1.2) and (2.5) that

$$b_3\varphi_2(0,\lambda_n) + b_4\varphi_1(0,\lambda_n) = -k_1$$

and

$$c_3\varphi_2(\pi,\lambda_n) + c_4\varphi_1(\pi,\lambda_n) = -\frac{k_2}{\beta_n}.$$

Thus,

$$\langle F(x), \varphi(x,\lambda_n)\rangle = \int_0^\pi \widetilde{\varphi}(t,\lambda_n)f(t)\rho(t)dt - f_3 - \frac{f_4}{\beta_n} = 0$$

is found. From here and (3.19), $Res_{\lambda=\lambda_n} y(x,\lambda) = 0$ is obtained. Hence, $y(x,\lambda)$ is entire function with respect to λ for each fixed $x \in [0,\pi]$. The following inequality is similarly obtained as in ([8], Lemma 1.3.2)

$$|\Delta(\lambda)| \ge |\lambda|^2 C_\delta \exp(|Im\lambda|\,\mu(\pi)) \tag{3.20}$$

which is valid in the domain

$$G_\delta := \left\{\lambda : \left|\lambda - \tilde{\lambda}_n\right| \ge \delta,\ \ n = 0, \pm 1, \pm 2...\right\},$$

where δ is a sufficiently small positive number. Taking into account the inequality (3.20) and the following equalities (see [1])

$$\lim_{|\lambda|\to\infty}\max_{0\le x\le\pi}\exp\left(-|Im\lambda|\,\mu(x)\right)\left|\int_0^x \widetilde{\varphi}(t,\lambda)f(t)\rho(t)dt\right| = 0, \tag{3.21}$$

$$\lim_{|\lambda|\to\infty}\max_{0\le x\le\pi}\exp\left(-|Im\lambda|\left(\mu(\pi)-\mu(x)\right)\right)\left|\int_x^\pi \widetilde{\psi}(t,\lambda)f(t)\rho(t)dt\right| = 0, \tag{3.22}$$

we have

$$\lim_{|\lambda|\to\infty}\max_{0\le x\le\pi}|y(x,\lambda)| = 0.$$

Consequently, $y(x,\lambda) \equiv 0$. From (3.14) and (3.15), $F(x) = 0$ a.e. on $(0,\pi)$ is obtained. □

Theorem 3.2. *Let $F(x) \in D(L)$. Then the following expansion formula holds:*

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \varphi(x, \lambda_n), \tag{3.23}$$

$$f_3 = \sum_{n=-\infty}^{\infty} a_n \left[b_3 \varphi_2(0, \lambda_n) + b_4 \varphi_1(0, \lambda_n)\right], \tag{3.24}$$

$$f_4 = \sum_{n=-\infty}^{\infty} a_n \left[b_3 \varphi_2(\pi, \lambda_n) + b_4 \varphi_1(\pi, \lambda_n)\right], \tag{3.25}$$

where

$$a_n = \frac{1}{\alpha_n} \langle f(x), \varphi(x, \lambda_n) \rangle .$$

The series converges uniformly with respect to $x \in [0, \pi]$. The series (3.23)-(3.25) converges in $L_{2,\rho}(0, \pi; \mathbb{C}^2) \oplus \mathbb{C}^2$ for $F(x) \in L_{2,\rho}(0, \pi; \mathbb{C}^2) \oplus \mathbb{C}^2$ and Parseval equality

$$\|F\|^2 = \sum_{n=-\infty}^{\infty} \alpha_n |a_n|^2 \tag{3.26}$$

is valid.

Proof. Since $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ are solution of the problem (1.1),(1.2),

$$\begin{aligned} y(x,\lambda) &= -\frac{1}{\lambda\Delta(\lambda)}\psi(x,\lambda)\int_0^x \left\{-\frac{\partial}{\partial t}\widetilde{\varphi}(t,\lambda)B + \widetilde{\varphi}(t,\lambda)\Omega(t)\right\} f(t)dt \\ &\quad -\frac{1}{\lambda\Delta(\lambda)}\varphi(x,\lambda)\int_x^\pi \left\{-\frac{\partial}{\partial t}\widetilde{\psi}(t,\lambda)B + \widetilde{\psi}(t,\lambda)\Omega(t)\right\} f(t)dt + \frac{f_4}{\Delta(\lambda)}\varphi(x,\lambda) + \frac{f_3}{\Delta(\lambda)}\psi(x,\lambda) \end{aligned}$$

can be written. Integrating by parts and using the expression of Wronskian

$$y(x,\lambda) = -\frac{1}{\lambda} f(x) - \frac{1}{\lambda} z(x,\lambda) + \frac{f_4}{\Delta(\lambda)}\varphi(x,\lambda) + \frac{f_3}{\Delta(\lambda)}\psi(x,\lambda) \tag{3.27}$$

is obtained, where

$$\begin{aligned} z(x,\lambda) = \frac{1}{\Delta(\lambda)} \Big\{ &\psi(x,\lambda)\int_0^x \widetilde{\varphi}(t,\lambda)Bf'(t)dt + \varphi(x,\lambda)\int_x^\pi \widetilde{\psi}(t,\lambda)Bf'(t)dt + \\ &+ \psi(x,\lambda)\int_0^x \widetilde{\varphi}(t,\lambda)\Omega(t)f(t)dt + \varphi(x,\lambda)\int_x^\pi \widetilde{\psi}(t,\lambda)\Omega(t)f(t)dt \Big\}. \end{aligned}$$

It follows from (3.21) and (3.22) that

$$\lim_{|\lambda|\to\infty} \max_{0\le x\le\pi} |z(x,\lambda)| = 0, \quad \lambda \in G_\delta. \tag{3.28}$$

Now, we integrate $y(x, \lambda)$ with respect to λ over the contour Γ_N with oriented counter clockwise as follows:

$$I_N(x) = \frac{1}{2\pi i}\oint_{\Gamma_N} y(x,\lambda)d\lambda,$$

where

$$\Gamma_N = \left\{\lambda : |\lambda| = \left(N + \frac{1}{\pi}\arctan\left(\frac{c_3 b_4 - c_4 b_3}{b_3 c_3 + c_4 b_4}\right)\right)\frac{\pi}{\mu(\pi)} + \frac{\pi}{2\mu(\pi)}\right\},$$

N is sufficiently large natural number. Applying residue theorem, we have

$$\begin{aligned} I_N(x) &= \sum_{n=-N}^{N} \underset{\lambda=\lambda_n}{Res}\; y(x,\lambda) \\ &= -\sum_{n=-N}^{N}\frac{1}{\alpha_n}\varphi(x,\lambda_n)\int_0^\pi \widetilde{\varphi}(t,\lambda_n)f(t)\rho(t)dt + \sum_{n=-N}^{N}\frac{f_4}{\dot{\Delta}(\lambda_n)}\varphi(x,\lambda_n) + \sum_{n=-N}^{N}\frac{f_3}{\dot{\Delta}(\lambda_n)}\psi(x,\lambda_n). \end{aligned}$$

On the other hand, taking into account the equation (3.27)

$$f(x) = \sum_{n=-N}^{N} a_n \varphi(x, \lambda_n) + \epsilon_N(x) \tag{3.29}$$

is found, where

$$\epsilon_N(x) = -\frac{1}{2\pi i} \oint_{\Gamma_N} \frac{1}{\lambda} z(x, \lambda) d\lambda$$

and

$$a_n = \frac{1}{\alpha_n} \int_0^{\pi} \widetilde{\varphi}(t, \lambda_n) f(t) \rho(t) dt.$$

From (3.28), $\lim_{N\to\infty} \max_{0\le x\le\pi} |\epsilon_N(x)| = 0$. Thus, by going over in (3.29) to the limit as $N \to \infty$ the expansion formula (3.23) with respect to eigenfunction is obtained. Since the system of $\{\varphi(x, \lambda_n)\}$, $(n \in \mathbb{Z})$ is complete and orthogonal in $L_{2,\rho}(0, \pi; \mathbb{C}^2) \oplus \mathbb{C}^2$, Parseval equality (3.26) is valid. □

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On the Probabilistic Stability of the 2-variable k-AC-mixed Type Functional Equation

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Abstract

In this paper, we obtain the general solution and the generalized Ulam-Hyers stability of the 2-variable k-AC mixed type functional equation

$$f(x+ky,z+kw)+f(x-ky,z-kw) \\ = k^2[f(x+y,z+w)+f(x-y,z-w)]+2(1-k^2)f(x,z).$$

for any $k \in Z - \{0, \pm 1\}$ in α-Šerstnev Menger Probabilistic normed spaces.

Keywords: Generalized Hyers-Ulam-Rassias stability, k-AC mixed type functional equation, α-Šerstnev Menger Probabilistic normed spaces.

2010 MSC: 39B55, 39B52, 39B82.

1 Introduction

Menger introduced probabilistic metric space in 1942 [16]. A probabilistic normed space (PN space) is a natural generalization of an ordinary normed linear space. Such spaces were first introduced by Šerstnev in 1963, (see, [28]). Alsina et al. generalized the definition of PN space [1]. This definition became the standard one and has been adopted by all researchers, who after them have investigated the properties of PN spaces. In this article, we adopt the new definition of α-Šerstnev PN spaces (or generalized Šerstnev PN spaces) given in the paper [14] by Lafuerza-Guillén and Rodríguez.

The problem of Ulam-Hyers stability for functional equations concerns deriving conditions under which, given an approximate solution of a functional equation, one may find an exact solution that is near it in some sense. The problem was first stated by Ulam [30] in 1940 for the case of group homomorphisms, and solved by Hyers [9] in the setting of Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution ([2, 7, 22]) and in terms of the methods used for the proof ([4, 6, 8, 10, 29]). Many interesting results concerning this problem can be found, for example, in [11–13, 15, 17–20, 23, 24].

The stability of generalized mixed type functional equation of the form

$$f(x+ky)+f(x-ky) = k^2[f(x+y)+f(x-y)]+2(1-k^2)f(x) \tag{1.1}$$

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for fixed integers k and $k \neq 0, \pm 1$ in quasi-Banach spaces was introduced by M. Eshaghi Gordji and H. Khodaie [5]. The mixed type functional equation (1.1) is having the property additive, quadratic and cubic.

J.H. Bae and W.G. Park proved the general solution and investigated the generalized Hyers-Ulam stability of the 2-variable quadratic functional equation

$$f(x+y,z+w)+f(x-y,z-w)=2f(x,z)+2(y,w). \tag{1.2}$$

The functional equation (1.2) has solution

$$f(x,y)=ax^2+bxy+cy^2 \tag{1.3}$$

The general solution and generalized Hyers-Ulam stability of a 3-variable quadratic functional equation

$$f(x+y,z+w,u+v)+f(x-y,z-w,u-v)=2f(x,z,u)+2(y,w,v) \tag{1.4}$$

was discussed by K. Ravi and M. Arun Kumar [25]. The solution of (1.4) is of the form

$$f(x,y,z)=ax^2+by^2+cz^2+dxy+eyz+fzx \tag{1.5}$$

Very recently, M. Aruk Kumar et al., introduced and investigated the solution and generalized Ulam-Hyers stability of a 2-varibale AC-mixed type functional equation

$$f(2x+y,2z+w)-f(2x-y,2z-w)=4[f(x+y,z+w)-f(x-y,z-w)]-6f(y,w) \tag{1.6}$$

having solutions

$$f(x,y)=ax+by \tag{1.7}$$

and

$$f(x,y)=ax^3+bx^2y+cxy^2+dy^3 \tag{1.8}$$

in Banach spaces [3] and Quasi-Beta normed space [21].

Following the same approach, in this paper, we investigate the general solution and establish that generalized Ulam-Hyers stability of the 2-variable k-AC mixed type functional equation

$$\begin{aligned} &f(x+ky,z+kw)+f(x-ky,z-kw)\\ &\quad=k^2[f(x+y,z+w)+f(x-y,z-w)]+2(1-k^2)f(x,z) \end{aligned} \tag{1.9}$$

having solutions

$$f(x,y)=ax+by \tag{1.10}$$

and

$$f(x,y)=ax^3+bx^2y+cxy^2+dy^3 \tag{1.11}$$

for fixed integers k with $k \neq 0, \pm 1$ in α-Šestnev (or generalized Šerstnev) Menger Probabilistic normed spaces.

Δ^+ is the space of distribution functions that is, the space of all mappings $F : R \cup \{-\infty, \infty\} \to [0,1]$ that is non-decreasing, left-continuous on R and such that $F(0)=0$ and $F(+\infty)=1$. D^+ is a subset of Δ^+ consisting of all functions F for which $\lim_{x\to+\infty} F(x)=1$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions. The maximal element for Δ^+ in this order is the distribution function ϵ_0 given by

$$\epsilon_0(t)=\begin{cases}0, \text{if } t\leq 0\\ 1, \text{if } t>0\end{cases}$$

Definition 1.1. *[26, 27] A triangle function is a mapping $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ such that, for all F, G, H, K in Δ^+,*

(1) $\tau(F,\epsilon_0)=F$,

(2) $\tau(F,G)=\tau(G,F)$,

(3) $\tau(F,G)\leq\tau(H,K)$ *whenever* $F\leq H$, $G\leq K$,

(4) $\tau(\tau(F,G),H)=\tau(F,\tau(G,H))$.

Moreover, a triangle function is continuous if it is continuous in the metric space (Δ^+, d_s).

Typical continuous triangle functions are

$$\tau_T(F,G)(x) := \sup_{s+t=x} T(F(s),G(t)) \tag{1.12}$$

and

$$\tau_{T^*}(F,G)(x) := \inf_{s+t=x} T^*(F(s),G(t)) \tag{1.13}$$

for all $F, G \in \Delta^+$ *and all* $x \in R$. *Here, T is a continuous t-norm and* T^* *is the corresponding continuous t-conorm, i.e., both are continuous binary operations on [0, 1] that are commutative, associative, and non decreasing in each variable; T has 1 as identity and* T^* *has 0 as identity. Also* $T^*(x,y) = 1 - T(1-x, 1-y)$.

Definition 1.2 (PN spaces redefined [1]). *A PN space is a quadruple* (V, ν, τ, τ^*), *where V is a real vector space,* τ *and* τ^* *are continuous triangle functions such that* $\tau \leq \tau^*$, *and the mapping* $\nu : V \to \Delta^+$ *satisfies, for all p and q in V, the conditions:*

(N1) $\nu_p = \epsilon_0$ *if, and only if,* $p = \theta$ *(θ is the null vector in V);*

(N2) $\forall\ p \in V, \nu_{-p} = \nu_p$;

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;

(N4) $\forall\ \alpha \in [0,1], \nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$.

A PN space is called a Šerstnev-space if it satisfies (N1), (N3) and the following condition:

$$(\check{S})\ \ \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right) \tag{1.14}$$

holds for every $\alpha \neq 0 \in \mathbb{R}$ *and* $x > 0$.

If $\tau = \tau_T$ *and* $\tau^* = \tau_{T^*}$ *for some continuous t-norm T and its t-conorm* T^*, *then the PN space* $(V, \nu, \tau_T, \tau_{T^*})$ *is called Menger PN space (briefly, MPN space), and is denoted by* (V, ν, T).

Let $\phi : [0, +\infty] \to [0, +\infty]$ *be a non-decreasing, left-continuous function with* $\phi(0) = 0$, $\phi(+\infty) = +\infty$ *and* $\phi(x) > 0$ *for* $x > 0$. *Let* $\hat{\phi}$ *be the (unique) quasi-inverse of* ϕ *which is left-continuous.* $\hat{\phi}$ *is defined by* $\hat{\phi}(0) = 0$, $\hat{\phi}(+\infty) = +\infty$ *and* $\hat{\phi}(t) = sup\{u : \phi(u) < t\}$ *for all* $0 < t < +\infty$. *It follows that* $\hat{\phi}(\phi(x)) \leq x$ *and* $\phi(\hat{\phi}(y)) \leq y$ *for all x and y.*

Definition 1.3. *[14] A quadruple* (V, ν, τ, τ^*) *satisfy the*

$$(\phi - \check{S})\ \ \nu_{\lambda p}(x) = \nu_p\left(\hat{\phi}\left(\frac{\phi(x)}{|\lambda|}\right)\right) \tag{1.15}$$

for all $x \in \mathbb{R}^+$, $p \in V$ *and* $\lambda \in \mathbb{R}\backslash\{0\}$ *is called a ϕ-Šerstnev PN space (generalized Šerstnev space).*

If $\phi(x) = x^{1/\alpha}$ *for a fixed positive real number* α, *the condition* $(\phi - \check{S})$ *takes the form*

$$(\alpha - \check{S})\ \ \nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|^\alpha}\right) \tag{1.16}$$

for every $p \in V$, *for every* $x > 0$ *and* $\lambda \in \mathbb{R}\backslash\{0\}$.

PN spaces satisfying the condition $(\alpha - \check{S})$ *are called α-Šerstnev PN spaces.*

Definition 1.4. *Let* (V, ν, τ) *be a PN space and* $\{x_n\}$ *be a sequence in V. Then* $\{x_n\}$ *is said to be convergent if there exists* $x \in V$ *such that*

$$\lim_{n\to\infty} \nu_{x_n - x}(t) = 1 \tag{1.17}$$

for all $t > 0$. *In this case x is called the limit of* $\{x_n\}$.

Definition 1.5. *The sequence* $\{x_n\}$ *in* (V, ν, τ) *is called a Cauchy sequence if, for every* $\epsilon > 0$ *and* $\delta > 0$, *there exists a positive integer* n_0 *such that* $\nu(x_n - x_m)(\delta) > 1 - \epsilon$ *for all* $m, n \geq n_0$. *Clearly, every convergent sequence in a PN-space is Cauchy. If every Cauchy sequence is convergent in a PN-space* (V, ν, τ), *then* (V, ν, τ) *is called a probabilistic Banach space (PB-space).*

2 General Solution

Through out this section let U and V be real vector spaces and we present the solution of (1.9) using Lemma 2.1, 2.2, 2.3.

Lemma 2.1. *If $f : U^2 \to V$ is a mapping satisfying (1.9) and let $g : U^2 \to V$ be a mapping given by*

$$g(x,x) = f(2x,2x) - 8f(x,x) \tag{2.18}$$

for all $x \in U$ then

$$g(2x,2x) = 2g(x,x) \tag{2.19}$$

for all $x \in U$ such that g is additive.

Proof. Letting (x,y,z,w) by $(0,0,0,0)$ in (1.9), we get

$$f(0,0) = 0 \tag{2.20}$$

Setting (x,y,z,w) by (y,x,w,z) in (1.9), we obtain

$$\begin{aligned} &f(y+kx, w+kz) + f(y-kx, w-kz) \\ &\quad = k^2[f(x+y, w+z) + f(y-x, w-z)] + 2(1-k^2)f(z,x) \end{aligned} \tag{2.21}$$

for all $x,y,z,w \in U$.
Replacing (x,y,z,w) by $(x,-y,z,-w)$ in (2.21), we get

$$\begin{aligned} &f(-y+kx, -w+kz) + f(-y-kz, -w-kz) \\ &\quad = k^2[f(x-y),(w-z)) + f(-y-x, -w-z)] + 2(1-k^2)f(z,x) \end{aligned} \tag{2.22}$$

for all $x,y,z,w \in U$.
From (2.21) and (2.22) we arrive at

$$\begin{aligned} &f(y+kx, w+kz) + f(y-kx, w-kz) + f(-y+kx, -w+kz) \\ &\quad + f(-y-kx, -w-kz) = k^2[f(x+y, w+z) + f(y-x, w-z) \\ &\quad + f(x-y, z-w) + f(-y-x, -w-z)] + 4(1-k^2)f(z,x) \end{aligned} \tag{2.23}$$

Now, letting (x,y,z,w) by $(0,y,0,y)$ in (2.23), we obtain

$$2[k^2-1][f(y,y) + f(-y,-y)] = 0$$

which implies

$$f(y,y) = -f(-y,-y) \tag{2.24}$$

for all $y \in U$.
Replacing (x,y,z,w) by (x,x,x,x) in (1.9), we get

$$\begin{aligned} &f((1+k)x, (1+k)x) + f((1-k)x, (1-k)x) \\ &\quad = k^2 f(2x,2x) + 2(1-k^2)f(x,x) \end{aligned} \tag{2.25}$$

for all $x \in U$. Now, replacing x by $2x$ in (2.25), we have

$$\begin{aligned} &f(2(1+k)x, 2(1+k)x) + f(2(1-k)x, 2(1-k)x) \\ &\quad = k^2 f(4x,4x) + 2(1-k^2)f(2x,2x) \end{aligned} \tag{2.26}$$

for all $x \in U$. Again replacing (x,y,z,w) by $(2x,x,2x,x)$ in (1.9), we obtain

$$\begin{aligned} &f((2+k)x, (2+k)x) + f((2-k)x, (2-k)x) \\ &\quad = k^2 f(3x,3x) + k^2 f(x,x) + 2(1-k^2)f(2x,2x) \end{aligned} \tag{2.27}$$

for all $x \in U$.

Replacing (x,y,z,w) by $(x,2x,x,2x)$ in (1.9), we get

$$f((1+2k)x,(1+2k)x)+f((1-2k)x,(1-2k)x) \\ = k^2 f(3x,3x) - k^2 f(x,x) + 2(1-k^2)f(x,x) \tag{2.28}$$

for all $x \in U$. Replacing (x,y,z,w) by $(x,3x,x,3x)$ in (1.9), we obtain

$$f((1+3k)x,(1+3k)x)+f((1-3k)x,(1-3k)x) \\ = k^2 f(4x,4x) - k^2 f(2x,2x) + 2(1-k^2)f(x,x) \tag{2.29}$$

for all $x \in U$. We substitute (x,y,z,w) by $((1+k)x,x,(1+k)x,x)$ in (1.9) and then (x,y,z,w) by $((1-k)x,x,(1-k)x,x)$ in (1.9) to obtain

$$f((1+2k)x,(1+2k)x)+f(x,x) = k^2 f((2+k)x,(2+k)x) \\ + k^2 f(kx,kx) + 2(1-k^2)f((1+k)x,(1+k)x) \tag{2.30}$$

and

$$f((1-2k)x,(1-2k)x)+f(x,x) = k^2 f((2-k)x,(2-k)x) \\ - k^2 f(kx,kx) + 2(1-k^2)f((1-k)x,(1-k)x) \tag{2.31}$$

for all $x \in U$. Then, by adding (2.30) to (2.31), we have

$$f((1+2k)x,(1+2k)x)+f((1-2k)x,(1-2k)x)+2f(x,x) \\ = k^2 f((2+k)x,(2+k)x) + k^2 f((2-k)x,(2-k)x) \\ + 2(1-k^2)[f((1+k)x,(1+k)x)+f((1-k)x,(1-k)x)] \tag{2.32}$$

for all $x \in U$. Now, substitute (x,y,z,w) by $((1+2k)x,x,(1+2k)x,x)$ in (1.9) and (x,y,z,w) by $((1-2k)x,x,(1-2k)x,x)$ in (1.9) to obtain

$$f((1+3k)x,(1+3k)x)+f((1+k)x,(1+k)x) \\ = k^2 f(2(1+k)x,2(1+k)x) + k^2 f(2kx,2kx) \\ + 2(1-k^2)f((1+2k)x,(1+2k)x) \tag{2.33}$$

and

$$f((1-3k)x,(1-3k)x)+f((1-k)x,(1-k)x) \\ = k^2 f(2(1-k)x,2(1-k)x) - k^2 f(2kx,2kx) \\ + 2(1-k^2)f((1-2k)x,(1-2k)x) \tag{2.34}$$

for all $x \in U$. Now, adding (2.33) to (2.34), we have,

$$f((1+3k)x,(1+3k)x)+f((1-3k)x,(1-3k)x)+f((1+k)x,(1+k)x) \\ + f((1-k)x,(1-k)x) = k^2 f(2(1+k)x,2(1+k)x) \\ + k^2 f(2(1-k)x,2(1-k)x) \\ + 2(1-k^2)[f((1+2k)x,(1+2k)x)+f((1-2k)x,(1-2k)x)] \tag{2.35}$$

for all $x \in U$. From (2.25), (2.27), (2.28) and (2.32), we arrive at

$$f(3x,3x) = 4f(2x,2x) - 5f(x,x) \tag{2.36}$$

for all $x \in U$. From (2.26), (2.28), (2.25), (2.29) and (2.35), we have

$$f(4x,4x) = 2f(2x,2x) + 2f(3x,3x) - 6f(x,x) \tag{2.37}$$

for all $x \in U$. Using (2.36) in (2.37), we obtain

$$f(4x,4x) = 10f(2x,2x) - 16f(x,x) \tag{2.38}$$

for all $x \in U$. From (2.38), we establish

$$f(4x,4x) - 8f(2x,2x) = 2f(2x,2x) - 16f(x,x) \tag{2.39}$$

for all $x \in U$. Using (2.18) in (2.39), we get our desired result. □

Lemma 2.2. *If $f : U^2 \to V$ be a mapping satisfying (1.9) and let $h : U^2 \to V$ be a mapping given by*

$$h(x,x) = f(2x,2x) - 2f(x,x) \tag{2.40}$$

for all $x \in U$ then

$$h(2x,2x) = 8h(x,x) \tag{2.41}$$

for all $x \in U$ such that h is cubic.

Proof. Proceeding as in Lemma 2.1, it follows from (2.38)

$$f(4x,4x) - 2f(2x,2x) = 8f(2x,2x) - 16f(x,x) \tag{2.42}$$

for all $x \in U$. Using (2.40) in (2.42), we arrive at our desired result. □

Remark 2.1. *If $f : U^2 \to V$ be a mapping satisfying (1.9) let $g, h : U^2 \to V$ be mappings defined by (2.18) and (2.40) then*

$$f(x,x) = \frac{1}{6}(h(x,x) - g(x,x)) \tag{2.43}$$

for all $x \in U$.

Lemma 2.3. *If $f : U^2 \to V$ is a mapping satisfying (1.9) and let $t : U \to V$ be a mapping given by*

$$t(x) = f(x,x) \tag{2.44}$$

for all $x \in U$, then t satisfies

$$t(x+ky) + t(x-ky) = k^2[t(x+y) + t(x-y)] + 2(1-k^2)t(x) \tag{2.45}$$

for all $x, y \in U$.

Proof. From (1.9) and (2.44), we get

$$\begin{aligned} t(x+ky) + t(x-ky) &= f(x+ky, x+ky) - f(x-ky, x-ky) \\ &= k^2[f(x+y, x+y) + f(x-y, x-y)] + 2(1-k^2)f(x,x) \\ &= k^2[t(x+y) + t(x-y)] + 2(1-k^2)t(x) \end{aligned}$$

for all $x, y \in U$. □

3 Stability Results : Direct Method

In this section, we investigate the generalized Ulam-Hyers stability problem of (1.9) using direct method. Let U be a real linear space and (Y, ν, τ_T) be a α-Šerstnev MPB space. Now, we define a difference operator $\Delta f : U^4 \to Y$ by

$$\begin{aligned} \Delta f(x,y,z,w) = f(x+ky, z+kw) + f(x-ky, z-kw) - k^2 f(x+y, z+w) \\ - k^2 f(x-y, z-w) - 2(1-k^2)f(x,z) \end{aligned} \tag{3.46}$$

$\forall\ x, y, z, w \in U$, where $f : U^2 \to Y$ is a mapping.

Theorem 3.1. *Let $f : U^2 \to Y$ be a mapping for which there exist a function $\xi : U^4 \to D^+$ with the condition*

$$\lim_{m\to\infty} \tau_T \left[\xi_{(2^m x, 2^m y, 2^m z, 2^m w)}(2^{m\alpha} t), \xi_{(2^m x, 2^m y, 2^m z, 2^m w)}(2^{(m-3)\alpha - 1} t)\right] = 1 \tag{3.47}$$

such that the functional inequality

$$\nu_{\Delta f(x,y,z,w)}(t) \geq \xi_{x,y,z,w}(t) \tag{3.48}$$

for all $x, y, z, w \in U$, $t > 0$ and $\alpha > 0$. Then there exists a unique 2-variable additive mapping $A(x,x) : U^2 \to Y$ satisfying (1.9) and

$$\nu_{f(2x,2x)-8f(x,x)-A(x,x)}(t) \geq \tilde{\Phi} \tag{3.49}$$

where

$$A(x,x) = \lim_{n\to\infty} \frac{f(2^{(n+1)}x, 2^{(n+1)}x) - 8f(2^n x, 2^n x)}{2^n} \tag{3.50}$$

$$\begin{cases} \tilde{\Phi} = \lim_{n\to\infty} \Phi_n = 1 \\ \Phi_n = \tau_T \left[\tilde{\tau}_{T(2^{n-1}x)}(t), \Phi_{n-1}\right], \ \textit{for } n > 1 \end{cases} \tag{3.51}$$

$$\Phi_1 = \tilde{\tau}_{T(x)}(t) \tag{3.52}$$

and

$$\begin{aligned} \tilde{\tau}_{T(x)}(t) = \tau_T \Bigg(\tau_T \Bigg(\tau_T \Bigg(\tau_T \Bigg(\xi_{(x,2x,x,2x)} \left(\frac{k^{2\alpha} t}{2^4 2^\alpha} \right), \\ \xi_{((1-2k)x,x,(1-2k)x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4} \right) \Bigg), \tau_T \Bigg(\xi_{((1+2k)x,x,(1+2k)x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4} \right), \\ \xi_{(x,x,x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4} \right) \Bigg) \Bigg), \tau_T \Bigg(\xi_{(2x,2x,2x,2x)} \left(\frac{|k^2-1|^\alpha t}{2^3} \right), \\ \xi_{(x,3x,x,3x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3} \right), \tau_T \Bigg(\tau_T \Bigg(\tau_T \Bigg(\xi_{(x,x,x,x)} \left(\frac{k^{2\alpha} t}{2^4 2^{2\alpha}} \right), \\ \xi_{((1-k)x,x,(1-k)x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4 2^\alpha} \right) \Bigg), \tau_T \Bigg(\xi_{((1+k)x,x,(1+k)x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4 2^\alpha} \right) \\ \xi_{(x,2x,x,2x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4 2^\alpha} \right) \Bigg) \Bigg), \xi_{(2x,x,2x,x)} \left(\frac{|k^2-1|^\alpha t}{2^4 2^\alpha} \right) \Bigg) \Bigg) \Bigg), \end{aligned} \tag{3.53}$$

for all $x \in U$, $t > 0$ and $\alpha > 0$.

Proof. Letting (x,y,z,w) by (x,x,x,x) in (3.48), we obtain

$$\begin{aligned} &\nu_{f((1+k)x,(1+k)x)+f((1-k)x,(1-k)x)-k^2 f(2x,2x)-2(1-k^2)f(x,x)}(t) \\ &\geq \xi_{(x,x,x,x)}(t), \ \forall\, x \in U, t > 0. \end{aligned} \tag{3.54}$$

It follows from (3.54) that

$$\begin{aligned} &\nu_{f(2(1+k)x,2(1+k)x)+f(2(1-k)x,2(1-k)x)-k^2 f(4x,4x)-2(1-k^2)f(2x,2x)}(t) \\ &\geq \xi_{(2x,2x,2x,2x)}(t), \ \forall\, x \in U, t > 0. \end{aligned} \tag{3.55}$$

Replacing (x,y,z,w) by $(2x,x,2x,x)$ in (3.48), respectively, we have

$$\begin{aligned} &\nu_{f((2+k)x,(2+k)x)+f((2-k)x,(2-k)x)-k^2 f(3x,3x)-k^2 f(x,x)-2(1-k^2)f(2x,2x)}(t) \\ &\geq \xi_{(2x,x,2x,x)}(t), \ \forall\, x \in U, t > 0. \end{aligned} \tag{3.56}$$

Setting (x,y,z,w) by $(x,2x,x,2x)$ in (3.48) gives

$$\nu_{f((1+2k)x,(1+2k)x)+f((1-2k)x,(1-2k)x)-k^2f(3x,3x)-k^2f(x,x)-2(1-k^2)f(x,x)}(t) \geq \xi_{(x,2x,x,2x)}(t),\ \forall\, x\in U, t>0. \tag{3.57}$$

Replacing (x,y,z,w) by $(x,3x,x,3x)$ in (3.48), we obtain

$$\nu_{f((1+3k)x,(1+3k)x)+f((1-3k)x,(1-3k)x)-k^2f(4x,4x)+k^2f(2x,2x)-2(1-k^2)f(x,x)}(t) \geq \xi_{(x,3x,x,3x)}(t),\ \forall\, x\in U, t>0. \tag{3.58}$$

Replacing (x,y,z,w) by $((1+k)x,x,(1+k)x,x)$ in (3.48), respectively, we get

$$\nu_{f((1+2k)x,(1+2k)x)+f(x,x)-k^2f((2+k)x,(2+k)x)-k^2f(kx,kx)-2(1-k^2)f((1+k)x,(1+k)x)}(t) \geq \xi_{((1+k)x,x,(1+k)x,x)}(t),\ \forall\, x\in U, t>0. \tag{3.59}$$

Replacing (x,y,z,w) by $((1-k)x,x,(1-k)x,x)$ in (3.48), respectively, one gets

$$\nu_{f((1-2k)x,(1-2k)x)+f(x,x)-k^2f((2-k)x,(2-k)x)+k^2f(kx,kx)-2(1-k^2)f((1-k)x,(1-k)x)}(t) \geq \xi_{((1-k)x,x,(1-k)x,x)}(t),\ \forall\, x\in U, t>0. \tag{3.60}$$

Replacing (x,y,z,w) by $((1+2k)x,x,(1+2k)x,x)$ in (3.48), respectively, we obtain

$$\nu_{f((1+3k)x,(1+3k)x)+f((1+k)x,(1+k)x)-k^2f(2(1+k)x,2(1+k)x)-k^2f(2kx,2kx)-2(1-k^2)f((1+2k)x,(1+2k)x)}(t) \geq \xi_{((1+2k)x,x,(1+2k)x,x)}(t),\ \forall\, x\in U, t>0. \tag{3.61}$$

Replacing (x,y,z,w) by $((1-2k)x,x,(1-2k)x,x)$ in (3.48), respectively, we have

$$\nu_{f((1-3k)x,(1-3k)x)+f((1-k)x,(1-k)x)-k^2f(2(1-k)x,2(1-k)x)+k^2f(2kx,2kx)-2(1-k^2)f((1-2k)x,(1-2k)x)}(t) \geq \xi_{((1-2k)x,x,(1-2k)x,x)}(t),\ \forall\, x\in U, t>0. \tag{3.62}$$

Thus it follows from (3.54), (3.56), (3.57), (3.59) and (3.60) that

$$\begin{aligned}&\nu_{f(3x,3x)-4f(2x,2x)+5f(x,x)}(t)\\ &\geq \tau_T\left(\tau_T\left(\tau_T\left(\xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}t}{2^3 2^\alpha}\right),\ \xi_{((1-k)x,x,(1-k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right),\right.\right.\\ &\left.\tau_T\left(\xi_{((1+k)x,x,(1+k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right),\xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right)\right)\\ &\left.\xi_{(2x,x,2x,x)}\left(\frac{|k^2-1|^\alpha t}{2}\right)\right),\ \forall\, x\in U, t>0 \text{ and } \alpha>0.\end{aligned} \tag{3.63}$$

Also, from (3.54), (3.55), (3.57), (3.58) (3.61) and (3.62), we have

$$\begin{aligned}&\nu_{f(4x,4x)-2f(3x,3x)-2f(2x,2x)+6f(x,x)}(t)\\ &\geq \tau_T\left(\tau_T\left(\tau_T\left(\xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}t}{2^3 2^\alpha}\right),\ ,\xi_{((1-2k)x,x,(1-2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right),\right.\right.\\ &\left.\tau_T\left(\xi_{((1+2k)x,x,(1+2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right),\xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right)\right)\\ &\left.\tau_T\left(\xi_{(2x,2x,2x,2x)}\left(\frac{|k^2-1|^\alpha t}{2^2}\right),\xi_{(x,3x,x,3x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^2}\right)\right)\right),\end{aligned} \tag{3.64}$$

for all $x\in U$, $t>0$ and $\alpha>0$.
Finally, by using (3.63) and (3.64), we obtain

$$\nu_{f(4x,4x)-10f(2x,2x)+16f(x,x)}(t) \geq \tilde{\tau}_{T(x)}(t) \tag{3.65}$$

where,

$$\begin{aligned}
&\tilde{\tau}_{T(x)}(t)\\
&= \tau_T\left(\tau_T\left(\tau_T\left(\tau_T\left(\xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}t}{2^4 2^{\alpha}}\right),\ \xi_{((1-2k)x,x,(1-2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4}\right)\right),\right.\right.\right.\\
&\tau_T\left(\xi_{((1+2k)x,x,(1+2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4}\right),\xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4}\right)\right)\Bigg)\Bigg),\\
&\tau_T\left(\xi_{(2x,2x,2x,2x)}\left(\frac{|k^2-1|^{\alpha}t}{2^3}\right),\xi_{(x,3x,x,3x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^3}\right)\right)\Bigg),\\
&\tau_T\left(\tau_T\left(\tau_T\left(\xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}t}{2^4 2^{2\alpha}}\right),\xi_{((1-k)x,x,(1-k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4 2^{2\alpha}}\right)\right),\right.\right.\\
&\tau_T\left(\xi_{((1+k)x,x,(1+k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4 2^{\alpha}}\right),\xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4 2^{\alpha}}\right)\right)\Bigg),\\
&\xi_{(2x,x,2x,x)}\left(\frac{|k^2-1|^{\alpha}t}{2^2 2^{\alpha}}\right)\Bigg)\Bigg),\ \forall\, x\in U, t>0 \text{ and } \alpha>0.
\end{aligned}\tag{3.66}$$

Let $g : U^2 \to Y$ be a function defined by

$$g(x,x) = f(2x,2x) - 8f(2x,2x) \ \text{ for all } x \in U. \tag{3.67}$$

From (3.65), we conclude that

$$\nu_{\frac{g(2x,2x)}{2}-g(x,x)}(t) \geq \tilde{\tau}_{T(x)}(2^{\alpha}t) \geq \tilde{\tau}_{T(x)}(t),\ \forall\, x \in U, t>0 \text{ and } \alpha>0 \tag{3.68}$$

which implies that

$$\nu_{\frac{g(2^{\ell+1}x,2^{\ell+1}x)}{2^{\ell+1}}-\frac{g(2^{\ell}x,2^{\ell}x)}{2^{\ell}}}(t) \geq \tilde{\tau}_{T(2^{\ell}x)}(2^{(\ell+1)\alpha}t) \tag{3.69}$$

for all $x \in U$, $t>0$, $\alpha>0$ and $\ell \in \mathbb{N}$. From the inequalities (3.68) and (3.69) we use iterative methods and induction on n and apply defined sequence in (3.51) and (3.52) to prove our next relation

$$\nu_{\frac{g(2^n x,2^n x)}{2^n}-g(x,x)}(t) \geq \tau_T\left[\tilde{\tau}_{T(2^{n-1}x)}(t),\Phi_{n-1}\right]\ \ \forall\, x \in U, t>0 \text{ and } \alpha>0. \tag{3.70}$$

So

$$\nu_{\frac{g(2^{m+n}x,2^{m+n}x)}{2^{m+n}}-\frac{g(2^m x,2^m x)}{2^m}}(t) \geq \tau_T\left[\tilde{\tau}_{T(2^{(m+n)-1}x)}(2^{m\alpha}t),\Phi_{(m+n)-1}\right] \tag{3.71}$$

for all non negative integers m and n and for all $x \in U$, $t>0$. By assumptions (3.71) shows that the sequence $\left\{\frac{g(2^n x,2^n x)}{2^n}\right\}$ is a Cauchy sequence in Y for all $x \in U$. Since Y is a α-Šerstnev MPB, it follows that the sequence $\left\{\frac{g(2^n x,2^n x)}{2^n}\right\}$ converges for all $x \in U$. Therefore, one can define the function $A(x,x) : U^2 \to Y$ by

$$A(x,x) = \lim_{n\to\infty}\frac{g(2^n x,2^n x)}{2^n} \ \text{ for all } x \in U. \tag{3.72}$$

Now, if we replace (x,y,z,w) by $(2^n x, 2^n y, 2^n z, 2^n w)$ in (3.48), respectively, then it follows that

$$\begin{aligned}
&\nu_{\frac{\Delta g(2^n x,2^n y,2^n z,2^n w)}{2^n}}(t) = \nu_{\frac{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}{2^n}-8\frac{\Delta f(2^n x,2^n y,2^n z,2^n w)}{2^n}}(t)\\
&\geq \tau_T\left[\nu_{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}(2^{n\alpha-1}t),\nu_{\Delta f(2^n x,2^n y,2^n z,2^n w)}(2^{(n-3)\alpha-1}t)\right]\\
&\geq \tau_T\left[\xi_{2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w}(2^{n\alpha-1}t),\xi_{2^n x,2^n y,2^n z,2^n w}(2^{(n-3)\alpha-1}t)\right]
\end{aligned}\tag{3.73}$$

for all $x,y,z,w \in U$, $t>0$ and $\alpha>0$. By letting $n\to\infty$ in (3.73), we have $\nu_{\Delta A(x,y,z,w)}(t) = 1$ for all $t>0$ and so $\Delta A(x,y,z,w) = 0$. Hence A satisfies (1.9) for all $x,y,z,w \in U$. To prove (3.49), if we take the limit as $n\to\infty$

in (3.70), then we can get (3.49). Finally, to prove the uniqueness of the additive function A subject to (3.49), assume that there exists another 2-variable additive mapping A' which satisfies (3.49) and (1.9), then

$$\begin{aligned} \nu_{A(x,x)-A'(x,x)}(t) &= \nu_{\frac{A(2^n x,2^n x)}{2^n}-\frac{A'(2^n x,2^n x)}{2^n}}(t) \\ &= \nu_{A(2^n x,2^n x)-A'(2^n x,2^n x)}(2^{n\alpha}t) \\ &\geq \nu_{A(2^n x,2^n x)-g(2^n x,2^n x)+g(2^n x,2^n x)-A'(2^n x,2^n x)}(2^{n\alpha}t) \\ &\geq \lim_{n\to\infty} \tau_T\left[\tau_T\left[\tilde{\tau}_{T(2^{2n-1}x)}(2^{n\alpha-1}t),\Phi_{n-1}\right],\tau_T\left[\tilde{\tau}_{T(2^{2n-1}x)}(2^{n\alpha-1}t),\Phi_{n-1}\right]\right. \end{aligned} \tag{3.74}$$

which tends to 1 as $n\to\infty$ for all $x \in U$. So we can conclude that $A = A'$. This completes the proof of the theorem. □

Theorem 3.2. *Let $f : U^2 \to Y$ be a mapping for which there exist a function $\xi : U^4 \to D^+$ with the condition*

$$\lim_{m\to\infty} \tau_T\left[\xi_{(2^m x,2^m y,2^m z,2^m w)}(2^{3m\alpha}t),\xi_{(2^m x,2^m y,2^m z,2^m w)}(2^{(3m-1)\alpha-1}t),\right] \tag{3.75}$$

such that the functional inequality (3.48) is satisfied for all $x,y,z,w \in U$, $t > 0$ and $\alpha > 0$. Then there exists a unique 2-variable cubic mapping $c(x,x) : U^2 \to Y$ satisfying (1.9) and

$$\nu_{f(2x,2x)-2f(x,x)-c(x,x)}(t) \geq \tilde{\Psi} \tag{3.76}$$

where

$$c(x,x) = \lim_{n\to\infty} \frac{f(2^{(n+1)}x,2^{(n+1)}x)-2f(2^n x,2^n x)}{2^{3n}} \tag{3.77}$$

$$\begin{cases} \tilde{\Psi} = \lim_{n\to\infty}\Psi_n = 1 \\ \Psi_n = \tau_T\left[\tilde{\tau}_{T(2^{n-1}x)}(2^{2n\alpha}t),\Psi_{n-1}\right] \end{cases} \tag{3.78}$$

$$\Psi_1 = \tilde{\tau}_{T(x)}(2^{2\alpha}t),\ \forall\, x \in U, t > 0, \alpha > 0, \tag{3.79}$$

where $\tilde{\tau}_{T(x)}(t)$ is defined as in Theorem 3.1.

Proof. By the similar approach as in the proof of Theorem 3.1, we can obtain

$$\nu_{f(4x,4x)-10f(2x,2x)+16f(x,x)}(t) \geq \tilde{\tau}_{T(x)}(t),\ \forall\, x \in U, t > 0.$$

Let $h : U^2 \to Y$ be a function defined by

$$h(x,x) = f(2x,2x) - 2f(x,x),\ \text{for all } x \in U \tag{3.80}$$

Thus from (3.65), we have

$$\nu_{\frac{h(2x,2x)}{2^3}-h(x,x)}(t) \geq \tilde{\tau}_{T(x)}(2^{3\alpha}t) \geq \overline{\tau}_{T(x)}(2^{2\alpha}t),\ \forall\, x \in U, t > 0, \alpha > 0 \tag{3.81}$$

which implies that

$$\nu_{\frac{h(2^{\ell+1}x,2^{\ell+1}x)}{2^{3(\ell+1)}}-\frac{h(2^\ell x,2^\ell x)}{2^{3\ell}}}(t) \geq \overline{\tau}_{T(2^\ell x)}(2^{3(\ell+1)\alpha}t) \tag{3.82}$$

for all $x \in U$, $t > 0$, $\alpha > 0$ and $\ell \in \mathbb{N}$. Thus it follows from (3.82) and (N3)

$$\nu_{\frac{h(2^n x,2^n x)}{2^{3n}}-h(x,x)}(t) \geq \tau_T\left[\tilde{\tau}_{T(2^{n-1}x)}(2^{2n\alpha}t),\Phi_{n-1}\right],\ \forall\, x \in U; t > 0, \alpha > 0. \tag{3.83}$$

In order to prove the convergence of the sequence $\left\{\frac{h(2^n x,2^n x)}{2^{3n}}\right\}$ if we replace x with $2^m x$ in (3.83), then we get

$$\nu_{\frac{h(2^{n+m}x,2^{n+m}x)}{2^{3(n+m)}}-\frac{h(2^m x,2^m x)}{2^{3m}}}(t) \geq \tau_T\left[\overline{\tau}_{T(2^{n+m-1}x)}(2^{(2n+3m)\alpha}t),\Phi_{n+m}\right] \tag{3.84}$$

for all non-negative integers m and n and $\forall\, x \in U$, $t>0$, $\alpha>0$.

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, by assumptions, the sequence $\left\{\frac{h(2^n x, 2^n x)}{2^{3n}}\right\}$ is a Cauchy sequence in Y for all $x \in U$. Since Y is a α-Šerstnev MPB, one can define the function $c(x,x) : U^2 \to Y$ by

$$c(x,x) = \lim_{n\to\infty} \frac{h(2^n x, 2^n x)}{2^{3n}} \text{ for all } x \in U. \tag{3.85}$$

Now, if we replace (x,y,z,w) by $(2^n x, 2^n y, 2^n z, 2^n w)$ in (3.48), respectively, then it follows that

$$\begin{aligned} &\nu_{\frac{\Delta h(2^n x,2^n y,2^n z,2^n w)}{2^{3n}}}(t) = \nu_{\frac{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}{2^{3n}} - 2\frac{\Delta f(2^n x,2^n y,2^n z,2^n w)}{2^{3n}}}(t) \\ &\geq \tau_T\left[\nu_{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}(2^{3n\alpha-1}t), \nu_{\Delta f(2^n x,2^n y,2^n z,2^n w)}(2^{(3n-2)\alpha-1}t)\right] \\ &\geq \tau_T\left[\xi_{(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}(2^{3n\alpha-1}t), \xi_{(2^n x,2^n y,2^n z,2^n w)}(2^{(3n-1)\alpha-1}t)\right] \end{aligned} \tag{3.86}$$

for all $x,y,z,w \in U$, $t>0$ and $\alpha>0$. By letting $n\to\infty$ in (3.86), we find that $\nu_{\Delta c(x,y,z,w)}(t) = 1$ for all $t>0$, which implies $\Delta c(x,y,z,w) = 0$ and so c satisfies (1.9) for all $x,y,z,w \in U$. To prove (3.76), if we take the limit as $n\to\infty$ in (3.83), then we get (3.76). The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. □

Theorem 3.3. *Let $\xi : U^2 \to D^+$ be a function with the conditions given in (3.47) and (3.75) and $f : U^2 \to Y$ be a function which satisfies (3.48) for all $x,y,z,w \in U$ and $t>0$. Then there exists a unique 2-variable additive mapping $A : U^2 \to Y$ and a unique 2-variable cubic mapping $C : U^2 \to Y$ satisfying (1.9) such that*

$$\begin{aligned} &\nu_{f(x,x)-A(x,x)-C(x,x)}(t) \geq \\ &\lim_{n\to\infty} \tau_T\left[\tau_T\left(\tilde{\tau}_{T(2^{n-1}x)}(3^\alpha 2^{\alpha-1}t), \Phi_{n-1}\right), \tau_T\left(\tilde{\tau}_{T(2^{n-1}x)}(2^{(2n+1)\alpha-1}3^\alpha t), \Psi_{n-1}\right)\right] \end{aligned} \tag{3.87}$$

for all $x \in U$, $t>0$ and $\alpha>0$, where Φ_n, $\tilde{\tau}_{T(x)}(t)$ is defined as in Theorem 3.1 and Ψ_n is defined as in Theorem 3.2.

Proof. By Theorems 3.1 and 3.2, there exist a unique 2-variable additive function $A_0 : U^2 \to Y$ and a unique 2-variable cubic function $C_0 : U^2 \to Y$ such that

$$\nu_{f(2x,2x)-8f(x,x)-A_0(x,x)}(t) \geq \tilde{\Phi} \tag{3.88}$$

and

$$\nu_{f(2x,2x)-2f(x,x)-C_0(x,x)}(t) \geq \tilde{\Psi},\ \forall\, x \in U, t>0. \tag{3.89}$$

Thus it follows from (3.88) and (3.89) that

$$\begin{aligned} &\nu_{f(x,x)+\frac{1}{6}A_0(x,x)-\frac{1}{6}C_0(x,x)}(t) \\ &\geq \tau_T\left[\nu_{f(2x,2x)-8f(x,x)-A_0(x,x)}(3^\alpha 2^{\alpha-1}t), \nu_{f(2x,2x)-2f(x,x)-C_0(x,x)}(3^\alpha 2^{\alpha-1}t)\right] \end{aligned} \tag{3.90}$$

for all $x \in U$, $t>0$ and $\alpha>0$. Thus we obtain (3.87) by letting $A(x,x) = -\frac{1}{6}A_0(x,x)$ and $C(x,x) = \frac{1}{6}C_0(x,x)$ for all $x \in U$. This completes the proof of the stability of the functional equation (1.9) in α-Šerstnev MPN spaces. □

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On Separation Axioms in Ideal Topological Spaces

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Abstract

Separation axioms in ideal topological spaces are discussed in the literature. In this paper we define the separation axioms in ideal topological spaces in a new way which is more natural than the previous versions and discuss some properties. Also we discuss the relationship of our definition with other definitions and prove some results in the context of separation axioms in ideal topological space. We show a property that holds in ideal topological theory which does not hold in the classical theory of topology; and also we show a property that holds in the classical theory that does not hold in the ideal topological theory.

Keywords: Hausdorff, regular, normal, ideal topological spaces.

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1 Introduction

An ideal on a set X is a nonempty collection $\mathcal{I}$ of subsets of X which is closed under finite union such that if A is in $\mathcal{I}$, then all subsets of A are also in $\mathcal{I}$. In 1944 Vaidyanathaswamy[12] introduced the concept of ideals in topological spaces. Later the concepts were further studied and discussed by Kuratowski[8], Noiri[3, 5] and many others[2, 6, 9]. If $\mathcal{I}$ is an ideal on a topological space $(X, \mathcal{T})$, then we can construct a topology on X, called $*$-topology, denoted by $\mathcal{T}^*$. The triplet $(X, \mathcal{T}, \mathcal{I})$ is called an ideal topological space.

In 1995, Dontchev[2] introduced the notion of I-Hausdorff space and Abd El-Monsef[1] developed the notion of quasi-I-Hausdorff space in 2000. Later Nasef[9] has improved the concepts of I-Hausdorff space and quasi- I-Hausdorff space. Further, Hatir and Noiri[4] introduced semi-I-Hausdorff space which is weaker than Hausdorff space.

In the above stated, and in many other works, these concepts were developed using notions like $\mathcal{I}$-open, semi-$\mathcal{I}$-open, quasi $\mathcal{I}$-open and so on. But a theory highlighting the topology $\mathcal{T}^*$ induced by an ideal $\mathcal{I}$ was developed in [11]. In[11], several ideals on the same topological space $(X, \mathcal{T})$ were considered and the relationship among the topologies generated by these ideals were discussed.

In this paper we define a concept called $\mathscr{I}$-Hausdorffness, in the context of ideal topological spaces slightly different from the definition available in the literature[2]. Also we define regular, normal spaces in the context of ideal topological spaces and prove certain results similar to results available in classical theory. We also prove that the intersection of two $\mathscr{I}$-Hausdorff topologies is an $\mathscr{I}$-Hausdorff topology, in contrast to the classical result which states that the intersection of two Hausdorff topologies need not be a Hausdorff topology. Further we show that the product of two $\mathscr{I}$-Hausdorff spaces need not be an $\mathscr{I}$-Hausdorff space, in contrast to the classical result which states that the product of two Hausdorff spaces is a Hausdorff space.

In Section 2 we recall some definitions and results from the literature and prove certain results which we need in the sequel; in Section 3 we define and discuss Hausdorffness in ideal topological spaces; in Section 4

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we define and discuss regularity and normality in the context of ideal topological spaces and finally we give some concluding remarks.

2 Preliminary Definitions and Results

Let us start with the definition of an ideal in a topological space.

Definition 2.1. *[12] Let X be any set. An ideal on X is a nonempty collection $\mathcal{I}$ of subsets of X satisfying the following.*

i. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

ii. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

If $(X, \mathcal{T})$ is a topological space and $\mathcal{I}$ is an ideal on X, then the triplet $(X, \mathcal{T}, \mathcal{I})$ is called an ideal topological space or ideal space.

Throughout this paper X, $\mathcal{T}$ and $\mathcal{I}$ will denote, respectively a nonempty set, a topology on X and an ideal on X. If $(X, \mathcal{T})$ is a topological space and $x \in X$, $\mathcal{T}(x)$ denote the collection $\{U \in \mathcal{T} / x \in U\}$ of all open sets in $(X, \mathcal{T})$ containing x. We denote the complement of A in X by A^c. If X is any set, by $\mathscr{P}(X)$ we denote the collection of all subsets of X and call it as the power set of X. The definitions and results which are not stated explicitly are as in [10].

A closure operator on a set X is a function from $\mathscr{P}(X)$ to $\mathscr{P}(X)$, taking A to $\overline{A}$, satisfying the following conditions: $\overline{\emptyset} = \emptyset$, $A \subseteq \overline{A}$, $\overline{\overline{A}} = \overline{A}$ for all A, and for any A and B, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. The above four conditions are called Kuratowski closure axioms [7]. If "$\overline{\ \ }$" is a closure operator on a set X, $\mathcal{F}$ is the family of all subsets A of X for which $\overline{A} = A$, and if $\mathcal{T}$ is the family of complements of members of $\mathcal{F}$, then $\mathcal{T}$ is a topology on X and $\overline{A}$ is the $\mathcal{T}$-closure of A for each subset A of X. This topology is called the topology generated by the closure operator "$\overline{\ \ }$".

Definition 2.2. *[8] For any subset A of X, define*

$$A^*_{(\mathcal{I},\mathcal{T})} = \{x \in X / U \cap A \notin \mathcal{I} \text{ for every } U \in \mathcal{T}(x)\}.$$

Let $\overline{A} = A \cup A^*_{(\mathcal{I},\mathcal{T})}$. Then "$\overline{\ \ }$" is a Kuratowski closure operator which gives a topology on X, called the topology generated by $\mathcal{I}$, and is denoted by $\mathcal{T}_{\mathcal{I}}$. This topology is also called $*$-topology or ideal topology.

Let $(X, \mathcal{T})$ be a topological space. Then the following results hold trivially.

i. If $\mathcal{I} = \{\emptyset\}$, then $\mathcal{T}_{\mathcal{I}} = \mathcal{T}$.

ii. If $A \in \mathcal{I}$, then $A^* = \emptyset$ and A is closed in $(X, \mathcal{T}_{\mathcal{I}})$.

iii. If A is closed in $\mathcal{T}_{\mathcal{I}}$, then $A^*_{(\mathcal{I},\mathcal{T})} \subseteq A$.

iv. If $\mathcal{I}_1 \subseteq \mathcal{I}_2$ are two ideals on X then $\mathcal{T}_{\mathcal{I}_1} \subseteq \mathcal{T}_{\mathcal{I}_2}$.

Now we prove a result which we use in the sequel.

Theorem 2.1. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two ideals on X. Then $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} = \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$.*

Proof. Since the intersection of two ideals is an ideal, $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2}$ is meaningful. As $\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq \mathcal{I}_1$ and $\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq \mathcal{I}_2$, we have $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} \subseteq \mathcal{T}_{\mathcal{I}_1}$ and $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} \subseteq \mathcal{T}_{\mathcal{I}_2}$. Therefore $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} \subseteq \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$.

Conversely, let us assume that $V \in \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$ and let $A = V^c$. Then A is closed in $\mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$ and hence A is closed in $\mathcal{T}_{\mathcal{I}_1}$ and A is closed in $\mathcal{T}_{\mathcal{I}_2}$. This implies that $A^*_{(\mathcal{I}_1,\mathcal{T})} \subseteq A$ and $A^*_{(\mathcal{I}_2,\mathcal{T})} \subseteq A$. We aim to prove that A is closed in $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2}$. It is enough to show that $A^*_{(\mathcal{I}_1 \cap \mathcal{I}_2,\mathcal{T})} \subseteq A$. Assume that $x \notin A$. This implies that $x \notin A^*_{(\mathcal{I}_1,\mathcal{T})}$ and $x \notin A^*_{(\mathcal{I}_2,\mathcal{T})}$. Then there exist open sets U and V in $\mathcal{T}$ containing x such that $U \cap A \in \mathcal{I}_1$ and $V \cap A \in \mathcal{I}_2$. Let us take $G = U \cap V$. Clearly $x \in G$ and $G \in \mathcal{T}$. Also

$$G \cap A = (U \cap V) \cap A = (U \cap A) \cap (V \cap A) \in \mathcal{I}_1 \cap \mathcal{I}_2.$$

Thus there exists an open set $G \in \mathcal{T}(x)$ such that $G \cap A \in \mathcal{I}_1 \cap \mathcal{I}_2$. Therefore $x \notin A^*_{(\mathcal{I}_1 \cap \mathcal{I}_2,\mathcal{T})}$ and hence $A^*_{(\mathcal{I}_1 \cap \mathcal{I}_2,\mathcal{T})} \subseteq A$. This implies that A is closed in $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2}$. Thus $\mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2} \subseteq \mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2}$. □

We note that this result is not true in case of intersection of infinitely many ideals. If $\{\mathcal{I}_\alpha\}$ is a collection of ideals on $(X, \mathcal{T})$, then we have only $\mathcal{T}_{\cap \mathcal{I}_\alpha} \subseteq \cap \mathcal{T}_{\mathcal{I}_\alpha}$. The equality holds if and only if $(X, \mathcal{T})$ is an Alexandroff space[11, Theorem 3.3].

3 Hausdorff Spaces in Ideal Topological Spaces

In this section we define Hausdorff space in the context of ideal topological spaces, compare it with the one available in the literature and prove some results. We start with the definition of $\mathcal{I}$-open sets and Hausdorff spaces in the context of ideal topological spaces given by Dontchev[2].

Definition 3.3. *[3] Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. A subset A of X is said to be $\mathcal{I}$-open if $A \subseteq int(A^*)$.*

Definition 3.4. *[2] An ideal topological space $(X, \mathcal{T}, \mathcal{I})$ is called $\mathcal{I}$-Hausdorff if for every two distinct points x, y of X, there exist disjoint $\mathcal{I}$-open sets U, V in $(X, \mathcal{T})$ such that $x \in U$ and $y \in V$.*

According to this definition, if $(X, \mathcal{T})$ is a topological space and if $\mathcal{I} = \mathscr{P}(X)$, then $(X, \mathcal{T})$ is not $\mathcal{I}$-Hausdorff even if $(X, \mathcal{T})$ is Hausdorff. Indeed, if A is any subset of X, then $A \in \mathcal{I}$ and hence $A^* = \emptyset$ which implies that no set other than the empty set is $\mathcal{I}$-open; so one cannot find two disjoint $\mathcal{I}$-open sets containing two distinct points. Furthermore, the set $X = \{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{0\}$ under the usual metric of $\mathbb{R}$ is a metric space and hence it is Hausdorff; but when $\mathcal{I} = \{\emptyset\}$, X is not $\mathcal{I}$-Hausdorff. Example 3.1 in [4] shows that an $\mathcal{I}$-Hausdorff space need not be Hausdorff. So an $\mathcal{I}$-Hausdorff space need not be a Hausdorff space and a Hausdorff space need not be an $\mathcal{I}$-Hausdorff space according to the definition available in the literature.

However, according to the theory of $\mathcal{I}$-Hausdorff space we are going to develop in this paper, every Hausdorff space is an $\mathcal{I}$-Hausdorff space and there are $\mathcal{I}$-Hausdorff spaces which are not Hausdorff space. To avoid confusions in the notations we write $\mathscr{I}$-Hausdorff instead of writing $\mathcal{I}$-Hausdorff in the new sense.

Definition 3.5. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{I}$ be an ideal on X. Then $(X, \mathcal{T})$ is said to be $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}$ if for every pair of distinct points x and y in X, there exist two open sets U_1 and U_2 in $\mathcal{T}$ such that $x \in U_1, y \in U_2$ and $U_1 \cap U_2 \in \mathcal{I}$.*

From the very definition itself, it follows that every Hausdorff space is $\mathscr{I}$-Hausdorff whatever be the ideal $\mathcal{I}$ on it, as $\emptyset \in \mathcal{I}$; if $X = \{1, 2\}$, $\mathcal{T} = \{\emptyset, X, \{1\}\}$ and $\mathcal{I} = \{\emptyset, \{1\}\}$, then X is $\mathscr{I}$-Hausdorff with respect to $\mathcal{I}$ whereas it is not Hausdorff in the classical sense. Thus the class of $\mathscr{I}$-Hausdorff spaces is strictly larger than the class of Hausdorff spaces. Whenever there is no ambiguity we just write $\mathscr{I}$-Hausdorff leaving the tail "with respect to the ideal $\mathcal{I}$".

In view of the discussion below Definition 3.4, there are many $\mathscr{I}$-Hausdorff spaces in our context which are not $\mathcal{I}$-Hausdorff space according to Definition 3.4. Example 3.1 in [4] shows that an $\mathcal{I}$-Hausdorff space need not be an $\mathscr{I}$-Hausdorff space in our context. From this we conclude that our definition of Hausdorffness is different from, and more natural than, the one available in the literature.

Theorem 3.2. *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. Then X is $\mathscr{I}$-Hausdorff with respect to $\mathcal{I}$ if and only if the following holds:*

> *If $x, y \in X$ with $x \neq y$, then there exist sets $V_1, V_2 \in \mathcal{T}$ and $I_1, I_2 \in \mathcal{I}$ such that $x \in V_1 - I_1, y \in V_2 - I_2$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$.*

Proof. Assume that X is $\mathscr{I}$-Hausdorff. Let $x, y \in X$ such that $x \neq y$. Since X is $\mathscr{I}$-Hausdorff, there exist open sets V_1 and V_2 in $\mathcal{T}$ such that $x \in V_1$, $y \in V_2$ and $V_1 \cap V_2 \in \mathcal{I}$. By taking $I_1 = I_2 = \emptyset$ we see that the statement holds.

To prove the converse, let $x, y \in X$ such that $x \neq y$. Then there exist sets $V_1, V_2 \in \mathcal{T}$ and $I_1, I_2 \in \mathcal{I}$ such that $x \in V_1 - I_1, y \in V_2 - I_2$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$. We claim that $V_1 \cap V_2 \in \mathcal{I}$.

As $(V_1 - I_1) \cap (V_2 - I_2) = (V_1 \cap V_2) - (I_1 \cup I_2)$, we have

$$V_1 \cap V_2 = [(V_1 - I_1) \cap (V_2 - I_2)] \cup [(V_1 \cap V_2) \cap (I_1 \cup I_2)].$$

Since I_1, $I_2 \in \mathcal{I}$, we have $I_1 \cup I_2 \in \mathcal{I}$. Since $(V_1 \cap V_2) \cap (I_1 \cup I_2) \subseteq (I_1 \cup I_2)$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$, we have $V_1 \cap V_2 \in \mathcal{I}$. Thus X is $\mathscr{I}$-Hausdorff. □

If $\mathcal{I}$ is an ideal on a set X and if $Y \subseteq X$, then $\mathcal{I}_Y = \{A \cap Y \ / \ A \in \mathcal{I}\}$ is an ideal on Y.

Theorem 3.3. *If $(X, \mathcal{T})$ is an $\mathscr{I}$-Hausdorff space with respect to the ideal $\mathcal{I}$ and if $Y \subseteq X$, then $(Y, \mathcal{T}_Y)$ is $\mathscr{I}$-Hausdorff space with respect to the ideal $\mathcal{I}_Y$ where $\mathcal{T}_Y$ is the subspace topology on Y inherited from $\mathcal{T}$.*

Proof. Let $y_1, y_2 \in Y$ be such that $y_1 \neq y_2$. Since X is $\mathscr{I}$-Hausdorff, there exist open sets G_1, G_2 in $\mathcal{T}$ such that $y_1 \in G_1$, $y_2 \in G_2$ and $G_1 \cap G_2 \in \mathcal{I}$. Let $H_1 = Y \cap G_1$ and $H_2 = Y \cap G_2$. Then H_1 and H_2 are open sets in $\mathcal{T}_Y$ containing y_1 and y_2 respectively. As $H_1 \cap H_2 = (G_1 \cap G_2) \cap Y$, we have $H_1 \cap H_2 \in \mathcal{I}_Y$. Therefore Y is $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}_Y$. □

In the crisp topological theory, the intersection of two Hausdorff topologies on a set X need not be Hausdorff. That is, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two Hausdorff topologies on X, then $\mathcal{T}_1 \cap \mathcal{T}_2$ need not be Hausdorff. But if $\mathcal{T}_1$ and $\mathcal{T}_2$ are Hausdorff topologies on a set X and if there exists a topology $\mathcal{T}$ and two ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ on X such that $\mathcal{T}_1 = \mathcal{T}_{\mathcal{I}_1}$ and $\mathcal{T}_2 = \mathcal{T}_{\mathcal{I}_2}$, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is a Hausdorff topology on X (See Theorem 3.6).

First we give a necessary and sufficient condition for a space $(X, \mathcal{T})$ to be $\mathscr{I}$-Hausdorff with respect to an ideal $\mathcal{I}$.

Theorem 3.4. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{I}$ be an ideal on X. Then $(X, \mathcal{T})$ is $\mathscr{I}$-Hausdorff if and only if $(X, \mathcal{T}_{\mathcal{I}})$ is Hausdorff.*

Proof. Let x and y be two distinct points in an $\mathscr{I}$-Hausdorff space $(X, \mathcal{T})$. Then there exist open sets U and V in $\mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V \in \mathcal{I}$. Let us take $U_1 = U - ((U \cap V) - \{x\})$ and $U_2 = V - ((U \cap V) - \{y\})$. Clearly $x \in U_1$ and $y \in U_2$. Since $U \cap V \in \mathcal{I}$, $(U \cap V) - \{x\} \in \mathcal{I}$; therefore it is closed in $\mathcal{T}_{\mathcal{I}}$ and hence U_1 is open in $\mathcal{T}_{\mathcal{I}}$. Similarly U_2 is open in $\mathcal{T}_{\mathcal{I}}$. Thus we get two open sets U_1 and U_2 in $\mathcal{T}_{\mathcal{I}}$ such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \emptyset$. Therefore, $(X, \mathcal{T}_{\mathcal{I}})$ is Hausdorff.

Conversely, let x and y be two distinct points in $(X, \mathcal{T})$. Since $(X, \mathcal{T}_{\mathcal{I}})$ is Hausdorff, there exist open sets U and V in $\mathcal{T}_{\mathcal{I}}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. As $U \in \mathcal{T}_{\mathcal{I}}$, U^c is closed in $\mathcal{T}_{\mathcal{I}}$ and hence $(U^c)^* \subseteq U^c$. Since $x \notin U^c$, we have $x \notin (U^c)^*$. Thus there exists $U_1 \in \mathcal{T}$ containing x such that $U_1 \cap U^c \in \mathcal{I}$. Let $I_1 = U_1 \cap U^c$. Clearly $x \in U_1 - I_1 \subseteq U$. Similarly there exists $U_2 \in \mathcal{T}$ and $I_2 \in \mathcal{I}$ such that $y \in U_2 - I_2 \subseteq V$. Since $U \cap V = \emptyset$, we have $(U_1 - I_1) \cap (U_2 - I_2) = \emptyset$. It follows that $U_1 \cap U_2 \subseteq I_1 \cup I_2$ and hence $U_1 \cap U_2 \in \mathcal{I}$. Thus we get open sets U_1 and U_2 in $\mathcal{T}$ such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 \in \mathcal{I}$. Therefore $(X, \mathcal{T})$ is $\mathscr{I}$-Hausdorff. □

Theorem 3.5. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is $\mathscr{I}$-Hausdorff with respect to $\mathcal{I}_1$ and $\mathcal{I}_2$, then $(X, \mathcal{T})$ is $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$.*

Proof. Let x and y be two distinct points in $(X, \mathcal{T})$. By Theorem 3.4, $(X, \mathcal{T}_{\mathcal{I}_1})$ and $(X, \mathcal{T}_{\mathcal{I}_2})$ are Hausdorff. Since $(X, \mathcal{T}_{\mathcal{I}_1})$ is Hausdorff, as in the proof of Theorem 3.4, there exist $U_1, V_1 \in \mathcal{T}$ and $I_1, J_1 \in \mathcal{I}_1$ such that $x \in U_1 - I_1$, $y \in V_1 - J_1$ and $(U_1 - I_1) \cap (V_1 - J_1) = \emptyset$. Similarly there exist $U_2, V_2 \in \mathcal{T}$ and $I_2, J_2 \in \mathcal{I}_2$ such that $x \in U_2 - I_2$, $y \in V_2 - J_2$ and $(U_2 - I_2) \cap (V_2 - J_2) = \emptyset$.

Let $W_1 = U_1 \cap U_2$ and $W_2 = V_1 \cap V_2$. Clearly $x \in W_1$, $y \in W_2$ and $W_1, W_2 \in \mathcal{T}$. Let $I = (I_1 \cup J_1) \cap (I_2 \cup J_2)$. Since $I \subseteq I_1 \cup J_1$ and $I \subseteq I_2 \cup J_2$, we have $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. We claim that $W_1 \cap W_2 \in \mathcal{I}_1 \cap \mathcal{I}_2$. As $(U_1 - I_1) \cap (V_1 - J_1) = \emptyset$ and $(U_2 - I_2) \cap (V_2 - J_2) = \emptyset$, we have $U_1 \cap V_1 \subseteq I_1 \cup J_1$ and $U_2 \cap V_2 \subseteq I_2 \cup J_2$. Since $W_1 \cap W_2 = (U_1 \cap V_1) \cap (U_2 \cap V_2)$, we have $W_1 \cap W_2 \subseteq I$ and hence $W_1 \cap W_2 \in \mathcal{I}_1 \cap \mathcal{I}_2$. Therefore $(X, \mathcal{T})$ is $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$.

□

Theorem 3.6. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be Hausdorff topologies on a set X. Let there be a topology $\mathcal{T}$ and two ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ on X such that $\mathcal{T}_1 = \mathcal{T}_{\mathcal{I}_1}$ and $\mathcal{T}_2 = \mathcal{T}_{\mathcal{I}_2}$. Then $\mathcal{T}_1 \cap \mathcal{T}_2$ is a Hausdorff topology on X.*

Proof. Since $\mathcal{T}_1 = \mathcal{T}_{\mathcal{I}_1}$, $\mathcal{T}_2 = \mathcal{T}_{\mathcal{I}_2}$, by Theorem 3.4, $(X, \mathcal{T})$ is $\mathscr{I}$-Hausdorff with respect to the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$. Also by Theorem 3.5, $(X, \mathcal{T})$ is $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$; by Theorem 3.4, $(X, \mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2})$ is Hausdorff. By Theorem 2.1, $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} = \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$. Therefore $(X, \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2})$ is Hausdorff. In other words, $\mathcal{T}_1 \cap \mathcal{T}_2$ is a Hausdorff topology on X. □

If $\mathscr{A}$ and $\mathscr{B}$ are collections of subsets of X and Y, then the collection $\mathscr{A} \times \mathscr{B} = \{A \times B \ / \ A \in \mathscr{A}, B \in \mathscr{B}\}$ is called the product of $\mathscr{A}$ and $\mathscr{B}$. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals on X_1 and X_2, then $\mathcal{I}_1 \times \mathcal{I}_2$ need not be an ideal on $X_1 \times X_2$. Indeed, if $X = \{1, 2, 3\}$, $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, then $\mathcal{I} \times \mathcal{I}$ is not an ideal on $X \times X$ as $\{(1, 1), (2, 2)\}$

is not in $\mathcal{I} \times \mathcal{I}$ whereas $\{(1,1)\}, \{(2,2)\}$ are in $\mathcal{I} \times \mathcal{I}$. However with ideals $\mathcal{I}_1$ on X and $\mathcal{I}_2$ on Y we can associate an ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ on $X \times Y$ in a natural way. The ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ is in fact the smallest ideal containing $\mathcal{I}_1 \times \mathcal{I}_2$ which can be obtained as the intersection of all ideals containing $\mathcal{I}_1 \times \mathcal{I}_2$.

It is well known that the product of two Hausdorff spaces is a Hausdorff space in crisp topological theory. But this is not true in ideal topological theory. That is, if $(X_1, \mathcal{T}_1)$ is $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}_1$ and $(X_2, \mathcal{T}_2)$ is $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}_2$, then $(X_1 \times X_2, \mathcal{T}_1 \times \mathcal{T}_2)$ need not be $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ as seen in the following example.

Example 3.1. *Let $X_1 = \{1,3\}, X_2 = \{1,4\}$; let $\mathcal{T}_1 = \{\emptyset, X_1, \{1\}\}, \mathcal{T}_2 = \{\emptyset, X_2, \{1\}\}$; then $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X_1 and X_2; let $\mathcal{I}_1 = \{\emptyset, \{1\}\}, \mathcal{I}_2 = \{\emptyset, \{1\}\}$; then $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals on X_1 and X_2. Clearly the product topology $\mathcal{T}_1 \times \mathcal{T}_2$ is the collection*

$$\{\emptyset, X_1 \times X_2, \{(1,1)\}, \{(1,1),(1,4)\}, \{(1,1),(3,1)\}, \{(1,1),(1,4),(3,1)\}\}$$

and the ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ is the collection $\{\emptyset, \{(1,1)\}\}$. As we cannot separate the points $(1,4)$ and $(3,4)$, $X_1 \times X_2$ is not $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$.

We note an interesting observation in the comparison of Hausdorff theory between the crisp and ideal topological theory. One point sets in Hausdorff spaces are closed in crisp theory whereas it is not so in the theory of ideal topology. For example, if $X = \{1,2,3\}$, $\mathcal{T} = \{\emptyset, X, \{1\}, \{1,2\}\}$ and $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, then $\{1\}, \{2\}$ are not closed in $(X, \mathcal{T})$; but X is $\mathscr{I}$-Hausdorff with respect to $\mathcal{I}$.

4 Regular and Normal Spaces

Now we define regular and normal space in the context of ideal topological spaces and prove some results.

Definition 4.6. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{I}$ be an ideal on X. Let singleton sets be closed in X. Then $(X, \mathcal{T})$ is said to be $\mathscr{I}$-regular with respect to the ideal $\mathcal{I}$ if given $x \in X$ and a closed set B not containing x, there exist two open sets U_1 and U_2 in $\mathcal{T}$ such that $x \in U_1, B \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$.*

As $\emptyset \in \mathcal{I}$, every regular space is $\mathscr{I}$-regular with respect to the ideal $\mathcal{I}$ whatever be the ideal $\mathcal{I}$. An $\mathscr{I}$-regular space with respect to an ideal $\mathcal{I}$ need not be regular. For example, any uncountable set X with cocountable topology is not regular; but it is $\mathscr{I}$-regular space with respect to the ideal $\mathcal{I}$ where $\mathcal{I} = \mathscr{P}(X)$.

In Theorem 3.4 we have proved that a space $(X, \mathcal{T})$ is $\mathscr{I}$-Hausdorff with respect to the ideal $\mathcal{I}$ if and only if $(X, \mathcal{T}_{\mathcal{I}})$ is Hausdorff. But in the case of regular spaces it is not so. That is, if $(X, \mathcal{T})$ is $\mathscr{I}$-regular with respect to an ideal $\mathcal{I}$, then $(X, \mathcal{T}_{\mathcal{I}})$ need not be regular. For example, in $\mathbb{R}$ with usual topology, let $\mathcal{I}$ be the collection of all subsets of $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. Since $\mathbb{R}$ with usual topology is regular, it is $\mathscr{I}$-regular with respect to the ideal $\mathcal{I}$; but it is not $\mathcal{T}_{\mathcal{I}}$-regular because we cannot separate the point 0 and a closed set $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. However, $(X, \mathcal{T}_{\mathcal{I}})$ is regular if the following additional condition is satisfied.

C1: For any $A \in \mathcal{I}$ and $x \notin A$, there exists U in $\mathcal{T}$ such that $x \in U$ and $\overline{U} \cap A = \emptyset$.

Theorem 4.7. *If $(X, \mathcal{T})$ is $\mathscr{I}$-regular with respect to the ideal $\mathcal{I}$ and if* **C1** *holds, then $(X, \mathcal{T}_{\mathcal{I}})$ is regular.*

Proof. Let F be closed in $\mathcal{T}_{\mathcal{I}}$ and $x \notin F$.

Suppose F is closed in $\mathcal{T}$, by $\mathscr{I}$-regularity, there exist $U_1, U_2 \in \mathcal{T}$ such that $x \in U_1$, $F \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$. If needed replacing U_1 by $U_1 \cap F^c$, we can assume $U_1 \cap F = \emptyset$. Let $I = U_1 \cap U_2$, $V_1 = U_1$ and $V_2 = U_2 - I$. Since $I \cap F = \emptyset$ and $F \subseteq U_2$, $F \subseteq V_2$. Clearly $V_1 \cap V_2 = \emptyset$. Thus we obtained two open sets V_1, V_2 in $\mathcal{T}_{\mathcal{I}}$ such that $x \in V_1$, $F \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$. Hence $(X, \mathcal{T}_{\mathcal{I}})$ is regular in this case.

Now we prove the general case. Since F is closed in $\mathcal{T}_{\mathcal{I}}$, we have $F^*_{(\mathcal{I},\mathcal{T})} \subseteq F$ and hence $x \notin F^*_{(\mathcal{I},\mathcal{T})}$. Then there exists $U \in \mathcal{T}(x)$ such that $U \cap F \in \mathcal{I}$. Let $I = U \cap F$ and $F_1 = F - I$. Clearly $U \cap F_1 = \emptyset$. Let F_2 be the closure of F_1 with respect to $\mathcal{T}$. Since $x \in U$ and $U \cap F_1 = \emptyset$, we have $x \notin F_2$. Therefore F_2 is closed set in $\mathcal{T}$ such that $x \notin F_2$. By the particular case discussed above, there exist $V_1, V_2 \in \mathcal{T}_{\mathcal{I}}$ such that $x \in V_1$, $F_2 \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$. Since $x \notin I \in \mathcal{I}$, by the condition **C1**, there exists an open set $U \in \mathcal{T}$ such that $x \in U$ and $\overline{U} \cap I = \emptyset$. Let us take $W_1 = U$ and $W_2 = \overline{U}^c$. Also $x \in W_1$, $I \subseteq W_2$ and $W_1 \cap W_2 = \emptyset$. Let $G_1 = V_1 \cap W_1$ and $G_2 = V_2 \cup W_2$. Clearly $x \in G_1$ and $G_1 \cap G_2 = \emptyset$. Since $F_2 \subseteq V_2$ and $I \subseteq W_2$, we have $F \subseteq G_2$. Thus we get open sets G_1, G_2 in $\mathcal{T}_{\mathcal{I}}$ such that $x \in G_1$, $F \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ and hence $(X, \mathcal{T}_{\mathcal{I}})$ is regular. $\square$

We can weaken condition **C1** by the following condition:

C2: For any $A \in \mathcal{I}$ and $x \notin A$, there exist $U, V \in \mathcal{T}_{\mathcal{I}}$ such that $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$.

Theorem 4.8. *If $(X, \mathcal{T})$ is $\mathscr{I}$-regular with respect to the ideal $\mathcal{I}$ and if* **C2** *holds, then $(X, \mathcal{T}_{\mathcal{I}})$ is regular.*

If $x \notin I \in \mathcal{I}$, by the condition **C2**, there exist W_1, W_2 in $\mathcal{T}_{\mathcal{I}}$ such that $x \in W_1$, $I \subseteq W_2$ and $W_1 \cap W_2 = \emptyset$. Replacing the sets W_1 and W_2 in the proof of Theorem 4.7 by these W_1 and W_2, we get the proof.

If U is a set that exists in condition **C1**, then U and $\overline{U}^c$ serve as the open sets in condition **C2**. Thus **C2** is weaker than **C1**. The following example shows that **C2** is strictly weaker than **C1**.

Example 4.2. *Let $X = \{1,2,3,4\}, \mathcal{T} = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}\}$ and $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$. Let $A = \{2\}$, $x = 3$. As the only open set containing x in $\mathcal{T}$ is X, condition* **C1** *is not satisfied whereas it is easy to verify that condition* **C2** *is satisfied.*

The following theorem can be proved analogous to Theorem 3.2.

Theorem 4.9. *Let $\mathcal{I}$ be an ideal on $(X, \mathcal{T})$. Then X is $\mathscr{I}$-regular with respect to $\mathcal{I}$ if and only if the following holds:*

If $x \in X$ and a closed set B not containing x, then there exist sets $V_1, V_2 \in \mathcal{T}$ and $I_1, I_2 \in \mathcal{I}$ such that $x \in V_1 - I_1, B \subseteq V_2 - I_2$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$.

Theorem 4.10. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is $\mathscr{I}$-regular with respect to $\mathcal{I}_1$ and $\mathcal{I}_2$, then $(X, \mathcal{T})$ is $\mathscr{I}$-regular with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$.*

Proof. Let $x \in X$ and B a closed set in $(X, \mathcal{T})$ not containing x. Since X is $\mathscr{I}$-regular with respect to $\mathcal{I}_1$, there exist two open sets U_1, V_1 in $\mathcal{T}$ such that $x \in U_1$, $B \subseteq V_1$ and $U_1 \cap V_1 \in \mathcal{I}_1$. Similarly there exist two open sets U_2, V_2 in $\mathcal{T}$ such that $x \in U_2$, $B \subseteq V_2$ and $U_2 \cap V_2 \in \mathcal{I}_2$. Let $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$. Clearly $x \in U$ and $B \subseteq V$. Since $U \cap V = (U_1 \cap V_1) \cap (U_2 \cap V_2)$, we have $U \cap V \in \mathcal{I}_1 \cap \mathcal{I}_2$. Thus there exist two open sets U, V in $\mathcal{T}$ such that $x \in U$, $B \subseteq V$ and $U \cap V \in \mathcal{I}_1 \cap \mathcal{I}_2$ and hence $(X, \mathcal{T})$ is $\mathscr{I}$-regular with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$.

□

Theorem 4.11. *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. If X is an $\mathscr{I}$-regular space with respect to $\mathcal{I}$ and if $Y \subseteq X$, then $(Y, \mathcal{T}_Y)$ is $\mathscr{I}$-regular space with respect to the ideal $\mathcal{I}_Y$ where $\mathcal{T}_Y$ is the subspace topology on Y inherited from $\mathcal{T}$.*

Proof. Let A be a closed set in $(Y, \mathcal{T}_Y)$ and $x \notin A$. Since A is closed in Y, we have $A = Y \cap F$ where F is closed in X. As F is closed in X and $x \notin F$, there exist U_1, U_2 in $\mathcal{T}$ such that $x \in U_1$, $F \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$. Let $V_1 = Y \cap U_1$ and $V_2 = Y \cap U_2$. Clearly $x \in V_1$, $A \subseteq V_2$ and V_1, V_2 are open sets in $\mathcal{T}_Y$. As $V_1 \cap V_2 = (U_1 \cap U_2) \cap Y$, we have $V_1 \cap V_2 \in \mathcal{I}_Y$. Therefore Y is $\mathscr{I}$-regular with respect to the ideal $\mathcal{I}_Y$. □

Theorem 4.12. *Every $\mathscr{I}$-regular space is $\mathscr{I}$-Hausdorff space with respect to the same ideal.*

Now we define normal space in the context of ideal topological spaces and prove some results.

Definition 4.7. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{I}$ be an ideal on X. Let singleton sets be closed in X. Then $(X, \mathcal{T})$ is said to be $\mathscr{I}$-normal with respect to the ideal $\mathcal{I}$ if given two disjoint closed sets A and B, there exist two open sets U_1 and U_2 in $\mathcal{T}$ such that $A \subseteq U_1, B \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$.*

As $\emptyset \in \mathcal{I}$, every normal space is $\mathscr{I}$-normal space with respect to the ideal $\mathcal{I}$ whatever be the ideal $\mathcal{I}$ on X. The converse is not true. For example, any infinite set X with cofinite topology is not normal; but it is $\mathscr{I}$-normal space with respect to the ideal $\mathcal{I}$ where $\mathcal{I} = \mathscr{P}(X)$.
The following theorem can be proved analogous to Theorem 3.2.

Theorem 4.13. *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. Then X is $\mathscr{I}$-normal with respect to $\mathcal{I}$ if and only if the following holds:*

If A and B be two closed sets such that $A \cap B = \emptyset$, then there exist sets $V_1, V_2 \in \mathcal{T}$ and $I_1, I_2 \in \mathcal{I}$ such that $A \subseteq V_1 - I_1, B \subseteq V_2 - I_2$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$.

The following theorem can be proved analogous to Theorem 4.10.

Theorem 4.14. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is $\mathscr{I}$-normal with respect to $\mathcal{I}_1$ and $\mathcal{I}_2$, then $(X, \mathcal{T})$ is $\mathscr{I}$-normal with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$.*

Theorem 4.15. *Every $\mathscr{I}$-normal space is $\mathscr{I}$-regular with respect to the same ideal.*

Theorem 4.16. *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. If X is an $\mathscr{I}$-normal space with respect to $\mathcal{I}$ and if Y is a closed subset of $(X, \mathcal{T})$, then $(Y, \mathcal{T}_Y)$ is $\mathscr{I}$-normal with respect to the ideal $\mathcal{I}_Y$ where $\mathcal{T}_Y$ is the subspace topology on Y inherited from $\mathcal{T}$.*

Proof. Let Y be a closed subset of X and let A and B be disjoint closed sets in $(Y, \mathcal{T}_Y)$. Then A and B are disjoint closed sets in $(X, \mathcal{T})$. By $\mathscr{I}$-normality, there exist U_1, U_2 in $\mathcal{T}$ such that $A \subseteq U_1$, $B \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$. Let $V_1 = Y \cap U_1$ and $V_2 = Y \cap U_2$. Clearly $A \subseteq V_1$, $B \subseteq V_2$ and V_1, V_2 are open sets in $\mathcal{T}_Y$. As $V_1 \cap V_2 = (U_1 \cap U_2) \cap Y$, we have $V_1 \cap V_2 \in \mathcal{I}_Y$. Therefore Y is $\mathscr{I}$-normal with respect to the ideal $\mathcal{I}_Y$. □

Conclusion

We defined and discussed the separation axioms in ideal topological spaces in a new way which is more natural than the previous versions. We proved a property that holds in ideal topological theory which does not hold in the classical theory of topology and also established a property that holds in the classical theory which does not hold in the ideal topological theory. This makes the ideal topological theory interesting and independent. Many concepts available in the classical theory may be discussed using the theory developed in this paper.

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Cotoal Edge Domination Number of a Graph

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Abstract

A set F of a graph $G(V,E)$ is an edge dominating set if every edge in $E-F$ is adjacent to some edge in F. An edge domination number $\gamma^{'}(G)$ of G is the minimum cardinality of an edge dominating set. An edge dominating set F is called a cototal edge dominating set if the induced subgraph $\langle E-F\rangle$ doesnot contain isolated edge. The minimum cardinality of the cototal edge dominating set in G is its domination number and is denoted by $\gamma^{'}_{cot}(G)$. We investigate several properties of cototal edge dominating sets and give some bounds on the cototal edge domination number.

Keywords: Edge domination number, cototal edge domination number.

2010 MSC: 34G20.

1 Introduction

Let $G(V,E)$ be a graph with $p=\mid V\mid$ and $q=\mid E\mid$ denoting the number of vertices and edges respectively. All the graphs considered here are finite, non-trivial, undirected and connected without loops or multiple edges.

The degree of a vertex u is denoted by $d(u)$. The degree of an edge $e=uv$ of a graph G is the number defined by $deg(e)=degu+degv-2$. The minimum(maximum) degree of an edge is denoted by $\delta^{'}(\Delta^{'})$. The induced subgraph of $X\subseteq E$ is denoted by $\langle X\rangle$. For a real number x, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x and $\lceil x\rceil$ denotes the smallest integer greater than or equal to x. An edge independence number $\beta_1(G)$ is defined to be the number of edges in a maximum independent set of edges of G. A vertex of degree one is called a pendant vertex. An edge incident to pendant vertex is called the pendant edge. Let $\Omega_1(G)$ be the set of all pendant edges of G. As usual, P_p, C_p and K_p are respectively the path, cycle and complete graph of order p. $K_{m,n}$ is the complete bipartite graph with two partite sets containing m and n vertices. Let $t\geq 3, n\geq 1$ be two integers. We denote by W_t^n the graph C_t+K_n as a generalized wheel. Note that for $n=1, W_t^1=K_1+C_{n-1}$ is a wheel. $B_{r,s}$ is a graph obtained by joining the centres of two stars $K_{1,r}$ and $K_{1,s}$ by an edge called as Bistar or double star. The subdivision graph of a graph G, denoted by $S(G)$, is a graph obtained from G by deleting every edge uv of G and replacing it by a vertex w of degree 2 that is joined to u and v.

Let $G(V,E)$ be a connected graph. A subset S of V is called a dominating set of G if every vertex in $V-S$ is adjacent to at least one vertex in S. The concept of edge domination was introduced by Mitchell and Hedetniemi [5, 7].

Definition 1.1. *A subset F of E is called an edge dominating set of G if every edge not in F is adjacent to some edge in F. The minimum cardinality of an edge dominating set of G is called an edge domination number and is denoted by*

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$\gamma^{'}(G)$.

Definition 1.2. *A dominating set D of G is a cototal dominating set if the induced subgraph $\langle V - D \rangle$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ of G is the minimum cardinality of a cototal dominating set.*

This concept was introduced by Kulli, Janakiram and Iyer in [6]. Any undefined term or notation in this paper can be found in Harary [4]. We need the following Theorems for our study on cototal edge domination number.

Theorem 1.1. *[3] For every n, $\gamma^{'}_{3,n} = n$.*

Theorem 1.2. *[2] For any connected graph G of even order $p, \gamma^{'}(G) = p/2$ if and only if G is isomorphic to K_p or $K_{p/2,p/2}$.*

In this paper, we determine the relation among the graph parameters, like edge independent number, maximum edge degree and cototal edge domination number of a graph. We have also derived some relations to determine cototal edge domination number of graph obtained by adding end edges to cycle, cartesian product, subdivision of graphs, corona and join of graphs.

Definition 1.3. *A set $F \subseteq E(G)$ is said to be cototal edge dominating set if F is an edge dominating set and induced subgraph $\langle E - F \rangle$ has no isolated edges. The minimum cardinality of cototal edge dominating set in G is the cototal edge domination number and is denoted by $\gamma^{'}_{cot}(G)$ of G.*

2 Main results

We list out cototal edge domination number of some standard graphs.

Theorem 2.1.

1. *For any spider G, the cototal edge domination number, $\gamma^{'}_{cot}(G) = |\Omega_1(G)|$.*
2. *For any octopus G, the cototal edge domination number, $\gamma^{'}_{cot}(G) = |\Omega_1(G)| + 2$.*
3. *For any generalized wheel W^n_t, with $t \geq 3, \gamma^{'}_{cot}(W^n_t) = \lceil t/2 \rceil$.*
4. *For any wheel W_n, with $n \geq 3, \gamma^{'}_{cot}(W_n) = \lceil n/2 \rceil$.*

Theorem 2.2. *For any path P_p with $p \geq 5$, the cototal edge domination number,*

$$\gamma^{'}_{cot}(P_p) = \begin{cases} \lfloor \frac{p}{3} \rfloor + 2 & \text{if } p \equiv 1(mod3), \\ \lfloor \frac{p}{3} \rfloor + 1 & \text{if } p \equiv 0 \text{ or } 2(mod3). \end{cases}$$

Proof. Let $P_p : v_1, v_2, ..., v_p$ be any path and let $e_i = v_i v_{i+1}$ be an edge. Let

$$S_1 = \begin{cases} S & \text{if } p \equiv 2(mod3), \\ S \cup \{e_{p-1}\} & \text{if } p \equiv 0(mod3), \\ S \cup \{e_{p-1}, e_{p-2}\} & \text{if } p \equiv 1(mod3). \end{cases}$$

be an edge set where $S = \{e_j : j = 3k+1 \text{ for } 0 \leq k \leq \lceil \frac{p}{3} \rceil - 1\}$. Clearly S_1 is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ has no isolated edges. Therefore $|S_1|$ will be the cototal edge dominating set with minimum cardinality. Hence the proof. □

Theorem 2.3. *For any cycle C_p with $p \geq 3$, the cototal edge domination number,*

$$\gamma^{'}_{cot}(C_p) = \begin{cases} \lfloor \frac{p}{3} \rfloor & \text{if } p \equiv 0(mod3), \\ \lfloor \frac{p}{3} \rfloor + 1 & \text{if } p \equiv 1(mod3), \\ \lfloor \frac{p}{3} \rfloor + 2 & \text{if } p \equiv 2(mod3). \end{cases}$$

Proof. Let $C_p : v_1, v_2, ..., v_p$ be any cycle and let $e_i = v_i v_{i+1}$ be an edge. Let

$$S_1 = \begin{cases} S & \text{if } p \equiv 0 \text{ or } 1(mod 3), \\ S \cup \{e_p\} & \text{if } p \equiv 2(mod 3). \end{cases}$$

be an edge set where $S = \{e_j : j = 3k+1 \text{ for } 0 \leq k \leq \lceil \frac{p}{3} \rceil - 1\}$. Clearly S_1 is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ has no isolated edges. Therefore $|S_1|$ will be the cototal edge dominating set with minimum cardinality. Hence the proof. □

Theorem 2.4. *For any complete graph K_p with $p \geq 3$ vertices, the cototal edge domination number, $\gamma'_{cot}(K_p) = \lfloor \frac{p}{2} \rfloor$.*

Proof. Let S be a maximum matching of K_p. We know that the edge independence number $\beta_1(G)$ of a graph G is the maximum cardinality of an independent set of edges. Furthermore for an integer $p \geq 3$ and integers m and n with $1 \leq m \leq n$, we have $\beta_1(K_p) = \lfloor \frac{p}{2} \rfloor$. Clearly S is an edge dominating set. Also the induced subgraph $\langle E - S \rangle$ has no isolated edges. Therefore S is a cototal edge dominating set of K_p.

$$\begin{aligned} \gamma'_{cot}(K_p) &= |S|, \\ &= \left\lfloor \frac{p}{2} \right\rfloor. \end{aligned}$$

Hence the proof. □

Corollary 2.1. *Let G' be a graph obtained from K_p by adding the pendant edges to the vertices of K_p. Then the cototal edge domination number,*

$$\gamma'_{cot}(G') = \begin{cases} \gamma'_{cot}(K_p) & \text{if } p \text{ is even,} \\ \gamma'_{cot}(K_p) + 1 & \text{if } p \text{ is odd.} \end{cases}$$

Theorem 2.5. *For any complete bipartite graph $K_{m,n}$ with $2 \leq m \leq n$ the cototal edge domination number, $\gamma'_{cot}(K_{m,n})$ = m.*

Proof. Let v be any vertex on $K_{m,n}$such that $deg(v) = min(m, n)$. Let S be the set containing all the edges incident on v. It is clear that S is an edge dominating set. Also the induced subgraph $\langle E - S \rangle$ does not contain an isolated edge. Thus S is a cototal edge dominating set and the minimal cardinality $|S|$ of S will be the cototal edge domination number. Thus

$$\begin{aligned} \gamma'_{cot}(K_{m,n}) &\leq |S|, \\ &= deg(v). \end{aligned}$$

Therefore $\gamma'_{cot}(K_{m,n}) = m$. Hence the proof. □

Theorem 2.6. *Let F be a cototal edge dominating set of a connected graph G with $p \geq 5$ vertices. Then every pendant edge $e = uv$ with degree(support vertex) = 2 is in F.*

Proof. Let $e = uv$ be a pendant edge with degree of support vertex v is 2. Suppose $e \notin F$ and $e_1 \in N(e)$. Since F is a dominating set we have $e_1 \in F$. Also the induced subgraph $\langle E - F \rangle$ contains an isolated edge, a contradiction. Therefore every pendant edge with degree of support vertex equal to two, is in F. Hence the proof. □

The following Theorem gives a sharp upper bound for the cototal edge domination number of G.

Theorem 2.7. *For any connected graph G with $\delta(G) > 2$, the cototal edge domination number, $\gamma'_{cot}(G) \leq \beta_1(G)$.*

Proof. Let G be a graph with $\delta(G) > 2$. Let S be an edge independent set in G such that $|S| = \beta_1(G)$. Clearly every edge $e_1 \in E - S$ is adjacent to atleast one edge of S. Hence S is an edge dominating set. Since $\delta(G) > 2$, the induced subgraph $\langle E - S \rangle$ is connected. Therefore S is a cototal edge dominating set. Hence $\gamma'_{cot}(G) \leq \beta_1(G)$. Hence the proof. □

Let $S = \{e_1, e_2, \cdots e_s\}$ be a set of non-adjacent edges to e. The characterization of all graphs for which $\gamma'_{cot}(G) = q - \triangle'$ seems to be a difficult problem. In the next Theorem, we characterize few graphs for which $\gamma'_{cot}(G) = q - \triangle'$.

Theorem 2.8. *Let G be a graph with $\delta(G) \geq 2$ and $e = uv$ with $d(u), d(v) \geq 3$ be an edge with maximum degree. Then $\gamma'_{cot}(G) \leq q - \triangle'(G)$.*

Proof. Let $e = uv$ be an edge with maximum degree. Let $E_1 = \{e_1, e_2, \cdots e_r\}$ be the set of edges adjacent to e. Then $F = E - E_1$ is an edge dominating set and since $\delta(G) \geq 2$ the induced subgraph $\langle E - F \rangle$ has no isolated edges. Therefore F is a cototal edge dominating set of G. Hence $|F| \leq |E| - |E_1|$. Therefore $\gamma'_{cot}(G) \leq q - \triangle'(G)$. Equality holds in the following cases:

Case i) $S = 0$.
Let $e = uv$ be an edge with maximum degree. Since $d(u), d(v) > 2$, the induced subgraph $\langle E - \{e\} \rangle$ has no isolated edges. Because there are no non-adjacent edges to e and all the edges in G are dominated by $\{e\}$. Therefore $F = \{e\}$ is a cototal edge dominating set. Hence
$\gamma'_{cot}(G) = q - \triangle'(G)$.

Case ii) $|S| = 1$.
Let $e = uv$ be an edge with maximum degree. Since there is only one edge e_1 not adjacent to e, we get $F = \{e\} \cup \{e_1\}$ as an edge dominating set. Also $\langle E - F \rangle$ has no isolated edges. Therefore F is a cototal edge dominating set. Hence $\gamma'_{cot}(G) = q - \triangle'(G)$.

Hence the proof. □

Corollary 2.2. *If degree of each vertex in $\langle S \rangle$ is one and for all $w - \{u, v\}$ in $V(G)$, the other end vertices of the edges not in S are either u or v then*
$\gamma'_{cot}(G) = q - \triangle'(G) - 1$.

□

Let $e = uv$ be an edge satisfying $d(e) = q - 1$. Let $\mathcal{H}$ be a family of graphs isomorphic to $K_{1,r}$, $B_{r,s}$, where $r, s \geq 3$, the graphs formed by adding P_3 to $B_{r,s}$ such that end vertices of P_3 is u and v or the graph $\mathcal{G}''$ formed by adding P_3 to $B_{r,s}$ and removing pendant edges adjacent to u or adjacent to v or both of $B_{r,s}$. Few graphs which belong to $\mathcal{G}'$ or $\mathcal{G}''$ are shown in Figure 1. Now we have the following Theorem.

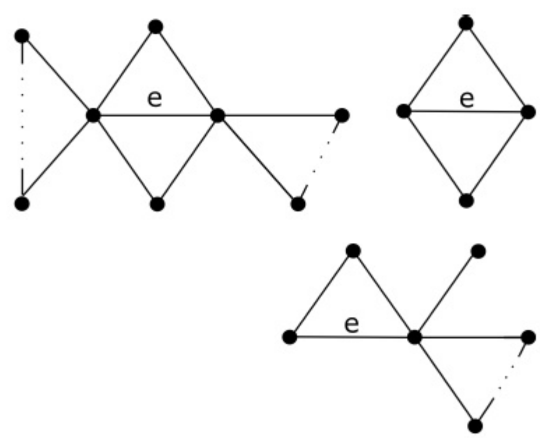

Figure 1: Illustration for the family $\mathcal{H}$

Theorem 2.9. *The cototal edge domination number of $\mathcal{H}$ is one if and only if the graph belongs to $\mathcal{H}$.*

Proof. The inequality $1 \leq \gamma'_{cot}(\mathcal{H})$ is obvious. If there exists an edge $e = uv \in \mathcal{H}$ where $d(e) = q - 1$ then $\gamma'_{cot}(\mathcal{H}) = 1$. Assume $\gamma'_{cot}(\mathcal{H}) = 1$. Then $F = \{e_1\}$ is a minimum cototal edge dominating set of G. Since F is a dominating set of G, e_1 must be an universal edge which belongs to the family $\mathcal{H}$. If $e \notin \mathcal{H}$ then there is a possibility where the induced subgraph $\langle E - F \rangle$ contains an isolated edge. It is a contradiction to the definition of cototal edge dominating set. Therefore $e \in \mathcal{H}$. Hence the proof. □

The following Theorem relates γ'_{cot} of few standard graphs G with their *subdivision* graph $S(G)$ in terms of vertices.

Theorem 2.10. *For any path P_p, $\gamma'_{cot}(P_p) + \gamma'_{cot}(S(P_p)) \leq p+3$.*

Proof. Let S be the γ'_{cot} set of P_p. Therefore by Theorem 2.2,

$$\gamma'_{cot}(P_p) = |S| \tag{2.1}$$

Let S' be its γ'_{cot} set of *subdivision* of P_p. Then by Theorem 2.2 we get $\gamma'_{cot}\Big(S(P_p)\Big) = |S'|$ that is

$$\gamma'_{cot}(S(P_p)) = \begin{cases} \frac{2p}{3} & \text{if } p \equiv 0(mod3), \\ \left\lfloor \frac{2p}{3} \right\rfloor + 2 & \text{if } p \equiv 1(mod3), \\ \left\lfloor \frac{2p}{3} \right\rfloor + 1 & \text{if } p \equiv 2(mod3). \end{cases} \tag{2.2}$$

Consider a particular case where $p \equiv 1(mod3)$. Adding equations (2.1) and (2.2), we get

$$\begin{aligned} \gamma'_{cot}(P_p) + \gamma'_{cot}(S(P_p)) &= |S| + |S'|, \\ &= \left\lfloor \frac{p}{3} \right\rfloor + \left\lfloor \frac{2p}{3} \right\rfloor + 4, \\ &\leq p+3. \end{aligned}$$

The other two cases are obvious. Hence the proof. □

Theorem 2.11. *For any Cycle C_p , $\gamma'_{cot}(C_p) + \gamma'_{cot}S((C_p)) \leq p+2$.*

Proof of this Theorem is similar to the Theorem 2.10.

□

The *corona* $G = H \circ K_1$ is a graph constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added. The following Theorem gives a sharp bound for the cototal edge domination number of $(G_{p_1} \circ G_{p_2})$.

Theorem 2.12. *Let G be a connected graph with p_1 vertices and K_{1,p_2} be any star. The cototal edge domination number of the corona of G and K_{1,p_2}, is given by $\gamma'_{cot}(G \circ K_{1,p_2}) = p_1$.*

Proof. Let $u_i \in V_1$ for $1 \leq i \leq p_1$ be a vertex set of G and v be a support vertex of K_{1,p_2}. Let $S = \{u_i v : \forall u_i \in G\}$ be an edge set in the *corona* $(G \circ K_{1,p_2})$. Then S is a minimum edge dominating set of $(G \circ K_{1,p_2})$. Clearly the induced subgraph $\langle E - S \rangle$ does not contain any isolated edge. Hence $S = \{e_1, e_2 \cdots e_{p_1}\}$ is a cototal edge dominating set with minimum cardinality. Therefore
$\gamma'_{cot}(G \circ K_{1,p_2}) = p_1$. Hence the proof. □

Theorem 2.13. *Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two connected graphs. The cototal edge domination number of the corona of G_1 and G_2, is given by $\gamma'_{cot}(G_1 \circ G_2) \leq p_1 + p_1\lfloor \frac{p_2}{2} \rfloor$. Equality holds if G_2 is either a cycle or a complete graph.*

Proof. Let $u_i \in V_1$ for $1 \leq i \leq p_1$ and let $v_j \in V_2$ for $1 \leq j \leq p_2$ be the vertex set of G_1 and G_2 respectively. Let $S = \{u_i v_k : \forall u_i \in G_1$ and any one vertex $v_k \in V_2\}$ be an edge set in the *corona* $(G_1 \circ G_2)$. Let E_1 be an edge dominating set for the graph G_2. Then set S along with p_1 copies of E_1 in the *corona* will form an edge dominating set F of $(G_1 \circ G_2)$. Clearly $\langle E - F \rangle$ doesnot contain isolated edges. Therefore

$$\begin{aligned} \gamma'_{cot}(G_1 \circ G_2) &\leq |S| + p_1(|E_1|), \\ &\leq p_1 + p_1 \left\lfloor \frac{p_2}{2} \right\rfloor. \end{aligned}$$

Hence the proof. □

For disjoint graphs G_1 and G_2, the *join* $G = G_1 + G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \wedge v \in V(G_2)\}$. In the following Theorem we give the result on the *join* of two paths.
Let $\varepsilon = 2p_1 - p_2$ be a constant used in the Theorem below.

Theorem 2.14. *Let P_{p_1} and P_{p_2} be any two paths of order p_1, p_2 respectively. Then the cototal edge domination number,*

$$\gamma'_{cot}(P_{p_1} + P_{p_2}) \leq \begin{cases} \lceil \frac{p_2}{2} \rceil + \left\lceil \frac{p_1 - \lfloor \frac{p_2}{2} \rfloor}{2} \right\rceil & \text{for } \varepsilon > 0, \\ p_1 + \frac{p_2 - 2p_1}{2} & \text{for } \varepsilon \leq 0. \end{cases}$$

Proof. Let P_{p_1} and P_{p_2} be the two paths labelled in order as $u_1 e_1 u_2 e_2 \cdots u_{p_1-1}$ $e_{p_1-1} u_{p_1}$ and $v_1 e'_1 v_2 e'_2 \cdots v_{p_2-1} e'_{p_2-1} v_{p_2}$ respectively. Let us consider the following cases:
Case i) Suppose $\varepsilon > 0$.
Choose an edge set $E_1 = \{u_i v_{2i} : 1 \leq i \leq \lfloor \frac{p_2}{2} \rfloor\}$. Then $|E_1| = \lfloor \frac{p_2}{2} \rfloor$. Let E_2 be the set of alternate edges in the path $u_{p_1}, u_{p_1-1}, \cdots, u_{\lfloor \frac{p_2}{2} \rfloor}$ in order such that $|E_2| = \left\lceil \frac{p_1 - \lfloor \frac{p_2}{2} \rfloor}{2} \right\rceil$. Then $E_1 \cup E_2$ forms a cototal edge dominating set in the *join* of the two paths P_{p_1} and P_{p_2}. Thus

$$\begin{aligned} \gamma'_{cot}(P_{p_1} + P_{p_2}) &\leq |E_1| + |E_2|, \\ &\leq \left\lfloor \frac{p_2}{2} \right\rfloor + \left\lceil \frac{p_1 - \lfloor \frac{p_2}{2} \rfloor}{2} \right\rceil. \end{aligned}$$

Case ii) Suppose $|\varepsilon| \leq 0$.
Choose an edge set $E'_1 = \{u_i v_{2i} : 1 \leq i \leq p_1\}$ then $|E'_1| = p_1$. Let E'_2 be the edge dominating set of the path $v_{2p_1+1}, \cdots v_{p_2}$ in order such that $|E'_2| = \lceil \frac{p_2 - 2p_1}{2} \rceil$. Thus $E'_1 \cup E'_2$ forms a cototal edge dominating set in the *join* of the two paths P_{p_1} and P_{p_2} . Thus

$$\begin{aligned} \gamma'_{cot}(P_{p_1} + P_{p_2}) &\leq |E'_1| + |E'_2|, \\ &\leq p_1 + \left\lceil \frac{p_2 - 2p_1}{2} \right\rceil. \end{aligned}$$

Hence the proof. □

In the following Theorem we give the result on the *join* of two cycles.

Theorem 2.15. *Let C_{p_1} and C_{p_2} be any two cycles of order p_1, p_2 respectively. Then the cototal edge domination number,*

$$\gamma'_{cot}(C_{p_1} + C_{p_2}) \leq \begin{cases} \lceil \frac{p_2}{2} \rceil + \left\lfloor \frac{p_1 - \lceil \frac{p_2}{2} \rceil + 1}{2} \right\rfloor & \text{for } \varepsilon \geq 0, \\ p_1 + \lceil \frac{\varepsilon}{3} \rceil & \text{for } \varepsilon < 0. \end{cases}$$

Proof. Let C_{p_1} and C_{p_2} be the two cycles labelled in order as, $u_1 e_1 u_2 e_2 \cdots u_{p_1}$ $e_{p_1} u_1$ and $v_1 e'_1 v_2 e'_2 \cdots v_{p_2} e'_{p_2} v_1$ respectively. Let us consider the following cases
Case i) Suppose $\varepsilon \geq 0$.
Consider the edge set $E_1 = \{u_i v_{2i-1} : 1 \leq i \leq \lceil \frac{p_2}{2} \rceil\}$. Let E_2 be the set of alternate edges in the path $u_{\lceil \frac{p_2}{2} \rceil + 1}, u_{\lceil \frac{p_2}{2} \rceil + 2}, \cdots u_{p_1}, u$ in order such that $|E_2| = \left\lceil \frac{p_1 - \lceil \frac{p_2}{2} \rceil + 1}{2} \right\rceil$. Then $E_1 \cup E_2$ forms a cototal edge dominating set in the *join* of two cycles C_{p_1} and C_{p_2}. Thus

$$\begin{aligned} \gamma'_{cot}(C_{p_1} + C_{p_2}) &\leq |E_1| + |E_2|, \\ &\leq \left\lceil \frac{p_2}{2} \right\rceil + \left\lfloor \frac{p_1 - \lceil \frac{p_2}{2} \rceil + 1}{2} \right\rfloor. \end{aligned}$$

Case ii) Suppose $|\varepsilon| < 0$.
Choose an edge set $E'_1 = \{u_i v_{2i-1} : 1 \leq i \leq p_1\}$ then $|E'_1| = p_1$. Let E'_2 be the edge dominating set in the path $v_{2p_1}, v_{2p_1+1}, \cdots v_{p_2}$ in order such that $|E'_2| = \lceil \frac{p_2 - 2p_1}{3} \rceil$. Thus $E'_1 \cup E'_2$ forms a cototal edge dominating set in the *join* of the two cycles C_{p_1} and C_{p_2}. Thus

$$\begin{aligned} \gamma'_{cot}(C_{p_1} + C_{p_2}) &\leq |E'_1| + |E'_2|, \\ &\leq p_1 + \left\lceil \frac{p_2 - 2p_1}{3} \right\rceil. \end{aligned}$$

Hence the proof. □

In the following two Theorems we are adding K_2 to a cycle C_p. For an edge $e = uv$ of a graph G with $deg(u) = 1$ and $deg(v) > 1$, we call e an end edge and u an end vertex.

Theorem 2.16. *Let $G^{'}$ be the graph obtained by adding k end edges u_1v_j for $j = 1, 2, \cdots k$ to a cycle C_p where $u_1 \in C_p$ and $\{v_1, v_2, \cdots v_k\} \notin C_p$. Then the cototal edge domination number, $\gamma^{'}_{cot}(G^{'}) = \lceil \frac{p}{3} \rceil$.*

Proof. Let $C_p : u_1, u_2, \cdots, u_p$ be a cycle with p vertices and $G^{'}$ be the graph obtained by adding k end edges $\{u_1v_1, u_1v_2, \cdots u_1v_k\}$ such that $u_1 \in C_p$ and $\{v_1, v_2, \cdots v_k\} \notin C_p$. Let $e_i = u_iu_{i+1}$ be an edge on cycle.
Let $S = \{e_j : j = 3l + 1$ for $0 \leq l \leq \lfloor \frac{p}{3} \rfloor - 1\}$ and

$$S_1 = \begin{cases} S & \text{if } p \equiv 0 \text{ or } 2(mod3), \\ S \cup \{e_{p-2}\} & \text{if } p \equiv 1(mod3). \end{cases}$$

be an edge set on S. Then S_1 is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ doesnot contain isolated edges. Therefore S_1 is a cototal edge dominating set and $|S_1|$ will be the cototal edge domination number for the graph $G^{'}$. Hence the proof. □

Theorem 2.17. *Let $G^{'}$ be the graph obtained by adding k end edges u_iv_j to a cycle C_p with $d(u_i) \geq 3$ where $u_i \in C_p$, for $i = 1, 2, \cdots p, v_j \notin C_p$ for $j = 1, 2, \cdots k$. Then the cototal edge domination number, $\gamma^{'}_{cot}(G^{'}) = \lceil \frac{p}{2} \rceil$.*

Proof. Let $C_p : u_1, u_2, \cdots, u_p$ be a cycle and $G^{'}$ be the graph obtained by adding k end edges u_iv_j where $u_i \in C_p$ for $i = 1, 2, \cdots p$ and $v_j \notin C_p$ for $j = 1, 2, \cdots k$. Let $e_i = u_iu_{i+1}$ be an edge of $G^{'}$.
Let $S = \{e_j : j = 2l + 1$ for $0 \leq l \leq \lfloor \frac{p}{2} \rfloor - 1\}$ and

$$S_1 = \begin{cases} S \cup \{e_{p-1}\} & \text{if } p \equiv 0(mod2), \\ S \cup \{u_pv_1\} & \text{if } p \equiv 1(mod2). \end{cases}$$

be an edge set of $G^{'}$. Then S_1 is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ doesnot contain isolated edges. Therefore S_1 is a cototal edge dominating set and $|S_1|$ will be the cototal edge domination number for the graph $G^{'}$. Hence the proof. □

The following corollary is the immediate consequence of the above Theorems.

Corollary 2.3. *Let $G^{'}$ be the graph obtained by adding k end edges u_iv_j to a cycle C_p of order $p \geq 3$ in any manner then from the above Theorems, we*
get $\lceil \frac{p}{3} \rceil \leq \gamma^{'}_{cot}(G^{'}) \leq \lceil \frac{p}{2} \rceil$.

3 Cartesian product of independent cototal edge domination number

In this section we define a new parameter "Independent cototal edge domination number" of a graph. An edge dominating set F is called an independent edge dominating set if no two edges of F are adjacent [1]. An independent edge domination number $\gamma^{'}_{i}(G)$ of G is the minimum cardinality taken over all independent edge dominating sets of G. The cototal edge dominating set is said to be an independent cototal edge dominating set if the induced subgraph $\langle F \rangle$ is an independent edge set.

The cartesian product of G and H, denoted $G \times H$, has vertex set $V(G) \times V(H)$. Two vertices (u, v), $(u^{'}, v^{'})$, in $V(G) \times V(H)$ are adjacent if either $u = u^{'}$ and $vv^{'} \in E(H)$, or $v = v^{'}$ and $uu^{'} \in E(G)$. The graph $P_n \times P_m$ has m copy of the graph P_n in m columns. Let $\gamma^{'}_{icot}(P_n \times P_m)$ denotes the size of minimum independent cototal edge dominating set of two paths $(P_n \times P_m$ where $n \leq m$. In the next Theorem, we calculate $\gamma^{'}_{cot}$ for the product of two paths P_3 and P_m.

Theorem 3.18. *Let P_3 be a path of length 3 and P_m be any path with $m \geq 3$. Then independent cototal edge dominating number of the product of these two paths, $\gamma^{'}_{icot}[P_3 \times P_m] = m$.*

Proof. Consider the independent edge set
$S = \Big\{\{(1, 2 + 3k), (1, 3 + 3k)\}, \{(2, 1 + 3k), (2, 2 + 3k)\}, \{(3, 2 + 3k), (3, 3 + 3k)\}/k = 0, 1, 2 \cdots \lfloor \frac{m}{3} \rfloor - 1\Big\}$, as shown in Figure 2.

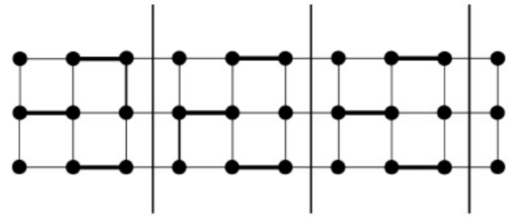

Figure 2: Cartesian product of $P_3 \times P_m$

Case i) Let $m \equiv 0(mod 3)$.
Then S is an independent edge dominating set and the induced subgraph $\langle E - S\rangle$ of $P_3 \times P_m$ has no isolated edges. By Theorem 1.1, S is an independent cototal edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|S| = m$.

Case ii) Let $m \equiv 1(mod 3)$.
Then $S_1 = S \cup \{(1, n), (2, n)\}$ is an independent edge dominating set and the induced subgraph $\langle E - S_1\rangle$ of $P_3 \times P_m$ has no isolated edges. By Theorem 1.1, S_1 is an independent cototal edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|S_1| = |S| + 1 = m$.

Case iii) Let $m \equiv 2(mod 3)$.
Then $S_2 = S \cup \Big\{\{(2, n-1), (3, n-1)\}, \{(1, n), (2, n)\}\Big\}$ is an independent edge dominating set and the induced subgraph $\langle E - S_2\rangle$ of $P_3 \times P_m$ has no isolated edges. By Theorem 1.1, S_2 is an independent cototal edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|S_2| = |S| + 2 = m$.

Thus $\gamma'_{icot}[P_3 \times P_m] = m$. Hence the proof. □

In the next Theorem, we generalize γ'_{cot} for the *product* of two paths P_n and P_m.

Theorem 3.19. *Let P_n be a path with n vertices and P_m be any path with m vertices where $n \leq m$. Then independent cototal edge domination number of the cartesian product of two paths,*

$$\gamma'_{icot}[P_n \times P_m] = \begin{cases} \frac{nm}{3} & \text{if } m \equiv 0(mod 3), \\ \left\lceil \frac{nm}{3} \right\rceil + 1 & \text{if } m \equiv 1 \text{ or } 2(mod 3). \end{cases}$$

Proof. Consider an independent edge set as shown in Figure 3.
If i is odd, then

$$S = \begin{cases} \bigcup_{i=1,3,\cdots n} & \{(i, 3k+2), (i, 3k+3)\} : k = 0, 1, \cdots \lfloor \frac{m}{3} \rfloor - 1, \\ \bigcup_{i=2,4,\cdots n-1} & \{(i, 3k+1), (i, 3k+2)\} : k = 0, 1, \cdots \lfloor \frac{m}{3} \rfloor - 1. \end{cases}$$

If i is even, then

$$S = \begin{cases} \bigcup_{i=1,3,\cdots n-1} & \{(i, 3k+2), (i, 3k+3)\} : k = 0, 1, \cdots \lfloor \frac{m}{3} \rfloor - 1, \\ \bigcup_{i=2,4,\cdots n} & \{(i, 3k+1), (i, 3k+2)\} : k = 0, 1, \cdots \lfloor \frac{m}{3} \rfloor - 1. \end{cases}$$

Case i) Let $m \equiv 0(mod 3)$.
Then S is an independent edge dominating set and the induced subgraph $\langle E - S\rangle$ of $P_n \times P_m$ has no isolated edges. Thus S is an independent cototal edge dominating set of $P_n \times P_m$. Therefore

$$\gamma'_{icot}[P_n \times P_m] \leq |S|, \\ \leq \frac{nm}{3}.$$

Case ii) Let $m \equiv 1(mod 3)$.
Let us discuss the following subcases.

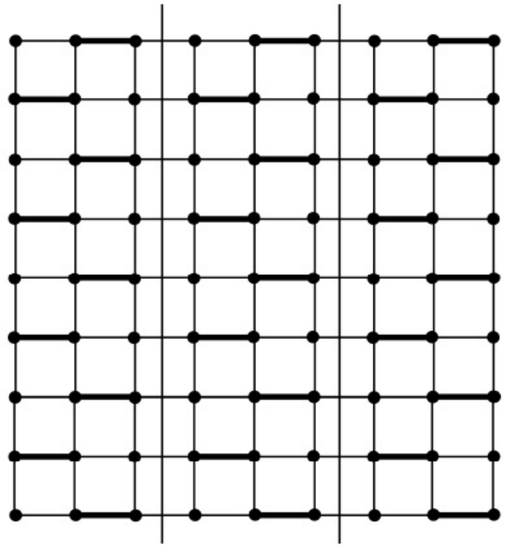

Figure 3: Cartesian product of $P_n \times P_m$

1. n is an odd number.
Let $S_1 = \Big\{\{(1,m),(2,m)\},\{(3,m),(4,m)\},\cdots\{(n-2,m),$
$(n-1,m)\}\Big\}$. Then $F_1 = S \cup S_1$ is an independent edge dominating set and the induced subgraph $\langle E - F_1\rangle$ of $P_n \times P_m$. Thus $|F_1| \leq |S| + |S_1|$.

2. n is an even number.
Let $S_2 = \Big\{\{(1,m),(2,m)\},\{(3,m),(4,m)\},\cdots\{(n-2,m),$
$(n-1,m)\}\Big\}$. Then $F_2 = S \cup S_2$ is an independent edge dominating set and the induced subgraph $\langle E - F_2\rangle$ of $P_n \times P_m$ has no isolated edges. Thus $|F_2| \leq |S| + |S_2|$.

Combining the above two subcases, we conclude that F_1 and F_2 form an independent edge dominating set of $P_n \times P_m$ in both the cases respectively. Therefore

$$\gamma'_{icot}[P_n \times P_m] \leq |S_1 \cup S_2| + \left\lfloor \frac{n}{2} \right\rfloor,$$
$$\leq \left\lceil \frac{nm}{3} \right\rceil + 1.$$

Case iii) Let $m \equiv 2(mod 3)$.
For $n \geq 4$, we can partition the set of m columns of $P_n \times P_m$ into $B_i, i = 1,2,\cdots \lfloor \frac{m}{3} \rfloor$ blocks at the beginning and two columns at the end. The set S will dominate B_i blocks. In addition we dominate m and $m-1$ columns by a set isomorphic to S_R as shown in Figure 4. Let $n = 4q + l : 1 \leq q \leq \lfloor \frac{n}{4} \rfloor, 0 \leq l \leq 3$. Consider the following two cases to find S_R

i) If $q = 1$ then $S_R = \{R_l : 0 \leq l \leq 3\}$

ii) If $q > 1$ then $S_R = \Big\{(\lfloor \frac{n}{4} \rfloor - 1)R_0 + R_l : 0 \leq l \leq 3\Big\}$

Therefore $S_3 = S \cup S_R$ is an independent edge dominating set and the induced subgraph $\langle E - S_3\rangle$ of $P_n \times P_m$ has no isolated edges. Thus S_3 is a independent cototal edge dominating set of $P_n \times P_m$. Therefore,

$$\gamma'_{icot}[P_n \times P_m] \leq |S_3|,$$
$$\leq |S| + |S_R|,$$
$$\leq \left\lceil \frac{nm}{3} \right\rceil + 1.$$

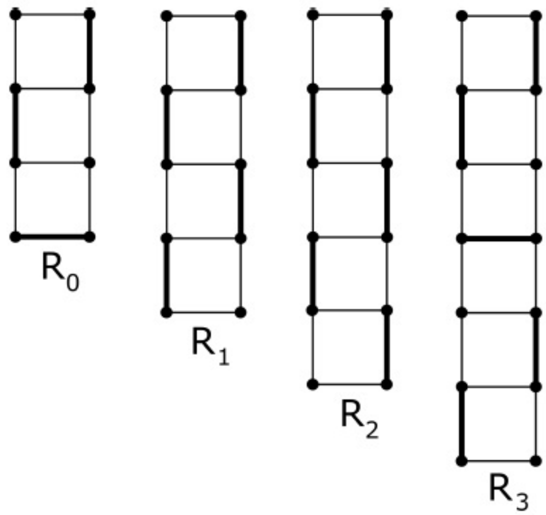

Figure 4: S_R

Hence the proof. □

Now let us study few more bounds on *cartesian product* of two cycles.

Theorem 3.20. *Let C_3 be a cycle of order 3 and C_m be any cycle of order m. Then independent cototal edge dominating number of the cartesian product of two cycles,*

$$\gamma'_{icot}[C_3 \times C_m] = \begin{cases} m & \text{if } m \text{ is even,} \\ m+1 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Consider an independent edge set.
$S = \Big\{\{(1,2k-1),(2,2k-1)\},\{(2,2k),(3,2k)\} : k = 1,2,\cdots \lceil \frac{m}{2} \rceil\Big\}$.

Case i) Suppose m is an even number.
Then S is an independent edge dominating set and the induced subgraph $\langle E - S \rangle$ of $C_3 \times C_m$ has no isolated edges. Thus S is an independent cototal edge dominating set of $C_3 \times C_m$. Therefore $|S| = m$.

Case ii) Suppose m is an odd number.
Then $S_1 = S \cup \Big\{(3,1),(3,m)\Big\}$ is an independent edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ of $C_3 \times C_m$ has no isolated edges. Thus S_1 is an independent cototal edge dominating set of $C_3 \times C_m$. Therefore $|S| = m+1$.

Hence the proof. □

Theorem 3.21. *Let C_4 be a cycle of order 4 and C_m be any cycle of order m where $m \geq 4$. Then independent cototal edge dominating number of the cartesian product of two cycles, $\gamma'_{icot}[C_4 \times C_m] \leq m + \lceil \frac{m}{2} \rceil$.*

Proof. Consider an independent edge set
$S = \Big\{\{(1,2k-1),(2,2k-1)\},\{(2,2k),(3,2k)\} : k = 1,2,\cdots \lceil \frac{m}{2} \rceil\Big\}$
and the set,

$$S_1 = \begin{cases} \Big\{(4,2i-1),(4,2i)\Big\} & \text{if } m \equiv 0(mod2), \\ \Big\{(4,2i-1),(4,2i)\Big\} \cup \Big\{(3,m),(4,m)\Big\} & \text{if } m \equiv 1(mod2), \end{cases}$$

for $i = 1,3,\cdots \lfloor \frac{m}{2} \rfloor$. Then $S \cup S_1$ is an independent edge dominating set and the induced subgraph $\langle E - (S \cup S_1) \rangle$ of $C_4 \times C_m$ has no isolated edges. Thus $S \cup S_1$ is an independent cototal edge dominating set of $C_4 \times C_m$.

$$\gamma'_{icot}[C_4 \times C_m] \leq |S \cup S_1|, \\ \leq m + \Big\lceil \frac{m}{2} \Big\rceil.$$

Hence the proof. □

Now let us study few more bounds for independent cototal edge dominating number on *cartesian product* of a path, cycle and complete graph.

Theorem 3.22. *Let P_3 be a path with 3 vertices and C_m be any cycle with m vertices where $m \leq 3$. Then independent cototal edge dominating number of the cartesian product,*

$$\gamma'_{icot}[P_3 \times C_m] \leq \begin{cases} m & \text{if } m \equiv 0(mod3), \\ m+1 & \text{if } m \equiv 1 \text{ or } 2(mod3). \end{cases}$$

Proof. Partition the set of m columns of $P_3 \times C_m$ into B_i blocks for $m \geq 6$. For $j = 1, 2, \cdots \lfloor \frac{m-3}{3} \rfloor$,

$$S = \begin{cases} \bigcup_{i=1,3} & \{(i, 3j-2), (i, 3j-1)\}, \\ \bigcup_{i=2} & \{(i, 3j-1), (i, 3j)\}. \end{cases}$$

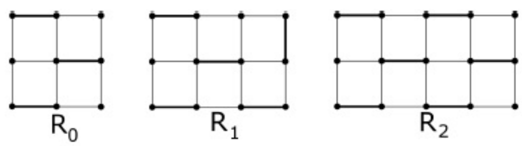

Figure 5: R_i

Case i) $m \equiv 0(mod3)$.
Then $S \cup R_0$, (as shown in R_0 of Figure 5) is an independent edge dominating set and the induced subgraph $\langle E - (S \cup R_0) \rangle$ of $P_3 \times C_m$ has no isolated edges. Thus $S \cup R_0$ is an independent cototal edge dominating set of $P_3 \times C_m$. Therefore

$$\gamma'_{icot}(P_3 \times C_m) \leq |S \cup R_0|, \\ \leq \frac{nm}{3} = m.$$

Case ii) $m \equiv 1(mod3)$,
Then $S \cup R_1$ (as shown in R_1 of Figure 5) is an independent edge dominating set and the induced subgraph $\langle E - (S \cup R_1) \rangle$ of $P_3 \times C_m$ has no isolated edges. Thus $S \cup R_1$ is an independent cototal edge dominating set of $P_3 \times C_m$. Therefore

$$\gamma'_{icot}[P_3 \times C_m] \leq |S| \cup |R_1|, \\ \leq \left\lceil \frac{nm}{3} \right\rceil, \\ \leq m+1.$$

Case iii) $m \equiv 2(mod3)$,
Then $S \cup R_2$ (as shown in R_2 of Figure 5) is an independent edge dominating set and the induced subgraph $\langle E - (S \cup R_2) \rangle$ of $P_3 \times C_m$ has no isolated edges. Thus $S \cup R_2$ is an independent cototal edge dominating set of $P_3 \times C_m$. Therefore

$$\gamma'_{icot}[P_3 \times C_m] \leq |S| + |R_2|, \\ \leq \lceil \frac{nm}{3} \rceil, \\ \leq m+1.$$

Hence the proof. □

Theorem 3.23. *Let P_n be a path with n vertices and K_m be any complete graph with m vertices where $n \leq m$. Then independent cototal edge domination number of the cartesian product,*

$$\gamma'_{icot}[P_n \times K_m] \leq n\left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. Consider the following two cases,

Case i) m is an even number.
Consider the set

$$S = \bigcup_{i=1,2,\cdots n} \Big\{(i,1),(i,2)\Big\},\Big\{(i,3),(i,4)\Big\},\cdots\Big\{(i,m-1),(i,m)\Big\}.$$

If m is even then by Theorem 1.2, edge domination number of the K_m is $\frac{m}{2}$. Here we have n copies of K_m. Also S is an independent edge dominating set and the induced subgraph $\langle E-S\rangle$ of $P_n \times K_m$ has no isolated edges. Thus S is an independent cototal edge dominating set of $P_n \times K_m$. Therefore $|S| = n(\frac{m}{2})$.

Case ii) m is an odd number.
Let us discuss the following subcases:

1. If n is an odd number, then

$$S = \begin{cases} \bigcup_{i=1,3,\cdots n} & \Big\{(i,1),(i,2)\Big\},\Big\{(i,3),(i,4)\Big\},\cdots,\Big\{(i,m-2),(i,m-1)\Big\} \\ \bigcup_{i=2,4,\cdots n-1} & \Big\{(i,2),(i,3)\Big\},\Big\{(i,4),(i,5)\Big\},\cdots,\Big\{(i,m-1),(i,m)\Big\}. \end{cases}$$

2. If n is an even number, then

$$S = \begin{cases} \bigcup_{i=1,3,\cdots n-1} & \Big\{(i,1),(i,2)\Big\},\Big\{(i,3),(i,4)\Big\},\cdots,\Big\{(i,m-2),(i,m-1)\Big\} \\ \bigcup_{i=2,4,\cdots n} & \Big\{(i,2),(i,3)\Big\},\Big\{(i,4),(i,5)\Big\},\cdots,\Big\{(i,m-1),(i,m)\Big\}. \end{cases}$$

If m is odd then by Theorem 1.2, edge domination number of the K_m is lesser than $\frac{m}{2}$. Here we have n copies of K_m. Also S is an independent edge dominating set and the induced subgraph $\langle E-S\rangle$ of $P_n \times K_m$ has no isolated edges. Thus S is an independent cototal edge dominating set of $P_n \times K_m$. Therefore $|S| < n\lfloor \frac{m}{2} \rfloor$.

Combining the above two cases, we obtain the following result.

$$\gamma'_{icot}[P_n \times K_m] \leq n\left\lfloor \frac{m}{2} \right\rfloor$$

Hence the proof. □

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