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Perturbation of Differential Linear System

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Abstract

The main theme studied concerns perturbation of differential linear system with constant coefficients:

$$\frac{dX}{dt} = AX + b. \tag{0.1}$$

The data of the system (0.1) provides the expression of a vector field X of $\mathbb{R}^n$, in the coordinates $X_1, X_2, ..., X_n$. The singularity of the system (0.1) or the field X, expressed by coordinates $X_1, X_2, ..., X_n$ is given by the solutions of the system of equations $AX + b = 0$.

In general, a small perturbation of a regular linear standard real matrix M is a matrix of the form:

$$M' = M + \epsilon.$$

where $\epsilon = (\epsilon_{ij})$is a matrix with elements infinitely small.
We study the regular linear perturbation when the singularity is a point with various situations and practical examples and in the case where the singular place is a line with various practical situations. we hope that our contribution is in fact to use certain technical of non standard Analysis (infinitesimal calculus) which simplify obviously the proves.

Keywords: perturbation singular, regular, critical points, exact solution, differential system, orbits, infinitely-small , infinitely-large.

2010 MSC: 39B55, 39B52, 39B82.

1 Introduction

The study of linear stability informs us about the stability of the system when the non-linear terms are taken into account.When the two eigenvalues have a strictly negative real part, linear stability implies non-linear stability.

In the case of unstable systems and when the two eigenvalues are strictly positive real parts. A system which is unstable by linear stability it remains when the non-linear contributions are taken into consideration. On the other hand, when at least one of the real part of the eigenvalues is zero, i.e. in the case of centers, taking into account the non-linear terms can lead to different results from those obtained by linearization. we hope that our contribution is in fact to use certain Technics of non standard Analysis (infinitesimal calculus) which simplify obviously the proves.

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2 Regular linear perturbations

A translation of the origin, we assume $b = 0$, that gives a homogeneous linear system:

$$\frac{dx}{dt} = Ax.$$

Since $rank(A) = 1$, there exist elements $(\alpha, \beta) \in \mathbb{R}^2$, $(a, b) \in \mathbb{R}^2$, $a^2 + b^2 \succ 0$ such as $a_{11} = \alpha a$, $a_{12} = \alpha b$, $a_{21} = \beta a$, $a_{22} = \beta b$.
The matrix A is written as:

$$A = \begin{pmatrix} \alpha a & \alpha b \\ \beta a & \beta b \end{pmatrix} \tag{2.2}$$

and the system becomes:

$$\begin{cases} x_1' = \alpha a x_1 + \alpha b x_2 \\ x_2' = \beta a x_1 + \beta b x_2 \end{cases}$$

3 Notation

The following abbreviations will be adopted.

NSO: Non singular orbits.
S: Singularity to indicate that a quantity does not take the value 0.
We will write indifferently $a \neq 0$ or $(\bar{a})$.
NSO: $x_1 = constant$ whence $x_1' = 0$ Thus $\alpha = 0$.
S: $x_2 = 0$ whence $x_2' = \beta b x_2$.

$$\begin{cases} x_1' = 0; & \alpha = 0, a = 0, \beta \neq 0, b \neq 0, \\ x_2' = \beta b x_2; & \beta b \prec 0. \end{cases}$$

$$\begin{cases} x_1' = 0; & \alpha = 0, a = 0, \beta \neq 0, b \neq 0, \\ x_2' = \beta b x_2; & \beta b \prec 0. \end{cases}$$

NSO: it is a line of positive slope.
S: $x_2 = 0$.

$$\begin{cases} x_1' = \alpha b x_2 & \alpha \neq 0, a = 0, \beta \neq 0, b \neq 0 \\ x_2' = \beta b x_2 & \alpha b \prec 0, \beta b \prec 0 \end{cases}$$

NSO: $x_2 = constant$ whence $x_2' = 0$ Thus $\beta = 0$
S: it is a line of positive slope.
NSO: They are line of negative slope.
S: it is a line of negative slope parallel with NOS.

$$\begin{cases} x_1' = \alpha\left(a x_1 + b x_2\right) & \alpha \neq 0, a \neq 0, \beta \neq 0, b \neq 0 \\ x_2' = \beta\left(a x_1 + b x_2\right) & \alpha a \prec 0, \beta a \prec 0 \end{cases}$$

The non singular orbits are perpendicular to the singularity, the system which makes it possible to describe them is

$$\begin{cases} x_1' = 0 \\ x_2' = 0 \end{cases}$$

The non singular orbits are parallel to the singularity described by the following system

$$\begin{cases} x_1' = x_2 \\ x_2' = 0 \end{cases} .$$

Since $rankA = 1$, there exist elements $(\alpha, \beta) \in \mathbb{R}^2$, $(a, b) \in \mathbb{R}^2$, $a^2 + b^2 \succ 0$ such as $a_{11} = \alpha a$, $a_{12} = \alpha b$,$a_{21} = \beta a$, $a_{22} = \beta b$.

The matrix (A, b) is written as:

$$(A, b) = \begin{pmatrix} \alpha a & \alpha b & b_1 \\ \beta a & \beta b & b_1 \end{pmatrix}$$

and the system (0.1) become:

$$\begin{cases} x_1' = \alpha (ax_1 + bx_2) + b_1 \\ x_2' = \beta (ax_1 + bx_2) + b_2 \end{cases}$$

first case $b_1 = b_2 = 0$ we must take α, β, a, b non zero so that the matrix (A, b) remains of $rank2$ second case $b_1 = 0, b_2 \neq 0$ we have:

$$\begin{cases} x_1' = \alpha (ax_1 + bx_2) \\ x_2' = \beta (ax_1 + bx_2) + b_2 \end{cases}$$

If $\alpha = 0, \beta \neq 0$ is impossible because $rank\,(A, b)$ will not be equal any more to 2

Remark 3.1. *For a linear differential connection of* $\mathbb{R}^2$ *with constant coefficients*

$$\frac{dX}{dt} = AX + b$$

with:

$$rank\,(A, b) = 1 + rank\,(A) = 2.$$

There exists two possible models.

Ame exotique or parabola The trajectories of the parabola of equation $x_2 = \frac{1}{2}x_1^2 + k, k \in \mathbb{R}$ corresponding with the system:

$$\begin{cases} x_1' = 1 \\ x_2' = x_1. \end{cases}$$

Ame stable The trajectories are exponential curves of equation $x_2 = k\exp(x_1), k \in \mathbb{R}$ corresponding with the system:

$$\begin{cases} x_1' = 1 \\ x_2' = x_2. \end{cases}$$

The axis of the traces represents the states with a comb type.
The axis of the traces represents the states of the heart type.

4 Regular linear perturbation when the singular place is a point

If B is a real matrix of order p and ϵ a real matrix of p order have infinitely small elements, then it exists a real infinitely small ϵ,such as $\det (B + \epsilon) = \det B + \epsilon$.
The various situations or the singularity is a point.
With a loss less of general information, it can be limited to the homogeneous systems:

$$\frac{dx}{dt} = Ax.$$

with $rankA = 2$, A standard matrix.
First case : A is not diagonal.
We work with the figure X, giving the qualitative states in term of trace.
Defined as $Tr : \mathbb{R}^2 \to \mathbb{R}$ is a continuous standard function.
Thus $Tr\,(B + \epsilon) = TrB + \epsilon$ where ϵ is Infinitesimal. If B is a standard function and ϵ a matrix of Infinitesimals.

The linear differential system $\frac{dx}{dt} = Ax$, A standard matrix.
In SDN the state (respectively IDN) ($\det A \succ 0, TrA \prec 0$ respectively $TrA \succ 0$).
$\det A = \frac{1}{4}(TrA)^2$, undergoing a small regular linear perturbation: $\frac{dx}{dt} = (A+\epsilon)\,x$ and ϵ a matrix of infinitesimals.
SDN the state (respectively IDN) changes into SDN(respectively IDN).
if $\det(A+\epsilon) = \frac{1}{4}(Tr\,(A+\epsilon))^2$ (it is said that states SDN and IDN resist).
The SDN state (respectively IDN) changes into SN (respectively IN).
if $\det(A+\epsilon) \prec \frac{1}{4}(Tr\,(A+\epsilon))^2$.
The linear differential system $\frac{dx}{dt} = Ax$, A standard.
In the state C $(detA \succ 0, TrA = 0)$ undergoing a small regular linear perturbation. $\frac{dx}{dt} = (A+\epsilon)\,x$ and ϵ a matrix of infinitesimals.Then the state C resist if $(Tr\,(A+\epsilon)) = 0$ and the state C transforms into FI if $(Tr(A+\epsilon)) \succ 0$ and the state C transform into FS if $(Tr(A+\epsilon)) \prec 0$

Example 4.1. *Let be the system* $\frac{dx}{dt} = Ax$ *in the state C*

$$A = \begin{pmatrix} -2 & 2 \\ -3 & 2 \end{pmatrix}$$

$\det A = 2, TrA = 0$
The C state resists if the matrix ϵ is chosen null.
The state C transforms it self into FI if we take

$$\epsilon = \begin{pmatrix} \epsilon_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

with $\epsilon_{11}\ i, p \succ 0$

$$A+\epsilon = \begin{pmatrix} -2+\epsilon_{11} & 2 \\ -3 & 2 \end{pmatrix}$$

$\det(A+\epsilon) = 2 + 2\epsilon_{11} \succ 0$, $Tr(A+\epsilon) = \epsilon_{11}$

The C state transforms it self into FS if we take $\epsilon = \begin{pmatrix} \epsilon_{11} & 0 \\ 0 & 0 \end{pmatrix}$ with $\epsilon_{11} i,\ p \succ 0$

We obtain:

$$\det(A+\epsilon) = 2 + 2\epsilon_{11} \succ 0, Tr\,(A+\epsilon) = \epsilon_{11} \succ 0. \tag{4.3}$$

$$\det(A+\epsilon) = 2 + 2\epsilon_{11} \succ \frac{1}{4}(Tr\,(A+\epsilon))^2 = \frac{1}{4}\epsilon_{11}^2 \succ 0. \tag{4.4}$$

qualitative state $colC$ with three answers over looked the small regular linear perturbation, to resist ,change into $colFS$, change into $colFI$

5 Conclusion

we used non-standard matrices infinitely close to standard matrices, then we try to see if the Poincares Classification looses its properties at the singular points. Our goal is to find, for non-linear systems when the linearized is a matrix close to a standard matrix, a possible link between what we do and to generalize our results.

References

[1] Arnold v, *ordinary differential equations Edition MIR Moscou* (1974).

[2] Arnold v, *chapter additional of the theory of the differential equations* ,Edition MIR Moscou 1978.

[3] Artigue Mr. Gautheron V, *Differential connections, graphic study*,Cedis , Paris,1983

[4] Bobo Seke,*Ombre des graphes de fonctions continues*,These, Strasbourg,1981

[5] Bobo Seke,*Optimisation:Mthode des fonctions boites*,I.R.M.A.Strasbourg,1983

[6] Bobo Seke,*Mthodes des rgionalisations quelques applications*,I.R.M.A.Strasbourg,1988.

[7] Ismail Tahir Hassan,*Etude macroscopique de certaines courbes Approximation infinitisimaledes polynomes convexe* ,These, Strasbourg,1984.

[8] Salvador SANCHEZ-PEDRENO G,*Equations diffrentielles hautement non linaires*,These, Mulhouse Strasbourg,1989

[9] Troesch A,*Etude macroscopique de systme diffrentielle*,London,Math.Soc.(3),48 (1984)p.121-160.

[10] Belaib L,*Regionalization Method for Nonlinear Differential Equation System in a cartesien Plan*,Journal of Mathematics and Statistics 2 (4) :464-468, 2006

[11] Belaib L; Bouarroudj N,*Study of Families of curves in the Euclidian plan*.Journal of Mathematics and Statistics 3 (3) :100-105, 2007

[12] Hadadine M.Z ; Belaib L,*A Non Generic Case of Differential System of R3*Applied Mathematical sciences,Vol, 6, 2012, no.100, 4955-4964

[13] J.Mole,*Ordinary differential equation* springer-verlag, 1969.

[14] Y.Saad,*Iterative method for sparse linear systems* PWS publishing, 1995.

[15] K.Burrage,,*A special family of Runge-Kutta methods for solving stiff differential equation,* volume 18, 22-41, 1978..

[16] J Guckenheimer,,*Singular Hopf Bifurcation in Systems with Two Slow Variables* SIAM J.Vol. 7, No. 4, pp. 1355-1377, Appl Dyna Sys,2008,Society for Industrial and Applied Mathematics..

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Pseudo Asymptotically Periodic Integral Solution of Partial Neutral Functional Differential Equations

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Abstract In this paper, we propose a new class of functions called μ-pseudo $\mathcal{S}$-asymptotically periodic function on $\mathbb{R}$ by the measure theory. Furthermore, the existence, uniqueness of μ-pseudo $\mathcal{S}$-asymptotically periodic integral solution to partial neutral functional differential equations with finite delay are investigated. Here we assume that the undelayed part is not necessarily densely defined and satisfies the Hille-Yosida condition.

Keywords: μ-pseudo $\mathcal{S}$-asymptotically periodic function, Partial neutral functional differential equations, Measure theory, Integral solution.

2010 MSC: 34K06, 34D05.

1 Introduction

The existence of periodic solution or asymptotically periodic solution is very important in the qualitative studies of many problems. Many authors have made important contributions to the theory of periodicity, asymptotic periodicity and applications to differential equations, integral equations, integro-differential equations and partial functional differential equations. More details on this topic can be found in [10, 12, 20, 21, 25, 30].

The notion of $\mathcal{S}$-asymptotic periodicity is an important generalization of asymptotic periodicity, which was introduced by Henríquez et al. in [18, 19]. Since then, it attracted the attention of many researchers [7, 11, 13, 22] and this concept has undergone several interesting, natural, and powerful generalizations, such as pseudo $\mathcal{S}$-asymptotic periodicity [23], weighted pseudo $\mathcal{S}$-asymptotic periodicity [28], and so on. On the other hand, Blot et al. [9] used some results of the measure theory to establish a new concept of μ-pseudo almost periodicity which generalizes weighted pseudo almost periodicity. Using the methods of [9], we introduce the concept of μ-pseudo $\mathcal{S}$-asymptotic periodicity by measure theory in this paper.

Partial neutral functional differential equations (PNFDEs), arising from many biological, chemical, and physical systems, become an interesting and important field in dynamical systems. In the standard framework of semilinear PNFDEs, one assumes that the operator A in the linear part is densely defined. However, there are many examples in which the density condition is not satisfied [3, 15, 17, 27, 29]. Here we assume that the linear part is not necessarily densely defined and satisfies the Hille-Yosida condition. Existence, uniqueness of μ-pseudo $\mathcal{S}$-asymptotically periodic integral solution to PNFDEs are investigated.

The paper is organized as follows. In Section 2, some notations are presented and we propose a new class of functions called μ-pseudo $\mathcal{S}$-asymptotically periodic function by the measure theory. In Section 3, we recall some fundamental results which include the variation of constants formula and spectral decomposition. Sections 4 is devoted to the existence and uniqueness of μ-pseudo $\mathcal{S}$-asymptotically periodic integral solution of PNFDEs. In Section 5, we provide an example to illustrate our main results.

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2 Preliminaries and basic results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ are two Banach spaces and $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{C}$ stand for the set of natural numbers, real numbers, nonnegative real numbers and complex numbers, respectively. For A being a linear operator on X, $D(A)$, $\rho(A)$, $R(\lambda, A)$, $\sigma(A)$ stand for the domain, the resolvent set, the resolvent and spectrum of A. In order to facilitate the discussion below, we further introduce the following notations:

- $C(\mathbb{R}, X)$ (resp. $C(\mathbb{R}\times Y, X)$): the set of continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R}\times Y$ to X).
- $\mathcal{C} = C([-r, 0], X)$: the space of continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology.
- $BC(\mathbb{R}, X)$ (resp. $BC(\mathbb{R}\times Y, X)$): the Banach space of bounded continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R}\times Y$ to X) with the supremum norm.
- $B(X, Y)$: the Banach space of bounded linear operators from X to Y endowed with the operator topology. In particular, we write $B(X)$ when $X = Y$.
- $L^p(\mathbb{R}, X)$: the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable functions $f : \mathbb{R} \to X$ such that $\|f\| \in L^p(\mathbb{R}, \mathbb{R})$.

For $\omega > 0$, define

$$C_0(\mathbb{R}, X) = \{x \in BC(\mathbb{R}, X) : \lim_{|t|\to\infty} \|x(t)\| = 0\}.$$

$$C_\omega(\mathbb{R}, X) = \{x \in BC(\mathbb{R}, X) : x \text{ is } \omega\text{-priodic}\}.$$

Definition 2.1. A function $f \in BC(\mathbb{R}, X)$ is called asymptotically ω-periodic if there exists $g \in C_\omega(\mathbb{R}, X)$, $\varphi \in C_0(\mathbb{R}, X)$ such that $f = g + \varphi$. Denote by $AP_\omega(\mathbb{R}, X)$ the set of such functions.

Definition 2.2. A function $f \in BC(\mathbb{R}, X)$ is said to be $\mathcal{S}$-asymptotically ω-periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. Denote by $\mathcal{S}AP_\omega(\mathbb{R}, X)$ the set of such functions.

Definition 2.3. A function $f \in BC(\mathbb{R}, X)$ is called pseudo $\mathcal{S}$-asymptotically ω-periodic if there exists $\omega > 0$ such that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(t+\omega) - f(t)\| dt = 0.$$

Denote by $P\mathcal{S}AP_\omega(\mathbb{R}, X)$ the set of such functions.

Let U be the set of all functions $\rho : \mathbb{R} \to (0, \infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $T > 0$ and each $\rho \in U$, set

$$m(T, \rho) := \int_{-T}^{T} \rho(t) \mathrm{d}t.$$

Define $U_\infty := \{\rho \in U : \lim_{T\to\infty} m(T, \rho) = \infty\}$.

Definition 2.4. Let $\rho \in U_\infty$. A function $f \in BC(\mathbb{R}, X)$ is called weighted pseudo $\mathcal{S}$-asymptotically ω-periodic if there exists $\omega > 0$ such that

$$\lim_{T\to\infty} \frac{1}{m(T,\rho)}\int_{-T}^{T} \rho(t)\|f(t+\omega) - f(t)\| dt = 0.$$

Denote by $WP\mathcal{S}AP_\omega(\mathbb{R}, X)$ the set of such functions.

Remark 2.1. Note that in the above definitions, if the function f is limited on $\mathbb{R}^+$, i.e., $AP_\omega(\mathbb{R}^+, X)$, $\mathcal{S}AP_\omega(\mathbb{R}^+, X)$, $P\mathcal{S}AP_\omega(\mathbb{R}^+, X)$, $WP\mathcal{S}AP_\omega(\mathbb{R}^+, X)$ is defined in [18], [23], [24], [28], respectively.

Next, we introduce the new class of functions called μ-pseudo $\mathcal{S}$-asymptotically periodic on $\mathbb{R}$ by the measure theory. $\mathcal{B}$ denotes the Lebesgue σ-field of $\mathbb{R}$, $\mathcal{M}$ stands for the set of all positive measure μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = \infty$ and $\mu([a, b]) < \infty$ for all $a, b \in \mathbb{R}$ $(a \le b)$. We formulate the following hypothesis:

(H_0) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau, a \in A\}) \le \beta\mu(A) \quad \text{if } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

Definition 2.5. Let $\mu \in \mathcal{M}$. A function $f \in BC(\mathbb{R}, X)$ is called μ-pseudo $\mathcal{S}$-asymptotically ω-periodic if there exists $\omega > 0$ such that

$$\lim_{T\to+\infty} \frac{1}{\mu([-T,T])} \int_{[-T,T]} \|f(t+\omega) - f(t)\| d\mu(t) = 0.$$

Denote by $P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$ the set of such functions.

Remark 2.2. (i) If the measure μ is the Lebesgue measure, $P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$ is $P\mathcal{S}AP_\omega(\mathbb{R}, X)$.

(ii) Let $\rho(t) > 0$ a.e. on $\mathbb{R}$ for the Lebesgue measure. μ denotes the positive measure defined by

$$\mu(A) = \int_A \rho(t)dt \quad \text{for } \; A \in \mathcal{B},$$

where dt denotes the Lebesgue measure on $\mathbb{R}$, then $P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$ is $WP\mathcal{S}AP_\omega(\mathbb{R}, X)$. One can see [4, 8, 9] for more details.

Similarly as the proof of [9], one has the following results for $P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$.

Lemma 2.1. *Let $\mu \in \mathcal{M}$, then the following properties hold:*

(i) *$f \pm g \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$ if $f, g \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$.*

(ii) *$\lambda f \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$ if $\lambda \in \mathbb{R}$, $f \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$.*

(iii) *$AP_\omega(\mathbb{R}, X) \subset \mathcal{S}AP_\omega(\mathbb{R}, X) \subset P\mathcal{S}AP_\omega(\mathbb{R}, X) \subset WP\mathcal{S}AP_\omega(\mathbb{R}, X) \subset P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$.*

(iv) *$P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$ is a Banach space with the supremum norm $\|\cdot\|$.*

Lemma 2.2. *Let $\mu \in \mathcal{M}$ and satisfies (H_0), then $P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$ is translation invariant.*

Theorem 2.1. *Assume that $\mu \in \mathcal{M}$. Let $f : \mathbb{R} \times X \to X$ be a function bounded on bounded sets of X, $f \in P\mathcal{S}AP_\omega(\mathbb{R} \times X, X, \mu)$, and there exists a constant $L_f > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le L_f \|x - y\|, \quad t \in \mathbb{R}, \;\; x, y \in X,$$

then $f(\cdot, u(\cdot)) \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$ if $u(\cdot) \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$.

.

Lemma 2.3. *Let $\mu \in \mathcal{M}$ and satisfies (H_0), if $f \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$, $G \in L^1(\mathbb{R}, B(X))$, then the convolution product $f * G$ defined by*

$$(f * G)(t) = \int_{-\infty}^{+\infty} G(s) f(t-s) ds, \quad t \in \mathbb{R}$$

lies in $P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$.

Proof. Let $f \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$, then by Lemma 2.2, one has $f(\cdot - s) \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$ for all $s \in \mathbb{R}$. It is not difficult to see that $f * G \in BC(\mathbb{R}, X)$. Since $\mu(\mathbb{R}) = +\infty$, then there exists $r_0 \ge 0$ such that $\mu([-r, r]) > 0$ for all $r \ge r_0$. Hence by Fubini's theorem, one has

$$\begin{aligned}
&\frac{1}{\mu([-T,T])} \int_{[-T,T]} \|(f*G)(t+\omega) - (f*G)(t)\| d\mu(t) \\
&\le \frac{1}{\mu([-T,T])} \int_{[-T,T]} \int_{-\infty}^{+\infty} \|G(s)\| \|f(t+\omega-s) - f(t-s)\| ds d\mu(t) \\
&\le \int_{-\infty}^{+\infty} \frac{\|G(s)\|}{\mu([-T,T])} \int_{[-T,T]} \|f(t-s+\omega) - f(t-s)\| d\mu(t) ds.
\end{aligned}$$

Moreover, since $G \in L^1(\mathbb{R}, B(X))$ and

$$0 \le \frac{\|G(s)\|}{\mu([-T,T])} \int_{[-T,T]} \|f(t-s+\omega) - f(t-s)\| d\mu(t) \le 2\|G\|\|f\| \quad \text{for all } \; s \in \mathbb{R},$$

then

$$\lim_{T\to+\infty} \int_{-\infty}^{\infty} \frac{\|G(s)\|}{\mu([-T,T])} \int_{[-T,T]} \|f(t-s+\omega) - f(t-s)\| d\mu(t) ds = 0,$$

by Lebesgue dominated convergence theorem and $f(\cdot - s) \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$, one has

$$\lim_{T\to+\infty} \frac{1}{\mu([-T,T])} \int_{[-T,T]} \|(f*G)(t+\omega) - (f*G)(t)\| d\mu(t) = 0,$$

that is $f * G \in P\mathcal{S}AP_\omega(\mathbb{R}, X, \mu)$. □

3 Variation of constants formula and spectral decomposition

In this paper, we will investigate the existence and uniqueness of μ-pseudo $\mathcal{S}$-asymptotically periodic integral solution for PNFDEs:

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where A is a linear operator on Banach space X, not necessarily densely defined and satisfies the Hille-Yosida condition. Fix $r \geq 0$, $u_t \in \mathcal{C}$ is defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r, 0]$. $\mathcal{D} \in B(\mathcal{C}, X)$, $L \in B(\mathcal{C}, X)$, $f \in PSAP_\omega(\mathbb{R}, X, \mu)$. For the well posedness of (3.1), we assume that $\mathcal{D}$ has the following form:

$$\mathcal{D}\psi = \psi(0) - \int_{-r}^{0} [d\eta(\theta)]\psi(\theta) \quad \text{for } \psi \in \mathcal{C},$$

for a mapping $\eta : [-r, 0] \to B(X)$ of bounded variation and nonatomic at zero, which means that there exists a continuous nondecreasing function $\delta : [0, r] \to [0, +\infty)$ such that $\delta(0) = 0$ and

$$\left| \int_{-s}^{0} [d\eta(\theta)]\psi(\theta) \right| \leq \delta(s) \sup_{-r \leq \theta \leq 0} |\psi(\theta)| \quad \text{for } \psi \in \mathcal{C}, s \in [0, r].$$

To (3.1), we associate the following initial value problem

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t), & t \geq \sigma, \\ u_\sigma = \varphi \in \mathcal{C}. \end{cases} \tag{3.2}$$

Definition 3.1. [16] $u \in C([-r+\sigma, +\infty], X)$ is said to be an integral solution of (3.2) if the following conditions hold:

(i) $\int_\sigma^t \mathcal{D}u_s ds \in D(A)$ for $t \geq \sigma$.

(ii) $\mathcal{D}u_t = \mathcal{D}\varphi + A\int_\sigma^t \mathcal{D}u_s ds + \int_\sigma^t (L(u_s) + f(s))ds$ for $t \geq \sigma$.

(ii) $u_\sigma = \varphi$.

If $\overline{D(A)} = X$, the integral solution coincide with the known mild solution. One can see that if u_t is an integral solution of (3.2), then $u_t \in D(A)$ for all $t \geq 0$, in particular $\mathcal{D}\varphi \in \overline{D(A)}$. Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ which defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ A_0 x = Ax \quad \text{for } x \in D(A_0). \end{cases}$$

We make the following assumption:

(H_1) A satisfies the Hille-Yosida condition: there exist $M \geq 1$, $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{M}{(\lambda - \omega)^n}, \quad \text{for } n \in \mathbb{N}, \lambda > \omega.$$

Lemma 3.1. *[6] A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

The phase space $\mathcal{C}_0$ of (3.2) is defined by

$$\mathcal{C}_0 = \{\varphi \in \mathcal{C} : \mathcal{D}\varphi \in \overline{D(A)}\}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on $\mathcal{C}_0$ by

$$\mathcal{U}(t) = v_t(\cdot, \varphi),$$

where $v(\cdot, \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}v_t = A\mathcal{D}v_t + L(v_t), & t \geq 0, \\ v_0 = \varphi \in \mathcal{C}. \end{cases}$$

Proposition 3.1. *[2] $(\mathcal{U}(t))_{t\geq 0}$ is a strongly continuous semigroup of linear operators on $\mathcal{C}_0$.*

Theorem 3.1. *[2] Let $\mathcal{A}_{\mathcal{U}}$ defined on $\mathcal{C}_0$ by*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \{\varphi \in C^1([-r,0],X) : \mathcal{D}\varphi \in D(A), \mathcal{D}\varphi' \in \overline{D(A)} \text{ and } \mathcal{D}\varphi' = A\mathcal{D}\varphi + L(\varphi)\} \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi' \text{ for } \varphi \in D(\mathcal{A}_{\mathcal{U}}). \end{cases}$$

Then $\mathcal{A}_{\mathcal{U}}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ on $\mathcal{C}_0$.

Let X_0 be the space defined by

$$\langle X_0 \rangle = \{X_0 c : c \in X\}$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r,0), \\ c & \text{if if } \theta = 0. \end{cases}$$

The space $\mathcal{C}_0 \oplus \langle X_0 \rangle$ equipped with the norm $|\phi + X_0 c| = |\phi|_{\mathcal{C}} + |c|$ for $(\phi, c) \in \mathcal{C}_0 \times X$ is a Banach space and consider the extension $\mathcal{A}_{\mathcal{U}}$ defined on $\mathcal{C}_0 \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}}_{\mathcal{U}}) = \{\varphi \in C^1([-r,0],X) : \mathcal{D}\varphi \in D(A) \text{ and } \mathcal{D}\varphi' \in \overline{D(A)}\} \\ \widetilde{\mathcal{A}}_{\mathcal{U}}\varphi = \varphi' + X_0(A\mathcal{D}\varphi + L(\varphi) - \mathcal{D}\varphi'). \end{cases}$$

In order to compute the resolvent operator $R(\lambda, \widetilde{\mathcal{A}_{\mathcal{U}}})$, we suppose the following assumption.

(H_2) $\mathcal{D}e^{\lambda}c \in D(A)$ for all $c \in D(A)$ and all complex λ, where $e^{\lambda}c \in \mathcal{C}$ is defined by

$$(e^{\lambda}c)(\theta) = e^{\lambda\theta}c, \quad \text{for } \theta \in [-r,0].$$

Lemma 3.2. *[2] Assume that (H_1)-(H_2) hold, then $\widetilde{\mathcal{A}}_{\mathcal{U}}$ satisfies the Hille-Yosida condition on $C_0 \otimes \langle X_0 \rangle$: there exists $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $(\widetilde{\omega}, +\infty) \subset \rho(\widetilde{\mathcal{A}}_{\mathcal{U}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}}_{\mathcal{U}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n}, \quad \text{for } n \in \mathbb{N}, \lambda > \widetilde{\omega}.$$

Moreover, the part of $\widetilde{\mathcal{A}}_{\mathcal{U}}$ on $D(\widetilde{\mathcal{A}}_{\mathcal{U}}) = \mathcal{C}_0$ is exactly the operator $\mathcal{A}_U$.

Now, we can state the variation of constants formula associated to (3.2)

Theorem 3.2. *[2] Assume that (H_1) and (H_2) hold, then for $\varphi \in \mathcal{C}_0$, the integral solution x of (3.2) is given by the following variation of constants formula*

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda\to+\infty} \int_0^t \mathcal{U}(t-s)\widetilde{B}_\lambda X_0 f(s) ds \quad \text{for } t \geq \sigma,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}}_{\mathcal{U}})^{-1}$ for $\lambda > \widetilde{\omega}$.

Definition 3.2. We say a semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \emptyset.$$

Definition 3.3. The operator $\mathcal{D}$ is said to be stable if there exist positive constants η, ν such that the solution of the homogenous equation

$$\begin{cases} \mathcal{D}y_t = 0 & \text{for } t \geq 0 \\ y_0 = \phi, \end{cases}$$

where $\phi \in \{\psi \in \mathcal{C} : \mathcal{D}\psi = 0\}$ satisfies

$$|y_t(\cdot, \phi)| \leq \nu e^{-\eta t}|\phi| \quad \text{for } t \geq 0.$$

Example 3.1. The operator $\mathcal{D}$ defined by

$$\mathcal{D}\varphi = \varphi(0) - q\varphi(-r)$$

is stable if and only if $|q| < 1$.

For the sequel, we make the following assumptions:

(H_3) $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

(H_4) The operator $\mathcal{D}$ is stable.

We get the following result on the spectral decomposition of the phase space $\mathcal{C}_0$.

Theorem 3.3. *[2] Assume that (H_1)-(H_4) hold. If the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic, then the space $\mathcal{C}_0$ is decomposed as a direct sum*

$$\mathcal{C}_0 = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restricted semigroup on $\mathcal{U}$ is a group and there exist positive constants $\overline{M}, \overline{\omega}$ such that

$$\begin{aligned} |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{-\overline{\omega}t}|\varphi| \quad \text{for } t \geq 0, \varphi \in S, \\ |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{\overline{\omega}t}|\varphi| \quad \text{for } t \leq 0, \varphi \in U, \end{aligned}$$

where S and U are called the stable and unstable space respectively.

Theorem 3.4. *[5] Assume that (H_1)-(H_4) hold and the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. If $f \in BC(\mathbb{R}, X)$, then there exists a unique bounded integral solution u of (3.1) which is given by*

$$\begin{aligned} u_t = &\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds \\ &+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \quad \text{for } t \in \mathbb{R}, \end{aligned} \tag{3.3}$$

where $\mathcal{U}^s(t)$, $\mathcal{U}^u(t)$ are the restrictions of $\mathcal{U}(t)$ on S, U respectively, Π^s, Π^u are the projections of $\mathcal{C}_0$ onto S, U, respectively.

4 Partial neutral functional differential equations

In what follows, we will investigate the existence, uniqueness of μ-pseudo $\mathcal{S}$-asymptotically periodic integral solution of PNFDEs. First, consider following partial neutral functional differential equations

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where A is a linear operator on Banach space X, satisfies the Hille-Yosida condition. $u_t \in \mathcal{C}$ is defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r, 0]$. $\mathcal{D} \in B(\mathcal{C}, X)$, $L \in B(\mathcal{C}, X)$, $f \in PSAP_\omega(\mathbb{R}, X, \mu)$, $\mu \in \mathcal{M}$.

Theorem 4.1. *Assume that (H_1)-(H_4) hold, $f \in PSAP_\omega(\mathbb{R}, X, \mu)$, $\mu \in \mathcal{M}$ and the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic, then (4.1) has a unique integral solution $u \in PSAP_\omega(\mathbb{R}, X, \mu)$ which is given by (3.3).*

Proof. By Theorem 3.4, (4.1) has a unique bounded integral solution u which is given by (3.3). Let

$$u_t = (\Gamma^s f)(t) + (\Gamma^u f)(t),$$

where

$$\begin{aligned} (\Gamma^s f)(t) &= \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds, \\ (\Gamma^u f)(t) &= \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds. \end{aligned}$$

By (H_1), there exists a constant $\widetilde{K} > 0$ such that

$$\|(\Gamma^s f)(t)\| \leq \widetilde{K} \int_{-\infty}^{t} e^{-\overline{\omega}(t-s)} \|f(s)\| ds. \tag{4.2}$$

Let $G : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$G(t) = e^{-\overline{\omega} t} \text{ for } t \geq 0 \quad \text{and} \quad G(t) = 0 \text{ for } t < 0,$$

hence

$$\int_{-\infty}^{t} e^{-\overline{\omega}(t-s)} \|f(s)\| ds = \int_{0}^{\infty} e^{-\overline{\omega} s} \|f(t-s)\| ds = \int_{-\infty}^{\infty} G(s) \|f(t-s)\| ds.$$

Sine $\|f(t)\| \in PSAP_\omega(\mathbb{R}, \mathbb{R}, \mu)$, by Lemma 2.3, one has

$$\int_{-\infty}^{t} e^{-\overline{\omega}(t-s)} \|f(s)\| ds \in PSAP_\omega(\mathbb{R}, \mathbb{R}, \mu),$$

so $\Gamma^s f \in PSAP_\omega(\mathbb{R}, X, \mu)$ by (4.2). Proceeding in a similar manner, we have $\Gamma^u f \in PSAP_\omega(\mathbb{R}, X, \mu)$. The proof is complete. □

Next, consider the nonlinear equation

$$\frac{d}{dt} \mathcal{D} u_t = A \mathcal{D} u_t + L(u_t), + f(t, u(t-r)), \quad t \in \mathbb{R}, \tag{4.3}$$

where A is a linear operator on Banach space X, satisfies the Hille-Yosida condition, $\mathcal{D} \in B(\mathcal{C}, X)$, $L \in B(\mathcal{C}, X)$, $f : \mathbb{R} \times X \to X$ is a function bounded on bounded sets of X.

We make the following assumption:

(H_5) $f \in PSAP_\omega(\mathbb{R} \times X, X, \mu)$, $\mu \in \mathcal{M}$ and and satisfies the Lipschitz condition

$$\|f(t, u) - f(t, v)\| \leq L_f \|u - v\|, \quad u, v \in X, \ t \in \mathbb{R},$$

where $L_f > 0$ is a constant.

Theorem 4.2. *Assume that (H_0)-(H_5) hold and the semigroup $(\mathcal{U}(t))_{t \geq 0}$ is hyperbolic, then (4.3) has a unique integral solution $u(t) \in PSAP_\omega(\mathbb{R}, X, \mu)$ if L_f is small enough.*

Proof. Let $v \in PSAP_\omega(\mathbb{R}, X, \mu)$, by Theorem 2.1, Lemma 2.2 and (H_5), it is easy to see that $f(\cdot, v(\cdot - r)) \in PSAP_\omega(\mathbb{R}, X, \mu)$. Consider the equation

$$\frac{d}{dt} \mathcal{D} u_t = A \mathcal{D} u_t + L(u_t) + f(t, v(t-r)), \quad t \in \mathbb{R}. \tag{4.4}$$

By Theorem 4.1, we deduce that (4.4) has a unique integral solution $\mathcal{F} v$ which is given by

$$\left[\lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s (\widetilde{B}_\lambda X_0 f(s, v(s-r))) ds + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s) \Pi^u (\widetilde{B}_\lambda X_0 f(s, v(s-r))) ds \right] (0).$$

The operator $\mathcal{F}$ is well defined on $PSAP_\omega(\mathbb{R}, X, \mu)$. By (H_5), there exists a constant σ_0 such that

$$\sup_{t \in \mathbb{R}} |(\mathcal{F} v_1)(t) - (\mathcal{F} v_2)(t)| \leq L_f \sigma_0 \sup_{t \in \mathbb{R}} |v_1(t) - v_2(t)|.$$

If we choose $L_f \sigma_0 < 1$, by Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $PSAP_\omega(\mathbb{R}, X, \mu)$, which is the μ-pseudo $\mathcal{S}$-asymptotically periodic integral solution to (4.3). □

5 Example

Consider the nonautonomous version of the model proposed in [26]

$$\begin{cases} \dfrac{\partial}{\partial t}[u(t,\xi)-qu(t-r,\xi)] = \dfrac{\partial^2}{\partial \xi^2}[u(t,\xi)-qu(t-r,\xi)] + \displaystyle\int_{-r}^{0}\gamma(\theta)u(t+\theta,\xi)d\theta \\ \qquad\qquad\qquad + \vartheta(u(t-r,\xi)) + \phi(t)g(\xi) \quad \text{for } t\in\mathbb{R},\ \xi\in[0,\pi], \\ u(t,0)-qu(t-r,0) = u(t,\pi)-qu(t-r,\pi) = 0 \qquad \text{for } t\in\mathbb{R}, \end{cases} \tag{5.1}$$

where $q\in(0,1)$, $\gamma\in C([-r,0],\mathbb{R})$, $g\in C([-r,0],\mathbb{R})$, $\vartheta:\mathbb{R}\to\mathbb{R}$ is a Lipschitzian continuous function with Lipschitz constant L_ϑ, and $\phi\in PSAP_\omega(\mathbb{R},X,\mu)$, $\mu\in\mathcal{M}$ satisfying (H_0).

Let $X=C([0,\pi],\mathbb{R})$ and define the operator A by

$$D(A)=\{u\in C^2([0,\pi],\mathbb{R}): u(0)=u(\pi)=0\}, \quad \text{and } Au:=u'',\ u\in D(A).$$

Lemma 5.1. *[14] The operator A satisfies the Hille-Yosida condition on X:*

$$(0,+\infty)\subset\rho(A) \quad \textit{and } |(\lambda I-A)^{-1}|\le\frac{1}{\lambda} \textit{ for } \lambda>0.$$

It is not difficult to see that (H_1) holds by Lemma 5.1. Let A_0 be the part of the operator A in $\overline{D(A)}$, A_0 is given by

$$\begin{aligned} &D(A_0)=\{u\in C^2([0,\pi],\mathbb{R}): u(0)=u(\pi)=u''(0)=u''(\pi)=0\},\\ &Au:=u'',\ u\in D(A_0). \end{aligned}$$

A_0 generates a strongly continuous compact semigroup $(T_0(t))_{t\ge0}$ on $\overline{D(A)}$, which implies that (H_3) holds and $\overline{D(A)}=\{u\in X: u(0)=u(\pi)=0\}$.

Define the bounded linear operator $\mathcal{D}:\mathcal{C}\to X$ by

$$\mathcal{D}\psi=\psi(0)-q\psi(\psi-r).$$

Since $0<q<1$, then $\mathcal{D}$ is stable and (H_4) holds. Moreover, by definitions of the operators $A,\mathcal{D}$, it follows that (H_2) is satisfied.

Let

$$\begin{aligned} &L(\psi)(\xi)=\int_{-r}^{0}\gamma(\theta)\psi(\theta)(\xi)d\theta \quad \text{for } \xi\in[0,\pi],\psi\in\mathcal{C}.\\ &f(t,y)(\xi)=\vartheta(y(\xi))+\phi(t)g(\xi) \quad \text{for } y\in X,t\in\mathbb{R},\xi\in[0,\pi], \end{aligned}$$

then $L\in B(\mathcal{C},X)$ and (H_5) holds with the Lipschitz constant L_ϑ. Let $u(t)=u(t,\cdot)$, (5.1) can be rewritten as an abstract system of the form (4.3). For the hyberbolicity, we suppose that

(H_6) $\int_{-r}^{0}|\gamma(\theta)|d\theta<1-q$.

Lemma 5.2. *[16] Assume that (H_6) holds, then the semigroup $(\mathcal{U}(t))_{t\ge0}$ is hyperbolic.*

By Theorem 4.2, one has

Theorem 5.1. *Under the above assumptions, (5.1) has a unique integral solution $u\in PSAP_\omega(\mathbb{R},X,\mu)$ if L_ϑ is small enough.*

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References

[1] M. Adimy and K. Ezzinbi, Existence and linearized stability for partial neutral functional differential equations, *Differential Equations and Dynamical Systems*, 7 (4) (1999), 371–417.

[2] M. Adimy, K. Ezzinbi and M. Laklach, Spectral decomposition for partial neutral functional differential equations, *Canadian Applied Mathematics Quarterly*, 9 (9) (2001), 1–34.

[3] M. Adimy, A. Elazzouzi and K. Ezzinbi, Bohr-Neugebauer type theoremfor some partial neutral functional differential equations, *Nonlinear Analysis*, 66 (5) (2007), 1145–1160.

[4] M. Adimy, K. Ezzinbi and C. Marquet, Ergodic and weighted pseudo-almost periodic solutions for partial functional differential equations in fading memory spaces, *Journal of Applied Mathematics and Computing*, 44 (1-2) (2014), 147–165.

[5] M. Alia, K. Ezzinbi and S. Fatajou, Exponential dichotomy and pseudo-almost automorphy for partial neutral functional differential equations, *Nonlinear Analysis*, 71 (5-6) (2009), 2210–2226.

[6] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, in: Monographs in Mathematics, vol. 96, *Birkhäuser Verlag,* 2001.

[7] J. Blot, P. Cieutat and G. M. N'Guérékata, $\mathcal{S}$-asymptocially ω-periodic functins and applications to evolution equations, *African Diaspora Journal of Mathematics*, 12 (2) (2011), 113–121.

[8] J. Blot, P. Cieutat and K. Ezzinbi, Measure theory and pseudo almost automorphic functions: New developments and applications, *Nonlinear Analysis*, 75 (4) (2012), 2426–2447.

[9] J. Blot, P. Cieutat and K. Ezzinbi, New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications, *Applicable Analysis*, 92 (3) (2013), 493–526.

[10] Y. K. Chang, R. Y. Ma and Z. H. Zhao, Almost periodic solutions to a stochastic differential equation in Hilbert spaces, *Results in Mathematics*, 63 (1-2) (2013), 435–449.

[11] C. Cuevas and C. Lizama, $\mathcal{S}$-asymptotically ω-periodic solutions for semilinear Volterra equations, *Mathematical Methods in the Applied Sciences*, 33 (13) (2010), 1628–1636.

[12] J. M. Cushing, Forced asymptotically periodic solutions of predator-prey systems with or without hereditary effects, *SIAM Journal on Applied Mathematics*, 30 (4) (1976), 665–674.

[13] B. de Andrade and C. Cuevas, $\mathcal{S}$-asymptotically ω-periodic and asymptotically ω-periodic solutions to semi-linear Cauchy problems with non-dense domain, *Nonlinear Analysis*, 72 (6) (2010), 3190–3208.

[14] K. Ezzinbi and G. M. N'Guérékata, Massera type theorem for almost automorphic solutions of functional differential equations of neutral type, *Journal of Mathematical Analysis and Applications*, 316 (2) (2006), 707–721.

[15] K. Ezzinbi and G. M. N'Guérékata, Almost automorphic solutions for some partial functional differential equations, *Journal of Mathematical Analysis and Applications*, 328 (1) (2007), 344–358.

[16] K. Ezzinbi, S. Fatajou and G. M. N'Guérékata, C^n-almost automorphic solutions for partial neutral functional differential equations, *Applicable Analysis*, 86 (9) (2007), 1127–1146,.

[17] K. Ezzinbi, H. Toure and I. Zabsonre, Pseudo almost periodic solutions of class r for neutral partial functional differential equations, *Afrika Matematika*, 24 (4) (2013), 691–704.

[18] H. R. Henríquez, M. Pierri and P. Táboas, On $\mathcal{S}$-asymptotically ω-periodic functions on Banach spaces and applications, *Journal of Mathematical Analysis and Applications*, 343 (2) (2008), 1119–1130.

[19] H. R. Henríquez, M. Pierri and P. Táboas, Existence of $\mathcal{S}$-asymptotically ω-periodic solutions for abstract neutral functional-differential equations, *Bulletin of the Australian Mathematical Society*, 78 (3) (2008), 365–382.

[20] H. R. Henríquez, C. Cuevas and A. Caicedo, Asymptotically periodic solutions of neutral partial differential equations with infinite delay, *Communications on Pure and Applied Analysis*, 12 (5) (2013), 2031–2068.

[21] X. X. Luo and Y. L. Wang, Asymptotic behavior of solutions to a nonautonomous semilinear evolution equation, *Malaya Journal of Matematik*, 2 (3) (2014), 277–286.

[22] M. Pierri, On $\mathcal{S}$-asymptotically ω-periodic functions and applications, *Nonlinear Analysis*, 75 (2) (2012), 651–661.

[23] M. Pierri and V. Rolnik, On pseudo $\mathcal{S}$-asymptotically periodic functions, *Bulletin of the Australian Mathematical Society*, 87 (2) (2013), 238–254.

[24] F. Y. Wei and K. Wang, Asymptotically periodic Logistic equation, *Journal of Biomathematics*, 20 (4) (2005), 399–405.

[25] F. Y. Wei and K. Wang, Global stability and asymptotically periodic solutions for nonautonomous cooperative Lotka-Volterra diffusion system, *Applied Mathematics and Computation*, 182 (1) (2006), 161–165.

[26] J. Wu and H. Xia, Self-sustained oscillations in a ring array of coupled lossless transmission lines, *Journal of Differential Equations*, 124 (1) (1996), 247–278.

[27] Z. N. Xia, M. Fan and H. Y. Wang, Weighted pseudo-almost automorphy of partial neutral functional differential equations with operator of nondense domain, *Canadian Applied Mathematics Quarterly*, 20 (4) (2012), 589–607.

[28] Z. N. Xia, Weighted pseudo asymptotically periodic solutions for differential and integro-differential equations in Banach spaces, (Submitted).

[29] I. Zabsonre and H. Toure, Pseudo-almost periodic and pseudo-almost automorphic solutions of class r under the light of measure theory, *African Diaspora Journal of Mathematics*, 19 (1) (2016), 58–86.

[30] Z. J. Zeng, Asymptotically periodic solution and optimal harvesting policy for Gompertz system, *Nonlinear Analysis: Real World Applications*, 12 (3) (2011), 1401–1409.

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Nondifferentiable Augmented Lagrangian, ε-Proximal penalty methods and Applications

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Abstract

The purpose of this work is to prove results concerning the duality theory and to give detailed study on the augmented Lagrangian algorithms and ε-proximal penalty method which are considered, today, as the most strong algorithms to solve nonlinear differentiable and nondifferentiable problems of optimization. We give an algorithm of primal-dual type, where we show that sequences $\left\{\lambda^k\right\}_k$ and $\left\{x^k\right\}_k$ generated by this algorithm converge globally, with at least the Slater condition, to $\overline{\lambda}$ and $\overline{x}$. Numerical simulations are given.

Keywords: Convex programming, augmented Lagrangian, ε-proximal penalty method, duality, Perturbation, Convergence of algorithms.

2010 MSC: 90C25, 49M29, 49N15.

1 Introduction

The augmented Lagrangian methods present a large inconvenience of point of view stability. If we have a sequence $\left\{\lambda^k\right\}_k$ who converges to an optimum $\overline{\lambda}$ of the dual function, the successive solutions x^k obtained converge to an optimal solution only if $L(x,\lambda)$ has an unique minimum at x in a neighborhood of $\overline{\lambda}$ (it will be the case for example if $L(x,\lambda)$ is strictly convex at x).

So the methods of exterior penalties present the inconvenience that, to obtain a feasible point, we make tighten the coefficient of penalty towards the infinity, then the penalized function becomes badly conditioned for which the methods of gradients have a slow convergence

In the case of the equality constraints, Hestenes (1969) and Powell (1969) suggested combining previous both approaches (penalties and dualities), and suggested solving a sequence of unconstrained problems of the following shape:

$$L_r(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + r\sum_{i=1}^{m} (g_i(x))^2 \tag{1.1}$$

A generalization of Hestenes and Powell function to inequality constraints will be after given.

So, the general principle of these methods consists in determining a saddle point of L_r instead of solving $(\mathcal{P})$. The first component of the saddle point is, also, an optimal solution of the problem $(\mathcal{P})$.

The augmented Lagrangian method can be considered as an improvement of the penalty methods, because it avoids having to use coefficients of penalties too big.

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Besides, the fact of adding the quadratic term $r(g(x))^2$ in the classical Lagrangian will improve the properties of convergence of Lagrangian algorithms because the augmented Lagrangian is strictly convex at x. It is the case where we find an unique primal solution in the neighborhood of the dual solution.

We can say that the augmented Lagrangian has a much more fundamental interest. Today, it is widely recognized that the algorithms of optimization based on the use of the augmented Lagrangian, are a part of the most effective general methods to solve differentiable and nondifferentiable mathematical programming problems.

The purpose of this work is to prove results concerning the duality theory and to give detailed study on the augmented Lagrangian algorithms and ε-proximal penalty methods which are considered, today, as the most strong algorithms to solve nonlinear differentiable and nondifferentiable problems of optimization. Numerical experiments are given.

2 Main Results

2.1 Results on the Augmented Lagrangian

Consider the following mathematical programming problem :

$$(\mathcal{P}) \quad \begin{cases} \alpha := Inf f(x) \\ \text{subject to } \ x \in C \end{cases} \tag{2.2}$$

where

- f is a convex function with finite values and non necessarily differentiable.
- $C := \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, ..., m\}$, $g_i \ (i = 1, ..., m)$ are $\mathcal{C}^1$-convex functions.

Suppose that

$$\lim_{(\|x\| \longrightarrow +\infty)} f(x) = +\infty \text{ (i.e., } f \text{ is inf-compact)} \tag{2.3}$$

and there exists x_0 such that

$$g_i(x_0) < 0, \quad (i = 1, ..., m) \tag{2.4}$$

Definition 2.1. *The augmented Lagrangian associated to the problem $(\mathcal{P})$ is defined as follows*

$$L_r(x, \lambda) := f(x) + \frac{1}{2r} \sum_{i=1}^{m} (\psi^+(\lambda_i + r g_i(x))^2 - \lambda_i^2) \text{ for all } x \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^m_+, \tag{2.5}$$

where $\psi^+(t) = Max(0, t)$. Or still

$$L_r(x, \lambda) := f(x) + \sum_{i=1}^{m} \begin{cases} \frac{r}{2} g_i^2(x) + \lambda_i g_i(x) & \text{if } \ g_i(x) > -\frac{\lambda_i}{r} \\ \\ -\frac{1}{2r}\lambda_i^2 & \text{if } \ g_i(x) \leq -\frac{\lambda_i}{r}. \end{cases} \tag{2.6}$$

Remark 2.1. *Put*

$$L_r(x, \lambda) = f(x) + \varphi(g(x), \lambda, r),$$

where

$$\varphi(u, \lambda, r) = \frac{1}{2r} \sum_{i=1}^{m} (\psi^+(\lambda_i + r u_i)^2 - \lambda_i^2), \quad u \in \mathbb{R}^m, \quad \lambda \in \mathbb{R}^m_+, \quad r > 0.$$

We notice well that

- *if $u \leq 0$, then $\varphi(u, \lambda, r) \leq 0$;*
- *if $u = 0$, then $\varphi(u, \lambda, r) = 0$.*

Corollary 2.1. *We have* $\lim\limits_{(r \longrightarrow 0)} L_r(x, \lambda) = L(x, \lambda).$

We have the following lemma :

Lemma 2.1. *We have*

$$\underset{x\in\mathbb{R}^n}{Inf}\ \underset{(\lambda,\, r)\in T}{Sup}\ L_r(x,\lambda) = \alpha,$$

where $T = \mathbb{R}^m_+ \times \mathbb{R}_+$.

Proof. At first, we notice that for all u and $c \geq 0$ there is a couple $(\lambda, r) \in T$ such that $\varphi(u,\lambda,r) > c$.

Indeed, we distinguish two cases :

- Ca **se** 1**:** If $u \nleq 0$, there exists at least one component $u_i > 0$. We note by

$$I := \left\{ i \in \{1,...,m\} : u_i > -\frac{\lambda_i}{r} \right\}.$$

$I \neq \emptyset$. Then

$$\varphi(u,\lambda,r) = \sum_{i\in I}(\frac{r}{2}u_i^2 + \lambda_i u_i) - \sum_{i\notin I}\frac{\lambda_i^2}{2r}.$$

If $I = \{1,...,m\}$ then $\varphi(u,\lambda,r) \longrightarrow +\infty$, as $(\lambda,r) \longrightarrow +\infty$.

Else, we have $\varphi(u,0,r) \longrightarrow +\infty$, as $(r \longrightarrow +\infty)$.

Then, in both cases there existe $(\lambda,r) \in T$ such that

$$\varphi(u,\lambda,r) > c. \tag{2.7}$$

- **Case 2:** If $u_i \leq 0$, for all $i \in \{1,...,m\}$, one has

$$\frac{1}{2r}(\psi^+(\lambda_i + ru_i)^2 - \lambda_i^2) = \left\{ \begin{array}{ll} \frac{r}{2}u_i^2 + \lambda_i u_i & \text{if } u_i > -\frac{\lambda_i}{r} \\ \\ -\frac{1}{2r}\lambda_i^2 & \text{if } u_i \leq -\frac{\lambda_i}{r} \end{array} \right\} \leq 0$$

then

$$\underset{(\lambda,\, r)\in T}{Sup}\ \varphi(u,\lambda,r) = 0 \tag{2.8}$$

By means of formulae (2.6) and (2.7), one has

$$\underset{(\lambda,\, r)\in T}{Sup}\ L_r(x,\lambda) = \left\{ \begin{array}{l} f(x) \ \text{ if } x \in C \\ \\ +\infty \ \text{ else;} \end{array} \right.$$

thus

$$\underset{x\in\mathbb{R}^n}{Inf}\ \underset{(\lambda,\, r)\in T}{Sup}\ L_r(x,\lambda) = \underset{x\in C}{Inf} f(x) = \alpha.$$

□

By definition, we put

$$d_r(\lambda) := \underset{x\in\mathbb{R}^n}{Inf}\, L_r(x,\lambda), \text{ for all } \lambda \in \mathbb{R}^m_+.$$

We have the following Lemma:

Lemma 2.2. *For all* $r > 0$, *we have*

$$d_r(\lambda) := \underset{z\geq 0}{Sup}\left\{ d(z) - \frac{1}{2r}\left\|z-\lambda\right\|^2 \right\} \text{ for all } \lambda \in \mathbb{R}^m_+. \tag{2.9}$$

Proof. We have

$$d_r(\lambda) := \underset{z\geq 0}{Sup}\left\{d(z) - \tfrac{1}{2r}\|z-\lambda\|^2\right\}$$

$$= \underset{z\geq 0}{Sup}\left\{\underset{x\in\mathbb{R}^n}{Inf}\left\{f(x)+\sum_{i=1}^{m} z_i g_i(x)\right\} - \tfrac{1}{2r}\|z-\lambda\|^2\right\}$$

$$= \underset{z\geq 0}{Sup}\left\{\underset{x\in\mathbb{R}^n}{Inf}\left\{f(x)+\sum_{i=1}^{m} z_i g_i(x) - \tfrac{1}{2r}\|z-\lambda\|^2\right\}\right\}.$$

The function

$$(x,z)\longrightarrow \delta(x,z) = f(x)+\sum_{i=1}^{m} z_i g_i(x) - \frac{1}{2r}\|z-\lambda\|^2$$

admits a saddle point because it verifies the following conditions :

. $\delta(x,z)$ is convex for x an concave for z;

. $\delta(x,z)$ tends to $+\infty$ as $\|x\|\longrightarrow+\infty$ (at a point $z=0$);

. $\delta(x,z)$ tends to $-\infty$ as $\|z\|\longrightarrow+\infty$ (at a point $x_0 : g(x_0)<0$).

Then, we can invert $SupInf$ by $InfSup$ and we have

$$d_r(\lambda) = \underset{z\geq 0}{Sup}\,\underset{x\in\mathbb{R}^n}{Inf}\left\{f(x)+\sum_{i=1}^{m} z_i g_i(x) - \tfrac{1}{2r}\|z-\lambda\|^2\right\}$$

$$= \underset{x\in\mathbb{R}^n}{Inf}\,\underset{z\geq 0}{Sup}\left\{f(x)+\sum_{i=1}^{m} z_i g_i(x) - \tfrac{1}{2r}\|z-\lambda\|^2\right\}.$$

The Sup is reached at $\bar{z}$ where

$$\bar{z}_i = \left\{\begin{array}{ll} rg_i(x)+\lambda_i & \text{if } g_i(x) > -\frac{\lambda_i}{r} \\ 0 & \text{if } g_i(x)\leq -\frac{\lambda_i}{r}\end{array}\right\} = \psi^+(rg_i(x)+\lambda_i).$$

For this notation, then the function $d_r(\lambda)$ spells

$$d_r(\lambda) := \underset{x\in\mathbb{R}^n}{Inf}\left\{f(x)+\sum_{i=1}^{m}\psi^+(rg_i(x)+\lambda_i)g_i(x) - \tfrac{1}{2r}\sum_{i=1}^{m}(\psi^+(rg_i(x)+\lambda_i)-\lambda_i)^2\right\}$$

$$= \underset{x\in\mathbb{R}^n}{Inf}\left\{f(x)+\sum_{i=1}^{m}(\psi^+(rg_i(x)+\lambda_i)g_i(x) - \tfrac{1}{2r}(\psi^+(rg_i(x)+\lambda_i)-\lambda_i)^2)\right\}$$

$$= \underset{x\in\mathbb{R}^n}{Inf}\left\{f(x)+\sum_{i=1}^{m}\left\{\begin{array}{ll}\lambda_i g_i(x)+\frac{r}{2}g_i^2(x) & \text{if } g_i(x) > -\frac{\lambda_i}{r} \\ -\frac{1}{2r}\lambda_i^2 & \text{if } g_i(x)\leq -\frac{\lambda_i}{r}\end{array}\right\}\right.$$

$$= \underset{x\in\mathbb{R}^n}{Inf}\left\{f(x)+\tfrac{1}{2r}\sum_{i=1}^{m}(\psi^+(rg_i(x)+\lambda_i)^2-\lambda_i^2))\right\} = \underset{x\in\mathbb{R}^n}{Inf} L_r(x,\lambda).$$

According to ([3], remark 2.1), d_r is the regularized function of d. It is, thus, differentiable at λ and we have

$$\nabla d_r(\lambda) = -\frac{1}{r}(\lambda - z_\lambda)$$

where z_λ realizes the Sup in the expression (2.8). We note, also, that d_r has the same optimal solutions as d. □

Definition 2.2. *The dual problem associated to the problem $(\mathcal{P})$ is the following one :*

$$(\mathcal{D}) \quad \beta := \underset{(\lambda,\,r)\in T}{Sup}\; d_r(\lambda), \tag{2.10}$$

where $T = \mathbb{R}_+^m \times \mathbb{R}_+$.

Definition 2.3. *We call perturbation function of* $(\mathcal{P})$ *the function* p *defined by*

$$p(u) := \underset{x\in\mathbb{R}^n}{Inf}\, F(x,u),$$

where

$$F(x,u) := \begin{cases} f(x) & \text{if } g(x) \le u, \\ +\infty & \text{else} \end{cases} \tag{2.11}$$

Remark 2.2. . *If* $u = 0$ *then* $p(0) = \alpha$
. *If* $u_1 \ge u_2$ *then* $p(u_2) \ge p(u_1)$.

The following lemma shows the relation which exists between L_r and F.

Lemma 2.3. *We have*

$$L_r(x,\lambda) = \underset{u\in\mathbb{R}^m}{Inf}\, \{F(x,u) + \varphi(u,\lambda,r)\} \tag{2.12}$$

for all $x \in \mathbb{R}^n$ *and* $(\lambda, r) \in T$.

Proof. Let $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.
. If $g(x) \le u$ we have

$$F(x,u) = f(x) \text{ and } \varphi(g(x),\lambda,r) \le \varphi(u,\lambda,r), \ \forall(\lambda,r) \in T.$$

. If $g(x) \nleq u$ we have $F(x,u) = +\infty$. Then

$$L_r(x,\lambda) = f(x) + \varphi(g(x),\lambda,r) \le F(x,u) + \varphi(u,\lambda,r), \ \forall u \in \mathbb{R}^m,$$

thus

$$L_r(x,\lambda) \le \underset{u\in\mathbb{R}^m}{Inf}\, \{F(x,u) + \varphi(u,\lambda,r)\},$$

but

$$L_r(x,\lambda) = F(x,g(x)) + \varphi(g(x),\lambda,r) \ge \underset{u\in\mathbb{R}^m}{Inf}\, \{F(x,u) + \varphi(u,\lambda,r)\}.$$

Then both inequalities give the expression (2.11). □

Lemma 2.4. *We have*

$$L_r(x,\lambda) = \underset{u\in\mathbb{R}^m}{Inf} \left\{F(x,u) + <\lambda,u> + \frac{r}{2}\|u\|^2\right\}, \tag{2.13}$$

where F *is given by the expression* (2.11).

Proof. Indeed, let us put

$$\varphi_r(x^k,\lambda) = \underset{u\in\mathbb{R}^m}{Inf} \left\{F(x^k,u) + <\lambda,u> + \frac{r}{2}\|u\|^2\right\}$$

The Inf in the expression of $\varphi_r(x^k,\lambda)$ exists and unique (the function at u is strongly convex). For every x, we indicate by C_x the following set :

$$C_x := \{u \in \mathbb{R}^m : u \ge g(x)\}.$$

Then, the expression (2.12) becomes

$$\varphi_r(x^k,\lambda) = \underset{u\in C_x}{Inf} \left\{F(x,u) + <\lambda,u> + \tfrac{r}{2}\|u\|^2\right\} = \underset{u\in C_x}{Inf} \left\{f(x) + <\lambda,u> + \tfrac{r}{2}\|u\|^2\right\}$$

$$= f(x) + \underset{u\in C_x}{Inf} \left\{<\lambda,u> + \tfrac{r}{2}\|u\|^2\right\}.$$

To calculate the solution of $\underset{u\in C_x}{Inf} \left\{<\lambda,u> + \tfrac{r}{2}\|u\|^2\right\}$ we look for a minimization according to every i. Let us put

$$w(u) = <\lambda,u> + \frac{r}{2}\|u\|^2,$$

then, $\nabla w(u) = \lambda + ru$.

For all i, if $g_i(x) < -\frac{\lambda_i}{r}$ then, $\overline{u}_i = -\frac{\lambda_i}{r}$, else $\overline{u}_i = g_i(x)$. Thus

$$\underset{x \in C_x}{Inf}\, w(x) = \sum_{i=1}^{m} \begin{cases} \lambda_i g_i(x) + \frac{1}{2r}(g_i(x))^2 & \text{if } g_i(x) \geq -\frac{\lambda_i}{r} \\ -\frac{1}{2r}\lambda_i^2 & \text{if } g_i(x) < -\frac{\lambda_i}{r} \end{cases} = \varphi(g(x), \lambda, r).$$

Then $\quad \varphi_r(x^k, \lambda) = L_r(x, \lambda)$. □

We notice that L_r is convex at x and concave at (λ, r), consequently d_r is concave.

We have the following weak duality theorem :

Theorem 2.1. *(Weak duality) We have*

$$\beta \leq \alpha. \tag{2.14}$$

Proof. We always have

$$\underset{(\lambda,r)\in T}{Sup}\ \underset{x\in\mathbb{R}^n}{Inf}\, L_r(x,\lambda) \leq \underset{x\in\mathbb{R}^n}{Inf}\ \underset{(\lambda,r)\in T}{Sup}\ L_r(x,\lambda),$$

thus

$$\beta \leq \alpha.$$

□

Another relation exists between d_r and p is given by the following lemma :

Lemma 2.5. *We have*

$$d_r(\lambda) = \underset{u\in\mathbb{R}^n}{Inf}\ \{p(u) + \varphi(u,\lambda,r)\}, \ \ \forall\ (\lambda, r) \in T. \tag{2.15}$$

Proof. We have, according to the Lemma 2.4,

$$d_r(\lambda) = \underset{x\in\mathbb{R}^n}{Inf}\, L_r(x,\lambda) = \underset{x\in\mathbb{R}^n}{Inf}\ \underset{u\in\mathbb{R}^n}{Inf}\ \{F(x,u) + \varphi(u,\lambda,r)\}$$

$$= \underset{u\in\mathbb{R}^n}{Inf}\ \underset{x\in\mathbb{R}^n}{Inf}\ \{F(x,u) + \varphi(u,\lambda,r)\} = \underset{u\in\mathbb{R}^n}{Inf}\ \{p(u) + \varphi(u,\lambda,r)\}.$$

□

Lemma 2.6. *There is a function Φ such that for all $(\lambda, r) \in T,\ (z, s) \in T,\ r > s$, we have*

$$\varphi(u,\lambda,r) - \varphi(u,z,s) \geq -\Phi(\lambda,z,s,r)$$

with

$$\lim_{(r\longrightarrow+\infty)} \Phi(\lambda,z,s,r) = 0. \tag{2.16}$$

Proof. We have

$$\varphi(u,\lambda,r) - \varphi(u,z,s) = \frac{1}{2r}\sum_{i=1}^{m}\Psi^+(ru_i+\lambda_i) - \frac{1}{2s}\sum_{i=1}^{m}\Psi^+(su_i+z_i).$$

We distinguish two cases :

Case 1:

. If $u_i \leq -\frac{\lambda_i}{r}$, then $\Psi^+(u_i+\lambda_i) = 0$.

. If $u_i \leq -\frac{z_i}{s}$, then $\Psi^+(su_i+z_i) = 0$, thus

$$\frac{1}{2r}\Psi^+(ru_i+\lambda_i) - \frac{1}{2s}\Psi^+(su_i+z_i) = 0.$$

. If $u_i > -\frac{z_i}{s}$, then $-\frac{z_i}{s} < u_i \leq -\frac{\lambda_i}{r}$, from hence

$$\frac{1}{2s}\Psi^+(su_i+z_i) = \frac{s}{2}u_i^2 + z_iu_i \leq \frac{s}{2}(-\frac{\lambda_i}{r})^2 + z_i(-\frac{\lambda_i}{r}) \leq \frac{s}{2}\frac{\lambda_i^2}{r^2} - \frac{z_i\lambda_i}{r}.$$

As $r > s$, then, we have

$$\frac{1}{2s}\Psi^{+}(su+z) \leq \frac{\lambda_i^2}{2r} - \frac{z_i\lambda_i}{r}.$$

It holds that

$$\frac{1}{2r}\Psi^{+}(ru_i+\lambda_i) - \frac{1}{2s}\Psi^{+}(su_i+z_i) \geq -(-\frac{\lambda_i^2}{2r} + \frac{z_i\lambda_i}{r}) \longrightarrow 0, \text{ as } r \longrightarrow +\infty.$$

Case 2:

. If $u_i > -\frac{\lambda_i}{r}$, then

$$\frac{1}{2r}\Psi^{+}(ru_i+\lambda_i) = \frac{r}{2}u_i^2 + \lambda_i u_i.$$

. If $u_i \leq -\frac{z_i}{s}$, then

$$\frac{1}{2s}\Psi^{+}(su_i+z_i) = 0,$$

thus $-\frac{\lambda_i}{r} < u_i \leq -\frac{z_i}{s}$, it holds that

$$\frac{1}{2s}\Psi^{+}(ru_i+\lambda_i) - \frac{1}{2s}\Psi^{+}(su_i+z_i) = \frac{r}{2}u_i^2 + \lambda_i u_i \geq -(-\frac{r}{2}(-\frac{z_i}{s})^2 - \lambda_i(-\frac{z_i}{s}))$$

$$\geq -(-\frac{z_i^2}{2r} + \lambda_i\frac{z_i}{r}) \longrightarrow 0, \text{ as } r \longrightarrow +\infty.$$

. If $u_i > -\frac{z_i}{s}$, then, we have

$$\frac{1}{2r}\Psi^{+}(ru_i+\lambda_i) - \frac{1}{2s}\Psi^{+}(su_i+z_i) = \frac{1}{2}u_i^2(r-s) + (\lambda_i - z_i)u_i$$

$$\geq \frac{1}{2}(\frac{z_i-\lambda_i}{r-s})^2 + (\lambda_i - z_i)\frac{(z_i-\lambda_i)}{r-s}$$

$$\geq \frac{1}{2}(\frac{z_i-\lambda_i}{r-s})^2 + \frac{(z_i-\lambda_i)^2}{r-s} \longrightarrow 0, \text{ as } r \longrightarrow +\infty.$$

Finally, in every cases there is a function Φ verifying

$$\varphi(u,\lambda,r) - \varphi(u,z,s) \geq -\Phi(\lambda,z,s,r), \text{ for all } (\lambda,r) \in T,\ (z,s) \in T,\ \ r > s$$

with

$$\lim_{(r \longrightarrow +\infty)} \Phi(\lambda,z,s,r) = 0.$$

□

It results from this lemma the following result :

Lemma 2.7. *For all* $(\lambda,r) \in T,\ (r > 0)$, *we have*

$$d_r(\lambda) \geq \underset{(z,s)\in T,\ (r>s>0)}{Sup} (d_s(z) - \Phi(\lambda,z,s,r)).$$

Proof. According to the Lemma 2.8, we have

$$\varphi(u,\lambda,r) \geq \varphi(u,z,s) - \Phi(\lambda,z,s,r).$$

Hence

$$p(u) + \varphi(u,\lambda,r) \geq p(u) + \varphi(u,z,s) - \Phi(\lambda,z,s,r) \ \ \forall\, u \in \mathbb{R}^m,$$

then

$$\underset{u\in\mathbb{R}^m}{Inf}\,(p(u) + \varphi(u,\lambda,r)) \geq \underset{u\in\mathbb{R}^m}{Inf}\,(p(u) + \varphi(u,z,s) - \Phi(\lambda,z,s,r)).$$

It holds that

$$d_r(\lambda) \geq d_s(z) - \Phi(\lambda,z,s,r), \ \ \forall (z,s) \in T,\ \forall r > s > 0$$

$$\Longrightarrow d_r(\lambda) \geq \underset{(z,s)\in T,\ (r>s>0)}{Sup} (d_s(z) - \Phi(\lambda,z,s,r)).$$

□

We have the following theorem :

Theorem 2.2. *We have*

$$\beta = \underset{(z,s)\in T}{Sup}\ d_s(z) = \lim_{(r\longrightarrow+\infty)} d_r(\lambda),\ \textit{for all}\ \lambda \in \mathbb{R}^m_+. \tag{2.17}$$

Proof. For all $(z,s) \in T,\ \varepsilon > 0$ and $\lambda \in \mathbb{R}^m_+$, it exists, according to the Lemma 2.8, r enough large, with $(r > s)$ such that $\Phi(\lambda, z, s, r) < \varepsilon$. Then

$$d_r(\lambda) \geq d_s(z) - \varepsilon,\ \ \forall\varepsilon > 0,$$

thus

$$\lim_{(r\longrightarrow+\infty)} d_r(\lambda) \geq d_s(z) - \varepsilon,\ \ \forall\varepsilon > 0,\ \forall(z,s) \in T.$$

And then, for every $\varepsilon > 0$

$$\lim_{(r\longrightarrow+\infty)} d_r(\lambda) \geq \underset{(z,s)\in T}{Sup}\ d_s(z) - \varepsilon,$$

thus

$$\lim_{(r\longrightarrow+\infty)} d_r(\lambda) \geq \underset{(z,s)\in T}{Sup}\ d_s(z).$$

On the other hand,

$$\underset{(z,s)\in T}{Sup}\ d_s(z) \geq d_r(\lambda),\ \ \forall\lambda \in \mathbb{R}^m_+$$

$$\underset{(z,s)\in T}{Sup}\ d_s(z) \geq \lim_{(r\longrightarrow+\infty)} d_r(\lambda),$$

where holds the result. □

This theorem gives a technique of resolution of $(\mathcal{D})$. Indeed; if we penalize the function d, by using the term of penalty $(-\frac{1}{2r}\left\|z-\lambda\right\|)$, then by making the resolution when $(r \longrightarrow +\infty)$, we are in front of a said penalty method.

The following algorithm shows the necessary steps for the resolution :

Algorithm 1:

Step 1: $(k = 0)$

Fixe λ and we choose a factor of penalty $r_0 > 0$ and $z_0 \in \mathbb{R}^m_+,\ (k = 0)$.

Step 2: $(k \geq 0)$

Find z_k solution of

$$d_{r_k}(\lambda) = \underset{z\geq 0}{Sup}\left\{d(z) - \frac{1}{2r_k}\left\|z-\lambda\right\|^2\right\}.$$

Step 3:

If z_k do not verify the stop test one makes $r_{k+1} > r_k$, $k \longrightarrow k+1$ and we return to the step 1.

2.2 Augmented Lagrangian Algorithms

Let $(\mathcal{P})$ be the following constrained mathematical programming problem :

$$(\mathcal{P}) \qquad \alpha := \underset{x\in C}{Inf} f(x),$$

where

. f is a non necessarely differentiable convex function with finite value ;

. $C := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, ..., m\}$;

. $g_i\ (i = 1, ..., m)$ are $\mathcal{C}^1$-convex functions.

Suppose that $\lim_{(\|x\|\longrightarrow+\infty)} f(x) = +\infty$ and there exists x_0 such that

$$g_i(x_0) \leq 0,\ i = 1, ..., m.$$

We give an algorithm with which we can calculate optimal solutions of $(\mathcal{P})$.

Algorithm 2:
Step 1: $(k = 0)$
Let us fix $r > 0$, let us determine one $\overline{\lambda}$ in which the function d_r reaches its maximum on $\mathbb{R}^m$.
Step 2:
Let us look for any point $\overline{x}$ which minimizes the convex function $L_r(.,\overline{\lambda})$ on $\mathbb{R}^n$.

Remark 2.3. *The essential difficulty in the previous method lies in the calculation of $(\overline{x},\overline{\lambda})$. This couple is not calculable with accuracy. But, if $\overline{\lambda}$ and $\overline{x}$ are approximately determined by the previous method, can we be sure that $\overline{x}$ is, approximately, an optimal solution of $(\mathcal{P})$?*

Another complication appears because of the non direct clarified of d_r at the wished way.

However, we can calculate $d_r(\lambda)$ and $\nabla d_r(\lambda)$, for every λ, by determining a point x which minimizes $L_r(.,\lambda)$ on $\mathbb{R}^n$. This operation is too expensive (from point of view cost) by repeating, every time, the process of iteration.

To by-pass this difficulty, let us suppose that for one λ given, we have one $x \in \mathbb{R}^n$ minimizing $L_r(.,\lambda)$ on $\mathbb{R}^n$ with a precision $\varepsilon \geq 0$, that is

$$L_r(x,\lambda) - d_r(\lambda) \leq \varepsilon.$$

We see that

$$d_r(\lambda') \leq L_r(x,\lambda') \leq L_r(x,\lambda) + < \lambda' - \lambda, \nabla_\lambda L_r(x,\lambda) > \quad \forall x \in \mathbb{R}^n,\ \lambda' \in \mathbb{R}^m$$

$$\Longrightarrow d_r(\lambda') \leq d_r(\lambda) + < \lambda' - \lambda, \nabla_\lambda L_r(x,\lambda) > + \varepsilon.$$

It holds that $\nabla_\lambda L_r(x,\lambda)$ is an ε -subgradient of d_r at λ.

Definition 2.4. *. A sequence $\left\{x^k\right\}_k$ of $\mathbb{R}^n$ is called asymptotically feasible for the problem $(\mathcal{P})$ if*

$$\lim_{(k\longrightarrow+\infty)} g_i(x^k) \leq 0,\ i = 1,...,m.$$

. A sequence which realizes the Sup of the problem $(\mathcal{D})$ is a sequence $\left\{\lambda^k\right\}_k$ of $\mathbb{R}^m$ such that

$$d_r(\lambda^k) \longrightarrow Sup d_r, \text{ as } (k \longrightarrow +\infty).$$

. An asymptotically minimizing sequence of $(\mathcal{P})$ is a sequence $\left\{x^k\right\}_k$ asymptotically feasible and such that

$$\lim_{(k\longrightarrow+\infty)} f(x^k) = \alpha.$$

Theorem 2.3. *Let $\left\{\lambda^k\right\}_k$ be a bounded sequence wich maximizes $(\mathcal{D})$, let $\left\{x^k\right\}_k$ be a sequence satisfying*

$$L_r(x^k,\lambda^k) - \underset{x\in\mathbb{R}^n}{Inf}\, L_r(x,\lambda^k) = L_r(x^k,\lambda^k) - d_r(\lambda^k) \leq \varepsilon_k,$$

where $\varepsilon_k \longrightarrow 0$ as $k \longrightarrow +\infty$.

Then $\left\{x^k\right\}_k$ is an asymptotically minimizing sequence of $(\mathcal{P})$.

For the proof of this theorem, we need to the following three lemmas:

Lemma 2.8. *The function d_r satisfies, for all $\lambda,\ \lambda' \in \mathbb{R}^m_+$*

$$\begin{aligned} &d_r(\lambda') \leq d_r(\lambda) + < \lambda' - \lambda, \nabla d_r(\lambda) > \\ &d_r(\lambda') \geq d_r(\lambda) + < \lambda' - \lambda, \nabla d_r(\lambda) > - \tfrac{1}{2r}\left\|\lambda' - \lambda\right\|^2 \end{aligned} \tag{2.18}$$

Proof. The first inequality is immediate from the concavity of $d_r(\lambda)$.

For the second inequality, we have

$$d_r(\lambda) = \underset{z\in\mathbb{R}^n_+}{Sup}\left\{d(z) - \frac{1}{2r}\left\|\lambda - z\right\|^2\right\}.$$

It exists an unique z_λ such that

$$d_r(\lambda) = d(z_\lambda) - \frac{1}{2r}\|\lambda - z_\lambda\|^2.$$

Let us put

$$q(\lambda^{'}) := d(z_\lambda) - \frac{1}{2r}\left\|\lambda^{'} - z_\lambda\right\|^2.$$

Or $q(\lambda^{'})$ is quadratic, we shall have

$$q(\lambda^{'}) = q(\lambda) + < \lambda^{'} - \lambda, \nabla q(\lambda) > + \frac{1}{2}(\lambda^{'} - \lambda)^t \nabla q(\lambda)(\lambda^{'} - \lambda).$$

Because

$$q(\lambda) = d_r(\lambda) \quad \text{and} \quad q(\lambda^{'}) \leq d_r(\lambda^{'}), \ \forall \lambda^{'},$$

it holds that

$$\nabla q(\lambda) = \nabla d_r(\lambda).$$

On the other hand,

$$\nabla^2 q(\lambda) = -\frac{1}{r} Id \ \ \text{(where } Id \text{ is an identity matrix)},$$

then

$$q(\lambda^{'}) = d_r(\lambda) + < \lambda^{'} - \lambda, \nabla d_r(\lambda) > - \frac{1}{2r}\left\|\lambda^{'} - \lambda\right\|^2.$$

So

$$d_r(\lambda) + < \lambda^{'} - \lambda, \nabla d_r(\lambda) > - \frac{1}{2r}\left\|\lambda^{'} - \lambda\right\|^2 \leq d_r(\lambda^{'}).$$

□

Lemma 2.9. *We have*

$$\frac{r}{2}\left\|\nabla d_r(\lambda^k)\right\|^2 \leq Sup d_r - d_r(\lambda^k). \tag{2.19}$$

Proof. According to Lemma 2.12, it holds that

$$Sup d_r \geq \underset{\lambda^{'} \in \mathbb{R}^m_+}{Sup} \left\{ d_r(\lambda^k) + < \lambda^{'} - \lambda, \nabla d_r(\lambda^k) > - \tfrac{1}{2r}\left\|\lambda^{'} - \lambda\right\|^2 \right\}$$

$$= d_r(\lambda^k) + \underset{\lambda^{'} \in \mathbb{R}^m_+}{Sup} \left\{ < \lambda^{'} - \lambda, \nabla d_r(\lambda^k) > - \tfrac{1}{2r}\left\|\lambda^{'} - \lambda\right\|^2 \right\}$$

$$= d_r(\lambda^k) + \tfrac{r}{2}\left\|\nabla d_r(\lambda^k)\right\|^2$$

what gives

$$\frac{r}{2}\left\|\nabla d_r(\lambda^k)\right\|^2 \leq Sup d_r - d_r(\lambda^k).$$

□

Lemma 2.10. *Let us consider following both properties:*
(**a**) $L_r(x^k, \lambda^k) - \underset{x \in \mathbb{R}^n}{Inf} L_r(x, \lambda^k) = L_r(x^k, \lambda^k) - d_r(\lambda^k) \leq \varepsilon_k$,
where $\varepsilon_k \longrightarrow 0$, *as* $k \longrightarrow +\infty$;
(**b**) $\frac{r}{2}\left\|\nabla_\lambda L_r(x^k, \lambda^k) - \nabla d_r(\lambda^k)\right\|^2 \leq \varepsilon_k$.
Then (**a**) $\Longrightarrow$ (**b**).

Proof. We use the Lemma 2.12 and the concavity of $L_r(x^k, .)$ then, we shall have for every $w \in \mathbb{R}^m$

$$d_r(w) \leq L_r(x^k, w) \leq L_r(x^k, \lambda^k) + < w - \lambda^k, \nabla_\lambda L_r(x^k, \lambda^k) >$$

and

$$d_r(w) \geq d_r(\lambda^k) + < w - \lambda^k, \nabla d_r(\lambda^k) > - \frac{1}{2r}\left\|w - \lambda^k\right\|^2$$

what gives

$$L_r(x^k,\lambda^k) - d_r(\lambda^k) \geq < w - \lambda^k, \nabla d_r(\lambda^k) - \nabla_\lambda L_r(x^k,\lambda^k) > -\frac{1}{2r}\left\|w - \lambda^k\right\|^2.$$

That is

$$L_r(x^k,\lambda^k) - d_r(\lambda^k) \geq \underset{w\in R^m_+}{Sup}\left\{< w - \lambda^k, \nabla d_r(\lambda^k) - \nabla_\lambda L_r(x^k,\lambda^k) > -\tfrac{1}{2r}\left\|w - \lambda^k\right\|^2\right\}$$

$$= r\left\|\nabla d_r(\lambda^k) - \nabla_\lambda L_r(x^k,\lambda^k)\right\|^2 - \tfrac{r}{2}\left\|\nabla d_r(\lambda^k) - \nabla_\lambda L_r(x^k,\lambda^k)\right\|^2$$

$$= \tfrac{r}{2}\left\|\nabla d_r(\lambda^k) - \nabla_\lambda L_r(x^k,\lambda^k)\right\|^2.$$

Where, according to (**a**), we have

$$\frac{r}{2}\left\|\nabla d_r(\lambda^k) - \nabla_\lambda L_r(x^k,\lambda^k)\right\|^2 \leq \varepsilon_k.$$

□

Proof. (**Theorem 2.11**) According to the Lemma 2.14 we have

$$L_r(x^k,\lambda) = Inf\left\{F(x^k,u) + < x,u > + \frac{r}{2}\|u\|^2\right\},$$

where F is given by

$$F(x,u) = \begin{cases} f(x) & \text{if } g_i(x) \leq u_i,\ i = 1,...,m \\ \\ +\infty & \text{else.} \end{cases}$$

For $\lambda = \lambda^k$, there is an unique point u^k such that

$$L_r(x^k,\lambda^k) = F(x^k,u^k) + < \lambda^k,u^k > + \frac{r}{2}\left\|u^k\right\|^2.$$

Let us put

$$q(\lambda) = F(x^k,u^k) + < \lambda,u^k > + \frac{r}{2}\left\|u^k\right\|^2.$$

We notice that

$$q(\lambda) \geq L_r(x^k,\lambda)\ \ \forall\lambda, \text{ and } q(\lambda^k) = L_r(x^k,\lambda^k),$$

thus

$$\nabla q(\lambda^k) = \nabla_\lambda L_r(x^k,\lambda^k).$$

Then,

$$u^k = \nabla_\lambda L_r(x^k,\lambda^k).$$

We have by hypothesis

$$L_r(x^k,\lambda^k) - d_r(\lambda^k) \leq \varepsilon_k$$

what implies that

$$\lim_k L_r(x^k,\lambda^k) = \lim_k d_r(\lambda^k) = Sup d_r.$$

According to the Lemma 2.13 and Lemma 2.14, we have

$$\left.\begin{array}{l} \lim_k \nabla d_r(\lambda^k) = 0 \\ \\ \lim_k \frac{r}{2}\left\|\nabla d_r(\lambda^k) - \nabla_\lambda L_r(x^k,\lambda^k)\right\|^2 = 0 \end{array}\right\} \Longrightarrow \lim_k \nabla d_r(\lambda^k) = \lim_k \nabla_\lambda L_r(x^k,\lambda^k) = 0.$$

Then, $\lim_k u^k = 0$.

The sequence $\left\{\lambda^k\right\}_k$ being bounded, then

$$F(x^k,u^k) = L_r(x^k,\lambda^k) - <\lambda^k,u^k> -\tfrac{r}{2}\left\|u^k\right\|^2$$

$$\Longrightarrow \lim_k F(x^k,u^k) = \lim_k (L_r(x^k,\lambda^k) - <\lambda^k,u^k> -\tfrac{r}{2}\left\|u^k\right\|^2) = Sup d_r.$$

We always have $d_r(\lambda) \leq f(x), \ \forall\lambda, \ \forall x,$ thus

$$\lim_k F(x^k,u^k) = Sup d_r(\lambda) \leq \alpha.$$

On the other hand,

$$\lim_k F(x^k,u^k) = \lim_k f(x^k) \ \text{ with } \lim_k g_i(x^k) \leq 0 \quad (i = 1,...,m).$$

Then

$$\lim_k f(x^k) = Sup d_r(\lambda) \leq \alpha \ \text{ with } \lim_k g_i(x^k) \leq 0 \quad (i = 1,...,m).$$

It holds $\lim_k f(x^k) = \alpha$. Consequently $\left\{x^k\right\}_k$ is an asymtotically minimizing sequence of $(\mathcal{P})$. □

2.3 Study of the Convergence

We are going to give an algorithm of primal-dual type, where we show that sequences $\left\{\lambda^k\right\}_k$ and $\left\{x^k\right\}_k$ generated by this algorithm converge globally, with at least the Slater condition, to $\overline{\lambda}$ and $\overline{x}$.

The algorithm to be studied depends on the initial choice of $r_0 > 0$, $\lambda^0 \in \mathbb{R}^m$ and the sequence $\{\varepsilon_k\}_k$ with

$$\varepsilon_k \geq 0 \ \text{ and } \ \lim_k \varepsilon_k = 0.$$

Algorithm 3:

Step 0: (initialization) $(k = 0)$

Choose a factor of penalty $r_k > 0$, a precision $\delta > 0$, a multiplier λ^0 and a sequence $\{\varepsilon_k\}_k$ with $\varepsilon_k \geq 0$ and $\lim_k \varepsilon_k = 0$

Step 1: $(k \geq 0)$

Find x^k such that

$$L_{r_k}(x^k,\lambda^k) - d_{r_k}(\lambda^k) \leq \varepsilon_k.$$

Step 2:

Define

$$\lambda_i^{k+1} = \max\left\{\lambda_i^k + r_k g_i(x^k), 0\right\};$$

or

$$\lambda^{k+1} = \lambda^k + r_k \nabla_\lambda L_{r_k}(x^k,\lambda^k).$$

Step 3:

If

$$\left\|\nabla_\lambda L_{r_k}(x^k,\lambda^k)\right\| \leq \delta \tag{2.20}$$

Stop and sets x^k as solution of $(\mathcal{P})$.

Else, $r_{k+1} \geq r_k$ (if need be) return to the step 1.

Lemma 2.11. *([2]) Suppose that the sequence $\left\{\lambda^k\right\}_k$ is bounded (bounded by M), then the expression* (2.20) *implies*

$$f(\overline{x}) \geq f(x^k) - \sigma_k,$$

where

$$\sigma_k = \delta(M + (2\varepsilon_k + \frac{3r_k\,\delta}{2})) + \varepsilon_k.$$

Proof. Let $\overline{x}$ be a solution of $(\mathcal{P})$. From the formula (2.17), we have

$$d_{r_k}(\lambda^{k+1}) \geq d_{r_k}(\lambda^k) + < \lambda^{k+1} - \lambda^k, \nabla_\lambda d_{r_k}(\lambda^k) > -\frac{1}{2r_k}\left\|\lambda^{k+1} - \lambda^k\right\|^2.$$

Thus

$$f(\overline{x}) \geq d_{r_k}(\lambda^{k+1})$$

$$\geq d_{r_k}(\lambda^k) - \left\|\lambda^{k+1} - \lambda^k\right\|\left\|\nabla_\lambda d_{r_k}(\lambda^k)\right\| - \tfrac{1}{2r_k}\left\|\lambda^{k+1} - \lambda^k\right\|^2.$$

According to the step 2 and the step 3 of the Algorithm 3, we have

$$f(\overline{x}) \geq d_{r_k}(\lambda^k) - r_k\left\|\nabla_\lambda L_r(x^k, \lambda^k)\right\|\left\|\nabla_\lambda d_{r_k}(\lambda^k)\right\| - \tfrac{r_k}{2}\left\|\nabla_\lambda L_r(x^k, \lambda^k)\right\|^2$$

$$\geq d_{r_k}(\lambda^k) - r_k\delta\left\|\nabla_\lambda d_{r_k}(\lambda^k)\right\| - \tfrac{r_k}{2}\delta^2.$$

From the Lemma 2.14, we have

$$\frac{r_k}{2}\left\|\nabla d_r(\lambda^k)\right\| - \frac{r_k}{2}\left\|\nabla_\lambda L_r(x^k, \lambda^k)\right\| \leq \frac{r_k}{2}\left\|\nabla_\lambda L_r(x^k, \lambda^k) - \nabla d_r(\lambda^k)\right\|^2 \leq \varepsilon_k.$$

What implies that

$$\left\|\nabla d_r(\lambda^k)\right\| \leq \frac{2\varepsilon_k}{r_k} + \delta \Longrightarrow -\left\|\nabla_\lambda d_{r_k}(\lambda^k)\right\| \geq -(\frac{2\varepsilon_k}{r_k} + \delta).$$

It results that

$$f(\overline{x}) \geq d_{r_k}(\lambda^k) - r_k\delta(\tfrac{2\varepsilon_k}{r_k} + \delta) - \tfrac{r_k}{2}\delta^2$$

$$= d_{r_k}(\lambda^k) - \delta(2\varepsilon_k + \tfrac{3r_k}{2}\delta).$$

On the other hand, according to the step 1 of the same Algorithm, we have

$$d_{r_k}(\lambda^k) \geq L_{r_k}(x^k, \lambda^k) - \varepsilon_k.$$

Then

$$d_{r_k}(\lambda^k) \geq f(x^k) + \tfrac{1}{2r_k}\sum_{i=1}^m (\Psi^+(\lambda_i^k + r_k g_i(x^k))^2 - (\lambda_i^k)^2) - \varepsilon_k$$

$$\geq f(x^k) + \tfrac{1}{2r_k}\sum_{i=1}^m ((\lambda_i^{k+1})^2 - (\lambda_i^k)^2) - \varepsilon_k$$

$$= f(x^k) + \tfrac{1}{2r_k}\sum_{i=1}^m (\lambda_i^{k+1} - \lambda_i^k)(\lambda_i^{k+1} + \lambda_i^k) - \varepsilon_k$$

namely,

$$d_{r_k}(\lambda^k) \geq f(x^k) + \tfrac{1}{2r_k}\sum_{i=1}^m r_k \tfrac{\partial L(x^k,\lambda^k)}{\partial \lambda_i}(\lambda_i^{k+1} + \lambda_i^k) - \varepsilon_k$$

$$= f(x^k) + \tfrac{1}{2}\sum_{i=1}^m \tfrac{\partial L(x^k,\lambda^k)}{\partial \lambda_i}(\lambda_i^{k+1} + \lambda_i^k) - \varepsilon_k$$

$$= f(x^k) + \tfrac{1}{2} < \nabla_\lambda L_r(x^k, \lambda^k), \lambda^{k+1} + \lambda^k > -\varepsilon_k.$$

Thus

$$d_{r_k}(\lambda^k) \geq f(x^k) - \tfrac{1}{2}\left\|\nabla_\lambda L_r(x^k, \lambda^k)\right\|\left\|\lambda^{k+1} + \lambda^k\right\| - \varepsilon_k$$

$$\geq f(x^k) - \tfrac{\delta}{2}\left\|\lambda^{k+1} + \lambda^k\right\| - \varepsilon_k.$$

Finally, we have

$$f(\overline{x}) \geq d_{r_k}(\lambda^k) - \delta(2\varepsilon_k + \tfrac{3r_k}{2}\delta)$$

$$\geq f(x^k) - \tfrac{\delta}{2}\left\|\lambda^{k+1} + \lambda^k\right\| - \varepsilon_k - \delta(2\varepsilon_k + \tfrac{3r_k}{2}\delta),$$

it holds that

$$f(\overline{x}) \geq f(x^k) - \delta(M + (2\varepsilon_k + \frac{3r_k}{2}\delta)) - \varepsilon_k.$$

□

The general result is given by the following theorem :

Theorem 2.4. *Let us suppose that $(\mathcal{P})$ possesses a K-T vector and that*

$$\sum_{k\geq 1} \sqrt{\varepsilon_k} < +\infty \tag{2.21}$$

Then, the following properties are satisfied :
($\mathbf{a}$) *the sequence $\left\{\lambda^k\right\}_k$ is bounded, and its cluster values are K-T vectors ;*
($\mathbf{b}$) *the sequence $\left\{x^k\right\}_k$ is an asymtotically minimizing of $(\mathcal{P})$.*

Proof. ($\mathbf{a}$) According to the Lemma 2.14 and by hypothesis (step 1), we have

$$\frac{r_k}{2}\left\|\nabla_\lambda L_{r_k}(x^k, \lambda^k) - \nabla d_{r_k}(x^k)\right\| \leq \varepsilon_k.$$

According ([3], remark 2.2), we have

$$\nabla d_{r_k}(\lambda^k) = \frac{1}{r_k}(z_{\lambda^k} - \lambda^k)$$

where z_{λ^k} realizes the Sup in the definition of d_{r_k}. But

$$\lambda^{k+1} = \lambda^k + r_k \nabla_\lambda L_{r_k}(x^k, \lambda^k).$$

From which it holds

$$\nabla_\lambda L_{r_k}(x^k, \lambda^k) = \frac{1}{r_k}(\lambda^{k+1} - \lambda^k).$$

Then

$$\frac{r_k}{2}\left\|\nabla_\lambda L_{r_k}(x^k, \lambda^k) - \nabla d_{r_k}(x^k)\right\|^2 = \frac{r_k}{2}\left\|\frac{1}{r_k}(\lambda^{k+1} - \lambda^k) - \frac{1}{r_k}(z_{\lambda^k} - \lambda^k)\right\|^2 \leq \varepsilon_k$$

Namely

$$\frac{1}{2r_k}\left\|\lambda^{k+1} - z_{\lambda^k}\right\|^2 \leq \varepsilon_k.$$

Taking the limit on k we find

$$\lim_k (\lambda^{k+1} - z_{\lambda^k}) = 0 \tag{2.22}$$

Consider the application $Prox$ defined by

$$z \longrightarrow Prox(z) = h(z) + \frac{1}{2}\|z - \lambda\|^2$$

where h is a convex function. Let us put

$$Prox(h; \lambda) = \arg\min_z \left\{h(z) + \frac{1}{2}\|z - \lambda\|^2\right\}.$$

We have, according ([5], Theo.31.5, p. 340),

$$\|Prox(h; u) - Prox(h; \lambda)\| \leq \|u - \lambda\|.$$

Let us put

$$h(z) = -r_k d(z)$$

(h is convex), then

$$Prox(h;\lambda) = \underset{z\in\mathbb{R}^m}{\arg\min}\left\{h(z) + \tfrac{1}{2}\|z-\lambda\|^2\right\} = \underset{z\in\mathbb{R}^m}{\arg\min}\left\{-r_k d(z) + \tfrac{1}{2}\|z-\lambda\|^2\right\}$$

$$= -r_k \underset{z\in\mathbb{R}^m}{\arg\min}\left\{d(z) - \tfrac{1}{2r_k}\|z-\lambda\|^2\right\} = -r_k z_\lambda.$$

It holds that

$$\|Prox(h;u) - Prox(h;\lambda)\| = \|-r_k z_u + r_k z_\lambda\|$$

$$\Longrightarrow r_k\|z_u - z_\lambda\| \le \|u-\lambda\| \Longrightarrow \|z_u - z_\lambda\| \le \tfrac{1}{r_k}\|u-\lambda\|.$$

Let $\overline{\lambda}$ be any K-T vector, then

$$\nabla d_{r_k}(\overline{\lambda}) = 0 \Longrightarrow r_k \nabla d_{r_k}(\overline{\lambda}) = 0 \Longrightarrow z_{\overline{\lambda}} = \overline{\lambda} + r_k \nabla d_{r_k}(\overline{\lambda}) = \overline{\lambda}.$$

Thus

$$\left\|z_{\lambda^{k+1}} - \overline{\lambda}\right\| = \left\|z_{\lambda^{k+1}} - z_{\overline{\lambda}}\right\| \le \frac{1}{r_k}\left\|\lambda^{k+1} - \overline{\lambda}\right\|.$$

Using the previous expressions, we shall have

$$\left\|\lambda^{k+1} - \overline{\lambda}\right\| = \left\|\lambda^{k+1} - z_{\lambda^k} + z_{\lambda^k} - \overline{\lambda}\right\| \le \left\|\lambda^{k+1} - z_{\lambda^k}\right\| + \left\|z_{\lambda^k} - \overline{\lambda}\right\|$$

$$\le \sqrt{2r_k\varepsilon_k} + \tfrac{1}{r_k}\left\|\lambda^k - \overline{\lambda}\right\|.$$

In particular

$$\left\|\lambda^{k+1} - \overline{\lambda}\right\| \le \Phi(r_k, \varepsilon_k) < +\infty.$$

Hence, $\left\{\lambda^k\right\}_k$ is a bounded sequence.

Let $\{\lambda^s\}_s$ be a convergent subsequence to $\overline{\lambda}$, according to the expression (2.21), we have

$$\lim_s(\lambda^{s+1} - z_{\lambda^s}) = 0.$$

We know that

$$z_{\lambda^s} = \lambda^s + r_k \nabla d_{r_k}(\lambda^s)$$

$$\Longrightarrow \lim_s(\lambda^{s+1} - \lambda^s - r_k \nabla d_{r_k}(\lambda^s)) = 0 \Longrightarrow \lim_s \nabla d_{r_k}(\lambda^s) = \nabla d_{r_k}(\overline{\lambda}) = 0.$$

As d_{r_k} is concave, then $\overline{\lambda}$ maximizes d_{r_k}, namely, $\overline{\lambda}$ is a K-T vector.

(**b**) According to the Theorem 2.11, $\{x^s\}_s$ is an asymtotically minimizing sequence of $(\mathcal{P})$. □

2.4 Numerical Experiments

In this paragraph, we propose some numerical experiments illustrating the methods of nondifferentiable convex programming problems that we had studied above and in ([3]). We established a comparative study with the results of ([3]).

Let us call back that the previous methods consist in solving a sequence of unconstrained problems. Every problem of which must be solved by the **Algorithm 4** of ([3]) by making the linear search given by the expression (20) in ([3]).

Example 2.1. *Consider the following mathematical programming problem :*

$$(\mathcal{P}) \quad \begin{cases} \alpha := Inf\left\{f(x) = \max_{i=1}^{3}(x^t A_i x + b_i^t x + c_i)\right\} \\ \quad subject\ to \quad x_1^2 + 3x_2 + 2x_1 \le 0, \end{cases}$$

where

$$A_1 = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad c_1 = 4;$$

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad c_2 = -5;$$

$$A_3 = \begin{pmatrix} 2.5 & 2 \\ 0.5 & 2 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \quad c_3 = 3;$$

x^0 *initial*	k	k *total*	x_k	$f(x_k)$	r_k	ε_k	$r_k h(x^k)$	$s_k = \|g_k\| \left\|x^{k+1} - x^k\right\|$	*time* *s*
$(2,0)$	4	196	$(-0.391, 0.210)$	3.49	10^4	10^{-4}	$10^{-8}6.0$	$10^{-7}2.0$	0.17
$(4,3)$	6	268	$(-0.460, 0.236)$	3.49	10^6	10^{-6}	10^{-8}	$10^{-2}2.0$	0.22
$(-2,1)$	5	379	$(-0.404, 0.215)$	3.48	10^5	10^{-5}	$10^{-9}2.0$	$10^{-8}7.0$	0.28

Table 1
"-Proximal Penalty method : $(\delta = 10^{-6})$

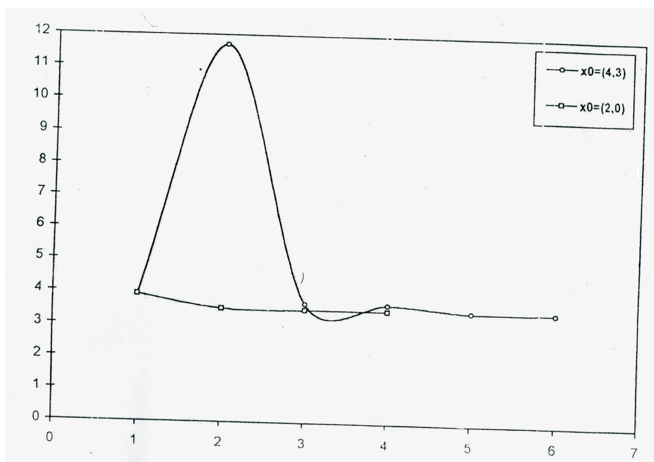

Figure 1: The objective function value at each step

λ_0 *initial*	k	k *total*	x_k	$f(x_k)$	r_k	ε_k	$s = \left\|\nabla L_{r_k}(x_k, \lambda_k)\right\|$	*time* *s*
5	14	79	$(-0.402, 0.214)$	3.49	14	10^{-14}	$10^{-5}5.0$	0.11
0.5	7	31	$(-0.402, 0.214)$	3.49	7	10^{-7}	$10^{-5}7.0$	0.06
12	15	90	$(-0.402, 0.214)$	3.49	15	10^{-15}	$10^{-5}6.0$	0.11
-1	8	36	$(-0.402, 0.214)$	3.49	8	10^{-8}	$10^{-5}5.0$	0.05
-8	2	13	$(-0.402, 0.214)$	3.49	2	10^{-2}	0.0	0.06

Table 2
augmented Lagrangian method : $(\delta = 10^{-4})$

-

Example 2.2. *Consider the following mathematical programming problem :*

$$(\mathcal{P}) \qquad \begin{cases} \alpha := Inf f(x) = \max(2x+2,\ (x+1)^2,\ x^2+1) \\ \quad subject\ to \quad 2x+3 \leq 0. \end{cases}$$

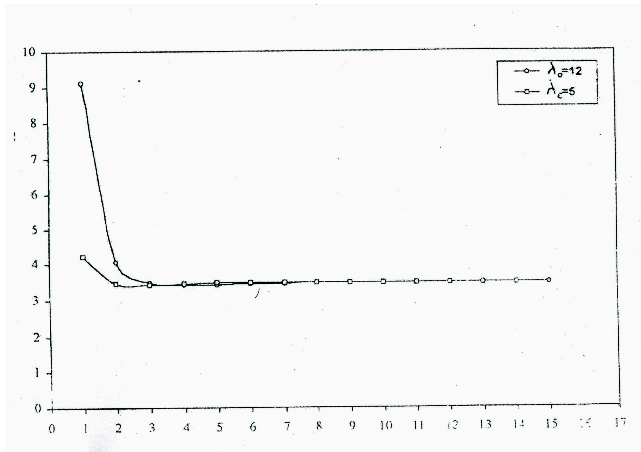

Figure 2: The objective function value at each step

x^0 *initial*	k	k *total*	x_k	$f(x_k)$	r_k	ε_k	$r_k h(x^k)$	$s_k = \lVert g_k \rVert \left\lVert x^{k+1} - x^k \right\rVert$	*time* s
5	4	15	-1.5	3.25	10^4	10^{-4}	$10^{-5}5.6$	$10^{-12}5.5$	0.06
62	7	26	-1.5	3.25	10^7	10^{-7}	$10^{-8}5.6$	$10^{-12}5.5$	0.06
-412	4	15	-1.5	3.25	10^4	10^{-4}	$10^{-5}5.6$	$10^{-12}5.5$	0.05

Table 3

ε-***Proximal Penalty method*** : $(\delta = 10^{-11})$

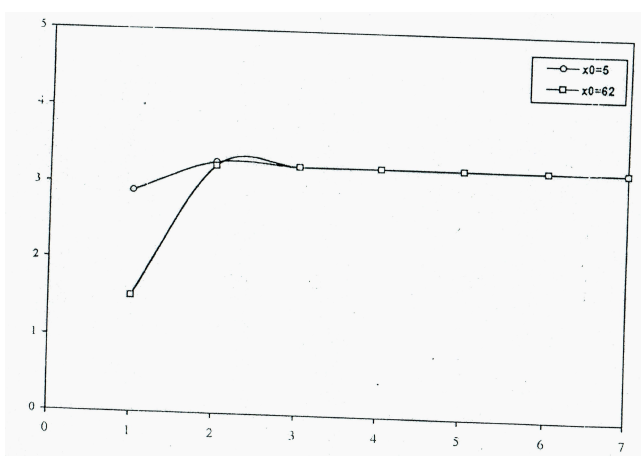

Figure 3: The objective function value at each step

λ_0 initial	k	k total	x_k	$f(x_k)$	r_k	ε_k	$s = \|\nabla L_{r_k}(x_k, \lambda_k)\|$	time s
2	16	47	-1.5	3.25	16	10^{-16}	$10^{-7}7.1$	0.06
20	21	68	-1.5	3.25	21	10^{-21}	$10^{-7}9.1$	0.05
35	22	72	-1.5	3.25	22	10^{-22}	$10^{-7}7.9$	0.05
-1	19	65	-1.5	3.25	19	10^{-19}	$10^{-7}5$	0.06
-5	19	67	-1.5	3.25	19	10^{-19}	$10^{-7}5.3$	0.05

Table 4
augmented Lagrangian method *:* $(\delta = 10^{-6})$

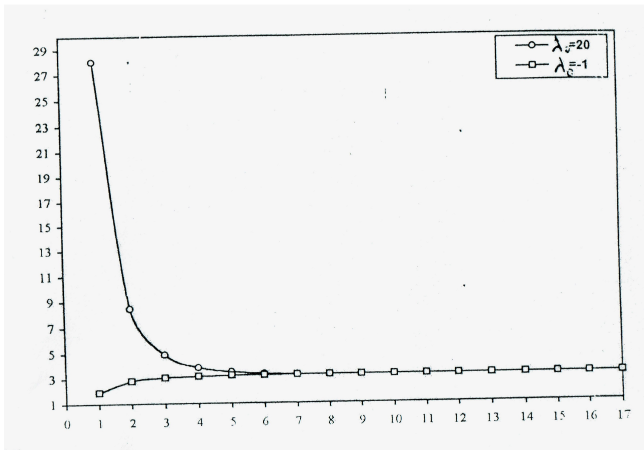

Figure 4: The objective function value at each step

Example 2.3. *Consider the following mathematical programming problem :*

$$(\mathcal{P}) \qquad \begin{cases} \alpha := Inf\,\{f(x) = \max(f_1(x), f_2(x))\} \\ \quad \textit{subject to} \begin{cases} x_1 + 2x_2 \le 0 \\ x_2 + 1 \le 0 \end{cases}, \end{cases}$$

where

$$f_1(x) = x_1^2 + x_2^2 - x_2 - x_1 - 1,$$

$$f_2(x) = 3x_1^2 + 2x_2^2 + 2x_1x_2 - 16x_1 - 14x_2 + 22$$

x^0 initial	k	k total	x_k	$f(x_k)$	r_k	ε_k	$r_k h(x^k)$	$s_k = \|g_k\| \left\|x^{k+1} - x^k\right\|$	time s
$(2,0)$	6	20	$(2,-1)$	14	10^6	10^{-6}	$10^{-6}9.0$	$10^{-9}5.0$	0.05
$(-4,3)$	6	24	$(2,-1)$	14	10^6	10^{-6}	$10^{-6}9.0$	$10^{-9}3.0$	0.05
$(6,-7)$	6	24	$(2,-1)$	14	10^6	10^{-6}	$10^{-6}9.0$	$10^{-9}3.0$	0.06

Table 5
ε- ***Proximal Penalty method*** : $(\delta = 10^{-8})$

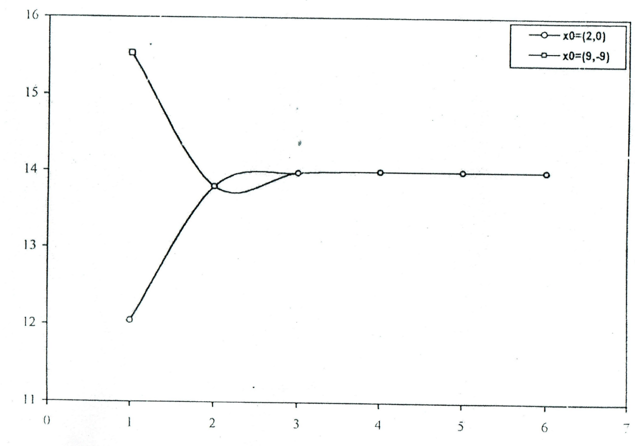

Figure 5: The objective function value at each step

λ_0 initial	k	k total	x_k	$f(x_k)$	r_k	ε_k	$s = \|\nabla L_{r_k}(x_k, \lambda_k)\|$	time s
$(2,6)$	22	280	$(2,-1)$	14	22	10^{-22}	$10^{-5}6.0$	0.16
$(3,0)$	20	220	$(2,-1)$	14	20	10^{-22}	$10^{-5}7.0$	0.11
$(5,3)$	18	185	$(2,-1)$	14	18	10^{18}	$10^{-5}7.0$	0.11
$(-5,-1)$	18	192	$(2,-1)$	14	18	10^{18}	$10^{-5}6.0$	0.11
$(-1,0)$	17	170	$(2,-1)$	14	17	10^{-17}	$10^{-5}8.0$	0.11

Table 6
augmented Lagrangian method : $(\delta = 10^{-4})$

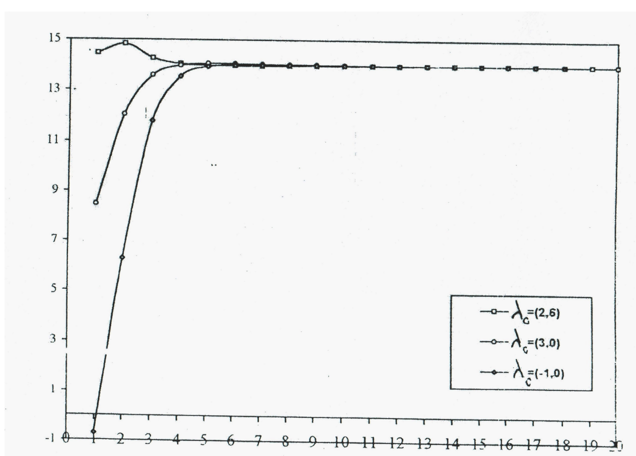

Figure 6: The objective function value at each step

Example 2.4. *Consider the following mathematical programming problem :*

$$(\mathcal{P}) \qquad \begin{cases} \alpha := Inf\,\{f(x) = \max(f_1(x), f_2(x), f_3(x))\} \\ \qquad subject\ to \quad \begin{cases} x_1 - x_2 + 1 \le 0 \\ 2x_2 - 1 \le 0 \end{cases} \end{cases},$$

where

$$f_1(x) = x_1^2 + x_2^2,$$

$$f_2(x) = (x_1 + x_2)^2$$

$$f_3(x) = (2x_1 + 3x_2)^2$$

x^0 initial	k	k total	x_k	$f(x_k)$	r_k	ε_k	$r_k h(x^k)$	$s_k = \|g_k\| \left\|x^{k+1} - x^k\right\|$	time s
$(3,2)$	6	35	$(-0.5, 0.5)$	0.5	10^6	10^{-6}	$10^{-7}6.7$	$10^{-7}5.7$	0.06
$(5,4)$	6	33	$(-0.5, 0.5)$	0.5	10^6	10^{-6}	$10^{-7}9.1$	$10^{-7}9.8$	0.05
$(-2,-4)$	6	27	$(-0.5, 05)$	0.5	10^6	10^{-6}	10^{-6}	$10^{-6}1.2$	0.05

Table 7
ε-***Proximal Penalty method*** : $(\delta = 10^{-5})$

λ_0 initial	k	k total	x_k	$f(x_k)$	r_k	ε_k	$s = \left\|\nabla L_{r_k}(x_k, \lambda_k)\right\|$	time s
$(3,1)$	11	119	$(-0.5, 0.5)$	0.5	11	10^{-11}	$10^{-5}7$	0.11
$(4,3)$	10	100	$(-0.5, 0.5)$	0.5	10	10^{-10}	$10^{-5}3$	0.11
$(2,5)$	10	134	$(-0.5, 0.5)$	0.5	10	10^{-10}	$10^{-5}8$	0.11
$(-1,0)$	11	126	$(-0.5, 0.5)$	0.5	11	10^{-11}	$10^{-5}3$	0.11
$(-2,-4)$	12	140	$(-0.5, 0.5)$	0.5	12	10^{-12}	$10^{-5}2$	0.11

Table 8
augmented Lagrangian method : $(\delta = 10^{-4})$

Example 2.5. *Consider the following mathematical programming problem :*

$$(\mathcal{P}) \qquad \begin{cases} \alpha := Inf \left\{ f(x) = \max_{i=1}^{3} (x^t A_i x + b_i^t x + c_i) \right\} \\ \qquad subject\ to \quad \begin{cases} x_1 + x_3 \leq 0 \\ 2x_1 + 1 \leq 0 \end{cases}, \end{cases}$$

where

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad c_1 = 0;$$

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad c_2 = -2;$$

$$A_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad c_3 = 2;$$

x^0 initial	k	k total	x_k	$f(x_k)$	r_k	ε_k	$r_k h(x^k)$	$s_k = \|g_k\| \left\|x^{k+1} - x^k\right\|$	time s
$(1,2,4)$	8	36	$(0,-0.5,0)$	2.25	10^9	10^{-9}	$10^{-11}6.3$	$10^{-10}3.4$	0.11
$(2,8,0)$	3	19	$(0,-0.5,0)$	2.25	10^4	10^{-4}	$10^{-6}6.3$	$10^{-10}1.4$	0.06
$(-2,-1,5)$	9	37	$(0,-0.5,0)$	2.25	10^{10}	10^{-10}	$10^{-10}1.4$	$10^{-13}6.9$	0.11

Table 9
ε-***Proximal Penalty method*** : $(\delta = 10^{-9})$

λ_0 initial	k	k total	x_k	$f(x_k)$	r_k	ε_k	$s = \|\nabla L_{r_k}(x_k, \lambda_k)\|$	time s
$(3, 2)$	9	57	$(0, -0.5, 0)$	2.25	9	10^{-9}	$10^{-6}3.4$	0.06
$(19, 2.58)$	9	139	$(0, -0.5, 0)$	2.25	9	10^{-9}	$10^{-6}1.8$	0.22
$(4, 6)$	9	69	$(0, -0.5, 0)$	2.25	9	10^{-9}	$10^{-6}6.9$	0.11
$(-1, -4)$	9	59	$(0, -0.5, 0)$	2.25	9	10^{-9}	$10^{-6}4.1$	0.11

Table 10
augmented Lagrangian method : $(\delta = 10^{-5})$

Example 2.6. *Consider the following mathematical programming problem :*

$$(\mathcal{P}) \qquad \begin{cases} \alpha := Inf \left\{ f(x) = \begin{cases} -x + |x| + e^{|x|} & if \;\; x \leq 0 \\ x^2 + |x| + e^{|x|} & else \end{cases} \right\} \\ \qquad \qquad subject\ to \quad x + 1 \leq 0. \end{cases}$$

x^0 initial	k	k total	x_k	$f(x_k)$	r_k	ε_k	$r_k h(x^k)$	$s_k = \|g_k\| \|x^{k+1} - x^k\|$	time s
1	11	33	-1	4.718	10^{11}	10^{-11}	$10^{-11}5.6$	$10^{-12}4.3$	0.05
-2	5	14	-1	4.718	10^5	10^{-5}	$10^{-5}5.6$	$10^{-12}4.3$	0.06
-1.5	5	14	-1	4.718	10^5	10^{-5}	$10^{-5}5.6$	$10^{-12}4.3$	0.05

Table 11
ε-Proximal Penalty method : $(\delta = 10^{-11})$

λ_0 initial	k	k total	x_k	$f(x_k)$	r_k	ε_k	$s = \|\nabla L_{r_k}(x_k, \lambda_k)\|$	time s
3	21	63	-1	4.718	21	10^{-21}	$10^{-7}6.5$	0.06
5	19	57	-1	4.718	19	10^{-19}	$10^{-7}6.9$	0.06
9	23	83	-1	4.718	23	10^{-23}	$10^{-7}5.3$	0.06
1.5	22	77	-1	4.718	22	10^{-22}	$10^{-7}6.3$	0.05
0.6	22	80	-1	4.718	22	10^{-22}	$10^{-7}8.3$	0.06

Table 12
augmented Lagrangian method : $(\delta = 10^{-6})$

2.5 Comments and Conclusions

Basing itself on the results obtained in the previous numerical experiments, we can make the following remarks :

1) for the ε-proximal penalty methods, we used the classical penalty functions :

$$h(x) = \sum_{i=1}^{m} (g_i(x))^2$$

and the sequence $(r_k)_k$ such that $r_{k+1} = 10 r_k$;

2) for the augmented Lagrangian method, we use the sequence $(r_k)_k$ such that

$$r_{k+1} = r_k + 1.$$

and for the sequence $(\varepsilon_k)_k$, we make it decrease in the following way :

$$\varepsilon_{k+1} = \frac{\varepsilon_k}{10}.$$

Generally, the obtained solutions are enough precise.

The number of iterations depends, on one hand of the algorithm used to solve the unconstrained subproblems, on the other hand on initial points.

The two previous approaches possess the property of the global convergence.

From a theoretical point of view, both approaches use the proximal regularization. The first one makes the regularity for the subproblems, the other one for the dual function associated with the ordinary Lagrangian. So the idea to return the resolution of primal problem to a sequence of auxiliary problems.

The algorithm that we had used requiet the knowledge at least of a subgradient in every step, and the value of the function to be minimized, then a difficulty concerning the determination of a subgradient which is, generally, difficult in practice.

From point of comparative view, we notice according to the previous numerical experiments that number of necessary iterations to obtain a minimum in the augmented Lagrangian method is higher than counts it of iterations in the ε-proximal penalty method. As well as the run time.

We also notice that the penalty factor is too much large in the ε-proximal penalty method, and enough small in the augmented Lagrangian method.

The stop test in the augmented Lagrangian method is more successful than the stop test in the ε-proximal penalty method.

2.6 General Conclusions

The ε-proximal penalty method is a method of nondifferentiable optimization. It is a member of algorithms whose the generated sequences are asymptotically minimizing. Thus, it is the technique which puts in connection the classical optimization and the asymptotic analysis.

It has advantages for the perturbed problems and in fluid mechanics.

From theoretical point of view, we think that this technique will be widened in problems of positive semidefinite optimization. Thing still is not made and raises open problems in this direction.

The augmented Lagrangien method is a well known technique by its efficiency in the theoretical and practical cases. It applies to differentiable and nondifferentiable optimization problems.

This technique will be widened in positive semidefinite optimization problems with large-sized matrices, thing still is not made and raises open problems still.

3 Acknowledgement

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References

[1] N. Daili, Some augmented Lagrangian methods applied to nondifferentiable optimization problems, J. Inf. Optim. Sci. 33 (2012), no. 4-5, 487-526. MR3060105.

[2] N. Daili, Applications de Lagrangien Augmenté à des problèmes de programmation convexe non nécessairement différentiables, Pub. 455/P-261, IRMA.- CNRS, Strasbourg *I*, France, 1986.

[3] N. Daili and Kh. Saadi, ε-Proximal Point Algorithms for Nondifferentiable Convex Optimization and Applications, Adv. Model. Optim. 14 (2012), no. 1, 175-195. MR2910314

[4] A. Gonen and M. Avriel, Duality in Nonlinear Programs using Augmented Lagrangian Function, J. Math. Analys. Appl. 121 (1987), pp. 39-56.

[5] R.T. Rockafellar, Convex analysis, Princeton Univ. Press, Princeton, 1970.

[6] R.T. Rockafellar, Augmented Lagrangians and applications of the proximal point algorithm in convex programming, Math. Oper. Resea. 1(1976), no. 2, pp. 97-116.

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On The Cohen p-Nuclear Positive Sublinear Operators

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Abstract

In the present paper, we will introduce the concept of Cohen p-nuclear positive sublinear operators. We give an analogue to "Pietsch's domination theorem" and we study some properties concerning this notion.

Keywords: Cohen p-nuclear operators, Pietsch's domination theorem, Strongly p-summing operators, Positive operator, Sublinear operators.

2010 MSC: 46B40, 46B42, 47B60, 47B65.

1 Introduction

For a Banach space X, X^* will denote its topological dual and B_X will denote its closed unit ball. For a Banach lattice E, E^+ will denote its positive cone. Throughout the paper, X, Y will be Banach spaces and E, F will be Banach lattices. Let $\mathcal{L}(X;Y)$ denote the Banach space of all continuous linear operators from X to Y. For $1 \leq p < \infty$, let p^* be its conjugate, that is, $1/p + 1/p^* = 1$.

The notion of Cohen p-nuclear operators $(1 \leq p \leq \infty)$ was initiated by Cohen in [9]. A linear operator u between two Banach spaces X, Y is Cohen p-nuclear for $(1 < p < \infty)$ if there is a positive constant C such that for all $n \in \mathbb{N}; x_1, ..., x_n \in X$ and $y_1, ..., y_n \in Y$ we have

$$\left|\sum_{i=1}^{n} \langle u(x_i), y_i^* \rangle\right| \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |x_i(x^*)|^p \, d\mu_1(x^*)\right)^{\frac{1}{p}} \times$$
$$\times \sup_{y \in B_Y} \left(\sum_{i=1}^{n} |y_i^*(y)|^{p^*} \, d\mu_2(y)\right)^{\frac{1}{p^*}}.$$

The smallest constant C which is noted by $n_p(u)$, such that the above inequality holds, is called the Cohen p-nuclear norm on the space $\mathcal{N}_p(X, Y)$ of all Cohen p-nuclear operators from X into Y which is a Banach space. We have $\mathcal{N}_1(X, Y) = \Pi_1(X, Y)$ (the Banach space of all 1-summing operators) and $\mathcal{N}_\infty(X, Y) = \mathcal{D}_\infty(X, Y)$ (the Banach space of all strongly ∞-summing operators).

In [9, Theorem 2.3.2], Cohen proves that, if u verifies a domination theorem then u is p-nuclear and he asked if the statement of this theorem characterizes p-nuclear operators. In [6], Achour et al. generalized this notion to the sublinear operators and they gave an analogue to "Pietsch's domination theorem" for this category of operators. Motivated by that, we study this notion with the positive sublinear maps and we propose, among others, an analogue to "Pietsch's domination theorem" for this category of operators which is one of the main results of this paper and we also discuss some properties concerning this class. It remains to prove the Pietsch's factorization theorem.

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This paper is organized as follows: In the first section, we give some basic definitions and terminology concerning Banach lattices. We also recall some standard notations. In the second section, we present some definitions and properties concerning positive sublinear operators. We give the definition of positive p-summing operators introduced by Blasco [7, 8] and we present the notion of strongly p-summing sublinear operators initiated in [6]. In Section 3, we generalize the class of Cohen p-nuclear operators to the positive sublinear operators. This category verifies a domination theorem, which is the principal result. We used another Technics than the Ky Fan's lemma. We end in Section 4, by studying a relation between some classes of positive sublinear operators (p-nuclear and p-summing).

2 Preliminary

We start by recalling the abstract definition of Banach lattices. Let E be a Banach space. If E is a vector lattice and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ we say that E is a Banach lattice. If the lattice is complete, we say that E is a complete Banach lattice and for all x in E, $\|x\| = \||x|\|$. The dual E^* of a Banach lattice E is a complete endowed with the natural order $x_1, x_2 \in E$

$$x_1^* \leq x_2^* \Longleftrightarrow \langle x_1^*, x \rangle \leq \langle x_2^*, x \rangle, \forall x \in E^+.$$

where $\langle ., . \rangle$ denotes the bracket of duality. If we consider E as a Sublattice of E^{**} we have for

$$x_1 \leq x_2 \Longleftrightarrow \langle x_1, x^* \rangle \leq \langle x_2, x^* \rangle, \forall x^* \in E^{*+}.$$

for more details on this, the interested reader can consult the references [11].

Given $1 \leq p < \infty$ we will write $\ell_p^n(X)$ for the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\left\|(x_i)_{i=1}^n\right\|_p = \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}},$$

and $\ell_p^{n,w}(X)$ for the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\left\|(x_i)_{i=1}^n\right\|_{p,w} = \sup_{\|\phi\|_{X^*} \leq 1} \left(\sum_{i=1}^n |\phi(x_i)|^p\right)^{\frac{1}{p}},$$

where X^* denotes the topological dual of X. The closed unit ball of X will be denoted by B_X. Let $\ell_p(X)$ be the Banach space of all absolutely p-summable sequences $(x_i)_{i=1}^\infty$ in X with the norm

$$\left\|(x_i)_{i=1}^\infty\right\|_p = \left(\sum_{i=1}^\infty \|x_i\|^p\right)^{\frac{1}{p}}.$$

We denote by $\ell_p^w(X)$ the Banach space of all weakly p-summable sequences $(x_i)_{i=1}^\infty$ in X with the norm

$$\left\|(x_i)_{i=1}^\infty\right\|_{p,w} = \sup_{\|\phi\|_{X^*} \leq 1} \left(\sum_{i=1}^\infty |\phi(x_i)|^p\right)^{\frac{1}{p}},$$

Note that $\ell_p^w(X) = \ell_p(X)$ for some $1 \leq p < \infty$ if, and only if, X is finite dimensional. We continue in specifying definitions of the convexity and the concavity.

Definition 2.1. *Let $1 \leq p \leq \infty$.*

(i) A sublinear operator $T : F \longrightarrow E$ is called a p-convex if there exists a constant C such that for every n in $\mathbb{N}$ the operators

$$\begin{array}{cccc} T_n : & \ell_p^n(F) & \longrightarrow & E\left(\ell_p^n\right) \\ & (x_1, ..., x_n) & \longmapsto & (T(x_1), ..., T(x_n)) \end{array}$$

are uniformly bounded by C.

(ii) *A sublinear operator $T : E \longrightarrow F$ is called a p-convex if there exists a constant C such that for every n in $\mathbb{N}$ the operators*

$$\begin{array}{llll} T_n: & E\left(\ell_p^n\right) & \longrightarrow & \ell_p^n\left(F\right) \\ & \left(x_1,...,x_n\right) & \longmapsto & \left(T\left(x_1\right),...,T\left(x_n\right)\right) \end{array}$$

are uniformly bounded by C.

The space E is p-convex (p-concave) if id_E is p-convex (p-concave).

3 Positive sublinear operators

We give in this section some elementary definitions and fundamental properties relative to positive sublinear operators, for example see [6].

Definition 3.1. *An operator T from X into F is said to be positive sublinear if we have for all x, y in X and λ in $\mathbb{R}_+$.*

$$\begin{array}{ll} i) & T\left(\lambda x\right) = \lambda T\left(x\right), \\ ii) & T\left(x+y\right) \leq T\left(x\right) + T\left(y\right), \\ iii) & T\left(x\right) \geq 0. \end{array}$$

Let us denote by

$$\mathcal{SL}^+\left(X,F\right) = \{\text{positive sublinear operators, } T : X \longrightarrow F\}.$$

A positive sublinear operator is continuous if, and only if, there is $C > 0$ such that for all $x \in X$, $\|T(x)\| \leq C\|x\|$. In this case, we said that T is bounded and we write

$$\|T\| = \sup_{x\in B_X} \|T(x)\|$$

and we put

$$\mathcal{SB}^+\left(X,F\right) = \{\text{bounded positive sublinear operators, } T : X \longrightarrow F\}.$$

Remark 3.2. If $u : X \longrightarrow F$ is a linear operator, then $|u|$ is a positive sublinear operator.

Proposition 3.3. *Let T be a symmetric sublinear operator between X and F. Then, T is positive.*

Proof. For every x in X

$$\begin{array}{lll} 0 & = & T\left(x-x\right) \\ & \leq & T\left(x\right) + T\left(-x\right) \\ & \leq & 2T\left(x\right). \quad \blacksquare \end{array}$$

Lemma 3.4. *Let $T : E \longrightarrow F$ be an increasing sublinear operator, if $|T|$ exist, then*

$$|T\left(x\right)| \leq |T|\left(|x|\right)$$

for all $x \in E$.

Proof. As $x \leq |x|$ and $-x \leq |x|$. Then by the monotonicity of T, we have

$$\forall x \in E, T\left(x\right) \leq T\left(|x|\right),$$

and

$$\forall x \in E, -T\left(x\right) \leq T\left(-x\right) \leq T\left(|x|\right),$$

and also

$$|T(x)| \leq T(|x|) \leq |T|(|x|)$$

for all $x \in E$. ■

Now, we study the continuity of an increasing positive sublinear operator. We adapt the same demonstration as in the linear case see [1, 12].

Theorem 3.5. *Let $T : E \longrightarrow F$ be an increasing positive sublinear operator. Then, T is continuous.*

Proof. We assume that T is not continuous. Then there exists a sequence $(x_n)_n$ in E with $\|x_n\| = 1$ such that $\|T(x_n)\| \geq n^3$ for all $n \in \mathbb{N}$. We have $|T(x_n)| \leq T(|x_n|)$, one can take $x_n \geq 0$ for all n. As $\sum\limits_{n\geq 1} \frac{\|x_n\|}{n^2} < \infty$ and E is complete, then the serie $\sum\limits_{n\geq 1} \frac{x_n}{n^2}$ converges in norm in E. Let $x = \sum\limits_{n\geq 1} \frac{x_n}{n^2}$. Then, it is clear that $0 \leq \frac{x_n}{n^2} \leq x$ for all n, and $T\left(\frac{x_n}{n^2}\right) \leq T(x)$ for all n, since T is increasing, we write $n \leq \left\|T\left(\frac{x_n}{n^2}\right)\right\| \leq \|T(x)\| < \infty$, for all n by the monotonicity of the norm of F, contradiction. Then T is continuous. ■

Remark 3.6. Without increase, we not know the answer. But we conjucture it's true.

Definition 3.7. *We said that a positive sublinear operator T between X, F is p-regular, $1 \leq p < \infty$, if there exist a constant $C > 0$ such that for all $(x_i)_1^n \subset X$, we have*

$$\left\|\left(\sum_{i=1}^{n} |T(x_i)|^p\right)^{\frac{1}{p}}\right\|_F \leq C \left\|\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}\right\|_X \tag{3.1}$$

if $p < +\infty$, and if $p = +\infty$, we take the sup.

We note by

$$\rho_p(X, F) = \{p\text{-regular positive sublinear operators } T : X \longrightarrow F\}$$

and

$$\rho_p(T) = \inf\{C, \text{verifying the inequality } (3.1)\}.$$

The above proposition is not true for positive sublinear operators.

Proposition 3.8 [11, Proposition $1.d.9$**].** *Let $T : E \longrightarrow F$ be a positive operator. Then, for every $1 \leq p \leq \infty$, T is p-regular.*

The following counterexample (communicated by Gilles Godefroy, 2002), shows that the positive sublinear operator T isn't 2-regular..

We define a function S_r by

$$S_r : L_2(T) \longrightarrow L_1(\Omega, \mu); \qquad T = \mathbb{R}/2\pi\mathbb{Z}$$

$$f \longrightarrow S_r(f) = \frac{1}{2r}\int_{x-r}^{x+r} |f(y)|^2 \, dy, \quad \forall x \in \mathbb{R} \text{ et } 0 < r \leq \pi.$$

We put $T_r f = \sqrt{S_r f}$, hence the operator T_r is sublinear, and the operator T defined by

$$Tf = \sup\{T_r f : 0 < r < \pi\}.$$

For more details, see [4].

Proposition 3.9. *Let $1 \leq p < \infty$. Then $i) \Longleftrightarrow ii)$. Such that:*

i) F is p-concave.

ii) Every p-regular positive sublinear operators $T : X \longrightarrow F$, is p-concave.

Proof.

$ii) \Longrightarrow i)$ We put $X = F$ and $T = Id_X$.

$i) \Longrightarrow ii)$ We suppose that F is p-concave, i.e.,

$$\forall f_1,...,f_n \in F, \left(\sum_{i=1}^{n} \|f_i\|^p\right)^{\frac{1}{p}} \leq K \left\| \left(\sum_{i=1}^{n} |f_i|^p\right)^{\frac{1}{p}} \right\|.$$

For all $x_1,...,x_n$ in X,

$$\begin{aligned}\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} &\leq K \left\| \left(\sum_{i=1}^{n} |T(x_i)|^p\right)^{\frac{1}{p}} \right\| \\ &\leq K' \left\| \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \right\|, \ K' = K\|T\|.\end{aligned}$$

Then T is concave. ■

Corollary 3.10. *Every p-regular positive sublinear operators $T : X \longrightarrow L_p$, $1 \leq p < \infty$, is bounded.*

Proof. It is easy.

Proposition 3.11. *Let $1 < p < \infty$. Then $i) \Longleftrightarrow ii)$. Such that:*
i) E is p-convex.
ii) Every p-regular positive sublinear operators $T : E \longrightarrow Y$, is p-convex.

Proof.
$i) \Longrightarrow ii)$ We have, for all $x_1,...,x_n$ in E

$$\begin{aligned}\left\| \left(\sum_{i=1}^{n} |T(x_i)|^p\right)^{\frac{1}{p}} \right\|_Y &\leq \|T\| \left\| \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \right\|_E, \ p\text{-regular} \\ &\leq C\|T\| \left(\sum_{i=1}^{n} \|x_i\|_E^p\right)^{\frac{1}{p}}.\end{aligned}$$

Then T is p-convex. The converse is obvious. ■

4 Cohen p-nuclear positive sublinear operators

To conclude this section, we recall the definition of positive p-summing sublinear operators, which was first stated in the linear case by Blasco in [7].

Definition 4.1. *Let $T : X \longrightarrow F$ be a positive sublinear operator. We will say that T is "p-summing" ($1 \leq p < +\infty$) (we write $T \in S\Pi_p^+(X, F)$), if there exists a positive constant C such that for all $n \in \mathbb{N}$ and all $\{x_1,...,x_n\} \subset X$, we have*

$$\|(T(x_i))\|_{\ell_p^n(F)} \leq C \|(x_i)\|_{\ell_p^{nw}(X)}. \tag{4.2}$$

We put $\pi_p^+(T) = \inf\{C$ verifying the inequality (4.2)$\}$.

We introduce the following extension of the class of Cohen p-nuclear operators. We give the domination theorem for such a category.

Definition 4.2. *Let $1 < p < \infty$. A positive sublinear operator T between X and F is p-nuclear if there is $C > 0$ such that for all $n \in \mathbb{N}$ and $x_1,...,x_n$ in X, $y_1^*,...,y_n^*$ in F^{*+} we have:*

$$\left|\sum_{i=1}^{n}\left\langle T\left(x_i\right), y_i^*\right\rangle\right| \leq C \sup_{x^*\in B_{X^*}^+}\left(\sum_{i=1}^{n}\left(\left|x_i\right|\left(x^*\right)\right)^p d\mu_1\left(x^*\right)\right)^{\frac{1}{p}} \times$$
$$\times \sup_{y^{**}\in B_{F^{**}}^+}\left(\sum_{i=1}^{n}\left(y_i^*\left(y^{**}\right)\right)^{p^*} d\mu_2\left(y^{**}\right)\right)^{\frac{1}{p^*}} \qquad (I)$$

We denote by $n_p^+(T)$ the smallest constant C which verified the inequality (I), called p-nuclear norm on $\mathcal{SN}_p^+(X,F)$, the Banach space of all p-nuclear positive sublinear operators. If $p=1$, we obtain the Banach space of all 1-summing positive sublinear operators.

Theorem 4.3. **(Composition theorem)**. *Let* X *be a Banach space,* E *and* F *two Banach lattices. Let* T *be in* $\mathcal{SB}^+(X,E)$, *u a positive operator in* $\mathcal{L}(E,F)$ *and* v *in* $\mathcal{L}(Y,X)$.

i) If T *is Cohen p-nuclear, then* $u\circ T$ *is p-nuclear positive sublinear operator and* $n_p^+(u\circ T)\leq \|u\|\, n_p^+(T)$.

ii) If T *is Cohen p-nuclear, then* $T\circ v$ *is p-nuclear positive sublinear operator and* $n_p^+(T\circ v)\leq \|v\|\, n_p^+(T)$.

Theorem 4.4. *A positive sublinear operator between* X, F *is p-summing* $(1\leq p<+\infty)$, *if, and only if, there exists a positive constant* $C>0$ *and a Borel probability* μ *on* $B_{X^*}^+$ *such that*

$$\|T(x)\| \leq \pi_p^+(T)\left(\int_{B_{E^*}^+}\left(|x|\left(x^*\right)\right)^p d\mu\left(x^*\right)\right)^{\frac{1}{p}} \tag{4.3}$$

for every $x\in X$. *Moreover, in this case* $\pi_p^+(T)=\inf\{C>0$*: for all C verifying the inequality* $(4.3)\}$.

Proof. It is similar to the linear case (see [7]).

The main result of this section is the next theorem.

Theorem 4.5. *Let* T *be a bounded positive sublinear operator from* X *into* F. *Then the two following properties are equivalent.*

1) *The operator* T *is in* $\mathcal{SN}_p^+(X,F)$.

2) *There are some Banach space* Z, *a positive p-summing sublinear operator* $u: X\longrightarrow Z$ *and a positive strongly p-summing operator* $v: Z\longrightarrow F$ *such that* $T=vu$.

Proof. 1) $\Longrightarrow$ 2) We consider the operator $u_0: x\in X\longrightarrow \langle |x|,.\rangle\in L_p\left(B_{X^*}^+,\mu\right)$, we notice that $\|Tx\|\leq C\|u_0(x)\|$, for all $x\in X$, let Z be a closed subspace of $L_p(\mu)$ such that $Z=\overline{u_0(X)}$, and let $u: X\longrightarrow Z$ the induite operator. Notice that u is a positive p-summing sublinear operator from X into Z with $\pi_p^+(u)\leq 1$. We write $T=vu$, for some $v\in\mathcal{L}(Z,F)$. If $y^*\in F^{*+}$, then

$$\begin{aligned}
\|v^*(y^*)\| &= \sup\{|\langle u(x), v^*(y^*)\rangle|\} : \|u(x)\|_p\leq 1\\
&= \sup|\langle T(x), y^*\rangle| : \int_{B_{X^*}^+}|\langle x^*,|x|\rangle|^p d\mu(x^*)\leq 1\\
&\leq C\left(\int_{B_{F^{**}}^+}|\langle y^{**},y^*\rangle|^{p^*} d\lambda(y^{**})\right)^{\frac{1}{P^*}}.
\end{aligned}$$

by Pietsch's domination theorem for positive p-summing operators, $v^*\in\Pi_{p^*}^+(F^*,Z^*)$ and $\pi_{p^*}^+(v^*)\leq C$. This implies that v is a positive strongly p-summing operator, see [2, Theorem 4.6].

2) $\Longrightarrow$ 1) It's clear. ■

5 Applications

The main result of this section is the next extension of the Pietsch's domination theorem to this class of operators. For proof, we will use Theorem 4.5. In [6], Achour et al. used Ky Fan's lemma to prove the domination theorem.

Theorem 5.1. *The following two conditions are equivalent.*
1) *$T : X \longrightarrow F$ is Cohen p-nuclear positive sublinear operator and $n_p^+(T) \leq C$.*
2) *There exists a constant $C \geq 0$ and two positives Radon measures μ_1 on $B_{X^*}^+$ and μ_2 on $B_{F^{**}}^+$, such that for all $x \in E$ and $y^* \in F^{*+}$, we have*

$$|\langle T(x), y^*\rangle| \leq$$
$$C\left(\int_{B_{X^*}^+} (|x|(x^*))^p \, d\mu_1(x^*)\right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} (y^*(y^{**}))^{p^*} \, d\mu_2(y^{**})\right)^{\frac{1}{p^*}} \qquad (J)$$

in this case

$$n_p(T) = \inf\{C > 0, \text{ for all } C, \text{verifying the inequality } (J)\}.$$

Proof. 2) $\Longrightarrow$ 1) Letting $x_1, ..., x_n \in X$ and $y_1^*, ..., y_n^* \in F^{*+}$ according to (J), we have

$$|\langle T(x_i), y_i^*\rangle| \leq C\left(\int_{B_{X^*}^+} (|x_i|(x^*))^p \, d\mu_1(x^*)\right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} (y_i^*(y^{**}))^{p^*} \, d\mu_2(y^{**})\right)^{\frac{1}{p^*}}.$$

We deduce,

$$\begin{aligned}
&\left|\sum_{i=1}^n \langle T(x_i), y_i^*\rangle\right| \leq \\
&\leq C\sum_{i=1}^n \left(\int_{B_{X^*}^+} (|x_i|(x^*))^p \, d\mu_1(x^*)\right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} (y_i^*(y^{**}))^{p^*} \, d\mu_2(y^{**})\right)^{\frac{1}{p^*}} \\
&\leq C\left(\sum_{i=1}^n \int_{B_{X^*}^+} (|x_i|(x^*))^p \, d\mu_1(x^*)\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \int_{B_{F^{**}}^+} (y_i^*(y^{**}))^{p^*} \, d\mu_2(y^{**})\right)^{\frac{1}{p^*}} \\
&\leq C \sup_{x^* \in B_{X^*}^+} \left(\sum_{i=1}^n (|x_i|(x^*))^p \, d\mu_1(x^*)\right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^n (y_i^*(y^{**}))^{p^*} \, d\mu_2(y^{**})\right)^{\frac{1}{p^*}}
\end{aligned}$$

This implies that T is a p-nuclear positive sublinear operator.

1) $\Longrightarrow$ 2) If $T \in S\mathcal{N}_p^+(E, F)$, thus, according to the above $T = vu$ where $u \in S\Pi_p^+(E, Z)$ and $v \in \mathcal{D}_p^+(Z, F)\left[v^* \in \pi_{p^*}^+(F^*, Z^*)\right]$. by [6, Thm 2.4] and [2, Theorem 4.13] there exist a constant $C > 0$, two positive Radon measures μ_1 on $B_{E^*}^+$ and μ_2 on $B_{F^{**}}^+$, endowed with their weak$*$ topologies, such that for all $x \in E$ and $y^* \in F^{*+}$,

$$\begin{aligned}
|\langle T(x), y^*\rangle| &= |\langle vu(x), y^*\rangle| \\
&= |\langle u(x), v^*(y^*)\rangle| \\
&\leq \|u(x)\| \, \|v^*(y^*)\| \\
&\leq C\left(\int_{B_{E^*}^+} (\langle |x|, x^*\rangle)^p \, d\mu_1(x^*)\right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**}\rangle^{p^*} \, d\mu_2(y^{**})\right)^{\frac{1}{p^*}}.
\end{aligned}$$

This was proven. ■

Now we are ready to use the Grothendieck–Maurey theorem in the positive sublinear case.

Theorem 5.2. *Let E, F and G be three Banach lattices where G is 2-concave space. Let $T : C(K) \longrightarrow E$ be 2-regular positive sublinear operator, $w : E \longrightarrow F$ a positive 2-concave operator and a positive strongly p-summing operator $v : F \longrightarrow G$. Then vwT is Cohen 2-nuclear positive sublinear operator and $n_2^+(vwT) \leq d_2^+(v)\, C_2^+(w)\, \rho_2(T)$.*

Proof. The operator wT is positive 2-summing sublinear [5, Theorem 3.6] and by Theorem 4.5, the operator vwT is Cohen 2-nuclear positive sublinear. ■

Proposition 5.3. *We have*

$$S\mathcal{N}_p^+(E,F) \subseteq S\Pi_p^+(E,F) \text{ and } \pi_p^+(T) \leq n_p^+(T).$$

Proof. Let T be an operator in $S\mathcal{N}_p^+(E,F)$. For all $x \in E$, we have

$$\begin{aligned}
\|T(x)\| &= \sup_{y^* \in B_{F^*}^+} |\langle T(x), y^* \rangle| \\
&\leq \sup_{y^* \in B_{F^*}^+} n_p^+(T) \left(\int_{B_{E^*}^+} (|x|(x^*))^p \, d\mu_1(x^*) \right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} (y^*(y^{**}))^{p^*} \, d\mu_2(y^{**}) \right)^{\frac{1}{p^*}} \\
&\leq n_p^+(T) \left(\int_{B_{E^*}^+} (|x|(x^*))^p \, d\mu_1(x^*) \right)^{\frac{1}{p}} \sup_{y^* \in B_{F^*}^+} \|y^*\| \\
&\leq n_p^+(T) \left(\int_{B_{E^*}^+} (|x|(x^*))^p \, d\mu_1(x^*) \right)^{\frac{1}{p}}.
\end{aligned}$$

Then, T is a positive p-summing sublinear operator and $\pi_p^+(T) \leq n_p^+(T)$. ■

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References

[1] D. H. Alipantis and O. Burkinshaw, Positive Operators, *Academic Press, INC* (1985).

[2] D. Achour and A. Belacel, Domination and factorization theorems for positive strongly p-summing operators, *Positivity* 18 (2014) 785-804.

[3] D. Achour and L. Mezrag, Factorisation des opérateurs sous-linéaires par $L_{p\infty}$ et L_{q1}, *Ann. Sci. Math. Quebec* 26 (2002), 109-121. (in French).

[4] D. Achour and L. Mezrag, Little Grothendieck's theorem for sublinear operators, *J. Math. Anal. Appli*. 296 (2004), 541-552.

[5] D. Achour, L. Mezrag and K. Saadi, On the Cohen p-nuclear sublinear operators, *Journal of inequalities in pure and applied mathematics*. Vol. 10 (2009), Issue 2, Article 46, 13pp.

[6] D. Achour, L. Mezrag and A. Tiaiba, On the strongly p-summing sublinear operators, *Taiwanese Journal of Mathematics*. 11(4) (2007), 959-973.

[7] O. Blacso, A class of operators from a Banach lattice into a Banach space, *Collect. Math.*, 37 (1986), 13–22.

[8] O. Blacso, Positive p-summing operators from L_p-spaces, *Proc. Amer. Math. Soc.*, 31 (1988), 275–280.

[9] J. S. Cohen, Absolutely p-summing, p-nuclear operators and their conjugates, *Math. Ann.*, 201 (1973), 177–200.

[10] J. L. Krivine, Théorèmes de factorisation dans les espaces réticulés, *Séminaire Maurey-Schwartz* 1973 − 1974, *exposés 22 et 23, École Polytechnique. Paris.*

[11] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I and II, *Springer-Verlag, Berlin* (1996) .

[12] H. Schaefer, Banach lattices and Positive Operators, *Springer-Verlag, Berlin Heidelberg New York* (1974) .

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Third Hankel Determinant for Certain Subclass of Analytic Functions

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Abstract

The third Hankel determinant, $H_3(1)$ for subclass of analytic functions satisfying geometric condition

$$Re\,\frac{zf'(z)}{f(z)}\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} > 0$$

for nonnegative real number α, in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ is derived in line with a method of classical analysis devised by Libera and Zlotkiewicz [9].

Keywords: Hankel determinant, caratheodory functions, product of geometric expression, analytic functions.

2010 MSC: 30C45.

1 Introduction

Let A denote the class of functions

$$f(z) = z + a_2z^2 + a_3z^3 + ... \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$. By S, S^*, C and R, we mean the well known subclasses of A which consist of univalent, starlike, convex and bounded turning functions respectively. In [8], Jimoh et-al introduced a subclass of analytic functions denoted by $\mathcal{J}_\alpha$ which satisfy the geometric condition:

$$Re\,\frac{zf'(z)}{f(z)}\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} > 0. \tag{1.2}$$

for non negative real number α,where estimates on the bounds of some coefficients were investigated. Also in [6], Ganiyu et-al obtained the bound on the second Hankel determinant, $H_2(2)$ for this same subclass of analytic functions, $\mathcal{J}_\alpha$. In [10], Noonan and Thomas defined the qth Hankel determinant of f for $q \geq 1$, $n \geq 0$ by:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

This determinant has been considered for specific choices of q and n by several authors with subject of inquiry ranging from rate of growth of $H_q(n)$ as $n \to \infty$ to the determination of precise bounds on $H_q(n)$ for some subclasses of analytic functions. It is well known that the Fekete-Szego functional is $|a_3 - a_2^2|$=$H_2(1)$. The

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second Hankel determinant defined by $H_2(2) = |a_2a_4 - a_3^2|$ also received a lot of attention by researchers among which is the notable work of Janteng, et-al, [7] where they obtained the second Hankel determinant for some subclasses of analytic functions. Other contributors in this regard include Abubaker [1], Al-Refai [2], Norlyda et-al [11], Vamshee [13].

Babalola [4], Shanmungam et-al [12], Vamshee et-al [14] have studied the third Hankel determinant, $H_3(1)$ for various classes of analytic and univalent functions. In the present investigation, our focus is on the third Hankel determinant, $H_3(1)$ for the subclass $\mathcal{J}_\alpha$ given by:

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

For $f \in A$, $a_1 = 1$ so that

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and by using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \tag{1.3}$$

In this paper, we seek to find the sharp upper bound on $|a_2a_3 - a_4|$, $|a_3 - a_2^2|$ and $|H_3(1)|$ respectively for the functions belonging to the subclass $\mathcal{J}_\alpha$. We shall make use of our earlier results on the bounds on each of the coefficients and the functional $|a_2a_4 - a_3^2|$=$H_2(2)$.

2 Preliminary Lemmas.

To prove the main results in the next section, we need the following lemmas. Let P denote the class of Caratheodory functions $p(z) = 1 + c_1z + c_2z^2 + \cdots$ which are analytic and satisfy $p(0) = 1$, Re $p(z) > 0$ in open unit disk U.

Lemma 2.1. *[5] Let $p \in P$. Then $|c_k| \leq 2, k = 1, 2, 3....$ Equality is attained by the moebius function*

$$L_0(z) = \frac{1+z}{1-z}.$$

Lemma 2.2. *[9] Let $p \in P$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.1}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.2}$$

for some value of x,z such that $|x| \leq 1$ and $|z| \leq 1$.

Lemma 2.3. *[3] Let $p \in P$. Then we have sharp inequalities*

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \leq \begin{cases} 2(1-\sigma), & \text{if } \sigma \leq 0, \\ 2, & \text{if } 0 \leq \sigma \leq 2, \\ 2(\sigma - 1), & \text{if } \sigma \geq 2. \end{cases}$$

Lemma 2.4. *[6] Let $f \in \mathcal{J}_\alpha$. Then*

$$|H_2(2)| \leq \frac{4}{(\alpha+4)^2}$$

Lemma 2.5. *[8] Let $f \in J_\alpha$. Then*

$$|a_2| \leq \frac{2}{\alpha+2}$$

$$|a_3| \leq \begin{cases} \frac{2(\alpha+6)}{(\alpha+2)^2(\alpha+4)} & \text{if } 0 < \alpha \leq \frac{-3+\sqrt{17}}{2}, \\ \frac{2}{\alpha+4} & \text{if } \alpha \geq \frac{-3+\sqrt{17}}{2}. \end{cases}$$

$$|a_4| \leq \begin{cases} \frac{52\alpha^4+472\alpha^3+1208\alpha^2+896\alpha+288}{6(\alpha+2)^3(\alpha+4)(\alpha+6)} & \text{if } \alpha \leq \frac{-5+\sqrt{33}}{2}, \\ \frac{14\alpha^2+96\alpha+232}{3(\alpha+2)(\alpha+4)(\alpha+6)} & \text{if } \alpha \geq \frac{-5+\sqrt{33}}{2}. \end{cases}$$

$$|a_5| \leq \begin{cases} \frac{14\alpha^5+236\alpha^4+1348\alpha^3+2976\alpha^2+2160\alpha+1024}{(\alpha+2)^2(\alpha+4)^2(\alpha+6)(\alpha+8)} & \text{if } \alpha \leq \frac{-7+\sqrt{57}}{2}, \\ \frac{4\alpha^4+74\alpha^3+584\alpha^2+2152\alpha+3072}{(\alpha+2)(\alpha+4)^2(\alpha+6)(\alpha+8)} & \text{if } \alpha \geq \frac{-7+\sqrt{57}}{2}. \end{cases}$$

3 Main Results.

Theorem 3.1. *let $f \in J_\alpha$. Then we have*

$$|a_2a_3 - a_4| \leq \frac{2(\alpha^2+6\alpha+16)}{3(\alpha+2)(\alpha+4)(\alpha+6)}\sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}}, \quad 0 \leq \alpha < 1$$

Proof. Using the results obtained earlier in [8], we have that if $f \in \mathcal{J}_\alpha$, then

$$a_2 = \frac{c_1}{\alpha+2} \tag{3.1}$$

$$a_3 = \frac{c_2}{\alpha+4} - \frac{\alpha^2+3\alpha-2}{2(\alpha+2)^2(\alpha+4)}c_1^2 \tag{3.2}$$

$$a_4 = \frac{c_3}{\alpha+6} + \frac{2-5\alpha-\alpha^2}{(\alpha+2)(\alpha+4)(\alpha+6)}c_1c_2 + \frac{2\alpha^4+17\alpha^3+31\alpha^2-8\alpha+12}{6(\alpha+2)^3(\alpha+4)(\alpha+6)}c_1^3 \tag{3.3}$$

so that

$$|a_2a_3 - a_4| = \left|\frac{\alpha^2+6\alpha+4}{(\alpha+2)(\alpha+4)(\alpha+6)}c_1c_2 - \frac{\alpha^4+10\alpha^3+29\alpha^2+20\alpha-12}{3(\alpha+2)^3(\alpha+4)(\alpha+6)}c_1^3 - \frac{c_3}{\alpha+6}\right| \tag{3.4}$$

substituting c_2 and c_3 in Lemma 2.2 into equation (3.4), we have

$$\begin{aligned}|a_2a_3 - a_4| = &\left|\frac{48-8\alpha-32\alpha^2-10\alpha^3-\alpha^4}{12(\alpha+2)^3(\alpha+4)(\alpha+6)}c_1^3 - \frac{2(4-c_1^2)}{(\alpha+2)(\alpha+4)(\alpha+6)}c_1x\right.\\ &\left.+\frac{(4-c_1^2)}{4(\alpha+6)}c_1x^2 - \frac{(4-c_1^2)(1-|x|^2)z}{2(\alpha+6)}\right|\end{aligned}$$

By Lemma 2.1, $|c_1| \leq 2$. Suppose that $c_1 = c$, we may assume without restriction that $c \in [0,2]$. By the use of triangle inequality with $\xi = |x|$ and noting that $48 - 8\alpha - 32\alpha^2 - 10\alpha^3 - \alpha^4 \geq 0$ for $0 \leq \alpha < 1$, we obtain

$$\begin{aligned}|a_2a_3 - a_4| &\leq \frac{48-8\alpha-32\alpha^2-10\alpha^3-\alpha^4}{12(\alpha+2)^3(\alpha+4)(\alpha+6)}c^3 + \frac{2(4-c^2)}{(\alpha+2)(\alpha+)(\alpha+6)}c\xi\\ &+\frac{(c-2)(4-c^2)}{4(\alpha+6)}\xi^2 + \frac{4-c^2}{2(\alpha+6)}\\ &= F(c,\xi)\end{aligned} \tag{3.5}$$

we assume the upper bound for equation (3.5) occurs at an interior point of the set $\{(\xi, c) : \xi \in [0,1]$ and $c \in [0,2]\}$. Differentiating $F(c,\xi)$ partially with respect to ξ, we get

$$F'(c,\xi) = \frac{2(4-c^2)c}{(\alpha+2)(\alpha+4)(\alpha+6)} + \frac{(c-2)(4-c^2)\xi}{2(\alpha+6)}$$

For $0 < \xi < 1$ and for fixed c with $0 < c < 2$, we observe that $F'(c,\xi) > 0$. Therefore, $F'(c,\xi)$ is an increasing function of ξ, which contracdicts our assumption that the maximum value of it occurs at an interior point of the set $\{(\xi, c) : \xi \in [0,1]$ and $c \in [0,2]\}$. Also for fixed $c \in [0,2]$, we have

$$\max_{0\leq\xi\leq1} F(c,\xi) = F(c,1) = G(c), \quad say$$

replacing ξ by 1 in equation (3.5), we obtain

$$G(c) = F(c,1) = \frac{\alpha^2+6\alpha+16}{(\alpha+2)(\alpha+4)(\alpha+6)}c - \frac{\alpha^4+10\alpha^3+41\alpha^2+68\alpha+36}{3(\alpha+2)^3(\alpha+4)(\alpha+6)}c^3$$

so that

$$G'(c) = \frac{\alpha^2+6\alpha+16}{(\alpha+2)(\alpha+4)(\alpha+6)} - \frac{\alpha^4+10\alpha^3+41\alpha^2+68\alpha+36}{(\alpha+2)^3(\alpha+4)(\alpha+6)}c^2$$

$G'(c) = 0$ implies

$$c = \pm\sqrt{\frac{(\alpha+2)(\alpha^2+6\alpha+16)}{\alpha^3+8\alpha^2+25\alpha+18}}$$

since $c \in [0,2]$, we have $c = \sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}}$ as the maximum point of $G(c)$. Therefore

$$G(c) \leq \frac{2(\alpha^2+6\alpha+16)}{3(\alpha+2)(\alpha+4)(\alpha+6)}\sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}}$$

That is the upper bound of equation (3.5) correspounds to $\xi = 1$ and $c = \sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}}$. □

Theorem 3.2. *Let $f \in \mathcal{J}_\alpha$.Then*

$$|a_3 - a_2^2| \leq \frac{2}{\alpha+4}$$

Proof. Using equations (3.1) and (3.2),

$$|a_3 - a_2^2| = \frac{1}{\alpha+4}\left|c_2 - \left(\frac{\alpha+3}{\alpha+2}\right)\frac{c_1^2}{2}\right|$$

Applying Lemma 2.3, with $\sigma = \frac{\alpha+3}{\alpha+2}$, we obtain

$$\left|c_2 - \left(\frac{\alpha+3}{\alpha+2}\right)\frac{c_1^2}{2}\right| \leq 2$$

hence the result. □

Corollary 3.1. *Let $f \in \mathcal{J}_\alpha$.Then*

$$|H_3(1)| \leq \begin{cases} \frac{J_1+J_2\sqrt{J_5}}{(\alpha+2)^2 J_6}, & \text{if } 0 \leq \alpha \leq \frac{-3+\sqrt{17}}{2}, \\ \frac{J_3+J_4\sqrt{J_5}}{J_6}, & \text{if } \alpha \geq \frac{-3+\sqrt{17}}{2}. \end{cases}$$

where,

$$J_1 = 252\alpha^{11} + 8784\alpha^{10} + 134316\alpha^9 + 1188072\alpha^8 + 6737328\alpha^7 + 25615584\alpha^6 + 66411072\alpha^5 + 117846144\alpha^4 + 143325504\alpha^3 + 41925888\alpha^2 + 64143360\alpha + 16920576,$$

$$J_2 = 52\alpha^8 + 1408\alpha^7 + 15944\alpha^6 + 99248\alpha^5 + 369248\alpha^4 + 818240\alpha^3 + 997120\alpha^2 + 569344\alpha + 147456,$$

$$J_3 = 72\alpha^8 + 2304\alpha^7 + 31176\alpha^6 + 233442\alpha^5 + 1057248\alpha^4 + 2945178\alpha^3 + 4854420\alpha^2 + 4258872\alpha + 1496880,$$

$$J_4 = 28\alpha^6 + 696\alpha^5 + 7280\alpha^4 + 42144\alpha^3 + 143744\alpha^2 + 276480\alpha + 237568,$$

$$J_5 = \alpha^6 + 16\alpha^5 + 117\alpha^4 + 474\alpha^3 + 1100\alpha^2 + 1304\alpha + 576,$$

$$J_6 = 9(\alpha+2)^2(\alpha+4)^3(\alpha+6)^2(\alpha+8)(\alpha^3+8\alpha^2+25\alpha+18).$$

Proof. By equation (1.3), we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|$$

using Lemma 4, the first inequality of the result in Lemma 5 together with the results obtained in Theorems 3.1 and 3.2,

$$\begin{aligned} |H_3(1)| \leq & \frac{8(\alpha+6)}{(\alpha+2)^2(\alpha+4)^3} + \frac{2(14\alpha^5 + 236\alpha^4 + 1348\alpha^3 + 2976\alpha^2 + 2160\alpha + 1024)}{(\alpha+2)^2(\alpha+4)^3(\alpha+6)(\alpha+8)} \\ & + \left(\frac{(\alpha^2+6\alpha+16)(52\alpha^4 + 472\alpha^3 + 1208\alpha^2 + 896\alpha + 288)}{9(\alpha+2)^4(\alpha+4)^2(\alpha+6)^2(\alpha^3+8\alpha^2+25\alpha+18)} \right) \\ & \left(\sqrt{(\alpha^3+8\alpha^2+28\alpha+32)(\alpha^3+8\alpha^2+25\alpha+18)} \right) \end{aligned}$$

simplifying, we have the first inequality.
Also by using Lemma 2.4, the second inequality of the result in Lemma 2.5 together with the results obtained in Theorems 3.1 and 3.2,

$$\begin{aligned} |H_3(1)| \leq & \frac{8}{(\alpha+4)^3} + \frac{2(4\alpha^4 + 74\alpha^3 + 584\alpha^2 + 2152\alpha + 3072)}{(\alpha+2)(\alpha+4)^3(\alpha+6)(\alpha+8)} \\ & + \frac{2(\alpha^2+6\alpha+16)(14\alpha^2+96\alpha+232)}{9(\alpha+2)^2(\alpha+4)^2(\alpha+6)^2} \sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}} \end{aligned}$$

By simplification, we obtain the other inequality.

□

4 Conclusion

We have been able to find the sharp upper bound on functionals $|a_2a_3 - a_4|$, $|a_3 - a_2^2|$ and the third Hankel determinant, $|H_3(1)|$ for the functions belonging to the subclass $\mathcal{J}_\alpha$.

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References

[1] Abubaker, A. and Darus, M., Hankel determinant for a class of analytic functions involving a generalized linear differential operator, *International Journal of Pure and Applied Mathematics*, 69 (4) (2011), 429–435.

[2] Al-Refai, O. and Darus, M., Second Hankel determinant for a class of analytic functions defined by a fractional operator, *European Journal of Science Resources* , 28 (2) (2009), 234–241.

[3] Babalola, K.O. and Opoola, T.O. , On the coefficients of certain analytic and univalent functions, *Advances in Inequalities for Series, Nova Science Publishers*, (2008), 5–17.

[4] Babalola, K.O., On $H_3(1)$ Hankel Determinant for some Classes of Univalent Functions, *Inequality Theory and Applications, Nova Science Publishers, Inc.*, 6 (2010), 1–7.

[5] Duren, P.L., Univalent functions , *Mariner Publishing Co. Inc. Tampa, Florida* , 1 (1983) .

[6] Ganiyu, M.A., Jimoh, F.M., Ejieji, C.N. and Babalola, K.O., Second Hankel determinant for Certain Subclasses of Analytic functions, *Submitted*.

[7] Janteng, A., Abdulhalim, S. and Darus, M., Hankel determinant for starlike and convex functions, *Int. Journal of Math. Analysis*, 1 (13) (2007), 619–625.

[8] Jimoh, F.M., Ganiyu, M.A., Ejieji, C.N., and Babalola, K.O., Bounds on coefficients of certain Analytic functions, *Nigeria Journal of Pure and Applied Science, Faculty of Science, University of Ilorin*, 26 (2013), 2411–2418.

[9] Libera, R.J and Zlotkiewicz, E.J., Coefficient bounds for the inverse of a function with derivative in P, *Proc. Amer. Math. Soc.*, 87 (2) (1983), 251–257.

[10] Noonan, J.W. and Thomas, D.K., Second Hankel determinant of areally mean p-valent functions, *Trans. Amer. Math. Soc.*, 223 (1976), 337–346.

[11] Norlyda, M., Daud, M. and Shaharuddin, C.S., Second Hankel determinant for certain generalized classes of analytic functions, *International Journal of Mathematical Analysis*, 6 (17) (2012), 807–812.

[12] Shanmugam, G., Adolf, Stephen B. and Babalola, K.O., Third Hankel determinant for α-starlike functions, *Gulf Journal of Mathematics*, 2 (2) (2014), 107–113.

[13] Vamshee, D.K. and Ramreddy, T. , Hankel determinant for p-valent starlike and convex functions of order α, *Novi sad J. Math.*, 42 (2) (2012),89–96.

[14] Vamshee, D. K., Venkateswarlu, B. and Ramreddy T., Third Hankel determinant for bounded turning functions of order alpha, *Journal of the Nigerian Mathematical Society*, 34 (2) (2015) 121–127.

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Approximate controllability of nonlocal impulsive fractional neutral stochastic integro-differential equations with state-dependent delay in Hilbert spaces

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Abstract

In this manuscript, we study the approximate controllability results for nonlocal impulsive fractional neutral stochastic integro-differential equations with state-dependent delay conditions in Hilbert spaces under the assumptions that the corresponding linear system is approximately controllable. The results are obtained by using fractional calculus, semigroup theory, stochastic analysis and fixed point theorem. An example is provided to show the application of our result.

Keywords: Fractional differential equations, approximate controllability, stochastic differential system, nonlocal condition, state-dependent delay, fixed point theorem, semigroup theory.

2010 MSC: 65C30, 26A33, 34A08, 34H05.

1 Introduction

In this manuscript, we set up the approximate controllability of mild solutions for nonlocal impulsive fractional neutral stochastic integro-differential systems (abbreviated, NIFNSIDS) with state-dependent delay (abbreviated, SDD) in Hilbert spaces through the utilization of the fixed point theorem thanks to Schauder [30]. We discuss the neutral integro-differential equations of fractional-order with SDD of the model

$$\begin{aligned} {}^C D_t^\alpha[u(t) - \mathscr{G}(t, u_{\varrho(t,u_t)})] &= \mathscr{A}u(t) + Bv(t) + \mathscr{F}\left(t, u_{\varrho(t,u_t)}, \int_0^t e_1(t,s,u_{\varrho(s,u_s)})ds\right) \\ &\quad + \Sigma\left(t, u_{\varrho(t,u_t)}, \int_0^t e_2(t,s,u_{\varrho(s,u_s)})ds\right)\frac{dw(t)}{dt}, \quad t \neq t_k, \quad k = 1,2,\cdots,n, \qquad (1.1) \\ \Delta u(t_k) &= \mathcal{I}_k(u(t_k^-)), \quad k = 1,2,\cdots,n, \qquad (1.2) \\ u(0) + h(u) &= \varphi \in \mathcal{B}, \qquad (1.3) \end{aligned}$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order α, $\alpha \in (0,1)$, the state variable u takes values in a Hilbert space $\mathcal{H}$; $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $k = 1,2,\ldots,n$ are impulsive function, which the solution is jump at impulsive point t_k, $0 < t_1 < t_2 < \cdots < t_n < T$; $\mathscr{A} : D(\mathscr{A}) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator $\{\mathbb{T}(t) : t \geq 0\}$. That is to say, $\|\mathbb{T}(t)\| \leq \mathcal{M}$ for some constant $\mathcal{M} \geq 1$ and every $t \geq 0$; the control function v is given in $\mathcal{L}^2(\mathscr{I}, U)$, U is a Hilbert space, B is a bounded linear operator from U into $\mathcal{H}$. The time history u_t: $(-\infty, 0] \to \mathcal{H}$, $u_t(\theta) = u(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ described axiomatically in section 2 and ϱ :

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$\mathscr{I} \times \mathcal{B} \to (-\infty, T]$ is a continuous function. Let $\mathcal{K}$ be a another Hilbert space, suppose $\{W(t)\}_{t\geq 0}$ is a given $\mathcal{K}$-valued Brownian motion or wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Denote $\mathcal{PC}(\mathscr{I}, \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H})) = \{u(t)$ is continuous everywhere except for some t_k at which $u(t_k^-)$ and $u(t_k^+)$ exist and $u(t_k^-) = u(t_k)\}$ be the Banach space of piece-wise continuous function from $\mathscr{I}$ into $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H})$ with the norm $\|u\|_{\mathcal{PC}} = \sup_{t\in\mathscr{I}} |u(t)| < \infty$, $\mathcal{PC}(\mathscr{I}, \mathcal{L}^2)$ is the closed subspace of $\mathcal{PC}(\mathscr{I}, \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H}))$ consisting of a measurable and $\mathcal{F}_t$-adapted $\mathcal{H}$-valued process $u(\cdot) \in \mathcal{PC}(\mathscr{I}, \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H}))$ with the norm defined $\|u\|^2 = \sup\{\mathbb{E}\|u(t)\|^2,\ t \in \mathscr{I}\}$. The functions $\mathscr{G}, \mathscr{F}, \Sigma, e_i, i = 1, 2; \mathcal{I}_k$ and h are suitable functions to be specified later.

The emergence of fractional calculus arise new questions in fundamental physics, which provides great challenging interest for the mathematicians and physicists in the theory of fractional calculus. The fractional differential equations (abbreviated, FDEs) have been considered to be the valuable tool, which can describe dynamical behavior of real life phenomena more accurately. For instance, the nonlinear oscillation of earthquake can be well modeled with fractional derivatives. We can find the numerous applications of FDEs in control theory, nonlinear oscillation of earthquake, the fluid-dynamic traffic model, aerodynamics and in almost every field of science and engineering. For more points of interest on this concept, we allude the reader to Pazy [27]. There has been a lot of enthusiasm toward the solutions of fractional differential equations in systematic and mathematical thoughts. For fundamental certainties about fractional systems, one can make reference to the books [6, 13, 38], and the papers [11, 15], and the references cited therein.

FDEs with delay features happen in several areas such as medical and physical with SDD or non-constant delay. These days, existence and controllability results of mild solutions for such problems became very attractive and several researchers working on it. As of late, few number of papers have been published on the fractional order problems with SDD [1, 2, 9, 20, 30, 35] and references therein. Especially, in [1], the authors analyzed the existence results for fractional integro-differential equations whereas Benchohra et al. [2] examined the existence of mild solutions for fractional integro-differential equations in Banach spaces.

An important feature of real-world dynamic processes that has attracted considerable interest by scientists is the effect of abrupt changes. Hereby, "abrupt" is meant in the sense of a multi-scale problem, i.e. the state of a system changes only slowly for a long time interval, and then undergoes a drastic change within a very short time interval. For example, a football may be flying through the air for several seconds before it changes its flight direction within milliseconds during a collision with a goal post. For the mathematical description of this system, the specification of two sets of equations is appropriate: one for the flight phase, and one for the collision phase.

Several mathematical models can be developed for the football example. In a simplified setting, the motion of the football could be described by the position and velocity of its center of mass, and the encounter with the goal post could be treated as an inelastic collision (i.e. by an immediate change of the football's velocity).

For the description of the collision of the ball with the goal post leads to differential equations in which the velocity experiences, at the time of the collision, a so-called impulse. There is really a noteworthy improvement in impulsive concept, particularly in the region of impulsive differential frameworks having fixed times; for the additional purposes of enthusiasm on this concept and on its uses, see for example the treatise by Lakshmikantham et al. [22], Ivanka M. Stamova [34], Bainov et al. [4], Benchohra et al. [7] and the papers [3, 8, 10, 15], and the references cited therein.

In addition, the investigation of stochastic differential comparisons has pulled in awesome enthusiasm because of its applications in portraying numerous issues in material science, biology, chemistry, mechanics, etc. As a matter of fact, the accurate analysis or assessment subjected to a realistic environment has to take into account the potential randomness in the system properties, such as fluctuations in the stock market or noise in a communication network. All these problems in mathematics are modeled and depicted by stochastic differential equations or stochastic integro-differential equations with delay and impulses.

On the other hand, controllability is one of the important fundamental concepts in mathematical control theory and plays an important role in both deterministic and stochastic control system. In many dynamical systems, the control does not affect the complete state of the dynamical system but only a part on it. Further, very often in real industrial processes it is possible to observe only a certain part of the complete state of the dynamical system. This, the dynamical systems must be treated by the weaker concept of controllability, namely approximate controllability.

The existence, controllability and other qualitative and quantitative attributes of stochastic FDEs are the most progressing area of pursuit, for instance, see [5, 12, 16, 17, 25, 39–44]. In particular, Toufik Guendouzi et al. [16, 17] reviewed existence and approximate controllability of different types of fractional stochastic differential and integro-differential systems with SDD in Hilbert spaces under different suitable fixed point theorems. Lately, Zhang et al. [44] derived a new set of sufficient conditions for approximate controllability of impulsive fractional stochastic differential equations with state-dependent delay in Hilbert spaces with the help of fractional calculus and stochastic analysis. Moreover, Yan et al. [39, 40] investigated for approximate controllability of impulsive partial neutral stochastic functional integro-differential inclusion with infinite delay. Recently, Sakthivel et al. [29, 31] reviewed the approximate controllability of fractional neutral stochastic differential inclusions with nonlocal conditions and infinite delay by utilizing the Krasnoselskii's fixed point theorem. Very recently, Vijayakumar et al. [24, 36, 37] derived the controllability and approximate controllability results for abstract neutral integro-differential inclusions with infinite delay in Hilbert spaces.

The best of our knowledge, it appears that little is thought about approximate controllability results for IFNSIDS with non-local and SDD conditions in Hilbert spaces. The point of this manuscript is to analyze this fascinating model (1.1)- (1.3).

The rest of this paper is organized as follows. In Section 2 is focused on call to mind of some crucial perspectives that will be utilized in this work to accomplish our primary results. In Section 3, we declare and present the existence results about by proposes of Schauder fixed point theorem. In Section 4, an example is given to illustrate our results.

2 Preliminaries

Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ and $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ denote two real separable Hilbert spaces. For our convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathcal{H}, \mathcal{K}$ and $(\cdot,\cdot)$ to denote the inner product without any confusion. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space furnished with a normal filtration $\mathcal{F}_t, t \in \mathscr{I}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathcal{P}$-null sets), and $\mathbb{E}(\cdot)$ denotes the expectation with respect to the measure $\mathcal{P}$. An $\mathcal{H}$-valued random variable is an $\mathcal{F}$ measurable function $u(t)\colon \Omega \to \mathcal{H}$, and a collection of random variable $\mathcal{W} = \{u(t,\omega)\colon \Omega \to \mathcal{H}|_{t\in T}\}$ is called a stochastic process. We suppress the dependence on $\omega \in \Omega$ and write $u(t)$ instead of $u(t,\omega)$ and $u(t)\colon \mathscr{I} \to \mathcal{H}$ in the place of $\mathcal{W}$. Assume that $\{\beta_n\}_{n\geq 1}$ be a sequence of real valued independent Brownian motions, defined by $W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)\chi_n$, $t \geq 0$, where $\{\chi_n\}_{n\geq 1}$ is complete orthonormal system in $\mathcal{K}$ and $\lambda_n \geq 0$ $(n = 1, 2, \dots)$ are non-negative real numbers. Let $Q \in \mathcal{L}(\mathcal{K},\mathcal{K})$ be an operator satisfying $Q\chi_n = \lambda_n\chi_n$ with $tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$. Then, the above $\mathcal{K}$-valued stochastic process $W(t)$ is a Q-wiener process. Let us assume $\mathcal{F}_t = \sigma(W(s) : 0 \leq s \leq t)$ is the σ-algebra generated by W and $\mathcal{F}_T = \mathcal{F}$.

Let $\mathcal{L}(\mathcal{K},\mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ equipped with the usual operator norm $\|\cdot\|$. For $\varphi \in \mathcal{L}(\mathcal{K},\mathcal{H})$ and define

$$\|\varphi\|_Q^2 = tr(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\varphi\chi_n\|^2.$$

If $\|\varphi\|_Q^2 < \infty$, then φ is called a Q-Hilbert-Schmidt operator. Let $\mathcal{L}_Q(\mathcal{K},\mathcal{H})$ denote the space of all Q-Hilbert-Schmidt operators φ. The completion $\mathcal{L}_Q(\mathcal{K},\mathcal{H})$ of $\mathcal{L}(\mathcal{K},\mathcal{H})$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\varphi\|_Q^2 = <\varphi,\varphi>$ is a Hilbert space with the above norm topology.

Without loss of generality, we assume that $0 \in \wp(\mathscr{A})$, the resolvent set of $\mathscr{A}$. Then for $0 < \eta \leq 1$, it is possible to define the fractional power $\mathscr{A}^\eta$ as a closed linear operator on its domain $D(\mathscr{A}^\eta)$, being dense in $\mathcal{H}$, and we denote by $\mathcal{H}_\eta$ the Banach space of $D(\mathscr{A}^\eta)$ endowed with the norm $\|u\|_\eta = \|\mathscr{A}^\eta u\|$, which is equivalent to the graph norm of $\mathscr{A}^\eta$.

Lemma 2.1. *[27] Suppose that the preceding conditions are satisfied.*

(i) Let $0 < \eta \le 1$*, then* $\mathcal{H}_\eta$ *is a Banach space.*

(ii) If $0 < \nu \le \eta$*, then the embedding* $\mathcal{H}_\nu \subset \mathcal{H}_\eta$ *is compact whenever the resolvent operator of* $\mathscr{A}$ *is compact.*

(iii) For every $\eta \in (0,1]$*, there exists a positive constant* $\mathcal{C}_\eta$ *such that*

$$\|\mathscr{A}^\eta \mathbb{T}(t)\| \le \frac{\mathcal{C}_\eta}{t^\eta}, \quad t > 0.$$

It needs to be outlined that, once the delay is infinite, then we should talk about the theoretical phase space $\mathcal{B}$ in a beneficial way.

We assume that the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a semi-normed linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty, 0]$ into $\mathcal{H}$ and fulfilling the subsequent elementary adages as a result of Hale and Kato (see the case in point in [15, 18, 19]).

If $u : (-\infty, T] \to \mathcal{H}, T > 0$, is continuous on $\mathscr{I}$ and $u_0 \in \mathcal{B}$, then for every $t \in \mathscr{I}$ the accompanying conditions hold:

(P_1) u_t is in $\mathcal{B}$;

(P_2) $\|u(t)\| \le H\|u_t\|_{\mathcal{B}}$;

(P_3) $\|u_t\|_{\mathcal{B}} \le \mathscr{E}_1(t)\sup\{\|u(s)\| : 0 \le s \le t\} + \mathscr{E}_2(t)\|u_0\|_{\mathcal{B}}$, where $H > 0$ is a constant and $\mathscr{E}_1(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous, $\mathscr{E}_2(\cdot) : [0, +\infty) \to [0, +\infty)$ is locally bounded and $\mathscr{E}_1, \mathscr{E}_2$ are independent of $u(\cdot)$.

(P_4) The function $t \to \varphi_t$ is well described and continuous from the set

$$\mathcal{R}(\varrho^-) = \{\varrho(s, \psi) : (s, \psi) \in \mathscr{I} \times \mathcal{B}\},$$

into $\mathcal{B}$ and there is a continuous and bounded function $J^\varphi : \mathcal{R}(\varrho^-) \to (0, \infty)$ to ensure that $\mathbb{E}\|\varphi_t\|^2_{\mathcal{B}} \le J^\varphi(t)\mathbb{E}\|\varphi\|^2_{\mathcal{B}}$ for every $t \in \mathcal{R}(\varrho^-)$.

(P_5) The space $\mathcal{B}$ is complete.

Let $u : (-\infty, T] \to \mathcal{H}$ be an $\mathcal{F}_t$-adapted measurable process such that we have the $\mathscr{F}_0$-adapted process $u_0 = \varphi(t) \in \mathcal{L}^2(\Omega, \mathcal{B})$, then

$$\mathbb{E}\|u_t\|^2_{\mathcal{B}} \le \mathscr{E}_1^{*2} \sup_{0 \le s \le T} \{\mathbb{E}\|u(s)\|^2\} + \mathscr{E}_2^{*2}\mathbb{E}\|\varphi\|^2_{\mathcal{B}},$$

where $\mathscr{E}_1^* = \sup_{s \in \mathscr{I}} \mathscr{E}_1(s)$ and $\mathscr{E}_2^* = \sup_{s \in \mathscr{I}} \mathscr{E}_2(s)$.

Lemma 2.2. *[14] Let u:* $(-\infty, T] \to \mathcal{H}$ *be a function in a way that* $u_0 = \varphi$ *and* $u \in \mathcal{PC}(\mathscr{I}, \mathcal{L}^2)$ *and if* (P_4) *hold, then*

$$\mathbb{E}\|u_s\|^2_{\mathcal{B}} \le \mathscr{E}_1^{*2} \sup\{\mathbb{E}\|u(\theta)\|^2_{\mathcal{H}} : \theta \in [0, \max\{0, s\}]\} + (\mathscr{E}_2^* + J^\varphi)^2 \mathbb{E}\|u_0\|^2_{\mathcal{B}}, \quad s \in \mathcal{R}(\varrho^-) \cup \mathscr{I},$$

where $J^\varphi = \sup_{t \in \mathcal{R}(\varrho^-)} J^\varphi(t)$*.*

Recognize the space

$$\mathscr{B}_T = \left\{u : (-\infty, T] \to \mathcal{H} \text{ such that } u_0 \in \mathcal{B} \text{ and the constraint } u|_{\mathscr{I}} \in \mathcal{PC}(\mathscr{I}, \mathcal{L}^2)\right\}.$$

The function $\|\cdot\|_{\mathscr{B}_T}$ to be a seminorm in $\mathscr{B}_T$, it is described by

$$\|u\|_{\mathscr{B}_T} = \|\varphi\|_{\mathcal{B}} + \sup\left\{\left(\mathbb{E}\|u(s)\|^2\right)^{\frac{1}{2}} : s \in [0, T]\right\}, \; u \in \mathscr{B}_T.$$

Now, we provide some fundamental definitions and results of the fractional calculus theory that happen to be utilized additionally within this manuscript.

Definition 2.1. *[21] The fractional integral of order γ with the lower limit zero for a function f is determined as*

$$I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \quad \gamma > 0,$$

offered the right part is point-wise described on $[0, +\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *[21] The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f \in \mathcal{L}^1(\mathscr{I}, \mathcal{H})$ is characterized as*

$$D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{1-n+\gamma}} ds, \quad t > 0, \quad n-1 < \gamma < n.$$

Definition 2.3. *[21, 28] The Caputo derivative of order γ for a function $f \in \mathcal{L}^1(\mathscr{I}, \mathcal{H})$ could be consisting as*

$${}^{C}D_t^{\gamma} f(t) = D_t^{\gamma}(f(t) - f(0)), \quad t > 0, \quad 0 < \gamma < 1.$$

Definition 2.4. *[45, Definition 4.59] The Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \alpha, \beta > 0, \ z \in \widetilde{C},$$

where $\widetilde{C}$ denotes the complex plane. When $\beta = 1$, fix $E_{\alpha}(z) = E_{\alpha,1}(z)$.

Definition 2.5. *[45] The Mainardi's function has the form*

$$\phi_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n - \alpha + 1)}, \ 0 < \alpha < 1, \ z \in \widetilde{C}.$$

Presently, we are in a position to characterize the mild solution for the system (1.1)-(1.3). For this, first we assume that the approximate controllability of its linear fractional differtial system

$${}^{C}D_t^{\alpha} x(t) = \mathscr{A} x(t) + Bv(t) + \mathscr{F}(t) \frac{dw(t)}{dt}, \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

where ${}^{C}D_t^{\alpha}$ and $\mathscr{A}$ are defined in (1.1)-(1.3). Now, we first consider the classical solutions to the problem (2.1)-(2.2). Then, based on the expression of such solutions, we define the mild solutions of the problem (2.1)-(2.2). At last, the relations between the analytic semigroup $\{\mathbb{T}(t)\}_{t \geq 0}$ and some solution operators is obtained.

For our convenient at this position to introduce the controllability operator associated with (2.1)-(2.2), thus

$$\Gamma_0^T = \int_0^T \mathcal{S}_{\alpha}(T-s) BB^* \mathcal{S}_{\alpha}^*(T-s) ds,$$

where B^* and $\mathcal{S}_{\alpha}^*$ are the adjoint of B and $\mathcal{S}_{\alpha}$ respectively. It is straightforward that the operator Γ_0^T is a linear bounded operator.

Let $u(T; u_0, v)$ be the state value of (1.1)-(1.3) at teriminal time T corresponding to the control v and the intial value u_0. Introduce the set $\mathscr{R}(T, u_0) = \{u(T; u_0, v) : v \in \mathcal{L}^2(\mathscr{I}, U)\}$, which is called the reachable set of the system (1.1)-(1.3) at terminal time T, its closure in $\mathcal{H}$ is denoted by $\overline{\mathscr{R}(T, u_0)}$.

Definition 2.6. *[44] The system (1.1)-(1.3) is said to be approximately controllable on $\mathscr{I}$ if $\overline{\mathscr{R}(T, u_0)} = \mathcal{L}^2(\Omega, \mathcal{H})$, that is, given an arbitrary $\epsilon > 0$, it is possible to steer from the point u_0 to within a distance ϵ from all points in the state space $\mathcal{H}$ at time T.*

Lemma 2.3. *[44] The linear fractional control system (2.1)-(2.2) is approximately controllable on $\mathscr{I}$ if and only if $\mu(\mu\mathcal{I} + \Gamma_0^T) \to 0$ as $\mu \to 0^+$ in the strong operator topology.*

Lemma 2.4. *([31, Lemma 3.2]) For any $\tilde{u}_T \in \mathcal{L}^2(\mathcal{F}_T, \mathcal{H})$, there exists $\tilde{\phi} \in \mathcal{L}^2_{\mathcal{F}}(\Omega; \mathcal{L}^2(0, T; \mathcal{L}^0_2))$ such that $\tilde{u}_T = \mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s)$.*

Now for any $\mu > 0$ and $\tilde{u}_T \in \mathcal{L}^2(\mathcal{F}_T, \mathcal{H})$, we define the control function

$$v^{\mu}(t) = \begin{cases} B^* S^*_{\alpha}(T-t)(\mu\mathcal{I} + \Gamma_0^T)^{-1}\left[\mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s) - \mathcal{T}_{\alpha}(T)[\varphi(0) - h(u_T) - \mathscr{G}(0,\varphi)]\right] \\ -B^*\mathcal{S}^*_{\alpha}(T-t)(\mu\mathcal{I} + \Gamma_0^T)^{-1}\mathscr{G}(T, u_{\varrho(T,u_T)}) \\ -B^*\mathcal{S}^*_{\alpha}(T-t)\int_0^T (\mu\mathcal{I} + \Gamma_s^T)^{-1}\mathscr{A}\mathcal{S}_{\alpha}(T-s)\mathscr{G}\left(s, u_{\varrho(s,u_s)}\right)ds \\ -B^*\mathcal{S}^*_{\alpha}(T-t)\int_0^T (\mu\mathcal{I} + \Gamma_s^T)^{-1}\mathcal{S}_{\alpha}(T-s)\mathscr{F}\left(s, u_{\varrho(s,u_s)}, \int_0^s e_1(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)ds \\ -B^* S^*_{\alpha}(T-t)\int_0^T (\mu\mathcal{I} + \Gamma_s^T)^{-1}\mathcal{S}_{\alpha}(T-s)\Sigma\left(s, u_{\varrho(s,u_s)}, \int_0^s e_2(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)dw(s) \\ -B^* S^*_{\alpha}(T-t)(\mu\mathcal{I} + \Gamma_0^T)^{-1}\sum_{0<t_k<t}\mathcal{T}_{\alpha}(T-t_k)\mathcal{I}_k(u(t_k^-)). \end{cases}$$

Lemma 2.5. *[33, Lemma 6] Using $\mathscr{A}$ to denote the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$, then if $\mathscr{F}$ satisfies a uniform Hölder condition with exponent $\beta \in (0,1]$, the solution of the Cauchy system (2.1)-(2.2) are fixed points of the subsequent operator equation:*

$$\Psi x(t) = \mathcal{T}_{\alpha}(t)x_0 + \int_0^t \mathcal{S}_{\alpha}(t-s)Bv(s)ds + \int_0^t \mathcal{S}_{\alpha}(t-s)\mathscr{F}(s)dw(s), \tag{2.3}$$

where

$$\mathcal{T}_{\alpha}(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}\lambda^{\alpha-1}R(\lambda^{\alpha}, \mathscr{A})d\lambda \quad \text{and} \quad \mathcal{S}_{\alpha}(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}R(\lambda^{\alpha}, \mathscr{A})d\lambda.$$

Here C is a suitable path satisfying $\lambda^{\alpha} \notin \mu + S_{\theta}$ for some $\lambda \in C$.

Proof. According to the Definitions of 2.1 and 2.2, we modify the Cauchy system (2.1)-(2.2) in the equivalent integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\mathscr{A}x(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{Bv(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\mathscr{F}(s)}{(t-s)^{1-\alpha}}dw(s). \tag{2.4}$$

Let $\lambda > 0$. Making use of the Laplace transform

$$(\mathscr{L}x)(\lambda) = \int_0^{\infty} e^{-\lambda s}x(s)ds, \quad (\mathscr{L}v(t))(\lambda) = \int_0^{\infty} e^{-\lambda s}v(s)ds,$$

$$\text{and} \quad (\mathscr{L}\mathscr{F}(t))(\lambda) = \int_0^{\infty} e^{-\lambda s}\mathscr{F}(s)dw(s)$$

to (2.4) we receive

$$\begin{aligned} (\mathscr{L}x)(\lambda) &= \int_0^{\infty} e^{-\lambda s}\left[x_0 + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{A}x(\theta)}{(s-\theta)^{1-\alpha}}d\theta + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{Bv(\theta)}{(s-\theta)^{1-\alpha}}d\theta \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{F}(\theta)}{(s-\theta)^{1-\alpha}}dw(\theta)\right]ds \\ &= \int_0^{\infty} e^{-\lambda s}x_0 ds + \int_0^{\infty} e^{-\lambda s}\left[\frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{A}x(\theta)}{(s-\theta)^{1-\alpha}}d\theta\right]ds \\ &\quad + \int_0^{\infty} e^{-\lambda s}\left[\frac{1}{\Gamma(\alpha)}\int_0^s \frac{Bv(\theta)}{(s-\theta)^{1-\alpha}}d\theta\right]ds \\ &\quad + \int_0^{\infty} e^{-\lambda s}\left[\frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{F}(\theta)}{(s-\theta)^{1-\alpha}}dw(\theta)\right]ds \\ &= \frac{1}{\lambda}\left[e^{-\lambda s}\right]_{\infty}^0 x_0 + \frac{1}{\lambda^{\alpha}}\mathscr{A}(\mathscr{L}x)(\lambda) + \frac{1}{\lambda^{\alpha}}B(\mathscr{L}v(t))(\lambda) + \frac{1}{\lambda^{\alpha}}(\mathscr{L}\mathscr{F}(t))(\lambda) \end{aligned}$$

$$
\begin{aligned}
(\mathscr{L}x)(\lambda) - \frac{1}{\lambda^\alpha}\mathscr{A}(\mathscr{L}x)(\lambda) &= \frac{1}{\lambda}x_0 + + \frac{1}{\lambda^\alpha}B(\mathscr{L}v(t))(\lambda) + \frac{1}{\lambda^\alpha}(\mathscr{L}\mathscr{F}(t))(\lambda)\\
(\lambda^\alpha I - \mathscr{A})(\mathscr{L}x)(\lambda) &= \frac{\lambda^\alpha}{\lambda}x_0 + B(\mathscr{L}v(t))(\lambda) + (\mathscr{L}\mathscr{F}(t))(\lambda)\\
(\mathscr{L}x)(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha I - \mathscr{A})^{-1}x_0 + (\lambda^\alpha I - \mathscr{A})^{-1}B(\mathscr{L}v(t))(\lambda)\\
&\quad + (\lambda^\alpha I - \mathscr{A})^{-1}(\mathscr{L}\mathscr{F}(t))(\lambda).
\end{aligned}
$$

Using $\lambda^\alpha(\lambda^\alpha - \mathscr{A})^{-1} = I + \mathscr{A}(\lambda^\alpha - \mathscr{A})^{-1}$, the above equation is then inverse Laplace transformed to obtain

$$x(t) = \mathcal{T}_\alpha(t)x_0 + \int_0^t \mathcal{S}_\alpha(t-s)Bv(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s)dw(s).$$

It is noted that $\mathscr{F}$ satisfy a uniform Hölder condition with exponent $\beta \in (0,1)$. Hence, the classical solutions of Cauchy system (2.1)-(2.2) are fixed points of the operator equation (2.3). □

In view of Lemma 2.5, we determine the mild solutions of the system (2.1)-(2.2).

Definition 2.7. *A function $x : \mathscr{I} \to \mathcal{H}$ is considered to be a mild solution of problem (2.1)-(2.2) if $x \in C(\mathscr{I}, \mathcal{H})$ fulfills the accompanying integral equation:*

$$x(t) = \mathcal{T}_\alpha(t)x_0 + \int_0^t \mathcal{S}_\alpha(t-s)Bv(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s)dw(s), \quad t \in \mathscr{I}.$$

Remark 2.1. *It is straightforward to confirm that the classical solution of the system (2.1)-(2.2) is a mild solution of the same system. Thus, Definition 2.7 is well defined (see [23, 27]).*

Lemma 2.6. *[33, Lemma 9] Assuming $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup, given by $\{\mathbb{T}(t)\}_{t\geq 0}$ and $0 \in \wp(\mathscr{A})$, then we have*

$$\mathcal{S}_\alpha(t) = \alpha\int_0^\infty r\phi_\alpha(r)t^{\alpha-1}\mathbb{T}(t^\alpha r)dr \quad \text{and} \quad \mathcal{T}_\alpha(t) = \int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr. \tag{2.5}$$

Here $\phi_\alpha(r)$ is the probability density function characterized on $(0,\infty)$ in such a way that its Laplace transform has the form

$$\int_0^\infty e^{-rx}\phi_\alpha(r)dr = \sum_{j=0}^\infty \frac{(-x)^j}{\Gamma(1+\alpha j)}, \quad x > 0,$$

which fulfills

$$\int_0^\infty \phi_\alpha(r)dr = 1 \quad \text{and} \quad \int_0^\infty r^\eta\phi_\alpha(r)dr \leq 1, \quad 0 \leq \eta \leq 1.$$

Proof. For all $x \in D(\mathscr{A}) \subset \mathcal{H}$, we have

$$(\lambda - \mathscr{A})^{-1}x = \int_0^\infty e^{-\lambda s}\mathbb{T}(s)xds.$$

Let

$$\int_0^\infty e^{-\lambda r}\psi_\alpha(r)dr = e^{-\lambda^\alpha},$$

where $\alpha \in (0,1)$, $\psi_\alpha(r) = \frac{1}{\pi}\sum_{1\leq n<\infty}(-1)^n r^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha)$, and $r \in (0,\infty)$(see[26]). Thus, we get

$$
\begin{aligned}
(\lambda^\alpha - \mathscr{A})^{-1}x &= \int_0^\infty e^{-\lambda^\alpha s}\mathbb{T}(s)xds\\
&= \int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^\alpha}\mathbb{T}(t^\alpha)xdt\\
&= \int_0^\infty \alpha t^{\alpha-1}\left[\int_0^\infty e^{-\lambda tr}\psi_\alpha(r)dr\right]\mathbb{T}(t^\alpha)xdt\\
&= \int_0^\infty \alpha\left[\int_0^\infty e^{-\lambda t}\psi_\alpha(r)dr\right]\mathbb{T}\left(\frac{t^\alpha}{r^\alpha}\right)x\frac{t^{\alpha-1}}{r^\alpha}dt\\
&= \int_0^\infty e^{-\lambda t}\left(\alpha\int_0^\infty r\phi_\alpha(r)t^{\alpha-1}\mathbb{T}(t^\alpha r)xdr\right)dt, \qquad (2.6)
\end{aligned}
$$

where $\phi_\alpha(r) = \left(\frac{1}{\alpha}\right) r^{-1-\frac{1}{\alpha}} \psi_\alpha\left(r^{\frac{-1}{\alpha}}\right)$ is the probability density function outlined on $(0,\infty)$ in such a way that

$$\int_0^\infty \phi_\alpha(r)dr = 1 \quad \text{and} \quad \int_0^\infty r^\eta \phi_\alpha(r)dr \le 1, \quad 0 \le \eta \le 1.$$

In perspective of Lemma 2.5 and equation (2.6), we sustain

$$\begin{aligned}
\mathcal{S}_\alpha(t) &= \frac{1}{2\pi i}\int_C e^{\lambda t} R(\lambda^\alpha, \mathscr{A}) d\lambda \\
&= \int_0^\infty e^{\lambda t}(\lambda^\alpha - \mathscr{A})^{-1} dt \\
&= \alpha \int_0^\infty r\phi_\alpha(r) t^{\alpha-1} \mathbb{T}(t^\alpha r) dr.
\end{aligned}$$

Further, we calculate the estimation of $\mathcal{S}_\alpha(t)$:

$$\begin{aligned}
\|\mathcal{S}_\alpha(t)\| &= \left\| \alpha \int_0^\infty r\phi_\alpha(r) t^{\alpha-1} \mathbb{T}(t^\alpha r) dr \right\| \\
&\le \alpha \left[\int_0^\infty r\phi_\alpha(r) dr \right] t^{\alpha-1} \|\mathbb{T}(t^\alpha r)\| \\
&\le \alpha \frac{\Gamma(2)}{\Gamma(1+\alpha)} t^{\alpha-1} \mathcal{M} \\
&\le \frac{\mathcal{M}}{\Gamma(\alpha)} t^{\alpha-1},
\end{aligned}$$

where $\int_0^\infty r^\beta \phi_\alpha(r) dr = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}$.

Then again, for all $x \in D(\mathscr{A}) \subset \mathcal{H}$, we notice that

$$\begin{aligned}
\lambda^{\alpha-1}(\lambda^\alpha - \mathscr{A})^{-1} x &= \int_0^\infty \lambda^{\alpha-1} e^{-\lambda^\alpha s} \mathbb{T}(s) x ds \\
&= \int_0^\infty \alpha(\lambda t)^{\alpha-1} e^{-(\lambda t)^\alpha} \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty \frac{-1}{\lambda} \frac{d}{dt} \left[e^{-(\lambda t)^\alpha} \right] \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty \frac{-1}{\lambda} \frac{d}{dt} \left[\int_0^\infty e^{-\lambda t r} \psi_\alpha(r) dr \right] \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty \left[\int_0^\infty \frac{-1}{\lambda} \left[-\lambda r e^{-\lambda t r} \right] \psi_\alpha(r) dr \right] \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty \int_0^\infty r e^{-\lambda t r} \psi_\alpha(r) dr \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \psi_\alpha(r) \mathbb{T}\left(\frac{t^\alpha}{r^\alpha} \right) x dr \right] dt \\
&= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \phi_\alpha(r) \mathbb{T}(t^\alpha r) x dr \right] dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{T}_\alpha(t) &= \frac{1}{2\pi i}\int_C e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, \mathscr{A}) d\lambda \\
&= \int_0^\infty e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - \mathscr{A})^{-1} dt \\
&= \int_0^\infty \phi_\alpha(r) \mathbb{T}(t^\alpha r) dr.
\end{aligned}$$

Moreover, the estimation of $\mathcal{T}_\alpha(t)$ is

$$\begin{aligned}\|\mathcal{T}_\alpha(t)\| &= \left\| \int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr \right\| \\ &\leq \left(\int_0^\infty \phi_\alpha(r)dr \right) \|\mathbb{T}(t^\alpha r)\| \\ &\leq \mathcal{M},\end{aligned}$$

where $\int_0^\infty \phi_\alpha(r)dr = 1$. □

Before we characterize the mild solution for the system (1.1)-(1.3), finally, we treat the following system:

$$^{C}D_t^\alpha \left[x(t) - \mathscr{G}(t, x(t))\right] = \mathscr{A}x(t) + Bv(t) + \mathscr{F}(t, x(t)) + \Sigma(t, x(t))\frac{dw(t)}{dt}, \quad t \neq t_k, \tag{2.7}$$

$$\Delta x(t_k) = \mathcal{I}_k(x(t_k^-)), \quad k = 1, 2, \cdots, n, \tag{2.8}$$

$$x(0) + h(x) = \varphi(0), \tag{2.9}$$

where $^{C}D_t^\alpha, B, v(t)$ and $\mathscr{A}$ are defined in (1.1)-(1.3) and $\mathscr{F}, \Sigma, \mathscr{G}$ are appropriate functions.

From the Definition of 2.1 and 2.2, the general integral equation of the system (2.7)-(2.9) can be expressed as

$$\begin{aligned}x(t) = {} & \varphi(0) - \mathscr{G}(0, \varphi) - h(x) + \mathscr{G}(t, x(t)) + \sum_{k=1}^n \mathcal{I}_k(x(t_k^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{A}x(s)ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bv(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{F}(s, x(s))ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Sigma(s, x(s))dw(s).\end{aligned} \tag{2.10}$$

Presently, we take after the thought utilized as a part of the paper [46] and apply the Laplace transformation for (2.10), we get

$$\begin{aligned}u(\lambda) = {} & \lambda^{\alpha-1}(\lambda^\alpha I - \mathscr{A})^{-1}[\varphi(0) - \mathscr{G}(0, \varphi) - h(x)] + \lambda^\alpha(\lambda^\alpha I - \mathscr{A})^{-1}w(\lambda) + (\lambda^\alpha I - \mathscr{A})^{-1}v(\lambda) \\ & + (\lambda^\alpha I - \mathscr{A})^{-1}By(\lambda) + (\lambda^\alpha I - \mathscr{A})^{-1}z(\lambda) + \lambda^{\alpha-1}(\lambda^\alpha I - \mathscr{A})^{-1} \sum_{k=1}^n \mathcal{I}_k(x(t_k^-)),\end{aligned}$$

where

$$\begin{aligned}& u(\lambda) = \int_0^\infty e^{-\lambda s}x(s)ds, \quad v(\lambda) = \int_0^\infty e^{-\lambda s}\mathscr{F}(s, x(s))ds, \quad w(\lambda) = \int_0^\infty e^{-\lambda s}\mathscr{G}(s, x(s))ds, \\ & y(\lambda) = \int_0^\infty e^{-\lambda s}v(s)ds, \quad z(\lambda) = \int_0^\infty e^{-\lambda s}\Sigma(s, x(s))dw(s).\end{aligned}$$

At that point by the same calculations in [46] and the properties of the Laplace transform, we obtain the mild solution of the system (2.7)-(2.9) as

$$x(t) = \begin{cases} \mathcal{T}_\alpha(t)[\varphi(0) - \mathscr{G}(0, \varphi) - h(x)] + \mathscr{G}(t, x(t)) + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, x(s))ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)Bv(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s, x(s))\,ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\Sigma(s, x(s))\,dw(s) + \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-)). \end{cases} \tag{2.11}$$

where $\mathcal{T}_\alpha$ and $\mathcal{S}_\alpha$ are same as defined in (2.5).

Next, we shall show that this mild solution satisfy the system (2.7)-(2.9). To prove this, first we prove the following crucial lemma.

Lemma 2.7. *[32, Lemma 3.3] Assuming $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup, given by $\{\mathbb{T}(t)\}_{t\geq 0}$ and if $0<\alpha<1$, then*

$$^{C}D_t^{\alpha}[\mathcal{T}_\alpha(t)x_0]=\mathscr{A}[\mathcal{T}_\alpha(t)x_0],$$

and

$$\begin{aligned}&^{C}D_t^{\alpha}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s))+Bv(s)+\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\right)\\&=\mathscr{A}\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s))+Bv(s)+\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\\&\quad+\mathscr{A}\mathscr{G}(t,x(t))+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt},\end{aligned}$$

where $\mathcal{T}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are same as defined in equation (2.5).

Proof. By the well known result from [32, Lemma 3.3], we have

$$^{C}D_t^{\alpha}[\mathcal{T}_\alpha(t)x_0]=\mathscr{A}[\mathcal{T}_\alpha(t)x_0].$$

Furthermore,

$$\begin{aligned}&\mathscr{L}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s))+Bv(s)+\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\right)\\&=\mathscr{L}(\mathcal{S}_\alpha(t))\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t))+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt}\right)\\&=R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t))+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt}\right)\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}&\mathscr{L}\left(\ ^{C}D_t^{\alpha}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s))+Bv(s)+\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\right)\right)\\&=\lambda^\alpha\left[R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t))+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt}\right)\right]-\lambda^{\alpha-1}.0\\&=(\lambda^\alpha I-\mathscr{A}+\mathscr{A})R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t))+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt}\right)\\&=(\lambda^\alpha I-\mathscr{A})R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t))+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt}\right)\\&\quad+\mathscr{A}R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t))+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt}\right).\end{aligned}\tag{2.13}$$

Thus, it follows from (2.12) and (2.13) that

$$\begin{aligned}&^{C}D_t^{\alpha}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s))+Bv(s)+\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\right)\\&=\mathscr{A}\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s))+Bv(s)+\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\\&\quad+\mathscr{A}\mathscr{G}(t,x(t))+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt}.\end{aligned}$$

□

Now, it is time to show that the mild solution satisfy the model (2.7)-(2.9). From the equation (2.11), we have

$$\begin{aligned}x(t)-\mathscr{G}(t,x(t))&=\mathcal{T}_\alpha(t)[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)]+\int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}(s,x(s))+Bv(s)\\&\quad+\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\Big]ds+\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-)).\end{aligned}$$

Taking Caputo derivative on both sides and with regard of above Lemma 2.7 , we have

$$\begin{aligned}
{}^{C}D_t^{\alpha}\Big(x(t)-\mathscr{G}(t,x(t))\Big) &= {}^{C}D_t^{\alpha}\Big(\mathcal{T}_\alpha(t)[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)]\Big) + {}^{C}D_t^{\alpha}\Big(\int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}(s,x(s)) \\
&\quad + Bv(s) + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\Big]ds\Big) \\
&\quad + {}^{C}D_t^{\alpha}\Big(\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-))\Big) \\
&= \mathscr{A}\mathcal{T}_\alpha(t)[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)] + \mathscr{A}\Big(\int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}(s,x(s)) + Bv(s) \\
&\quad + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\Big]ds\Big) + \mathscr{A}\mathscr{G}(t,x(t)) + Bv(t) + \mathscr{F}(t,x(t)) \\
&\quad + \Sigma(t,x(t))\frac{dw(t)}{dt} + \mathscr{A}\Big(\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-))\Big) \\
&= \mathscr{A}\Big(\mathcal{T}_\alpha(t)[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)] + \mathscr{G}(t,x(t)) + \int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}(s,x(s)) \\
&\quad + Bv(s) + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\Big]ds + \sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-))\Big) \\
&\quad + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt} \\
&= \mathscr{A}x(t) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt}.
\end{aligned}$$

That is

$$ {}^{C}D_t^{\alpha}\big(x(t)-\mathscr{G}(t,x(t))\big) = \mathscr{A}x(t) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt}. $$

From the above discussion, we observe that our definition of a mild solution satisfies the given system (2.7)-(2.9).

In accordance with the above discussion, we determine the mild solution of the model (1.1)-(1.3).

Definition 2.8. *[44, Definition 2.1] A stochastic process $u:(-\infty,T]\to\mathcal{H}$ is called a mild solution of the system (1.1)-(1.3) if*

(i) $u(t)$ *is measurable and* $\mathcal{F}_t$*-adapted for each* $t\in\mathscr{I}$;

(ii) $\Delta u(t_k) = u(t_k^+) - u(t_k^-) = \mathcal{I}_k(x(t_k^-)), \quad k=1,2,\ldots n;$

(iii) $u(0)+h(u)=\varphi;$

(iv) $u(t)$ *is continuous on* $\mathscr{I}$, *the function* $\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,u_{\varrho(s,u_s)})$ *is integrable and the following stochastic integral equation is satisfied,*

$$u(t) = \begin{cases} \mathcal{T}_\alpha(t)[\varphi(0)-h(u)-\mathscr{G}(0,\varphi)] + \mathscr{G}(t,u_{\varrho(t,u_t)}) + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,u_{\varrho(s,u_s)})ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)Bv^{\mu}ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\left(s,u_{\varrho(s,u_s)},\int_0^s e_1(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\Sigma\left(s,u_{\varrho(s,u_s)},\int_0^s e_2(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)dw(s) \\ + \sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(u(t_k^-)). \end{cases} \tag{2.14}$$

(v) $u_0(\cdot)=\varphi\in\mathcal{B}$ *on* $(-\infty,0]$ *satisfying* $\|\varphi\|_{\mathcal{B}}^2<\infty$.

3 The main results

In this segment, we show and demonstrate the controllability of solutions for the model (1.1)-(1.3) under Schauder [30] fixed point theorem together with operator semigroups and fractional calculus.

Presently, we itemizing the subsequent suppositions:

(H0) $\mathcal{S}_\alpha(t)$, $t > 0$ is compact.

(H1) The function $\mathscr{G} : \mathscr{I} \times \mathcal{B} \to \mathcal{H}$ is continuous and there exist some constants $\beta \in (0,1)$ and $\mathcal{M}_\mathscr{G} > 0$ such that $\mathscr{G}$ is $\mathcal{H}_\beta$-valued and it satisfies the following conditions.

$$\mathbb{E}\|\mathscr{A}^\beta \mathscr{G}(t,x)\|^2 \leq \mathcal{M}_\mathscr{G}(1+\|x\|^2_\mathcal{B}), \quad t \in \mathscr{I}, \quad x \in \mathcal{B}.$$

(H2) The function $\mathscr{F} : \mathscr{I} \times \mathcal{B} \times \mathcal{H} \to \mathcal{H}$ is continuous and there exist two continuous functions $\mathscr{F}_1, \mathscr{F}_2 : \mathscr{I} \to (0,\infty)$ such that
$$\mathbb{E}\|\mathscr{F}(t,x,\phi)\|^2_\mathcal{H} \leq \mathscr{F}_1(t)\|x\|^2_\mathcal{B} + \mathscr{F}_2(t)\mathbb{E}\|\phi\|^2_\mathcal{H}, \qquad (t,x,\phi) \in \mathscr{I} \times \mathcal{B} \times \mathcal{H},$$
and $\mathscr{F}_1^* = \sup_{s\in[0,t]} \mathscr{F}_1(s)$, $\quad \mathscr{F}_2^* = \sup_{s\in[0,t]} \mathscr{F}_2(s)$.

(H3) The function $e_i : \mathcal{D} \times \mathcal{B} \to \mathcal{H}$, where $\mathcal{D} = \{(t,s) \in \mathscr{I} \times \mathscr{I}; 0 \leq s \leq t \leq T\}$ satisfies:

(i) For each $(t,s) \in \mathcal{D}$, the function $e_i(t,s,.) : \mathcal{B} \to \mathcal{H}$ is continuous, and for each $\phi \in \mathcal{B}$, the function $e_i(\cdot,\cdot,\phi) : \mathcal{D} \to \mathcal{H}$ is strongly measurable.

(ii) There exist constants $\widetilde{\mathcal{M}}_0$, $\widetilde{\mathcal{M}}_1 > 0$ such that for all $t,s \in \mathscr{I}$ and $x \in \mathcal{B}$,

$$\mathbb{E}\|e_i(t,s,x)\|^2 \leq \widetilde{\mathcal{M}}_j(1+\|x\|^2_\mathcal{B}), \quad \text{for} \quad i = 1,2 \quad \text{and} \quad j = 0,1.$$

(H4) The function Σ: $\mathscr{I} \times \mathcal{B} \times \mathcal{H} \to \mathcal{L}(\mathcal{K},\mathcal{H})$ is continuous and there exist two continuous functions $\Sigma_1, \Sigma_2 : \mathscr{I} \to (0,\infty)$ such that
$$\mathbb{E}\|\Sigma(t,x,\phi)\|^2_\mathcal{H} \leq \Sigma_1(t)\|x\|^2_\mathcal{B} + \Sigma_2(t)\mathbb{E}\|\phi\|^2_\mathcal{H}, \qquad (t,x,\phi) \in \mathscr{I} \times \mathcal{B} \times \mathcal{H},$$
and $\Sigma_1^* = \sup_{s\in[0,t]} \Sigma_1(s)$, $\quad \Sigma_2^* = \sup_{s\in[0,t]} \Sigma_2(s)$.

(H5) The function $\mathcal{I}_k : \mathcal{B} \to \mathcal{H}, k = 1,2,\ldots,n$ are continuous and there exist non-decreasing continuous functions $\mathcal{M}_{\mathcal{I}_k} : \mathcal{R}^+ \to \mathcal{R}^+$ such that for each $x \in \mathcal{B}$,

$$\mathbb{E}\|\mathcal{I}_k(x)\|^2 \leq \mathcal{M}_{\mathcal{I}_k}(\mathbb{E}\|x\|^2_\mathcal{B}), \qquad \liminf_{r\to\infty} \frac{\mathcal{M}_{\mathcal{I}_k}(r)}{r} = \gamma_k < \infty.$$

(H6) The function $h : \mathcal{B} \to \mathcal{H}$ is continuous and there exists a constant $\mathcal{M}_h > 0$ such that for each $x \in \mathcal{B}$, we sustain

$$\mathbb{E}\|h(x)\|^2 \leq \mathcal{M}_h\|x\|^2_\mathcal{B}.$$

Presently, we are in a position to derive the controllability results for the model (1.1)-(1.3).

Theorem 3.1. *Assume that the assumptions $(H0)$-$(H6)$ hold. Then the system (1.1)-(1.3) has a mild solution on $\mathscr{I}$ provided that*

$$32\left(1+\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^4\frac{T^{4\alpha-2}}{\alpha^2}\right)\left[\mathcal{M}^2\left(\mathcal{M}_h + H^2 n\sum_{k=1}^n \gamma_k\right) + \mathcal{M}_\mathscr{G}\left(\mathcal{N}_0{}^2 + \left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right)^2\right)\right.$$
$$\left.+\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^* + tr(Q)\Sigma_1^* + \left(\mathscr{F}_2^*\widetilde{\mathcal{M}}_0 + \Sigma_2^* tr(Q)\widetilde{\mathcal{M}}_1\right)T\right]\right]\mathscr{E}_1^{*2} < 1 \tag{3.1}$$

where $\mathcal{N}_0 = \|\mathscr{A}^{-\beta}\|$.

Proof. We will transform the model (1.1)-(1.3) into a fixed-point problem. Recognize the operator Υ : $\mathscr{B}_T \to \mathscr{B}_T$ specified by

$$(\Upsilon u)(t) = \begin{cases} \varphi(t), \quad t \in (-\infty, 0], \\ \mathcal{T}_\alpha(t)[\varphi(0) - h(u) - \mathscr{G}(0,\varphi)] + \mathscr{G}(t, u_{\varrho(t,u_t)}) + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, u_{\varrho(s,u_s)})ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)Bv^\mu(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\left(s, u_{\varrho(s,u_s)}, \int_0^s e_1(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\Sigma\left(s, u_{\varrho(s,u_s)}, \int_0^s e_2(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)dw(s) \\ + \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(u(t_k^-)), \quad t \in \mathscr{I}. \end{cases}$$

In perspective of Lemma 2.1 and for any $u \in \mathcal{H}$ and $\beta \in (0,1)$, we have

$$\begin{aligned} &\|\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,u_{\varrho(s,u_s)})\|^2_{\mathcal{H}} \\ &= \|\mathscr{A}^{1-\beta}\mathcal{S}_\alpha(t-s)\mathscr{A}^\beta\mathscr{G}(s,u_{\varrho(s,u_s)})\|^2_{\mathcal{H}} \\ &\le \left\| \left[\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathscr{A}^{1-\beta}\mathbb{T}((t-s)^\alpha r)dr\right]\mathscr{A}^\beta\mathscr{G}(s,u_{\varrho(s,u_s)})\right\|^2_{\mathcal{H}} \\ &\le \left(\alpha\mathcal{C}_{1-\beta}(t-s)^{\alpha\beta-1}\right)^2\left[\int_0^\infty r^\beta\phi_\alpha(r)dr\right]^2\|\mathscr{A}^\beta\mathscr{G}(s,u_{\varrho(s,u_s)})\|^2_{\mathcal{H}}. \end{aligned} \tag{3.2}$$

On the other hand, from $\int_0^\infty r^{-q}\psi_\alpha(r)dr = \frac{\Gamma(1+\frac{q}{\alpha})}{\Gamma(1+q)}$, for all $q \in [0,1]$ (see [46, Lemma 3.2]), we have

$$\int_0^\infty r^\beta\phi_\alpha(r)dr = \int_0^\infty \frac{1}{r^{\beta\alpha}}\psi_\alpha(r)dr = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}. \tag{3.3}$$

Then, by (3.2) and (3.3) , it is easy to see that

$$\|\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,u_{\varrho(s,u_s)})\|^2_{\mathcal{H}} \le \left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)(t-s)^{1-\alpha\beta}}\right)^2\|\mathscr{A}^\beta\mathscr{G}(s,u_{\varrho(s,u_s)})\|^2_{\mathcal{H}}. \tag{3.4}$$

It is obvious that the function $s \to \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,u_{\varrho(s,u_s)})$ is integrable on [0, t) for every $t > 0$.

It is evident that the fixed points of the operator Υ are mild solutions of the model (1.1)-(1.3). We express the function $x(\cdot) : (-\infty, T] \to \mathcal{H}$ by

$$x(t) = \begin{cases} \varphi(t), \quad t \le 0; \\ \mathcal{T}_\alpha(t)\varphi(0), \quad t \in \mathscr{I}, \end{cases}$$

then $x_0 = \varphi$. For every function $z \in C(\mathscr{I}, \mathcal{R}^+)$ with $z(0) = 0$, we allocate as $\bar{z}$ is characterized by

$$\bar{z}(t) = \begin{cases} 0, \quad t \le 0; \\ z(t), \quad t \in \mathscr{I}. \end{cases}$$

If $u(\cdot)$ fulfills (2.14), we are able to split it as $u(t) = z(t) + x(t)$, $t \in \mathscr{I}$, which suggests $u_t = z_t + x_t$, for each $t \in \mathscr{I}$ and also the function $z(\cdot)$ fulfills

$$z(t) = \begin{cases} \mathcal{T}_\alpha(t)[-h(z_t+x_t) - \mathscr{G}(0,\varphi)] + \mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}) \\ + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\right)ds + \int_0^t \mathcal{S}_\alpha(t-s)Bv^\mu(s)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s) \\ \quad (\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s) \\ \quad (\times)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s) \\ + \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z(t_k^-) + x(t_k^-)), \quad t \in \mathscr{I}. \end{cases}$$

Let $\mathscr{B}_T^0 = \{z \in \mathscr{B}_T : z_0 = 0 \in \mathcal{B}\}$. Let $\|\cdot\|_{\mathscr{B}_T^0}$ be the seminorm in $\mathscr{B}_T^0$ described by

$$\|z\|_{\mathscr{B}_T^0} = \sup_{s\in\mathscr{I}} \left(\mathbb{E}\|z(s)\|^2\right)^{\frac{1}{2}} + \|z_0\|_{\mathcal{B}} = \sup_{s\in\mathscr{I}} \left(\mathbb{E}\|z(s)\|^2\right)^{\frac{1}{2}}, \quad z \in \mathscr{B}_T^0,$$

as a result $(\mathscr{B}_T^0, \|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. Set $B_r = \{z \in \mathscr{B}_T^0 : \|z\|^2 \le r\}$ for some $r \ge 0$; then for each r, $B_r \subset \mathscr{B}_T^0$ is clearly a bounded closed convex set. For $z \in B_r$, from Lemma 2.2 and along with the above discussion, we get

$$\begin{aligned}
&\mathbb{E}\|z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}\|_{\mathcal{B}}^2 \\
&\le 2\left(\mathbb{E}\|z_{\varrho(t,z_t+x_t)}\|_{\mathcal{B}}^2 + \mathbb{E}\|x_{\varrho(t,z_t+x_t)}\|_{\mathcal{B}}^2\right) \\
&\le 4\Bigg(\mathscr{E}_1^{*2} \sup_{\substack{0\le s\le \max(0,t)\\ t\in\mathcal{R}(\varrho^-)\cup\mathscr{I}}} \mathbb{E}\|z(s)\|^2 + (\mathscr{E}_2^* + J^{\varphi})^2\mathbb{E}\|z_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2} \sup_{\substack{0\le s\le \max(0,t)\\ t\in\mathcal{R}(\varrho^-)\cup\mathscr{I}}} \mathbb{E}\|x(s)\|^2 \\
&\quad + (\mathscr{E}_2^* + J^{\varphi})^2\mathbb{E}\|x_0\|_{\mathcal{B}}^2\Bigg) \\
&\le 4\left(\mathscr{E}_1^{*2} r + (\mathscr{E}_2^* + J^{\varphi})^2\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\mathbb{E}\|\mathcal{T}_\alpha(t)\varphi(0)\|^2\right) \\
&\le 4\left(\mathscr{E}_1^{*2} r + (\mathscr{E}_2^* + J^{\varphi})^2\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\left\|\int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr\right\|^2 \mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2\right) \\
&\le 4\mathscr{E}_1^{*2}\left(r + \mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2\right) + 4(\mathscr{E}_2^* + J^{\varphi})^2\mathbb{E}\|\varphi\|_{\mathcal{B}}^2 \\
&\le 4\mathscr{E}_1^{*2} r + c_n = r^*,
\end{aligned} \tag{3.5}$$

where $c_n = 4\left[\mathscr{E}_1^{*2}\mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2 + (\mathscr{E}_2^* + J^{\varphi})^2\mathbb{E}\|\varphi\|_{\mathcal{B}}^2\right]$ and

$$\begin{aligned}
\mathbb{E}\|z_t + x_t\|_{\mathcal{B}}^2 &\le 2(\mathbb{E}\|z_t\|_{\mathcal{B}}^2 + \mathbb{E}\|x_t\|_{\mathcal{B}}^2) \\
&\le 4\left(\mathscr{E}_2^{*2}\mathbb{E}\|z_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\sup_{s\in\mathscr{I}}\mathbb{E}\|z(s)\|^2 + \mathscr{E}_2^{*2}\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\sup_{s\in\mathscr{I}}\mathbb{E}\|x(s)\|^2\right) \\
&\le 4\left(\mathscr{E}_1^{*2} r + \mathscr{E}_2^{*2}\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\mathbb{E}\|\mathcal{T}_\alpha(t)\varphi(0)\|^2\right) \\
&\le 4\left(\mathscr{E}_1^{*2} r + \mathscr{E}_2^{*2}\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\left\|\int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr\right\|^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2\right) \\
&\le 4\left(\mathscr{E}_1^{*2}(r + \mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2) + \mathscr{E}_2^{*2}\mathbb{E}\|\varphi\|_{\mathcal{B}}^2\right) \\
&\le 4\mathscr{E}_1^{*2} r + \widetilde{c}_n = \widetilde{r},
\end{aligned} \tag{3.6}$$

where $\widetilde{c}_n = 4\left[\mathscr{E}_1^{*2}\mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2 + \mathscr{E}_2^{*2}\mathbb{E}\|\varphi\|_{\mathcal{B}}^2\right]$. We delimit the operator $\overline{\Upsilon} : \mathscr{B}_T^0 \to \mathscr{B}_T^0$ by

$$\begin{aligned}
(\overline{\Upsilon}z)(t) = {}& \mathcal{T}_\alpha(t)[-h(z_t + x_t) - \mathscr{G}(0,\varphi)] + \mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}) \\
&+ \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\right)ds + \int_0^t \mathcal{S}_\alpha(t-s)Bv^\mu(s)ds \\
&+ \int_0^t \mathcal{S}_\alpha(t-s) \\
&\quad (\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds \\
&+ \int_0^t \mathcal{S}_\alpha(t-s) \\
&\quad (\times)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s) \\
&+ \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z(t_k^-) + x(t_k^-)), \quad t \in \mathscr{I}.
\end{aligned}$$

It is vindicated that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ has a fixed point. Thus, let us demonstrate that $\overline{\Upsilon}$ has a fixed point.

The facts of the theorem is lengthy and technical. Therefore it is practical to split it into several steps.
Step 1: $\overline{\Upsilon}(B_r) \subset B_r$ for some $r > 0$.

We assert that there exists a positive integer r in ways that $\overline{\Upsilon}(B_r) \subset B_r$. If it is not true, then for each positive number r, we can find a function $z^r(\cdot) \in B_r$, but $\overline{\Upsilon}(z^r) \notin B_r$, i.e., $\mathbb{E}\|\overline{\Upsilon}(z^r)(t)\|^2 > r$ for some $t \in \mathscr{I}$, we sustain

$$
\begin{aligned}
&\mathbb{E}\|v^{\mu}(s)\|^2 \\
&\leq 10\mathbb{E}\left\| B^*\mathcal{S}_{\alpha}^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\left[\mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s) - \mathcal{T}_{\alpha}(T)[\varphi(0) - h(z_T + x_T)]\right] \right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_{\alpha}^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathcal{T}_{\alpha}(T)\mathscr{G}(0,\varphi) \right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_{\alpha}^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathscr{G}(T, z_{\varrho(T,z_T+x_T)} + x_{\varrho(T,z_T+x_T)}) \right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_{\alpha}^*(T-t)\int_0^T (\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathscr{A}\mathcal{S}_{\alpha}(T-s)\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\right) ds \right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_{\alpha}^*(T-t)\int_0^T (\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathcal{S}_{\alpha}(T-s) \right. \\
&\qquad \left. (\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right) ds \right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_{\alpha}^*(T-t)\int_0^T (\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathcal{S}_{\alpha}(T-s) \right. \\
&\qquad \left. (\times)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right) dw(s) \right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_{\alpha}^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\sum_{0<t_k<t}\mathcal{T}_{\alpha}(T-t_k)\mathcal{I}_k(z(t_k^-) + x(t_k^-)) \right\|^2 \\
&= \sum_{i=1}^{7} J_i. \qquad (3.7)
\end{aligned}
$$

By using (3.4), (3.5), (3.6),(H1)-(H6) and Holder's inequality, we receive

$$
\begin{aligned}
J_1 &= 10\mathbb{E}\left\| B^*\mathcal{S}_{\alpha}^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\left[\mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s) - \mathcal{T}_{\alpha}(T)[\varphi(0) - h(z_T + x_T)]\right] \right\|^2 \\
&\leq 10\mathcal{M}_B^2 \left\| \alpha\int_0^{\infty} r\phi_{\alpha}(r)(T-t)^{\alpha-1}\mathbb{T}((T-t)^{\alpha}r)dr \right\|^2 \frac{1}{\mu^2}\left[\mathbb{E}\|\tilde{u}_T\|^2 + \int_0^T \mathbb{E}\|\tilde{\phi}(s)\|^2 ds \right. \\
&\quad \left. + \left\| \int_0^{\infty}\phi_{\alpha}(r)\mathbb{T}(T^{\alpha}r)dr \right\|^2 \left[\mathbb{E}\|\varphi(0)\|^2 + \mathbb{E}\|h(z_T + x_T)\|^2\right]\right] \\
&\leq 10\mathcal{M}_B^2\left(\frac{\alpha\mathcal{M}T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\frac{1}{\mu^2}\left[\mathbb{E}\|\tilde{u}_T\|^2 + \int_0^T \mathbb{E}\|\tilde{\phi}(s)\|^2 ds + \mathcal{M}^2\left[\mathbb{E}\|\varphi(0)\|^2 + \mathbb{E}\|h(z_T + x_T)\|^2\right]\right] \\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left[\mathbb{E}\|\tilde{u}_T\|^2 + \int_0^T \mathbb{E}\|\tilde{\phi}(s)\|^2 ds + \mathcal{M}^2\left[\mathbb{E}\|\varphi(0)\|^2 + \mathcal{M}_h\tilde{r}\right]\right], \\
&\qquad \text{where}\|B^*\| = \mathcal{M}_B. \\
J_2 &= 10\mathbb{E}\left\| B^*\mathcal{S}_{\alpha}^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathcal{T}_{\alpha}(T)\mathscr{G}(0,\varphi) \right\|^2 \\
&\leq 10\mathcal{M}_B^2\left(\frac{\alpha\mathcal{M}T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\frac{1}{\mu^2}\mathbb{E}\|\mathscr{G}(0,\varphi)\|^2
\end{aligned}
$$

$$\leq 10\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \frac{1}{\mu^2}\|\mathscr{A}^{-\beta}\|^2\mathbb{E}\|\mathscr{A}^{\beta}\mathscr{G}(0,\varphi)\|^2$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{M}^2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2).$$

$$J_3 = 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathscr{G}(T, z_{\varrho(T,z_T+x_T)} + x_{\varrho(T,z_T+x_T)})\right\|^2$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \mathbb{E}\left\|\mathscr{G}(T, z_{\varrho(T,z_T+x_T)} + x_{\varrho(T,z_T+x_T)})\right\|^2$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \|A^{-\beta}\|^2\mathbb{E}\left\| A^{\beta}\mathscr{G}(T, z_{\varrho(T,z_T+x_T)} + x_{\varrho(T,z_T+x_T)})\right\|^2$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+r^*).$$

$$J_4 = 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\right)ds\right\|^2$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \left\|\int_0^T \mathscr{A}^{1-\beta}\mathcal{S}_\alpha(T-s)ds\right\|^2\mathbb{E}\|\mathscr{A}^{\beta}\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\right)\|^2$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\right)^2\int_0^T(T-s)^{\alpha\beta-1}ds\int_0^T(T-s)^{\alpha\beta-1}\mathcal{M}_{\mathscr{G}}(1+r^*)ds$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2\mathcal{M}_{\mathscr{G}}(1+r^*).$$

$$J_5 = 10\mathbb{E}\Bigg\| B^*\mathcal{S}_\alpha^*(T-t)\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathcal{S}_\alpha(T-s)$$

$$(\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds\Bigg\|^2$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\frac{T^\alpha}{\alpha}\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\int_0^T(T-s)^{\alpha-1}\Bigg[\mathscr{F}_1(s)\|z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\|_{\mathcal{B}}^2$$

$$+\mathscr{F}_2(s)\mathbb{E}\left\|\int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right\|_{\mathcal{H}}^2\Bigg]ds$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^* r^* + \mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\right].$$

$$J_6 = 10\mathbb{E}\Bigg\| B^*\mathcal{S}_\alpha^*(T-t)\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathcal{S}_\alpha(T-s)$$

$$(\times)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s)\Bigg\|^2$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\frac{T^\alpha}{\alpha}\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\int_0^T(T-s)^{\alpha-1}tr(Q)\Bigg[\Sigma_1(s)\|z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\|_{\mathcal{B}}^2$$

$$+\Sigma_2(s)\mathbb{E}\left\|\int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right\|_{\mathcal{H}}^2\Bigg]ds$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\left[\Sigma_1^* r^* + \Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\right].$$

$$J_7 = 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\sum_{0<t_k<t}\mathcal{T}_\alpha(T-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-))\right\|^2$$

$$\leq \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \| \mathcal{T}_\alpha(T-t_k)\|^2 n \sum_{k=1}^{n} \mathbb{E}\| \mathcal{I}_k(z(t_k^-)+x(t_k^-))\|^2$$
$$\leq \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \mathcal{M}^2 n \sum_{k=1}^{n} \mathcal{M}_{\mathcal{I}_k} H^2 \|z_t+x_t\|^2$$
$$\leq \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{M}^2 H^2 n \sum_{k=1}^{n} \mathcal{M}_{\mathcal{I}_k}\tilde{r}.$$

By combining the estimations $(J_1)-(J_7)$ together with (3.7), we sustain

$$\begin{aligned}
\mathbb{E}\| v^\mu(s)\|^2 \leq{}& \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left[\mathbb{E}\|\tilde{u}_T\|^2 + \int_0^T \mathbb{E}\|\tilde{\phi}(s)\|^2 ds + \mathcal{M}^2\left[\mathbb{E}\|\varphi(0)\|^2 + \mathcal{M}_h \tilde{r}\right]\right] \\
&+ \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{M}^2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|^2_{\mathcal{B}}) \\
&+ \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+r^*) \\
&+ \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2 \mathcal{M}_{\mathscr{G}}(1+r^*) \\
&+ \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^* r^* + \mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\right] \\
&+ \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\left[\Sigma_1^* r^* + \Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\right] \\
&+ \frac{10}{\mu^2}\left(\frac{\alpha \mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{M}^2 H^2 n \sum_{k=1}^{n} \mathcal{M}_{\mathcal{I}_k}\tilde{r}.
\end{aligned}$$

$$\begin{aligned}
r \leq{}& \mathbb{E}\|\overline{\Upsilon}(z^r)(t)\|^2 \\
\leq{}& 8\mathbb{E}\|\mathcal{T}_\alpha(t)[-h(z_t^r+x_t)-\mathscr{G}(0,\varphi)]\|^2 + 8\mathbb{E}\|\mathscr{G}(t, z^r_{\varrho(t,z_t^r+x_t)} + x_{\varrho(t,z_t^r+x_t)})\|^2 \\
&+ 8\mathbb{E}\left\| \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, z^r_{\varrho(s,z_s^r+x_s)} + x_{\varrho(s,z_s^r+x_s)}\right) ds\right\|^2 \\
&+ 8\mathbb{E}\left\| \int_0^t \mathcal{S}_\alpha(t-s)Bv^\mu(s)ds\right\|^2 + 8\mathbb{E}\left\| \int_0^t \mathcal{S}_\alpha(t-s)\right. \\
&\left.(\times)\mathscr{F}\left(s, z^r_{\varrho(s,z_s^r+x_s)} + x_{\varrho(s,z_s^r+x_s)}, \int_0^s e_1(s,\tau, z^r_{\varrho(\tau,z_\tau^r+x_\tau)} + x_{\varrho(\tau,z_\tau^r+x_\tau)})d\tau\right) ds\right\|^2 \\
&+ 8\mathbb{E}\left\| \int_0^t \mathcal{S}_\alpha(t-s)\right. \\
&\left.(\times)\Sigma\left(s, z^r_{\varrho(s,z_s^r+x_s)} + x_{\varrho(s,z_s^r+x_s)}, \int_0^s e_2(s,\tau, z^r_{\varrho(\tau,z_\tau^r+x_\tau)} + x_{\varrho(\tau,z_\tau^r+x_\tau)})d\tau\right) dw(s)\right\|^2 \\
&+ 8\mathbb{E}\|\sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z^r(t_k^-)+x(t_k^-))\|^2 \\
={}& \sum_{i=8}^{14} J_i. \qquad (3.8)
\end{aligned}$$

$$\begin{aligned}
J_8 ={}& 8\mathbb{E}\|\mathcal{T}_\alpha(t)[-h(z_t^r+x_t)-\mathscr{G}(0,\varphi)]\|^2 \\
\leq{}& 8\|\mathcal{T}_\alpha(t)\|^2\left[\mathbb{E}\|h(z_t^r+x_t)\|^2_{\mathcal{B}} + \mathbb{E}\|\mathscr{G}(0,\varphi)\|^2\right] \\
\leq{}& 8\left\|\int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr\right\|^2\left[\mathbb{E}\|h(z_t^r+x_t)\|^2_{\mathcal{B}} + \mathbb{E}\|\mathscr{G}(0,\varphi)\|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq 8\mathcal{M}^2\left[\mathcal{M}_h\|z_t^r+x_t\|_{\mathcal{B}}^2+\|\mathscr{A}^{-\beta}\|^2\mathbb{E}\|\mathscr{A}^{\beta}\mathscr{G}(0,\varphi)\|^2\right]\\
&\leq 8\mathcal{M}^2\left[\mathcal{M}_h\left(4\mathscr{E}_1^{*2}r+\widetilde{c}_n\right)+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)\right]\\
&\leq 32\mathcal{M}^2\mathcal{M}_h\mathscr{E}_1^{*2}r+C_1,
\end{aligned}
$$

where $\mathcal{N}_0 = \|A^{-\beta}\|$ and $C_1 = 8\mathcal{M}^2\mathcal{M}_h\widetilde{c}_n + 8\mathcal{M}^2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)$.

$$
\begin{aligned}
J_9 &= 8\mathbb{E}\|\mathscr{G}(t,z^r_{\varrho(t,z_t^r+x_t)}+x_{\varrho(t,z_t^r+x_t)})\|^2\\
&\leq 8\|\mathscr{A}^{-\beta}\|^2\mathbb{E}\|\mathscr{A}^{\beta}\mathscr{G}(t,z^r_{\varrho(t,z_t^r+x_t)}+x_{\varrho(t,z_t^r+x_t)})\|^2\\
&\leq 8\|\mathscr{A}^{-\beta}\|^2\mathcal{M}_{\mathscr{G}}\left(1+\|z^r_{\varrho(t,z_t^r+x_t)}+x_{\varrho(t,z_t^r+x_t)}\|_{\mathcal{B}}^2\right)\\
&\leq 32\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r+C_2,
\end{aligned}
$$

where $C_2 = 8\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+c_n)$.

$$
\begin{aligned}
J_{10} &= 8\mathbb{E}\left\|\int_0^t\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s,z^r_{\varrho(s,z_s^r+x_s)}+x_{\varrho(s,z_s^r+x_s)}\right)ds\right\|^2\\
&\leq 8\left\|\int_0^t\left\{\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathscr{A}^{1-\beta}\mathbb{T}((t-s)^\alpha r)dr\right\}ds\right\|^2\\
&\quad(\times)\mathbb{E}\left\|\mathscr{A}^{\beta}\mathscr{G}\left(s,z^r_{\varrho(s,z_s^r+x_s)}+x_{\varrho(s,z_s^r+x_s)}\right)\right\|^2\\
&\leq 8\mathcal{M}_{\mathscr{G}}\left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\right)^2\int_0^t(t-s)^{\alpha\beta-1}ds\int_0^t(t-s)^{\alpha\beta-1}\left(1+\|z^r_{\varrho(s,z_s^r+x_s)}+x_{\varrho(s,z_s^r+x_s)}\|_{\mathcal{B}}^2\right)ds\\
&\leq 8\mathcal{M}_{\mathscr{G}}\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2\left(1+4\mathscr{E}_1^{*2}r+c_n\right)\\
&\leq 32\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2+C_3,
\end{aligned}
$$

where $C_3 = 8\mathcal{M}_{\mathscr{G}}\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2(1+c_n)$.

$$
\begin{aligned}
J_{11} &= 8\mathbb{E}\left\|\int_0^t\mathcal{S}_\alpha(t-s)Bv^\mu(s)ds\right\|^2\\
&\leq 8\left\|\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha r)dr\right\|^2\mathbb{E}\left\|\int_0^t Bv^\mu(s)ds\right\|^2\\
&\leq 8\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\mathcal{M}_B^2\frac{T^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|v^\mu(s)\|^2ds\\
&\leq 8\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^2\frac{T^{2\alpha}}{\alpha^2}(\times)\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_BT^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\mathcal{M}_v,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{M}_v =&\mathbb{E}\|\tilde{u}_T\|^2+\int_0^T\mathbb{E}\|\tilde{\phi}(s)\|^2ds+\mathcal{M}^2\left[\mathbb{E}\|\varphi(0)\|^2+\mathcal{M}_h\tilde{r}\right]+\mathcal{M}^2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+r^*)\\
&+\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2\mathcal{M}_{\mathscr{G}}(1+r^*)+\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^*r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\right]\\
&+\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2tr(Q)\left[\Sigma_1^*r^*+\Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\right]+\mathcal{M}^2H^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\tilde{r}.
\end{aligned}
$$

$$\begin{aligned}
J_{12} &= 8\mathbb{E}\left\| \int_0^t \mathcal{S}_\alpha(t-s) \right. \\
&\quad (\times)\mathscr{F}\left(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}, \int_0^s e_1(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})d\tau\right) ds \Big\|^2 \\
&\leq 8\left\|\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha r)dr\right\|^2 \\
&\quad (\times)\mathbb{E}\left\| \int_0^t \mathscr{F}\left(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}, \int_0^s e_1(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})d\tau\right) ds \right\|^2 \\
&\leq 8\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)\frac{T^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}\Big[\mathscr{F}_1(s)\|z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}\|^2_{\mathcal{B}} \\
&\quad + \mathscr{F}_2(s)\int_0^s \mathbb{E}\|e_1(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})\|^2_{\mathcal{H}}d\tau\Big]ds \\
&\leq 32\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 \mathscr{E}_1^{*2} r(\mathscr{F}_1^* + \mathscr{F}_2^*\widetilde{\mathcal{M}}_0 T) + C_4,
\end{aligned}$$

where $C_4 = 8\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left(\mathscr{F}_1^* c_n + \mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+c_n)T\right)$.

$$\begin{aligned}
J_{13} &= 8\mathbb{E}\left\| \int_0^t \mathcal{S}_\alpha(t-s) \right. \\
&\quad (\times)\Sigma\left(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}, \int_0^s e_2(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})d\tau\right) dw(s) \Big\|^2 \\
&\leq 8\left\|\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha r)dr\right\|^2 tr(Q) \\
&\quad (\times)\mathbb{E}\left\| \int_0^t \Sigma\left(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}, \int_0^s e_2(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})d\tau\right) ds \right\|^2 \\
&\leq 8\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)\frac{T^\alpha}{\alpha}\frac{T^\alpha}{\alpha}tr(Q)\left[\Sigma_1^*(4\mathscr{E}_1^{*2}+c_n) + \Sigma_2^*\widetilde{\mathcal{M}}_1(1+4\mathscr{E}_1^{*2}+c_n)T\right] \\
&\leq 32\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\mathscr{E}_1^{*2} r(\Sigma_1^* + \Sigma_2^*\widetilde{\mathcal{M}}_1 T) + C_5,
\end{aligned}$$

where $C_5 = 8\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\left(\Sigma_1^* c_n + \Sigma_2^*\widetilde{\mathcal{M}}_1(1+c_n)T\right)$.

$$\begin{aligned}
J_{14} &= 8n\sum_{k=1}^n \|\mathcal{T}_\alpha(t-t_k)\|^2\mathbb{E}\|\mathcal{I}_k(z^r(t_k^-)+x(t_k^-))\|^2 \\
&\leq 8n\sum_{k=1}^n \left\|\int_0^\infty \mathbb{T}((t-t_k)^\alpha r)\phi_\alpha(r)dr\right\|^2\mathbb{E}\|\mathcal{I}_k(z^r(t_k^-)+x(t_k^-))\|^2 \\
&\leq 8\mathcal{M}^2 n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}\mathbb{E}\|(z^r(t_k^-)+x(t_k^-))\|^2 \\
&\leq 8\mathcal{M}^2 n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}\left(\sup_{t\in\mathscr{I}}\mathbb{E}\|z^r(t)+x(t)\|^2\right) \\
&\leq 8\mathcal{M}^2 H^2 n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}\mathbb{E}\|z^r_t + x_t\|^2_{\mathcal{B}} \\
&\leq 8\mathcal{M}^2 H^2 n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}\left[4\mathscr{E}_1^{*2} r + \widetilde{c}_n\right] \\
&\leq 32\mathcal{M}^2 H^2\mathscr{E}_1^{*2} n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k} r + C_6,
\end{aligned}$$

where $C_6 = 8\mathcal{M}^2 H^2\widetilde{c}_n n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}$.

By combining the estimations $(J_8)-(J_{14})$ together with (3.8), we sustain

$$\begin{aligned}
r &\leq \mathbb{E}\|\overline{\Upsilon}(z^r)(t)\|^2 \\
&\leq 32\mathcal{M}^2\mathcal{M}_h\mathscr{E}_1^{*2}r + C_1 + 32\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r + C_2 + 32\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2 + C_3 \\
&\quad + 8\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^2\frac{T^{2\alpha}}{\alpha^2}(\times)\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\mathcal{M}_v + 32\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\mathscr{E}_1^{*2}r(\mathscr{F}_1^* + \mathscr{F}_2^*\widetilde{\mathcal{M}}_0 T) \\
&\quad + C_4 + 32\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\mathscr{E}_1^{*2}r(\Sigma_1^* + \Sigma_2^*\widetilde{\mathcal{M}}_1 T) + C_5 + 32\mathcal{M}^2H^2\mathscr{E}_1^{*2}n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}r + C_6,
\end{aligned}$$

where C_1-C_6 are independent of r. Dividing both sides by r and taking the limit as $r\to\infty$, we sustain

$$\begin{aligned}
&32\left(1+\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^4\frac{T^{4\alpha-2}}{\alpha^2}\right)\Bigg[\mathcal{M}^2\left(\mathcal{M}_h + H^2 n\sum_{k=1}^{n}\gamma_k\right) + \mathcal{M}_{\mathscr{G}}\left({\mathcal{N}_0}^2 + \left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right)^2\right) \\
&\qquad + \left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^* + tr(Q)\Sigma_1^* + \left(\mathscr{F}_2^*\widetilde{\mathcal{M}}_0 + \Sigma_2^* tr(Q)\widetilde{\mathcal{M}}_1\right)T\right]\Bigg]\mathscr{E}_1^{*2} \geq 1
\end{aligned}$$

which is a contradiction to (3.1). For this reason for some positive number r in a way that $\overline{\Upsilon}(B_r)\subset B_r$.

Step 2: Now we prove that for each $\mu>0$, the operator $\overline{\Upsilon}$ maps B_r into a relatively compact subset of B_r. First we prove that the set $\mathcal{V}(t)=\{(\overline{\Upsilon}z)(t): z\in B_r\}$ is relatively compact in $\mathcal{H}$ for every $t\in\mathscr{I}$. The case $t=0$ is obvious. For $0<\epsilon<t\leq T$, define $(\overline{\Upsilon}^\epsilon z)(t)=\mathcal{S}_\alpha(\epsilon)Q(t-\epsilon)$, where

$$\begin{aligned}
Q(t-\epsilon) &= \mathcal{T}_\alpha(t)[-h(z_t+x_t)-\mathscr{G}(0,\varphi)] + \mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}) \\
&\quad + \int_0^{t-\epsilon}\mathscr{A}\mathcal{S}_\alpha(t-\epsilon-s)\mathscr{G}(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)})ds + \int_0^{t-\epsilon}\mathcal{S}_\alpha(t-\epsilon-s)Bv^\mu(s)ds \\
&\quad + \int_0^{t-\epsilon}\mathcal{S}_\alpha(t-\epsilon-s)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\right)ds \\
&\quad + \int_0^{t-\epsilon}\mathcal{S}_\alpha(t-\epsilon-s)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\right)dw(s) \\
&\quad + \sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-)).
\end{aligned}$$

Since $\mathcal{S}_\alpha(t)$ is compact and $Q(t-\epsilon)$ is bounded on B_r, the set $\mathcal{V}_\epsilon(t)=\{(\overline{\Upsilon}^\epsilon z)(t): z(\cdot)\in B_r\}$ is relatively compact in $\mathcal{H}$. Also for every $z\in B_r$, we have

$$\begin{aligned}
&\mathbb{E}\|(\overline{\Upsilon}z)(t)-(\overline{\Upsilon}^\epsilon z)(t)\|^2_{\mathcal{H}} \\
&\leq 4\mathbb{E}\left\|\int_{t-\epsilon}^{t}\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)})ds\right\|^2 \\
&\quad + 4\mathbb{E}\left\|\int_{t-\epsilon}^{t}\mathcal{S}_\alpha(t-s)Bv^\mu(s)ds\right\|^2 + 4\mathbb{E}\left\|\int_{t-\epsilon}^{t}\mathcal{S}_\alpha(t-s)\right. \\
&\quad (\times)\left.\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds\right\|^2 \\
&\quad + 4\mathbb{E}\left\|\int_{t-\epsilon}^{t}\mathcal{S}_\alpha(t-s)\right. \\
&\quad (\times)\left.\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s)\right\|^2 \\
&\leq 4\left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\right)^2\frac{\epsilon^{\alpha\beta}}{\alpha\beta}\int_{t-\epsilon}^{t}(t-s)^{\alpha\beta-1}\mathcal{M}_{\mathscr{G}}(1+r^*)ds + 4\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^2\frac{\epsilon^\alpha}{\alpha} \\
&\quad (\times)\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}\mathcal{M}_v ds + 4\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\frac{\epsilon^\alpha}{\alpha}\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}\left[\mathscr{F}_1 r^* + \mathscr{F}_2^*(1+\widetilde{\mathcal{M}}_0)T\right]ds
\end{aligned}$$

$$+4\bigg(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\bigg)^2\frac{\epsilon^\alpha}{\alpha}tr(Q)\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}\big[\Sigma_1 r^*+\Sigma_2^*(1+\widetilde{\mathcal{M}}_1)T\big]ds$$
$$\to 0,\quad \text{as}\quad \epsilon\to 0^+.$$

This implies that there are relatively compact sets arbitrarily close to the set $\mathcal{V}(t), t>0$. As a result, $\mathcal{V}(t)=\{(\overline{\Upsilon}z)(t): z\in B_r\}$ is also relatively compact in $\mathcal{H}$.

Step 3: Next we shall show that $\mathcal{V}(t)=\{(\overline{\Upsilon}z)(t): z\in B_r\}$ is equicontinuous in $[0,T]$. For $0\le t_1\le t_2\le T$ such that $\|\mathbb{T}(t_1^\alpha)-\mathbb{T}(t_2^\alpha)\|<\epsilon$, we get

$$\begin{aligned}
&\mathbb{E}\|(\overline{\Upsilon}z)(t_2)-(\overline{\Upsilon}z)(t_1)\|^2\\
&\le 16\mathbb{E}\,\big\|\big[\mathbb{T}(t_2^\alpha r)-\mathbb{T}(t_1^\alpha r)\big]\big[-h(z_t+x_t)-\mathscr{G}(0,\varphi)\big]\big\|^2\\
&\quad+16\mathbb{E}\,\big\|\mathscr{G}\big(t_2,z_{\varrho(t_2,z_{t_2}+x_{t_2})}+x_{\varrho(t_2,z_{t_2}+x_{t_2})}\big)-\mathscr{G}\big(t_1,z_{\varrho(t_1,z_{t_1}+x_{t_1})}+x_{\varrho(t_1,z_{t_1}+x_{t_1})}\big)\big\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)^2\int_0^{t_1}(t_1-s)^{\alpha-1}\mathscr{A}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]\\
&\qquad(\times)\mathscr{G}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)^2\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\mathscr{A}\mathbb{T}((t_2-s)^\alpha r)\\
&\qquad(\times)\mathscr{G}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)^2\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathscr{A}\mathbb{T}((t_2-s)^\alpha r)\\
&\qquad(\times)\mathscr{G}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]Bv^\mu(s)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\mathbb{T}((t_2-s)^\alpha r)Bv^\mu(s)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{T}((t_2-s)^\alpha r)Bv^\mu(s)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]\\
&\qquad(\times)\mathscr{F}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{T}((t_2-s)^\alpha r)\\
&\qquad(\times)\mathscr{F}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{T}((t_2-s)^\alpha r)\\
&\qquad(\times)\mathscr{F}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(s,z_s+x_s)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\bigg)ds\bigg\|^2
\end{aligned}$$

$$
\begin{aligned}
&+16\mathbb{E}\Big\|\Big(\frac{\alpha}{\Gamma(1+\alpha)}\Big)\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]\\
&\quad(\times)\Sigma\Big(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\Big)dw(s)\Big\|^2\\
&+16\mathbb{E}\Big\|\Big(\frac{\alpha}{\Gamma(1+\alpha)}\Big)\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{T}((t_2-s)^\alpha r)\\
&\quad(\times)\Sigma\Big(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\Big)dw(s)\Big\|^2\\
&+16\mathbb{E}\Big\|\Big(\frac{\alpha}{\Gamma(1+\alpha)}\Big)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{T}((t_2-s)^\alpha r)\\
&\quad(\times)\Sigma\Big(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(s,z_s+x_s)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\Big)dw(s)\Big\|^2\\
&+16\mathbb{E}\Big\|\sum_{0<t_k<t_1}\big[\mathbb{T}((t_2-t_k)^\alpha r)-\mathbb{T}((t_1-t_k)^\alpha r)\big]\mathbb{E}\big[\mathcal{I}_k(z(t_k^-)+x(t_k^-))\big]\Big\|^2\\
&+16\mathbb{E}\Big\|\sum_{t_1<t_k<t_2}\mathbb{T}((t_2-t_k)^\alpha r)\mathbb{E}\big[\mathcal{I}_k(z(t_k^-)+x(t_k^-))\big]\Big\|^2\\
\le\ &32\epsilon^2\big[\mathcal{M}_h\widetilde{r}+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)\big]+16\mathcal{N}_0^2\mathbb{E}\Big\|\mathscr{A}^\beta\mathscr{G}\big(t_2,z_{\varrho(t_2,z_{t_2}+x_{t_2})}+x_{\varrho(t_2,z_{t_2}+x_{t_2})}\big)\\
&\quad-\mathscr{A}^\beta\mathscr{G}\big(t_1,z_{\varrho(t_1,z_{t_1}+x_{t_1})}+x_{\varrho(t_1,z_{t_1}+x_{t_1})}\big)\Big\|^2\\
&+16\Big(\frac{\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{t_1^\alpha}{\alpha}\mathcal{M}_{\mathscr{G}}(1+r^*)\int_0^{t_1}(t_1-s)^{\alpha-1}\big\|\mathscr{A}^{1-\beta}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]\big\|^2ds\\
&+16\Big(\frac{\alpha}{\Gamma(1+\alpha)}\Big)^2\|\mathscr{A}^{1-\beta}\mathbb{T}((t_2-s)^\alpha r)\|^2\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds\\
&\quad(\times)\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathcal{M}_{\mathscr{G}}(1+r^*)ds\\
&+16\Big(\frac{\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|\mathscr{A}^{1-\beta}\mathbb{T}((t_2-s)^\alpha r)\|^2\mathcal{M}_{\mathscr{G}}(1+r^*)ds\\
&+16\Big(\frac{\alpha\epsilon\mathcal{M}_B}{\Gamma(1+\alpha)}\Big)^2\frac{t_1^\alpha}{\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}\mathbb{E}\|v^\mu(s)\|^2ds\\
&+16\Big(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\Big)^2\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\mathbb{E}\|v^\mu(s)\|^2ds\\
&+16\Big(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\Big)^2\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{E}\|v^\mu(s)\|^2ds\\
&+16\Big(\frac{\alpha\epsilon}{\Gamma(1+\alpha)}\Big)^2\frac{t_1^\alpha}{\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathscr{F}_1^*r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\big]ds\\
&+16\Big(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\Big)^2\big[\mathscr{F}_1^*r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\big]\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds\\
&\quad(\times)\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds\\
&+16\Big(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\Big)^2\frac{(t_2-t_1)^\alpha}{\alpha}\big[\mathscr{F}_1^*r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\big]\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&+16\Big(\frac{\alpha\epsilon}{\Gamma(1+\alpha)}\Big)^2\frac{t_1^\alpha}{\alpha}tr(Q)\big[\Sigma_1^*r^*+\Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\big]\int_0^{t_1}(t_1-s)^{\alpha-1}ds\\
&+16\Big(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\Big)^2tr(Q)\big[\Sigma_1^*r^*+\Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\big]\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds\\
&\quad(\times)\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds
\end{aligned}
$$

$$+16\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\frac{(t_2-t_1)^\alpha}{\alpha}tr(Q)\left[\Sigma_1^* r^*+\Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\right]\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds$$
$$+16\epsilon^2H^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\widetilde{r}+16\mathcal{M}^2H^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\widetilde{r}.$$

Therefore, for ϵ sufficiently small, the right-hand side of the above inequality tends to zero as $t_1\to t_2$. Since the compactness of $\mathcal{T}_\alpha(t)$ for $t>0$ implies the continuity in the uniform operator topology. This proves that $\mathcal{V}$ is right equicontinuous at $t\in(0,T)$. Similarly, we can prove that the right equicontinuity at zero and the left equicontinuity at $t\in(0,T]$. Thus $(\overline{\Upsilon}z)$ is equicontinuous on $[0,T]$. By using a procedure similar to that used in [1], we can easily prove that the map $(\overline{\Upsilon}z)$ is continuous on z which completes the proof that $\Upsilon(\cdot)$ is completely continuos. Hence from the schauder fixed point theorem Υ has a fixed point and consequently the systems (1.1) -(1.3) has a mild solution on $[0,T]$. □

Theorem 3.2. *Assume that the conditions of above theorem hold and, in addition, the function $\mathscr{G},\mathscr{F},\Sigma,e_i,\{i=1,2\}$ and h are uniformly bounded on their respective domains. If $\mathbb{T}(t)$ is compact, then the impulsive fractional neutral stochastic integro-differential equations (1.1)-(1.3) is approximately controllable on $\mathscr{I}$*

Proof. Let $u^\mu(\cdot)$ be fixed point of $\overline{\Upsilon}$. By using the stochastic Fubini theorem, any fixed point of $\overline{\Upsilon}$ is a mild solution of (1.1)-(1.3), if the control $v^\mu(t)$ satisfies

$$u^\mu(T)=\tilde{u}_T-\mu\Phi(v^\mu(\cdot)),\tag{3.9}$$

where

$$\Phi v^\mu(t)=\begin{cases}(\mu\mathcal{I}+\Gamma_0^T)^{-1}\left[\mathbb{E}\tilde{u}_T+\int_0^T\tilde{\phi}(s)dw(s)-\mathcal{T}_\alpha(T)[\varphi(0)-h(z_T+x_T)-\mathscr{G}(0,\varphi)]\right]\\ -(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathscr{G}(T,z_{\varrho(T,z_T+x_T)}+x_{\varrho(T,z_T+x_T)})\\ -\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\right)ds\\ -\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\\ \quad(\times)\mathscr{F}\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds\\ -\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\\ \quad(\times)\Sigma\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s)\\ -(\mu\mathcal{I}+\Gamma_0^T)^{-1}\sum_{0<t_k<t}\mathcal{T}_\alpha(T-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-)).\end{cases}$$

Further, by assumption, $\mathscr{G},\mathscr{F},\Sigma,e_i,\{i=1,2\}$ and h are uniformly bounded on $\mathscr{I}$. Then there are subsequences still denoted by

$$\left\{\mathscr{A}^\beta\mathscr{G}(s,u^\mu_{\varrho(s,u_s)}),\mathscr{F}\left(s,u^\mu_{\varrho(s,u_s)},\int_0^s e_1(s,\tau,u^\mu_{\varrho(\tau,u_\tau)})d\tau\right),\Sigma\left(s,u^\mu_{\varrho(s,u_s)},\int_0^s e_2(s,\tau,u^\mu_{\varrho(\tau,u_\tau)})d\tau\right)\right\},$$

which converge weakly to $\{\mathscr{G}(s),\mathscr{F}(s),\Sigma(s)\}$, respectively. Thus from the (3.9), we have

$$\mathbb{E}\|u^\mu(T)-\tilde{u}_T\|^2$$
$$\le 9\mathbb{E}\left\|\mu(\mu\mathcal{I}+\Gamma_0^T)^{-1}\left[\mathbb{E}\tilde{u}_T+\int_0^T\tilde{\phi}(s)dw(s)-\mathcal{T}_\alpha(T)[\varphi(0)-h(z_T+x_T)-\mathscr{G}(0,\varphi)]\right]\right\|^2$$
$$+9\mathbb{E}\left\|\mu(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathscr{G}(T,z_{\varrho(T,z_T+x_T)}+x_{\varrho(T,z_T+x_T)})\right\|^2$$

$$
\begin{aligned}
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\left[\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\right)-\mathscr{G}(s)\right]ds\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}(s)ds\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\right. \\
&\quad(\times)\left.\left[\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)-\mathscr{F}(s)\right]ds\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\mathscr{F}(s)ds\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\right. \\
&\quad(\times)\left.\left[\Sigma\left(s, z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)-\Sigma(s)\right]dw(s)\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\Sigma(s)dw(s)\right\|^2 \\
&+9\mathbb{E}\left\| \mu(\mu\mathcal{I}+\Gamma_0^T)^{-1}\sum_{0<t_k<t}\mathcal{T}_\alpha(T-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-))\right\|^2.
\end{aligned}
$$

On the other hand, by lemma 2.3 for all $0 \le s \le T$, the operator $\mu(\mu\mathcal{I}+\Gamma_s^T)^{-1} \to 0$ strongly as $\mu \to 0^+$, and moreover $\|\mu(\mu\mathcal{I}+\Gamma_0^T)^{-1}\| \le 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $\mathcal{S}_\alpha(t)$, we obtain $\mathbb{E}\|u^\mu(T)-\tilde{u}_T\|^2 \to 0$ as $\mu \to 0^+$. This gives the approximate controllability of (1.1)-(1.3). The proof is now completed. □

4 Application

In this section an illustration is provided for the existence results to the following IFNSIDS with SDD of the structure

$$
\begin{aligned}
&D_t^\alpha\left[u(t,x)-\int_{-\infty}^t \mu_1(s-t)u(s-\varrho_1(t)\varrho_2(\|u(t)\|),x)ds\right] \\
&=\frac{\partial^2}{\partial x^2}u(t,x)+\mu(t,x)+\int_{-\infty}^t \mu_2(t,x,s-t)P_1\Big(u(s-\varrho_1(t)\varrho_2(\|u(t)\|),x)\Big)ds \\
&\quad+\int_0^t\int_{-\infty}^s k_1(s-\tau)P_2\Big(u(\tau-\varrho_1(\tau)\varrho_2(\|u(\tau)\|),x)\Big)d\tau ds \\
&\quad+\left[\int_{-\infty}^t \mu_3(t,x,s-t)Q_1\Big(u(s-\varrho_1(t)\varrho_2(\|u(t)\|),x)\Big)ds\right. \\
&\quad+\left.\int_0^t\int_{-\infty}^s k_2(s-\tau)Q_2\Big(u(\tau-\varrho_1(\tau)\varrho_2(\|u(\tau)\|),x)\Big)d\tau ds\right]\frac{d\beta(t)}{dt},\ x\in[0,\pi],\ 0\le t\le T, \qquad (4.1)
\end{aligned}
$$

$$u(t,0)=0=u(t,\pi), \quad t\ge 0, \qquad (4.2)$$

$$u(0,x)+\int_0^\pi k_3(x,z)u(t,z)dz=\varphi(t,x), \quad t\in(-\infty,0], \quad 0\le x\le\pi, \qquad (4.3)$$

$$\Delta u(t_k,x)=\int_{-\infty}^{t_k}\eta_k(s-t_k)u(s,x)ds, \quad k=1,2,\ldots,n, \qquad (4.4)$$

where $\beta(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathcal{P})$; D_t^α is Caputo's fractional derivative of order $0<\alpha<1$; φ is continuous; and $0<t_1<t_2<\cdots<t_n<T$

are prefixed numbers. We consider $\mathcal{H} = \mathcal{K} = L^2[0,\pi]$ having the norm $\|\cdot\|_{\mathcal{L}^2}$ and define the operator $\mathscr{A} : D(\mathscr{A}) \subset \mathcal{H} \to \mathcal{H}$ by $\mathscr{A}w = w''$ with the domain

$$D(\mathscr{A}) = \{w \in \mathcal{H} : w, w' \quad \text{are absolutely continuous}, \quad w'' \in \mathcal{H},\, w(0) = w(\pi) = 0\}.$$

Then

$$\mathscr{A}w = \sum_{n=1}^{\infty} n^2 \langle w, w_n \rangle w_n, \quad w \in D(\mathscr{A}),$$

in which $w_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns)$, $n = 1, 2, \ldots, .$ is the orthogonal set of eigenvectors of $\mathscr{A}$. It is long familiar that $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$ in $\mathcal{H}$ and is provided by

$$\mathbb{T}(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}\langle w, w_n \rangle w_n, \quad \text{for all} \quad w \in \mathcal{H}, \quad \text{and every} \quad t > 0.$$

If we fix $\beta = \frac{1}{2}$, then the operator $(\mathscr{A})^{\frac{1}{2}}$ is given by

$$(\mathscr{A})^{\frac{1}{2}} w = \sum_{n=1}^{\infty} n \langle w, w_n \rangle w_n, \;\; w \in (D(\mathscr{A})^{\frac{1}{2}}),$$

in which $(D(\mathscr{A})^{\frac{1}{2}}) = \left\{ \omega(\cdot) \in \mathcal{H} : \sum_{n=1}^{\infty} n\langle \omega, w_n \rangle w_n \in \mathcal{H} \right\}$ and $\|(\mathscr{A})^{-\frac{1}{2}}\| = 1$. Let $\gamma < 0$, define the phase space

$$\mathcal{B} = \left\{ \varphi \in C((-\infty, 0], \mathcal{H}) : \lim_{\theta \to -\infty} e^{\gamma\theta}\varphi(\theta) \quad \text{exists in} \quad \mathcal{H} \right\},$$

and let $\|\varphi\|_{\mathcal{B}} = \sup_{\theta \in (-\infty, 0]} \left\{ e^{\gamma\theta}\|\varphi(\theta)\|_{\mathcal{L}^2} \right\}$, then $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space satisfies $(P_1) - (P_3)$ with $H = 1, \mathscr{E}_1(t) = \max\{1, e^{-\gamma t}\}, \mathscr{E}_2(t) = e^{-\gamma t}$. Therefore, for $(t, \varphi) \in [0, T] \times \mathcal{B}$, where $\varphi(\theta)(x) = \varphi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$. Set

$$u(t)(x) = u(t, x), \quad \varrho(t, \varphi) = \varrho_1(t)\varrho_2(\|\varphi(0)\|),$$

we have

$$\mathscr{G}(t, \varphi)(x) = \int_{-\infty}^{0} \mu_1(\theta)\varphi(\theta)(x)d\theta,$$

$$\mathscr{F}(t, \varphi, \mathscr{H}\varphi)(x) = \int_{-\infty}^{0} \mu_2(t, x, \theta)P_1(\varphi(\theta)(x))d\theta + \mathscr{H}\varphi(x),$$

$$\Sigma(t, \varphi, \overline{\mathscr{H}}\varphi)(x) = \int_{-\infty}^{0} \mu_3(t, x, \theta)Q_1(\varphi(\theta)(x))d\theta + \overline{\mathscr{H}}\varphi(x)$$

and

$$\mathcal{I}_k(\varphi)(x) = \int_{-\infty}^{0} \eta_k(\theta)\varphi(\theta)(x)d\theta, \quad k = 1, 2, \ldots, n,$$

where

$$\mathscr{H}\varphi(x) = \int_0^t \int_{-\infty}^{0} k_1(s-\theta)P_2(\varphi(\theta)(x))d\theta ds, \quad \overline{\mathscr{H}}\varphi(x) = \int_0^t \int_{-\infty}^{0} k_2(s-\theta)Q_2(\varphi(\theta)(x))d\theta ds.$$

Further, define the bounded linear operator $B : U \to \mathcal{H}$ by $Bv(t)(x) = \mu(t, x)$, $0 \leq x \leq \pi, u \in U$, where $\mu : [0,1] \times [0, \pi] \to [0, \pi]$ is continuous. Now, under the above conditions, we can represent the system (4.1) - (4.4) in the abstract form (1.1) - (1.3) . Hence , according to Theorem 3.2, system (4.1) - (4.4) is approximately controllable on $[0, T]$.

5 Conclusion

In this manuscript, we have studied the approximate controllability results for impulsive stochastic fractional neutral integro-differential systems with non-local and state-dependent delay conditions in Hilbert space. More precisely, by utilizing the stochastic analysis theory, fractional powers of operators and Schauder fixed point theorem, we investigate the IFNSIDS with NLCs and SDD in Hilbert space. To validate the obtained theoretical results, one example is analyzed. The FDEs are very efficient to describe the real-life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and controllability.

There are two direct issues which require further study. First, we will investigate the approximate controllability of fractional neutral stochastic integro-differential systems with state-dependent delay both in the case of a Poisson jumps and a normal topological space. Secondly, we will be devoted to studying the approximate controllability of a new class of impulsive fractional stochastic differential equations with state-dependent delay and non-instantaneous impulses as discussed in [15].

References

[1] R. P. Agarwal, D. B. Andrade and G . Siracusa, On fractional integro-differential equations with state-dependent delay, *Computers and Mathematics with Applications,* 62(2011), 1143-1149.

[2] K. Aissani and M. Benchohra, Fractional integro-differential equations with state-dependent delay, *Advances in Dynamical Systems and Applications*, 9(1)(2014), 17-30.

[3] K. Aissani and M. Benchohra, Impulsive fractional differential inclusions with infinite delay, *Electronic Journal of Differential Equations*, Vol. 2013(2013), No. 265, 1-13.

[4] D. Bainov and V. Covachev, *Impulsive Differential Equations with a Small Parameter*, World Scientific Publishing Corporation, Singapore, 1995.

[5] P. Balasubramanian and P. Tamilalagan, Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function, *Applied Mathematics and Computation*, 256(2015), 232-246.

[6] D. Baleanu, J. A. T. Machado and A. C. J. Luo, *Fractional Dynamics and Control*, Springer, New York, USA, 2012.

[7] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions, in: Contemporary Mathematics and its Applications*, Vol. 2, Hindawi Publishing Corporation, New York, 2006.

[8] G. Bonanno, R. Rodriguez-Lopez and S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations, *Fractional Calculus and Applied Analysis*, 17(3)(2014), 717-744.

[9] J. P. Carvalho dos Santos, M. Mallika Arjunan and C. Cuevas, Existence results for fractional neutral integrodifferential equations with state-dependent delay, *Computers and Mathematics with Applications,* 62 (2011), 1275–1283.

[10] A. Chadha and D. N. Pandey, Existence results for an impulsive neutral stochastic fractional integro-differential equation with infinite delay, *Nonlinear Analysis: Theory, Methods and Applications*, 128(2015), 149–175.

[11] B. C. Dhage, S. B. Dhage and S. K. Ntouyas, Existence and approximate solutions for fractional differential equations with nonlocal conditions, *Journal of Fractional Calculus and Applications*, 7(1)(2016), 24-35.

[12] A. Debbouche and D.F.M. Torres, Approximate controllability of fractional nonlocal delay semilinear systems in Hilbert spaces, *International Journal of Control*, 86(2013), 1577-1580.

[13] D. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[14] X. Fu and R. Huang, Existence of solutions for neutral integro-differential equations with state-dependent delay, *Applied Mathematics and Computation,* 224(2013), 743-759.

[15] G. Gautam and J. Dabas, Mild solutions for class of neutral fractional functional differential equations with not instantaneous impulses, *Applied Mathematics and Computation*, 259(2015), 480–489.

[16] T. Guendouzi and L. Bousmaha, Existence of solutions for fractional partial neutral stochastic functional integro-differential inclusions with state-dependent delay and analytic resolvent operators, *Vietnam Journal of Mathematics,* 43(4)(2015), 687-704.

[17] T. Guendouzi and L. Bousmaha, Approximate controllability of fractional neutral stochastic functional integro-differential inclusions with infinite delay, *Qualitative Theory of Dynamical Systems,* 13(2014), 89-119.

[18] J. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcialaj Ekvacioj,* 21(1978), 11-41.

[19] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Unbounded Delay,* Springer-Verlag, Berlin, 1991.

[20] P. Kalamani, D. Baleanu, S.Selvarasu and M. Mallika Arjunan, On existence results for impulsive fractional neutral stochastic integro-differential equations with nonlocal and state-dependent delay conditions, *Advances in Difference Equations,* (2016), 2016:163.

[21] A. Kilbas, H. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amesterdam, 2006.

[22] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[23] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkh*ä*user, Basel, 1995.

[24] N.I. Mahmudov, V. Vijayakumar and R. Murugesu, Approximate controllability of second-order evolution differential inclusions in Hilbert spaces, *Mediterranean Journal of Mathematics,* 13(5) (2016), 3433-3454.

[25] N. I. Mahmudov, Approximate controllability of fractional neutral evolution equations in Banach spaces, *Abstract and Applied Analysis,* (2013) Article ID 531894, 11 pages.

[26] F. Mainardi, P. Paradisi and R. Gorenflo, *Probability distributions generated by fractional difusion equations*, In: Kertesz, J, Kondor, I(eds.) Econophysics: An emerging Science . Kluwer Academic, Dordrecht, 2000.

[27] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[28] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[29] R. Sakthivel, Y. Ren, A. Debbouche and N.I. Mahmudov, Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions, *Applicable Analysis* 95(11), 2016, 2361-2382.

[30] R. Sakthivel and Y. Ren, Approximate controllability of fractional differential equations with state-dependent delay, *Results in Mathematics,* 63(3)(2013), 949-963.

[31] R. Sakthivel, R. Ganesh and S. Suganya, Approximate controllability of fractional neutral stochastic system with infinite delay, *Reports on Mathematical Physics*, 70(3)(2012), 291-311.

[32] X. B. Shu and Y. Shi, A study on the mild solution of impulsive fractional evolution equations, *Applied Mahtematics and Computation*, 273(2016), 465-476.

[33] X. B. Shu and F. Xu, The existence of solutions for impulsive fractional partial neutral differential equations, *Journal of Mathematics*, Volume 2013, Article ID 147193, 9 pages.

[34] I. M. Stamova, *Stability Analysis of Impulsive Functional Differential Equations*, De Gruyter, 2009.

[35] Suganya. S, Baleanu. D, Kalamani. P and Mallika Arjunan. M, On fractional neutral integro-differential systems with state-dependent delay and non-instantaneous impulses, *Advances in Difference Equations*, (2015) 2015:372.

[36] V. Vijayakumar, Approximate controllability results for abstract neutral integro-differential inclusions with infinite delay in Hilbert spaces, *IMA Journal of Mathematical Control and Information*, (2016), 1-18. doi: 10.1093/imamci/dnw049.

[37] V. Vijayakumar, C. Ravichandran, R. Murugesu and J.J. Trujillo, Controllability results for a class of fractional semilinear integro-differential inclusions via resolvent operators, *Applied Mathematics and Computation*, 247 (2014), 152-161.

[38] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[39] Z. Yan and F. Lu, Approximate controllability of impulsive partial neutral stochastic functional integro-differential inclusions with infinite delay, *Advances in Difference Equations*, (2015):106.

[40] Z. Yan and X. Jia, Approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-dependent delay, *Collectanea Mathematica*, 66(2015), 93-124.

[41] Z. Yan and H. Zhang, Asymptotic stability of fractional impulsive stochastic partial integro-differential equations with state-dependent delay, *Electronic Journal of Differential Equations*, Vol. 2013 , No. 206, 1-29.

[42] Z. Yan, Approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces, *IMA Journal of Mathematical Control and Information*, (2012), 1-20.

[43] Y. Zhang and J. Li, Approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions, *Boundary Value Problems*, (2013), 2013:193.

[44] X. Zhang, C. Zhu and C. Yuan, Approximate controllability of impulsive fractional stochastic differential equations with state-dependent delay, *Advances in Difference Equations*, (2015), 2015:91.

[45] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[46] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Computers and Mathematics with Applications*, 59(3)(2010), 1063-1077.

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Oscillation theorems for higher order neutral nonlinear dynamic equations on time scales

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Abstract

In this paper, we will establish some oscillation criteria for the even-order nonlinear dynamic equation

$$\left(a\left(x^{\Delta^{n-2}}\right)^{\gamma}\right)^{\Delta^2}(t)+f\left(t,x^{\alpha}(t)\right)=0, \qquad t\in[t_0,\infty)_{\mathbb{T}}$$

on a time scales $\mathbb{T}$ with n is an even integer ≥ 3, where γ and α are the ratios of positive odd integer and a is areal valued rd-continuous function defined on $\mathbb{T}$.

Keywords: Time scale, Oscillation, Neutral delay differential equation.

2010 MSC: 34K11, 39A10, 39A99.

1 Introduction

The theory of time scales was introduced by Hilger [1] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus. The books on the subjects of time scale, that is, measure chain, by Bohner and Peterson [2], [3], summarize and organize much of time scale calculus.

The theory of oscillations is an important branch of the applied theory of dynamic equations related to the study of oscillatory phenomena in technology and natural and social sciences. In recent years, there has been much research activity concerning the oscillation of solutions of various dynamic equations on time scales.

In this paper, we deal with the oscillation of all solutions of the even-order nonlinear delay dynamic equation

$$\left(a\left(x^{\Delta^{n-2}}\right)^{\gamma}\right)^{\Delta^2}(t)+f\left(t,x^{\alpha}(t)\right)=0, \qquad t\in[t_0,+\infty)_{\mathbb{T}} \tag{1.1}$$

on a time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$, n is an even integer ≥ 3. Where α, γ are a quotient of odd positive integer, $a\in\mathcal{C}^1\left(\mathbb{T},\mathbb{R}^+\right)$ such that $a^{\Delta}(t)>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$ and f satisfies the following conditions:

$(\mathcal{H}_1)$ $f:\mathbb{T}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous,

$(\mathcal{H}_2)$ $f(t,-x)=-f(t,x)$ for all $t\in[t_0,\infty)_{\mathbb{T}}$, $x\in\mathbb{R}$,

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$(\mathcal{H}_3)$ There exist a function $r : \mathbb{T} \longrightarrow \mathbb{R}$ positive and rd-continuous, such that

$$\frac{f(t,x)}{x} \geq r(t), \qquad \text{for all } t \in [t_0,\infty)_{\mathbb{T}}, x \in \mathbb{R} - \{0\}. \tag{1.2}$$

In order to prove our theorems we shall need the following two lemmas.

Lemma 1.1. *[4] If $n \in \mathbb{N}$, $\sup \mathbb{T} = \infty$ and $f \in C^n_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ then the following statements are true.*

1. $\liminf_{t\to\infty} f^{\Delta^n}(t) > 0$ *implies* $\lim_{t\to\infty} f^{\Delta^k}(t) = \infty$ *for all* $k \in [0,n)_{\mathbb{Z}}$.
2. $\limsup_{t\to\infty} f^{\Delta^n}(t) < 0$ *implies* $\lim_{t\to\infty} f^{\Delta^k}(t) = -\infty$ *for all* $k \in [0,n)_{\mathbb{Z}}$.

Lemma 1.2. *[7] Assume that $\sup \mathbb{T} = \infty$, $f \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $\lambda > 0$. Then*

$$f^{\Delta}(f^{\sigma})^{-\lambda} \leq \frac{\left(f^{1-\lambda}\right)^{\Delta}}{1-\lambda} \leq f^{\Delta} f^{-\lambda}, \quad \text{on } [t_0,\infty)_{\mathbb{T}}.$$

2 Main results

In this section, we establish some sufficient conditions which guarantee that every solution x of (1.1) oscillates on $[t_0,\infty)_{\mathbb{T}}$.

Before stating the main results, we begin with the following lemma.

Lemma 2.3. *Suppose that x is an eventually positive solution of (1.1) and*

$$\lim_{t\to\infty} \frac{1}{a(t)} \in \mathbb{R}^*_+, \qquad \lim_{t\to\infty} \frac{t}{a(t)} \int_t^{\infty} r(s)\,\Delta s = \infty. \tag{2.3}$$

Then there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$\left(a\left(x^{\Delta^{n-2}}\right)^{\gamma}\right)^{\Delta}(t) > 0, \qquad x^{\Delta^{n-2}}(t) > 0, \text{ for all } t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.4}$$

Lemma 2.4. *Assume that x is an eventually positive solution of (1.1) and (2.3) hold. Suppose there exists a sequence functions $\phi_1, \phi_2, \cdots, \phi_{n-2} \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$. Let $A_1, A_2, \cdots A_{n-2}$ are functions defined by*

$$A_1(t,t_1) := \left\{\frac{a(t)}{\phi_1(t)}\right\}^{\frac{1}{\gamma}} \int_{t_1}^{t} \left\{\frac{\phi_1(s)}{a(s)}\right\}^{\frac{1}{\gamma}} \Delta s, \qquad \text{for } t \in [t_1,\infty)_{\mathbb{T}},$$

and

$$A_k(t,t_1) := \frac{1}{\phi_k(t)} \int_{t_1}^{t} \phi_k(s)\,\Delta s, \quad \text{for all } t \in [t_1,\infty)_{\mathbb{T}} \text{ and all } k \in [2,n-1)_{\mathbb{Z}}.$$

where $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Moreover, suppose that

$$\phi_1(t) - \phi_1^{\Delta}(t)(t-t_1) \leq 0, \qquad \text{for } t \in [t_1,\infty)_{\mathbb{T}}, \tag{2.5}$$

and

$$\phi_k(t) - \phi_k^{\Delta}(t) A_{k-1}(t,t_1) \leq 0, \quad \text{for all } t \in [t_1,\infty)_{\mathbb{T}} \text{ and all } k \in [2,n-1)_{\mathbb{Z}}. \tag{2.6}$$

Then

$$x^{\Delta^k}(t) \geq E_k(t,t_1)\, x^{\Delta^{n-2}}(t), \quad \text{for all } t \in [t_1,\infty)_{\mathbb{T}} \text{ and all } k \in [0,n-2)_{\mathbb{Z}},$$

where

$$E_k(t,t_1) := \prod_{m=1}^{m=n-k-2} A_m(t,t_1), \quad \text{for all } t \in [t_1,\infty)_{\mathbb{T}}.$$

Theorem 2.1. *Let (2.3) hold and $\alpha > \gamma$. Assume that there exist sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that*

$$\int_{t_1}^{\infty} E_1(t, t_1) \left(\frac{t - t_1}{a(t)} \int_{\sigma(t)}^{\infty} r(u)\, \Delta u \right)^{\frac{1}{\gamma}} \Delta t = \infty, \tag{2.7}$$

where E_1 is defined as in Lemma 2.4.
Then equation (1.1) is oscillatory.

Proof. Suppose the contrary, that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1.1), since the substitution $y(t) = -x(t)$ transforms equation (1.1) into an equation of the same form. Say $x(t) > 0$ for $t \geq t_1 \geq t_0$.

By (1.2) , we get

$$\left(a \left(x^{\Delta^{n-2}} \right)^{\gamma} \right)^{\Delta^2} (t) \leq -r(t)\, x^{a}(t), \qquad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.8}$$

Integrating (2.8) form t to ∞, we have

$$\left(a \left(x^{\Delta^{n-2}} \right)^{\gamma} \right)^{\Delta} (t) \geq \int_{t}^{\infty} r(s)\, x^{\alpha}(s)\, \Delta s, \qquad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.9}$$

By (2.8), we have that $\left(a \left(x^{\Delta^{n-2}} \right)^{\gamma} \right)^{\Delta}$ is nonincreasing in $[t_1, \infty)_{\mathbb{T}}$. Then, for all $t \in [t_1, \infty)_{\mathbb{T}}$, we obtain

$$a(t) \left(x^{\Delta^{n-2}}(t) \right)^{\gamma} \geq \int_{t_1}^{t} \left(a \left(x^{\Delta^{n-2}} \right)^{\gamma} \right)^{\Delta} (s)\, \Delta s \geq (t - t_1) \left(a \left(x^{\Delta^{n-2}} \right)^{\gamma} \right)^{\Delta} (t).$$

As above we see that

$$x^{\Delta^{n-2}}(t) \geq \left(\frac{t - t_1}{a(t)} \int_{t}^{\infty} r(s)\, x^{\alpha}(s)\, \Delta s \right)^{\frac{1}{\gamma}}, \qquad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

By lemma 2.4, we have

$$x^{\Delta}(t) \geq \left(\frac{t - t_1}{a(t)} \int_{t}^{\infty} r(s)\, x^{\alpha}(s)\, \Delta s \right)^{\frac{1}{\gamma}} E_1(t, t_1), \qquad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

Clearly $x^{\Delta}(t) > 0$, for $t \in [t_1, \infty)_{\mathbb{T}}$, then

$$x^{\Delta}(t)\, x^{\frac{-\alpha}{\gamma}}(\sigma(t)) \geq \left(\frac{t - t_1}{a(t)} \int_{\sigma(t)}^{\infty} r(s)\, \Delta s \right)^{\frac{1}{\gamma}} E_1(t, t_1), \qquad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

By lemma 1.2, we get

$$\frac{\gamma}{\gamma - \alpha} \left(x^{1 - \frac{\alpha}{\gamma}} \right)^{\Delta} (t) \geq \left(\frac{t - t_1}{a(t)} \int_{\sigma(t)}^{\infty} r(s)\, \Delta s \right)^{\frac{1}{\gamma}} E_1(t, t_1), \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.10}$$

Integrating (2.10) from t_1 to t and letting $t \to \infty$, we have

$$\int_{t_1}^{\infty} E_1(t, t_1) \left(\frac{t - t_1}{a(t)} \int_{\sigma(t)}^{\infty} r(s)\, \Delta s \right)^{\frac{1}{\gamma}} \Delta t \leq -\frac{\gamma}{\gamma - \alpha} x^{1 - \frac{\alpha}{\gamma}}(t_1).$$

This result is in contradiction with (2.7) . □

Theorem 2.2. *Let (2.3) holds and $\alpha = \gamma \geq 1$. Assume that there exist positive function $\delta \in C^1_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}\right)$ such that for all sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$, for some $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that*

$$\int_{t_2}^{\infty} \delta(t)\, r(t) - \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{\left(\delta_+^\Delta(t)\right)^{\gamma+1} a(t)}{\delta^\gamma(t)\, E_1^\gamma(t,t_1)\,(t-t_1)} \Delta t = \infty, \tag{2.11}$$

where $\delta_+^\Delta(t) = \max\left(0, \delta^\Delta(t)\right)$ and E_1 is defined as in Lemma 2.4.
Then equation (1.1) is oscillatory.

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$.
We define the function $w(t)$ by

$$w(t) = \delta(t) \frac{\left(a\left(x^{\Delta^{n-2}}\right)^\gamma\right)^\Delta(t)}{x^\gamma(t)}, \qquad t \in [t_1,\infty)_{\mathbb{T}}.$$

Then $w(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$ and by (2.8) which implies that

$$\begin{aligned} w^\Delta(t) &\leq -\delta(t)\, r(t) + \frac{w^\sigma(t)}{\delta^\sigma(t)} x^\gamma(\sigma(t)) \left\{ \frac{\delta^\Delta(t)\, x^\gamma(t) - \delta(t)\left(x^\gamma\right)^\Delta(t)}{x^\gamma(t)\, x^\gamma(\sigma(t))} \right\} \\ &\leq -\delta(t)\, r(t) + \frac{\delta^\Delta(t)}{\delta^\sigma(t)} w^\sigma(t) - w^\sigma(t) \frac{\delta(t)\left(x^\gamma\right)^\Delta(t)}{\delta^\sigma(t)\, x^\gamma(t)}. \end{aligned} \tag{2.12}$$

By Pötzsche's chain rule [2], we get

$$\begin{aligned} \left(x^\gamma(t)\right)^\Delta &= \gamma x^\Delta(t) \int_0^1 \left(hx(t) + (1-h)x^\sigma(t)\right)^{\gamma-1} dh \\ &\geq x^\Delta(t) x^{\gamma-1}(t). \end{aligned} \tag{2.13}$$

Substituting (2.13) in (2.12), we find

$$w^\Delta(t) \leq -\delta(t)\, r(t) + \frac{\delta^\Delta(t)}{\delta^\sigma(t)} w^\sigma(t) - w^\sigma(t) \frac{\delta(t)\, x^\Delta(t)}{\delta^\sigma(t)\, x(t)}. \tag{2.14}$$

By lemma 2.4, we find

$$\begin{aligned} x^\Delta(t) &\geq \frac{E_1(t,t_1)}{(a(t))^{\frac{1}{\gamma}}} \left[a(t)\left(x^{\Delta^{n-2}}(t)\right)^\gamma\right]^{\frac{1}{\gamma}} \\ &\geq E_1(t,t_1) \left[\frac{t-t_1}{a(t)}\right]^{\frac{1}{\gamma}} \left[\left(a\left(x^{\Delta^{n-2}}\right)^\gamma\right)^\Delta(t)\right]^{\frac{1}{\gamma}} \\ &\geq E_1(t,t_1)\, x(t) \left(\frac{t-t_1}{a(t)\,\delta^\sigma(t)}\right)^{\frac{1}{\gamma}} \left(w^\sigma(t)\right)^{\frac{1}{\gamma}}. \end{aligned} \tag{2.15}$$

Substituting (2.15) in (2.14), we get

$$w^\Delta(t) \leq -\delta(t)\, r(t) + \frac{\delta^\Delta(t)}{\delta^\sigma(t)} w^\sigma(t) - \frac{\delta(t)\, E_1(t,t_1)}{\delta^\sigma(t)} \left(\frac{t-t_1}{a(t)\,\delta^\sigma(t)}\right)^{\frac{1}{\gamma}} \left(w^\sigma(t)\right)^{1+\frac{1}{\gamma}}.$$

Using the inequality [10]

$$By - Ay^{1+\frac{1}{\beta}} \leq \frac{\beta^\beta B^{\beta+1}}{(\beta+1)^{\beta+1} A^\beta}, \qquad A > 0,\ B > 0 \text{ and } \beta > 0.$$

which yields

$$w^\Delta(t) \leq -\delta(t)\, r(t) + \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{\left(\delta_+^\Delta(t)\right)^{\gamma+1} a(t)}{\delta^\gamma(t)\, E_1^\gamma(t,t_1)\,(t-t_1)}.$$

Integrating the last inequality from t_2 to t, we have

$$\int_{t_2}^{t} \delta(s) r(s) - \frac{\gamma^{\gamma} \left(\delta_{+}^{\Delta}(s)\right)^{\gamma+1} a(s)}{(\gamma+1)^{\gamma+1} \delta^{\gamma}(s) E_1^{\gamma}(s,t_1)(s-t_1)} \Delta s \leq w(t_2) - w(t) \leq w(t_2).$$

which contradicts (2.11). This completes the proof. □

Theorem 2.3. *Let (2.3) holds and $\gamma > \alpha$. Assume that there exist positive function $\delta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that for all sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$, such that*

$$\int_{t_1}^{\infty} \delta^{\sigma}(t) r(t) E_0^{\alpha}(t,t_1) \left(\frac{t-t_1}{a(t)\delta(t)}\right)^{\frac{\alpha}{\gamma}} \Delta t = \infty, \tag{2.16}$$

where $\delta^{\Delta}(t) \leq 0$, for all $t \in [t_1,\infty)_{\mathbb{T}}$ and E_0 is defined as in Lemma 2.4.
Then every solution of (1.1) is either oscillatory.

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$.
Let

$$w(t) = \delta(t)\left(a\left(x^{\Delta^{n-2}}\right)^{\gamma}\right)^{\Delta}(t), \qquad t \in [t_1,\infty)_{\mathbb{T}}.$$

Then $w(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$ and by (1.2), we obtain

$$w^{\Delta}(t) \leq -\delta^{\sigma}(t) r(t) x^{\alpha}(t). \tag{2.17}$$

By lemma 2.4, we get

$$\begin{aligned} x(t) &\geq E_0(t,t_1) x^{\Delta^{n-2}}(t) \\ &\geq E_0(t,t_1)\left(\frac{t-t_1}{a(t)\delta(t)}\right)^{\frac{1}{\gamma}} w^{\frac{1}{\gamma}}(t). \end{aligned} \tag{2.18}$$

Substituting (2.18) in (2.17), we find

$$-w^{\Delta}(t) w^{\frac{-\alpha}{\gamma}}(t) \geq \delta^{\sigma}(t) r(t) E_0^{\alpha}(t,t_1)\left(\frac{t-t_1}{a(t)\delta(t)}\right)^{\frac{\alpha}{\gamma}}.$$

By Lemma 1.2 we have

$$-\frac{\gamma}{\gamma-\alpha}\left(w^{1-\frac{\alpha}{\gamma}}\right)^{\Delta}(t) \geq \delta^{\sigma}(t) r(t) E_0^{\alpha}(t,t_1)\left(\frac{t-t_1}{a(t)\delta(t)}\right)^{\frac{\alpha}{\gamma}}.$$

Integrating this inequality from t_1 to t we obtain

$$\int_{t_1}^{t} \delta^{\sigma}(s) r(s) E_0^{\alpha}(s,t_1)\left(\frac{s-t_1}{a(s)\delta(s)}\right)^{\frac{\alpha}{\gamma}} \Delta s \leq \frac{\gamma}{\gamma-\alpha} w^{1-\frac{\alpha}{\gamma}}(t_1),$$

for all large t. This result is in contradiction with (2.16). This completes the proof. □

3 Example

As some application of the main results, we present the following example.

Example 3.1. *On the quantum set $\mathbb{T} = \overline{2^{\mathbb{Z}}}$. Consider the following n-order neutral differential equation*

$$x^{\Delta^n}(t) + t^{-\frac{3}{2}} x^{\alpha}(t) = 0, \quad t \in [1,\infty)_{\overline{2^{\mathbb{Z}}}}. \tag{3.19}$$

where $n \geq 3$ is even integer. Here $a(t) = 1$, $r(t) = t^{-\frac{3}{2}}$, $\gamma = 1$ and α is a quotient of odd positive integer.
It is easy to see that (2.3) hold.
Set

$$\phi_1(t) := h_k(t, t_1), \qquad \text{for all } k \in [1, n-1)_{\mathbb{Z}} \text{ and for } t \in [t_1, \infty)_{\overline{2^{\mathbb{Z}}}}.$$

Then (2.6) and (2.5) holds.
Moreover, for all $k \in [1, n-1)_{\mathbb{Z}}$, we have

$$A_k(t, t_1) = \frac{h_{k+1}(t, t_1)}{h_k(t, t_1)}, \qquad \text{for all } t \in [t_1, \infty)_{\overline{2^{\mathbb{Z}}}}.$$

Then

$$E_1(t, t_1)\left(\frac{(t-t_1)}{a(t)}\int_{\sigma(t)}^{\infty} r(u)\,\Delta u\right)^{\frac{1}{\gamma}} \geq \frac{h_{n-2}(t, t_1)}{\sqrt{t}}, \quad \text{for all } t \in [t_1, \infty)_{\overline{2^{\mathbb{Z}}}}.$$

By Theorem 2.1, every solution x of (3.19) is either oscillatory.

References

[1] S. Hilger, *Analysis on measure chains-a unified approach to continuous and discrete calculus, Results Math,* 8(1990), 18 − 56.

[2] M. Bohner, and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston,* (2001) .

[3] M. Bohne, and A. Peterson, *Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston,* (2001) .

[4] B. Karpuz, *Sufficient conditions for the oscillation and asymptotic beaviour of higher-order dynamic equations of neutral type, Applied Mathematics and Computation,* 221(2013), 453 − 462.

[5] B. Baculíkovà, and J. Dzǔurina, *Oscillation theorems for higher order neutral differential equations, Applied Mathematics and Computation,* 219(2012), 3769 − 3778.

[6] R. P. Agarwal , M. Bohner, T. Li, and C. Zhang, *A new approach in the study of oscillatory behavior of even-order neutral delay differential equations, Applied Mathematics and Computation,* 225(2013), 787 − 794.

[7] S.R. Grace, *On the Oscillation of nth Order Dynamic Equations on Time-Scales, Mediterr. J. Math,* 10(2013), 147 − 156.

[8] Y. Shi, *Oscillation criteria for nth order nonlinear neutral differential equations, Applied Mathematics and Computation,* 235(2014), 423 − 429.

[9] C. Zhanga, R. P. Agarwal , M. Bohner, and T. Li, *New results for oscillatory behavior of even-order half-linear delay differential equations, Applied Mathematics Letters,* 26(2013), 179 − 183.

[10] S. H. Saker, *Oscillation of second-order nonlinear neutral delay dynamic equations on time scales, Journal of Computational and Applied Mathematics,* 187(2006), 123 − 141.

[11] S. H. Saker, *Oscillation of second-order nonlinear neutral delay dynamic equations on time scales, Journal of Computational and Applied Mathematics,* 187(2006), 123 − 141.

[12] L. Erbe, A. Peterson, and S. H. Saker, *Hille and Nehari type criteria for third dorder dynamic equations, J. Math. Anal. Appl,* 329(2007), 112 − 131.

[13] L. Erbe, T.S. Hassan, and A. Peterson, *Oscillation criteria for nonlinear damped dynamic equations on time scales, Appl. Math. Comput,* 203(2008), 343 − 357.

[14] S. R. Grace, M. Bohner, and S. Sun, *Oscillation of fourth-order dynamic equations. Hacettepe Journal of Mathematics and Statistics*, 39(2010), 545 − 553.

[15] F. Z. Ladrani, A. Hammoudi, and A. Benaissa Cherif, *Oscillation theorems for fourth-order nonlinear dynamic equations on time scales, Electronic Journal of Mathematical Analysis and Applications*, 3(2015), 46 − 58.

[16] T. Li, E. Thandapani, and S. Tang, *Oscillation theorems for fourth-order delay dynamic equations on time scales, Bulletin of Mathematical Analysis and Applications*, 3(2011), 190 − 199.

[17] Y. Qi, and J. Yu, *Oscillation criteria for fourth-order nonlinear delay dynamic equations, Electronic Journal of Differential Equations*, 79(2013), 1 − 17.

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Existence results for non-autonomous neutral integro-differential systems with impulsive and nonlocal conditions

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Abstract

In accordance with semigroup theory, fractional powers of operators, approximation techniques and Banach contraction principle fixed point theorem, this manuscript is primarily involved with the existence results for an impulsive non-autonomous neutral integro-differential systems with nonlocal conditions in Banach space $\mathbb{E}$.

Keywords: Integro-differential equations, Semigroup theory, Impulsive and nonlocal conditions, Evolution equations, Fixed point theorem.

2010 MSC: 34A37, 34K37, 35R11, 34G20.

1 Introduction

The dynamics of many processes in engineering, physics, population dynamics, biology, medicine and other fields are subject to sudden changes just like shocks or perturbations. These perturbations may be considered as impulses. In particular, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. This sort of models can be defined by impulsive differential equations. For more details on this theory and its applications, we suggest the reader to refer the books [1, 2] and the papers [3–8], and the references cited therein. These days, impulsive integro-differential equations have become an significant area of research because of their uses to numerous problems arising in communications, control technology, impact mechanics and electrical engineering, etc.

The nonlocal condition, which is a speculation of the standard initial condition, was inspired by physical issues. On many instances, problems under consideration, primarily coming up from physics phenomena, advise that the initial condition is an estimation via solving the problem in some finite sequence of times, and then we say that the initial condition is nonlocal. Evolution problems with nonlocal initial conditions in Banach spaces are now perfectly realized due to the fact it was initiated by Byszewski [9, 10], where the author demonstrated the existence and uniqueness of mild, strong and classical solution to the first-order initial value problem by utilizing the techniques of semigroups and the Banach fixed point theorem. For the importance of nonlocal conditions in diverse areas, we suggest [9, 10] and references cited therein.

Moreover, a class of equations depends on past as well as present values but which involve derivatives with delays as well as the function itself. Such equations historically have been referred to as neutral functional differential equations. For systems with neutral type, the existence of the solution has been investigated in Tsoi [11]. A great information to the literature for neutral functional differential equations is the book by Hale and Lunel [12] and the references therein.

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The existence, controllability and other qualitative and quantitative properties of non-autonomous differential and integro-differential equations with impulsive conditions) are the most advancing area of pursuit, in particular, see [13–15]. In [13], the authors studied the existence of the mild solutions to a class of abstract non-autonomous impulsive functional integro-differential equations. The existence and Ulam-Hyers-Rassias stability of mild solution of impulsive non-autonomous differential equations are studied by authors in [14]. In particular, in [15], author has demonstrated the controllability of a system of impulsive semilinear non-autonomous differential equations via Rothe's type fixed-point theorem. By applying approximation techniques and fractional operator, the existence of the mild solution for different class of impulsive functional integro-differential equations have been established by many authors[16–21]. Recently, in [19], the authors investigate the existence of a mild solution for an impulsive nonlocal non-autonomous neutral functional differential equation in Banach space by utilizing the approximation techniques, fractional powers of operators and Krasnoselskii's fixed-point theorem.

Motivated by above mentioned works [13, 19], the main purpose of this paper is to prove the existence of mild solutions for the following impulsive non-autonomous neutral partial integro-differential equations in a Banach space $\mathbb{E}$:

$$\frac{d}{dt}\left[z(t)-\mathscr{F}\left(t,z(h_1(t)),\int_0^t a_1(t,s,z(h_2(s)))ds\right)\right]=-B(t)z(t)+\mathscr{G}\left(t,z(h_3(t)),\int_0^t a_2(t,s,z(h_4(s)))ds\right)$$
$$+\mathscr{H}\left(t,z(h_5(t)),\int_0^t a_3(t,s,z(h_6(s)))ds\right),\ t\in\mathscr{I},\ t\neq t_i, \tag{1.1}$$

$$z(0)=z_0+g(z)\in\mathbb{E}, \tag{1.2}$$

$$\Delta z(t_i)=\mathcal{I}_i(z(t_i)),\quad i=1,2,\ldots,q,\ q\in\mathbb{N}, \tag{1.3}$$

where $\mathscr{I}=[0,T], 0<T<\infty, -B(t):\mathcal{D}(B(t))\subseteq\mathbb{E}\to\mathbb{E}, t\geq 0$ is a closed densely defined linear operator. Here, $h_j:\mathscr{I}\to\mathscr{I}, j=1,2,\ldots,6$ and $0=t_0<t_1<t_2<\cdots<t_q<t_{q+1}=T$ are fixed numbers, $\Delta z|_{t=t_i}=z(t_i^+)-z(t_i^-)$ and $z(t_i^-)=\lim_{\epsilon\to 0-}z(t_i+\epsilon)$ and $z(t_i^+)=\lim_{\epsilon\to 0+}z(t_i+\epsilon)$ denotes the left and right limits of $z(t)$ at $t=t_i$, respectively. Let $B(t)$ be the infinitesimal generator of a compact analytic semigroup of bounded linear operators on a Banach space $\mathbb{E}$. The functions $\mathscr{F},\mathscr{G},\mathscr{H},a_i, i=1,2,3$ and $I_i:\mathbb{E}\to\mathbb{E}(i=1,2,\ldots,q)$ are appropriate functions fulfilling some suitable conditions to be specified later.

The rest of this paper is organized as follows: In section 2, we recall some basic definitions and preliminary facts which will be utilized throughout this paper. Existence theorems and their proofs are given in section 3. Finally, in Section 4 an example is presented to illustrate the application of the obtained results.

2 Preliminaries

In this section, we recall some basic definitions, preliminaries, theorems and lemmas and assumptions required for establishing our results.

Throughout this manuscript, we assume that $(\mathbb{E},\|\cdot\|)$ is a Banach space and the notation $\mathcal{C}([0,T],\mathbb{E})$ stands for the space of $\mathbb{E}$-valued continuous functions on $[0,T]$ with the norm $\|y\|=\sup\{\|y(\tau)\|,\tau\in[0,T]\}$ and $\mathscr{L}^1([0,T],\mathbb{E})$ denotes the space of $\mathbb{E}$-valued Bochner integrable functions on $[0,T]$ endowed with the norm $\|\mathcal{F}\|_{\mathscr{L}^1}=\int_0^T\|\mathcal{F}(t)\|dt, \mathcal{F}\in([0,T],\mathbb{E})$. We denote by $\mathcal{C}^\beta([0,T],\mathbb{E})$ the space of all uniformly H$o$lder continuous functions from $[0,T]$ into $\mathbb{E}$ with exponent $\beta>0$. We can easily confirm that $\mathcal{C}^\beta([0,T],\mathbb{E})$ is a Banach space with the norm

$$\|z\|_{\mathcal{C}^\beta([0,T],\mathbb{E})}=\sup_{0\leq t\leq T}\|z(t)\|+\sup_{0\leq t,s\leq T,t\neq s}\frac{\|z(t)-z(s)\|}{|t-s|^\beta}.$$

To be able to define the mild solution for the impulsive problem, we define the space $PC([0,T];\mathbb{E})=\{z:[0,T]\to\mathbb{E}: y$ is continuous at $t\neq t_i$ and left continuous at $t=t_i$ and $z(t_i^+)$ exists, for all $i=1,2,\cdots,q\}$. Clearly, $PC([0,T];\mathbb{E})$ is a Banach space endowed the norm $\|z\|_{PC}=\sup_{t\in[0,T]}\|z(s)\|$. For a function $z \in PC([0,T];\mathbb{E})$ and $i \in \{0,1,\cdots,q\}$, we define the function $\widetilde{z}_i \in \mathcal{C}([t_i,t_{i+1}],\mathbb{E})$ such that

$$\widetilde{z}_i(t) = \begin{cases} z(t), & \text{for} \quad t \in (t_i,t_{i+1}], \\ z(t_i^+), & \text{for} \quad t = t_i. \end{cases}$$

For $W \subset PC([0,T],\mathbb{E})$ and $i \in \{0,1,\cdots,q\}$, we have $\widetilde{W}_i = \{\widetilde{z}_i : z \in W\}$ and following Accoli-Arzela type criteria.

Lemma 2.1. *[18] A set $W \subset PC([0,T];\mathbb{E})$ is relatively compact in $PC([0,T];\mathbb{E})$ if and only if each set $\widetilde{W}_j (j = 1,2,\cdots,q)$ is relatively compact in $\mathcal{C}([t_j,t_{j+1}],\mathbb{E})(j = 0,1,2,\cdots,q)$.*

Let $\{B(t) : 0 \le t \le T\}, 0 < T < \infty$ be a family of closed linear operators on the Banach space $\mathbb{E}$. We impose following restrictions ([22]) as:

(P1) The domain $\mathcal{D}(B)$ of $\{B(t) : t \in [0,T]\}$ is dense in $\mathbb{E}$ and $\mathcal{D}(B)$ is independent of t.

(P2) For each $0 \le t \le T$ and $Re\,\lambda \le 0$, the resolvent $R(\lambda; B(t))$ exists and there exists a positive constant K (independent of t and λ) such that

$$\|R(\lambda; B(t))\| \le \frac{K}{(|\lambda| + 1)}, \quad Re\lambda \le 0, \quad t \in [0,T].$$

(P3) For each fixed $\xi \in [0,T]$, there exists a constant $K > 0$ and $0 < \mu \le 1$ such that

$$\|[B(\tau) - B(s)]B^{-1}(\xi)\| \le K|\tau - s|^{\mu}, \quad \text{for any } \tau, s \in [0,T],$$

where μ and K are independent of τ, s and ξ.

(P4) For every $t \in [0,T]$, the resolvent set of $B(t)$, the resolvent $R(\lambda, B(t))$, is a compact operator for some $\lambda \in \rho(B(t))$.

The assumptions $(P1) - (P3)$ permit that there is a unique linear evolution system (linear evolution operator) $\mathcal{S}(t,s), 0 \le s \le t \le T$ which is generated by family $\{B(t) : t \in [0,T]\}$ and there exists a family of bounded linear operators $\{\Phi(t,s) : 0 \le t \le s \le T\}$ such that $\|\Phi(t,s)\| \le \dfrac{K}{|t-s|^{1-\mu}}$. We also have that $\mathcal{S}(t,s)$ can be written as

$$\mathcal{S}(t,s) = e^{-(t-s)B(t)} + \int_0^t e^{-(t-\tau)B(\tau)}\Phi(\tau,s)d\tau.$$

The assumption $(P2)$ guarantees that $-B(s), s \in [0,T]$ is the infinitesimal generator of a strongly continuous compact analytic semigroup $\{e^{-tB(s)} : t \ge 0\}$ in $\mathbb{B}(\mathbb{E})$, where the symbol $\mathbb{B}(\mathbb{E})$ stands for the Banach algebra of all bounded linear operators on $\mathbb{E}$.

By the assumptions $(P1) - (P4)$ see [[22]], it follows that there is a unique fundamental solution $\{\mathcal{S}(t,s) : 0 \le t \le s \le T\}$ for the homogeneous Cauchy problem such that

(i) $\mathcal{S}(t,s) \in \mathbb{B}(\mathbb{E})$ and the mapping $(t,s) \to \mathcal{S}(t,s)y$ is continuous for $y \in \mathbb{E}$, i.e $\mathcal{S}(t,s)$ is strongly continuous in t,s for all $0 \le s \le t \le T$.

(ii) For each $y \in \mathbb{E}, \mathcal{S}(t,s)y \in \mathcal{D}(B)$, for all $0 \le s \le t \le T$.

(iii) $\mathcal{S}(t,\tau)\mathcal{S}(\tau,s) = \mathcal{S}(t,s)$ for all $0 \le s \le \tau \le t \le T$.

(iv) For each $0 \le s < t \le T$, the derivative $\dfrac{\partial \mathcal{S}(t,s)}{\partial t}$ exists in the stong operator topology and an element of $\mathbb{B}(\mathbb{E})$, and strongly continuous in t, where $s < t \le T$.

(v) $\mathcal{S}(t,t) = I$.

(vi) $\dfrac{\partial \mathcal{S}(t,s)}{\partial t} + B(t)\mathcal{S}(t,s) = 0$ for all $0 \le s < t \le T$.

Further, we have also the following assumptions:

$$\|e^{-tB(\tau)}\| \le Ke^{-dt}, \quad t \ge 0;$$
$$\|B(\tau)e^{-tB(t)}\| \le \frac{Ke^{-dt}}{t}, \quad t > 0;$$
$$\|B(t)\mathcal{S}(t,\tau)\| \le K|t-\tau|^{-1}, \quad 0 \le s \le t \le T,$$

for all $\tau \in [0,T]$, where d is a positive constant. For $\alpha > 0$, we may define negative fractional powers $B(t)^{-\alpha}$ as

$$B(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}e^{-sB(t)}ds.$$

Then, the operator $B(t)^{-\alpha}$ is bounded linear and one to one operator on $\mathbb{E}$ and $B^{-\alpha}(t)B^{-\beta}(t) = B^{-(\alpha+\beta)}(t)$. Therefore, it implies that there exists an inverse of the operator $B(t)^{-\alpha}$. We can define $B(t)^{\alpha} \equiv [B(t)^{-\alpha}]^{-1}$ which is the positive fractional powers of $B(t)$. The operator $B(t)^{\alpha} \equiv [B(t)^{-\alpha}]^{-1}$ is closed densely defined linear operator with domain $\mathcal{D}(B(t)^{\alpha}) \subset \mathbb{E}$ and for $\alpha < \beta$, we get $\mathcal{D}(B(t)^{\beta}) \subset \mathcal{D}(B(t)^{\alpha})$. Let $E_\alpha(t_0) = \mathcal{D}(B(t_0)^{\alpha})$ be a Banach space with a norm $\|z\|_\alpha = \|B(t_0)^{\alpha}z\|, t_0 \in [0,T]$. For $0 < \omega_1 \le \omega_2$, we have that embedding $E_{\omega_2}(t_0) \hookrightarrow E_{\omega_1}(t_0)$ is continuous and dense. For each $\alpha > 0$, we may define $E_{-\alpha}(t_0) = (E_\alpha)^*(t_0)$, which is the dual space of $E_\alpha(t_0)$. The dual space is a Banach space with natural norm $\|z\|_{-\alpha} = \|B(t_0)^{-\alpha}z\|$. In particular, by the assumption $(P3)$, we conclude a constant $K > 0$, such that

$$\|B(t)B(s)^{-1}\| \le K, \quad \text{for all} \quad 0 \le s,t \le T. \tag{2.1}$$

Now, we also have following results:

$$\|B^{\alpha}(t)B^{-\beta}(s)\| \le \mathcal{N}_{\alpha,\beta}, \tag{2.2}$$
$$\|B^{\beta}(t)e^{-sB(t)}\| \le \frac{\mathcal{N}_\beta}{s^\beta}e^{-ws}, \quad t>0, \quad \beta \le 0, \quad w > 0, \tag{2.3}$$
$$\|B^{\beta}(t)\mathcal{S}(t,s)\| \le \mathcal{N}_\beta|t-s|^{-\beta}, \quad 0 < \beta < \mu+1, \tag{2.4}$$
$$\|B^{\beta}(t)\mathcal{S}(t,s)B^{-\beta}(s)\| \le \mathcal{N}'_\beta, \quad 0 < \beta < \mu+1, \tag{2.5}$$

for $s,t \in [0,T], 0 \le \alpha < \beta$ and $t > 0$, where $\mathcal{N}_{\alpha,\beta}$ is a constant related to T and μ and $\mathcal{N}_{\alpha,\beta}, \mathcal{N}_\beta, \mathcal{N}'_\beta$ show their dependence on the constants α, β. We also have following results.

Lemma 2.2. *([23, Lemma II.14.1]) Suppose that* $(P1)-(P3)$ *are satisfied. If* $0 \le \gamma \le 1, 0 \le \beta \le \alpha < 1+\mu, 0 < \alpha-\gamma \le 1$, *then for any* $0 \le \tau < t < t+\Delta t \le t_0, 0 \le \varsigma \le t_0$,

$$\|B^{\gamma}(\varsigma)[\mathcal{S}(t+\Delta t,\tau) - \mathcal{S}(t,\tau)]B^{-\beta}(\tau)\| \le \mathcal{N}_{\gamma,\beta,\alpha}(\Delta t)^{\alpha-\gamma}|t-\tau|^{\beta-\alpha}.$$

For additional details about the above mentioned concept, we refer to monographs [22–24].

Our main existence results are based on Banach contraction principle and the Kransnoselskii's fixed point theorem.

Lemma 2.3. *If* $\mathbb{E}$ *is a Banach space and* $\Gamma : \mathbb{E} \to \mathbb{E}$ *is a contraction mapping, then* Γ *has a unique fixed point.*

3 Existence Results

In this section, we present and prove the existence results for the problem (1.1)-(1.3) under different fixed point theorem. Initially, we prove the existence and uniqueness for the problem (1.1)-(1.3) on the Banach subspace $E_\alpha(t_0)$ for some $0 < \alpha < 1$ and $t_0 \in [0,T]$ under Banach fixed point theorem. To be able to use this theorem, we need to list the following conditions:

(H1) $\mathscr{F} : \mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0) \to \mathbb{E}$ is a Lipschitz continuous function then there exists $\mathscr{L}_{\mathscr{F}}, \mathscr{L}^*_{\mathscr{F}} > 0$ and for all $t, s, \in [0,T]$ and $x, y, \overline{x}, \overline{y} \in E_\alpha(t_0)$ such that

$$\|B(t)\mathscr{F}(t,x,y) - B(t)\mathscr{F}(s,\overline{x},\overline{y})\| \le \mathscr{L}_{\mathscr{F}}\left[|t-s| + \|x-\overline{x}\|_\alpha + \|y-\overline{y}\|_\alpha\right],$$

$$\|B(t)\mathscr{F}(t,x,0)\| \le \mathscr{L}_{\mathscr{F}}\|x\| + \mathscr{L}_{\mathscr{F}}^*, \quad x \in E_\alpha(t_0),$$

and

$$\mathscr{L}_{\mathscr{F}}^* = \sup_{t\in\mathscr{I}} \|B(t)\mathscr{F}(t,0,0)\|.$$

(H2) The nonlinear function $\mathscr{G} : \mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0) \to \mathbb{E}$ is a Lipschitz continuous function with $\mathscr{G}(\mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0)) \subset \mathcal{D}(B)$. Then there exist constants $\mathscr{L}_{\mathscr{G}}, \mathscr{L}_{\mathscr{G}}^* > 0$ and for all $t, s, \in [0,T]$ and $x, y, \overline{x}, \overline{y} \in E_\alpha(t_0)$, such that

$$\|\mathscr{G}(t,x,y) - \mathscr{G}(s,\overline{x},\overline{y})\| \le \mathscr{L}_{\mathscr{G}}\left[|t-s| + \|x-\overline{x}\|_\alpha + \|y-\overline{y}\|_\alpha\right],$$

and

$$\mathscr{L}_{\mathscr{G}}^* = \sup_{t\in\mathscr{I}} \|\mathscr{G}(t,0,0)\|.$$

(H3) The nonlinear function $\mathscr{H} : \mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0) \to \mathbb{E}$ is a Lipschitz continuous function with $\mathscr{H}(\mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0)) \subset \mathcal{D}(B)$. Then there exist constants $\mathscr{L}_{\mathscr{H}}, \mathscr{L}_{\mathscr{H}}^* > 0$ and for all $t, s, \in [0,T]$ and $x, y, \overline{x}, \overline{y} \in E_\alpha(t_0)$, such that

$$\|\mathscr{H}(t,x,y) - \mathscr{H}(s,\overline{x},\overline{y})\| \le \mathscr{L}_{\mathscr{H}}\left[|t-s| + \|x-\overline{x}\|_\alpha + \|y-\overline{y}\|_\alpha\right],$$

and

$$\mathscr{L}_{\mathscr{H}}^* = \sup_{t\in\mathscr{I}} \|\mathscr{H}(t,0,0)\|.$$

(H4) The map $a_i : \mathscr{D} \times E_\alpha(t_0) \to E_\alpha(t_0), i = 1,2,3$; where $\mathscr{D} = \{(t,s) \in \mathscr{I} \times \mathscr{I} : t \ge s\}$ and $i = 1,2,3$ are continuous and there exist positive constants $\mathscr{L}_{a_i}, \mathscr{L}_{a_i}^* > 0$ such that

$$\left\| \int_0^t [a_i(t,s,x) - a_i(t,s,y)]ds \right\|_\alpha \le \mathscr{L}_{a_i}\|x-y\|_\alpha, \quad x, y \in E_\alpha(t_0),$$

and

$$\mathscr{L}_{a_i}^* = \sup_{t\in[0,T]} \int_0^t \|a_1(t,s,0)\|ds.$$

(H5) The functions $\mathcal{I}_i : E_\alpha(t_0) \to E_\alpha(t_0), i = 1,2,\cdots,q$ are continuous functions and there exists a positive constant $\mathscr{L}_I > 0$ such that

$$\|B(t)\mathcal{I}_i x - B(t)\mathcal{I}_i\overline{x}\| \le \mathscr{L}_I\|x-\overline{x}\|_\alpha.$$

(H6) The function $g : PC([0,T], E_\alpha(t_0)) \to \mathcal{D}(B)$ is a nonlinear function which satisfies that $B(t)g$ is continuous on $PC([0,T], E_\alpha(t_0))$ and there exists a constant $\mathscr{L}_g$ such that

$$\|B(t)g(z) - B(t)g(\overline{z})\| \le \mathscr{L}_g\|z-\overline{z}\|_{PC},$$
$$\|B(t)g(z)\| \le \mathscr{L}_g\|z\|_{PC(E_\alpha(t_0))}, \text{ for each } z \in PC(\mathscr{I}, E_\alpha(t_0)).$$

Consider the sets $\mathscr{B}_r = \{z \in E_\alpha(t_0) : \|z\|_\alpha \le r\}$ and $\mathscr{W}_r = \{z \in PC([0,T], E_\alpha(t_0)) : z(t) \in \mathscr{B}_r, \text{for all } t \in [0,T]\}$ for each finite constant $r > 0$.

Now, we are in a position to define the mild solution for the problem (1.1)-(1.3).

Definition 3.1. *A PC fucntion $z(\cdot) : \mathscr{I} \to \mathbb{E}$ is called a mild solution for the problem if $z(0) = z_0 + g(z)$ and the following integral equation*

$$z(t) = \begin{cases} \mathcal{S}(t,0)\left[z_0 + g(z) - \mathscr{F}(0, z(h_1(0)), 0)\right] \\ +\mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s))ds\right) \\ -\int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\left(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)d\tau \\ +\int_0^t \mathcal{S}(t,\tau)\mathscr{G}\left(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)d\tau \\ +\int_0^t \mathcal{S}(t,\tau)\mathscr{H}\left(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)d\tau \\ +\sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{I}_i(z(t_i^-)), \quad t \in [0,T]. \end{cases}$$

is fulfilled.

Theorem 3.1. *Let (H1)-(H6) holds, $z_0 \in E_\beta(t_0)$ for some $\beta \in (0,1]$ and*

$$\Upsilon = \Bigg[\mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}[\mathscr{L}_\mathscr{F} + Kq\mathscr{L}_I] + \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g + \mathcal{N}_{\alpha,1}\mathscr{L}_\mathscr{F}(1+\mathscr{L}_{a_1}) + \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\{\mathscr{L}_\mathscr{F}(1+\mathscr{L}_{a_1}) + \mathscr{L}_\mathscr{G}(1+\mathscr{L}_{a_2}) + \mathscr{L}_\mathscr{H}(1+\mathscr{L}_{a_3})\}\Bigg] < 1 \tag{3.1}$$

then the impulsive problem (1.1)-(1.3) has a unique mild solution $x \in \mathbb{E}$.

Proof. First, we will transform the problem (1.1)-(1.3) into a fixed point problem. Recognize the operator $\Gamma : PC(J, E_\alpha(t_0)) \to PC(J, E_\alpha(t_0))$ by

$$(\Gamma z)(t) = \begin{cases} \mathcal{S}(t,0)\left[z_0 + g(z) - \mathscr{F}(0, z(h_1(0)), 0)\right] \\ +\mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s))ds\right) \\ -\int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\left(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)d\tau \\ +\int_0^t \mathcal{S}(t,\tau)\mathscr{G}\left(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)d\tau \\ +\int_0^t \mathcal{S}(t,\tau)\mathscr{H}\left(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)d\tau \\ +\sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{I}_i(z(t_i^-)), \quad t \in [0,T]. \end{cases}$$

It is evident that the fixed points of the operator Γ are mild solutions of the model (1.1)-(1.3).

Now, let us demonstrating that Γ has a unique fixed point. Initially, we show that Γ maps $\mathscr{W}_r$ into $\mathscr{W}_r$. For any $z(\cdot) \in \mathscr{W}_r$, we have

$$\begin{aligned} \|(\Gamma z)(t)\|_\alpha \le\ & \|\mathcal{S}(t,0)z_0\|_\alpha + \|\mathcal{S}(t,0)g(z)\|_\alpha + \|\mathcal{S}(t,0)\mathscr{F}(0,z(h_1(0)),0)\|_\alpha \\ & + \left\|\mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s))ds\right)\right\|_\alpha \\ & + \left\|\int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\left(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)d\tau\right\|_\alpha \\ & + \left\|\int_0^t \mathcal{S}(t,\tau)\mathscr{G}\left(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)d\tau\right\|_\alpha \\ & + \left\|\int_0^t \mathcal{S}(t,\tau)\mathscr{H}\left(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)d\tau\right\|_\alpha \\ & + \left\|\sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{I}_i(z(t_i^-))\right\|_\alpha \\ \le\ & \sum_{k=1}^{8} I_k. \end{aligned} \tag{3.2}$$

Now, with the help of the above discussions along with (2.1)-(2.5), we can find the following estimations:

$$\begin{aligned}
I_1 &= \|\mathcal{S}(t,0)z_0\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,0)B^{-\beta}(0)\|\|B^\beta(0)z_0\| \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\|B^\beta(0)z_0\| \\
I_2 &= \|\mathcal{S}(t,0)g(z)\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-\beta}(t)\|B^\beta(t)\mathcal{S}(t,0)B^{-1}(0)\|\|B(0)g(z)\| \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g\|z\| \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g r \\
I_3 &= \|\mathcal{S}(t,0)\mathscr{F}(0,z(h_1(0)),0)\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,0)B^{-\beta}(0)\|\|B^\beta(0)B^{-1}(t)\|[\|B(t)\mathscr{F}(0,z(h_1(0)),0)\|] \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}(\mathscr{L}_\mathscr{F} r+\mathscr{L}^*_\mathscr{F}) \\
I_4 &= \left\|\mathscr{F}\left(t,z(h_1(t)),\int_0^t a_1(t,s,z(h_2(s))ds\right)\right\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-1}(t)\|\left[\left\|B(t)\mathscr{F}\left(t,z(h_1(t)),\int_0^t a_1(t,s,z(h_2(s))ds\right)\right\|\right] \\
&\leq \|B^\alpha(t_0)B^{-1}(t)\|\left[\left\|B(t)\mathscr{F}\left(t,z(h_1(t)),\int_0^t a_1(t,s,z(h_2(s))ds\right)\right.\right. \\
&\quad \left.\left. - B(t)\mathscr{F}(t,0,0)+B(t)\mathscr{F}(t,0,0)\right\|\right] \\
&\leq \mathcal{N}_{\alpha,1}\left[\mathscr{L}_\mathscr{F}\left(\|z(h_1(t))\|+\left\|\int_0^t a_1(t,s,z(h_2(s)))ds\right\|\right)+\mathscr{L}^*_\mathscr{F}\right] \\
&\leq \mathcal{N}_{\alpha,1}\left[\mathscr{L}_\mathscr{F}\left(\|z(h_1(t))\|+\left\|\int_0^t a_1(t,s,z(h_2(s)))ds-\int_0^t a_1(t,s,0)ds\right\|\right.\right. \\
&\quad \left.\left.+\left\|\int_0^t a_1(t,s,0)ds\right\|\right)+\mathscr{L}^*_\mathscr{F}\right] \\
&\leq \mathcal{N}_{\alpha,1}\left[\mathscr{L}_\mathscr{F}\left(\|z(h_1(t))\|+\mathscr{L}_{a_1}\|z(h_2(s))\|+\mathscr{L}^*_{a_1}\right)+\mathscr{L}^*_\mathscr{F}\right] \\
&\leq \mathcal{N}_{\alpha,1}\left[\mathscr{L}_\mathscr{F}[(1+\mathscr{L}_{a_1})r+\mathscr{L}^*_{a_1}]+\mathscr{L}^*_\mathscr{F}\right] \\
I_5 &= \left\|\int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\left(\tau,z(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)d\tau\right\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,\tau)\|\left[\left\|\mathscr{F}\left(\tau,z(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)\right\|\right]d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\left[\mathscr{L}_\mathscr{F}(1+\mathscr{L}_{a_1})r+\mathscr{L}_\mathscr{F}\mathscr{L}^*_{a_1}+\mathscr{L}^*_\mathscr{F}\right] \\
I_6 &= \left\|\int_0^t \mathcal{S}(t,\tau)\mathscr{G}\left(\tau,z(h_3(\tau)),\int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)d\tau\right\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,\tau)\|\left[\left\|\mathscr{G}\left(\tau,z(h_3(\tau)),\int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)\right\|\right]d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\left[\mathscr{L}_\mathscr{G}(1+\mathscr{L}_{a_2})r+\mathscr{L}_\mathscr{G}\mathscr{L}^*_{a_2}+\mathscr{L}^*_\mathscr{G}\right] \\
I_7 &= \left\|\int_0^t \mathcal{S}(t,\tau)\mathscr{H}\left(\tau,z(h_5(\tau)),\int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)d\tau\right\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,\tau)\|\left[\left\|\mathscr{H}\left(\tau,z(h_5(\tau)),\int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)\right\|\right]d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\left[\mathscr{L}_\mathscr{H}(1+\mathscr{L}_{a_3})r+\mathscr{L}_\mathscr{H}\mathscr{L}^*_{a_3}+\mathscr{L}^*_\mathscr{H}\right]
\end{aligned}$$

$$
\begin{aligned}
I_8 &= \left\| \sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{I}_i(z(t_i^-)) \right\|_\alpha \\
&\le \sum_{0<t_i<t} \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,t_i)B^{-\beta}(t_i)\| \|B^\beta(t_i)B^{-1}(0)\| [\|B(0)B^{-1}(t)\| \|B(t)\mathcal{I}_i(z(t_i^-))\|] \\
&\le \sum_{i=1}^{q} \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta \mathcal{N}_{\beta,1} K \mathscr{L}_I(\|z(t_i)\|) \\
&\le \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta \mathcal{N}_{\beta,1} K q \mathscr{L}_I r.
\end{aligned}
$$

Now, we substitute the estimations $(I_1) - (I_8)$ in (3.2), we obtain

$$
\begin{aligned}
&\|(\Gamma z)(t)\|_\alpha \\
&\le \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta \|B^\beta(0)z_0\| + [\mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1} + \mathcal{N}_{\alpha,1}]\mathscr{L}^*_{\mathscr{F}} + \mathcal{N}_{\alpha,1}\mathscr{L}_{\mathscr{F}}\mathscr{L}^*_{a_1} + \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta \frac{T^{1-\beta}}{1-\beta}\Big[\mathscr{L}_{\mathscr{F}}\mathscr{L}^*_{a_1} + \mathscr{L}_{\mathscr{G}}\mathscr{L}^*_{a_2} \\
&\quad + \mathscr{L}_{\mathscr{H}}\mathscr{L}^*_{a_3} + \mathscr{L}^*_{\mathscr{F}} + \mathscr{L}^*_{\mathscr{G}} + \mathscr{L}^*_{\mathscr{H}}\Big] + r\Big[\mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}[\mathscr{L}_{\mathscr{F}} + Kq\mathscr{L}_I] + \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g + \mathcal{N}_{\alpha,1}\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1}) \\
&\quad + \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\{\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1}) + \mathscr{L}_{\mathscr{G}}(1+\mathscr{L}_{a_2}) + \mathscr{L}_{\mathscr{H}}(1+\mathscr{L}_{a_3})\}\Big] \\
&\le r.
\end{aligned}
$$

Therefore, the operator Γ maps $\mathscr{W}_r$ into $\mathscr{W}_r$. Finally, we show that Γ is a contraction on $PC([0,T], E_\alpha(t_0))$.

Remark 3.1. *For better readability, we find the contraction estimations are below:*
Let us consider $z, \overline{z} \in PC([0,T], E_\alpha(t_0))$ and $t \in [0,T]$, then we obtain

$$
\|\Gamma z(t) - \Gamma\overline{z}(t)\|_\alpha = \sum_{k=9}^{16} I_k,
$$

where

$$
\begin{aligned}
I_9 &= \|\mathcal{S}(t,0)z_0 - \mathcal{S}(t,0)z_0\|_\alpha \\
&\le 0 \\
I_{10} &= \|\mathcal{S}(t,0)g(z) - \mathcal{S}(t,0)g(\overline{z})\|_\alpha \\
&\le \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,0)B^{-\beta}(0)\| \Big[\|B^\beta(0)B^{-1}(0)\| \|B(0)[g(z)-g(\overline{z})]\|\Big] \\
&\le \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}\mathscr{L}_g\|z-\overline{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{11} &= \|\mathcal{S}(t,0)\mathscr{F}(0,z(h_1(0)),0) - \mathcal{S}(t,0)\mathscr{F}(0,\overline{z}(h_1(0)),0)\|_\alpha \\
&\le \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,0)B^{-\beta}(0)\| \|B^\beta(0)B^{-1}(t)\| \Big[\|B(t)\mathscr{F}(0,z(h_1(0)),0) - B(t)\mathscr{F}(0,\overline{z}(h_1(0)),0)\|\Big] \\
&\le \mathcal{N}_{\alpha,\beta}\mathcal{N}^1_\beta\mathcal{N}_{\beta,1}\mathscr{L}_{\mathscr{F}}\|z-\overline{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{12} &= \left\| \mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s)))ds\right) - \mathscr{F}\left(t, \overline{z}(h_1(t)), \int_0^t a_1(t,s,\overline{z}(h_2(s)))ds\right) \right\|_\alpha \\
&\le \|B^\alpha(t_0)B^{-1}(t)\| \left\| B(t)\mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s)))ds\right) - B(t)\mathscr{F}\left(t, \overline{z}(h_1(t)), \int_0^t a_1(t,s,\overline{z}(h_2(s)))ds\right) \right\| \\
&\le \mathcal{N}_{\alpha,1}\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1})\|z-\overline{z}\|_{PC([0,T],E_\alpha(t_0))}
\end{aligned}
$$

$$
\begin{aligned}
I_{13} &= \Big\| \int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\Big(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\Big) \\
&\quad - \int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\Big(\tau, \bar{z}(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,\bar{z}(h_2(\xi))d\xi\Big) \Big\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,\tau)\| \Big[\Big\| B(\tau)\mathscr{F}\Big(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\Big) \\
&\quad - B(\tau)\mathscr{F}\Big(\tau, \bar{z}(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,\bar{z}(h_2(\xi))d\xi\Big) \Big\| \Big] d\tau \\
&\leq \int_0^t \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta (t-\tau)^{-\beta} \mathscr{L}_{\mathscr{F}} \Big[\|z(h_1(\tau)) - \bar{z}(h_1(\tau))\| + \Big\| \int_0^\tau a_1(\tau,\xi,z(h_2(\xi)))d\xi - \int_0^\tau a_1(\tau,\xi,\bar{z}(h_2(\xi)))d\xi \Big\| \Big] d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta \int_0^t (t-s)^{-\beta} \mathscr{L}_{\mathscr{F}} \big(\|z-\bar{z}\| + \mathscr{L}_{a_1}\|z-\bar{z}\| \big) d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta \frac{T^{1-\beta}}{1-\beta} \mathscr{L}_{\mathscr{F}} (1+\mathscr{L}_{a_1}) \|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{14} &= \Big\| \int_0^t \mathcal{S}(t,\tau)\mathscr{G}\Big(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\Big) d\tau \\
&\quad - \int_0^t \mathcal{S}(t,\tau)\mathscr{G}\Big(\tau, \bar{z}(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,\bar{z}(h_4(\xi)))d\xi\Big) d\tau \Big\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,\tau)\| \Big[\Big\| \mathscr{G}\Big(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\Big) \\
&\quad - \mathscr{G}\Big(\tau, \bar{z}(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,\bar{z}(h_4(\xi)))d\xi\Big) \Big\| \Big] d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta \frac{T^{1-\beta}}{1-\beta} \mathscr{L}_{\mathscr{G}} (1+\mathscr{L}_{a_2}) \|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{15} &= \Big\| \int_0^t \mathcal{S}(t,\tau)\mathscr{H}\Big(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\Big) d\tau \\
&\quad - \int_0^t \mathcal{S}(t,\tau)\mathscr{H}\Big(\tau, \bar{z}(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,\bar{z}(h_6(\xi))d\xi\Big) \Big\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,\tau)\| \Big[\Big\| \mathscr{H}\Big(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\Big) \\
&\quad - \mathscr{H}\Big(\tau, \bar{z}(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,\bar{z}(h_6(\xi))d\xi\Big) \Big\| \Big] d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta \frac{T^{1-\beta}}{1-\beta} \mathscr{L}_{\mathscr{H}} (1+\mathscr{L}_{a_3}) \|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{16} &= \Big\| \sum_{0<t_i<t} \mathcal{S}(t,t_i)[\mathcal{I}_i(z(t_i^-)) - \mathcal{I}_i(\bar{z}(t_i^-))] \Big\|_\alpha \\
&\leq \sum_{i=1}^q \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,t_i)B^{-\beta}(t_i)\| \|B^\beta(t_i)B^{-1}(0)\| \big[\|B(0)B^{-1}(t)\| \|B(t)[\mathcal{I}_i(z(t_i^-)) - \mathcal{I}_i(\bar{z}(t_i^-))]\| \big] \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1} Kq\mathscr{L}_I \|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))}.
\end{aligned}
$$

Now, we enter into the main proof of this theorem. From Remark 3.1, we obtain

$$
\begin{aligned}
&\|\Gamma z(t) - \Gamma\bar{z}(t)\|_\alpha \\
&\leq \Big[\mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}[\mathscr{L}_{\mathscr{F}} + Kq\mathscr{L}_I] + \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g + \mathcal{N}_{\alpha,1}\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1}) + \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta \frac{T^{1-\beta}}{1-\beta} \big\{ \mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1}) \\
&\quad + \mathscr{L}_{\mathscr{G}}(1+\mathscr{L}_{a_2}) + \mathscr{L}_{\mathscr{H}}(1+\mathscr{L}_{a_3}) \big\} \Big] \|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))}
\end{aligned}
$$

Therefore, we take the supremum of t over $[0,T]$ and we have

$$\|\Gamma z - \Gamma\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \leq \Upsilon \|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))}.$$

Since $\Upsilon < 1$ by the inequality (3.1), it indicates that the map Γ is contraction on $PC([0,T], E_\alpha(t_0))$. Hence, by Banach contraction principle, there exists a unique fixed point $z \in PC([0,T], E_\alpha(t_0))$ such that $\Gamma z(t) = z(t)$ which is a mild solution of the problem (1.1)-(1.3). The proof is now completed. □

References

[1] V. Lakshmikantham, D. D Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, Teaneck, London, 1989.

[2] M. Benchohra, J. Henderson and S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Contemporary Mathematics and its Applications, Vol.2, Hindawi Publishing Corporation, New York, 2006.

[3] Y.-K. Chang, A. Anguraj and M. Mallika Arjunan, Existence results for impulsive neutral functional differential equations with infinite delay, *Nonlinear Analysis: Hybrid Systems*, 2(2008), 209–218.

[4] B. Liu, Controllability of impulsive neutral functional differential inclusions with infinite delay, *Nonlinear Analysis,* 60(2005), 1533–1552.

[5] N. Abada, M. Benchohra, H. Hammouche, Nonlinear impulsive partial functional differential inclusions with state-dependent delay and multivalued jumps, *Nonlinear Analysis: Hybrid Systems*, 4(4)(2010), 791–803.

[6] B. Radhakrishnan and K. Balachandran, Controllability of impulsive neutral functional evolution integrodifferential systems with infinite delay, *Nonlinear Analysis: Hybrid Systems*, 5(4)(2011), 655–670.

[7] J.Y. Park, K. Balachandran and G. Arthi, Controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces, *Nonlinear Analysis: Hybrid systems*, 3(2009), 184–194.

[8] E. Hernandez, M. Rabello and H.R. Henriquez, Existence of solutions for impulsive partial neutral functional differential equations, *J. Math. Anal. Appl.*, 331(2)(2007), 1135–1158.

[9] L. Byszewski, Theorems about existence and uniqueness of solutions of a semi-linear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.,* 162(1991), 494–505.

[10] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.,* 40(1991), 11–19.

[11] A.C. Tsoi, Explicit solution to a class of neutral delay-differential equations, *Int. J. Control*, 22(1975), 787–799.

[12] J.K. Hale and S.M.V. Lunel, *Introduction to Functional Differential Equations*, New York, Springer, 1993.

[13] M.N. Rabelo, M. Henrique and G. Siracusa, Existence of integro-differential solutions for a class of abstract partial impulsive differential equations, *J. Inequal. Appl.,* 2011, 19(2011).

[14] J.-R. Wang and Z. Lin, A class of impulsive non-autonomous differential equations and Ulam-Hyers-Rassias stability, *Math. Methods Appl. Sci.,* 38(5)(2014), 868–880.

[15] H. Leiva, Controllability of semilinear impulsive non-autonomous systems, *Int. J. Cont.,* 88(3)(2015), 585–592.

[16] Z. Fan and G. Li, Existence results for semilinear differential equations with nonlocal and impulsive conditions, *J. Funct. Anal.,* 258(2010), 1709–1727.

[17] Z. Yan, Existence of solutions for nonlocal impulsive partial functional integro-differential equations via fractional operators, *J. Comput. Appl. Math.,* 235(2011), 2252–2262.

[18] Z. Yan and X. Yan, Existence of solutions for a impulsive nonlocal stochastic functional integro-diiferetial inclusions in Hilbert spaces, *Z. Angew Math. Phys.,* 64(2013), 573–590.

[19] A. Chadha and D.N. Pandey, Existence of a mild solution for an impulsive nonlocal non-autonomous neutral functional differential equation, *Ann. Univ. Ferrara*, 62(2016), 1-21.

[20] R.N. Wang and P.X. Zhu, Non-autonomous evolution inclusions with nonlocal history conditions: global integral solutions, *Nonlinear Anal: TMA*, 85(2013), 180–191.

[21] X. Fu, Existence of solutions for non-autonomous functional evolution equations with nonlocal conditions, *Electron. J. Differ. Equ.*, 2012(110), 1–15.

[22] A. Friedman, *Partial Differential Equations*, Dover Publication, New York, 1997.

[23] A. Friedman and M. Shinbrot, Volterra integral equations in a Banach space, *Trans. Am. Math. Sci.*, 126(1967), 131–179.

[24] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

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Exact soliton solutions of the generalized combined and the generalized double combined sinh-cosh-Gordon equations

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Abstract

In this paper, the extended tanh method is used to construct exact solutions of the generalized combined sinh-cosh-Gordon equations and the generalized double combined sinh-cosh-Gordon equations which arises in mathematical physics and has a wide range of scientific applications that range from chemical reactions to water surface gravity waves. The extended tanh method is an efficient method for obtaining exact solutions of nonlinear partial differential equations. This method can be applied to nonintegrable equations as well as to integrable ones.

Keywords: Extended tanh method, Combined sinh-cosh-Gordon equations, Double combined sinh-cosh-Gordon equation, soliton.

2010 MSC: 47F05, 35QXX.

1 Introduction

Phenomena in physics and other fields are often described by nonlinear evolution equations. When we want to understand the physical mechanism of phenomena in nature, described by nonlinear evolution equations, exact solutions for the nonlinear evolution equations have to be explored. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media and optical fibers, etc.
Thus, the methods for deriving exact solutions for the governing equations have to be developed. Recently, many powerful methods have been established and improved. Among these methods, we cite the tanh and extended tanh methods [1-9], $(\frac{G'}{G})$-expansion method [10-13], the homogeneous balance method [14], the Jacobi elliptic function method [15, 16], the exp-function method [17], the first-integral method [18-20], the sine-cosine method [21] and so on.

The pioneer work Malfiet in [2, 3] introduced the powerful tanh method for a reliable treatment of the nonlinear wave equations. The useful tanh method is widely used by many work and by the references therein. Later, the extended tanh method, developed by Wazwaz [4, 5], is a direct and effective algebraic method for handling nonlinear equations. Various extensions of the method were developed as well.
The aim of this paper is to find exact soliton solutions of the generalized combined and the generalized double combined sinh-cosh-Gordon equations [22], by using the extended tanh method.
The paper is arranged as follows. In Section 2, we describe briefly the extended tanh method. In Section 3 and 4, we apply this method to find exact soliton solutions of the generalized combined and the generalized double combined sinh-cosh-Gordon equations. In Section 5, some conclusions are given.

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2 The extended tanh method and tanh method

A PDE

$$F(u,u_x,u_t,u_{xx},u_{xt},u_{xxx},....)=0, \tag{2.1}$$

can be converted to an ODE

$$G(u,u',u'',u''',....)=0, \tag{2.2}$$

upon using a wave variable $\xi = x-ct$. Eq. (2.2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.
Introducing a new independent variable

$$Y=tanh(\mu\xi), \quad \xi=x-ct, \tag{2.3}$$

leads to the change of derivatives:

$$\frac{d}{d\xi}=\mu(1-Y^2)\frac{d}{dY},$$

$$\frac{d^2}{d\xi^2}=-2\mu^2Y(1-Y^2)\frac{d}{dY}+\mu^2(1-Y^2)^2\frac{d^2}{dY^2}. \tag{2.4}$$

The extended tanh method admits the use of the finite expansion

$$u(\mu\xi)=S(Y)=\sum_{k=0}^{M}a_kY^k+\sum_{k=1}^{M}b_kY^{-k}, \tag{2.5}$$

where M is a positive integer, in most cases, that will be determined. Expansion (2.5) reduces to the standard tanh method for $b_k=0,(k=1,...,M)$. Substituting (2.5) into the ODE (2.2) results in an algebraic equation in powers of Y.

To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. We then collect all coefficients of powers of Y in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters $a_k(k=0,..,M)$, $b_k(k=1,..,M)$, μ and c. Having determined these parameters we obtain an analytic solution $u(x,t)$ in a closed form.

3 The generalized combined sinh-cosh-Gordon equation

Let us consider the generalized combined sinh-cosh-Gordon equations

$$u_{tt}-ku_{xx}+\alpha\sinh(nu)+\beta\cosh(nu)=0. \tag{3.6}$$

Using the variable $u(x,t)=u(\mu\xi)$, $\xi=x-ct$, carries Eq. (3.6) into the ODE

$$(c^2-k)u''+\alpha\sinh(nu)+\beta\cosh(nu)=0. \tag{3.7}$$

We use the Painleve property

$$v=e^{nu}, \tag{3.8}$$

or equivalently

$$u=\frac{1}{n}lnv, \tag{3.9}$$

from which we find

$$u'=\frac{1}{n}\frac{v'}{v}, \quad u''=\frac{1}{n}\frac{vv''-(v')^2}{v^2}. \tag{3.10}$$

The transformation (3.8) also gives

$$\sinh(nu)=\frac{v-v^{-1}}{2}, \quad \cosh(nu)=\frac{v+v^{-1}}{2}, \tag{3.11}$$

that also gives

$$u = \frac{1}{n} arccosh[\frac{v + v^{-1}}{2}]. \tag{3.12}$$

Substituting the transformations introduced above into Eq. (3.7) gives the ODE

$$(\alpha+\beta)nv^3 - (\alpha-\beta)nv + 2(c^2-k)vv'' - 2(c^2-k)(v')^2 = 0. \tag{3.13}$$

Balancing vv'' with v^3 in Eq. (3.13) gives

$$2M + 2 = 3M,$$

then

$$M = 2.$$

In this case, the extended tanh method the form (2.5) admits the use of the finite expansion

$$v(x,t) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \tag{3.14}$$

Substituting the form (3.14) into Eq. (3.13) and using (2.4), collecting the coefficients of Y we obtain:

Coefficient of Y^6: $n(\alpha+\beta)a_2^3 + 4(c^2-k)\mu^2 a_2^2$.

Coefficient of Y^5: $3n(\alpha+\beta)a_1 a_2^2 + 8(c^2-k)\mu^2 a_1 a_2$.

Coefficient of Y^4: $3n(\alpha+\beta)(a_0 a_2^2 + a_1^2 a_2) + 2(c^2-k)\mu^2(6a_0 a_2 + a_1^2)$.

Coefficient of Y^3: $n(\alpha+\beta)(3b_1 a_2^2 + 6a_0 a_1 a_2 + a_1^3) + 4(c^2-k)\mu^2(a_0 a_1 - a_1 a_2 + 5a_2 b_1)$.

Coefficient of Y^2: $3n(\alpha+\beta)(a_0 a_1^2 + a_0^2 a_2 + 2a_1 a_2 b_1 + b_2 a_2^2) - n(\alpha-\beta)a_2$
$+ 4(c^2-k)\mu^2(2a_1 b_1 - 4a_0 a_2 + 8a_2 b_2 - a_2^2)$.

Coefficient of Y^1: $3n(\alpha+\beta)(a_0^2 a_1 + a_1^2 b_1 + 2a_0 a_2 b_1 + 2a_1 a_2 b_2) - n(\alpha-\beta)a_1$
$+ 4(c^2-k)\mu^2(-a_0 a_1 - a_1 a_2 + 4a_1 b_2 - 9a_2 b_1)$.

Coefficient of Y^0: $3n(\alpha+\beta)(a_1^2 b_2 + a_2 b_1^2 + 2a_0 a_1 b_1 + 2a_0 a_2 b_2) + n(\alpha+\beta)a_0^3 - n(\alpha-\beta)a_0$
$2(c^2-k)\mu^2(2a_0 a_2 + 2a_0 b_2 - 32a_2 b_2 - a_1^2 - 8a_1 b_1 - b_1^2)$.

Coefficient of Y^{-1}: $3n(\alpha+\beta)(a_0^2 b_1 + a_1 b_1^2 + 2a_0 a_1 b_1 + 2a_2 b_1 b_2) - n(\alpha-\beta)b_1$
$+ 4(c^2-k)\mu^2(-a_0 b_1 - b_1 b_2 + 4a_2 b_1 - 9a_1 b_2)$.

Coefficient of Y^{-2}: $3n(\alpha+\beta)(a_0 b_1^2 + a_0^2 b_2 + 2a_1 b_1 b_2 + a_2 b_2^2) - n(\alpha-\beta)b_2$
$+ 4(c^2-k)\mu^2(2a_1 b_1 - 4a_0 b_2 + 8a_2 b_2 - b_2^2)$.

Coefficient of Y^{-3}: $n(\alpha+\beta)(3a_1 b_2^2 + 6a_0 b_1 b_2 + b_1^3) + 4(c^2-k)\mu^2(a_0 b_1 - b_1 b_2 + 5a_1 b_2)$.

Coefficient of Y^{-4}: $3n(\alpha+\beta)(a_0 b_2^2 + b_1^2 b_2) + 2(c^2-k)\mu^2(6a_0 b_2 + b_1^2)$.

Coefficient of Y^{-5}: $3n(\alpha+\beta)b_1 b_2^2 + 8(c^2-k)\mu^2 b_1 b_2$.

Coefficient of Y^{-6}: $n(\alpha+\beta)b_2^3 + 4(c^2-k)\mu^2 b_2^2$.

Setting these coefficients equal to zero, and solving the resulting system, by using Maple, we find the following sets of solutions:

$$a_0 = 0,\ a_1 = 0,\ a_2 = -\sqrt{\frac{\alpha-\beta}{\alpha+\beta}},\ b_1 = 0,\ b_2 = 0,\ \mu = \frac{\sqrt{n}}{2}\frac{\sqrt[4]{\alpha^2-\beta^2}}{\sqrt{c^2-k}}. \tag{3.15}$$

$$a_0=0,\ \ a_1=0,\ \ a_2=0,\ \ b_1=0,\ \ b_2=-\sqrt{\frac{\alpha-\beta}{\alpha+\beta}},\ \ \mu=\frac{\sqrt{n}}{2}\frac{\sqrt[4]{\alpha^2-\beta^2}}{\sqrt{c^2-k}}. \tag{3.16}$$

$$a_0=\frac{1}{2}\sqrt{\frac{\alpha-\beta}{\alpha+\beta}},\ \ a_1=0,\ \ a_2=\frac{1}{4}\sqrt{\frac{\alpha-\beta}{\alpha+\beta}},\ \ b_1=0,\ \ b_2=\frac{1}{4}\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}, \tag{3.17}$$

$$\mu=\frac{\sqrt{n}}{4}\frac{\sqrt[4]{\alpha^2-\beta^2}}{\sqrt{c^2-k}}.$$

Recall that

$$u=\frac{1}{n}arccosh[\frac{v+v^{-1}}{2}].$$

The sets (3.15)-(3.17) give the solitons solutions for $\alpha>\beta,\ c^2>k$

$$u_1(x,t)=\frac{1}{n}arccosh\{\frac{-(\alpha-\beta)\tanh^2[\mu(x-ct)]-(\alpha+\beta)\coth^2[\mu(x-ct)]}{2\sqrt{\alpha^2-\beta^2}}\}, \tag{3.18}$$

$$u_2(x,t)=\frac{1}{n}arccosh\{\frac{-(\alpha-\beta)\coth^2[\mu(x-ct)]-(\alpha+\beta)\tanh^2[\mu(x-ct)]}{2\sqrt{\alpha^2-\beta^2}}\}, \tag{3.19}$$

where $\mu=\frac{\sqrt{n}}{2}\frac{\sqrt[4]{\alpha^2-\beta^2}}{\sqrt{c^2-k}}$,

$$u_3(x,t)=\frac{1}{n}arccosh\{\frac{(\alpha-\beta)(2+\tanh^2[\mu(x-ct)]+\coth^2[\mu(x-ct)])^2+16(\alpha+\beta)}{8\sqrt{\alpha^2-\beta^2}(2+\tanh^2[\mu(x-ct)]+\coth^2[\mu(x-ct)])}\}, \tag{3.20}$$

where $\mu=\frac{\sqrt{n}}{4}\frac{\sqrt[4]{\alpha^2-\beta^2}}{\sqrt{c^2-k}}$.

However for $c^2<k$, we obtain the travelling wave solutions

$$u_4(x,t)=\frac{1}{n}arccosh\{\frac{(\alpha-\beta)\tan^2[\mu(x-ct)]+(\alpha+\beta)\cot^2[\mu(x-ct)]}{2\sqrt{\alpha^2-\beta^2}}\}, \tag{3.21}$$

$$u_5(x,t)=\frac{1}{n}arccosh\{\frac{(\alpha-\beta)\cot^2[\mu(x-ct)]+(\alpha+\beta)\tan^2[\mu(x-ct)]}{2\sqrt{\alpha^2-\beta^2}}\}, \tag{3.22}$$

where $\mu=\frac{\sqrt{n}}{2}\frac{\sqrt[4]{\alpha^2-\beta^2}}{\sqrt{c^2-k}}$,

$$u_6(x,t)=\frac{1}{n}arccosh\{\frac{(\alpha-\beta)(2+\tan^2[\mu(x-ct)]+\cot^2[\mu(x-ct)])^2+16(\alpha+\beta)}{8\sqrt{\alpha^2-\beta^2}(2+\tan^2[\mu(x-ct)]+\cot^2[\mu(x-ct)])}\}, \tag{3.23}$$

where $\mu=\frac{\sqrt{n}}{4}\frac{\sqrt[4]{\alpha^2-\beta^2}}{\sqrt{c^2-k}}$.

4 The generalized double combined sinh-cosh-Gordon equation

In this section we study the generalized double combined sinh-cosh-Gordon equation

$$u_{tt}-ku_{xx}+\alpha\sinh(nu)+\alpha\cosh(nu)+\beta\sinh(2nu)+\beta\cosh(2nu)=0. \tag{4.24}$$

We take the transformation

$$u(x,t)=u(\mu\xi),\ \ \xi=x-ct.$$

The substitution of the transformation into (4.24) yields the ODE

$$(c^2-k)u''+\alpha\sinh(nu)+\alpha\cosh(nu)+\beta\sinh(2nu)+\beta\cosh(2nu)=0. \tag{4.25}$$

We use the Painleve property

$$v=e^{nu}, \tag{4.26}$$

or equivalently

$$u = \frac{1}{n} lnv, \tag{4.27}$$

from which we find

$$u' = \frac{1}{n}\frac{v'}{v}, \quad u'' = \frac{1}{n}\frac{vv'' - (v')^2}{v^2}. \tag{4.28}$$

The transformation (4.26) also gives

$$\sinh(nu) = \frac{v - v^{-1}}{2}, \quad \cosh(nu) = \frac{v + v^{-1}}{2}, \sinh(2nu) = \frac{v^2 - v^{-2}}{2}, \tag{4.29}$$

$$\cosh(2nu) = \frac{v^2 + v^{-2}}{2},$$

that also gives

$$u = \frac{1}{n} arccosh[\frac{v + v^{-1}}{2}]. \tag{4.30}$$

Substituting the transformations introduced above into Eq. (4.25) gives the ODE

$$2\beta n v^4 + 2\alpha n v^3 + 2(c^2 - k)vv'' - 2(c^2 - k)(v')^2 = 0. \tag{4.31}$$

Balancing vv'' with v^4 in Eq. (4.31) gives

$$2M + 2 = 4M,$$

then

$$M = 1.$$

In this case, the extended tanh method the form (2.5) admits the use of the finite expansion

$$v(x,t) = S(Y) = a_0 + a_1 Y + \frac{b_1}{Y}. \tag{4.32}$$

Substituting the form (4.32) into Eq. (4.31) and using (2.4), collecting the coefficients of Y we obtain:

Coefficient of Y^4: $2n\beta a_1^4 + 2(c^2 - k)\mu^2 a_1^2$.

Coefficient of Y^3: $2na_1^3(4\beta a_0 + \alpha) + 4(c^2 - k)\mu^2 a_0 a_1$.

Coefficient of Y^2: $8n\beta a_1^3 b_1 + 6na_0 a_1^2(2\beta a_0 + \alpha) + 8(c^2 - k)\mu^2 a_1 b_1$.

Coefficient of Y^1: $6na_1^2 b_1(4\beta a_0 + \alpha) + 2na_0^2 a_1(4\beta a_0 + 3\alpha) - 4(c^2 - k)\mu^2 a_0 a_1$.

Coefficient of Y^0: $2na_0^3(\beta a_0 + \alpha) - 2(c^2 - k)\mu^2(a_1^2 + b_1^2 + 8a_1 b_1) + 12n\beta a_1 b_1(2a_0^2 + a_1 b_1 + \alpha n a_0)$.

Coefficient of Y^{-1}: $6na_1 b_1^2(4\beta a_0 + \alpha) + 2na_0^2 b_1(4\beta a_0 + 3\alpha) - 4(c^2 - k)\mu^2 a_0 b_1$.

Coefficient of Y^{-2}: $8n\beta b_1^3 a_1 + 6na_0 b_1^2(2\beta a_0 + \alpha) + 8(c^2 - k)\mu^2 a_1 b_1$.

Coefficient of Y^{-3}: $2nb_1^3(4\beta a_0 + \alpha) + 4(c^2 - k)\mu^2 a_0 b_1$.

Coefficient of Y^{-4}: $2n\beta b_1^4 + 2(c^2 - k)\mu^2 b_1^2$.

Setting these coefficients equal to zero, and solving the resulting system, by using Maple, we find the following sets of solutions:

$$a_0 = -\frac{\alpha}{2\beta}, \quad a_1 = 0, \quad b_1 = \pm\frac{\alpha}{2\beta}, \quad \mu = \pm\frac{\alpha}{2}\sqrt{\frac{n}{\beta(k - c^2)}}. \tag{4.33}$$

$$a_0 = -\frac{\alpha}{2\beta}, \quad a_1 = \pm\frac{\alpha}{2\beta}, \quad b_1 = 0, \quad \mu = \pm\frac{\alpha}{2}\sqrt{\frac{n}{\beta(k - c^2)}}. \tag{4.34}$$

$$a_0 = -\frac{\alpha}{2\beta}, \quad a_1 = \pm\frac{\alpha}{4\beta}, \quad b_1 = \pm\frac{\alpha}{4\beta}, \quad \mu = \pm\frac{\alpha}{4}\sqrt{\frac{n}{\beta(k-c^2)}}. \tag{4.35}$$

Recall that

$$u = \frac{1}{n}arccosh[\frac{v+v^{-1}}{2}].$$

The sets (4.33)-(4.35) give the soliton solutions

$$u_1(x,t) = \frac{1}{n}arccosh\{\frac{-\alpha^2(1\pm\coth[\mu(x-ct)])^2 - 4\beta^2}{4\alpha\beta(1\pm\coth[\mu(x-ct)])}\}, \tag{4.36}$$

$$u_2(x,t) = \frac{1}{n}arccosh\{\frac{-\alpha^2(1\pm\tanh[\mu(x-ct)])^2 - 4\beta^2}{4\alpha\beta(1\pm\tanh[\mu(x-ct)])}\}, \tag{4.37}$$

where $\mu = \pm\frac{\alpha}{2}\sqrt{\frac{n}{\beta(k-c^2)}}$, $k > c^2$.

$$u_3(x,t) = \frac{1}{n}arccosh\{\frac{-\alpha^2(2\pm\tanh[\mu(x-ct)]\pm coth[\mu(x-ct)])^2 - 16\beta^2}{8\alpha\beta(2\pm\tanh[\mu(x-ct)]\pm coth[\mu(x-ct)])}\}, \tag{4.38}$$

where $\mu = \pm\frac{\alpha}{4}\sqrt{\frac{n}{\beta(k-c^2)}}$.
However, for $k < c^2$, complex solutions can be obtained that are not needed in this work.

5 Conclusion

In this paper, the extended tanh method has been successfully applied to find the exact solutions for the generalized combined and the generalized double combined sinh-cosh-Gordon equations. The results indicate the efficiency and reliability of the method. Thus, we can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

References

[1] M. A. Abdou, The extended tanh method and its applications for solving nonlinear physical models. *Applied Mathematics and Computation*, 190 (2007), 988-996.

[2] W. Malfliet and W. Hereman, The tanh method: I. Exact solutions of nonlinear evolution and wave equations. *Physica Scripta*, 54 (1996), 563-568.

[3] W. Malfliet and W. Hereman, The tanh method: II. Perturbation technique for conservative systems. *Physica Scripta*, 54 (1996), 569-575.

[4] A. M. Wazwaz, New solitary wave solutions to modified forms of Degasperis-Procesi and Camassa-Holm equations. *Applied Mathematics and Computation*, 186(1) (2006), 130-141.

[5] A. M. Wazwaz, The extended tanh method for new soliton solutions for many forms of the fifth-order KdV equations. *Applied Mathematics and Computation*, 184(2) (2007), 1002-1014.

[6] A. M. Wazwaz, The extended tanh method for abundant solitary wave solutions of nonlinear wave equations. *Applied Mathematics and Computation*, 187 (2007), 1131-1142.

[7] K. Krishnakumar, The extended tanh method for certain system of nonlinear ordinary differential equations, *Malaya journal of matematik*, 2(2) (2014), 133-140.

[8] N. Taghizadeh, M. Mirzazadeh and S. R. Moosavi Noori, Exact solutions of (2+1)-dimensional nonlinear evolution equations by using the extended tanh method. *Australian Journal of Basic and Applied Sciences*, 5(6) (2011), 1474-1481.

[9] N. Taghizadeh and S. R. Moosavi Noori, Two Reliable Methods for Solving the Modified Improved Kadomtsev-Petviashvili Equation. *Applications and Applied Mathematics: An International Journal*, 7(2) (2012), 658-671.

[10] M. L. Wang, X. Z. Li and J. L. Zhang, The $(\frac{G'}{G})$ -expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters A*, 372 (2008), 417-423.

[11] A. Bekir, Application of the $(\frac{G'}{G})$-expansion method for nonlinear evolution equations. *Physics Letters A*, 372 (2008), 3400-3406.

[12] N. Taghizadeh, S. R. Moosavi Noori and S. B. Moosavi Noori, Application of the extended $(\frac{G'}{G})$-expansion method to the Improved Eckhaus Equation. *Applications and Applied Mathematics: An International Journal*, 9(1) (2014), 371-387.

[13] S. D. Zhu, The extended $(\frac{G'}{G})$ -expansion method and traveling wave solutions of nonlinear evolution equations. *Mathematical and Computational Applications*, 15(5) (2010), 924-929.

[14] M. Wang, Solitary wave solutions for variant Boussinesq equations. *Physics Letters A*, 199 (1995), 169-172.

[15] M. Inc and M. Ergut, Periodic wave solutions for the generalized shallow water wave equation by the improved Jacobi elliptic function method. *Applied Mathematics E-Notes*, 5 (2005), 89-96.

[16] E. H. M. Zahran and M. M. A. Khater, Exact traveling wave solutions for the system of shallow water wave equations and modified Liouville equation using extended Jacobian elliptic function expansion method. *American Journal of Computational Mathematics*, 4 (2014), 455-463.

[17] Y. Z. Li, K. M. Li and C. Lin, Exp-function method for solving the generalized-Zakharov equations. *Applied Mathematics and Computation*, 205 (2008), 197-201.

[18] Z. S. Feng, The first integral method to study the Burgers-Korteweg-de Vries equation. *Journal of Physics A: Mathematical and General*, 35(2) (2002), 343-349.

[19] N. Taghizadeh and M. Mirzazadeh, The first integral method to some complex nonlinear partial differential equations. *Journal of Computational and Applied Mathematics*, 235 (2011), 4871-4877.

[20] M. Eslami, M. Mirzazadeh, B. Fathi Vajargah and Anjan Biswas, Optical solitons for the resonant nonlinear Schrdingers equationwith time-dependent coefficients by the first integral method, *Optik - International Journal for Light and Electron Optics*, 125(13) (2014), 3107-3116.

[21] A. M. Wazwaz, The sine-cosine method for obtaining solutions with compact and noncompact structures. *Applied Mathematics and Computation*, 159(2) (2004), 559-576.

[22] A. M. Wazwaz, The variable separated ODE and the tanh methods for solving the combined and the double combined sinh-cosh-Gordon equations. *Applied Mathematics and Computation*, 177 (2006), 745-754.

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Study on combinatorial dual graph in intuitionistic fuzzy environment

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Abstract

In this paper, intuitionistic fuzzy planar graphs are defined and various properties are studied. The intuitionistic fuzzy graphs are more efficient than fuzzy graphs, since it was found that one component is not sufficient to illustrate some special types of information. The notion of intuitionistic fuzzy dual graph and one of its close association namely intuitionistic fuzzy combinatorial dual graph is presented here. Some properties on intuitionistic fuzzy combinatorial dual graphs are investigated here.

Keywords: Intuitionistic fuzzy graphs, intuitionistic fuzzy planar graphs, intuitionistic fuzzy dual graphs, intuitionistic fuzzy combinatorial dual graphs.

2010 MSC: 03E72.

1 Introduction

Graph theory has a numerous applications in different research areas to structuring and designing of several models, its structures are used to represent various networking problems namely traffic network, telephone network, railway network, communication problems etc. The notion of fuzzy set (FS) was first introduced by Zadeh [11] (1965) to handle uncertainty in real life problems. After that it was found that one component is not sufficient to represent some special types of information. In this situations, a component namely non-membership value is needed to illustrate the information completely. To overcome this limitation of FS Atanassov [2] (1986) introduced the notion of intuitionistic fuzzy set (IFS) in addition to a new component known as degree of non-membership. Fuzzy graph (FG) theory was introduced by Rosenfeld [5] in 1975. Samanta et al. [6–8] defined fuzzy planar graph (FPG) in a different way where crossing between edges are allowed. Some related works are also found in [3, 4]. The idea of intuitionistic fuzzy graph (IFG) discussed by Shannon et al. [10]. Alshehri et al. [1] introduced the notion of intuitionistic fuzzy planar graphs (IFPG). Shriram et al. [9] defined fuzzy combinatorial dual graph.

In this work, we present IFPG, intuitionistic fuzzy faces, intuitionistic fuzzy dual graphs (IFDG), intuitionistic fuzzy combinatorial dual graphs (IFCDG) which is one of the classification of IFDGs. Also, introduced the terms strong (weak) IFPGs, strength of an edge, intersecting value between the edges. The IFMGs, IFPGs, IFDGs and IFCDGs are illustrated by an examples and lot of are presented of these graphs.

2 Preliminaries

This section, we give some related terminologies and results.

Definition 2.1. [5] *A FG is of the form* $\zeta = (\tilde{V}, \sigma, \mu)$ *where* $\tilde{V}$ *is the vertex set,* $\sigma : \tilde{V} \to [0,1]$ *and* $\mu : \tilde{V} \times \tilde{V} \to [0,1]$ *denote the degree of membership of* $r \in \tilde{V}$ *and edge* $(r,s) \in \zeta$*, respectively such that* $\mu(r,s) \leq min(\sigma(r), \sigma(s))$ $\forall$ $r, s \in \tilde{V}$.

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Definition 2.2. [2] *Let χ be the universe. Then a IFS $\tilde{A}$ is defined on X as $\tilde{A} = \{r, (\mu_{\tilde{A}}(r), \nu_{\tilde{A}}(r)) : r \in X\}$, where $\mu_{\tilde{A}}(r)$ and $\nu_{\tilde{A}}(r)$ are independent denote the degree of membership (DMS) and degree of non-membership (DNS) of $r \in \tilde{A}$, respectively with $0 \leq \mu_{\tilde{A}}(r) + \nu_{\tilde{A}}(r) \leq 1 \; \forall \; r \in X$. Also $\forall \; r \in X$, $D_{\tilde{A}}(r) = 1 - (\mu_{\tilde{A}}(r) + \nu_{\tilde{A}}(r))$ represent denial degree of r in $\tilde{A}$.*

Definition 2.3. [1] *A intuitionistic fuzzy relation (IFR) R is a intuitionistic fuzzy (IF) subset of $X \times Y$ is given by $\mathrm{R} = \{(r,s), \mu_{\mathrm{R}}(r,s), \nu_{\mathrm{R}}(r,s) | (r,s) \in X \times Y\}$, where $\mu_{\mathrm{R}}, \nu_{\mathrm{R}} : X \times Y \to [0,1]$ denote DMS and DNS of an edge (r,s) in R, respectively with $0 \leq \mu_{\mathrm{R}}(r,s) + \nu_{\mathrm{R}}(r,s) \leq 1$ for every $(r,s) \in X \times Y$.*

Definition 2.4. [1] *A IFG is of the form $\tilde{G} = (\tilde{V}, \tilde{A}, \tilde{B})$ where $\tilde{A} = (\mu_{\tilde{A}}, \nu_{\tilde{A}})$, $\tilde{B} = (\mu_{\tilde{B}}, \nu_{\tilde{B}})$ and*
(i) $\tilde{V} = \{r_1, r_2, .., r_n\}$ such that that $\mu_{\tilde{A}}, \nu_{\tilde{A}} : \tilde{V} \to [0,1]$ denote the DMS and DNS of $r_i \in \tilde{V}$, respectively with $0 \leq \mu_{\tilde{A}}(r_i) + \nu_{\tilde{A}}(r_i) \leq 1 \; \forall r_i \in \tilde{V}$, $(i = 1,2,..,n)$.
(ii) $\mu_{\tilde{B}}, \nu_{\tilde{B}} : \tilde{V} \times \tilde{V} \to [0,1]$ denote the DMS and DNS of an edge (r_i, r_j), respectively such that $\mu_{\tilde{B}}(r_i, r_j) \leq min\{\mu_{\tilde{A}}(r_i), \mu_{\tilde{A}}(r_j)\}$ and $\nu_{\tilde{B}}(r_i, r_j) \leq max\{\nu_{\tilde{A}}(r_i), \nu_{\tilde{A}}(r_j)\}$ with $\mu_{\tilde{B}}(r_i, r_j) + \nu_{\tilde{B}}(r_i, r_j) \leq 1$ for every (r_i, r_j), $(i, j = 1, 2, .., n)$.

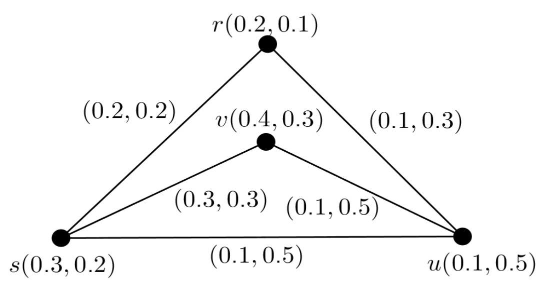

Figure 1: Example of a IFG

Definition 2.5. [1] *A intuitionistic fuzzy multiset (IFMS) M is given by $\mathrm{M} = \{(r, \mu^i_{\mathrm{M}}(r), \nu^i_{\mathrm{M}}(r)) : i = 1,2,..,n | r \in \tilde{V}\}$, where $n = max\{i : \mu^i_{\mathrm{M}}(r) \neq 0$ or $\nu^i_{\mathrm{M}}(r) \neq 0\}$ and $\mu^i_{\mathrm{M}}(r), \nu^i_{\mathrm{M}}(r) \in [0,1]$ are the DMS and DNS of $r \in \tilde{V}$, respectively with $0 \leq \mu^i_{\mathrm{M}}(r) + \nu^i_{\mathrm{M}}(r) \leq 1 \; \forall \; r \in \tilde{V}$.*

Now, we introduce the notion of IFPG, for that it needs to define Intuitionistic fuzzy multigraph (IFMG) using the concept of IFMS.

Definition 2.6. [1] *Let $\tilde{A} = (\mu_{\tilde{A}}, \nu_{\tilde{A}})$ be a IFS on a non-empty set $\tilde{V}$ and $\tilde{B} = \{(rs, \mu^i_{\tilde{B}}(rs), \nu^i_{\tilde{B}}(rs)) : i = 1, 2, \ldots, n_{rs} | rs \in \tilde{V} \times \tilde{V}\}$ be a IFMS on $\tilde{V} \times \tilde{V}$ such that $\mu^i_{\tilde{B}}(rs) \leq min\{\mu_{\tilde{A}}(r), \mu_{\tilde{A}}(s)\}$, $\nu^i_{\tilde{B}}(rs) \leq max\{\nu_{\tilde{A}}(r), \nu_{\tilde{A}}(s)\}$ for all $i = 1, 2, \ldots, n_{rs}$, where $n_{rs} = max\{i : \mu^i_{\tilde{B}}(rs) \neq 0$ or $\nu^i_{\tilde{B}}(rs) \neq 0\}$ is the number of edges between r and s. Then $\tilde{G} = (\tilde{V}, \tilde{A}, \tilde{B})$ is called IFMG where $\mu_{\tilde{A}}(r), \nu_{\tilde{A}}(r)$ and $\mu^i_{\tilde{B}}(rs), \nu^i_{\tilde{B}}(rs)$ represent the DMS and DNS of vertex r and the i^{th} edge between r and s in $\tilde{G}$, respectively.*

Definition 2.7. [1] *Let $\tilde{G} = (\tilde{V}, \tilde{A}, \tilde{B})$ be IFMG, where $\tilde{B} = \{(rs, \mu^i_{\tilde{B}}(rs), \nu^i_{\tilde{B}}(rs)) : i = 1, 2, \ldots, n_{rs} | rs \in \tilde{V} \times \tilde{V}\}$ and $n_{rs} = max\{i : \mu^i_{\tilde{B}}(rs) \neq 0$ or $\nu^i_{\tilde{B}}(rs) \neq 0\}$. A multiedge rs is strong in $\tilde{G}$ if $\frac{1}{2}min\{\mu_{\tilde{A}}(r), \mu_{\tilde{A}}(s)\} \leq \mu^i_{\tilde{B}}(rs)$, $\frac{1}{2}max\{\nu_{\tilde{A}}(r), \nu_{\tilde{A}}(s)\} \leq \nu^i_{\tilde{B}}(rs)$ for all $i = 1, 2, \ldots, n_{rs}$.*

Example 2.1. *Consider a MG $\tilde{G}^* = (\tilde{V}, E)$, where $\tilde{V} = \{r, s, u, v\}$ and $E = \{rs, su, sv, sv, uv\}$. Let $\tilde{A} = (\mu_{\tilde{A}}, \nu_{\tilde{A}})$ be a IFS on $\tilde{V}$ and $\tilde{B} = (\mu_{\tilde{B}}, \nu_{\tilde{B}})$ be a IFMS on $\tilde{V} \times \tilde{V}$ given in Table 1 and 2. Fig.2 is a IFMG.*

Table 1: IFS $\tilde{A}$

$\tilde{A}$	r	s	u	v
$\mu_{\tilde{A}}$	0.4	0.45	0.3	0.3
$\nu_{\tilde{A}}$	0.4	0.1	0.25	0.4

Table 2: IFMS $\tilde{B}$

$\tilde{B}$	rs	su	sv	sv	uv
$\mu_{\tilde{B}}$	0.35	0.3	0.3	0.2	0.3
$\nu_{\tilde{B}}$	0.2	0.25	0.3	0.4	0.25

Here, rs and uv be two strong edges as $\frac{1}{2}min\{0.4, 0.45\} \leq 0.35$, $\frac{1}{2}max\{0.4, 0.1\} = 0.2$ and $\frac{1}{2}min\{0.3, 0.3\} \leq 0.3$, $\frac{1}{2}max\{0.25, 0.4\} \leq 0.25$.

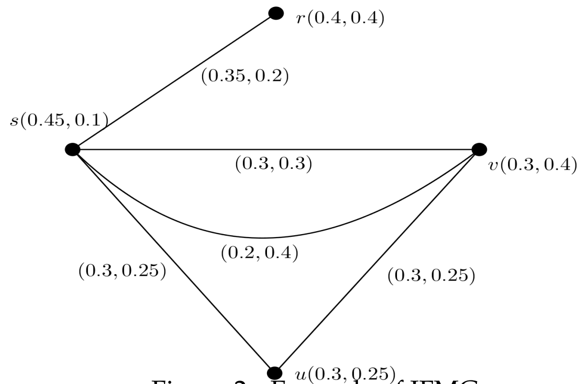

Figure 2: Example of IFMG

Definition 2.8. [1] *Let in $\tilde{G}$, P is the intersecting point between the edges $(rs, \mu^i_{\tilde{B}}(rs), \nu^i_{\tilde{B}}(rs))$ and $(uv, \mu^j_{\tilde{B}}(uv), \nu^j_{\tilde{B}}(uv))$, where i, j are fixed integers. The strength of the edge rs is defined as $I_{rs} = (t_{rs}, f_{rs}) = \left(\frac{\mu^i_{\tilde{B}}(rs)}{min(\mu_{\tilde{A}}(r), \mu_{\tilde{A}}(s))}, \frac{\nu^i_{\tilde{B}}(rs)}{max(\nu_{\tilde{A}}(r), \nu_{\tilde{A}}(s))}\right)$. The edge rs is strong if $t_{rs} \geq 0.5$ and $f_{rs} \geq 0.5$ otherwise weak. At P the intersecting value is $\widetilde{I}_P = (t_p, f_p) = \left(\frac{t_{rs}+t_{uv}}{2}, \frac{f_{rs}+f_{uv}}{2}\right)$.*

Example 2.2. *In Fig.3, strength of the edges (r, u) and (s, v) are $I_{ru} = (0.8, 0.8)$ and $I_{sv} = (0.66, 0.88)$, respectively. Thus at P intersecting value is $\widetilde{I}_P = (0.73, 0.84)$.*

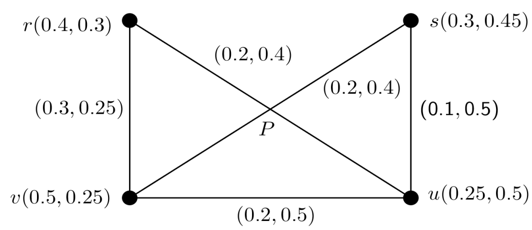

Figure 3: Intersecting value between two edges

Definition 2.9. [1] *Let $P_1, P_2, \ldots, P_k$ be k (integer) intersecting points between the edges of IFMG $\tilde{G}$. Then $\tilde{G}$ is IFPG with DP $f = (f_t, f_f)$, where $f_t = \frac{1}{1+\{t_{P_1}+t_{P_2}+\ldots+t_{P_k}\}}$ and $f_f = \frac{1}{1+\{f_{P_1}+f_{P_2}+\ldots+f_{P_k}\}}$. Clearly, $f = (f_t, f_f)$ is bounded as $0 < f_t \leq 1$ and $0 < f_f \leq 1$. DP increases if intersecting points decreases.*

Example 2.3. *Consider a IFMG $\tilde{G}^* = (\tilde{V}, E)$, where $\tilde{V} = (r, s, u, v)$ and $E = \{rs, ru, ru, su, sv, sv, rv, uv\}$. Let $\tilde{A} = (\mu_{\tilde{A}}, \nu_{\tilde{A}})$ be a IFS of $\tilde{V}$ and $\tilde{B} = (\mu_{\tilde{B}}, \nu_{\tilde{B}})$ be a IFMS of $\tilde{V} \times \tilde{V}$ given in Table 3 and 4.*

Table 3: IFS $\tilde{A}$

$\tilde{A}$	r	s	u	v
$\mu_{\tilde{A}}$	0.5	0.4	0.6	0.3
$\nu_{\tilde{A}}$	0.2	0.3	0.1	0.4

Table 4: IFMS $\tilde{B}$

$\tilde{B}$	rs	ru	ru	su	sv	sv	rv	uv
$\mu_{\tilde{B}}$	0.4	0.3	0.3	0.4	0.2	0.2	0.2	0.2
$\nu_{\tilde{B}}$	0.2	0.1	0.2	0.1	0.3	0.2	0.2	0.3

In Fig.4, a IFPG is considered with two intersecting points P_1 and P_2, between the edges $(ru, (0.3, 0.1))$, $(sv, (0.2, 0.3))$ and $(ru, (0.3, 0.2))$, $(sv, (0.2, 0.2))$, respectively. The strength of $(ru, (0.3, 0.1))$, $(sv, (0.2, 0.3))$, $(ru, (0.3, 0.2))$ and $(sv, (0.2, 0.2))$ are respectively $I_{ru} = (0.6, 0.5)$, $I_{sv} = (0.66, 0.75)$, $I_{ru} = (0.6, 1)$ and $I_{sv} = (0.66, 0.5)$. At P_1, intersecting value is $\widetilde{I}_{P_1} = (0.63, 0.62)$ and at P_2, $\widetilde{I}_{P_2} = (0.63, 0.75)$. Thus, the DP of $\tilde{G}^*$ is $f = (0.44, 0.42)$.

Definition 2.10. [1] *A IFPG $\tilde{G}$ is strong if its DP $f = (f_t, f_f)$ is such that $f_t > 0.5$ and $f_f > 0.5$. Otherwise weak.*

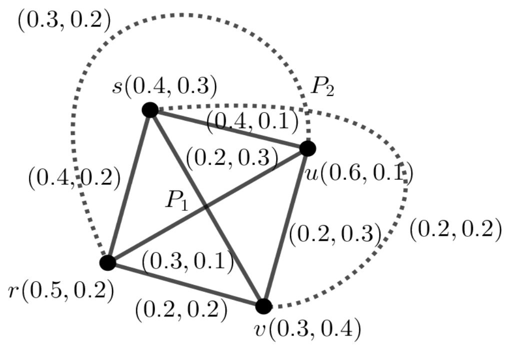

Figure 4: Example of IFPG

Now we present a special type of IFPG called 0.67-IFPG with DP $f = (f_t, f_f)$, where $f_t \geq 0.67$ and $f_f \geq 0.67$. When DP is $(1, 1)$, its geometrical representation is like as crisp planar graph. The above theorem state that, if DP is $f = (f_t, f_f)$, where $f_t \geq 0.67$ and $f_f \geq 0.67$, then two strong edges not intersect in $\tilde{G}$ and if there is any crossing, this is the crossing between the edges except both are strong. Thus any IFPG having no intersecting point between the edges is a IFPG with DP $(1, 1)$. Therefore, it is a 0.67-IFPG.

3 Intuitionistic fuzzy dual graph (IFDG)

At first we present intuitionistic fuzzy face (IFF) of a IFPG. Face is a region bounded by IF edges in a IFG. The presence of a IFF depending on minimum strength of its boundary edges. Because if all boundary edges of a IFF have DMS and DNS 1 and 0, respectively, it turn out crisp face but if we removed one of such edge or has membership degrees 0 and 1, respectively, the IFF does not exit. A IFF with its membership degrees are defined below.

Definition 3.11. *Let $\tilde{G}$ be a IFPG and $\tilde{B} = \{(rs, \mu^i_{\tilde{B}}(rs), \nu^i_{\tilde{B}}(rs)), i = 1, 2, \ldots, n_{rs} | rs \in \tilde{V} \times \tilde{V}\}$, where $n_{rs} = max\{i : \mu^i_{\tilde{B}}(rs) \neq 0$ or $\nu^i_{\tilde{B}}(rs) \neq 0\}$. A IFF of $\tilde{G}$ is a region, enclosed by the set of IF edges $\mathrm{E}' \subset E$. The DMS and DNS of IFF are, respectively $min\{\frac{\mu^i_{\tilde{B}}(rs)}{min(\mu_{\tilde{A}}(r), \mu_{\tilde{A}}(s))}, i = 1, 2, \ldots, n_{rs} | rs \in \mathrm{E}'\}$ and $max\{\frac{\nu^i_{\tilde{B}}(rs)}{max(\nu_{\tilde{A}}(r), \nu_{\tilde{A}}(s))}, i = 1, 2, \ldots, n_{rs} | rs \in \mathrm{E}'\}$.*

Definition 3.12. *A IFF strong if its DMS > 0.5 and DNS < 0.5 and weak otherwise. Each IFPG has an outer face with an infinite region and inner faces with finite region.*

Example 3.4. *In Fig.5, the IFPG has the faces: $\tilde{F}_1$ (inner face) is enclosed by the edges $(r_1r_2, 0.4, 0.1), (r_2r_3, 0.6, 0.1)$ and $(r_1r_3, 0.4, 0.1)$. $\tilde{F}_2$ (outer face) is enclosed by the edges $(r_1r_4, 0.4, 0.1)$, $(r_1r_3, 0.4, 0.1), (r_2r_3, 0.6, 0.1)$ and $(r_2r_4, 0.5, 0.1)$. $\tilde{F}_3$ (inner face) is enclosed by the edges $(r_1r_2, 0.4, 0.1)$, $(r_1r_4, 0.4, 0.1)$ and $(r_2r_4, 0.5, 0.1)$. The IFFs $\tilde{F}_1, \tilde{F}_2$ and $\tilde{F}_3$ are strong as all have same DMS and DNS 0.8 and 0.33, respectively.*

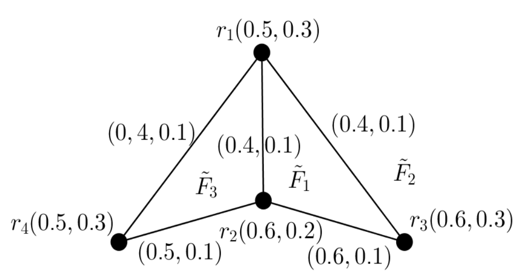

Figure 5: Example of faces in IFPG

Now we introduce dual of IFPG with DP $(1, 1)$. The vertices of IFDG are imposed corresponding to strong IFFs and edges are imposed corresponding to common border edges of IFFs.

Definition 3.13. *Let $\tilde{G} = (\tilde{V}, \tilde{A}, \tilde{B})$ be a 0.67-IFPG and $\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_n$ be its strong IFFs. The IFDG of $\tilde{G}$ is a IFPG $\tilde{G}_1 = (\tilde{V}_1, \tilde{A}_1, \tilde{B}_1)$, where $\tilde{V}_1 = \{t_i, i = 1, 2, \ldots, n\}$ and each t_i in $\tilde{G}_1$ is considered corresponding to the face $\tilde{F}_i$ of $\tilde{G}$. The DMS and DNS of vertices are given by the mapping $\tilde{A}_1 = (\mu_{\tilde{A}_1}, \nu_{\tilde{A}_1}) : \tilde{V}_1 \to [0,1] \times [0,1]$ such that*
$\mu_{\tilde{A}_1}(t_i) = max\{\mu^i_{\tilde{B}}(rs), i = 1, 2, \ldots, n_{rs} | rs \text{ is the border edge of } \tilde{F}_i\}$,
$\nu_{\tilde{A}_1}(t_i) = min\{\nu^i_{\tilde{B}}(rs), i = 1, 2, \ldots, n_{rs} | rs \text{ is the border edge of } \tilde{F}_i\}$.
In IFDG $\tilde{G}_1$, may exists more than one edge between t_i and t_j as two faces $\tilde{F}_i$ and $\tilde{F}_j$ of $\tilde{G}$ may exists more than one common edge. Let $\mu^l_{\tilde{B}}(t_i t_j)$ and $\nu^l_{\tilde{B}}(t_i t_j)$ denotes the DMS and DNS of the l-th edge between t_i and t_j, respectively. The DMS and DNS of IF edges in IFDG are given by $\mu^l_{\tilde{B}_1}(t_i t_j) = \mu^i_{\tilde{B}}(rs)^l$ and $\nu^l_{\tilde{B}_1}(t_i t_j) = \nu^i_{\tilde{B}}(rs)^l$, where $(rs)^l$ is border edge between $\tilde{F}_i$ and $\tilde{F}_j$ and $l = 1, 2, \ldots, p$, where p is the number of common border edges between $\tilde{F}_i$ and $\tilde{F}_j$ or the edges between t_i and t_j. If in a IFDG present any strong pendent edge, then for that there is a self-loop in $\tilde{G}_1$. The DMS and DNS of the self-loop of $\tilde{G}_1$ and pendent edge of $\tilde{G}$ are same.

Example 3.5. *In Fig.6, consider a IFPG $\tilde{G} = (\tilde{V}, \tilde{A}, \tilde{B})$, where $\tilde{V} = \{r, s, u, v\}$, $\tilde{A} = \{(r, 0.4, 0.3), (s, 0.6, 0.2), (u, 0.7, 0.3), (v, 0.3, 0.3)\}$ and $\tilde{B} = \{(rs, 0.4, 0.1), (ru, 0.3, 0.1), (rv, 0.3, 0.1), (su, 0.6, 0.1), (uv, 0.3, 0.1), (rv, 0.2, 0.1)\}$.*

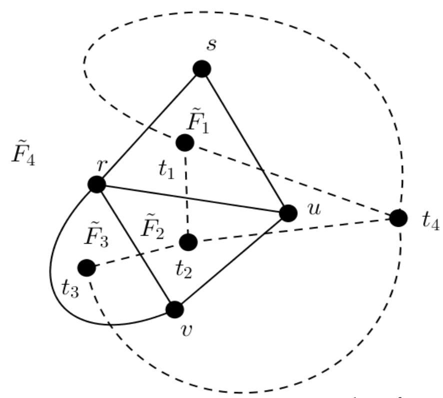

Figure 6: Example of IFDG

This graph has four faces $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$ and $\tilde{F}_4$, where $\tilde{F}_1$ is enclosed by the edges $(rs, 0.4, 0.1)$, $(ru, 0.3, 0.1)$ and $(su, 0.6, 0.1)$, $\tilde{F}_2$ is enclosed by $(ru, 0.3, 0.1), (rv, 0.3, 0.1)$ and $(uv, 0.3, 0.1)$, $\tilde{F}_3$ is enclosed by $(rv, 0.3, 0.1), (rv, 0.2, 0.1)$ and outer face $\tilde{F}_4$ is enclosed by $(rs, 0.4, 0.1), (su, 0.6, 0.1)$, $(uv, 0.3, 0.1)$ and $(rv, 0.2, 0.1)$. Since IFFs are strong, the vertex set of IFDG is $\tilde{V}_1 = \{t_1, t_2, t_3, t_4\}$, where each t_i is assigned corresponding to each $\tilde{F}_i, i = 1, 2, 3, 4$. Thus $\mu_{\tilde{A}_1}(t_1) = 0.6$, $\nu_{\tilde{A}_1}(t_1) = 0.1$, $\mu_{\tilde{A}_1}(t_2) = 0.3$, $\nu_{\tilde{A}_1}(t_2) = 0.1$, $\mu_{\tilde{A}_1}(t_3) = 0.3$, $\nu_{\tilde{A}_1}(t_3) = 0.1$, $\mu_{\tilde{A}_1}(t_4) = 0.6$, $\nu_{\tilde{A}_1}(t_4) = 0.1$.
It is seen that rs and su are the common edges between $\tilde{F}_1$ and $\tilde{F}_4$. So in IFDG $\tilde{G}_1$ there exists two edges between t_1 and t_4. The DMS and DNS of these edges are given by
$\mu_{\tilde{B}_1}(t_1 t_4) = \mu_{\tilde{B}}(rs) = 0.4$, $\nu_{\tilde{B}_1}(t_1 t_4) = \nu_{\tilde{B}}(rs) = 0.1$,
$\mu_{\tilde{B}_1}(t_1 t_4) = \mu_{\tilde{B}}(su) = 0.6$, $\nu_{\tilde{B}_1}(t_1 t_4) = \nu_{\tilde{B}}(su) = 0.1$.
Also,
$\mu_{\tilde{B}_1}(t_1 t_2) = \mu_{\tilde{B}}(ru) = 0.3$, $\nu_{\tilde{B}_1}(t_1 t_2) = \nu_{\tilde{B}}(ru) = 0.1$,
$\mu_{\tilde{B}_1}(t_2 t_3) = \mu_{\tilde{B}}(rv) = 0.3$, $\nu_{\tilde{B}_1}(t_2 t_3) = \nu_{\tilde{B}}(rv) = 0.1$,
$\mu_{\tilde{B}_1}(t_2 t_4) = \mu_{\tilde{B}}(uv) = 0.3$, $\nu_{\tilde{B}_1}(t_2 t_4) = \nu_{\tilde{B}}(uv) = 0.1$,
$\mu_{\tilde{B}_1}(t_3 t_4) = \mu_{\tilde{B}}(rv) = 0.2$, $\nu_{\tilde{B}_1}(t_3 t_4) = \nu_{\tilde{B}}(rv) = 0.1$.
Therefore, the edge set of IFDG is $\tilde{B}_1 = \{(t_1 t_4, 0.4, 0.1), (t_1 t_4, 0.6, 0.1), (t_1 t_2, 0.3, 0.1), (t_2 t_3, 0.3, 0.1), (t_2 t_4, 0.3, 0.1), (t_3 t_4, 0.2, 0.1)\}$. The IFDG $\tilde{G}_1$ of $\tilde{G}$ is drawn by dotted line in Fig.6.

4 Intuitionistic fuzzy combinatorial dual graph (IFCDG)

In this section, we define one of the classification of IFDG known as IFCDG and give some theorems of it.

Definition 4.14. *Let* $\tilde{G} = (\tilde{V}, \tilde{A}, \tilde{B})$ *be a* 0.67*-IFPG. The IFCDG of* $\tilde{G}$ *is* $\tilde{G}'_1 = (\tilde{V}'_1, \tilde{A}'_1, \tilde{B}'_1)$*, where* $\tilde{V}'_1 = \{t'_i, i = 1, 2, ..., n\}$ *is the vertex set of* $\tilde{G}'_1$*. The DMS and DNS of the vertices of* $\tilde{G}'_1$ *are given by the mapping* $\tilde{A}'_1 = (\mu_{\tilde{A}'_1}, \nu_{\tilde{A}'_1}) : \tilde{V}'_1 \to [0,1] \times [0,1]$ *such that*
$\mu_{\tilde{A}'_1}(t'_i) = max\{\mu^r(t'_i t'_j), r = 1, 2, ..., n_{t'_i t'_j} | t'_i t'_j \text{ is an edge adjacent to } t'_i\}$,
$\nu_{\tilde{A}'_1}(t'_i) = min\{\nu^r(t'_i t'_j), r = 1, 2, ..., n_{t'_i t'_j} | t'_i t'_j \text{ is an edge adjacent to } t'_i\}$.
Between the edges of $\tilde{G}$ *and* $\tilde{G}'_1$ *there is a one-to-one correspondence such that the DMS and DNS of the edges of* $\tilde{G}'_1$ *are the DMS and DNS of the edges in* $\tilde{G}$ *with the condition each cycle of* $\tilde{G}$ *is cut set of* $\tilde{G}'_1$*.*

Example 4.6. *Consider a* 0.67*-IFPG* $\tilde{G} = (\tilde{V}, \tilde{A}, \tilde{B})$*, where* $\tilde{V} = \{r, s, u, v, w\}$, $\tilde{A} = \{(r, 0.5, 0.3), (s, 0.4, 0.2), (u, 0.6, 0.3), (v, 0.3, 0.2), (w, 0.5, 0.3)\}$ *and* $\tilde{B} = \{(e_1, 0.4, 0.3), (e_2, 0.4, 0.3), (e_3, 0.3, 0.3), (e_4, 0.3, 0.3), (e_5, 0.4, 0.3), (e_6, 0.3, 0.2)\}$ *(see Fig.7). The cycles of* $\tilde{G}$ *are* $\{e_1, e_2, e_3, e_4, e_5\}$, $\{e_2, e_3, e_6\}$ *and* $\{e_1, e_6, e_4, e_5\}$ *form cut sets in IFCDG* $\tilde{G}'_1 = (\tilde{V}'_1, \tilde{A}'_1, \tilde{B}'_1)$*, where* $\tilde{V}'_1 = \{t'_1, t'_2, t'_3\}$, $\tilde{A}'_1 = \{(t'_1, 0.4, 0.2), (t'_3, 0.4, 0.3)\}$ *and* $\tilde{B}'_1 = \{(e'_1, 0.4, 0.3), (e'_2, 0.4, 0.3), (e'_3, 0.3, 0.3), (e'_4, 0.3, 0.3), (e'_5, 0.4, 0.3), (e'_6, 0.3, 0.2)\}$.

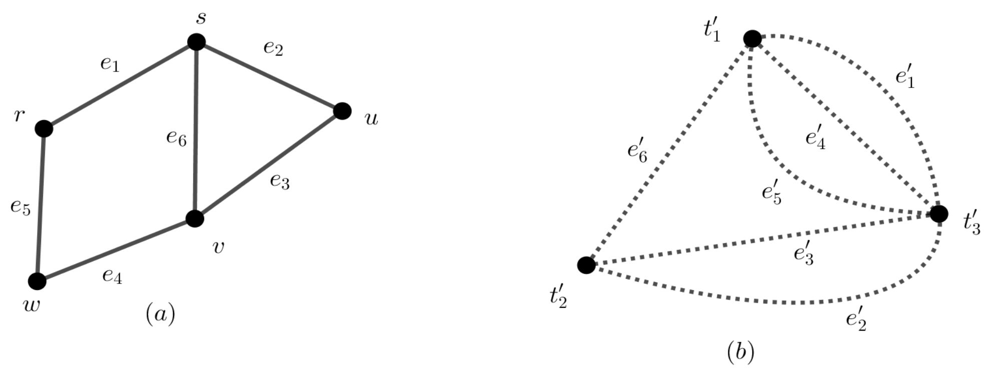

Figure 7: (a) A IFPG $\tilde{G}$ and (b) its IFCDG $\tilde{G}'_1$

Theorem 4.1. *Every* 0.67*-IFPG has a IFCDG.*

Proof. Let $\tilde{G}$ be 0.67-IFPG and $\tilde{G}'$ be the IFCDG. Then between the edges of $\tilde{G}$ and $\tilde{G}'_1$ there is a one-to-one correspondence such that the DMS and DNS of the edges of $\tilde{G}'_1$ are known. Let $\tilde{C}$ be a cycle of $\tilde{G}$ and it divides $\tilde{G}$ into two regions. Then we isolate the vertices of $\tilde{G}'_1$ into two non-empty subsets $\tilde{A}'$ and $\tilde{B}'$ (say), both are determined by the boundary of the cycle inside and outside $\tilde{C}$, respectively in $\tilde{G}$.
Let corresponding to the edges of $\tilde{C}$, we have a set of edges $\tilde{Ç}$ in $\tilde{G}'_1$ and removal of $\tilde{Ç}$ two subsets $\tilde{A}'$ and $\tilde{B}'$ becomes disjoint and $\tilde{G}'_1$ is disconnected. Thus $\tilde{Ç}$ is a cut set of $\tilde{G}'_1$.
Hence, each cycle of $\tilde{G}$ forms a cut set in $\tilde{G}'_1$. This proves the theorem. □

Example 4.7. *Consider a* 0.67*-IFPG* $\tilde{G} = (\tilde{V}, \tilde{A}, \tilde{B})$*, where* $\tilde{V} = \{r, s, u, v, w\}$, $\tilde{A} = \{(r, 0.6, 0.1), (s, 0.5, 0.4), (u, 0.4, 0.3), (v, 0.3, 0.4), (w, 0.7, 0.2)\}$ *and* $\tilde{B} = \{(e_1, 0.5, 0.4), (e_2, 0.4, 0.3), (e_3, 0.3, 0.4), (e_4, 0.4, 0.4), (e_5, 0.3, 0.4), (e_6, 0.5, 0.4), (e_7, 0.3, 0.4), (e_8, 0.3, 0.4)\}$ *and its IFCDG is* $\tilde{G}'_1 = (\tilde{V}'_1, \tilde{A}'_1, \tilde{B}'_1)$*, where* $\tilde{V}'_1 = \{t'_1, t'_2, t'_3, t'_4, t'_5\}$, $\tilde{A}'_1 = \{(t'_1, 0.5, 0.3), (t'_2, 0.5, 0.4), (t'_3, 0.4, 0.4), (t'_4, 0.5, 0.3), (t'_5, 0.5, 0.4)\}$ *and* $\tilde{B}'_1 = \{(e'_1, 0.5, 0.4), (e'_2, 0.4, 0.3), (e'_3, 0.3, 0.4), (e'_4, 0.4, 0.4), (e'_5, 0.3, 0.4), (e'_6, 0.5, 0.4), (e'_7, 0.3, 0.4), (e'_8, 0.3, 0.4)\}$ *(see Fig.8). Let* $\tilde{C} = \{e_1, e_2, e_3, e_4\}$ *be any cycle of* $\tilde{G}$ *such that* $\tilde{A}' = \{t'_1, t'_3\}$ *and and* $\tilde{B}' = \{t'_2, t'_4, t'_5\}$ *in* $\tilde{G}'$*. If we remove the corresponding edges of* $\tilde{C}$*, then* $\tilde{G}'_1$ *becomes disconnected. Hence, cycles of* $\tilde{G}$ *forms the cut sets in* $\tilde{G}'_1$*.*

Theorem 4.2. *Every IFCDG of a IFG has a* 0.67*-IFPG.*

Proof. Let K_5 or $K_{3,3}$ has a IFCDG. Both graphs has finite number of edges and one intersecting point can not be avoid for any representation of them.
Case-I: Let K_5 or $K_{3,3}$ has at least one weak edge in $\tilde{G}$ and this edge is not considered in IFG $\tilde{G}$. Then $\tilde{G}$ has no intersecting point between its edges and has a IFCDG $\tilde{G}'_1$. Thus K_5 or $K_{3,3}$ is a 0.67-IFPG.

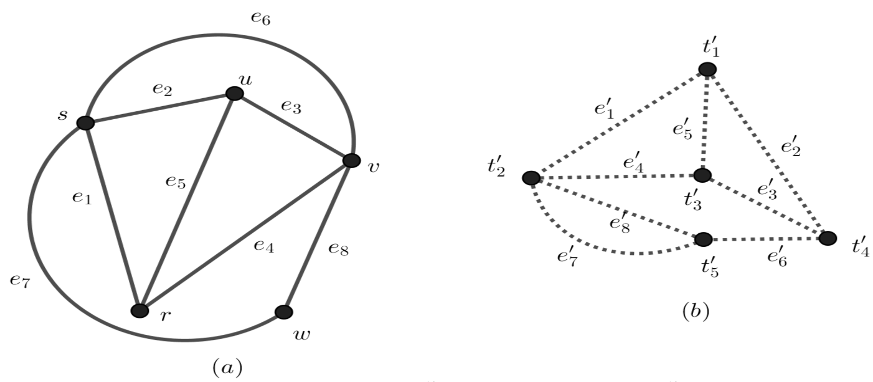

Figure 8: (a) A IFPG $\tilde{G}$ and (b) its IFCDG $\tilde{G}'_1$

Case-II: Let in K_5 or $K_{3,3}$ all edges are strong and there is only one intersecting point between strong edges. Then the DP is $f = (f_t, f_f)$, where $f_t < 0.67, f_f < 0.67$. Thus no dual graph can be drawn. Therefore, K_5 or $K_{3,3}$ does not have any 0.67-IFPG and IFCDG. □

Theorem 4.3. *A 0.67-IFPG is planar iff it has a IFCDG.*

Proof. Combining theorem 7.3 and theorem 7.5, we conclude it. □

5 Conclusion

This study relates the IFPGs and discussed its important consequences known as IFDGs and IFCDGs both are closely associated. For the 0.67-IFPG we define IFDG. But, when DP of IFPG is less than 0.67, then some modifications are needed to define it. IFMG, DP of a IFPG and IFF have also been introduced here and some corresponding results have been studied. This work can be viewed as the generalization of the study on fuzzy combinatorial dual graph.

References

[1] N. Alshehri and M. Akram, Intuitionistic Fuzzy Planar Graphs, *Discrete Dynamics in Nature and Society*, vol. 2014, 2014, 1–9.

[2] K. T. Atanassov, Intuitionistic Fuzzy Sets, *Fuzzy sets and Systems*, 20(1)(1986), 87–96.

[3] G. Nirmala and K. Dhanabal, Special Planar Fuzzy Graph Configurations, *International Journal of Scientific and Research Publications*, 2(7)(2012), 1–4.

[4] T. Pramanik, S. Samanta, M. Pal, Interval-Valued Fuzzy Planar Graphs, *Int. J. Mach. Learn. and Cyber*, 7(4)(2014), 653–664.

[5] A. Rosenfeld, Fuzzy Graphs, in: L. A. Zadeh, K. S. Fu, M. Shimura (Eds.), *Fuzzy Sets and Their Applications, Academic Press, New York*, pp. 77-95, 1975.

[6] S. Samanta, A. Pal and M. Pal, Concept of Fuzzy Planar Graphs, *Proceedings of the Science and Information Conference, London, UK*, pp. 557-563, October 7-9, 2013.

[7] S. Samanta, A. Pal and M. Pal, New Concept of fuzzy planar graph, *International Journal of Advance Research in Artificial Intelligence*, 3(1)(2014), 52–59.

[8] S. Samanta, M. Pal, Fuzzy Planar Graphs, *IEEE Transections on Fuzzy Systems*, 23(6)(2015), 1936–1942.

[9] S. Shriram, R. Sujatha and S. Narasimman, Fuzzy combinatorial dual graph, *International Journal of Pure and Applied Mathematics*, 109(7)(2016), 35–41.

[10] Shannon, A., Atanassov, K. T.: A first step to a theory of the intuitionistic fuzzy graphs, *Proceeding of FUBEST*, 1994, 59–61.

[11] L.A. Zadeh, Fuzzy sets, *Information and Control*, 8(3)(1965), 338–353.

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