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The Natural Lift of the Fixed Centrode of a Non-null Curve in Minkowski 3-Space

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Abstract

In this study, we dealt with the natural lift curves of the fixed centrode of a non-null curve.Furthermore, some interesting result about the original curve were obtained, depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ and $T\left(H_0^2\right)$.

Keywords: Natural lift, geodesic spray, Darboux vector.

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1 Introduction

Thorpe gave the concepts of the natural lift curve and geodesic spray in [12]. Thorpe provied the natural lift $\bar{\alpha}$ of the curve α is an integral curve of the geodesic spray iff α is an geodesic on M. Çalışkan at al. studied the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and fixed centrode of a curve in [11]. They gave some interesting results about the original curve, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2\right)$. Some properties of $\overline{M}$-vector field Z defined on a hypersurface M of $\overline{M}$ were studied by Agashe in [1]. $\overline{M}$-integral curve of Z and $\overline{M}$-geodesic spray are defined by Çalışkan and Sivridağ. They gave the main theorem: The natural lift $\bar{\alpha}$ of the curve α (in $\overline{M}$) is an $\overline{M}$-integral curve of the geodesic spray Z iff α is an $\overline{M}$-geodesic in [5]. Bilici et al. have proposed the natural lift curves and the geodesic sprays for the spherical indicatrices of the the involute evolute curve couple in Euclidean 3-space. They gave some interesting results about the evolute curve, depending on the assumption that the natural lift curve of the spherical indicatrices of the involute should be the integral curve on the tangent bundle $T\left(S^2\right)$ in [3]. Then Bilici applied this problem to involutes of a timelike curve in Minkowski 3-space (see [4]). Ergün and Çalışkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space in [7]. The anologue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan in [7]. Çalışkan and Ergün defined $\overline{M}$-vector field Z, $\overline{M}$-geodesic spray, $\overline{M}$-integral curve of Z, $\overline{M}$-geodesic in [6].The anologue of the theorem of Sivridağ and Çalışkan was given in Minkowski 3-space by Ergün and Çalışkan in [5]. Walrave characterized the curve with constant curvature in Minkowski 3-space in [12]. In differential geometry, especially the theory of space curve, the Darboux vector is the areal velocity vector of the Frenet frame of a spacere curve. It is named after Gaston Darboux who discovered it. In term of the Frenet-Serret apparatus, the darboux vector W can be expressed as $W = \tau T + \kappa B$, details are given in Lambert et al. in [8].

In this study,we studied the fixed centrode curve of a curve and characterized the curve if the natural lift of the fixed centrode curve is an integral curve of the geodesic sprays.

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Let Minkowski 3-space $\mathbb{R}_1^3$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ and lightlike (or null) if $g(X,X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(t)$ in $\mathbb{R}_1^3$ where t is a pseudo-arclength parameter, can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\dot{\alpha}(t)$ are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset \mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ ($resp. x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ ($resp.$ $x_1 < 0$). The norm of a vector X is defined by $\|X\|_{IL} = \sqrt{|g(X,X)|}$, [9].

The Lorentzian sphere and hyperbolic sphere of radius 1 in $\mathbb{R}_1^3$ are given by

$$S_1^2 = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \, : \, g(X,X) = 1 \right\}$$

and

$$H_0^2 = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \, : \, g(X,X) = -1 \right\}$$

respectively,[8].The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}_1^3$ are orthogonal if and only if $g(X,X) = 0$, [9].

Now let X and Y be two vectors in $\mathbb{R}_1^3$, then the Lorentzian cross product is given by

$$X \times Y = (x_3y_2 - x_2y_3, \, x_1y_3 - x_3y_1, \, x_1y_2 - x_2y_1), \text{[2]}.$$

We denote by $\{T(t), N(t), B(t)\}$ the moving Frenet frame along the curve α. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively.

Let α be a unit speed timelike space curve with curvature κ and torsion τ. Let Frenet vector fields of α be $\{T, N, B\}$. In this trihedron, T is timelike vector field, N and B are spacelike vector fields.For this vectors, we can write

$$T \times N = B, \quad N \times B = -T, \quad B \times T = N,$$

where $\times$ is the Lorentzian cross product, [2]. in space $\mathbb{R}_1^3$ Then, Frenet formulas are given by

$$\dot{T} = \kappa N, \; \dot{N} = \kappa T + \tau B, \; \dot{B} = -\tau N, \text{[13]}.$$

The Frenet instantaneous rotation vector for the timelike curve is given by $W = \tau T + \kappa B$.

Let α be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that T and B are spacelike vector fields and N is a timelike vector field In this situation,.

$$T \times N = B, \quad N \times B = T, \quad B \times T = -N,$$

Then, Frenet formulas are given by

$$\dot{T} = \kappa N, \; \dot{N} = \kappa T + \tau B, \; \dot{B} = \tau N, \text{[13]}.$$

The Frenet instantaneous rotation vector for the spacelike space curve with a spacelike binormal is given by $W = \tau T - \kappa B$.

Lemma 1.1. *Let Xand Y be nonzero Lorentz orthogonal vectors in $\mathbb{R}_1^3$. If X is timelike, then Y is spacelike,* [10].

Lemma 1.2. *Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}_1^3$. Then*

$$g(X,Y) \leq \|X\| \, \|Y\|$$

whit equality if and only if X and Y are linearly dependent, [10].

Lemma 1.3. *i) Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}_1^3$. By the Lemma 2, there is unique nonnegative real number $\varphi(X,Y)$ such that*

$$g(X,Y) = \|X\|\,\|Y\|\cosh\varphi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$.
ii) Let X and Y be spacelike vektors in $\mathbb{R}_1^3$ that span a spacelike vector subspace. Then we have

$$|g(X,Y)| \leq \|X\|\,\|Y\|\,.$$

Hence, there is a unique real number $\varphi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\|\,\|Y\|\cos\varphi(X,Y)$$

the Lorentzian spacelike angle between X and Y is defined to be $\varphi(X,Y)$.
iii) Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a timelike vector subspace. Then we have

$$g(X,Y) > \|X\|\,\|Y\|\,.$$

Hence, there is a unique pozitive real number $\varphi(X,Y)$ between 0 and π such that

$$|g(X,Y)| = \|X\|\,\|Y\|\cosh\varphi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$.
iv) Let X be a spacelike vector and Y be a pozitive timelike vector in $\mathbb{R}_1^3$. Then there is a unique nonnegative reel number $\varphi(X,Y)$ such that

$$|g(X,Y)| = \|X\|\,\|Y\|\sinh\varphi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$, [10].

Theorem 1.1. *Let α be a unit speed timelike space curve.Then we have*

1. *$\kappa = 0$ if and only if α is a part of a timelike straight line;*
2. *$\tau = 0$ if and only if α is a planar timelike curve;*
3. *$\tau = 0$ and $\kappa =$constant > 0 if and only if α is a part of a orthogonal hyperbola;*
4. *$\kappa =$constant > 0, $\tau =$constant $\neq 0$ and $|\tau| > \kappa$ if and only if α is a part of a timelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa\cos\left(\sqrt{K}s\right), \kappa\sin\left(\sqrt{K}s\right)\right)$$

   with $K = \tau^2 - \kappa^2$;

5. *$\kappa =$constant > 0, $\tau =$constant $\neq 0$ and $|\tau| < \kappa$ if and only if α is a timelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\cosh\left(\sqrt{K}s\right)\right)$$

   with $K = \kappa^2 - \tau^2$;

6. *$\kappa =$constant > 0, $\tau =$constant $\neq 0$ and $|\tau| = \kappa$ if and only if α can be parameterized by*

$$\alpha(s) = \frac{1}{6}\left(\kappa^2 s^3 + 6s, 3\kappa s^2, \kappa\tau s^3\right)$$

   [13].

Theorem 1.2. *Let α be a unit speed spacelike space curve with a spacelike binormal. Then we have*

1. *$\tau = 0$ and $\kappa =$constant > 0 if and only if α is a part of a orthogonal hyperbola;*

2. κ =*constant* > 0, τ =*constant* $\neq 0$ *if and only if* α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(, \kappa\cosh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\sinh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 + \tau^2$, [13].

Theorem 1.3. *Let* α *be a unit speed spacelike space curve with a timelike binormal. Then we have*

1. $\tau = 0$ *and* κ =*constant* > 0 *if and only if* α *is a part of a circle;*
2. κ =*constant* > 0, τ =*constant* $\neq 0$ *and* $|\tau| > \kappa$ *if and only if* α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\cosh\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$;

3. κ =*constant* > 0, τ =*constant* $\neq 0$ *and* $|\tau| < \kappa$ *if and only if* α *is a part of a spacelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa\cos\left(\sqrt{K}s\right), \kappa\sin\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$;

4. κ =*constant* > 0, τ =*constant* $\neq 0$ *and* $|\tau| = \kappa$ *if and only if* α *can be parameterized by*

$$\alpha(s) = \frac{1}{6}\left(\kappa\tau s^3, -\kappa^2 s^3 + 6s, 3\kappa s^2\right)$$

[13].

2 The Natural Lift of the Fixed Centrode of a Non-null Curve in Minkowski 3-Space

Definition 2.1. *Let* M *be a hypersurface in* $\mathbb{R}^3_1$ *and let* $\alpha : I \longrightarrow M$ *be a parametrized curve.* α *is called an integral curve of* X *if*

$$\frac{d}{dt}(\alpha(t)) = X(\alpha(t)) \ \ (\text{for all } t \in I)$$

where X *is a smooth tangent vector field on* M, [9]. *We have*

$$TM = \underset{P \in M}{} T_P M = \chi(M)$$

where $T_P M$ *is the tangent space of* M *at* P *and* $\chi(M)$ *is the space of vector fields of* M.

Definition 2.2. *For any parametrized curve* $\alpha : I \longrightarrow M$, $\overline{\alpha} : I \longrightarrow TM$ *given by*

$$\overline{\alpha}(t) = \left(\alpha(t), \dot{\alpha}(t)\right) = \dot{\alpha}(t)\,|_{\alpha(t)}$$

is called the natural lift of α *on* TM.*Thus, we can write*

$$\frac{d\overline{\alpha}}{dt} = \frac{d}{dt}\left(\dot{\alpha}(t)\,|_{\alpha(t)}\right) = D_{\dot{\alpha}(t)}\dot{\alpha}(t)$$

where D *is the Levi-Civita connection on* $\mathbb{R}^3_1$,[7].

Definition 2.3. *A* $X \in \chi(TM)$ *is called a geodesic spray if for* $V \in TM$ $X(V) = +\varepsilon g(S(V), V)N$ *,where* $\varepsilon = g(N, N)$,[7].

Theorem 2.1. *The natural lift* $\overline{\alpha}$ *of the curve* α *is an integral curve of geodesic spray* X *if and only if* α *is a geodesic on* M*,*[7].

Definition 2.4. *(Unit Vector C of Direction W for Non-null Curves):*

1. *For the curve* α *with a timelike tanget,* θ *being a Lorentzian timelike angle between the spacelike binormal unit* $-B$ *and the Frenet instantaneous rotation vector* W.

   (i)If $|\kappa| > |\tau|$*, then* W *is a spacelike vector. In this situation, from Lemma 1.3 iii) we can write*

$$\begin{aligned}\kappa &= \|W\| \cosh\theta \\ \tau &= \|W\| \sinh\theta\end{aligned}$$

   $\|W\|^2 = g(W,W) = \kappa^2 - \tau^2$ *and* $C = \frac{W}{\|W\|} = \sinh\theta T + \cosh\theta B$*, where* C *is unit vector of direction* W.

   (ii)If $|\kappa| < |\tau|$*, then* W *is a timelike vector. In this situation, from Lemma 1.3 iv) we can write*

$$\begin{aligned}\kappa &= \|W\| \sinh\theta \\ \tau &= \|W\| \cosh\theta\end{aligned}$$

   $\|W\|^2 = -g(W,W) = -\left(\kappa^2 - \tau^2\right)$ *and* $C = \cosh\theta T + \sinh\theta B$.

2. *For the curve* α *with a timelike principal normal,* θ *being an angle between the* B *and the* W*, if* B *and* W *spacelike vectors that span a spacelike vektor subspace then by the Lemma 3 ii) we can write*

$$\begin{aligned}\kappa &= \|W\| \cos\theta \\ \tau &= \|W\| \sin\theta\end{aligned}$$

   $\|W\|^2 = g(W,W) = \kappa^2 + \tau^2$ *and* $C = \sin\theta T - \cos\theta B$.

3. *For the curve* α *with a timelike binormal,* θ *being a Lorentzian timelike angle between the* $-B$ *and the* W.

   (i)If $|\kappa| < |\tau|$*, then* W *is a spacelike vector. In this situation, from Lemma 3 iv) we can write*

$$\begin{aligned}\kappa &= \|W\| \sinh\theta \\ \tau &= \|W\| \cosh\theta\end{aligned}$$

   $\|W\|^2 = g(W,W) = \tau^2 - \kappa^2$ *and* $C = -\cosh\theta T + \sinh\theta B$.

   (ii)If $|\kappa| > |\tau|$*, then* W *is a timelike vector. In this situation, from Lemma 3 i) we have*

$$\begin{aligned}\kappa &= \|W\| \cosh\theta \\ \tau &= \|W\| \sinh\theta\end{aligned}$$

   $\|W\|^2 = -g(W,W) = -\left(\tau^2 - \kappa^2\right)$ *and* $C = -\sinh\theta T + \cosh\theta B$.

Let D, $\overline{D}$ and $\overline{\overline{D}}$ be connections in $\mathbb{R}^3_1$, S^2_1 and H^2_0 respectively and ξ be a unit normal vector field of S^2_1 and H^2_0. Then Gauss Equations are given by the followings

$$\begin{aligned}D_X Y &= \overline{D}_X Y + \varepsilon g\left(S(X), Y\right)\xi, \\ D_X Y &= \overline{\overline{D}}_X Y + \varepsilon g\left(S(X), Y\right)\xi,\end{aligned}$$

where $\varepsilon = g(\xi, \xi)$ and S is the shape operator of S^2_1 and H^2_0 .

Let α_C be the fixed centrode of the motion described by the curve α. Then the curve is given by $\alpha_C = C(s)$ and $C = \frac{W}{\|W\|}$,where W being the Darboux vector.

We have investigate how α must be curve satifying the condition that $\overline{\alpha}_C$ is an integral curve of the geodesic spray, where $\overline{\alpha}_C$ is the natural lift of the curve α_C.

(i) **Let α be a unit speed timelike space curve.**

(a) Let W is a spacelike vector. If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{D}_{\dot{\alpha_C}}\dot{\alpha_C} = 0$$

that is

$$D_{\dot{\alpha_C}}\dot{\alpha_C} = \bar{D}_{\dot{\alpha_C}}\dot{\alpha_C} + \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C} = \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)C$$

where $\epsilon = g(\xi, \xi)$ and $\xi = C$. Since T, N, B are linearly independent, we have $\dot{\theta} = 0$ or $\tau = \kappa = 0$.

Corollary 2.1. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ then α is a part of a timelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa \sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa \cosh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$.

(b) Let W is a timelike vector. If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{\bar{D}}_{\dot{\alpha_C}}\dot{\alpha_C} = 0$$

that is

$$D_{\dot{\alpha}_{CF}}\dot{\alpha_C} = \bar{\bar{D}}_{\dot{\alpha_C}}\dot{\alpha_C} + \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C} = \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)C$$

where $\varepsilon = g(\xi, \xi)$ and $\xi = C$. Since T, N, B are linearly independent, we have $\dot{\theta} = 0$ or $\tau = \kappa = 0$.

Corollary 2.2. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$ then α is a part of a timelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa \cos\left(\sqrt{K}s\right), \kappa \sin\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$.

(ii) **Let α be a unit speed spacelike space curve with a spacelike binormal.**

W is a spacelike vector. If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{D}_{\dot{\alpha_C}}\dot{\alpha_C} = 0$$

that is

$$D_{\dot{\alpha}_{CF}}\dot{\alpha_C} = \bar{D}_{\dot{\alpha_C}}\dot{\alpha_C} + \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C} = \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)C$$

where $\varepsilon = g(\xi, \xi)$ and $\xi = C$. Because T, N, B are linearly independent, we have $\dot{\theta} = 0$ or $\tau = \kappa = 0$.

Corollary 2.3. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ then α is a part of a spacelike hyperbolic helix,*

$$\alpha\left(s\right)=\frac{1}{K}\left(\kappa\cosh\left(\sqrt{K}s\right),\sqrt{\tau^2K}s,\kappa\sinh\left(\sqrt{K}s\right)\right)$$

with $K=\kappa^2+\tau^2$.

(iii) **Let α be a unit speed spacelike space curve with a timelike binormal.**

(a) Let W is a spacelike vector. If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{D}_{\dot{\alpha_C}}\dot{\alpha_C}=0$$

that is

$$D_{\dot{\alpha}_{CF}}\dot{\alpha_C}=\bar{D}_{\dot{\alpha_C}}\dot{\alpha_C}+\varepsilon g\left(S\left(\dot{\alpha_C}\right),\dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C}=\varepsilon g\left(S\left(\dot{\alpha_C}\right),\dot{\alpha_C}\right)C$$

where $\varepsilon=g\left(\xi,\xi\right)$ and $\xi=C$. Because T, N, B are linearly independent, we have $\dot{\theta}=0$ or $\tau=\kappa=0$.

Corollary 2.4. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ then α is a part of a spacelike hyperbolic helix,*

$$\alpha\left(s\right)=\frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right),\sqrt{\tau^2K}s,\kappa\cosh\left(\sqrt{K}s\right)\right)$$

with $K=\tau^2-\kappa^2$.

(b) Let W is a timelike vector.If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{\bar{D}}_{\dot{\alpha_C}}\dot{\alpha_C}=0$$

that is

$$D_{\dot{\alpha}_{CF}}\dot{\alpha_C}=\bar{\bar{D}}_{\dot{\alpha_C}}\dot{\alpha_C}+\varepsilon g\left(S\left(\dot{\alpha_C}\right),\dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C}=\varepsilon g\left(S\left(\dot{\alpha_C}\right),\dot{\alpha_C}\right)C$$

where $\varepsilon=g\left(\xi,\xi\right)$ and $\xi=C$. Since T, N, B are linearly independent, we have $\dot{\theta}=0$ or $\tau=\kappa=0$.

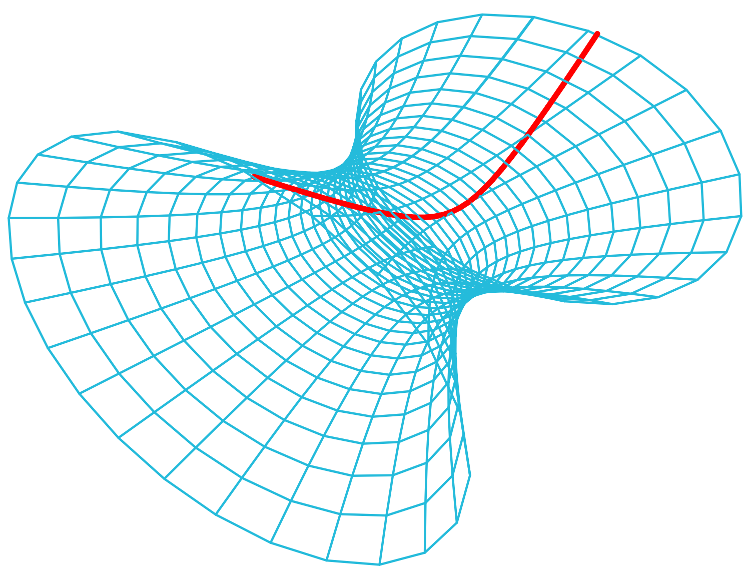
Corollary 2.5. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$ then α is a part of a spacelike circular helix,*

$$\alpha\left(s\right)=\frac{1}{K}\left(\sqrt{\tau^2K}s,\kappa\cos\left(\sqrt{K}s\right),\kappa\sin\left(\sqrt{K}s\right)\right)$$

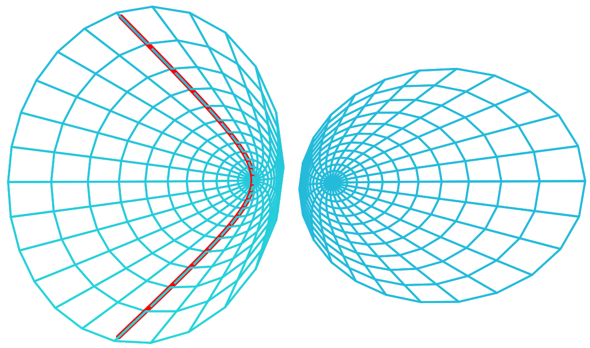
with $K=\kappa^2-\tau^2$.

Example 2.1. *Let* $\alpha(s) = \left(\cosh\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, \sinh\left(\frac{s}{\sqrt{2}}\right)\right)$ *be a unit speed spacelike hyperbolic helix with*

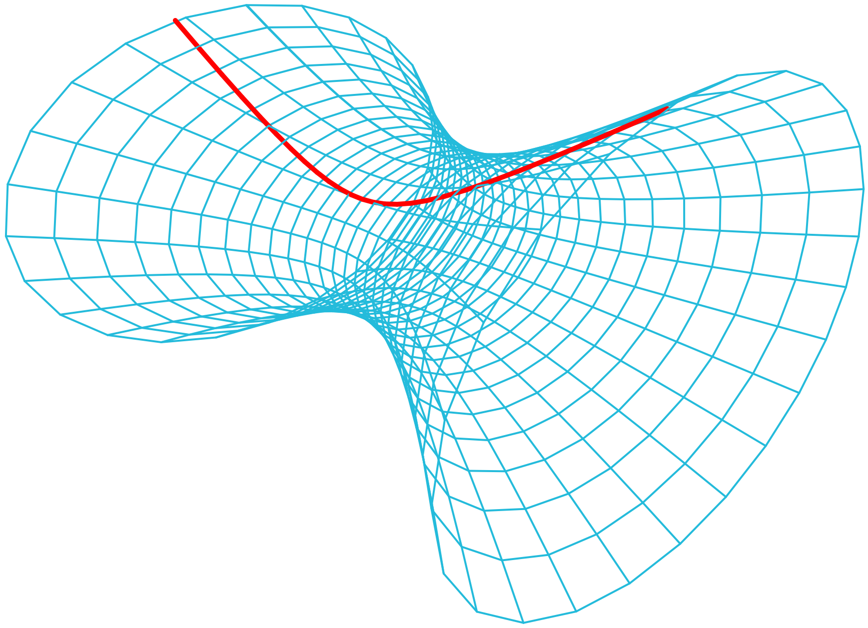
$$\begin{aligned}
T(s) &= \left(\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
N(s) &= \left(\cosh\left(\frac{s}{\sqrt{2}}\right), 0, \sinh\left(\frac{s}{\sqrt{2}}\right)\right), \\
B(s) &= \left(-\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
C(s) &= \left(\sinh\left(\frac{s}{\sqrt{2}}\right), 0, \cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
\alpha_T(s) &= \left(\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
\alpha_B(s) &= \left(-\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
\alpha_C(s) &= \left(\sinh\left(\frac{s}{\sqrt{2}}\right), 0, \cosh\left(\frac{s}{\sqrt{2}}\right)\right),
\end{aligned}$$



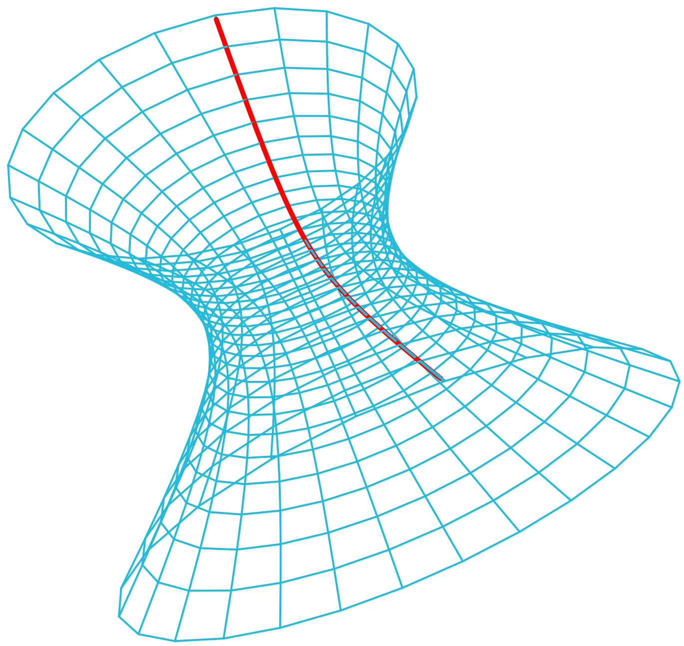
Tangent indicatrix of α.



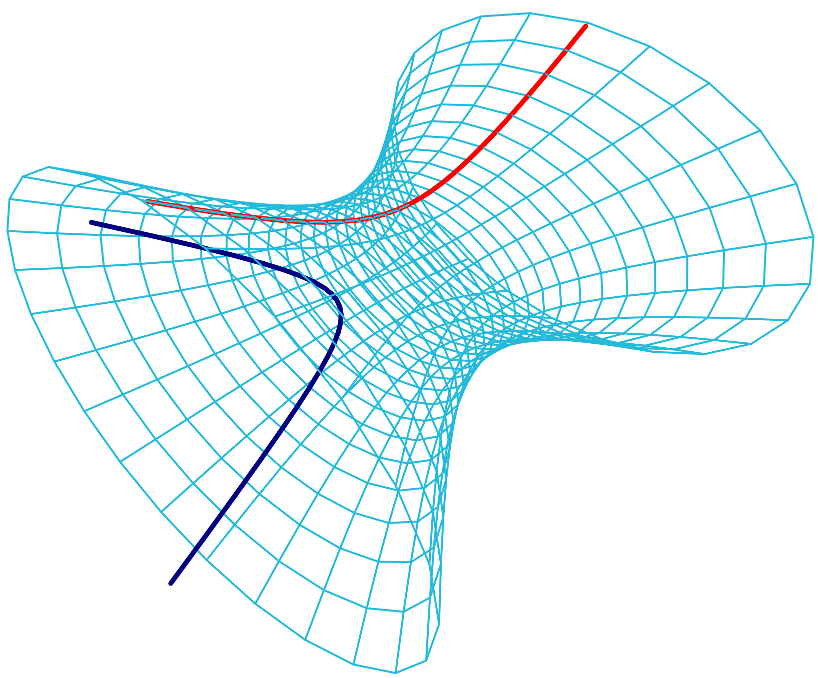
Principal normal indicatrix of α.



Binormal indicatrix of α.



Fixed cenroid of α.



Fixed cenroid of α and its natural lift curve.

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On the Probabilistic Stability of the 2-variable k-AC-mixed Type Functional Equation

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Abstract

In this paper, we obtain the general solution and the generalized Ulam-Hyers stability of the 2-variable k-AC mixed type functional equation

$$\begin{aligned} &f(x+ky,z+kw)+f(x-ky,z-kw) \\ &\quad = k^2[f(x+y,z+w)+f(x-y,z-w)]+2(1-k^2)f(x,z). \end{aligned}$$

for any $k \in Z - \{0, \pm 1\}$ in α-Šerstnev Menger Probabilistic normed spaces.

Keywords: Generalized Hyers-Ulam-Rassias stability, k-AC mixed type functional equation, α-Šerstnev Menger Probabilistic normed spaces.

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1 Introduction

Menger introduced probabilistic metric space in 1942 [16]. A probabilistic normed space (PN space) is a natural generalization of an ordinary normed linear space. Such spaces were first introduced by Šerstnev in 1963, (see, [28]). Alsina et al. generalized the definition of PN space [1]. This definition became the standard one and has been adopted by all researchers, who after them have investigated the properties of PN spaces. In this article, we adopt the new definition of α-Šerstnev PN spaces (or generalized Šerstnev PN spaces) given in the paper [14] by Lafuerza-Guillén and Rodríguez.

The problem of Ulam-Hyers stability for functional equations concerns deriving conditions under which, given an approximate solution of a functional equation, one may find an exact solution that is near it in some sense. The problem was first stated by Ulam [30] in 1940 for the case of group homomorphisms, and solved by Hyers [9] in the setting of Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution ([2, 7, 22]) and in terms of the methods used for the proof ([4, 6, 8, 10, 29]). Many interesting results concerning this problem can be found, for example, in [11–13, 15, 17–20, 23, 24].

The stability of generalized mixed type functional equation of the form

$$f(x+ky)+f(x-ky) = k^2[f(x+y)+f(x-y)]+2(1-k^2)f(x) \tag{1.1}$$

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for fixed integers k and $k \neq 0, \pm 1$ in quasi-Banach spaces was introduced by M. Eshaghi Gordji and H. Khodaie [5]. The mixed type functional equation (1.1) is having the property additive, quadratic and cubic.

J.H. Bae and W.G. Park proved the general solution and investigated the generalized Hyers-Ulam stability of the 2-variable quadratic functional equation

$$f(x+y,z+w)+f(x-y,z-w)=2f(x,z)+2(y,w). \tag{1.2}$$

The functional equation (1.2) has solution

$$f(x,y)=ax^2+bxy+cy^2 \tag{1.3}$$

The general solution and generalized Hyers-Ulam stability of a 3-variable quadratic functional equation

$$f(x+y,z+w,u+v)+f(x-y,z-w,u-v)=2f(x,z,u)+2(y,w,v) \tag{1.4}$$

was discussed by K. Ravi and M. Arun Kumar [25]. The solution of (1.4) is of the form

$$f(x,y,z)=ax^2+by^2+cz^2+dxy+eyz+fzx \tag{1.5}$$

Very recently, M. Aruk Kumar et al., introduced and investigated the solution and generalized Ulam-Hyers stability of a 2-varibale AC-mixed type functional equation

$$f(2x+y,2z+w)-f(2x-y,2z-w)=4[f(x+y,z+w)-f(x-y,z-w)]-6f(y,w) \tag{1.6}$$

having solutions

$$f(x,y)=ax+by \tag{1.7}$$

and

$$f(x,y)=ax^3+bx^2y+cxy^2+dy^3 \tag{1.8}$$

in Banach spaces [3] and Quasi-Beta normed space [21].

Following the same approach, in this paper, we investigate the general solution and establish that generalized Ulam-Hyers stability of the 2-variable k-AC mixed type functional equation

$$\begin{aligned} &f(x+ky,z+kw)+f(x-ky,z-kw)\\ &\quad =k^2[f(x+y,z+w)+f(x-y,z-w)]+2(1-k^2)f(x,z) \end{aligned} \tag{1.9}$$

having solutions

$$f(x,y)=ax+by \tag{1.10}$$

and

$$f(x,y)=ax^3+bx^2y+cxy^2+dy^3 \tag{1.11}$$

for fixed integers k with $k \neq 0, \pm 1$ in α-Šestnev (or generalized Šerstnev) Menger Probabilistic normed spaces.

Δ^+ is the space of distribution functions that is, the space of all mappings $F : R \cup \{-\infty, \infty\} \to [0,1]$ that is non-decreasing, left-continuous on R and such that $F(0)=0$ and $F(+\infty)=1$. D^+ is a subset of Δ^+ consisting of all functions F for which $\lim_{x\to+\infty} F(x)=1$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions. The maximal element for Δ^+ in this order is the distribution function ϵ_0 given by

$$\epsilon_0(t)=\begin{cases}0, \text{if } t\leq 0\\ 1, \text{if } t>0\end{cases}$$

Definition 1.1. *[26, 27] A triangle function is a mapping $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ such that, for all F, G, H, K in Δ^+,*

(1) $\tau(F,\epsilon_0)=F$,

(2) $\tau(F,G)=\tau(G,F)$,

(3) $\tau(F,G)\leq\tau(H,K)$ *whenever* $F\leq H$, $G\leq K$,

(4) $\tau(\tau(F,G),H)=\tau(F,\tau(G,H))$.

Moreover, a triangle function is continuous if it is continuous in the metric space (Δ^+, d_s).

Typical continuous triangle functions are

$$\tau_T(F,G)(x) := \sup_{s+t=x} T(F(s),G(t)) \tag{1.12}$$

and

$$\tau_{T^*}(F,G)(x) := \inf_{s+t=x} T^*(F(s),G(t)) \tag{1.13}$$

for all $F, G \in \Delta^+$ *and all* $x \in R$. *Here,* T *is a continuous t-norm and* T^* *is the corresponding continuous t-conorm, i.e., both are continuous binary operations on [0, 1] that are commutative, associative, and non decreasing in each variable;* T *has 1 as identity and* T^* *has 0 as identity. Also* $T^*(x,y) = 1 - T(1-x, 1-y)$.

Definition 1.2 (PN spaces redefined [1]). *A PN space is a quadruple* (V, ν, τ, τ^*), *where* V *is a real vector space,* τ *and* τ^* *are continuous triangle functions such that* $\tau \leq \tau^*$, *and the mapping* $\nu : V \to \Delta^+$ *satisfies, for all* p *and* q *in* V, *the conditions:*

(N1) $\nu_p = \epsilon_0$ *if, and only if,* $p = \theta$ *(*θ *is the null vector in* V*);*

(N2) $\forall\ p \in V, \nu_{-p} = \nu_p$;

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;

(N4) $\forall\ \alpha \in [0,1], \nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$.

A PN space is called a Šerstnev-space if it satisfies (N1), (N3) and the following condition:

$$(\check{S})\ \ \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right) \tag{1.14}$$

holds for every $\alpha \neq 0 \in \mathbb{R}$ *and* $x > 0$.

If $\tau = \tau_T$ *and* $\tau^* = \tau_{T^*}$ *for some continuous t-norm* T *and its t-conorm* T^*, *then the PN space* $(V, \nu, \tau_T, \tau_{T^*})$ *is called Menger PN space (briefly, MPN space), and is denoted by* (V, ν, T).

Let $\phi : [0, +\infty] \to [0, +\infty]$ *be a non-decreasing, left-continuous function with* $\phi(0) = 0$, $\phi(+\infty) = +\infty$ *and* $\phi(x) > 0$ *for* $x > 0$. *Let* $\hat{\phi}$ *be the (unique) quasi-inverse of* ϕ *which is left-continuous.* $\hat{\phi}$ *is defined by* $\hat{\phi}(0) = 0$, $\hat{\phi}(+\infty) = +\infty$ *and* $\hat{\phi}(t) = sup\{u : \phi(u) < t\}$ *for all* $0 < t < +\infty$. *It follows that* $\hat{\phi}(\phi(x)) \leq x$ *and* $\phi(\hat{\phi}(y)) \leq y$ *for all* x *and* y.

Definition 1.3. *[14] A quadruple* (V, ν, τ, τ^*) *satisfy the*

$$(\phi - \check{S})\ \ \nu_{\lambda p}(x) = \nu_p\left(\hat{\phi}\left(\frac{\phi(x)}{|\lambda|}\right)\right) \tag{1.15}$$

for all $x \in \mathbb{R}^+$, $p \in V$ *and* $\lambda \in \mathbb{R}\backslash\{0\}$ *is called a* ϕ*-Šerstnev PN space (generalized Šerstnev space).*

If $\phi(x) = x^{1/\alpha}$ *for a fixed positive real number* α, *the condition* $(\phi - \check{S})$ *takes the form*

$$(\alpha - \check{S})\ \ \nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|^\alpha}\right) \tag{1.16}$$

for every $p \in V$, *for every* $x > 0$ *and* $\lambda \in \mathbb{R}\backslash\{0\}$.

PN spaces satisfying the condition $(\alpha - \check{S})$ *are called* α*-Šerstnev PN spaces.*

Definition 1.4. *Let* (V, ν, τ) *be a PN space and* $\{x_n\}$ *be a sequence in* V. *Then* $\{x_n\}$ *is said to be convergent if there exists* $x \in V$ *such that*

$$\lim_{n\to\infty} \nu_{x_n - x}(t) = 1 \tag{1.17}$$

for all $t > 0$. *In this case* x *is called the limit of* $\{x_n\}$.

Definition 1.5. *The sequence* $\{x_n\}$ *in* (V, ν, τ) *is called a Cauchy sequence if, for every* $\epsilon > 0$ *and* $\delta > 0$, *there exists a positive integer* n_0 *such that* $\nu(x_n - x_m)(\delta) > 1 - \epsilon$ *for all* $m, n \geq n_0$. *Clearly, every convergent sequence in a PN-space is Cauchy. If every Cauchy sequence is convergent in a PN-space* (V, ν, τ), *then* (V, ν, τ) *is called a probabilistic Banach space (PB-space).*

2 General Solution

Through out this section let U and V be real vector spaces and we present the solution of (1.9) using Lemma 2.1, 2.2, 2.3.

Lemma 2.1. *If $f : U^2 \to V$ is a mapping satisfying (1.9) and let $g : U^2 \to V$ be a mapping given by*

$$g(x,x) = f(2x,2x) - 8f(x,x) \tag{2.1}$$

for all $x \in U$ then

$$g(2x,2x) = 2g(x,x) \tag{2.2}$$

for all $x \in U$ such that g is additive.

Proof. Letting (x,y,z,w) by $(0,0,0,0)$ in (1.9), we get

$$f(0,0) = 0 \tag{2.3}$$

Setting (x,y,z,w) by (y,x,w,z) in (1.9), we obtain

$$\begin{aligned} &f(y+kx, w+kz) + f(y-kx, w-kz) \\ &\quad = k^2[f(x+y, w+z) + f(y-x, w-z)] + 2(1-k^2)f(z,x) \end{aligned} \tag{2.4}$$

for all $x,y,z,w \in U$.
Replacing (x,y,z,w) by $(x,-y,z,-w)$ in (2.4), we get

$$\begin{aligned} &f(-y+kx, -w+kz) + f(-y-kz, -w-kz) \\ &\quad = k^2[f(x-y),(w-z)) + f(-y-x, -w-z)] + 2(1-k^2)f(z,x) \end{aligned} \tag{2.5}$$

for all $x,y,z,w \in U$.
From (2.4) and (2.5) we arrive at

$$\begin{aligned} &f(y+kx, w+kz) + f(y-kx, w-kz) + f(-y+kx, -w+kz) \\ &\quad + f(-y-kx, -w-kz) = k^2[f(x+y, w+z) + f(y-x, w-z) \\ &\quad + f(x-y, z-w) + f(-y-x, -w-z)] + 4(1-k^2)f(z,x) \end{aligned} \tag{2.6}$$

Now, letting (x,y,z,w) by $(0,y,0,y)$ in (2.6), we obtain

$$2[k^2-1][f(y,y) + f(-y,-y)] = 0$$

which implies

$$f(y,y) = -f(-y,-y) \tag{2.7}$$

for all $y \in U$.
Replacing (x,y,z,w) by (x,x,x,x) in (1.9), we get

$$\begin{aligned} &f((1+k)x,(1+k)x) + f((1-k)x,(1-k)x) \\ &\quad = k^2 f(2x,2x) + 2(1-k^2)f(x,x) \end{aligned} \tag{2.8}$$

for all $x \in U$. Now, replacing x by $2x$ in (2.8), we have

$$\begin{aligned} &f(2(1+k)x, 2(1+k)x) + f(2(1-k)x, 2(1-k)x) \\ &\quad = k^2 f(4x,4x) + 2(1-k^2)f(2x,2x) \end{aligned} \tag{2.9}$$

for all $x \in U$. Again replacing (x,y,z,w) by $(2x,x,2x,x)$ in (1.9), we obtain

$$\begin{aligned} &f((2+k)x,(2+k)x) + f((2-k)x,(2-k)x) \\ &\quad = k^2 f(3x,3x) + k^2 f(x,x) + 2(1-k^2)f(2x,2x) \end{aligned} \tag{2.10}$$

for all $x \in U$.

Replacing (x, y, z, w) by $(x, 2x, x, 2x)$ in (1.9), we get

$$\begin{aligned} f((1+2k)x, (1+2k)x) + f((1-2k)x, (1-2k)x) \\ = k^2 f(3x, 3x) - k^2 f(x, x) + 2(1-k^2) f(x, x) \end{aligned} \tag{2.11}$$

for all $x \in U$. Replacing (x, y, z, w) by $(x, 3x, x, 3x)$ in (1.9), we obtain

$$\begin{aligned} f((1+3k)x, (1+3k)x) + f((1-3k)x, (1-3k)x) \\ = k^2 f(4x, 4x) - k^2 f(2x, 2x) + 2(1-k^2) f(x, x) \end{aligned} \tag{2.12}$$

for all $x \in U$. We substitute (x, y, z, w) by $((1+k)x, x, (1+k)x, x)$ in (1.9) and then (x, y, z, w) by $((1-k)x, x, (1-k)x, x)$ in (1.9) to obtain

$$\begin{aligned} f((1+2k)x, (1+2k)x) + f(x, x) = k^2 f((2+k)x, (2+k)x) \\ + k^2 f(kx, kx) + 2(1-k^2) f((1+k)x, (1+k)x) \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} f((1-2k)x, (1-2k)x) + f(x, x) = k^2 f((2-k)x, (2-k)x) \\ - k^2 f(kx, kx) + 2(1-k^2) f((1-k)x, (1-k)x) \end{aligned} \tag{2.14}$$

for all $x \in U$. Then, by adding (2.13) to (2.14), we have

$$\begin{aligned} f((1+2k)x, (1+2k)x) + f((1-2k)x, (1-2k)x) + 2f(x, x) \\ = k^2 f((2+k)x, (2+k)x) + k^2 f((2-k)x, (2-k)x) \\ + 2(1-k^2)[f((1+k)x, (1+k)x) + f((1-k)x, (1-k)x)] \end{aligned} \tag{2.15}$$

for all $x \in U$. Now, substitute (x, y, z, w) by $((1+2k)x, x, (1+2k)x, x)$ in (1.9) and (x, y, z, w) by $((1-2k)x, x, (1-2k)x, x)$ in (1.9) to obtain

$$\begin{aligned} f((1+3k)x, (1+3k)x) + f((1+k)x, (1+k)x) \\ = k^2 f(2(1+k)x, 2(1+k)x) + k^2 f(2kx, 2kx) \\ + 2(1-k^2) f((1+2k)x, (1+2k)x) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} f((1-3k)x, (1-3k)x) + f((1-k)x, (1-k)x) \\ = k^2 f(2(1-k)x, 2(1-k)x) - k^2 f(2kx, 2kx) \\ + 2(1-k^2) f((1-2k)x, (1-2k)x) \end{aligned} \tag{2.17}$$

for all $x \in U$. Now, adding (2.16) to (2.17), we have,

$$\begin{aligned} f((1+3k)x, (1+3k)x) + f((1-3k)x, (1-3k)x) + f((1+k)x, (1+k)x) \\ + f((1-k)x, (1-k)x) = k^2 f(2(1+k)x, 2(1+k)x) \\ + k^2 f(2(1-k)x, 2(1-k)x) \\ + 2(1-k^2)[f((1+2k)x, (1+2k)x) + f((1-2k)x, (1-2k)x)] \end{aligned} \tag{2.18}$$

for all $x \in U$. From (2.8), (2.10), (2.11) and (2.15), we arrive at

$$f(3x, 3x) = 4f(2x, 2x) - 5f(x, x) \tag{2.19}$$

for all $x \in U$. From (2.9), (2.11), (2.8), (2.12) and (2.18), we have

$$f(4x, 4x) = 2f(2x, 2x) + 2f(3x, 3x) - 6f(x, x) \tag{2.20}$$

for all $x \in U$. Using (2.19) in (2.20), we obtain

$$f(4x,4x) = 10f(2x,2x) - 16f(x,x) \tag{2.21}$$

for all $x \in U$. From (2.21), we establish

$$f(4x,4x) - 8f(2x,2x) = 2f(2x,2x) - 16f(x,x) \tag{2.22}$$

for all $x \in U$. Using (2.1) in (2.22), we get our desired result. □

Lemma 2.2. *If $f : U^2 \to V$ be a mapping satisfying (1.9) and let $h : U^2 \to V$ be a mapping given by*

$$h(x,x) = f(2x,2x) - 2f(x,x) \tag{2.23}$$

for all $x \in U$ then

$$h(2x,2x) = 8h(x,x) \tag{2.24}$$

for all $x \in U$ such that h is cubic.

Proof. Proceeding as in Lemma 2.1, it follows from (2.21)

$$f(4x,4x) - 2f(2x,2x) = 8f(2x,2x) - 16f(x,x) \tag{2.25}$$

for all $x \in U$. Using (2.23) in (2.25), we arrive at our desired result. □

Remark 2.1. *If $f : U^2 \to V$ be a mapping satisfying (1.9) let $g,h : U^2 \to V$ be mappings defined by (2.1) and (2.23) then*

$$f(x,x) = \frac{1}{6}(h(x,x) - g(x,x)) \tag{2.26}$$

for all $x \in U$.

Lemma 2.3. *If $f : U^2 \to V$ is a mapping satisfying (1.9) and let $t : U \to V$ be a mapping given by*

$$t(x) = f(x,x) \tag{2.27}$$

for all $x \in U$, then t satisfies

$$t(x+ky) + t(x-ky) = k^2[t(x+y) + t(x-y)] + 2(1-k^2)t(x) \tag{2.28}$$

for all $x,y \in U$.

Proof. From (1.9) and (2.27), we get

$$\begin{aligned} t(x+ky) + t(x-ky) &= f(x+ky,x+ky) - f(x-ky,x-ky) \\ &= k^2[f(x+y,x+y) + f(x-y,x-y)] + 2(1-k^2)f(x,x) \\ &= k^2[t(x+y) + t(x-y)] + 2(1-k^2)t(x) \end{aligned}$$

for all $x,y \in U$. □

3 Stability Results : Direct Method

In this section, we investigate the generalized Ulam-Hyers stability problem of (1.9) using direct method. Let U be a real linear space and (Y,ν,τ_T) be a α-Šerstnev MPB space. Now, we define a difference operator $\Delta f : U^4 \to Y$ by

$$\begin{aligned} \Delta f(x,y,z,w) = f(x+ky,z+kw) + f(x-ky,z-kw) - k^2 f(x+y,z+w) \\ - k^2 f(x-y,z-w) - 2(1-k^2)f(x,z) \end{aligned} \tag{3.1}$$

$\forall\ x,y,z,w \in U$, where $f : U^2 \to Y$ is a mapping.

Theorem 3.1. *Let $f : U^2 \to Y$ be a mapping for which there exist a function $\xi : U^4 \to D^+$ with the condition*

$$\lim_{m\to\infty} \tau_T \left[\xi_{(2^m x, 2^m y, 2^m z, 2^m w)}(2^{m\alpha} t), \xi_{(2^m x, 2^m y, 2^m z, 2^m w)}(2^{(m-3)\alpha-1} t)\right] = 1 \tag{3.2}$$

such that the functional inequality

$$\nu_{\Delta f(x,y,z,w)}(t) \geq \xi_{x,y,z,w}(t) \tag{3.3}$$

for all $x, y, z, w \in U$, $t > 0$ and $\alpha > 0$. Then there exists a unique 2-variable additive mapping $A(x,x) : U^2 \to Y$ satisfying (1.9) and

$$\nu_{f(2x,2x)-8f(x,x)-A(x,x)}(t) \geq \tilde{\Phi} \tag{3.4}$$

where

$$A(x,x) = \lim_{n\to\infty} \frac{f(2^{(n+1)}x, 2^{(n+1)}x) - 8f(2^n x, 2^n x)}{2^n} \tag{3.5}$$

$$\begin{cases} \tilde{\Phi} = \lim_{n\to\infty} \Phi_n = 1 \\ \Phi_n = \tau_T \left[\tilde{\tau}_{T(2^{n-1}x)}(t), \Phi_{n-1}\right], \ \textit{for } n > 1 \end{cases} \tag{3.6}$$

$$\Phi_1 = \tilde{\tau}_{T(x)}(t) \tag{3.7}$$

and

$$\begin{aligned}
\tilde{\tau}_{T(x)}(t) = \tau_T \Bigg(&\tau_T \Bigg(\tau_T \Bigg(\tau_T \Bigg(\xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha} t}{2^4 2^\alpha}\right),\\
&\xi_{((1-2k)x,x,(1-2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4}\right)\Bigg)\Bigg), \tau_T \Bigg(\xi_{((1+2k)x,x,(1+2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4}\right),\\
&\xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4}\right)\Bigg)\Bigg)\Bigg), \tau_T \Bigg(\xi_{(2x,2x,2x,2x)}\left(\frac{|k^2-1|^\alpha t}{2^3}\right),\\
&\xi_{(x,3x,x,3x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right), \tau_T \Bigg(\tau_T \Bigg(\tau_T \Bigg(\xi_{(x,x,x,x)}\left(\frac{k^{2\alpha} t}{2^4 2^{2\alpha}}\right),\\
&\xi_{((1-k)x,x,(1-k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4 2^\alpha}\right)\Bigg)\Bigg), \tau_T \Bigg(\xi_{((1+k)x,x,(1+k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4 2^\alpha}\right)\\
&\xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4 2^\alpha}\right)\Bigg)\Bigg)\Bigg), \xi_{(2x,x,2x,x)}\left(\frac{|k^2-1|^\alpha t}{2^4 2^\alpha}\right)\Bigg)\Bigg)\Bigg),
\end{aligned} \tag{3.8}$$

for all $x \in U$, $t > 0$ and $\alpha > 0$.

Proof. Letting (x, y, z, w) by (x, x, x, x) in (3.3), we obtain

$$\begin{aligned} &\nu_{f((1+k)x,(1+k)x)+f((1-k)x,(1-k)x)-k^2 f(2x,2x)-2(1-k^2)f(x,x)}(t) \\ &\geq \xi_{(x,x,x,x)}(t), \ \forall\, x \in U, t > 0. \end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\begin{aligned} &\nu_{f(2(1+k)x,2(1+k)x)+f(2(1-k)x,2(1-k)x)-k^2 f(4x,4x)-2(1-k^2)f(2x,2x)}(t) \\ &\geq \xi_{(2x,2x,2x,2x)}(t), \ \forall\, x \in U, t > 0. \end{aligned} \tag{3.10}$$

Replacing (x, y, z, w) by $(2x, x, 2x, x)$ in (3.3), respectively, we have

$$\begin{aligned} &\nu_{f((2+k)x,(2+k)x)+f((2-k)x,(2-k)x)-k^2 f(3x,3x)-k^2 f(x,x)-2(1-k^2)f(2x,2x)}(t) \\ &\geq \xi_{(2x,x,2x,x)}(t), \ \forall\, x \in U, t > 0. \end{aligned} \tag{3.11}$$

Setting (x, y, z, w) by $(x, 2x, x, 2x)$ in (3.3) gives

$$\nu_{f((1+2k)x,(1+2k)x)+f((1-2k)x,(1-2k)x)-k^2 f(3x,3x)-k^2 f(x,x)-2(1-k^2)f(x,x)}(t) \geq \xi_{(x,2x,x,2x)}(t),\ \forall\, x \in U, t > 0. \tag{3.12}$$

Replacing (x, y, z, w) by $(x, 3x, x, 3x)$ in (3.3), we obtain

$$\nu_{f((1+3k)x,(1+3k)x)+f((1-3k)x,(1-3k)x)-k^2 f(4x,4x)+k^2 f(2x,2x)-2(1-k^2)f(x,x)}(t) \geq \xi_{(x,3x,x,3x)}(t),\ \forall\, x \in U, t > 0. \tag{3.13}$$

Replacing (x, y, z, w) by $((1+k)x, x, (1+k)x, x)$ in (3.3), respectively, we get

$$\nu_{f((1+2k)x,(1+2k)x)+f(x,x)-k^2 f((2+k)x,(2+k)x)-k^2 f(kx,kx)-2(1-k^2)f((1+k)x,(1+k)x)}(t) \geq \xi_{((1+k)x,x,(1+k)x,x)}(t),\ \forall\, x \in U, t > 0. \tag{3.14}$$

Replacing (x, y, z, w) by $((1-k)x, x, (1-k)x, x)$ in (3.3), respectively, one gets

$$\nu_{f((1-2k)x,(1-2k)x)+f(x,x)-k^2 f((2-k)x,(2-k)x)+k^2 f(kx,kx)-2(1-k^2)f((1-k)x,(1-k)x)}(t) \geq \xi_{((1-k)x,x,(1-k)x,x)}(t),\ \forall\, x \in U, t > 0. \tag{3.15}$$

Replacing (x, y, z, w) by $((1+2k)x, x, (1+2k)x, x)$ in (3.3), respectively, we obtain

$$\nu_{f((1+3k)x,(1+3k)x)+f((1+k)x,(1+k)x)-k^2 f(2(1+k)x,2(1+k)x)-k^2 f(2kx,2kx)-2(1-k^2)f((1+2k)x,(1+2k)x)}(t) \geq \xi_{((1+2k)x,x,(1+2k)x,x)}(t),\ \forall\, x \in U, t > 0. \tag{3.16}$$

Replacing (x, y, z, w) by $((1-2k)x, x, (1-2k)x, x)$ in (3.3), respectively, we have

$$\nu_{f((1-3k)x,(1-3k)x)+f((1-k)x,(1-k)x)-k^2 f(2(1-k)x,2(1-k)x)+k^2 f(2kx,2kx)-2(1-k^2)f((1-2k)x,(1-2k)x)}(t) \geq \xi_{((1-2k)x,x,(1-2k)x,x)}(t),\ \forall\, x \in U, t > 0. \tag{3.17}$$

Thus it follows from (3.9), (3.11), (3.12), (3.14) and (3.15) that

$$\begin{aligned}
&\nu_{f(3x,3x)-4f(2x,2x)+5f(x,x)}(t) \\
&\geq \tau_T\left(\tau_T\left(\tau_T\left(\xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}t}{2^3 2^\alpha}\right),\ \xi_{((1-k)x,x,(1-k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right),\right.\right. \\
&\tau_T\left(\xi_{((1+k)x,x,(1+k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right), \xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right)\Bigg) \\
&\xi_{(2x,x,2x,x)}\left(\frac{|k^2-1|^\alpha t}{2}\right)\Bigg),\ \forall\, x \in U, t > 0 \text{ and } \alpha > 0.
\end{aligned} \tag{3.18}$$

Also, from (3.9), (3.10), (3.12), (3.13) (3.16) and (3.17), we have

$$\begin{aligned}
&\nu_{f(4x,4x)-2f(3x,3x)-2f(2x,2x)+6f(x,x)}(t) \\
&\geq \tau_T\left(\tau_T\left(\tau_T\left(\xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}t}{2^3 2^\alpha}\right),\ ,\xi_{((1-2k)x,x,(1-2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right),\right.\right. \\
&\tau_T\left(\xi_{((1+2k)x,x,(1+2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right), \xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right)\Bigg) \\
&\tau_T\left(\xi_{(2x,2x,2x,2x)}\left(\frac{|k^2-1|^\alpha t}{2^2}\right), \xi_{(x,3x,x,3x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^2}\right)\right)\Bigg),
\end{aligned} \tag{3.19}$$

for all $x \in U$, $t > 0$ and $\alpha > 0$.
Finally, by using (3.18) and (3.19), we obtain

$$\nu_{f(4x,4x)-10f(2x,2x)+16f(x,x)}(t) \geq \tilde{\tau}_{T(x)}(t) \tag{3.20}$$

where,

$$\begin{aligned}
&\tilde{\tau}_{T(x)}(t)\\
&= \tau_T\Bigg(\tau_T\Bigg(\tau_T\Bigg(\tau_T\Bigg(\xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}t}{2^4 2^{\alpha}}\right),\ \xi_{((1-2k)x,x,(1-2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4}\right)\Bigg),\\
&\tau_T\Bigg(\xi_{((1+2k)x,x,(1+2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4}\right), \xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4}\right)\Bigg)\Bigg),\\
&\tau_T\Bigg(\xi_{(2x,2x,2x,2x)}\left(\frac{|k^2-1|^{\alpha}t}{2^3}\right), \xi_{(x,3x,x,3x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^3}\right)\Bigg)\Bigg),\\
&\tau_T\Bigg(\tau_T\Bigg(\tau_T\Bigg(\xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}t}{2^4 2^{2\alpha}}\right),\ \xi_{((1-k)x,x,(1-k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4 2^{2\alpha}}\right)\Bigg),\\
&\tau_T\Bigg(\xi_{((1+k)x,x,(1+k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4 2^{\alpha}}\right), \xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}|k^2-1|^{\alpha}t}{2^4 2^{\alpha}}\right)\Bigg)\Bigg),\\
&\xi_{(2x,x,2x,x)}\left(\frac{|k^2-1|^{\alpha}t}{2^2 2^{\alpha}}\right)\Bigg)\Bigg),\ \forall\ x\in U, t>0 \text{ and } \alpha>0. \qquad (3.21)
\end{aligned}$$

Let $g: U^2 \to Y$ be a function defined by

$$g(x,x) = f(2x,2x) - 8f(2x,2x) \ \text{ for all } x \in U. \qquad (3.22)$$

From (3.20), we conclude that

$$\nu_{\frac{g(2x,2x)}{2}-g(x,x)}(t) \geq \tilde{\tau}_{T(x)}(2^{\alpha}t) \geq \tilde{\tau}_{T(x)}(t),\ \forall\ x\in U, t>0 \text{ and } \alpha>0 \qquad (3.23)$$

which implies that

$$\nu_{\frac{g(2^{\ell+1}x,2^{\ell+1}x)}{2^{\ell+1}}-\frac{g(2^{\ell}x,2^{\ell}x)}{2^{\ell}}}(t) \geq \tilde{\tau}_{T(2^{\ell}x)}(2^{(\ell+1)\alpha}t) \qquad (3.24)$$

for all $x \in U$, $t > 0$, $\alpha > 0$ and $\ell \in \mathbb{N}$. From the inequalities (3.23) and (3.24) we use iterative methods and induction on n and apply defined sequence in (3.6) and (3.7) to prove our next relation

$$\nu_{\frac{g(2^n x,2^n x)}{2^n}-g(x,x)}(t) \geq \tau_T\left[\tilde{\tau}_{T(2^{n-1}x)}(t), \Phi_{n-1}\right]\ \forall\ x\in U, t>0 \text{ and } \alpha>0. \qquad (3.25)$$

So

$$\nu_{\frac{g(2^{m+n}x,2^{m+n}x)}{2^{m+n}}-\frac{g(2^m x,2^m x)}{2^m}}(t) \geq \tau_T\left[\tilde{\tau}_{T(2^{(m+n)-1}x)}(2^{m\alpha}t), \Phi_{(m+n)-1}\right] \qquad (3.26)$$

for all non negative integers m and n and for all $x \in U$, $t > 0$. By assumptions (3.26) shows that the sequence $\left\{\frac{g(2^n x,2^n x)}{2^n}\right\}$ is a Cauchy sequence in Y for all $x \in U$. Since Y is a α-Šerstnev MPB, it follows that the sequence $\left\{\frac{g(2^n x,2^n x)}{2^n}\right\}$ converges for all $x \in U$. Therefore, one can define the function $A(x,x): U^2 \to Y$ by

$$A(x,x) = \lim_{n\to\infty} \frac{g(2^n x,2^n x)}{2^n} \ \text{ for all } x \in U. \qquad (3.27)$$

Now, if we replace (x,y,z,w) by $(2^n x, 2^n y, 2^n z, 2^n w)$ in (3.3), respectively, then it follows that

$$\begin{aligned}
&\nu_{\frac{\Delta g(2^n x,2^n y,2^n z,2^n w)}{2^n}}(t) = \nu_{\frac{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}{2^n}-8\frac{\Delta f(2^n x,2^n y,2^n z,2^n w)}{2^n}}(t)\\
&\geq \tau_T\left[\nu_{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}(2^{n\alpha-1}t), \nu_{\Delta f(2^n x,2^n y,2^n z,2^n w)}(2^{(n-3)\alpha-1}t)\right]\\
&\geq \tau_T\left[\xi_{2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w}(2^{n\alpha-1}t), \xi_{2^n x,2^n y,2^n z,2^n w}(2^{(n-3)\alpha-1}t)\right] \qquad (3.28)
\end{aligned}$$

for all $x,y,z,w \in U$, $t > 0$ and $\alpha > 0$. By letting $n \to \infty$ in (3.28), we have $\nu_{\Delta A(x,y,z,w)}(t) = 1$ for all $t > 0$ and so $\Delta A(x,y,z,w) = 0$. Hence A satisfies (1.9) for all $x,y,z,w \in U$. To prove (3.4), if we take the limit as

$n \to \infty$ in (3.25), then we can get (3.4). Finally, to prove the uniqueness of the additive function A subject to (3.4), assume that there exists another 2-variable additive mapping A' which satisfies (3.4) and (1.9), then

$$\begin{aligned} \nu_{A(x,x)-A'(x,x)}(t) &= \nu_{\frac{A(2^n x,2^n x)}{2^n} - \frac{A'(2^n x,2^n x)}{2^n}}(t) \\ &= \nu_{A(2^n x,2^n x)-A'(2^n x,2^n x)}(2^{n\alpha} t) \\ &\geq \nu_{A(2^n x,2^n x)-g(2^n x,2^n x)+g(2^n x,2^n x)-A'(2^n x,2^n x)}(2^{n\alpha} t) \\ &\geq \lim_{n\to\infty} \tau_T \left[\tau_T \left[\tilde{\tau}_{T(2^{2n-1}x)}(2^{n\alpha-1}t), \Phi_{n-1}\right], \tau_T \left[\tilde{\tau}_{T(2^{2n-1}x)}(2^{n\alpha-1}t), \Phi_{n-1}\right]\right. \end{aligned} \tag{3.29}$$

which tends to 1 as $n \to \infty$ for all $x \in U$. So we can conclude that $A = A'$. This completes the proof of the theorem. □

Theorem 3.2. *Let $f : U^2 \to Y$ be a mapping for which there exist a function $\xi : U^4 \to D^+$ with the condition*

$$\lim_{m\to\infty} \tau_T \left[\xi_{(2^m x,2^m y,2^m z,2^m w)}(2^{3m\alpha} t), \xi_{(2^m x,2^m y,2^m z,2^m w)}(2^{(3m-1)\alpha-1} t),\right] \tag{3.30}$$

such that the functional inequality (3.3) is satisfied for all $x, y, z, w \in U$, $t > 0$ and $\alpha > 0$. Then there exists a unique 2-variable cubic mapping $c(x, x) : U^2 \to Y$ satisfying (1.9) and

$$\nu_{f(2x,2x)-2f(x,x)-c(x,x)}(t) \geq \tilde{\Psi} \tag{3.31}$$

where

$$c(x,x) = \lim_{n\to\infty} \frac{f(2^{(n+1)}x, 2^{(n+1)}x) - 2f(2^n x, 2^n x)}{2^{3n}} \tag{3.32}$$

$$\begin{cases} \tilde{\Psi} = \lim_{n\to\infty} \Psi_n = 1 \\ \Psi_n = \tau_T \left[\tilde{\tau}_{T(2^{n-1}x)}(2^{2n\alpha} t), \Psi_{n-1}\right] \end{cases} \tag{3.33}$$

$$\Psi_1 = \tilde{\tau}_{T(x)}(2^{2\alpha} t),\ \forall\, x \in U, t > 0, \alpha > 0, \tag{3.34}$$

where $\tilde{\tau}_{T(x)}(t)$ is defined as in Theorem 3.1.

Proof. By the similar approach as in the proof of Theorem 3.1, we can obtain

$$\nu_{f(4x,4x)-10f(2x,2x)+16f(x,x)}(t) \geq \tilde{\tau}_{T(x)}(t),\ \forall\, x \in U, t > 0.$$

Let $h : U^2 \to Y$ be a function defined by

$$h(x,x) = f(2x,2x) - 2f(x,x), \text{ for all } x \in U \tag{3.35}$$

Thus from (3.20), we have

$$\nu_{\frac{h(2x,2x)}{2^3} - h(x,x)}(t) \geq \tilde{\tau}_{T(x)}(2^{3\alpha} t) \geq \overline{\tau}_{T(x)}(2^{2\alpha} t),\ \forall\, x \in U, t > 0, \alpha > 0 \tag{3.36}$$

which implies that

$$\nu_{\frac{h(2^{\ell+1}x,2^{\ell+1}x)}{2^{3(\ell+1)}} - \frac{h(2^\ell x,2^\ell x)}{2^{3\ell}}}(t) \geq \overline{\tau}_{T(2^\ell x)}(2^{3(\ell+1)\alpha} t) \tag{3.37}$$

for all $x \in U, t > 0, \alpha > 0$ and $\ell \in \mathbb{N}$. Thus it follows from (3.37) and (N3)

$$\nu_{\frac{h(2^n x,2^n x)}{2^{3n}} - h(x,x)}(t) \geq \tau_T \left[\tilde{\tau}_{T(2^{n-1}x)}(2^{2n\alpha} t), \Phi_{n-1}\right],\ \forall\, x \in U; t > 0, \alpha > 0. \tag{3.38}$$

In order to prove the convergence of the sequence $\left\{\frac{h(2^n x,2^n x)}{2^{3n}}\right\}$ if we replace x with $2^m x$ in (3.38), then we get

$$\nu_{\frac{h(2^{n+m}x,2^{n+m}x)}{2^{3(n+m)}} - \frac{h(2^m x,2^m x)}{2^{3m}}}(t) \geq \tau_T \left[\overline{\tau}_{T(2^{n+m-1}x)}(2^{(2n+3m)\alpha} t), \Phi_{n+m}\right] \tag{3.39}$$

for all non-negative integers m and n and $\forall\, x \in U$, $t > 0$, $\alpha > 0$.

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, by assumptions, the sequence $\left\{\frac{h(2^n x, 2^n x)}{2^{3n}}\right\}$ is a Cauchy sequence in Y for all $x \in U$. Since Y is a α-Šerstnev MPB, one can define the function $c(x,x) : U^2 \to Y$ by

$$c(x,x) = \lim_{n\to\infty} \frac{h(2^n x, 2^n x)}{2^{3n}} \text{ for all } x \in U. \tag{3.40}$$

Now, if we replace (x,y,z,w) by $(2^n x, 2^n y, 2^n z, 2^n w)$ in (3.3), respectively, then it follows that

$$\begin{aligned} &\nu_{\frac{\Delta h(2^n x,2^n y,2^n z,2^n w)}{2^{3n}}}(t) = \nu_{\frac{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}{2^{3n}} - 2\frac{\Delta f(2^n x,2^n y,2^n z,2^n w)}{2^{3n}}}(t) \\ &\geq \tau_T\left[\nu_{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}(2^{3n\alpha-1}t), \nu_{\Delta f(2^n x,2^n y,2^n z,2^n w)}(2^{(3n-2)\alpha-1}t)\right] \\ &\geq \tau_T\left[\xi_{(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}(2^{3n\alpha-1}t), \xi_{(2^n x,2^n y,2^n z,2^n w)}(2^{(3n-1)\alpha-1}t)\right] \end{aligned} \tag{3.41}$$

for all $x,y,z,w \in U$, $t > 0$ and $\alpha > 0$. By letting $n \to \infty$ in (3.41), we find that $\nu_{\Delta c(x,y,z,w)}(t) = 1$ for all $t > 0$, which implies $\Delta c(x,y,z,w) = 0$ and so c satisfies (1.9) for all $x,y,z,w \in U$. To prove (3.31), if we take the limit as $n \to \infty$ in (3.38), then we get (3.31). The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. □

Theorem 3.3. *Let $\xi : U^2 \to D^+$ be a function with the conditions given in (3.2) and (3.30) and $f : U^2 \to Y$ be a function which satisfies (3.3) for all $x,y,z,w \in U$ and $t > 0$. Then there exists a unique 2-variable additive mapping $A : U^2 \to Y$ and a unique 2-variable cubic mapping $C : U^2 \to Y$ satisfying (1.9) such that*

$$\begin{aligned} &\nu_{f(x,x)-A(x,x)-C(x,x)}(t) \geq \\ &\lim_{n\to\infty} \tau_T\left[\tau_T\left(\tilde{\tau}_{T(2^{n-1}x)}(3^\alpha 2^{\alpha-1}t), \Phi_{n-1}\right), \tau_T\left(\tilde{\tau}_{T(2^{n-1}x)}(2^{(2n+1)\alpha-1}3^\alpha t), \Psi_{n-1}\right)\right] \end{aligned} \tag{3.42}$$

for all $x \in U$, $t > 0$ and $\alpha > 0$, where Φ_n, $\tilde{\tau}_{T(x)}(t)$ is defined as in Theorem 3.1 and Ψ_n is defined as in Theorem 3.2.

Proof. By Theorems 3.1 and 3.2, there exist a unique 2-variable additive function $A_0 : U^2 \to Y$ and a unique 2-variable cubic function $C_0 : U^2 \to Y$ such that

$$\nu_{f(2x,2x)-8f(x,x)-A_0(x,x)}(t) \geq \tilde{\Phi} \tag{3.43}$$

and

$$\nu_{f(2x,2x)-2f(x,x)-C_0(x,x)}(t) \geq \tilde{\Psi},\ \forall\, x \in U, t > 0. \tag{3.44}$$

Thus it follows from (3.43) and (3.44) that

$$\begin{aligned} &\nu_{f(x,x)+\frac{1}{6}A_0(x,x)-\frac{1}{6}C_0(x,x)}(t) \\ &\geq \tau_T\left[\nu_{f(2x,2x)-8f(x,x)-A_0(x,x)}(3^\alpha 2^{\alpha-1}t), \nu_{f(2x,2x)-2f(x,x)-C_0(x,x)}(3^\alpha 2^{\alpha-1}t)\right] \end{aligned} \tag{3.45}$$

for all $x \in U$, $t > 0$ and $\alpha > 0$. Thus we obtain (3.42) by letting $A(x,x) = -\frac{1}{6}A_0(x,x)$ and $C(x,x) = \frac{1}{6}C_0(x,x)$ for all $x \in U$. This completes the proof of the stability of the functional equation (1.9) in α-Šerstnev MPN spaces. □

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Certain properties of a subclass of harmonic convex functions of complex order defined by Multiplier transformations

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Abstract

In this paper, we investigate some properties of harmonic univalent functions of complex order using multiplier transformation.Such as Coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combination are determined for functions in this family. Further, we obtain the closure property of this class under integral operator. Consequently, many of our results are either extensions or new approaches to those corresponding to previously known results.

Keywords: Harmonic functions, analytic functions, univalent functions, starlike functions of complex order,Multiplier transformation..

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1 Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $D \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D. See Clunie and Sheil-Small [3].

Denote by $\mathcal{S_H}$ the family of functions $f = h + \overline{g}$ which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{S_H}$, the functions h and g analytic $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \;\; g(z) = \sum_{k=1}^{\infty} b_k z^k \;\; (|b_1| < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad (|b_1| < 1). \tag{1.1}$$

We note that the family $\mathcal{S_H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$.

Also, we denote by $T\mathcal{S_H}$ the subfamily of $\mathcal{S_H}$ consisting of harmonic functions of the form $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.2}$$

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In [3] Clunie and Sheil-Small, investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and its properties. Since then, there have been several studies related to the class $\mathcal{S}_{\mathcal{H}}$ and its subclasses. In particular, Avci and Zlotkiewicz [2], Silverman [9], Jahangiri [5, 6] and others have investigated various subclasses of $\mathcal{S}_{\mathcal{H}}$ and its properties. Furthermore, Yalçin and Öztürk [11] and Murugusundaramoorthy[7] have considered a class $T\mathcal{S}^*_{\mathcal{H}}(\gamma)$ of harmonic starlike functions of complex order based on a corresponding study of Nasr and Aouf [8] for analytic case. (see [4, 13]).

For $f \in S$ the differential operator $D^n(n \in N_0)$ of f was introduced by salagean for $f = h + \overline{g}$ Jagangiri et al[] defined the modified salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)} \tag{1.3}$$

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k. \tag{1.4}$$

Next,for functions $f \in A$ Cho and Srivastava defined Multiplier transformation.For $f = h + \overline{g}$ given by (1) we define the modified Multiplier transformation of f.

$$I^0_\gamma f(z) = D^0 f(z) = h(z) + \overline{g(z)} \tag{1.5}$$

$$I^1_\gamma f(z) = \frac{\gamma D^0 f(z) + D^1 f(z)}{\gamma + 1} \tag{1.6}$$

$$I^n_\gamma f(z) = I^1_\gamma (I^{n-1}_\gamma f(z)), \, (n \in N_0) \tag{1.7}$$

$$I^n_\gamma f(z) = z + \sum_{k=2}^{\infty} (\frac{k+\gamma}{1+\gamma})^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} (\frac{k-\gamma}{1+\gamma})^n \overline{b_k z^k} \tag{1.8}$$

Also if f is given by (1) then we have

$$I^n_\gamma f(z) = f \widetilde{*} \underbrace{(\phi_1(z) + \overline{\phi_2(z)}) \widetilde{*} ......\widetilde{*} (\phi_1(z) + \overline{\phi_2(z)})}_{n-times} = h * \underbrace{(\phi_1(z) * ...(\phi_1(z)}_{n-times} + \overline{g + \underbrace{(\phi_2(z) * ...(\phi_2(z)))}_{n-times}} \tag{1.9}$$

Where $*$ denotes the usual Hadamard product or convolution of power series and

$$\phi_1(z) = \frac{(1+\gamma)z - \gamma z^2}{(1+\gamma)(1-z)^2}, \; \phi_2(z) = \frac{(\gamma-1)z - \gamma z^2}{(1+\gamma)(1-z)^2} \tag{1.10}$$

By specializing the parameters γ and n we obtain the following operators studied by various authors for $f \in A$

$$(i) I^n_0 f(z) = D^n f(z) \; (ii) I^n_\lambda f(z) \; (iii) I^n_1 = I^n f(z) \tag{1.11}$$

Motivated by the earlier works of [4, 7, 11–13] now we define the class of harmonic convex functions of complex order in the following definition.

Definition 1.1. *For $0 \le \gamma < 1, 0 \le \lambda \le \frac{\gamma}{(1+\gamma)}$ or $\lambda \ge \frac{1}{1+\gamma}$ and $b \in \mathbb{C} \setminus \{0\}$, let $\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$ denote the family of harmonic functions $f \in \mathcal{S}_{\mathcal{H}}$ of the form (1.1) which satisfy the condition*

$$\Re \left(1 + \frac{1}{b} \left(\frac{\mathcal{F}(z)}{\mathcal{G}(z)} - 1 \right) \right) \ge \gamma, \tag{1.12}$$

where

$$\mathcal{F}(z) = \lambda(z^3 (I^n_\gamma h(z))''' - \overline{z^3 (I^n_\gamma g(z))'''}) + (2\lambda+1) z^2 (I^n_\gamma (h(z))'' + (1-4\lambda)\overline{z^2 (I^n_\gamma g(z))''} + z(I^n_\gamma h(z))' + (1-2\lambda)\overline{z(I^n_\gamma g(z))'}$$

and

$$\mathcal{G}(z) = \lambda(z^2 (I^n_\gamma (h(z))'' + \overline{z^2 (I^n_\gamma g(z))''}) + z(I^n_\gamma h(z))' + (2\lambda - 1)\overline{z(I^n_\gamma g(z))'}$$

for $z \in \mathcal{U}$. Further, we define the subclass $T\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$ of $\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$ consisting of functions $f = h + \bar{g}$ of the form (1.2).

We observe that for $b = 1$ the class was introduced and studied by first author with Öztürk [12], the class $\mathcal{SC}_{\mathcal{H}}(1,\gamma,0,0) = \mathcal{SC}_{\mathcal{H}}(\gamma)$ is given in [5, 6] and $\mathcal{SC}_{\mathcal{H}}(1,0,0,0) = \mathcal{SC}_{\mathcal{H}}$ see [2].

In this paper, we investigate coefficient conditions, extreme points and distortion bounds for functions in the families $T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$. We also examine their convolution and convex combination properties and neighborhood result. Further, we obtain the closure property of this class under integral operator. We remark that the results so obtained for these general families can be viewed as extensions and generalizations for various subclasses of $\mathcal{S}_{\mathcal{H}}$ as listed previously in this section.

2 Main results

3 Coefficient inequalities

Our first theorem gives a sufficient condition for functions in $\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$.

Theorem 3.1. *Let $f = h + \bar{g}$ be so that h and g are given by (1.1). If*

$$\sum_{k=1}^{\infty} \frac{(\frac{k+\gamma}{1+\gamma})^n k(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty} \frac{(\frac{k-\gamma}{1+\gamma})^n k(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \le 2, \tag{3.13}$$

where $a_1 = 1,\ 0 \le \gamma < 1,\ 0 \le \lambda \le \frac{\gamma}{1+\gamma}$ or $\lambda \ge \frac{1}{1+\gamma}$. Then $f \in \mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ and f is sense preserving, univalent harmonic in $\mathcal{U}$.

Proof. We show that $f \in \mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$. We only need to show that if (3.13) holds then the condition (1.12) is satisfied. In view of (1.1) the condition (1.12) takes the form

$$\Re\left(\frac{(1-\gamma)+\sum\limits_{k=2}^{\infty}\frac{(\frac{k+\gamma}{1+\gamma})^n k(k\lambda-\lambda+1)[(k-1)+b(1-\gamma)]}{b}|a_k|\frac{z^k}{z} - \sum\limits_{k=1}^{\infty}\frac{(\frac{k-\gamma}{1+\gamma})^n k(k\lambda+\lambda-1)[(k+1)-b(1-\gamma)]}{b}|b_k|\frac{\bar{z}^k}{z}}{1+\sum\limits_{k=2}^{\infty}(\frac{k+\gamma}{1+\gamma})^n k(k\lambda-\lambda+1)|a_k|\frac{z^k}{z}+\sum\limits_{k=1}^{\infty}(\frac{k-\gamma}{1+\gamma})^n k(k\lambda+\lambda-1)|b_k|\frac{\bar{z}^k}{z}}\right) = \Re\frac{1+A(z)}{1+B(z)}.$$

Setting

$$\frac{1+A(z)}{1+B(z)} = \frac{1+\omega(z)}{1-\omega(z)},$$

we will have $\Re\frac{1+A(z)}{1+B(z)} > 0$ if $|\omega(z)| < 1$,

$$\begin{aligned} \omega(z) &= \frac{A(z)-B(z)}{2+A(z)+B(z)} \\ &= \frac{-\gamma+\sum\limits_{k=2}^{\infty}(\frac{k+\gamma}{1+\gamma})^n k(k\lambda-\lambda+1)\left[\frac{[(k-1)+b(1-\gamma)]}{b}-1\right]|a_k|z^{k-1} - \sum\limits_{k=1}^{\infty}(\frac{k-\gamma}{1+\gamma})^n k(k\lambda+\lambda-1)\left[\frac{[(k+1)-b(1-\gamma)}{b}+1\right]|b_k|\frac{\bar{z}^k}{z}}{2-\gamma+\sum\limits_{k=2}^{\infty}(\frac{k+\gamma}{1+\gamma})^n k(k\lambda-\lambda+1)\left[\frac{[(k-1)+b(1-\gamma)]}{b}+1\right]|a_k|z^{k-1} - \sum\limits_{k=1}^{\infty}(\frac{k-\gamma}{1+\gamma})^n k(k\lambda+\lambda-1)\left[\frac{[(k+1)-b(1-\gamma)}{b}-1\right]|b_k|\frac{\bar{z}^k}{z}} \end{aligned}$$

This last expression is bounded above by 1 if and only if

$$\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{|b|}|a_k| + \sum_{k=1}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{|b|}|b_k| \le (1-\gamma).$$

Or, equivalently

$$\sum_{k=1}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \le 2$$

If $z_1 \neq z_2$, then for $\lambda \ge \frac{1}{1+\gamma}$ or $0 \le \lambda \le \frac{\gamma}{1+\gamma}$

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k(z_1^k-z_2^k)}\right| \\ &\ge 1-\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|} \ge 1-\frac{\sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|}{1-\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|} \\ &\ge 0,\end{aligned}$$

which proves univalence. Note that f is sense preserving in $\mathcal{U}$, for $0 \le \lambda \le \frac{\gamma}{1+\gamma}$ or $\lambda \ge \frac{1}{1+\gamma}$. This is because

$$\begin{aligned}|h'(z)| &\ge 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1-\sum_{k=2}^{\infty} k|a_k| \\ &\ge 1-\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| \\ &\ge \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \\ &> \sum_{k=1}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k||z|^{k-1} \\ &\ge \sum_{k=1}^{\infty} k(\frac{k+\gamma}{1+\gamma})^n|b_k||z|^{k-1} \ge |g'(z)|.\end{aligned}$$

The function

$$\begin{aligned}f(z) = z &+ \sum_{k=2}^{\infty}\frac{(1-\gamma)|b|}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}x_k z^k \\ &+ \sum_{k=1}^{\infty}\frac{(1-\gamma)|b|}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}\overline{y_k z^k},\end{aligned} \tag{3.14}$$

where $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$, show that the coefficient bound given by (3.13) is sharp. The functions of the form (3.14) are in $\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ because

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| + \frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|\right) \\ &= 1+\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 2.\end{aligned}$$

□

Theorem 3.2. *Let $f = h + \bar{g}$ be so that h and g are given by (1.2). Then $f \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \le 2, \tag{3.15}$$

where $a_1 = 1,\ 0 \le \gamma < 1,\ 0 \le \lambda \le \frac{\gamma}{1+\gamma}$ or $\lambda \ge \frac{1}{1+\gamma}$ and $b \in \mathbb{C} \setminus \{0\}$.

Proof. The 'if part' follows from Theorem 3.1 upon noting that the functions $\mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n) \subset \mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$. For the 'only if' part, we show that $f \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$. Then for $z = re^{i\theta}$ in $\mathcal{U}$ we obtain

$$\begin{aligned} &\Re\left(1+\frac{1}{b}\left(\frac{\mathcal{F}(z)}{\mathcal{G}(z)}-1\right)-\gamma\right) \\ = &\Re\left(\frac{(1-\gamma)z - \sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+b(1-\gamma)]}{b}|a_k|z^k - \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-b(1-\gamma)]}{b}|b_k|\overline{z}^k}{z-\sum_{k=2}^{\infty}k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)|a_k|z^k+\sum_{k=1}^{\infty}k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)|b_k|\overline{z}^k}\right) \\ \geq &\frac{(1-\gamma) - \sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+b(1-\gamma)]}{b}|a_k|r^{k-1} - \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-b(1-\gamma)]}{b}|b_k|r^{k-1}}{z-\sum_{k=2}^{\infty}k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)|a_k|r^{k-1}+\sum_{k=1}^{\infty}k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)|b_k|r^{k-1}} > 0, \end{aligned}$$

The above inequality must hold for all $z \in \mathcal{U}$. In particular, letting $z = r \to 1^-$ yields the required condition. □

As special cases of Theorem 3.2, we obtain the following two corollaries.

Corollary 3.1. *Let $f = h + \overline{g} \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,0,n)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{n(\frac{k+\gamma}{1+\gamma})^n[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty}\frac{n(\frac{k-\gamma}{1+\gamma})^n[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \leq 2.$$

Corollary 3.2. *Let $f = h + \overline{g} \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,1,n)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{n^2(\frac{k+\gamma}{1+\gamma})^n[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty}\frac{n^2(\frac{k-\gamma}{1+\gamma})^n[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \leq 2.$$

4 Extreme points and Distortion bounds

In this section, our first theorem gives the extreme points of the closed convex hulls of $\mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$.

Theorem 4.3. *Let $f = h + \overline{g} \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ if and only if f can be expressed as*

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k(z) + Y_k g_k(z)), \;\; z \in \mathcal{U}, \tag{4.16}$$

where $h_1(z) = z$,

$$h_k(z) = z - \frac{(1-\gamma)|b|}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}z^k \quad (k = 2,3,...)$$

and

$$g_k(z) = z + \frac{(1-\gamma)|b|}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}\overline{z}^k \quad (k = 1,2,3,...),$$

$$\sum_{k=1}^{\infty}(X_k + Y_k) = 1, \;\; X_k \geq 0, \;\; Y_k \geq 0.$$

In particular, the extreme points of $\mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (4.16), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}(X_k h_k(z) + Y_k g_k(z)) \\ &= \sum_{k=1}^{\infty}(X_k+Y_k)\,z - \sum_{k=2}^{\infty}\frac{(1-\gamma)|b|}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}X_k z^k \\ &\quad + \sum_{k=1}^{\infty}\frac{(1-\gamma)|b|}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}Y_k\overline{z}^k. \end{aligned}$$

Then

$$\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}\left(\frac{(1-\gamma)|b|}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}\right)X_k$$
$$+\sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}\left(\frac{(1-\gamma)|b|}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}\right)Y_k$$
$$=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\leq 1$$

and so $f\in clcoT\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$.

Conversely, suppose that $f\in clcoT\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$. Letting

$$X_1=1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k$$

where

$$X_k=\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|,\ k=2,3,\dots,$$

and

$$Y_k=\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|,\ k=1,2,\dots,$$

we obtain the require representation, since

$$\begin{aligned}
f(z) &= z-\sum_{k=2}^{\infty}|a_k|z^k+\sum_{k=1}^{\infty}|b_k|\overline{z}^k\\
&= z-\sum_{k=2}^{\infty}\frac{(1-\gamma)|b|X_k}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}z^k\\
&+ \sum_{k=1}^{\infty}\frac{(1-\gamma)|b|Y_k}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}\overline{z}^k\\
&= z-\sum_{k=2}^{\infty}(z-h_k(z))X_k-\sum_{k=1}^{\infty}(z-g_k(z))Y_k\\
&= \left(1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k\right)z+\sum_{k=2}^{\infty}h_k(z)X_k+\sum_{k=1}^{\infty}g_k(z)Y_k\\
&= \sum_{k=1}^{\infty}(X_kh_k(z)+Y_kg_k(z)).
\end{aligned}$$

□

The following theorem gives the distortion bounds for functions in $T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ which yields a covering result for this family.

Theorem 4.4. *Let* $f\in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ *then*

$$|f(z)|\leq(1+|b_1|)r+r^2\left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right)$$

and

$$|f(z)|\geq(1-|b_1|)r-r^2\left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right).$$

Proof. Let $f \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$, Taking the absolute value of f and then by Theorem3.14, we obtain

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|) \\
&\leq (1+|b_1|)r + \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}\left(\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|\right. \\
&\qquad \left. + \frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}\left(1-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{(1-\gamma)|b|}|b_1|\right)r^2 \\
&= (1+|b_1|)r + \left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]} - \frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right)r^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\geq (1-|b_1|)r - r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|) \\
&\leq (1-|b_1|)r - \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}\left(\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|\right. \\
&\qquad \left. + \frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|\right)r^2 \\
&\geq (1-|b_1|)r - \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}\left(1-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{(1-\gamma)|b|}|b_1|\right)r^2 \\
&= (1-|b_1|)r - \left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]} - \frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right)r^2.
\end{aligned}$$

The upper and lower bounds given in Theorem 4.4 are respectively attained for the following functions.

$$f(z) = z + |b_1|\bar{z} + \frac{1}{\Gamma(2)}\left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]} - \frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right)\bar{z}^2$$

and

$$f(z) = (1-|b_1|)z - \frac{1}{\Gamma(2)}\left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]} - \frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right)z^2,$$

□

The following covering result follows from the left hand inequality in Theorem 4.4.

Corollary 4.3. *If $f \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$, then*

$$\left\{\omega : |\omega| < 1 - \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]} - \left[1 - \frac{(2\lambda-1)[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}\right]|b_1|\right\}.$$

5 Convolution and Convex Combinations

In this section we show that the class $TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k$, we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty} |a_k||A_k| z^k + \sum_{k=1}^{\infty} |b_k||B_k| \overline{z}^k. \tag{5.17}$$

Using the definition, we show that the class $TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ is closed under convolution.

Theorem 5.5. *For $0 \le \delta < \gamma < 1$, let $f \in TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ and $F \in TSC_{\mathcal{H}}(b,\delta,\lambda,n)$. Then $f * F \in TSC_{\mathcal{H}}(b,\gamma,\lambda,n) \subset TSC_{\mathcal{H}}(b,\delta,\lambda,n)$.*

Proof. Let $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k$ be in $TSC_{\mathcal{H}}(b,\delta,\lambda)$. Then the convolution $f * F$ is given by (5.17). From the assertion that $f * F \in TSC_{\mathcal{H}}(b,\delta,\lambda)$, we note that $|A_k| \le 1$ and $|B_k| \le 1$. In view of Theorem 3.2 and the inequality $0 \le \delta \le \gamma < 1$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n (k\lambda - \lambda + 1)[(k-1) + |b|(1-\delta)]}{(1-\delta)|b|} |a_k||A_k| \\
&\qquad + \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n (k\lambda + \lambda - 1)[(k+1) - |b|(1-\delta)]}{(1-\delta)|b|} |b_k||B_k| \\
&\le \sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n (k\lambda - \lambda + 1)[(k-1) + |b|(1-\delta)]}{(1-\delta)|b|} |a_k| \\
&\qquad + \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n (k\lambda + \lambda - 1)[(k+1) - |b|(1-\delta)]}{(1-\delta)|b|} |b_k| \\
&\le \sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n (k\lambda - \lambda + 1)[(k-1) + |b|(1-\gamma)]}{(1-\gamma)|b|} |a_k| \\
&\qquad + \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n (k\lambda + \lambda - 1)[(k+1) - |b|(1-\gamma)]}{(1-\gamma)|b|} |b_k| \\
&\le 1.
\end{aligned}$$

by Theorem 3.2, $f \in TSC_{\mathcal{H}}(b,\gamma,\lambda)$. By the same token, we then conclude that $f * F \in TSC_{\mathcal{H}}(b,\gamma,\lambda,n) \subset TSC_{\mathcal{H}}(b,\delta,\lambda,n)$. □

Next, we show that the class $TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ is closed under convex combination of its members.

Theorem 5.6. *The class $TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ is closed under convex combinations.*

Proof. For i=1,2,3,.... Suppose that $f_i(z) \in TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ where f_i given by

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{i,k}| z^k + \sum_{k=1}^{\infty} |b_{i,k}| \overline{z}^k.$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \le t_i \le 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i,k}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i,k}| \right) \overline{z}^k.$$

Since,

$$\sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n (k\lambda - \lambda + 1)[(k-1) + |b|(1-\gamma)]}{(1-\gamma)|b|} |a_{i,k}|$$
$$+ \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n (k\lambda + \lambda - 1)[(k+1) - |b|(1-\gamma)]}{(1-\gamma)|b|} |b_{i,k}| \leq 1.$$

from the above equation we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n (k\lambda - \lambda + 1)[(k-1) + |b|(1-\gamma)]}{(1-\gamma)|b|} \sum_{i=1}^{\infty} t_i |a_{i,k}| \\
&\qquad + \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n (k\lambda + \lambda - 1)[(k+1) - |b|(1-\gamma)]}{(1-\gamma)|b|} \sum_{i=1}^{\infty} t_i |b_{i,k}| \\
&= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n (k\lambda - \lambda + 1)[(k-1) + |b|(1-\gamma)]}{(1-\gamma)|b|} |a_{i,k}| \right. \\
&\qquad \left. + \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n (k\lambda + \lambda - 1)[(k+1) - |b|(1-\gamma)]}{(1-\gamma)|b|} |b_{i,k}| \right\} \\
&\leq \sum_{i=1}^{\infty} t_i = 1
\end{aligned}$$

This is the condition required by (3.14) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in T\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$. □

6 Class Preserving Integral Operator

In this section, we consider the closure property of the class $T\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$ under the Bernardi integral operator $\mathcal{L}_c[f(z)]$ which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c} \int_0^z \xi^{c-1} f(\xi) d\xi \ (c > -1).$$

Theorem 6.7. *Let $f(z) \in T\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$, then $\mathcal{L}_c[f(z)] \in T\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$.*

Proof. From the representation of $\mathcal{L}_c[f(z)]$, if follows that

$$\begin{aligned}
\mathcal{L}_c[f(z)] &= \frac{c+1}{z^c} \int_0^z \xi^{c-1} h(\xi) d\xi + \overline{\frac{c+1}{z^c} \int_0^z \xi^{c-1} g(\xi) d\xi} \\
&= \frac{c+1}{z^c} \int_0^z \xi^{c-1} \left(\xi - \sum_{k=2}^{\infty} |a_k| \xi^k \right) d\xi + \overline{\frac{c+1}{z^c} \int_0^z \xi^{c-1} \left(\sum_{k=1}^{\infty} |b_k| \xi^k \right) d\xi} \\
&= z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k,
\end{aligned}$$

where $A_k = \frac{c+1}{c+k}\,|a_k|$ and $B_k = \frac{c+1}{c+k}\,|b_k|$. Hence

$$\sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}\left(\frac{c+1}{c+k}|a_k|\right)$$
$$+\sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n|k\lambda+\lambda-1|[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}\left(\frac{c+1}{c+k}|b_k|\right)$$
$$\leq \sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|$$
$$+\sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n|k\lambda+\lambda-1|[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \leq 1,$$

since $f \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$, therefore by Theorem 3.2, $\mathcal{L}_c[f(z)] \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$. □

Remark 6.1. *Specializing the parameter, the result discussed in this paper leads many subclasses discussed in [4, 5, 7, 11–13].*

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Existence of solutions of q-functional integral equations with deviated argument

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Abstract

In this paper, we study the existence of solutions for q-functional integral equations in Banach space $C[0,T]$. The existence and uniqueness of solutions for the problems are proved by means of the Banach contraction principle.

Keywords: q-functional integral equations; Banach contraction principle; Deviated argument; existence.

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1 Introduction

The quantum calculus or q-difference calculus is an old subject that was first developed by Jackson ([12],[13]), while basic definitions and properties can be found in [15]. Studies on q-difference equations appeared already at the beginning of the last century in intensive works especially by F H Jackson [14], R D Carmichael [6], T E Mason [19], C R Adams [1], W J Trjitzinsky [21] and other authors [5].
Recently, q-calculus has served as abridge between mathematics and physics. It has a lot of applications in mathematics and physics([7]-[9],[17],[22]).

In this paper, we are concerned with the q-functional integral equations

$$x(t) \; = \; g(t) \; + \; \int_0^t f_1 \, (t, \; s, \; x(\phi(s))) \, d_q s, \quad t \in [0,T] \tag{1.1}$$

and

$$x(t) \; = \; g(t) \; + \; f_2 \, (t, \; \int_0^t g(s, \; x(\phi(s))) \, d_q s), \quad t \in [0,T] \tag{1.2}$$

where ϕ is deviated function. The existence of continuous solutions of the q-functional integral equation (1.1) in the Banach space $C[0,T]$ will be proved. The monotonicity of the solution of the equation (1.1) will be studied. The existence of continuous solutions of the q-functional integral equation (1.2) in Banach space $C[0,T]$ will be proved.

2 preliminaries

Here, we give the definition of q-derivative and q-integral and some of their properties which is referred to ([2],[15]).

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Let $q \in (0,1)$ and define

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbf{R}$$

which is called The q- analogue of n.

Definition 2.1. *The q-derivative of a real valued function f is defined by*

$$D_q f(t) = \frac{d_q f(t)}{d_q t} = \frac{f(qt) - f(t)}{qt - t}, \qquad D_q f(0) = \lim_{t \to 0} D_q f(t)$$

Note that $\lim_{q \to 1} D_q f(t) = f'(t)$ if $f(t)$ is differentiable.
The higher order q-derivative are defined as

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \qquad n \in \mathbf{N}.$$

Definition 2.2. *Suppose $0 < a < b$. The definite q-integral is defined as*

$$I_q f(x) = \int_0^b f(x)\, d_q x = (1-q) b \sum_{j=0}^{\infty} q^j f(q^j b).$$

and

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x.$$

Similarly, we have

$$I_q^0 f(t) = f(t), \qquad I_q^n f(t) = I_q I_q^{n-1} f(t), \qquad n \in \mathbf{N}.$$

Theorem 2.1 (see [15]). **(Fundamental Theorem of q-Calculus)**
If $F(x)$ is an antiderivative of $f(x)$, and $F(x)$ is continuous at $x = 0$, then

$$\int_a^b f(x) d_q x = F(b) - F(a), \qquad 0 \le a < b \le \infty.$$

Theorem 2.2. *(see [4],[15]) For any function f one has*

$$D_q I_q f(x) = f(x). \tag{2.3}$$

Theorem 2.3. *(see [2]) Let f be a function defined on $[a,b]$, $0 \le a \le b$, and c is a fixed point in $[a,b]$. Assume that there exists, $0 \le \gamma < 1$ such that $x^\gamma f(x)$ is continuous on $[a,b]$. Let*

$$F(x) = \int_c^x f(t)\, d_q t, \qquad x \in [a,b].$$

Then $F(x)$ is a continuous function on $[a,b]$.

Lemma 2.1. *If*

$$F(t) = \int_0^t f(s)\, d_q s, \qquad \textit{for} \quad t \in [a,b],$$

is continuous, then for every $\epsilon > 0 \; \exists \; \delta > 0$, such that $t_2, t_2 \in [0,T], \mid t_2 - t_1 \mid < \delta$, then

$$|F(t_2) - F(t_1)| < \epsilon$$

i.e.,

$$\left| \int_0^{t_2} f(s)\, d_q s - \int_0^{t_1} f(s)\, d_q s \right| < \epsilon.$$

Lemma 2.2. *(see [18])*
(1) *If f and g are q-integrable on $[a,b]$, $\alpha \in R$, $c \in [a,b]$, then*

(i) $\int_a^b [f(x) + g(x)]\, d_q x = \int_a^b f(x)\, d_q x + \int_a^b g(x)\, d_q x,$

(ii) $\int_a^b \alpha\, f(x)\, d_q x = \alpha \int_a^b f(x)\, d_q x,$

(iii) $\int_a^b f(x)\, d_qx = \int_a^c f(x)\, d_qx + \int_c^b f(x)\, d_qx.$

(2) *If* $|f|$ *is* q*-integrable on the interval* $[0, x]$*, then*

$$\left| \int_0^x f(x)\, d_qx \right| \leq \int_0^x |f(x)|\, d_qx.$$

.

(3) *If* f *and* g *are* q*-integrable on* $[0, x]$, $f(x) \leq g(x)$, *for all* $x \in [0, x]$*, then*

$$\int_0^x f(x)\, d_qx \leq \int_0^x g(x)\, d_qx.$$

3 Main results

Let X be the class of all continuous functions, $x \in C[0, T]$ with the norm

$$\|x\| = \sup_{t \in [0,T]} |\, x(t)\, |\, .$$

First, we study the existence and uniqueness of the solution of the q-functional integral equation (1.1) and then we proved the monotonicity for the solution.

Consider the q-functional integral equation (1.1) under the following assumptions

(i) $g : [0, T] \to R$ is continuous.

(ii) $f_1 : [0, T] \times [0, T] \times R \to R$ is continuous.

(iii) f_1 satisfies the Lipschitz condition

$$|f_1(t, s, x) - f_1(t, s, y)| \leq k(t, s)\, |x - y|.$$

(iv)

$$\sup_t \int_0^t k(t, s)\, d_qs \leq K$$

Now for the existence of a unique continuous solution of the q-functional integral equation (1.1) we have the following theorem.

Theorem 3.4. *Let the assumptions (i)-(iv) be satisfied. If* $K < 1$*, then the* q*-functional integral equation (1.1) has a unique solution* $x \in C[0, T]$.

Proof. Define the operator F associated with the q-functional integral equation (1.1) by

$$Fx(t) = g(t) + \int_0^t f_1(t,\, s,\, x(\phi(s)))\, d_qs.$$

To show that $F : C[0, T] \to C[0, T]$, let $x \in C[0, T]$, $t_1, t_2 \in [0, T]$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |g(t_2) - g(t_1) + \int_0^{t_2} f_1(t_2,\, s,\, x(\phi(s)))d_qs - \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))d_qs| \\
&\leq |g(t_2) - g(t_1)| + |\int_0^{t_2} f_1(t_2,\, s,\, x(\phi(s)))\, d_qs - \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))\, d_qs| \\
&\leq |g(t_2) - g(t_1)| + |\int_0^{t_2} f_1(t_2,\, s,\, x(\phi(s)))\, d_qs - \int_0^{t_2} f_1(t_1,\, s,\, x(\phi(s)))\, d_qs| \\
&\quad + |\int_0^{t_2} f_1(t_1,\, s,\, x(\phi(s)))d_qs - \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))d_qs| \\
&\leq |g(t_2) - g(t_1)| + \int_0^{t_2} |\, f_1(t_2,\, s,\, x(\phi(s))) - f_1(t_1,\, s,\, x(\phi(s)))\, |\, d_qs \\
&\quad + |\int_0^{t_2} f_1(t_1,\, s,\, x(\phi(s)))d_qs - \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))d_qs\, |
\end{aligned}$$

applying Theorem (2.3) and Lemma (2.1), then we deduce that

$$F : C[0,T] \rightarrow C[0,T].$$

Let $x, y \in C[0,T]$, we have

$$\begin{aligned}
|Fx(t) - Fy(t)| &= |g(t) + \int_0^t f_1(t,\, s,\, x(\phi(s)))\, d_qs - g(t) - \int_0^t f_1(t,\, s,\, y(\phi(s)))\, d_qs| \\
&= |\int_0^t f_1(t,\, s,\, x(\phi(s)))\, d_qs - \int_0^t f_1(t,\, s,\, y(\phi(s)))\, d_qs| \\
&\leq \int_0^t |f_1(t,\, s,\, x(\phi(s))) - f_1(t,\, s,\, y(\phi(s)))|\, d_qs \\
&\leq \int_0^t k(t,s)\, |x(\phi(s)) - y(\phi(s))|\, d_qs \\
&\leq \|x - y\| \int_0^t k(t,s)\, d_qs \\
&\leq K\, \|x - y\|.
\end{aligned}$$

This means that F is contraction.
Applying Banach contraction principle ([10],[16]), then we deduce that there exists a unique solution $x \in C[0,T]$ of the q-functional integral equation (1.1). □

The following theorem prove the monotonicity for the solution of the q-functional integral equation (1.1).

Theorem 3.5. *Let the assumptions (i)-(iv) of Theorem (3.1) be satisfied. If $f_1(t,s,x(\phi(s)))$ and $g(t)$ are monotonic nonincreasing(nondecreasing) in t for each $t \in [0,T]$, then the q-integral equation (1.1) has a unique monotonic nonincreasing(nondecreasing) solution $x \in C[0,T]$.*

Proof. Let f, g be monotonic nonincreasing functions in $t \in [0,T]$, then
for $t_2 > t_1$

$$\begin{aligned}
x(t_2) &= g(t_2) + \int_0^{t_2} f_1(t_2,\, s,\, x(\phi(s)))\, d_qs \\
&\leq g(t_1) + \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))\, d_qs \\
&= x(t_1).
\end{aligned}$$

Hence,

$$x(t_2) \leq x(t_1).$$

Also, If f_1, g are monotonic nondecreasing functions in $t \in [0,T]$, then
for $t_2 > t_1$

$$\begin{aligned}
x(t_2) &= g(t_2) + \int_0^{t_2} f_1(t_2,\, s,\, x(\phi(s)))\, d_qs \\
&\geq g(t_1) + \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))\, d_qs \\
&= x(t_1).
\end{aligned}$$

Hence

$$x(t_2) \geq x(t_1).$$

□

Now, we study the existence and uniqueness of the solution of the q-functional integral equation

$$x(t) = g(t) + f_2\,(t, \int_0^t g(s,\ x(\phi(s)))\,d_q s), \quad t \in [0,T]$$

Consider the q-functional integral equation (1.2) under the following assumptions

(i) $g : [0,T] \to R$ is continuous.

(ii) $f_2 : [0,T] \times R \to R$ is continuous.

(iii) f_2 satisfies the Lipschitz condition

$$|f_2(t,x(t)) - f_2(t,y(t))| \leq k\,|x(t) - y(t)|.$$

(iv) g satisfies the Lipschitz condition

$$|g(s,x(t)) - g(s,y(t))| \leq l\,|x(t) - y(t)|.$$

For the existence of a unique continuous solution of the q-functional integral equation (1.2), we have the following theorem.

Theorem 3.6. *Let the assumptions (i)-(iv) be satisfied. If $klT < 1$, then the q-functional integral equation (1.2) has a unique solution $x \in C[0,T]$.*

Proof. Define the operator F associated with the q-functional integral equation (1.2) by

$$Fx(t) = g(t) + f_2(t, \int_0^t g(s,\ x(\phi(s)))\,d_q s).$$

To show that $F : C[0,T] \to C[0,T]$, let $x \in C[0,T]$, $t_1, t_2 \in [0,T]$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |(g(t_2) - g(t_1)) + (f_2(t_2, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_1} g(s,x(\phi(s)))d_q s))| \\
&\leq |\,g(t_2) - g(t_1)\,| + |f_2(t_2, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_1} g(s,x(\phi(s)))d_q s)| \\
&\leq |\,g(t_2) - g(t_1)\,| + |f_2(t_2, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_2} g(s,x(\phi(s)))d_q s)| \\
&\quad + |f_2(t_1, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_1} g(s,x(\phi(s)))d_q s)| \\
&\leq |\,g(t_2) - g(t_1)\,| + |f_2(t_2, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_2} g(s,x(\phi(s)))d_q s)| \\
&\quad + |\int_0^{t_2} g(s,x(\phi(s)))d_q s - \int_0^{t_1} g(s,x(\phi(s)))d_q s|
\end{aligned}$$

applying Theorem (2.3) and Lemma (2.1), then we deduce that

$$F : C[0,T] \to C[0,T].$$

Let $x, y \in C[0,T]$, we have

$$\begin{aligned}
|Fx(t) - Fy(t)| &= |\, g(t) + f_2(t, \int_0^t g(s, x(\phi(s)))\, d_q s) - g(t) - f_2(t, \int_0^t g(s, y(\phi(s)))\, d_q s)\,| \\
&= |\, f_2(t, \int_0^t g(s, x(\phi(s)))\, d_q s) - f_2(t, \int_0^t g(s, y(\phi(s)))\, d_q s)\,| \\
&\leq k\, |\int_0^t g(s, x(\phi(s)))\, d_q s - \int_0^t g(s, y(\phi(s)))\, d_q s\,| \\
&\leq k \int_0^t |\, g(s, x(\phi(s))) - g(s, y(\phi(s)))\,|\, d_q s \\
&\leq kl \int_0^t |\, x(\phi(s)) - y(\phi(s))\,|\, d_q s \\
&\leq klT\, \|x - y\|.
\end{aligned}$$

This means that F ([10]) is contraction .
Then F has a fixed point $x \in C[0,T]$ which proves that there exists a unique solution of the q-functional integral equation (1.2). □

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Reciprocal Graphs

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Abstract

Eigenvalue of a graph is the eigenvalue of its adjacency matrix. A graph G is reciprocal if the reciprocal of each of its eigenvalue is also an eigenvalue of G. The Wiener index $W(G)$ of a graph G is defined by $W(G) = \frac{1}{2}\sum\limits_{d\in D} d$ where D is the distance matrix of G. In this paper some new classes of reciprocal graphs and an upperbound for their energy are discussed. Pairs of equienergetic reciprocal graphs on every $n \equiv 0 \bmod (12)$ and $n \equiv 0 \bmod (16)$ are constructed. The Wiener indices of some classes of reciprocal graphs are also obtained.

Keywords: Eigenvalue, Energy, Reciprocal graphs, splitting graph, Wiener index.

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1 Introduction

Let G be a graph of order n and size m with the vertex set $V(G)$ labelled as $\{v_1, v_2, \ldots, v_n\}$. The set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of an adjacency matrix A of G is called its spectrum and is denoted by $spec(G)$. Non-isomorphic graphs with the same spectrum are called cospectral. Studies on graphs with a specific pattern in their spectrum have been of interest. Gutman and Cvetkovic studied the spectral structure of graphs having a maximal eigenvalue not greater than 2 in [5] and Balinska et.al have studied graphs with integral spectra in [2]. In [12] some new constructions of integral graphs are provided. Dias in [6] has identified graphs with complementary pairs of eigenvalues(eigenvalues λ_1 and λ_2 with $\lambda_1 + \lambda_2 = -1$). A graph G is reciprocal [20] if the reciprocal of each of its eigenvalue is also an eigenvalue of G. The first reference of a reciprocal graph appeared in the work of J.R. Dias in [6, 7] and the chemical molecules of Dendralene and Radialene have been discussed there in. In [20] some classes of reciprocal graphs have been identified. In [3] reciprocal graphs are also referred to as graphs with property R.

The energy of a graph G [1], denoted by $E(G)$ is the sum of the absolute values of its eigenvalues. Non-cospectral graphs with the same energy are called equienergetic. In [8, 9, 15] some bounds on energy are described. In [1] and [22, 23] a pair of equienergetic graphs are constructed for every $n \equiv 0 \,(\bmod\ 4)$ and $n \equiv 0 \,(\bmod\ 5)$ and in [10] we have extended it for $n = 6,\ 14,\ 18$ and $n \geq 20$. In [17] a pair of equienergetic graphs within the family of iterated line graphs of regular graphs and in [11] a pair of equienergetic graphs obtained from the cross product of graphs are described. In [13] a pair of equienergetic self-complementary graphs on n vertices is constructed for every $n = 4k$ and $n = 24t + 1$, $k \geq 2,\ t \geq 3$. A plethora of papers have been appeared dealing with this parameter in recent years.

The distance matrix of a connected graph G, denoted by $D(G)$ is defined as $D(G) = [d(v_i, v_j)]$ where $d(v_i, v_j)$ is the distance between v_i and v_j. The Wiener index $W(G)$ is defined by

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$W(G) = \frac{1}{2}\sum_{d \in D} d$. The chemical applications of this index are well established in [16, 18].

In this paper, we construct some new classes of reciprocal graphs and an upperbound for their energy is obtained. Pairs of equienergetic reciprocal graphs on $n \equiv 0 \bmod (12)$ and $n \equiv 0 \bmod (16)$ are constructed. The Wiener indices of some classes of reciprocal graphs are also obtained. These results are not found so far in literature.

2 Some new classes of reciprocal graphs

If A and B are two matrices then $A \otimes B$ denote the tensor product of A and B. We use the following properties of block matrices[4].

Lemma 2.1. *Let M, N, P and Q be matrices with M invertible. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$. Then $|S| = |M|\,|Q - PM^{-1}N|$. Moreover if M and P commutes then $|S| = |MQ - PN|$ where the symbol $|.|$ denotes the determinant.*

We consider the following operations on G.

Operation 1. *Attach a pendant vertex to each vertex of G. The resultant graph is called the pendant join graph of G.[Also referred to as G corona K_1 in [3].]*

Operation 2. *[19] Introduce n isolated vertices u_i, $i = 1$ to n and join u_i to the neighbors of v_i. The resultant graph is called the splitting graph of G.*

Operation 3. *In addition to G introduce two sets of n isolated vertices $U = \{u_i\}$ and $W = \{w_i\}$ corresponding to $V = \{v_i\}$, $i = 1$ to n. Join u_i and w_i to the neighbors of v_i and then w_i to the vertices in U corresponding to the neighbors of v_i in G for each $i = 1$ to n. The resultant graph is called the double splitting graph of G.*

Operation 4. *In addition to G introduce two more copies of G on $U = \{u_i\}$ and $W = \{w_i\}$ corresponding to $V = \{v_i\}$, $i = 1$ to n. Join u_i to the neighbors of v_i and then w_i to u_i for each $i = 1$ to n. The resultant graph is called the composition graph of G.*

Operation 5. *In addition to G introduce two more copies of G on $U = \{u_i\}$ and $W = \{w_i\}$ corresponding to $V = \{v_i\}$, $i = 1$ to n. Join w_i to the neighbors of v_i and vertices in U corresponding to the neighbors of v_i in G for each $i = 1$ to n.*

Lemma 2.2. *Let G be a graph on n vertices with $spec(G) = \{\lambda_1, \ldots, \lambda_n\}$ and H_i be the graph obtained from Operation i, $i = 1$ to 5. Then*

$$spec(H_1) = \left\{\frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2}\right\}_{i=1}^{n}$$

$$spec(H_2) = \left\{\left(\frac{1 \pm \sqrt{5}}{2}\right)\lambda_i\right\}_{i=1}^{n}$$

$$spec(H_3) = \left\{-\lambda_i, \left(1 \pm \sqrt{2}\right)\lambda_i\right\}_{i=1}^{n}$$

$$spec(H_4) = \left\{\lambda_i, \lambda_i \pm \sqrt{\lambda_i^2 + 1}\right\}_{i=1}^{n}$$

$$spec(H_5) = \left\{\lambda_i, \left(1 \pm \sqrt{2}\right)\lambda_i\right\}_{i=1}^{n}$$

Proof. The proof follows from Table 1 which gives the adjacency matrix of $H_i s$ for $i = 1$ to 5 and its spectrum, obtained using Lemma 2.1 and the spectrum of tensor product of matrices.

Table 1

Graph	Adjacency matrix	Spectrum
H_1	$\begin{bmatrix} A & I \\ I & 0 \end{bmatrix}$	$\left\{\frac{\lambda_i \pm \sqrt{\lambda_i^2+4}}{2}\right\}_{i=1}^{n}$
H_2	$\begin{bmatrix} A & A \\ A & 0 \end{bmatrix} = A \otimes \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\left\{\left(\frac{1\pm\sqrt{5}}{2}\right)\lambda_i\right\}_{i=1}^{n}$
H_3	$\begin{bmatrix} A & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix} = A \otimes \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$\left\{-\lambda_i, \left(1 \pm \sqrt{2}\right)\lambda_i\right\}_{i=1}^{n}$
H_4	$\begin{bmatrix} A & A & 0 \\ A & A & I \\ 0 & I & A \end{bmatrix}$	$\left\{\lambda_i, \lambda_i \pm \sqrt{\lambda_i^2+1}\right\}_{i=1}^{n}$
H_5	$\begin{bmatrix} A & 0 & A \\ 0 & A & A \\ A & A & A \end{bmatrix} = A \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\left\{\lambda_i, \left(1 \pm \sqrt{2}\right)\lambda_i\right\}_{i=1}^{n}$

□

Note: $H_3 = H_5$ when G is bipartite.

Theorem 2.1. *The pendant join graph of a graph G is reciprocal if and only if G is bipartite.*

Proof. Let G be a bipartite graph and H, its pendant join graph. Then, corresponding to a non-zero eigenvalue λ of G, $-\lambda$ is also an eigenvalue of G [4].
By Lemma 2.2, $spec(H) = \{\frac{\lambda\pm\sqrt{\lambda^2+4}}{2}, \lambda \in spec(G)\}$. Let $\alpha = \frac{\lambda+\sqrt{\lambda^2+4}}{2}$ be an eigenvalue of H. Then

$$\begin{aligned}\frac{1}{\alpha} &= \frac{2}{\lambda+\sqrt{\lambda^2+4}} \\ &= \frac{2\left(\lambda-\sqrt{\lambda^2+4}\right)}{\left(\lambda+\sqrt{\lambda^2+4}\right)\left(\lambda-\sqrt{\lambda^2+4}\right)} \\ &= \frac{2\left(\lambda-\sqrt{\lambda^2+4}\right)}{-4} \\ &= \frac{(-\lambda)+\sqrt{(-\lambda)^2+4}}{2}\end{aligned}$$

is an eigenvalue of H as $-\lambda$ is an eigenvalue of G. Similarly for $\alpha = \frac{\lambda-\sqrt{\lambda^2+4}}{2}$ also. The eigenvalues of H corresponding to the zero eigenvalues of G if any, are 1 and -1 which are self reciprocal. Therefore H is a reciprocal graph.
The converse can be proved by retracing the argument. □

Note 1. *This theorem enlarges the classes of reciprocal graphs mentioned in [20]. The claim in [20] that the pendant join graph of C_n is reciprocal for every n is not correct as C_n is not bipartite for odd n.*

Definition 2.1. *A graph G is partially reciprocal if $\frac{-1}{\lambda} \in spec(G)$ for every $\lambda \in spec(G)$.*

Examples:-

- Pendant join graph of any graph.
- Splitting graph of any reciprocal graph.

Theorem 2.2. *The splitting graph of G is reciprocal if and only if G is partially reciprocal.*

Proof. Let G be partially reciprocal and H be its splitting graph. Let $\alpha \in spec(H)$. Then by Lemma 3, $\alpha = \left(\frac{1\pm\sqrt{5}}{2}\right)\lambda$, $\lambda \in spec(G)$. Without loss of generality, take $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)\lambda$. Then $\frac{1}{\alpha} = \left(\frac{1-\sqrt{5}}{2}\right)\frac{-1}{\lambda}$. Thus $\frac{1}{\alpha} \in spec(H)$ as G is partially reciprocal and hence H is reciprocal.
Conversely assume that H is reciprocal. Then by the structure of $spec(H)$ as given by Lemma 2.2, G is partially reciprocal. □

Theorem 2.3. *Let G be a reciprocal graph. Then the double splitting graph and the composition graph of G are reciprocal if and only if G is bipartite.*

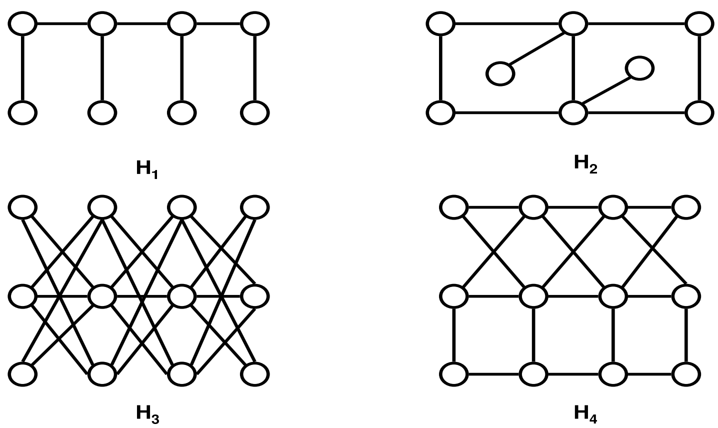
Proof. Let G be a bipartite reciprocal graph. Then $\lambda \in spec(G) \Rightarrow -\lambda, \frac{1}{\lambda}, \frac{-1}{\lambda} \in spec(G)$. Let H and H' respectively denote the double splitting graph and composition graph of G. Then using Lemma 2.2 and Table 2 it follows that H and H' are reciprocal.

Table 2

$Spec(H)$	$\frac{1}{spec(H)}$	$Spec(H')$	$\frac{1}{spec(H')}$
$\left\{-\lambda, \left(1 \pm \sqrt{2}\right) \lambda\right\}$	$\left\{-\frac{1}{\lambda}, \left(1 \pm \sqrt{2}\right) \frac{-1}{\lambda}\right\}$	$\left\{\lambda, \lambda \pm \sqrt{\lambda^2+1}\right\}$	$\left\{\frac{1}{\lambda}, -\lambda \pm \sqrt{(-\lambda)^2+1}\right\}$

Converse also follows. □

Illustration: The following graphs are reciprocal when $G = P_4$.



3 An upperbound for the energy of reciprocal graphs

The following bounds on the energy of a graph are known.

1. [15] $\sqrt{2m + n(n-1)\left|\det A\right|^{\frac{2}{n}}} E(G) \sqrt{2mn}$

2. [8] $E(G) \frac{2m}{n} + \sqrt{(n-1)\left(2m - 4\frac{m^2}{n^2}\right)}$

3. [9] $E(G) \frac{4m}{n} + \sqrt{(n-2)\left(2m - 8\frac{m^2}{n^2}\right)}$, if G is bipartite.

In this section we derive a better upperbound for the energy of a reciprocal graph and prove that the bound is best possible. A graph of order n and size m is referred to as an (n, m) graph.

Theorem 3.4. *Let G be an (n, m) reciprocal graph. Then $E(G) \leq \sqrt{\frac{n(2m+n)}{2}}$ and the bound is best possible for $G = tK_2$ and tP_4.*

Proof. Let G be an (n, m) reciprocal graph with $spec(G) = \{\lambda_1, \ldots, \lambda_n\}$.
Therefore $\sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} \frac{1}{|\lambda_i|} = E$ and $\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \frac{1}{\lambda_i^2} = 2m$.

Now we have [21]the following inequality for real sequences a_i, b_i and c_i, $1 \leq i \leq n$

$$\sum_{i=1}^{n} a_i c_i \sum_{i=1}^{n} b_i c_i \leq \frac{1}{2} \left\{ \sum_{i=1}^{n} a_i b_i + \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} b_i^2 \right)^{1/2} \right\} \sum_{i=1}^{n} c_i^2$$

Taking $a_i = |\lambda_i|, b_i = \frac{1}{|\lambda_i|}$ and $c_i = 1\ \forall i = 1, 2, \ldots, n,$

we have $[E(G)]^2 \leq \frac{1}{2}[n + 2m]\, n$ and hence $E(G) \leq \sqrt{\frac{n(2m+n)}{2}}$.
When $G = tK_2$, $n = 2t$, $m = t$, $E(G) = 2t$ and when $G = tP_4$, $n = 4t$, $m = 3t$, $E(G) = 2t\sqrt{5}$. □

4 Equienergetic reciprocal graphs

In this section we prove the existence of a pair of equienergetic reciprocal graphs on every $n = 12p$ and $n = 16p$, $p \geq 3$.

Theorem 4.5. *Let G be K_p and F_1 be the graph obtained by applying Operations 3, 1 and 2 on G and F_2, the graph obtained by applying Operations 5, 1 and 2 on G successively. Then F_1 and F_2 are reciprocal and equienergetic on $12p$ vertices.*

Proof. Let $G = K_p$. We have $spec(K_p) = \begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}$.
Let G_3 be the graph obtained by applying Operation 3 on G. Then by Lemma 2.2,
$spec(G_3) = \begin{pmatrix} -(p-1) & 1 & \left(1 \pm \sqrt{2}\right)(p-1) & -\left(1 \pm \sqrt{2}\right) \\ 1 & p-1 & \text{each once} & \text{each } p-1 \text{ times} \end{pmatrix}$.
Now, let G_{31} be the graph obtained by applying Operation 1 on G_3. Then by Lemma 2.2 $spec(G_{31})$

$$= \begin{pmatrix} \frac{p-1\pm\sqrt{(p-1)^2+4}}{2} & \frac{-1\pm\sqrt{5}}{2} & \frac{(1+\sqrt{2})(p-1)\pm\sqrt{\left\{(1+\sqrt{2})(p-1)\right\}^2+4}}{2} \\ \text{each once} & \text{each } p-1 \text{ times} & \text{each once} \\ \\ \frac{(1-\sqrt{2})(p-1)\pm\sqrt{\left\{(1-\sqrt{2})(p-1)\right\}^2+4}}{2} & \frac{(1+\sqrt{2})\pm\sqrt{\left\{(1+\sqrt{2})\right\}^2+4}}{2} & \frac{(1-\sqrt{2})\pm\sqrt{\left\{(1-\sqrt{2})\right\}^2+4}}{2} \\ \text{each once} & \text{each } p-1 \text{ times} & \text{each } p-1 \text{ times} \end{pmatrix}$$

Then

$$\begin{aligned} E(G_{31}) =& \sqrt{(p-1)^2+4} + \sqrt{5}\,(p-1) + \sqrt{\left\{\left(1+\sqrt{2}\right)(p-1)\right\}^2+4} \\ &+ \sqrt{\left\{\left(1-\sqrt{2}\right)(p-1)\right\}^2+4} + (p-1)\left[\sqrt{\left(1+\sqrt{2}\right)^2+4} + \sqrt{\left(1-\sqrt{2}\right)^2+4}\right] \\ =& \sqrt{(p-1)^2+4} + \sqrt{5}\,(p-1) + (p-1)\sqrt{14+2\sqrt{41}} \\ &+ \sqrt{6\,(p-1)^2+8+2\sqrt{(p-1)^4+24\,(p-1)^2+16}} \end{aligned}$$

Now, let F_1 be the graph obtained by applying Operation 2 on G_{31}. Then by Lemma 2.2,
$E(F_1) = \sqrt{5}E(G_{31})$. Let G_{51} be the graph obtained by applying Operations 5 and 1 on G successively and F_2 be that obtained by applying Operation 2 on G_{51}. Then we have
$E(F_2) = \sqrt{5}E(G_{51}) = \sqrt{5}E(G_{31}) = E(F_1)$. Also by Theorem 2, F_1 and F_2 are reciprocal. Thus the theorem follows. □

Lemma 4.3. *Let G be a non-bipartite graph on p vertices with $spec(G) = \{\lambda_1, \ldots, \lambda_p\}$ and an adjacency matrix A. Then the spectra of graphs whose adjacency matrices are*

$F' = \begin{bmatrix} A & A & A & A \\ A & A & 0 & A \\ A & 0 & A & A \\ A & A & A & 0 \end{bmatrix}$ *and* $H' = \begin{bmatrix} 0 & A & A & A \\ A & 0 & A & A \\ A & A & A & A \\ A & A & A & 0 \end{bmatrix}$ *are*
$\left\{\lambda_i, -\lambda_i, \left(\frac{3\pm\sqrt{13}}{2}\right)\lambda_i\right\}_{i=1}^{p}$ *and* $\left\{-\lambda_i, -\lambda_i, \left(\frac{3\pm\sqrt{13}}{2}\right)\lambda_i\right\}_{i=1}^{p}$ *respectively .*

Theorem 4.6. *Let G be K_p. Let T_1 and T_2 be the graphs obtained by applying Operations 1 and 2 successively on graphs associated with F' and H' respectively. Then T_1 and T_2 are reciprocal and equienergetic on $16p$ vertices.*

Proof. Let the graph associated with F' be also denoted by F' and F_1', the graph obtained by applying Operation 1 on F'. Then by a similar computation as in Theorem 5,

$$E(F_1') = 2\sqrt{(p-1)^2+4} + 2\sqrt{5}\,(p-1) + \sqrt{\left(\frac{11+3\sqrt{13}}{2}\right)(p-1)^2+4} + \sqrt{\left(\frac{11-3\sqrt{13}}{2}\right)(p-1)^2+4} + (p-1)\left[\sqrt{\left(\frac{11+3\sqrt{13}}{2}\right)+4} + \sqrt{\left(\frac{11-3\sqrt{13}}{2}\right)+4}\right]$$

and $E(T_1) = \sqrt{5}E(F_1') = \sqrt{5}E(H_1') = E(T_2)$, by Lemma 2.2. Also by Theorem 2, T_1 and T_2 are reciprocal. Hence the theorem. □

5 Wiener index of some reciprocal graphs

In this section we derive the Wiener indices of some classes of reciprocal graphs described in the earlier section. We shall denote by $D(G) = D$, the distance matrix of G and d_i, the sum of entries in the i^{th} row of D. The following theorem generalizes the results in [14].

Theorem 5.7. *Let G be a graph with Wiener index $W(G)$. Let H be the pendant join graph of G. Then $W(H) = 4W(G) + n(2n-1)$.*

Proof. We have, $W(G) = \frac{1}{2}\sum_{i=1}^{n} d_i$.
Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $U = \{u_1, u_2, \ldots, u_n\}$ be the corresponding vertices used in the pendant join of G. Then the distance matrix of H is as follows.

$$\left[\begin{array}{cccc|cccc} 0 & d(v_1,v_2) & ... & d(v_1,v_n) & 1 & 1+d(v_1,v_2) & ... & 1+d(v_1,v_n) \\ ... & ... & ... & ... & ... & ... & ... & ... \\ d(v_n,v_1) & ... & ... & 0 & 1+d(v_n,v_1) & ... & ... & 1 \\ \hline 1 & 1+d(v_1,v_2) & ... & 1+d(v_1,v_n) & 0 & 2+d(v_1,v_2) & ... & 2+d(v_1,v_n) \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 1+d(v_n,v_1) & ... & ... & ... & 2+d(v_n,v_1) & ... & ... & 0 \end{array}\right]$$

$$\begin{aligned} \text{since } d(v_i,u_j) &= 1; \text{ if } i = j \\ &= 1 + d(v_i,v_j); i \neq j \text{and} \\ d(u_i,u_j) &= d(u_i,v_i) + d(v_i,v_j) + d(v_j,u_j) \\ &= 2 + d(v_i,v_j) \end{aligned}$$

The row sum matrix of H is $\begin{bmatrix} 2d_1+n \\ \vdots \\ 2d_n+n \\ 2d_1+3n-2 \\ \vdots \\ 2d_n+3n-2 \end{bmatrix}$.

$$\text{Then } W(H) = \frac{1}{2}\left[\sum_{i=1}^{n}(2d_i+n)+\sum_{i=1}^{n}(2d_i+3n-2)\right]$$
$$= 4W(G)+n(2n-1). \text{ Hence the theorem.}$$

□

The proof techniques of the following theorems are on similar lines.

Theorem 5.8. *Let G be a triangle free* (n, m) *graph and H, its splitting graph. Then* $W(H) = 4W(G) + 2(m + n)$.

Corollory 5.1. *Let G be a triangle free* (n, m) *graph and F, the splitting graph of the pendant join graph of G. Then* $W(F) = 2[8W(G) + 4n^2 + (m + n)]$.

Theorem 5.9. *Let G be a triangle free* (n, m) *graph and H, its double splitting graph. Then* $W(H) = 9W(G) + 4m + 6n$.

Theorem 5.10. *Let G be a triangle free* (n, m) *graph and H, its composition graph. Then* $W(H) = 9W(G) + 2n^2 + 4n$.

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A Generalization of Natural Density

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Abstract

The concept of natural density is generalized. It is proved that the new theory is consistent with the existing theory in the literature. Many new results were obtained. A theorem analogous to the Riemann's theorem on rearrangement of non-absolutely convergent series is proved in the sense of generalized natural density. Some more possible generalizations are suggested.

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1 Introduction

We know that the set of even natural numbers and the set of natural numbers have same cardinality. In other words both the sets have equal number of elements and they have the same size. But intuitively we feel that the set of natural numbers is one half of the set of integers. This intuition is made into a mathematical concept called natural density [1]. In this paper we generalize this concept and derive some interesting results. We also suggest some more possible generalizations. Now we give some preliminary concepts which are available in the literature. As usual we use $\mathbb{N}$ to denote the set of natural numbers and $|S|$ to denote the cardinality of the set S.

Definition 1.1. *Let $A \subseteq \mathbb{N}$. Let $A(n) = \{1, 2, \ldots, n\} \cap A$ for all $n \in \mathbb{N}$. The upper density and the lower density of A are defined as $\limsup_{n\to\infty} \frac{|A(n)|}{n}$ and $\liminf_{n\to\infty} \frac{|A(n)|}{n}$ respectively; they are denoted by $\bar{d}(A)$ and $\underline{d}(A)$ respectively. The natural density $d(A)$ of A is defined as $\lim_{n\to\infty} \frac{|A(n)|}{n}$ if the limit exists.*

A has natural density if and only if $\bar{d}(A) = \underline{d}(A)$. We have some classical results:

- For any finite set A, $d(A) = 0$.
- for any $k \in \mathbb{N}$, $d(k\mathbb{N}) = \frac{1}{k}$ where $n\mathbb{N}$ is the set of all positive multiples of k.
- the infinite set $\{n^2 : n \in \mathbb{N}\}$ has density 0.

Further for any subsets A and B of $\mathbb{N}$, if $d(A)$ and $d(B)$ exist, then

- $d(A^c) = 1 - d(A)$.
- for any finite set F, $d(A - F) = d(A)$.
- $d(A \cup B) = d(A) + d(B) - d(A \cap B)$.

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- $d(kA) = \frac{1}{k}d(A)$, for $k \in \mathbb{N}$.
- $d(A+c) = d(A)$ for all constant $c \in \mathbb{N}$ where

$$A + c = \{a + c/a \in A\}.$$

- If

$$A = \bigcup_{n=0}^{\infty} \{2^{2n}, 2^{2n}+1, \ldots, 2^{2n+1}-1\},$$

  then $\overline{d}(A) = \frac{2}{3}$ and $\underline{d}(A) = \frac{1}{3}$; this shows the existence of a set for which natural density does not exist.

In Section 2, we give a generalization of the concept of natural density and in Section 3, we prove a theorem very similar to the Riemann's theorem on rearrangement of nonabsolutely convergent series. This very interesting theorem suggests us the generalization is a natural one and also that many classical theorems may have similar interpretations.

2 Generalization of Natural Density

We observe that the expression $\frac{|A(n)|}{n}$ is equal to $\frac{|A \cap X_n|}{|X_n|}$ where X_n is the set $\{1, 2, \ldots, n\}$ and that the sets X_n form an increasing sequence of subsets of the natural numbers whose union is the whole set of natural numbers. This motivates us the following definitions.

Definition 2.2. *Let $\mathscr{C} = \{X_n\}$ be any sequence of subsets of $\mathbb{N}$ such that $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$ and $\cup X_n = \mathbb{N}$. Then $\mathscr{C}$ is called a cover for $\mathbb{N}$.*

We simply write 'cover' instead of writing 'cover for $\mathbb{N}$'. We define the natural density in a generalized form in the following definition.

Definition 2.3. *The Upper density $\overline{d}_{\mathscr{C}}(A)$ and the lower density $\underline{d}_{\mathscr{C}}(A)$ of a subset A of $\mathbb{N}$ with respect to a cover $\mathscr{C}$ are defined as*

$$\overline{d}_{\mathscr{C}}(A) = \limsup_{n\to\infty} \frac{|A \cap X_n|}{|X_n|} \text{ and } \underline{d}_{\mathscr{C}}(A) = \liminf_{n\to\infty} \frac{|A \cap X_n|}{|X_n|}.$$

The density $d_{\mathscr{C}}(A)$ of A with respect to $\mathscr{C}$ is defined as

$$d_{\mathscr{C}}(A) = \lim_{n\to\infty} \frac{|A \cap X_n|}{|X_n|}$$

provided the limit exists.

If $X_n = \{1, 2, \ldots, n\}$, then we get the theory of natural density which is available in the literature. So the concept of natural density becomes a particular case of the new concept and the new theory is consistent with that available in the literature.

If $\mathscr{C}$ is any cover for $\mathbb{N}$ and if A and B are subsets of $\mathbb{N}$ such that $d_{\mathscr{C}}(A)$ and $d_{\mathscr{C}}(B)$ exist, then the following results follow from the definition.

- $d_{\mathscr{C}}(\mathbb{N}) = 1$.
- $d_{\mathscr{C}}(A^c) = 1 - d_{\mathscr{C}}(A)$ where A^c denote the complement of A in $\mathbb{N}$.
- for any finite set F, $d_{\mathscr{C}}(F) = 0$.
- for any finite set F, $d_{\mathscr{C}}(A - F) = d_{\mathscr{C}}(A)$.
- $d_{\mathscr{C}}(A \cup B) = d_{\mathscr{C}}(A) + d_{\mathscr{C}}(B) - d_{\mathscr{C}}(A \cap B)$.

Example 2.1. *Let $A = 2\mathbb{N}$. Let $X_n = \{1, 2, \ldots, n\}$ and $\mathscr{C}$ be the cover $\{X_n\}$. Then $d_{\mathscr{C}}(A)$ is the natural density, which is equal to $\frac{1}{2}$. Let $\mathscr{D}$ be the cover $\{X_n\}$ where*

$$X_n = \{1, 2, 3, \ldots, 2n+1, 2n+3, \ldots, 4n-1\}.$$

Then the sequence $\left(\frac{|A \cap X_n|}{|X_n|}\right)$ is, $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots$ which converge to $\frac{1}{3}$. That is, $d_{\mathscr{D}}(A) = \frac{1}{3}$.

This example shows that the density of a set may vary as the cover varies. In Theorem 3.2, we prove that for any real number α, $0 \leq \alpha \leq 1$, if A is an infinite set whose complement is also infinite, there is a cover $\mathscr{C}$ so that $d_{\mathscr{C}}(A) = \alpha$.

Let us consider another example.

Example 2.2. *Let $A = \{1, 3, 5, \dots\}$ and let $\mathscr{C}$ be the cover $\{X_n\}$ where*

$$X_n = \{1, 2, 3, \dots, 2n, 2(n+1), 2(n+2), \dots, 4n\}.$$

Then $d_{\mathscr{C}}(A)$ is $\frac{1}{3}$ and $d_{\mathscr{C}}(2A) = \frac{1}{3}$.

This example shows that, $d_{\mathscr{C}}(kA)$ need not be equal to $\frac{1}{k} d_{\mathscr{C}}(A)$ in contrast with the classical result $d(kA) = \frac{1}{k} d(A)$. Also it is easy to verify that $d_{\mathscr{C}}(A+1) = \frac{2}{3}$ which shows that, $d_{\mathscr{C}}(A)$ need not be equal to $d_{\mathscr{C}}(A+1)$ in contrast with the classical result $d(A+c) = d(A)$ for all constant $c \in \mathbb{N}$.

Theorem 2.1. *Let A be a subset of $\mathbb{N}$. Let $m_1 < m_2 < m_3 < \dots$ be an increasing sequence of natural numbers. Let $X_n = \{1, 2, \dots, m_n\}$ and $\mathscr{C} = \{X_n\}$. Then $\mathscr{C}$ is a cover of $\mathbb{N}$ and $d_{\mathscr{C}}(A) = d(A)$ provided $d(A)$ exists.*

Proof. Let $d(A)$ exist. Let $a_n = \frac{|A \cap \{1,2,\dots,n\}|}{n}$ and $b_n = \frac{|A \cap \{1,2,\dots,m_n\}|}{m_n}$. Then $d(A) = \lim_{n\to\infty} a_n$ which exists by our assumption. As (b_n) is a subsequence of (a_n), $d_{\mathscr{C}}(A) = \lim_{n\to\infty} b_n$ exists and is equal to $d(A)$. □

3 The Major Theorem

In this section, we prove a theorem which resembles the Riemann's theorem on rearrangements of series. First we recall Riemann's theorem on rearrangement of Series: If Σa_n is a nonabsolutely convergent series (Σa_n is convergent and $\Sigma |a_n|$ is not convergent) of real numbers and $-\infty \leq \alpha \leq \beta \leq \infty$, then there exists a rearrangement Σb_n of Σa_n with partial sum sequence (t_n) such that $\liminf_{n\to\infty} t_n = \alpha$ and $\limsup_{n\to\infty} t_n = \beta$.

We now state our main theorem.

Theorem 3.2. *If A is an infinite subset of $\mathbb{N}$ whose complement is also an infinite set and $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$, then there exists a cover $\mathscr{C}$ such that $\underline{d}_{\mathscr{C}}(A) = \alpha$ and $\overline{d}_{\mathscr{C}}(A) = \beta$.*

Proof. There exists a sequence of rational numbers in $[0,1]$ whose limit infimum is α limit supremum is β. Indeed if, $p_1, p_2, \dots$ and $q_1, q_2, \dots$ are sequences of rational numbers in $[0,1]$ converging to α and β respectively, then the sequence $p_1, q_1, p_2, q_2, \dots$ has the required property.

Let a and b be two rational numbers in $[0,1]$. Let a representation $\frac{m}{n}$ for a be given. Then we claim that there exists a representation $\frac{m'}{n'}$ for b such that $m \leq m'$ and $n < n'$. If $b = \frac{p}{q}$ is any representation of b, and if $m' = pmn$ and $n' = qmn$, then $b = \frac{m'}{n'}$ is a required representation of b, if at least one of m and n is different from 1. If $m = n = 1$, then $\frac{2p}{2q}$ will be a representation of b with the required property.

We claim that there exists a sequence

$$\frac{m_1}{n_1}, \frac{m_2}{n_2}, \frac{m_3}{n_3}, \dots$$

of rational numbers such that

$$m_1 \leq m_2 \leq m_3 \leq \dots, \quad n_1 < n_2 < n_3 < \dots,$$

and $m_i \leq n_i$ for all i, so that

$$\liminf_{k\to\infty} \frac{m_k}{n_k} = \alpha \text{ and } \limsup_{k\to\infty} \frac{m_k}{n_k} = \beta.$$

To prove this claim let $\alpha_1, \alpha_2, \alpha_3, \dots$ be a sequence of rational numbers in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n = \alpha$ and $\limsup_{n\to\infty} \alpha_n = \beta$. Taking α_1 and α_2 as a and b with the representation $\alpha_1 = \frac{m_1}{n_1}$ in our first claim, we get a representation $\alpha_2 = \frac{m_2}{n_2}$ such that $m_1 \leq m_2$ and $n_1 < n_2$. Taking α_2 and α_3 as a and b with the representation $\alpha_2 = \frac{m_2}{n_2}$ in the same claim we get a representation $\alpha_3 = \frac{m_3}{n_3}$ such that $m_2 \leq m_3$ and $n_2 < n_3$. Continuing in this way, we get a sequence with the required properties.

Let $B = \mathbb{N} - A$. Since A and B are infinite subset of $\mathbb{N}$, we can write the elements of the sets as infinite sequences:

$$A: \ a_1 < a_2 < a_3 < \ldots, \text{ and } B: \ b_1 < b_2 < b_3 < \ldots.$$

Let

$$X_k = \{a_1, a_2, a_3, \ldots, a_{m_k}, b_1, b_2, \ldots, b_{n_k - m_k}\}$$

for $k = 1, 2, 3, \ldots$. Then $\mathscr{C} = \{X_k\}$ is a cover with $\underline{d}_{\mathscr{C}}(A) = \alpha$ and $\overline{d}_{\mathscr{C}}(A) = \beta$. □

Corollary 3.1. *If A an infinite subset of $\mathbb{N}$ whose complement is also an infinite set and if $\alpha \in [0,1]$, then there exists a cover $\mathscr{C}$ such that $d_{\mathscr{C}}(A) = \alpha$.*

Conclusion

The theory developed here can be viewed as way to find the density of a set after assigning some weights to the natural numbers. If for some k and ℓ in $\mathbb{N}$, there is an n such that $k \in X_n$ and $\ell \notin X_n$, we may consider the weight of k is larger (or equal) than the weight of ℓ.

Moreover, in the existing literature our intuition that the set of positive even integers is half of the set of positive integers, is given a mathematical meaning. In the new theory the intuition by which the theory started fails. This is not an odd one in mathematics.

We started topology generalizing the concept of metric spaces. In the metric space $\mathbb{R}$, with usual topology the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \ldots$$

converges to 0 and only to 0. But the same sequence on $\mathbb{R}$ with the topology $\tau = \{\mathbb{R}, \emptyset, \{0\}\}$ converges to all real numbers other than 0 and it does not converge to 0, breaking our intuition that the sequence tends to 0. Likewise our theory also breaks some intuitions. Through this happens, the theory developed in this work has many similarities with the theory available in the literature of other branches of mathematics like Riemann's theorem on rearrangement of non-absolutely converging series. Some other types of densities and many open problems were discussed in [2, 3] and some of them can be studied in this new context.

We have discussed a generalization of the concept of natural density by replacing $\frac{|A\cap\{1,2,3,\ldots,n\}|}{n}$ by $\frac{|A\cap X_n|}{|X_n|}$ where $\{X_n\}$ is a sequence of subsets of $\mathbb{N}$ satisfying certain properties. Replacing $\frac{|A\cap\{1,2,3,\ldots,n\}|}{n}$ by $\frac{\mu(A\cap X_n)}{\mu(X_n)}$ where A and X_n are subsets of a measure space (X, μ), we can further generalize the concept of natural density to a very large setup. For example one may take $X = \mathbb{R}$, the Lebesgue measure on $\mathbb{R}$ as μ, and $\{X_n\}$ as an increasing sequence of sets with finite measure whose union is $\mathbb{R}$, and obtain new results like the set of positive real numbers is one half of the set of all real numbers and so on.

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Existence results for nonlinear fractional differential equation with nonlocal integral boundary conditions

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Abstract

In this paper, we shall study a nonlinear fractional differential equation with nonlocal integral boundary conditions. We have used fixed point theorems and Laray-Schauder nonlinear alternative to study the existence and uniqueness of solutions to the given equation. In the last, we have given examples to illustrate the applications of the abstract results.

Keywords: Fractional differential equations, Fixed point theorems, Laray-Schauder nonlinear alternative, Nonlocal boundary conditions.

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1 Introduction

Fractional differential equations are the generalization of ordinary differential equations to arbitrary non integer orders. The fact, that the fractional derivative(integral) is an operator which includes integer order derivatives(integrals) as special cases, is the reason why in present fractional differential equations becomes very popular and many applications are available. The fractional differential equations are of great importance because these are more precise in the modeling of many phenomenon, for instance, the nonlinear oscillations of earthquake can be described by the fractional differential equations. These differential equations are also very important to describe the memory and hereditary properties of various materials and phenomenon, this characteristic of fractional differential equations makes the fractional-order models more realistic and practical than the classical integer-order models. Recent work on fractional differential equations shows an overwhelming interest in this direction, for instance see [1–12] and the references cited therein. There have been many good books and monographs available on this field see [13–17].

On the other hand, the differential equations with a deviating argument are generalization of differential equations in which we permit the unknown function and its derivative to appear under different values of the argument. It is very important and significant branch of nonlinear analysis with numerous applications to physics, mechanics, control theory, biology, ecology, economics, theory of nuclear reactors, engineering, natural sciences and many other areas of science and technology. For a good introduction see [8, 18–21] and references cited therein.

The boundary value problem of fractional differential equations have been one of the hottest problems. Many problems related to blood flow, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so on can be reduced to nonlocal integral boundary problems. As a matter of fact, there are many papers dealing with the investigations on boundary value problems for some kinds of fractional differential equation with specific configurations covering theoretical as well as application aspects

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of the subject. In this consequence, Bai and Lu [12] studied the existence of positive solutions for the fractional boundary value problem using Krasnoselskii's fixed point theorem and the Leggett-William's fixed point theorem. They established the criteria on the existence of at least one or three positive solutions for the boundary value problem. Later on, Kaufmann and Mboumi[4] discussed the existence of positive solutions for the fractional boundary value problem and provide sufficient conditions for the existence of at least one and at least three positive solutions to the nonlinear fractional boundary value problem. In [23] Ahmad et. al investigated a boundary value problem of Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions using Krasnoselskii's fixed point theorem. In [7] Yan et. al studied the boundary value problems for fractional differential equations subject to nonlocal boundary condition using Banach's fixed point theorem and Schaefer's fixed point theorem. In [11] Zhong et. al investigated nonlocal and multiple-point boundary value problem for fractional differential equations and establish the conditions for the uniqueness of solutions as well as the existence of at least one solution. In [9] Murad et. al investigated the existence and uniqueness of solutions to the nonlinear fractional differential equation of an arbitrary order with integral boundary condition using Schauder fixed point theorem and the Banach contraction principle. In [1] Ahmad et. al discussed a new class of fractional boundary value problems and establish the results using Banach and Krasnoselskii's fixed point theorem. Authors in [1] also studied Riemann-Liouville fractional nonlocal integral boundary value problems in [2] by means of classical fixed point theorems. In [10] Ntouyas et. al. studied the boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions and obtained some new existence and uniqueness results by using fixed point theorems. In [6] Nyamoradi et. al investigate the existence of solutions for the multipoint boundary value problem of a fractional order differential inclusion on an infinite interval using suitable fixed point theorems. In [3] Ahmad et. al investigate the existence of solutions for higher order fractional differential inclusions with fractional integral boundary conditions involving nonintersecting finite many strips of arbitrary length using some standard fixed point theorems for multivalued maps. Akiladevi et.al [5] discuss the existence and uniqueness of solutions to the nonlinear neutral fractional boundary value problem using fixed point theorems. Recently, Zhao [25] studied triple positive solutions for two classes of delayed nonlinear fractional differential equation with nonlinear integral boundary value conditions using Leggett-Williams fixed point theorem and a generalization of Leggett-Williams fixed point theorem.

Motivated by the aforementioned techniques and papers, we have come to the conclusion that, although the fractional boundary value problems have been studied by many authors, but there is few gap in the literature on the boundary value problems with integral boundary conditions. In order to enhance the theoretical knowledge of the above, in this paper we intend to investigate the existence and uniqueness of solutions to the following Caputo-type fractional differential equation with deviated argument and nonlocal integral boundary conditions:

$$\begin{cases} {}^c\mathbf{D}^\gamma[z(t)-\mathcal{G}(t,z(t))] = \mathcal{F}(t,z(t),z[k(t,z(t))]), \quad 1<\gamma\leq 2, \quad t\in(0,1) \\ z(0)=0, \quad z(\tau)=\alpha\int_\eta^1 z(v)dv, \qquad 0<\tau<\eta<1, \end{cases} \tag{1.1}$$

where ${}^c\mathbf{D}^\gamma$ is the Caputo fractional derivative of order γ. $\mathcal{F}$, $\mathcal{G}$ and k are suitably defined functions satisfying certain conditions to be stated later and α is a positive real constant. The nonlocal integral boundary condition $z(\tau)=\alpha\int_\eta^1 z(v)dv$ shows that the integration over a sub-strip $(\eta,1)$ of an unknown function is proportional to the value of the unknown function at a nonlocal point $\tau\in(0,1)$ with $\tau<\eta<1$.

In this work, our main aim is to establish some existence and uniqueness results for the system (1.1) by using fixed point techniques which will provide an effective way to deal with such problems. Most of the existing articles are only devoted to study of fractional differential equation with nonlocal integral boundary conditions up until now Caputo-type fractional differential equation with deviated argument and nonlocal integral boundary conditions, has not been considered in the literature. In this paper, the first sufficient condition proving existence and uniqueness of the mild solution of (1.1) is derived by utilizing Banach fixed point theorem under Lipschitz continuity of nonlinear terms. The second sufficient condition proving existence of the mild solution of (1.1) is obtained via Krasnoselskii's fixed point theorem. The third sufficient condition is obtained by using Laray-Schauder nonlinear alternative under non-Lipschitz continuity of nonlinear terms.

2 Preliminaries

In this segment we discuss some basic definitions of fractional integration and differentiation and some lemmas which plays an important role in the further sections.

Definition 2.1. *[17] For a function $f \in L^1(\mathbb{R}^+)$, the fractional integral of order γ is described by*

$$I^{\gamma}_{0+}f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-v)^{\gamma-1}f(v)dv, \quad t>0, \quad \gamma>0.$$

Definition 2.2. *[13] For a function $f \in C^{m-1}(\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$, the Caputo fractional derivative of order γ is described by*

$${}^{c}\boldsymbol{D}^{\gamma}_{0+}f(t) = \frac{1}{\Gamma(m-\gamma)}\int_0^t (t-v)^{m-\gamma-1}f^{m}(v)dv,$$

where $m-1<\gamma<m$, $m=[\gamma]+1$ and $[\gamma]$ denotes the integral part of the real number γ.

Lemma 2.1. *[14] Let $q>0$, then*

$$D^{-\gamma}D^{\gamma}f(t) = f(t) + C_1t^{\gamma-1} + C_2t^{\gamma-2} + \ldots + C_nt^{\gamma-1},$$

for arbitrary $C_i \in \mathbb{R}$, $i=1,2,\ldots,n$, $n=[\gamma]+1$.

Lemma 2.2. *For any functions $\mathcal{F} \in C([0,1],\mathbb{R})$ and $\mathcal{G} \in C^1([0,1],\mathbb{R})$, the solution of following linear fractional boundary value problem*

$$\begin{aligned} {}^{c}\boldsymbol{D}^{\gamma}[z(t)-\mathcal{G}(t)] &= \mathcal{F}(t), \quad 1<\gamma\leq 2, \quad t\in(0,1) && (2.2)\\ z(0) &= 0, \quad z(\tau) = \alpha\int_{\eta}^{1} z(v)dv, \qquad 0<\eta<1, && (2.3)\end{aligned}$$

is defined by

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-v)^{\gamma-1}\mathcal{F}(v)dv - \mathcal{G}(0) + \mathcal{G}(t)\\ &+\frac{t}{\Lambda}\Big\{\mathcal{G}(0)(1-\alpha(1-\eta)) - \mathcal{G}(\tau) - \frac{1}{\Gamma(\gamma)}\int_0^{\tau}(\tau-v)^{\gamma-1}\mathcal{F}(v)dv\\ &+\alpha\int_{\eta}^{1}\mathcal{G}(v)dv + \frac{\alpha}{\Gamma(\gamma)}\int_{\eta}^{1}\Big(\int_0^{v}(v-u)^{\gamma-1}\mathcal{F}(u)du\Big)dv\Big\}, \end{aligned} \tag{2.4}$$

where

$$\Lambda = \tau - \frac{\alpha}{2}(1-\eta^2) \neq 0. \tag{2.5}$$

Proof. Using Lemma(2.1), the solution z of (2.2) given by

$$z(t) = I^{\gamma}\mathcal{F}(t) - \mathcal{G}(0) + \mathcal{G}(t) + C_2t + C_1, \tag{2.6}$$

for some constants $C_1, C_2 \in \mathbb{R}$.
On applying the boundary conditions (2.3), we get $C_1 = 0$ and

$$\begin{aligned} C_2 = \frac{1}{(\tau-\frac{\alpha}{2}(1-\eta^2))}&\Big\{\mathcal{G}(0)(1-\alpha(1-\eta)) - \mathcal{G}(\tau) - \frac{1}{\Gamma(\gamma)}\int_0^{\tau}(\tau-v)^{\gamma-1}\mathcal{F}(v)dv\\ &+\alpha\int_{\eta}^{1}\mathcal{G}(v)dv + \frac{\alpha}{\Gamma(\gamma)}\int_{\eta}^{1}\Big(\int_0^{v}(v-u)^{\gamma-1}\mathcal{F}(u)du\Big)dv\Big\}.\end{aligned}$$

Substituting the values of C_1 and C_2 in (2.6), we get (2.4). □

3 Existence and Uniqueness Results

Let $\mathcal{C} = C([0,1], \mathbb{R})$ be the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the norm

$$\|z\| = \sup_{t\in[0,1]} |z(t)|, \quad z \in \mathcal{C}.$$

Set,

$$\mathfrak{B} = \{z \in \mathcal{C} : |z(t) - z(v)| \leq L|t - v| \, \forall \, t, v \in [0,1]\},$$

where L is a positive constant.
With the help of Lemma (2.2), we introduce an operator $\Phi : \mathfrak{B} \to \mathfrak{B}$ as

$$\begin{aligned}(\Phi z)(t) = & \frac{1}{\Gamma(\gamma)} \int_0^t (t-l)^{\gamma-1} \mathcal{F}(l, z(l), z[k(l, z(l))]) dl + [\frac{t}{\Lambda}(1 - \alpha(1-\eta)) - 1]\mathcal{G}(0, z(0)) \\ & + \mathcal{G}(t, z(t)) + \frac{t}{\Lambda}\Big\{ - \mathcal{G}(\tau, z(\tau)) - \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau - l)^{\gamma-1} \mathcal{F}(l, z(l), z[k(l, z(l))]) dl \\ & + \alpha \int_\eta^1 \mathcal{G}(l, z(l)) dl + \frac{\alpha}{\Gamma(\gamma)} \int_\eta^1 \Big(\int_0^l (l-y)^{\gamma-1} \mathcal{F}(y, z(y), z[k(y, z(y))]) dy \Big) dl \Big\}, \end{aligned} \tag{3.7}$$

where Λ is given by (2.5). Here note that the boundary value problem (1.1) has solutions if and only if the operator Φ has fixed points.
Now, we introduce some assumptions which are required for the existence and uniqueness of the solution to boundary value problem (1.1).

(H1) The continuous function k is defined from $[0,1] \times \mathbb{R}$ to $\mathbb{R}$ with a constant $L_k > 0$ such that

$$|k(t, z) - k(t, x)| \leq L_k |z - x|.$$

(H2) The continuous function $\mathcal{F}$ is defined from $[0,1] \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ with a constant $L_f > 0$ such that

$$|\mathcal{F}(t, z, z[k(t, z(t))]) - \mathcal{F}(t, x, x[k(t, x(t))])| \leq L_f(2 + LL_k)|z - x|.$$

(H3) The continuously differentiable function $\mathcal{G}$ is defined from $[0,1] \times \mathbb{R}$ to $\mathbb{R}$ with a constant $L_g > 0$ such that

$$|\mathcal{G}(t, z) - \mathcal{G}(t, x)| \leq L_g |z - x|.$$

(H4) There exists $M_1(t)$ and $M_2(t) \in \mathcal{C}$ such that

$$|\mathcal{F}(t, z, z[k(t, z(t))])| \leq M_1(t),$$

and

$$|\mathcal{G}(t, z)| \leq M_2(t).$$

Theorem 3.1. *Suppose* $(H1) - (H3)$ *hold with* $\delta_1 = L_f(2 + LL_k)\mu_1 + L_g\mu_2 < 1$, *where*

$$\mu_1 = \frac{1}{|\Lambda|}\Big(\frac{(|\Lambda| + \tau^\gamma)}{\Gamma(\gamma+1)} + \frac{\alpha(1 - \eta^{\gamma+1})}{\Gamma(\gamma+2)} \Big) \text{ and } \mu_2 = \Big(1 + \frac{1}{|\Lambda|}(1 + \alpha(1-\eta))\Big).$$

Then the boundary value problem (1.1) has a unique solution.

Proof. Let $\sup_{t\in[0,1]} |\mathcal{F}(t, 0, 0)| = N_1$, $\sup_{t\in[0,1]} |\mathcal{G}(t, 0)| = N_2$ and $B_r = \{z \in \mathfrak{B} : \|z\| \leq r\}$, where $r \geq \frac{\delta_2}{1-\delta_1}$ with

$$\delta_2 = N_1\mu_1 + N_2\mu_2 + \frac{1}{|\Lambda|}((1 - \alpha(1-\eta)) - 1)|\mathcal{G}(0, z(0))|).$$

Now we will show that $\Phi B_r \subset B_r$. For $z \in B_r, 0 \leq t \leq 1$, we have

$$\begin{aligned}
\|(\Phi z)(t)\| \leq & \sup_{t\in[0,1]} \Big\{ \frac{1}{\Gamma(\gamma)} \int_0^t (t-l)^{\gamma-1} |\mathcal{F}(l,z(l),z[k(l,z(l))]) - \mathcal{F}(l,0,0) + \mathcal{F}(l,0,0)| dl \\
& + [\frac{t}{|\Lambda|}(1-\alpha(1-\eta)) - 1] |\mathcal{G}(0,z(0))| + |\mathcal{G}(t,z(t)) - \mathcal{G}(t,0) + \mathcal{G}(t,0)| \\
& + \frac{t}{|\Lambda|} \Big\{ |\mathcal{G}(\tau,z(\tau)) - \mathcal{G}(\tau,0) + \mathcal{G}(\tau,0)| + \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau-l)^{\gamma-1} |\mathcal{F}(l,z(l),z[k(l,z(l))]) \\
& - \mathcal{F}(l,0,0) + \mathcal{F}(l,0,0)| dl + \alpha \int_\eta^1 |\mathcal{G}(l,z(l)) - \mathcal{G}(l,0) + \mathcal{G}(l,0)| dl \\
& + \frac{\alpha}{\Gamma(\gamma)} \int_\eta^1 \Big(\int_0^l (l-y)^{\gamma-1} |\mathcal{F}(y,z(y),z[k(y,z(y))]) - \mathcal{F}(y,0,0) + \mathcal{F}(y,0,0)| dy \Big) dl \Big\} \\
\leq & (L_f(2+LL_k)r + N_1)\mu_1 + (L_g r + N_2)\mu_2 + \frac{1}{|\Lambda|}((1-\alpha(1-\eta)) - 1)|\mathcal{G}(0,z(0))| \\
\leq & (L_f(2+LL_k)\mu_1 + L_g\mu_2)r + \Big[N_1\mu_1 + N_2\mu_2 + \frac{1}{|\Lambda|}((1-\alpha(1-\eta)) - 1)|\mathcal{G}(0,z(0))| \Big] \\
\leq & \delta_1 r + \delta_2 \leq r.
\end{aligned}$$

Thus $\Phi B_r \subset B_r$. Now for $z, x \in B_r$ and $t \in [0,1]$, we have

$$\begin{aligned}
\|\Phi z - \Phi x\| \leq & \sup_{t\in[0,1]} \Big\{ \frac{1}{\Gamma(\gamma)} \int_0^t (t-l)^{\gamma-1} |\mathcal{F}(l,z(l),z[k(l,z(l))]) - \mathcal{F}(l,x(l),x[k(l,x(l))])| dl \\
& + |\mathcal{G}(t,z(t)) - \mathcal{G}(t,x(t))| + \frac{t}{|\Lambda|} \Big\{ |\mathcal{G}(\tau,z(\tau)) - \mathcal{G}(\tau,x(\tau))| \\
& + \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau-l)^{\gamma-1} |\mathcal{F}(l,z(l),z[k(l,z(l))]) - \mathcal{F}(l,x(l),x[k(l,x(l))])| dl \\
& + \alpha \int_\eta^1 |\mathcal{G}(l,z(l)) - \mathcal{G}(l,x(l))| dl + \frac{\alpha}{\Gamma(\gamma)} \int_\eta^1 \Big(\int_0^l (l-y)^{\gamma-1} |\mathcal{F}(y,z(y),z[k(y,z(y))]) \\
& - \mathcal{F}(y,x(y),x[k(y,x(y))])| dy \Big) dl \Big\} \\
\leq & [L_f(2+LL_k)\mu_1 + L_g\mu_2]|z-x| \\
\leq & \delta_1 |z-x|.
\end{aligned}$$

Since $\delta_1 < 1$, $\|\Phi z - \Phi x\| < |z-x|$ i.e. Φ is a contraction mapping. Therefore by Banach contraction principle, the boundary value problem (1.1) has a unique solution. □

Krasnoselskii combined two main result(Schauder's theorem and the contraction mapping principle) of fixed-point theory and gave a new theorem called Krasnoselskii's fixed point theorem. Now we show existence of solution with the help of Krasnoselskii's fixed point theorem [24].

Theorem 3.2. ***(Krasnoselskii fixed point theorem [24])*** *Let X be a Banach space and B be a nonempty, closed and convex subset of X. Let Q_1 and Q_2 be two operators which maps B into X such that*

1. *$Q_1 x + Q_2 y \in B$, whenever $x, y \in B$,*
2. *Q_1 is completely continuous,*
3. *Q_2 is a contraction mapping.*

Then there exists $z \in B$ such that $z = Q_1 z + Q_2 z$.

Theorem 3.3. *Let $(H1) - (H4)$ hold with*

$$\delta = \left(\frac{(L_f(2+LL_k))}{|\Lambda|} \left[\frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{\alpha(1-\eta^{\gamma+1})}{\Gamma(\gamma+2)} \right] + L_g \left[1 + \frac{1}{|\Lambda|}(1+\alpha(1-\eta)) \right] \right) < 1.$$

Then there exists at least one solution on $[0,1]$ of the given boundary value problem (1.1).

Proof. Let $\sup_{t\in[0,1]} |M_i(t)| = \|M_i\|$ for $i = 1,2$, $M = \max\{M_1, M_2, \mathcal{G}(0,z(0))\}$ and $B_r = \{z \in \mathfrak{B} : \|z\| \leq r\}$, choose r such that

$$r \geq \|M\|\left[\mu_1 + \mu_2 + \frac{1}{|\Lambda|}(1-\alpha(1-\eta)) - 1\right].$$

Now, introduce the decomposition of the map Φ into Φ_1 and Φ_2 on B_r for $t \in [0,1]$ such that

$$\begin{aligned}
(\Phi_1 z)(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-l)^{\gamma-1}\mathcal{F}(l,z(l),z[k(l,z(l))])dl,\\
(\Phi_2 z)(t) &= [\frac{t}{\Lambda}(1-\alpha(1-\eta)) - 1]\mathcal{G}(0,z(0)) + \mathcal{G}(t,z(t))\\
&+\frac{t}{\Lambda}\Big\{ -\mathcal{G}(\tau,z(\tau)) - \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-l)^{\gamma-1}\mathcal{F}(l,z(l),z[k(l,z(l))])dl\\
&+\alpha\int_\eta^1 \mathcal{G}(l,z(l))dl + \frac{\alpha}{\Gamma(\gamma)}\int_\eta^1\Big(\int_0^l (l-y)^{\gamma-1}\mathcal{F}(y,z(y),z[k(y,z(y))])dy\Big)dl\Big\}.
\end{aligned}$$

For $y, x \in B_r$, we have

$$\begin{aligned}
\|\Phi_1 z + \Phi_2 x\| &\leq \sup_{t\in[0,1]}\Big\{\frac{1}{\Gamma(\gamma)}\int_0^t (t-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))])|dl + [\frac{t}{|\Lambda|}(1-\alpha(1-\eta)) - 1]|\mathcal{G}(0,x(0))|\\
&\quad +|\mathcal{G}(t,x(t))| + \frac{t}{|\Lambda|}\Big[|\mathcal{G}(\tau,x(\tau))| + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-l)^{\gamma-1}|\mathcal{F}(l,x(l),x[k(l,x(l))])|dl\\
&\quad +\alpha\int_\eta^1 |\mathcal{G}(l,x(l))|dl + \frac{\alpha}{\Gamma(\gamma)}\int_\eta^1\Big(\int_0^l (l-y)^{\gamma-1}|\mathcal{F}(y,x(y),x[k(y,x(y))])|dy\Big)dl\Big]\Big\}\\
&\leq \|M_1\|\mu_1 + \|M_2\|\mu_2 + [\frac{1}{|\Lambda|}(1-\alpha(1-\eta)) - 1]|\mathcal{G}(0,z(0))|\\
&\leq \|M\|\left[\mu_1 + \mu_2 + \frac{1}{|\Lambda|}(1-\alpha(1-\eta)) - 1\right]\\
&\leq r.
\end{aligned}$$

Thus $\Phi_1 z + \Phi_2 x \in B_r$. Now to show Φ_1 is continuous and compact. The continuity of $\mathcal{F}$ implies the continuity of Φ_1. Also

$$\begin{aligned}
\|(\Phi_1 z)(t)\| &\leq \sup_{t\in[0,1]}\Big\{\frac{1}{\Gamma(\gamma)}\int_0^t (t-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))])|dl\Big\}\\
&\leq \frac{\|M_1\|}{\Gamma(\gamma+1)},
\end{aligned}$$

i.e. map Φ_1 is uniformly bounded on B_r.
Now, we show that $\{\Phi_1 z(t) : z \in B_r\}$ is equicontinuous. Clearly $\{\Phi_1 z(t) : z \in B_r\}$ are equicontinuous at $t = 0$. For $t < t + h \leq 1$, $h > 0$, we have

$$\begin{aligned}
\|\Phi_1 z(t+h) - \Phi_1 z(t)\| &\leq \frac{1}{\Gamma(\gamma)}\|\int_0^{t+h}(t+h-l)^{\gamma-1}\mathcal{F}(l,z(l),z[k(l,z(l))])dl\\
&\quad - \int_0^t (t-l)^{\gamma-1}\mathcal{F}(l,z(l),z[k(l,z(l))])dl\|\\
&\leq \frac{1}{\Gamma(\gamma)}\int_0^t\Big[(t+h-l)^{\gamma-1} - (t-l)^{\gamma-1}\Big]\|\mathcal{F}(l,z(l),z[k(l,z(l))])\|dl\\
&\quad +\frac{1}{\Gamma(\gamma)}\int_t^{t+h}(t+h-l)^{\gamma-1}\|\mathcal{F}(l,z(l),z[k(l,z(l))])\|dl,
\end{aligned}$$

which tends to zero as $h \to 0$, thus the set $\{\Phi_1 z(t) : z \in B_r\}$ is equicontinuous. Therefore by Arzelà-Ascoli's theorem Φ_1 is completely continuous.

Next we prove that Φ_2 is a contraction. For this

$$\begin{aligned}
\|\Phi_2 z - \Phi_2 x\| \leq & \sup_{t\in[0,1]} \Big\{ |\mathcal{G}(t,z(t)) - \mathcal{G}(t,x(t))| + \frac{t}{|\Lambda|}\Big\{ |\mathcal{G}(\tau,z(\tau)) - \mathcal{G}(\tau,x(\tau))| + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-l)^{\gamma-1} \\
& |\mathcal{F}(l,z(l),z[k(l,z(l))]) - \mathcal{F}(l,x(l),x[k(l,x(l))])|dl + \alpha\int_\eta^1 |\mathcal{G}(l,z(l)) - \mathcal{G}(l,x(l))|dl \\
& + \frac{\alpha}{\Gamma(\gamma)}\int_\eta^1 \Big(\int_0^l (l-y)^{\gamma-1} |\mathcal{F}(y,z(y),z[k(y,z(y))]) - \mathcal{F}(y,x(y),x[k(y,x(y))])|dy \Big) dl \Big\} \\
\leq & \Big(\frac{(L_f(2+LL_k))}{|\Lambda|}\Big[\frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{\alpha(1-\eta^{\gamma+1})}{\Gamma(\gamma+2)}\Big] + L_g\Big[1 + \frac{1}{|\Lambda|}(1+\alpha(1-\eta))\Big]\Big)|z-x| \\
\leq & \delta|z-x|.
\end{aligned}$$

Since $\delta < 1$, $\|\Phi_2 z - \Phi_2 x\| < |z-x|$ i.e. Φ_2 is a contraction. Therefore by Krasnoselskii fixed point theorem, there exists at least one solution on $[0,1]$ of boundary value problem (1.1). □

In our next result we show the existence of solution with the help of Laray-Schauder nonlinear alternative [22].

Theorem 3.4. ***(Laray-Schauder nonlinear alternative [22])*** *Let U and $\overline{U}$ denote respectively the open and closed subset of a nonempty, closed and convex set B of a Banach space X such that $0 \in U$. Let $T : \overline{U} \to B$ be a continuous and compact operator. Then either*

(i) *T has a fixed point in $\overline{U}$, or*

(ii) *there exists a point $u \in \partial U$ such that $u = \varepsilon Tu$ for some $\varepsilon \in (0,1)$, where ∂U is the boundary of U.*

Theorem 3.5. *Let the following assumptions hold.*

(H5) *There exists continuous nondecreasing functions $\psi_1, \psi_2 : [0,\infty) \to (0,\infty)$ and $\theta_1, \theta_2 \in L^1([0,1], \mathbb{R}^+)$ such that*

- **(i)** $|\mathcal{F}(t,z,x)| \leq \theta_1(t)\psi_1(\|z\| + \|x\|)$,
- **(ii)** $|\mathcal{G}(t,z)| \leq \theta_2(t)\psi_2(\|z\|)$.

(H6) *There exists a constant $P > 0$ such that $\frac{P}{\Theta} \geq 1$, where*

$$\begin{aligned}
\Theta = \psi(\|P\|)\Big[& \theta_2(1) + I^\gamma\Big(\theta_2(1) + \frac{1}{|\Lambda|}(\theta_1(\tau) + \alpha\int_\eta^1 \theta_1(l)dl)\Big) \\
& + \frac{1}{|\Lambda|}\Big(((1-\alpha(1-\eta)) - 1) + \theta_2(\tau)\Big) + \alpha\int_\eta^1 \theta_2(l)dl\Big].
\end{aligned}$$

Then there exists at least one solution on $[0,1]$ of the given boundary value problem (1.1).

Proof. Clearly the operator $\Phi : \mathfrak{B} \to \mathfrak{B}$ defined by (3.7) is continuous. Firstly we show that the bounded sets in $\mathfrak{B}$ are mapped into the bounded sets in $\mathfrak{B}$ by the mapping Φ. For $r > 0$, let $B_r = \{z \in \mathfrak{B} : \|z\| \leq r\}$ be a bounded set in $\mathfrak{B}$. Thus for $z \in B_r$, we get

$$\begin{aligned}
\|(\Phi z)(t)\| \leq & \sup_{t\in[0,1]} \Big\{ \frac{1}{\Gamma(\gamma)}\int_0^t (t-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))])|dl + [\frac{t}{|\Lambda|}(1-\alpha(1-\eta)) - 1]|\mathcal{G}(0,z(0))| \\
& + |\mathcal{G}(t,z(t))| + \frac{t}{|\Lambda|}\Big\{ |\mathcal{G}(\tau,z(\tau))| + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))])|dl \\
& + \alpha\int_\eta^1 |\mathcal{G}(l,z(l))|dl + \frac{\alpha}{\Gamma(\gamma)}\int_\eta^1 \Big(\int_0^l (l-y)^{\gamma-1}|\mathcal{F}(y,z(y),z[k(y,z(y))])|dy\Big)dl \Big\} \\
\leq & \psi_1(2\|r\|)\int_0^1 \frac{(1-l)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(l)dl + \frac{1}{|\Lambda|}((1-a(1-\eta)) - 1)|\mathcal{G}(0,z(0))| + \psi_2(\|r\|)\theta_2(1) \\
& + \frac{1}{|\Lambda|}\Big\{ \psi_2(\|r\|)\theta_2(\tau) + \psi_1(2\|r\|)\int_0^\tau \frac{(\tau-l)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(l)dl \\
& + \alpha\psi_2(\|r\|)\int_\eta^1 \theta_2(l)dl + \alpha\psi_1(2\|r\|)\int_\eta^1 \Big(\int_0^l \frac{(l-y)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(y)dy\Big)dl \Big\},
\end{aligned}$$

choose $\psi(r) \leq \max\{\psi_1(2\|r\|), \mathcal{G}(0,z(0)), \psi_2(\|r\|)\}$, we obtain

$$\|(\Phi z)(t)\| \leq \psi(r)\Big[\theta_2(1) + I^\gamma\Big(\theta_2(1) + \frac{1}{|\Lambda|}(\theta_1(\tau) + \alpha\int_\eta^1 \theta_1(l)dl)\Big) + \frac{1}{|\Lambda|}\Big(((1-\alpha(1-\eta))-1) + \theta_2(\tau)\Big) + \alpha\int_\eta^1 \theta_2(l)dl\Big]. \tag{3.8}$$

Next, we will show that Φ maps bounded sets into equicontinuous sets in B_r. For this, let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $z \in B_r$, then

$$\begin{aligned}\|(\Phi z)(t_2) - (\Phi z)(t_1)\| \leq\;& \int_0^{t_2} \frac{(t_2-l)^{\gamma-1}}{\Gamma(\gamma)}|\mathcal{F}(l,z(l),z[k(l,z(l))])|dl + |\mathcal{G}(t_2,z(t_2))| \\ &- \int_0^{t_1} \frac{(t_1-l)^{\gamma-1}}{\Gamma(\gamma)}|\mathcal{F}(l,z(l),z[k(l,z(l))])|dl - |\mathcal{G}(t_1,z(t_1))| \\ &+ \frac{(t_2-t_1)}{|\Lambda|}\Big[(1-\alpha(1-\eta))\mathcal{G}(0,z(0)) + \mathcal{G}(\tau,z(\tau)) + \alpha\int_\eta^1 |\mathcal{G}(l,z(l))|dl \\ &+ \int_0^\tau (\tau-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))])|dl \\ &+ \alpha\int_\eta^1 \Big(\int_0^l (l-y)^{\gamma-1}|\mathcal{F}(y,z(y),z[k(y,z(y))])|dy\Big)dl\Big] \\ \leq\;& \psi_1(2\|r\|)\Big[\int_0^{t_1} \frac{(t_2-l)^{\gamma-1} - (t_1-l)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(l)dl + \int_{t_1}^{t_2} \frac{(t_2-l)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(l)dl \\ &+ \frac{|t_2-t_1|}{|\Lambda|}\Big(\int_0^\tau \frac{(\tau-l)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(l)dl + \alpha\int_\eta^1 \Big(\int_0^l \frac{(l-y)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(y)dy\Big)dl\Big)\Big] \\ &+ (\theta_2(t_2) - \theta_1(t_1))\psi_2(\|r\|) + \frac{|t_2-t_1|}{|\Lambda|}\Big(1-\alpha(1-\eta))|\mathcal{G}(0,z(0))| \\ &+ |\mathcal{G}(\tau,z(\tau))| + \alpha\psi_2(\|r\|)\int_\eta^1 \theta_2(l)dl\Big).\end{aligned}$$

Clearly, the right hand side does not depend on $z \in B_r$ and tends to zero as $t_2 \to t_1$. Thus by Arzelà-Ascoli theorem, Φ is compact and continuous.
Now, suppose z be the solution of the given problem. Then for $\varepsilon \in (0,1)$ and using (3.8), we get

$$\|z(t)\| = \|\varepsilon(\Phi z)(t)\| \leq \psi(\|z\|)\Big[\theta_2(1) + I^\gamma\Big(\theta_2(1) + \frac{1}{|\Lambda|}(\theta_1(\tau) + \alpha\int_\eta^1 \theta_1(l)dl)\Big) + \frac{1}{|\Lambda|}\Big(((1-\alpha(1-\eta))-1) + \theta_2(\tau)\Big) + \alpha\int_\eta^1 \theta_2(l)dl\Big],$$

which implies

$$\|z\| \leq \psi(\|z\|)\Big[\theta_2(1) + I^\gamma\Big(\theta_2(1) + \frac{1}{|\Lambda|}(\theta_1(\tau) + \alpha\int_\eta^1 \theta_1(l)dl)\Big) + \frac{1}{|\Lambda|}\Big(((1-\alpha(1-\eta))-1) + \theta_2(\tau)\Big) + \alpha\int_\eta^1 \theta_2(l)dl\Big].$$

Using assumption $(H6)$, we get P such that $\|z\| \neq P$. Set $V = \{z \in \mathcal{C} : \|z\| < P\}$.
Here the operator $\Phi : \overline{V} \to \mathcal{C}$ is continuous and completely continuous. For any V, there is no $z \in \partial V$ such that $z = \varepsilon\Phi z$ for some $\varepsilon \in (0,1)$. Using Laray-Schauder nonlinear alternative, we conclude that there exists a fixed point $z \in \overline{V}$ of operator Φ and this z is a solution of boundary value problem (1.1). □

4 Examples

In this section, we present some examples, which indicate how our abstract result can be applied to the problem.

Example(1): Consider the following fractional boundary value problem

$$\begin{cases} {}^c\mathbf{D}^{3/2}\Big[z(t) - \frac{e^{-t}}{1+16e^{-t}}\frac{|z(t)|+1}{2+|z(t)|}\Big] = \frac{1}{(t+7)^2}\Big(|z(t)| + |t(|z(t)|+1)| + 2\Big), \\ z(0) = 0, \quad z(1/4) = \int_{1/2}^{1} z(l)dl. \end{cases} \tag{4.9}$$

Here $\gamma = 3/2$, $\tau = 1/4$, $\alpha = 2$, $\eta = 1/2$, $\mathcal{G}(t,z(t)) = \frac{e^{-t}}{1+16e^{-t}}\frac{(|z(t)|+1)}{(2+|z(t)|)}$, $k(t,z(t)) = \frac{t}{(t+7)^2}(|z(t)|+1)$ and $\mathcal{F}(t,z(t),z[k(t,z(t))]) = \frac{1}{(t+7)^2}\Big(|z(t)| + |t(|z(t)|+1)| + 2\Big)$. Here $\Lambda = \tau - \frac{\alpha}{2}(1-\eta^2) = -1/2 \neq 0$.
Observe that

$$\begin{aligned} |k(t,z(t)) - k(t,x(t))| &\leq \frac{1}{49}|z-x|, \\ |\mathcal{F}(t,z,z[k(t,z(t))]) - \mathcal{F}(t,x,x[k(t,x(t))])| &\leq \frac{1}{(t+7)^2}\Big[|z| - |x| + |t|(|z| - |x|)\Big] \\ &\leq \frac{2}{49}|z-x|, \\ |\mathcal{G}(t,z(t)) - \mathcal{G}(t,x(t))| &\leq \Big|\frac{e^{-t}}{1+16e^{-t}}\Big|\Big|\frac{|z(t)|+1}{2+|z(t)|} - \frac{|x(t)|+1}{2+|x(t)|}\Big| \\ &\leq \frac{1}{17}|z-x|. \end{aligned}$$

Thus assumptions $(H1)$-$(H3)$ holds with $L_f(2+LL_k) = 2/49$ and $L_g = 1/17$ and we get $\delta_1 = .2210 < 1$. Using Theorem (3.1) we get (4.9) has a unique solution.
Example(2): Consider the fractional boundary value problem given by

$$\begin{cases} {}^c\mathbf{D}^{3/2}\Big[z(t) - \frac{1}{(t+7)^2}\sin z\Big] = \frac{1}{\pi^2\sqrt{(1+t)}}\Big(\sin z + \sin(t\sin z)\Big), \\ z(0) = 0, \quad z(1/4) = \int_{1/2}^{1} z(l)dl. \end{cases} \tag{4.10}$$

Here $\gamma = 3/2$, $\tau = 1/4$, $\alpha = 1$, $\eta = 1/2$, $\mathcal{G}(t,z(t)) = \frac{1}{(t+7)^2}\sin z$, $k(t,z(t)) = \frac{1}{\pi^2\sqrt{(1+t)}}t\sin z$ and
$\mathcal{F}(t,z(t),z[k(t,z(t))]) = \frac{1}{\pi^2\sqrt{(1+t)}}\Big(\sin z + \sin(t\sin z)\Big)$. Here $\Lambda = \tau - \frac{\alpha}{2}(1-\eta^2) = -1/8 \neq 0$.
Observe that

$$\begin{aligned} |k(t,z(t)) - k(t,x(t))| &\leq \frac{1}{\pi^2}|z-x|, \\ |\mathcal{F}(t,z,z[k(t,z(t))]) - \mathcal{F}(t,x,x[k(t,x(t))])| &\leq \frac{2}{\pi^2}|z-x|, \\ |\mathcal{G}(t,z(t)) - \mathcal{G}(t,x(t))| &\leq \frac{1}{49}|z-x|, \\ |\mathcal{F}(t,z,z[k(t,z(t))])| &\leq \frac{2}{\pi^2\sqrt{(1+t)}} = M_1(t), \\ |\mathcal{G}(t,z(t))| &\leq \frac{1}{(t+7)^2} = M_2(t). \end{aligned}$$

Thus conditions $(H1)$-$(H4)$ holds with $L_f(2+LL_k) = 2/\pi^2$ and $L_g = 1/49$ and we get $\delta = .8186 < 1$. Clearly the assumptions $(H1)$-$(H4)$ of Theorem (3.3) are satisfied. Therefore (4.10) has at least one solution on $[0,1]$.
Example(3): Consider the following fractional boundary value problem

$$\begin{cases} {}^c\mathbf{D}^{3/2}\Big[z(t) - \frac{1}{(t+11)^2}(|z|+1)\Big] = \frac{1}{(t+7)^2}\Big[|z| + |\sin(|z|+1)| + 2\Big], \\ z(0) = 0, \quad z(1/2) = \int_{3/4}^{1} z(l)dl. \end{cases} \tag{4.11}$$

Here $\gamma = 3/2$, $\tau = 1/4$, $\alpha = 1$, $\eta = 3/4$, $\mathcal{G}(t,z(t)) = \frac{1}{(t+11)^2}(|z|+1)$, $k(t,z(t)) = \frac{1}{(t+7)^2}\sin(|z|+1)$ and $\mathcal{F}(t,z(t),z[k(t,z(t))]) = \frac{1}{(t+7)^2}\Big[|z| + |\sin(|z|+1)| + 2\Big]$. Here $\Lambda = \tau - \frac{a}{2}(1-\eta^2) = 9/32 \neq 0$.

Observe that

$$
\begin{aligned}
|\mathcal{F}(t,z,z[k(t,z(t))])| &\leq \frac{1}{49}(2|z|+3),\\
|\mathcal{G}(t,z(t))| &\leq \frac{1}{121}(|z|+1).
\end{aligned}
$$

From $(H5)$ we get $\theta_1(t)=1$, $\psi_1(\|z\|+\|x\|)=\frac{1}{49}(2|z|+3)$, $\theta_2(t)=1$ and $\psi_2(\|z\|)=\frac{1}{121}(|z|+1)$. Also

$$
\begin{aligned}
\Theta &= \psi(\|M\|)\Big[\theta_2(1)+I^{\gamma}\Big(\theta_2(1)+\frac{1}{|\Lambda|}(\theta_1(\tau)+\alpha\int_{\eta}^{1}\theta_1(l)dl)\Big)\\
&\quad +\frac{1}{|\Lambda|}\Big(((1-\alpha(1-\eta))-1)+\theta_2(\tau)\Big)+\alpha\int_{\eta}^{1}\theta_2(l)dl\Big]\\
&= \psi(\|M\|)(8.0012).
\end{aligned}
$$

Using condition $\frac{P}{\Theta}\geq 1$, we found that there exists a constant P such that $P\geq .7274>0$, therefore assumptions $(H5)$ and $(H6)$ of Theorem (3.5) are fulfilled. Therefore (4.11) has at least one solution on $[0,1]$.

5 Conclusion

This paper has investigated the existence and uniqueness of solution to the Caputo-type fractional differential equation with deviated argument and nonlocal integral boundary conditions. The first sufficient condition proving existence and uniqueness of the mild solution of (1.1) is derived by utilizing Banach fixed point theorem under Lipschitz continuity of nonlinear terms. The second sufficient condition proving existence of the mild solution of (1.1) is obtained via Krasnoselskii's fixed point theorem. The third sufficient condition is obtained by using Laray-Schauder nonlinear alternative under non-Lipschitz continuity of nonlinear terms. At last, examples are provided to illustrate the applications of the abstract results.

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Interval criteria for oscillation of second-order impulsive delay differential equation with mixed nonlinearities

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Abstract

We obtain interval oscillation criteria for the second-order impulsive delay differential equation

$$(r(t)\Phi_\alpha(x'(t)))' + p(t)\Phi_\alpha(x(t-\tau)) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t-\tau)) = e(t),\ t \geq t_0,\ t \neq t_k,$$
$$x(t_k^{\ +}) = a_k x(t_k), \quad x'(t_k^{\ +}) = b_k x'(t_k),\ k = 1, 2, 3, \dots .$$

The results obtained in this paper extend some of the existing results. We have given some examples to illustrate our results.

Keywords: Interval oscillation; Impulse; Delay; Mixed nonlinearities.

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1 Introduction

Consider the second-order impulsive delay differential equation with mixed nonlinearities

$$(r(t)\Phi_\alpha(x'(t)))' + p(t)\Phi_\alpha(x(t-\tau)) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t-\tau)) = e(t),\ t \geq t_0,\ t \neq t_k,$$
$$x(t_k^{\ +}) = a_k x(t_k), \quad x'(t_k^{\ +}) = b_k x'(t_k),\ k = 1, 2, 3, \dots \tag{1.1}$$

where

$$x(t_k^-) := \lim_{t \to t_k^-} x(t), \quad x(t_k^+) := \lim_{t \to t_k^+} x(t),$$
$$x'(t_k^-) := \lim_{h \to 0^-} \frac{x(t_k + h) - x(t_k)}{h}, \quad x'(t_k^+) := \lim_{h \to 0^+} \frac{x(t_k + h) - x(t_k)}{h}.$$

$\Phi_*(s) := |s|^{*-1}s$, τ is a non negative constant, $\{t_k\}$ denotes the impulsive moment sequence with $0 \leq t_0 < t_1 < \dots < t_k < \dots$, $\lim_{k\to\infty} t_k = \infty$ and $t_{k+1} - t_k > \tau$ for $k = 1, 2, 3, \dots$.
Let $J \subset \mathbb{R}$ be an interval, we define

$$PLC(J, \mathbb{R}) := \{h : J \to \mathbb{R} \mid h \text{ is continuous on each interval } (t_k, t_{k+1}),$$
$$h(t_k^{\pm}) \text{ exists and } h(t_k) = h(t_k^-) \text{ for all } k \in \mathbb{N}\}.$$

For given t_0 and $\phi \in PLC([t_0 - \tau, t_0], \mathbb{R})$, we say $x \in PLC([t_0 - \tau, \infty), \mathbb{R})$ is a solution of equation(1.1) with the initial value ϕ if $x(t)$ satisfies equation(1.1) for $t \geq t_0$ and $x(t) = \phi(t)$ for $t \in [t_0 - \tau, t_0]$.

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A nontrivial solution of equation(1.1) is called *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*.

The theory of impulsive differential equations is an important branch of differential equations. The first paper in this theory is related to V. D. Milman and A. D. Mishkis in 1960 [14]. In recent years the oscillation theory of impulsive differential equations emerging as an important area of research, since such equations have applications in control theory, physics, biology, population dynamics, economics, etc. For further applications and questions concerning existence and uniqueness of solutions of impulsive differential equation, see for example Lakshmigantham et. al [10] and the references cited therein.

During the last decades, several oscillation results were established for different kinds of impulsive delay differential equations (see Agarwal and Karakoc [2]). Recently, interval oscillation of impulsive delay differential equations was attracting the interest of many researchers, see Guo et. al[5, 6] and Li and Cheung [11]. However, only very few interval oscillation results are available in the literature for " second order impulsive differential equations with delay ". For example, Huang and Feng [8] considered the second order delay differential equations with impulses

$$x''(t) + p(t)f(x(t-\tau)) = e(t),\ t \geq t_0,\ t \neq t_k,$$
$$x(t_k{}^+) = a_k x(t_k), \quad x'(t_k{}^+) = b_k x'(t_k),\ k = 1, 2, ...$$

and established some interval oscillation criteria which developed some known results for the equations without delay or impulses [4, 12, 18].

In [5], Guo et. al considered the second order mixed nonlinear impulsive differential equations with delay

$$\big(r(t)\Phi_\alpha(x'(t))\big)' + p_0(t)\Phi_\alpha(x(t)) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(t-\sigma)) = e(t),\ t \geq t_0,\ t \neq \tau_k,$$
$$x(\tau_k{}^+) = a_k x(\tau_k), \quad x'(\tau_k{}^+) = b_k x'(\tau_k),\ k = 1, 2, ...$$

and obtained some interval oscillation criteria which generalized the results in [13, 15, 17].

In [11], Li and Cheung established some interval oscillation criteria for the second order impulsive delay differential equations of the form

$$\big(p(t)(x'(t))\big)' + q(t)(x(t-\tau)) + \sum_{i=1}^{n} q_i(t)\Phi_{\alpha_i}(x(t-\tau)) = e(t),\ t \geq t_0,\ t \neq t_k,$$
$$x(t_k{}^+) = a_k x(t_k), \quad x'(t_k{}^+) = b_k x'(t_k),\ k = 1, 2, ...$$

Motivated mainly by [5, 6, 11], we establish some interval oscillation criteria for equation (1.1). We also provide two examples to illustrate the effectiveness of our results.

2 Main results

Throughout this paper, assume that the following conditions hold without further mention:

(A1) $r(t) \in C([t_0, \infty), (0, \infty))$ is non-decreasing, $p, q_i, e \in PLC([t_0, \infty), \mathbb{R})$, $i = 1, 2 \ldots, n$;

(A2) $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$ are constants;

(A3) α is a quotient of odd positive integers, $b_k \geq a_k > 0$, $k \in \mathbb{N}$ are constants.

let $k(s) := \max\{i : t_0 < t_i < s\}$ and for $c_j < d_j$, let $M_j := \max\{r(t) : t \in [c_j, d_j]\}$, $j = 1, 2$, $\Omega_j := \{\omega \in C^1[c_j, d_j] : \omega(t) \not\equiv 0,\ \omega(c_j) = \omega(d_j) = 0\}$, $j = 1, 2$. For two constants $c, d \notin \{t_k\}$ with $c < d$ and a function $\phi \in C([c, d], \mathbb{R})$, we define an operator $\Psi : C([c, d], \mathbb{R}) \to \mathbb{R}$ by

$$\Psi_c^d[\phi] = \begin{cases} 0, & \text{for } k(c) = k(d), \\ \phi(t_{k(c)+1})\theta(c) + \sum_{i=k(c)+2}^{k(d)} \phi(t_i)\varepsilon(t_i), & \text{for } k(c) < k(d), \end{cases}$$

where

$$\theta(c) = \frac{b_{k(c)+1}^\alpha - a_{k(c)+1}^\alpha}{(a_{k(c)+1}^\alpha (t_{k(c)+1} - c)^\alpha)}, \quad \varepsilon(t_i) = \frac{b_i^\alpha - a_i^\alpha}{(a_i^\alpha (t_i - t_{i-1})^\alpha)}.$$

where $\sum_s^t = 0$ if $s > t$.

In the discussion of the impulse moments of $x(t)$ and $x(t-\tau)$, we need to consider the following four cases for $k(c_j) < k(d_j)$,

$$(s_1)\ t_{k(c_j)} + \tau < c_j \text{ and } t_{k(d_j)} + \tau > d_j;\ (s_2)\ t_{k(c_j)} + \tau < c_j \text{ and } t_{k(d_j)} + \tau < d_j;$$
$$(s_3)\ t_{k(c_j)} + \tau > c_j \text{ and } t_{k(d_j)} + \tau > d_j;\ (s_4)\ t_{k(c_j)} + \tau > c_j \text{ and } t_{k(d_j)} + \tau < d_j,\ j = 1,2$$

and the three cases for $k(c_j) = k(d_j)$,

$$(\tilde{s_1})\ t_{k(c_j)} + \tau < c_j;\ (\tilde{s_2})\ t_{k(d_j)} + \tau < d_j;\ (\tilde{s_3})\ t_{k(d_j)} + \tau > d_j,\ j = 1,2.$$

Combining (s_*) with $(\tilde{s_*})$, we can get 12 cases. Throughout the paper, we study equation(1.1) under the case of combination of (s_1) with $(\tilde{s_1})$ only. The discussions for other cases are similar and so omitted.

Let us see some lemmas which will be useful to prove our main results.

Lemma 2.1. *[1] For any given n-tuple $\{\beta_1, \beta_2, \ldots, \beta_n\}$ satisfying $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$, there corresponds an n-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ such that*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \qquad \sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1. \tag{2.2}$$

Lemma 2.2. *[1] For any given n-tuple $\{\beta_1, \beta_2, \ldots, \beta_n\}$ satisfying $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$, there corresponds an n-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ such that*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \qquad \sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1. \tag{2.3}$$

Lemma 2.3. *[7] Suppose X and Y are non-negative, then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda - 1) Y^{\lambda}, \lambda > 1 \tag{2.4}$$

where equality holds if and only if $X = Y$.

Lemma 2.4. *Assume that for any $T \geq t_0$, there exists $c_j, d_j \notin \{t_k\}$, $j = 1,2$ such that $T < c_1 < d_1 \leq c_2 < d_2$ and*

$$\begin{aligned} p(t), q_i(t) &\geq 0, \quad t \in [c_1 - \tau, d_1] \cup [c_2 - \tau, d_2] \backslash \{t_k\},\ i = 1,2,3,...,n \\ e(t) &\leq 0, \quad t \in [c_1 - \tau, d_1] \backslash \{t_k\}, \\ e(t) &\geq 0, \quad t \in [c_2 - \tau, d_2] \backslash \{t_k\}. \end{aligned} \tag{2.5}$$

If $x(t)$ is a non-oscillatory solution of equation(1.1), then there exist the following estimations of $x(t-\tau)/x(t)$;

$$\begin{aligned} &(a)\, for \quad t \in (t_i + \tau, t_{i+1}], && \frac{x(t-\tau)}{x(t)} > \left(\frac{t - t_i - \tau}{t - t_i} \right), \\ &(b)\, for \quad t \in (t_i, t_i + \tau), && \frac{x(t-\tau)}{x(t)} > \left(\frac{t - t_i}{b_i(t + \tau - t_i)} \right), \\ &(c)\, for \quad t \in [c_j, t_{k(c_j)+1}], && \frac{x(t-\tau)}{x(t)} > \left(\frac{t - t_{k(c_j)} - \tau}{t - t_{k(c_j)}} \right), \\ &(d)\, for \quad t \in (t_{k(d_j)}, d_j], && \frac{x(t-\tau)}{x(t)} > \left(\frac{t - t_{k(d_j)}}{b_{k(d_j)}(t + \tau - t_{k(d_j)})} \right), \end{aligned} \tag{2.6}$$

where $i = k(c_j), ..., k(d_j) - 1$, $j = 1,2$.

Proof. Without loss of generality, we assume that $x(t) > 0$ and $x(t-\tau) > 0$ for $t \geq t_0$. In this case the selected interval of t is $[c_1, d_1]$. From equation(1.1) and (2.5), we obtain

$$\left[r(t) \Phi_{\alpha}(x'(t)) \right]' = e(t) - p(t) \Phi_{\alpha}(x(t-\tau)) - \sum_{i=1}^{n} q_i(t) \Phi_{\beta_i}(x(t-\tau)) \leq 0 \tag{2.7}$$

Hence $r(t)\Phi_\alpha(x'(t))$ is non-increasing on the interval $[c_1, d_1]\backslash\{t_k\}$.
Case(a): $t_i + \tau < t \le t_{i+1}$.
Then $(t-\tau, t) \subset (t_i, t_{i+1}]$ and hence there is no impulsive moment in $(t-\tau, t)$. For any $s \in (t-\tau, t)$, we have

$$x(s) - x(t_i^+) = x'(\xi_1)(s - t_i),\ \xi_1 \in (t_i, s).$$

Because of the facts that $x(t_i^+) > 0, \phi_\alpha(*)$ is an increasing function and $r(s)\Phi_\alpha(x'(s))$ is non-increasing on (t_i, t_{i+1}), we have

$$\phi_\alpha(x(s)) > \phi_\alpha(x'(\xi_1)(s-t_i)) = \frac{r(\xi_1)}{r(\xi_1)}\phi_\alpha(x'(\xi_1))(s-t_i)^\alpha,$$

and hence

$$\Phi_\alpha(x(s)) \ge \frac{r(s)\Phi_\alpha(x'(s))}{r(\xi_1)}(s-t_i)^\alpha.$$

Since $r(s)$ is positive and non-decreasing, the above inequality becomes

$$\phi_\alpha(x(s)) \ge \phi_\alpha(x'(s)(s-t_i)),\ \xi_1 \in (t_i, s).$$

Thus, we have

$$\frac{x'(s)}{x(s)} < \frac{1}{(s-t_i)}.$$

Integrating both sides of the above inequality from $t-\tau$ to t, we obtain

$$\frac{x(t-\tau)}{x(t)} > \left(\frac{t-t_i-\tau}{t-t_i}\right),\ t \in (t_i+\tau, t_{i+1}]. \tag{2.8}$$

Case(b): $t \in (t_i, t_i + \tau)$.
Then $t - \tau \in (t_i - \tau, t_i)$. ie, $t_i - \tau < t - \tau < t_i < t < t_i + \tau$. Then there is an impulsive moment t_i in $(t-\tau, t)$. Then we have,

$$x(t) - x(t_i^+) = x'(\xi_2)(t-t_i),\ \xi_2 \in (t_i, t).$$

Using the impulsive condition of equation(1.1) and the monotone properties of $r(t)$, $\phi_\alpha(t)$ and $r(t)\phi_\alpha(x'(t))$, we get

$$\begin{aligned}\phi_\alpha(x(t) - a_i x(t_i)) &\le \frac{r(t_i^+)\phi_\alpha(x'(t_i^+))}{r(\xi_2)}(t-t_i)^\alpha \\ &= \phi_\alpha(b_i x'(t_i))(t-t_i)^\alpha \\ \Rightarrow \phi_\alpha\left(\frac{x(t)}{x(t_i)} - a_i\right) &\le \phi_\alpha\left(b_i\frac{x'(t_i)}{x(t_i)}(t-t_i)\right)\end{aligned} \tag{2.9}$$

In addition, by mean value theorem on $[t_i - \tau, t_i]$, we have

$$x(t_i) - x(t_i - \tau) = x'(\xi_3)\tau,\ \xi_3 \in (t_i - \tau, t_i)$$
$$\text{and hence,}\quad \phi_\alpha(x(t_i)) > \phi_\alpha(x'(\xi_3)\tau)$$

By using the monotone properties of $r(t), \phi_\alpha(t)$ and $r(t)\phi_\alpha(x'(t))$, we have

$$\begin{aligned}&\phi_\alpha(x(t_i)) \ge \phi_\alpha(x'(t_i)\tau) \\ \Rightarrow\quad &\frac{x'(t_i)}{x(t_i)} < \frac{1}{\tau}\end{aligned} \tag{2.10}$$

From (2.9) and (2.10), we have,

$$\begin{aligned}&\phi_\alpha\left(\frac{x(t)}{x(t_i)} - a_i\right) \le \phi_\alpha\left(\frac{b_i(t-t_i)}{\tau}\right) \\ \Rightarrow\quad &\frac{x(t)}{x(t_i)} \le \frac{b_i(t-t_i+\tau)}{\tau}\end{aligned} \tag{2.11}$$

For some $s \in (t_i - \tau, t_i)$, we have

$$\begin{aligned}&x(s) - x(t_i - \tau) = x'(\xi_4)(s - t_i + \tau),\ \xi_4 \in (t_i - \tau, s) \\ \Rightarrow\quad &\phi_\alpha(x(s)) > \frac{r(\xi_4)\phi_\alpha(x'(\xi_4))}{r(\xi_4)}(s-t_i+\tau)^\alpha.\end{aligned}$$

Again by using the monotone properties of $r(t)$, $\phi_\alpha(t)$ and $r(t)\phi_\alpha(x'(t))$, we have

$$\begin{aligned} &\phi_\alpha(x(s)) \geq \phi_\alpha(x'(s)(s-t_i+\tau)) \\ \Rightarrow\ &\frac{x'(s)}{x(s)} < \frac{1}{(s-t_i+\tau)}. \end{aligned}$$

Integrating both sides of the above inequality from $t-\tau$ to t_i where $t \in (t_i, t_i+\tau)$, we have

$$\frac{x(t-\tau)}{x(t_i)} > \frac{t-t_i}{\tau}, \ t \in (t_i, t_i+\tau). \tag{2.12}$$

Hence, from (2.11) and (2.12), we have

$$\frac{x(t-\tau)}{x(t)} > \left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right), \ t \in (t_i, t_i+\tau).$$

Case(c): $t \in [c_1, t_{k(c_1)+1}]$.
Then $t-\tau \in [c_1-\tau, t_{k(c_1)+1}-\tau]$ and hence there is no impulsive moment in $(t-\tau, t)$.
For any $s \in (t-\tau, t)$ as in Case(a), we have

$$\phi_\alpha(x(s)) > \phi_\alpha(x'(\xi_5)(s-t_{k(c_1)}))$$

By the monotone properties of $\phi_\alpha(*)$ and $r(s)\Phi_\alpha(x'(s))$, we have

$$\Phi_\alpha(x(s)) \geq \frac{r(s)\Phi_\alpha(x'(s))}{r(\xi_5)}(s-t_{k(c_1)})^\alpha.$$

Since $r(s)$ is positive and non decreasing, the above inequality becomes

$$\begin{aligned} &\phi_\alpha(x(s)) \geq \phi_\alpha(x'(s)(s-t_{k(c_1)})), \ \xi_5 \in (t_{k(c_1)}, s) \\ \Rightarrow\ &\frac{x'(s)}{x(s)} < \frac{1}{(s-t_{k(c_1)})} \end{aligned}$$

Integrating both sides of the above inequality from $t-\tau$ to t, we obtain

$$\frac{x(t-\tau)}{x(t)} > \left(\frac{t-t_{k(c_1)}-\tau}{t-t_{k(c_1)}}\right), \ t \in [c_1, t_{k(c_1)+1}].$$

Case(d): $t \in (t_{k(d_1)}, d_1]$.
Then $t-\tau \in (t_{k(d_1)}-\tau, d_1-\tau]$. ie, $t_{k(d_1)}-\tau < t-\tau < t_{k(d_1)} < t < t_{k(d_1)}+\tau$. Then there is an impulsive moment $t_{k(d_1)}$ in $(t-\tau, t)$. Making a similar analysis of Case(b), we obtain

$$\frac{x(t-\tau)}{x(t)} > \left(\frac{t-t_{k(d_1)}}{b_{k(d_1)}(t+\tau-t_{k(d_1)})}\right), \ t \in (t_{k(d_1)}, d_1].$$

When $x(t) < 0$, we can choose interval $[c_2, d_2]$ to study equation(1.1). The proof is similar and hence omitted. This completes the proof. □

Theorem 2.1. *Assume that for any $T \geq t_0$, there exists $c_j, d_j \notin \{t_k\}$, $j = 1, 2$, such that $T < c_1 < d_1 \leq c_2 < d_2$ and (2.5) holds. If there exists $\omega_j(t) \in \Omega_j(c_j, d_j)$, $j = 1, 2$ such that, for $k(c_j) < k(d_j)$,*

$$\begin{aligned} &\int_{c_j}^{t_{k(c_j)+1}} W_j(t)\left(\frac{t-t_{k(c_j)}-\tau}{t-t_{k(c_j)}}\right)^\alpha dt \\ &+ \sum_{i=k(c_j)+1}^{k(d_j)-1}\left[\int_{t_i}^{t_i+\tau} W_j(t)\left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right)^\alpha dt + \int_{t_i+\tau}^{t_{i+1}} W_j(t)\left(\frac{t-t_i-\tau}{t-t_i}\right)^\alpha\right] \\ &+ \int_{t_{k(d_j)}}^{d_j} W_j(t)\left(\frac{t-t_{k(d_j)}}{b_{k(d_j)}(t+\tau-t_{k(d_j)})}\right)^\alpha dt - \int_{c_j}^{d_j}(r(t)\left|\omega_j'(t)\right|^{\alpha+1})dt \\ &\geq M_j\Psi_{c_j}^{d_j}[\omega_j^{\alpha+1}], \end{aligned} \tag{2.13}$$

and for $k(c_j) = k(d_j)$,

$$\int_{c_j}^{d_j} \left(W_j(t) \left(\frac{t - c_j}{t - c_j + \tau} \right)^{\alpha} - r(t) \left| \omega_j'(t) \right|^{\alpha+1} \right) dt \geq 0, \tag{2.14}$$

where, $W_j(t) = Q(t)\omega_j^{\alpha+1}$, $j = 1, 2.$, *and*

$$Q(t) = \left(p(t) + \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) |e(t)|^{\eta_0} \right),$$

then equation (1.1) *is oscillatory.*

Proof. To arrive at a contradiction, let us suppose that $x(t)$ is a non-oscillatory solution of equation(1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(t - \tau) > 0$ for $t \geq t_0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$. We define

$$u(t) = r(t) \frac{\phi_\alpha(x'(t))}{x^\alpha(t)}, \; t \in [c_1, d_1]. \tag{2.15}$$

It follows that for $t \neq t_k$,

$$u'(t) = -\left(p(t) \frac{x^\alpha(t-\tau)}{x^\alpha(t)} + \frac{\sum_{i=1}^{n} q_i(t) \phi_{\beta_i}(x(t-\tau))}{x^\alpha(t)} + \frac{|e(t)|}{x^\alpha(t)} \right) - \alpha u(t) \frac{x'(t)}{x(t)} \tag{2.16}$$

for all $t \neq t_k$, $t \geq t_0$, and $u(t_k^+) = \frac{b_k}{a_k} u(t_k)$ for all $k \in \mathbb{N}$.
From the assumptions, we can choose $c_1, d_1 \geq t_0$ such that $p(t) \geq 0$ and $q_i(t) \geq 0$ for $t \in [c_1 - \tau, d_1]$, $i = 1, 2, \ldots, n$, and $e(t) \leq 0$ for $t \in [c_1 - \tau, d_1]$. By Lemma 2.1, there exist $\eta_i > 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \beta_i \eta_i = \alpha$ and $\sum_{i=1}^{n} \eta_i < 1$.
Define $\eta_0 := 1 - \sum_{i=1}^{n} \eta_i$ and let

$$\begin{aligned} u_0 &:= \eta_0^{-1} \left| \frac{e(t) x(t-\tau)}{x^\alpha(t)} \right| x^{-1}(t-\tau), \\ u_i &:= \eta_i^{-1} q_i(t) \frac{x(t-\tau)}{x^\alpha(t)} \, x^{\beta_i - 1}(t-\tau), \quad i = 1, 2, \ldots, n. \end{aligned}$$

Then by the arithmetic-geometric mean inequality (see Beckenbach and Bellman [3])

$$\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i}, \; u_i \geq 0, \text{ and } \eta_i > 0$$

we have

$$\begin{aligned} u'(t) \leq &- p(t) \frac{x^\alpha(t-\tau)}{x^\alpha(t)} - \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \frac{x^{\eta_i}(t-\tau)}{(x^{\eta_i}(t))^\alpha} x^{(\beta_i - 1)\eta_i}(t-\tau) |e(t)|^{\eta_0} \times \frac{x^{\eta_0}(t-\tau)}{(x^{\eta_0}(t))^\alpha} x^{-\eta_0}(t-\tau) \\ &- \frac{\alpha}{r^{1/\alpha}} u(t) \left(\frac{r(t)\phi_\alpha(x'(t))}{x^\alpha(t)} \right)^{1/\alpha} \frac{x'(t)}{\phi_\alpha(x'(t))^{1/\alpha}}, \; t \neq t_k. \end{aligned} \tag{2.17}$$

Since, by using Lemma(2.2), we have

$$\prod_{i=0}^{n} \frac{x^{\eta_i}(t-\tau)}{(x^{\eta_i}(t))^\alpha} = \frac{x^{\eta_0 + \eta_1 + \cdots + \eta_n}(t-\tau)}{(x^{\eta_0 + \eta_1 + \cdots + \eta_n}(t))^\alpha} = \frac{x(t-\tau)}{x^\alpha(t)}$$

and

$$\prod_{i=1}^{n} x^{(\beta_i - 1)\eta_i}(t-\tau) x^{-\eta_0}(t-\tau) = x^{\alpha - 1}(t-\tau),$$

the inequality (2.17) becomes

$$\begin{aligned} u'(t) &\leq -\left[p(t) + \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) |e(t)|^{\eta_0} \right] \times \frac{x^\alpha(t-\tau)}{x^\alpha(t)} - \frac{\alpha}{r^{1/\alpha}(t)} u^{\frac{1+\alpha}{\alpha}}(t), \\ &= -Q(t) \left(\frac{x(t-\tau)}{x(t)} \right)^\alpha - \frac{\alpha}{r^{1/\alpha}(t)} u^{1+\alpha/\alpha}(t), \; t \neq t_k \end{aligned} \tag{2.18}$$

where

$$Q(t)=\left(p(t)+\eta_0^{-\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}q_i^{\eta_i}(t)|e(t)|^{\eta_0}\right).$$

First we consider the case $k(c_1)<k(d_1)$. In this case the impulsive moments in $[c_1,d_1]$ are $t_{k(c_1)+1},t_{k(c_1)+2},\ldots,t_{k(d_1)}$. Choosing a $\omega_1(t)\in\Omega_1(c_1,d_1)$, multiplying both sides of (2.18) by $\omega_1^{\alpha+1}(t)$, and then integrating it from c_1 to d_1, we have

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)}\omega_1^{\alpha+1}(t_i)[u(t_i)-u(t_i^+)]\\
&\quad\le\int_{c_1}^{t_{k(c_1)+1}}\left[(\alpha+1)\left|\omega_1^{\alpha}(t)\omega_1'(t)\right||u(t)|-\frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{(1+\alpha)/\alpha}\omega_1^{\alpha+1}(t)\right]dt\\
&\quad+\sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{t_i}^{t_{i+1}}\left[(\alpha+1)\left|\omega_1^{\alpha}(t)\omega_1'(t)\right||u(t)|-\frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{(1+\alpha)/\alpha}\omega_1^{\alpha+1}(t)\right]dt\\
&\quad+\int_{t_{k(d_1)}}^{d_1}\left[(\alpha+1)\left|\omega_1^{\alpha}(t)\omega_1'(t)\right||u(t)|-\frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{(1+\alpha)/\alpha}\omega_1^{\alpha+1}(t)\right]dt\\
&\quad-\int_{c_1}^{t_{k(c_1)+1}}\left(\frac{x(t-\tau)}{x(t)}\right)^{\alpha}W_1(t)dt\\
&\quad-\sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{t_i}^{t_i+\tau}\left(\frac{x(t-\tau)}{x(t)}\right)^{\alpha}W_1(t)dt+\int_{t_i+\tau}^{t_{i+1}}\left(\frac{x(t-\tau)}{x(t)}\right)^{\alpha}W_1(t)dt\right]\\
&\quad-\int_{t_{k(d_1)}}^{d_1}\left(\frac{x(t-\tau)}{x(t)}\right)^{\alpha}W_1(t)dt.
\end{aligned}\tag{2.19}$$

where $W_1(t)=Q(t)\omega_1^{\alpha+1}$.
Letting

$$\lambda=1+\frac{1}{\alpha},\ X=\left(\frac{\alpha}{r^{1/\alpha}(t)}\right)^{\alpha/\alpha+1}|\omega_1^{\alpha}(t)||u(t)|\ \text{ and }\ Y=[\alpha r(t)]^{\alpha/\alpha+1}\left|\omega_1'(t)\right|^{\alpha},$$

and then by using Lemma(2.3), we get

$$(\alpha+1)\left|\omega_1^{\alpha}(t)\omega_1'(t)\right||u(t)|-\frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{(1+\alpha)/\alpha}\omega_1^{\alpha+1}(t)\le r(t)\left|\omega_1'(t)\right|^{\alpha+1}.\tag{2.20}$$

Meanwhile, for $t=t_k$, $k=1,2,\ldots$

$$u(t_k^+)=\left(\frac{b_k}{a_k}\right)^{\alpha}u(t_k).\tag{2.21}$$

Then the left hand side of the inequality(2.19) becomes

$$\sum_{i=k(c_1)+1}^{k(d_1)}\omega_1^{\alpha+1}(t_i)[u(t_i)-u(t_i^+)]=\sum_{i=k(c_1)+1}^{k(d_1)}\frac{a_i^{\alpha}-b_i^{\alpha}}{a_i^{\alpha}}\omega_1^{\alpha+1}(t_i)u(t_i).\tag{2.22}$$

Substituting (2.20) and (2.22) in (2.19), we get

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)}\frac{a_i^{\alpha}-b_i^{\alpha}}{a_i^{\alpha}}\omega_1^{\alpha+1}(t_i)u(t_i)\\
&\quad\le\int_{c_1}^{d_1}r(t)\left|\omega_1'(t)\right|^{\alpha+1}dt-\int_{c_1}^{t_{k(c_1)+1}}\left(\frac{x(t-\tau)}{x(t)}\right)^{\alpha}W_1(t)dt\\
&\quad-\sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{t_i}^{t_i+\tau}\left(\frac{x(t-\tau)}{x(t)}\right)^{\alpha}W_1(t)dt+\int_{t_i+\tau}^{t_{i+1}}\left(\frac{x(t-\tau)}{x(t)}\right)^{\alpha}W_1(t)dt\right]\\
&\quad-\int_{t_{k(d_1)}}^{d_1}\left(\frac{x(t-\tau)}{x(t)}\right)^{\alpha}W_1(t)dt.
\end{aligned}\tag{2.23}$$

On the other hand, for $t \in (t_{i-1}, t_i] \subset [c_1, d_1]$, $i = k(c_1) + 2, \dots, k(d_1)$, we have

$$x(t) - x(t_{i-1}) = x'(\xi)(t - t_{i-1}), \ \xi \in (t_{i-1}, t).$$

In view of $x(t_{i-1}) > 0$ and the monotone properties of $\phi_\alpha(t)$, $r(t)\phi_\alpha(x'(t))$ and $r(t)$ we obtain

$$\phi_\alpha(x(t)) > \phi_\alpha x'(\xi)\phi_\alpha(t - t_{i-1}) \geq \frac{r(t)}{r(\xi)}\phi_\alpha x'(t)\phi_\alpha(t - t_{i-1})$$
$$\Longrightarrow \qquad \frac{r(t)\phi_\alpha(x'(t))}{\phi_\alpha(x(t))} < \frac{r(\xi)}{(t - t_{i-1})^\alpha}.$$

Let $t \to t_i^-$, it follows that

$$u(t_i) = \frac{r(t_i)\phi_\alpha(x'(t_i))}{\phi_\alpha(x(t_i))} < \frac{M_1}{(t_i - t_{i-1})^\alpha}, \ i = k(c_1) + 2, \dots, k(d_1). \tag{2.24}$$

Making a similar analysis on $(c_1, t_{k(c_1)+1}]$, we get

$$u(t_{k(c_1)+1}) = \frac{r(t_{k(c_1)+1})\phi_\alpha(x'(t_{k(c_1)+1}))}{\phi_\alpha(x(t_{k(c_1)+1}))} < \frac{M_1}{(t_{k(c_1)+1} - c_1)^\alpha}. \tag{2.25}$$

Then from (2.24), (2.25) and (A_3), we have

$$\begin{aligned} \sum_{i=k(c_1)+1}^{k(d_1)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha} \omega_1^{\alpha+1}(t_i) u(t_i) &< M_1 \left[\omega_1^{\alpha+1}(t_{k(c_1)+1})\theta(c_1) + \sum_{i=k(c_1)+2}^{k(d_1)} \omega_1^{\alpha+1}(t_i)\varepsilon(t_i) \right] \\ &= M_1 \Psi_{c_1}^{d_1}\left[\omega_1^{\alpha+1}\right]. \end{aligned} \tag{2.26}$$

Hence, from (2.23) and (2.26) and applying Lemma (2.4), we obtain

$$\begin{aligned} &\int_{c_1}^{t_{k(c_1)+1}} W_1(t) \left(\frac{t - t_{k(c_1)} - \tau}{t - t_{k(c_1)}} \right)^\alpha dt \\ &+ \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{t_i}^{t_i+\tau} W_1(t) \left(\frac{t - t_i}{b_i(t + \tau - t_i)} \right)^\alpha dt + \int_{t_i+\tau}^{\tau_{i+1}} W_1(t) \left(\frac{t - t_i - \tau}{t - t_i} dt \right)^\alpha \right] \\ &+ \int_{t_{k(d_1)}}^{d_1} W_1(t) \left(\frac{t - t_{k(d_1)}}{b_{k(d_1)}(t + \tau - t_{k(d_1)})} \right)^\alpha dt - \int_{c_1}^{d_1} r(t) \left|\omega_1'(t)\right|^{\alpha+1} dt \\ &< M_1 \Psi_{c_1}^{d_1}\left[\omega_1^{\alpha+1}\right]. \end{aligned} \tag{2.27}$$

This contradicts (2.13).

Next we consider the case $k(c_1) = k(d_1)$. By the condition $(\tilde{s_1})$ we know there is no impulse moments in $[c_1, d_1]$. Multipling both sides of (2.18) by $\omega_1^{\alpha+1}(t)$, with ω as prescribed in the hypothesis of the theorem, and then integrating it from c_1 to d_1, we obtain

$$\int_{c_1}^{d_1} u'(t)\omega_1^{\alpha+1} dt \leq -\int_{c_1}^{d_1} \frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(\alpha+1)/\alpha} \omega_1^{\alpha+1}(t) dt - \int_{c_1}^{d_1} \left(\frac{x(t-\tau)}{x(t)} \right)^\alpha W_1(t) dt. \tag{2.28}$$

Using integration by parts on the left hand side and noting the condition $\omega_1(c_1) = \omega_1(d_1) = 0$, we obtain

$$\int_{c_1}^{d_1} \left[(\alpha+1)\omega_1^\alpha \omega_1'(t) u(t) - \frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(\alpha+1)/\alpha} \omega_1^{\alpha+1}(t) \right] dt - \int_{c_1}^{d_1} \left(\frac{x(t-\tau)}{x(t)} \right)^\alpha W_1(t) dt \geq 0. \tag{2.29}$$

It follows that

$$\int_{c_1}^{d_1} \left[(\alpha+1) \left|\omega_1^\alpha \omega_1'(t)\right| |u(t)| - \frac{\alpha}{r^{1/\alpha}(t)} \omega_1^{\alpha+1}(t) |u(t)|^{(\alpha+1)/\alpha} \right] dt - \int_{c_1}^{d_1} \left(\frac{x(t-\tau)}{x(t)} \right)^\alpha W_1(t) dt \geq 0. \tag{2.30}$$

Letting

$$\lambda = 1 + \frac{1}{\alpha}, \ X = \left(\frac{\alpha}{r^{1/\alpha}(t)} \right)^{\alpha/\alpha+1} \left|\omega_1^\alpha(t)\right| |u(t)| \ \text{ and } Y = [\alpha r(t)]^{\alpha/\alpha+1} \left|\omega_1'(t)\right|^\alpha$$

and applying the Lemma(2.3), we get

$$\int_{c_1}^{d_1}\left[r(t)\left|\omega_1'(t)\right|^{\alpha+1}-\left(\frac{x(t-\tau)}{x(t)}\right)^{\alpha}W_1(t)\right]dt\geq 0. \tag{2.31}$$

Now to estimate $\frac{x(t-\tau)}{x(t)}$ on $[c_1,d_1]$.
If $t\in[c_1,d_1]$ then $t-\tau\in[c_1-\tau,d_1-\tau]$ and then there is no impulsive moment in $(t-\tau,t)$. For any $t\in(t-\tau,t)$, we have

$$x(t)-x(c_1-\tau)=x'(\xi)(t-c_1+\tau),\ \xi\in(c_1-\tau,t).$$

By using the monotone properties of $r(t),\phi_\alpha(*)$ and $r(t)\Phi_\alpha(x'(t))$, we get

$$\begin{aligned}\phi_\alpha(x(t)) &> \phi_\alpha(x'(\xi))(t-c_1+\tau)=\frac{r(\xi)}{r(\xi)}\phi_\alpha(x'(\xi))(t-c_1+\tau)^{\alpha}\\ &\geq \frac{r(t)\Phi_\alpha(x'(t))}{r(t)}(t-c_1+\tau)^{\alpha}=\phi_\alpha(x'(t))(t-c_1+\tau).\end{aligned}$$

Therefore,

$$\frac{x'(t)}{x(t)}<\frac{1}{(t-c_1+\tau)}.$$

Integrating both sides of the above inequality from $t-\tau$ to t, we obtain

$$\frac{x(t-\tau)}{x(t)}>\left(\frac{t-c_1}{t-c_1+\tau}\right),\ t\in[c_1,d_1]. \tag{2.32}$$

From (2.31) and (2.32) we obtain

$$\int_{c_1}^{d_1}\left[W_1(t)\left(\frac{t-c_1}{t-c_1+\tau}\right)^{\alpha}-r(t)\left|\omega_1'(t)\right|^{\alpha+1}\right]dt<0. \tag{2.33}$$

This again contradicts our assumption.

When $x(t)$ is eventually negative, we can consider the interval $[c_2,d_2]$ and reach a similar contradiction.Thus the proof is complete. □

Following Kong [9] and Philos [16], we introduce a class of functions:
Let $D=\{(t,s):t_0\leq s\leq t\}$, $H_1,H_2\in C^1(D,\mathbb{R})$. A pair of functions (H_1,H_2) is said to belong to a function class $\mathcal{H}$, if $H_1(t,t)=H_2(t,t)=0$, $H_1(t,s)>0$, $H_2(t,s)>0$ for $t>s$ and there exist $h_1,h_2\in L_{\rm loc}(D,\mathbb{R})$ such that

$$\frac{\partial H_1(t,s)}{\partial t}=h_1(t,s)H_1(t,s),\quad \frac{\partial H_2(t,s)}{\partial s}=-h_2(t,s)H_2(t,s).$$

We assume there exists c_j, d_j, $\delta_j\notin\{t_k\}$, $k=1,2,\ldots,$ $(j=1,2)$ which satisfy $T<c_1<\delta_1<d_1\leq c_2<\delta_2<d_2$ for any $T\geq t_0$. Noticing whether or not there are impulse moments of $x(t)$ in $[c_j,\delta_j]$ and $[\delta_j,d_j]$, we should consider the following four cases,

$$\begin{aligned}&(S_1)\ k(c_j)<k(\delta_j)<k(d_j);\quad (S_2)\ k(c_j)=k(\delta_j)<k(d_j);\\ &(S_3)\ k(c_j)<k(\delta_j)=k(d_j);\quad (S_4)\ k(c_j)=k(\delta_j)=k(d_j),\ j=1,2.\end{aligned}$$

Moreover in the discussion of impulse moments of $x(t-\tau)$, it is necessary to consider the following two cases,

$$(\bar{S}_1)\ t_{k(\delta_j)}+\tau>\delta_j;\quad (\bar{S}_2)\ t_{k(\delta_j)}+\tau\leq\delta_j,\ j=1,2.$$

In the following theorem, we only consider the case of combination of (S_1) with $(\bar{S_1})$. For the other cases, similar conclusions can be given and hence their proof is omitted.

For our convenience, we define

$$\begin{aligned}\Pi_{1,j} =: \frac{1}{H_1(\delta_j, c_j)} \Bigg\{ & \int_{c_j}^{t_{k(c_j)+1}} \tilde{H_1}(t, c_j) \left(\frac{t - t_{k(c_j)} - \tau}{t - t_{k(c_j)}} \right)^{\alpha} dt \\ & + \sum_{i=k(c_j)+1}^{k(\delta_1)-1} \left[\int_{t_i}^{t_i+\tau} \tilde{H_1}(t, c_j) \left(\frac{t - t_i}{b_i(t + \tau - t_i)} \right)^{\alpha} dt + \int_{t_i+\tau}^{t_{i+1}} \tilde{H_1}(t, c_j) \left(\frac{t - t_i - \tau}{t - t_i} \right)^{\alpha} dt \right] \\ & + \int_{t_{k(\delta_j)}}^{\delta_j} \tilde{H_1}(t, c_j) \left(\frac{t - t_{k(\delta_j)}}{b_{k(\delta_j)}(t + \tau - t_{k(\delta_j)})} \right)^{\alpha} dt \\ & - \frac{1}{(\alpha+1)^{\alpha+1}} \int_{c_j}^{\delta_j} r(t) H_1(t, c_j) \left| h_1(t, c_j) \right|^{\alpha+1} dt \Bigg\}\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}\Pi_{2,j} =: \frac{1}{H_2(d_j, \delta_j)} \Bigg\{ & \int_{\delta_j}^{t_{k(\delta_j)}+\tau} \tilde{H_2}(d_j, t) \left(\frac{t - t_{k(\delta_j)}}{b_{k(\delta_j)}(t + \tau - t_{k(\delta_j)})} \right)^{\alpha} dt + \int_{t_{k(\delta_j)}+\tau}^{t_{k(\delta_j)+1}} \tilde{H_2}(d_j, t) \left(\frac{t - t_{k(\delta_j)} - \tau}{t - t_{k(\delta_j)}} \right)^{\alpha} dt \\ & + \sum_{i=k(\delta_j)+1}^{k(d_j)-1} \left[\int_{t_i}^{t_i+\tau} \tilde{H_2}(d_j, t) \left(\frac{t - t_i}{b_i(t + \tau - t_i)} \right)^{\alpha} dt + \int_{t_i+\tau}^{t_{i+1}} \tilde{H_2}(d_j, t) \left(\frac{t - t_i - \tau}{t - t_i} \right)^{\alpha} dt \right] \\ & + \int_{t_{k(d_j)}}^{d_j} \tilde{H_2}(d_j, t) \left(\frac{t - t_{k(d_j)}}{b_{k(d_j)}(t + \tau - t_{k(d_j)})} \right)^{\alpha} dt \\ & - \frac{1}{(\alpha+1)^{\alpha+1}} \int_{\delta_j}^{d_j} r(t) H_2(d_j, t) \left| h_2(d_j, t) \right|^{\alpha+1} dt \Bigg\},\end{aligned} \tag{2.35}$$

where $\tilde{H_1}(t, c_j) = H_1(t, c_j) Q(t)$, $\quad \tilde{H_2}(d_j, t) = H_2(d_j, t) Q(t)$, $(j = 1, 2)$ and

$$Q(t) = \left(p(t) + \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) |e(t)|^{\eta_0} \right).$$

Theorem 2.2. *Assume that for any $T \geq t_0$, there exist c_j, d_j, $\delta_j \notin \{t_k\}$, $j = 1, 2$ such that $c_1 < \delta_1 < d_1 \leq c_2 < \delta_2 < d_2$, and (2.5) holds. If there exists $(H_1, H_2) \in \mathcal{H}$ such that*

$$\Pi_{1,j} + \Pi_{2,j} > \frac{M_j}{H_1(\delta_j, c_j)} \Psi_{c_j}^{\delta_j}[H_1(., c_j)] + \frac{M_j}{H_2(d_j, \delta_j)} \Psi_{\delta_j}^{d_j}[H_2(d_j, .)], \ j = 1, 2, \tag{2.36}$$

then equation(1.1) is oscillatory.

Proof. To arrive at a contradiction, let us suppose that $x(t)$ is a non-oscillatory solution of equation(1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(t - \tau) > 0$ for $t \geq t_0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$. Continuing as in Theorem(2.5), we can get (2.18). Multiplying both sides of (2.18) by $H_1(t, c_1)$ and integrating it from c_1 to δ_1, we have

$$\begin{aligned}\int_{c_1}^{\delta_1} H_1(t, c_1) u'(t) dt \leq & - \int_{c_1}^{\delta_1} H_1(t, c_1) \frac{\alpha}{r^{1/\alpha}(t)} \left| u(t) \right|^{(1+\alpha)/\alpha} dt \\ & - \int_{c_1}^{\delta_1} \tilde{H_1}(t, c_1) \left(\frac{x(t - \tau)}{x(t)} \right)^{\alpha} dt\end{aligned} \tag{2.37}$$

Since the impulsive moments $t_{k(c_1)+1}, t_{k(c_1)+2}, \ldots, t_{k(\delta_1)}$ are in $[c_1, \delta_1]$, using the integration by parts on the left-hand side of the above inequality, we obtain

$$\begin{aligned}
\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt &= \left(\int_{c_1}^{t_{k(c_1)+1}} + \int_{t_{k(c_1)+1}}^{t_{k(c_1)+2}} + \ldots + \int_{t_{k(\delta_1)}}^{\delta_1}\right) H_1(t,c_1)du(t) \\
&= \sum_{i=k(c_1)+1}^{k(\delta_1)} [u(t_i) - u(t_i^+)]H_1(t_i,c_1) + u(\delta_1)H(\delta_1,c_1) \\
&\quad - \left(\int_{c_1}^{t_{k(c_1)+1}} + \int_{t_{k(c_1)+1}}^{t_{k(c_1)+2}} + \ldots + \int_{t_{k(\delta_1)}}^{\delta_1}\right) u(t)h_1(t,c_1)H_1(t,c_1)dt \\
&= \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{a_i^\alpha - b_i^\alpha}{a_i^\alpha} H_1(t_i,c_1)u(t_i) + H_1(\delta_1,c_1)u(\delta_1) \\
&\quad - \left(\int_{c_1}^{t_{k(c_1)+1}} + \int_{t_{k(c_1)+1}}^{t_{k(c_1)+2}} \ldots + \int_{t_{k(\delta_1)}}^{\delta_1}\right) u(t)h_1(t,c_1)H_1(t,c_1)dt.
\end{aligned} \tag{2.38}$$

Substituting (2.38) into (2.37), we have

$$\begin{aligned}
\int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau)}{x(t)}\right)^\alpha dt \le &\sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha} H_1(t_i,c_1)u(t_i) - H_1(\delta_1,c_1)u(\delta_1) \\
&+ \int_{c_1}^{\delta_1} H_1(t,c_1)\left[|h_1(t,c_1)|\,|u(t)| - \frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{(1+\alpha)/\alpha}\right] dt.
\end{aligned} \tag{2.39}$$

Letting

$$\lambda = 1 + \frac{1}{\alpha},\ X = \frac{\alpha^{\alpha/\alpha+1}|u(t)|}{[r(t)]^{1/\alpha+1}} \text{ and } Y = \left[\alpha(\alpha+1)^{-(\alpha+1)}r(t)\right]^{\alpha/\alpha+1}|h_1(t,c_1)|^\alpha,$$

and then by using Lemma(2.3), the above inequality becomes

$$\begin{aligned}
\int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau)}{x(t)}\right)^\alpha dt \le &\sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha} H_1(t_i,c_1)u(t_i) - H_1(\delta_1,c_1)u(\delta_1) \\
&+ \frac{1}{(1+\alpha)^{1+\alpha}}\int_{c_1}^{\delta_1} r(t)H_1(t,c_1)\,|h_1(t,c_1)|^{\alpha+1} dt.
\end{aligned} \tag{2.40}$$

To estimate $\frac{x(t-\tau)}{x(t)}$, we have to divide the interval $[c_1, \delta_1]$ into several sub intervals and by using Lemma(2.4), we get estimation for the left hand side of the above inequality as follows,

$$\begin{aligned}
&\int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau)}{x(t)}\right)^\alpha dt \\
&> \int_{c_1}^{t_{k(c_1)+1}} \tilde{H}_1(t,c_1)\left(\frac{t - t_{k(c_1)} - \tau}{t - t_{k(c_1)}}\right)^\alpha dt \\
&+ \sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{t_i}^{t_i+\tau} \tilde{H}_1(t,c_1)\left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right)^\alpha dt + \int_{t_i+\tau}^{t_{i+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_i-\tau}{t-t_i}\right)^\alpha dt\right] \\
&+ \int_{t_k(\delta_1)}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{t - t_{k(\delta_1)}}{b_{k(\delta_1)}(t+\tau-t_{k(\delta_1)})}\right)^\alpha dt.
\end{aligned} \tag{2.41}$$

From (2.40) and (2.41), we have

$$\begin{aligned}&\int_{c_1}^{t_{k(c_1)+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(c_1)}-\tau}{t-t_{k(c_1)}}\right)^{\alpha}dt\\&+\sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{t_i}^{t_i+\tau}\tilde{H}_1(t,c_1)\left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right)^{\alpha}dt+\int_{t_i+\tau}^{t_{i+1}}\tilde{H}_1(t,c_1)\left(\frac{t-t_i-\tau}{t-t_i}\right)^{\alpha}dt\right]\\&+\int_{t_{k(\delta_1)}}^{\delta_1}\tilde{H}_1(t,c_1)\left(\frac{t-t_{k(\delta_1)}}{b_{k(\delta_1)}(t+\tau-t_{k(\delta_1)})}\right)^{\alpha}dt-\frac{1}{(1+\alpha)^{1+\alpha}}\int_{c_1}^{\delta_1}r(t)H_1(t,c_1)\left|h_1(t,c_1)\right|^{\alpha+1}dt\\&<\sum_{i=k(c_1)+1}^{k(\delta_1)}\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}H_1(t_i,c_1)u(t_i)-H_1(\delta_1,c_1)u(\delta_1).\end{aligned}\tag{2.42}$$

Multiplying both sides of (2.18) by $H_2(d_1,t)$ and using similar analysis as above, we can obtain

$$\begin{aligned}&\int_{\delta_1}^{t_{k(\delta_1)}+\tau}\tilde{H}_2(d_1,t)\left(\frac{t-t_{k(\delta_1)}}{b_{k(\delta_1)}(t+\tau-t_{k(\delta_1)})}\right)^{\alpha}dt+\int_{t_{k(\delta_1)}+\tau}^{t_{k(\delta_1)+1}}\tilde{H}_2(d_1,t)\left(\frac{t-t_{k(\delta_1)}-\tau}{t-t_{k(\delta_1)}}\right)^{\alpha}dt\\&+\sum_{i=k(\delta_1)+1}^{k(d_1)-1}\left[\int_{t_i}^{t_i+\tau}\tilde{H}_2(d_1,t)\left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right)^{\alpha}dt+\int_{t_i+\tau}^{t_{i+1}}\tilde{H}_2(d_1,t)\left(\frac{t-t_i-\tau}{t-t_i}\right)^{\alpha}dt\right]\\&+\int_{t_{k(d_1)}}^{d_1}\tilde{H}_2(d_1,t)\left(\frac{t-t_{k(d_1)}}{b_{k(d_1)}(t+\tau-t_{k(d_1)})}\right)^{\alpha}dt-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{\delta_1}^{d_1}r(t)H_2(d_1,t)\left|h_2(d_1,t)\right|^{\alpha+1}dt\\&<\sum_{i=k(\delta_1)+1}^{k(d_1)}\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}H_2(d_1,t_i)u(t_i)+H_2(d_1,\delta_1)u(\delta_1).\end{aligned}\tag{2.43}$$

Dividing (2.42) and (2.43) by $H_1(\delta_1,c_1)$ and $H_2(d_1,\delta_1)$ respectively, and adding them, we get

$$\begin{aligned}\Pi_{1,1}+\Pi_{2,1}\ <\ &\frac{1}{H_1(\delta_1,c_1)}\sum_{i=k(c_1)+1}^{k(\delta_1)}\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}H_1(t_i,c_1)u(t_i)\\&+\frac{1}{H_2(d_1,\delta_1)}\sum_{i=k(\delta_1)+1}^{k(d_1)}\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}H_2(d_1,t_i)u(t_i).\end{aligned}\tag{2.44}$$

On the other hand, similar to (2.26), we have

$$\sum_{i=k(c_1)+1}^{k(\delta_1)}\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}H_1(t_i,c_1)u(t_i)\le M_1\Psi_{c_1}^{\delta_1}[H_1(.,c_1)]\tag{2.45}$$

and

$$\sum_{i=k(\delta_1)+1}^{k(d_1)}\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}H_2(d_1,t_i)u(t_i)\le M_1\Psi_{\delta_1}^{d_1}[H_2(d_1,.)].\tag{2.46}$$

Substituting (2.45)and (2.46) in (2.44), we obtain a contradiction to the condition (2.36).

When $x(t)$ is eventually negative, we can consider $[c_2,d_2]$ and reach a similar contradiction. Hence the proof is complete. □

Remark 2.1. *When $\alpha=1$, our results reduces to Theorem(2.2) and Theorem(2.4) of [11].*

Remark 2.2. *When $\tau=0$ and $\alpha=1$, Theorem(2.5) reduces to Theorem(2.1) of [13].*

Remark 2.3. *When $a_k=b_k=1$ for all $k=1,2,3,\ldots,\tau=0$ and $\alpha=1$, our results reduces to Theorem(1) of [17] for the case $\rho(t)=1$.*

3 Examples

In this section we give two examples to illustrate our main results.

Example 3.1. Consider the impulsive differential equation

$$\begin{aligned} &(\Phi_\alpha(x'(t)))' + \gamma_0 \sin t \Phi_\alpha\left(x(t-\frac{\pi}{12})\right) + \gamma_1 e^{-t/2}\Phi_{\beta_1}\left(x(t-\frac{\pi}{12})\right) \\ &\quad + \gamma_2\cos^2 t\Phi_{\beta_2}\left(x(t-\frac{\pi}{12})\right) = \sin 2t,\ t \ge t_0,\ t \ne t_{k,i}, \\ &x(t_{k,i}^+) = a_k x(t_{k,i}), \quad x'(t_{k,i}^+) = b_k x'(t_{k,i}), \\ &\text{where} \quad t_{k,i} = 2k\pi + \frac{3\pi}{8} + (-1)^{i-2}\left(\frac{\pi}{4}\right), \quad i = 1,2 \text{ and } k = 1,2,\dots \end{aligned} \tag{3.47}$$

Here,

$$r(t) = 1, p(t) = \gamma_0 \sin t, q_1(t) = \gamma_1 e^{-t/2}, q_2(t) = \gamma_2\cos^2 t \text{ and } e(t) = \sin 2t,\ t \ge t_0 > 0,$$

where γ_0, γ_1 and γ_2 are positive constants. If we choose $\eta_0 = 1/2$, $\beta_1 = 19/2$, $\beta_2 = 5/2$ and $\alpha = 3$, then by Lemma (2.1), we can easily find $\eta_1 = \eta_2 = 1/4$. For any $T > 0$, we can choose n large enough such that $T < c_1 = 2n\pi + \frac{\pi}{12} < d_1 = 2n\pi + \frac{\pi}{6}$ and $c_2 = 2n\pi + \frac{\pi}{4} < d_2 = 2n\pi + \frac{2\pi}{3}$, then there are impulsive moments $t_{n,1} = 2n\pi + \frac{\pi}{8}$ in $[c_1, d_1]$ and $t_{n,2} = 2n\pi + \frac{5\pi}{8}$ in $[c_2, d_2]$.
Let

$$\omega_j(t) = \sin 12t \in \Omega_j(c_j, d_j),\ j = 1,2.$$

Then we have,

$$Q(t) = \gamma_0 \sin t + (1/2)^{-1/2}(1/4)^{-1/4}(1/4)^{-1/4}\gamma_1^{1/4}(e^{-t/2})^{1/4}\gamma_2^{1/4}(\cos t)^{1/2}|\sin 2t|^{1/2},$$

and

$$W_j(t) = Q(t)\omega_j^{\alpha+1}(t),\ j = 1,2.$$

In view of $\sum_{i=k(c_j)+1}^{k(d_j)-1} = 0$ as $k(c_j)+1 > k(d_j)-1$, $j = 1,2$, the left hand side of (2.13) is the following

$$\begin{aligned} &\int_{c_1}^{t_{k(c_1)+1}} W_1(t)\left(\frac{t - t_{k(c_1)} - \tau}{t - t_{k(c_1)}}\right)^\alpha dt \\ &+ \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{t_i}^{t_i+\tau} W_1(t)\left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right)^\alpha dt + \int_{t_i+\tau}^{t_{i+1}} W_1(t)\left(\frac{t-t_i-\tau}{t-t_i}\right)^\alpha\right] \\ &+ \int_{t_{k(d_1)}}^{d_1} W_1(t)\left(\frac{t - t_{k(d_1)}}{b_{k(d_1)}(t+\tau-t_{k(d_1)})}\right)^\alpha dt - \int_{c_1}^{d_1}(r(t)\left|\omega_1'(t)\right|^{\alpha+1})dt \\ &= \int_{2n\pi+\pi/12}^{2n\pi+\pi/8} W_1(t)\left(\frac{t-(2(n-1)\pi+5\pi/8)-\pi/12}{t-(2(n-1)\pi+5\pi/8))}\right)^3 dt \\ &+ \int_{2n\pi+\pi/8}^{2n\pi+\pi/6} W_1(t)\left(\frac{t-(2n\pi+\pi/8)}{b_{n,1}(t+\pi/12-(2n\pi+\pi/8))}\right)^3 dt - 12^4\int_{2n\pi+\pi/12}^{2n\pi+\pi/6}\left(\cos^4 12t\right)dt \\ &= \int_{\pi/12}^{\pi/8} W_1(t)\left(\frac{t+31\pi/24}{t+11\pi/8)}\right)^3 dt + \int_{\pi/8}^{\pi/6} W_1(t)\left(\frac{t-\pi/8}{b_{n,1}(t-\pi/24)}\right)^3 dt - 12^4\int_{\pi/12}^{\pi/6}\left(\cos^4 12t\right)dt. \\ &\approx \left[0.01464\gamma_0 + 0.0878\gamma_1^{1/4}\gamma_2^{1/4}\right] + b_{n,1}^{-3}\left[0.00004889\gamma_0 + 0.0002811\gamma_1^{1/4}\gamma_2^{1/4}\right] - 648\pi. \end{aligned} \tag{3.48}$$

On the other hand , the right hand side of (2.13)

$$\begin{aligned} \Psi_{c_1}^{d_1}[\omega_1^{\alpha+1}] &= \omega_1^{\alpha+1}(t_{k(c_1)+1})\frac{b_{k(c_1)+1}^\alpha - a_{k(c_1)+1}^\alpha}{(a_{k(c_1)+1}^\alpha(t_{k(c_1)+1}-c_1)^\alpha)} + \sum_{i=k(c_1)+2}^{k(d_1)}\omega_1^{\alpha+1}(t_i)\frac{b_i^\alpha - a_i^\alpha}{(a_i^\alpha(t_i-t_{i-1})^\alpha)}. \\ &= \sin^4 12(2n\pi+\pi/8)\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}(2n\pi+\pi/8-(2n\pi+\pi/12))}\right)^3 \\ &= \left(\frac{24}{\pi}\right)^3\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}}\right)^3. \end{aligned} \tag{3.49}$$

Thus for $t \in [c_1, d_1]$, if we choose γ_0, γ_1 and γ_2 large enough so that

$$\begin{aligned} &0.01464\gamma_0 + 0.0878\gamma_1^{1/4}\gamma_2^{1/4} + b_{n,1}^{-3}\left(0.00004889\gamma_0 + 0.0002811\gamma_1^{1/4}\gamma_2^{1/4}\right) - 648\pi \\ &\geq \left(\frac{24}{\pi}\right)^3 \left(\frac{b_{n,1} - a_{n,1}}{a_{n,1}}\right)^3, \end{aligned} \tag{3.50}$$

then (2.13) will be satisfied.

Similarly for $t \in [c_2, d_2]$, we can get the following condition

$$\begin{aligned} &0.153651\gamma_0 + 0.02648\gamma_1^{1/4}\gamma_2^{1/4} + b_{n,2}^{-3}\left(0.00010044\gamma_0 - 0.000143\gamma_1^{1/4}\gamma_2^{1/4}\right) - 3240\pi \\ &\geq \left(\frac{8}{3\pi}\right)^3 \left(\frac{b_{n,2} - a_{n,2}}{a_{n,2}}\right)^3. \end{aligned} \tag{3.51}$$

Hence by Theorem (2.1) for suitable γ_0, γ_1 and γ_2, equation(3.47) becomes oscillatory.

Example 3.2. Consider the impulsive differential equation

$$\begin{aligned} &(\Phi_\alpha(x'(t)))' + \kappa_0 p(t)\Phi_\alpha\left(x(t - \frac{\pi}{12})\right) + \kappa_1 q_1(t)\Phi_{\beta_1}\left(x(t - \frac{\pi}{12})\right) \\ &\quad + \kappa_2 q_2(t)\Phi_{\beta_2}\left(x(t - \frac{\pi}{12})\right) = e(t),\ t \geq t_0,\ t \neq t_{k,i}, \\ &x(t_{k,i}^+) = a_k x(t_{k,i}), \quad x'(t_{k,i}^+) = b_k x'(t_{k,i}) \end{aligned} \tag{3.52}$$

where κ_0, κ_1, and κ_2 are positive constants, and

$$t_{n,1} = 2n\pi + \pi/8,\ t_{n,2} = 2n\pi + 3\pi/8,\ t_{n,3} = 2n\pi + 13\pi/8 \text{ and } t_{n,4} = 2n\pi + 17\pi/8.$$

In addition let, $q_1(t) = e^{t/2}$, $q_2(t) = e^{t/4}$,

$$p(t) = \begin{cases} e^4 t, & t \in [2n\pi + \pi/12, 2n\pi + \pi/2], \\ \sin^2 t, & t \in [2n\pi + 3\pi/2, 2n\pi + 5\pi/2] \end{cases}$$

and

$$e(t) = \begin{cases} -\sin 2t, & t \in [2n\pi + \pi/12, 2n\pi + \pi/2], \\ \cos^2 t, & t \in [2n\pi + 3\pi/2, 2n\pi + 5\pi/2]. \end{cases}$$

For any $t_0 > 0$, we choose n large enough such that $t_0 < 2n\pi + \pi/12$ and let $[c_1, d_1] = [2n\pi + \pi/12, 2n\pi + \pi/2]$, $[c_2, d_2] = [2n\pi + 3\pi/2, 2n\pi + 5\pi/2]$, $\delta_1 = 2n\pi + \pi/6$, $\delta_2 = 2n\pi + 5\pi/3$. Then $p(t), q(t)$ and $e(t)$ satisfy (2.5) on $[c_1, d_1]$ and $[c_2, d_2]$. Let $H_1(t,s) = H_2(t,s) = (t-s)^3$ then $h_1(t,s) = -h_2(t,s) = 3/(t-s)$. Now choose $\eta_0 = 1/2$, $\beta_1 = 5/2$, $\beta_2 = 1/2$, and $\alpha = 1$.
Then one can easily find $\eta_1 = 3/8$, $\eta_2 = 1/8$.

$$Q(t) = p(t) + (1/2)^{-1/2}(3/8)^{-3/8}(1/8)^{-1/8} q_1^{3/8}(t) q_2^{1/8}(t)|e(t)|^{1/2}.$$

Also by a simple calculation, we get

$$\begin{aligned} \Pi_{1,1} = &\frac{1}{H_1(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{\pi}{12})} \\ &\left\{ \int_{2n\pi+\pi/12}^{2n\pi+\pi/8} H_1(t, 2n\pi + \pi/12) Q(t) \left(\frac{t - (2(n-1)\pi + 3\pi/8) - \pi/12}{t - (2(n-1)\pi + 3\pi/8)}\right) dt \right. \\ &+ \int_{2n\pi+\pi/8}^{2n\pi+\pi/6} H_1(t, 2n\pi + \pi/12) Q(t) \left(\frac{t - (2n\pi + \pi/8)}{b_{n,1}(t + \pi/12 - (2n\pi + \pi/8))}\right) dt \\ &\left. - \frac{1}{2^2} \int_{2n\pi+\pi/12}^{2n\pi+\pi/6} H_1(t, 2n\pi + \pi/12) \left|h_1(t, 2n\pi + \pi/12)\right|^2 dt \right\} \\ &\approx \kappa_0 \left(0.0169 + \frac{0.1042}{b_{n,1}}\right) + \kappa_1{}^{3/8} \kappa_2{}^{1/8} \left(0.0101) + \frac{0.0411}{b_{n,1}}\right) - 4.2971 \end{aligned} \tag{3.53}$$

and

$$\begin{aligned}
\Pi_{2,1} =& \frac{1}{H_2(2n\pi+\frac{\pi}{2},2n\pi+\frac{\pi}{6})} \\
& \left\{ \int_{2n\pi+\pi/6}^{2n\pi+\pi/8+\pi/12} \tilde{H}_2(2n\pi+\pi/2,t)\left(\frac{t-(2n\pi+\pi/8)}{b_{n,1}(t+\pi/12-(2n\pi+\pi/8))}\right)dt \right. \\
& + \int_{2n\pi+\pi/8+\pi/12}^{2n\pi+3\pi/8} \tilde{H}_2(2n\pi+\pi/2,t)\left(\frac{t-(2n\pi+\pi/8)-\pi/12}{t-(2n\pi+\pi/8)}\right)dt \\
& + \int_{2n\pi+3\pi/8}^{2n\pi+\pi/2} \tilde{H}_2(2n\pi+\pi/2,t)\left(\frac{t-(2n\pi+3\pi/8)}{b_{n,2}(t+\pi/12-(2n\pi+3\pi/8))}\right)dt \\
& \left. - \frac{1}{(2)^2}\int_{2n\pi+\pi/6}^{2n\pi+\pi/2} H_2(2n\pi+\pi/2,t)\left|h_2(2n\pi+\pi/2,t)\right|^2 dt \right\}. \\
\approx & \,\kappa_0\left(2.0198+\frac{0.4843}{b_{n,1}}+\frac{0.1987}{b_{n,2}}\right)+{\kappa_1}^{3/8}{\kappa_2}^{1/8}\left(0.1597+\frac{0.1340}{b_{n,1}}+\frac{0.0031}{b_{n,2}}\right)-1.0742.
\end{aligned} \tag{3.54}$$

From (3.53) and (3.54), we get

$$\Pi_{1,1}+\Pi_{2,1} \approx \kappa_0\left(2.0367+\frac{0.5885}{b_{n,1}}+\frac{0.1987}{b_{n,2}}\right)+{\kappa_1}^{3/8}{\kappa_2}^{1/8}\left(0.1698+\frac{0.1751}{b_{n,1}}+\frac{0.0031}{b_{n,2}}\right)-5.3713. \tag{3.55}$$

which gives the left hand side of (2.36).
On the other hand, the right hand side of the inequality (2.36) is

$$\begin{aligned}
\frac{M_1}{H_1(\delta_1,c_1)}\Psi_{c_1}^{\delta_1}[H_1(.,c_1)] =& \frac{1}{H_1(2n\pi+\pi/6,2n\pi+\pi/12)}H_1(2n\pi+\pi/8,2n\pi+\pi/12) \\
& \times\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}(2n\pi+\pi/8-(2n\pi+\pi/12))}\right) \\
\approx & (0.9549)\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}}\right),
\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}
\frac{M_1}{H_2(d_1,\delta_1)}\Psi_{\delta_1}^{d_1}[H_2(d_1,.)] =& \frac{1}{(2n\pi+\pi/2-2n\pi-\pi/6)^3}(2n\pi+\pi/2-2n\pi-3\pi/8)^3 \\
& \times\left(\frac{b_{n,2}-a_{n,2}}{a_{n,2}(2n\pi+3\pi/8-2n\pi-\pi/6)}\right) \\
\approx & (0.0805)\left(\frac{b_{n,2}-a_{n,2}}{a_{n,2}}\right).
\end{aligned} \tag{3.57}$$

From (3.56) and (3.57), we have the right hand side of (2.36) as

$$\begin{aligned}
& \frac{M_1}{H_1(\delta_1,c_1)}\Psi_{c_1}^{\delta_1}[H_1(.,c_1)]+\frac{M_1}{H_2(d_1,\delta_1)}\Psi_{\delta_1}^{d_1}[H_2(d_1,.)] \\
& \approx (0.9549)\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}}\right)+(0.0805)\left(\frac{b_{n,2}-a_{n,2}}{a_{n,2}}\right).
\end{aligned} \tag{3.58}$$

Thus (2.36) is satisfied for $j=1$ if

$$\begin{aligned}
& \kappa_0\left(2.0367+\frac{0.5885}{b_{n,1}}+\frac{0.1987}{b_{n,2}}\right)+{\kappa_1}^{3/8}{\kappa_2}^{1/8}\left(0.1698+\frac{0.1751}{b_{n,1}}+\frac{0.0031}{b_{n,2}}\right) \\
& > 5.3713+(0.9549)\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}}\right)+(0.0805)\left(\frac{b_{n,2}-a_{n,2}}{a_{n,2}}\right).
\end{aligned} \tag{3.59}$$

Similarly for $[c_2,d_2]$, we have

$$\Pi_{1,2}+\Pi_{2,2} \approx \kappa_0\left(0.0887+\frac{0.0501}{b_{n,3}}+\frac{0.0046}{b_{n,4}}\right)+{\kappa_1}^{3/8}{\kappa_2}^{1/8}\left(2.6583+\frac{0.4302}{b_{n,3}}+\frac{0.1122}{b_{n,4}}\right)-2.5782. \tag{3.60}$$

and

$$\begin{aligned}\frac{M_2}{H_1(\delta_2, c_2)}\Psi_{c_2}^{\delta_2}[H_1(.,c_2)] + \frac{M_2}{H_2(d_2,\delta_2)}\Psi_{\delta_2}^{d_2}[H_2(d_2,.)] \\ \approx (1.0742)\left(\frac{b_{n,3}-a_{n,3}}{a_{n,3}}\right) + (0.0632)\left(\frac{b_{n,4}-a_{n,4}}{a_{n,4}}\right).\end{aligned} \tag{3.61}$$

Thus (2.36) is satisfied for $j = 2$ if

$$\begin{aligned}\kappa_0\left(0.0887 + \frac{0.0501}{b_{n,3}} + \frac{0.0046}{b_{n,4}}\right) + {\kappa_1}^{3/8}{\kappa_2}^{1/8}\left(2.6583 + \frac{0.4302}{b_{n,3}} + \frac{0.1122}{b_{n,4}}\right) \\ > 2.5782 + (1.0742)\left(\frac{b_{n,3}-a_{n,3}}{a_{n,3}}\right) + (0.0632)\left(\frac{b_{n,4}-a_{n,4}}{a_{n,4}}\right).\end{aligned} \tag{3.62}$$

Hence, by Theorem (2.2), equation(3.52) is oscillatory if (3.59) and (3.62) hold.

4 Conclusion

In this paper, we have established interval oscillation results for equation (1.1) using Riccati transformation, some classical inequalities and Kong's technique. These results extend some well-known results in [11, 13, 17].

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Semi-invariant submanifolds of a Kenmotsu manifold with a generalized almost r-contact structure admitting a semi-symmetric metric connection

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Abstract

We consider an almost r-contact Kenmotsu manifold admitting a semi-symmetric metric connection and study semi-invariant submanifolds of an almost r-contact Kenmotsu manifold endowed with a semi-symmetric meric connection. We obtain Gauss and Weingarten formuale for such a connection and also discuss the integrability conditions of the distributions on a generalized Kenmotsu manifold.

Keywords: Kenmotsu manifolds, almost r-contact structures, semi-invariant submanifolds, semi-symmetric metric connection, integrability conditions, parallel horizontal distribution.

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1 Introduction

A Let M be an n-dimensional differentiable manifold. The torsion tensor T of a linear connection ∇ in M is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection. In 1924, A. friedmann and J. A. Schouten introduced the notion of semi-symmetric linear connection [8]. In 1932, H. A. Hayden [10] introduced semi-symmetric metric connection in a Riemannian manifold and this was studied systematically by K. Yano [14]. In 1975, S. Golab studied some properties of semi-symmetric and quarter-symmetric linear connections [9]. A linear connection ∇ is said to be semi-symmetric if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form.

On the other hand, A. Bejancu, introduced the notion of semi-invariant submanifolds [6] or contact CR-submanifolds [5], as a generalization of invariant and anti-invariant submanifolds of an almost contact metric manifold and was followed by several geometers in [1, 2, 4, 7, 11, 12]. Semi-invariant submanifolds of a Kenmotsu manifold immersed in a generalized almost r-contact metric structure was defined and studied by R. Nivas and S. Yadav [13]. The first author, M. D. Siddiqi and J. P. ojha studied some characteristic

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properties of semi-invariant submanifolds of a Kenmotsu manifold immersed in a generalized almost r-contact structure admitting a quarter-symmetric non-metric connection [3].

Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jaruselam or Mekka or the North pole, then this displacement is semi-symmetric and metric [8]

Motivated by the above studies, in this paper we study semi-invariant submanifolds of a Kenmotsu manifold immersed in a generalized almost r-contact structure admitting a semi-symmetric metric connection. The paper is organized as follows : In Section 2, we give a brief account of a Kenmotsu manifold immersed in a generalized almost r-contact metric manifold. In Section 3, semi-invariant submanifolds, semi-symmetric metric connection are defined and also Gauss and Weingarten equations are obtained. In Section 4, some lemmas on semi-invariant submanifolds are proved and integrability conditions of certain distributions on semi-invariant submanifolds are discussed. In the last Section 5, semi-invariant submanifolds of a generalized Kenmotsu manifold with parallel horizontal distributions for semi-symmetric metric connection are investigated.

2 Preliminaries

Let $\bar{M}$ be a $(2n+r)$-dimensional Kenmotsu manifold with a generalized almost r-contact structure (ϕ, ξ_p, η_p, g), where ϕ is a tensor field of type $(1,1)$, ξ_p are r-vector fields, η_p are r 1-forms and g is the associated Riemannian metric, satisfying

$$\phi^2 = a^2 I + \sum_{p=1}^{r} \eta_p \otimes \xi_p, \tag{2.1}$$

$$\eta_p(\xi_q) = \delta_{pq}, \qquad p, q \in (r) := 1, 2, 3......r, \tag{2.2}$$

$$\phi(\xi_p) = 0, \;\; p \in (r), \tag{2.3}$$

$$\eta_p(\phi X) = 0, \;\; p \in (r), \tag{2.4}$$

$$g(\phi X, \phi Y) + a^2 g(X, Y) + \sum_{p=1}^{r} \eta_p(X)\eta_p(Y) = 0, \tag{2.5}$$

$$\eta_p(X) = g(X, \xi_p), \tag{2.6}$$

$$(\bar{\bar{\nabla}}_X \phi)Y = -\sum_{p=1}^{r} \eta_p(Y)\phi X - g(X, \phi Y)\sum_{p=1}^{r} \xi_p, \tag{2.7}$$

$$\bar{\bar{\nabla}}_X \xi_p = X - \sum_{p=1}^{r} \eta_p(X)\xi_p, \tag{2.8}$$

where I is the identity tensor field and X, Y are vector fields on $\bar{M}$ and $\bar{\bar{\nabla}}$ denotes the Riemannian connection.

3 Semi-invariant Submanifolds

An n-dimensional Riemannian submanifold M of a Kenmotsu manifold $\bar{M}$ with an almost r-contact structure is called a semi-invariant submanifold, if ξ_p is tangent to M and there exists on M a pair of orthogonal distributions $(D, D^{\perp})$ such that

(i) $TM = D \oplus D^{\perp} + \{\xi_p\}$,

(ii) the distribution D is invariant under ϕ, that is, $\phi D_x = D_x$ for all $x \in M$,

(ii) the distribution $D^{\perp}$ is anti-invariant under ϕ, that is, $\phi D_x^{\perp} \subset T_x^{\perp} M$ for all $x \in M$,

where T_xM and $T_x^{\perp}M$ are respectively the tangent and normal space of M at x.
The distribution D $(resp., D^{\perp})$ can be defined by projection P $(resp., Q)$ which satisfies the conditions

$$P^2 = P, Q^2 = Q, PQ = QP = 0. \tag{3.9}$$

The pair of distributions $(D, D^{\perp})$ is called the ξ-horizontal (resp., ξ-vertical), if $\xi_x \in D_x$ (resp., $\xi_x \in D_x^{\perp}$). A semi-invariant submanifold M is said to be an invariant (resp., anti-invariant) submanifold if $D_x^{\perp} = 0$ (resp., $D_x = 0$) for each $x \in M$, we also call M proper, if neither D nor $D^{\perp}$ is null. It is easy to check that each hypersurface of M which is tangent to ξ_p inherits a structure of the semi-invariant submanifold of $\bar{M}$.
Owing due to the existence of 1-form η_p, we define a semi-symmetric metric connection $\bar{\nabla}$ in a Kenmotsu manifold with a generalized almost r-contact structure by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \sum_{p=1}^{r} \eta_p(Y)X - g(X,Y)\sum_{p=1}^{r} \xi_p \tag{3.10}$$

for any $X, Y \in TM$, where $\bar{\bar{\nabla}}$ is the induced connection on M. From (2.7) and (3.10), we get

$$(\bar{\nabla}_X \phi)Y = -2\sum_{p=1}^{r} \eta_p(Y)\phi X - g(X,\phi Y)\sum_{p=1}^{r} \xi_p. \tag{3.11}$$

We denote the metric tensor of $\bar{M}$ as well as that is induced on M by g. Let $\bar{\nabla}$ be the semi-symmetric metric connection on $\bar{M}$ and ∇ be the induced connection on M with respect to the unit normal N.

Theorem 3.1. *The connection induced on the semi-invariant submanifolds of a generalized Kenmotsu manifold with a semi-symmetric metric connection is also a semi-symmetric metric connection.*

Proof. Let ∇ be the induced connection with respect to the unit normal N on semi-invariant submanifolds of a generalized Kenmotsu manifold with a semi-symmetric metric connection $\bar{\nabla}$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + m(X,Y), \tag{3.12}$$

where m is a tensor field of type $(0,2)$ on semi-invariant submanifold M. If ∇^* is the induced connection on semi-invariant submanifolds from the Riemannian connection $\bar{\bar{\nabla}}$, then

$$\bar{\bar{\nabla}}_X Y = \nabla^*_X Y + h(X,Y), \tag{3.13}$$

where h is the second fundamental tensor. Now from (3.10), (3.12) and (3.13), we have

$$\nabla_X Y + m(X,Y) = \nabla^*_X Y + h(X,Y) + \eta_p(Y)\phi X - g(X,Y)\sum_{p=1}^{r} \xi_p.$$

Equating the tangential and normal components from both the sides of the above equation, we get

$$h(X,Y) = m(X,Y),$$

$$\nabla_X Y = \nabla^*_X Y + \eta_p(Y)\phi X - g(X,Y)\sum_{p=1}^{r} \xi_p.$$

Thus the connection ∇ is also a semi-symmetric metric connection. □

Now, the Gauss formula for semi-invariant submanifolds of a generalized Kenmotsu manifold with a semi-symmetric metric connection is

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3.14}$$

and Weingarten formula for M is given by

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{3.15}$$

for $X, Y \in TM, N \in T^{\perp}M$, where h and A are called the second fundamental tensors of M and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X,Y),N) = g(A_N X, Y). \tag{3.16}$$

Any vector field X tangent to M is given as

$$X = PX + QX + \eta_p(X)\xi_p, \tag{3.17}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. For any vector field N normal to M, we have

$$\phi N = BN + CN, \tag{3.18}$$

where BN (resp., CN) denotes the tangential (resp., normal) component of ϕN.

4 Integrability of distributions

Lemma 4.1. *Let M be a semi-invariant submanifolds of a generalized Kenmotsu manifold with a semi-symmetric metric connection. Then*

$$2(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]$$

for each $X, Y \in D$.

Proof. Using Gauss formula, we have

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X). \tag{4.19}$$

Also the covariant differentiation yields

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y]. \tag{4.20}$$

From (4.19) and (4.20), we get

$$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]. \tag{4.21}$$

Using $\eta_p(X) = 0$ for each $X \in D$ in (3.11), we get

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = 0. \tag{4.22}$$

On adding (4.21) and (4.22), we get the result. □

Similar computations also yields the following:

Lemma 4.2. *Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold with a semi-symmetric metric connection. Then*

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

for each $X \in D$, $Y \in D^{\perp}$.

Lemma 4.3. *Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold with a semi-symmetric metric connection. Then*

$$P\nabla_X\phi PY - PA_{\phi QY}X = \phi P\nabla_X Y - 2\sum_{p=1}^{r}\eta_p(Y)\phi PX, \tag{4.23}$$

$$Q\nabla_X\phi PY - QA_{\phi QY}X = Bh(X,Y), \tag{4.24}$$

$$h(X,\phi PY) + \nabla^{\perp}_X\phi QY = \phi Q\nabla_X Y + Ch(X,Y) - 2\sum_{p=1}^{r}\eta_p(Y)\phi QX, \tag{4.25}$$

$$\eta_P(\nabla_X\phi PY) - \eta_P(A_{\phi QY}X) = -2g(X,\phi Y) \tag{4.26}$$

for all $X, Y \in TM$.

Proof. By the covariant differentiation of ϕY, we have

$$\bar{\nabla}_X \phi Y = (\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_X Y).$$

Using (3.14) and (3.17) in the above equation, we get

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi PY + \bar{\nabla}_X \phi QY - \phi \nabla_X Y - \phi h(X,Y). \tag{4.27}$$

By the use of Gauss and Weingarten formulae and (3.18) in (4.27), we have

$$(\bar{\nabla}_X \phi)Y = P\nabla_X \phi PY + Q\nabla_X \phi PY + \eta_P(\nabla_X \phi PY)\xi_p + h(X,\phi PY) - PA_{\phi QY}X \tag{4.28}$$

$$-QA_{\phi QY}X - \eta_P(A_{\phi QY}X)\xi_p + \nabla^{\perp}_X \phi QY - \phi P\nabla_X Y - \phi Q\nabla_X Y - Bh(X,Y) - Ch(X,Y).$$

On comparing (4.27) and (4.28) and equating horizontal, vertical and normal components, we get (4.23), (4.24), (4.25) and (4.26) respectively. □

Definition 4.1. *The horizontal distribution D is said to be parallel with respect to the connection ∇ on M, if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.*

Theorem 4.2. *Let M be semi-invariant submanifolds of a generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric metric connection. If M is ξ_p- horizontal, then the distribution D is integrable if and only if*

$$h(X,\phi Y) = h(\phi X, Y) \tag{4.29}$$

for all $X, Y \in D$.

Proof. Let M be ξ_p- horizontal and $X, Y \in D$, then (4.25) reduces to

$$h(X,\phi Y) = \phi Q\nabla_X Y + Ch(X,Y) \tag{4.30}$$

from which we get

$$h(X,\phi Y) - h(\phi X, Y) = \phi Q[X,Y].$$

Thus if M is ξ_p horizontal, then we have

$$h(X,\phi Y) = h(\phi X, Y).$$

Hence D is integrable. □

Theorem 4.3. *Let M be semi-invariant submanifolds of a generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric metric connection. If M is ξ_p-vertical, then the distribution $D^{\perp}$ is integrable if and only if $A_{\phi X}Y = A_{\phi Y}X$.*

Proof. Let M be ξ_p-vertical and $X, Y \in D^{\perp}$, then (4.25) reduces to

$$\nabla^{\perp}_X \phi Y = \phi Q\nabla_X Y + Ch(X,Y) - 2\sum_{p=1}^{r} \eta_p(Y)\phi QX. \tag{4.31}$$

By using (3.11), (3.15) and (4.31), we get

$$\bar{\nabla}_X \phi Y = -2\sum_{p=1}^{r} \eta_p(Y)\phi X - 2g(X,\phi Y)\sum_{p=1}^{r} \xi_p + \phi P\nabla_X Y \tag{4.32}$$

$$+\phi Q\nabla_X Y + Bh(X,Y) + Ch(X,Y).$$

Since M is ξ_p-verticle, Weingarten formula is given by

$$\nabla^{\perp}_X \phi Y = \bar{\nabla}_X \phi Y + A_{\phi Y}X$$

which by using (4.32) becomes

$$\nabla^{\perp}_X \phi Y = -2\sum_{p=1}^{r} \eta_p(Y)\phi X + \phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X,Y) \tag{4.33}$$

$$+Ch(X,Y) + A_{\phi Y}X.$$

From (4.31) and (4.33), we get

$$\phi P\nabla_X Y = -A_{\phi Y}X - Bh(X,Y).$$

Similarly, $\phi P\nabla_Y X = -A_{\phi X}Y - Bh(X,Y)$, which gives

$$\phi P[X,Y] = A_{\phi X}Y - A_{\phi Y}X.$$

Thus if M is ξ_p-verticle, we see that $[X,Y] \in D^{\perp}$, that is, $P[X,Y] = 0$ if and only if $A_{\phi X}Y = A_{\phi Y}X$. □

5 Parallel horizontal distribution

Definition 5.2. *A non-zero normal vector field N is said to be D-parallel normal section if*

$$\nabla^{\perp}_X N = 0 \quad for \;\; all \;\; X \in D. \tag{5.34}$$

Definition 5.3. *A semi-invariant submanifold M is said to be totally r-contact umbilical if there exists a normal vector H on M such that*

$$h(X,Y) = g(\phi X, \phi Y)H + \sum_{p=1}^{r} \eta_p(X)h(Y,\xi_p) + \sum_{p=1}^{r} \eta_p(Y)h(X,\xi_p) \tag{5.35}$$

for all vector fields X, Y tangent to M.

If $H = 0$, then the fundamental form is given by

$$h(X,Y) = \sum_{p=1}^{r} \eta_p(X)h(Y,\xi_p) + \sum_{p=1}^{r} \eta_p(Y)h(X,\xi_p), \tag{5.36}$$

then M is called totally r-contact geodesic.

Theorem 5.4. *If M is totally r-contact umbilical semi-invariant submanifolds of a generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric metric connection with parallel horizontal distribution, then M is totally r-contact geodesic.*

Proof. Let M be semi-invariant submanifolds of a generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric metric connection. Then from (4.23) and (4.24), we have

$$P\nabla_X \phi PY - PA_{\phi QY}X = \phi P\nabla_X Y - 2\sum_{p=1}^{r} \eta_p(Y)\phi PX,$$

$$Q\nabla_X \phi PY - QA_{\phi QY}X = Bh(X,Y).$$

Adding the last two equations, we have

$$\nabla_X \phi PY - A_{\phi QY}X = \phi P\nabla_X Y + Bh(X,Y). \tag{5.37}$$

Interchanging X and Y in (5.37), we get

$$\nabla_Y \phi PX - A_{\phi QX}Y = \phi P\nabla_Y X + Bh(X,Y). \tag{5.38}$$

Adding (5.37) and (5.38), we get

$$\nabla_X \phi PY + \nabla_Y \phi PX - A_{\phi QY}X - A_{\phi QX}Y = \phi P\nabla_X Y + \phi P\nabla_Y X + 2Bh(X,Y).$$

Taking inner product with Z, we get

$$g(\nabla_X \phi PY + \nabla_Y \phi PX - A_{\phi QY}X - A_{\phi QX}Y, Z) = g(\phi P\nabla_X Y + \phi P\nabla_Y X + 2Bh(X,Y), Z).$$

Splitting the above equation, we get

$$g(\nabla_X \phi PY, Z) + g(\nabla_Y \phi PX, Z) - g(A_{\phi QY}X, Z) - g(A_{\phi QX}Y, Z) = g(\phi P\nabla_X Y, Z)$$

$$+g(\phi P\nabla_Y X, Z) + g[2B(g(\phi X, \phi Y)H + \sum_{p=1}^{r}\eta_p(X)h(Y, \xi_p) + \sum_{p=1}^{r}\eta_p(Y)h(X, \xi_p), Z)].$$

$$g(\nabla_X \phi PY, Z) + g(\nabla_Y \phi PX, Z) - g(h(X, Z), \phi QY) - g(h(Y, Z), \phi QX) = g(\phi P\nabla_X Y, Z)$$

$$+g(\phi P\nabla_Y X, Z) + 2g(\phi X, \phi Y)g(BH, Z) + 2\sum_{p=1}^{r}\eta_p(X)g(Bh(Y, \xi_p), Z) + 2\sum_{p=1}^{r}\eta_p(Y)g(Bh(X, \xi_p), Z).$$

$$= g(\phi P\nabla_X Y, Z) + g(\phi P\nabla_Y X, Z) - 2a^2 g(X, Y)g(BH, Z) - 2\sum_{p=1}^{r}\eta_p(X)\eta_p(Y)g(BH, Z)$$

$$+2\sum_{p=1}^{r}\eta_p(X)g(h(Y, \xi_p), \phi Z) + 2\sum_{p=1}^{r}\eta_p(Y)g(h(X, \xi_p), \phi Z)$$

which by replacing Y by BH and Z by X and then using (5.35), we get

$$g(\nabla_X \phi PBH, X) + g(\nabla_{BH}\phi PX, X) - g(X, X)g(H, \phi QBH) - g(BH, X)g(H, \phi QX) \tag{5.39}$$

$$= g(\phi P\nabla_X BH, X) + g(\phi P\nabla_{BH} X, X) - 2a^2 g(X, BH)g(BH, X) - 2\sum_{p=1}^{r}\eta_p(X)\eta_p(BH)g(BH, X)$$

$$+2\sum_{p=1}^{r}\eta_p(X)g(h(BH, \xi_p), \phi X) + 2\sum_{p=1}^{r}\eta_p(BH)g(h(X, \xi_p), \phi X).$$

For any $X \in D$, we have

$$g(X, BH) = g(\phi X, BH) = 0.$$

Taking covariant differentiation along vector X, we get

$$g(\nabla_X \phi X, BH) + g(\phi X, \nabla_X BH) = 0.$$

As the horizontal distribution D is parallel, so we have

$$g(\phi X, \nabla_X BH) = 0. \tag{5.40}$$

From (5.39) and (5.40), we get

$$g(\nabla_{BH}\phi PX, X) - g(H, \phi QBH) = g(\phi P\nabla_{BH} X, X).$$

For any unit vector $X \in D$, we have

$$g((\nabla_{BH}\phi P)X, X) + g(\phi P\nabla_{BH} X, X) - g(H, \phi QBH) = g(\phi P\nabla_{BH} X, X).$$

$$g((\nabla_{BH}\phi P)X, X) - g(H, \phi QBH) = 0. \tag{5.41}$$

From (5.41), we have

$$g(BH, QBH) + \sum_{p=1}^{r}\eta_p(PH)g(\phi X, X) = 0.$$

Thus we have

$$g((\nabla_{BH}\phi P)X, X) = g(H, \phi QBH) = -g(\phi H, QBH) = -g(BH, QBH) = 0.$$

provided $BH = 0$.
Since $\phi H \in D^{\perp}$, we have $CH = 0$, hence $\phi H = 0$, thus $H = 0$.
Hence M is totally r-contact geodesic. □

Remark 5.1. *For a generalized Kenmotsu manifold with a semi-symmetric metric connection, we have*

$$\bar{\nabla}_X\xi_p = \bar{\bar{\nabla}}_X\xi_p + \sum_{p=1}^{r}\eta_p(\xi_p)X - g(X,\xi_p)\sum_{p=1}^{r}\xi_p \tag{5.42}$$
$$= 2PX + 2QX.$$

Equating the tangential and normal components, we have

$$\bar{\nabla}_X\xi_p = 2PX + 2QX = 2X, \tag{5.43}$$

$$h(X,\xi_p) = 0, \tag{5.44}$$

$$\eta_p(X)\xi_p = 0. \tag{5.45}$$

Also for any $X \in D$, we have

$$g(A_N\xi_p, X) = g(h(X,\xi_p), N) = 0. \tag{5.46}$$

Thus if $X \in D$, then $A_N\xi_p \in D^\perp$ and if $X \in D^\perp$, then $A_N\xi_p \in D$.

Theorem 5.5. *Let M be D-umbilic (resp., $D^\perp$-umbilic) semi-invariant submanifolds of a generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric metric connection. If M is ξ_p-horizontal (resp., ξ_p-verticle) , then it is D-totally geodesic (resp., $D^\perp$-totally geodesic).*

Proof. If M is D-umbilic semi-invariant submanifolds of a generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric metric connection with ξ_p-horizontal, then we have

$$h(X,\xi_p) = g(X,\xi_p)L \tag{5.47}$$

which means that $L = 0$, from which we get $h(X,\xi_p) = 0$. Hence M is D-totally geodesic.
Similarly, we can prove that if M is a $D^\perp$-umbilic semi-invariant submanifold with ξ_p-verticle, then M is $D^\perp$-totally geodesic. □

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Riemann-Liouville Fractional Hermite-Hadamard Inequalities for differentiable $\lambda\varphi$-preinvex functions

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Abstract

In this work, we demonstrate Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via once differentiable and twice differentiable defined using $\lambda\varphi$-preinvex functions.

Keywords: Fractional Hermite-Hadamard ineauqualities, φ-preinvex functions, Riemann-Liouville Fractional Integral.

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1 INTRODUCTION

The recently, Fractional calculus and generalizations is handled much. In especially the issue of fractional calculus is done various applications. These areas is physical sciences, economics, engineering, medicine and biological sciences$[1-8]$.

In this work, we give some Hermite-Hadamard type inequalities and the results via classical Riemann-Liouville fractional integrals for $\lambda\varphi-$preinvex functions by considering recent studies about this field.

2 Preliminaries

In this section, we will give some definitions, lemmas and notations which we use later in this work.

Definition 2.1. (*see* [3]) *Let* $f \in L[a,b]$ *.The Riemann-Liouville fractional integral* $J^{\alpha}_{a+}f$ *and* $J^{\alpha}_{b-}f$ *of order* $\alpha > 0$ *with* $a > 0$ *are defined by*

$$\begin{aligned} J^{\alpha}_{a+}f(x) &= \tfrac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt \quad ,0 \le a < x \le b \\ J^{\alpha}_{b-}f(x) &= \tfrac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt \quad ,0 \le a < x \le b \end{aligned} \tag{2.1}$$

Where Γ *is the gamma function.*

Definition 2.2. (*see* [9]) *The incomplete beta function is defined as follows:*

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt, \tag{2.2}$$

Here $x \in [0,1]\,, a, b > 0.$

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Definition 2.3. (*see* [10]) *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class $MT(I)$ if f is positive and $\forall x, y \in I$ and $t \in (0,1)$ satisfies the inequality:*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{2.3}$$

Definition 2.4. (*see* [11]) *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class $m - MT(I)$ if f is positive and $\forall x, y \in I$ and $t \in (0,1)$, with $m \in [0,1]$ satisfies the inequality:*

$$f(tx + m(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{2.4}$$

Definition 2.5. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to a $\lambda - MT-$convex function or said to belong to the class $\lambda - MT(I)$ if f is positive and $\forall x, y \in I$, $\lambda \in \left(0, \frac{1}{2}\right]$ and $t \in (0,1)$ satisfies the inequality:*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(y). \tag{2.5}$$

Lemma 2.0. (*see* [12]) *Let $f : [a,b] \to \mathbb{R}$ be a once differentiable mapping on (a,b) for $a < b$. If $f' \in L[a,b]$, there is a following equality for fractional integrals*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \\ &= \frac{b-a}{2} \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha} \right] f'(ta + (1-t)b)\, dt. \end{aligned} \tag{2.6}$$

Lemma 2.0. (*see* [13]) *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) for $a < b$. If $f'' \in L[a,b]$, there is following equality for fractional integrals*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \\ &= \frac{(b-a)^2}{2} \int_0^1 \left[\frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] f''(ta + (1-t)b)\, dt. \end{aligned} \tag{2.7}$$

Lemma 2.0. (*see* [14]) *For $t \in [0,1]$,we have*

$$\begin{aligned} (1-t)^m &\leq 2^{1-m} - t^m \quad for\ m \in [0,1], \\ (1-t)^m &\geq 2^{1-m} - t^m \quad for\ m \in [1,\infty). \end{aligned}$$

Let $\mathbb{R}^n$ be Euclidian space and K is said to a nonempty closed in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$, $\varphi : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be a continuous functions.

Definition 2.6. ([15]) *Let $u \in K$. The set K is said to be $\varphi-$invex at u according to η and φ if*

$$u + te^{i\varphi}\eta(v,u) \in K \tag{2.8}$$

for all $u, v \in K$ and $t \in [0,1]$.

Remark 2.1. *Some special cases of Definition 6 are as follows.*

(1) If $\varphi = 0$, there K is defined an invex set.
(2) If $\eta(v,u) = v - u$, there K is defined a $\varphi-$convex set.
(3) If $\varphi = 0$ and $\eta(v,u) = v - u$, there K is defined a convex set.

Definition 2.7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function.Afunction f on the set $K_{\varphi\eta}$ is said to be λ_{φ} − preinvex function according to φ and bifunction η and $\forall u, v \in I$, $t \in (0,1)$ and $0 \leq \varphi \leq \frac{\pi}{2}$ then*

$$f\left(u + te^{i\varphi}\eta(v,u)\right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(u). \tag{2.9}$$

Remark 2.2. *In Definition 7, if $\lambda = \frac{1}{2}$, $\varphi = 0$ and $\eta(v,u) = v - u$. Definition 7 reduces to Definition 3;*

$$f(tv + (1-t)u) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(u).$$

Remark 2.3. *By considering Definition 7, if $\lambda = \frac{1}{2}$, $\varphi = 0$, and $\eta(v,u) = v - u$. for $m \in [0,1]$, we can write;*

$$f\left(mu + te^{i\varphi}\eta(v,mu)\right) = f(tv + m(1-t)u) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(u).$$

Remark 2.4. *In Definition 7, if $\varphi = 0$ and $\eta(v,u) = v - u$. Definition 7 reduces to Definition 5;*

$$f(tv + (1-t)u) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(u).$$

3 Main Results

Lemma 3.0. *Let $f : [a,b] \to \mathbb{R}$ be a once differentiable mappings on (a,b) with $a < b$, $\eta(b,a) > 0$. If $f' \in L\left[a, a+e^{i\varphi}\eta(b,a)\right]$, then the following equality for fractional integral holds:*

$$\begin{aligned}\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} &- \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha} f(a)\right] \\ &= \frac{e^{i\varphi}\eta(b,a)}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right] f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt.\end{aligned} \tag{3.10}$$

Proof. By using Definition 7 and via the partial integration method ,we have following equality.

$$\begin{aligned}&\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right] f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt \\ &= \frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{e^{i\varphi}\eta(b,a)} - \frac{\alpha}{e^{i\varphi}\eta(b,a)} \\ &\quad\times\left[\frac{1}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\int_a^{a+e^{i\varphi}\eta(b,a)}(x-a)^{\alpha-1} f(x)\,dx\right. \\ &\quad\left.+\frac{1}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\int_a^{a+e^{i\varphi}\eta(b,a)}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{\alpha-1} f(x)\,dx\right] \\ &= \frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{e^{i\varphi}\eta(b,a)} - \frac{\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha+1}} \\ &\quad\times\left[J_{a^+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha} f(a)\right].\end{aligned} \tag{3.11}$$

By multiplying the both sides of (3.2) by $\frac{e^{i\varphi}\eta(b,a)}{2}$, we have:

$$\begin{aligned}\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} &- \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha} f(a)\right] \\ &= \frac{e^{i\varphi}\eta(b,a)}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right] f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt.\end{aligned}$$

The proof is done. □

Remark 3.5. *In Lemma 4, if $\varphi = 0$ and $\eta(b,a) = b-a$, Lemma 4 reduces to Lemma 1;*

$$\begin{aligned}\frac{f(a)+f(b)}{2} &- \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b)+J_{b^-}^{\alpha} f(a)\right] \\ &= \frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right] f'(ta+(1-t)b)\,dt.\end{aligned}$$

Theorem 3.1. *Let $I \subseteq \mathbb{R} \to \mathbb{R}$ be a open invex set with respect to bifunction $\eta : I \times I \to \mathbb{R}$ where $\eta(b,a) > 0$. Let $f : [0,b] \to \mathbb{R}$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ decreasing and $\lambda_{\varphi}-$ preinvex function on I for $\alpha > 0$ and $0 \le a < b$, then:*

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha} f(a)\right]\right| \\ &\quad\le \frac{e^{i\varphi}\eta(b,a)}{4}\left[\left|f'(a)\right|+\frac{1-\lambda}{\lambda}\left|f'(b)\right|\right]\left(B_{\frac{1}{2}}\left(\frac{1}{2},\alpha+\frac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\frac{1}{2},\frac{1}{2}\right)\right).\end{aligned}$$

Proof. By using Definition 7 and Lemma 4,we have:

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha} f(a)\right]\right| \\ &\le \frac{e^{i\varphi}\eta(b,a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt \\ &\le \frac{e^{i\varphi}\eta(b,a)}{2}\left[\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt\right. \\ &\quad\left.+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt\right] \\ &\le \frac{e^{i\varphi}\eta(b,a)}{2}\left[\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(a)\right|+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f'(b)\right|\right)dt\right. \\ &\quad\left.+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(a)\right|+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f'(b)\right|\right)dt\right] \\ &\le \frac{e^{i\varphi}\eta(b,a)}{2}\left[\left|f'(a)\right|\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\frac{1}{2\sqrt{t(1-t)}}dt\right. \\ &\quad\left.+\frac{(1-\lambda)}{\lambda}\left|f'(b)\right|\int_0^{\frac{1}{2}}\left[t^{\alpha}-(1-t)^{\alpha}\right]\frac{1}{2\sqrt{t(1-t)}}dt\right] \\ &\le \frac{e^{i\varphi}\eta(b,a)}{4}\left[\left|f'(a)\right|+\frac{1-\lambda}{\lambda}\left|f'(b)\right|\right]\left(B_{\frac{1}{2}}\left(\frac{1}{2},\alpha+\frac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\frac{1}{2},\frac{1}{2}\right)\right)\end{aligned}$$

The proof is done. □

Theorem 3.2. *Let $I = [a, b] \to \mathbb{R}$ be a open invex set with respect to bifunction $\eta : I \times I \to \mathbb{R}$ and $f : [0, b] \to \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$. If $|f'|^q$ is measurable and $|f'|^q$ decreasing and $\lambda_\varphi -$ preinvex function on I for $0 \le a < b$ and $\eta(b, a) > 0$ then:*

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right) + J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha} f(a)\right]\right| \le \frac{e^{i\varphi}\eta(b,a)}{2}\left[\frac{\pi}{4}|f'(a)|^q + \frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)|f'(b)|^q\right]^{\frac{1}{q}}\left(\frac{2-2^{1-\alpha p}}{p\alpha+1}\right)^{\frac{1}{p}}$$

where $\alpha > 0$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Definition 7, Lemma 4 and Hölder's inequality, we have:

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right) + J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha} f(a)\right]\right| \\
&\le \frac{e^{i\varphi}\eta(b,a)}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right| dt \\
&\le \frac{e^{i\varphi}\eta(b,a)}{2}\left(\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{e^{i\varphi}\eta(b,a)}{2}\left(\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|^p dt\right)^{\frac{1}{p}} \\
&\qquad \times\left(\int_0^1\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^q + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f'(b)|^q\right)dt\right)^{\frac{1}{q}} \\
&\le \frac{e^{i\varphi}\eta(b,a)}{2}\left[\frac{\pi}{4}|f'(a)|^q + \frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)|f'(b)|^q\right]^{\frac{1}{q}} \\
&\qquad \times\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha p} - t^{\alpha p}\right]dt + \int_{\frac{1}{2}}^1\left[t^{\alpha p} - (1-t)^{\alpha p}\right]dt\right)^{\frac{1}{p}} \\
&\le \frac{e^{i\varphi}\eta(b,a)}{2}\left[|f'(a)|^q + \frac{1-\lambda}{\lambda}|f'(b)|^q\right]^{\frac{1}{q}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left(\frac{2-2^{1-\alpha p}}{p\alpha+1}\right)^{\frac{1}{p}}.
\end{aligned}$$

Here, we $(A_1 - A_2)^P \le A_1^P - A_2^P$ for any $A_1 > A_2 \ge 0$ and $p \ge 1$. The proof is done. □

Theorem 3.3. *Let $I = [0, b] \to \mathbb{R}$ be a open invex set with respect to bifunction $\eta : I \times I \to \mathbb{R}$ and $f : [0, b] \to \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$, $f' \in L\left[a + e^{i\varphi}\eta(b, a)\right]$. If $|f'|^q$ is measurable and $|f'|^q$ decreasing and $\lambda_\varphi -$ preinvex function on I for $0 \le a < b$ and $\eta(b, a) > 0$ then:*

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right) + J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha} f(a)\right]\right| \\
&\le 2^{-\frac{1}{q}} e^{i\varphi}\eta(b,a)\left(\frac{1-2^{-\alpha}}{\alpha+1}\right)^{\frac{q-1}{q}}\left[\frac{|f'(a)|^q}{2}\left(B_{\frac{1}{2}}\left(\frac{1}{2},\alpha+\frac{1}{2}\right) - B_{\frac{1}{2}}\left(\alpha+\frac{1}{2},\frac{1}{2}\right)\right)\right. \\
&\qquad\left. + \left(\frac{1-\lambda}{\lambda}\right)\frac{|f'(b)|^q}{2}\left(B_{\frac{1}{2}}\left(\frac{1}{2},\alpha+\frac{1}{2}\right) - B_{\frac{1}{2}}\left(\alpha+\frac{1}{2},\frac{1}{2}\right)\right)\right]^{\frac{1}{q}}
\end{aligned}$$

where $\alpha > 0$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Definition 7, Lemma 4 and Power Mean inequality, we have:

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right) + J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha} f(a)\right]\right| \\
&\le \frac{e^{i\varphi}\eta(b,a)}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right| dt \\
&\le \frac{e^{i\varphi}\eta(b,a)}{2}\left(\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right| dt\right)^{1-\frac{1}{q}} \\
&\qquad \times\left(\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{e^{i\varphi}\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha} - t^{\alpha}\right]dt + \int_{\frac{1}{2}}^1\left[t^{\alpha} - (1-t)^{\alpha}\right]dt\right)^{1-\frac{1}{q}} \\
&\qquad \times\left(\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{e^{i\varphi}\eta(b,a)}{2}\left(\frac{2-2^{1-\alpha}}{\alpha+1}\right)^{\frac{q-1}{q}}\left[\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(a)\right|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f'(b)\right|^q\right)dt\right.\\
&\qquad\left.+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(a)\right|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f'(b)\right|^q\right)dt\right]^{\frac{1}{q}}\\
&\leq 2^{-\frac{1}{q}}e^{i\varphi}\eta(b,a)\left(\frac{1-2^{-\alpha}}{\alpha+1}\right)^{\frac{q-1}{q}}\left[\frac{\left|f'(a)\right|^q}{2}\left(B_{\frac{1}{2}}\left(\frac{1}{2},\alpha+\frac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\frac{1}{2},\frac{1}{2}\right)\right)\right.\\
&\qquad\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{\left|f'(b)\right|^q}{2}\left(B_{\frac{1}{2}}\left(\frac{1}{2},\alpha+\frac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\frac{1}{2},\frac{1}{2}\right)\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

The proof is done. □

Lemma 3.0. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mappings on (a,b) with $a<b$, $\eta(b,a)>0$. If $f''\in L\left[a,a+e^{i\varphi}\eta(b,a)\right]$, then the following equality for fractional integral holds:*

$$
\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right|\\
&=\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt.
\end{aligned}
\tag{3.12}
$$

Proof. By using Definition 7 and Lemma 2, if use twice the partial integration method, we have:

$$
\begin{aligned}
&\int_0^1\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right]f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt\\
&=-\left.\frac{\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)}{(\alpha+1)e^{i\varphi}\eta(b,a)}\right|_0^1\\
&\quad+\frac{1}{e^{i\varphi}\eta(b,a)}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt\\
&=\frac{1}{e^{i\varphi}\eta(b,a)}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt
\end{aligned}
\tag{3.13}
$$

Motivated by Lemma 4, then:

$$
\begin{aligned}
&\frac{1}{e^{i\varphi}\eta(b,a)}\left(\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{e^{i\varphi}\eta(b,a)}-\frac{\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha+1}}\right.\\
&\qquad\left.\times\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right)\\
&=\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{\left(e^{i\varphi}\eta(b,a)\right)^2}-\frac{\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha+2}}\\
&\qquad\times\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\bigg).
\end{aligned}
$$

By multipling the both sides of (3.5) by $\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2}$, we have:

$$
\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right|\\
&\qquad=\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2}\int_0^1\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right]f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt
\end{aligned}
$$

The proof is done. □

Remark 3.6. *In Lemma* 5*, if $\varphi=0$ and $\eta(b,a)=b-a$. Lemma* 5 *reduces to Lemma* 2*;*

$$
\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\\
&\qquad=\frac{(b-a)^2}{2}\int_0^1\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right]f''(ta+(1-t)b)\,dt.
\end{aligned}
$$

Theorem 3.4. *Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping. If $|f''|$ is measurable and $|f''|$ is decreasing and $\lambda-$ preinvex function on $[0,b]$ for $0\leq a<b$, $\eta(b,a)>0$ and $\alpha>0$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right|\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{4(\alpha+1)}\left\{\left|f''(a)\right|\left[\frac{\pi}{2}-B\left(\frac{3}{2},\alpha+\frac{3}{2}\right)-B\left(\alpha+\frac{5}{2},\frac{1}{2}\right)\right]\right.\\
&\qquad\left.+\left(\frac{1-\lambda}{\lambda}\right)\left|f''(b)\right|\left[\frac{\pi}{2}-B\left(\frac{1}{2},\alpha+\frac{5}{2}\right)-B\left(\alpha+\frac{3}{2},\frac{3}{2}\right)\right]\right\}.
\end{aligned}
$$

Proof. By using Definition 7 and Lemma 5, we have:

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}+\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f\left(a\right)\right]\right|\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2}\int_0^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\left(a+\left(1-t\right)e^{i\varphi}\eta\left(b,a\right)\right)\right|dt\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\int_0^1\left[1-\left(1-t\right)^{\alpha+1}-t^{\alpha+1}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''\left(a\right)\right|+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''\left(b\right)\right|\right)dt\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left\{\frac{|f''(a)|}{2}\left(\int_0^1t^{\frac{1}{2}}\left(1-t\right)^{\frac{-1}{2}}dt-\int_0^1t^{\frac{1}{2}}\left(1-t\right)^{\alpha+\frac{1}{2}}dt-\int_0^1t^{\alpha+\frac{3}{2}}\left(1-t\right)^{\frac{-1}{2}}dt\right)\right.\\
&\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|}{2}\left(\int_0^1t^{\frac{-1}{2}}\left(1-t\right)^{\frac{1}{2}}dt-\int_0^1t^{\frac{-1}{2}}\left(1-t\right)^{\alpha+\frac{3}{2}}dt-\int_0^1t^{\alpha+\frac{1}{2}}\left(1-t\right)^{\frac{1}{2}}dt\right)\right\}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{4(\alpha+1)}\left\{\left|f''\left(a\right)\right|\left[\frac{\pi}{2}-B\left(\frac{3}{2},\alpha+\frac{3}{2}\right)-B\left(\alpha+\frac{5}{2},\frac{1}{2}\right)\right]\right.\\
&\quad\left.+\left(\frac{1-\lambda}{\lambda}\right)\left|f''\left(b\right)\right|\left[\frac{\pi}{2}-B\left(\frac{1}{2},\alpha+\frac{5}{2}\right)-B\left(\alpha+\frac{3}{2},\frac{3}{2}\right)\right]\right\}.
\end{aligned}$$

The proof is done. □

Theorem 3.5. *Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f''|^q$ is measurable and $|f''|^q$ is decreasing and $\lambda_{\varphi}-$ preinvex function on $[0,b]$ for $\eta\left(b,a\right)>0$ and $0\leq a<b$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f\left(a\right)\right]\right|\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left(1-2^{1-\alpha}\right)\left(\frac{\pi}{4}\left|f''\left(a\right)\right|^q+\frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)\left|f''\left(b\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 7, Lemma 5 and Hölder's inequality we have:

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f\left(a\right)\right]\right|\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2}\int_0^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\left(a+\left(1-t\right)e^{i\varphi}\eta\left(b,a\right)\right)\right|dt\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left(\int_0^1\left[1-\left(1-t\right)^{\alpha+1}-t^{\alpha+1}\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(a+\left(1-t\right)e^{i\varphi}\eta\left(b,a\right)\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left(\int_0^1\left[1-2^{-\alpha}\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''\left(a\right)\right|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''\left(b\right)\right|^q\right)^q dt\right)^{\frac{1}{q}}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left(1-2^{-\alpha}\right)\left(\frac{\pi}{4}\left|f''\left(a\right)\right|^q+\frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)\left|f''\left(b\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof is done. □

Theorem 3.6. *Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f''|^q$ is measurable and $|f''|^q$ is decreasing and $\lambda_{\varphi}-$ preinvex function on $[0,b]$ for $0\leq a<b$ and $\eta\left(b,a\right)>0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f\left(a\right)\right]\right|\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left(1-2^{-\alpha}\right)^{\frac{q-1}{q}}\left(\frac{|f''(a)|^q}{2}\left[B\left(\frac{3}{2},\alpha+\frac{3}{2}\right)+B\left(\alpha+\frac{5}{2},\frac{1}{2}\right)-\frac{\pi}{2}\right]\right.\\
&\qquad\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|^q}{2}\left[B\left(\frac{1}{2},\alpha+\frac{5}{2}\right)+B\left(\alpha+\frac{3}{2},\frac{3}{2}\right)-\frac{\pi}{2}\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 7, Lemma 5 and Power Mean's inequality, we have:

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a+}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha}f\left(a\right)\right]\right|\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2}\int_{0}^{1}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\left(a+(1-t)\,e^{i\varphi}\eta\left(b,a\right)\right)\right|dt\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(\int_{0}^{1}\left|1-(1-t)^{\alpha+1}-t^{\alpha+1}\right|dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{0}^{1}\left|1-(1-t)^{\alpha+1}-t^{\alpha+1}\right|\left|f''\left(a+(1-t)\,e^{i\varphi}\eta\left(b,a\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(\int_{0}^{1}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{0}^{1}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''\left(a\right)\right|^{q}+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''\left(b\right)\right|^{q}\right)dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(1-2^{-\alpha}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\frac{|f''(a)|^{q}}{2}\left(\int_{0}^{1}t^{\frac{1}{2}}\left(1-t\right)^{-\frac{1}{2}}dt-\int_{0}^{1}t^{\frac{1}{2}}\left(1-t\right)^{\alpha+\frac{1}{2}}dt-\int_{0}^{1}t^{\alpha+\frac{3}{2}}\left(1-t\right)^{-\frac{1}{2}}dt\right)\right.\\
&\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|^{q}}{2}\left(\int_{0}^{1}t^{-\frac{1}{2}}\left(1-t\right)^{\frac{1}{2}}dt-\int_{0}^{1}t^{-\frac{1}{2}}\left(1-t\right)^{\alpha+\frac{3}{2}}dt-\int_{0}^{1}t^{\alpha+\frac{1}{2}}\left(1-t\right)^{\frac{1}{2}}dt\right)\right)^{\frac{1}{q}}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(1-2^{-\alpha}\right)^{1-\frac{1}{q}}\left(\frac{|f''(a)|^{q}}{2}\left(\frac{\pi}{2}-B\left(\frac{3}{2},\alpha+\frac{3}{2}\right)-B\left(\alpha+\frac{5}{2},\frac{1}{2}\right)\right)\right.\\
&\qquad\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|^{q}}{2}\left(\frac{\pi}{2}-B\left(\frac{1}{2},\alpha+\frac{5}{2}\right)-B\left(\alpha+\frac{3}{2},\frac{3}{2}\right)\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof is done. □

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A Note on Global Bipartite Domination in Graphs

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Abstract

In this paper we introduce the concept of the *global bipartite domination number* $\gamma_{gb}(G)$ of a connected bipartite graph G and study some of its general properties. Moreover we determine the global bipartite domination number of certain classes of graphs.

Keywords: Domination, global bipartite domination, global bipartite domination number.

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1 Introduction

In this paper we consider simple, connected and bipartite graphs. All notations and definitions not given here can be found in [1, 3]. A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite nonempty set and $E(G)$ is a collection of 2- point subsets of V. The sets $V(G)$ and $E(G)$ are the vertex set and edge set of G respectively. The *degree* of a vertex v in G is the number of edges incident at v. The set of all neighbors of v is the *open neighborhood* of v, denoted by $N(v)$. Let $P_n,\ C_n,\ K_n$ and $K_{m,n}$denote path, cycle, complete graph and complete bipartite graph respectively. The sudivision of the graph G is the graph $S(G)$ obtained from G by subdividing each edge of G. The corona $G \circ K_1$ of G is the graph obtained from G by adding a pendant edge to each vertex of G. A set $A \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is called a *dominating set*, if every vertex $v \in V$ is either an element of A or adjacent to an element of A. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G.

2 Main results

We introduce a new concept, namely, **Global Bipartite Dominating Set** of a simple bipartite graph. Then we define the global bipartite domination number of G.

Definition 2.1. *Let G be a connected bipartite graph with bipartition (X, Y), with $|X| = m$ and $|Y| = n$. The relative complement of G in $K_{m,n}$ denoted by $\widehat{G}$ is the graph obtained by deleting all edges of G from $K_{m,n}$ (i.e., $K_{m,n} \setminus G$). A global bipartite dominating set (GBDS) of G is a set S of vertices of G such that it dominates G and its relative complement $\widehat{G}$. The global bipartite domination number, $\gamma_{gb}(G)$ is the minimum cardinality of a global bipartite dominating set of G.*

Theorem 2.1. *For any connected spanning subgraph G of $K_{m,n}$, $\gamma(G) \leq \gamma_{gb}(G) \leq m + n$.*

Proof. A global bipartite dominating set of G is a dominating set of G and so $\gamma(G) \leq \gamma_{gb}(G)$. The set of all vertices of G is clearly a GBDS of G so, $\gamma_{gb}(G) \leq m + n$. Therefore $\gamma(G) \leq \gamma_{gb}(G) \leq m + n$. □

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Remark 2.1. *The bounds in Theorem 2.1 are sharp. For the complete bipartite graph $G = K_{m,n}$, $\gamma_{gb}(K_{m,n}) = m + n$. For P_4, $\gamma(P_4) = \gamma_{gb}(P_4) = 2$. So $K_{m,n}$ has the largest possible GBD number. Also the bounds in Theorem 2.1 are strict. For the graph $K_{2,3} - e$, $\gamma(K_{2,3} - e) = 2$ and $\gamma_{gb}(K_{2,3} - e) = 4$.*

Theorem 2.2. *If G and $\widehat{G}$ does not contain isolated vertices, then $\gamma_{gb}(G) \leq min\{m, n\}$, where G is a spanning subgraph of $K_{m,n}$.*

Proof. Let (X, Y) be the bipartition of G with $|X| = m \leq |Y| = n$. Since G and $\widehat{G}$ does not contain isolated vertices, X is a G.B.D.S. of G. Therefore $\gamma_{gb}(G) \leq m$. □

Theorem 2.3. *For any positive integers m and n, $\gamma_{gb}(K_{m,n}) = m + n$.*

Proof. Let G be a complete bipartite graph with partitions X and Y. Then $uv \in E(G)$ for every $u \in X$ and $v \in Y$. Let $\widehat{G}$ denotes the relative complement of G in $K_{m,n}$. Then $\widehat{G}$ contains $m + n$ isolated vertices. Hence every global bipartite dominating set of G must contain all vertices of $\widehat{G}$ and so $\gamma_{gb}(G) \geq slant m + n$. Now $V(G)$ is a global bipartite dominating set of G. Hence $\gamma_{gb}(G) = m + n$. □

Theorem 2.4. *For a spanning subgraph G of $K_{m,n}$, a vertex v is in every global bipartite dominating set of G if and only if v is an isolated vertex in $\widehat{G}$.*

Proof. If $|V(G)| \leq 3$, the proof is trivial. So let $|V(G)| > 3$. If v is an isolated vertex in $\widehat{G}$, then v is in every global bipartite dominating set of G. Conversely if v is not an isolated vertex in $\widehat{G}$, then there exist atleast two vertices u and w such that u is adjacent to v in G and w is adjacent to v in $\widehat{G}$. So $V(G) \setminus \{v\}$ is a global bipartite dominating set of G. □

Theorem 2.5. *Let G be a connected bipartite graph with partite sets X and Y. Let $S = V_1 \cup V_2$ be a GBDS of G, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. Then if $V_1 = \phi$, then $V_2 = Y$ and if $V_2 = \phi$, then $V_1 = X$.*

Proof. Let $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. If $V_1 = \phi$, then $S \subseteq Y$. Since G is bipartite, the vertices in Y are not adjacent and so $S \supseteq Y$. Therefore $S = V_2 = Y$. Similarly, we can prove that if $V_2 = \phi$ then $V_1 = X$. □

Theorem 2.6. *Let (X, Y) be the bipartition of a connected graph G. Then X is a GBDS of G if and only if $|N(y)| < |X|$, $\forall y \in Y$.*

Proof. Let X be a GBDS of G. If possible assume that there exists a vertex $y \in Y$ such that $|N(y)| = |X|$. Then y is an isolated vertex in $\widehat{G}$, contradiction to the fact that X is a GBDS of G. Conversely, since G is connected, X is dominating set of G. So it is sufficient to show that X dominates $\widehat{G}$ also. Let $y \in Y$, then $N(y)$ is a proper subset of X. So y is adjacent to at least one vertex of X in $\widehat{G}$. This completes the proof. □

Theorem 2.7. *Let G be a connected sub graph of $K_{m,n}$. Then $\gamma_{gb}(G) = m + n - 1$ if and only if $G \cong K_{m,n} - e$.*

Proof. Let $G \cong K_{m,n} - e$. where $e = uv \in E(K_{m,n})$. So $uv \notin E(G)$ and hence $uv \in E(\widehat{G})$. Since $\widehat{G}$ contains $m + n - 2$ isolated vertices, every global bipartite dominating set of G contains all vertices of $V(G) - \{u, v\}$ and at least one of u and v. Thus

$$\gamma_{gb}(G) \geq m + n - 1 \tag{2.1}$$

Since $V(G) - \{u\}$ is a GBDS of G, it follows that

$$\gamma_{gb}(G) \leq m + n - 1 \tag{2.2}$$

Thus by (1) and (2)we obtain $\gamma_{gb}(G) = m + n - 1$.
Conversely assume that $\gamma_{gb}(G) = m + n - 1$. To prove $G \cong K_{m,n} - e$. We observe that $\gamma_{gb}(K_{m,n}) = m + n$ and $\gamma_{gb}(K_{m,n} - e) = m + n - 1$. Let G be a proper subgraph of $K_{m,n} - e$ containing $m + n$ vertices. Then $\widehat{G}$ contains atmost $m + n - 3$ isolated vertices. In that case $\widehat{G}$ contains a path uvw. Then $V(G) - \{u, w\}$ is a GBDS of G. So $\gamma_{gb}(G) \leq m + n - 2$. This completes the proof. □

Theorem 2.8. *Let G be a graph with bipartition (X, Y). If G has a γ-set $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ then S is a γ_{gb}-set of G if and only if $\bigcap_{x \in V_1} N(x) \subseteq V_2$ and $\bigcap_{y \in V_2} N(y) \subseteq V_1$.*

Proof. Let $\bigcap_{x\in V_1} N(x) \subseteq V_2$ and $\bigcap_{y\in V_2} N(y) \subseteq V_1$. Since S is a γ- set of G, it suffices to show that S dominates the relative compliment of G. Let $u \in X$. If $u \in \bigcap_{y\in V_2} N(y)$, then $u \in V_1$. If $u \notin \bigcap_{y\in V_2} N(y)$ then u is adjacent to atleast one vertex of V_2 in $\widehat{G}$. Similarly, we can prove that if $v \in Y$ then $v \in V_2$ or v is adjacent to atleast one vertex of V_1 in $\widehat{G}$. Conversely, let S dominates $\widehat{G}$. Let x be an arbitrary vertex in X. If $x \in \bigcap_{y\in V_2} N(y)$, then in $\widehat{G}$, x is not adjacent to any vertex of V_2. Since S dominates $\widehat{G}$, we can deduce that $x \in V_1$. If $x \notin \bigcap_{y\in V_2} N(y)$, then x is adjacent to atleast one element of V_2 in $\widehat{G}$. Hence the proof. □

Corollary 1. *Let G be a connected bipartite graph with n vertices, $n \geq 4$. Then $\gamma_{gb}(G \circ K_1) = n$, where $G \circ K_1$ denotes the corona of the graphs G and K_1.*

Proof. If $G \cong K_{1,n}$, the proof is trivial. Otherwise, let (X, Y) be the bipartition of $G \circ K_1$. Let $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$, be the set of all pendant vertices of $G \circ K_1$. Clearly S is γ-set of $G \circ K_1$. Also $\bigcap_{x\in V_1} N(x) = \phi$ and $\bigcap_{y\in V_2} N(y) = \phi$. Therefore the proof follows immediately from theorem 2.8. □

Corollary 2. *For $n \geq 10$, $\gamma_{gb}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. Let $V(P_n) = \{1, 2, 3, \ldots, n\}$. Then $X = \{x : x \text{ is even}, x \leq n\}$, $Y = \{y : y \text{ is odd}, y \leq n\}$ is the bipartition of P_n. Let $S_1 = \{i : i \equiv 1(mod\, 3), i \leq n\}$ and $S_2 = \{i : i + 1 \equiv 0(mod\, 3), i \leq n\}$. Then either S_1 or S_2 is a γ-set of P_n. Also for $i = 1, 2$, $\bigcap_{x\in S_i\cap X} N(x) = \phi$ and $\bigcap_{y\in S_i\cap Y} N(y) = \phi$. Thus the proof follows from theorem 2.8. □

Corollary 3. *For an even integer $n \geq 10$, $\gamma_{gb}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

Proof. The proof is exactly similar to corollary 2. □

Theorem 2.9. *For any two positive integers a and b with $a < b$, there exists a graph G such that $\gamma(G) = a$ and $\gamma_{gb}(G) = b$.*

Proof. Consider the graph $K_{b-a,a}$, with partite sets $W = \{w_1, w_2, \ldots, w_{b-a}\}$ and $U = \{u_1, u_2, \ldots, u_a\}$. Let G be the graph obtained from $K_{b-a,a}$ by adding new vertices $v_1, v_2, \ldots, v_a$and join v_i with u_i for $i = 1, 2, \ldots, a$. Let S be a dominating set of G. Since for each i, v_i is adjacent to u_i only, $|S| \geq a$. Now U is a dominating set of G. So $|S| \leq a$. Hence $\gamma(G) = a$. In $\widehat{G}$, the vertices $w_1, w_2, \ldots, w_{b-a}$ are isolated. So W is a subset of every γ_{gb}-set of G. Therefore the set $\{u_1, u_2, \ldots, u_a, w_1, w_2, \ldots, w_{b-a}\}$ is a γ_{gb}-set of G. Hence $\gamma_{gb}(G) = b$.

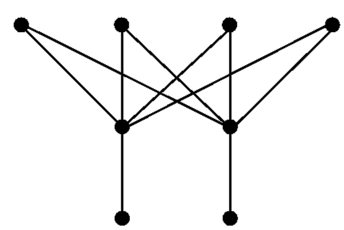

Figure 1: Graph G with $\gamma = 2$ and $\gamma_{gb} = 6$

□

Lemma 2.1. *If G is an r-regular connected bipartite graph with bipartition (X, Y) then $|X| = |Y|$.*

Proof. Each edge in G contributes exactly one to the degree sums $r|X|$ and $r|Y|$. Therefore $r|X| = r|Y| = |E| \Rightarrow |X| = |Y|$. □

Theorem 2.10. *If G is an $n - 1$-regular bipartite graph, then $\gamma_{gb}(G) = n$.*

Proof. Since G is $n-1$ regular, $\widehat{G}$ has n components and all of them are P_2. So $\gamma(\widehat{G}) = n$. Then by theorem 2.8, we can find a γ-set of $\widehat{G}$ such that it dominates G also. Therefore $\gamma_{gb}(G) = n$. □

Theorem 2.11. *Let G be a healthy spider with* $2n+1$ *vertices, then* $\gamma_{gb}(G) = n+1$.

Proof. Let S be a γ-set of G, then $|S| = n$ and $u \notin S$ (see Figure 2). So S dominates all vertices except u in $\widehat{G}$. So $S \cup \{u\}$ is a γ_{gb}-set of G. This completes the proof.

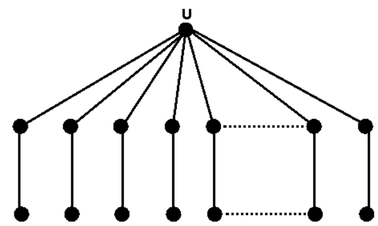

Figure 2: Healthy Spider

□

Theorem 2.12. *If G is a wounded spider with* $n+k+1$ *vertices, then* $\gamma_{gb}(G) = k+1$.

Proof. Observe that $\gamma(G) = k+1$. Also the set $S = \{1, 2, 3, \ldots, k, u\}$ is a γ_{gb}-set of G (see Figure 3).

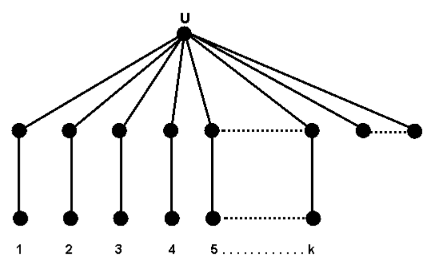

Figure 3: Wounded Spider

□

Theorem 2.13. $\gamma_{gb}(B_n) = 4$, *where* B_n *is the book graph on* $2n+1$ *vertices.*

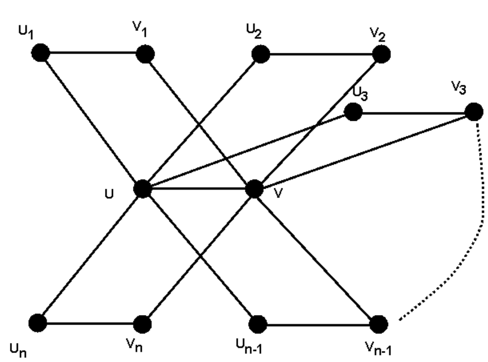

Figure 4: Book Graph

Proof. Let the vertices of B_n be labelled as shown in figure 4. Then $X = \{v, u_1, u_2, \dots, u_n\}$, $Y = \{u, v_1, v_2, \dots, v_n\}$ is the bipartition of B_n. Clearly the set $\{u, v\}$ is the γ-set of B_n. Also $\{u, v, u_1, v_1\}$ is a γ-set of $\widehat{B_n}$. Therefore $\gamma_{gb}(B_n) = 4$. □

Theorem 2.14. *$\gamma_{gb}(S(K_n)) = n$, where $S(K_n)$ is the subdivision of the complete graph K_n.*

Proof. Let X be the set of all old vertices and Y be the set of all new vertices of $S(K_n)$. Then (X, Y) is a bipartition of $S(K_n)$. In $S(K_n)$, the degree of each vertex in X is $n - 1$ and the degree of each vertex in Y is 2. We construct a γ-set of $S(K_n)$ as follows: Let $S \subseteq X$ such that $|S| = n - 2$. Then S dominates all but one vertex u in Y. Also $N(u) = \{x, y\}$ and $X - S = \{x, y\}$. So $S \cup \{u\}$ is a γ-set of $S(K_n)$. Since $S \cup \{u\}$ does not dominate x and y in $\widehat{G}$, this set is not a γ_{gb}-set. So $S \cup \{u, v\}$, where $v \notin N(x) \cup N(y)$, is a γ_{gb}-set of $S(K_n)$. Therefore $\gamma_{gb}(S(K_n)) = n$. □

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Erratum: Certain properties of a subclass of harmonic convex functions of complex order defined by Multiplier transformations-Malaya J. Mat. 4(3)2016, 362-372

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In the paper entitled Certain properties of a subclass of harmonic convex functions of complex order defined by Multiplier transformations- Malaya J. Mat. 4(3)2016, 362-372, the presentation of definition of modified Multiplier transformation of harmonic function $f = h + \overline{g}$ as given below.

$$I_\gamma^0 f(z) = D^0 f(z) = h(z) + \overline{g(z)} \tag{1}$$

$$I_\gamma^1 f(z) = \frac{\gamma D^0 f(z) + D^1 f(z)}{\gamma + 1} \tag{2}$$

$$I_\gamma^n f(z) = I_\gamma^1 (I_\gamma^{n-1} f(z)),\ (n \in N_0) \tag{3}$$

$$I_\gamma^n f(z) = z + \sum_{k=2}^{\infty} (\frac{k+\gamma}{1+\gamma})^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} (\frac{k-\gamma}{1+\gamma})^n \overline{b_k z^k}. \tag{4}$$

Also if f is given by (1) then,

$$I_\gamma^n f(z) = f \tilde{*} \underbrace{(\phi_1(z) + \overline{\phi_2(z)}) \tilde{*} ...... \tilde{*} (\phi_1(z) + \overline{\phi_2(z)})}_{n-times} = h * \underbrace{(\phi_1(z) * ...(\phi_1(z)}_{n-times} + \overline{g + \underbrace{(\phi_2(z) * ...(\phi_2(z)))}_{n-times}}, \tag{5}$$

where $*$ denotes the usual Hadamard product or convolution of power series and

$$\phi_1(z) = \frac{(1+\gamma)z - \gamma z^2}{(1+\gamma)(1-z)^2},\ \phi_2(z) = \frac{(\gamma-1)z - \gamma z^2}{(1+\gamma)(1-z)^2} \tag{6}$$

is taken from the article by Yasar and S. Yalçin [1].

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Caratheodory's Theorem for $\mathbb{B}^{-1}$-convex Sets

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Abstract

In this article, our main concept is $\mathbb{B}^{-1}-$convexity that is a new abstract convexity type. For the $\mathbb{B}^{-1}-$convex sets, Caratheodory's Theorem which is one of the most important results in convexity theory is proved and its corollary is given.

Keywords: Caratheodory's Theorem, $\mathbb{B}^{-1}-$convexity, $\mathbb{B}^{-1}-$convex sets, abstract convexity.

2010 MSC: 52A20, 52A35, 52A05.

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1 Introduction

Caratheodory's Theorem is the fundamental dimensionality result in convexity theory, and it is the source of many other results in which dimensionality is prominent. It is used to prove Helly's Theorem, concerning intersections of convex sets, as well as various results about infinite systems of linear inequalities.

If S is a subset of $\mathbb{R}^n$, the convex hull of S can be obtained by forming all convex combinations of elements of S. According to the classical theorem of Caratheodory, it is not really necessary to form combinations involving more than $n+1$ elements at a time. One can limit attention to convex combinations $\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_m x_m$ such that $m \leq n+1$ (or even to combinations such that $m = n+1$, if one does not insist on the vectors x_i being distinct).

$\mathbb{B}^{-1}$-convexity is an abstract convexity type ([5-7]). In 2012, $\mathbb{B}^{-1}$-convexity is introduced in [1]. Then, $\mathbb{B}^{-1}-$convex sets and their properties examined in [2, 4]. The applications of $\mathbb{B}^{-1}$-convexity to Mathematical Economy is investigated in [3]. Separation of $\mathbb{B}^{-1}-$convex sets by $\mathbb{B}^{-1}-$measurable maps is studied in [8].

In this paper, we examine Caratheodory's Theorem for $\mathbb{B}^{-1}-$convex sets. As being in classic convexity, this theorem is significant in $\mathbb{B}^{-1}-$convexity and it has applications to the Optimization Theory and Mathematical Economy. Since it is used for proving Helly's and Radon Theorems which are thought to be examined for $\mathbb{B}^{-1}-$convexity in next studies, we need to express Caratheodory's Theorem for $\mathbb{B}^{-1}-$convex sets.

The outline of this article is as follows: In Section 2, we recall some definitions and theorems about $\mathbb{B}^{-1}-$convexity. Then, we prove the Caratheodory's Theorem for $\mathbb{B}^{-1}-$convex sets and its corollary in last section.

2 $\mathbb{B}^{-1}-$convexity

For $r \in \mathbb{Z}^-$, the map $x \to \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $K = \mathbb{R} \setminus \{0\}$ to itself; $\boldsymbol{x} = (x_1, x_2, ..., x_n) \to \Phi_r(\boldsymbol{x}) = (\varphi_r(x_1), \varphi_r(x_2), ..., \varphi_r(x_n))$ is homeomorphism from K^n to itself.

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For a finite nonempty set $A = \left\{x^{(1)}, x^{(2)}, ..., x^{(m)}\right\} \subset K^n$ the Φ_r-convex hull (shortly r-convex hull) of A, which we denote $Co^r(A)$ is given by

$$Co^r(A) = \left\{\Phi_r^{-1}\left(\sum_{i=1}^{m} t_i \Phi_r(x^{(i)})\right) : t_i \geq 0, \sum_{i=1}^{m} t_i = 1\right\}.$$

We denote by $\overset{m}{\underset{i=1}{\wedge}} x^{(i)}$ the greatest lower bound with respect to the coordinate-wise order relation of $x^{(1)}, x^{(2)}, ..., x^{(m)} \in \mathbb{R}^n$, that is:

$$\overset{m}{\underset{i=1}{\wedge}} x^{(i)} = \left(\min\left\{x_1^{(1)}, x_1^{(2)}, ..., x_1^{(m)}\right\}, ..., \min\left\{x_n^{(1)}, x_n^{(2)}, ..., x_n^{(m)}\right\}\right)$$

where, $x_j^{(i)}$ denotes jth coordinate of the point $x^{(i)}$.

Thus, we can define $\mathbb{B}^{-1}$-polytopes as follows:

Definition 2.1. *[1] The Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r\in\mathbb{Z}^-}$, denoted by $Co^{-\infty}(A)$ where A is a finite subset of K^n, is called $\mathbb{B}^{-1}$-polytope of A.*

The definition of $\mathbb{B}^{-1}$-polytope can be expressed in the following form in $\mathbb{R}^n_{++} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i > 0,\ i = 1, 2, ..., n\}$.

Theorem 2.1. *[1] For all nonempty finite subsets $A = \left\{x^{(1)}, x^{(2)}, ..., x^{(m)}\right\} \subset \mathbb{R}^n_{++}$ we have*

$$Co^{-\infty}(A) = \lim_{r\to-\infty} Co^r(A) = \left\{\overset{m}{\underset{i=1}{\wedge}} t_i x^{(i)} : t_i \geq 1, \min_{1\leq i\leq m} t_i = 1\right\}.$$

Next, we give the definition of $\mathbb{B}^{-1}$-convex sets.

Definition 2.2. *[1] A subset U of K^n is called a $\mathbb{B}^{-1}$-convex if for all finite subsets $A \subset U$ the $\mathbb{B}^{-1}$-polytope $Co^{-\infty}(A)$ is contained in U.*

By Theorem 2.1, we can reformulate the above definition for subsets of $\mathbb{R}^n_{++}$:

Theorem 2.2. *[1] A subset U of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}$-convex if and only if for all $x^{(1)}, x^{(2)} \in U$ and all $\lambda \in [1, \infty)$ one has $\lambda x^{(1)} \wedge x^{(2)} \in U$.*

Definition 2.3. *Given a set $S \subset K^n$, the intersection of all the B^{-1}-convex subsets of K^n containing S is called the B^{-1}-convex hull of S and is denoted by $\mathbb{B}^{-1}[S]$.*

3 Caratheodory's Theorem for $\mathbb{B}^{-1}$-convex Sets

Lemma 3.1. *In $\mathbb{R}^n_{++}$, a set of the form $\prod_{i=1}^n [x_i, y_i]$ is a $\mathbb{B}^{-1}$-convex set.*

Proof. If $A \subset \prod_{i=1}^n [x_i, y_i]$ then $\Phi_r(A) \subset \prod_{i=1}^n \left[x_i^{2r+1}, y_i^{2r+1}\right]$, from the convexity of a product of intervals we obtain, after taking the inverse image by Φ_r, $Co^r(A) \subset \prod_{i=1}^n [x_i, y_i]$ and therefore $Co^{-\infty}(A) \subset \prod_{i=1}^n [x_i, y_i]$. □

We denote by $\langle L\rangle_m$, the family of nonempty subsets of L of cardinality at most m.

Theorem 3.3. *(Carathedory's Theorem) If L is a compact subset of $\mathbb{R}^n_{++}$ then*

$$Co^{-\infty}(L) = \bigcup_{A\in\langle L\rangle_{n+1}} Co^{-\infty}(A)$$

Consequently, for all subsets L of $\mathbb{R}^n_{++}$,

$$\mathbb{B}^{-1}[L] = \bigcup_{A\in\langle L\rangle_{n+1}} \mathbb{B}^{-1}[A] = \bigcup_{A\in\langle L\rangle_{n+1}} Co^{-\infty}(A);$$

and, if L is compact, $\mathbb{B}^{-1}[L] = Co^{-\infty}(L)$.

Proof. If $x \in Co^{-\infty}(L)$ then there is a sequence $\left(x_{r_k}\right)_{r_k \in \mathbb{N}}$ with $x_{r_k} \in Co^{-r_k}(L)$, $\forall k \in \mathbb{N}$ which converges to x. But from Caratheodorys theorem, there is, for each k, a set of points $x_k^1, x_k^2, ..., x_k^{n+1}$ in L and a set of numbers $\rho_k^1, \rho_k^2, ..., \rho_k^{n+1}$ in $[1, +\infty)$ such that

$$\sum_{j=1}^{n+1} \left(\rho_k^j\right)^{-2r_k+1} = 1$$

and

$$\Phi_{-r_k}\left(x_{r_k}\right) = \sum_{j=1}^{n+1} \left(\rho_k^j\right)^{-2r_k+1} \Phi_{-r_k}\left(x_k^j\right)$$

or, for $i = 1, 2, ..., n$,

$$x_{r_k,i} = \left(\sum_{j=1}^{n+1} \left(\rho_k^j x_{k,i}^j\right)^{-2r_k+1}\right)^{\frac{1}{-2r_k+1}}$$

Since L is compact we can without loss of generality assume that each of the sequences $\left(x_k^j\right)_{k \in \mathbb{N}}$, $j = 1, 2, ..., n+1$ converges in L to a point x^j, and also that each of the sequences ρ_k^j, $j = 1, 2, ..., n+1$ converges in L to a point ρ^j in $[1, +\infty)$. Taking into account that all the numbers involved are positive we have

$$\lim_{k \to \infty} \left(\sum_{j=1}^{n+1} \left(\rho_k^j x_{k,i}^j\right)^{-2r_k+1}\right)^{\frac{1}{-2r_k+1}} = \min_{1 \le j \le n+1} \left\{\rho^j x_i^j\right\}$$

moreover

$$\min_{1 \le j \le n+1} \left\{\rho^j\right\} = 1 \, .$$

Taking the limit componentwise we obtain $x = \wedge_{j=1}^{n+1} \rho^j x^j$, with $\rho^j \ge 1$ for all j and $\min_{1 \le j \le n+1} \left\{\rho^j\right\} = 1$. We have shown that $x \in Co^{-\infty}(A)$ with $A = \left\{x^1, x^2, ..., x^{n+1}\right\} \subset L$. The last formula follows from $\mathbb{B}^{-1}[A] = Co^{-\infty}(A)$ for all finite sets A, $\mathbb{B}^{-1}[L] = \bigcup_{A \in \langle L \rangle} Co^{-\infty}(A)$ and the first part applied to the finite sets $A \in \langle L \rangle$. □

Corollary 3.1. *If L is a compact subset of* $\mathbb{R}_{++}^n$ *then* $\mathbb{B}^{-1}[L]$ *is compact.*

Proof. If $L \subset \prod_{i=1}^n [a_i, b_i]$ then $Co^{-\infty}(L) \subset \prod_{i=1}^n [x_i, y_i]$; $Co^{-\infty}(L)$ is therefore compact. The equality $\mathbb{B}^{-1}[L] = Co^{-\infty}(L)$ concludes the proof. □

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Fractional integral Chebyshev inequality without synchronous functions condition

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Abstract

In this paper, we use the Riemann-Liouville fractional integrals to establish some new integral inequalities related to the Chebyshev inequality in the case where the synchronicity of the given functions is replaced by another condition. This paper generalises some recent results in the paper of [C.P. Niculescu and I. Roventa: An extension of Chebyshev's algebraic inequality, Math. Reports, 2013].

Keywords: Integral inequalities, Riemann-Liouville integral, Chebyshev inequality.

2010 MSC: 26D10, 26A33. ©2012 MJM. All rights reserved.

1 Introduction

Let us consider the Chebyshev inequality [10]

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1.1}$$

where f and g are two integrable and synchronous functions on $[a,b]$ i.e. $(f(x)-f(y)(g(x)-g(y))\geq 0, x,y\in [a,b]$.
Many researchers have given considerable attention to (1.1), see [2, 4, 7, 11–13, 15] and the references therein. For the fractional integration case, it has been proved in [1] that for any synchronous functions f and g on $[a,b]$, the fractional inequality

$$J^{\alpha}(1)J^{\alpha}fg(x)\geq J^{\alpha}f(x)J^{\alpha}g(x), x\in[a,b] \tag{1.2}$$

is valid.
For more information and applications on Chebyshev inequality, we refer the reader to [3, 5, 6, 9, 14, 16].
On the other hand, recently in [11], C.P. Niculescu and L. Roventa have proved that for two functions f and g of the space $L^{\infty}([a,b])$, the Chebyshev's inequality still works by assuming the condition:

$$\left(f(x)-\frac{1}{x-a}\int_a^b f(x)dx\right)\left(g(x)-\frac{1}{x-a}\int_a^b g(x)dx\right)\geq 0. \tag{1.3}$$

The main purpose of this paper is to establish some new results for (1.1) by using the Riemann-Liouville fractional integrals. We present our results in the case where the synchronicity of the given functions is replaced by another condition that is more general than that presented in [11]. For our results, Theorem 1 of [11] can be deduced as a special case.

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2 Preliminaries

In this section, we present some preliminaries on Riemann-Liouville fractional integration.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a,b]$ is defined as*

$$\begin{aligned} J_a^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \ \alpha > 0, \ a < t \leq b, \\ J_a^{0} f(t) &= f(t), \end{aligned} \tag{2.4}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

For $\alpha > 0$, $\beta > 0$, we have the following two properties:

$$J_a^{\alpha} J_a^{\beta} f(t) = J_a^{\alpha+\beta} f(t) \tag{2.5}$$

and

$$J_a^{\alpha} J_a^{\beta} f(t) = J_a^{\beta} J_a^{\alpha} f(t). \tag{2.6}$$

For more details, one can consult [8].

3 Main Results

Lemma 3.1. *Let f and g be two functions belonging to $L^{\infty}([a,b])$, then for all $x \in \,]a,b]\,, \alpha \geq 1$, we have*

$$\begin{aligned} &\frac{1}{x-a} J_a^{\alpha} f(x) g(x) \\ = &\left(\frac{1}{x-a}\int_a^x f(s)ds\right)\left(\frac{1}{x-a} J_a^{\alpha} g(x)\right) \\ &+\frac{1}{(x-a)\,\Gamma(\alpha)}\int_a^x \left[\left(f(t)-\frac{1}{t-a}\int_a^t f(s)ds\right)\left((x-t)^{\alpha-1}g(t)-\frac{1}{t-a}\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\right] dt. \end{aligned} \tag{3.7}$$

Proof. We have:

$$\begin{aligned} &\int_a^x f(t)(x-t)^{\alpha-1} g(t)dt \\ = &\left(f(t)\int_a^t (x-s)^{\alpha-1} g(s)ds\right)_{t=a}^{t=x} - \int_a^x \left(f'(t)\int_a^t (x-s)^{\alpha-1}g(s)ds\right) dt \\ = &f(x)\int_a^x \left((x-s)^{\alpha-1}g(s)ds - \int_a^x ((t-a)f'(t))\left(\frac{1}{t-a}\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\right) dt. \end{aligned} \tag{3.8}$$

To integrate by part, let us take the quantities

$$u(t) = \frac{1}{t-a}\int_a^t (x-s)^{\alpha-1}g(s)ds, u(t)' = \frac{-1}{(t-a)^2}\int_a^t (x-s)^{\alpha-1}g(s)ds + \frac{(x-t)^{\alpha-1}}{(t-a)}g(t)$$

and

$$v'(t) = (t-a)f'(t), v(t) = \int_a^t (s-a)f'(s)ds = (t-a)f(t) - \int_a^t f(s)ds.$$

So, it yields that

$$\begin{aligned} &\int_a^x f(t)(x-t)^{\alpha-1}g(t)dt \\ = &f(x)\int_a^x (x-s)^{\alpha-1}g(s)ds \\ &-\left[\left(\frac{1}{t-a}\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\left((t-a)f(t)-\int_a^t f(s)ds\right)\right]_{t=a}^{t=x} \\ &-\int_a^x \left[\left(\frac{1}{(t-a)^2}\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\left((t-a)f(t)-\int_a^t f(s)ds\right)\right] dt \\ &+\int_a^x \left[\left(\frac{(x-t)^{\alpha-1}}{(t-a)}g(t)\right)\left((t-a)f(t)-\int_a^t f(s)ds\right)\right] dt. \end{aligned} \tag{3.9}$$

Consequently,

$$\begin{aligned}
&\int_a^x f(t)(x-t)^{\alpha-1}g(t)dt\\
=\ &\frac{1}{x-a}\left(\int_a^x f(s)ds\right)\left(\int_a^x (x-s)^{\alpha-1}g(s)ds\right)\\
&-\int_a^x \frac{1}{(t-a)^2}\left(\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\left((t-a)f(t)-\int_a^t f(s)ds\right)dt\\
&+\int_a^x \left(\frac{1}{t-a}(x-t)^{\alpha-1}g(t)\right)\left((t-a)f(t)-\int_a^t f(s)ds\right)dt.
\end{aligned}\tag{3.10}$$

Therefore,

$$\begin{aligned}
&\int_a^x f(t)(x-t)^{\alpha-1}g(t)dt\\
=\ &\frac{1}{x-a}\left(\int_a^x f(s)ds\right)\left(\int_a^x (x-s)^{\alpha-1}g(s)ds\right)\\
&-\int_a^x \left[\frac{1}{(t-a)}\left(\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\frac{1}{(t-a)}\left((t-a)f(t)-\int_a^t f(s)ds\right)\right]dt\\
&+\int_a^x \left[\left(\frac{1}{t-a}(x-t)^{\alpha-1}g(t)\right)\left((t-a)f(t)-\int_a^t f(s)ds\right)\right]dt.
\end{aligned}\tag{3.11}$$

Hence,

$$\begin{aligned}
&\int_a^x f(t)(x-t)^{\alpha-1}g(t)dt\\
=\ &\frac{1}{x-a}\left(\int_a^x f(s)ds\right)\left(\int_a^x (x-s)^{\alpha-1}g(s)ds\right)\\
&-\int_a^x \left[\frac{1}{(t-a)}\left(\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\left(f(t)-\frac{1}{(t-a)}\int_a^t f(s)ds\right)\right]dt\\
&+\int_a^x \left[(x-t)^{\alpha-1}g(t)\left(f(t)-\frac{1}{t-a}\int_a^t f(s)ds\right)\right]dt. \qquad (3.12)\\
=\ &\frac{1}{x-a}\left(\int_a^x f(s)ds\right)\left(\int_a^x (x-s)^{\alpha-1}g(s)ds\right)\\
&+\int_a^x \left[\left(f(t)-\frac{1}{t-a}\int_a^t f(s)ds\right)\left((x-t)^{\alpha-1}g(t)-\frac{1}{t-a}\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\right]dt,
\end{aligned}$$

and then,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}g(t)dt\\
=\ &\frac{1}{x-a}\left(\int_a^x f(s)ds\right)\left(\frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}g(s)ds\right) \qquad (3.13)\\
&+\frac{1}{\Gamma(\alpha)}\int_a^x \left[\left(f(t)-\frac{1}{t-a}\int_a^t f(s)ds\right)\left((x-t)^{\alpha-1}g(t)-\frac{1}{t-a}\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\right]dt.
\end{aligned}$$

So,

$$\begin{aligned}
&J_a^\alpha f(x)g(x)\\
=\ &\frac{1}{x-a}\left(\int_a^x f(s)ds\right)J_a^\alpha g(x) \qquad (3.14)\\
&+\frac{1}{\Gamma(\alpha)}\int_a^x \left[\left(f(t)-\frac{1}{t-a}\int_a^t f(s)ds\right)\left((x-t)^{\alpha-1}g(t)-\frac{1}{t-a}\int_a^t (x-s)^{\alpha-1}g(s)ds\right)\right]dt.
\end{aligned}$$

Consequently, we obtain (3.7). □

An immediate consequence of the previous Lemma is the following result:

Theorem 3.1. *Let f and g be two functions of the space $L^{\infty}([a,b])$ and suppose that for any $\alpha \geq 1$ and for any $t, x \in]a,b]; t \leq x \leq b$, the inequality*

$$\left(f(t) - \frac{1}{t-a}\int_a^t f(s)ds\right)\left((x-t)^{\alpha-1}g(t) - \frac{1}{t-a}\int_a^t (x-s)^{\alpha-1}g(s)ds\right) \geq 0$$

is satisfied.
Then, we have:

$$\begin{aligned} &\frac{1}{x-a}J_a^{\alpha}f(x)g(x) \\ \geq\ &\left(\frac{1}{x-a}\int_a^x f(s)ds\right)\left(\frac{1}{x-a}J_a^{\alpha}g(x)\right). \end{aligned} \tag{3.15}$$

Remark 3.1. *Taking $\alpha = 1, x = b$ in Theorem 3.1, we obtain Theorem 1 of [11].*

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An efficient modification of PIM by using Chebyshev polynomials

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Abstract

In this article, an efficient modification of the Picard iteration method (PIM) is presented by using Chebyshev polynomials. Special attention is given to study the convergence of the proposed method. The proposed modification is tested for some examples to demonstrate reliability and efficiency of the introduced method. A comparison between our numerical results against the conventional numerical method, fourth-order Runge-Kutta method (RK4) is given. From the presented examples, we found that the proposed method can be applied to wide class of non-linear ordinary differential equations.

Keywords: Picard iteration method, Chebyshev polynomials, Runge-Kutta method, Convergence analysis.

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1 Introduction

Many different approximate methods have recently introduced to solve non-linear problems of differential equations, such as, variational iteration method ([3], [8], [18], [19], [22]), Adomian decomposition method ([1], [10], [23]), homotopy perturbation method ([6], [20]) and spectral collocation method ([6], [17]). The Adomian decomposition method provides solutions as a series by employing the so-called Adomian's polynomials which are related to the derivatives of the nonlinearities; therefore, these nonlinearities must be analytical functions of the dependent variables and this has often been ignored in the literature, for the existence and the uniqueness of solutions to, for example, initial-value problems in ODEs is ensured under much milder conditions ([4], [14]). However, the decomposition method may be formulated in a manner that does not require that the nonlinearities be differentiable with respect to the dependent variables and their derivatives [15]. Other techniques also require that the nonlinearities be analytical functions of the dependent variable and provide either convergent series or asymptotic expansions to the solution include perturbation methods [13], the homotopy perturbation technique and the homotopy analysis procedure [21].

By way of contrast, iterative techniques for solving a large class of linear or non-linear differential equations without the tangible restriction of sensitivity to the degree of the non-linear term and also it reduces the size of calculations besides, its interactions are direct and straightforward. These techniques include the well-known Picard fixed-point iterative procedure.

In this paper, we present a modification of PIM. This modification depends on the useful properties of the Chebyshev polynomials. Special attention is given to study the convergence analysis of the proposed method. Convergence analysis is reliable enough to estimate the maximum absolute error of the solution given by PIM. To guarantee this study, effectively employ this modification to a certain class of non-linear ODEs. Therefore, this modification of PIM has been widely used for solving non-linear problems to overcome the shortcoming of other methods.

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The rest of this paper is organized as follows: Section 2 is assigned to the analysis of the standard PIM. Section 3 is assigned to the convergence study of the proposed method. In section 4, some test problems have been solved by the modified PIM, to illustrate the efficiency of the proposed method. In finally, the paper ends with the conclusions in section 5.

2 Picard iteration method

To illustrate the analysis of PIM, we limit ourselves to consider the following non-linear first order ODE in the type ([5], [9], [16])

$$u'(x) = R\,u + N(u), \qquad u(0) = c, \quad 0 < x < a, \tag{2.1}$$

here R is a linear bounded operator i.e., it is possible to find a number $m_1 > 0$ such that $||R\,u|| \leq m_1||u||$. The non-linear term $N(u)$ is Lipschitz continuous with $|N(u) - N(v)| \leq m_2\,|u - v|, \forall\, x \in J = [0, a]$, for any constant $m_2 > 0$.
The PIM gives the possibility to write the solution of Eq.(2.1) in the following iteration formula

$$u_p(x) = u(0) + \int_0^x [R\,u_{p-1}(\tau) + N(u_{p-1}(\tau))\,]d\tau, \qquad p \geq 1. \tag{2.2}$$

The successive approximations u_p, $p \geq 0$, of the solution $u(x)$ will be readily obtained upon using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0. In this technique we obtain a sequence of components of the solution $u(x)$. Consequently, the exact solution may be obtained by using

$$u(x) = \lim_{p\to\infty} u_p(x). \tag{2.3}$$

3 Convergence analysis

In this section, the sufficient conditions are presented to guarantee the convergence of PIM, when applied to solve non-linear ODEs, where the main point is that we prove the convergence of the recurrence sequence ([2], [12]), which is generated by using PIM.

Lemma 3.1. *Let $A : U \to V$ be a bounded linear operator and let $\{u_p\}$ be a convergent sequence in U with limit u, then $u_p \to u$ in U implies that $A(u_p) \to A(u)$ in V [12].*

Now, to prove the convergence of the sequence of solution using the Picard iteration method, we will rewrite Eq.(2.2) in an operator form as follows

$$u_p = A[u_{p-1}], \tag{3.4}$$

where the operator A takes the following form

$$A\,[u] = u(0) + \int_0^x [R\,u + N(u)\,]\,d\tau. \tag{3.5}$$

Theorem 3.1. *Assume that X be a Banach space and $A : X \to X$ is a nonlinear mapping, and suppose that*

$$||\,A[u] - A[v]\,|| \leq \alpha\,||u - v||, \quad \forall\;\, u, v \in X, \tag{3.6}$$

for any constant $\alpha = (m_1 + m_2)a$ $(0 < \alpha < 1)$ where m_1, m_2 and a are defined above. Then A has a unique fixed point. Furthermore, the sequence (2.2) using PIM with an arbitrary choice of $u(0) \in X$, converges to the fixed point of A and

$$||u_p - u_q|| \leq \frac{\alpha^q}{1-\alpha}\,||u_1 - u_0||. \tag{3.7}$$

Proof. Denoting $(C[J], ||.||)$ Banach space of all continuous functions on J with the norm defined by

$$||u(x)|| = \max_{x\in J} |u(x)|.$$

We are going to prove that the sequence $\{u_p\}$ is a Cauchy sequence in this Banach space

$$\begin{aligned}\|u_p - u_q\| &= \max_{x\in J} \mid u_p - u_q \mid \\ &= \max_{x\in J} \Big| \int_0^x [\, R\,(u_{p-1} - u_{q-1}) + N(u_{p-1}) - N(u_{q-1})\,]\, d\tau \Big| \\ &\le \max_{x\in J} \int_0^x [\, |R\,(u_{p-1} - u_{q-1})| + |N(u_{p-1}) - N(u_{q-1})|\,]\, d\tau \\ &\le \max_{x\in J} \int_0^x [\, (m_1 + m_2)(u_{p-1} - u_{q-1})\,]\, d\tau \\ &\le \alpha ||u_{p-1} - u_{q-1}||.\end{aligned}$$

Let, $p = q + 1$ then

$$\|u_{q+1} - u_q\| \le \alpha\, \|u_q - u_{q-1}\| \le \alpha^2\, \|u_{q-1} - u_{q-2}\| \le ... \le \alpha^q\, \|u_1 - u_0\|.$$

From the triangle inequality we have

$$\begin{aligned}||u_p - u_q|| &\le ||u_{q+1} - u_q|| + ||u_{q+2} - u_{q+1}|| + ... + ||u_p - u_{p-1}|| \\ &\le [\, \alpha^q + \alpha^{q+1} + ... + \alpha^{p-1}\,]\, ||u_1 - u_0|| \\ &\le \alpha^q [\, 1 + \alpha + \alpha^2 + ... + \alpha^{p-q-1}\,]\, ||u_1 - u_0|| \\ &\le \alpha^q [\, \frac{1 - \alpha^{p-q-1}}{1 - \alpha}\,]\, ||u_1 - u_0||.\end{aligned}$$

Since $0 < \alpha < 1$ so, $(1 - \alpha^{p-q-1}) < 1$ then

$$||u_p - u_q|| \le \frac{\alpha^q}{1 - \alpha} ||u_1 - u_0||.$$

But $||u_1 - u_0|| < \infty$ so, as $q \to \infty$ then $||u_p - u_q|| \to 0$. We conclude that $\{u_p\}$ is a Cauchy sequence in $C[J]$ so, the sequence converges and the proof is complete. □

Theorem 3.2. *The maximum absolute error of the approximate solution u_p to problem (2.1) is estimated to be*

$$\max_{t\in J} |u_{exact} - u_p| \le \beta, \tag{3.8}$$

where $\beta = \dfrac{\alpha^q\, a\, [\, m_1\, ||u_0|| + k\,]}{1 - \alpha}$, $\qquad k = \max_{x\in J} |N(u_0)|.$

Proof. From Theorem 1 and inequality (3.7) we have

$$||u_p - u_q|| \le \frac{\alpha^q}{1 - \alpha} ||u_1 - u_0||,$$

as $p \to \infty$ then $u_p \to u_{\text{exact}}$ and

$$||u_1 - u_0|| = \max_{x\in J} \Big| \int_0^x [\, R\, u_0 + N(u_0)\,]\, d\tau \Big| \le \max_{x\in J} \int_0^x [\, |R\, u_0| + |N(u_0)|\,]\, d\tau \le a\, [\, m_1\, ||u_0|| + k\,],$$

so, the maximum absolute error in the interval J is

$$||u_{\text{exact}} - u_p|| = \max_{x\in J} |u_{\text{exact}} - u_p| \le \beta.$$

This completes the proof. □

Our main goal in this paper is concerned with the implementation of PIM and its modification which have efficiently used to solve a certain class of ODEs. To achieve this goal, at the beginning of implementation of PIM, we use the orthogonal Chebyshev polynomials to expand the functions in the non-homogeneous term in the considered differential equation [17].

4 Solution procedure using the modified PIM

In this section, an efficient modification of PIM is presented by using Chebyshev polynomials. The well known Chebyshev polynomials [17] are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula

$$T_{n+1}(z) = 2z\,T_n(z) - T_{n-1}(z), \qquad n = 1,2,\dots.$$

The first three Chebyshev polynomials are $T_0(z) = 1,\ T_1(z) = z,\ T_2(z) = 2z^2 - 1$.

Theorem 4.3. *The error in approximating $f(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$f_m(x) = \sum_{k=0}^{m} c_k\,T_k(x), \tag{4.9}$$

then, for all $f(x)$, all m, and all $x \in [-1,1]$, we have

$$E_T(m) \equiv |f(x) - f_m(x)| \leq \sum_{k=m+1}^{\infty} |c_k|. \tag{4.10}$$

Proof. The Chebyshev polynomials are bounded by one, that is, $|T_k(x)| \leq 1$ for all $x \in [-1,1]$ and for all k. This implies that the k-th term is bounded by $|c_k|$. Subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds gives the theorem. □

For more details about the definition of the Chebyshev polynomials and its properties see ([7], [11], [17]). Now, in order to use these polynomials on the interval $[0,1]$ we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = 2x - 1$. Let the shifted Chebyshev polynomials $T_n(2x-1)$ be denoted by $T_n^*(x)$. Then $T_n^*(x)$ can be obtained as follows

$$T_{n+1}^*(x) = 2(2x-1)\,T_n^*(x) - T_{n-1}^*(x), \qquad n = 1,2,\dots. \tag{4.11}$$

Now, we use the shifted Chebyshev expansion to expand $f(x)$ in the following form

$$f(x) \approx f_m(x) = \sum_{k=0}^{m} c_k\,T_k^*(x), \tag{4.12}$$

where the constant coefficients c_k are defined by

$$c_k = \frac{2}{\pi\,h_k}\int_0^1 \frac{f(x)\,T_k^*(x)}{\sqrt{x-x^2}}\,dx, \qquad h_0 = 2, \quad h_k = 1, \quad k = 1,2,\dots. \tag{4.13}$$

Now, the proposed modification will implement to solve the following two initial non-linear ordinary differential equations.

Model problem 1

Consider the following non-linear ordinary differential equation

$$u''(x) + x\,u'(x) + x^2\,u^3(x) = f(x), \quad x \in [0,1], \tag{4.14}$$

where $f(x) = (2+6x^2)e^{x^2} + x^2\,e^{3x^2}$ and subject to the following initial conditions

$$u(0) = 1, \qquad u'(0) = 0. \tag{4.15}$$

The exact solution of this problem is $u(x) = e^{x^2}$.
The procedure of the solution follows the following two steps:
Step 1. Expand the function $f(x)$ using shifted Chebyshev polynomials:

Using the above consideration, the function $f(x)$ can be approximated by eight terms ($m = 8$) of the shifted Chebyshev expansion (4.12) as follows

$$f_C(x) \approx 2.00232 - 0.358488\,x + 18.0328\,x^2 - 86.4534\,x^3 + 416.556\,x^4 - 1042.66\,x^5 + 1502.72x^6 - 1134.64x^7 + 366.624x^8.$$

Step 2. Implementation of PIM:

To solve Eq.(4.14) by the PIM we reduce this equation to the following system of first order ODEs

$$u'(x) = v(x), \tag{4.16}$$

$$v'(x) = -x\,v(x) - x^2\,u^3(x) + f(x), \tag{4.17}$$

with the following initial conditions $u(0) = 1,\ v(0) = 0$.
Now, the PIM gives the possibility to write the solution of the system (4.16)-(4.17) with the aid of the following iteration formula

$$u_{n+1}(x) = u_0 + \int_0^x v_n(\tau)d\tau, \qquad n \geq 0, \tag{4.18}$$

$$v_{n+1}(x) = v_0 - \int_0^x [\tau\,v_n(\tau) + \tau^2\,u_n^3(\tau) - f(\tau)\,]d\tau, \qquad n \geq 0. \tag{4.19}$$

We start with initial approximations $u_0 = 1$, $v_0 = 0$, and by using the above iteration formulae (4.18)-(4.19), we can directly obtain the components of the solution.
Now, the first three components of the solution $u(x)$ of Eq.(4.14) by using (4.18)-(4.19) are

$$u_0(x) = 1,$$

$$u_1(x) = 1,$$

$$u_2(x) = 1 + 1.00116x^2 - 0.059748x^3 + 1.4194x^4 - 4.32267x^5 + 13.8852x^6 - 24.8252x^7$$
$$+26.8343x^8 - 15.7589x^9 + 4.0736x^{10} + ...,$$

$$u_3(x) = 1 + 1.00116x^2 - 0.059748x^3 + 1.25254x^4 - 4.31371x^5 + 13.6959x^6 - 24.3106x^7 + 25.3466x^8$$
$$-13.3453x^9 + 1.68833x^{10} + 1.28936x^{11} - 0.308606x^{12} + ...$$

Now, also to perform PIM, we can expand the function $f(x)$ using Taylor series at the point $x = x_0$ as follows

$$f(x) \approx \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}\,(x - x_0)^k, \tag{4.20}$$

for an arbitrary integer number m.
If we expand the function $f(x)$ by the Taylor series (4.20) about the point $x_0 = 0$ with eight terms, we have

$$f_T(x) \approx 2 + 9\,x^2 + 10\,x^4 + 7.83\,x^6 + 5.58333\,x^8 + O(x^9).$$

So, the first three components of the solution by using (4.18)-(4.19) are

$$u_0(x) = 1,$$
$$u_1(x) = 1,$$
$$u_2(x) = 1 + x^2 + 0.666667x^4 + 0.333333x^6 + 0.139881x^8 + 0.062037x^{10},$$
$$u_3(x) = 1 + x^2 + 0.5x^4 + 0.244444x^6 + 0.104167x^8 + 0.0496032x^{10} - 0.00469978x^{12}\,.$$

Also, to solve the same problem (4.14) using the fourth-order Runge-Kutta method, we used its corresponding system of ODEs (4.16)-(4.17).

The absolute errors between the function $f(x)$ and its approximation by using the Taylor expansion (Top) and the Chebyshev expansion (Bottom) are presented in figure 1.
The absolute error between the exact solution $u(x)$ and the approximate solution $u_C(x) = u_4(x)$ (after four iterations) and using the Chebyshev expansion for $f(x)$ with $m = 8$ is presented in figure 2(Right). Also, the absolute error between the exact solution $u(x)$ and the approximate solution $u_T(x) = u_4(x)$ (after four iterations) using the Taylor expansion for $f(x)$ with eight terms is presented in figure 2(Left).

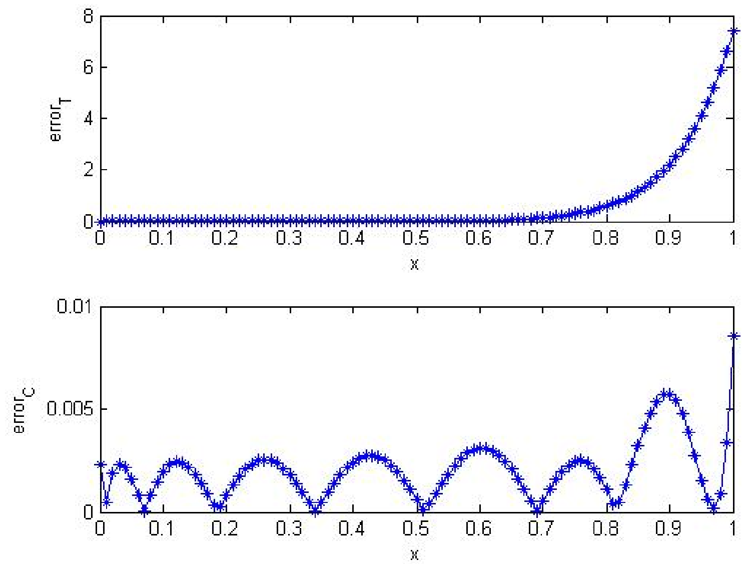

Figure 1: The absolute error: $|f(x) - f_T(x)|$ (Top) and $|f(x) - f_C(x)|$ (Bottom).

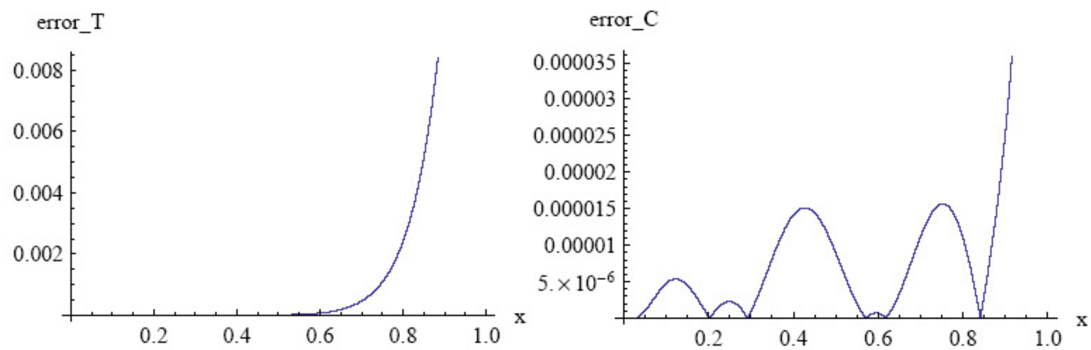

Figure 2: The absolute error $|u(x) - u_T(x)|$ (Left) and $|u(x) - u_C(x)|$ (Right).

Also, the figure 3 presents a comparison between the exact solution $u(x)$, with the numerical solution u_{RK4} using fourth-order Runge-Kutta and the approximate solution of our proposed method $u_C(x)$. From this figure, we can see that the two methods are in excellent agreement with the exact solution.

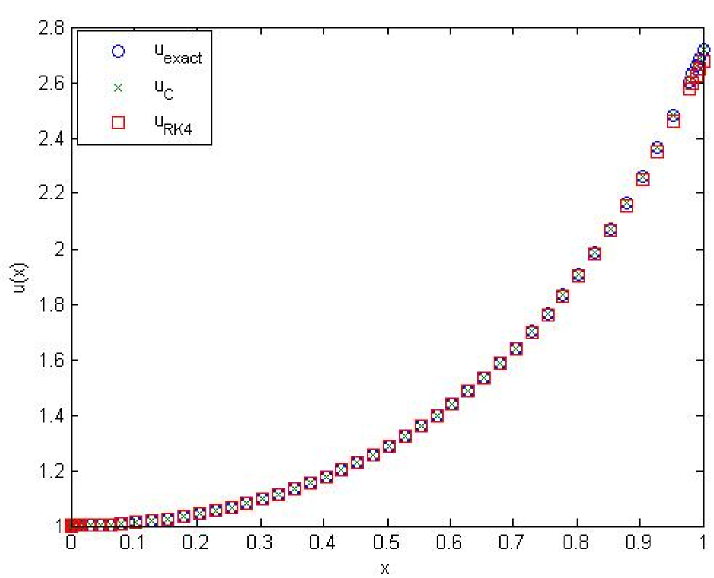

Figure 3: Comparison between the exact solution $u(x)$, u_{RK4} and the approximate solution of the proposed method $u_C(x)$.

Model problem 2

Consider the following non-linear ordinary differential equation

$$u'' + u\,u' = f(x), \quad x \in [0,1], \tag{4.21}$$

where $f(x) = x\sin(2x^2) - 4x^2\sin(x^2) + 2\cos(x^2)$ with the following initial conditions

$$u(0) = 0, \quad u'(0) = 0. \tag{4.22}$$

The exact solution of this problem is $u(x) = \sin(x^2)$.
The procedure of the solution follows the following two steps:
Step 1. Expand the function $f(x)$ using shifted Chebyshev polynomials:
Using the above consideration, the function $f(x)$ can be approximated by eight terms ($m = 8$) of the expansion (4.12) as follows

$$f_C(x) \approx 2 - 0.0003\,x + 0.008\,x^2 + 1.892\,x^3 - 4.308\,x^4 - 2.399\,x^5 + 4.682\,x^6 - 6.276\,x^7 + 3.025\,x^8.$$

Step 2. Implementation of PIM:
To solve Eq.(4.21) by the PIM we reduce this equation to the following system of ODEs

$$u'(x) = v(x), \tag{4.23}$$

$$v'(x) = -u(x)\,v(x) + f(x), \tag{4.24}$$

with the following initial conditions $u(0) = 0,\ v(0) = 0$.
According to PIM we can construct the following iteration formula

$$u_{n+1}(x) = u_0 + \int_0^x [v_n(\tau)]d\tau, \qquad n \geq 0. \tag{4.25}$$

$$v_{n+1}(x) = v_0 - \int_0^x [u_n(\tau)\,v_n(\tau) - f(\tau)\,]d\tau, \qquad n \geq 0. \tag{4.26}$$

Therefore, the first three components of the solution $u(x)$ of Eq.(4.21) using (4.25)-(4.26) are

$$\begin{aligned}
&u_0(x) = 0,\\
&u_1(x) = x^2 + 0.1\,x^5 - 0.166667\,x^6 - 0.0185185\,x^9 + 0.00833333\,x^{10} + \ldots,\\
&u_2(x) = x^2 - 0.166667\,x^6 - 0.012\,x^8 + 0.008333\,x^{10} - 0.0004545\,x^{11} + 0.002932\,x^{12} + \ldots,\\
&u_3(x) = x^2 - 0.1667\,x^6 + 0.0083\,x^{10} + 0.0011\,x^{11} - 0.0017\,x^{13} + 0.00003\,x^{14} - 0.0003\,x^{15} + \ldots,
\end{aligned}$$

Now, if we expand the function $f(x)$ by the Taylor series (4.20) with eight terms, we have

$$f_T(x) \approx 2 + 2\,x^3 - 5\,x^4 - 1.33333\,x^7 + 0.75\,x^8 + O(x^9).$$

So, the first three components of the solution $u(x)$ of Eq.(4.21) using (4.25)-(4.26) are

$$\begin{aligned}
&u_0(x) = 0,\\
&u_1(x) = x^2 - 0.00004\,x^3 + 0.0007\,x^4 + 0.0946\,x^5 - 0.1436\,x^6 - 0.0571\,x^7 + 0.0836\,x^8 + \ldots,\\
&u_2(x) = x^2 - 0.00004\,x^3 + 0.0007\,x^4 - 0.0054\,x^5 - 0.143585\,x^6 - 0.0572\,x^7 + 0.0718\,x^8 + \ldots,\\
&u_3(x) = x^2 - 0.00004\,x^3 + 0.0007\,x^4 - 0.0054\,x^5 - 0.1436\,x^6 - 0.0572\,x^7 + 0.0843\,x^8 + \ldots.
\end{aligned}$$

Figure 4 presents the absolute error between the function $f(x)$ and its approximation by using the Taylor expansion (Top) and the Chebyshev expansion (Bottom).

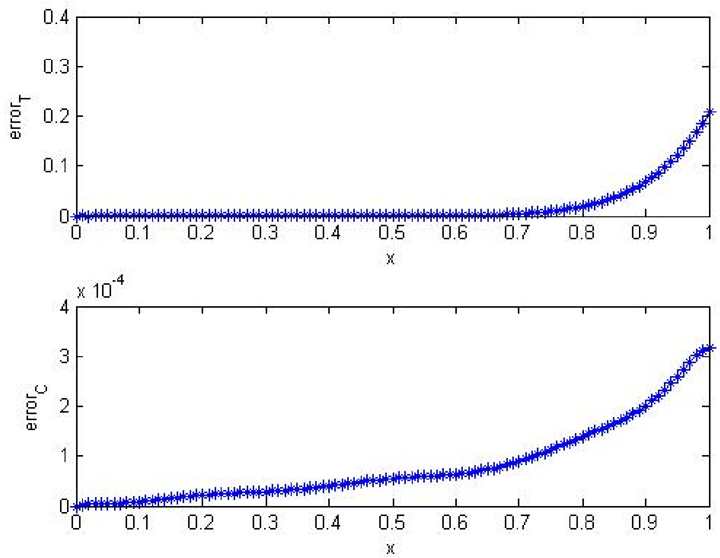

Figure 4: The absolute error: $|f(x) - f_T(x)|$ (Top) and $|f(x) - f_C(x)|$ (Bottom).

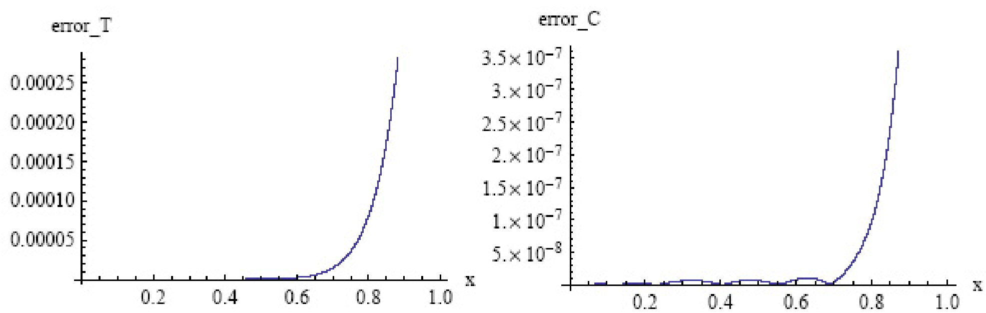

Figure 5: The absolute error: $|u(x) - u_T(x)|$ (Left) and $|u(x) - u_C(x)|$ (Right).

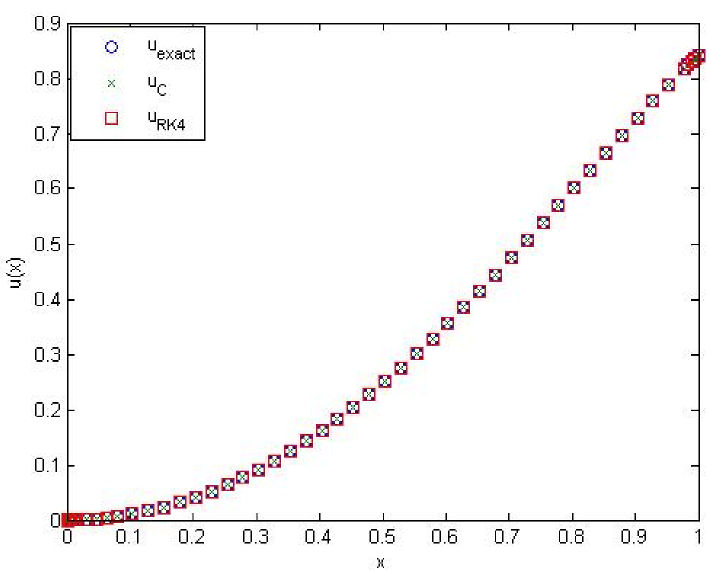

Figure 6: Comparison between the exact solution $u(x)$, u_{RK4} and the approximate solution of the proposed method $u_C(x)$.

The absolute error between the exact solution $u(x)$ and the approximate solution $u_C(x) \simeq u_4(x)$ (after four iterations) using the Chebyshev expansion for $f(x)$ with $m = 8$ is presented in figure 5(Right). Also, the absolute error between the exact solution $u(x)$ and the approximate solution $u_T(x) \simeq u_4(x)$ (after four iterations) using the Taylor expansion for $f(x)$ with eight terms is presented in figure 5(Left). Also, the figure 6 presents a comparison between exact solution $u(x)$, with the numerical solution u_{RK4} using fourth-order Runge-Kutta and the approximate solution of the proposed method $u_C(x)$. From these figures, we can conclude that the proposed method is in excellent agreement with the exact solution.

5 Conclusion

In this article, we used the properties of the shifted Chebyshev polynomials to introduce an efficient modification of PIM. Also, we presented comparative solutions with the proposed method and fourth-order Runge-Kutta method. From the introduced model problems, we can conclude that the proposed idea can be applied to solve the non-linear models of ordinary differential equations. Also, the obtained results demonstrate reliability and efficiency of the proposed method and achieve the convergence study of the method. From the resulting numerical solution we can conclude that the solution using this modification converges faster and is in excellent conformance with the exact solution. An interesting point about PIM is that only few iterations or, even in some special cases, one iteration, lead to exact solution or solution with high accuracy. Finally, all the obtained numerical results are done by using Matlab 8.

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On the Biordered set of Rings

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Abstract

In [4] K.S.S. Nambooripad introduced biordered sets as a partial algebra (E, ω^r, ω^l) where ω^r and ω^l are two quasiorders on the set E satisfying biorder axioms; to study the structure of a regular semigroup. Later in [2] David Esdown showed that the set of idempotents of a regular semigroup forms a regular biordered set. Here we extend the idea of biordered sets into rings and discussed some of its properties.

Keywords: Biordered set, Sandwitch set.

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1 Introduction

The set of idempotent elements in a semigroup S usually denoted as $E(S)$ and is important structural objects which can be used effectively in analyzing the structure of the semigroup. The concept of biordered set was originally introduced by Nambooripad[1972, 1979] to describe the structure of the set of idempotents of a semigroup in general and that of a regular semigroup in particular. A biordered set is a partial algebra (partial semigroup) together with two quasi orders on the domain of definition of the partial binary operation. Nambooripad identified a partial binary operation on the set of idempotents $E(S)$ of a semigroup S arising from the binary operation in S, defined two quasi orders on $E(S)$ and the resulting structure is abstracted as a biordered set. later on david Esdown showed that any biordeed set arises as the set of idempotents of a semigroup (see[2]).

In this paper we discuss the biordered sets which are the set of idempotents of a ring and we provide certain examples of such biordered sets.

2 Preliminaries

First we recall some basic definitions regarding semigroups, biorderede sets and rings needed in the sequel. A set S in which for every pair of elements $a, b \in S$ there is an element $a \cdot b \in S$ which is called the product of a by b is called a groupoid. A groupoid S is a semigroup if the binary operation on S is associative. An element $a \in S$ is called regular if there exists an element $a' \in S$ such that $aa'a = a$, if every element of S is regular then S is a regular semigroup. An element $e \in S$ such that $e \cdot e = e$ is called an idempotent and the set of all idempotents in S will be denoted by $E(S)$.

2.1 Biordered Sets

By a partial algebra E we mean a set together with a partial binary operation on E. Then $(e, f) \in D_E$ if and only if the product ef exists in the partial algebra E. If E is a partial algebra, we shall often denote the underlying set by E itself; and the domain of the partial binary operation on E will then be denoted by

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D_E. Also, for brevity, we write $ef = g$, to mean $(e, f) \in D_E$ and $ef = g$. The dual of a statement T about a partial algebra E is the statement T^* obtained by replacing all products ef by its left-right dual fe. When D_E is symmetric, T^* is meaningful whenever T is. On E we define

$$\omega^r = \{(e, f) : fe = e\} \; \omega^l = \{(e, f) : ef = e\}$$

and $\mathcal{R} = \omega^r \cap (\omega^r)^{-1}$, $\mathcal{L} = \omega^l \cap (\omega^l)^{-1}$, $\omega = \omega^r \cap \omega^l$. The data required to specify a biordered set E consists of a pair of quasiorders ω^r and ω^l. We will refer to ω^r as the right quasiorder of E and, ω^l as the left quasiorder of E.

Definition 2.1. *Let E be a partial algebra. Then E is a biordered set if the following axioms and their duals hold:*

1. *ω^r and ω^l are quasi orders on E and*

$$D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$$

2. $f \in \omega^r(e) \Rightarrow f\mathcal{R}fe\omega e$
3. $g\omega^l f$ *and* $f, g \in \omega^r(e) \Rightarrow ge\omega^l fe.$
4. $g\omega^r f\omega^r e \Rightarrow gf = (ge)f$
5. $g\omega^l f$ and $f, g \in \omega^r(e) \Rightarrow (fg)e = (fe)(ge).$

We shall often write $E =< E, \omega^l, \omega^r >$ to mean that E is a biordered set with quasiorders ω^l, ω^r. The relation ω defined is a partial order and

$$\omega \cap (\omega)^{-1} \subset \omega^r \cap (\omega^l)^{-1} = 1_E.$$

Definition 2.2. *Let $\mathcal{M}(e, f)$ denote the quasi ordered set $(\omega^l(e) \cap \omega^r(f), <)$ where $<$ is defined by $g < h \Leftrightarrow eg\omega^r eh$,* and *$gf\omega^l hf$. Then the set*

$$S(e, f) = \{h \in M(e, f) : g < h \text{ forall } g \in M(e, f)\}$$

is called the sandwich set of e and f.

1. $f, g \in \omega^r(e) \Rightarrow S(f, g)e = S(fe, ge)$

   The biordered set E is said to be regular if $S(e, f) \neq \emptyset \; \forall e, f \in E$

A ring is a set R together with two binary operations $'+', '\cdot'$ with the following properties.

1. The set $(R, +)$ is an abelian group.
2. The set $(R, \cdot)$ is a semigroup.
3. The operation $\cdot$ is distributive over $+$.

3 Biordered set of a Ring

Let $(R, +, .)$ be a ring. An element $e \in R$ is a multiplicative idempotent if $e \cdot e = e$ and an additive idempotent if $e + e = e$ and e is an idempotent in the ring R if and only if e is both an additive and a multiplicative idempotent. Denote E as the set of all multiplicative idempotents in R. In $(R, +, \cdot)$ define

$$a \oplus b = a + b - ab.$$

It is easy to see that $\oplus$ is an associative binary operation on R and both the additive reduct $(R, \oplus)$ and the multiplicative reduct $(R, \cdot)$ are semigroups. Further it can be seen that every multiplicative idempotent in $(R, \cdot)$ is an additive idempotent in $(R, \oplus)$ and hence the set of multiplicative idempotents E of $(R, \cdot)$ coinsides with the set of additive idempotents of $E^{\oplus}$ $(R, \oplus)$.

Lemma 3.1. *Let e, f be idempotents in R then,*

$$e \oplus f = e \Longleftrightarrow f\omega^r e$$
$$e \oplus f = f \Longleftrightarrow e\omega^l f$$

Proof. Suppose $e \oplus f = e$, then

$$e + f - ef = e \Rightarrow f - ef = 0 \Rightarrow f = ef \Rightarrow f\omega^r e.$$

Conversely, let $f\omega^r e$ then, $ef = f$. Consider $e \oplus f$, we have

$$e \oplus f = e + f - ef = e + f - f = e.$$

Similarly, let $e \oplus f = f$ then by definition,

$$e + f - ef = f \Rightarrow e - ef = 0 \Rightarrow ef = e \Rightarrow e\omega^l f.$$

Conversely, assume that $e\omega^l F$ then $ef = e$. Therefore,

$$e \oplus f = e + f - ef = e + f - e = f$$

□

It is easy to observe that the domain of both the binary operations $\cdot$ and $\oplus$ coincides and we denote this domain by D, for $(e, f) \in D$ either $(e, f) \in \omega^r \cup \omega^l$ or $(f, e) \in \omega^r \cup \omega^l$. In the first case either $f \oplus e = e$ or $e \oplus f = e$. If $f \oplus e = e, (e \oplus f)^2 = (e \oplus f) \oplus (e \oplus f) = e \oplus (f \oplus e) \oplus f = e \oplus e \oplus f = e \oplus f$ and so $e \oplus f \in E^{\oplus}$. Thus $e \oplus f \in E^{\oplus}$ whenever $(e, f) \in \omega^r \cup \omega^l$. Similarly, it can be seen that $e \oplus f \in E^{\oplus}$ when $(f, e) \in \omega^r \cup \omega^l$. Thus, by restricting the operation in $(R, \oplus, \cdot)$ to D we obtain the partial algebra $(D, \oplus)$ defining the operations in the ring R to $(D, \oplus)$, we obtain a partial algebra on $E^{\oplus}$. Now in the light of the biorder axioms we have the following Proposition.

Proposition 3.1. *Let e, f, g be idempotents in R. Then*

1. $e\omega^l f \Rightarrow e\omega f \oplus e\mathcal{L} f$
2. $g\omega^l f,\ e \in \omega^l(f) \cap \omega^l(g) \Rightarrow e \oplus g\omega^l e \oplus f$
3. $e\omega^l f\omega^l g \Rightarrow (f \oplus e) \oplus g = f \oplus g$
4. $f\omega^r g,\ e \in \omega^l(f) \cap \omega^l(g) \Rightarrow e \oplus (f \oplus g) = (e \oplus f) \oplus (e \oplus g)$

Proof. (1) $e\omega^l f$, so $e(f \oplus e) = e(f + e - fe) = e$ and $(f \oplus e)e = (f + e - fe)e = e$ that is $e\omega(f \oplus e)$. Also $(f \oplus e)f = (f + e - fe)f = f + ef - fef = f + e - fe = f \oplus e$ and $f(f \oplus e) = f(f + e - fe) = f$ that is $f \oplus e\mathcal{L} f$.

(2) $g\omega^l f$ and $e \in \omega^l(f) \cap \omega^l(g)$. Therefore,

$$(e \oplus g) \cdot (e \oplus f) = (e + g - eg) \cdot f = e \oplus g$$

Thus, $(e \oplus g)\omega^l(e \oplus f)$.

(3) $e\omega^l f\omega^l g$, we have $e \oplus f = f$, $f \oplus g = g$ and $e \oplus g = g$. Therefore,

$$f \oplus g = f \oplus (e \oplus g) = (f \oplus e) \oplus g.$$

(4) Since $f\omega^r g$, $e \in \omega^l(f) \cap \omega^l(g)$ we have, $f \oplus g = g$, $e \oplus f = f$ and $e \oplus g = g$. Therefore,

$$e \oplus (f \oplus g) = (e \oplus f) \oplus g = (e \oplus f) \oplus (e \oplus g).$$

□

Next we proceed to define the addictive sandwich set of the biordered set $E^{\oplus}$.

Proposition 3.2. *For $e, f \in E^{\oplus}$, let*

$$\tilde{M}(e,f) = \{g \in E_{\mathscr{R}} \,:\, e \in \omega^r(g) \text{ and } f \in \omega^l(g), \prec\}$$

where $\prec$ is defined by $h \prec g \iff hg = gh = h$. Then $\tilde{M}(e,f)$ is a quasiordered set and the set

$$\tilde{S}(e,f) = \{h \in \tilde{M}(e,f) : h \prec g \text{ for all } g \in \tilde{M}(e,f)\}$$

is called the addictive sandwich set of e and f (in that order).

Proof. For $g, h \in \tilde{M}(e,f)$, then both gh and hg in $\tilde{M}(e,f)$ also $h \prec h$ and if $h \prec g$, $g \prec k$ then $h \prec k$. Thus $\tilde{M}(e,f)$ is a quasiordered set and $\tilde{S}(e,f)$ are minimal elements of $\tilde{M}(e,f)$. □

Lemma 3.2. *For any idempotents $e, f \in R$ and $h \in \tilde{S}(e,f)$ then $f \oplus h \oplus e = h$.*

Proof. Since $h \in \tilde{S}(e,f)$, we have $he = e$ and $fh = f$ thus

$$\begin{aligned} f \oplus h \oplus e &= (f \oplus h) + e - (f \oplus h)e \\ &= f + h - f + e - (f + h - fh)e \\ &= h. \end{aligned}$$

□

Remark 3.1. *For any two idempotents $e, f \in R$ and $e \neq f$ then $\tilde{S}(e,f)$ and $S(e,f)$ are disjoint.*

Example 3.1. *A complemented distributive lattice is called a Boolean lattice. Let $(L, \vee, \wedge)$ be a Boolean lattice. Then $(L, +, \cdot)$ where $e + f = e \vee f$ and $e \cdot f = e \wedge f$ is a ring. Now define $\oplus$ on $(L, +, \cdot)$ as follows*

$$e \oplus f = (e \wedge f') \vee (e' \wedge f)$$

so $e \oplus f = (e + f) - ef$ and $\mathscr{L} = (L, \oplus)$ is a semigroup and we denote the addictive idempotent set by $E^{\oplus}$. It should be noted that the set of multiplicative idempotents E and the set of all addictive idempotent set $E^{\oplus}$ coincides with L and $\mathscr{L}$ (ie., the lattice is a band with respect to both $\cdot$ and $\oplus$. Let us now describe the biordered set E as follows:
ω^r and ω^l, defined by $e\omega^r f \Rightarrow f \wedge e = e$ and $e\omega^l f \Rightarrow e \wedge f = e$ are quasiorders and $\omega = \omega^r \cap \omega^l$ is a partial order. Since $e \wedge f = f \wedge e$ we have $\omega^r = \omega^l = \omega$ on E. Also $M(e,f) = (\omega^l(e) \cap \omega^r(f), <)$ where $g < h \Leftrightarrow eg\omega^r eh$, $gf\omega^l hf$, and $S(e,f)$ the maximal elements of $M(e,f)$, thus $S(e,f) = \{e \wedge f\}$.
Next we define the addictive sandwitch set $E^{\oplus}$ as follows

$$\tilde{M}(e,f) = \{g \,:\, e\omega^r g \text{ and } f\omega^l g, \prec\}$$

where $h \prec g$ means $hg = gh = h$, thus we have $\tilde{M}(e,f) = \{e \vee f\}$ and

$$\tilde{S}(e,f) = \{e \vee f\}.$$

Example 3.2. *Consider the real quarternions $Q = \{q = q_0 + q_1 i + q_2 j + q_3 j \mid q_i \in R\}$. It is well known that with respect to the usual additin and multplication defined by the rule $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ is a noncommutative skewfield. The idempotent set is*

$$E_Q = \{e = (0,0,0,0), f = (1,0,0,0)\}$$

then $\omega^l(e) = \{e\}$ and $\omega^r(f) = \{e, f\}$, so $M(e,f) = \{e\} = S(e,f)$.
Now for $q, r \in Q$ define $q \oplus r = q + r - qr$, it is easy to obseve that $\mathscr{Q} = (Q, \oplus)$ is a semigroup and $E_{\mathscr{Q}} = E_Q$. The additive sandwich set of $\mathscr{Q}$ is described as follows.

$$\tilde{M}(e,f) = \{g \in E_{\mathscr{R}} \,:\, e \in \omega^r(g) \text{ and } f \in \omega^l(g), \prec\}$$

since $e \in \omega^l(f)$ and $f \in \omega^r(f)$, we have $\tilde{M}(e,f) = \{f\}$ Also since

$$\tilde{S}(e,f) = \{h \in \tilde{M}(e,f) : h \prec g \text{ for all } g \in \tilde{M}(e,f)\}$$

we have $\tilde{S}(e, f) = \{f\}$.

Example 3.3. *Consider the set* $\mathscr{M}_2(\mathcal{Z})$ *of* 2×2 *matrices with integer entries. This is a non-commutative ring with usual addition and multiplication of matrices. The possible idempotents* $E_{\mathscr{R}}$ *in this ring are*

$$\begin{bmatrix} 0 & 0 \\ 0 & 0, \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Let $e = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \in (E_{\mathscr{R}}, \cdot)$. *then*

$$\omega^l(e) = \{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\} \text{ and } \omega^r(f) = \{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

Thus $M(e, f) = \{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, <\}$ *and so* $S(e, f) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,
Now we proceed to describe the addictive sandwitch set, we have

$$\tilde{M}(e, f) = \{g : e \in \omega^r(g), f \in \omega^l(g), \prec\}$$

where $h \prec g$ *means* $hg = gh = h$. *Thus* $\tilde{M}(e, f) = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \prec\}$.
Thus

$$\tilde{S}(e, f) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Zagreb Indices of a Graph and its Common Neighborhood Graph

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Abstract

A complete set of relations is established between the first and second Zagreb index of a graph and of its congraph. Formulas for the Zagreb indices of several derived graphs are also obtained.

Keywords: Vertex degree, Zagreb indices, Common neighborhood graph.

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1 Introduction

The graphs considered in this paper are assumed to be simple, i.e., to possess no directed or weighted edges and no self–loops. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. If $|V(G)| = p$ and $|E(G)| = q$, then we say that G is a (p,q)-graph. The edge connecting the vertices x and y will be denoted by xy.

The set of vertices of G, adjacent to a vertex v will be denoted by $N_G(v)$. The degree of the vertex v, denoted by $d(v) = d_G(v)$, is the number of first neighbors of v, that is $d_G(v) = |N_G(v)|$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *common neighborhood graph* (*congraph*) of G, denoted by $con(G)$, is the graph with vertex set $V(con(G)) = V(G)$, in which two vertices are adjacent if and only if they have a common neighbor in G. In other words, for every $x, y \in V(G)$,

$$xy \in E(con(G)) \Longleftrightarrow N_G(x) \cap N_G(y) \neq \emptyset .$$

The concept of common neighborhood graphs originates from the study of a special kind of graph energy [2]. The basic properties of these derived graphs were established soon after that [1, 3]. Also, various mathematical properties of congraphs have been discovered [8, 13, 14].

Two old and most studied degree–based graph invariants are the so-called *first and second Zagreb indices*, defined as

$$M_1(G) = \sum_{v \in V(G)} d(v)^2 \qquad \text{and} \qquad M_2(G) = \sum_{uv \in E(G)} d(u)\,d(v) .$$

For details on their history, mathematical properties and chemical applications, we refer to [4, 5, 9–12] and the references cited therein.

The so-called *forgotten topological index* is defined as [6, 7]

$$F = F(G) = \sum_{v \in V(G)} d(v)^3 .$$

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In [15], Li and Zheng introduced the first general Zagreb index as

$$M_1^k(G) = \sum_{v \in V(G)} d(v)^k$$

where $k \in \mathbb{N} \cup \{0\}$. Obviously, $M_1^0(G) = |V(G)|$, $M_1^1(G) = 2|E(G)|$, $M_1^2(G) = M_1(G)$, and $M_1^3(G) = F(G)$. Also, in [16], the the second general Zagreb index was defined as

$$M_2^k(G) = \sum_{uv \in E(G)} \big[d(u)\, d(v)\big]^k$$

where $k \in \mathbb{N} \cup \{0\}$. Obviously $M_2^0(G) = |E(G)|$ and $M_2^1(G) = M_2(G)$.

We now define two new degree–based graph invariants, pertaining to congraphs:

$$\Xi_1(G) = \sum_{v \in V(G)} d_G(v)\, d_{con(G)}(v) \qquad \text{and} \qquad \Xi_2(G) = \sum_{uv \in E(con(G))} d_G(u)\, d_G(v)\,.$$

Throughout this paper, we use standard graph–theoretical notation. $\overline{G}$ denoted the complement of the graph G. As usual, P_n, C_n, and K_n, are, respectively, the n-vertex path, cycle, and complete graph. In addition, $K_{n,m}$ is the complete bipartite graph with $n+m$ vertices. Recall that $K_{1,n-1}$ is called the star and often denoted by S_n.

In this paper, we investigate some properties of congraphs and the Zagreb indices of congraphs and establish relations between the Zagreb indices of congraphs and several degree–based invariants of the parent graphs.

2 Degree–related properties of common neighborhood graph

Lemma 2.1. *Let G be a simple (p,q)-graph and let $con(G)$ be a (p,q')-graph. Then, for every $v \in V(G)$ the following holds.*

(1) $d_{con(G)}(v) = \left| \bigcup_{u \in N_G(v)} N_G(u) \setminus \{v\} \right| = |N_{con(G)}(v)|\,.$

(2) *If G has no cycles of size 4, then* $d_{con(G)}(v) + d_G(v) = \sum_{u \in N_G(v)} d_G(u)$.

(3) *If $d_G(u) + d_G(v) > p$ holds for every $u, v \in V(G)$, then $con(G) \cong K_p$.*

(4) *If G has no cycles of size 3, then $con(G)$ is a subgraph of $\overline{G}$.*

Proof.
(1) From the definition of a congraph we have

$$u \in N_{con(G)}(v) \Longleftrightarrow uv \in E(con(G)) \Longleftrightarrow N_G(u) \cap N_G(v) \neq \emptyset\,.$$

Then there exists $a \in N_G(v)$ and $a \in N_G(u)$ such that

$$a \in N_G(v) \text{ and } u \in N_G(a)$$

implies

$$N_{con(G)}(v) = \bigcup_{u \in N_G(v)} N_G(u) \setminus \{v\}\,.$$

(2) For every $u, w \in N_G(v)$, we have $v \in N_G(u) \cap N_G(w)$. We can easily see that $N_G(u) \cap N_G(w) = \{v\}$, since, if there exist $a \in N_G(u) \cap N_G(w)$ such that $a \neq v$, it would follow that $au, vu, aw, vw \in E(G)$, that is we would

have a cycle of size 4, which is a contradiction. Also, by

$$\begin{aligned} d_{con(G)}(v) &= \left| \bigcup_{u \in N_G(v)} N_G(u) \setminus \{v\} \right| = \left| \bigcup_{u \in N_G(v)} (N_G(u) \setminus \{v\}) \right| \\ &= \sum_{u \in N_G(v)} |N_G(u) \setminus \{v\}| = \sum_{u \in N_G(v)} (|N_G(u)| - 1) \\ &= \left(\sum_{u \in N_G(v)} d(u) \right) - |N_G(v)| = \left(\sum_{u \in N_G(v)} d(u) \right) - d(v) \end{aligned}$$

the claim (2) in Lemma 2.1 follows.

(3) It suffices to show that $N_G(u) \cap N_G(v) \neq \emptyset$ for every $u, v \in V(G)$. Otherwise, we would have

$$p \geq |N_G(u) \cup N_G(v)| = |N_G(u)| + |N_G(v)| = d(u) + d(v) > p$$

which is a contradiction. Hence, it follows that $uv \in E(con(G))$ that is $con(G) \cong K_p$.

(4) It is enough to show that $E(con(G)) \subseteq E(\overline{G})$. Hence, for every $uv \in E(con(G))$, we have $N_G(u) \cap N_G(v) \neq \emptyset$. That is there exist $a \in N_G(u) \cap N_G(v)$. Then $au, av \in E(G)$, but $uv \notin E(G)$, otherwise G would have a cycle of size 3. Hence, $uv \in E(\overline{G})$. □

Theorem 2.1. *Let G be a (p, q)-graph. In the congraph of G, for every $u, v \in V(G)$, if $d(u) + d(v) > p$ then:*

(1) $\Xi_1(G) = 2q\,(p-1)$

(2) $\Xi_2(G) = 2q^2 - \frac{1}{2}\,M_1(G)$.

Proof. By Lemma 2.1, $con(G) \cong K_p$.

(1)

$$\begin{aligned} \Xi_1(G) &= \sum_{v \in V(G)} d_G(v)\, d_{con(G)}(v) = \sum_{v \in V(G)} d_G(v)\,(p-1) \\ &= (p-1) \sum_{v \in V(G)} d_G(v) = 2q(p-1). \end{aligned}$$

(2)

$$\begin{aligned} \Xi_2(G) &= \sum_{uv \in E(con(G))} d(u)\,d(v) = \sum_{uv \in E(K_p)} d(u)\,d(v) = \frac{1}{2} \sum_{u,v \in V(G),\, u \neq v} d(u)\,d(v) \\ &= \frac{1}{2} \left[\sum_{u \in V(G)} d(u) \sum_{v \in V(G)} d(v) - \sum_{v \in V(G)} d(v)^2 \right] = \frac{1}{2} [2q \cdot 2q - M_1(G)] \\ &= 2q^2 - \frac{1}{2}\,M_1(G). \end{aligned}$$

□

Theorem 2.2. *Let G be a (p, q)-graph and have no cycles of size 4. Also, let $con(G)$ be a (p, q')-graph. Then,*

$$q' = \frac{1}{2} \sum_{v \in V(G)} d_G(v)^2 - q = \frac{1}{2}\,M_1(G) - q. \tag{2.1}$$

Proof. First we show that $N_G(u) \cap N_G(w) = \{v\}$ holds for every $u, w \in N_G(v)$. Otherwise, if there would exist $a \in N_G(u) \cap N_G(v)$, then it is easy to see that G has a cycle of size 4, which is a contradiction. Hence, by Lemma 2.1 we get $d_{con(G)}(v) + d_G(v) = \sum_{u \in N_G(v)} d_G(u)$. Thus,

$$\sum_{v \in V(G)} d_{con(G)}(v) + \sum_{v \in V(G)} d_G(v) = \sum_{v \in V(G)} \sum_{u \in N_G(v)} d_G(u)$$

and

$$2q' + 2q = \sum_{v \in V(G)} d_G(v)^2$$

from which Eq. (2.1) follows. □

Theorem 2.3. *Let G be a (p,q)-graph having no cycles of size 4. Also, let $con(G)$ be a (p,q')-graph. Then,*

(1) $M_1(con(G)) = F + 2\,\Xi_2(G) - 4M_2(G) + M_1(G)$;

(2) $M_2(G) = \frac{1}{2}\left[\Xi_1(G) + M_1(G)\right]$.

Proof. By Lemma 2.1, we have:
(1)

$$\begin{aligned} M_1(con(G)) &= \sum_{v \in V(con(G))} d_{con(G)}(v)^2 = \sum_{v \in V(G)} \left(\sum_{u \in N_G(v)} d(u) - d(v) \right)^2 \\ &= \sum_{v \in V(G)} \left(\sum_{u \in N_G(v)} d(u) \right)^2 - 2 \sum_{v \in V(G)} \left(\sum_{u \in N_G(v)} d(u) \right) d(v) + \sum_{v \in V(G)} d(v)^2 \\ &= F + 2\,\Xi_2(G) - 4M_2(G) + M_1(G)\,. \end{aligned}$$

(2)

$$\begin{aligned} \Xi_1(G) &= \sum_{v \in V(G)} d(v)\, d_{con(G)}(v) = \sum_{v \in V(G)} d(v) \left(\sum_{u \in N_G(v)} d(u) - d(v) \right) \\ &= \sum_{v \in V(G)} d(v) \left(\sum_{u \in N_G(v)} d(u) \right) - \sum_{v \in V(G)} d(v)^2 \\ &= 2 \sum_{uv \in E(G)} d(v)\, d(u) - \sum_{v \in V(G)} d(v)^2 = 2M_2(G) - M_1(G) \end{aligned}$$

□

If there is a cycle of size 4, then we can change it into a square. Two cycles of order 4 in a graph are said to be disjoint, if they have no common diagonals in their corresponding squares.

Definition 2.1. *A graph G is called type S, if any two cycles of size 4 are disjoint.*

Example 2.1. (1) *Every graph which has at most one cycle of size 4 is a graph of type S.*

(2) *Every graph, such that every two cycles of order 4 have at most one common edge in their corresponding squares, is a graph of type S.*

(3) *K_4 is a graph of type S.*

(4) *$K_{2,3}$ is not a graph of type S.*

Theorem 2.4. *Let G be a (p,q)-graph and s be the number corresponding squares of cycles of size 4. Also, let $con(G)$ be a (p,q')-graph. Then,*

(1) *If G is a graph of type S, then $M_1(G) = 2q + 2q' + 4s$.*

(2) *If G is a any graph, $M_1(G) \leq 2q + 2q' + 4s$.*

(3) *If G has no cycles of size 4, then $M_1(G) = 2q + 2q'$.*

Proof. (1) Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $A = [a_{ij}]_{p\times p}$ be the adjacency matrix of graph G. Since $d(v_i) = \sum_{k=1}^{p} a_{ik}$, we get

$$\begin{aligned} M_1(G) &= \sum_{v_i\in V(G)} d(v_i)^2 = \sum_{v_i\in V(G)} \left(\sum_{k=1}^{p} a_{ik}\right)^2 \\ &= \sum_{v_i\in V(G)} \sum_{k=1}^{p} a_{ik}^2 + 2 \sum_{v_i\in V(G)} \sum_{1\leq k\leq k'\leq p} a_{ik}\, a_{ik'} \\ &= \sum_{v_i\in V(G)} \sum_{k=1}^{p} a_{ik} + 2 \sum_{v_i\in V(G)} \sum_{1\leq k\leq k'\leq p} a_{ik}\, a_{ik'} \\ &= \sum_{v_i\in V(G)} d(v_i) + 2 \sum_{v_i\in V(G)} \sum_{1\leq k\leq k'\leq p} a_{ik}\, a_{ik'}\ . \end{aligned}$$

Since $a_{ik}\, a_{ik'} = 0 \ \ or \ \ 1$. Hence it is equal with one if $a_{ik} = 1$ and $a_{ik'} = 1$. Therefore, for some $k \neq k'$ there exist $v_k, v_{k'} \in V(G)$ such that $v_i\, v_k \in E(G)$ and $v_i\, v_{k'} \in E(G)$. Hence $v_k\, v_{k'} \in E(con(G))$ and this edge appears only once, since G has no cycles of size 4. But, if G has any cycle of size 4, then this edge is appear only twice. Since every cycle of size 4 corresponds to a square and every square, have two diagonals. Thus $\sum_{v\in V(G)} \sum_{1\leq k\leq k'\leq p} a_{ik}\, a_{ik'} = q' + 2s$. Therefore, $M_1(G) = 2q + 2q' + 4s$.

(2) The proof of this part is similar to part (1) but since edge $v_i\, v_k \in E(G)$ appears at most twice, hence $M_1(G) \leq 2q + 2q' + 4s$.

(3) It directly follows from part (1).

□

Corollary 2.1. *Let G be a tree. Then,*

$$M_1(G) = 2q + 2q'\ .$$

Corollary 2.2. *Let G be a (p,q)-graph and s be the number corresponding squares of cycles of size 4. Also, let $con(G)$ be a (p,q')-graph. In this case, if G is graph of type S, then $q' = \frac{1}{2}M_1(G) - q - 2s$.*

The following theorem is well known.

Theorem 2.5. *Let G be a graph with vertices labeled $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let A be its corresponding adjacency matrix. For any positive integer k, the (i, j) entry $a_{ij}^{(k)}$ of $A^k = [a_{ij}^{(k)}]$ is equal to the number of walks from v_i to v_j that use exactly k edges.*

Remark 2.1. *For a simple (p,q)-graph, we have*

(1) *For every $i \neq j$ entry $a_{ij}^{(2)}$ of $A^2 = [a_{ij}^{(2)}]$ is equal to the number paths of order 2 from v_i to v_j.*

(2) *$tr A^2 = \sum_{i=1}^{p} a_{ii}^{(2)} = 2q$.*

(3) *$\sum_{\substack{1\leq i,j\leq p \\ i\neq j}} a_{ij}^{(2)}$ is equal to the number paths of order 2 from u to v for every disjoint $u, v \in V(G)$.*

Lemma 2.2. *Let $A = [a_{ij}]$ be the adjacency matrix of the graph G. Define $B = [b_{ij}]$ such that* $b_{ij} = \begin{cases} 1 & a_{ij}^{(2)} \neq 0 \ \ for \ \ i \neq j \\ 0 & otherwise \end{cases}$

Then B is the adjacency matrix of $con(G)$. In particular, if G has no cycles of size 4, then $B = A^2 - C$ where C is degree matrix of G.

Proof. For every $v_iv_j \in E(con(G))$, it is enough that $b_{ij} = 1$ and otherwise it is equal zero. By definition from b_{ij} we have b_{ij} is equal one if $a_{ij}^{(2)} \neq 0$ for $i \neq j$. This implies that $a_{ij}^{(2)} = |N_G(v_i) \cap N_G(v_j)| \neq 0$, that is $N_G(v_i) \cap N_G(v_j) \neq \emptyset$. Hence $v_iv_j \in E(con(G))$. In particular, if G has no cycle of size 4, then $a_{ij}^{(2)} = 1 \ or \ 0$ for $i \neq j$. Otherwise, we get $|N_G(v_i) \cap N_G(v_j)| \geq 2$. Then G has a cycle of size 4, which is a contradiction. Thus, $B = A^2 - C$. □

Remark 2.2. *For a (p,q)-graph, let r be the number paths of order 3 from u to v for every $\{u,v\} \subseteq V(G)$, and t_i the number of cycles of size 3 containing the vertex v_i. Then,*

(1) *For every $i \neq j$, the entry $a_{ij}^{(3)}$ of $A^3 = [a_{ij}^{(3)}]$ is equal to the number of walks from v_i to v_j of order 3.*

(2) $trA^3 = \sum_{i=1}^{p} a_{ii}^{(3)} = \sum_{i=1}^{p} 2t_i = 6\,\ell$, *where ℓ is the number of triangle.*

(3) *Let r_{ij} be the number of paths from v_i to v_j of order 3, then*

$$a_{ij}^{(3)} = \begin{cases} d(v_i) + d(v_j) - 1 + r_{ij} & v_iv_j \in E(G) \\ r_{ij} & v_iv_j \notin E(G) \\ 2t_i & i = j \end{cases}$$

(4)

$$\begin{aligned} \sum_{1 \leq i,j \leq p} a_{ij}^{(3)} &= 6\,\ell + 2\left(\sum_{v_iv_j \in E(G)} (d(v_i) + d(v_j) - 1 + r_{ij})\right) + 2\left(\sum_{v_iv_j \notin E(G)} r_{ij}\right) \\ &= 6\,\ell + 2M_1(G) - 2q + 2r\,. \end{aligned}$$

Theorem 2.6. *Let G be a (p,q)-graph and $con(G)$ a (p,q')-graph. Also, let $A = [a_{ij}]_{p\times p}$ and $B = [b_{ij}]_{p\times p}$ be the adjacency matrices of G and $con(G)$, respectively.*

Then,

(1) $\Xi_1(G) = \sum_{1\leq i,j \leq p} c_{ij}$ *where $A\,B = [c_{ij}]_{p\times p}$.*

(2) *If G has no cycle of size 4, then $\Xi_1(G)$ is equal to the number of paths of order 2 or 3 from u to v for every $u,v \in V(G)$.*

(3) *If G has no cycle of size 3 and 4, then $\Xi_1(G) = 2|L| + 2|L'|$, where $L = \{\{u,v\} \subseteq V(G) \,|\, d(u,v) = 2\}$ and $L' = \{\{u,v\} \subseteq V(G)|\ d(u,v) = 3\}$.*

Proof. (1) Let $V(G) = \{v_1, v_2, \ldots, v_p\}$. Since $d_G(v_k) = \sum_{i=1}^{p} a_{ik}$ and $d_{con(G)}(v_k) = \sum_{j=1}^{p} b_{kj}$, we have

$$\begin{aligned} \sum_{1\leq i,j\leq p} c_{ij} &= \sum_{1\leq i,j \leq p} \sum_{k=1}^{p} a_{ik}\, b_{kj} = \sum_{k=1}^{p} \sum_{1\leq i,j\leq p} a_{ik}\, b_{kj} \\ &= \sum_{k=1}^{p} \left(\sum_{i=1}^{p} a_{ik}\right)\left(\sum_{j=1}^{p} b_{kj}\right) = \sum_{k=1}^{p} d(v_k)\, d_{con(G)}(v_k) \\ &= \sum_{v\in V(G)} d_G(v)\, d_{con(G)}(v) = \Xi_1(G)\,. \end{aligned}$$

(2)

$$\begin{aligned}\sum_{1\leq i,j\leq p} c_{ij} = \Xi_1(G) &= \sum_{v\in V(G)} d_G(v)\, d_{con(G)}(v) = \sum_{v_i\in V(G)} \sum_{k=1}^{p} a_{ik} \sum_{k'=1}^{p} b_{ik'} \\ &= \sum_{v_i\in V(G)} \sum_{1\leq k,k'\leq p} a_{ik}\, b_{ik'}\,.\end{aligned}$$

For $a_{ik} = 1$ and $b_{ik'} = 1$ we have $v_iv_k \in E(G)$ and $v_iv_{k'} \in E(con(G))$, respectively. Thus we have three cases:

case(1): For $k = k'$ and $i \neq j$, if $v_iv_j, v_iv_k \in E(G)$, then $a_{ik}\, b_{ik'} = 1$.

case(2): For $k = k'$ and $i = j$, if $av_i, av_k, v_iv_k \in E(G)$, then $a_{ik}\, b_{ik'} = 1$.

case(3): For $k \neq k'$ and $i \neq j$ if $v_iv_k, v_iv_j, v_jv_{k'} \in E(G)$,, then $a_{ik}\, b_{ik'} = 1$.

Since the graph G has no cycles of size 4, in every of the above cases only once appear. Thus, $\sum_{1\leq i,j\leq p} c_{ij} = \Xi_1(G)$ is the number all of paths of order 2 or 3 from u to v for every $u, v \in V(G)$.

(3) This part can be obtained easily from part (2).

□

Theorem 2.7. *Let G be a (p,q)-graph. Then, $2\,M_2(G) - 2\,M_1(G) + 2q = r + 6\,\ell$ where $r =$ the number of all paths of order 3 from u to v for every $\{u,v\} \subseteq V(G)$ and ℓ is the number of triangles.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ then

$$\begin{aligned}M_2(G) &= \sum_{v_iv_j\in E(G)} d(v_i)\, d(v_j) = \sum_{v_iv_j\in E(G)} \sum_{k=1}^{p} a_{ik} \sum_{k'=1}^{p} a_{k'j} \\ &= \sum_{k=1}^{p}\sum_{k'=1}^{p}\sum_{v_iv_j\in E(G)} a_{ik}\, a_{k'j} = \frac{1}{2}\sum_{\{k,k'\}\subseteq V(G)} \left(\sum_{v_iv_j\in E(G)} a_{ik}\, a_{k'j}\right).\end{aligned}$$

Since $v_iv_j \in E(G)$, if $a_{ik}a_{k'j} = 1$, then $v_iv_j \in E(G)$, $a_{ik} = 1$, and $a_{k'j} = 1$. In this case, there exist vertices v_k and $v_{k'}$ such that we have following four cases:

case(1): If $k' = i$ and $v_iv_j, v_iv_k \in E(G)$, then $a_{ik}\, a_{k'j} = 1$.

case(2): If $k = j$ and $v_iv_j, v_jv_{k'} \in E(G)$, then $a_{ik}\, a_{k'j} = 1$.

case(3): If $k = k'$ and $v_iv_k, v_iv_j \in E(G)$, then $a_{ik}\, a_{k'j} = 1$.

case(4): If $k \neq k'$ and $v_iv_k, v_iv_j, v_jv_{k'} \in E(G)$, then $a_{ik}\, a_{k'j} = 1$.

Thus, in every above cases determine all of the number of walks of order 3. Thus, by Remark 2.2,

$$M_2(G) = \frac{1}{2}\sum_{1\leq i,j\leq p} a_{ij}^{(3)} = \frac{1}{2}(6\,\ell + 2\,M_1(G) - 2q + 2\,r) = 3\ell + M_1(G) - q + r\,.$$

□

Example 2.2.

Let G be a $(4,4)$-graph with $V(G) = \{a,b,c,d\}$ and $E(G) = \{ab, ac, bc, bd\}$. Then, $M_2(G) = 19$, $M_1(G) = 18$ where $q = 4, r = 2$ and $\ell = 1$. Then

$$19 = M_2(G) = 3 + 18 - 4 + 2 = 3\ell + M_1(G) - q + r\,.$$

3 Conclusion

In this paper, we defined the Zagreb indices of congraphs and investigate the degree–related properties of the congraphs and the Zagreb indices of congraphs. Moreover, we obtained relations between Zagreb indices of parent graphs and graph invariants such as number of edges of parent graph, number of edges of congraph, the number of all paths of order 3, number of triangles and the number of cycles of size 4 by using adjacency matrix of the parent graph.

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θ-local function and ψ_θ-operator

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Abstract

In this paper, we introduce the notion of a θ-local function and investigate some of their properties. Also, we define two operators $()^{*\theta}$ and ψ_θ in an ideal topological space.

Keywords: θ-local function, $()^{*\theta}$-operator, θ-compatible and ψ_θ-operator.

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1 Introduction

In 1968, Velicko[22] introduced the notions of θ-open subsets, θ-closed subsets and θ-closure, for the sake of studying the important class of H-closed spaces in terms of arbitrary filterbases. In 1990, Jankovic and Hamlett[7,8] defined the concept of $\mathcal{I}$-open set via local function which was given by Vaidyanathaswamy. O.Njastad[16,17] introduced the concept of compatible ideals in 1966. This ideal was also called as supercompact by Vaidyanathaswamy[20,21]. In an ideal topological space, the local function was introduced by Kuratowski[11]. After that so many mathematicians like Hayashi [7], Natkaniec[15] and Modak and Bandyopadhyay[14] have studied this field and proved some new results in an ideal topological spaces. In 2009, Jeong Gi Kang and Chang Su Kim [10] defined pre-local function, semi-local function and α -local function. In 2011, Shyamapada Modak [16] introduced δ-local function and an operator ψ_δ in the ideal topological spaces. In 2013, Arokia Rani and Nithya[2] introduced precompatible ideals, Al-Omari and Noiri[1] defined the local closure function and an operator ψ_Γ and K. Bhavani[3,4] introduced g-local function and ψ_g -operator in the ideal topological spaces.

In this paper, we introduce the notion of a θ-local function and investigate some of their properties. We also introduce two operators $()^{*\theta}$ and ψ_θ a $*\theta$-closure operator in lines with kuratowski. Also, we discuss θ-compatibility of topological spaces.

2 Preliminaries

Let (X,τ) be a topological space with no separation properties assumed. For a subset A of a space (X,τ), cl(A) and int(A) denote the closure of A and the interior of A respectively. (X,τ) and (Y,σ) will be replaced by X and Y if there is no chance of confusion. A subset A of X is said to be semi open[9] (resp. pre open[10] and α -open[13] if A$\subset$ cl(int(A)) (resp. A $\subset$ int(cl(A)) and A $\subset$ int(cl(int(A)))). The complement of semi open (resp. pre open and α -open) is called semi closed (resp. pre closed and α-closed).

A set A is said to be θ-open[1] if every point of A has an open neighborhood whose closure is contained in A. It is very well known that the family of all θ-open subsets of (X,τ) are topologies on X which we shall denote by τ^θ. From the definitions it follows immediately that $\tau^\theta \subset \tau$. A space (X,τ) is regular if and only if $\tau^\theta = \tau$. A point x $\in$ X is said to be in the θ-closure of a subset A$\subseteq X$[6] if for each open neighbourhood U of x

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we have cl(U)∩ A $\neq \phi$. We shall denote θ-closure by $cl_\theta(A)$. A subset $A \subseteq X$ is called θ-closed if A = $cl_\theta(A)$. In general, the θ- closure of a given set need not be a θ-closed set. But it is always closed. A point x ∈ A is said to be a θ-limit point of A[5] in X if for each θ-open set U containing x, such that U ∩ (A - $\{x\}$)$\neq \phi$ .The set all θ-limit points of A is called a θ-derived set of A and is denoted by $D_\theta(A)$.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A ∈ $\mathcal{I}$ and B⊆ A implies B∈ $\mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $(A \cup B) \in \mathcal{I}$. A topological space (X, τ) with an ideal $\mathcal{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathcal{I})$. For a subset A $\subseteq X$, $A^*(\mathcal{I})$ = $\{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ is called the local function of A with respect to $\mathcal{I}$ and τ[4]. We simply write A* in case there is no chance for confusion. A Kuratowski[11] closure operator cl*(.) for a topology $\tau^*(\mathcal{I})$ called the τ^*-topology finer than τ is defined $cl^*(A)$ = $A \cup A^*$. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is τ^*-closed [18] (resp. $*$ -dense in itself [18], $*$-perfect [18]) if $A^* \subset A$ (resp. $A \subset A^*$, $A = A^*$). Clearly, A is $*$-perfect if and only if A is τ^*-closed and $*$-dense in itself. An ideal $\mathcal{I}$ in a space (X, τ) is said to be compatible with respect to τ[9], denoted by $\mathcal{I} \sim \tau$, if for every subset A of X and for each $x \in A$, there exists a neighborhood U of x such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. Let (X, τ) be a topological space with $\mathcal{I}$ an ideal on X, then τ is pre-compatible[2] with $\mathcal{I}$, if for every $A \subseteq X$, and for every $x \in A$, there exists a $U \in PO(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$ and is denoted by $\mathcal{I} \sim_P \tau$. An operator[8] $\psi : \wp(X) \to \tau$ is defined as: $\psi(A)$ = $\{x \in X$: there exists an open set O_x such that $O_x - A \in \mathcal{I}\}$, for every $A \in \wp(X)$. Its equivalent definition is $\psi(A)$ = $X - (X - A)^*$. Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then the set (1) ${A^*}_p(\mathcal{I}, \tau)$ = $\{x \in X : U \cap A \notin \mathcal{I}$ for each $U \in \tau^p(x)\}$ is called the pre-local function with respect to $\mathcal{I}$ and τ . (2) ${A^*}_s(\mathcal{I}, \tau)$ = $\{x \in X : U \cap A \notin \mathcal{I}$ for each $U \in \tau^s(x)\}$ is called the semi-local function with respect to $\mathcal{I}$ and τ . (3) ${A^*}_\alpha(\mathcal{I}, \tau)$ = $\{x \in X : U \cup A \notin \mathcal{I}$ for each $U \in \tau^\alpha(x)\}$ is called the α-local function with respect to $\mathcal{I}$ and τ . Al-Omari and Noiri[1] defined the local closure function and an operator ψ_Γ in an ideal topological spaces as follows:$\Gamma(A)(\mathcal{I}, \tau)$ = $\{x \in X : A \cap cl(U) \notin \mathcal{I}$ for every $U \in \tau(x)\}$ and $\psi_\Gamma(A)$ = X - $\Gamma(X - A)$ where $\psi : \wp(X) \to \tau$. K. Bhavani[3,4] introduced g-local function and ψ_g-operator in the ideal topological spaces as: : $A^*(\mathcal{I}, \tau_g) = \{x \in X : U \cap A \notin \mathcal{I}$ for every g-open set U containing $x\}$ and $\psi_g(A) = \{x \in X$: there exists a g-open set U_x containing x such that $U_x - A \in \mathcal{I}\}$ for every $A \in \wp(X)$ where $\psi_g : \wp(X) \to \wp(X)$.

Result 2.1 Let A be a subset of a topological space (X, τ). If $A \in \tau^\theta$, then $cl_\theta(A)$ = A

Lemma 2.1. *[1]. Let A be a subset of a topological space (X, τ). Then*

1. *if A is open, then cl(A) = $cl_\theta(A)$*
2. *if A is closed, then int(A) = $int_\theta(A)$*

Lemma 2.2. *If $(X, \tau, \mathcal{I})$ is an ideal topological space, then $\mathcal{I}$ is codense[18] if and only in $A \subset A^*$ for every open set A of X.*

Lemma 2.3. *[18]. If $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If $A \subset A^*$, then A^* = $cl(A^*)$ = $cl(A)$ = $cl^*(A)$.*

3 The Operator() $^{*\theta}$

In this section we shall introduce an operator $()^{*\theta}$ and discuss various properties of this operator.

Definition 3.1. *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then, the θ-local function of $\mathcal{I}$ on X is defined as $A^{*\theta}(\mathcal{I}, \tau)$ = $\{x \in X : U_x \cap A \notin \mathcal{I}$ for every $U_x \in \theta O(X, x)\}$ with respect to $\mathcal{I}$ and τ and is denoted as $A^{*\theta}$ for $A^{*\theta}(\mathcal{I}, \tau)$.*

Lemma 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every subset A of X,*

(1) ${A^*}_p(\mathcal{I}, \tau) \subseteq A^{*\theta}(\mathcal{I}, \tau)$.

(2) ${A^*}_s(\mathcal{I}, \tau) \subseteq A^{*\theta}(\mathcal{I}, \tau)$.

(3) ${A^*}_\alpha(\mathcal{I}, \tau) \subseteq A^{*\theta}(\mathcal{I}, \tau)$.

(4) $\Gamma(A)(\mathcal{I}, \tau) \subseteq A^{*\theta}(\mathcal{I}, \tau)$.

(5) ${A^*}_g(\mathcal{I}, \tau) \subseteq A^{*\theta}(\mathcal{I}, \tau)$.

Proof. Straight forward. □

Remark 3.1. *The converse of the Lemma 3.1 need not be true as seen in the following examples.*

Example 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $I = \{\phi, \{c\}\}$. If $A = \{a, b\}$, then $A^{*\theta} = \{a, b, c\} \not\subset \{a, b\} = A^*{}_p$.*

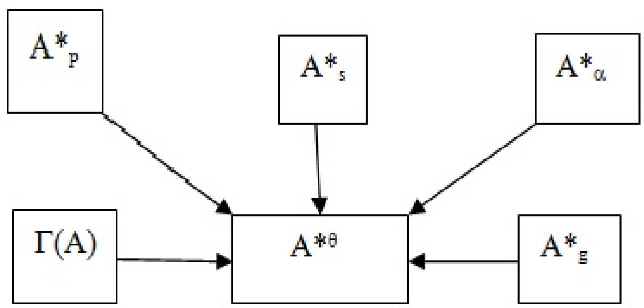
Example 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $X = \{a, b, c, d\}$, $\tau = \{\phi ,\{a\},\{b\},\{c\},\{a, b\}, \{a, c\},\{b, c\},\{a, b, c\}, \{a, c, d\},X\}$ and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. If $A = \{a, b, c\}$, then $A^{*\theta} = \{a, c, d\} \not\subset \{a, d\} = A^*{}_s$.*

Example 3.3. *In example 3.2, if $A = \{b, c, d\}$ then, $A^{*\theta} = \{a, c, d\} \not\subset \{d\} = A^*{}_\alpha$.*

Example 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $X = \{a, b, c, d\}$, $\tau = \{\phi, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $I = \{\phi, \{c\}\}$. If $A = \{a\}$, then $A^{*\theta} = \{a, b, c, d\} \not\subset \{a, b, c\} = \Gamma(A)$.*

Example 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $X = \{a, b, c, d, e\}$, $\tau = \{\phi ,\{a\},\{c, d\},\{a, c, d\}, \{b, c, d, e\},X\}$ and $\mathcal{I} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. If $A = \{a, b, c, d\}\}$, then $A^{*\theta} = X \not\subset \{a, b\} = A^*{}_g$.*

Remark 3.2. *The above discussions are summarized in the following diagram.*



Remark 3.3. *$A \subset A^{*\theta}$ and $A^{*\theta} \subset A$ are not true in general as shown in the following example.*

Example 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $X = \{a, b, c, d, e\}$, $\tau = \{\phi ,\{a\},\{c, d\},\{a, c, d\}, \{b, c, d, e\},X\}$ and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. (i) If $A = \{a, b\}$, then $A^{*\theta} = \{a\}$. Therefore, $A \not\subset A^{*\theta}$. (ii) If $A = \{a, b, d\}$, then $A^{*\theta} = X$. Therefore, $A^{*\theta} \not\subset A$.*

Remark 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. Then, $cl^{*\theta}(A) = A \cup A^{*\theta}$ is a $*\theta$-closure operator.*

Remark 3.5. *Open sets of $\tau^{*\theta}$. Let (X, τ) be a topological space and $\mathcal{I}$ an ideal on X and observe that A is $\tau^{*\theta}$-closed iff $\tau^{*\theta} \subset A$. Now we have $U \in \tau^{*\theta}$ iff $X - U$ is $\tau^{*\theta}$-closed iff $(X - U)^{*\theta} \subseteq X - U$ iff $U \subseteq X - (X - U)^{*\theta}$. Therefore, $x \in U \rightarrow x \notin (X - U)^{*\theta} \rightarrow$ there exists a θ-neighbourhood V such that $V \cap (X - U) \in \mathcal{I}$. Now let $I = V \cap (X - U)$ and we have $x \in V - I \subseteq U$, where $I \in \mathcal{I}$. We shall denote $\beta(\mathcal{I}, \tau^\theta) = \{V - I : V \in \tau^\theta, I \in \mathcal{I}\}$.*

Theorem 3.1. *Let (X, τ) be a topological space and $\mathcal{I}$ an ideal on X. Then β is a basis for $\tau^{*\theta}$.*

Lemma 3.2. *If $(X, \tau, \mathcal{I})$ is an ideal topological space and $A \subset X$. If $A \subset A^{*\theta}$, then $A^{*\theta}= cl_\theta(A) = cl^{*\theta}(A)$.*

Proof. Always $cl^{*\theta}(A) \subset cl_\theta(A)$. Let $x \notin cl^{*\theta}(A)$. Then, there exists a $\tau^{*\theta}$-open set G containing x such that $G \cap A = \phi$. By Remark 3.5, there exists $V \in \tau^\theta$ and $I \in \mathcal{I}$ such that $x \in V - I \subset G$. Since $G \cap A = \phi \Rightarrow (V - I) \cap A = \phi \Rightarrow (V \cap A) - I = \phi \Rightarrow ((V \cap A) - I)^{*\theta} = \phi^{*\theta} \Rightarrow (V \cap A)^{*\theta} - I^{*\theta} = \phi \Rightarrow (V \cap A)^{*\theta} = \phi \Rightarrow V \cap A^{*\theta} = \phi \Rightarrow x \notin cl_\theta(A)$. Therefore,$cl_\theta(A) \subset cl^{*\theta}(A)$. Hence $cl^{*\theta}(A) = cl_\theta(A) - - - (1)$. We know that $cl^{*\theta}(A) = A \cup A^{*\theta} = A^{*\theta}$—(2), since $A \subset A^{*\theta}$. From (1) and (2), $A^{*\theta} = cl_\theta(A) = cl^{*\theta}(A)$. □

Definition 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If $A \subset A^{*\theta}$, then A is said to be $*\theta$-dense in itself.*

Definition 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If $A^{*\theta} \subset A$, then A is said to be $*\theta$-closed.*

Remark 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. Then, $\tau^{*\theta} = \{X - A : cl^{*\theta}(A) = A\}$.*

Proposition 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. Then A is $\tau^{*\theta}$-closed if and only if $A^{*\theta} \subset A$.*

Proof. Let A be $\tau^{*\theta}$-closed. Then, A = $cl^{*\theta}(A) \Rightarrow A = A \cup A^{*\theta} \Rightarrow A^{*\theta} \subset A$. Conversely, let $A^{*\theta} \subset A$. By assumption, $A \cup A^{*\theta} = A$. i.e. $cl^{*\theta}(A) = A$. Hence, A is $\tau^{*\theta}$ -closed. □

Proposition 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following hold for every subset A of X, $cl^{*\theta}(A) \subset cl_\theta(A)$;*

Proof. Let $x \in cl^{*\theta}(A)$. Then, $x \in A$ or $x \in A^{*\theta}$. If $x \in A^{*\theta}$, then there exists a θ-open set U_x containing x such that $U_x \cap A \notin \mathcal{I}$. That is $U_x \cap A \neq \phi$. This implies that $x \in cl_\theta(A)$. Thus, $cl^{*\theta}(A) \subset cl_\theta(A)$. □

Proposition 3.3. *Let $x \in cl^{*\theta}(A)$ if and only if $V \cap A \neq \phi$ for every $*\theta$-open set $V \subseteq X$.*

Properties of $()^{*\theta}$ operator

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A,B be subsets of X. Then for θ-local functions the following properties hold:*

(i) $\phi^{*\theta} = \phi$.

(ii) $A \subset B$ *implies* $A^{*\theta} \subset B^{*\theta}$.

(iii) *For an another ideal* $\mathcal{J} \supset \mathcal{I}$ *on X,* $A^{*\theta}(\mathcal{J}) \subset A^{*\theta}(\mathcal{I})$.

(iv) $A^* \subset A^{*\theta}$.

(v) $A^{*\theta} \subset cl_\theta(A)$.

(vi) $(A^{*\theta})^{*\theta} \subset A^{*\theta}$, *if A is θ-closed.*

(vii) $A^{*\theta} \cup B^{*\theta} = (A \cup B)^{*\theta}$.

(viii) $(A \cap B)^{*\theta} \subset A^{*\theta} \cap B^{*\theta}$.

(ix) *for a θ-open set U,* $U \cap A^{*\theta} = U \cap (U \cap A)^{*\theta} \subset (U \cap A)^{*\theta}$.

(x) *For* $I \in \mathcal{I}$, $(A \cup I)^{*\theta} = A^{*\theta} = (A - I)^{*\theta}$.

(xi) $(A - B)^{*\theta} - B^{*\theta} = (A^{*\theta} - B^{*\theta}) \subset (A - B)^{*\theta}$.

(xii) $(A - A^{*\theta}) \cap (A - A^{*\theta})^{*\theta} = \phi$.

(xiii) *If* $A \in \mathcal{I}$, *then* $A^{*\theta} = \phi$.

(xiv) $A^{*\theta}(\mathcal{I} \cap \mathcal{J}) \supset A^{*\theta}(\mathcal{I}) \cup A^{*\theta}(\mathcal{J})$.

Proof. (i) From the definition of θ-local function, $\phi^{*\theta} = \phi$ is obvious.

(ii) Let $x \in A^{*\theta}$. Then for every θ-open set U_x containing x, $U_x \cap A \notin \mathcal{I}$. Since $A \subset B$ implies that $U_x \cap A \subset U_x \cap B \notin \mathcal{I}$. Therefore, $U_x \cap B \notin \mathcal{I}$. This implies that $x \in B^{*\theta}$. Hence, $A^{*\theta} \subset B^{*\theta}$.

(iii) Let $x \in A^{*\theta}(\mathcal{J})$. Then for every θ-open set U_x containing x, such that $U_x \cap A \notin \mathcal{J}$. This implies that $U_x \cap A \notin \mathcal{I}$, since $\mathcal{I} \subset \mathcal{J}$. So, $x \in A^{*\theta}(\mathcal{I})$. Hence, $A^{*\theta}(\mathcal{J}) \subset A^{*\theta}(\mathcal{I})$.

(iv) Let $x \in A^*$. We assert that $x \in A^{*\theta}$. If not, then there is a θ-open set U_x containing x such that $U_x \cap A \in \mathcal{I}$. Since every θ-open is open, U_x is open and since, $U_x \cap A \in \mathcal{I}$ contradicts the assumption $x \in A^*$. Therefore, $x \in A^{*\theta}$. This implies that $A^* \subset A^{*\theta}$.

(v) Let $x \in A^{*\theta}$. Then for every θ-open set U_x containing x, $U_x \cap A \notin \mathcal{I}$. Since every θ-open is open, U_x is open. This implies that $U_x \cap A \neq \phi$ for every θ-open set containing x. Hence, $x \in cl_\theta(A)$.

(vi) From (v) $A^{*\theta} \subset cl_\theta(A)$. $(A^{*\theta})^{*\theta} \subset (cl_\theta(A))^{*\theta}$. But $A = cl_\theta(A)$, since A is θ-closed. This implies that $(A^{*\theta})^{*\theta} \subset A^{*\theta}$.

(vii) Since $A \subset A \cup B$ and $B \subset A \cup B$. Then from (ii) $A^{*\theta} \subset (A \cup B)^{*\theta}$ and $B^{*\theta} \subset (A \cup B)^{*\theta}$. Hence, $A^{*\theta} \cup B^{*\theta} \subset (A \cup B)^{*\theta}$. Conversely suppose that $x \notin A^{*\theta} \cup B^{*\theta}$. Then, $x \notin A^{*\theta}$ and $x \notin B^{*\theta}$. If $x \notin A^{*\theta}$, then there exists θ-open set U_x containing x such that $U_x \cap A \in \mathcal{I}$. Similarly since $x \notin B^{*\theta}$, there exists θ-open set V_x containing x such that $V_x \cap A \in \mathcal{I}$. Then by the hereditary property of ideal, $A \cap (U_x \cap V_x) \in \mathcal{I}$ and $B \cap (U_x \cap V_x) \in \mathcal{I}$. Again, by the finite additivity of the ideal, $(A \cup B) \cap (U_x \cap V_x) \in \mathcal{I}$. Hence, $x \notin (A \cup B)^{*\theta}$. So, $(A \cup B)^{*\theta} \subset A^{*\theta} \cup B^{*\theta}$. Hence $A^{*\theta} \cup B^{*\theta} = (A \cup B)^{*\theta}$.

(viii) Since $A \cap B \subset A$ and $A \cap B \subset B$, from (2), $(A \cap B)^{*\theta} \subset A^{*\theta}$ and $(A \cap B)^{*\theta} \subset B^{*\theta}$. Hence, $(A \cap B)^{*\theta} \subset A^{*\theta} \cap B^{*\theta}$.

(ix) Let $x \in U \cap A^{*\theta}$. Let V_x be a θ-open set containing x, then $A \cap (U \cap V_x) \notin \mathcal{I}$, since $x \in A^{*\theta}$ and $U \cap V_x$ is a θ-open set containing x . Hence, $x \in (U \cap A)^{*\theta}$. Therefore, $U \cap A^{*\theta} \subset (U \cap A)^{*\theta}$.
Therefore, $U \cap A^{*\theta} \subset U \cap (U \cap A)^{*\theta} ----(1)$. Again for $U \cap A \subset A$, $(U \cap A)^{*\theta} \subset A^{*\theta}$. So, $U \cap (U \cap A)^{*\theta} \subset U \cap A^{*\theta} ----(2)$. From (1) and (2) we have $U \cap A^{*\theta} = U \cap (U \cap A)^{*\theta}$. Hence, $U \in \tau^{\theta}$, $U \cap A^{*\theta} = U \cap (U \cap A)^{*\theta} \subset (U \cap A)^{*\theta}$.

(x) Since $A \subset A \cup I$, $A^{*\theta} \subset (A \cup I)^{*\theta}[by(i)] ----- (1)$. Let $x \in (A \cup I)^{*\theta}$. Then for every θ-open set U_x containing x, $U_x \cap (A \cup I) \notin \mathcal{I}$. Since $U_x \cap I \in \mathcal{I}$, it follows that $U_x \cap A \notin \mathcal{I}$. Hence $x \in A^{*\theta}$ which implies that $(A \cup I)^{*\theta} \subset A^{*\theta} ----- (2)$. From (1) and (2), we have $(A \cup I)^{*\theta} = A^{*\theta} ----- (3)$. Since $(A - I) \subset A$, then $(A - I)^{*\theta} \subset A^{*\theta} ----- (4)$. Now, for reverse inclusion, let $x \in A^{*\theta}$. We claim that $x \in (A - I)^{*\theta}$. If not, then there is some θ-open set U_x containing x such that $U_x \cap (A - I) \in \mathcal{I}$. Since, $I \in \mathcal{I}$, $I \cup (U_x \cap (A - I)) \in \mathcal{I}$. This implies $I \cup (U_x \cap A) \in \mathcal{I}$. So, $U_x \cap A \in \mathcal{I}$, a contradiction to the fact that $x \in A^{*\theta}$. Hence, $A^{*\theta} \subset (A - I)^{*\theta} -------(5)$. From (4) and (5), we have, $A^{*\theta} = (A - I)^{*\theta}$. Again from (3) and (6) we have $(A \cup I)^{*\theta} = A^{*\theta} = (A - I)^{*\theta}$.

(xi) Let $x \in A^{*\theta} - B^{*\theta}$. Then, $x \in A^{*\theta}$ and $x \notin B^{*\theta}$. This implies that $U_x \cap A \notin \mathcal{I}$, for every θ-open set U_x containing x and $V_x \cap B \in \mathcal{I}$, for some θ-open set V_x containing x. Hence $V_x \cap A \notin \mathcal{I}$ and $V_x \cap B \in \mathcal{I}$. Suppose that $(A - B) \cap V_x \in \mathcal{I}$. Since $((A - B) \cap V_x) \cup (B \cap V_x) = (A \cup B) \cap V_x$, by finite additivity property of ideal, $(A \cup B) \cap V_x \in \mathcal{I}$. Since $A \cap V_x \subset (A \cup B) \cap V_x$, $A \cap V_x \in \mathcal{I}$, which is a contradiction to the fact that $V_x \cap A \notin \mathcal{I}$. Therefore, $(A - B) \cap V_x \notin \mathcal{I}$ and so, $x \in (A - B)^{*\theta} --- (1)$. Therefore, $A^{*\theta} - B^{*\theta} \subset (A - B)^{*\theta} ----(2)$.
Also, $x \notin B^{*\theta}$ implies that $x \in (A - B)^{*\theta} - B^{*\theta}$. Therefore, $A^{*\theta} - B^{*\theta} \subset (A - B)^{*\theta} - B^{*\theta} ----(3)$. Let $x \in (A - B)^{*\theta} - B^{*\theta}$. Then $x \in (A - B)^{*\theta}$ and $x \notin B^{*\theta}$. If $x \in (A - B)^{*\theta}$, then for every θ-open set U_x containing x such that $(A - B) \cap U_x \notin \mathcal{I}$. Suppose that $x \notin A^{*\theta}$, then there is some θ-open set V_x containing x, $A \cap V_x \in \mathcal{I}$. Since, $x \notin B^{*\theta}$, then there is some θ-open set W_x containing x, such that $B \cap W_x \in \mathcal{I}$. Since $((A - B) \cap V_x) \cup (B \cap V_x) = (A \cup B) \cap V_x = (A \cap V_x) \cup (B \cap V_x)$ by finite additive property of the ideal, $(A \cup B) \cap V_x \in \mathcal{I}$. Since $(A - B) \cap V_x \subset (A \cup B) \cap V_x$, $(A - B) \cap V_x \in \mathcal{I}$ which is a contradiction. Therefore, $A \cap V_x \notin \mathcal{I}$, $x \in A^{*\theta}$ and $x \notin B^{*\theta}$. Therefore, $x \in A^{*\theta} - B^{*\theta}$. Thus $(A - B)^{*\theta} - B^{*\theta} \subset A^{*\theta} - B^{*\theta} --- (4)$. From (3) and (4), we have $(A^{*\theta} - B^{*\theta}) = (A - B)^{*\theta} - B^{*\theta}$. Using (2), we have $(A - B)^{*\theta} - B^{*\theta} = (A^{*\theta} - B^{*\theta}) \subset (A - B)^{*\theta}$.

(xii) Since $A - A^{*\theta} \subset X - A^{*\theta}$. So, $(A - A^{*\theta}) \cap A^{*\theta} = \phi$. Since $(A - A^{*\theta}) \subset A$, $(A - A^{*\theta})^{*\theta} \subset A^{*\theta}$. It follows that $(A - A^{*\theta}) \cap (A - A^{*\theta})^{*\theta} = \phi$.

(xiii) Suppose that $x \in A^{*\theta}$. Then, there exists some θ-open set containing x such that $U_x \cap A \notin \mathcal{I}$. But, since $A \in \mathcal{I}$, $U_x \cap A \in \mathcal{I}$ for every $U_x \in \tau^{\theta}$. This is a contradiction. Hence, $A^{*\theta} = \phi$.

□

Remark 3.7. *In Theorem 3.2, the reverse inclusions of (iii), (viii) are not valid as in the following example.*

Example 3.7. *Let* $X = \{a, b, c, d\}$ *with* $\tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$, $\mathcal{J} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$ *and* $\mathcal{I} = \{\phi, \{c\}\}$.

(1) Let $A = \{a, b\}$. *Then,* $A^{*\theta}(\mathcal{I}) = \{a, b, c, d\} \not\subset \{a\} = A^{*\theta}(\mathcal{J})$.

(2) Let $A = \{a, b, c, d\}$, $A^{*\theta} = X$, $B = \{a, b, c, e\}$, $B^{*\theta} = X$, $A \cap B = \{a, b, c\}$, $(A \cap B)^{*\theta} = \{a\}$. *Therefore* $A^{*\theta} \cap B^{*\theta} = X \not\subset \{a\} = (A \cap B)^{*\theta}$.

Proposition 3.4. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $A \subset X$ *where* $\mathcal{I} = \{\phi\}$. *Then* $A^{*\theta} = cl_{\theta}(A)$.

Proof. Let $\mathcal{I} = \{\phi\}$. We know that $cl_\theta(A) = A \cup D_\theta(A)$ where $D_\theta(A)$ is the θ-derived set of A. Let $x \in A \cup D_\theta(A)$ and let U_x be a θ-open set containing x. Then $x \in A$ or $x \in D_\theta(A)$. If $x \in A$ then $x \in U_x \cap A$ and so $U_x \cap A \neq \phi$. If $x \in D_\theta(A)$, then $\phi \neq [U_x - \{x\}] \cap A \subset U_x \cap A$ and thus $U_x \cap A \neq \phi$. Hence, $cl_\theta(A) = A \cup D_\theta(A) \subset A^{*\theta}$. By Theorem 3.2(v), $A^{*\theta} \subset cl_\theta(A)$. Therefore, $A^{*\theta} = cl_\theta(A)$. □

Proposition 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$ where $\mathcal{I} = \wp(X)$. Then $A^{*\theta} = \phi$ for every $A \subset X$.*

Proof. Since $A^{*\theta} = \{x \in X : U_x \cap A \notin \wp(X)$ for every θ-open set U_x containing $x\} = \phi$. Therefore, $A^{*\theta} = \phi$ for every $A \subset X$. □

Theorem 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A,B be subsets of X. Then for θ-local functions the following properties hold:*

1. *$A^{*\theta} = cl_\theta(A^{*\theta}) \subseteq cl_\theta(A)$ and $A^{*\theta}$ is θ-closed.*
2. *If $A \subseteq A^{*\theta}$ and $A^{*\theta}$ is open, then, $A^{*\theta} = cl_\theta(A)$.*

Proof.

1. Always $A^{*\theta} \subseteq cl_\theta(A^{*\theta})$. Let $x \in cl_\theta(A^{*\theta})$. Then , there exists some open set U_x containing x such that $A^{*\theta} \cap U_x \neq \phi$. Therefore, there exists some $y \in A^{*\theta} \cap U_x$ and $U_x \in \tau^\theta(x)$. Since $y \in A^{*\theta}$, there exists some θ-open set V_x such that $A \cap V_x \cap U_x = A \cap V_x \notin \mathcal{I}$. Therefore, $x \in A^{*\theta}$. Hence, $A^{*\theta} = cl_\theta(A^{*\theta})$ and $A^{*\theta} = cl_\theta(A^{*\theta}) \subseteq cl_\theta(A)$ by Theorem 3.2 (v).
2. For any subset A of X, by(1) we have $A^{*\theta} = cl_\theta(A^{*\theta}) \subseteq cl_\theta(A)$. Since $A \subseteq A^{*\theta}$ and $A^{*\theta}$ is open, by Lemma 1.2, $cl_\theta(A) \subseteq cl_\theta(A^{*\theta}) = cl(A^{*\theta}) = A^{*\theta} \subseteq cl_\theta(A)$ and hence, $A^{*\theta} = cl_\theta(A)$.

□

Theorem 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then, $A^{*\theta} \supset A - \cup\{U \subset X : U \in \mathcal{I}\}$ for all $A \subset X$.*

Proof. Let $B = \cup\{U \subset X : U \in \mathcal{I}\}$ and let $x \in (A - B)$. Then $x \in A$ and $x \notin B$. This implies that $x \notin U$ for all $U \in \mathcal{I}$ so that $\{x\} = \{x\} \cap A \notin \mathcal{I}$ because $x \in A$. For every $G \in \tau^\theta(x)$, we have $\{x\} \cap A \subset G \cap A \notin \mathcal{I}$ by the heredity of ideal. Hence, $x \in A^{*\theta}$. □

Remark 3.8. *The converse of the theorem 3.4 need not be true as seen in the following example.*

Example 3.8. *Let $X = \{a, b, c, d\}$ with $\tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$ and $I = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Let $A = \{a, b, c, d\}$. $B = \cup\{U \subset X : U \in I\} = \{b, c\}$. A - B = $\{a, d\}$. $A^{*\theta}$ =X $\not\subset \{a, d\} = A - B$.*

Theorem 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $B = \cup\{U \subset X : U \in \mathcal{I}\}$. If $B \in \mathcal{I}$ then $(A^{*\theta})^{*\theta} = A^{*\theta}$ for all $A \subset X$.*

Proof. Let A be a subset of X. Then, $(A^{*\theta})^{*\theta} \subset A^{*\theta}$ by Theorem 3.2(vi) . Furthermore, $A^{*\theta} \supset A - B$ by Theorem 3.4. It follows from Theorem 3.2(ii) that $(A^{*\theta})^{*\theta} \supset (A - B)^{*\theta}$. Since $B \in \mathcal{I}$, by Theorem 3.2 (x) implies that $(A^{*\theta})^{*\theta} \supset (A - B)^{*\theta} = A^{*\theta}$. Therefore, $(A^{*\theta})^{*\theta} = A^{*\theta}$. □

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which $\tau^\theta = \wp(X)$. Then $A^{*\theta} = A - \cup\{U \subset X : U \in \mathcal{I}\}$ for all $A \subset X$.*

Proof. Let B = $A - \cup\{U \subset X : U \in \mathcal{I}\}$ and let $x \in A^{*\theta}$. Then $\{x\} \cap A \notin \mathcal{I}$ because $\{x\} \in \tau^\theta = \wp(X)$. Since ideal $\mathcal{I}$ always contains ϕ, $\{x\} \cap A \neq \phi$ and so $x \in A$. It follows that $\{x\} = \{x\} \cap A \notin \mathcal{I}$ so that $x \notin U$ for all $U \in \mathcal{I}$. Hence, $x \notin B$ and therefore, $x \in A - B$. Hence, $A^{*\theta} \subset A - B$. The reverse inclusion is obvious by Theorem 3.4. □

Remark 3.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which every member of τ is clopen. Then $A^{*\theta} = A - \cup\{U \subset X : U \in \mathcal{I}\}$ for all $A \subset X$.*

Proof. Let B = $A - \cup\{U \subset X : U \in \mathcal{I}\}$ and let $A \in \wp(X)$. Then every clopen set is θ-open. Hence $A \in \tau^\theta$, which means that $\wp(X) \subset \tau^\theta$ so that $\wp(X) = \tau^\theta$. By Theorem 3.6 $A^{*\theta} = A - B$. □

Theorem 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then, the following properties holds.*

1. *If $\mathcal{I} = \{\phi\}$, then $cl^{*\theta}(A) = cl_\theta(A)$.*

2. If $\mathcal{I} = \wp(X)$, then $cl^{*\theta}(A) = A$.
3. If $A \in \mathcal{I}$, then $cl^{*\theta}(A) = A$.

Proof. Obvious. □

Theorem 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A,B be subsets of X. Then for $*\theta$-local functions the following properties hold:*

(i) $cl^{*\theta}(\phi) = \phi$.

(ii) If $A \subset B$, then $cl^{*\theta}(A) \subset cl^{*\theta}(B)$.

(iii) For an another ideal $\mathcal{J} \supseteq \mathcal{I}$ on X, $cl^{*\theta}(A, \tau, \mathcal{J}) \subset cl^{*\theta}(A, \tau, \mathcal{I})$.

(iv) $cl^{*}(A) \subset cl^{*\theta}(A)$.

(v) $cl^{*\theta}(A) \subset cl_{\theta}(A)$.

(vi) $cl^{*\theta}(cl^{*\theta}(A)) \subset cl^{*\theta}(A)$ if A is θ-closed.

(vii) $cl^{*\theta}(A) \cup cl^{*\theta}(B) = cl^{*\theta}(A \cup B)$.

(viii) $cl^{*\theta}(A \cap B) \subset cl^{*\theta}(A) \cap cl^{*\theta}(B)$.

Proof. It is obvious by using Remark 3.5 and Theorem 3.7. □

Remark 3.10. *In Theorem 3.8, The reverse inclusions of (ii), (iv), (v) and the converse of (iii) and (viii) are not valid as seen in the following examples.*

Example 3.9. *(iii) Let $X = \{a, b, c, d\}$ with $\tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{a\}\}$, $\mathcal{J} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Let $A = \{a, d\}$, $cl^{*\theta}(A, \tau, \mathcal{J}) = \{a, c, d\} \supset \{a, d\} = cl^{*\theta}(A, \tau, \mathcal{I})$ but $\mathcal{J} \not\subset \mathcal{I}$.*

Example 3.10. *Let $X = \{a, b, c, d\}$ with $\tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$.*

(ii) Let $A = \{c\}$,$B = \{a, b\}$. Then $cl^{*\theta}(A) = \{c\} \subset X = cl^{*\theta}(B)$, but$A \not\subset B$.

(iv) Let $A = \{a\}$. Then $cl^{*\theta}(A) = \{a, c, d\} \not\subset \{a, d\} = cl^{*}(A)$.

(viii) Let $A = \{b, c\}$, $B = \{b, d\}$. Then $cl^{*\theta}(A) = \{b, c\}$, $cl^{*\theta}(B) = X$, $A \cap B = \{b\}$. $cl^{*\theta}(A \cap B) = \{b\}$. So, $cl^{*\theta}(A) \cap cl^{*\theta}(B) = \{b, c\} \not\subset \{b\} = cl^{*\theta}(A \cap B)$.

(v) Let $A = \{b, c\}$. Then, $cl_{\theta}(A) = X \not\subset \{b, c\} = cl^{*\theta}(A)$.

Remark 3.11. *$D_{\theta}(A) \subset cl^{*\theta}(A)$ and $cl^{*\theta}(A) \subset D_{\theta}(A)$ are not true in general as shown in the following example.*

Example 3.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $X = \{a, b, c, d, e\}$, $\tau = \{\phi ,\{a\},\{c, d\},\{a, c, d\}, \{b, c, d, e\},X\}$ and $\mathcal{I} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$.*

(i) If $A = \{c, d\}$, then $A^{*\theta} = \phi$. Therefore, $D_{\theta}(A) = \{b, c, d, e\} \not\subset \{c, d\} = cl^{*\theta}(A)$.

(ii) If $A = \{a, b, d\}$, then $A^{*\theta} = X$. Therefore, $cl^{*\theta}(A) = X \not\subset \{b, c, d, e\} = D_{\theta}(A)$.

Proposition 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For any subset A of X, the following properties are hold.*

(i) $A^{*\theta} - A \subset cl_{\theta}(A) - A \subset D_{\theta}(A)$.

(ii) If $\mathcal{I} = \{\phi\}$, then $A^{*\theta} - A = cl_{\theta}(A) - A \subset D_{\theta}(A)$.

(iii) If $\mathcal{I} = \wp(X)$, then $A^{*\theta} = D_{\theta}(A)$.

Proof. (i)From Theorem 3.2(v), we have $A^{*\theta} \subset cl_{\theta}(A)$. Then, $A^{*\theta} - A \subset cl_{\theta}(A) - A$. Since $cl_{\theta}(A) = A \cup D_{\theta}(A)$, $cl_{\theta}(A) - A \subset D_{\theta}(A)$. It follows that $A^{*\theta} - A \subset cl_{\theta}(A) - A \subset D_{\theta}(A)$.
(ii) and (iii) are straight forward by Proposition 3.4 and Proposition 3.5. □

4 θ - Compatibility

Definition 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then τ is θ-compatible with the ideal $\mathcal{I}$, if for every $A \subseteq X$ and if for every $x \in A$, there exists $U \in \tau^{\theta}(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$ and it is denoted by $\tau \sim^{\theta} \mathcal{I}$.*

Theorem 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then the following properties are equivalent:*

1. *$\tau \sim^{\theta} \mathcal{I}$;*
2. *If a subset A of X has a cover of θ-open sets each of whose intersection with A is in $\mathcal{I}$, then $A \in \mathcal{I}$;*
3. *For every $A \subseteq X$, $A \cap A^{*\theta} = \phi$ implies that $A \in \mathcal{I}$;*
4. *For every $A \subseteq X$, $A - A^{*\theta} \in \mathcal{I}$.*
5. *For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B^{*\theta}$, then $A \in \mathcal{I}$.*

Proof. $(1) \Rightarrow (2)$. The proof is obvious.

$(2) \Rightarrow (3)$. Let$A \subseteq X$ and $x \in A$. Since $A \cap A^{*\theta} = \phi$, $x \notin A^{*\theta}$ and there exists some θ-open set $V_x \in \tau^{\theta}$ such that $V_x \cap A \in \mathcal{I}$. Therefore, we have $A \subseteq \bigcup\{V_x : x \in A\}$ and $V_x \in \tau^{\theta}$ and by (2) $A \in \mathcal{I}$.

$(3) \Rightarrow (4)$. For any $A \subseteq X$, $A - A^{*\theta} \subseteq A$ and $(A - A^{*\theta}) \cap (A - A^{*\theta})^{*\theta} \subseteq (A - A^{*\theta}) \cap A^{*\theta} = \phi$. By (3), $A - A^{*\theta} \in \mathcal{I}$.

$(4) \Rightarrow (5)$. By (4), for every $A \subseteq X$, $A - A^{*\theta} \in \mathcal{I}$. Let $A - A^{*\theta} = J \in \mathcal{I}$, $A = J \cup (A \cap A^{*\theta})$ and by Theorem 3.17 (vii) and (xiii), $A^{*\theta} = J^{*\theta} \cup (A \cap A^{*\theta})^{*\theta} = (A \cap A^{*\theta})^{*\theta}$. Therefore, we have $(A \cap A^{*\theta}) = A \cap (A \cap A^{*\theta})^{*\theta} \subseteq (A \cap A^{*\theta})^{*\theta}$ and $(A \cap A^{*\theta}) \subseteq A$. By the assumption $A \cap A^{*\theta} = \phi$ and hence $A = (A - A^{*\theta}) \in \mathcal{I}$.

$(5) \Rightarrow (1)$. Let $A \subseteq X$ and assume that for every $x \in A$, there exists some θ-open set U_x containing x, $U_x \cap A \in \mathcal{I}$. Then $A \cap A^{*\theta} = \phi$. Suppose that A contains B such that $B \subseteq B^{*\theta}$. Then $B = B \cap B^{*\theta} \subseteq A \cap A^{*\theta} = \phi$. Therefore, A contains no nonempty subset B with $B \subseteq B^{*\theta}$. Hence $A \in \mathcal{I}$. □

Lemma 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\tau \sim^{\theta} \mathcal{I}$, then for every $A \subseteq X$, $A \cap A^{*\theta} = \phi$ implies that $A^{*\theta} = \phi$.*

Proof. Let A be any subset of X and $A \cap A^{*\theta} = \phi$. By Theorem 4.1, $A \in \mathcal{I}$ and by Theorem 3.2 (xiii), $A^{*\theta} = \phi$. □

Theorem 4.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\tau \sim^{\theta} \mathcal{I}$ then the following properties are equivalent:*

1. *For every $A \subseteq X$, $A \cap A^{*\theta} = \phi$ implies that $A^{*\theta} = \phi$.*
2. *For every $A \subseteq X$, $(A - A^{*\theta})^{*\theta} = \phi$.*
3. *For every $A \subseteq X$, $(A \cap A^{*\theta})^{*\theta} = A^{*\theta}$.*

Proof. $(1) \Rightarrow (2)$. Assume that every $A \subseteq X$, $A \cap A^{*\theta} = \phi$ implies that $A^{*\theta} = \phi$. Let $B = A - A^{*\theta}$, then $B \cap B^{*\theta} = (A - A^{*\theta}) \cap (A - A^{*\theta})^{*\theta} = (A \cap (X - A^{*\theta})) \cap (A \cap (X - A^{*\theta}))^{*\theta} \subseteq (A \cap (X - A^{*\theta})) \cap (A^{*\theta} \cap (X - A^{*\theta})^{*\theta}) = \phi$. By (1), we have $B^{*\theta} = \phi$. Hence $(A - A^{*\theta})^{*\theta} = \phi$.

$(2) \Rightarrow (3)$ Assume for every $A \subseteq X$, $(A - A^{*\theta})^{*\theta} = \phi$. $A = (A - A^{*\theta}) \cup (A \cap A^{*\theta})$. $A^{*\theta} = [(A - A^{*\theta}) \cup (A \cap A^{*\theta})]^{*\theta} = (A - A^{*\theta})^{*\theta} \cup (A \cap A^{*\theta})^{*\theta} = (A \cap A^{*\theta})^{*\theta}$.

$(3) \Rightarrow (1)$ Assume for every $A \subseteq X$, $A \cap A^{*\theta} = \phi$ and $(A \cap A^{*\theta})^{*\theta} = A^{*\theta}$. This implies that $\phi = \phi^{*\theta} = A^{*\theta}$. □

Definition 4.2. *If $(X, \tau, \mathcal{I})$ is an ideal topological space, then $\mathcal{I}$ is $*\theta$-codense if and only if $A \subset A^{*\theta}$ for every θ-open set A of X.*

Characterization of θ -local function in $*\theta$-codense ideal topological space.

Theorem 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent:*

1. *$X = X^{*\theta}$.*
2. *$\tau^{\theta} \cap \mathcal{I} = \{\phi\}$.*

3. *If $I \in \mathcal{I}$, then $int_\theta(I) = \phi$.*

4. *For every $U \in \tau^\theta$, $U \subset U^{*\theta}$.*

Proof. (1) $\Rightarrow$ (2): Let $U \in \tau^\theta \cap \mathcal{I}$. Then $U \in \tau^\theta$ and $U \in \mathcal{I}$. Suppose that $x \in U$. Since $x \in X$, this implies $x \in X^{*\theta}$. Since U is a θ-open set containing x, $U \cap X \notin \mathcal{I}$ implies that $U \notin \mathcal{I}$ which is a contradiction. Therefore, $x \notin U$ for every $x \in X$. This implies that $U = \phi$ and so $\tau^\theta \cap \mathcal{I} = \{\phi\}$.

(2) $\Rightarrow$ (3): Suppose that (2) holds. Let $I \in \mathcal{I}$ be such that $I \neq \phi$. Then $int_\theta(I) \in \tau^\theta$ and $int_\theta(I) \subset I$ implies that $int_\theta(I) \in \mathcal{I}$. Therefore, by (2), $int_\theta(I) = \phi$.

(3) $\Rightarrow$ (4): $U \in \tau^\theta$ and $x \in U$. Suppose that $x \notin U^{*\theta}$. Then there exists a θ-open set V_x containing x such that $V_x \cap U \in \mathcal{I}$. Since $U \cap V_x$ is a θ-open set containing x, $U \cap V_x = int_\theta(U \cap V_x) = \phi$ by (3). Since $x \in V_x$,$x \notin U$. Thus $U \subset U^{*\theta}$ for every $U \in \tau^\theta$.

(4) $\Rightarrow$ (1): Since X is θ-open, by (4), $X \subset X^{*\theta}$, $X = X^{*\theta}$.

□

Theorem 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $I \in \mathcal{I}$. Then, I is $\tau^{*\theta}$-closed.*

Proof. Let $I \in \mathcal{I}$. By Theorem 3.22 (x) $I^{*\theta} = (I - I)^{*\theta} = \phi^{*\theta} = \phi$. Hence $cl^{*\theta}(I) = I \cup I^{*\theta} =$ I which implies that I is $\tau^{*\theta}$ -closed. □

Theorem 4.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. Then $A^{*\theta}(\tau^{*\theta}, \mathcal{I}) \subset A^{*\theta}(\tau^\theta, \mathcal{I})$.*

Proof. Let $x \in A^{*\theta}(\tau^{*\theta}, \mathcal{I})$. Suppose that $x \notin A^{*\theta}(\tau^{*\theta}, \mathcal{I})$. Then there exists a θ-open set U_x containing x, such that $A \cap U_x \in \mathcal{I}$. Since $U_x \in \tau^\theta \subset \tau^{*\theta}$, $A \cap U_x \in \mathcal{I}$ for a $\tau^{*\theta}$-open set U_x containing x. Therefore, $x \notin A^{*\theta}(\tau^{*\theta}, \mathcal{I})$ which implies that $A^{*\theta}(\tau^{*\theta}, \mathcal{I}) \subset A^{*\theta}(\tau^\theta, \mathcal{I})$. □

Theorem 4.6. *Let (X, τ) be an ideal topological space where $\mathcal{I}$ and $\mathcal{J}$ are ideals on X and $A \subset X$. Then the following hold:*

(i) $A^{*\theta}(\mathcal{I} \cap \mathcal{J}) = A^{*\theta}(\mathcal{I}) \cup A^{*\theta}(\mathcal{J})$.

(ii) *If $\mathcal{I} \subset \mathcal{J}$, then $\tau^{*\theta}(\mathcal{I}) \subset \tau^{*\theta}(\mathcal{J})$.*

(iii) $\tau^{*\theta}(\mathcal{I} \cap \mathcal{J}) = \tau^{*\theta}(\mathcal{I}) \cap \tau^{*\theta}(\mathcal{J})$.

Proof. (i) Let $x \notin A^{*\theta}(\mathcal{I} \cap \mathcal{J})$ if and only if there exists a θ-open set U_x containing x, such that $A \cap U_x \in \mathcal{I} \cap \mathcal{J}$ if and only if $A \cap U_x \in \mathcal{I}$ and $A \cap U_x \in \mathcal{J}$ if and only if $x \notin A^{*\theta}(\mathcal{I})$ and $x \notin A^{*\theta}(\mathcal{J})$ if and only if $x \notin A^{*\theta}(\mathcal{I}) \cup A^{*\theta}(\mathcal{J})$. Hence, $A^{*\theta}(\mathcal{I} \cap \mathcal{J}) = A^{*\theta}(\mathcal{I}) \cup A^{*\theta}(\mathcal{J})$ for every subset $A \subset X$.

(ii) Let $\mathcal{I} \subset \mathcal{J}$. Now if $X - A \in \tau^{*\theta}(\mathcal{I})$, then $A \cup A^{*\theta}(\mathcal{I}) = A$ which implies that $A^{*\theta}(\mathcal{I}) \subset A$. Since $\mathcal{I} \subset \mathcal{J}$, $A^{*\theta}(\mathcal{J}) \subset A^{*\theta}(\mathcal{I}) \subset A$ by Theorem 3.17 (iii). Therefore, $X - A \in \tau^{*\theta}(\mathcal{J})$ which implies that $\tau^{*\theta}(\mathcal{I}) \subset \tau^{*\theta}(\mathcal{J})$.

(iii) Let $A \subset X$ and $X - A \in \tau^{*\theta}(\mathcal{I} \cap \mathcal{J})$. Since $\mathcal{I} \cap \mathcal{J}$ is a subset of $\mathcal{I}$ and $\mathcal{J}$, $X - A \in \tau^{*\theta}(\mathcal{I})$ and $X - A \in \tau^{*\theta}(\mathcal{J})$ if and only if A is $\tau^{*\theta}(\mathcal{I})$- closed and $\tau^{*\theta}(\mathcal{J})$- closed if and only if $A^{*\theta}(\mathcal{I}) \subset A$ and $A^{*\theta}(\mathcal{J}) \subset A$. Hence, $A^{*\theta}(\mathcal{I}) \cup A^{*\theta}(\mathcal{J}) \subset A$ if and only if $A^{*\theta}(\mathcal{I} \cap \mathcal{J}) \subset A$ by (i). This implies that A is $\tau^{*\theta}(\mathcal{I} \cap \mathcal{J})$-closed. Therefore, $\tau^{*\theta}(\mathcal{I} \cap \mathcal{J}) = \tau^{*\theta}(\mathcal{I}) \cap \tau^{*\theta}(\mathcal{J})$.

□

5 The operator ψ_θ

Definition 5.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. An operator $\psi_\theta : \wp(X) \to \tau$ is defined as $\psi_\theta(A) = \{x \in X:$ there exists a θ-open set U_x containing x such that $U_x - A \in \mathcal{I}\}$, for every $A \in \wp(X)$. We observe that $\psi_\theta(A) = X - (X - A)^{*\theta}$.*

Theorem 5.1. *Let $(X, \tau, \mathcal{I})$ be a ideal topological space. Then, for $A \in \wp(X)$, $\psi_\theta(A) = X - (X - A)^{*\theta}$.*

Proof. Let $x \in \psi_\theta(A)$. Then there exists a θ-open set U_x containing x such that $U_x - A \in \mathcal{I}$. Then $X \cap (U_x - A) \in \mathcal{I}$, implies that $U_x \cap (X - A) \in \mathcal{I}$. So $x \notin (X - A)^{*\theta}$ and hence, $x \in X - (X - A)^{*\theta}$. Therefore, $\psi_\theta(A) \subset X - (X - A)^{*\theta}$. For reverse inclusion, if $x \in X - (X - A)^{*\theta}$, then $x \notin (X - A)^{*\theta}$ and so there exists a θ-open set U_x containing x such that $U_x \cap (X - A) \in \mathcal{I}$ which implies that $U_x - A \in \mathcal{I}$. Hence $x \in \psi_\theta(A)$. Thus $X - (X - A)^{*\theta} \subset \psi_\theta(A)$ and so $\psi_\theta(A) = X - (X - A)^{*\theta}$. □

Theorem 5.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A , B be subsets of X, then the following hold:*

(i) If $A \subseteq B$, then $\psi_\theta(A) \subseteq \psi_\theta(B)$.

(ii) If $A, B \in \wp(X)$, then $\psi_\theta(A) \cup \psi_\theta(B) \subset \psi_\theta(A \cup B)$

(iii) If $A, B \in \wp(X)$, then $\psi_\theta(A) \cap \psi_\theta(B)$=$\psi_\theta(A \cap B)$.

(iv) If $A \subseteq X$, $\psi_\theta(A) \subset \psi(A)$.

(v) If $U \in \tau^\theta$, then $U \subseteq \psi_\theta(U)$. Also, if $U \in \tau^{*\theta}$, then $U \subseteq \psi_\theta(U)$.

(vi) If $A \subseteq X$, then $\psi_\theta(A) \subseteq \psi_\theta(\psi_\theta(A))$.

(vii) If $A \subseteq X$, then $\psi_\theta(A) = \psi_\theta(\psi_\theta(A))$ if and only if $((X - A)^{*\theta})^{*\theta} = (X - A)^{*\theta}$.

(viii) If $A \subseteq X$ and $I \in \mathcal{I}$, then $\psi_\theta(A - I) = \psi_\theta(A) = \psi_\theta(A \cup I)$.

(ix) If $(A - B) \cup (B - A) \in \mathcal{I}$, then $\psi_\theta(A) = \psi_\theta(B)$.

Proof. (i) Since $A \subseteq B$, then $(X - A) \supseteq (X - B)$. Then by Theorem 3.22 (ii), $(X - A)^{*\theta} \supseteq (X - B)^{*\theta}$ and hence $\psi_\theta(A) \subseteq \psi_\theta(B)$.

(ii) Since $A \subset A \cup B$ and $B \subset A \cup B$, by (i) $\psi_\theta(A) \cup \psi_\theta(B) \subset \psi_\theta(A \cup B)$.

(iii) $\psi_\theta(A \cap B) = X - (X - (A \cap B))^{*\theta} = X - ((X - A) \cup (X - B))^{*\theta}$. This implies that $\psi_\theta(A \cap B) = X - ((X - A)^{*\theta} \cup (X - B)^{*\theta})$, from Theorem 3.22(xi). Therefore, $\psi_\theta(A \cap B) = (X - (X - A)^{*\theta}) \cup (X - (X - B)^{*\theta})$ and hence, $\psi_\theta(A \cap B) = \psi_\theta(A) \cap \psi_\theta(B)$.

(iv) From Theorem 3.17 (iv), we have that $(X - A)^* \subset (X - A)^{*\theta}$. This implies that $X - (X - A)^* \supset X - (X - A)^{*\theta}$ and $\psi_\theta(A) \subset \psi(A)$.

(v) Since $U \in \tau^\theta$, then $X - U$ is a θ-closed set. So, $cl_\theta(X - U) = X - U$. By theorem 3.22 (vi), $(X - U)^{*\theta} \subseteq cl_\theta(X - U) = (X - U)$. Then, $U \subseteq X - (X - U)^{*\theta} = \psi_\theta(U)$ for every $U \in \tau^\theta$. If $U \in \tau^{*\theta}$, then $X - U$ is a $\tau^{*\theta}$-closed which implies that $(X - U)^{*\theta} \subseteq (X - U)$ and so, $U \subseteq X - (X - U)^{*\theta} = \psi_\theta(U)$.

(vi) This follows from (i) and (v).

(vii) Since $\psi_\theta(\psi_\theta(A)) = X - (X - \psi_\theta(A))^{*\theta} = X - (X - (X - (X - A)^{*\theta}))^{*\theta} = X - ((X - A)^{*\theta})^{*\theta} = X - (X - A)^{*\theta} = \psi_\theta(A)$ if and only if $((X - A)^{*\theta})^{*\theta} = (X - A)^{*\theta}$.

(viii) We know that $X - (X - (A - \mathcal{I}))^{*\theta} = X - ((X - A) \cup \mathcal{I})^{*\theta} = X - (X - A)^{*\theta}$, (Theorem3.22(xvi)). So, $\psi_\theta(A - \mathcal{I}) = \psi_\theta(A)$. Also, we know that $X - (X - (A \cup \mathcal{I}))^{*\theta} = X - ((X - A) - \mathcal{I})^{*\theta} = X - (X - A)^{*\theta}$, (from Theorem 3.22(xvi)). So, $\psi_\theta(A - \mathcal{I}) = \psi_\theta(A)$. Also, $\psi_\theta(A \cup \mathcal{I}) = \psi_\theta(A)$.

(ix) Given that $(A - B) \cup (B - A) \in \mathcal{I}$, and let $A - B = I_1$, $B - A = I_2$. We observe that I_1 and $I_2 \in \mathcal{I}$ by heredity. Also, observe that, $B = ((A - I_1) \cup I_2)$. Thus, $\psi_\theta(A) = \psi_\theta((A - I_1) \cup I_2) = \psi_\theta(B)$.

□

Corollary 5.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $U \subseteq \psi_\theta(U)$ for every θ-open set $U \subseteq X$.*

Proof. We know that $\psi_\theta(U) = X - (X - U)^{*\theta}$. Now $(X - U)^{*\theta} \subseteq cl_\theta(X - U) = X - U$, since $X - U$ is θ-closed. Therefore, $U = X - (X - U) \subseteq X - (X - U)^{*\theta} = \psi_\theta(U)$. □

Remark 5.1. *The following example shows that a set A is not θ-open but satisfies $A \subseteq \psi_\theta(A)$.*

Example 5.1. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}, \{c, d\}\}$. Let $A = \{b\}$. Then $\psi_\theta(\{b\}) = X - (X - \{b\})^{*\theta} = X - (\{a, c, d\})^{*\theta} = X - \{a\} = \{b, c, d\}$. Therefore, $A \subseteq \psi_\theta(A)$, But A is not θ-open.*

Theorem 5.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $A \subseteq X$, then, $A \cap \psi_\theta(A) = int_\theta(A)$.*

Proof. If $x \in A \cap \psi_\theta(A)$, then $x \in A$ and there exists a θ-open set U_x containing x, such that $U_x - A \in \mathcal{I}$. Then, by Remark 3.5, $U_x - (U_x - A) \in \tau^\theta$-open neighborhood of x and $x \in int_\theta(A)$. On the other hand, if $x \in int_\theta(A)$ there exists a basic τ^θ-open neighborhood $V_x - A$ of x, where $V_x - A \in \tau$ and $I \in \mathcal{I}$, such that $x \in V_x - I \subseteq A$ which implies $V_x - A \subseteq I$ and hence $V_x - A \in \mathcal{I}$. Hence, $x \in A \cap \psi_\theta(A)$. □

Theorem 5.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following properties hold:*

1. $\psi_\theta(A) = \bigcup\{U \in \tau^\theta \colon U - A \in \mathcal{I}\}$.
2. $\psi_\theta(A) \supseteq \bigcup\{U \in \tau^\theta \colon (U - A) \cup (A - U) \in \mathcal{I}\}$.

Proof. (1) This follows immediately from the definition of ψ_θ-operator.

(2) Since $\mathcal{I}$ is heredity, it is obvious that $\bigcup\{U \in \tau^\theta \colon (U - A) \cup (A - U) \in \mathcal{I}\} \subseteq \bigcup\{U \in \tau^\theta \colon U - A \in \mathcal{I}\} = \psi_\theta(A)$ for every $A \subseteq X$.

□

Theorem 5.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\tau \sim^\theta \mathcal{I}$ if and only if $\psi_\theta(A) - A \in \mathcal{I}$ for every $A \subseteq X$.*

Proof. **Necessity:**
Assume $\tau \sim^\theta \mathcal{I}$ and let $A \subseteq X$. Observe that $x \in \psi_\theta(A) - A \in \mathcal{I}$ if and only if $x \notin A$ and $x \notin (X - A)^{*\theta}$ if and only if $x \notin A$ and there exists some θ-open set $U_x \in \tau^\theta(x)$ such that $U_x - A \in \mathcal{I}$ if and only if there exists some θ-open set $U_x \in \tau^\theta(x)$ such that $x \in U_x - A \in \mathcal{I}$. Now, for each $x \in \psi_\theta(A) - A$ and $U_x \in \tau^\theta(x)$, $U_x \cap (\psi_\theta(A) - A) \in \mathcal{I}$ by heredity and hence, $\psi_\theta(A) - A \in \mathcal{I}$ by assumption that $\tau \sim^\theta \mathcal{I}$.
Sufficiency:
Let $A \subseteq X$ and assume that for each $x \in A$ there exists some θ-open set $U_x \in \tau^\theta(x)$ such that $U_x \cap A \in \mathcal{I}$. Observe that $\psi_\theta(X - A) - (X - A) = A - A^{*\theta} = \{x :$ there exists some θ-open set $U_x \in \tau^\theta(x)$ such that $U_x \cap A \in \mathcal{I}\}$. Thus, we have $A \subseteq \psi_\theta(X - A) - (X - A) \in \mathcal{I}$ and hence, $A \in \mathcal{I}$ by heredity of $\mathcal{I}$. □

Theorem 5.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^\theta \mathcal{I}$, $A \subseteq X$. If N is a nonempty θ-open subset of $A^{*\theta} \cap \psi_\theta(A)$, then $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof. If $N \subseteq A^{*\theta} \cap \psi_\theta(A)$, then $N - A \subseteq \psi_\theta(A) - A \in \mathcal{I}$ by Theorem 5.5 and hence $N - A \in \mathcal{I}$ by heredity. Since $N \in \tau^\theta - \{\phi\}$ and $N \subseteq A^{*\theta}$, we have $N \cap A \notin \mathcal{I}$ by the definition of $A^{*\theta}$. □

Remark 5.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^\theta \mathcal{I}$. Then $\psi_\theta(A) = \psi_\theta(\psi_\theta(A))$ for every $A \subseteq X$.*

Proof. $\psi_\theta(A) \subseteq \psi_\theta(\psi_\theta(A))$ follows from Theorem 5.2(vi). Since $\tau \sim^\theta \mathcal{I}$, it follows from Theorem 5.5 that $\psi_\theta(A) \subseteq A \cup \mathcal{I}$ for some $I \in \mathcal{I}$, and hence $\psi_\theta(A) = \psi_\theta(\psi_\theta(A))$ by Theorem 5.2 (viii). □

Theorem 5.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^\theta \mathcal{I}$. Then $\psi_\theta(A) = \bigcup\{\psi_\theta(U) \colon U \in \tau^\theta , \psi_\theta(U) - A \in \mathcal{I}\}$.*

Proof. Let $\Phi(A) = \bigcup\{\psi_\theta(U) \colon U \in \tau^\theta , \psi_\theta(U) - A \in \mathcal{I}\}$. Clearly $\Phi(A) \subseteq \psi_\theta(A)$. Now let $x \in \psi_\theta(A)$. Then, there exists a θ-open set U, such that $U - A \in \mathcal{I}$. By Corollary 5.1, $U \subseteq \psi_\theta(U)$and $\psi_\theta(U) - A \subseteq [\psi_\theta(U) - U] \cup [U - A]$. By Theorem 5.5 $\psi_\theta(U) - U \in \mathcal{I}$. Hence, $x \in \Phi(A)$ and $\Phi(A) \supseteq \psi_\theta(A)$. Consequently, we obtain $\Phi(A) = \psi_\theta(A)$. □

Theorem 5.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^\theta \mathcal{I}$, where $\tau^\theta \cap \mathcal{I} = \phi$. Then for $A \subseteq X$, $\psi_\theta(A) \subseteq A^{*\theta}$.*

Proof. Suppose $x \in \psi_\theta(A)$ and $x \notin A^{*\theta}$. Then, there exists a θ-open set $U_x \in \tau(x)$ such that $U_x \cap A \in \mathcal{I}$. Since $x \in \psi_\theta(A)$, by Theorem 5.4 $x \in \bigcup\{U \in \tau^\theta : U - A \in \mathcal{I}\}$ and there exists a θ-open set $V_x \in \tau^\theta(x)$ such that $V_x - A \in \mathcal{I}$. Now, we have $U_x \cap V_x \in \tau^\theta(x)$, $U_x \cap V_x \cap A \in \mathcal{I}$ and $U_x \cap V_x - A \in \mathcal{I}$ by heredity. Hence, by finite additivity, we have $(U_x \cap V_x \cap A) \cup (U_x \cap V_x - A) = U_x \cap V_x \in \mathcal{I}$. Since $(U_x \cap V_x) \in \tau^\theta$, this is contrary to $\tau^\theta \cap \mathcal{I}$ $= \phi$. Therefore, $x \in A^{*\theta}$. This implies that $\psi_\theta(A) \subseteq A^{*\theta}$. □

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Continuous dependence of the solution of a stochastic differential equation with nonlocal conditions

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Abstract

In this paper we are concerned with a nonlocal problem of a stochastic differential equation that contains a Brownian motion. The solution contains both of mean square Riemann and mean square Riemann-Steltjes integrals, so we study an existence theorem for unique mean square continuous solution and its continuous dependence of the random data X_0 and the (non-random data) coefficients of the nonlocal condition a_k. Also, a stochastic differential equation with the integral condition will be considered.

Keywords: Integral condition, Brownian motion, unique mean square solution, continuous dependence, random data, non-random data, integral condition.

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1 Introduction

Many authors in the last decades studied a nonlocal problems of ordinary differential equations, the reader is referred to ([3]-[8]), and references therein.
Also the theory of stochastic differential equations, random fixed point theory, existence of solutions of stochastic differential equations by using successive approximation method and properties of these solutions have been extensively studied by several authors, especially those contain the Brownian motion as a formal derivative of the Gausian white noise, the Brownian motion $W(t), t \in R$, is defined as a stochastic process such that

$$W(0) = 0,\ E(W(t)) = 0,\ E(W(t))^2 = t$$

and $[W(t_1) - W(t_2)]$ is a Gaussian random variable for all $t_1, t_2 \in R$. The reader is referred to ([1]-[2]) and ([9]-[13]) and references therein.
Here we are concerned with the stochastic differential equation

$$dX(t) = f(t, X(t))dt + g(t)dW(t), \quad t \in (0, T] \tag{1.1}$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \quad a_k > 0, \tau_k \in (0, T), \tag{1.2}$$

where X_0 is a second order random variable independent of the Brownian motion $W(t)$ and a_k are positive real integers.

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The existence of a unique mean square solution will be studied. The continuous dependence on the random data X_0 and the non-random data a_k will be established. The problem (1.1) with the integral condition

$$X(0) + \int_0^T X(s)dv(s) = X_0. \tag{1.3}$$

will be considered.

2 Integral representation

Let $I = [0, T]$ and $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic process with the norm

$$\| X \|_C = \sup_{t\in[0,T]} \| X(t) \|_2 = \sup_{t\in[0,T]} \sqrt{E(X(t))^2}.$$

Throughout the paper we assume that the following assumptions hold

(H1) The function $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is mean square continuous.

(H2) There exists an integrable function $k : [0, T] \to R^+$, where

$$\sup_{t\in[0,T]} \int_0^t k(s)ds \le m$$

such that the function f satisfies the mean square Lipschitz condition

$$\| f(t, X_1(t)) - f(t, X_2(t)) \|_2 \le k(t) \| X_1(t) - X_2(t) \|_2 .$$

(H3) There exists a positive real number m_1 such that

$$\sup_{t\in[0,T]} | f(t, 0) | \le m_1.$$

Now we have the following lemmas.

Lemma 2.1. *For a deterministic function $g(t) : I \to \Re^+$ and a Brownian motion $W(t)$*

$$\left\| \int_0^t g(s)dW(s) \right\|^2 = \int_0^t g^2(s)ds$$

Proof.

$$\begin{aligned}
\left\| \int_0^t g(s)dW(s) \right\|^2 &= E\left(\int_0^t g(s)dW(s) \right)^2 \\
&= E\left(\int_0^t g(s)dW(s) \right)\left(\int_0^t g(s)dW(s) \right) \\
&= E\left(\lim_{n\to\infty} \sum_{k=0}^{n-1} g(t_k)\Delta W(t_k) \right)\left(\lim_{n\to\infty} \sum_{k=0}^{n-1} g(t_k)\Delta W(t_k) \right) \\
&= \left(\lim_{n\to\infty} \sum_{k=0}^{n-1} g^2(t_k) E(\Delta W(t_k))^2 \right) \\
&= \left(\lim_{n\to\infty} \sum_{k=0}^{n-1} g^2(t_k)(\Delta t_k) \right) \\
&= \int_0^t g^2(s)ds
\end{aligned}$$

This complete the proof. □

Lemma 2.2. *The solution of the problem (1.1)and(1.2) can be expressed by the integral equation*

$$X(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s), \tag{2.1}$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$.

Proof. . Integrating equation (1.1), we obtain

$$X(t) = X(0) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s)$$

and

$$X(\tau_k) = X(0) + \int_0^{\tau_k} f(s, X(s))ds + \int_0^{\tau_k} g(s)dW(s),$$

then

$$\sum_{k=1}^{n} a_k X(\tau_k) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)$$

$$X_0 - X(0) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)$$

and

$$\left(1 + \sum_{k=1}^{n} a_k\right) X(0) = X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s),$$

then

$$X(0) = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1} \left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} \int_0^{\tau_k} g(s)dW(s)\right).$$

Hence

$$X(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s).$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$. □

Now define the mapping

$$FX(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s). \tag{2.2}$$

Then we can prove the following lemma.

Lemma 2.3. $F : C \to C$.

Proof. . Let $X \in C$, t_1 , $t_2 \in [0, T]$ such that $| t_2 - t_1 | < \delta$, then

$$FX(t_2) - FX(t_1) = \int_{t_1}^{t_2} f(s, X(s))ds + \int_{t_1}^{t_2} g(s)dW(s).$$

From assumption (H2) we have

$$\| f(t, X(t)) \|_2 - | f(t,0) | \leq \| f(t, X(t)) - f(t,0) \|_2 \leq k(t) \| X(t) \|_2,$$

then we have

$$\| f(t, X(t)) \|_2 \leq k(t) \| X(t) \|_2 + | f(t,0) | \leq k(t) \; \| X \|_C + m_1.$$

So,

$$\| F X(t_2) - F X(t_1) \|_2 \leq \int_{t_1}^{t_2} ||f(s, X(s))||_2 \, ds + \| \int_{t_1}^{t_2} g(s) dW(s) \|_2,$$

using assumptions and lemma 2.1, we get

$$\| F X(t_2) - F X(t_1) \|_2 \leq \| X \|_C \int_{t_1}^{t_2} k(s) ds + m_1(t_2 - t_1) + \sqrt{\int_{t_1}^{t_2} g^2(s) ds}$$

which proves that $F : C \to C$. □

3 Existence and uniqueness

For the existence of a unique continuous solution $X \in C$ of the problem (1.1)-(1.2), we have the following theorem.

Theorem 3.1. *Let the assumptions $(H1) - (H3)$ be satisfied. If $2m < 1$, then the problem (1.1)-(1.2) has a unique solution $X \in C$.*

Proof. Let X and $X^* \in C$, then

$$\begin{aligned}
& \| FX(t) - FX^*(t) \|_2 \\
= & \left\| \int_0^t [f(s, X(s)) - f(s, X^*(s))] ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} [f(s, X(s)) - f(s, X^*(s))] ds \right\|_2 \\
\leq & \int_0^t ||f(s, X(s)) - f(s, X^*(s))||_2 \, ds + a \sum_{k=1}^{n} a_k \int_0^{\tau_k} ||f(s, X(s)) - f(s, X^*(s))||_2 \, ds \\
\leq & \; m \| X - X^* \|_C + \left[a \sum_{k=1}^{n} a_k \right] m \| X - X^* \|_C, \\
\leq & \left[1 + a \sum_{k=1}^{n} a_k \right] m \| X - X^* \|_C \\
\leq & \; 2m \| X - X^* \|_C .
\end{aligned}$$

Hence

$$\| FX - FX^* \|_C \leq 2m \| X - X^* \|_C .$$

If $2m < 1$, then F is contraction and there exists a unique solution $X \in C$ of the nonlocal stochastic problem (1.1)-(1.2), [2]. This solution is given by (2.1). □

4 Continuous dependence

Consider the stochastic differential equation (1.1) with the nonlocal condition

$$X(0) + \sum_{k=1}^{n} a_k \, X(\tau_k) = \tilde{X}_0 \quad , \tau_k \in (0, T) \tag{4.1}$$

Definition 4.1. *The solution* $X \in C$ *of the nonlocal problem (1.1)-(1.2) is continuously dependent (on the data* X_0*) if* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that* $\| X_0 - \tilde{X}_0 \|_2 \leq \delta$ *implies that* $\| X - \tilde{X} \|_C \leq \epsilon$

Here, we study the continuous dependence (on the random data X_0) of the solution of the stochastic differential equation (1.1) and (1.2).

Theorem 4.2. *Let the assumptions* $(H1) - (H3)$ *be satisfied. Then the solution of the nonlocal problem (1.1)-(1.2) is continuously dependent on the random data* X_0*.*

Proof. Let

$$X(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s)$$

be the solution of the nonlocal problem (1.1)-(1.2) and

$$\tilde{X}(t) = a\left(\tilde{X}_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, \tilde{X}(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, \tilde{X}(s))ds + \int_0^t g(s)dW(s)$$

be the solution of the nonlocal problem (1.1) and (4.1). Then

$$\begin{aligned} X(t) - \tilde{X}(t) &= a[X_0 - \tilde{X}_0] - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds \\ &+ \int_0^t [f(s, X(s)) - f(s, \tilde{X}(s))]ds. \end{aligned}$$

Using our assumptions, we get

$$\begin{aligned} \| X(t) - \tilde{X}(t) \|_2 &\leq a \| X_0 - \tilde{X}_0 \|_2 + a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \| f(s, X(s)) - f(s, \tilde{X}(s)) \|_2 \, ds \\ &+ \int_0^t \| f(s, X(s)) - f(s, \tilde{X}(s)) \|_2 \, ds \\ &\leq a\delta + 2m \| X - \tilde{X} \|_2, \end{aligned}$$

then

$$\| X - \tilde{X} \|_C \leq \frac{a\delta}{1 - 2m} = \epsilon$$

This complete the proof. □

Now consider the stochastic differential equation (1.1) with the nonlocal condition

$$X(0) + \sum_{k=1}^{n} \tilde{a}_k X(\tau_k) = X_0 \quad , \tau_k \in (0, T) \tag{4.2}$$

Definition 4.2. *The solution* $X \in C$ *of the nonlocal problem (1.1)-(1.2) is continuously dependent (on the coefficient* a_k *of the nonlocal condition) if* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that* $| a_k - \tilde{a}_k | \leq \delta$ *implies that* $\| X - \tilde{X} \|_C \leq \epsilon$

Here, we study the continuous dependence (on the coefficient a_k of the nonlocal condition) of the solution of the stochastic differential equation (1.1) and (1.2).

Theorem 4.3. *Let the assumptions* $(H1) - (H3)$ *be satisfied. Then the solution of the nonlocal problem (1.1)-(1.2) is continuously dependent on the coefficient* a_k *of the nonlocal condition.*

Proof. Let

$$X(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s)$$

be the solution of the nonlocal problem (1.1)-(1.2) and

$$\tilde{X}(t) = \tilde{a}\left(X_0 - \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} f(s, \tilde{X}(s))ds - \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, \tilde{X}(s))ds + \int_0^t g(s)dW(s)$$

be the solution of the nonlocal problem (1.1) and (4.2).

Then

$$\begin{aligned} X(t) - \tilde{X}(t) &= [a - \tilde{a}]X_0 + \int_0^t [f(s, X(s)) - f(s, \tilde{X}(s))]ds - \left[\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} \tilde{a}_k\right] \int_0^{\tau_k} g(s)dW(s) \\ &- a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \tilde{a}\sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} f(s, \tilde{X}(s))ds. \end{aligned}$$

Now

$$| a - \tilde{a} | = \left| \frac{1}{1 + \sum_{k=1}^{n} a_k} - \frac{1}{1 + \sum_{k=1}^{n} \tilde{a}_k} \right| = \left| \frac{\sum_{k=1}^{n} (\tilde{a}_k - a_k)}{\left(1 + \sum_{k=1}^{n} a_k\right)\left(1 + \sum_{k=1}^{n} \tilde{a}_k\right)} \right| \leq \left| \sum_{k=1}^{n} (\tilde{a}_k - a_k) \right| \leq n\delta$$

and

$$\begin{aligned} & \tilde{a}\sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} f(s, \tilde{X}(s))ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds \\ = \ & \tilde{a}\left(1 + \sum_{k=1}^{n} \tilde{a}_k\right) \int_0^{\tau_k} f(s, \tilde{X}(s))ds - a\left(1 + \sum_{k=1}^{n} a_k\right) \int_0^{\tau_k} f(s, X(s))ds \\ - \ & \tilde{a}\int_0^{\tau_k} f(s, \tilde{X}(s))ds + a\int_0^{\tau_k} f(s, X(s))ds \\ = \ & \tilde{a}(\tilde{a}^{-1}) \int_0^{\tau_k} f(s, \tilde{X}(s))ds - a(a^{-1}) \int_0^{\tau_k} f(s, X(s))ds \\ - \ & \tilde{a}\int_0^{\tau_k} f(s, \tilde{X}(s))ds + a\int_0^{\tau_k} f(s, X(s))ds \\ = \ & -\int_0^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds + a\int_0^{\tau_k} f(s, X(s))ds - \tilde{a}\int_0^{\tau_k} f(s, \tilde{X}(s))ds \\ - \ & \tilde{a}\int_0^{\tau_k} f(s, X(s))ds + \tilde{a}\int_0^{\tau_k} f(s, X(s))ds \\ = \ & -\int_0^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds + [a - \tilde{a}]\int_0^{\tau_k} f(s, X(s))ds \\ + \ & \tilde{a}\int_0^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds \end{aligned}$$

and

$$\begin{aligned}\left[a\sum_{k=1}^{n}a_k-\tilde{a}\sum_{k=1}^{n}\tilde{a}_k\right]\int_0^{\tau_k}g(s)dW(s) &= \left[a\left(1+\sum_{k=1}^{n}a_k\right)-\tilde{a}\left(1+\sum_{k=1}^{n}\tilde{a}_k\right)\right]\int_0^{\tau_k}g(s)dW(s)\\ &- [a-\tilde{a}]\int_0^{\tau_k}g(s)dW(s)\\ &= \left[aa^{-1}-\tilde{a}\tilde{a}^{-1}\right]\int_0^{\tau_k}g(s)dW(s)-[a-\tilde{a}]\int_0^{\tau_k}g(s)dW(s)\\ &= -[a-\tilde{a}]\int_0^{\tau_k}g(s)dW(s).\end{aligned}$$

Then

$$\begin{aligned}\| X(t)-\tilde{X}(t)\|_2 &\leq n\delta \| X_0\|_2+\int_{\tau_k}^{t}\| f(s,X(s))-f(s,\tilde{X}(s))\|_2\,ds+n\delta\left\|\int_0^{\tau_k}g(s)dW(s)\right\|_2\\ &+ n\delta\,[m\| X\|_C+m_1T]+\tilde{a}\int_0^{\tau_k}\| f(s,X(s))-f(s,\tilde{X}(s))\|_2\,ds.\end{aligned}$$

Using our assumptions we get

$$\begin{aligned}\| X-\tilde{X}\|_C &\leq n\delta\| X_0\|_2+m\| X-\tilde{X}\|_C+n\delta\sqrt{\int_0^{\tau_k}g^2(s)ds}+n\delta\,[m\| X\|_C+m_1T]\\ &+ \tilde{a}m\| X-\tilde{X}\|_C,\end{aligned}$$

then

$$\begin{aligned}\| X-\widetilde{X}\|_C &\leq n\delta\left[\| X_0\|_2+m\| X\|_C+m_1T+\sqrt{\int_0^{\tau_k}g^2(s)ds}\right]+(1+\tilde{a})m\| X-\tilde{X}\|_C\\ &\leq n\delta\left[\| X_0\|_2+m\| X\|_C+m_1T+\sqrt{\int_0^{\tau_k}g^2(s)ds}\right]+2m\| X-\tilde{X}\|_C\,.\end{aligned}$$

Hence

$$\| X-\widetilde{X}\|_C\leq\frac{n\delta\left[\| X_0\|_2+m\| X\|_C+m_1T+\sqrt{\int_0^{\tau_k}g^2(s)ds}\right]}{1-2m}=\epsilon.$$

This complete the proof. □

5 Nonlocal Integral Condition

Let

$$a_k=v(t_k)-v(t_{k-1}),\ \tau_k\in\ (t_{k-1},t_k),$$

where

$$0\ <\ t_1\ <\ t_2\ <\ t_3\ <\ ....\ <\ T.$$

Then, the nonlocal condition (1.2) will be in the form

$$X(0) \;+\; \sum_{k=1}^{n} \; X(\tau_k)\;(v(t_k) - v(t_{k-1})) = \; X_0.$$

From the mean square continuity of the solution of the nonlocal problem (1.1)-(1.2), we obtain from [13]

$$\underset{n\to\infty}{l.i.m} \sum_{k=1}^{n} X(\tau_k)\;(v(t_k) - v(t_{k-1})) = \int_0^T X(s)dv(s),$$

that is, the nonlocal conditions (1.2) is transformed to the mean square Riemann-Steltjes integral condition

$$X(0) \;+\; \int_0^T X(s)dv(s) \;=\; X_0.$$

Now, we have the following theorem.

Theorem 5.4. *Let the assumptions (H1)-(H3) be satisfied, then the stochastic differential equation (1.1) with the nonlocal integral condition (1.3) has a unique mean square continuous solution represented in the form*

$$X(t) = a^\star \left(X_0 - \int_0^T\int_0^s f(\theta, X(\theta))d\theta dv(s) - \int_0^T\int_0^s g(\theta)dW(\theta)dv(s) \right) + \int_0^t f(\theta, X(\theta))d\theta + \int_0^t g(\theta)dW(\theta),$$

where $a^\star \; = (1 + v(T) - v(0))^{-1}$.

Proof. Taking the limit of equation (2.1) we get the proof. □

6 Conclusion

Here we defined the mean square continuous solution for the stochastic differential equation and proved the existence of unique solution of the problem (1.1)-(1.2), then we studied the continuous dependence of the solution of (1.1)-(1.2) on the initial random data and the nonrandom coefficient of the nonlocal condition .

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On the maximal and minimal solutions of a nonlocal problem of a delay stochastic differential equation

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Abstract

In this paper we are concerned with a problem of of a delay stochastic differential equation with nonlocal condition, the solution is represented as stochastic integral equation that contain mean square Riemann integral. We study the existence of at least mean square continuous solution for this problem. The existence of the maximal and minimal solutions will be proved.

Keywords: Nonlocal condition, delay equation, random Caratheodory function, stochastic Lebesgue dominated convergence theorem, at least mean square continuous solution, maximal solution, minimal solution.

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1 Introduction

The problems of differential equation with nonlocal condition studied recently by some authors, see ([3]-[5]) and ([7]-[8]) and references therein. Problems of the stochastic differential equations have been extensively studied by several authors in the last decades The reader is referred to ([1]-[2]), ([6]) and ([9]-[14]) and references therein.
Let $\phi : [0, T] \to [0, T]$ be continuous real-valued function such that $\phi(t) \leq t,\ t \in [0, T]$.
Here we are concerned with the delay stochastic differential equation

$$\frac{dX(t)}{dt} = f(t, X(\phi(t))), \quad t \in (0, T] \tag{1.1}$$

with the random nonlocal initial condition

$$X(0) + \sum_{k=1}^{m} a_k X(\tau_k) = X_0, \quad \tau_k \in (0, T), \tag{1.2}$$

where X_0 is a second order random variable and a_k are positive real numbers.
Our aim is to study the existence of at least mean square continuous solution of the problem (1.1)-(1.2). Also we define the maximal and minimal solution of the stochastic differential equation. Hence we study the existence of maximal and minimal solution of the problem (1.1)-(1.2).

2 Preliminaries

Here we give some preliminaries which will be needed in our work.

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Definition 2.1. *[13][Random Caratheodory function]*
Let X be a stochastic process and let $t \in I = [a, b]$, a and b are real numbers. A stochastic function $f(t, X(\omega))$ is called a Caratheodory function if it satisfies the following conditions

1. *$f(t, X(.))$ is measurable for every t,*

2. *$f(., X(\omega))$ is continuous for a.e. stochastic process X.*

Theorem 2.1. *[12][Schauder and Tychonoff theorem]*
Let Q be a closed bounded convex set in a Banach space and Let T be a completely continuous operator on Q such that $T(Q) \subset Q$. Then T has at least one fixed point in Q. That is, there is at least one $x^* \in Q$ such that $T(x^*) = x^*$.

Definition 2.2. *[10] A family of real random functions $(X_1(t), X_2(t), ..., X_k(t))$ is uniformly bounded in mean square sense if there exist a $\beta \in R$ (β is finite) such that $E(X_n^2(t)) < \beta$ for all $n \geq 1$ and all $t \in I = [a, b]$, where a, b are real numbers.*

Definition 2.3. *[10] A family of real random functions $(X_1(t), X_2(t), ..., X_k(t))$ is equicontinuous in mean square sense if for each $t \in I = [a, b]$, where a, b are real numbers and $\epsilon > 0$, there exist a $\delta > 0$ such that*

$$E([X_n(t_2) - X_n(t_1)]^2) < \epsilon, \;\; \forall\, n \geq 1 \quad \text{when ever} \quad | t_2 - t_1 | < \delta.$$

Theorem 2.2. *[10][Arzela theorem]*
Every uniformly bounded equicontinuous family (sequence) of functions $(f_1(x), f_2(x), ..., f_k(x))$ has at least one subsequence which converges uniformly on the $I = [a, b]$, where a, b are real numbers

Theorem 2.3. *[11][Stochastic Lebesgue dominated convergence theorem]*
Let $X_n(t)$ be a sequence of random vectors (or functions) is converging to $X(t)$ such that

$$X(t) = \lim_{n \to \infty} X_n(t), \quad a.s.,$$

and $X_n(t)$ is dominated by an integrable function $a(t)$ such that $\| X_n(t) \|_2 \leq a(t)$. Then

1. *$E[\lim_{n \to \infty} X_n] = \lim_{n \to \infty} E[X_n]$ and*

2. *$E[X_n(t) - X(t)] \to 0 \quad as \quad n \to \infty$*

where a.s. means that it happens with probability one.

3 Integral representation

Let $I = [0, T]$ and $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic process with the norm

$$\| X \|_C = \sup_{t \in [0,T]} \| X(t) \|_2 = \sup_{t \in [0,T]} \sqrt{E\,(X(t))^2}.$$

Throughout the paper we assume that the following assumptions hold

i- The functions $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is Caratheodory function in mean square sense.

ii- There exists an integrable function $l(t) \in L^1$ such that

$$\| f(t, X(t)) \|_2 \leq l(t)\,, \quad \forall (t, X) \in I \times L_2(\Omega)$$

with $\left[\sup_{t \in [0,T]} \int_0^t l(s) ds \leq M \right]$, where M is a positive real number.

Now we have the following lemma.

Lemma 3.1. *The solution of the nonlocal stochastic problem (1.1) and (1.2) can be expressed by the stochastic integral equation*

$$X(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds\right) + \int_0^t f(s, X(\phi(s)))ds \tag{3.1}$$

where $a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1}$.

Proof. . Integrating equation (1.1), we obtain

$$X(t) = X(0) + \int_0^t f(s, X(\phi(s)))ds$$

and

$$X(\tau_k) = X(0) + \int_0^{\tau_k} f(s, X(\phi(s)))ds,$$

then

$$\sum_{k=1}^{m} a_k X(\tau_k) = \sum_{k=1}^{m} a_k X(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds,$$

$$X_0 - X(0) = \sum_{k=1}^{m} a_k X(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds$$

and

$$\left(1 + \sum_{k=1}^{m} a_k\right) X(0) = X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds,$$

then

$$X(0) = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1} \left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds\right).$$

Hence

$$X(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds\right) + \int_0^t f(s, X(\phi(s)))ds,$$

where $a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1}$. □

4 Existence of at least mean square continuous solution

For the existence of at least continuous solution $X \in C$ of the stochastic problem (1.1) and (1.2), we have the following theorem.

Theorem 4.4. *Let the assumptions (i)-(ii) be satisfied, then the problem (1.1)-(1.2) has at least a solution* $X \in C$ *given by the stochastic integral equation (3.1).*

Proof. . Consider in the space C, the set Q such that

$$Q = \{X \in C : \| X \|_C \leq \beta; \ \beta \text{ is a positive real number}\}$$

Now for each $X(t) \in Q$ we can define the operator H by

$$HX(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds\right) + \int_0^t f(s, X(\phi(s)))ds$$

we shall prove that $HX(t) \in Q$. For that let $X(t) \in Q$, then

$$\begin{aligned} \| HX(t) \|_2 &\leq a \| X_0 \|_2 + a \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, X(\phi(s))) \|_2 \, ds + \int_0^t \| f(s, X(\phi(s))) \|_2 \, ds \\ &\leq a \| X_0 \|_2 + a \sum_{k=1}^{m} a_k \int_0^{\tau_k} l(\phi(s)) ds + \int_0^t l(\phi(s)) ds \\ &\leq a \| X_0 \|_2 + a \sum_{k=1}^{m} a_k \int_0^{\tau_k} l(s) ds + \int_0^t l(s) ds \\ &\leq a \| X_0 \|_2 + a \sum_{k=1}^{m} a_k M + M. \end{aligned}$$

Let $a \| X_0 \|_2 + a \sum_{k=1}^{m} a_k M + M = \beta$, β is clearly a positive real number, then ($\| HX \|_C \leq \beta$), so $HX \in Q$ and hence $HQ \subset Q$ and is also uniformly bounded.
For $t_1, t_2 \in R^+$, $t_1 < t_2$, let $| t_2 - t_1 | < \delta$, then

$$\| HX(t_2) - HX(t_1) \|_2 \leq \int_{t_1}^{t_2} \| f(s, X(\phi(s))) \|_2 \, ds \leq \int_{t_1}^{t_2} l(s) ds \leq M.$$

Then $\{HX\}$ is a class of equicontinuous functions. Therefore the operator H is equicontinuous and uniformly bounded.
Suppose that $\{X_n\} \in C$ such that $X_n \to X$ in mean square sense.
So,

$$\begin{aligned} \underset{n\to\infty}{l.i.m}\, HX_n(t) &= \underset{n\to\infty}{l.i.m} \left[aX_0 - a \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X_n(\phi(s))) ds \right] + \underset{n\to\infty}{l.i.m} \left[\int_0^t f(s, X_n(\phi(s))) ds \right] \\ &= aX_0 - \left(a \sum_{k=1}^{m} a_k \right) \underset{n\to\infty}{l.i.m} \left[\int_0^{\tau_k} f(s, X_n(\phi(s))) ds \right] + \underset{n\to\infty}{l.i.m} \left[\int_0^t f(s, X_n(\phi(s))) ds \right]. \end{aligned}$$

Using our assumptions and then applying stochastic Lebesgue dominated convergence theorem, we get

$$\begin{aligned} \underset{n\to\infty}{l.i.m}\, HX_n(t) &= aX_0 - a \sum_{k=1}^{m} a_k \int_0^{\tau_k} \underset{n\to\infty}{l.i.m} [f(s, X_n(s))] ds + \int_0^t \underset{n\to\infty}{l.i.m} [f(s, X_n(\phi(s)))] ds \\ &= aX_0 - a \sum_{k=1}^{m} a_k \int_0^{\tau_k} [f(s, \underset{n\to\infty}{l.i.m} X_n(\phi(s)))] ds + \int_0^t [f(s, \underset{n\to\infty}{l.i.m} X_n(\phi(s)))] ds \\ &= aX_0 - a \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s))) ds + \int_0^t f(s, x(\phi(s))) ds \\ &= HX(t) \end{aligned}$$

This proves that H is continuous operator, then H is continuous and compact.

Then H has a fixed point $X \in C$ which proves that there exists at least one solution of the stochastic differential equation (1.1)-(1.2) given by (3.1). □

5 Maximal and minimal solution

Now we give the following definition.

Definition 5.4. *Let $q(t)$ be a solution of the problem (1.1)-(1.2), then $q(t)$ is said to be a maximal solution of (1.1)-(1.2) if every solution $X(t)$ of (1.1)-(1.2) satisfies the inequality*

$$\| X(t) \|_2 < \| q(t)) \|_2 .$$

A minimal solution $s(t)$ can be defined by similar way by reversing the above inequality i.e.

$$\| X(t) \|_2 > \| s(t) \|_2 .$$

In this section f assumed to satisfy the following definition.

Definition 5.5. *The functions $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically decreasing if for any $X, Y \in L_2(\Omega)$ satisfying*

$$\| X(t) \|_2 < \| Y(t) \|_2$$

implies that

$$\| f(t, X(t)) \|_2 < \| f(t, Y(t)) \|_2 .$$

Now we have the following lemma.

Lemma 5.2. *Let the assumptions (i)-(ii) be satisfied and let $X, Y \in L_2(\Omega)$ satisfying*

$$\| X(t) \|_2 \le a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, X(\phi(s))) \|_2 \, ds \right) + \int_0^t ||f(s, X(\phi(s)))||_2 \, ds$$

and

$$\| Y(t) \|_2 \ge a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, Y(\phi(s))) \|_2 \, ds \right) + \int_0^t ||f(s, Y(\phi(s)))||_2 \, ds.$$

If $f(t; x)$ is stochastically decreasing function . Then

$$\| X(t) \|_2 < \| Y(t) \|_2 \tag{5.1}$$

Proof. . Let the conclusion (5.1) be false, then there exists t_1 such that

$$\| X(t_1) \|_2 = \| Y(t_1) \|_2, \ \ t_1 > 0 \tag{5.2}$$

and

$$\| X(t) \|_2 < \| Y(t) \|_2, \ \ 0 < t < t_1 \tag{5.3}$$

since $f(t; x)$ satisfies the definition (5.5)and using equation (5.3), we get

$$\begin{aligned} \| X(t_1) \|_2 \ \ &\le \ \ a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, X(\phi(s))) \|_2 \, ds \right) + \int_0^{t_1} \| f(s, X(\phi(s))) \|_2 \, ds \\ &< \ \ a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, Y(\phi(s))) \|_2 \, ds \right) + \int_0^{t_1} \| f(s, Y(\phi(s))) \|_2 \, ds \\ &< \ \ \| Y(t) \|_2, \ \ 0 < t < t_1, \end{aligned}$$

which contradicts equation (5.2), then

$$\| X(t) \|_2 < \| Y(t) \|_2 .$$

□

Now we have the following theorem.

Theorem 5.5. *Let the assumptions (i)-(ii) be satisfied. If $f(t, X(t))$ satisfies the definition (5.5), then there exist a maximal solution of the problem (1.1)-(1.2).*

Proof. . Firstly we shall prove the existence of the maximal solution of the problem.
Let $\epsilon > 0$ be given. Now consider the integral equation

$$X_\epsilon(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_\epsilon(s, X_\epsilon(\phi(s)))ds\right) + \int_0^t f_\epsilon(s, X_\epsilon(\phi(s)))ds, \tag{5.4}$$

where

$$f_\epsilon(t, X_\epsilon(t)) = f(s, X_\epsilon(t)) + \epsilon$$

Clearly the function $f_\epsilon(t, X_\epsilon(t))$ satisfies the conditions (i)-(ii) and

$$\| f_\epsilon(t, X_\epsilon(t)) \|_2 \leq l(t) + \epsilon = \grave{l}(t),$$

then equation (5.4) is a solution of the problem (1.1)-(1.2) according to Theorem (4.4).
Now let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$ Then

$$\begin{aligned} X_{\epsilon_1}(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_{\epsilon_1}(s, X_{\epsilon_1}(\phi(s)))ds\right) + \int_0^t f_{\epsilon_1}(s, X_{\epsilon_1}(\phi(s)))ds, \\ &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_1)\, ds\right) + \int_0^t (f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_1)\, ds, \end{aligned}$$

this implies that

$$\begin{aligned} \| X_{\epsilon_1}(t) \|_2 &\geq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_1||_2\, ds + \int_0^t ||f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_2||_2\, ds \\ &\geq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_2||_2\, ds + \int_0^t ||f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_2||_2\, ds, \;\; \epsilon_2 < \epsilon_1 \end{aligned} \tag{5.5}$$

and

$$\| X_{\epsilon_2}(t) \|_2 \leq a\left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} ||(f(s, X_{\epsilon_2}(\phi(s))) + \epsilon_2)||_2\, ds\right) + \int_0^t ||(f(s, X_{\epsilon_2}(\phi(s))) + \epsilon_2)||_2\, ds. \tag{5.6}$$

Using Lemma (5.2), then equations (5.5) and (5.6) implies

$$\| X_{\epsilon_2}(t) \|_2 < \| X_{\epsilon_1}(t) \|_2$$

As shown before in the proof of Theorem (4.4) the family of functions $x_\epsilon(t)$ defined by equation (3.1) is uniformly bounded and equicontinuous functions. Hence by Arzela Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} X_{\epsilon_n}(t)$ exists uniformly in C and denote this limit by $q(t)$, then from the continuity of the function f_{ϵ_n} in the second argument and applying Lebesgue dominated convergence Theorem, we get

$$q(t) = \lim_{n\to\infty} X_{\epsilon_n}(t)$$

which proves that $q(t)$ is a solution of the problem (1.1)-(1.2)
Finally, we shall show that $q(t)$ is the maximal solution of the problem (1.1)-(1.2). To do this, let $X(t)$ be any solution of the problem (1.1)-(1.2).
Then

$$\| X_\epsilon(t) \|_2 \geq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||f(s, X_\epsilon(\phi(s))) + \epsilon||_2\, ds + \int_0^t ||f(s, X_\epsilon(\phi(s))) + \epsilon||_2\, ds$$

and

$$\| X(t) \|_2 \leq a \left(||X_0||_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, X_\epsilon(\phi(s))) \|_2 \, ds \right) + \int_0^t \| f(s, X(\phi(s))) \|_2 \, ds.$$

Applying Lemma (5.2), we get

$$\| X_\epsilon(t) \|_2 > \| X(t) \|_2$$

from the uniqueness of the maximal solution (see [6]), it is clear that $X_\epsilon(t)$ tends to $q(t)$ uniformly as $\epsilon \to 0$. □

By similar way as done above we can prove that $s(t)$ is the minimal solution of the problem (1.1)-(1.2). The maximal and minimal solutions of the problem (1.1)-(1.2) can be defined in the same fashion as done above. If the function f assumed to satisfy the following definition.

Definition 5.6. *The functions $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically increasing if for any $X, Y \in L_2(\Omega)$ satisfying*

$$\| X(t) \|_2 < \| Y(t) \|_2$$

implies that

$$\| f(t, X(t)) \|_2 > \| f(t, Y(t)) \|_2 .$$

Now we have the following theorem.

Theorem 5.6. *Let the assumptions (i)-(ii) be satisfied. If $f(t, X)$ satisfies the definition (5.6), then there exist a minimal solution of the problem (1.1)-(1.2).*

6 Examples

Here, as an application of our results, we give the following two examples.

Example 6.1. *Let $\beta \in (0, 1]$. As ϕ, one can take, for example $\phi(t) = \beta t$.*

Let the assumptions of Theorem (4.4) be satisfied. Then the problem

$$\frac{dX(t)}{dt} = f(t, X(\beta t)), \quad t \in (0, T]$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \quad \tau_k \in (0, T),$$

has at least one solution $X \in C([0, T], L^2(\Omega))$.

Example 6.2. *Let the assumptions of Theorem (4.4) be satisfied, let $\gamma \geq 1$. As ϕ, one can tack, for example $\phi(t) = t^\gamma$. Then the problem*

$$\frac{dX(t)}{dt} = f(t, X(t^\gamma)), \quad t \in (0, 1]$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \quad \tau_k \in (0, 1),$$

has at least one solution $X \in C([0, 1], L^2(\Omega))$.

7 Conclusion

Here we defined the mean square solution for the stochastic differential equation and proved the existence of at least one solution of the problem (1.1)-(1.2), then we proved the existence of the maximal and minimal solution of (1.1)-(1.2).

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Coefficient Estimates for Bazilevič Ma-Minda Functions in the Space of Sigmoid Function

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Abstract

In this work, the authors investigated the coefficient estimates for Bazilevič Ma-Minda Functions for the class $T_n^{\alpha}(\lambda,\beta,l,\Phi)$. The first few coefficient bounds for this class were obtained and also the relevant connection to Fekete-Szegö theorem and were briefly discussed. Our results serve as a new generalization in this direction and gives birth to many corollaries.

Keywords: Analytic Function, Univalent Function, Starlike Function, Convex Function, Bazilevič Function, Subordination, Sigmoid Function, Fekete-Szegö Inequality.

2010 MSC: 30C45, 33E99.

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1 Introduction

In the twentieth century, the theory of special functions was overshadowed by other fields like functional analysis, real analysis, algebra, topology, differential equations and so on. These functions do not have specific definitions but they constitute an information process that is inspired by the way biological nervous system such as the brain processes information. This information process contains large numbers of highly interconnected elements (neurons) working together to perform specific tasks.

Special functions can be categorized into three, namely ramp function, sigmoid function and threshold function. The most popular of the functions is the sigmoid function because of its gradient descent algorithm. It can be evaluated by truncated series expansion (see details in [5], [9] and [11]).

The sigmoid function of the form

$$g(z)=\frac{1}{1+e^{-z}} \tag{1.1}$$

is differentiable and has the following properties:

(i) it outputs real numbers between 0 and 1.

(ii) it maps a very large input domain to a small range of outputs.

(iii) it never loses information because it is a one-to-one function.

(iv) it increases monotonically.

The four properties show that sigmoid function is very useful in geometric functions theory.

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Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U) \tag{1.2}$$

which are analytic in the open disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

A domain $U \subset \mathbb{C}$ is convex if the line segment joining any two points in U lies entirely in U, while a domain is starlike with respect to a point $\omega_0 \in U$ if the line segment joining any point of U to ω_0 lies inside U. A function $f \in A$ is starlike if $f(U)$ is a starlike domain with respect to the origin and convex if $f(U)$ is convex.

Recall that starlike and convex functions are denoted by ST and CV respectively and analytically written as $Re\frac{zf'(z)}{f(z)} > 0$ and $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. Starlike and convex functions of type α are denoted by $ST(\alpha)$ and $CV(\alpha)$ respectively and characterized by $Re\frac{zf'(z)}{f(z)} > \alpha$ and $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$ where $\alpha : 0 \le \alpha < 1$ (see detail in [2]).

The two functions f and g are analytic in the open unit disk U. We say f is subordinate to g written as $f < g \in U$ if there exists a Schwarz function $w(z)$ which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. It follows from Schwarz lemma that $f(z) < g(z) \quad (z \in U) \implies f(0) = g(0)$ and $f(U) \subset g(U)$ (see details in [8]).

Ma and Minda[7] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the open unit disk U, $\varphi(0) = 1$ and $\varphi'(0) > 0$ and φ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike function consists of functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} < \varphi(z)$ and Ma-Minda convex function consists of functions $f \in A$ satisfying subordination $1 + \frac{zf''(z)}{f'(z)} < \varphi(z)$ (detail in [2]).

Lemma 1.1 (Pommerenke[13]). If a function $p \in P$ is given by

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \quad (z \in U) \tag{1.3}$$

then $|p_k| \le 2 \quad (k \in N)$, where P is the class of Caratheodory function, analytic in U for which $p(0) = 1$ and Re $p(z) > 0 \quad (z \in U)$.

Let $\alpha > 0$ (α is real), then

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^\alpha \tag{1.4}$$

which gives

$$f(z)^\alpha = (z + a_2 z^2 + a_3 z^3 + a_4 z^4 + ...)^\alpha \tag{1.5}$$

Or, equivalently

$$f(z)^\alpha = (z(1 + a_2 z + a_3 z^2 + a_4 z^3 + ...))^\alpha \tag{1.6}$$

Using simple expansion for (1.6), we have

$$f(z)^\alpha = z^\alpha \left(1 + \alpha(a_2 z + a_3 z^2 + a_4 z^3 + ...) + \frac{\alpha(\alpha - 1)}{2!}(a_2 z + a_3 z^2 + a_4 z^3 + ...)^2 + ...\right) \tag{1.7}$$

Since the expansion continues, then

$$f(z)^\alpha = z^\alpha \left(1 + \alpha(a_2 z + a_3 z^2 + a_4 z^3 + ...)\right)$$

which implies

$$f(z)^\alpha = z^\alpha + \alpha a_2 z^{\alpha+1} + \alpha a_3 z^{\alpha+2} + \alpha a_4 z^{\alpha+3} + ...$$

This finally gives

$$f(z)^{\alpha} = z^{\alpha} + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1} \tag{1.8}$$

Catas et al.[3] defined the Catas Operator as follows:

$$\begin{aligned} I^0(\lambda, l) &: A \to A \\ I^0(\lambda, l) f(z) &= f(z) \\ I^1(\lambda, l) f(z) &= (I(\lambda, l) f(z)) \left(\frac{1-\lambda+l}{1+l}\right) + (I(\lambda, l) f(z)) \left(\frac{\lambda z}{1+l}\right) = z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right) a_k z^k \end{aligned}$$

and

$$I^2(\lambda, l) f(z) = (I^1(\lambda, l) f(z)) \left(\frac{1-\lambda+l}{1+l}\right) + (I^1(\lambda, l) f(z)) \left(\frac{\lambda z}{1+l}\right) = z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right)^2 a_k z^k$$

In general,

$$I^n(\lambda, l) f(z) = I(\lambda, l)(I^{n-1}(\lambda, l) f(z)) = z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n a_k z^k \tag{1.9}$$

Applying (1.9) in (1.8), we have

$$I^n(\lambda, l) f(z)^{\alpha} = \left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^n z^{\alpha} + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^n a_k(\alpha) z^{\alpha+k-1} \tag{1.10}$$

where $n \in N_0$, $\alpha > 0$ (α is real), $\lambda \geq 0$, $l \geq 0$.

Oladipo and Olatunji[10] used (1.10) to define a class $T_n^{\alpha}(\lambda, \beta, l)$ with geometric condition satisfying

$$Re \frac{I^n(\lambda, l) f(z)^{\alpha}}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^n z^{\alpha}} > \beta \tag{1.11}$$

where $n \in N_0$, $\alpha > 0$ (α is real), $\lambda \geq 0$, $l \geq 0$ and $0 \leq \beta < 1$. The first few coefficient bounds for the class were obtained and the coefficient inequalities for the class were derived by employing Hayami's method [6]. By specializing the parameters involved in (1.11), we obtain various subclasses of analytic functions studied by [1], [12], [14], [15] and so on.

In this work, the authors defined a new class of functions denoted by $T_n^{\alpha}(\lambda, \beta, l, \Phi)$ as related to modified sigmoid function with geometric condition satisfying

$$\frac{Re \frac{I^n(\lambda, l) f(z)^{\alpha}}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^n z^{\alpha}} - \beta}{1-\beta} < \varphi(z) \tag{1.12}$$

where $n \in N_0$, $\alpha > 0$ (α is real), $\lambda \geq 0$, $l \geq 0$ and $0 \leq \beta < 1$. The first few coefficient estimates for the class are obtained. Also, the relevant connection to Fekete-Szegö theorem are briefly discussed.

For the purpose of our results, we require the following lemmas.

Lemma 1.2 (Fadipe-Joseph et al.[5]). Let g be a sigmoid function and

$$\Phi(z) = 2g(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m \tag{1.13}$$

then $\Phi(z) \in P$, $|z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 1.3 (Fadipe-Joseph et al.[5]). Let

$$\Phi_{m,n}(z) = 2g(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right) \tag{1.14}$$

then $|\Phi_{m,n}(z)| < 2$.

Lemma 1.4 (Fadipe-Joseph et al.[5]). If $\Phi(z) \in P$ and it is starlike, then f is a normalized univalent function of the form (1.2).

Setting $m = 1$, Fadipe-Joseph et al.[5] remarked that $\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ where $c_n = \frac{(-1)^{n+1}}{(2n)!}$. As such, $|c_n| \leq 2$, $n = 1, 2, 3, ...$ and the result is sharp for each n.

2 Coefficient Estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the open unit disk U, with $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \beta_4 z^4 + ... \ (\beta_1 > 0) \tag{2.15}$$

For functions in the class $T_n^{\alpha}(\lambda, \beta, l, \Phi)$, the following results are obtained.

Theorem 2.1. *If $f(z)^{\alpha} \in T_n^{\alpha}(\lambda, \beta, l, \Phi)$ is given by (1.12), then*

$$|a_2(\alpha)| \leq \frac{(1-\beta)B_1}{4\alpha\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.16}$$

$$|a_3(\alpha)| \leq \frac{(1-\beta)\left[2\alpha(B_2 - B_1)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha-1)(1-\beta)B_1^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{32\alpha^2\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.17}$$

$$\begin{aligned}|a_4(\alpha)| \leq &\frac{2(1-\beta)(3B_3 - 6B_2 - B_1)}{384\alpha\left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n} \\ &- \frac{(\alpha-1)(1-\beta)^3 B_1}{384\alpha^3\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{3n}}\left\{\frac{3\left[2\alpha(B_2 - B_1)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha-1)(1-\beta)B_1^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n}\right\} \\ &+ \frac{(\alpha-2)(1-\beta)^2 B_1^3}{384\alpha}\end{aligned} \tag{2.18}$$

Proof. Let $f(z)^{\alpha} \in T_n^{\alpha}(\lambda, \beta, l, \Phi)$. Then there are analytic functions $u : U \to U$ with $u(0) = 0$ satisfying

$$\frac{\frac{I^n(\lambda,l)f(z)^{\alpha}}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^n z^{\alpha}} - \beta}{1-\beta} = \varphi(u(z)) \tag{2.19}$$

Define the function $\Phi(z)$ by

$$\Phi(z) = \frac{1+u(z)}{1-u(z)} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - ... \tag{2.20}$$

or, equivalently

$$u(z) = \frac{\Phi(z)-1}{\Phi(z)+1} = \frac{1}{4}z - \frac{1}{16}z^2 - \frac{1}{192}z^3 - \frac{5}{768}z^4 - \frac{13}{15360}z^5 + ... \tag{2.21}$$

In view of (2.19), (2.20) and (2.21), clearly

$$\frac{\frac{I^n(\lambda,l)f(z)^{\alpha}}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^n z^{\alpha}} - \beta}{1-\beta} = \varphi\left(\frac{\Phi(z)-1}{\Phi(z)+1}\right) \tag{2.22}$$

Using (2.21) together with (2.15), it is evident that

$$\varphi\left(\frac{\Phi(z)-1}{\Phi(z)+1}\right) = 1 + \frac{B_1}{4}z + \frac{B_2 - B_1}{16}z^2 - \frac{B_1 + 6B_2 - 3B_3}{192}z^3 + \frac{5B_1 + B_2 - 9B_3 + 3B_4}{768}z^4 + ... \tag{2.23}$$

Recall that

$$\frac{I^n(\lambda,l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^n z^\alpha} = 1+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right)^n a_k(\alpha)z^{k-1}$$

which has the expansion

$$\begin{aligned}&1+\alpha\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n a_2 z+\left(\alpha a_3+\frac{\alpha(\alpha-1)}{2}a_2^2\right)\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n z^2\\&+\left(\alpha a_4+\alpha(\alpha-1)a_2a_3+\frac{\alpha(\alpha-1)(\alpha-2)}{6}a_2^3\right)\left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n z^3\\&+\left(\alpha a_5+\frac{\alpha(\alpha-1)}{2!}(2a_2a_4+a_3^2)+\frac{\alpha(\alpha-1)(\alpha-2)}{3}a_2^2a_3+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24}a_2^4\right)\left(\frac{1+\lambda(\alpha+3)+l}{1+\lambda(\alpha-1)+l}\right)^n z^4\\&+\ldots\end{aligned} \tag{2.24}$$

Therefore (2.22) yields

$$\begin{aligned}&1+\alpha\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n a_2 z+\left(\alpha a_3+\frac{\alpha(\alpha-1)}{2}a_2^2\right)\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n z^2\\&+\left(\alpha a_4+\alpha(\alpha-1)a_2a_3+\frac{\alpha(\alpha-1)(\alpha-2)}{6}a_2^3\right)\left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n z^3\\&+\left(\alpha a_5+\frac{\alpha(\alpha-1)}{2!}(2a_2a_4+a_3^2)+\frac{\alpha(\alpha-1)(\alpha-2)}{3}a_2^2a_3+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24}a_2^4\right)\left(\frac{1+\lambda(\alpha+3)+l}{1+\lambda(\alpha-1)+l}\right)^n z^4\\&+\ldots=\beta+(1-\beta)\left[1+\frac{B_1}{4}z+\frac{B_2-\beta_1}{16}z^2-\frac{B_1+6B_2-3B_3}{192}z^3+\frac{5B_1+B_2-9B_3+3B_4}{768}z^4+\ldots\right]\end{aligned} \tag{2.25}$$

Comparing the L.H.S. and R.H.S. of (2.25), it gives

$$\alpha\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n a_2(\alpha)=\frac{(1-\beta)B_1}{4} \tag{2.26}$$

$$\left(\alpha a_3(\alpha)+\frac{\alpha(\alpha-1)}{2}a_2^2(\alpha)\right)\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n=\frac{(1-\beta)(B_2-B_1)}{16} \tag{2.27}$$

$$\left(\alpha a_4(\alpha)+\alpha(\alpha-1)a_2(\alpha)a_3(\alpha)+\frac{\alpha(\alpha-1)(\alpha-2)}{6}a_2^3(\alpha)\right)\left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n=-\frac{(1-\beta)(B_1+6B_2-3B_3)}{192} \tag{2.28}$$

So, by simple computation, we obtain

$$|a_2(\alpha)|\le\frac{(1-\beta)B_1}{4\alpha\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.29}$$

$$|a_3(\alpha)|\le\frac{(1-\beta)\left[2\alpha(B_2-B_1)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}-(\alpha-1)(1-\beta)B_1^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{32\alpha^2\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.30}$$

$$\begin{aligned}|a_4(\alpha)|\le&\frac{2(1-\beta)(3B_3-6B_2-B_1)}{384\alpha\left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n}\\&-\frac{(\alpha-1)(1-\beta)^3B_1}{384\alpha^3\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{3n}}\left\{\frac{3\left[2\alpha(B_2-B_1)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}-(\alpha-1)(1-\beta)B_1^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n}\right\}\\&+\frac{(\alpha-2)(1-\beta)^2B_1^3}{384\alpha}\end{aligned} \tag{2.31}$$

□

and this completes the proof of Theorem (2.1).

By specializing some parameters that are involved, we obtain some corollaries.

Setting $\beta = 0$, it gives the following corollary

Corollary 2.1. If $f(z)^{\alpha} \in T_n^{\alpha}(\lambda, 0, l, \Phi)$ is given by (1.12), then

$$|a_2(\alpha)| \leq \frac{B_1}{4\alpha\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.32}$$

$$|a_3(\alpha)| \leq \frac{\left[2\alpha(B_2-B_1)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha-1)B_1^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{32\alpha^2\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.33}$$

$$\begin{aligned} |a_4(\alpha)| \leq & \frac{2(3B_3-6B_2-B_1)}{384\alpha\left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n} \\ & - \frac{(\alpha-1)B_1}{384\alpha^3\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{3n}}\left\{\frac{3\left[2\alpha(B_2-B_1)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha-1)B_1^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n}\right\} \\ & + \frac{(\alpha-2)B_1^3}{384\alpha}. \end{aligned} \tag{2.34}$$

Setting $\alpha = 1$ in Corollary (2.1) gives

Corollary 2.2. If $f(z) \in T_n^1(\lambda, 0, l, \Phi)$ is given by (1.12), then

$$|a_2(1)| \leq \frac{B_1}{4\left(\frac{1+\lambda+l}{1+l}\right)^n} \tag{2.35}$$

$$|a_3(1)| \leq \frac{\left[2(B_2-B_1)\left(\frac{1+\lambda+l}{1+l}\right)^{2n}\right]}{32\left(\frac{1+\lambda+l}{1+l}\right)^{2n}\left(\frac{1+2\lambda+l}{1+l}\right)^n} \tag{2.36}$$

$$|a_4(1)| \leq \frac{2(3B_3-6B_2-B_1)}{384\left(\frac{1+3\lambda+l}{1+l}\right)^n} - \frac{B_1^3}{384}. \tag{2.37}$$

Putting $\lambda = 1$ in Corollary (2.2) yields

Corollary 2.3. If $f(z) \in T_n^1(1, 0, l, \Phi)$ is given by (1.12), then

$$|a_2(1)| \leq \frac{B_1}{4\left(\frac{2+l}{1+l}\right)^n} \tag{2.38}$$

$$|a_3(1)| \leq \frac{\left[2(B_2-B_1)\left(\frac{2+l}{1+l}\right)^{2n}\right]}{32\left(\frac{2+l}{1+l}\right)^{2n}\left(\frac{3+l}{1+l}\right)^n} \tag{2.39}$$

$$|a_4(1)| \leq \frac{2(3B_3-6B_2-B_1)}{384\left(\frac{4+l}{1+l}\right)^n} - \frac{B_1^3}{384}. \tag{2.40}$$

Taking $l = 0$ in Corollary (2.3) it is seen that

Corollary 2.4. If $f(z) \in T_n^1(1,0,0,\Phi)$ is given by (1.12), then

$$|a_2(1)| \leq \frac{B_1}{4(2)^n} \tag{2.41}$$

$$|a_3(1)| \leq \frac{\left[2(B_2 - B_1)(2)^{2n}\right]}{32(2)^{2n}3^n} \tag{2.42}$$

$$|a_4(1)| \leq \frac{2(3B_3 - 6B_2 - B_1)}{384(4)^n} - \frac{B_1^3}{384}. \tag{2.43}$$

If $n = 0$ in Corollary (2.4) we get

Corollary 2.5. If $f(z) \in T_0^1(1,0,0,\Phi)$ is given by (1.12), then

$$|a_2(1)| \leq \frac{B_1}{4} \tag{2.44}$$

$$|a_3(1)| \leq \frac{(B_2 - B_1)}{16} \tag{2.45}$$

$$|a_4(1)| \leq \frac{(3B_3 - 6B_2 - B_1)}{192} - \frac{B_1^3}{384}. \tag{2.46}$$

3 The Fekete-Szegö Inequality

In order to obtain the Fekete-Szegö Inequalities, we shall employ the Deniz and Orhan[4] and Ma and Minda[7] approach.

Theorem 3.1. *If* $f(z)^\alpha \in T_n^\alpha(\lambda,\beta,l,\Phi)$ *is given by* (1.12)*, then*

$$|a_3 - \mu a_2^2| \leq \frac{1-\beta}{32}\left|\frac{B_1^2(\beta-1)(\alpha+2\mu-1)\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n - 2\alpha(B_1-B_2)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}}{\alpha^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}}\right|. \tag{3.47}$$

Proof. From (2.29) and (2.30), we have

$$a_3 - \mu a_2^2 = \frac{(1-\beta)\left[2\alpha(B_2-B_1)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha-1)(1-\beta)B_1^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{32\alpha^2\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} - \mu\left[\frac{(1-\beta)B_1}{4\alpha\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n}\right]^2. \tag{3.48}$$

Simplifying (3.48), we have

$$a_3 - \mu a_2^2 = \frac{1-\beta}{32}\left[\frac{B_1^2(\beta-1)(\alpha+2\mu-1)\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n - 2\alpha(B_1-B_2)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}}{\alpha^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}}\right] \tag{3.49}$$

which completes the proof. □

Taking $\mu = 1$, we obtain

Corollary 3.6. If $f(z)^\alpha \in T_n^\alpha(\lambda,\beta,l,\Phi)$ is given by (1.12), then

$$|a_3 - a_2^2| \leq \frac{1-\beta}{32}\left|\frac{B_1^2(\beta-1)(\alpha+1)\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n - 2\alpha(B_1-B_2)\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}}{\alpha^2\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n}}\right|. \tag{3.50}$$

4 Conclusion

By varying other parameters that are involved, many corollaries can be generated.

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Donoho-Stark Uncertainty Principle for the Generalized Bessel Transform

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Abstract

The generalized Bessel transform satisfies some uncertainty principles similar to the Euclidean Fourier transform. A generalization of Donoho-Stark uncertainty principle is obtained for the generalized Bessel transform.

Keywords: Generalized Bessel transform; Donoho-stark's uncertainty principle.

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1 Introduction

There are many theorems known which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized. That it is impossible for a nonzero function and its Fourier transform to be simultaneously small. There are several manifestations of this principle. We refer the reader to the excellent survey article by Folland and Sitaram [3], and also the monograph by S. Thangavelu [5]. In this paper we are interested in a variant of Donoho-Stark's uncertainty principle. Recall that Donoho and Stark [2] paid attention to the supports of functions and gave qualitative uncertainty principles for the Fourier transforms. The purpose of this paper is to obtain uncertainty principle similar to Donoho-Stark's principle for the generalized Bessel transform. The outline of the content of this paper is as follows.
Section 2 is dedicated to some properties and results concerning the Generalized Bessel transform.
Section 3 is devoted to the Donoho-Stark's uncertainty principle for the Generalized Bessel transform.

2 Preliminaries

In this section we recapitulate some facts about harmonic analysis related to the generalized Bessel operator. We cite here, as briefly as possible, some properties. For more details we refer to [1].
Throughout this paper we assume that $\alpha > \frac{-1}{2}$.

We consider the second-order singular differential operator on the half line

$$\mathcal{L}_{\alpha,n} f(x) = \frac{d^2}{dx^2} f(x) + \frac{2\alpha+1}{x}\frac{d}{dx} f(x) - \frac{4n(\alpha+n)}{x^2} f(x).$$

The generalized Bessel transform is defined for a function $f \in L^1_{\alpha,n}(\mathbb{R}^+)$ by

$$\mathcal{F}_{\alpha,n}(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x) x^{2\alpha+1} dx, \quad \lambda \geq 0, \tag{2.1}$$

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where

$$\varphi_\lambda(x) = a_{\alpha+2n} x^{2n} \int_0^1 cos(\lambda t x)(1-t^2)^{\alpha+2n-\frac{1}{2}} dt$$

and

$$\varphi_\lambda(x) = a_{\alpha+2n} x^{2n} \int_0^1 cos(\lambda t x)(1-t^2)^{\alpha+2n-\frac{1}{2}} dt$$

and

$$a_{\alpha+2n} = \frac{2\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+\frac{1}{2})}. \tag{2.2}$$

- The function φ_λ satisfies the differential equation

$$\mathcal{L}_{\alpha,n}\varphi_\lambda = -\lambda^2\varphi_\lambda$$

- For all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}^+$,

$$|\varphi_\lambda(x)| \leq x^{2n} e^{|Im\lambda||x|}. \tag{2.3}$$

- For all $\lambda \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$,

$$\lambda^{2n}\varphi_\lambda(x) = x^{2n}\varphi_x(\lambda). \tag{2.4}$$

We denote by

- $L^p_\alpha(\mathbb{R}^+)$ the class of measurable functions f on $[0, +\infty[$ for which

$$\|f\|_{L^p_\alpha(\mathbb{R}^+)} < \infty$$

  where

$$\|f\|_{L^p_\alpha(\mathbb{R}^+)} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{\frac{1}{p}}, \quad \text{if } p < \infty,$$

  and $\|f\|_{L^\infty_\alpha(\mathbb{R}^+)} = ess\ sup_{x\geq 0}|f(x)|$.

- $L^p_{\alpha,n}(\mathbb{R}^+)$ the class of measurable functions f on $\mathbb{R}^+$ for which

$$\|f\|_{L^p_{\alpha,n}(\mathbb{R}^+)} = \|x^{-2n}f\|_{L^p_{\alpha+2n}(\mathbb{R}^+)} < \infty.$$

For every $f \in L^1_{\alpha,n}(\mathbb{R}^+) \bigcap L^2_{\alpha,n}(\mathbb{R}^+)$ we have the Plancherel formula

$$\int_0^\infty |\ f(x)\ |^2\ x^{2\alpha+1} dx = \int_0^\infty |\ \mathcal{F}_{\alpha,n}(f)(\lambda)\ |^2\ d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = \frac{1}{4^{\alpha+2n}(\Gamma(\alpha+2n+1))^2}\lambda^{2\alpha+4n+1} d\lambda. \tag{2.5}$$

The generalized Bessel transform $\mathcal{F}_{\alpha,n}$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}(\mathbb{R}^+)$ onto $L^2_{\alpha+2n}(\mathbb{R}^+)$.
The inverse transform is given by

$$\mathcal{F}^{-1}_{\alpha,n}(f)(x) = \int_0^\infty f(\lambda)\varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda), \tag{2.6}$$

where the integral converge in $L^2_{\alpha,n}(\mathbb{R}^+)$.

Let $f \in L^1_{\alpha,n}(\mathbb{R}^+)$ such that $\mathcal{F}_{\alpha,n}(f) \in L^1_{\alpha+2n}(\mathbb{R}^+)$, then the inverse generalized Fourier-Bessel transform is given by the formula

$$f(x) = \int_0^\infty \mathcal{F}_{\alpha,n}(f)(\lambda)\varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda). \tag{2.7}$$

3 Donoho-Stark for the Fourier generalized transform

Throughout this section we denote by $\|.\|$ the operator norm on $L^2_{\alpha,n}(\mathbb{R}_+)$. More precisely if T is an operator then

$$\|T\| = sup_{f\in L^2_{\alpha,n}(\mathbb{R}^+)} \frac{\|Tf\|_{L^2_{\alpha,n}(\mathbb{R}^+)}}{\|f\|_{L^2_{\alpha,n}(\mathbb{R}^+)}}.$$

We say that f is ϵ-concentrated on a measurable set E if

$$||f - \mathcal{X}_E f||_{L^2_{\alpha,n}(\mathbb{R}^+)} < \epsilon,$$

where χ_E is the characteristic function of the set E.
Donoho and Stark [3] have shown that if f of unit $L^2(\mathbb{R}^+)$ norm is ϵ_T concentrated on a measurable set T and its Fourier transform $\mathcal{F}(f)$ is ϵ_W, on a measurable set W, then

$$|W|.|T| \geq (1 - \epsilon_T - \epsilon_W)^2.$$

Here, $|T|$ is the Lebesque measure of the set T. This inequality has been slightly improved in ref.[4] to

$$|W|.|T| \geq (1 - (\epsilon_T^2 + \epsilon_W^2)^{\frac{1}{2}})^2.$$

In this section, we will extend the Donoho-Stark uncertainty principle to the generalized Bessel transform.
Let P_E denote the time-limiting operator

$$(P_E f)(x) = \begin{cases} f(x), & x \in E \\ 0, & x \in \mathbb{R}^+ \backslash E \end{cases}. \tag{3.8}$$

This operator cuts off the part of f outside E. Let us now be more precise, we need to introduce some notations, so f is ϵ-concentrated on a set E if, and only if

$$||f - P_E f||_{L^2_{\alpha,n}(\mathbb{R}^+)} \leq \epsilon.$$

For simplicity, we will use P_X to $P_{[0,X]}$. Clearly $||P_E|| = 1$ because P_E is a projection. The second operator is the frequency-limiting operator

$$(Q_E f)(x) = \int_E \varphi_y(x)\mathcal{F}_{\alpha,n}(f)(y)d\mu_{\alpha+2n}(y), \tag{3.9}$$

From (2.6) we can also write Q_E as follows

$$Q_E f(x) = \mathcal{F}^{-1}_{\alpha,n}(P_E(\mathcal{F}_{\alpha,n}(f)))(x).$$

Then by (2.6) and (2.7) we deduce that $\mathcal{F}_{\alpha,n}(f)$ is ε-concentrated on F if and only if $\|f - Q_F f\|_{L^2_{\alpha,n}(\mathbb{R})} \leq \varepsilon\|f\|_{L^2_{\alpha,n}(\mathbb{R})}$.

We have from (3.8) and (3.9)

$$\begin{aligned} (P_X Q_Y f)(x) &= P_X \int_0^Y \varphi_y(x)\mathcal{F}_{\alpha,n}(f)(y)d\mu_{\alpha+2n}(y) \\ &= P_X \int_0^Y \varphi_y(x) \int_0^\infty \varphi_y(t)f(t)d\mu_\alpha(t)d\mu_{\alpha+2n}(y) \\ &= P_X \int_0^\infty f(t) \int_0^Y \varphi_y(x)\varphi_y(t)d\mu_{\alpha+2n}(y)d\mu_\alpha(t) \\ &= \int_0^\infty f(t)q(x,t)d\mu_\alpha(t), \end{aligned}$$

where

$$q(x,t) = \begin{cases} \int_0^Y \varphi_y(x)\varphi_y(t)d\mu_{\alpha+2n}(y), & x < X \\ 0, & x \geq X \end{cases}.$$

The Hilbert-Schmidt norm of P_XQ_Y is

$$\|P_XQ_Y\|_{HS} = \left(\int_0^\infty \int_0^\infty |q(x,t)|^2 d\mu_\alpha(x) d\mu_\alpha(t)\right)^{\frac{1}{2}}.$$

The norm $||P_XQ_Y||$ does not exceed the Hilbert-Schmidt norm of P_XQ_Y, therefore

$$\begin{aligned} ||P_XQ_Y||^2 &\leq \|P_XQ_Y\|_{HS}^2 \\ &= \int_0^\infty \int_0^\infty |q(x,t)|^2 d\mu_\alpha(x) d\mu_\alpha(t) \\ &= \int_0^X \int_0^\infty |q(x,t)|^2 d\mu_\alpha(x) d\mu_\alpha(t). \end{aligned}$$

Notice that

$$\begin{aligned} q(x,t) &= \int_0^Y \varphi_y(x)\varphi_y(t) d\mu_{\alpha+2n}(y) \\ &= \int_0^Y y^{2n}\varphi_y(x) y^{2n}\varphi_y(t) d\mu_\alpha(y). \end{aligned}$$

From (2.4) we deduce that

$$\begin{aligned} &= \int_0^Y x^{2n}\varphi_x(y) t^{2n}\varphi_t(y) d\mu_\alpha(y) \\ &= \int_0^Y x^{2n} t^{2n}\varphi_x(y)\varphi_t(y) d\mu_\alpha(y) \\ &= x^{2n} t^{2n} \mathcal{F}_{\alpha,n}(\varphi_t(.)\mathcal{X}_{[0,Y]})(x), \end{aligned}$$

the Plancherel formula for the generalized Bessel transform yields

$$\begin{aligned} \int_0^\infty |q(x,t)|^2 d\mu_\alpha(x) &= \int_0^\infty |x^{2n} t^{2n} \mathcal{F}_{\alpha,n}(\varphi_t(.)\mathcal{X}_{[0,Y]})(x)|^2 d\mu_\alpha(x) \\ &= \frac{a_\alpha}{a_{\alpha+2n}} \int_0^\infty |t^{2n} \mathcal{F}_{\alpha,n}(\varphi_t(.)\mathcal{X}_{[0,Y]})(x)|^2 d\mu_{\alpha+2n}(x) \\ &= \frac{a_\alpha}{a_{\alpha+2n}} \int_0^\infty |\mathcal{F}_{\alpha,n}(t^{2n}\varphi_t(.)\mathcal{X}_{[0,Y]})(x)|^2 d\mu_{\alpha+2n}(x), \end{aligned}$$

by Plancherel formula we have

$$\begin{aligned} \frac{a_\alpha}{a_{\alpha+2n}} \int_0^\infty |\mathcal{F}_{\alpha,n}(t^{2n}\varphi_t(.)\mathcal{X}_{[0,Y]})(x)|^2 d\mu_{\alpha+2n}(x) &= \frac{a_\alpha}{a_{\alpha+2n}} \int_0^Y |t^{2n}\varphi_t(x)|^2 d\mu_\alpha(x) \\ &= \frac{a_\alpha}{a_{\alpha+2n}} \int_0^Y |x^{2n}\varphi_x(t)|^2 d\mu_\alpha(x) \\ &= \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^2 \int_0^Y |\varphi_x(t)|^2 d\mu_{\alpha+2n}(x). \end{aligned}$$

Consequently,

$$\begin{aligned} ||P_XQ_Y||^2 &\leq \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^2 \int_0^X \int_0^Y |\varphi_x(t)|^2 d\mu_{\alpha+2n}(x) d\mu_\alpha(t) \\ &\leq \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^2 \int_0^X \int_0^Y |t^{2n}|^2 d\mu_{\alpha+2n}(x) d\mu_\alpha(t) \\ &= \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^3 \int_0^X \int_0^Y d\mu_{\alpha+2n}(x) d\mu_{\alpha+2n}(t) \\ &= \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^3 \int_0^X \int_0^Y d\mu_{\alpha+2n}(x) d\mu_{\alpha+2n}(t) \\ &= \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^3 \frac{(XY)^{\alpha+2n+1}}{\alpha+2n+1}. \end{aligned}$$

We put

$$b_{\alpha,n} = \left(\frac{a_{\alpha+2n}}{a_\alpha}\right)^3 (\alpha + 2n + 1). \tag{3.10}$$

Let $XY < (b_{\alpha,n})^{\frac{1}{\alpha+2n+1}}$. Then $||P_X Q_Y|| < 1$ and therefore $I - P_X Q_Y$ is invertible with

$$\begin{aligned} ||(I - P_X Q_Y)^{-1}|| &\leq \sum_{k=0}^{\infty} ||P_X Q_Y||^k \\ &\leq \sum_{k=0}^{\infty} \left[\frac{(XY)^{\alpha+2n+1}}{b_{\alpha,n}}\right]^k \\ &= \frac{b_{\alpha,n}}{b_{\alpha,n} - (XY)^{\alpha+2n+1}}. \end{aligned}$$

We have

$$I = P_X + P_{(X,\infty)} = P_X Q_Y + P_X Q_{(Y,\infty)} + P_{(X,\infty)}.$$

The orthogonality of P_X and $P_{(X,\infty)}$ gives

$$||P_X Q_{(Y,\infty)} f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)} + ||P_{(X,\infty)} f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)} = ||P_X Q_{(Y,\infty)} f + P_{(X,\infty)} f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}.$$

Together with $||P_X|| = 1$

$$\begin{aligned} ||f||^2_{2,\alpha,n} &\leq ||(I - P_X Q_Y)^{-1}||^2 ||(I - P_X Q_Y) f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)} \\ &\leq \left(\frac{b_{\alpha,n}}{b_{\alpha,n} - (XY)^{\alpha+2n+1}}\right)^2 \left[||P_X Q_{(Y,\infty)} f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)} + ||P_{(X,\infty)} f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}\right] \\ &\leq \left(\frac{b_{\alpha,n}}{b_{\alpha,n} - (XY)^{\alpha+2n+1}}\right)^2 \left[||Q_{(Y,\infty)} f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)} + ||P_{(X,\infty)} f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}\right]. \end{aligned}$$

If f of unit norm is ϵ_X-time-limited on $[0, X]$, then $||P_{(X,\infty)} f||_{L^2_{\alpha,n}(\mathbb{R}^+)} \leq \epsilon_X$, If f of unit norm is ϵ_Y-bandlimited on $[0, Y]$, then $||Q_{(Y,\infty)} f||_{L^2_{\alpha,n}(\mathbb{R}^+)} \leq \epsilon_Y$. Then if f of unit norm is both ϵ_X-time-limited and ϵ_Y-bandlimited,

$$1 \leq \left(\frac{b_{\alpha,n}}{b_{\alpha,n} - (XY)^{\alpha+2n+1}}\right)^2 (\epsilon_X^2 + \epsilon_Y^2)$$

or

$$XY \geq (b_{\alpha,n})^{\frac{1}{\alpha+2n+1}} \left(1 - (\epsilon_X^2 + \epsilon_Y^2)^{\frac{1}{2}}\right)^{\frac{1}{\alpha+2n+1}}.$$

We arrive at the Donoho-Stark uncertainty principle for the generalized Bessel transform.

Theorem 3.1. *Let a unit norm signal f be ϵ_X-time-limited on $[0, X]$ and ϵ_Y-bandlimited on $[0, Y]$. Then*

$$XY \geq (b_{\alpha,n})^{\frac{1}{\alpha+2n+1}} \left(1 - (\epsilon_X^2 + \epsilon_Y^2)^{\frac{1}{2}}\right)^{\frac{1}{\alpha+2n+1}}$$

where $b_{\alpha,n}$ is given by (3.10).

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