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Compactly Coinvariant Subspaces

Batool Rostamzadeh Kasrineh[a,*]

[a]Department of Mathematics, Islamic Azad University, Rafsanjan Branch, Rafsanjan, Iran.

Abstract

First, we obtain some results on compactly invariant subspaces in normed spaces. Then we introduce the notion of compactly coinvariant pair. Also, a sufficient condition for a normed space X has a compactly coinvariant pair, is given.

Keywords: Compactly invariant subspace, Compactly coinvariant subspace, Reflexive space.

2010 MSC: 46B20, 46B50.

1 Introduction

The notion of an invariant subspace is fundamental to the subject of operator theory (see [1]-[3]). Given a linear operator T on a Banach space X, a closed subspace M of X is said to be a non-trivial invariant subspace for T if $T(M) \subset M$ and $M \neq \emptyset$ and X. This generalizes the idea of eigenspaces of $n \times n$ matrices. A famous unsolved problem, called the invariant subspace problem, "asks whether every bounded linear operator on a Hilbert space (more generally, a Banach space) admits a non-trivial invariant subspace"?

Definition 1.1. *[5]. Let X and Y be normed spaces and $T : X \to Y$ a linear operator. T is compact if for every bounded set A of X, $\overline{T(A)}$ is compact.*
The space of all compact operators of X into Y is denoted by $K(X, Y)$.

Uniformly invariant normed spaces are an another class of normed spaces which introduced by A. M. Forouzanfar et.al (see [4]). Authors of [4] have also introduced the useful notion compactly invariant subspace.

Definition 1.2. *A normed space X is called to be compactly invariant when for each closed subspace Y, there exists nonzero $T \in K(X)$ such that $T(Y) \subseteq Y$.*

In this paper we give more details on compactly invariant subspaces and then introduce the concept of compactly coinvariant subspaces.

2 Main Results

The purpose of this section mainly consists in proving some results on compactly invariant subspaces. To do so, we first give the following proposition. It shows that an isometrically isomorphisms preserve compactly invariant subspaces.

Proposition 2.1. *Let X and Y be normed spaces such that $X \cong Y$. If $Z \subset X$ is a compactly invariant subspace, then its isomorphic image is a compactly invariant subspace in Y.*

*Corresponding author.
: f.rostamzadeh_88@yahoo.com (Batool Rostamzadeh Kasrineh).

Proof. Let $f : X \to Y$ be an isometrically isomorphism and T a nonzero compact operator on X such that $T(Z) \subset Z$. Define $S : Y \to Y$ by $S(y) =: (f \circ T \circ f^{-1})(y)$. S is compact, since T is compact. Moreover, $S(f(Z)) \subset f(Z)$. □

The next theorem is deal with the existence of a compactly invariant subspace in the dual space of a normed space.

Theorem 2.1. *If Y is a compactly invariant subspace in X, then $Y^{\perp}$ is a compactly invariant subspace in X^*, where $Y^{\perp} = \{f \in X^* : \ f(y) = 0 \ \forall y \in Y\}$.*

Proof. By definition, there exist a nonzero compact operator T on X such that $T(Y) \subset Y$. It is well-known and easy to show that the linear operator $T^* : X^* \to X^*$ defined by $T^*(f) = f \circ T$ is compact [5]. Also, it can be seen easily that $T^*(Y^{\perp}) \subset Y^{\perp}$, as desired. □

Corollary 2.1. *If X has a compactly invariant subspace, then X^* has also.*

The previous corollary can be generalized for finite-dimensional spaces as follow:

Theorem 2.2. *Let X be a finite-dimensional space. Then $X^{(n)}$, nth dual of X, has a compactly invariant subspace, for every $n \in \mathbb{N}$.*

Proof. X has a compactly invariant subspace [4]. Since X is reflexive, so the canonical embedding $J : X \to X^{**}$ is surjective and so by Proposition 2.1, X^{**} has a compactly invariant subspace. In exactly the same way, since X^* has a compactly invariant subspace (Corollary 2.1), so X^{***} has also. By continuing the process, we get the desired conclusion. □

A natural question which arises here is that "is the intersection of two compactly invariant subspace, compactly invariant"?
This question inspire us to define the following notion.

Definition 2.3. *A pair (Y, Z) of subspaces of a normed space X is called compactly coinvariant, when there is a nonzero compact operator T on X such that $T(Y) \subset Y$ and $T(Z) \subset Z$.*

It is clear that each of the component of a pair (Y, Z) is compactly invariant.

Remark 2.1. *If Y is a compactly invariant subspace in X, then $(Y, \{0\})$ and (Y, X) are compactly coinvariant pairs. These pairs are called trivial. Note that if (Y, Z) is a compactly coinvariant pair, then $Y \cap Z$ is a compactly invariant subspace and so this can be a partial answer to the question which presented before of Definition 2.3.*

Proposition 2.2. *Let X has a compactly coinvariant pair, then X^* has also.*

Proof. Suppose that (Y, Z) is a compactly coinvariant pair of X. It can be seen easily that $(Y^{\perp}, Z^{\perp})$ is a compactly coinvariant pair of X^*. □

Corollary 2.2. *Let X be a reflexive space which has a compactly coinvariant pair. Then $X^{(n)}$ has a compactly coinvariant pair.*

Proof. Is similar to the proof of Corollary 2.1. □

Finally, we give a sufficient conditions for a normed space X which has a compactly coinvariant pair.

Theorem 2.3. *Let X be a normed space with a compactly invariant and complemented subspace Y such that $Y \cap Z = \{0\}$, for every subspace Z in X. Then X has a compactly coinvariant pair.*

Proof. Let $X = Y \oplus W$, where W is a subspace of X. There exist a nonzero operator T on X such that $T(Y) \subset Y$. On the other hand,

$$T(Y) \oplus T(W) \subset Y \oplus T(W)$$

Now since $T(W) \cap Y = \{0\}$, then $T(W) \subset W$. □

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General Solution and Generalized Ulam - Hyers Stability of A Additive Functional Equation Originating From N Observations of An Arithmetic Mean In Banach Spaces Using Various Substitutions In Two Different Approaches

M. Arunkumar[a,*], E. Sathya[b] and S. Ramamoorthi[c]

[a,b]*Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.*

[c]*Department of Mathematics, Arunai Engineering College, Tiruvannamalai - 606 604, TamilNadu, India.*

Abstract

In this paper, we introduce and investigate the general solution and generalized Ulam- Hyers stability of a additive functional equation

$$f\left(\frac{\sum\limits_{k=1}^{N} x_k}{N}\right) = \frac{1}{N}\sum_{k=1}^{N} f(x_k)$$

originating from N observations of an arithmetic mean in Banach spaces using various substitutions in two different approaches with $N \geq 2$.

Keywords: Arithmetic mean, additive functional equation, Generalized Hyers-Ulam stability, fixed point.

2010 MSC: 39B52, 32B72, 32B82.

1 Introduction

In [39], Ulam proposed the general Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In [19], Hyers gave the first affirmative answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution [4, 17, 28, 31].

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function.

*Corresponding author.
E-mail addresses: annarun2002@yahoo.co.in (M. Arunkumar), sathya24mathematics@gmail.com(E. Sathya), ramsdmaths@yahoo.com (S. Ramamoorthi).

The second famous Jensen functional equation is

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}\left(f(x) + f(y)\right) \tag{1.2}$$

its solution and stability was investigated by Jensen [21], Aczel [2], Aczel et.al., [3].

The Jensen functional equation (1.2) makes sense in algebraic systems that are 2-divisible (that is, division by 2 is permissible), by replacing x by $x+y$ and y by $x-y$, (1.2) goes over to

$$f(x+y) + f(x-y) = 2f(x), \tag{1.3}$$

which in a way eliminates this problem and makes sense in algebraic systems that need not be 2-divisible. The equations (1.2) and (1.3) are equivalent in 2-divisible systems. Both equations can be solved by relating them to the additive equation (1.1) (see P.K.Sahoo, Pl.Palaniappan. [37]).

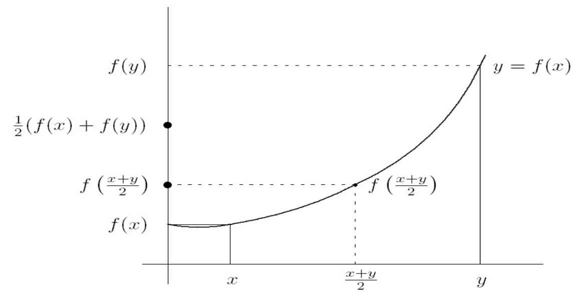

Fig: 1.1 Geometrical Interpretation of Functional Equation (1.2)

The solution and stability of various additive functional equation in various normed spaces were introduced and discussed in [5–11, 15, 26, 35] and reference cited there in.

Definition 1.1. ***Arithmetic Mean (A.M.):*** *Arithmetic mean is the total of all the items divided by their total number of items*

$$A.M. = \frac{X_1 + X_2 + \cdots + X_N}{N}.$$

In this paper, the authors introduce and investigate the general solution and generalized Ulam- Hyers stability of a additive functional equation

$$f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) = \frac{1}{N}\sum_{k=1}^{N} f(x_k) \tag{1.4}$$

originating from N observations of an arithmetic mean in Banach spaces using various substitutions in two different approaches with $N \geq 2$. In particular when $N = 2$, we arrive the Jensen functional equation (1.2).

2 General Solution of The Functional Equation (1.4)

In this section, we present the general solution of the functional equation (1.4). For this assume U and V be real vector spaces.

Lemma 2.1. *If $f : U \to V$ be a mapping satisfying (1.1) if and only if $f : U \to V$ satisfies (1.2) for all $x, y \in U$.*

Lemma 2.2. *If $f : U \to V$ be a mapping satisfying (1.2) if and only if $f : U \to V$ satisfies (1.3) for all $x, y \in U$.*

Lemma 2.3. *If $f : U \to V$ be a mapping satisfying (1.1) if and only if $f : U \to V$ satisfies (1.3) for all $x, y \in U$.*

Remark 2.1. *If $f : U \to V$ be a mapping satisfying (1.1), (1.2) and (1.3) for all $x, y \in U$ then they are equivalent*

Theorem 2.1. *If $f : U \to V$ be a mapping satisfying (1.1) for all $x, y \in U$ if and only if $f : U \to V$ satisfies (1.4) for all $x_1, \cdots, x_N \in U$.*

Theorem 2.2. *If $f : U \to V$ be a mapping satisfying (1.2) for all $x, y \in U$ if and only if $f : U \to V$ satisfies (1.4) for all $x_1, \cdots, x_N \in U$.*

Theorem 2.3. *If $f : U \to V$ be a mapping satisfying (1.3) for all $x, y \in U$ if and only if $f : U \to V$ satisfies (1.4) for all $x_1, \cdots, x_N \in U$.*

Remark 2.2. *If $f : U \to V$ be a mapping satisfying (1.1), (1.2), (1.3) and (1.4) then they are equivalent.*

3 Generalized Ulam - Hyers Stability of (1.4) In Banach Space : Direct Method

In this section, we test the generalized Ulam - Hyers stability of the functional equation (1.4) in Banach space. To prove the stability results throughout this section, we assume Y be a Normed space and Z be a Banach space.

3.1 Substitution - 1: $N \geq 2$ N is an Integer

Theorem 3.1. *Let $\lambda : Y^N \to [0, \infty)$ and $f : Y \to Z$ are functions fulfilling the inequalities*

$$\lim_{a\to\infty} \frac{\lambda(N^{ad}x_1, N^{ad}x_2 \cdots, N^{ad}x_{N-1}, N^{ad}x_N)}{N^{ad}} = 0 \tag{3.1}$$

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k) \right\| \leq \lambda(x_1, x_2 \cdots, x_{N-1}, x_N) \tag{3.2}$$

for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function fulfilling the functional equation (1.4) and

$$\|f(x) - A_A(x)\| \leq \sum_{b=\frac{1-d}{2}}^{\infty} \frac{\Lambda_A(N^{bd}x)}{N^{bd}} \tag{3.3}$$

for all $x \in Y$, where $A_A(x)$ and $\Lambda_A(N^b x)$ is defined as

$$A_A(x) = \lim_{a\to\infty} \frac{f(N^{ad}x)}{N^{ad}} \tag{3.4}$$

and

$$\Lambda_A(N^{bd}x) = \lambda(N^{(b+1)d}x, 0 \cdots, 0, 0) \tag{3.5}$$

respectively, for all $x \in Y$ with $d = \pm 1$.

Proof. Replacing $(x_1, x_2, \cdots, x_N)$ by $(x, 0, \cdots, 0)$ in (3.2), we obtain

$$\left\| f\left(\frac{x}{N}\right) - \frac{1}{N}f(x) \right\| \leq \lambda(x, 0 \cdots, 0, 0) \tag{3.6}$$

for all $x \in Y$. Setting x by Nx in (3.6), we get

$$\left\| f(x) - \frac{1}{N}f(Nx) \right\| \leq \lambda(Nx, 0 \cdots, 0, 0) \tag{3.7}$$

for all $x \in Y$. Define $\Lambda_A(x) = \lambda(Nx, 0 \cdots, 0, 0)$ in (3.7), we have

$$\left\| f(x) - \frac{1}{N}f(Nx) \right\| \leq \Lambda_A(x) \tag{3.8}$$

for all $x \in Y$. Replacing x by Nx and divided by N in (3.8), we arrive

$$\left\| \frac{1}{N}f(Nx) - \frac{1}{N^2}f(N^2x) \right\| \leq \frac{\Lambda_A(Nx)}{N} \tag{3.9}$$

for all $x \in Y$. Combining (3.8) and (3.9), we reach

$$\left\| f(x) - \frac{1}{N^2}f(N^2x) \right\| \leq \Lambda_A(x) + \frac{\Lambda_A(Nx)}{N} \tag{3.10}$$

for all $x \in Y$. Generalizing, one can get

$$\left\| f(x) - \frac{1}{N^a}f(N^a x) \right\| \leq \sum_{b=0}^{a-1} \frac{\Lambda_A(N^b x)}{N^b} \tag{3.11}$$

for all $x \in Y$. If we put x by $N^c x$ and divided by N^c in (3.11), we obtain

$$\left\| \frac{1}{N^c} f(N^c x) - \frac{1}{N^{a+c}} f(N^{a+c} x) \right\| \rightarrow 0 \quad as \quad c \rightarrow \infty \tag{3.12}$$

for all $x \in Y$. Thus $\left\{ \frac{1}{N^a} f(N^a x) \right\}$ is a Cauchy sequence in Z. Since Z is complete, define a mapping such that

$$A_A(x) = \lim_{a \to \infty} \frac{f(N^a x)}{N^a}$$

for all $x \in Y$. Letting a approaches to infinity in (3.11), we get (3.3) as desired. Letting $(x_1, \cdots, x_N) = (N^a x_1, \cdots, N^a x_N)$ and divided by N^a in (3.2) and using the definition the $A_A(x)$, we see that $A_A(x)$ satisfies the functional equation (1.4) for all $x_1, \cdots, x_n \in Y$. To show $A_A(x)$ is unique, let $A_B(x)$ be another mapping satisfying the functional equation (1.4) and (3.3) for all $x \in Y$. Now $\|A_A(x) - A_B(x)\| = \frac{1}{N^c} \|A_A(N^c x) - A_B(N^c x)\| \leq \frac{1}{N^c} \left\{ ||A_A(N^c x) - f(N^c x)|| + ||f(N^c x) - A_B(N^c x)|| \right\} \leq 2 \sum_{b=0}^{\infty} \frac{\Lambda_A(N^{b+c} x)}{N^{b+c}} \rightarrow 0 \ as \ c \rightarrow \infty$ for all $x \in Y$. Thus $A_A(x)$ is unique. Hence the theorem is true for $d = 1$.

Replacing x by $\frac{x}{N}$ and multiply by N in (3.8), we arrive

$$\left\| N f\left(\frac{x}{N} \right) - f(x) \right\| \leq N \Lambda_A \left(\frac{x}{N} \right) \tag{3.13}$$

for all $x \in Y$. The rest of the proof is similar to that of previous case. Hence the proof is complete. □

From Theorem 3.1, we prove the following corollaries concerning the Hyers- Ulam, Hyers - Ulam - TRassias and Hyers - Ulam - JRassias stabilities of the functional equation (1.4).

Corollary 3.1. *Let $f : Y \rightarrow Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \tag{3.14}$$

where $\beta > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_A(x)\| \leq \frac{\beta N}{|N-1|} \tag{3.15}$$

for all $x \in Y$.

Corollary 3.2. *Let $f : Y \rightarrow Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \sum_{k=1}^{N} ||x_k||^t \tag{3.16}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_A(x)\| \leq \frac{\beta ||x||^t N^{t+1}}{|N - N^t|} \tag{3.17}$$

where $t \neq 1$ for all $x \in Y$.

Corollary 3.3. *Let $f : Y \rightarrow Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \left\{ \prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt} \right\} \tag{3.18}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_A(x)\| \leq \frac{\beta ||x||^{Nt} N^{Nt+1}}{|N - N^{Nt}|} \tag{3.19}$$

where $Nt \neq 1$ for all $x \in Y$.

3.2 Substitution - 2: $N \geq 2$, N is Odd Integer

Theorem 3.2. *Let $\lambda : Y^N \to [0, \infty)$ and $f : Y \to Z$ are functions fulfilling the inequalities*

$$\lim_{a\to\infty} \frac{\lambda(N^{ad}x_1, N^{ad}x_2 \cdots, N^{ad}x_{N-1}, N^{ad}x_N)}{N^{ad}} = 0 \tag{3.20}$$

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k) \right\| \leq \lambda(x_1, x_2 \cdots, x_{N-1}, x_N) \tag{3.21}$$

for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function fulfilling the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \sum_{b=\frac{1-d}{2}}^{\infty} \frac{\Lambda_O(N^{bd}x)}{N^{bd}} \tag{3.22}$$

for all $x \in Y$, where $A_O(x)$ and $\Lambda_O(N^b x)$ is defined as

$$A_O(x) = \lim_{a\to\infty} \frac{f(N^{ad}x)}{N^{ad}} \tag{3.23}$$

and

$$\Lambda_O(N^{bd}x) = \lambda(N^{(b+1)d}x, \underbrace{-N^{bd}x, \cdots, -N^{bd}x}_{\frac{N-1}{2}}, \underbrace{N^{bd}x, \cdots, N^{bd}x}_{\frac{N-1}{2}}) \tag{3.24}$$

respectively, for all $x \in Y$ with $d = \pm 1$.

Proof. Replacing $(x_1, x_2, \cdots, x_N)$ by $(x, \underbrace{-x, \cdots, -x}_{\frac{N-1}{2}}, \underbrace{x, \cdots, x}_{\frac{N-1}{2}})$ in (3.21), we obtain

$$\left\| f\left(\frac{x}{N}\right) - \frac{1}{N}f(x) \right\| \leq \lambda(x, \underbrace{-x, \cdots, -x}_{\frac{N-1}{2}}, \underbrace{x, \cdots, x}_{\frac{N-1}{2}}) \tag{3.25}$$

for all $x \in Y$. Setting x by Nx in (3.25), we get

$$\left\| f(x) - \frac{1}{N}f(Nx) \right\| \leq \lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-1}{2}}, \underbrace{x, \cdots, x}_{\frac{N-1}{2}}) \tag{3.26}$$

for all $x \in Y$. Define $\Lambda_O(x) = \lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-1}{2}}, \underbrace{x, \cdots, x}_{\frac{N-1}{2}})$ in (3.26), we have

$$\left\| f(x) - \frac{1}{N}f(Nx) \right\| \leq \Lambda_O(x) \tag{3.27}$$

for all $x \in Y$. Replacing x by Nx and divided by N in (3.27), we arrive

$$\left\| \frac{1}{N}f(Nx) - \frac{1}{N^2}f(N^2x) \right\| \leq \frac{\Lambda_O(Nx)}{N} \tag{3.28}$$

for all $x \in Y$. Combining (3.27) and (3.28), we reach

$$\left\| f(x) - \frac{1}{N^2}f(N^2x) \right\| \leq \Lambda_O(x) + \frac{\Lambda_O(Nx)}{N} \tag{3.29}$$

for all $x \in Y$. Generalizing, one can get

$$\left\| f(x) - \frac{1}{N^a}f(N^a x) \right\| \leq \sum_{b=0}^{a-1} \frac{\Lambda_O(N^b x)}{N^b} \tag{3.30}$$

for all $x \in Y$. If we put x by $N^c x$ and divided by N^c in (3.30), we obtain

$$\left\| \frac{1}{N^c} f(N^c x) - \frac{1}{N^{a+c}} f(N^{a+c} x) \right\| \rightarrow \quad 0 \quad as \quad c \quad \rightarrow \quad \infty \tag{3.31}$$

for all $x \in Y$. Thus $\left\{ \frac{1}{N^a} f(N^a x) \right\}$ is a Cauchy sequence in Z. Since Z is complete, define a mapping such that

$$A_O(x) = \lim_{a \to \infty} \frac{f(N^a x)}{N^a}$$

for all $x \in Y$. The rest of proof is similar to that of Theorem 3.1. □

From Theorem 3.2, we prove the following corollaries concerning the Hyers- Ulam, Hyers - Ulam - TRassias and Hyers - Ulam - JRassias stabilities of the functional equation (1.4).

Corollary 3.4. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^N x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^N f(x_k) \right\| \leq \beta \tag{3.32}$$

where $\beta > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \frac{\beta N}{|N-1|} \tag{3.33}$$

for all $x \in Y$.

Corollary 3.5. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^N x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^N f(x_k) \right\| \leq \beta \sum_{k=1}^N ||x_k||^t \tag{3.34}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \frac{\beta ||x||^t (N^t + N + 1)}{|N - N^t|} \tag{3.35}$$

where $t \neq 1$ for all $x \in Y$.

Corollary 3.6. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^N x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^N f(x_k) \right\| \leq \beta \prod_{k=1}^N ||x_k||^t \tag{3.36}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \frac{\beta ||x||^{Nt} N^{t+1}}{|N - N^{Nt}|} \tag{3.37}$$

where $Nt \neq 1$ for all $x \in Y$.

Corollary 3.7. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^N x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^N f(x_k) \right\| \leq \beta \left\{ \prod_{k=1}^N ||x_k||^t + \sum_{k=1}^N ||x_k||^{Nt} \right\} \tag{3.38}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \frac{\beta ||x||^{Nt} (N^{t+1} + N^{Nt} + N + 1)}{|N - N^{Nt}|} \tag{3.39}$$

where $Nt \neq 1$ for all $x \in Y$.

3.3 Substitution - 3: $N \geq 2$, N is Even Integer

Theorem 3.3. *Let $\lambda : Y^N \to [0, \infty)$ and $f : Y \to Z$ are functions fulfilling the inequalities*

$$\lim_{a\to\infty} \frac{\lambda(N^{ad}x_1, N^{ad}x_2 \cdots, N^{ad}x_{N-1}, N^{ad}x_N)}{N^{ad}} = 0 \tag{3.40}$$

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right\| \leq \lambda(x_1, x_2 \cdots, x_{N-1}, x_N) \tag{3.41}$$

for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function fulfilling the functional equation (1.4) and

$$\|f(x) - A_E(x)\| \leq \sum_{b=\frac{1-d}{2}}^{\infty} \frac{\Lambda_E(N^{bd}x)}{N^{bd}} \tag{3.42}$$

for all $x \in Y$, where $A_E(x)$ and $\Lambda_E(N^b x)$ is defined as

$$A_E(x) = \lim_{a\to\infty} \frac{f(N^{ad}x)}{N^{ad}} \tag{3.43}$$

and

$$\Lambda_E(N^{bd}x) = \lambda(N^{(b+1)d}x, \underbrace{-N^{bd}x, \cdots, -N^{bd}x}_{\frac{N-2}{2}}, \underbrace{N^{bd}x, \cdots, N^{bd}x}_{\frac{N-2}{2}}, 0) \tag{3.44}$$

respectively, for all $x \in Y$ with $d = \pm 1$.

Proof. Replacing $(x_1, x_2, \cdots, x_N)$ by $(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-2}{2}}, \underbrace{x, \cdots, x}_{\frac{N-2}{2}}, 0)$ in (3.41), we obtain

$$\left\| f\left(\frac{x}{N}\right) - \frac{1}{N}f(x)\right\| \leq \lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-2}{2}}, \underbrace{x, \cdots, x}_{\frac{N-2}{2}}, 0) \tag{3.45}$$

for all $x \in Y$. Setting x by Nx in (3.45), we get

$$\left\| f(x) - \frac{1}{N}f(Nx)\right\| \leq \lambda\lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-2}{2}}, \underbrace{x, \cdots, x}_{\frac{N-2}{2}}, 0) \tag{3.46}$$

for all $x \in Y$. Define $\Lambda_E(x) = \lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-2}{2}}, \underbrace{x, \cdots, x}_{\frac{N-2}{2}}, 0)$ in (3.46), we have

$$\left\| f(x) - \frac{1}{N}f(Nx)\right\| \leq \Lambda_E(x) \tag{3.47}$$

for all $x \in Y$. Replacing x by Nx and divided by N in (3.47), we arrive

$$\left\| \frac{1}{N}f(Nx) - \frac{1}{N^2}f(N^2x)\right\| \leq \frac{\Lambda_E(Nx)}{N} \tag{3.48}$$

for all $x \in Y$. Combining (3.47) and (3.48), we reach

$$\left\| f(x) - \frac{1}{N^2}f(N^2x)\right\| \leq \Lambda_E(x) + \frac{\Lambda_E(Nx)}{N} \tag{3.49}$$

for all $x \in Y$. Generalizing, one can get

$$\left\| f(x) - \frac{1}{N^a}f(N^a x)\right\| \leq \sum_{b=0}^{a-1} \frac{\Lambda_E(N^b x)}{N^b} \tag{3.50}$$

for all $x \in Y$. If we put x by $N^c x$ and divided by N^c in (3.50), we obtain

$$\left\| \frac{1}{N^c} f(N^c x) - \frac{1}{N^{a+c}} f(N^{a+c} x) \right\| \rightarrow \quad 0 \quad as \quad c \quad \rightarrow \quad \infty \tag{3.51}$$

for all $x \in Y$. Thus $\left\{ \frac{1}{N^a} f(N^a x) \right\}$ is a Cauchy sequence in Z. Since Z is complete, define a mapping such that

$$A_E(x) = \lim_{a \to \infty} \frac{f(N^a x)}{N^a}$$

for all $x \in Y$. The rest of proof is similar to that of Theorem 3.1. □

From Theorem 3.3, we prove the following corollaries concerning the Hyers- Ulam, Hyers - Ulam - TRassias and Hyers - Ulam - JRassias stabilities of the functional equation (1.4).

Corollary 3.8. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \tag{3.52}$$

where $\beta > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_E(x)\| \leq \frac{\beta}{|N-1|} \tag{3.53}$$

for all $x \in Y$.

Corollary 3.9. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \sum_{k=1}^{N} ||x_k||^t \tag{3.54}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_E(x)\| \leq \frac{\beta ||x||^t (N^t + N - 2)}{|N - N^t|} \tag{3.55}$$

where $t \neq 1$ for all $x \in Y$.

Corollary 3.10. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \left\{ \prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt} \right\} \tag{3.56}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_E(x)\| \leq \frac{\beta ||x||^{Nt} (N^{Nt} + N - 2)}{|N - N^{Nt}|} \tag{3.57}$$

where $Nt \neq 1$ for all $x \in Y$.

4 Generalized Ulam - Hyers Stability of (1.4) In Banach Space : Fixed Point Method

In this section, the generalized Ulam - Hyers stability of the functional equation (1.4) is proved using fixed point method.

Now, we present the following theorem due to B. Margolis and J.B. Diaz [25] for the fixed point theory.

Theorem 4.1. *[25] Suppose that for a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty, \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that the properties hold:

(FP1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T;

(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;

(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using Theorem 4.1, we obtain the Hyers - Ulam stability of (1.4). To prove the stability results throughout this section, we assume $\mathcal{Y}$ be a Normed space and $\mathcal{Z}$ be a Banach space.

4.1 Substitution - 1: $N \geq 2$ N is an Integer

Theorem 4.2. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\lambda : \mathcal{Y}^N \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\alpha_i^n} \lambda(\alpha_i^n x_1, \cdots, \alpha_i^n x_N) = 0 \tag{4.1}$$

where

$$\alpha_i = \begin{cases} N & if \quad i = 0, \\ \frac{1}{N} & if \quad i = 1 \end{cases} \tag{4.2}$$

such that the functional inequality

$$\left\| f\left(\frac{\sum_{k=1}^N x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^N f(x_k) \right\| \leq \lambda(x, y) \tag{4.3}$$

holds for all $x_1, \cdots, x_N \in \mathcal{Y}$. Assume that there exists $L = L(i)$ such that the function

$$x \to T(x) = \Lambda_A\left(\frac{x}{N}\right) \tag{4.4}$$

with the property

$$\frac{1}{\alpha_i} T(\alpha_i x) = L\, T(x) \tag{4.5}$$

for all $x \in \mathcal{Y}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ satisfying the functional equation (1.4) and

$$\| f(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right) T(x) \tag{4.6}$$

for all $x \in \mathcal{Y}$.

Proof. Consider the set

$$\Psi = \{h/h : \mathcal{Y} \to \mathcal{Z},\ h(0) = 0\}$$

and introduce the generalized metric on Ψ,

$$\inf\{\rho \in (0, \infty) : \| h(x) - g(x) \| \leq \rho\, T(x), x \in \mathcal{Y}\}. \tag{4.7}$$

It is easy to see that (4.7) is complete with respect to the defined metric. Define $J : \Psi \to \Psi$ by

$$Jh(x) = \frac{1}{\alpha_i} h(\alpha_i x),$$

for all $x \in \mathcal{Y}$. Now, from (4.7) and $h, g \in \Psi$, we arrive

$$\inf\{L\rho \in (0, \infty) : \| Jh(x) - Jg(x) \| \leq L\rho T(x), x \in \mathcal{Y}\}.$$

This implies J is a strictly contractive mapping on Ψ with Lipschitz constant L (see [25]). It follows from (4.7),(4.5) and (3.8) for the case $i = 1$, we reach

$$\begin{aligned} &\inf\left\{1 \in (0,\infty) : \left\| f(x) - \frac{f(Nx)}{N} \right\| \leq \Lambda_A(x), x \in \mathcal{Y} \right\} \quad or \\ &\inf\left\{1 \in (0,\infty) : \| f(x) - Jf(x) \| \leq L\, T(x), x \in \mathcal{Y} \right\} \quad or \\ &\inf\left\{L^0 \in (0,\infty) : \| f(x) - Jf(x) \| \leq L\, T(x), x \in \mathcal{Y} \right\} \quad or \\ &\inf\left\{L^{1-1} \in (0,\infty) : \| f(x) - Jf(x) \| \leq L\, T(x), x \in \mathcal{Y} \right\}. \end{aligned} \tag{4.8}$$

Again replacing $x = \frac{x}{N}$ in (3.8) and it follows from (4.7), (4.5) for the case $i = 0$, we get

$$\begin{aligned} &\inf\left\{1 \in (0,\infty) : \left\| Nf\left(\frac{x}{N}\right) - f(x) \right\| \leq N\Lambda_A\left(\frac{x}{N}\right), x \in \mathcal{Y} \right\} \quad or \\ &\inf\left\{1 \in (0,\infty) : \| Jf(x) - f(x) \| \leq LT(x), x \in \mathcal{Y} \right\} \quad or \\ &\inf\left\{L^1 \in (0,\infty) : \| Jf(x) - f(x) \| \leq LT(x), x \in \mathcal{Y} \right\} \quad or \\ &\inf\left\{L^{1-0} \in (0,\infty) : \| Jf(x) - f(x) \| \leq LT(x), x \in \mathcal{Y} \right\}. \end{aligned} \tag{4.9}$$

Thus, from (4.8) and (4.9), we arrive

$$\inf\left\{L^{1-i} \in (0,\infty) : \| f(x) - Jf(x) \| \leq L^{1-i}T(x), x \in \mathcal{Y} \right\}. \tag{4.10}$$

Hence property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{A}$ of J in Ψ such that

$$\mathcal{A}(x) = \lim_{n\to\infty} \frac{1}{\alpha_i^n} f(\alpha_i^n x) \tag{4.11}$$

for all $x \in \mathcal{Y}$. In order to show that $\mathcal{A}$ satisfies (1.4), replacing $(x_1, \cdots, x_N)$ by $(\alpha_i^n x_1, \cdots, \alpha_i^n x_n)$ and dividing by α_i^n in (4.3), we have

$$\|\mathcal{A}(x_1, \cdots, x_N)\| = \lim_{n\to\infty} \frac{1}{\alpha_i^n} \|f(\alpha_i^n x_1, \cdots, \alpha_i^n x_n)\| \leq \lim_{n\to\infty} \frac{1}{\alpha_i^n} \lambda\left(\alpha_i^n x, \alpha_i^n y\right) = 0$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$ and so the mapping $\mathcal{A}$ is additive. i.e., $\mathcal{A}$ satisfies the functional equation (1.4). By property (FP3), $\mathcal{A}$ is the unique fixed point of J in the set

$$\Delta = \{\mathcal{A} \in \Psi : d(f, \mathcal{A}) < \infty\},$$

$\mathcal{A}$ is the unique function such that

$$\inf\left\{\rho \in (0,\infty) : \| f(x) - \mathcal{A}(x) \| \leq \rho T(x), x \in \mathcal{Y} \right\}.$$

Finally by property (FP4), we obtain

$$\| f(x) - \mathcal{A}(x) \| \leq \| f(x) - Jf(x) \|,$$

implying

$$\| f(x) - \mathcal{A}(x) \| \leq \frac{L^{1-i}}{1-L},$$

which yields

$$\inf\left\{\frac{L^{1-i}}{1-L} \in (0,\infty) : \| f(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right) T(x), x \in \mathcal{Y} \right\}.$$

This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.2 concerning the stability of (1.4).

Corollary 4.11. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping. If there exist real numbers β and t such that*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k) \right\| \leq \begin{cases} \beta, & \\ \beta \sum_{k=1}^{N} ||x_k||^t & t \neq 1; \\ \beta \left\{ \prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt} \right\} & Nt \neq 1; \end{cases} \tag{4.12}$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(x) - \mathcal{A}(x)\| \leq \begin{cases} \dfrac{\beta}{|N-1|}, \\ \dfrac{\beta\, N^t}{|N-N^t|}, \\ \dfrac{\beta\, N^{Nt}}{|N-N^{Nt}|} \end{cases} \tag{4.13}$$

for all $x \in \mathcal{Y}$.

Proof. Let

$$\lambda(x_1, \cdots, x_n) = \begin{cases} \beta, \\ \beta \sum_{k=1}^{N} ||x_k||^t \\ \beta \left\{ \prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt} \right\} \end{cases} \tag{4.14}$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$. Now

$$\frac{1}{\alpha_i^n} \lambda(\alpha_i^n x_1, \alpha_i^n x_n) = \begin{cases} \dfrac{\beta}{\alpha_i^n}, \\ \dfrac{\beta}{\alpha_i^n} \displaystyle\sum_{k=1}^{N} ||\alpha_i^n x_k||^t, \\ \dfrac{\beta}{\alpha_i^n} \left\{ \displaystyle\prod_{k=1}^{N} ||\alpha_i^n x_k||^t + \sum_{k=1}^{N} ||\alpha_i^n x_k||^{Nt} \right\} \end{cases} = \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (4.1) holds. It follows from (4.4), (4.5) and (4.14), we have

$$T(x) = \Lambda_A\left(\frac{x}{N}\right) = \lambda(x, 0 \cdots, 0, 0) = \begin{cases} \beta \\ \beta||x||^t \\ \beta||x||^{Nt} \end{cases} \tag{4.15}$$

and

$$\frac{1}{\alpha_i} T(\alpha_i x) = \frac{1}{\alpha_i} \Lambda_A\left(\frac{\alpha_i x}{N}\right) = \lambda(\alpha_i x, 0 \cdots, 0, 0) = \begin{cases} \alpha_i^{-1} \beta, \\ \alpha_i^{t-1} \beta ||x||^t, \\ \alpha_i^{Nt-1} \beta ||x||^{Nt} \end{cases} = LT(x) \tag{4.16}$$

for all $x \in \mathcal{Y}$. Hence, in view of (4.16) the inequality (4.6) holds for

(i). $L = \alpha_i^{-1}$ if $i = 0$ and $L = \frac{1}{\alpha_i^{-1}}$ if $i = 1$,

(ii). $L = \alpha_i^{t-1}$ for $t < 1$ if $i = 0$ and $L = \frac{1}{\alpha_i^{t-1}}$ for $t > 1$ if $i = 1$,

(iii). $L = \alpha_i^{Nt-1}$ for $Nt > 1$ if $i = 0$ and $L = \frac{1}{\alpha_i^{Nt-1}}$ for $Nt > 1$ if $i = 1$.

Hence the proof is complete. □

The proof of the following theorems and corollaries of Sections 4.2 and 4.3 are similar tracing to that of Theorem 4.2 and Corollary 4.11. Hence the details of the proof are omitted.

4.2 Substitution - 2: $N \geq 2$, N is Odd Integer

Theorem 4.3. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\lambda : \mathcal{Y}^N \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\alpha_i^n} \lambda(\alpha_i^n x_1, \cdots, \alpha_i^n x_N) = 0 \tag{4.17}$$

where

$$\alpha_i = \begin{cases} N & if \quad i = 0, \\ \frac{1}{N} & if \quad i = 1 \end{cases} \tag{4.18}$$

such that the functional inequality

$$\left\| f\left(\frac{\sum_{k=1}^N x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^N f(x_k) \right\| \leq \lambda(x, y) \tag{4.19}$$

holds for all $x_1, \cdots, x_N \in \mathcal{Y}$. Assume that there exists $L = L(i)$ such that the function

$$x \to T(x) = \Lambda_O\left(\frac{x}{N}\right) \tag{4.20}$$

with the property

$$\frac{1}{\alpha_i} T(\alpha_i x) = L\, T(x) \tag{4.21}$$

for all $x \in \mathcal{Y}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ satisfying the functional equation (1.4) and

$$\| f(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right) T(x) \tag{4.22}$$

for all $x \in \mathcal{Y}$.

Corollary 4.12. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping. If there exist real numbers β and t such that*

$$\left\| \left\| f\left(\frac{\sum_{k=1}^N x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^N f(x_k) \right\| \right\| \leq \begin{cases} \beta, & \\ \beta \sum_{k=1}^N ||x_k||^t & t \neq 1; \\ \beta \prod_{k=1}^N ||x_k||^t & Nt \neq 1; \\ \beta \left\{ \prod_{k=1}^N ||x_k||^t + \sum_{k=1}^N ||x_k||^{Nt} \right\} & Nt \neq 1; \end{cases} \tag{4.23}$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(x) - \mathcal{A}(x)\| \leq \begin{cases} \dfrac{\beta}{|N-1|}, \\ \dfrac{\beta\, N^{t+1}}{|N - N^t|}, \\ \dfrac{\beta\, N^{Nt}}{|N - N^{Nt}|} \\ \dfrac{\beta\, (N^{2Nt}+1)}{|N - N^{Nt}|} \end{cases} \tag{4.24}$$

for all $x \in \mathcal{Y}$.

4.3 Substitution - 3: $N \geq 2$, N is Even Integer

Theorem 4.4. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\lambda : \mathcal{Y}^N \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\alpha_i^n} \lambda(\alpha_i^n x_1, \cdots, \alpha_i^n x_N) = 0 \tag{4.25}$$

where

$$\alpha_i = \begin{cases} N & if \quad i = 0, \\ \frac{1}{N} & if \quad i = 1 \end{cases} \tag{4.26}$$

such that the functional inequality

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right\| \leq \lambda(x,y) \tag{4.27}$$

holds for all $x_1, \cdots, x_N \in \mathcal{Y}$. *Assume that there exists* $L = L(i)$ *such that the function*

$$x \to T(x) = \Lambda_E\left(\frac{x}{N}\right) \tag{4.28}$$

with the property

$$\frac{1}{\alpha_i} T(\alpha_i x) = L\, T(x) \tag{4.29}$$

for all $x \in \mathcal{Y}$. *Then there exists a unique additive mapping* $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ *satisfying the functional equation* (1.4) *and*

$$\| f(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right) T(x) \tag{4.30}$$

for all $x \in \mathcal{Y}$.

Corollary 4.13. *Let* $f : \mathcal{Y} \to \mathcal{Z}$ *be a mapping. If there exist real numbers* β *and* t *such that*

$$\left\|\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right\|\right\| \leq \begin{cases} \beta, & \\ \beta \sum_{k=1}^{N} ||x_k||^t & t \neq 1; \\ \beta \left\{\prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt}\right\} & Nt \neq 1; \end{cases} \tag{4.31}$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$, *then there exists a unique additive function* $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ *such that*

$$\|f(x) - \mathcal{A}(x)\| \leq \begin{cases} \dfrac{\beta}{|N-1|}, \\ \dfrac{\beta\, N^t (N-1)}{|N - N^t|}, \\ \dfrac{\beta\, N^{Nt}(N-1)}{|N - N^{Nt}|} \end{cases} \tag{4.32}$$

for all $x \in \mathcal{Y}$.

5 Application

Consider the additive functional equation

$$f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) = \frac{1}{N}\sum_{k=1}^{N} f(x_k).$$

Since $f(x) = x$ is the solution of the above functional equation, we arrive

$$f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) = f\left(\frac{x_1 + x_2 + x_3 + \cdots\cdots + x_N}{N}\right) = \frac{x_1 + x_2 + x_3 + \cdots\cdots + x_N}{N}$$

This gives the N observations of an arithmetic mean.

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Second kind shifted Chebyshev polynomials and power series method for solving multi-order non-linear fractional differential equations

Amr M. S. Mahdy[1]* and Ali A. A. El-dahdouh[2]

[1,2]Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt.

Abstract

In this paper, we use shifted Chebyshev approximations with the second kind [25] and fractional power series method (FPSM) ([3], [8]) to solve the multi-order non-linear fractional differential equations. The fractional derivative is described in the Caputo sense. The properties of shifted Chebyshev polynomials with the second kind are utilized to reduce multi-order NFDEs. The system of non-linear of algebraic equations which solved by using Newton iteration method. We compared with FPSM. The results are compared with the traditional methods [23].

Keywords: Shifted Chebyshev polynomials with the second kind; Fractional power series method; Caputo derivative; Multi-order nonlinear fractional differential equations.

2010 MSC: 65N20, 41D15.

1 Introduction

Fractional differential equations have recently been applied in various area of engineering, science, finance, applied mathematics, bio-engineering and others ([1], [4]). However, many researchers remain unaware of this field using numerical different methods ([5], [9], [10], [11]-[14], [18], [20], [24], [26]).

The collocation methods in ([6], [7], [15], [23]) based on the Chebyshev polynomials for solving multi-term linear and nonlinear fractional differential equations subject to non-homogeneous initial conditions.

The organization of this paper is as follows. In the next section, we give the definitions of fractional derivatives in fractional calculus. In the section 3, we give the fractional power series method. In the section 4, we give some properties of Chebyshev polynomials of the second kind. In section 5, we procedure of solution for the multi-order NFDEs. In section 6, numerical simulation and comparison are given to clarify the method. Also a conclusion is given in section 7. Note that we have computed the numerical results using Matlab programming.

Now, we describe some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

2 Definitions of fractional derivative

Definition 2.1.

The Caputo fractional derivative operator D^{α} of order α is defined in the following form [19]

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0,$$

*Corresponding author.
E-mail address: amr_mahdy85@yahoo.com (Amr M. S. Mahdy), ali.abdelaziz010b@yahoo.com (Ali A. A. El-dahdouh).

where $m-1<\alpha<m,\ m\in\mathbb{N},\ x>0$.
Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^{\alpha}\left(\lambda f(x)+\mu g(x)\right)=\lambda D^{\alpha} f(x)+\mu D^{\alpha} g(x),$$

where λ and μ are constants.
For the Caputo's derivative we have [19]

$$D^{\alpha} C=0, \qquad \text{C is a constant}, \tag{2.1}$$

$$D^{\alpha} x^{n}=\begin{cases} 0, & \text{for} \quad n\in\mathbb{N}_0 \text{ and } n<\lceil\alpha\rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}, & \text{for} \quad n\in\mathbb{N}_0 \text{ and } n\geq\lceil\alpha\rceil. \end{cases} \tag{2.2}$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α. Also $\mathbb{N}_0=\{0,1,2,...\}$. Recall that for $\alpha\in\mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.
For more details on fractional derivatives definitions and its properties see ([16], [19], [21]).

The main goal in this article is concerned with the application of Chebyshev pseudo-spectral method for the second kind [25] and Power series method [3] to obtain the numerical solution of multi-order fractional differential equation of the form

$$D^{\alpha}y(x)=F\left(x,y(x),D^{\beta_1}y,...,D^{\beta_n}y\right), \tag{2.3}$$

with the following initial conditions

$$y^{(k)}(0)=y_k, \quad k=0,1,...,m, \tag{2.4}$$

where $m<\alpha\leq m+1$, $0<\beta_1<\beta_2<...<\beta_n<\alpha$ and D^{α} denotes Caputo fractional derivative of order α. It should be noted that F can be nonlinear in general.

The main idea of this work is to apply the Chebyshev collocation method for the second kind to discretize (2.3) to reduce multi-order NFDEs to a system of nonlinear of algebraic equations, and use Newton iteration method to solve the resulting system.
Chebyshev polynomials of the second kind are well known family of orthogonal polynomials on the interval $[-1,1]$ that have many applications ([2], [17], [22]). They are widely used because of their good properties in the approximation of functions [17]. However, with our best knowledge, very little work was done to adapt this polynomials to the solution of fractional differential equations.

3 Fractional power series method

In this section, we use fractional power series method (FPSM) ([3], [8]) to solve multi-order fractional differential equation. Compared to the above method, the FPSM is more simple and effective.

Definition 3.2. *([3], [8])*

A power series representation of the form

$$\sum_{n=0}^{\infty}c_n(t-t_0)^{n\alpha}=c_0+c_1(t-t_0)^{\alpha}+c_2(t-t_0)^{2\alpha}+\cdots, \tag{3.5}$$

where $0\leq m-1<\alpha\leq m,\ m\in N^{+}$ and $t\geq t_0$ is called a fractional power series (FPS) about t_0 where t is a variable and c_n are the coefficients of the series.

Theorem 3.1. *([3], [8])*

Suppose that the FPS $\sum_{n=0}^{\infty}c_n t^{n\alpha}$ has radius of convergence $R>0$. If $f(t)$ is a function defined by $f(t)=\sum_{n=0}^{\infty}c_n t^{n\alpha}$, $0\leq t<R$, then for $m-1<\alpha\leq m$ and $0<t\leq R$, we have

$$D^{\alpha}f(t)=\sum_{n=1}^{\infty}c_n\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha}. \tag{3.6}$$

4 Some properties of Chebyshev polynomials of the second kind

4.1 Chebyshev polynomials of the second kind

The Chebyshev polynomials $U_n(x)$of the second kind are orthogonal polynomials of degree n in x defined on the $[-1,1]$ ([17], [25])

$$U_n(x) = \frac{\sin(n+1)^{\theta}}{\sin\theta}$$

where $x = \cos\theta$ and $\theta \in [0,\pi]$.

The polynomials $U_n(x)$ are orthogonal on $[-1,1]$ with respect to the inner products

$$(U_n(x), U_m(x)) = \int_{-1}^{1} \sqrt{1-x^2}U_n(x)U_m(x)dx = \{^{0,\ n\neq m,}_{\frac{\pi}{2},\ n=m,} \tag{4.7}$$

where $\sqrt{1-x^2}$ is weight function.

$U_n(x)$ may be generated by using the recurrence relations

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2,3,...$$

with

$U_0(x) = 1. \quad U_1(x) = 2x.$

The analytical form of the Chebyshev polynomials of the second kind $U_n(x)$ of degree n is given by:

$$U_n(x) = \sum_{i=0}^{\lceil\frac{\pi}{2}\rceil}(-1)^i\binom{n-i}{i}(2x)^{n-2i}, \ \ n > 0$$

Using the properties of Gamma function the previous equation can be rewritten as:

$$U_n(x) = \sum_{i=0}^{\lceil\frac{\pi}{2}\rceil}(-1)^i 2^{n-2i}\frac{\Gamma(n-i+1)x^{n-2i}}{\Gamma(i+1)\Gamma(n-2i+1)}, \ \ n > 0, \tag{4.8}$$

where $\lceil\frac{\pi}{2}\rceil$ denotes the integral part of $n/2$.

4.2 Shifted Chebyshev polynomials of the second kind

In order to use these polynomials on the interval $x \in [0,1]$ ([17], [25]). We define the so called shifted Chebyshev polynomials of the second kind $U_n^*(x)$ by introducing the change variable $z = 2x-1$. This means that the shifted Chebyshev polynomials of the second kind defined as:

$$U_n^*(x) = U_n(2x-1),$$

also there are important relation between the shifted and second kind Chebyshev polynomials as follows:

$$2xU_{n-1}^*(x^2) = U_{2n-1}(x),$$

these polynomials are orthogonal on the support interval $[0,1]$ as the following inner product:

$$(U_n^*(x), U_m^*(x)) = \int_{0}^{1} \sqrt{x-x^2}U_n^*(x)U_m^*(x)dx = \{^{0,\ n\neq m,}_{\frac{\pi}{8},\ n=m,} \tag{4.9}$$

where $\sqrt{x-x^2}$ is weight function.

$U_n^*(x)$ may be generated by using the recurrence relations

$$U_n^*(x) = 2(2x-1)U_{n-1}^*(x) - U_{n-2}^*(x), \ \ n = 2,3,...$$

with start values

$U_0^*(x) = 1, \ \ U_1^*(x) = 4x-2.$

The analytical form of the shifted Chebyshev polynomials of the second kind $U_n^*(x)$ of degree n is given by

$$U_n^*(x) = \sum_{i=0}^{n}(-1)^i 2^{2n-2i}\frac{\Gamma(2n-i+2)x^{n-i}}{\Gamma(i+1)\Gamma(2n-2i+2)}, \ \ n > 0, \tag{4.10}$$

The function which may be appear in solution of the model problem (nonlinear multi-order fractional differential equations) can be written as series of $U^*(x)$.

Let $g(x)$ be a square integrable in $[0,1]$ it can be expressed in terms of the shifted Chebyshev polynomials of the second kind as follows:

$$g(x) = \sum_{i=0}^{\infty} a_i U_i^*(x), \tag{4.11}$$

where the coefficients $a_i, i = 0, 1, ...,$ are given by:

$$a_i = \frac{2}{\pi} \int_{-1}^{1} g(\frac{x+1}{2}) \sqrt{1-x^2} U_i(x) dx, \tag{4.12}$$

or

$$a_i = \frac{8}{\pi} \int_{0}^{1} g(x) \sqrt{x-x^2} U_i^*(x) dx, \tag{4.13}$$

In practice, only the first $(m+1)$ terms of shifted Chebyshev polynomials of the second kind are considered in the approximate case. Then we have:

$$g_m(x) = \sum_{i=0}^{m} a_i U_i^*(x), \tag{4.14}$$

The main approximate formula of the fractional derivative of $g_m(x)$ is given in the following theorem.

Theorem 4.2.

Let $g(x)$ be approximated by shifted Chebyshev polynomials of the second kind as (4.14) and also suppose $\alpha > 0$, then

$$D^{\alpha}(g_m(x)) = \sum_{i=\lceil \alpha \rceil}^{m} \sum_{k=0}^{i-\lceil \alpha \rceil} b_i\, N_{i,k}^{(\alpha)}\, x^{i-k-\alpha}, \tag{4.15}$$

where $N_{i,k}^{(\alpha)}$ is given by

$$N_{i,k}^{(\alpha)} = (-1)^k \frac{2^{2i-2k}\,(2n+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\alpha)}. \tag{4.16}$$

Proof. see([25]). □

5 Procedure of solution for the multi-order NFDEs

Consider the multi-order nonlinear fractional differential equation of type given in Eq.(2.3). In order to use Chebyshev collocation method for the second kind, we first approximate $y(x)$ as

$$y_m(x) = \sum_{i=0}^{m} c_i\, U_i^*(x). \tag{5.17}$$

From Eqs.(2.3), (5.17) and Theorem 2 we have

$$\begin{aligned} &\sum_{i=\lceil \alpha \rceil}^{m} \sum_{k=0}^{i-\lceil \alpha \rceil} c_i\, N_{i,k}^{(\alpha)}\, x^{i-k-\alpha} = \\ &F\left(x, \sum_{i=0}^{m} c_i\, U_i^*(x), \sum_{i=\lceil \beta_1 \rceil}^{m} \sum_{k=0}^{i-\lceil \beta_1 \rceil} c_i\, N_{i,k}^{(\beta_1)}\, x^{i-k-\beta_1}, ..., \sum_{i=\lceil \beta_n \rceil}^{m} \sum_{k=0}^{i-\lceil \beta_n \rceil} c_i\, N_{i,k}^{(\beta_n)}\, x^{i-k-\beta_n}\right), \end{aligned} \tag{5.18}$$

we now collocate Eq.(5.18) at $(m+1-\lceil \alpha \rceil)$ points x_p as

$$\begin{aligned} &\sum_{i=\lceil \alpha \rceil}^{m} \sum_{k=0}^{i-\lceil \alpha \rceil} c_i\, N_{i,k}^{(\alpha)}\, x_p^{i-k-\alpha} = \\ &F\left(x_p, \sum_{i=0}^{m} c_i\, U_i^*(x_p), \sum_{i=\lceil \beta_1 \rceil}^{m} \sum_{k=0}^{i-\lceil \beta_1 \rceil} c_i\, N_{i,k}^{(\beta_1)}\, x_p^{i-k-\beta_1}, ..., \sum_{i=\lceil \beta_n \rceil}^{m} \sum_{k=0}^{i-\lceil \beta_n \rceil} c_i\, N_{i,k}^{(\beta_n)}\, x_p^{i-k-\beta_n}\right). \end{aligned} \tag{5.19}$$

For suitable collocation points we use roots of shifted Chebyshev polynomial $U^*_{m+1-\lceil\alpha\rceil}(x)$.
Also, by substituting Eqs.(5.17) in the initial conditions, we can find $\lceil\alpha\rceil$ equations. By substituting Eqs.(5.17) in the initial conditions (2.4) we obtain

$$\sum_{i=0}^{m}(-1)^i c_i = 0, \qquad \sum_{i=0}^{m} c_i U^{*(k)}(0) = 0, \qquad k = 1,2,...,m \tag{5.20}$$

Equation (5.19), together with $\lceil\alpha\rceil$ equations of the initial conditions (5.20), give $(m+1)$ of nonlinear algebraic equations which can be solved, for the unknown $c_i, i = 0,1,...,m$, using Newton iteration method, as described in the following section.

6 large Numerical simulation and comparison

In this section, we implement the proposed method to solve the muti-order NFDEs (2.3)-(2.4) with different two examples.

Example 1

Consider the following nonlinear initial value problem [23]

$$D^3y(x) + D^{2.5}y(x) + y^2(x) = x^4, \tag{6.21}$$

with the following initial conditions

$$y(0) = y'(0) = 0, \quad y''(0) = 2. \tag{6.22}$$

6.1 SCP2K

We apply the suggested method with $m = 3$, and approximate the solution $y(x)$ as follows

$$y_3(x) = \sum_{i=0}^{3} c_i\, U^*_i(x). \tag{6.23}$$

Using Eq.(5.19), $\alpha = 2$, $\beta_1 = 1.5$, and for $p = 0$, we have

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i\, N^{(\alpha)}_{i,k}\, x_p^{i-k-\alpha} + \sum_{i=\lceil\beta_1\rceil}^{m}\sum_{k=0}^{i-\lceil\beta_1\rceil} c_i\, N^{(\beta_1)}_{i,k}\, x_p^{i-k-\beta_1} + \left(\sum_{i=0}^{m} c_i\, U^*_i(x_p)\right)^2 = x_p^4, \tag{6.24}$$

where x_p are roots of the shifted Chebyshev polynomial for the second kind $U^*_1(x)$, i.e., $x_0 = 0.5$.
By using Eqs.(6.24) and (5.20) we obtain the following nonlinear system of algebraic equations

$$c_3(N^{(\alpha)}_{3,0} + N^{(\beta_1)}_{3,0}\, x_0^{0.5}) + (s_0c_0 + s_1c_1 + s_2c_2 + s_3c_3)^2 = x_0^4, \tag{6.25}$$

$$c_1 - c_1 + c_2 - c_3 = 0, \tag{6.26}$$

$$k_0c_0 + k_1c_1 + k_2c_2 + k_3c_3 = 0, \tag{6.27}$$

$$r_0c_0 + r_1c_1 + r_2c_2 + r_3c_3 = 2, \tag{6.28}$$

where

$$s_i = U^*_i(x_0), \qquad k_i = U^{*(1)}_i(0), \qquad r_i = U^{*(2)}_i(0).$$

By solving Eqs.(6.25)-(6.28) we obtain

$$c_0 = \frac{3}{8}, \qquad c_1 = \frac{4}{8}, \qquad c_2 = \frac{1}{8}, \qquad c_3 = 0.$$

Therefore

$$y(x) = \begin{pmatrix} \frac{3}{8}, & \frac{4}{8}, & \frac{1}{8}, & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2x-1 \\ 8x^2-8x+1 \\ 32x^3-48x^2+18x-1 \end{pmatrix} = x^2,$$

which is the exact solution of this problem [23].

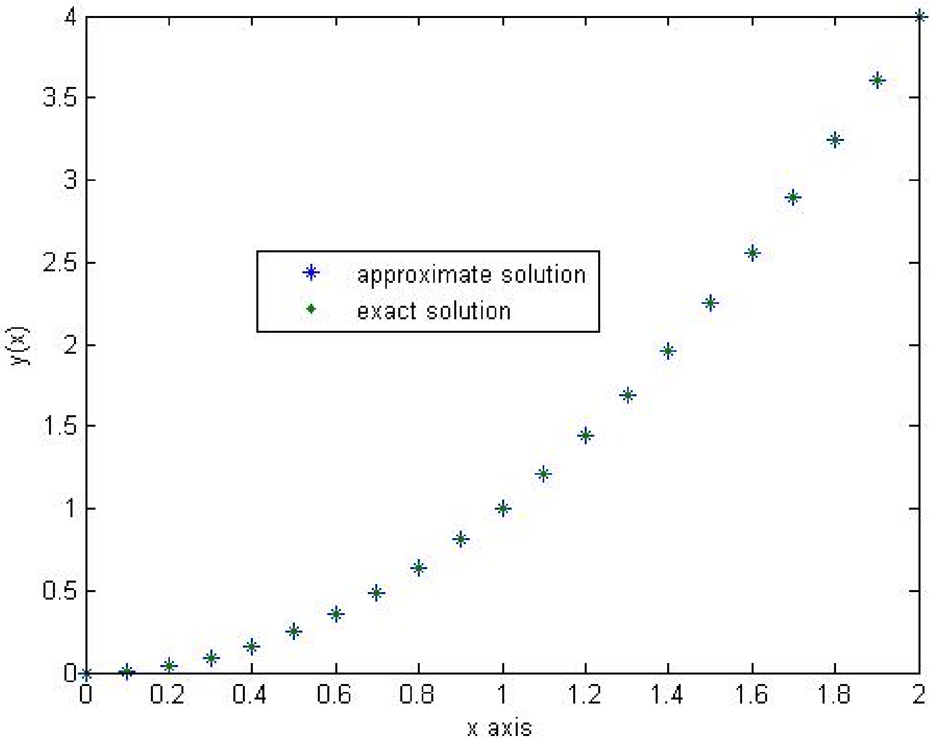

Figure 4.1. Comparison between the exact solution and the numerical solution.

From this Figure 4.1, we can conclude that the numerical results are excellent agreement with the exact solution. Also, it is evident that the overall errors can be made smaller by adding new terms from the series (5.17).

6.2 FPSM

To apple FPSM, we suppose that the solution the form

$$y(x)=\sum_{k=0}^{\infty}a_k x^{\alpha k}=a_0+a_1x^{\alpha}+a_2x^{2\alpha}+.... \tag{6.29}$$

$$\begin{aligned} y^2(x) &= (\sum_{k=0}^{\infty}a_k x^{\alpha k})^2=(a_0+a_1x^{\alpha}+a_2x^{2\alpha}+...)^2 \\ &= a_0^2+2a_0a_1x^{\alpha}+(2a_0a_2+a_1^2)x^{2\alpha}+2a_1a_2x^{3\alpha}+a_2^2x^{4\alpha}+... \end{aligned} \tag{6.30}$$

$$D^3y(x)=\alpha(\alpha-1)(\alpha-2)a_1x^{\alpha-3}+2\alpha(2\alpha-1)(2\alpha-2)a_2x^{2\alpha-3}+... \tag{6.31}$$

by theorem 1

$$D^{\alpha}y(x)=\sum_{k=1}^{\infty}a_k\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}x^{\alpha(k-1)} \tag{6.32}$$

the equation (6.21) can be written as

$$D^{2.5}y(x)=x^4-D^3y(x)-y^2(x) \tag{6.33}$$

substituting (6.30), (6.31), (6.32) into (6.33)and comparing the cofficients of x^{α},we get

$$\sum_{k=1}^{\infty}a_k\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}x^{\alpha(k-1)}=$$

$$\begin{aligned} & x^4-(\alpha(\alpha-1)(\alpha-2)a_1x^{\alpha-3}+2\alpha(2\alpha-1)(2\alpha-2)a_2x^{2\alpha-3}+...) \\ & -(a_0^2+2a_0a_1x^{\alpha}+(2a_0a_2+a_1^2)x^{2\alpha}+2a_1a_2x^{3\alpha}+a_2^2x^{4\alpha}+...) \end{aligned}$$

using initial condition $y(o) = 0$

we have $\quad a_0 = y(0) = 0$

Next, we determine the $a_k(k = 1, 2, ...)$.
For example, if $\quad a_0 = 0$ then
$a_1\Gamma(\alpha + 1) = x^4 - a_0$

$$a_1 = \frac{x^4}{\Gamma(\alpha + 1)}$$

$a_2 = 0$
therefore we obtain the approximate solution of equation

$$y(x) = \sum_{k=0}^{\infty} a_k x^{\alpha k} = 0 + \frac{x^4}{\Gamma(\alpha + 1)} x^{\alpha} + 0 + ...$$

Example 2

In this example, we consider the following nonlinear differential equation [23]

$$D^4 y(x) + D^{3.5} y(x) + y^3(x) = x^9, \tag{6.34}$$

subject to the initial conditions

$$y^{(k)}(0) = 0, \quad k = 0, 1, 2, 3. \tag{6.35}$$

6.3 SCP2K

To solve the above problem, by applying the proposed technique described in Section 5 with $m = 4$, we approximate the solution as

$$y(x) = c_0 U_0^*(x) + c_1 U_1^*(x) + c_2 U_2^*(x) + c_3 U_3^*(x) + c_4 U_4^*(x).$$

Using Eq.(5.19), $\alpha = 4$, $\beta_1 = 3.5$, and for $p = 0$, we have

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=0}^{i-\lceil\alpha\rceil} c_i\, N_{i,k}^{(\alpha)}\, x_p^{i-k-\alpha} + \sum_{i=\lceil\beta_1\rceil}^{m} \sum_{k=0}^{i-\lceil\beta_1\rceil} c_i\, N_{i,k}^{(\beta_1)}\, x_p^{i-k-\beta_1} + \left(\sum_{i=0}^{m} c_i\, U_i^*(x_p)\right)^3 = x_p^9, \tag{6.36}$$

where x_p are roots of the shifted Chebyshev polynomial for the second kind $U_1^*(x)$, i.e., $x_0 = 0.5$.
By using Eqs.(6.36) and (5.20) we obtain the following nonlinear system of algebraic equations

$$c_4 \left(N_{4,0}^{(\alpha)} + N_{4,0}^{(\beta_1)}\, x_0^{0.5}\right) + (s_0 c_0 + s_1 c_1 + s_2 c_2 + s_3 c_3 + s_4 c_4)^3 = x_0^9, \tag{6.37}$$

$$c_1 - c_1 + c_2 - c_3 + c_4 = 0, \tag{6.38}$$

$$k_0 c_0 + k_1 c_1 + k_2 c_2 + k_3 c_3 + k_4 c_4 = 0, \tag{6.39}$$

$$r_0 c_0 + r_1 c_1 + r_2 c_2 + r_3 c_3 + r_4 c_4 = 0, \tag{6.40}$$

$$z_0 c_0 + z_1 c_1 + z_2 c_2 + z_3 c_3 + z_4 c_4 = 0, \tag{6.41}$$

where

$$s_i = U_i^*(x_0), \qquad k_i = U_i^{*(1)}(0), \qquad r_i = U_i^{*(2)}(0), \qquad z_i = U_i^{*(3)}(0).$$

By solving Eqs.(6.37)-(6.41) we obtain

$$c_0 = \frac{10}{32}, \qquad c_1 = \frac{15}{32}, \qquad c_2 = \frac{6}{32}, \qquad c_3 = \frac{1}{32}, \qquad c_4 = 0.$$

Therefore

$$y(x) = \begin{pmatrix} \frac{10}{32}, & \frac{15}{32}, & \frac{6}{32}, & \frac{1}{32}, & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2x - 1 \\ 8x^2 - 8x + 1 \\ 32x^3 - 48x^2 + 18x - 1 \\ 128x^4 - 256x^3 + 160x^2 - 32x + 1 \end{pmatrix} = x^3,$$

which is the exact solution of this problem [23].

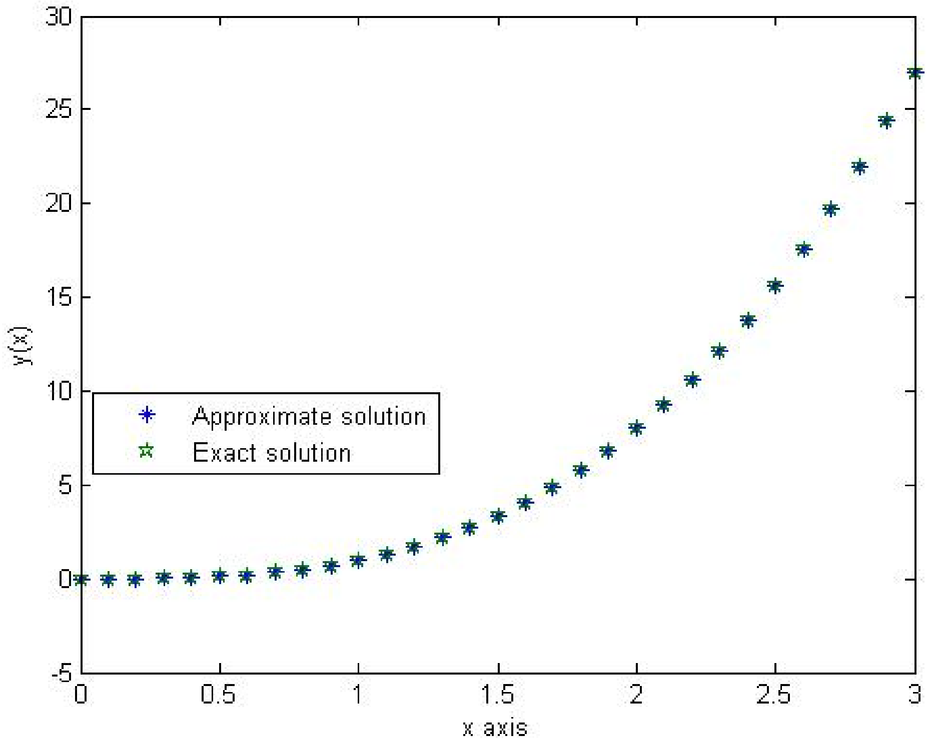

Figure 4.2. Comparison between the exact solution and the numerical solution.

From this Figure 4.2, we can conclude that the numerical results are excellent agreement with the exact solution. Also, it is evident that the overall errors can be made smaller by adding new terms from the series (5.17).

6.4 FPSM

To apple FPSM,we suppose that the solution the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{\alpha k} = a_0 + a_1 x^{\alpha} + a_2 x^{2\alpha} + ..., \tag{6.42}$$

$$\begin{aligned} y^3(x) &= (\sum_{k=0}^{\infty} a_k x^{\alpha k})^3 = (a_0 + a_1 x^{\alpha} + a_2 x^{2\alpha} + ...)^3, \\ &= a_0^3 + 3a_0^2 a_1 x^{\alpha} + (3a_0^2 a_2 + 3a_0 a_1^2) x^{2\alpha} + (6a_0 a_1 a_2 + a_1^3) x^{3\alpha} \\ &\quad + (3a_1^2 a_2 + 3a_0 a_2^2) x^{4\alpha} + 3a_1 a_2^2 x^{5\alpha} + a_2^3 x^{6\alpha} + ... \end{aligned} \tag{6.43}$$

$$D^4 y(x) = \alpha(\alpha-1)(\alpha-2)(\alpha-3) a_1 x^{\alpha-4} + 2\alpha(2\alpha-1)(2\alpha-2)(2\alpha-3) a_2 x^{2\alpha-4} + ..., \tag{6.44}$$

by theorem

$$D^{\alpha} y(x) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} x^{\alpha(k-1)}, \tag{6.45}$$

the equation(6.34)can be written as

$$D^{3.5} y(x) = x^9 - D^4 y(x) - y^3(x), \tag{6.46}$$

substituting (6.43),(6.44),(6.45) into (6.46)and comparing the cofficients of x^{α},we get

$\sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} x^{\alpha(k-1)} =$

$$x^9 - [\alpha(\alpha-1)(\alpha-2)(\alpha-3)a_1x^{\alpha-4} + 2\alpha(2\alpha-1)(2\alpha-2)(2\alpha-3)a_2x^{2\alpha-4} + ...]$$
$$-[a_0^3 + 3a_0^2a_1x^\alpha + (3a_0^2a_2 + 3a_0a_1^2)x^{2\alpha} + (6a_0a_1a_2 + a_1^3)x^{3\alpha} +$$
$$+(3a_1^2a_2 + 3a_0a_2^2)x^{4\alpha} + 3a_1a_2^2x^{5\alpha} + a_2^3x^{6\alpha} + ...]$$

using initial condition $y^k(o) = 0$

we have $a_0 = y(0) = 0$

Next, we determine the $a_k (k = 1, 2, ...)$.
For example, if $a_0 = 0$ then
$a_1\Gamma(\alpha + 1) = x^9 - a_0^3$

$$a_1 = \frac{x^9}{\Gamma(\alpha+1)}$$

therefore we obtain the approximate solution of equation

$$y(x) = \sum_{k=0}^{\infty} a_k x^{\alpha k} = 0 + \frac{x^9}{\Gamma(\alpha+1)}x^\alpha + ...$$

7 Conclusion

In this paper, we use shifted Chebyshev approximations with the second kind [25] and fractional power series method (FPSM) ([3], [8]) to solve the multi-order nonlinear fractional differential equations. The properties of the shifted Chebyshev polynomials for the second kind are used to reduce the nonlinear multi-order fractional differential equations to the solution of non-linear system of algebraic equations. The resulting system is solved by using Newton iteration method. The fractional derivative is considered in the Caputo sense. From the solutions obtained using the suggested method, we can conclude that these solutions are in excellent agreement with the already existing ones and show that this approach can be solve the problem effectively. Comparisons are made between approximate solutions and exact solutions and other methods to illustrate the validity and the great potential of the technique. All numerical results are obtained using Matlab 12b.

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Numerical Method For Variable-order Space Fractional Diffusion Equation and Applications

Bhausaheb R. Sontakke[a], Abhijeet Shelke[b∗]

[a]Department of Mathematics, Pratishthan Mahavidyalya, Paithan, Aurangabad(MS), India.

[b]Research Student, Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad (MS), India.

Abstract

The purpose of this paper is to develop the explicit fractional order finite difference scheme for variable-order space fractional diffusion equation (VOSFDE). Furthermore, the stability and convergence of the scheme in a bounded domain are discussed. As an application of the scheme, we solve some test problems and their solutions are represented graphically by Mathematica software.

Keywords: Space fractional, diffusion equation, finite difference scheme, stability analysis, Mathematica.

2010 MSC: 46B40, 46B42, 47B60, 47B65.

1 Introduction

Now a days, the research on variable-order fractional partial differential equations is relatively new, and numerical approximation of these equations is still at an early stage of development. Lin, Liu, Anh, and Turner [9] established an equality between the variable-order RiemannLiouville fractional derivative and its GrunwaldLetnikovexpansion. Using this relationship, they defined and obtained some properties of the operator $(\frac{d^2}{dx^2})^{\alpha(x,t)}$ and devised an explicit finite difference approximation scheme for a corresponding variable-order nonlinear fractional di?usion equation. Ilic et al.[7] proposed a new matrix method for a fractional-in-space diffusion equation with homogeneous and nonhomogeneous boundary conditions on a bounded domain.

The number of scientific and engineering problems involving fractional calculus is already very large and still growing. One of the main advantages of the fractional calculus is that the fractional derivatives provide an excellent approach for the description of memory and hereditary properties of various materials and processes [6]. Many of the numerical methods using different kinds of fractional derivative operators for solving fractional partial differential equations have been proposed and are available in the literature. Anh and Leonenko presented a spectral representation of the mean-square solution of the fractional diffusion equation with random initial condition, from which the Caputo-Djrbashian regularied fractional derivative was adopted [1]. Odibat proposed two algorithms for numerical fractional integration and Caputo fractional differentiation. Using the new modification derive an algorithm to approximate fractional derivatives of arbitrary order for a given function by a weighted sum of function and its ordinary derivative values at specified points [11]. Blaszczyk focused on a numerical scheme applied for a fractional oscillator equation which includes a complex form of left- and right-sided fractional derivatives in a finite time interval [2]. Recently, more and more researchers are finding that a variety of important dynamical problems exhibit fractional order behavior that may vary with time or space. This fact indicates that variable order calculus is a natural candidate to provide an effective mathematical framework for the description of complex

∗Abhijeet S. Shelke

: brsontakke@rediffmail.com (Bhausaheb R. Sontakke).

dynamical problems. The concept of a variable order operator is a much more recent development, which is a new paradigm in science. Samko and Ross directly generalized the Riemann-Liouvile and Marchaud fractional integration and differentiation of the case of variable order, and then showed some properties and an inversion formula. Lorenzo and Hartley suggested the concept of a variable order operator is allowed to vary either as a function of the independent variable of integration or differentiation (t), or as a function of some other (perhaps spatial) variable (x), they also explored more deeply the concept of variable order integration and differentiation and sought the relationship between the mathematical concepts and physical processes. Di?erent authors have proposed different definitions of variable order differential operators, each of these with a specific meaning to suit desired goals. Coimbra [4] took the Laplace-transform of Caputos definition of the fractional derivative as the starting point to suggest a novel definition for the variable order differential operator. Because of its meaningful physical interpretation, Coimbras definition is better suited for modeling physical problems. Ingman et al. [8] employed the time dependent variable order operator to model the viscoelastic deformation process. Pedro et al. studied the motion of particles suspended in a viscous fluid with drag force using variable order calculus. Sun et al. introduced a classification of variable-order fractional diffusion [2] models based on the possible physical origins that prompt the variable-order.

The variable order operator definitions recently proposed in the literature include the Riemann-Liouvile-definition, Caputo-definition, Marchaud-definition, Coimbra-definition and Grunwald-definition [4, 12]. However, to the best of the authors knowledge, detailed studies of the Grunwald-type variable order operator have not yet been performed. Samko et al. [24] compared the Riemann-Liouvile-definition and Marchaud-definition variable order operators, and noted the loss of certain properties of the Riemann-Liouvile definitions, with the Marchaud-definition being more suitable than the RiemannLiouvile-type. Ramirezetal, also compared the Riemann-Liouvile-definition, Caputo-definition, Marchaud-definition and Coimbra-definition variable order operators based on a very simple criteria: the variable order operator must return the correct fractional derivative that corresponds to the argument of the functional order. Ramirez et al. found that only the Marchaud-definition and Coimbra-definition satisfy the above elementary requirement, and the Coimbra-de?nition variable order operator is more efficient from the numerical standpoint. Soon et al. also showed that the Coimbra-definition variable order operator satisfies a mapping requirement, and it is the only definition that correctly describes position-dependent transitions between elastic and viscous regimes because it correctly returns the appropriate derivatives as a function of x(t). Ramirez, showed that the Coimbra definition is the most appropriate definition having fundamental characteristics that are desirable for physical modeling.

In our paper we consider the variable-order space fractional diffusion equation

$$\frac{\partial U(x,t)}{\partial t} = D\frac{\partial^{\alpha(x,t)}U(x,t)}{\partial x^{\alpha(x,t)}}; (x,t) \in \Omega = [0,L] \times [0,T],$$
$$1 < \alpha(x,t) \leq 2,\ t > 0$$

with initial and boundary conditions

$$\textit{initial condition}:\ U(x,0) = f(x),\ 0 \leq x \leq L$$

$$\textit{boundary conditions}:\ U(0,t) = U_L,\ U_(L,t) = U_R,\ t \geq 0$$

where $D > 0$ is the diffusivity coefficient.

We organize the paper as follows: In section 2 , we develop the explicit fractional order finite difference scheme for variable-order space fractional diffusion equation. The stability solution of VOSFDE by develop scheme is discussed in section 3 and in section 4 the convergence of the scheme is discussed upto the length. The numerical solution of variable-order space fractional diffusion equation is obtained using Mathematica software and it is represented graphically in the last section.

2 FINITE DIFFERENCE SCHEME

In this section, we develop the explicit fractional order finite difference scheme for variable-order space fractional diffusion equation. We consider the following variable-order space fractional diffusion equation

(VOSFDE) with initial and boundary conditions

$$\frac{\partial U(x,t)}{\partial t} = D\frac{\partial^{\alpha(x,t)}U(x,t)}{\partial x^{\alpha(x,t)}}; (x,t) \in \Omega = [0,L] \times [0,T], \qquad 1 < \alpha(x,t) \leq 2,\ t > 0 \tag{2.1}$$

$$initial\ condition:\ U(x,0) = f(x),\ 0 \leq x \leq L \tag{2.2}$$

$$boundary\ conditions:\ U(0,t) = U_L,\ U_(L,t) = U_R,\ t \geq 0 \tag{2.3}$$

where $D > 0$ is the diffusivity coefficient. Note that for $\alpha(x,t) = 2$, we recover the original diffusion equation.

$$\frac{\partial U(x,t)}{\partial t} = D\frac{\partial^2 U(x,t)}{\partial x^2};\ x\epsilon R;\ t \geq 0$$

Definition 2.1. *The Caputo space fractional variable-order derivative of order $\alpha(x)$, $(1 < \alpha(x) \leq 2)$ is denoted by ${}_0^C D_x^{\alpha(x)} U(x,t)$ and is defined as follows*

$${}_0^C D_x^{\alpha(x)} U(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha(x))}\int_0^x \frac{1}{(x-\xi)^{\alpha(x)-n+1}}\frac{\partial^n U(\xi,t)}{\partial \xi^n}d\xi, & n < \alpha(x) < n+1 \\ \frac{\partial^2 U(x,t)}{\partial t^2}, & \alpha(x) = 2 \end{cases}$$

where $\Gamma(.)$ is a Gamma function. Adopting the discrete scheme given in [3], we discretise the variable order space fractional derivative as follows

$$\begin{aligned} \frac{\partial^{\alpha(x_j,t_n)}U(x_j,t_n)}{\partial x^{\alpha(x_j,t_n)}} &= \frac{1}{\Gamma(2-\alpha(x,t))}\int_0^x \frac{1}{(x-\xi)^{\alpha(x)-1}}\frac{\partial^2 U(\xi,t)}{\partial \xi^2}d\xi \\ &= \frac{1}{\Gamma(2-\alpha(x,t))}\sum_{k=0}^{j-1}\int_{kh}^{(k+1)h}\eta^{1-\alpha(x,t)}\frac{\partial^2 U(x-\eta,t)}{\partial \eta^2}d\eta \\ &\approx \frac{1}{\Gamma(2-\alpha(x,t))}\sum_{k=0}^{j-1}\frac{U(x-(k-1)h,t)-2U(x-kh,t)+U(x-(k+1)h)}{h^2}\times \\ &\qquad \int_{kh}^{(k+1)h}\eta^{1-\alpha(x,t)}d\eta \\ &= \frac{1}{\Gamma(2-\alpha(x,t))}\sum_{k=0}^{j-1}\frac{U(x-(k-1)h,t)-2U(x-kh,t)+U(x-(k+1)h)}{h^2} \\ &\qquad \times [(k+1)^{2-\alpha(x,t)} - k^{2-\alpha(x,t)}]h^{2-\alpha(x,t)} \\ &= \frac{h^{-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))}\sum_{k=0}^{j-1}[U(x-(k-1)h,t)-2U(x-kh,t)+U(x-(k+1)h)] \\ &\qquad \times [(k+1)^{2-\alpha(x,t)} - k^{2-\alpha(x,t)}] \end{aligned} \tag{2.4}$$

For the explicit numerical approximation scheme, we take equally spaced mesh of M points for the spatial domain $0 \leq x \leq L$, N constant time steps for the time direction domain $0 \leq t \leq T$ and denote the spatial grid points by

$$x_j = jh\ j = 1,2,...,M$$

and the time direction grid points by

$$t_n = n\tau\ n = 0,1,2,...,N,$$

where the grid spacing is $h = \frac{L}{M}$ in the spatial domain and $\tau = \frac{T}{N}$ in the temporal domain. Therefore, we have

$$U_j^n = U(x_j,t_n) = (j\Delta x, n\Delta t) = (jh, n\tau)\ j = 1,2,...,M-1,\ n = 0,1,2,...,N.$$

At the grid points $(x-j,t_n)$ equation (2.1) becomes

$$\frac{\partial U(x_j,t_n)}{\partial t} = {}_0^C D_x^{\alpha(x_j,t_n)} U(x_j,t_n) = \frac{\partial^{\alpha(x_j,t_n)}U(x_j,t_n)}{\partial x^{\alpha(x_j,t_n)}} \tag{2.5}$$

We denote U_j^n for the numerical approximation to $U(x_j, t_n)$. From equations (2.4), (2.5) and using the forward difference to $\frac{\partial U(x,t)}{\partial t}$, we approximate the variable-order space fractional diffusion equation (2.1) as follows

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \frac{Dh^{-\alpha_j^n}}{\Gamma(3-\alpha_j^n)} \sum_{k=0}^{j-1} [U_{j-k+1}^n - 2U_{j-k}^n + U_{j-k-1}^n][(k+1)^{2-\alpha_j^n} - k^{2-\alpha_j^n}] \tag{2.6}$$

For simplicity, we define

$$b_j^n = \frac{\tau D h^{-\alpha_j^n}}{\Gamma(3-\alpha_j^n)} \text{ and } g_k^n = (k+1)^{2-\alpha_j^n} - k^{2-\alpha_j^n}.$$

Then after simplification, from equation (2.6), we get

$$U_j^{n+1} = b_j^n U_{j+1}^n + (1-2b_j^n)U_j^n + b_j^n U_{j-1}^n + b_j^n \sum_{k=1}^{j-1} g_k^n [U_{j-k+1}^n - 2U_{j-k}^n + U_{j-k-1}^n] \tag{2.7}$$

where $b_j^n = \frac{\tau D h^{-\alpha_j^n}}{\Gamma(3-\alpha_j^n)}$ and $g_k^n = (k+1)^{2-\alpha_j^n} - k^{2-\alpha_j^n}$, $j = 1, 2, ..., M-1,\ n = 0, 1, 2, ..., N$.
The initial condition is approximated as $U_j^0 = f(x_j)$, $j = 1, 2, ..., M-1$.
The boundary conditions are approximated as $U_0^n = U_L$ and $U_M^n = U_R$, $n = 0, 1, 2, ..., N$.
Therefore, the complete fractional approximated IBVP is

$$U_j^{n+1} = b_j^n U_{j+1}^n + (1-2b_j^n)U_j^n + b_j^n U_{j-1}^n + b_j^n \sum_{k=1}^{j-1} g_k^n [U_{j-k+1}^n - 2U_{j-k}^n + U_{j-k-1}^n] \tag{2.8}$$

$$\textit{initial condition}: \ U_j^0 = f(x_j),\ j = 1, 2, ..., M-1 \tag{2.9}$$

$$\textit{boundary conditions}: \ U_0^n = U_L,\ U_M^n = U_R,\ n = 0, 1, 2, ..., N. \tag{2.10}$$

where

$$b_j^n = \frac{\tau D h^{-\alpha_j^n}}{\Gamma(3-\alpha_j^n)} \text{ and } g_k^n = (k+1)^{2-\alpha_j^n} - k^{2-\alpha_j^n},\ j = 1, 2, ..., M,\ n = 0, 1, 2, ..., N.$$

Therefore, the fractional approximated IBVP (2.8) − (2.10) can be written in the following matrix equation form

$$U^{n+1} = AU^n + B \tag{2.11}$$

where $U^n = [U_1^n, U_2^n, ..., U_{M-1}^n]^T$, $B = [b_1^n U_0^n, b_2^n g_1^n U_0^n, ..., b_{M-1}^n g_{M-1}^n U_0^n]^T$ and
$A = (a_{ij})$ is an $(M-1) \times (M-1)$ matrix given by

$$a_{ij} = \begin{cases} 0, & \text{when } j \geq i+2 \\ b_i^n, & \text{when } j = i+1 \\ 1-(2-g_1^n)b_i^n, & \text{when } j = i = 2, 3, ..., M-1 \\ b_i^n(1-2g_1^n+g_2^n), & \text{when } j = i-1\ i \geq 3, 4, ..., M-1 \\ b_i^n(g_{i-j-1}^n - 2g_{i-j}^n + g_{i-j+1}^n), & \text{when } j \leq i-2\ i \geq 4, 5, ..., M-1 \end{cases}$$

while $a_{11} = 1-2b_1^n$, $a_{21} = b_2^n(1-2g_1^n)$ and $a_{i1} = b_i^n(g_{i-2}^n - 2g_{i-1}^n + g_{i-j+1}^n)$, $3 \leq i \leq M-1$. The above system of algebraic equations is solved by using Mathematica software in section 5.
In the next section, we discuss the stability of the solution of fractional order explicit finite difference scheme (2.8) − (2.10) developed for the time fractional diffusion equation (2.1) − (2.3).

3 STABILITY

Theorem 3.1. *The solution of the explicit type fractional order finite difference scheme (2.8) − (2.10) for the variable-order time fractional diffusion equation (2.1) − (2.3) is conditionally stable.*

Proof: We consider the equation (2.11)

$$U^{n+1} = AU^n + B. \tag{3.12}$$

Now B includes a column matrix of known values and known source term values. By Gerschgorin's first theorem [1], let λ_i be an eigenvalue of matrix A to linear system of equations (2.8) and x be the corresponding eigenvector then $Ax = \lambda x$. Choose i such that

$$|x_i| = max.\{|x_j|, j = 1, 2, ..., M-1\}$$

then

$$\sum_{j=1}^{N-1} a_{ij} x_j = \lambda x_i$$

and therefore,

$$\lambda = a_{ii} + \sum_{j=1, j\neq 1}^{N-1} a_{ij} \frac{x_j}{x_i} \tag{3.13}$$

We substitute the values of a_{ij} in equation (3.13), we get
(i) When $i = 1$

$$\begin{aligned}
\lambda &= 1 - 2b_1^n + b_1^n \frac{x_2}{x_1} \\
&\leq 1 - 2b_1^n + b_1^n = 1 - b_1^n \leq 1 \\
\lambda &\geq 1 - 2b_1^n - b_1^n = 1 - 3b_1^n \geq -1 \ when \ b_1^n \leq \frac{2}{3} \\
&\therefore 1 \leq \lambda \leq 1 \\
&\Rightarrow |\lambda| \leq 1 \ when \ b_1^n \leq \frac{2}{3}
\end{aligned}$$

(ii) When $i = 2$

$$\begin{aligned}
\lambda &= a_{22} + a_{21} \frac{x_1}{x_2} + a_{23} \frac{x_3}{x_2} \\
&= 1 - (2 - g_1^n) b_2^n + b_2^n (1 - 2g_1^n) \frac{x_1}{x_2} + b_2^n \frac{x_3}{x_2} \\
&\leq 1 - (2 - g_1^n) b_2^n + b_2^n (1 - 2g_1^n) + b_2^n \\
&\leq 1 - b_2^n g_1^n \leq 1 \\
\lambda &\geq 1 - (2 - g_1^n) b_2^n - b_2^n (1 - 2g_1^n) - b_2^n \geq 1 - b_2^n (4 - 3g_1^n) \geq -1 \\
&\qquad when \ b_2^n \leq \frac{2}{(4 - 3g_1^n)} \\
&\therefore 1 \leq \lambda \leq 1 \\
&\Rightarrow |\lambda| \leq 1 \ when \ b_2^n \leq \frac{2}{(4 - 3g_1^n)}
\end{aligned}$$

(iii) When $3 \leq i \leq M-1$

$$\lambda = 1 - b_i^n(2-g_1^n) + b_i^n(g_{i-2}^n - 2g_{i-1}^n)\frac{x_{i-2}}{x_i} + b_i^n \sum_{j=2}^{i-1}(g_{i-j-1}^n - 2g_{i-j}^n + g_{i-j+1}^n)\frac{x_{i-1}}{x_i} + b_i^n \frac{x_{i+1}}{x_i}$$
$$\leq 1 - b_i^n(2-g_1^n) + b_i^n(g_{i-2}^n - 2g_{i-1}^n) + b_i^n(g_{i-1}^n - g_{i-2}^n + g_0^n - g_1^n) + b_i^n$$
$$\leq 1 - b_i^n + b_i^n g_0^n \leq 1$$
$$\lambda \geq 1 - b_i^n(2-g_1^n) - b_i^n(g_{i-2}^n - 2g_{i-1}^n) - b_i^n(g_{i-1}^n - g_{i-2}^n + g_0^n - g_1^n) - b_i^n$$
$$\geq 1 - 4b_i^n + 2b_i^n g_1^n = 1 - 2b_i^n(2-g_1^n) \geq -1 \; when \; b_i^n \leq \frac{1}{(2-g_1^n)}$$
$$\therefore 1 \leq \lambda \leq 1$$
$$\Rightarrow |\lambda| \leq 1 \; when \; b_i^n \leq \frac{1}{(2-g_1^n)}.$$

Therefore, from $(i)-(iii)$, we prove

$$Max_{(3\leq i\leq M-1)}\{b_1^n, b_2^n, b_i^n\} \leq \frac{1}{2}$$

then the spectral radius $\rho(A)$ of matrix A satisfies $\rho(A) \leq 1$.
If

$$Max_{(3\leq i\leq M-1)}\{b_1^n, b_2^n, b_i^n\} \leq \frac{1}{2},$$

where

$$b_1^n \leq \frac{2}{3}, \; b_2^n \leq \frac{2}{(4-3g_1^n)} \; b_i^n \leq \frac{1}{(2-g_1^n)}$$

then there exist a positive number $\epsilon \leq C\tau$ such that

$$\|A\|_m \leq \rho(A) + C\tau \leq 1 + O(\tau) \leq 1 + \epsilon.$$

Hence, this prove that the fractional order finite difference scheme is conditionally stable.
The next section is devoted for convergence of the finite difference scheme.

4 CONVERGENCE

Consider the another vector

$$\bar{U}^n = [U(x_0,t_n), U(x_1,t_n), \cdots, U(x_M,t_n)]^T$$

which represents the exact solution at time level t_n. The finite difference equation (2.11) becomes

$$\bar{U}^{n+1} = A\bar{U}^n + B + \tau^n \tag{4.14}$$

where τ^n is the error vector of the truncation error at time level t_n.

Theorem 4.2. *Suppose that the continuous problem (2.1) − (2.3) has a smooth solution $U(x,t) \in C_{x,t}^{1+\bar{\alpha},2}$. Let U_j^n be the numerical solution computed by (2.8) − (2.10). If $\xi \leq \frac{1}{2}$ where $\xi = \min_{(3\leq i\leq M-1)}\{\; b_1^n, b_2^n, b_i^n \;\}$, then the fractional order explicit finite difference scheme (2.8) − (2.10) for IBVP (2.1) − (2.3) is convergent.*

Proof: Let Ω be the region $0 \leq x \leq L$ and $0 \leq t \leq T$. Take $U_j^n = U(x_j,t_n) = (j\Delta x, n\Delta t) = (jh, n\tau)$, $j = 1,2,...,M-1,\; n = 0,1,2,...,N.$ with $h = \frac{L}{M}$ and $\tau = \frac{T}{N}$.
We introduced another vector

$$\bar{U}^n = [U(x_0,t_n), U(x_1,t_n), \cdots, U(x_M,t_n)]^T$$

satisfying the finite difference equation (2.11), we get

$$\bar{U}^{n+1} = A\bar{U}^n + B + \tau^n \tag{4.15}$$

where τ^n is the error vector of the truncation error at time level t_n. Now we subtract (2.11) from (4.15), we have

$$(\bar{U}^{n+1} - U^{n+1}) = A(\bar{U}^n - U^n) + \tau^n \tag{4.16}$$

where

$$U^n = [u_1, u_2, \cdots, u_M]^T \; \bar{U}^n[\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N]^T$$

and set

$$E^n = \bar{U}^n - U^n.$$

Then from equation (4.16), we get

$$E^{n+1} = AE^n + \tau^n \tag{4.17}$$

Clearly, E^n satisfies (2.11), and if $\xi \leq \frac{1}{2}$ then from equation (4.17), we have

$$\begin{aligned} \|E^1\|_m &= \|AE^0\|_m + \|\tau^1\|_m \\ &\leq \|A\|_m \|E^0\|_m + \|\tau^1\|_m \\ &\leq (1+\epsilon)\|E^0\|_m + \|\tau^1\|_m \\ &\leq K\|E^0\|_m + \|\tau^1\|_m \qquad (K = (1+\epsilon)) \end{aligned}$$

Suppose that

$$\|E^n\|_m \leq K\|E^n\|_m + \|\tau^n\|_m$$

Therefore,

$$\begin{aligned} \|E^{n+1}\|_m &= \|AE^n\|_m + \|\tau^n\|_m \\ &\leq \|A\|_m \|E^n\|_m + \|\tau^n\|_m \\ &\leq (1+\epsilon)\|E^0\|_m + \|\tau^n\|_m \\ &\leq K\|E^0\|_m + \|\tau^n\|_m \end{aligned}$$

Hence, by induction we prove that

$$\|E^n\|_m \leq K\|E^0\|_m + \|\tau^n\|_m \; for \; all \; n.$$

Since,

$$\lim_{(h,\tau)\to(0,0)} \|\tau^n\|_m = 0, \; (1 \leq M \leq n),$$

implies that $\|E^n\|_m$tends to zero uniformly in Ω as $(h,\tau) \to (0,0)$.
This show that the scheme is conditionally convergent.
Hence, the proof of the theorem is completed.

5 NUMERICAL SOLUTIONS

In this section, we obtain the approximated solution of variable-order space fractional diffusion equation (VOSFDE) with initial and boundary conditions. To obtain the numerical solution of the variable-order space fractional diffusion equation (VOSFDE) by the finite difference scheme, it is important to use some analytical model. Therefore, we present an example to demonstrate that VOSFDE can be applied to simulate behavior of a fractional diffusion equation by using Mathematica Software.
We consider the following one-dimensional variable-order space fractional diffusion equation (VOSFDE) with initial and boundary conditions

$$\begin{aligned} \frac{\partial U(x,t)}{\partial t} &= \frac{\partial^{\alpha(x,t)} U(x,t)}{\partial x^{\alpha(x,t)}} \, 0 < x < 1, 1 < \alpha(x,t) \leq 2, \; t > 0 \\ \text{initial condition} &: U(x,0) = x(1-x), \; 0 \leq x \leq 1 \\ \text{boundary conditions} &: U(0,t) = 0 = U(x,t), \, t \geq 0 \end{aligned}$$

where $\alpha(x,t) = \frac{2+sin(xt)}{4}$ The numerical solutions are obtained at $t = 0.025$ by considering the parameters $\tau = 0.0025$, $h = 0.1$, is simulated in the following figure.

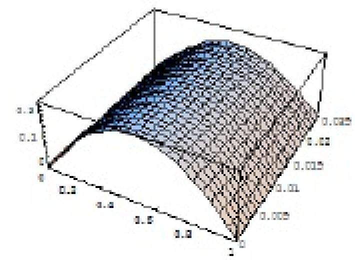

Fig.1.1 : The exact solution of variable-order diffusion equation, $t = 0.025$.

Fig.1.1 : The e

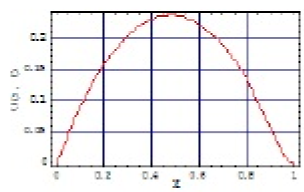

Fig.1.2 : The approximate solution of variable-order diffusion

equation with $h = 0.1$, $\tau = 0.0025$, $t = 0.025$.

Fig.1.2 : The approximate solution of variable-order diffusion

equation with $h = 0.1$, $\tau = 0.0025$, $t = 0.025$.

CONCLUSIONS

In the present paper, we develop a new numerical scheme for the variable order space fractional diffusion equation with the Caputo variable order space fractional operator. The proposed explicit difference approximation for space fractional variable-order diffusion equation can be reliably applied to solve any fractional order dynamical systems and controllers, minding the conditions for stability and convergence of the scheme. The numerical results are also compatible with theoretical analysis, hence showing the numerical stability of the finite difference scheme.

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3 Dimensional Additive Quadratic Functional Equation

M. Arunkumar[a]*, P. Narasimman[b], E.Sathya[c], N. Mahesh Kumar[d]

[a,c]Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.

[b]Department of Mathematics, Thiruvalluvar University College of Arts and Science, Gajalnaickanpatti, Tirupattur-635 901,TamilNadu, India.

[d]Department of Mathematics, Arunai Engineering College, Tiruvannamalai - 606 603, TamilNadu, India.

Abstract

In this paper, the authors established the general solution and generalized Ulam - Hyers stability of an 3 dimensional additive quadratic functional equation

$$\begin{aligned} &h(x+2y+3z)+h(x+2y-3z)+h(x-2y+3z)+h(-x+2y+3z)\\ &=h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z)\\ &\quad+2h(y)+4h(z)+5[h(y)+h(-y)]+14[h(z)+h(-z)] \end{aligned}$$

via Banach space and non-Archimedean fuzzy Banach Space using direct and fixed point methods.

Keywords: Additive functional equation, quadratic functional equation, mixed additive-quadratic functional equations, generalized Ulam - Hyers stability, Banach space, non-Archimedean fuzzy Banach space, fixed point

2010 MSC: 39B52, 32B72, 32B82.

1 Introduction

The study of stability problems for functional equations is related to a question of Ulam [51] concerning the stability of group homomorphisms was affirmatively answered for Banach spaces by Hyers [18]. Subsequently, this result of Hyers was generalized by Aoki [2] for additive mappings and by Rassias [42] for linear mappings by considering an unbounded Cauchy difference. The article of Rassias [42] has provided a lot of influence in the development of what we now call generalized Ulam-Hyers stability of functional It was further generalized via excellent results obtained by a number of authors [2, 17, 40, 42, 44]. The terminology generalized Ulam - Hyers stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [1, 14, 19, 23, 25, 41, 43, 45] .

K.W. Jun and H.M. Kim [21] introduced the following generalized quadratic and additive type functional equation

$$f\left(\sum_{i=1}^{n} x_i\right)+(n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i+x_j) \tag{1.1}$$

in real vector spaces. For $n=3$, Pl.Kannappan proved that a function f satisfies the functional equation (1.1) if and only if there exists a symmetric bi-additive function B and an additive function A such that $f(x)=B(x,x)+A(x)$ (see [24]). The Hyers-Ulam stability for the equation (1.1) when $n=3$ was proved by S.M.

*Corresponding author.
E-mail addresses: annarun2002@yahoo.co.in (M. Arunkumar), drpnarasimman81@gmail.com (P. Narasimman) sathya24mathematics@gmail.com (E.Sathya), mrnmahesh@yahoo.com (N. Mahesh Kumar).

Jung [22]. The Hyers-Ulam-Rassias stability for the equation (1.1) when $n = 4$ was also investigated by I.S. Chang et al., [13].

The general solution and the generalized Hyers-Ulam stability for the **quadratic and additive type functional equation**

$$f(x+ay) + af(x-y) = f(x-ay) + af(x+y) \tag{1.2}$$

for any positive integer a with $a \neq -1, 0, 1$ was discussed by K.W. Jun and H.M. Kim [20]. Also, A. Najati and M.B. Moghimi [33] investigated the generalized Hyers-Ulam-Rassias stability for the **quadratic and additive type functional equation** of the form

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 2f(2x) - 4f(x) \tag{1.3}$$

Recently, the general solution and generalized Ulam - Hyers stability of a mixed type Additive Quadratic(AQ)-functional equation

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y). \tag{1.4}$$

was investigated by M. Arunkumar and J.M. Rassias [5].

In this paper, the authors established the general solution and generalized Ulam - Hyers stability of an 3 dimensional additive quadratic functional equation

$$\begin{aligned} h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ = h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) \\ + 2h(y) + 4h(z) + 5[h(y) + h(-y)] + 14[h(z) + h(-z)] \end{aligned} \tag{1.5}$$

via Banach space and non-Archimedean fuzzy Banach Space using direct and fixed point methods. The above functional equation having

$$h(x) = ax + bx^2 \tag{1.6}$$

In Section 2 the the general solution of (1.5) is provided.

In Section 3 the generalized Ulam - Hyers stability of (1.5) is proved via Banach spaces using direct and fixed point methods.

In section 4 the generalized Ulam - Hyers stability of (1.5) is given in non-Archimedean fuzzy Banach Space using direct and fixed point methods.

2 General Solution of (1.5)

In this section, the general solution of the 3 dimensional additive quadratic functional equation (1.5) is given. For this, let us consider A and B be real vector spaces.

Theorem 2.1. *If $h : A \to B$ be an odd mapping satisfying (1.5) for all $x, y, z \in A$ if and only if $h : A \to B$ satisfying*

$$h(x+y) = h(x) + h(y) \tag{2.1}$$

for all $x, y \in A$.

Proof. Assume $h : A \to B$ be an odd mapping satisfying (1.5). Setting (x, y, z) by $(0, 0, 0)$ in (1.5), we get $h(0) = 0$. Letting z by 0 in (1.5) and using oddness of h, we obtain

$$h(x+2y) = h(x+y) + h(y) \tag{2.2}$$

for all $x, y \in A$. Setting x by 0 and x by y in (2.2) respectively, we reach

$$h(2y) = 2h(y) \qquad h(3y) = 3h(y) \tag{2.3}$$

for all $y \in A$. In general, for any positive integer a, we have

$$h(ay) = ah(y) \tag{2.4}$$

for all $y \in A$. Replacing x by $x-y$ in (2.2), we arrive (2.1) as desired.

Conversely, assume $h : A \to B$ be an odd mapping satisfying (2.1). Setting (x, y) by $(0, 0)$ in (2.1), we get $h(0) = 0$. Setting x by y and x by $2y$ in (2.1) respectively, we reach

$$h(2y) = 2h(y) \qquad h(3y) = 3h(y) \tag{2.5}$$

for all $y \in A$. In general, for any positive integer b, we have

$$h(by) = bh(y) \tag{2.6}$$

for all $y \in A$. Replacing (x, y) by $(x+2y, 3z)$ in (2.1), using (2.1) and (2.5), we have

$$h(x+2y+3z) = h(x+2y) + h(3z) = h(x) + 2h(y) + 3h(z) \tag{2.7}$$

for all $x, y, z \in A$. Again replacing (x, y) by $(x-2y, 3z)$ in (2.1), using (2.1), oddness of h and (2.5), we get

$$h(x-2y+3z) = h(x-2y) + h(3z) = h(x) - 2h(y) + 3h(z) \tag{2.8}$$

for all $x, y, z \in A$. Setting (x, y) by $(x+2y, -3z)$ in (2.1), using (2.1), oddness of h and (2.5), we get

$$h(x+2y-3z) = h(x+2y) + h(-3z) = h(x) + 2h(y) - 3h(z) \tag{2.9}$$

for all $x, y, z \in A$. Again setting (x, y) by $(-x+2y, 3z)$ in (2.1), using (2.1), oddness of h and (2.5), we get

$$h(-x+2y+3z) = h(-x+2y) + h(3z) = -h(x) + 2h(y) + 3h(z) \tag{2.10}$$

for all $x, y, z \in A$. Adding (2.7), (2.8), (2.9) and (2.10), we reach

$$\begin{aligned} h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ = 2h(x) + 4h(y) + 6h(z) \end{aligned} \tag{2.11}$$

for all $x, y, z \in A$. Replacing (x, y) by $(x+y, z)$ in (2.1), using (2.1) and (2.5), we have

$$h(x+y+z) = h(x+y) + h(z) = h(x) + h(y) + h(z) \tag{2.12}$$

for all $x, y, z \in A$. Again replacing (x, y) by $(x-y, z)$ in (2.1), using (2.1), oddness of h and (2.5), we get

$$h(x-y+z) = h(x-y) + h(z) = h(x) - h(y) + h(z) \tag{2.13}$$

for all $x, y, z \in A$. Setting (x, y) by $(x+y, -z)$ in (2.1), using (2.1), oddness of h and (2.5), we get

$$h(x+y-z) = h(x+y) + h(-z) = h(x) + h(y) - h(z) \tag{2.14}$$

for all $x, y, z \in A$. Again setting (x, y) by $(-x+y, z)$ in (2.1), using (2.1), oddness of h and (2.5), we get

$$h(-x+y+z) = h(-x+y) + h(z) = -h(x) + h(y) + h(z) \tag{2.15}$$

for all $x, y, z \in A$. Adding (2.12), (2.13), (2.14) and (2.15), we reach

$$\begin{aligned} h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) \\ = 2h(x) + 2h(y) + 2h(z) \end{aligned} \tag{2.16}$$

for all $x, y, z \in A$. Using (2.16) in (2.11), we arrive

$$\begin{aligned} & h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ & = h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) + 2h(y) + 4h(z) \end{aligned} \tag{2.17}$$

for all $x, y, z \in A$. Adding $5f(y) + 14f(z)$ on both sides of (2.17), we get

$$\begin{aligned} & h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) + 5f(y) + 14f(z) \\ & = h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) \\ & \qquad + 2h(y) + 4h(z) + 5f(y) + 14f(z) \end{aligned} \tag{2.18}$$

for all $x, y, z \in A$. It follows from (2.18) that

$$\begin{aligned} h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ = h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) \\ + 2h(y) + 4h(z) + 5[f(y) - f(y)] + 14[f(z) - f(z)] \end{aligned} \tag{2.19}$$

for all $x, y, z \in A$. Using oddness of h in (2.19), we desired our result. □

Theorem 2.2. *If $h : A \to B$ be an even mapping satisfying (1.5) for all $x, y, z \in A$ if and only if $h : A \to B$ satisfying*

$$h(x+y) + h(x-y) = 2h(x) + 2h(y) \tag{2.20}$$

for all $x, y \in A$.

Proof. Assume $h : A \to B$ be an even mapping satisfying (1.5). Setting (x, y, z) by $(0, 0, 0)$ in (1.5), we get $h(0) = 0$. Letting z by 0 in (1.5) and using evenness of h, we obtain

$$h(x+2y) + h(x-2y) = h(x+y) + h(x-y) + 6h(y) \tag{2.21}$$

for all $x, y \in A$. Setting x by 0 and x by y in (2.21) respectively, we reach

$$h(2y) = 4h(y) \qquad h(3y) = 9h(y) \tag{2.22}$$

for all $y \in A$. In general, for any positive integer c, we have

$$h(cy) = c^2 h(y) \tag{2.23}$$

for all $y \in A$. Interchanging x and y in (2.21) and using evenness of h, we arrive

$$h(2x+y) + h(2x-y) = h(x+y) + h(x-y) + 6h(x) \tag{2.24}$$

for all $x, y \in A$. By Theorem 2.1 of [12], we desired our result.

Conversely, assume $h : A \to B$ be an even mapping satisfying (2.20). Setting (x, y) by $(0, 0)$ in (2.20), we get $h(0) = 0$. Setting x by y and x by $2y$ in (2.20) respectively, we reach

$$h(2y) = 4h(y) \qquad h(3y) = 9h(y) \tag{2.25}$$

for all $y \in A$. In general, for any positive integer d, we have

$$h(dy) = d^2 h(y) \tag{2.26}$$

for all $y \in A$. Replacing (x, y) by $(x+2y, 3z)$ in (2.20) and using (2.25), we have

$$h(x+2y+3z) + h(x+2y-3z) = 2h(x+2y) + 2h(3z) = 2h(x+2y) + 18h(z) \tag{2.27}$$

for all $x, y, z \in A$. Again replacing (x, y) by $(x-2y, 3z)$ in (2.20)and using (2.25), we get

$$h(x-2y+3z) + h(x-2y-3z) = 2h(x-2y) + 2h(3z) = 2h(x-2y) + 18h(z) \tag{2.28}$$

for all $x, y, z \in A$. Adding (2.27) and (2.28) and using (2.24), we reach

$$\begin{aligned} h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ = 2h(x+2y) + 2h(x-2y) + 36h(z) \\ = 2h(x+y) + 2h(x-y) + 12h(y) + 36h(z) \end{aligned} \tag{2.29}$$

for all $x, y, z \in A$. Replacing (x, y) by $(x+y, z)$ in (2.20), we have

$$h(x+y+z) + h(x+y-z) = 2h(x+y) + 2h(z) \tag{2.30}$$

for all $x, y, z \in A$. Again replacing (x, y) by $(x-y, z)$ in (2.20), we get

$$h(x-y+z) + h(x-y-z) = 2h(x-y) + 2h(z) \tag{2.31}$$

for all $x, y, z \in A$. Adding (2.30) and (2.31), we reach

$$\begin{aligned} &h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z) \\ &\quad = 2h(x+y)+2h(x-y)+4h(z) \end{aligned} \tag{2.32}$$

for all $x, y, z \in A$. Using (2.32) in (2.29) and using evenness of h, we arrive

$$\begin{aligned} &h(x+2y+3z)+h(x+2y-3z)+h(x-2y+3z)+h(-x+2y+3z) \\ &= h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z)+12h(y)+32h(z) \\ &= h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z)+2h(y)+4h(z) \\ &\qquad + 5[h(y)+h(y)]+14[h(z)+h(z)] \end{aligned} \tag{2.33}$$

for all $x, y, z \in A$. Using evenness of h in (2.33), we desired our result. □

3 Stability Results for (1.5) in Banach Space

In this section, the generalized Ulam - Hyers stability of the functional equation (1.5) is provided. Throughout this section, let us consider X and Y to be a normed space and a Banach space, respectively. Define a mapping $Dh : X \to Y$ by

$$\begin{aligned} Dh(x,y,z) =& h(x+2y+3z)+h(x+2y-3z)+h(x-2y+3z)+h(-x+2y+3z) \\ &- [h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z) \\ &+ 2h(y)+4h(z)+5[h(y)+h(-y)]+14[h(z)+h(-z)]] \end{aligned}$$

for all $x, y, z \in X$.

3.1 Banach Spaces: Stability Results: Hyers Direct Method

Theorem 3.1. *Let $j \in \{-1, 1\}$ and $\psi : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\psi\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{6^{nj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{n\to\infty} \frac{\psi\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{6^{nj}} = 0 \tag{3.1}$$

for all $x, y, z \in X$. Let $h_a : X \to Y$ be an odd function satisfying the inequality

$$\|Dh_a(x,y,z)\| \le \psi(x,y,z) \tag{3.2}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ which satisfies (1.5) and

$$\|h_a(x) - A(x)\| \le \frac{1}{6} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi_A(6^{kj}x)}{6^{kj}} \tag{3.3}$$

where Ψ_A and $A(x)$ are respectively defined by

$$\Psi_A(6^{kj}x) = \psi\left(6^{kj}x, 6^{kj}x, 6^{kj}x\right) + \frac{1}{2}\psi\left(6^{kj}2x, 6^{kj}x, 0\right) + \frac{1}{2}\psi\left(0, 6^{kj}x, 0\right) \tag{3.4}$$

$$A(x) = \lim_{n\to\infty} \frac{h_a(6^{nj}x)}{6^{nj}} \tag{3.5}$$

for all $x \in X$.

Proof. Assume $j = 1$. Replacing (x, y, z) by (x, x, x) in (3.2) and using oddness of h_a, we obtain

$$\|h_a(6x) + h_a(2x) + h_a(4x) - h_a(3x) - 9h_a(x)\| \le \psi(x,x,x) \tag{3.6}$$

for all $x \in X$. Again replacing (x, y, z) by $(2x, x, 0)$ in (3.2) and using oddness of h_a, we have

$$\|h_a(4x) - h_a(3x) - h_a(x)\| \le \frac{1}{2}\psi(2x,x,0) \tag{3.7}$$

for all $x \in X$. Finally replacing (x, y, z) by $(0, x, 0)$ in (3.2) and using oddness of h_a, we get

$$\|h_a(2x) - 2h_a(x)\| \leq \frac{1}{2}\psi(0, x, 0) \tag{3.8}$$

for all $x \in X$. Now from (3.6), (3.7) and (3.8), we arrive

$$\begin{aligned} \|h_a(6x) - 6h_a(x)\| &= \|h_a(6x) + h_a(2x) + h_a(4x) - h_a(3x) - 9h_a(x) \\ &\qquad - h_a(4x) + h_a(3x) + h_a(x) - h_a(2x) + 2h_a(x)\| \\ &\leq \|h_a(6x) + h_a(2x) + h_a(4x) - h_a(3x) - 9h_a(x)\| \\ &\qquad \|h_a(4x) - h_a(3x) - h_a(x)\| + \|h_a(2x) - 2h_a(x)\| \\ &\leq \psi(x, x, x) + \frac{1}{2}\psi(2x, x, 0) + \frac{1}{2}\psi(0, x, 0) \end{aligned} \tag{3.9}$$

for all $x \in X$. It follows from (3.9) that

$$\|h_a(6x) - 6h_a(x)\| \leq \Psi_A(x) \tag{3.10}$$

where

$$\Psi_A(x) = \psi(x, x, x) + \frac{1}{2}\psi(2x, x, 0) + \frac{1}{2}\psi(0, x, 0) \tag{3.11}$$

for all $x \in X$. Divide (3.10) by 6, we get

$$\left\| h_a(x) - \frac{h_a(x)}{6} \right\| \leq \frac{1}{6}\Psi_A(x) \tag{3.12}$$

for all $x \in X$. Now replacing x by $6x$ and dividing by 6 in (3.12), we obtain

$$\left\| \frac{h_a(6x)}{6} - \frac{h_a(6^2x)}{6^2} \right\| \leq \frac{\Psi_A(6x)}{6^2} \tag{3.13}$$

for all $x \in X$. It follows from (3.12) and (3.13) that

$$\begin{aligned} \left\| h_a(x) - \frac{h_a(6^2x)}{6^2} \right\| &\leq \left\| h_a(x) - \frac{h_a(6x)}{6} \right\| + \left\| \frac{h_a(6x)}{6} - \frac{h_a(6^2x)}{6^2} \right\| \\ &\leq \frac{1}{6}\left[\Psi_A(x) + \frac{\Psi_A(6x)}{6} \right] \end{aligned} \tag{3.14}$$

for all $x \in X$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} \left\| h_a(x) - \frac{h_a(6^nx)}{6^n} \right\| &\leq \frac{1}{6}\sum_{k=0}^{n-1} \frac{\Psi_A(6^kx)}{6^k} \\ &\leq \frac{1}{6}\sum_{k=0}^{\infty} \frac{\psi(6^kx)}{6^k} \end{aligned} \tag{3.15}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{ \frac{h_a(6^nx)}{6^n} \right\},$$

replace x by 6^mx and divide by 6^m in (3.15), for any $m, n > 0$, we arrive the sequence $\left\{ \frac{h_a(6^nx)}{6^n} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{n\to\infty} \frac{h_a(6^nx)}{6^n}, \quad \forall\ x \in X.$$

Letting $n \to \infty$ in (3.15) we see that (3.3) holds for all $x \in X$. To prove A satisfies (1.5), replacing (x, y, z) by $(6^nx, 6^ny, 6^nz)$ and dividing by 6^n in (3.2), we obtain

$$\frac{1}{6^n}\left\| Dh_a(6^nx, 6^ny, 6^nz) \right\| \leq \frac{1}{6^n}\psi(6^nx, 6^ny, 6^nz)$$

for all $x, y, z \in X$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$DA(x, y, z) = 0.$$

for all $x, y, z \in X$. Hence A satisfies (1.5) for all $x, y, z \in X$. To show A is unique, let $B(x)$ be another additive mapping satisfying (1.5) and (3.3), then

$$\begin{aligned}
\|A(x) - B(x)\| &= \frac{1}{6^n} \|A(6^n x) - B(6^n x)\| \\
&\leq \frac{1}{6^n} \{\|A(6^n x) - h_a(6^n x)\| + \|h_a(6^n x) - B(6^n x)\|\} \\
&\leq \frac{1}{3} \sum_{k=0}^{\infty} \frac{\Psi_A(6^{k+n} x)}{6^{(k+n)}} \\
&\to 0 \quad as \quad n \to \infty
\end{aligned}$$

for all $x \in X$. Hence A is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.5).

Corollary 3.1. *Let θ and s be non negative real numbers. Let an odd function $h_a : X \to Y$ satisfy the inequality*

$$\|Dh_a(x, y, z)\| \leq \begin{cases} \theta, & \\ \theta \left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1; \\ \theta \left\{||x||^s ||y||^s ||z||^s\right\}, & s \neq 1; \\ \theta \left\{||x||^s ||y||^s ||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1; \end{cases} \tag{3.16}$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|h_a(x) - A(x)\| \leq \begin{cases} \dfrac{2\theta}{5}, \\ \dfrac{(8 + 2^s)\theta||x||^s}{2|6 - 6^s|}, \\ \dfrac{\theta||x||^{3s}}{|6 - 6^{3s}|}, \\ \dfrac{(10 + 2^s)\theta||x||^s}{2|6 - 6^{3s}|}, \end{cases} \tag{3.17}$$

for all $x \in X$.

Theorem 3.2. *Let $j \in \{-1, 1\}$ and $\psi : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\psi\left(6^{nj} x, 6^{nj} y, 6^{nj} z\right)}{36^{nj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{n \to \infty} \frac{\psi\left(6^{nj} x, 6^{nj} y, 6^{nj} z\right)}{36^{nj}} = 0 \tag{3.18}$$

for all $x, y, z \in X$. Let $h_q : X \to Y$ be an even function satisfying the inequality

$$\left\|Dh_q(x, y, z)\right\| \leq \psi\left(x, y, z\right) \tag{3.19}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies (1.5) and

$$\left\|h_q(x) - Q(x)\right\| \leq \frac{1}{36} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi_Q(6^{kj} x)}{36^{kj}} \tag{3.20}$$

where Ψ_Q and $A(x)$ are respectively defined by

$$\Psi_Q(6^{kj} x) = \psi\left(6^{kj} x, 6^{kj} x, 6^{kj} x\right) + \frac{1}{2}\psi\left(6^{kj} 2x, 6^{kj} x, 0\right) + \frac{1}{4}\psi\left(0, 6^{kj} x, 0\right) \tag{3.21}$$

$$Q(x) = \lim_{n \to \infty} \frac{h_q(6^{nj} x)}{36^{nj}} \tag{3.22}$$

for all $x \in X$.

Proof. Assume $j = 1$. Replacing (x, y, z) by (x, x, x) in (3.19) and using evenness of h_q, we get

$$\left\|h_q(6x) + h_q(2x) + h_q(4x) - h_q(3x) - 47h_q(x)\right\| \leq \psi(x, x, x) \tag{3.23}$$

for all $x \in X$. Again replacing (x, y, z) by $(2x, x, 0)$ in (3.19) and using evenness of h_q, we get

$$\left\|h_q(4x) - h_q(3x) - 7h_q(x)\right\| \leq \frac{1}{2}\psi(2x, x, 0) \tag{3.24}$$

for all $x \in X$. Finally replacing (x, y, z) by $(0, x, 0)$ in (3.19) and using oddness of h_q, we get

$$\left\|h_q(2x) - 4h_q(x)\right\| \leq \frac{1}{4}\psi(0, x, 0) \tag{3.25}$$

for all $x \in X$. Now from (3.23), (3.24) and (3.25), we arrive

$$\begin{aligned}
\left\|h_q(6x) - 36h_q(x)\right\| &= \left\|h_q(6x) + h_q(2x) + h_q(4x) - h_q(3x) - 47h_q(x)\right. \\
&\qquad \left. -h_q(4x) + h_q(3x) + 7h_q(x) - h_q(2x) + 4h_q(x)\right\| \\
&\leq \left\|h_q(6x) + h_q(2x) + h_q(4x) - h_q(3x) - 47h_q(x)\right\| \\
&\qquad \left\|h_q(4x) - h_q(3x) - 7h_q(x)\right\| + \left\|h_q(2x) - 4h_q(x)\right\| \\
&\leq \psi(x, x, x) + \frac{1}{2}\psi(2x, x, 0) + \frac{1}{4}\psi(0, x, 0)
\end{aligned} \tag{3.26}$$

for all $x \in X$. It follows from (3.26) that

$$\left\|h_q(6x) - 36h_q(x)\right\| \leq \Psi_Q(x) \tag{3.27}$$

where

$$\Psi_Q(x) = \psi(x, x, x) + \frac{1}{2}\psi(2x, x, 0) + \frac{1}{4}\psi(0, x, 0) \tag{3.28}$$

for all $x \in X$. Divide (3.27) by 6, we get

$$\left\|h_q(x) - \frac{h_q(6x)}{36}\right\| \leq \frac{1}{36}\Psi_Q(x) \tag{3.29}$$

for all $x \in X$. The rest of the proof is similar lines to that of Theorem 3.1. □

The following Corollary is an immediate consequence of Theorem 3.2 concerning the stability of (1.5).

Corollary 3.2. *Let θ and s be non negative real numbers. Let an even function $h_q : X \to Y$ satisfy the inequality*

$$\left\|Dh_q(x, y, z)\right\| \leq \begin{cases} \theta, & \\ \theta\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 2; \\ \theta\left\{||x||^s||y||^s||z||^s\right\}, & 3s \neq 2; \\ \theta\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 2; \end{cases} \tag{3.30}$$

for all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\left\|h_q(x) - Q(x)\right\| \leq \begin{cases} \dfrac{9\theta}{35}, \\ \dfrac{(15 + 2^{s+1})\theta||x||^s}{4|36 - 6^s|}, \\ \dfrac{\theta||x||^{3s}}{|36 - 6^{3s}|}, \\ \dfrac{(19 + 2^{s+1})\theta||x||^{3s}}{4|36 - 6^{3s}|}, \end{cases} \tag{3.31}$$

for all $x \in X$.

Now we are ready to prove our main theorem.

Theorem 3.3. *Let $j \in \{-1,1\}$ and $\psi : X^3 \to [0,\infty)$ be a function satisfying (3.1) and (3.18) for all $x,y,z \in X$. Let $h : X \to Y$ be a function satisfying the inequality*

$$\|Dh(x,y,z)\| \leq \psi(x,y,z) \tag{3.32}$$

for all $x,y,z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ which satisfies (1.5) and

$$\begin{aligned}\|h(x)-A(x)-Q(x)\| \leq \frac{1}{2}\Bigg[&\frac{1}{6}\sum_{k=\frac{1-j}{6}}^{\infty}\left(\frac{\Psi_A(6^{kj}x)}{6^{kj}}+\frac{\Psi_A(-6^{kj}x)}{6^{kj}}\right)\\ &+\frac{1}{36}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_Q(6^{kj}x)}{36^{kj}}+\frac{\Psi_Q(-6^{kj}x)}{36^{kj}}\right)\Bigg]\end{aligned} \tag{3.33}$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ are defined in (3.4) and (3.21) respectively for all $x \in X$.

Proof. Let $h_o(x) = \dfrac{h_a(x)-h_a(-x)}{2}$ for all $x \in X$. Then $h_o(0)=0$ and $h_o(-x) = -h_o(x)$ for all $x \in X$. Hence

$$\|Dh_o(x,y,z)\| \leq \frac{\psi(x,y,z)}{2}+\frac{\psi(-x,-y,-z)}{2} \tag{3.34}$$

for all $x,y,z \in X$. By Theorem 3.1, we have

$$\|h_o(x)-A(x)\| \leq \frac{1}{12}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_A(6^{kj}x)}{6^{kj}}+\frac{\Psi_A(-6^{kj}x)}{6^{kj}}\right) \tag{3.35}$$

for all $x \in X$. Also, let $h_e(x) = \dfrac{h_q(x)+h_q(-x)}{2}$ for all $x \in X$. Then $h_e(0)=0$ and $h_e(-x)=h_e(x)$ for all $x \in X$. Hence

$$\|Dh_e(x,y,z)\| \leq \frac{\psi(x,y,z)}{2}+\frac{\psi(-x,-y,-z)}{2} \tag{3.36}$$

for all $x,y,z \in X$. By Theorem 3.2, we have

$$\|h_e(x)-Q(x)\| \leq \frac{1}{72}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_Q(6^{kj}x)}{36^{kj}}+\frac{\Psi_Q(-6^{kj}x)}{36^{kj}}\right) \tag{3.37}$$

for all $x \in X$. Define

$$h(x) = h_e(x)+h_o(x) \tag{3.38}$$

for all $x \in X$. From (3.35),(3.37) and (3.38), we arrive

$$\begin{aligned}\|h(x)-A(x)-Q(x)\| &= \|h_e(x)+h_o(x)-A(x)-Q(x)\|\\ &\leq \|h_o(x)-A(x)\|+\|h_e(x)-Q(x)\|\\ &\leq \frac{1}{12}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_A(6^{kj}x)}{6^{kj}}+\frac{\Psi_A(-6^{kj}x)}{6^{kj}}\right)\\ &\quad+\frac{1}{72}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_Q(6^{kj}x)}{36^{kj}}+\frac{\Psi_Q(-6^{kj}x)}{36^{kj}}\right)\end{aligned}$$

for all $x \in X$. Hence the theorem is proved. □

Using Corollaries 3.1 and 3.2, we have the following Corollary concerning the stability of (1.5).

Corollary 3.3. *Let θ and s be non negative real numbers. Let a function $h : X \to Y$ satisfy the inequality*

$$\|Dh(x,y,z)\| \leq \begin{cases} \theta, & \\ \theta\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1,2; \\ \theta\left\{||x||^s||y||^s||z||^s\right\}, & 3s \neq 1,2; \\ \theta\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1,2; \end{cases} \tag{3.39}$$

for all $x, y, z \in X$. Then there exists a unique additive function $\mathcal{A} : X \to Y$ and a unique quadratic function $\mathcal{Q} : X \to Y$ such that

$$\|h(x) - \mathcal{A}(x) - \mathcal{Q}(x)\| \leq \begin{cases} \theta\left(\frac{2}{5} + \frac{9\theta}{35}\right), \\ \theta||x||^s\left(\frac{(8+2^s)}{2|6-6^s|} + \frac{(15+2^{s+1})}{4|36-6^s|}\right), \\ \theta||x||^{3s}\left(\frac{1}{|6-6^{3s}|} + \frac{1}{|36-6^{3s}|}\right), \\ \theta||x||^{3s}\left(\frac{(10+2^s)}{2|6-6^{3s}|} + \frac{(19+2^{s+1})}{4|36-6^{3s}|}\right) \end{cases} \tag{3.40}$$

for all $x \in X$.

3.2 Banach Space : Stability Results : Fixed Point Method

In this section, the generalized Ulam - Hyers stability of the functional equation (1.5) is proved using fixed point method provided.

Throughout this section let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ be a Banach space Define a mapping $Dh : \mathcal{X} \to \mathcal{Y}$ by

$$\begin{aligned} Dh(x,y,z) =& h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ &- h(x+y+z) - h(x+y-z) - h(x-y+z) - h(-x+y+z) \\ &- 2h(y) - 4h(z) - 5[h(y)+h(-y)] - 14[h(z)+h(-z)] \end{aligned}$$

for all $x, y, z \in \mathcal{X}$.

Now, we present the following theorem due to B. Margolis and J.B. Diaz [27] for fixed point theory.

Theorem 3.4. *[27] Suppose that for a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty, \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that the properties hold:

(FP1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;

(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T;

(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;

(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using the above theorem, we obtain the Hyers - Ulam stability of (1.5).

Theorem 3.5. *Let $Dh_a : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\psi : \mathcal{X}^3 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\rho_i^n} \psi(\rho_i^n x, \rho_i^n y, , \rho_i^n z) = 0 \tag{3.41}$$

where

$$\rho_i = \begin{cases} 6 & if \quad i = 0, \\ \frac{1}{6} & if \quad i = 1 \end{cases} \tag{3.42}$$

such that the functional inequality

$$\|Dh_a(x,y,z)\| \leq \psi(x,y,z) \tag{3.43}$$

holds for all $x, y \in \mathcal{X}$. Assume that there exists $L = L(i)$ such that the function

$$x \to \Phi_{AQ}(x) = \Psi_A\left(\frac{x}{6}\right)$$

where $\Psi_A(x)$ is defined in (3.11) with the property

$$\frac{1}{\rho_i}\Phi_{AQ}(\rho_i x) = L\,\Phi_{AQ}(x) \tag{3.44}$$

for all $x \in \mathcal{X}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ satisfying the functional equation (1.5) and

$$\| h_a(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right)\Phi_{AQ}(x) \tag{3.45}$$

for all $x \in \mathcal{X}$.

Proof. Consider the set

$$\Omega = \{h/h : \mathcal{X} \to \mathcal{Y},\ h(0) = 0\}$$

and introduce the generalized metric on Ω,

$$\inf\{M \in (0,\infty) :\| h(x) - g(x) \|\leq M\,\Phi_{AQ}(x), x \in \mathcal{X}\}. \tag{3.46}$$

It is easy to see that (3.46) is complete with respect to the defined metric. Define $J : \Omega \to \Omega$ by

$$Jh(x) = \frac{1}{\rho_i}h(\rho_i x),$$

for all $x \in \mathcal{X}$. Now, from (3.46) and $h, g \in \Omega$

$$\begin{aligned}
&\inf\left\{M \in (0,\infty) :\| h(x) - g(x) \|\leq M\,\Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{M \in (0,\infty) : \left\|\frac{1}{\rho_i}h(\rho_i x) - \frac{1}{\rho_i}g(\rho_i x)\right\| \leq \frac{M}{\rho_i}\Phi_{AQ}(\rho_i x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{LM \in (0,\infty) : \left\|\frac{1}{\rho_i}h(\rho_i x) - \frac{1}{\rho_i}g(\rho_i x)\right\| \leq LM\Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{LM \in (0,\infty) :\| Jh(x) - Jg(x) \|_{\mathcal{Y}}\leq LM\Phi_{AQ}(x), x \in \mathcal{X}\right\}.
\end{aligned}$$

This implies J is a strictly contractive mapping on Ω with Lipschitz constant L. It follows from (3.46),(3.10) and (3.44) for the case $i = 0$, we reach

$$\inf\{1 \in (0,\infty) : \|h_a(6x) - 6h_a(x)\| \leq \Psi_A(x), x \in \mathcal{X}\} \quad or \tag{3.47}$$

$$\begin{aligned}
&\inf\left\{1 \in (0,\infty) : \left\|\frac{h_a(x)}{6} - h_a(x)\right\| \leq \frac{1}{6}\Psi_A(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{L \in (0,\infty) : \|Jh_a f(x) - h_a(x)\| \leq L\,\Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{L^1 \in (0,\infty) : \|Jh_a f(x) - h_a(x)\| \leq L\,\Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or
\end{aligned}$$

$$\inf\left\{L^{1-0} \in (0,\infty) : \|Jh_a f(x) - h_a(x)\| \leq L\,\Phi_{AQ}(x), x \in \mathcal{X}\right\}. \tag{3.48}$$

Again replacing $x = \frac{x}{6}$ in (3.47) and (3.44) for the case $i = 1$ we get

$$\begin{aligned}
&\inf\left\{1 \in (0,\infty) : \left\|h_a(x) - 6h_a\left(\frac{x}{6}\right)\right\| \leq \Psi_A\left(\frac{x}{6}\right), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{1 \in (0,\infty) : \|h_a f(x) - Jh_a(x)\| \leq \Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{L^0 \in (0,\infty) : \|h_a f(x) - Jh_a(x)\| \leq \Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or
\end{aligned}$$

$$\inf\left\{L^{1-1} \in (0,\infty) : \|h_a f(x) - Jh_a(x)\| \leq \Phi_{AQ}(x), x \in \mathcal{X}\right\}. \tag{3.49}$$

Thus, from (4.8) and (3.49), we arrive

$$\inf\left\{L^{1-i}\in(0,\infty):\|h_af(x)-Jh_a(x)\|\le L^{1-i}\Phi_{AQ}(x),x\in\mathcal{X}\right\}. \tag{3.50}$$

Hence property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{A}$ of J in Ω such that

$$\mathcal{A}(x)=\lim_{n\to\infty}\frac{1}{\rho_i^n}h_a(\rho_i^n x) \tag{3.51}$$

for all $x\in\mathcal{X}$. In order to show that $\mathcal{A}$ satisfies (1.5), replacing (x,y,z) by $(\rho_i^n x,\rho_i^n y,\rho_i^n z)$ and dividing by ρ_i^n in (3.43), we have

$$\|\mathcal{A}(x,y,z)\|=\lim_{n\to\infty}\frac{1}{\rho_i^n}\|Dh_a(\rho_i^n x,\rho_i^n y,\rho_i^n z)\|\le\lim_{n\to\infty}\frac{1}{\rho_i^n}\psi(\rho_i^n x,\rho_i^n y,\rho_i^n z)=0$$

for all $x,y,z\in\mathcal{X}$, and so the mapping $\mathcal{A}$ is Additive. i.e., $\mathcal{A}$ satisfies the functional equation (1.5).

By property (FP3), $\mathcal{A}$ is the unique fixed point of J in the set $\Delta=\{\mathcal{A}\in\Omega:d(h_a,\mathcal{A})<\infty\}$, $\mathcal{A}$ is the unique function such that

$$\inf\{M\in(0,\infty):\|h_a(x)-\mathcal{A}(x)\|\le M\Phi_{AQ}(x),x\in\mathcal{X}\}.$$

Finally by property (FP4), we obtain

$$\|h_a(x)-\mathcal{A}(x)\|\le\|h_a(x)-Jh_a(x)\|$$

this implies

$$\|h_a(x)-\mathcal{A}(x)\|\le\frac{L^{1-i}}{1-L}$$

which yields

$$\inf\left\{\frac{L^{1-i}}{1-L}\in(0,\infty):\|h_a(x)-\mathcal{A}(x)\|\le\left(\frac{L^{1-i}}{1-L}\right)\Phi_{AQ}(x),x\in\mathcal{X}\right\}$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.5 concerning the stability of (1.5).

Corollary 3.4. *Let $Dh_a:\mathcal{X}\to\mathcal{Y}$ be a mapping. If there exist real numbers λ and s such that*

$$\|Dh_af(x,y,z)\|\le\begin{cases}\lambda, & \\ \lambda\left\{||x||^s+||y||^s+||z||^s\right\}, & s\ne1;\\ \lambda\left\{||x||^s||y||^s||z||^s\right\}, & 3s\ne1;\\ \lambda\left\{||x||^s||y||^s||z||^s+\left\{||x||^{3s}+||y||^{3s}+||z||^{3s}\right\}\right\}, & 3s\ne1;\end{cases} \tag{3.52}$$

for all $x,y,z\in\mathcal{X}$, then there exists a unique additive function $\mathcal{A}:\mathcal{X}\to\mathcal{Y}$ such that

$$\|f(x)-\mathcal{A}(x)\|\le\begin{cases}\dfrac{2\lambda}{5},\\ \dfrac{(8+2^s)\lambda||x||^s}{2|6-6^s|},\\ \dfrac{\lambda||x||^{3s}}{|6-6^{3s}|},\\ \dfrac{(10+2^s)\lambda||x||^s}{2|6-6^{3s}|},\end{cases} \tag{3.53}$$

for all $x\in\mathcal{X}$.

Proof. Let

$$\psi(x,y,z)=\begin{cases}\lambda,\\ \lambda\left\{||x||^s+||y||^s+||z||^s\right\},\\ \lambda\left\{||x||^s||y||^s||z||^s\right\},\\ \lambda\left\{||x||^s||y||^s||z||^s+\left\{||x||^{3s}+||y||^{3s}+||z||^{3s}\right\}\right\}\end{cases}$$

for all $x, y, z \in \mathcal{X}$. Now

$$\frac{1}{\rho_i^n}\psi(\rho_i^n x, \rho_i^n y, \rho_i^n z) = \begin{cases} \frac{\lambda}{\rho_i^n}, \\ \frac{\lambda}{\rho_i^n}\{||\rho_i^n x||^s + ||\rho_i^n y||^s + ||\rho_i^n z||^s\}, \\ \frac{\lambda}{\rho_i^n}||\rho_i^n x||^s\,||\rho_i^n y||^s\,||\rho_i^n z||^s, \\ \frac{\lambda}{\rho_i^n}\Big\{||\rho_i^n x||^s\,||\rho_i^n y||^s\,||\rho_i^n z||^s \\ \quad +\Big\{||\rho_i^n x||^{3s} + ||\rho_i^n y||^{3s} + ||\rho_i^n z||^{3s}\Big\}\Big\}, \end{cases} = \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (3.41) holds. But, we have

$$\Phi_{AQ}(x) = \Psi_A(\frac{x}{6})$$

has the property

$$\frac{1}{\rho_i}\Phi_{AQ}(\rho_i x) = L\,\Phi_{AQ}(x)$$

for all $x \in \mathcal{X}$. Hence

$$\Phi_{AQ}(x) = \Psi_A\left(\frac{x}{6}\right) = \psi\left(\frac{x}{6}, \frac{x}{6}, \frac{x}{6}\right) + \frac{1}{2}\psi\left(\frac{2x}{6}, \frac{x}{6}, 0\right) + \frac{1}{2}\psi\left(0, \frac{x}{6}, 0\right)$$

$$= \begin{cases} 2\lambda, \\ \frac{\lambda}{6^s}\left(\frac{8+2^s}{2}\right)\|x\|^s, \\ \frac{\lambda}{6^{3s}}\|x\|^{3s}, \\ \frac{\lambda}{6^{3s}}\left(\frac{10+2^{3s}}{2}\right)\|x\|^{3s}. \end{cases}$$

Now,

$$\frac{1}{\rho_i}\Phi_{AQ}(\rho_i x) = \begin{cases} \rho_i^{-1}\,2\lambda, \\ \rho_i^{s-1}\frac{\lambda}{6^s}\left(\frac{8+2^s}{2}\right)\|x\|^s, \\ \rho_i^{3s-1}\frac{\lambda}{6^{3s}}\|x\|^{3s} \\ \rho_i^{3s-1}\frac{\lambda}{6^{3s}}\left(\frac{10+2^{3s}}{2}\right)\|x\|^{3s}, \end{cases} = \begin{cases} \rho_i^{-1}\,\Phi_{AQ}(x), \\ \rho_i^{s-1}\Phi_{AQ}(x), \\ \rho_i^{3s-1}\Phi_{AQ}(x), \\ \rho_i^{3s-1}\Phi_{AQ}(x). \end{cases}$$

Hence, the inequality (3.45) holds for

(i). $L = \rho_i^{-1}$ if $i = 0$ and $L = \frac{1}{\rho_i^{-1}}$ if $i = 1$;

(ii). $L = \rho_i^{s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{\rho_i^{s-1}}$ for $s > 1$ if $i = 1$;

(iii). $L = \rho_i^{3s-1}$ for $3s > 1$ if $i = 0$ and $L = \frac{1}{\rho_i^{3s-1}}$ for $3s > 1$ if $i = 1$;

(iv). $L = \rho_i^{3s-1}$ for $3s > 1$ if $i = 0$ and $L = \frac{1}{\rho_i^{3s-1}}$ for $3s > 1$ if $i = 1$.

Now, from (3.45), we prove the following cases for condition (i).

$$L = \rho_i^{-1}, i = 0 \qquad\qquad L = \frac{1}{\rho_i^{-1}}, i = 1$$

$$L = 6^{-1}, i = 0 \qquad\qquad L = 6, i = 1$$

$$\begin{aligned} \| f(x) - \mathcal{A}(x) \| &\le \left(\tfrac{L^{1-i}}{1-L}\right)\Phi_{AQ}(x) \\ &= \left(\tfrac{(6^{-1})^{1-0}}{1-6^{-1}}\right)2\lambda \\ &= \left(\tfrac{6^{-1}}{1-6^{-1}}\right)2\lambda \\ &= \left(\tfrac{2\lambda}{5}\right) \end{aligned} \qquad\qquad \begin{aligned} \| f(x) - \mathcal{A}(x) \| &\le \left(\tfrac{L^{1-i}}{1-L}\right)\Phi_{AQ}(x) \\ &= \left(\tfrac{6^{1-1}}{1-6}\right)2\lambda \\ &= \left(\tfrac{1}{1-6}\right)2\lambda \\ &= \left(\tfrac{2\lambda}{-5}\right) \end{aligned}$$

Also, from (3.45), we prove the following cases for condition (ii).

$$L = \rho_i^{s-1}, s < 1, i = 0 \qquad L = \frac{1}{\rho_i^{s-1}}, s > 1, i = 1$$

$$L = 6^{s-1}, s < 1, i = 0 \qquad L = 6^{1-s}, s > 1, i = 1$$

$$\begin{aligned} \| f(x) - \mathcal{A}(x) \| &\leq \left(\frac{L^{1-i}}{1-L}\right) \Phi_{AQ}(x) \\ &= \left(\frac{(6^{s-1})^{1-0}}{1-6^{s-1}}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{6^{s-1}}{1-6^{s-1}}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{6^s}{6-6^s}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{\lambda}{6-6^s}\right) \left(\frac{8+2^s}{2}\right) \|x\|^s \end{aligned}$$

$$\begin{aligned} \| f(x) - \mathcal{A}(x) \| &\leq \left(\frac{L^{1-i}}{1-L}\right) \Phi_{AQ}(x) \\ &= \left(\frac{(6^{1-s})^{1-1}}{1-6^{1-s}}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{1}{1-6^{1-s}}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{6^s}{6^s-6}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{\lambda}{6^s-6}\right) \left(\frac{8+2^s}{2}\right) \|x\|^s \end{aligned}$$

Again, from (3.45), we prove the following cases for condition (iii).

$$L = \rho_i^{3s-1}, 3s < 1, i = 0 \qquad L = \frac{1}{\rho_i^{3s-1}}, 3s > 1, i = 1$$

$$L = 6^{3s-1}, 3s < 1, i = 0 \qquad L = 6^{1-3s}, 3s > 1, i = 1$$

$$\begin{aligned} \| f(x) - \mathcal{A}(x) \| &\leq \left(\frac{L^{1-i}}{1-L}\right) \Phi_{AQ}(x) \\ &= \left(\frac{(6^{3s-1})^{1-0}}{1-6^{3s-1}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{6^{3s-1}}{1-6^{3s-1}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{6^{3s}}{6-6^{3s}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{\lambda}{6-6^{3s}}\right) \end{aligned}$$

$$\begin{aligned} \| f(x) - \mathcal{A}(x) \| &\leq \left(\frac{L^{1-i}}{1-L}\right) \Phi_{AQ}(x) \\ &= \left(\frac{(6^{1-3s})^{1-1}}{1-6^{1-3s}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{1}{1-6^{1-3s}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{6^{3s}}{6^{3s}-6}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{\lambda}{6^{3s}-6}\right) \end{aligned}$$

Finally, to prove (3.45) for condition (iv), the proof is similar to that of condition (iii). Hence the proof is complete. □

The proof of the following theorems and corollaries is similar to that of Theorems 3.5, 3.3 and Corollaries 3.4, 3.3.

Theorem 3.6. *Let $Dh_q : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\psi : \mathcal{X}^3 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\rho_i^{2n}} \psi(\rho_i^n x, \rho_i^n y, \rho_i^n z) = 0 \tag{3.54}$$

where

$$\rho_i = \begin{cases} 6 & if \quad i = 0, \\ \frac{1}{6} & if \quad i = 1 \end{cases} \tag{3.55}$$

such that the functional inequality

$$\left\| Dh_q(x, y, z) \right\|_{\mathcal{Y}} \leq \psi(x, y, z) \tag{3.56}$$

holds for all $x, y \in \mathcal{X}$. Assume that there exists $L = L(i) = \frac{1}{6}$ such that the function

$$x \to \Phi_{AQ}(x) = \frac{1}{6} \Psi_Q\left(\frac{x}{6}\right)$$

where

$$\Psi_Q(x) = \psi(x, x, x) + \frac{1}{2}\psi(2x, x, 0) + \frac{1}{4}\psi(0, x, 0)$$

has the property

$$\frac{1}{\rho_i^2} \Phi_{AQ}(\rho_i x) = L\, \Phi_{AQ}(x) \tag{3.57}$$

for all $x \in \mathcal{X}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ satisfying the functional equation (1.5) and

$$\| h_q(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right)^p \Phi_{AQ}(x, x, x) \tag{3.58}$$

for all $x \in \mathcal{X}$.

Corollary 3.5. *Let $Dh_q : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there exist real numbers λ and s such that*

$$\|Dh_q f(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\} & s \neq 2; \\ \rho\left\{||x||^s ||y||^s ||z||^s\right\} & 3s \neq 2; \\ \rho\left\{||x||^s ||y||^s ||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\} & 3s \neq 2; \end{cases} \tag{3.59}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique quadratic function $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\|h_q(x) - \mathcal{Q}(x)\| \leq \begin{cases} \dfrac{7\lambda}{28 \cdot 6}, \\ \dfrac{(15 + 2^{s+1})\lambda||x||^s}{4 \cdot 6|36 - 6^s|}, \\ \dfrac{\lambda||x||^{3s}}{6|36 - 6^{3s}|}, \\ \dfrac{(19 + 2^s)\lambda||x||^s}{4 \cdot 6|36 - 6^{3s}|}, \end{cases} \tag{3.60}$$

for all $x \in \mathcal{X}$.

Theorem 3.7. *Let $Dh : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\psi : \mathcal{X}^3 \to [0, \infty)$ with the conditions (3.41), (3.54), (3.42), (3.55) such that the functional inequality*

$$\|Dh(x,y,z)\|_{\mathcal{Y}} \leq \psi(x,y,z) \tag{3.61}$$

holds for all $x, y, z \in \mathcal{X}$. Assume that there exists $L = L(i)$ such that the function

$$\frac{1}{6}\Psi_A\left(\frac{x}{6}\right) = \Phi_{AQ}(x) = \frac{1}{6}\Psi_Q\left(\frac{x}{6}\right)$$

where $\Psi_A(x), \Psi_Q(x)$ are defined in (3.11), (3.28) has the properties (3.44), (3.57) for all $x \in \mathcal{X}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ and a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying the functional equation (1.5) and

$$\| h(x) - \mathcal{A}(x) - \mathcal{Q}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right)^p \Phi_{AQ}(x,x,x) \tag{3.62}$$

for all $x \in \mathcal{X}$.

Corollary 3.6. *Let $Dh : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there exist real numbers λ and s such that*

$$\|Dhf(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\} & s \neq 2; \\ \rho\left\{||x||^s ||y||^s ||z||^s\right\} & 3s \neq 2; \\ \rho\left\{||x||^s ||y||^s ||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\} & 3s \neq 2; \end{cases} \tag{3.63}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ and a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - \mathcal{Q}(x) - \mathcal{A}(x)\| \leq \begin{cases} \theta\left(\dfrac{2}{5} + \dfrac{7}{28 \cdot 6}\right), \\ \theta||x||^s\left(\dfrac{(8 + 2^s)}{2|6 - 6^s|} + \dfrac{(15 + 2^{s+1})}{4 \cdot 6|36 - 6^s|}\right), \\ \theta||x||^{3s}\left(\dfrac{1}{|6 - 6^{3s}|} + \dfrac{1}{6|36 - 6^{3s}|}\right), \\ \theta||x||^{3s}\left(\dfrac{(10 + 2^s)}{2|6 - 6^{3s}|} + \dfrac{(19 + 2^{s+1})}{4 \cdot 6|36 - 6^{3s}|}\right) \end{cases} \tag{3.64}$$

for all $x \in \mathcal{X}$.

4 Preliminaries of Non-Archimediean Fuzzy Normed Space

It is to be noted that, Mirmostafaee and Moslehian [28] initiate a notion of a non-Archimedean fuzzy norm and studied the stability of the Cauchy equation in the context of non-Archimedean fuzzy spaces. They presented an interdisciplinary relation between the theory of fuzzy spaces, the theory of non-Archimedean spaces, and the theory of functional equations.

The most important paradigm of non-Archimedean spaces are $p-$adic numbers. A key property of $p-$adic numbers is that they do not fulfill the Archimedean axiom: for all $x, y > 0$, there exists an integer n, such that $x > ny$. It turned out that non-Archimedean spaces have many nice applications [26, 46, 50].

During the last three decades, theory of non-Archimedean spaces has prolonged the interest of physicists for their research, in particular, in problems coming from quantum physics, $p-$adic strings, and superstrings [26]. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles and there may be no unit vector in a non-Archimedean normed space (cf [26]). These facts show that the non-Archimedean framework is of special interest.

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, such that population dynamics, chaos control, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence, etc. [31, 47, 50]. The fuzzy topology proves to be a very valuable tool to deal with such situations where the use of classical theories breaks down. The most fascinating application of fuzzy topology in quantum particle physics arises in string and E-infinity theory of EI Naschie [34]- [38].

The definition of non-Archimedean fuzzy normed spaces was given in [30].

Definition 4.1. *Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$, such that for any $a, b \in \mathbb{K}$, we have*

(NA1) $|a| \geq 0$ *and equality holds if and only if* $a = 0$;

(NA2) $|ab| = |a|\,|b|$;

(NA3) $|a + b| \leq \max\{|a|, |b|\}$.

The condition $(NA3)$ is called the strong triangle inequality. Clearly, $|1| = |-1| = |1|$ and $n \leq 1$ for all $n \geq \mathbb{N}$. We always assume in addition that $|\cdot|$ is non trivial, i.e., that

(NA4) *there is an $a_0 \in \mathbb{K}$, such that $|a_0| \neq 0, 1$.*

The most important examples of non-Archimedean spaces are $p-$adic numbers.

Example 4.1. *Let p be a prime number. For any nonzero rational number x, there exists a unique integer n_x, such that $x = \frac{a}{b}p^n x$, where a and b are integers not divisible by p. Then $|x|_p = p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the $p-$adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\left| \sum_{k \geq n_x}^{\infty} a_k p^k \right|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field (see [46]). Note that if $p > 2$ then $|2^n|_p = 1$ for each integer n but $|2|_2 < 1$.*

Now we give the definition of a non-Archimedean fuzzy normed space.

Definition 4.2. *Let X be a linear space over a non-Archimedean field $\mathbb{K}$. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a non-Archimedean fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(NAF1) $N(x, c) = 0$ *for all* $c \leq 0$;

(NAF2) $x = 0$ *if and only if* $N(x, c) = 1$ *for all* $c > 0$;

(NAF3) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

(NAF4) $N(x + y, \max\{s, t\}) \geq min\{N(x, s), N(y, t)\}$;

(NAF5) $\lim_{t\to\infty} N(x,t) = 1.$

A non-Archimedean fuzzy norm is a pair (X, N) where X be a linear space and N is non-Archimedean fuzzy norm on X. If $(NAF4)$ holds then

(NAF6) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}.$

Recall that a classical vector space over the complex or real field satisfying $(NAF1) - (NAF5)$ is called a fuzzy normed space in the literature. We repeatedly use the fact $N(-x,t) = N(x,t),\ \ x \in X, t > 0$, which is deduced from $(NAF3)$. It is easy to see that $(NAF4)$ is equivalent to the following condition:

(NAF7) $N(x+y, t) \geq \min\{N(x,t), N(y,t)\}.$

Example 4.2. *Let $(X, ||\cdot||)$ be a non-Archimedean normed space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

Then (X, N) is a non-Archimedean fuzzy normed space.

Example 4.3. *Let $(X, ||\cdot||)$ be a non-Archimedean normed space. Then*

$$N(x,t) = \begin{cases} 0, & t \leq ||x||, \\ 1, & t > ||x||. \end{cases}$$

Then (X, N) is a non-Archimedean fuzzy normed space.

Definition 4.3. *Let (X, N) be a non-Archimedean fuzzy normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 4.4. *A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$. Due to this fact that*

$$N(x_n - x_m, t) \geq \min\{N(x_{j+1} - x_j, t)/m \leq j \leq n-1, \quad n > m\},$$

a sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n\to\infty} N(x_{n+1} - x_n, t) = 1$ for all $t > 0$.

Definition 4.5. *Every convergent sequence in a non-Archimedean fuzzy normed space is a Cauchy sequence. If every Cauchy sequence is convergent, then the non-Archimedean fuzzy normed space is called a non-Archimedean fuzzy Banach space.*

Here after, throughout this paper, assume that $\mathbb{K}$ non-Archimedean field, X be vector space over $\mathbb{K}$, (Y, N') be a non-Archimedean fuzzy Banach space over $\mathbb{K}$ and (Z, N') be an (Archimedean or non-Archimedean) fuzzy normed space. Also we use the following notation for a given mapping $Dh : X \to Y$ by such that

$$\begin{aligned} Dh(x,y,z) &= h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ &\quad - h(x+y+z) - h(x+y-z) - h(x-y+z) - h(-x+y+z) - 2h(y) \\ &\quad - 4h(z) - 5[h(y) + h(-y)] - 14[h(z) + h(-z)] \end{aligned}$$

for all $x, y, z \in X$.

In this section, the non-Archimedean fuzzy stability of a 3 dimensional additive quadratic functional equation (1.5) is provided using direct and fixed point methods.

4.1 NAFNS: Stability Results: Direct Method

Theorem 4.1. *Let $\kappa = \pm 1$ be fixed and let $\vartheta : X^3 \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{2}\right)^{\kappa} < 1$*

$$N'\left(\vartheta\left(6^{\kappa n}x, 6^{\kappa n}x, 6^{\kappa n}x\right), r\right) \geq N'\left(d^{\kappa n}\vartheta\left(x, x, x\right), r\right) \tag{4.1}$$

for all $x \in X$, all $r > 0$, and

$$\lim_{n\to\infty} N'\left(\vartheta\left(6^{\kappa n}x, 6^{\kappa n}y, 6^{\kappa n}z\right), 6^{\kappa n}r\right) = 1 \tag{4.2}$$

for all $x, y, z \in X$ and all $r > 0$. Suppose that an odd function $h_a : X \to Y$ satisfies the inequality

$$N\left(D\, h_a(x, y, z), r\right) \geq N'\left(\vartheta(x, y, z), r\right) \tag{4.3}$$

for all $x, y, z \in X$ and all $r > 0$. Then the limit

$$A(x) = N - \lim_{n\to\infty} \frac{h_a(6^{\kappa n}x)}{6^{\kappa n}} \tag{4.4}$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping satisfying (1.5) and

$$N\left(h_a(x) - A(x), r\right) \geq N'\left(\vartheta(2x), \frac{r|6-d|}{2}\right) \tag{4.5}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $\kappa = 1$. Replacing (x, y, z) by (x, x, x) in (4.3) and using oddness of h_a, we get

$$\begin{aligned} &N\left(h_a(6x) + h_a(2x) + h_a(4x) - h_a(3x) - 9h_a(x), r\right) \\ &\qquad\qquad \geq N'\left(\vartheta(x, x, x), r\right), \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.6}$$

It follows from (4.6) and (NAF3)that

$$\begin{aligned} &N\left(2h_a(6x) + 2h_a(2x) + 2h_a(4x) - 2h_a(3x) - 18h_a(x), 2r\right) \\ &\qquad\qquad \geq N'\left(\vartheta(x, x, x), r\right), \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.7}$$

Again replacing (x, y, z) by $(2x, x, 0)$ in (4.3) and using oddness of h_a, we get

$$N\left(2h_a(4x) - 2h_a(3x) - 2h_a(x), r\right) \geq N'\left(\vartheta(2x, x, 0), r\right), \quad \forall\, x \in X, r > 0. \tag{4.8}$$

Finally replacing (x, y, z) by $(0, x, 0)$ in (4.3) and using oddness of h_a, we get

$$N\left(2h_a(2x) - 4h_a(x), r\right) \geq N'\left(\vartheta(0, x, 0), r\right), \quad \forall\, x \in X, r > 0. \tag{4.9}$$

With the help of (NAF3), (NAF4), (4.7), (4.8) and (4.9) we arrive

$$\begin{aligned} &N\left(2h_a(6x) - 12h_a(x), 4r\right) \\ &= N\left(2h_a(6x) + 2h_a(2x) + 2h_a(4x) - 2h_a(3x) - 18h_a(x)\right. \\ &\qquad \left. -2h_a(4x) + 2h_a(3x) + 2h_a(x) - 2h_a(2x) + 4h_a(x), 4r\right) \\ &\geq \min\left\{N\left(2h_a(6x) + 2h_a(2x) + 2h_a(4x) - 2h_a(3x) - 18h_a(x), 2r\right),\right. \\ &\qquad \left. N\left(2h_a(4x) - 2h_a(3x) - 2h_a(x), r\right), N\left(2h_a(2x) - 4h_a(x), r\right)\right\} \\ &\geq \min\left\{N'\left(\vartheta(x, x, x), r\right), N'\left(\vartheta(2x, x, 0), r\right), N'\left(\vartheta(0, x, 0), r\right)\right\} \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.10}$$

Using (NAF3) in (4.10), we have

$$N\left(\frac{h_a(6x)}{6} - h_a(x), \frac{r}{3}\right) \geq N_1'\left(\vartheta(x), r\right), \quad \forall\, x \in X, r > 0, \tag{4.11}$$

where

$$N_1'\left(\vartheta(x), r\right) = \min\left\{N'\left(\vartheta(x, x, x), r\right), N'\left(\vartheta(2x, x, 0), r\right), N'\left(\vartheta(0, x, 0), r\right)\right\}$$

for all $x \in X$ and $r > 0$. Replacing x by $6^n x$ in (4.6), we obtain

$$N\left(\frac{h_a(6^{n+1}x)}{6} - h_a(6^n x), \frac{r}{3}\right) \geq N_1'\left(\vartheta(6^n x), r\right), \quad \forall\, x \in X, r > 0. \tag{4.12}$$

Using (4.1) and $(NAF3)$ in (4.12), we have

$$N\left(\frac{h_a(6^{n+1}x)}{6} - h_a(6^n x), \frac{r}{3}\right) \geq N_1'\left(\vartheta(x), \frac{r}{d^n}\right), \quad \forall\, x \in X, r > 0. \tag{4.13}$$

One can easy to verify from (4.13), that

$$N\left(\frac{h_a(6^{n+1}x)}{6^{n+1}} - \frac{h_a(6^n x)}{6^n}, \frac{r}{3 \cdot 6^n}\right) \geq N_1'\left(\vartheta(x), \frac{r}{d^n}\right), \quad \forall\, x \in X, r > 0. \tag{4.14}$$

Replacing r by $d^n r$ in (4.14), we obtain

$$N\left(\frac{h_a(6^{n+1}x)}{6^{n+1}} - \frac{h_a(6^n x)}{6^n}, \frac{d^n\, r}{3 \cdot 6^n}\right) \geq N_1'\left(\vartheta(x), r\right), \quad \forall\, x \in X, r > 0. \tag{4.15}$$

It is easy to see that

$$\frac{h_a(6^n x)}{6^n} - h_a(x) = \sum_{i=0}^{n-1}\left[\frac{h_a(6^{i+1}x)}{6^{i+1}} - \frac{h_a(6^i x)}{6^i}\right], \quad \forall\, x \in X. \tag{4.16}$$

It follows from (4.15) and (4.16), we have

$$\begin{aligned} N\left(\frac{h_a(6^n x)}{6^n} - h_a(x), \sum_{i=0}^{n-1}\frac{d^i\, r}{3 \cdot 6^i}\right) &\geq min \bigcup_{i=0}^{n-1}\left\{N\left(\frac{h_a(6^{i+1}x)}{6^{i+1}} - \frac{h_a(6^i x)}{6^i}, \frac{d^i\, r}{3 \cdot 6^i}\right)\right\} \\ &\geq min \bigcup_{i=0}^{n-1}\left\{N_1'\left(\vartheta(x), r\right)\right\} \\ &\geq N_1'\left(\vartheta(x), r\right), \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.17}$$

Replacing x by $6^m x$ in (4.17) and using (4.1), $(NAF3)$, we get

$$N\left(\frac{h_a(6^{n+m}x)}{6^{n+m}} - \frac{h_a(6^m x)}{6^m}, \sum_{i=0}^{n-1}\frac{d^{i+m}\, r}{3 \cdot 6^{i+m}}\right) \geq N_1'\left(\vartheta(x), \frac{r}{d^m}\right) \tag{4.18}$$

. for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Replacing r by $d^m r$ in (4.18), we get

$$N\left(\frac{h_a(6^{n+m}x)}{6^{n+m}} - \frac{h_a(6^m x)}{6^m}, \sum_{i=0}^{n-1}\frac{d^i\, r}{3 \cdot 6^{i+m}}\right) \geq N_1'\left(\vartheta(x), r\right) \tag{4.19}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. It follows from (4.19) that

$$N\left(\frac{h_a(6^{n+m}x)}{6^{n+m}} - \frac{h_a(6^m x)}{6^m}, r\right) \geq N_1'\left(\vartheta(x), \frac{r}{\sum_{i=m}^{m+n-1}\frac{d^i}{3 \cdot 6^i}}\right) \tag{4.20}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Since $0 < d < 6$ and $\sum_{i=0}^{n}\left(\frac{d}{6}\right)^i < \infty$, using $(NAF5)$ implies that $\left\{\frac{h_a(6^n x)}{6^n}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a non-Archimedean fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. So, we can define a mapping $A : X \to Y$ by

$$A(x) = N - \lim_{n \to \infty}\frac{h_a(6^n x)}{6^n}, \quad \forall\, x \in X.$$

Putting $m = 0$ in (4.20), we get

$$N\left(\frac{h_a(6^n x)}{6^n} - h_a(x), r\right) \geq N_1'\left(\vartheta(x), \frac{r}{\sum_{i=0}^{n-1}\frac{d^i}{3 \cdot 6^i}}\right), \; \forall\, x \in X, r > 0. \tag{4.21}$$

Letting $n \to \infty$ in (4.21) and using $(NAF5)$, we arrive

$$N\left(A(x)-h_a(x),r\right) \geq N_1'\left(\vartheta(x),\frac{r(6-d)}{2}\right) \quad \forall\, x\in X, r>0.$$

To prove A satisfies (1.5), replacing (x,y,z) by $(6^nx,6^ny,6^nz)$ in (4.3), respectively , we obtain

$$N\left(\frac{1}{6^n}(Dh_a(6^nx,6^ny,6^nz)),r\right) \geq N'\left(\vartheta(6^nx,6^ny,6^nz),6^nr\right) \tag{4.22}$$

for all $x,y,z\in X$ and all $r>0$. Now,

$$\begin{aligned}
&N\Big(A(x+2y+3z)+A(x+2y-3z)+A(x-2y+3z)+A(-x+2y+3z)\\
&\quad -A(x+y+z)-A(x+y-z)-A(x-y+z)-A(-x+y+z)\\
&\qquad -2A(y)-4A(z)-5[A(y)+A(-y)]-14[A(z)+A(-z)],r\Big)\\
&\geq \min\Bigg\{N\left(A(x+2y+3z)-\frac{1}{6^n}h_a(6^n(x+2y+3z)),\frac{r}{13}\right),\\
&\quad N\left(A(x+2y-3z)-\frac{1}{6^n}h_a(6^n(x+2y-3z)),\frac{r}{13}\right),\\
&\quad N\left(A(x-2y+3z)-\frac{1}{6^n}h_a(6^n(x-2y+3z)),\frac{r}{13}\right),\\
&\quad N\left(A(-x+2y+3z)-\frac{1}{6^n}h_a(6^n(-x+2y+3z)),\frac{r}{13}\right),\\
&\quad N\left(-A(x+y+z)+\frac{1}{6^n}h_a(6^n(x+y+z)),\frac{r}{13}\right),\\
&\quad N\left(-A(x+y-z)+\frac{1}{6^n}h_a(6^n(x+y-z)),\frac{r}{13}\right),\\
&\quad N\left(-A(x-y+z)+\frac{1}{6^n}h_a(6^n(x-y+z)),\frac{r}{13}\right),\\
&\quad N\left(-A(-x+y+z)+\frac{1}{6^n}h_a(6^n(-x+y+z)),\frac{r}{13}\right),\\
&\quad N\left(-2A(y)+\frac{2}{6^n}h_a(6^ny),\frac{r}{13}\right),N\left(-4A(z)+\frac{4}{6^n}h_a(6^nz),\frac{r}{13}\right),\\
&\quad N\left(-5[A(y)+A(-y)]+\frac{5}{6^n}[h_a(6^ny)+h_a(-6^ny),\frac{r}{13}\right),\\
&\quad N\left(-14[A(z)+A(-z)]+\frac{14}{6^n}[h_a(6^nz)+h_a(-6^nz)],\frac{r}{13}\right),\\
&\quad N\left(\frac{1}{6^n}h_a(6^n(x+2y+3z))+\frac{1}{6^n}h_a(6^n(x+2y-3z))\right.\\
&\qquad +\frac{1}{6^n}h_a(6^n(x-2y+3z))+\frac{1}{6^n}h_a(6^n(-x+2y+3z))\\
&\qquad +\frac{1}{6^n}h_a(6^n(x+y+z))+\frac{1}{6^n}h_a(6^n(x+y-z))\\
&\qquad +\frac{1}{6^n}h_a(6^n(x-y+z))+\frac{1}{6^n}h_a(6^n(-x+y+z))\\
&\qquad -\frac{2}{6^n}h_a(6^ny)-\frac{4}{6^n}h_a(6^nz)-\frac{5}{6^n}[h_a(6^ny)+h_a(-6^ny)]\\
&\qquad \left.-\frac{14}{6^n}[h_a(6^nz)+h_a(-6^nz)],\frac{r}{13}\right),\Bigg\}
\end{aligned} \tag{4.23}$$

for all $x,y,z\in X$ and all $r>0$. Using (4.22) and $(NAF5)$ in (4.23), we arrive

$$
\begin{aligned}
N\Big(&A(x+2y+3z) + A(x+2y-3z) + A(x-2y+3z) + A(-x+2y+3z) - A(x+y+z) \\
&- A(x+y-z) - A(x-y+z) - A(-x+y+z) \\
&\qquad - 2A(y) - 4A(z) - 5[A(y)+A(-y)] - 14[A(z)+A(-z)], r\Big) \\
&\geq \min\left\{1,1,1,1,1,1,1, N'\left(\vartheta(6^n x, 6^n y, 6^n z), 6^n r\right)\right\} \\
&\geq N'\left(\vartheta(6^n x, 6^n y, 6^n z), 6^n r\right)
\end{aligned} \tag{4.24}
$$

for all $x, y, z \in X$ and all $r > 0$. Letting $n \to \infty$ in (4.24) and using (4.2), we see that

$$
\begin{aligned}
N\Big(&A(x+2y+3z) + A(x+2y-3z) + A(x-2y+3z) + A(-x+2y+3z) \\
&- A(x+y+z) - A(x+y-z) - A(x-y+z) - A(-x+y+z) \\
&\qquad - 2A(y) - 4A(z) - 5[A(y)+A(-y)] - 14[A(z)+A(-z)], r\Big) = 1
\end{aligned}
$$

for all $x, y, z \in X$ and all $r > 0$. Using $(NAF2)$ in the above inequality, we get

$$
\begin{aligned}
&A(x+2y+3z) + A(x+2y-3z) + A(x-2y+3z) + A(-x+2y+3z) \\
&= A(x+y+z) + A(x+y-z) + A(x-y+z) + A(-x+y+z) \\
&\qquad + 2A(y) + 4A(z) + 5[A(y)+A(-y)] + 14[A(z)+A(-z)]
\end{aligned}
$$

for all $x, y, z \in X$. Hence A satisfies the functional equation (1.5). In order to prove $A(x)$ is unique, let $A'(x)$ be another additive function satisfying (1.5) and (4.4). Hence,

$$
\begin{aligned}
N(A(x) - A'(x), r) &= N\left(\frac{A(6^n x)}{6^n} - \frac{A'(6^n x)}{6^n}, r\right) \\
&= N\left(A(6^n x) - A'(6^n x), 6^n r\right) \\
&\geq min\left\{N\left(A(6^n x) - \frac{h_a(6^n x)}{6^n}, \frac{6^n r}{2}\right), N\left(\frac{h_a((6^n x)}{6^n} - A'(6^n x), \frac{6^n r}{2}\right)\right\} \\
&\geq min\left\{N_1'\left(\vartheta(6^n x), \frac{r\, 6^n (6-d)}{2\cdot 2}\right), N_1'\left(\vartheta(6^n x), \frac{r\, 6^n (6-d)}{2\cdot 2}\right)\right\} \\
&= N_1'\left(\vartheta(6^n x), \frac{r\, 6^n (6-d)}{4}\right) \\
&= N_1'\left(\vartheta(x), \frac{r\, 6^n (6-d)}{4d^n}\right), \quad \forall\, x \in X, r > 0.
\end{aligned}
$$

Since $\lim\limits_{n\to\infty} \dfrac{r\, 6^n (6-d)}{4d^n} = \infty$, we obtain

$$
\lim_{n\to\infty} N_1'\left(\vartheta(x), \frac{r\, 6^n (6-d)}{4d^n}\right) = 1.
$$

Thus

$$
N(A(x) - A'(x), r) = 1, \quad \forall\, x \in X, r > 0,
$$

Hence $A(x) = A'(x)$. Therefore $A(x)$ is unique.

For $\kappa = -1$, we can prove the result by a similar method. This completes the proof of the theorem. □

From Theorem 4.1, we obtain the following corollary concerning the stability for the functional equation (1.5).

Corollary 4.1. *Suppose that a odd function $h_a : X \to Y$ satisfies the inequality*

$$
N\left(D\, h_a(x,y,z,t), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), \\ N'\left(\epsilon\, ||x||^s\, ||y||^s\, ||z||^s, r\right), \\ N'\left(\epsilon\left\{||x||^s\, ||y||^s\, ||z||^s + \left(||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right)\right\}, r\right), \end{cases} \tag{4.25}
$$

for all $r > 0$ and all $x, y, z \in X$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(h_a(x) - A(x), r\right) \geq \begin{cases} N'\left(\epsilon, \frac{r|6-d|}{4}\right), & \\ N'\left((8+2^s)\epsilon||x||^s, r|6-d|\right), & s < 1 \quad or \quad s > 1; \\ N'\left(\epsilon||x||^{3s}, r|6-d|\right), & s < \frac{1}{3} \quad or \quad s > \frac{1}{3}; \\ N'\left((10+2^s)\epsilon||x||^{3s}, r|6-d|\right), & s < \frac{1}{3} \quad or \quad s > \frac{1}{3}; \end{cases} \tag{4.26}$$

for all $x \in X$ and all $r > 0$.

Proof. Setting

$$\vartheta(x,y,z) = \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), \\ N'\left(\epsilon\ ||x||^s\ ||y||^s\ ||z||^s, r\right), \\ N'\left(\epsilon\left\{||x||^s\ ||y||^s\ ||z||^s + \left(||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right)\right\}, r\right), \end{cases}$$

then the corollary is followed from Theorem 4.1. If we define

$$d = \begin{cases} 6^0, \\ 6^s, \\ 6^{3s} \\ 6^{3s}. \end{cases}$$

□

Example 4.4. *Let X be a normed space and N and N' be non-archimedian fuzzy norms on X and $\mathbb{R}$ defined by*

$$N(x,r) = \begin{cases} \dfrac{r}{r + \|x\|} & r > 0, \quad x \in X, \\ 0, & r \leq 0, \quad x \in X. \end{cases} \tag{4.27}$$

$$N'(x,r) = \begin{cases} \dfrac{r}{r + \|x\|} & r > 0, \quad x \in \mathbb{R}, \\ 0, & r \leq 0, \quad x \in \mathbb{R}. \end{cases} \tag{4.28}$$

Let $\vartheta : (0, \infty) \to (0, \infty)$ be a function such that $\vartheta(6l) < d\ \vartheta(l)$ for all $l > 0$ and $0 < d < 6$. Define

$$\begin{aligned} \beta(x,y,z) = {} & \vartheta(\|x+2y+3z\|) + \vartheta(\|x+2y-3z\|) + \vartheta(\|x-2y+3z\|) \\ & + \vartheta(\|-x+2y+3z\|) - \vartheta(\|x+y+z\|) - \vartheta(\|x+y-z\|) \\ & - \vartheta(\|x-y+z\|) - \vartheta(\|-x+y+z\|) - 2\vartheta(\|y\|) - 4\vartheta(\|z\|) \\ & - 5\vartheta(\|y\| + \|-y\|) - 14\vartheta(\|z\| + \|-z\|) \end{aligned}$$

for all $x, y, z \in X$. Let $x_0 \in X$ be a unit vector and define $h_a : X \to X$ by $h_a(x) = x + \vartheta(\|x\|)\, x_0$. Now for any $x, y, z \in X$ and $r > 0$, we have

$$\begin{aligned} N\left(Dh_a(x,y,z), r\right) &= \frac{r}{r + \|\beta(x,y,z)\| \cdot \|x_0\|} \\ &\geq \frac{r}{r + \|\beta(x,y,z)\|} \\ &= N'\left(\beta(x,y,z), r\right). \end{aligned}$$

For any $x, y, z \in X$ and $r > 0$, we have

$$\begin{aligned} N'\left(\beta(6x,6y,6z), r\right) &= \frac{r}{r + \beta(6x,6y,6z)} \\ &\geq \frac{r}{r + d\,\beta(x,y,z)} \\ &= N'\left(d\,\beta(x,y,z), r\right). \end{aligned}$$

Hence the inequalities (4.1) and (4.3) are satisfied. Using Theorem 4.1, there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(A\left(x\right)-h_{a}\left(x\right),\ r\right)\ \geq\ N_{1}'\left(\frac{2\beta\left(x\right)}{|6-d|},r\right)$$

$x \in X$ and $r > 0$.

The following theorem and corollary provide the stability result of (1.5) for h is even function.

Theorem 4.2. *Let $\kappa = \pm 1$ be fixed and let $\vartheta : X^3 \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{36}\right)^{\kappa} < 1$*

$$N'\left(\vartheta\left(6^{\kappa}x,6^{\kappa}x,6^{\kappa}x,6^{\kappa}x\right),r\right)\geq N'\left(d^{\kappa}\vartheta\left(x,x,x\right),r\right) \tag{4.29}$$

for all $x \in X$ and all $d > 0$, and

$$\lim_{n\to\infty}N'\left(\vartheta\left(6^{\kappa n}x,6^{\kappa n}y,6^{\kappa n}z\right),36^{\kappa n}r\right)=1 \tag{4.30}$$

for all $x,y,z \in X$ and all $r > 0$. Suppose that a even function $h_q : X \to Y$ satisfies the inequality

$$N\left(D\,h_{q}(x,y,z),r\right)\geq N'\left(\vartheta(x,y,z),r\right) \tag{4.31}$$

for all $x,y,z \in X$ and all $r > 0$. Then the limit

$$Q(x)=N-\lim_{n\to\infty}\frac{h_{q}(6^{\beta n}x)}{36^{\beta n}} \tag{4.32}$$

exists for all $x \in X$ and the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (1.5) and

$$N\left(h_{q}(x)-Q(x),r\right)\geq N_{2}'\left(\vartheta(x),\frac{r|36-d|}{2}\right) \tag{4.33}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $\kappa = 1$. Replacing (x,y,z) by (x,x,x) in (4.31) and using evenness of h_q, we get

$$\begin{aligned}N\big(h_{q}(6x)+h_{q}(2x)+h_{q}(4x)-h_{q}(3x)-47h_{q}(x),r\big)&\\ \geq N'\left(\vartheta(x,x,x),r\right),\quad\forall\,x\in X,r>0.&\end{aligned} \tag{4.34}$$

It follows from (4.34) and (NAF3)that

$$\begin{aligned}N\big(2h_{q}(6x)+2h_{q}(2x)+2h_{q}(4x)-2h_{q}(3x)-94h_{q}(x),2r\big)&\\ \geq N'\left(\vartheta(x,x,x),r\right),\quad\forall\,x\in X,r>0.&\end{aligned} \tag{4.35}$$

Again replacing (x,y,z) by $(2x,x,0)$ in (4.31) and using oddness of h_q, we get

$$N\left(2h_{q}(4x)-2h_{q}(3x)-14h_{q}(x),r\right)\geq N'\left(\vartheta(2x,x,0),r\right),\quad\forall\,x\in X,r>0. \tag{4.36}$$

Finally replacing (x,y,z) by $(0,x,0)$ in (4.31) and using oddness of h_q, we get

$$N\left(4h_{q}(2x)-16h_{q}(x),r\right)\geq N'\left(\vartheta(0,x,0),r\right),\quad\forall\,x\in X,r>0. \tag{4.37}$$

It follows from (4.37) and (NAF3) that

$$N\left(2h_{q}(2x)-8h_{q}(x),r\right)\geq N'\left(\vartheta(0,x,0),\frac{r}{2}\right),\quad\forall\,x\in X,r>0. \tag{4.38}$$

With the help of (NAF3), (NAF4), (4.35), (4.36) and (4.38) we arrive

$$\begin{aligned}&N\left(2h_{q}(6x)-72h_{q}(x),4r\right)\\&=N\big(2h_{q}(6x)+2h_{q}(2x)+2h_{q}(4x)-2h_{q}(3x)-94h_{q}(x)\\&\qquad-2h_{q}(4x)+2h_{q}(3x)+14h_{q}(x)-2h_{q}(2x)+8h_{q}(x),4r\big)\\&\geq\min\big\{N\left(2h_{q}(6x)+2h_{q}(2x)+2h_{q}(4x)-2h_{q}(3x)-18h_{q}(x),2r\right),\\&\qquad N\left(2h_{q}(4x)-2h_{q}(3x)-2h_{q}(x),r\right),N\left(2h_{q}(2x)-8h_{q}(x),r\right)\big\}\\&\geq\min\left\{N'\left(\vartheta(x,x,x),r\right),N'\left(\vartheta(2x,x,0),r\right),N'\left(\vartheta(0,x,0),\frac{r}{2}\right)\right\}\quad\forall\,x\in X,r>0.\end{aligned} \tag{4.39}$$

Using (NAF3) in (4.39), we have

$$N\left(\frac{h_q(6x)}{6} - h_q(x), \frac{r}{18}\right) \geq N_2'(\vartheta(x), r), \quad \forall\, x \in X, r > 0, \tag{4.40}$$

where

$$N_2'(\vartheta(x), r) = \min\left\{N'(\vartheta(x,x,x), r), N'(\vartheta(2x,x,0), r), N'\left(\vartheta(0,x,0), \frac{r}{2}\right)\right\}$$

for all $x \in X$ and $r > 0$. The rest of the proof is similar tracing to that of Theorem 4.1. □

From Theorem 4.2, we obtain the following corollary concerning the stability for the functional equation (1.5).

Corollary 4.2. *Suppose that a even function $h_q : X \to Y$ satisfies the inequality*

$$N\left(D\, h_q(x,y,z,t), r\right) \geq \begin{cases} N'(\epsilon, r), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), \\ N'\left(\epsilon\, ||x||^s\, ||y||^s\, ||z||^s, r\right), \\ N'\left(\epsilon\left\{||x||^s\, ||y||^s\, ||z||^s + \left(||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right)\right\}, r\right), \end{cases} \tag{4.41}$$

for all $x, y, z \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(h_q(x) - Q(x), r\right) \geq \begin{cases} N'\left(7\epsilon, r|36-d|\right), \\ N'\left((15+2^{s+1})\epsilon||x||^s, 2r|36-d|\right), & s < 2 \quad or \quad s > 2; \\ N'\left(\epsilon||x||^{3s}, \frac{r|36-d|}{2}\right), & s < \frac{2}{3} \quad or \quad s > \frac{2}{3}; \\ N'\left((19+2^{s+1})\epsilon||x||^{3s}, 2r|36-d|\right), & s < \frac{2}{3} \quad or \quad s > \frac{2}{3}; \end{cases} \tag{4.42}$$

for all $x \in X$ and all $r > 0$.

Example 4.5. *Let X be a normed space and N and N' be non-archimedian fuzzy norms on X and $\mathbb{R}$ defined by (4.27),(4.28). Let $\vartheta : (0,\infty) \to (0,\infty)$ be a function such that $\vartheta(2l) < d\,\vartheta(l)$ for all $l > 0$ and $0 < d < 36$. Define*

$$\begin{aligned} \beta(x,y,z) = {} & \vartheta(\|x+2y+3z\|) + \vartheta(\|x+2y-3z\|) + \vartheta(\|x-2y+3z\|) \\ & + \vartheta(\|-x+2y+3z\|) - \vartheta(\|x+y+z\|) - \vartheta(\|x+y-z\|) \\ & - \vartheta(\|x-y+z\|) - \vartheta(\|-x+y+z\|) - 2\vartheta(\|y\|) - 4\vartheta(\|z\|) \\ & - 5\vartheta(\|y\| + \|-y\|) - 14\vartheta(\|z\| + \|-z\|) \end{aligned}$$

for all $x, y, z \in X$. Let $x_0 \in X$ be a unit vector and define $h_q : X \to X$ by $h_q(x) = x + \vartheta(\|x\|)\, x_0$. Now for any $x, y, z \in X$ and $r > 0$, we have

$$\begin{aligned} N\left(Dh_q(x,y,z), r\right) &= \frac{r}{r + \|\beta(x,y,z)\| \cdot \|x_0\|} \\ &\geq \frac{r}{r + \|\beta(x,y,z)\|} \\ &= N'(\beta(x,y,z), r). \end{aligned}$$

For any $x, y, z \in X$ and $r > 0$, we have

$$\begin{aligned} N'(\beta(2x,2y,2z), r) &= \frac{r}{r + \beta(2x,2y,2z)} \\ &\geq \frac{r}{r + d\,\beta(x,y,z)} \\ &= N'(d\,\beta(x,y,z), r). \end{aligned}$$

Hence the inequalities (4.29) and (4.31) are satisfied. Using Theorem 4.2, there exists a unique additive mapping $A : X \to Y$ such that

$$N(Q(x) - h_q(x),\ r) \ \geq\ N_2'\left(\frac{\beta(x)}{|36-d|}, r\right)$$

$x \in X$ and $r > 0$.

The following theorem provide the stability result of (1.5) for mixed case.

Theorem 4.3. *Let* $\kappa = \pm 1$ *be fixed and let* $\vartheta : X^3 \to Z$ *be a mapping such that for some* d *with* $0 < \left(\frac{d}{6}\right)^{\kappa} < 1$ *and satisfying (4.1),(4.2),(4.29) and (4.30). Suppose that a function* $h : X \to Y$ *satisfies the inequality*

$$N\left(D\, h(x,y,z), r\right) \geq N'\left(\vartheta(x,y,z), r\right),\ \forall\, x,y,z \in X, r > 0. \tag{4.43}$$

Then there exists a unique additive mapping $A : X \to Y$ *and unique quadratic mapping* $Q : X \to Y$ *satisfying (1.5) and*

$$N\left(h(x) - A(x) - Q(x), r\right) \geq N_3\left(\vartheta(x), r\right) \tag{4.44}$$

where

$$N_3\left(\vartheta(x), r\right) = \min\left\{N_1''\left(\vartheta(x), \frac{r|6-d|}{2}\right), N_2''\left(\vartheta(x), \frac{r|36-d|}{2}\right)\right\} \tag{4.45}$$

for all $x \in X$ *and all* $r > 0$.

Proof. Clearly

$$|36| \leq |6| \leq d.$$

Let $h_o(x) = \dfrac{h_a(x) - h_a(-x)}{2}$ for all $x \in X$. Then $h_o(0) = 0$ and $h_o(-x) = -h_o(x)$ for all $x \in X$. Hence

$$\begin{aligned} N\left(D\, h_o(x,y,z), 2r\right) &= N\left(D\, h_a(x,y,z) - D\, h_a(-x,-y,-z), 2r\right) \\ &\geq \min\left\{N'\left(D\, h_a(x,y,z), r\right), N'\left(D\, h_a(-x,-y,-z), r\right)\right\} \\ &\geq \min\left\{N'\left(\vartheta(x,y,z), r\right), N'\left(\vartheta(-x,-y,-z), r\right)\right\} \end{aligned} \tag{4.46}$$

for all $x,y,z \in X$ and all $r > 0$. Let

$$N_1''(\vartheta(x,y,z), r) = \min\left\{N'\left(\vartheta(x,y,z), r\right), N'\left(\vartheta(-x,-y,-z), r\right)\right\} \tag{4.47}$$

for all $x,y,z \in X$ and all $r > 0$. By Theorems 4.1, there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(h_o(x) - A(x), r\right) \geq N_1''\left(\vartheta(x), \frac{r|6-d|}{2}\right) \tag{4.48}$$

for all $x \in X$ and all $r > 0$.

Also, let $h_e(x) = \dfrac{h_q(x) + h_q(-x)}{2}$ for all $x \in X$. Then $h_e(0) = 0$ and $h_e(-x) = h_e(x)$ for all $x \in X$. Hence

$$\begin{aligned} N\left(D\, h_e(x,y,z), r\right) &= N\left(D\, h_q(x,y,z) - D\, h_q(-x,-y,-z), 2r\right) \\ &\geq \min\left\{N'\left(\vartheta(x,y,z), r\right), N'\left(\vartheta(-x,-y,-z), r\right)\right\} \end{aligned} \tag{4.49}$$

for all $x,y,z \in X$ and all $r > 0$. Let

$$N_2(\vartheta(x,y,z), r) = \min\left\{N'\left(\vartheta(x,y,z), r\right), N'\left(\vartheta(-x,-y,-z), r\right)\right\} \tag{4.50}$$

for all $x,y,z \in X$ and all $r > 0$. By Theorem 4.2, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(h_e(x) - Q(x), r\right) \geq N_2\left(\vartheta(x), \frac{r|36-d|}{2}\right) \tag{4.51}$$

for all $x \in X$ and all $r > 0$. Define

$$h(x) = h_e(x) + h_o(x) \tag{4.52}$$

for all $x \in x$. From (4.44),(4.47) and (4.48), we arrive

$$\begin{aligned} N\left(h(x) - A(x) - Q(x), r\right) &= N\left(h_e(x) + h_o(x) - A(x) - Q(x), r\right) \\ &\geq \min\left\{N\left(h_o(x) - A(x), \frac{r}{2}\right), N\left(h_e(x) - Q(x), \frac{r}{2}\right)\right\} \\ &\geq \min\left\{N_1''\left(\vartheta(x), \frac{r|6-d|}{4}\right), N_2''\left(\vartheta(x), \frac{r|36-d|}{4}\right)\right\} \\ &= N_3\left(\vartheta(x), r\right) \end{aligned}$$

where

$$N_3\left(\vartheta(x), r\right) = \min\left\{N_1''\left(\vartheta(x), \frac{r|6-d|}{4}\right), N_2''\left(\vartheta(x), \frac{r|36-d|}{4}\right)\right\} \tag{4.53}$$

for all $x \in X$ and all $r > 0$. Hence the theorem is proved. □

The following corollary is the immediate consequence of Corollaries 4.1, 4.2 and Theorem 4.3 concerning the stability for the functional equation (1.5).

Corollary 4.3. *Suppose that a function $h : X \to Y$ satisfies the inequality*

$$N\left(D\,h(x,y,z), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), \\ N'\left(\epsilon\, ||x||^s\, ||y||^s\, ||z||^s, r\right), \\ N'\left(\epsilon\left\{||x||^s\, ||y||^s\, ||z||^s + \left(||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right)\right\}, r\right), \end{cases} \tag{4.54}$$

for all $x, y, z, t \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - A(x) - Q(x), r\right) \geq \begin{cases} N'\left(8\epsilon, r\left[\frac{|6-d|}{4} + |36-d|\right]\right), & \\ N'\left((23 + 2^s + 2^{s+1})\epsilon||x||^s, r\left[|6-d| + 2|36-d|\right]\right), & s \neq 1, 2; \\ N'\left(2\epsilon||x||^{3s}, r\left[|6-d| + \frac{|36-d|}{2}\right]\right), & s \neq \frac{1}{3}, \frac{2}{3}; \\ N'\left((29 + 2^s + 2^{s+1})\epsilon||x||^{4s}, r\left[|6-d| + 2|36-d|\right]\right), & s \neq \frac{1}{3}, \frac{2}{3}; \end{cases} \tag{4.55}$$

for all $x \in X$ and all $r > 0$.

4.2 NAFNS : Stability Results: Fixed Point Method

The following theorem provide the stability result of (1.5) for h is odd function.

Theorem 4.4. *Let $h_a : X \to Y$ be a mapping for which there exist a function $\vartheta : X^3 \to Z$ with the condition*

$$\lim_{n\to\infty} N'\left(\vartheta\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right), \mu_i^n r\right) = 1, \;\; \forall\, x, y, z \in X, r > 0 \tag{4.56}$$

where

$$\mu_i = \begin{cases} 6 & if \quad i = 0 \\ \frac{1}{6} & if \quad i = 1 \end{cases} \tag{4.57}$$

and satisfying the functional inequality

$$N\left(D\,h_a(x,y,z), r\right) \geq N'\left(\vartheta(x,y,z), r\right), \;\forall\, x, y, z \in X, r > 0. \tag{4.58}$$

If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \vartheta\left(\frac{x}{6}\right),$$

has the property

$$N'\left(L\frac{\beta(\mu_i x)}{\mu_i}, r\right) = N'\left(\beta(x), r\right), \;\forall\, x \in X, r > 0. \tag{4.59}$$

Then there exists a unique additive function $A : X \to Y$ satisfying the functional equation (1.5) and

$$N\left(h_a(x) - A(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-i}}{1-L} r\right), \;\forall\, x \in X, r > 0. \tag{4.60}$$

Proof. Let Ω is the set such that

$$\Omega = \left\{g | g : X \to Y, g(0) = 0\right\}.$$

Let d be a general metric on Ω, such that

$$d(g, h) = \inf\left\{K \in (0, \infty) | N\left(g(x) - h(x), r\right) \geq N'\left(\beta(x), Kr\right), x \in X\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \dfrac{1}{\mu_i} g(\mu_i x)$, for all $x \in X$. For $g, h \in \Omega$, we have

$$\begin{aligned}
&\inf \left\{K \in (0\infty) : N\left(g(x) - h(x), r\right) \geq N'\left(\beta(x), Kr\right), x \in X\right\} \quad or \\
&\inf \left\{K \in (0,\infty) : N\left(\frac{g(\mu_i x)}{\mu_i} - \frac{h(\mu_i x)}{\mu_i}, r\right) \geq N'\left(\beta(\mu_i x), K\mu_i r\right), x \in X\right\} \quad or \\
&\inf \left\{LK \in (0,\infty) : N\left(Tg(x) - Th(x), r\right) \geq N'\left(\beta(x), KLr\right), x \in X\right\}
\end{aligned}$$

This implies J is a strictly contractive mapping on Ω with Lipschitz constant L. We already proved the result (4.10)

$$N\left(2h_a(6x) - 12h_a(x), 4r\right) \geq N_1'\left(\vartheta(x), r\right) \tag{4.61}$$

where

$$N_1'\left(\vartheta(x), r\right) = \min\left\{N'\left(\vartheta(x,x,x), r\right), N'\left(\vartheta(2x,x,0), r\right), N'\left(\vartheta(0,x,0), r\right)\right\}$$

for all $x \in X$ and $r > 0$. Using (NAF3) in (4.61), we arrive

$$N\left(\frac{h_a(6x)}{6} - h_a(x), \frac{4r}{6\cdot 2}\right) \geq N_1'\left(\vartheta(x), r\right), \quad \forall\, x \in X, r > 0, \tag{4.62}$$

With the help of (4.59), when $i = 0$, it follows from (4.62), that

$$\begin{aligned}
&\inf \left\{1 \in (0,\infty) : N\left(\frac{h_a(6x)}{6} - h_a(x), \frac{4r}{6\cdot 2}\right) \geq N_1'\left(\vartheta(x), r\right), x \in X\right\} \quad or \\
&\inf \left\{1 \in (0,\infty) : N\left(\frac{h_a(6x)}{6} - h_a(x), r\right) \geq N_1'\left(\vartheta(x), 3r\right), x \in X\right\} \quad or \\
&\inf \left\{L \in (0,\infty) : N\left(Jh_a f(x) - h_a(x), r\right) \geq N_1'\left(\beta(x), r\right), x \in X\right\} \quad or \\
&\inf \left\{L^1 \in (0,\infty) : N\left(Jh_a f(x) - h_a(x), r\right) \geq N_1'\left(\beta(x), r\right), x \in X\right\} \quad or \\
&\inf \left\{L^{1-0} \in (0,\infty) : N\left(Jh_a f(x) - h_a(x), r\right) \geq N_1'\left(\beta(x), r\right), x \in X\right\}
\end{aligned} \tag{4.63}$$

Again replacing $x = \frac{x}{6}$ in (4.61), we obtain

$$N\left(h_a(x) - 6h_a\left(\frac{x}{6}\right), 2r\right) \geq N'\left(\vartheta\left(\frac{x}{6}\right), r\right), \quad \forall\; x \in X, r > 0. \tag{4.64}$$

When $i = 1$, it follows from (4.64), that

$$\begin{aligned}
&\inf \left\{1 \in (0,\infty) : N\left(h_a(x) - 6h_a\left(\frac{x}{6}\right), 2r\right), \geq N'\left(\vartheta\left(\frac{x}{6}\right), r\right), x \in X\right\} \quad or \\
&\inf \left\{1 \in (0,\infty) : N\left(h_a f(x) - Jh_a(x), r\right) \geq N'\left(\vartheta\left(\frac{x}{6}\right), \frac{r}{2}\right), x \in X\right\} \quad or \\
&\inf \left\{L^0 \in (0,\infty) : N\left(h_a f(x) - Jh_a(x), r\right) \geq N'\left(\beta(x), r\right), x \in X\right\} \quad or \\
&\inf \left\{L^{1-1} \in (0,\infty) : N\left(h_a f(x) - Jh_a(x), r\right) \geq N'\left(\beta(x), r\right), x \in X\right\}
\end{aligned} \tag{4.65}$$

Then from (4.63) and (4.65), we can conclude,

$$\inf \left\{L^{1-i} \in (0,\infty) : N\left(h_a f(x) - Jh_a(x), r\right) \geq N'\left(\beta(x), r\right), x \in \mathcal{X}\right\}$$

Hence property (FP1) holds. The rest of the proof is similar lines to the of Theorem 3.5. This completes the proof of the theorem. □

From Theorem 4.4, we obtain the following corollary concerning the stability for the functional equation (1.5).

Corollary 4.4. *Suppose that a odd function $h_a : X \to Y$ satisfies the inequality*

$$N\left(D\,h_a(x,y,z,t),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s\right),r\right), \\ N'\left(\epsilon\,||x||^s\,||y||^s\,||z||^s,r\right), \\ N'\left(\epsilon\left\{||x||^s\,||y||^s\,||z||^s+\left(||x||^{3s}+||y||^{3s}+||z||^{3s}\right)\right\},r\right), \end{cases} \tag{4.66}$$

for all $x,y,z,t \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique aditive mapping $A : X \to Y$ such that

$$N\left(f_a(x)-A(x),r\right) \geq \begin{cases} N'\left(\epsilon,\frac{r|6-d|}{4}\right), & \\ N'\left((8+2^s)\epsilon||x||^s,r|6-d|\right), & s<1 \quad or \quad s>1; \\ N'\left(\epsilon||x||^{3s},r|6-d|\right), & s<\frac{1}{3} \quad or \quad s>\frac{1}{3}; \\ N'\left((10+2^s)\epsilon||x||^{3s},r|6-d|\right), & s<\frac{1}{3} \quad or \quad s>\frac{1}{3}; \end{cases} \tag{4.67}$$

for all $x \in X$ and all $r > 0$.

The following theorem provide the stability result of (1.5) for h is even function using fixed point method. The proof of the following Theorem and Corollary is similar to that of Theorem 4.4 and Corollary 4.4. Hence the details of the proof is omitted.

Theorem 4.5. *Let $h_q : X \to Y$ be a even mapping for which there exist a function $\vartheta : X^3 \to Z$ with the condition*

$$\lim_{n\to\infty} N'\left(\vartheta\left(\mu_i^n x,\mu_i^n y,\mu_i^n z\right),\mu_i^{2n} r\right) = 1 \;\; \forall\, x,y,z \in X, r>0 \tag{4.68}$$

where

$$\mu_i = \begin{cases} 6 & if \quad i=0 \\ \frac{1}{6} & if \quad i=1 \end{cases} \tag{4.69}$$

and satisfying the functional inequality

$$N\left(D\,h_q(x,y,z),r\right) \geq N'\left(\vartheta(x,y,z),r\right) \;\forall\, x,y,z \in X, r>0. \tag{4.70}$$

If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \vartheta\left(x,\frac{x}{2},0,0\right),$$

has the property

$$N'\left(L\frac{1}{\mu_i^2}\beta(\mu_i x),r\right) = N'\left(\beta(x),r\right) \;\forall\, x \in X, r>0. \tag{4.71}$$

Then there exists a unique quadratic function $Q : X \to Y$ satisfying the functional equation (1.5) and

$$N\left(h_q(x)-Q(x),r\right) \geq N'\left(\beta(x),\frac{L^{1-i}}{1-L}r\right) \;\forall\, x \in X, r>0. \tag{4.72}$$

Corollary 4.5. *Suppose that a even function $h_q : X \to Y$ satisfies the inequality*

$$N\left(D\,h_q(x,y,z),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s\right),r\right), \\ N'\left(\epsilon\,||x||^s\,||y||^s\,||z||^s,r\right), \\ N'\left(\epsilon\left\{||x||^s\,||y||^s\,||z||^s+\left(||x||^{3s}+||y||^{3s}+||z||^{3s}\right)\right\},r\right), \end{cases} \tag{4.73}$$

for all $r > 0$ and all $x,y,z \in X$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(h_q(x)-A(x),r\right) \geq \begin{cases} N'\left(7\epsilon,r|36-d|\right), & \\ N'\left((15+2^{s+1})\epsilon||x||^s,2r|36-d|\right), & s<2 \quad or \quad s>2; \\ N'\left(\epsilon||x||^{3s},\frac{r|36-d|}{2}\right), & s<\frac{2}{3} \quad or \quad s>\frac{2}{3}; \\ N'\left((19+2^{s+1})\epsilon||x||^{3s},2r|36-d|\right), & s<\frac{2}{3} \quad or \quad s>\frac{2}{3}; \end{cases} \tag{4.74}$$

for all $x \in X$ and all $r > 0$.

The following theorem provide the stability result of (1.5) for mixed case in fixed point method.

Theorem 4.6. *Let $f : X \to Y$ be a mapping for which there exist a function $\vartheta : X^3 \to Z$ with the condition (4.56) and (4.68) satisfying the functional inequality*

$$N\left(D\, f(x,y,z),r\right) \geq N'\left(\vartheta(x,y,z),r\right), \ \forall\, x,y,z,t \in X, r > 0. \tag{4.75}$$

If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \vartheta\left(x, \frac{x}{2}, 0, 0\right),$$

has the properties (4.59) and (4.71) for all $x \in X$. Then there exists a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ satisfying the functional equation (1.5) and

$$N\left(f(x) - A(x) - Q(x), r\right) \geq N_3\left(\beta(x), r\right), \ \forall\, x \in X, r > 0. \tag{4.76}$$

The following corollary is the immediate consequence of Corollaries 4.4, 4.5 and Theorem 4.6 concerning the stability for the functional equation (1.5) in fixed point method.

Corollary 4.6. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(D\, f(x,y,z,t),r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), \\ N'\left(\epsilon\, ||x||^s\, ||y||^s\, ||z||^s, r\right), \\ N'\left(\epsilon\left\{||x||^s\, ||y||^s\, ||z||^s + \left(||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right)\right\}, r\right), \end{cases} \tag{4.77}$$

for all $r > 0$ and all $x, y, z, t \in X$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - A(x) - Q(x), r\right) \geq \begin{cases} N'\left(8\epsilon, r\left[\frac{|6-d|}{4} + |36-d|\right]\right), & \\ N'\left((23 + 2^s + 2^{s+1})\epsilon||x||^s, r\left[|6-d| + 2|36-d|\right]\right), & s \neq 1, 2; \\ N'\left(2\epsilon||x||^{3s}, r\left[|6-d| + \frac{|36-d|}{2}\right]\right), & s \neq \frac{1}{3}, \frac{2}{3}; \\ N'\left((29 + 2^s + 2^{s+1})\epsilon||x||^{4s}, r\left[|6-d| + 2|36-d|\right]\right), & s \neq \frac{1}{3}, \frac{2}{3}; \end{cases} \tag{4.78}$$

for all $x \in X$ and all $r > 0$.

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A Note on connectedness in a bispace

Amar Kumar Banerjee[a,*], Rahul Mondal[b**]

[a,b]Department of Mathematics,The University of Burdwan,Golapbag, Burdwan-713104,West Bengal, India.

Abstract

Here we have studied the idea of connectedness and totally disconnectedness in a bispace and some of its basic properties. Also it has been investigated how far several results as valid in a bitopological space are affected in a bispace.

Keywords: σ-space, bispace, connectedness, total disconnectedness.

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1 Introduction

J.C Kelly [4] introduced the notion of a bitopological space $(X, \mathcal{P}, \mathcal{Q})$ to be a set X with two topologies $\mathcal{P}$ and $\mathcal{Q}$ defined on it. It is with such spaces that some of the symmetric properties can be recovered when treating unsymmetric metrics(quasi-metrics). Other authors have expanded his results by considering analogous notions for uniform spaces. The notion of a σ- space(or simply space) was introduced by A.D Alexandroff [1] in 1940 generalizing the notion of a topological space where only countable union of open sets were taken to be open. The notion of a bispace was introduced by Lahiri and Das [7] generalizing the idea of a bitopological space in 2001. The concept of connectedness in a bitopological space was introduced by Pervin [8]. Pervin also introduced the idea of total disconnectedness in a bitopological space.The systematic study of bitopological spaces was begun by J.C Kelly [4], who introduced various separation properties into bitopological spaces and obtained generalizations of some important classical results and various other authors have contributed to the development of the theory. Here we have introduced the notion of connectedness in a bispace and studied some of its basic properties. Also we have studied some properties of totally disconnectedness in a bispace.

2 Preliminary

Definition 2.1.[1] A set X is called an Alexandroff space or σ- space(or simply space) if it is chosen a system $\mathcal{F}$ of subsets of X, satisfying the following axioms
(i)The intersection of countable number sets in $\mathcal{F}$ is a set in $\mathcal{F}$.
(ii)The union of finite number of sets from $\mathcal{F}$ is a set in $\mathcal{F}$.
(iii)The void set and X is a set in $\mathcal{F}$.
Sets of $\mathcal{F}$ are called closed sets. There complementary sets are called open.It is clear that instead of closed sets in the definition of a space, one may put open sets with subject to the conditions of countable summability, finite intersectability and the condition that X and the void set should be open.
The collection of such open sets will sometimes be denoted by $\mathcal{P}$ and the σ-space by $(X, \mathcal{P})$. It can be noted that $\mathcal{P}$ is not a topology in general as can be seen by taking $X = \mathbb{R}$, the set of real numbers and $\mathcal{P}$ as the collection of all F_σ sets in $\mathbb{R}$.

*Corresponding author. .
: akbanerjee1971@gmail.com (Amar Kumar Banerjee).

Definition 2.2.[1] To every set M we correlate its closure $\overline{M}$= the intersection of all closed sets containing M. Generally the closure in a space is not a closed set. We denote the closure of a set M in a space $(X, \mathcal{P})$ by $\mathcal{P}$-cl(M) or cl(M) or simply $\overline{M}$ when there is no confusion about $\mathcal{P}$.

The idea of limit points, derived set, interior of a set etc. in a σ-space are similar as in the case of a topological space which have been thoroughly discussed in [6].

Definition 2.3. [2] Let $(X, \mathcal{P})$ be a space. A family of open sets B is said to form a base (open) for $\mathcal{P}$ if and only if every open set can be expressed as countable union of members of B.
Theorem 2.1. [2] A collection of subsets B of a set X forms an open base of a suitable space structure $\mathcal{P}$ of X if and only if
1) the null set $\phi \in B$
2) X is the countable union of some sets belonging to B.
3) intersection of any two sets belonging to B is expressible as countable union of some sets belonging to B.
Definition 2.4. [7] Let X be a non-empty set. If $\mathcal{P}$ and $\mathcal{Q}$ be two collection of subsets of X such that $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ are two spaces, then X is called a bispace.
Definition 2.5. [7] A bispace $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise T_1 if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.
Definition 2.6. [7] A bispace $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise Hausdorff if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U$, $y \in V$, $U \cap V = \phi$.

3 Connectedness and Total Disconnectedness

The following two results are very useful in an arbitrary space $(X, \mathcal{P})$. For the sake of convenience we give the details proof also.
Theorem 3.1. In a σ-space $(X, \mathcal{P})$ if $A \subset X$ and $x \in \overline{A}$ then every open set U containing x intersects A.
Proof. Let $x \in \overline{A}$. If possible let there there exists an open set U containing x which does not intersect A. Then it implies that $A \subset X \setminus U$, where $X \setminus U$ is a closed set. Now $x \notin X \setminus U$ and $X \setminus U$ is a closed set containing A. This contradicts to the fact that $x \in \overline{A}$. ■
Theorem 3.2. In a σ-space $(X, \mathcal{P})$ if $A \subset X$ then $\overline{A} = A \cup A'$.
Proof. First suppose $x \in \overline{A}$. Then every open set containing x intersects A. Now if $x \notin A$, then for every open set U containing x we have, $(A \setminus \{x\}) \cap U = A \cap U \neq \emptyset$. So x becomes a limit point of A. Therefore $x \in A'$ and $x \in A \cup A'$. Hence $\overline{A} \subset A \cup A'$.
Conversely, let $y \in A \cup A'$. Now if $y \in A$ then $y \in \overline{A}$. If $y \in A'$ then if it happens that $y \notin \overline{A}$, there exists a closed set M containing A such that $y \notin M$ i.e., $y \in X \setminus M$. Now $X \setminus M$ is an open set containing y which does not intersect A and this leads to a contradiction to the fact that $y \in A'$. Therefore we must have $y \in \overline{A}$. Hence $A \cup A' \subset \overline{A}$. Thus $\overline{A} = A \cup A'$. ■

Throughout our discussion $(X, \mathcal{P}, \mathcal{Q})$ or simply X stands for a bispace, $\mathbb{R}$ stands for the set of real numbers, $\mathbb{Q}$ for the set of all rational numbers and $\mathbb{N}$ for the set of all natural numbers and sets are always subsets of X unless otherwise stated.
Definition3.1. (cf. [8]) Two non empty subsets A and B in $(X, \mathcal{P}, \mathcal{Q})$ are said to be pairwise separated if there exists a $\mathcal{P}$-open set U and a $\mathcal{Q}$-open set V such that $A \subset U$ and $B \subset V$ and $A \cap V = B \cap U = \phi$ or there exists a $\mathcal{Q}$-open set U and a $\mathcal{P}$-open set V such that $A \subset U$ and $B \subset V$ and $A \cap V = B \cap U = \phi$.
For the sake of convenience we use only the first condition.
Definition3.2. (cf. [8]) A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be connected if and only if X can not be express as the union of two non empty separated sets.

The following theorem is a characterization of connectedness in term of open sets. Though the proof is straightforward we give the details proof for the sake of completeness.
Theorem 3.3. The following statements are equivalent for any bispace $(X, \mathcal{P}, \mathcal{Q})$:
(i) $(X, \mathcal{P}, \mathcal{Q})$ is connected.
(ii) X can not be expressed as the union of two nonempty disjoint sets A and B such that A is $\mathcal{P}$ -open and B

is $\mathcal{Q}$ -open.
(iii) X contains no nonempty proper subset which is both $\mathcal{P}$ -open and $\mathcal{Q}$ -closed (or $\mathcal{P}$ -closed and $\mathcal{Q}$ -open).
Proof. **(i)** $\Rightarrow$ **(ii)** First suppose that $(X, \mathcal{P}, \mathcal{Q})$ is connected. Now if possible let X can be expressed as a union of two nonempty disjoint sets A and B such that A is $\mathcal{P}$-open B is $\mathcal{Q}$-open. Therefore $X = A \cup B$, A is $\mathcal{P}$-open B is $\mathcal{Q}$-open. Then clearly X can be expressed as a union of two non-empty disjoint separated sets A and B. This is a contradiction to the fact that X is connected. Therefore X cannot be expressed as a union of two non-empty disjoint sets A and B such that A is $\mathcal{P}$-open and B is $\mathcal{Q}$-open.

(ii) $\Rightarrow$ **(i)** Let X can not be expressed as the union of two nonempty disjoint sets A and B such that A is $\mathcal{P}$ -open and B is $\mathcal{Q}$ -open. If possible let X be not connected then X can be expressed as the union of two non empty separated sets P and Q. So there exists $\mathcal{P}$ -open set U and $\mathcal{Q}$ -open set V such that $P \subset U$ and $Q \subset V$ and $P \cap V = Q \cap U = \phi$. If possible let $y \in U \cap V$. Then $y \in X = P \cup Q$. Now if $y \in P$ then $y \in P \cap V$, a contradiction. If $y \in Q$ we will arrive at a similar kind of contradiction. Hence $U \cap V = \phi$. Therefore X can be expressed as the union of two non empty disjoint sets U and V where U is $\mathcal{P}$ -open and V is $\mathcal{Q}$ -open, which is a contradiction. Hence X must be connected.

(ii) $\Rightarrow$ **(iii)** Let (ii) holds, let X contains a nonempty proper subset M which is both $\mathcal{P}$ -open and $\mathcal{Q}$ -closed. Now $X = M \cup (X \setminus M)$, where M is $\mathcal{P}$ -open and $(X \setminus M)$ is $\mathcal{Q}$ -open, which is a contradiction. Therefore X contains no nonempty proper subset proper subset which is both $\mathcal{P}$ -open and $\mathcal{Q}$ -closed.

(iii) $\Rightarrow$ **(ii)** Let X contains no non-empty proper subset which is both $\mathcal{P}$-open and $\mathcal{Q}$-closed. Now if possible, let X can be expressed as the union of two non-empty disjoint sets A and B, where A is $\mathcal{P}$-open and B is $\mathcal{Q}$-open. Since $X = A \cup B$ therefore $A = X \setminus B$. Since B is $\mathcal{Q}$-open, $X \setminus B$ is $\mathcal{Q}$-closed. Therefore A becomes both $\mathcal{P}$-open and $\mathcal{Q}$-closed which is a contradiction. Hence **(ii)** holds. ■

Note 3.1 We see that $(X, \mathcal{P}, \mathcal{Q})$ is connected iff X can not be expressed as the union of two non empty disjoint sets A and B such that A is $\mathcal{P}$ -open and B is $\mathcal{Q}$ -open. When X can be so expressed we write $X = A|B$ and call this separation (or disconnection) of X. ■

We see that if $(X, \mathcal{P})$ is a σ-space and if $A \subset X$ then as in the case of topological spaces the collection $\mathcal{P}_A = \{U \cap A : U \in \mathcal{P}\}$ forms a σ-space structure on A and $(A, \mathcal{P}_A)$ is called a σ-subspace of $(X, \mathcal{P})$.

Definition 3.3.(cf. [8]) A function f mapping a bispace $(X, \mathcal{P}, \mathcal{Q})$ into a bispace $(X, \mathcal{P}^*, \mathcal{Q}^*)$ is said to be continuous if and only if induced mappings $f_1 : (X, \mathcal{P}) \to (X, \mathcal{P}^*)$ and $f_2 : (X, \mathcal{Q}) \to (X, \mathcal{Q}^*)$ are continuous.

We shall denote the left-hand and right-hand topologies on $\mathbb{R}$ (having bases (∞, x) and (y, ∞) respectively) by $\mathcal{L}$ and $\mathcal{R}$. As every bitopological space is a bispace we may treat $(\mathbb{R}, \mathcal{L}, \mathcal{R})$ as a bispace also.

Theorem 3.4. A bispace $(X, \mathcal{P}, \mathcal{Q})$ is connected if and only if every continuous mapping of $(X, \mathcal{P}, \mathcal{Q})$ into $(D, \mathcal{L}_D, \mathcal{R}_D)$ is constant, where $D = \{0, 1\}$.

Proof. Let $(X, \mathcal{P}, \mathcal{Q})$ be connected. Now if such a non-constant continuous mapping f exists, then $X = f^{-1}\{0\}|f^{-1}\{1\}$ is a separation of X which is a contradiction to the fact that X is connected.
Conversely, if $(X, \mathcal{P}, \mathcal{Q})$ is not connected then let $X = A|B$ be a separation of X. So we may define a non-constant continuous function f such that $f(A) = \{0\}$ and $f(B) = \{1\}$ which is a contradiction. Hence $(X, \mathcal{P}, \mathcal{Q})$ must be connected. ■

Theorem 3.5. A bispace $(X, \mathcal{P}, \mathcal{Q})$ is connected if and only if every continuous mapping of $(X, \mathcal{P}, \mathcal{Q})$ into $(\mathbb{R}, \mathcal{L}, \mathcal{R})$ has the Durbox property (i.e,its range is an interval).

Proof. Let us suppose every such continuous function has Durbox's property. Now if X is not connected then let $X = A|B$ is a separation of X. Then the continuous function f defined by $f(A) = \{0\}$ and $f(B) = \{1\}$ does not possess the Durbox's property, which is a contradiction.
Conversely, let $(X, \mathcal{P}, \mathcal{Q})$ be connected. Now if f is continuous but does not possess the Durbox's property, then there exists a point c in (a, b) such that $\{a, b\} \subset f(X)$ but $c \notin f(X)$. Then obviously $f^{-1}((-\infty, c))|f^{-1}((c, \infty))$ is a separation of X, which is a contradiction. ■

Theorem 3.6. The continuous image of a connected set is connected.

Proof. The proof is straightforward and so is omitted. ■

Definition 3.4. (cf. [8]) A subset E in a bispace $(X, \mathcal{P}, \mathcal{Q})$ is called connected if and only if $(E, \mathcal{P}_E, \mathcal{Q}_E)$ is connected and by component we mean maximal connected set in the bispace.

Theorem 3.7. If C is connected subset of a bispace $(X, \mathcal{P}, \mathcal{Q})$ and $X = A \cup B$ where A and B are non empty pairwise separated sets, then either $C \subset A$ or $C \subset B$.

Proof. We have $X = A \cup B$, where A and B are non empty pairwise separated sets. So there exists $\mathcal{P}$-open set U and $\mathcal{Q}$-open set V such that $A \subset U$ and $B \subset V$ and $A \cap V = \emptyset = B \cap U$. Now if neither $C \subset A$ nor $C \subset B$ is true then $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$. Now $C = (C \cap A) \cup (C \cap B)$. Again $C \cap A \subset C \cap U$, where $C \cap U$ is $\mathcal{P}_C$-open and $C \cap B \subset C \cap V$, where $C \cap V$ is $\mathcal{Q}_C$-open. We also have $(C \cap A) \cap (C \cap V) = \emptyset = (C \cap B) \cap (C \cap U)$. So we arrive at a contradiction, because C is connected. Therefore either $C \subset A$ or $C \subset B$ holds. ■

From the above result it immediately follows the corollaries.

Corollary 3.1. If every two points of a set E are contained in some connected subset of E, then E is connected.

Corollary 3.2. The union of any family of connected sets having a nonempty intersection is a connected set.

Theorem 3.8. If C is a connected set and $C \subset E \subset \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$, then E is is connected.

Proof. If $E = C$ then E is connected. If $E \neq C$ then if possible let E be not connected. Then there exists a $\mathcal{P}_E$-open set $U \neq \emptyset$ and $\mathcal{Q}_E$-open set $V \neq \emptyset$ such that $E = U \cup V$, $U \cap V = \emptyset$. Since C is connected, either $C \subset U$ or $C \subset V$. If $C \subset U$ then there exists a point $x \in E$ such that $x \in V$ and $x \notin U$. Hence $x \notin \mathcal{Q}_E\text{-}cl(C)$. Therefore $x \notin \mathcal{Q}\text{-}cl(C)$. Hence we have $x \notin \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$, which is a contradiction as $E \subset \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$. If $C \subset V$ then we will arrive at a similar contradiction. ■

Theorem 3.9. If C be a component in a bispace $(X, \mathcal{P}, \mathcal{Q})$, then $C = \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$.

Proof. Let C be a component and suppose that $p \notin C$. Then $C \cup \{p\}$ is not connected and let A, B be two non empty disjoint sets such that $C \cup \{p\} = A \cup B$, where A and B are open in $(C \cup \{p\}, \mathcal{P}_{C \cup \{p\}})$ and $(C \cup \{p\}, \mathcal{Q}_{C \cup \{p\}})$ respectively. Now since C is component, either $C \subset A$ or $C \subset B$. Hence either $\{p\} \subset A$ or $\{p\} \subset B$. So we see that $\{p\}$ is either $\mathcal{P}_{C \cup \{p\}}$-open or $\mathcal{Q}_{C \cup \{p\}}$-open. Hence either $p \notin \mathcal{P}_{C \cup \{p\}}\text{-}cl(C)$ or $p \notin \mathcal{Q}_{C \cup \{p\}}\text{-}cl(C)$. Therefore either $p \notin \mathcal{P}\text{-}cl(C)$ or $p \notin \mathcal{Q}\text{-}cl(C)$. Therefore $p \notin \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$. Hence we have $\mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C) \subset C$. Clearly $C \subset \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$. ■

Definition 3.5. (cf. [11]) A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be totally disconnected if for any two distinct points x and y there exists a disconnection $X = A|B$ with $x \in A$ and $y \in B$, where A is $\mathcal{P}$ open and B is $\mathcal{Q}$ open.

Definition 3.6. (cf. [11]) A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be weakly totally disconnected if for any two distinct points x and y there exists a disconnection $X = A|B$, where A is $\mathcal{P}$ open and B is $\mathcal{Q}$ open such that one point belongs to A and the other point belongs to B. In this case the roles of the points need not be interchangeable.

Example3.1. Let $X = \mathbb{R}$, the set of real numbers. We now consider the collections $\mathcal{P}$ and $\mathcal{Q}$ of subsets of X as follows.

$\mathcal{P} = \{\phi, X\} \cup \{$sub sets of X whose complement is finite$\}$

$\mathcal{Q} = \{\phi, X\} \cup \{$countable sub sets of X $\}$

It is easy to examine that $(X, \mathcal{P}, \mathcal{Q})$ is a bispace. If x and y be any two distinct points in X then $\{x\}$ is a $\mathcal{Q}$-open set containing x and $(X \setminus \{x\})$ is a $\mathcal{P}$-open set containing y with $X = \{x\} \cup (X \setminus \{x\})$ and $\{x\} \cap (X \setminus \{x\}) = \phi$. Again $(X \setminus \{y\})$ is a $\mathcal{P}$-open set containing x and $\{y\}$ is a $\mathcal{Q}$-open set containing y with $X = (X \setminus \{y\}) \cup \{y\}$ and $\{y\} \cap (X \setminus \{y\}) = \phi$.

Hence $(X, \mathcal{P}, \mathcal{Q})$ is totally disconnected bispace. ■

We now give an example of a weakly totally disconnected bispace which is not totally disconnected.

Example3.2. Let $X = \mathbb{R}$, the set of real numbers. We now consider the collections $\mathcal{P}$ and $\mathcal{Q}$ of subsets of X as follows.

$\mathcal{P} = \{\phi, X\} \cup \{$sub sets of X whose complement is finite$\}$

$\mathcal{Q} = \{\phi, X\} \cup \{$countable sub sets of X not containing 2$\}$

It is easy to examine that $(X, \mathcal{P}, \mathcal{Q})$ is a weakly totally disconnected bispace. Let p be an arbitrary point in X such that $p \neq 2$. Then $\{p\} \in \mathcal{Q}$ containing p and $(X \setminus \{p\}) \in \mathcal{P}$ containing 2, such that $\{p\} \cup (X \setminus \{p\}) = X$, $\{p\} \cap (X \setminus \{p\}) = \phi$.

But as there does not exists any $\mathcal{Q}$-open set B other then X which contains 2, X can not be totally disconnected. ■

Theorem 3.10. The component of a weakly totally disconnected bispace are its points.

Proof. The proof is parallel as in the case of a bitopological space [11] and so is omitted. ■

Definition 3.7. (cf.[11]) A bispace $(X, \mathcal{P}, \mathcal{Q})$ is pairwise weakly Hausdorff if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that one contains x and the other contains y and $U \cap V = \phi$. The roles of the points need not be interchangeable.

Theorem 3.11. Let $(X, \mathcal{P}, \mathcal{Q})$ be a pairwise weakly Hausdorff bispace. If $\mathcal{P}$ has a base whose sets are $\mathcal{Q}$ closed or $\mathcal{Q}$ has a base whose sets are $\mathcal{P}$ closed, then $(X, \mathcal{P}, \mathcal{Q})$ is weakly totally disconnected.

Proof. The proof is parallel to the proof of theorem 2.5 [11] and so is omitted. ■

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Further nonlinear integral inequalities in two independent variables on time scales and their applications

K. Boukerrioua[a,*], D.Diabi[b] and T.Chiheb[c]

[a]Lanos Laboratory, University of Badji-Mokhtar, Annaba, Algeria

[b]University of Badji-Mokhtar, Annaba, Algeria

[c]University of 8 Mai 1945 Guelma. Algeria.

Abstract

Using ideas from [15], some nonlinear integral inequalities on time scales in two independent variables are established. Also, some examples are presented to show the feasibility of these results.

Keywords: Dynamic equations, time scale, integral inequality.

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1 Introduction

During the few years, a lot of research related to studies and the extension of some fundamental integral inequalities used in the theory of differential and integral equations on time scales. For example, we refer the reader to the papers [1-5, 8-19]. The purpose of this note is to illustrate some time scale Pachpatte-type inequalities by extending some continuous inequalities given in [15]. Inequalities of this form have in particular dominated the study of certain classes of integral equations on time scales. Throughout this work a knowledge and understanding of time scales notation is assumed; for an excellent bibliography to the time scales, see monographs of M. Bohner [6, 7] for a general review.

2 Preliminaries on time scales

In this section, we begin by giving some necessary materials for our study.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. The forward jump operator σ on $\mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T}$ for all $t \in \mathbb{T}$, C_{rd} denotes the set of rd-continuous functions and the set $\mathbb{T}^k$ which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Throughout this paper, we always assume that $\mathbb{T}_1$and $\mathbb{T}_2$ are time scales, and consider the time scales intervals $\overline{\mathbb{T}}_1 = [a_1, \infty) \cap \mathbb{T}_1$ and $\overline{\mathbb{T}}_2 = [a_2, \infty) \cap \mathbb{T}_2$, for $a_1 \in \mathbb{T}_1$, and $a_2 \in \mathbb{T}_2$, Ω denote the set $\overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2$.we write $x^{\Delta_1 s}(s,t)$ the partial delta derivative of $x(s,t)$ with respect to the first variable and $x^{\Delta_2 t}(t,s)$ for the second variable.

Lemma 2.1. *[13, lemma 2] Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}, \tag{2.1}$$

for any $K > 0$.

*Corresponding author.

K. Boukerrioua: khaledv2004@yahoo.fr.

Lemma 2.2. *[11, Theorem 2.1] Let $u(t_1,t_2), a(t_1,t_2), f(t_1,t_2) \in C(\overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2, \mathbb{R}_0^+)$ with $a(t_1,t_2)$ nondecreasing in each of its variables. If*

$$u(t_1,t_2) \le a(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\,u(s_1,s_2)\,\Delta_2 s_2 \Delta_1 s_1, \tag{2.2}$$

for $(a_1,a_2), (t_1,t_2) \in \overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2$, then

$$u(t_1,t_2) \le a(t_1,t_2)\, e_{\int_{a_2}^{t_2} f(t_1,s_2)\,\Delta_2 s_2}(t_1,a_1), \qquad (t_1,t_2) \in \overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2 \tag{2.3}$$

where $\mathbb{T}_1, \mathbb{T}_2$ are time scales and $\overline{\mathbb{T}}_1 = [a_1,\infty) \cap \mathbb{T}_1$, $\overline{\mathbb{T}}_2 = [a_2,\infty) \cap \mathbb{T}_2$

Lemma 2.3. *[6, Theorem 1.117] Let $a \in \mathbb{T}^k$, $b \in \mathbb{T}$ and assume $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^\Delta(t,.)$ is rd-continuous on $[a,\sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [a,\sigma(t)]$, such that*

$$\left| f(\sigma(t),\tau) - f(s,\tau) - f^\Delta(t,\tau)(\sigma(t)-s) \right| < \varepsilon\,|\sigma(t)-s| \ \text{ for all } s \in U,$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

$$(i)\ g(t) := \int_a^t f(t,\tau)\Delta\tau \ \text{ implies } g^\Delta(t) = \int_a^t f^\Delta(t,\tau)\Delta\tau + f(\sigma(t),t);$$

Now we state the main results of this work.

3 Main result

Theorem 3.1. *Let $u(x,y), f(x,y)$ be nonnegative functions defined for $(x,y) \in \Omega$ that are right-dense continuous for $(x,y) \in \Omega$, and $L(x,y,s,t) \in C_{rd}(\Omega \times \Omega, \mathbb{R}^+)$. c, p, q, $r \in \mathbb{R}_0^+$ such that $p \ge q > 0, p \ge r > 0$.Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $]0,+\infty[$ with continuous decreasing first derivative on $]0,+\infty[$. If*

$$u^p(x,y) \le c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left[u^q(s,t) + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) g(u^r(\tau,\eta))\Delta_2\eta\Delta_1\tau\right]\Delta_2 t\Delta_1 s\,, \tag{3.4}$$

hold for all $(x,y) \in \Omega$, then

$$u(x,y) \le \left\{ P(x,y)\, e_{\int_{y_0}^{y} Q(\tau,\eta)\,\Delta_2\eta}(x,x_0) \right\}^{\frac{1}{p}}, \tag{3.5}$$

where

$$P(x,y) = c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left[\frac{p-q}{p}K^{\frac{q}{p}} + g\left(\frac{p-r}{p}K^{\frac{r}{p}}\right)\int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)\Delta_2\eta\Delta_1\tau\right]\Delta_2 t\Delta_1 s, \tag{3.6}$$

$$Q(s,t) = f(s,t)\left[\frac{q}{p}K^{\frac{q-p}{p}} + \frac{r}{p}g'\left(\frac{p-r}{p}K^{\frac{r}{p}}\right)K^{\frac{r-p}{p}}\int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)\Delta_2\eta\Delta_1\tau\right], \tag{3.7}$$

and $K > 0$.

Proof. Define a function $z(x,y)$ as follows

$$z(x,y) = c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left[u^q(s,t) + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) g(u^r(\tau,\eta))\Delta_2\eta\Delta_1\tau\right]\Delta_2 t\Delta_1 s \tag{3.8}$$

then

$$z(x_0,y) = z(x,y_0) = c \tag{3.9}$$

and

$$u^p(x,y) \le z(x,y) \tag{3.10}$$

then (3.10) implies

$$u(x,y) \leq z^{\frac{1}{p}}(x,y) \leq \frac{1}{p}K^{\frac{1-p}{p}} z(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}, \tag{3.11}$$

using (3.11) in (3.8), we get

$$z(x,y) \leq c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[z^{\frac{q}{p}}(s,t) + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)g(z^{\frac{r}{p}}(\tau,\eta))\Delta_2\eta\Delta_1\tau\right]\Delta_2 t\Delta_1 s. \tag{3.12}$$

By Lemma 2.1, the inequality (3.12) become

$$\begin{aligned} z(x,y) \quad \leq \quad & c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[\frac{q}{p}K^{\frac{q-p}{p}} z(s,t) + \frac{p-q}{p}K^{\frac{q}{p}} \right. \\ & \left. + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)g\left(\frac{r}{p}K^{\frac{r-p}{p}} z(\tau,\eta) + \frac{p-r}{p}K^{\frac{r}{p}}\right)\Delta_2\eta\Delta_1\tau\right]\Delta_2 t\Delta_1 s, \end{aligned} \tag{3.13}$$

Applying the mean value theorem for the function g, then for every $x_1 \geq y_1 > 0$, there exists $c \in]y_1, x_1[$ such that

$$g(x_1) - g(y_1) = g'(c)(x_1 - y_1) \leq g'(y_1)(x_1 - y_1),$$

the inequality (3.13) can be rewrite as follows

$$\begin{aligned} z(x,y) \quad \leq \quad & c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left[\frac{p-q}{p}K^{\frac{q}{p}} + g(\frac{p-r}{p}K^{\frac{r}{p}})\int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)\Delta_2\eta\Delta_1\tau\right]\Delta_2 t\Delta_1 s \\ & + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)z(s,t)\left[\frac{q}{p}K^{\frac{q-p}{p}} + \frac{r}{p}K^{\frac{r-p}{p}} g'(\frac{p-r}{p}K^{\frac{r}{p}})\int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)\Delta_2\eta\Delta_1\tau\right]\Delta_2 t\Delta_1 s, \end{aligned} \tag{3.14}$$

replace (3.6) and (3.7) in (3.14), we obtain

$$z(x,y) \leq P(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} Q(s,t)z(s,t)\Delta_2 t\Delta_1 s, \tag{3.15}$$

using Lemma 2.2 for (3.15), we get

$$z(x,y) \leq P(x,y)e_{\int_{y_0}^{y} Q(s,t)\Delta_2 t}(x,x_0). \tag{3.16}$$

The required inequality (3.5) follows from (3.11) and (3.16) . □

Remark 3.1. *If we take $g(x) = x$, Theorem 3.1 will be reduced to Theorem 3.1 in [15].*

Theorem 3.2. *Assume that $u(x,y)$, $f(x,y)$ are nonnegative functions defined for $(x,y) \in \Omega$, that are right-dense continuous for $(x,y) \in \Omega$, and $L(x,y,s,t) \in C_{rd}(\Omega\times\Omega, \mathbb{R}^+)$ .Let g_1 and $g_2 : \mathbb{R}_{+\to}\mathbb{R}_+$ are a differentiable increasing functions on $]0,+\infty[$ with continuous decreasing first derivative on $]0,+\infty[$. If*

$$u^p(x,y) \leq c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[g_1(u(s,t)) + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)g_2(u(\tau,\eta))\Delta_2\eta\Delta_1\tau\right]\Delta_2 t\Delta_1 s\,, \tag{3.17}$$

hold for all $(x,y) \in \Omega$, then

$$u(x,y) \leq \left\{P_*(x,y)e_{\int_{y_0}^{y} Q_*(\tau,\eta)\,\Delta_2\eta}(x,x_0)\right\}^{\frac{1}{p}} \tag{3.18}$$

where

$$P_*(x,y) = c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left[g_1(\frac{p-1}{p}K^{\frac{1}{p}}) + g_2(\frac{p-1}{p}K^{\frac{1}{p}})\int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)\Delta_2\eta\Delta_1\tau\right]\Delta_2 t\Delta_1 s \tag{3.19}$$

$$Q_*(s,t) = f(s,t)\left[\frac{1}{p}K^{\frac{1-p}{p}} g_1'(\frac{p-1}{p}K^{\frac{1}{p}}) + \frac{1}{p}K^{\frac{1-p}{p}} g_2'(\frac{p-1}{p}K^{\frac{1}{p}})\int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)\Delta_2\eta\Delta_1\tau\right]. \tag{3.20}$$

For $K > 0$.

Proof. Define a function $z(x,y)$ as follows

$$z(x,y)=c+\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)\left[\begin{array}{c} g_1(u(s,t))+ \\ \int_{s_0}^{s}\int_{t_0}^{t}L(s,t,\tau,\eta)g_2(u(\tau,\eta))\Delta_2\eta\Delta_1\tau \end{array}\right]\Delta_2t\Delta_1s\,, \tag{3.21}$$

Applying the mean value theorem for the functions g_1 and g_2, from (3.11) and (3.21), we obtain

$$z\,(x,y)\leq c+\int_{x_0}^{x}\int_{y_0}^{y}f\,(s,t)\left[\begin{array}{c} g_1(\frac{1}{p}K^{\frac{1-p}{p}}z(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}})+ \\ \int_{s_0}^{s}\int_{t_0}^{t}L(s,t,\tau,\eta)g_2(\frac{1}{p}K^{\frac{1-p}{p}}z(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}})\Delta_2\eta\Delta_1\tau \end{array}\right]\Delta_2t\Delta_1s. \tag{3.22}$$

The above inequality can be reformulated as

$$z\,(x,y)\leq P_*(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}Q_*(s,t)z(s,t)\Delta_2t\Delta_1s, \tag{3.23}$$

where P_* and Q_* are defined by (3.19)-(3.20).
Using Lemma 2, from (3.23) we obtain

$$u(x,y)\leq\left\{P_*(x,y)e_{\int_{y_0}^{y}Q_*(\tau,\eta)\ \Delta_2\eta}(x,x_0)\right\}^{\frac{1}{p}}. \tag{3.24}$$

The required inequality (3.18) follow from (3.11) and (3.24). □

Remark 3.2. *If we take $g_1(x)=x$, Theorem 3.2 will be reduced to Theorem 3.1 for $q=r=1$.*

4 An Application

In this section we give an application of Theorem 3.1. We consider the following partial dynamic equation on time scales

$$(u^p(x,y))^{\Delta_2y\Delta_1x}=F(x,y,u^q(x,y),\int_{x_0}^{x}\int_{y_0}^{y}h(s,t,\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau), \tag{4.25}$$

with the initial boundary conditions

$$u(x,y_0)=\alpha(x),u(x_0,y)=\beta(y),\ \alpha(0)=\beta(0)=0. \tag{4.26}$$

where $u\in C_{rd}(\Omega,\mathbb{R}),h\in C_{rd}(\Omega\times\Omega\times\mathbb{R},\mathbb{R})$ and $F\in C_{rd}(\Omega\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

Proposition 4.1. *Assume that*

$$\begin{aligned} |h(x,y,s,t,u(s,t)| &\leq L(x,y,s,t)\arctan(|u(s,t)|^r) \\ |F(x,y,u,v)| &\leq f(x,y)(|u|+|v|), \\ |\alpha(x)+\beta(y)| &\leq c, \end{aligned} \tag{4.27}$$

where L,f,c,p,q,r are defined as in Theorem 3.1.

If $u(x,y)$ is a solution of (4.25)-(4.26), then

$$u(x,y)\leq\left\{P(x,y)e_{\int_{y_0}^{y}Q(\tau,\eta)\ \Delta\eta}(x,x_0)\right\}^{\frac{1}{p}}, \tag{4.28}$$

where $P(x,y)$, $Q(x,y)$ are defined as in (3.6)-(3.7) respectively (by replacing $g(x)$ by $\arctan(x)$ and $g'(x)$ by $\frac{1}{1+x^2}$).

Proof. The solution $u(x,y)$ can be written as

$$u^p(x,y) = \alpha(x) + \beta(y) + \int_{x_0}^{x}\int_{y_0}^{y} F(s,t,u^q(s,t), \int_{s_0}^{s}\int_{t_0}^{t} h(s,t,\tau,\eta,u(\tau,\eta))\Delta_2\eta\Delta_1\tau)\Delta_2 t\Delta_1 s, \tag{4.29}$$

using (4.27) in (4.29), we have

$$|u^p(x,y)| \le c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)(|u^q(s,t)| + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)\arctan|u(\tau,\eta)|^r \Delta_2\eta\Delta_1\tau)\Delta_2 t\Delta_1 s, \tag{4.30}$$

Now, a suitable application of Theorem 3.1 for (4.30), yields the inequality (4.28). □

Remark 4.3. *We can also replace the function* $\arctan(|u(s,t)|^r)$ *by* $\ln(|u(s,t)|^r + 1)$ *in* (4.27) *to obtain another estimate of the solution of* (4.25) $-$ (4.26).

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On generalized α regular-interior and generalized α regular-closure in Topological Spaces

S. Sekar[a,*] and G. Kumar[b]

[a]Department of Mathematics, Government Arts College (Autonomous), Salem – 636 007, Tamil Nadu, India.

[b]Department of Mathematics, A.E.T. College, Attur – 636 108, Tamil Nadu, India.

Abstract

In this paper, the authors introduce a new class of generalized α regular-interior and generalized α regular-closure in topological spaces. Some characterizations and several properties concerning generalized α regular-interior and generalized α regular-closure are obtained.

Keywords: $g\alpha r$-closed sets, $g\alpha r$-closed map, $g\alpha r$-continuous map, contra $g\alpha r$-continuity.

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1 Introduction

Levine introduced generalized closed sets in topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. Many researchers like Arya et al[5], Balachandran et al[6], Bhattarcharya et al[7], Arockiarani et al[4], Gnanambal [8], Nagaveni[14] and Palaniappan et al[15] have worked on generalized closed sets. Andrjivic[3] gave a new type of generalized closed set in topological space called b closed sets. A.A.Omari and M.S.M. Noorani [2] made an analytical study and gave the concepts of generalized b closed sets in topological spaces.

Sekar and Mariappa [18] gave rgb-interior and rgb-closure in topological spaces. In this paper, the notion of $g\alpha r$-interior is defined and some of its basic properties are investigated. Also we introduce the idea of $g\alpha r$-closure in topological spaces using the notions of $g\alpha r$-closed sets and obtain some related results. Through out this paper (X,τ) and (Y,σ) represent the non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $A \subseteq X$, the closure of A and interior of A will be denoted by $cl(A)$ and $int(A)$ respectively, union of all $g\alpha r$-open sets X contained in A is called $g\alpha r$-interior of A and it is denoted by $g\alpha rint(A)$, the intersection of all $g\alpha r$-closed sets of X containing A is called $g\alpha r$-closure of A and it is denoted by $g\alpha rcl(A)$ [17].

2 Preliminaries

Definition 2.1. *Let a subset A of a topological space (X,τ), is called*

1) a α-open set [13] if $A \subseteq int(cl(int(A)))$.

2) a generalised-closed set (briefly g-closed) [10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

3) a weakly-closed set(briefly w-closed) [16] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open.

*Corresponding author.
E-mail address: sekar_nitt@rediffmail.com (S. Sekar), kumargmaths@gmail.com (G. Kumar).

4) *a generalized ∗-closed set (briefly g∗-closed) [20] if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is g-open in* X.

5) *a generalized α-closed set (briefly gα-closed)[12] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is α open in* X.

6) *an α generalized-closed set (briefly αg-closed)[11] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

7) *a generalized b- closed set (briefly gb- closed) [1] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

8) *a semi generalized b-closed set (briefly sgb- closed) [9] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi open in* X.

9) *a generalized αb- closed set (briefly gαb- closed) [19] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is α open in* X.

10) *a regular generalized b- closed set (briefly rgb- closed) [13] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.

11) *a generalized pre regular-closed set (briefly gpr-closed) [8] if* $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.

12) *a generalized α regular-closed set (briefly gαr-closed) [17] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.

3 Generalized α regular - interior in Topological space

Definition 3.2. *Let* A *be a subset of* X. *A point* $x \in A$ *is said to be* $g\alpha r$ *- interior point of* A *is* A *is a* $g\alpha r$ *- neighbourhood of* x. *The set of all* $g\alpha r$ *- interior points of* A *is called the* $g\alpha r$ *- interior of* A *and is denoted by* $g\alpha r - int(A)$.

Theorem 3.1. *If* A *be a subset of* X. *Then* $g\alpha r - int(A) = \{\cup G : G \text{ is a } g\alpha r \text{ - open}, G \subset A\}$.

Proof. Let A be a subset of X.

$$\begin{aligned} x \in g\alpha r - int(A) &\Leftrightarrow x \text{ is a } g\alpha r - \text{ interior point of } A \\ &\Leftrightarrow A \text{ is a } g\alpha r \text{ - nbhd of point } x \\ &\Leftrightarrow \text{ there exists } g\alpha r \text{ - open set } G \text{ such that } x \in G \subset A \\ &\Leftrightarrow x \in \{\cup G : G \text{ is a } g\alpha r\text{-open }, G \subset A\} \\ \text{Hence } g\alpha r - int(A) &= \{\cup G : G \text{ is a } g\alpha r\text{-open }, G \subset A\} \end{aligned}$$

□

Theorem 3.2. *Let* A *and* B *be subsets of* X. *Then*

(i) $g\alpha r - int(X) = X$ *and* $g\alpha r - int(\varphi) = \varphi$.

(ii) $g\alpha r - int(A) \subset A$.

(iii) *If* B *is any* $g\alpha r$ *- open set contained in* A, *then* $B \subset g\alpha r - int(A)$.

(iv) *If* $A \subset B$, *then* $g\alpha r - int(A) \subset g\alpha r - int(B)$.

(v) $g\alpha r - int(g\alpha r - int(A)) = g\alpha r - int(A)$.

Proof. (i) Since X and φ are $g\alpha r$ open sets, by Theorem 3.2

$$\begin{aligned} g\alpha r - int(X) &= \{\cup G : G \text{ is a } g\alpha r \text{ - open}, G \subset X\} \\ &= X \cup \{\text{ all } g\alpha r \text{ open sets }\} \\ &= X \end{aligned}$$

(i.e.,) $g\alpha r\, int(X) = X$. Since φ is the only $g\alpha r$ - open set contained in φ, $g\alpha r - int(\varphi) = \varphi$.

(ii) Let $x \in g\alpha r - int(A)$

$$\begin{aligned} x \in g\alpha r - int(A) &\Rightarrow x \text{ is a interior point of } A. \\ &\Rightarrow A \text{ is a nbhd of } x. \\ &\Rightarrow x \in A \\ \text{Thus, } x \in g\alpha r - int(A) &\Rightarrow x \in A \\ \text{Hence } g\alpha r - int(A) &\subset A. \end{aligned}$$

(iii) Let B be any $g\alpha r$ - open sets such that $B \subset A$. Let $x \in B$. Since B is a $g\alpha r$ - open set contained in A. x is a $g\alpha r$ - interior point of A.
(i.e.,) $x \in g\alpha r - int(A)$. Hence $B \subset g\alpha r - int(A)$.

(iv) Let A and B be subsets of X such that $A \subset B$. Let $x \in g\alpha r - int(A)$. Then x is a $g\alpha r$ - interior point of A and so A is a $g\alpha r$ - nbhd of x. Since $B \supset A$, B is also $g\alpha r$ - nbhd of $x \Rightarrow x \in g\alpha r - int(B)$. Thus we have shown that $x \in g\alpha r - int(A) \Rightarrow x \in g\alpha r - int(B)$.

(v) Proof is obvious.

□

Theorem 3.3. *If a subset A of space X is $g\alpha r$ - open, then $g\alpha r - int(A) = A$.*

Proof. Let A be $g\alpha r$ - open subset of X. We know that $g\alpha r - int(A) \subset A$. Also, A is $g\alpha r$ - open set contained in A. From Theorem 3.3 (iii) $A \subset g\alpha r - int(A)$. Hence $g\alpha r - int(A) = A$. □

The converse of the above theorem need not be true, as seen from the following example.

Example 3.1. *Let $X = \{a,b,c\}$ with $\tau = \{X, \varphi, \{b\}, \{c\}, \{b,c\}\}$. Then $g\alpha r - O(X) = \{X, \varphi, \{a\}, \{b\}, \{c\}, \{b,c\}\}$. $g\alpha r - int(\{a,c\} = \{a\} \cup \{c\} \cup \{\varphi\} = \{a,c\}$. But $\{a,c\}$ is not $g\alpha r$ - open set in X.*

Theorem 3.4. *If A and B are subsets of X, then*
$g\alpha r - int(A) \cup g\alpha r - int(B) \subset g\alpha r - int(A \cup B)$.

Proof. We know that $A \subset A \cup B$ and $B \subset A \cup B$. We have Theorem 3.3 (iv) $g\alpha r - int(A) \subset g\alpha r - int(A \cup B)$, $g\alpha r - int(B) \subset g\alpha r - int(A \cup B)$. This implies that $g\alpha r - int(A) \cup g\alpha r - int(B) \subset g\alpha r - int(A \cup B)$. □

Theorem 3.5. *If A and B are subsets of X, then*
$g\alpha r - int(A \cap B) = g\alpha r - int(A) \cap g\alpha r - int(B)$.

Proof. We know that $A \cap B \subset A$ and $A \cap B \subset B$. We have $g\alpha r - int(A \cap B) \subset g\alpha r - int(A)$ and $g\alpha r - int(A \cap B) \subset g\alpha r - int(B)$.
This implies that

$$g\alpha r - int(A \cap B) \subset g\alpha r - int(A) \cap g\alpha r - int(B). \tag{3.1}$$

Again let $x \in g\alpha r - int(A) \cap g\alpha r - int(B)$. Then $x \in g\alpha r - int(A)$ and $x \in g\alpha r - int(B)$. Hence x is a $g\alpha r$ - int point of each of sets A and B. It follows that A and B is $g\alpha r$ - nbhds of x, so that their intersection $A \cap B$ is also a $g\alpha r$ - nbhds of x. Hence $x \in g\alpha r - int(A \cap B)$. Thus $x \in g\alpha r - int(A) \cap g\alpha r - int(A)$ implies that $x \in g\alpha r - int(A \cap B)$. Therefore

$$g\alpha r - int(A) \cap g\alpha r - int(B) \subset g\alpha r - int(A \cap B) \tag{3.2}$$

From (3.1) and (3.2),
We get $g\alpha r - int(A \cap B) = g\alpha r - int(A) \cap g\alpha r - int(B)$. □

Theorem 3.6. *If A is a subset of X, then $int(A) \subset g\alpha r - int(A)$.*

Proof. Let A be a subset of X.

$$\begin{aligned}
\text{Let } x \in int(A) &\Rightarrow x \in \{\cup G : G \text{ is open, } G \subset A\} \\
&\Rightarrow \text{there exists an open set } G \\
&\quad\ \text{such that } x \in G \subset A \\
&\Rightarrow \text{there exist a } g\alpha r \text{ - open set } G \\
&\quad\ \text{such that } x \in G \subset A, \text{ as every open set is} \\
&\quad\ \text{a } g\alpha r \text{ - open set in } X \\
&\Rightarrow x \in \{\cup G : G \text{ is } g\alpha r \text{ - open, } G \subset A\} \\
&\Rightarrow x \in g\alpha r - int(A) \\
\text{Thus } x \in int(A) &\Rightarrow x \in g\alpha r - int(A) \\
\text{Hence } int(A) &\subset g\alpha r - int(A).
\end{aligned}$$

This completes the proof. □

Remark 3.1. *Containment relation in the above theorem may be proper as seen from the following example.*

Example 3.2. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}\}$. *Then* $g\alpha r - O(X) = \{X, \varphi, \{a\}, \{b\}, \{c\}, \{b, c\}\}$. *Let* $A = \{b, c\}$. *Now* $g\alpha r - int(A) = \{b, c\}$ *and* $int(A) = \{b\}$. *It follows that* $int(A) \subset g\alpha r - int(A)$ *and* $int(A) \neq g\alpha r - int(A)$.

Theorem 3.7. *If* A *is a subset of* X, *then* $g - int(A) \subset g\alpha r - int(A)$, *where* $g - int(A)$ *is given by* $g - int(A) = \cup\{G : G \text{ is } g \text{ - open, } G \subset A\}$.

Proof. Let A be a subset of X.

$$\begin{aligned}
\text{Let } x \in int(A) &\Rightarrow x \in \{\cup G : G \text{ is } g \text{ - open, } G \subset A\} \\
&\Rightarrow \text{there exists an } g \text{ - open set } G \\
&\quad\ \text{such that } x \in G \subset A \\
&\Rightarrow \text{there exist a } g\alpha r \text{ - open set } G \\
&\quad\ \text{such that } x \in G \subset A, \text{ as every } g \text{ open set} \\
&\quad\ \text{is a } g\alpha r \text{ - open set in } X \\
&\Rightarrow x \in \{\cup G : G \text{ is } g\alpha r \text{ - open, } G \subset A\} \\
&\Rightarrow x \in g\alpha r - int(A) \\
\text{Hence } g - int(A) &\subset g\alpha r - int(A).
\end{aligned}$$

This completes the proof. □

Remark 3.2. *Containment relation in the above theorem may be proper as seen from the following example.*

Example 3.3. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$. *Then* $g\alpha r - O(X) = \{X, \varphi, \{a\}, \{b, c\}\}$. *and g - open* $(X) = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$. *Let* $A = \{b, c\}$, $g\alpha r - int(A) = \{b, c\}$ *and* $g - int(A) = \{b\}$. *It follows that* $g - int(A) \subset g\alpha r - int(A)$ *and* $g - int(A) \neq g\alpha r - int(A)$.

4 Generalized α regular - closure in Topological space

Definition 4.3. *Let* A *be a subset of a space* X. *We define the* $g\alpha r$ *- closure of* A *to be the intersection of all* $g\alpha r$ *- closed sets containing* A.
In symbols, $g\alpha r - cl(A) = \{\cap F : A \subset F \in g\alpha rc(X)\}$.

Theorem 4.8. *If* A *and* B *are subsets of a space* X. *Then*

(i) $g\alpha r - cl(X) = X$ *and* $g\alpha r - cl(\varphi) = \varphi$

(ii) $A \subset g\alpha r - cl(A)$

(iii) If B is any $g\alpha r$ - closed set containing A, then $g\alpha r - cl(A) \subset B$

(iv) If $A \subset B$ then $g\alpha r - cl(A) \subset g\alpha r - cl(B)$

Proof. (i) By the definition of $g\alpha r$ - closure, X is the only $g\alpha r$ - closed set containing X. Therefore $g\alpha r - cl(X) =$ Intersection of all the $g\alpha r$ - closed sets containing $X = \cap\{X\} = X$. That is $g\alpha r - cl(X) = X$. By the definition of $g\alpha r$ - closure, $g\alpha r - cl(\varphi) =$ Intersection of all the $g\alpha r$ - closed sets containing $\varphi = \{\varphi\} = \varphi$. That is $g\alpha r - cl(\varphi) = \varphi$.

(ii) By the definition of $g\alpha r$ - closure of A, it is obvious that $A \subset g\alpha r - cl(A)$.

(iii) Let B be any $g\alpha r$ - closed set containing A. Since $g\alpha r - cl(A)$ is the intersection of all $g\alpha r$ - closed sets containing A, $g\alpha r - cl(A)$ is contained in every $g\alpha r$ - closed set containing A. Hence in particular $g\alpha r - cl(A) \subset B$.

(iv) Let A and B be subsets of X such that $A \subset B$. By the definition
$g\alpha r - cl(B) = \{\cap F : B \subset F \in g\alpha r - c(X)\}$. If $B \subset F \in g\alpha r - c(X)$, then $g\alpha r - cl(B) \subset F$. Since $A \subset B, A \subset B \subset F \in g\alpha r - c(X)$,
we have $g\alpha r - cl(A) \subset F$. Therefore $g\alpha r - cl(A) \subset \{\cap F : B \subset F g\alpha r - c(X)\} = g\alpha r - cl(B)$. (i.e.,) $g\alpha r - cl(A) \subset g\alpha r - cl(B)$.

□

Theorem 4.9. *If $A \subset X$ is $g\alpha r$ - closed, then $g\alpha r - cl(A) = A$.*

Proof. Let A be $g\alpha r$ - closed subset of X. We know that $A \subset g\alpha r - cl(A)$. Also $A \subset A$ and A is $g\alpha r$ - closed. By Theorem 4.2 (iii) $g\alpha r - cl(A) \subset A$. Hence $g\alpha r - cl(A) = A$. □

Remark 4.3. *The converse of the above theorem need not be true as seen from the following example.*

Example 4.4. *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \varphi, \{b\}, \{b, c\}\}$. Then $g\alpha r - C(X) = \{X, \varphi, \{a\}, \{b, c\}\}$. $g\alpha r - cl(\{c\}) = \{b, c\}$. But $\{c\}$ is not $g\alpha r$ - closed set in X.*

Theorem 4.10. *If A and B are subsets of a space X, then
$g\alpha r - cl(A \cap B) \subset g\alpha r - cl(A) \cap g\alpha r - cl(B)$.*

Proof. Let A and B be subsets of X. Clearly $A \cap B \subset A$ and $A \cap B \subset B$. By Theorem $g\alpha r - cl(A \cap B) \subset g\alpha r - cl(A)$ and $g\alpha r - cl(A \cap B) \subset g\alpha r - cl(B)$. Hence $g\alpha r - cl(A \cap B) \subset g\alpha r - cl(A) \cap g\alpha r - cl(B)$. □

Theorem 4.11. *If A and B are subsets of a space X then
$g\alpha r - cl(A \cup B) = g\alpha r - cl(A) \cup g\alpha r - cl(B)$.*

Proof. Let A and B be subsets of X. Clearly $A \subset A \cup B$ and $B \subset A \cup B$. We have

$$g\alpha r - cl(A) \cup g\alpha r - cl(B) \subset g\alpha r - cl(A \cup B) \tag{4.3}$$

Now to prove $g\alpha r - cl(A \cup B) \subset g\alpha r - cl(A) \cup g\alpha r - cl(B)$.
Let $x \in g\alpha r - cl(A \cup B)$ and suppose $x \notin g\alpha r - cl(A) \cup g\alpha r - cl(B)$. Then there exists $g\alpha r$ - closed sets A_1 and B_1 with $A \subset A_1, B \subset B_1$ and $x \notin A_1 \cup B_1$. We have $A \cup B \subset A_1 \cup B_1$ and $A_1 \cup B_1$ is $g\alpha r$ - closed set by Theorem such that $x \notin A_1 \cup B_1$. Thus $x \notin g\alpha r - cl(A \cup B)$ which is a contradiction to $x \in g\alpha r - cl(A \cup B)$. Hence

$$g\alpha r - cl(A \cup B) \subset g\alpha r - cl(A) \cup g\alpha r - cl(B) \tag{4.4}$$

From (4.3) and (4.4), we have $g\alpha r - cl(A \cup B) = g\alpha r - cl(A) \cup g\alpha r - cl(B)$. □

Theorem 4.12. *For an $x \in X$, $x \in g\alpha r - cl(A)$ if and only if $V \cap A \neq \varphi$ for every $g\alpha r$ - open sets V containing x.*

Proof. Let $x \in X$ and $x \in g\alpha r - cl(A)$. To prove $V \cap A \neq \varphi$ for every $g\alpha r$ - open set V containing x.
Prove the result by contradiction. Suppose there exists a $g\alpha r$ - open set V containing x such that $V \cap A = \varphi$. Then $A \subset X - V$ and $X - V$ is $g\alpha r$-closed. We have $g\alpha r - cl(A) \subset X - V$. This shows that $x \notin g\alpha r - cl(A)$, which is a contradiction. Hence $V \cap A \neq \varphi$ for every $g\alpha r$ - open set V containing x.
Conversely, let $V \cap A = \varphi$ for every $g\alpha r$ - open set V containing x. To prove $x \in g\alpha r - cl(A)$. We prove the result by contradiction. Suppose $x \notin g\alpha r - cl(A)$. Then $x \in X - F$ and $S - F$ is $g\alpha r$ - open. Also $(X - F) \cap A = \varphi$, which is a contradiction. Hence $x \in g\alpha r - cl(A)$. □

Theorem 4.13. *If A is a subset of a space X, then $g\alpha r - cl(A) \subset cl(A)$.*

Proof. Let A be a subset of a space S. By the definition of closure,
$cl(A) = \{\cap F : A \subset F \in C(X)\}$. If $A \subset F \in C(X)$, Then $A \subset F \in g\alpha r - C(X)$, because every closed set is $g\alpha r$ - closed. That is $g\alpha r - cl(A) \subset F$. Therefore $g\alpha r - cl(A) \subset \{\cap F \subset X : F \in C(X)\} = cl(A)$. Hence $g\alpha r - cl(A) \subset cl(A)$. □

Theorem 4.14. *If A is a subset of X, then $g\alpha r - cl(A) \subset g - cl(A)$, where $g - cl(A)$ is given by $g - cl(A) = \{\cap F \subset X : A \subset F$ and f is a g - closed set in $X\}$.*

Proof. Let A be a subset of X. By definition of $g - cl(A) = \{\cap F \subset X : A \subset F$ and f is a g - closed set in $X\}$. If $A \subset F$ and F is g - closed subset of x, then $A \subset F \in g\alpha r - cl(X)$, because every g closed is $g\alpha r$ - closed subset in X. That is $g\alpha r - cl(A) \subset F$. Therefore $g\alpha r - cl(A) \subset \{\cap F \subset X : A \subset F$ and f is a g - closed set in $X\} = g - cl(A)$. Hence $g\alpha r - cl(A) \subset g - cl(A)$. □

Corollary 4.1. *Let A be any subset of X. Then*

(i) $\left(g\alpha r - int(A)\right)^c = g\alpha r - cl(A^c)$

(ii) $g\alpha r - int(A) = \left(g\alpha r - cl(A^c)\right)^c$

(iii) $g\alpha r - cl(A) = \left(g\alpha r - int(A^c)\right)^c$

Proof. (i) Let $x \in \left(g\alpha r - int(A)\right)^c$. Then $x \notin g\alpha r - int(A)$. That is every $g\alpha r$ - open set U containing x is such that U not subset of A. That is every $g\alpha r$ - open set U containing x is such that $U \cap A^c \neq \varphi$. By Theorem $x \in \left(g\alpha r - cl(A^c)\right)$ and therefore $\left(g\alpha r - int(A)\right)^c \subset g\alpha r - cl(A^c)$.
Conversely, let $x \in g\alpha r - cl(A^c)$. Then by theorem, every $g\alpha r$ - open set U containing x is such that $U \cap A^c \neq \varphi$. That is every $g\alpha r$ - open set U containing x is such that U not subset of A. This implies by definition of $g\alpha r$ - interior of A, $x \notin g\alpha r - int(A)$. That is $x \in \left(g\alpha r - int(A)\right)^c$ and $g\alpha r - cl(A^c) \subset \left(g\alpha r - int(A)\right)^c$. Thus $\left(g\alpha r - int(A)\right)^c = g\alpha r - cl(A^c)$.

(ii) Follows by taking complements in (i).

(iii) Follows by replacing A by A^c in (i).

□

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Generalized Ulam - Hyers Stability of on (AQQ): Additive - Quadratic - Quartic Functional Equation

John M. Rassias[a], M. Arunkumar[b], E.Sathya[c], N. Mahesh Kumar[d] *

[a]Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, Athens 15342, Greece.

[b,c]Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.

[d]Department of Mathematics, Arunai Engineering College, Tiruvannamalai - 606 603, TamilNadu, India.

Abstract

In this paper, the authors obtain the general solution and generalized Ulam - Hyers stability of an (AQQ): additive - quadratic - quartic functional equation of the form

$$\begin{aligned} &f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z)\\ &\quad=2\left[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)\right]\\ &\qquad-4f(x)-4f(y)-2\left[f(z)+f(-z)\right]\end{aligned}$$

by using the classical Hyers' direct method. Counter examples for non stability are discussed also.

Keywords: Additive functional equations, Quadratic functional equations, Quartic functional equations, Mixed type functional equations, Ulam - Hyers stability, Ulam - Hyers - Rassias stability, Ulam - Gavruta - Rassias stability, Ulam - JMRassias stability.

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1 Introduction

One of the most interesting questions in the theory of functional equations concerning the famous Ulam stability problem is, as follows: when is it true that a mapping satisfying a functional equation approximately, must be close to an exact solution of the given functional equation?

The first stability problem was raised by S.M. Ulam [35] during his talk at the University of Wisconsin in 1940. In fact we are given a group $(G_1,\cdot)$ and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x\cdot y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$?

D.H. Hyers [16] gave the first affirmative partial answer to the question of Ulam for Banach spaces. It was further generalized via excellent results obtained by a number of authors [3, 12, 26, 30, 33].

The solution and stabilities of the following functional equations

1. Additive Functional Equation

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

*Corresponding author.

E-mail addresses: jrassias@primedu.uoa.gr (John M. Rassias), annarun2002@yahoo.co.in (M. Arunkumar), sathya24mathematics@gmail.com (E.Sathya), mrnmahesh@yahoo.com (N. Mahesh Kumar).

2. Quadratic Functional Equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.2}$$

3. Cubic Functional Equation

$$g(x+2y)+3g(x)=3g(x+y)+g(x-y)+6g(y) \tag{1.3}$$

4. Quartic Functional Equation

$$F(x+2y)+F(x-2y)+6F(x)=4[F(x+y)+F(x-y)+6F(y)] \tag{1.4}$$

5. Additive - Quadratic Functional Equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x). \tag{1.5}$$

6. Additive - Cubic Functional Equation

$$\begin{aligned}3f(x+y+z)&+f(-x+y+z)+f(x-y+z)+f(x+y-z)\\ &+4[f(x)+f(y)+f(z)]=4[f(x+y)+f(x+z)+f(y+z)]\end{aligned} \tag{1.6}$$

7. Additive - Quartic Functional Equation

$$\begin{aligned}f(2x+y)+f(2x-y)&=4[f(x+y)+f(x-y)]+12[f(x)+f(-x)]\\ &\quad-3[f(y)+f(-y)]-2[f(x)-f(-x)]\end{aligned} \tag{1.7}$$

8. Additive - Quadratic - Cubic Functional Equation

$$f(x+ky)+f(x-ky)=k^2[f(x+y)+f(x-y)]+2\left(1-k^2\right)f(x) \tag{1.8}$$

were investigated by [1], [21], [28], [27], [22], [29], [8], [13] and references cited there in.

Motivated by the above results, in this paper, the authors obtain the general solution and generalized Ulam - Hyers stability of additive - quadratic - quartic functional equations

$$\begin{aligned}f(x+y+z)&+f(x+y-z)+f(x-y+z)+f(x-y-z)\\ &=2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)]\\ &\quad-4f(x)-4f(y)-2[f(z)+f(-z)]\end{aligned} \tag{1.9}$$

having solution

$$f(x)=ax+bx^2+cx^4 \tag{1.10}$$

using Hyers direct method.

2 General Solution for the Functional Equation (1.9)

In this section, we present the solution of the functional equation (1.9). Throughout this section let $\mathcal{G}$ and $\mathcal{H}$ be real vector spaces.

Theorem 2.1. *Let $f:\mathcal{G}\to\mathcal{H}$ be an odd mapping. Then $f:\mathcal{G}\to\mathcal{H}$ satisfies the functional equation (1.9) for all $x,y,z\in\mathcal{G}$, if and only if $f:\mathcal{G}\to\mathcal{H}$ satisfies the functional equation (1.1) for all $x,y\in\mathcal{G}$.*

Proof. Assume $f : \mathcal{G} \to \mathcal{H}$ be an odd mapping satisfying (1.9). Replacing (x, y, z) by $(0, 0, 0)$, we get $f(0) = 0$. Again replacing (x, y, z) by $(0, x, x)$ and (x, y, z) by (x, x, x) in (1.9), we obtain

$$f(2x) = 2f(x) \qquad f(3x) = 3f(x) \tag{2.1}$$

for all $x \in \mathcal{G}$. In general for any positive integer m, we have

$$f(mx) = mf(x)$$

for all $x \in \mathcal{G}$. Putting z by x in (1.9), using oddness of f and (2.1), we get

$$f(2x + y) + f(2x - y) = 4f(x + y) - 4f(y) \tag{2.2}$$

for all $x, y \in \mathcal{G}$. Setting x by $\frac{x}{2}$ in (2.2) and using (2.1), we have

$$f(x + y) + f(x - y) = 2f(x + 2y) - 4f(y) \tag{2.3}$$

for all $x, y \in \mathcal{G}$. Interchanging x and y in (2.3) and using oddness of f, we get

$$f(x + y) - f(x - y) = 2f(2x + y) - 4f(x) \tag{2.4}$$

for all $x, y \in \mathcal{G}$. Replacing y by $-y$ in (2.4), we obtain

$$f(x - y) - f(x + y) = 2f(2x - y) - 4f(x) \tag{2.5}$$

for all $x, y \in \mathcal{G}$. Adding (2.4) and (2.5), we arrive

$$f(2x + y) + f(2x - y) = 4f(x) \tag{2.6}$$

for all $x, y \in \mathcal{G}$. Using (2.6) in (2.2), we derive our desired result.

Conversely, assume $f : \mathcal{G} \to \mathcal{H}$ be an odd mapping satisfying (1.1). Letting y by $y + z$ in (1.1) and using (1.1), we have

$$f(x + y + z) = f(x) + f(y) + f(z) \tag{2.7}$$

for all $x, y, z \in \mathcal{G}$. Setting z by $-z$ in (2.7), we arrive

$$f(x + y - z) = f(x) + f(y) + f(-z) \tag{2.8}$$

for all $x, y, z \in \mathcal{G}$. Replacing (x, y) by $(x + y, z)$ in (1.1), we get

$$f(x + y + z) = f(x + y) + f(z) \tag{2.9}$$

for all $x, y, z \in \mathcal{G}$. Again replacing z by $-z$ in (2.9), we obtain

$$f(x + y - z) = f(x + y) + f(-z) \tag{2.10}$$

for all $x, y, z \in \mathcal{G}$. Setting y by $-y$ in (2.9), we get

$$f(x - y + z) = f(x - y) + f(z) \tag{2.11}$$

for all $x, y, z \in \mathcal{G}$. Again setting z by $-z$ in (2.11), we obtain

$$f(x - y - z) = f(x - y) + f(-z) \tag{2.12}$$

for all $x, y, z \in \mathcal{G}$. Adding (2.9), (2.10), (2.11), (2.12) and using oddness of f, we arrive

$$f(x + y + z) + f(x + y - z) + f(x - y + z) + f(x - y - z) = 2f(x + y) + 2f(x - y) \tag{2.13}$$

for all $x, y, z \in \mathcal{G}$. Replacing (x, y, z) by (y, z, x) in (2.13) and using oddness of f, we get

$$f(x + y + z) - f(x - y - z) + f(x + y - z) - f(x - y + z) = 2f(y + z) + 2f(y - z) \tag{2.14}$$

for all $x, y, z \in \mathcal{G}$. Replacing (x, y, z) by (x, z, y) in (2.13) and using oddness of f, we obtain

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(x-y-z)=2f(x+z)+2f(x-z) \tag{2.15}$$

for all $x, y, z \in \mathcal{G}$. Adding (2.13), (2.14) and (2.15), we arrive

$$\begin{aligned} &3f(x+y+z)+3f(x+y-z)+f(x-y+z)+f(x-y-z) \\ &= 2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)] \end{aligned} \tag{2.16}$$

for all $x, y, z \in \mathcal{G}$. It follows from (2.16) that

$$\begin{aligned} &f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z) \\ &= 2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)] \\ &\quad -2[f(x+y+z)+f(x+y-z)] \end{aligned} \tag{2.17}$$

for all $x, y, z \in \mathcal{G}$. Using (2.7), (2.8) in (2.17), we arrive

$$\begin{aligned} &f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z) \\ &= 2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)] \\ &\quad -2[f(x)+f(y)+f(z)+f(x)+f(y)+f(-z)] \\ &= 2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)] \\ &\quad -4f(x)-4f(y)-2[f(z)+f(-z)] \end{aligned} \tag{2.18}$$

for all $x, y, z \in \mathcal{G}$. Hence the proof is complete. □

Lemma 2.1. *Let $f : \mathcal{G} \to \mathcal{H}$ be an odd mapping. Then $f : \mathcal{G} \to \mathcal{H}$ satisfies the functional equation (1.9) for all $x, y, z \in \mathcal{G}$, if and only if $f : \mathcal{G} \to \mathcal{H}$ satisfies the functional equation (1.1)*

Proof. Assume $f : \mathcal{G} \to \mathcal{H}$ be an odd mapping satisfying (1.9). Replacing (x, y) by $(0, 0)$, we get $f(0) = 0$. Again replacing x by 0 in (1.9) and using oddness of f, we obtain

$$f(y+z)+f(y-z)=2f(y) \tag{2.19}$$

for all $y, z \in \mathcal{G}$. By Theorem 2.1 of [4], we derive our desired result. □

Theorem 2.2. *If $f : \mathcal{G} \to \mathcal{H}$ is an even mapping satisfying the functional equation (1.9) for all $x, y, z \in \mathcal{G}$, then f is quadratic-quartic for all $x, y \in \mathcal{G}$.*

Proof. Replacing z by x in (1.9), we arrive

$$f(2x+y)+f(2x-y)=4\left[f(x+y)+f(x-y)\right]+2[f(2x)-4f(x)]-6f(y) \tag{2.20}$$

By Lemma 2.1 of [14], we see that f is quadratic-quartic. □

Theorem 2.3. *Let $f : \mathcal{G} \to \mathcal{H}$ be an even mapping. Then $f : \mathcal{G} \to \mathcal{H}$ satisfies (1.9) for all $x, y, z \in \mathcal{G}$ if and only if there exist a unique symmetric multiadditive mapping $M : \mathcal{G}^4 \to \mathcal{H}$ and a unique symmetric bi-additive mapping $B : \mathcal{G}^2 \to \mathcal{H}$ such that*

$$f(x)=M(x,x,x,x)+B(x,x) \tag{2.21}$$

for all $x \in \mathcal{G}$.

Proof. The proof follows from Theorem 2.2 and Theorem 2.2 of [14], we derive our desired result. □

The following Lemmas are important to prove our stability results.

Lemma 2.2. *If $f : \mathcal{G} \to \mathcal{H}$ is an odd mapping satisfying (1.9), then*

$$f(2x)=2f(x) \tag{2.22}$$

for all $x \in \mathcal{G}$, such that f is additive.

Proof. Letting (x,y,z) by (0,0,0) in (1.9), we get $f(0) = 0$. Replacing (x,y,z) by (x,x,x) in (1.9) and using oddness of f, we obtain

$$f(3x) = 6f(2x) - 9f(x) \tag{2.23}$$

for all $x \in \mathcal{G}$. Again replacing (x,y,z) by $(-x,x,x)$ in (1.9) and using oddness of f, we get

$$f(3x) = 2f(2x) - f(x) \tag{2.24}$$

for all $x \in \mathcal{G}$. It follows from (2.23) and (2.24), we derive our desired result. □

Lemma 2.3. *If $f : \mathcal{G} \to \mathcal{H}$ is an even mapping satisfying (1.9) and if $q_2 : \mathcal{G} \to \mathcal{H}$ is a mapping given by*

$$q_2(x) = f(2x) - 16f(x) \tag{2.25}$$

for all $x \in \mathcal{G}$, then

$$q_2(2x) = 4q_2(x) \tag{2.26}$$

for all $x \in \mathcal{G}$, such that q_2 is quadratic.

Proof. Letting (x,y,z) by (x,x,x) in (1.9), we get

$$f(3x) = 6f(2x) - 15f(x) \tag{2.27}$$

for all $x \in \mathcal{G}$. Again replacing (x,y,z) by $(x,x,2x)$ in (1.9) and using evenness of f, we have

$$f(4x) = 4f(3x) - 4f(2x) - 4f(x) \tag{2.28}$$

for all $x \in \mathcal{G}$. Using (2.27) in (2.28), we get

$$f(4x) = 20f(2x) - 64f(x) \tag{2.29}$$

for all $x \in \mathcal{G}$. From (2.25), we establish

$$q_2(2x) - 4q_2(x) = f(4x) - 20f(2x) + 64f(x) \tag{2.30}$$

for all $x \in \mathcal{G}$. Using (2.29) in (2.30), we derive our desired result. □

Lemma 2.4. *If $f : \mathcal{G} \to \mathcal{H}$ is an even mapping satisfying (1.9) and if $q_4 : \mathcal{G} \to \mathcal{H}$ is a mapping given by*

$$q_4(x) = f(2x) - 4f(x) \tag{2.31}$$

for all $x \in \mathcal{G}$, then

$$q_4(2x) = 16q_4(x) \tag{2.32}$$

for all $x \in \mathcal{G}$, such that q_4 is quartic.

Proof. It follows from (2.31) that

$$q_4(2x) - 16q_4(x) = f(4x) - 20f(2x) + 64f(x) \tag{2.33}$$

for all $x \in \mathcal{G}$. Using (2.29) in (2.33), we derive our desired result. □

Remark 2.1. *Let $f : \mathcal{G} \to \mathcal{H}$ be a mapping satisfying (1.9) and let $q_2, q_4 : \mathcal{G} \to \mathcal{H}$ be a mapping defined in (2.25) and (2.31) then*

$$f(x) = \frac{1}{12}(q_4(x) - q_2(x)) \tag{2.34}$$

for all $x \in \mathcal{G}$.

Hereafter, through out this paper, let we consider $\mathcal{G}$ be a normed space and $\mathcal{H}$ be a Banach space. Define a mapping $Df : \mathcal{G} \to \mathcal{H}$ by

$$\begin{aligned} Df(x,y,z) = {} & f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) \\ & - 2\left[f(x+y) + f(x-y) + f(y+z) + f(y-z) + f(x+z) + f(x-z)\right] \\ & - 4f(x) - 4f(y) - 2\left[f(z) + f(-z)\right] \end{aligned}$$

for all $x, y, z \in \mathcal{G}$.

3 Stability Results: Odd Case

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.9) for odd case.

Theorem 3.1. *Let $j = \pm 1$ and $\psi, \xi : \mathcal{G}^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{nj}} = 0 \tag{3.1}$$

for all $x, y, z \in \mathcal{G}$. Let $f_a : \mathcal{G} \to \mathcal{H}$ be an odd function satisfying the inequality

$$\|Df_a(x, y, z)\| \le \psi(x, y, z) \tag{3.2}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique additive mapping $A : \mathcal{G} \to \mathcal{H}$ which satisfies (1.9) and

$$\|f_a(x) - A(x)\| \le \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi\left(2^{kj}x\right)}{2^{kj}} \tag{3.3}$$

where $\xi\left(2^{kj}x\right)$ and $A(x)$ are defined by

$$\xi\left(2^{kj}x\right) = \frac{1}{4}\left[\psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) + \psi\left(-2^{kj}x, 2^{kj}x, 2^{kj}x\right)\right] \tag{3.4}$$

and

$$A(x) = \lim_{n\to\infty} \frac{f_a(2^{nj}x)}{2^{nj}} \tag{3.5}$$

for all $x \in \mathcal{G}$, respectively.

Proof. Replacing (x, y, z) by (x, x, x) in (3.2) and using oddness of f_a, we get

$$\|f_a(3x) - 6f_a(2x) + 9f_a(x)\| \le \psi(x, x, x) \tag{3.6}$$

for all $x \in \mathcal{G}$. Again replacing (x, y, z) by $(-x, x, x)$ in (3.2) and using oddness of f_a, we obtain

$$\|-f_a(3x) + 2f_a(2x) - f_a(x)\| \le \psi(-x, x, x) \tag{3.7}$$

for all $x \in \mathcal{G}$. It follows form (3.6) and (3.7) that

$$\begin{aligned}\|8f_a(x) - 4f_a(2x)\| &\le \|f_a(3x) - 6f_a(2x) + 9f_a(x)\| + \|-f_a(3x) + 2f_a(2x) - f_a(x)\| \\ &\le \psi(x, x, x) + \psi(-x, x, x)\end{aligned} \tag{3.8}$$

for all $x \in \mathcal{G}$. Dividing the above inequality by 8, we obtain

$$\left\|\frac{f_a(2x)}{2} - f_a(x)\right\| \le \frac{\xi(x)}{2} \tag{3.9}$$

where

$$\xi(x) = \frac{1}{4}\left[\psi(x, x, x) + \psi(-x, x, x)\right]$$

for all $x \in \mathcal{G}$. Now replacing x by $2x$ and dividing by 2 in (3.9), we get

$$\left\|\frac{f_a(2^2x)}{2^2} - \frac{f_a(2x)}{2}\right\| \le \frac{\xi(2x)}{2 \cdot 2} \tag{3.10}$$

for all $x \in \mathcal{G}$. From (3.9) and (3.10), we obtain

$$\begin{aligned}\left\|\frac{f_a(2^2x)}{2^2} - f_a(x)\right\| &\le \left\|\frac{f_a(2x)}{2} - f_a(x)\right\| + \left\|\frac{f_a(2^2x)}{2^2} - \frac{f_a(2x)}{2}\right\| \\ &\le \frac{1}{2}\left[\xi(x) + \frac{\xi(2x)}{2}\right]\end{aligned} \tag{3.11}$$

for all $x \in \mathcal{G}$. Proceeding further and using induction on a positive integer n, we get

$$\left\| \frac{f_a(2^n x)}{2^n} - f_a(x) \right\| \leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\xi(2^k x)}{2^k} \leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\xi(2^k x)}{2^k} \tag{3.12}$$

for all $x \in \mathcal{G}$. In order to prove the convergence of the sequence $\left\{ \frac{f_a(2^n x)}{2^n} \right\}$, replace x by $2^m x$ and dividing by 2^m in (3.12), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| \frac{f_a(2^{n+m} x)}{2^{(n+m)}} - \frac{f_a(2^m x)}{2^m} \right\| &= \frac{1}{2^m} \left\| \frac{f_a(2^n \cdot 2^m x)}{2^n} - f_a(2^m x) \right\| \\ &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\xi(2^{k+m} x)}{2^{k+m}} \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\xi(2^{k+m} x)}{2^{k+m}} \\ &\to 0 \quad as \quad m \to \infty \end{aligned}$$

for all $x \in \mathcal{G}$. Hence the sequence $\left\{ \frac{f_a(2^n x)}{2^n} \right\}$ is a Cauchy sequence. Since $\mathcal{H}$ is complete, there exists a mapping $A : \mathcal{G} \to \mathcal{H}$ such that

$$A(x) = \lim_{n \to \infty} \frac{f_a(2^n x)}{2^n}, \quad \forall \ x \in \mathcal{G}.$$

Letting $n \to \infty$ in (3.12), we see that (3.3) holds for all $x \in \mathcal{G}$. To prove that A satisfies (1.9), replacing (x, y, z) by $(2^n x, 2^n y, 2^n z)$ and dividing by 2^n in (3.2), we obtain

$$\frac{1}{2^n} \left\| D f_a(2^n x, 2^n y, 2^n z) \right\| \leq \frac{1}{2^n} \psi(2^n x, 2^n y, 2^n z)$$

for all $x, y, z \in \mathcal{G}$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$DA(x, y, z) = 0.$$

Hence A satisfies (1.9) for all $x, y, z \in \mathcal{G}$. To prove that A is unique, let $B(x)$ be another additive mapping satisfying (1.9) and (3.3), then

$$\begin{aligned} \|A(x) - B(x)\| &= \frac{1}{2^n} \|A(2^n x) - B(2^n x)\| \\ &\leq \frac{1}{2^n} \{\|A(2^n x) - f_a(2^n x)\| + \|f_a(2^n x) - B(2^n x)\|\} \\ &\leq \sum_{k=0}^{\infty} \frac{\xi(2^{k+n} x)}{2^{(k+n)}} \\ &\to 0 \quad as \quad n \to \infty \end{aligned}$$

for all $x \in \mathcal{G}$. Thus A is unique. Hence, for $j = 1$ the theorem holds.

Now, replacing x by $\frac{x}{2}$ in (3.8), we reach

$$\left\| 8 f_a\left(\frac{x}{2}\right) - 4 f_a(x) \right\| \leq \psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + \psi\left(-\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \tag{3.13}$$

for all $x \in \mathcal{G}$. Dividing the above inequality by 4, we obtain

$$\left\| 2 f_a\left(\frac{x}{2}\right) - f_a(x) \right\| \leq \xi\left(\frac{x}{2}\right) \tag{3.14}$$

where

$$\xi\left(\frac{x}{2}\right) = \frac{1}{4} \left[\psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + \psi\left(-\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \right]$$

for all $x \in \mathcal{G}$. The rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem holds. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 3.1. *Let ρ and s be nonnegative real numbers. Let an odd function $f_a : \mathcal{G} \to \mathcal{H}$ satisfy the inequality*

$$\|Df_a(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 1; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1; \end{cases} \tag{3.15}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique additive function $A : \mathcal{G} \to \mathcal{H}$ such that

$$\|f_a(x) - A(x)\| \leq \begin{cases} \frac{\rho}{2}, \\ \frac{3\rho||x||^s}{2|2-2^s|}, \\ \frac{\rho||x||^{3s}}{2|2-2^{3s}|}, \\ \frac{2\rho||x||^{3s}}{|2-2^{3s}|} \end{cases} \tag{3.16}$$

for all $x \in \mathcal{G}$.

Now, we provide an example to illustrate that the functional equation (1.9) is not stable for $s = 1$ in condition (ii) of Corollary 3.1.

Example 3.1. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant and define a function $f_a : \mathbb{R} \to \mathbb{R}$ by

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_a satisfies the functional inequality

$$|Df_a(x,y,z)| \leq 56\, \mu(|x| + |y| + |z|) \tag{3.17}$$

for all $x \in \mathbb{R}$. Then there do not exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_a(x) - A(x)| \leq \kappa|x| \qquad for\ all \quad x \in \mathbb{R}. \tag{3.18}$$

Proof. Now

$$|f_a(x)| \leq \sum_{n=0}^{\infty} \frac{|\psi(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\,\mu.$$

Therefore, we see that f_a is bounded. We are going to prove that f_a satisfies (3.17).

If $x = y = z = 0$ then (3.17) is trivial. If $|x| + |y| + |z| \geq \frac{1}{2}$ then the left hand side of (3.17) is less than 56μ. Now suppose that $0 < |x| + |y| + |z| < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^{k-1}} \leq |x| + |y| + |z| < \frac{1}{2^k}, \tag{3.19}$$

so that $2^{k-1}x < \frac{1}{2}, 2^{k-1}y < \frac{1}{2}, 2^{k-1}z < \frac{1}{2}$ and consequently

$$2^{k-1}(x+y+z), 2^{k-1}(x+y-z), 2^{k-1}(x-y+z), 2^{k-1}(x-y-z), 2^{k-1}(x+y), 2^{k-1}(x-y),$$
$$2^{k-1}(y+z), 2^{k-1}(y-z), 2^{k-1}(x+z), 2^{k-1}(x-z), 2^{k-1}(x), 2^{k-1}(y), 2^{k-1}(z), 2^{k-1}(-z) \in (-1,1).$$

Therefore for each $n = 0, 1, \dots, k-1$, we have

$$\begin{aligned}&2^n(x+y+z), 2^n(x+y-z), 2^n(x-y+z), 2^n(x-y-z), 2^n(x+y), 2^n(x-y),\\ &\quad 2^n(y+z), 2^n(y-z), 2^n(x+z), 2^n(x-z), 2^n(x), 2^n(y), 2^n(z), 2^n(-z) \in (-1,1)\end{aligned}$$

and

$$\begin{aligned}&\psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z))\\ &\quad -2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right.\\ &\quad\left. +\psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^nx) + 4\psi(2^ny) + 2\left[\psi(2^nz) + \psi(2^n-z)\right] = 0\end{aligned}$$

for $n = 0, 1, \dots, k-1$. From the definition of f_a and (3.19), we obtain that

$$\begin{aligned}&\Big| f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) - 2\left[f(x+y) + f(x-y)\right.\\ &\quad\left. + f(y+z) + f(y-z) + f(x+z) + f(x-z)\right] + 4f(x) + 4f(y) + 2\left[f(z) + f(-z)\right]\Big|\\ &\le \sum_{n=0}^{\infty} \frac{1}{2^n}\Big|\psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z))\\ &\quad -2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right.\\ &\quad\left. +\psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^nx) + 4\psi(2^ny) + 2\left[\psi(2^nz) + \psi(2^n-z)\right]\Big|\\ &\le \sum_{n=k}^{\infty} \frac{1}{2^n}\Big|\psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z))\\ &\quad -2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right.\\ &\quad\left. +\psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^nx) + 4\psi(2^ny) + 2\left[\psi(2^nz) + \psi(2^n-z)\right]\Big|\\ &\le \sum_{n=k}^{\infty} \frac{1}{2^n} 28\mu = 28\,\mu \times \frac{2}{2^k} = 56\,\mu(|x|+|y|+|z|).\end{aligned}$$

Thus f_a satisfies (3.17) for all $x \in \mathbb{R}$ with $0 < |x|+|y|+|z| < \dfrac{1}{2}$.

We claim that the additive functional equation (1.9) is not stable for $s = 1$ in condition (ii) Corollary 3.1. Suppose on the contrary that there exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.18). Since f_a is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x) = cx$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|f_a(x)| \le (\kappa + |c|)\,|x|. \tag{3.20}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \dots, m-1$. For this x, we get

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \ge \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\kappa + |c|)\,x$$

which contradicts (3.20). Therefore the additive functional equation (1.9) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality condition (ii) of (3.16). □

A counter example to illustrate the non stability in condition (iii) of Corollary 3.1 is given in the following example.

Example 3.2. *Let s be such that $0 < s < \dfrac{1}{3}$. Then there is a function $f_a : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying*

$$|Df_a(x,y,z)| \le \lambda |x|^{\frac{s}{3}}\, |y|^{\frac{s}{3}}\, |z|^{\frac{1-2s}{3}} \tag{3.21}$$

for all $x, y, z \in \mathbb{R}$ *and*

$$\sup_{x\neq 0} \frac{|f_a(x) - A(x)|}{|x|} = +\infty \tag{3.22}$$

for every additive mapping $A(x) : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f(x) = \begin{cases} x\ln|x|, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (3.22), it follows that

$$\begin{aligned} \sup_{x\neq 0} \frac{|f_a(x) - A(x)|}{|x|} &\geq \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}} \frac{|f_a(n) - A(n)|}{|n|} \\ &= \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}} \frac{|n\ln|n| - n\,A(1)|}{|n|} \\ &= \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}} |\ln|n| - A(1)| = \infty. \end{aligned}$$

We have to prove (3.21) is true.
Case (i): If $x, y, z > 0$ in (3.21) then,

$$\begin{aligned} &\Big| f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) - 2\left[f(x+y) + f(x-y) + f(y+z)\right. \\ &\qquad \left. + f(y-z) + f(x+z) + f(x-z)\right] - 4f(x) - 4f(y) - 2\left[f(z) + f(-z)\right] \Big| \\ &= |(x+y+z)\ln|x+y+z| + (x+y-z)\ln|x+y-z| + (x-y+z)\ln|x-y+z| \\ &\quad + (x-y-z)\ln|x-y-z| - 2[(x+y)\ln|x+y| + (x-y)\ln|x-y| + (y+z)\ln|y+z| \\ &\qquad + (y-z)\ln|y-z] - 4(x)\ln|x| - 4(y)\ln|y| - 2[(z)\ln|z| + (-z)\ln|-z|]|\,. \end{aligned}$$

Set $x = v_1, y = v_2, z = v_3$ it follows that

$$\begin{aligned} &\Big| f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) - 2\left[f(x+y) + f(x-y) + f(y+z)\right. \\ &\quad \left. + f(y-z) + f(x+z) + f(x-z)\right] - 4f(x) - 4f(y) - 2\left[f(z) + f(-z)\right] \Big| \\ &= |(v_1+v_2+v_3)\ln|v_1+v_2+v_3| + (v_1+v_2-v_3)\ln|v_1+v_2-v_3| \\ &\quad + (v_1-v_2+v_3)\ln|v_1-v_2+v_3| + (v_1-v_2-v_3)\ln|v_1-v_2-v_3| \\ &\qquad - 2[(v_1+v_2)\ln|v_1+v_2| + (v_1-v_2)\ln|v_1-v_2| + (v_2+v_3)\ln|v_2+v_3| \\ &\qquad\quad + (v_2-v_3)\ln|v_2-v_3] - 4(v_1)\ln|v_1| - 4(v_2)\ln|v_2| - 2[(v_3)\ln|v_3| + (-v_3)\ln|-v_3|]|\,. \\ &= \Big| f(v_1+v_2+v_3) + f(v_1+v_2-v_3) + f(v_1-v_2+v_3) + f(v_1-v_2-v_3) \\ &\qquad - 2\left[f(v_1+v_2) + f(v_1-v_2) + f(v_2+v_3) + f(v_2-v_3) + f(v_1+v_3) + f(v_1-v_3)\right] \\ &\qquad\quad - 4f(v_1) - 4f(v_2) - 2\left[f(v_3) + f(-v_3)\right] \Big| \\ &\leq \lambda |v_1|^{\frac{s}{3}}\, |v_2|^{\frac{s}{3}}\, |v_3|^{\frac{1-2s}{3}} \\ &= \lambda |x|^{\frac{s}{3}}\, |y|^{\frac{s}{3}}\, |z|^{\frac{1-2s}{3}}. \end{aligned}$$

For the cases (ii) $x, y, z < 0$, (iii) $x, y > 0, z < 0$, (iv) $x, y < 0, z > 0$ and (v) $x = y = z = 0$, the proof is similar tracing to that of Case (i). □

Now, the authors provide an example to illustrate that the functional equation (1.9) is not stable for $s = \frac{1}{3}$ in condition (iv) of Corollary 3.1.

Example 3.3. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \frac{\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_a : \mathbb{R} \to \mathbb{R}$ by

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_a satisfies the functional inequality

$$|Df_a(x,y,z)| \leq \frac{56\mu}{3}\{|x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|)\} \tag{3.23}$$

for all $x \in \mathbb{R}$. Then there do not exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_a(x) - A(x)| \leq \kappa |x| \qquad for\ all \quad x \in \mathbb{R}. \tag{3.24}$$

Proof. Now

$$|f_a(x)| \leq \sum_{n=0}^{\infty} \frac{|\psi(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\mu}{3} \frac{1}{2^n} = \frac{2\mu}{3}.$$

Therefore, we see that f_a is bounded. We are going to prove that f_a satisfies (3.17).

If $x = y = z = 0$ then (3.17) is trivial. If $|x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|) \geq \frac{1}{2}$ then the left hand side of (3.17) is less than $\frac{56\mu}{3}$. Now suppose that $0 < |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|) < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^{k-1}} \leq |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|) < \frac{1}{2^k}, \tag{3.25}$$

so that $2^{k-1}x^{\frac{1}{3}} < \frac{1}{2}, 2^{k-1}y^{\frac{1}{3}} < \frac{1}{2}, 2^{k-1}z^{\frac{1}{3}} < \frac{1}{2}, 2^{k-1}x < \frac{1}{2}, 2^{k-1}y < \frac{1}{2}, 2^{k-1}z < \frac{1}{2}$ and consequently

$$\begin{aligned} &2^{k-1}(x+y+z), 2^{k-1}(x+y-z), 2^{k-1}(x-y+z), 2^{k-1}(x-y-z), 2^{k-1}(x+y), 2^{k-1}(x-y), \\ &2^{k-1}(y+z), 2^{k-1}(y-z), 2^{k-1}(x+z), 2^{k-1}(x-z), 2^{k-1}(x), 2^{k-1}(y), 2^{k-1}(z), 2^{k-1}(-z) \in (-1,1). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} &2^n(x+y+z), 2^n(x+y-z), 2^n(x-y+z), 2^n(x-y-z), 2^n(x+y), 2^n(x-y), \\ &2^n(y+z), 2^n(y-z), 2^n(x+z), 2^n(x-z), 2^n(x), 2^n(y), 2^n(z), 2^n(-z) \in (-1,1) \end{aligned}$$

and

$$\begin{aligned} &\psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z)) \\ &\quad - 2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right. \\ &\qquad \left. + \psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^n x) + 4\psi(2^n y) + 2\left[\psi(2^n z) + \psi(2^n - z)\right] = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f_a and (3.19), we obtain that

$$\begin{aligned}
&\Big| f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) - 2\left[f(x+y) + f(x-y)\right.\\
&\qquad \left. + f(y+z) + f(y-z) + f(x+z) + f(x-z)\right] + 4f(x) + 4f(y) + 2\left[f(z) + f(-z)\right] \Big|\\
&\le \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z))\\
&\qquad - 2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right.\\
&\qquad \left. + \psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^n x) + 4\psi(2^n y) + 2\left[\psi(2^n z) + \psi(2^n - z)\right] \Big|\\
&\le \sum_{n=k}^{\infty} \frac{1}{2^n} \Big| \psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z))\\
&\qquad - 2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right.\\
&\qquad \left. + \psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^n x) + 4\psi(2^n y) + 2\left[\psi(2^n z) + \psi(2^n - z)\right] \Big|\\
&\le \sum_{n=k}^{\infty} \frac{1}{2^n} \frac{28\mu}{3} = \frac{28\mu}{3} \times \frac{2}{2^k} = \frac{56\mu}{3}(|x| + |y| + |z|).
\end{aligned}$$

Thus f_a satisfies (3.17) for all $x \in \mathbb{R}$ with $0 < |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|) < \dfrac{1}{2}$.

We claim that the additive functional equation (1.9) is not stable for $s = 1$ in condition (iv) Corollary 3.1. Suppose on the contrary that there exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.18). Since f_a is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x) = cx$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|f_a(x)| \le (\kappa + |c|)\,|x|. \tag{3.26}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \ge \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\kappa + |c|)\, x$$

which contradicts (3.26). Therefore the additive functional equation (1.9) is not stable in sense of Ulam, Hyers and Rassias if $s = \frac{1}{3}$, assumed in the inequality condition (ii) of (3.16). □

4 Stability Results: Even Case

In this section, we present the generalized Ulam-Hyers stability of the functional equation (1.9) for even case.

Theorem 4.1. *Let $j = \pm 1$ and $\psi, \zeta : \mathcal{G}^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{4^{nj}} = 0 \tag{4.1}$$

for all $x, y, z \in \mathcal{G}$. Let $f_q : \mathcal{G} \to \mathcal{H}$ be an even function satisfying the inequality

$$\left\| Df_q(x,y,z) \right\| \le \psi(x,y,z) \tag{4.2}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique quadratic mapping $Q_2 : \mathcal{G} \to \mathcal{H}$ which satisfies (1.9) and

$$\left\| f_q(2x) - 16 f_q(x) - Q_2(x) \right\| \le \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta\left(2^{kj}x\right)}{4^{kj}} \tag{4.3}$$

where $\zeta\left(2^{kj}x\right)$ and $Q_2(x)$ are defined by

$$\zeta\left(2^{kj}x\right) = 4\psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) + \psi\left(2^{(k+1)j}x, 2^{kj}x, 2^{kj}x\right) \tag{4.4}$$

and

$$Q_2(x) = \lim_{n\to\infty} \frac{1}{4^{nj}} \left(f_q(2^{(n+1)j}x) - 16 f_q(2^{nj}x) \right) \tag{4.5}$$

for all $x \in \mathcal{G}$, respectively.

Proof. Replacing (x, y, z) by (x, x, x) in (4.2) and using evenness of f_q, we get

$$\|f_q(3x) - 6f_q(2x) + 15f_q(x)\| \le \psi(x, x, x) \tag{4.6}$$

for all $x \in \mathcal{G}$. Again replacing (x, y, z) by $(2x, x, x)$ in (4.2) and using oddness of f_q, we obtain

$$\|f_q(4x) + 4f_q(2x) - 4f_q(3x) + 4f_q(x)\| \le \psi(2x, x, x) \tag{4.7}$$

for all $x \in \mathcal{G}$. It follows from (4.6) and (4.7) that

$$\begin{aligned} &\|f_q(4x) - 20f_q(2x) + 64f_q(x)\| \\ &\quad \le 4\|f_q(3x) - 6f_q(2x) + 15f_q(x)\| + \|f_q(4x) + 4f_q(2x) - 4f_q(3x) + 4f_q(x)\| \\ &\quad \le 4\psi(x, x, x) + \psi(2x, x, x) \end{aligned} \tag{4.8}$$

for all $x \in \mathcal{G}$. From (4.8), we arrive

$$\|f_q(4x) - 20f_q(2x) + 64f_q(x)\| \le \zeta(x) \tag{4.9}$$

where

$$\zeta(x) = 4\psi(x, x, x) + \psi(2x, x, x)$$

for all $x \in \mathcal{G}$. It is easy to see from (4.9) that

$$\|f_q(4x) - 16f_q(2x) - 4(f_q(2x) - 16f_q(x))\| \le \zeta(x) \tag{4.10}$$

for all $x \in \mathcal{G}$. Using (2.25) in (4.10), we obtain

$$\|q_2(2x) - 4q_2(x)\| \le \zeta(x) \tag{4.11}$$

for all $x \in \mathcal{G}$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 4.1 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 4.1. *Let ρ and s be nonnegative real numbers. Let an even function $f_q : \mathcal{G} \to \mathcal{H}$ satisfy the inequality*

$$\|Df_q(x, y, z)\| \le \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \ne 2; \\ \rho||x||^s||y||^s||z||^s, & 3s \ne 2; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \ne 2; \end{cases} \tag{4.12}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique quadratic function $Q_2 : \mathcal{G} \to \mathcal{H}$ such that

$$\|f_q(2x) - 16f_q(x) - Q_2(x)\| \le \begin{cases} \dfrac{5\rho}{3}, \\ \dfrac{(2^s + 14)\rho||x||^s}{2|4 - 2^s|}, \\ \dfrac{(2^s + 4)\rho||x||^{3s}}{2|4 - 2^{3s}|}, \\ \dfrac{(2^s + 2^{3s} + 18)\rho||x||^{3s}}{|4 - 2^{3s}|} \end{cases} \tag{4.13}$$

for all $x \in \mathcal{G}$.

Now, the authors provide an example to illustrate that the functional equation (1.9) is not stable for $s = 2$ in condition (ii) of Corollary 4.1.

Example 4.4. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x^2, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{4^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|Df_q(x,y,z)| \le \frac{112\,\mu}{3}(|x|^2 + |y|^2 + |z|^2) \tag{4.14}$$

for all $x \in \mathbb{R}$. Then there do not exist a quadratic mapping $Q_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_q(2x) - 16 f_q(x) - Q_2(x)| \le \kappa |x|^2 \qquad for\ all \quad x \in \mathbb{R}. \tag{4.15}$$

Proof. The proof of the example is similar to that of Example 3.1. □

A counter example to illustrate the non stability in condition (iii) of Corollary 4.1 is given in the following example.

Example 4.5. *Let s be such that $0 < s < \frac{2}{3}$. Then there is a function $f_q : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying*

$$\left|Df_q(x,y,z)\right| \le \lambda |x|^{\frac{s}{3}} \, |y|^{\frac{s}{3}} \, |z|^{\frac{2-2s}{3}} \tag{4.16}$$

for all $x, y, z \in \mathbb{R}$ and

$$\sup_{x\neq 0} \frac{\left|f_q(2x) - 16 f_q(x) - Q_2(x)\right|}{|x|^2} = +\infty \tag{4.17}$$

for every quadratic mapping $Q_2(x) : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f(x) = \begin{cases} x^2 \ln|x|, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (3.22), it follows that

$$\begin{aligned} \sup_{x\neq 0} \frac{\left|f_q(2x) - 16 f_q(x) - Q_2(x)\right|}{|x|^2} &\ge \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}} \frac{\left|f_q(2n) - 16 f_q(n) - Q_2(n)\right|}{|n|^2} \\ &= \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}} \frac{\left|4n^2 \ln|n| - n^2 16 \ln|n| - n^2\, Q_2(1)\right|}{|n|^2} \\ &= \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}} |4\ln|n| - 16\ln|n| - Q_2(1)| = \infty. \end{aligned}$$

The proof is similar tracing to that of Example 3.2. □

Now, the authors provide an example to illustrate that the functional equation (1.9) is not stable for $s = \frac{2}{3}$ in condition (iv) of Corollary 4.1.

Example 4.6. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \frac{2\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|Df_q(x,y,z)| \leq \frac{28 \times 8\mu}{3} \{|x|^{\frac{2}{3}} + |y|^{\frac{2}{3}} + |z|^{\frac{2}{3}} + (|x|^2 + |y|^2 + |z|^2)\} \tag{4.18}$$

for all $x \in \mathbb{R}$. Then there do not exist a quadratic mapping $Q_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_q(2x) - 16f_q(x) - Q_2(x)| \leq \kappa|x| \qquad for\ all \quad x \in \mathbb{R}. \tag{4.19}$$

Proof. The proof of the example is similar to that of Example 3.3. □

Theorem 4.2. *Let $j = \pm 1$ and $\psi, \zeta : \mathcal{G}^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{16^{nj}} = 0 \tag{4.20}$$

for all $x, y, z \in \mathcal{G}$. Let $f_q : \mathcal{G} \to \mathcal{H}$ be an even function satisfying the inequality

$$\left\|Df_q(x,y,z)\right\| \leq \psi\left(x,y,z\right) \tag{4.21}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique quartic mapping $Q_4 : \mathcal{G} \to \mathcal{H}$ which satisfies (1.9) and

$$\left\|f_q(2x) - 4f_q(x) - Q_4(x)\right\| \leq \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta\left(2^{kj}x\right)}{16^{kj}} \tag{4.22}$$

where $\zeta\left(2^{kj}x\right)$ is defined in (4.4) and $Q_4(x)$ is defined by

$$Q_4(x) = \lim_{n\to\infty} \frac{1}{16^{nj}} \left(f_q(2^{(n+1)j}x) - 4f_q(2^{nj}x)\right) \tag{4.23}$$

for all $x \in \mathcal{G}$.

Proof. It follows from (4.8), we have

$$\left\|f_q(4x) - 20f_q(2x) + 64f_q(x)\right\| \leq \zeta(x) \tag{4.24}$$

where

$$\zeta(x) = 4\psi\left(x,x,x\right) + \psi\left(2x,x,x\right)$$

for all $x \in \mathcal{G}$. It is easy to see from (4.24) that

$$\left\|f_q(4x) - 4f_q(2x) - 16(f_q(2x) - 4f_q(x))\right\| \leq \zeta(x) \tag{4.25}$$

for all $x \in \mathcal{G}$. Using (2.31) in (4.25), we obtain

$$\|q_4(2x) - 16q_4(x)\| \leq \zeta(x) \tag{4.26}$$

for all $x \in \mathcal{G}$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 4.2 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 4.2. *Let ρ and s be nonnegative real numbers. Let an even function $f_q : \mathcal{G} \to \mathcal{H}$ satisfy the inequality*

$$\left\|Df_q(x,y,z)\right\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{4.27}$$

for all $x, y, z \in \mathcal{G}$. *Then there exists a unique quartic function* $Q_4 : \mathcal{G} \to \mathcal{H}$ *such that*

$$\left\| f_q(2x) - 4f_q(x) - Q_4(x) \right\| \leq \begin{cases} \dfrac{\rho}{3}, \\ \dfrac{(2^s + 14)\rho||x||^s}{2|16 - 2^s|}, \\ \dfrac{(2^s + 4)\rho||x||^{3s}}{2|16 - 2^{3s}|}, \\ \dfrac{(2^s + 2^{3s} + 18)\rho||x||^{3s}}{|16 - 2^{3s}|} \end{cases} \tag{4.28}$$

for all $x \in \mathcal{G}$.

Now, the author provide an example to illustrate that the functional equation (1.9) is not stable for $s = 4$ in condition (ii) of Corollary 4.2.

Example 4.7. *Let* $\psi : \mathbb{R}^3 \to \mathbb{R}$ *be a function defined by*

$$\psi(x) = \begin{cases} \mu x^4, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ *is a constant, and define a function* $f_q : \mathbb{R} \to \mathbb{R}$ *by*

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{16^n} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then f_q *satisfies the functional inequality*

$$|Df_q(x, y, z)| \leq \frac{28 \times 16\,\mu}{15}(|x|^4 + |y|^4 + |z|^4) \tag{4.29}$$

for all $x \in \mathbb{R}$. *Then there do not exist a quartic mapping* $Q_4 : \mathbb{R} \to \mathbb{R}$ *and a constant* $\kappa > 0$ *such that*

$$|f_q(2x) - 4f_q(x) - Q_4(x)| \leq \kappa |x|^4 \qquad \text{for all} \quad x \in \mathbb{R}. \tag{4.30}$$

Proof. The proof of the example is similar to that of Example 3.1. □

A counter example to illustrate the non stability in condition (iii) of Corollary 4.2 is given in the following example.

Example 4.8. *Let* s *be such that* $0 < s < \frac{4}{3}$. *Then there is a function* $f : \mathbb{R} \to \mathbb{R}$ *and a constant* $\lambda > 0$ *satisfying*

$$\left|Df_q(x, y, z)\right| \leq \lambda |x|^{\frac{s}{3}} |y|^{\frac{s}{3}} |z|^{\frac{4-2s}{3}} \tag{4.31}$$

for all $x, y, z \in \mathbb{R}$ *and*

$$\sup_{x \neq 0} \frac{\left|f_q(2x) - 4f_q(x) - Q_4(x)\right|}{|x|^2} = +\infty \tag{4.32}$$

for every quartic mapping $Q_4(x) : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f(x) = \begin{cases} x^4 \ln |x|, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (3.22), it follows that

$$\begin{aligned} \sup_{x \neq 0} \frac{\left|f_q(2x) - 4f_q(x) - Q_4(x)\right|}{|x|^4} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|f_q(2n) - 4f_q(n) - Q_4(n)\right|}{|n|^4} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|16n^4 \ln |n| - n^4 4 \ln |n| - n^4\, Q_4(1)\right|}{|n|^4} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \left|16 \ln |n| - 4 \ln |n| - Q_4(1)\right| = \infty. \end{aligned}$$

The proof is similar tracing to that of Example 3.2. □

Now, the authors provide an example to illustrate that the functional equation (1.9) is not stable for $s = \frac{4}{3}$ in condition (iv) of Corollary 4.2.

Example 4.9. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \frac{4\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{16^n} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|Df_q(x,y,z)| \leq \frac{112 \times 16\mu}{45}\{|x|^{\frac{4}{3}} + |y|^{\frac{4}{3}} + |z|^{\frac{4}{3}} + (|x|^4 + |y|^4 + |z|^4)\} \tag{4.33}$$

for all $x \in \mathbb{R}$. Then there do not exist a quartic mapping $Q_4 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_q(2x) - 4f_q(x) - Q_4(x)| \leq \kappa|x| \qquad \text{for all} \quad x \in \mathbb{R}. \tag{4.34}$$

Proof. The proof of the example is similar to that of Example 3.3. □

Theorem 4.3. *Let $j = \pm 1$. Let $f_q : \mathcal{G} \to \mathcal{H}$ be a mapping for which there exists a function $\psi, \zeta : \mathcal{G}^3 \to [0,\infty)$ with the conditions given in (4.1) and (4.20) respectively, such that the functional inequality*

$$\|Df_q(x,y,z)\| \leq \psi(x,y,z) \tag{4.35}$$

for all $x,y,z \in \mathcal{G}$. Then there exists a unique quadratic mapping $Q_2(x) : \mathcal{G} \to \mathcal{H}$ and a unique quartic mapping $Q_4(x) : \mathcal{G} \to \mathcal{H}$ satisfying the functional equation (1.9) and

$$\|f_q(x) - Q_2(x) - Q_4(x)\| \leq \frac{1}{12}\left\{\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\zeta(2^{kj}x)}{4^{kj}} + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\zeta(2^{kj}x)}{16^{kj}}\right\} \tag{4.36}$$

for all $x \in \mathcal{G}$, where $\zeta(2^{kj}x)$, $Q_2(x)$ and $Q_4(x)$ are respectively defined in (4.4), (4.5) and (4.23) for all $x \in \mathcal{G}$.

Proof. By Theorems 4.1 and 4.2, there exists a unique quadratic function $Q_{2_1}(x) : \mathcal{G} \to \mathcal{H}$ and a unique quartic function $Q_{4_1}(x) : \mathcal{G} \to \mathcal{H}$ such that

$$\|f_q(2x) - 16f_q(x) - Q_{2_1}(x)\| \leq \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\zeta\left(2^{kj}x\right)}{4^{kj}} \tag{4.37}$$

and

$$\|f_q(2x) - 4f_q(x) - Q_{4_1}(x)\| \leq \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\zeta(2^{kj}x)}{16^{kj}} \tag{4.38}$$

for all $x \in \mathcal{G}$. Now from (4.37) and (4.38), one can see that

$$\begin{aligned}
&\left\|f_q(x) + \frac{1}{12}Q_{2_1}(x) - \frac{1}{12}Q_{4_1}(x)\right\| \\
&= \left\|\left\{-\frac{f_q(2x)}{12} + \frac{16f_q(x)}{12} + \frac{Q_{2_1}(x)}{12}\right\} + \left\{\frac{f_q(2x)}{12} - \frac{4f_q(x)}{12} - \frac{Q_{4_1}(x)}{12}\right\}\right\| \\
&\leq \frac{1}{12}\{\|f_q(2x) - 16f_q(x) - Q_{2_1}(x)\| + \|f_q(2x) - 4f_q(x) - Q_{4_1}(x)\|\} \\
&\leq \frac{1}{12}\left\{\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\zeta(2^{kj}x)}{4^{kj}} + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\zeta(2^{kj}x)}{16^{kj}}\right\}
\end{aligned}$$

for all $x \in \mathcal{G}$. Thus, we obtain (4.36) by defining $Q_2(x) = \frac{-1}{12}Q_{2_1}(x)$ and $Q_4(x) = \frac{1}{12}Q_{4_1}(x)$, where $\zeta(2^{kj}x)$, $Q_2(x)$ and $Q_4(x)$ are respectively defined in (4.4), (4.5) and (4.23) for all $x \in \mathcal{G}$. □

The following corollary is the immediate consequence of Theorem 4.3, using Corollaries 4.1 and 4.2 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 4.3. *Let $f_q : \mathcal{G} \to \mathcal{H}$ be a mapping and there exists real numbers ρ and s such that*

$$\|Df_q(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 2,4; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 2,4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 2,4; \end{cases} \tag{4.39}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique quadratic function $Q_2 : \mathcal{G} \to \mathcal{H}$ and a unique quartic function $Q_4 : \mathcal{G} \to \mathcal{H}$ such that

$$\|f_q(x) - Q_2(x) - Q_4(x)\| \leq \begin{cases} \dfrac{\rho}{6}, \\ \dfrac{(2^s+14)\rho||x||^s}{24}\left(\dfrac{1}{|16-2^s|} + \dfrac{1}{|4-2^s|}\right), \\ \dfrac{(2^s+4)\rho||x||^{3s}}{24}\left(\dfrac{1}{|16-2^{3s}|} + \dfrac{1}{|4-2^{3s}|}\right), \\ \dfrac{(2^s+2^{3s}+18)\rho||x||^{3s}}{12}\left(\dfrac{1}{|16-2^{3s}|} + \dfrac{1}{|4-2^{3s}|}\right) \end{cases} \tag{4.40}$$

for all $x \in \mathcal{G}$.

5 Stability Results: Mixed Case

Theorem 5.1. *Let $j = \pm 1$. Let $f : \mathcal{G} \to \mathcal{H}$ be a mapping for which there exists a function $\psi : \mathcal{G}^3 \to [0, \infty)$ with the conditions given in (3.1), (4.1) and (4.20) respectively, satisfying the functional inequality*

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{5.1}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique additive mapping $A(x) : \mathcal{G} \to \mathcal{H}$, a unique quadratic mapping $Q_2(x) : \mathcal{G} \to \mathcal{H}$, a unique quartic mapping $Q_4(x) : \mathcal{G} \to \mathcal{H}$ satisfying the functional equation (1.9) and

$$\begin{aligned} &\|f(x) - A(x) - Q_2(x) - Q_4(x)\| \\ &\leq \frac{1}{2}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\xi(2^{kj}x)}{2^{kj}} + \frac{\xi(-2^{kj}x)}{2^{kj}}\right)\right. \\ &\quad \left.+\frac{1}{12}\left[\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{4^{kj}} + \frac{\zeta(-2^{kj}x)}{4^{kj}}\right) + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{16^{kj}} + \frac{\zeta(-2^{kj}x)}{16^{kj}}\right)\right]\right\} \end{aligned} \tag{5.2}$$

for all $x \in \mathcal{G}$, where $\xi(2^{kj}x)$, $\zeta(2^{kj}x)$, $A(x)$, $Q_2(x)$ and $Q_4(x)$ are respectively defined in (3.4), (4.4), (3.5), (4.5) and (4.23) for all $x \in \mathcal{G}$.

Proof. Let $f_o(x) = \dfrac{f_a(x) - f_a(-x)}{2}$ for all $x \in \mathcal{G}$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in \mathcal{G}$. Hence

$$\|Df_o(x,y,z)\| \leq \frac{1}{2}\left\{\psi(x,y,z) + \psi(-x,-y,-z)\right\} \tag{5.3}$$

for all $x, y, z \in \mathcal{G}$. By Theorem 3.1, there exists a unique additive function $A(x) : \mathcal{G} \to \mathcal{H}$ such that

$$\|f_o(x) - A(x)\| \leq \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\xi(2^{kj}x)}{2^{kj}} + \frac{\xi(-2^{kj}x)}{2^{kj}}\right) \tag{5.4}$$

for all $x \in \mathcal{G}$. Also, let $f_e(x) = \dfrac{f_q(x) + f_q(-x)}{2}$ for all $x \in \mathcal{G}$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in \mathcal{G}$. Hence

$$\|Df_e(x,y,z)\| \leq \frac{1}{2}\left\{\psi(x,y,z) + \psi(-x,-y,-z)\right\} \tag{5.5}$$

for all $x,y,z \in \mathcal{G}$. By Theorem 4.3, there exists a unique quadratic mapping $Q_2(x) : \mathcal{G} \to \mathcal{H}$ and a unique quartic mapping $Q_4(x) : \mathcal{G} \to \mathcal{H}$ such that

$$\begin{aligned} &\|f_e(x) - Q_2(x) - Q_4(x)\| \\ &\leq \frac{1}{24}\left\{\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{4^{kj}} + \frac{\zeta(-2^{kj}x)}{4^{kj}}\right) + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{16^{kj}} + \frac{\zeta(-2^{kj}x)}{16^{kj}}\right)\right\} \end{aligned} \tag{5.6}$$

for all $x \in \mathcal{G}$. Define

$$f(x) = f_o(x) + f_e(x) \tag{5.7}$$

for all $x \in \mathcal{G}$. Now from (5.7), (5.6) and (5.4), we arrive

$$\begin{aligned} &\|f(x) - A(x) - Q_2(x) - Q_4(x)\| \\ &= \|f_o(x) + f_e(x) - A(x) - Q_2(x) - Q_4(x)\| \\ &\leq \|f_o(x) - A(x)\| + \|f_e(x) - Q_2(x) - Q_4(x)\| \\ &\leq \frac{1}{2}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\xi(2^{kj}x)}{2^{kj}} + \frac{\xi(-2^{kj}x)}{2^{kj}}\right)\right. \\ &\left. + \frac{1}{12}\left[\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{4^{kj}} + \frac{\zeta(-2^{kj}x)}{4^{kj}}\right) + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{16^{kj}} + \frac{\zeta(-2^{kj}x)}{16^{kj}}\right)\right]\right\} \end{aligned} \tag{5.8}$$

for all $x \in \mathcal{G}$, where $\xi(2^{kj}x)$, $\zeta(2^{kj}x)$, $A(x)$, $Q_2(x)$ and $Q_4(x)$ are respectively defined in (3.4), (4.4), (3.5), (4.5) and (4.23) for all $x \in \mathcal{G}$. □

The following corollary is the immediate consequence of Theorem 5.1, using Corollaries 3.1 and 4.3 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 5.1. *Let $f : \mathcal{G} \to \mathcal{H}$ be a mapping and there exists real numbers ρ and s such that*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1,2,4; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 1,2,4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1,2,4; \end{cases} \tag{5.9}$$

for all $x,y,z \in \mathcal{G}$, then there exists a unique additive mapping $A(x) : \mathcal{G} \to \mathcal{H}$, a unique quadratic mapping $Q_2(x) : \mathcal{G} \to \mathcal{H}$ and a unique quartic mapping $Q_4(x) : \mathcal{G} \to \mathcal{H}$ such that

$$\begin{aligned} &\|f(x) - A(x) - Q_2(x) - Q_4(x)\| \\ &\leq \begin{cases} \frac{2\rho}{3}, \\ \frac{\rho||x||^s}{2}\left(\frac{3}{|2-2^s|} + \frac{(2^s+14)}{12|16-2^s|} + \frac{(2^s+14)}{12|4-2^s|}\right), \\ \frac{\rho||x||^{3s}}{2}\left(\frac{1}{|2-2^{3s}|} + \frac{(2^s+4)}{12|16-2^{3s}|} + \frac{(2^s+4)}{12|4-2^{3s}|}\right), \\ \frac{\rho||x||^{3s}}{2}\left(\frac{4}{|2-2^{3s}|} + \frac{(2^s+2^{3s}+18)}{6|16-2^{3s}|} + \frac{(2^s+2^{3s}+18)}{6|4-2^{3s}|}\right) \end{cases} \end{aligned} \tag{5.10}$$

for all $x \in \mathcal{G}$.

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On semi generalized star b - Connectedness and semi generalized star b - Compactness in Topological Spaces

S. Sekar[a,*] and B. Jothilakshmi[b]

[a]Department of Mathematics, Government Arts College (Autonomous), Salem – 636 007, Tamil Nadu, India.

[b]Department of Mathematics, Government Arts College (Autonomous), Coimbatore – 641 045, Tamil Nadu, India.

Abstract

In this paper, the authors introduce a new type of connected spaces called semi generalized star b - connected spaces (briefly sg^*b-connected spaces) in topological spaces. The notion of semi generalized star b - compact spaces is also introduced (briefly sg^*b-compact spaces) in topological spaces. Some characterizations and several properties concerning sg^*b-connected spaces and sg^*b-compact spaces are obtained.

Keywords: sg^*b-closed sets, sg^*b-closed map, sg^*b-continuous map, contra sg^*b-continuity.

2010 MSC: 54C05, 54C08, 54C10.

1 Introduction

Topological spaces are mathematical structures that allow the formal definitions of concepts such as connectedness, compactness, interior and closure. In 1974, Das [4] defined the concept of semi-connectedness in topology and investigated its properties. Compactness is one of the most important, useful and fundamental concepts in topology. In 1981, Dorsett [6] introduced and studied the concept of semi-compact spaces. In 1990, Ganster [7] defined and investigated semi-Lindelof spaces. Since then, Hanna and Dorsett [10], Ganster and Mohammad S. Sarsak [8] investigated the properties of semi-compact spaces.

The notion of connectedness and compactness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Ganster and Steiner [9] introduced and studied the properties of gb-closed sets in topological spaces. Benchalli et al [2] introduced gb - compactness and gb - connectedness in topological spaces. Dontchev and Ganster[5] analyzed sg - compact space. Later, Shibani [13] introduced and analyzed rg - compactnes and rg - connectedness. Crossely et al [3] introduced semi - closure. Vadivel et al [14] studied $rg\alpha$ - interior and $rg\alpha$ - closure sets in topological spaces. The aim of this paper is to introduce the concept of sg^*b-connected and sg^*b-compactness in topological spaces.

2 Preliminaries

Definition 2.1. *A subset A of a topological space (X,τ), is called sg closed, if $scl(A) \subseteq U$. The complement of sg closed set is said to be sg open set . The family of all sg open sets (respectively semi generalised closed sets) of (X,τ) is denoted by $SG-O(X,\tau)$[respectively $SG-CL(X,\tau)$].*

Definition 2.2. *A subset A of a topological space (X,τ), is called semi generalized star b -closed set [11] (briefly sg^*b-closed set) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg open in X. The complement of sg^*b-closed set is*

*Corresponding author.
E-mail address: sekar_nitt@rediffmail.com (S. Sekar), balujothi87@gmail.com (B. Jothilakshmi).

called sg^*b-open. The family of all sg^*b-open [respectively sg^*b-closed] sets of (X,τ) is denoted by $sg^*b-O(X,\tau)$ [respectively $sg^*b-CL(X,\tau)$].

Definition 2.3. *A subset A of a topological space (X,τ) is called b-open set[1] if $A\subseteq cl(int(A))\cup int(cl(A))$. The complement of b-open set is b-closed sets. The family of all b-open sets (respectively b-closed sets) of (X,τ) is denoted by $bO(X,\tau)$ (respectively $bCL(X,\tau)$)*

Definition 2.4. *The sg^*b-closure of a set A, denoted by $sg^*b-Cl(A)$[12] is the intersection of all sg^*b-closed sets containing A.*

Definition 2.5. *The sg^*b-interior of a set A, denoted by $sg^*b-int(A)$[12] is the union of all sg^*b-open sets containing A.*

Definition 2.6. *A topological space X is said to be gb-connected [2] if X cannot be expressed as a disjoint of two non-empty gb-open sets in X. A sub set of X is gb-connected if it is gb-connected as a subspace.*

Definition 2.7. *A subset A of a topological space (X,τ) is called semi generalized star b-closed set[11] (briefly sg^*b-closed set) if $bcl(A)\subseteq U$ whenever $A\subseteq U$ and U is sg open in X.*

3 Semi Generalized Star b - Connectedness

Definition 3.8. *A topological space X is said to be sg^*b-connected if X cannot be expressed as a disjoint of two non - empty sg^*b-open sets in X. A subset of X is sg^*b-connected if it is sg^*b-connected as a subspace.*

Example 3.1. *Let $X=\{a,b,c\}$ and let $\tau=\{X,\varphi,\{b\},\{a,c\}\}$. It is sg^*b-connected.*

Theorem 3.1. *For a topological space X, the following are equivalent.*
(i) X is sg^*b-connected.
(ii) X and φ are the only subsets of X which are both sg^*b-open and sg^*b-closed.
(iii) Each sg^*b-continuous map of X into a discrete space Y with at least two points is constant map.

Proof. (i) $\Rightarrow$ (ii) : Suppose X is sg^*b - connected. Let S be a proper subset which is both sg^*b - open and sg^*b - closed in X. Its complement $X-S$ is also sg^*b - open and sg^*b - closed. $X=S\cup(X-S)$, a disjoint union of two non empty sg^*b - open sets which is contradicts (i). Therefore $S=\varphi$ or X.
(ii) $\Rightarrow$ (i) : Suppose that $X = A\cup B$ where A and B are disjoint non empty sg^*b - open subsets of X. Then A is both sg^*b - open and sg^*b - closed. By assumption $A=\varphi$ or X. Therefore X is sg^*b - connected.
(ii) $\Rightarrow$ (iii) : Let $f:X\rightarrow Y$ be a sg^*b - continuous map. X is covered by sg^*b - open and sg^*b - closed covering $\{f^{-1}(y):y\in Y\}$. By assumption $f^{-1}(y)=\varphi$ or X for each $y\in Y$. If $f^{-1}(y)=\varphi$ for all $y\in(Y)$, then f fails to be a map. Then there exists only one point $y\in Y$ such that $f^{-1}(y)\neq\varphi$ and hence $f^{-1}(y)=X$. This shows that f is a constant map.
(iii) $\Rightarrow$ (ii) : Let S be both sg^*b - open and sg^*b - closed in X. Suppose $S\neq\varphi$. Let $f:X\rightarrow Y$ be a sg^*b - continuous function defined by $f(S)=\{y\}$ and $f(X-S)=\{w\}$ for some distinct points y and w in Y. By (iii) f is a constant function. Therefore $S=X$. □

Theorem 3.2. *Every sg^*b - connected space is connected.*

Proof. Let X be sg^*b - connected. Suppose X is not connected. Then there exists a proper non empty subset B of X which is both open and closed in X. Since every closed set is sg^*b - closed, B is a proper non empty subset of X which is both sg^*b - open and sg^*b - closed in X. Using by Theorem 3.1, X is not sg^*b - connected. This proves the theorem. □

The converse of the above theorem need not be true as shown in the following example.

Example 3.2. *Let $X=\{a,b,c\}$ and let $\tau=\{X,\varphi,\{a\},\{b,c\}\}$. X is connected but not sg^*b - connected. Since $\{b\},\{a,c\}$ are disjoint sg^*b - open sets and $X=\{b\}\cup\{a,c\}$.*

Theorem 3.3. *If $f:X\rightarrow Y$ is a sg^*b - continuous and X is sg^*b - connected, then Y is connected.*

Proof. Suppose that Y is not connected. Let $Y = A \cup B$ where A and B are disjoint non - empty open set in Y. Since f is sg^*b - continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non - empty sg^*b - open sets in X. This contradicts the fact that X is sg^*b - connected. Hence Y is connected. □

Theorem 3.4. *If $f : X \to Y$ is a sg^*b - irresolut and X is sg^*b - connected, then Y is sg^*b - connected.*

Proof. Suppose that Y is not sg^*b connected. Let $Y = A \cup B$ where A and B are disjoint non - empty sg^*b open set in Y. Since f is sg^*b - irresolut and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non - empty sg^*b - open sets in X. This contradicts the fact that X is sg^*b - connected. Hence Y is sg^*b - connected. □

Definition 3.9. *A topological space X is said to be T_{sg^*b} - space if every sg^*b - closed set of X is closed subset of X.*

Theorem 3.5. *Suppose that X is T_{sg^*b} - space then X is connected if and only if it is sg^*b - connected.*

Proof. Suppose that X is connected. Then X cannot be expressed as disjoint union of two non - empty proper subsets of X. Suppose X is not a sg^*b - connected space. Let A and B be any two sg^*b - open subsets of X such that $X = A \cup B$, where $A \cap B = \varphi$ and $A \subset X, B \subset X$. Since X is T_{sg^*b} - space and A, B are sg^*b - open. A, B are open subsets of X, which contradicts that X is connected. Therefore X is sg^*b - connected.

Conversely, every open set is sg^*b - open. Therefore every sg^*b - connected space is connected. □

Theorem 3.6. *If the sg^*b - open sets C and D form a separation of X and if Y is sg^*b - connected subspace of X, then Y lies entirely within C or D.*

Proof. Since C and D are both sg^*b - open in X, the sets $C \cap Y$ and $D \cap Y$ are sg^*b - open in Y. These two sets are disjoint and their union is Y. If they were both non - empty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely C or D. □

Theorem 3.7. *Let A be a sg^*b - connected subspace of X. If $A \subset B \subset sg^*b - cl(A)$ then B is also sg^*b - connected.*

Proof. Let A be sg^*b - connected and let $A \subset B \subset sg^*b - cl(A)$. Suppose that $B = C \cup D$ is a separation of B by sg^*b - open sets. By using Theorem 3.6, A must lie entirely in C or D. Suppose that $A \subset C$, then $sg^*b - cl(A) \subset sg^*b - cl(B)$. Since $sg^*b - cl(C)$ and D are disjoint, B cannot intersect D. This contradicts the fact that C is non empty subset of B. So $D = \varphi$ which implies B is sg^*b - connected. □

Theorem 3.8. *A contra sg^*b - continuous image of an sg^*b - connected space is connected.*

Proof. Let $f : X \to Y$ is a contra sg^*b - continuous function from sg^*b - connected space X on to a space Y. Assume that Y is disconnected. Then $Y = A \cup B$, where A and B are non empty clopen sets in Y with $A \cap B = \varphi$. Since f is contra sg^*b - continous, we have $f^{-1}(A)$ and $f^{-1}(B)$ are non empty sg^*b - open sets in X with $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\varphi) = \varphi$. This shows that X is not sg^*b - connected, which is a contradiction. This proves the theorem. □

4 Semi Generalized Star b - Compactness

Definition 4.10. *A collection $\{A_\alpha : \alpha \in \Lambda\}$ of sg^*b -open sets in a topological space X is called a sg^*b - open cover of a subset B of X if $B \subset \bigcup \{A_\alpha : \alpha \in \Lambda\}$ holds.*

Definition 4.11. *A topological space X is sg^*b - compact if every sg^*b - open cover of X has a finite sub - cover.*

Definition 4.12. *A subset B of a topological space X is said to be sg^*b - compact relative to X, if for every collection $\{A_\alpha : \alpha \in \Lambda\}$ of sg^*b - open subsets of X such that $B \subset \bigcup \{A_\alpha : \alpha \in \Lambda\}$ there exists a finite subset Λ_0 of Λ such that $B \subset \bigcup \{A_\alpha : \alpha \in \Lambda_0\}$.*

Definition 4.13. *A subset B of a topological space X is said to be sg^*b - compact if B is sg^*b - compact as a subspace of X.*

Theorem 4.9. *Every sg^*b - closed subset of sg^*b - compact space is sg^*b - compact relative to X.*

Proof. Let A be sg^*b - closed subset of a sg^*b - compact space X. Then A^c is sg^*b - open in X. Let $M = \{G_\alpha : \alpha \in \Lambda\}$ be a cover of A by sg^*b - open sets in X. Then $M^* = M \cup A^c$ is a sg^*b - open cover of X. Since X is sg^*b - compact, M^* is reducible to a finite sub cover of X, say $X = G_{\alpha 1} \cup G_{\alpha 2} \cup G_{\alpha 3} \cup \ldots \cup G_{\alpha m} \cup A^c, G_{\alpha k} \in M$. But A and A^c are disjoint. Hence $A \subset G_{\alpha 1} \cup G_{\alpha 2} \cup G_{\alpha 3} \cup \ldots \cup G_{\alpha m}\ G_{\alpha k} \in M$, this implies that any sg^*b open cover M of A contains a finite sub - cover. Therefore A is gb - compact relative to X. That is, every sg^*b - closed subset of a sg^*b - compact space X is sg^*b - compact. □

Definition 4.14. *A function $f : X \to Y$ is said to be sg^*b - continuous if $f^{-1}(V)$ is sg^*b - closed in X for every closed set V of Y.*

Theorem 4.10. *A sg^*b - continuous image of a sg^*b - compact space is compact.*

Proof. Let $f : X \to Y$ be a sg^*b - continuous map from a sg^*b - compact space X onto a topological space Y. Let $\{A_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in \Lambda\}$ is a sg^*b - open cover of X. Since X is sg^*b - compact, it has a finite sub - cover say $\{f^{-1}(A_1), f^{-1} : i \in \Lambda(A_2), \ldots, f^{-1}(A_n)\}$. Since f is onto $\{A_1, A_2, \ldots, A_n\}$ is a cover of Y, which is finite. Therefore Y is compact. □

Definition 4.15. *A function $f : X \to Y$ is said to be sg^*b - irresolute if $f^{-1}(V)$ is sg^*b - closed in X for every sg^*b - closed set V of Y.*

Theorem 4.11. *If a map $f : X \to Y$ is sg^*b - irresolute and a subset B of X is sg^*b - compact relative to X, then the image $f(B)$ is sg^*b - compact relative to Y.*

Proof. Let$\{A_\alpha : \alpha \in \Lambda\}$ be any collection of sg^*b - open subsets of Y such that $f(B) \subset \bigcup\{A_\alpha : \alpha \in \Lambda\} \subset$. Then $B \subset \bigcup\{f^{-1}(A_\alpha) : \alpha \in \Lambda\}$. Since by hypothesis B is sg^*b - compact relative to X, there exists a finite subset $\Lambda_0 \in \Lambda$ such that $B \subset \bigcup\{f^{-1}(A_\alpha) : \alpha \in \Lambda_0\}$. Therefore we have $f(B) \bigcup \subset \{(A_\alpha) : \alpha \in \Lambda_0\}$, it shows that $f(B)$ is sg^*b - compact relative to Y. □

Theorem 4.12. *A space X is sg^*b - compact if and only if each family of sg^*b - closed subsets of X with the finite intersection property has a non - empty intersection.*

Proof. Given a collection A of subsets of X, let $C = \{X - A : A \in A\}$ be the collection of their complements. Then the following statements hold.
(a)A is a collection of sg^*b - open sets if and only if C is a collection of sg^*b - closed sets.
(b) The collection A covers X if and only if the intersection $\bigcap_{c \in C} C$ of all the elements of C is empty.
(c) The finite sub collection $\{A_1, A_2, \ldots A_n\}$ of A covers X if and only if the intersection of the corresponding elements $C_i = X - A_i$ of C is empty.The statement (a) is trivial, while the (b) and (c) follow from De Morgan's law.
$X - (\bigcup_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in J}(X - A_\alpha)$. The proof of the theorem now proceeds in two steps, taking contra positive of the theorem and then the complement.

The statement X is sg^*b - compact is equivalent to : Given any collection A of sg^*b - open subsets of X, if A covers X, then some finite sub collection of A covers X. This statement is equivalent to its contra positive, which is the following.

Given any collection A of sg^*b - open sets, if no finite sub - collection of A of covers X, then A does not cover X. Let C be as earlier, the collection equivalent to the following:

Given any collection C of sg^*b - closed sets, if every finite intersection of elements of C is not - empty, then the intersection of all the elements of C is non - empty. This is just the condition of our theorem. □

Definition 4.16. *A space X is said to be sg^*b - Lindelof space if every cover of X by sg^*b - open sets contains a countable sub cover.*

Theorem 4.13. *Let $f : X \to Y$ be a sg^*b - continuous surjection and X be sg^*b - Lindelof, then Y is Lindelof Space.*

Proof. Let $f : X \to Y$ be a sg^*b - continuous surjection and X be sg^*b - Lindelof. Let $\{V_\alpha\}$ be an open cover for Y. Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by sg^*b - open sets. Since X is sg^*b - Lindelof, $\{f^{-1}(V_\alpha)\}$ contains a countable sub cover, namely $\{f^{-1}(V_{\alpha n})\}$. Then $\{V_{\alpha n}\}$ is a countable subcover for Y. Thus Y is Lindelof space. □

Theorem 4.14. *Let $f : X \rightarrow Y$ be a sg^*b - irresolute surjection and X be sg^*b - Lindelof, then Y is sg^*b - Lindelof Space.*

Proof. Let $f : X \rightarrow Y$ be a sg^*b - irresolute surjection and X be sg^*b - Lindelof. Let $\{V_\alpha\}$ be an open cover for Y. Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by sg^*b - open sets. Since X is sg^*b - Lindelof, $\{f^{-1}(V_\alpha)\}$ contains a countable sub cover, namely $\{f^{-1}(V_{\alpha n})\}$. Then $\{V_{\alpha n}\}$ is a countable subcover for Y. Thus Y is sg^*b - Lindelof space. □

Theorem 4.15. *If $f : X \rightarrow Y$ is a sg^*b - open function and Y is sg^*b -Lindelof space, then X is Lindelof space.*

Proof. Let $\{V_\alpha\}$ be an open cover for X. Then $\{f(V_\alpha)\}$ is a cover of Y by sg^*b - open sets. Since Y is sg^*b Lindelof, $\{f(V_\alpha)\}$ contains a countable sub cover, namely $\{f(V_{\alpha n})\}$. Then $\{V_{\alpha n}\}$ is a countable sub cover for X. Thus X is Lindelof space. □

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General Solution and Generalized Ulam-Hyers Stability of a Generalized 3-Dimensional AQCQ Functional Equation

John M. Rassias,[a] M. Arunkumar,[b,*] and N. Mahesh Kumar[c]

[a]Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, Athens 15342, Greece.

[b]Department of Mathematics, Government Arts College, Tiruvannamalai, TamilNadu, India - 606 603.

[c]Department of Mathematics, Arunai Engineering College, Tiruvannamalai, TamilNadu, India - 606 603.

Abstract

In this paper, we achieve the general solution and generalized Ulam-Hyers stability of a generalized 3-dimensional AQCQ functional equation

$$\begin{aligned} f\left(x+r(y+z)\right)+f\left(x-r(y+z)\right) &= r^2\left[f\left(x+y+z\right)+f\left(x-y-z\right)\right]+2\left(1-r^2\right)f\left(x\right) \\ &+\frac{(r^4-r^2)}{12}\left[f\left(2(y+z)\right)+f\left(-2(y+z)\right)-4f\left(y+z\right)-4f\left(-(y+z)\right)\right] \end{aligned}$$

for all positive integers r with $r \geq 2$ in Banach Space using two different methods.

Keywords: Additive functional equations, quadratic functional equations, cubic functional equations, Quartic functional equations, mixed type functional equations, generalized Ulam - Hyers stability, fixed point.

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1 Introduction and Preliminaries

A basic question in the theory of functional equations is as follows: *When is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation?*. If the problem accepts a unique solution, we say the equation is stable.

The research of stability problems for functional equations was linked to the renowned Ulam problem [74] (in 1940), concerning the stability of group homomorphisms, which was first elucidated by D.H. Hyers [29], in 1941. This stability problem was more widespread by quite a lot of creators [2, 13, 55, 60, 63]. Other pertinent research works are also cited (see [1, 7, 13, 14, 17, 21, 24, 25, 30]).

The principal equation in the study of stability of functional equation is the equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

which is additive functional equation having solution $f(x) = cx$. Many researchers have their results about the stability of (1.1) in various spaces.

The functional equation

$$f\left(x+y\right)+f\left(x-y\right) = 2f\left(x\right)+2f\left(y\right) \tag{1.2}$$

is said to be a quadratic functional equation because the quadratic function $f\left(x\right) = x^2$ is a solution of the functional equation (1.2). Every solution of the quadratic functional equation is said to be a quadratic mapping. A quadratic functional equation was used to characterize inner product spaces.

*Corresponding author.

E-mail addresses: jrassias@primedu.uoa.gr (John M. Rassias) annarun2002@yahoo.co.in (M. Arunkumar), mrnmahesh@yahoo.com (N. Mahesh Kumar).

In 2001, J. M. Rassias introduced the cubic functional equation

$$C(x+2y) + 3C(x) = 3C(x+y) + C(x-y) + 6C(y) \tag{1.3}$$

and established the solution of the Ulam stability problem for cubic mappings. It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.3) which is called a cubic functional equation, and every solution of the cubic functional equation is said to be a cubic mapping.

The quartic functional equation

$$F(x+2y) + F(x-2y) + 6F(x) = 4[F(x+y) + F(x-y) + 6F(y)] \tag{1.4}$$

was introduced by J. M. Rassias . It is easy to show that the function $f(x) = x^4$ is the solution of (1.4). Every solution of the quartic functional equation is said to be a quartic mapping.

C.Park [51] proved the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation briefly, AQCQ-functional equation

$$\begin{aligned} f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] - 6f(x) \\ + f(2y) + f(-2y) - 4f(y) - 4f(-y) \end{aligned} \tag{1.5}$$

in non-Archimedean normed spaces.

In [64], Ravi et.al., introduced a general mixed-type AQCQ- functional equation

$$\begin{aligned} f(x+ay) + f(x-ay) = a^2[f(x+y) + f(x-y)] + 2\left(1-a^2\right)f(x) \\ + \frac{\left(a^4-a^2\right)}{12}[f(2y) + f(-2y) - 4f(y) - 4f(-y)] \end{aligned} \tag{1.6}$$

which is a generalized form of the AQCQ-functional equation (1.6) and obtained its general solution and generalized Hyers-Ulam stability for a fixed integer a with $a \neq 0, \pm 1$ in Banach spaces.
Now, we recall the following theorem which are useful to prove our fixed point stability results.

Theorem 1.1. *[12](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $\Gamma : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
$(B1)$ $\quad d(\Gamma^n x, \Gamma^{n+1} x) = \infty \quad \forall \; n \geq 0,$
or
$(B2)$ there exists a natural number n_0 such that:
(i) $d(\Gamma^n x, \Gamma^{n+1} x) < \infty$ for all $n \geq n_0$;
(ii)The sequence $(\Gamma^n x)$ is convergent to a fixed point y^* of Γ
(iii) y^* is the unique fixed point of Γ in the set $Y = \{y \in X : d(\Gamma^{n_0} x, y) < \infty\}$;
(iv) $d(y^*, y) \leq \frac{1}{1-L}\ d(y, \Gamma y)$ for all $y \in Y$.

In this paper, we obtain the general solution and generalized Ulam-Rassias stability of the generalized 3 dimensional AQCQ functional equation

$$\begin{aligned} & f(x+r(y+z)) + f(x-r(y+z)) \\ & \quad = r^2[f(x+y+z) + f(x-y-z)] + 2\left(1-r^2\right)f(x) \\ & \qquad + \frac{\left(r^4-r^2\right)}{12}[f(2(y+z)) + f(-2(y+z)) - 4f(y+z) - 4f(-(y+z))] \end{aligned} \tag{1.7}$$

for all positive integers r with $r \geq 2$ in Banach Space using two different methods.

2 General Solution

In this section, we present the general solution of the functional equation (1.6). Throughout this section, let U and V be real vector spaces.

Lemma 2.1. *Let $f : U \to V$ be a function satisfying (1.7) for all $x, y, z \in U$ then f satisfies (1.7) for all $x, y \in U$.*

Proof. Assume $f : U \to V$ satisfies (1.7). Replacing (r, z) by $(a, 0)$ in (1.7), we arrive our result. □

Theorem 2.1. *Let $f : U \to V$ be a function satisfying (1.7) for all $x, y, z \in U$and if f is even then f is quadratic - quartic.*

Proof. The proof follows from Lemma 2.1 and Theorem 2.2 of [64]. □

Theorem 2.2. *Let $f : U \to V$ be a function satisfying (1.7) for all $x, y, z \in U$and if f is odd then f is additive - cubic.*

Proof. The proof follows from Lemma 2.1 and Theorem 2.3 of [64]. □

Theorem 2.3. *Let $f : U \to V$ be a function satisfying (1.7) for all $x, y, z \in U$ if and only if there exists functions $A : U \to V$, $B : U^2 \to V$, $C : U^3 \to V$ and $D : U^4 \to V$ such that*

$$f(x) = A(x) + B(x,x) + C(x,x,x) + D(x,x,x,x)$$

for all $x \in U$, where A is additive, B is symmetric bi-additive, C is symmetric for each fixed one variable and is additive for fixed two variables and D is symmetric multi-additive.

Proof. The proof follows from Lemma 2.1 and Theorem 2.4 of [64]. □

Hereafter throughout this paper, let us consider U be a real normed space and V be a Banach space. Define a function $Df : U \to V$ by

$$\begin{aligned} Df(x,y,z) &= f(x + r(y+z)) + f(x - r(y+z)) - r^2 [f(x+y+z) + f(x-y-z)] - 2(1-r^2) f(x) \\ &\quad - \tfrac{(r^4-r^2)}{12} [f(2(y+z)) + f(-2(y+z)) - 4f(y+z) - 4f(-(y+z))] \end{aligned}$$

for all $x, y, z \in U$ and $r \geq 2$.

3 STABILITY RESULTS: EVEN CASE-DIRECT METHOD

In this section, we investigate the generalized Ulam - Hyers stability for the functional equation (1.7) for even case.

Theorem 3.1. *Let $j = \pm 1$. Let $\psi : U^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\psi(2^n x, 2^n y, 2^n z)}{4^n} \text{ converges and } \lim_{n\to\infty} \frac{\psi(2^n x, 2^n y, 2^n z)}{4^n} = 0 \tag{3.1}$$

for all $x, y, z \in U$ and let $f : U \to V$ be an even function satisfying the inequality

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{3.2}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ which satisfies (1.7) and

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{4^{kj}} \tag{3.3}$$

for all $x \in U$, where $Q_2(x)$ and $\Psi(2^{kj}x, 2^{kj}x, 2^{kj}x)$ are defined by

$$Q_2(x) = \lim_{n\to\infty} \frac{1}{4^{nj}} \left\{ f\left(2^{(n+1)j}x\right) - 16f\left(2^{nj}x\right) \right\} \tag{3.4}$$

$$\begin{aligned} \Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) = \tfrac{1}{r^4-r^2} \Big[& 12(1-r^2)\,\psi\left(0, 2^{kj}x, 0\right) + 12r^2 \psi\left(2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) \\ & + 6\psi\left(0, 2^{(k+1)j}x, 0\right) + 12\psi\left(2^{kj}rx, 2^{(k+1)j}x, -2^{kj}x\right) \Big] \end{aligned} \tag{3.5}$$

for all $x \in U$.

Proof. **Case (i) :** $j = 1$. Next using the evenness of f in (3.2), we get

$$\Big\| f\left(x+r(y+z)\right) + f\left(x-r(y+z)\right) - r^2\left[f\left(x+y+z\right) + f\left(x-y-z\right)\right] - 2\left(1-r^2\right) f\left(x\right) - \frac{\left(r^4-r^2\right)}{12}\left[2f\left(2(y+z)\right) - 8f\left(y+z\right)\right] \Big\| \le \psi\left(x,y,z\right) \tag{3.6}$$

for all $x,y,z \in U$. Interchanging x and y in (3.6), we obtain

$$\Big\| f\left(y+r(x+z)\right) + f\left(y-r(x+z)\right) - r^2[f\left(x+y+z\right) + f\left(y-x-z\right)] - 2\left(1-r^2\right) f\left(y\right) - \frac{\left(r^4-r^2\right)}{12}\left[2f\left(2(x+z)\right) - 8f\left(x+z\right)\right] \Big\| \le \psi\left(y,x,z\right) \tag{3.7}$$

for all $x,y,z \in U$. Letting $(y,z)$by $(0,0)$in (3.7)and using evenness of f, we have

$$\left\| 2f\left(rx\right) - 2\,r^2 f\left(x\right) - \frac{\left(r^4-r^2\right)}{12}\left[2f\left(2x\right) - 8f\left(x\right)\right] \right\| \le \psi\left(0,x,0\right) \tag{3.8}$$

for all $x \in U$. Putting (x,y,z) by $(2x,x,-x)$ in (3.7), we get

$$\| f\left((r+1)x\right) + f\left((r-1)x\right) - r^2 f\left(2x\right) - 2\left(1-r^2\right) f\left(x\right) - \frac{\left(r^4-r^2\right)}{12}\left[2f\left(2x\right) - 8f\left(x\right)\right]\| \le \psi\left(x,2x,-x\right) \tag{3.9}$$

for all $x \in U$. If we replace x by $2x$ in (3.8), we reach

$$\left\| 2f\left(2rx\right) - 2\,r^2 f\left(2x\right) - \frac{\left(r^4-r^2\right)}{12}\left[2f\left(4x\right) - 8f\left(2x\right)\right] \right\| \le \psi\left(0,2x,0\right) \tag{3.10}$$

for all $x \in U$. Setting (x,y,z) by $(2x,rx,-x)$ in (3.7), we obtain

$$\| f\left(2rx\right) - r^2\left[f(r+1)x + f(r-1)x\right] - 2\left(1-r^2\right) f\left(rx\right) - \frac{\left(r^4-r^2\right)}{12}\left[2f\left(2x\right) - 8f\left(x\right)\right]\| \le \psi\left(rx,2x,-x\right) \tag{3.11}$$

for all $x \in U$. Multiplying (3.8), (3.9), (3.10)and (3.11)by $12(1-r^2)$, $12r^2$, 6 and 12 respectively, we arrive

$$\begin{aligned}
&\left(r^4-r^2\right)\|f\left(4x\right) - 20f\left(2x\right) + 64f\left(x\right)\| \\
&= \Big\| \Big\{ 24\left(1-r^2\right) f\left(rx\right) - 24r^2\left(1-r^2\right) f\left(x\right) - \tfrac{12\left(1-r^2\right)\left(r^4-r^2\right)}{12}\left[2f\left(2x\right) - 8f\left(x\right)\right] \Big\} \\
&\quad + \left\{ 12r^2 f\left((r+1)x\right) + 12r^2 f\left((r-1)x\right) - 12r^4 f\left(2x\right) - 24r^2\left(1-r^2\right) f\left(x\right) - \tfrac{12r^2\left(r^4-r^2\right)}{12}\left[2f\left(2x\right) - 8f\left(x\right)\right] \right\} \\
&\quad + \left\{ -12f\left(2rx\right) + 12r^2 f\left(2x\right) + \tfrac{6\left(r^4-r^2\right)}{12}\left[2f\left(4x\right) - 8f\left(2x\right)\right] \right\} \\
&\quad + \left\{ 12f\left(2rx\right) - 12r^2[f\left((1+r)x\right) + f\left((1-r)x\right)] - 24\left(1-r^2\right) f\left(rx\right) - \tfrac{12\left(r^4-r^2\right)}{12}\left[2f\left(2x\right) - 8f\left(x\right)\right] \right\} \Big\| \\
&\le 12\left(1-r^2\right)\psi\left(0,x,0\right) + 12r^2\psi\left(x,2x,-x\right) + 6\psi\left(0,2x,0\right) + 12\psi\left(rx,2x,-x\right)
\end{aligned}$$

for all $x \in U$. It follows from above inequality that

$$\|f\left(4x\right) - 20f\left(2x\right) + 64f\left(x\right)\| \le \Psi\left(x,x,x\right) \tag{3.12}$$

where

$$\Psi\left(x,x,x\right) = \frac{1}{r^4-r^2}\left[12\left(1-r^2\right)\psi\left(0,x,0\right) + 12r^2\psi\left(x,2x,-x\right) + 6\psi\left(0,2x,0\right) + 12\psi\left(rx,2x,-x\right)\right]$$

for all $x \in U$. It is easy to see from (3.12)that

$$\|f\left(4x\right) - 16f\left(2x\right) - 4\left\{f(2x) - 16f(x)\right\}\| \le \Psi\left(x,x,x\right), \tag{3.13}$$

for all $x \in U$. Define a mapping $f_2 : U \to V$ by (See Theorem 2.2)

$$f_2(x) = f(2x) - 16f(x) \tag{3.14}$$

for all $x \in U$. Using (3.14)in (3.13), we get

$$\|f_2(2x) - 4f_2(x)\| \le \Psi(x, x, x) \tag{3.15}$$

for all $x \in U$. From (3.15), we have

$$\left\|\frac{f_2(2x)}{4} - f_2(x)\right\| \le \frac{\Psi(x, x, x)}{4} \tag{3.16}$$

for all $x \in U$. Now replacing x by $2x$ and dividing by 4 in (3.16), we obtain

$$\left\|\frac{f_2(2^2x)}{4^2} - \frac{f_2(2x)}{4}\right\| \le \frac{\Psi(2x, 2x, 2x)}{4^2} \tag{3.17}$$

for all $x \in U$. From (3.16)and (3.17), we arrive

$$\begin{aligned}\left\|\frac{f_2(2^2x)}{4^2} - f_2(x)\right\| &\le \left\|\frac{f_2(2^2x)}{4^2} - \frac{f_2(2x)}{4}\right\| + \left\|\frac{f_2(2x)}{4} - f_2(x)\right\| \\ &\le \frac{1}{4}\left[\Psi(x, x, x) + \frac{\Psi(2x,2x,2x)}{4}\right]\end{aligned} \tag{3.18}$$

for all $x \in U$. Proceeding further and using induction on a positive integer $'n'$, we get

$$\left\|\frac{f_2(2^nx)}{4^n} - f_2(x)\right\| \le \frac{1}{4}\sum_{k=0}^{n-1}\frac{\Psi\left(2^kx, 2^kx, 2^kx\right)}{4^k} \le \frac{1}{4}\sum_{k=0}^{\infty}\frac{\Psi\left(2^kx, 2^kx, 2^kx\right)}{4^k} \tag{3.19}$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{\frac{f_2(2^nx)}{4^n}\right\}$, replace x by 2^mx and dividing by 4^m in (3.19), for any $m, n > 0$, we deduce

$$\begin{aligned}\left\|\frac{f_2(2^{m+n}x)}{4^{m+n}} - \frac{f_2(2^mx)}{4^m}\right\| &= \frac{1}{4^m}\left\|\frac{f_2(2^n2^mx)}{4^n} - f_2(2^mx)\right\| \\ &\le \frac{1}{4}\sum_{k=0}^{n-1}\frac{\Psi\left(2^{k+m}x,2^{k+m}x,2^{k+m}x\right)}{4^{k+m}} \\ &\le \frac{1}{4}\sum_{k=0}^{\infty}\frac{\Psi\left(2^{k+m}x,2^{k+m}x,2^{k+m}x\right)}{4^{k+m}} \\ &\to 0 \quad as \quad m \to \infty\end{aligned}$$

for all $x \in U$. Hence the sequence $\left\{\frac{f_2(2^nx)}{4^n}\right\}$ is a Cauchy sequence. Since V is complete, there exists a quadratic mapping $Q_2 : U \to V$ such that

$$Q_2(x) = \lim_{n\to\infty}\frac{f_2(2^nx)}{4^n}, \qquad \forall\ x \in U.$$

Letting $n \to \infty$ in (3.19)and using (3.14), we see that (3.3)holds for all $x \in U$. To prove that Q_2 satisfies (1.7), replace (x, y, z) by $(2^nx, 2^ny, 2^nz)$ and dividing by 4^n in (3.2), we get

$$\begin{aligned}\frac{1}{4^n}\|f(2^n(x + r(y+z))) + f(2^n(x - r(y+z))) - r^2(f(2^n(x+y+z)) + f(2^n(x-y-z))) \\ -2\left(1-r^2\right)f(2^nx) - \frac{(r^4-r^2)}{12}[f(2^n(2(y+z))) + f(2^n(-2(y+z)))] \\ -\frac{(r^4-r^2)}{12}[-4f(2^n(y+x)) - 4f(2^n(-(y+z)))]\| \le \frac{\psi(2^nx,2^ny,2^nz)}{4^n}\end{aligned}$$

for all $x, y, z \in U$. Letting $n \to \infty$ in above inequality and using the definition of $Q_2(x)$, we see that

$$\begin{aligned}\|Q_2(x + r(y+z)) + Q_2(x - r(y+z)) - r^2(Q_2(x+y+z) + Q_2(x-y-z)) - 2\left(1-r^2\right)Q_2(x) \\ -\frac{(r^4-r^2)}{12}[Q_2(2(y+z)) + Q_2(-2(y+z)) - 4Q_2(y+z) - 4Q_2(-(y+z))]\| = 0\end{aligned}$$

which gives

$$\begin{aligned}Q_2(x + r(y+z)) + Q_2(x - r(y+z)) = r^2(Q_2(x+y+z) + Q_2(x-y-z)) + 2\left(1-r^2\right)Q_2(x) \\ +\frac{(r^4-r^2)}{12}[Q_2(2(y+z)) + Q_2(-2(y+z)) - 4Q_2(y+z) - 4Q_2(-(y+z))]\end{aligned}$$

for all $x, y, z \in U$. Hence Q_2 satisfies (1.7) for all $x, y, z \in U$. To show that Q_2 is unique, let Q_2' be another quadratic function satisfying (1.7) and (3.3). Now

$$\begin{aligned} \left\| Q_2(x) - Q_2'(x) \right\| &= \tfrac{1}{4^n} \left\| Q_2(2^n x) - Q_2'(2^n x) \right\| \\ &\le \tfrac{1}{4^n} \left\{ \| Q_2(2^n x) - f_2(2^n x) \| + \left\| f_2(2^n x) - Q_2'(2^n x) \right\| \right\} \\ &\le \tfrac{1}{4^n} \tfrac{1}{2} \sum_{k=0}^{\infty} \tfrac{\Psi(2^k x)}{4^k} \\ &\to 0 \quad as \ \ n \to \infty \end{aligned}$$

for all $x \in U$. Hence $Q_2$is unique. This completes the proof of the theorem.

Case (ii): Assume $j = -1$. Put $x = \frac{x}{2}$ in (3.15), we obtain

$$\left\| f_2(x) - 4 f_2\left(\frac{x}{2}\right) \right\| \le \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \tag{3.20}$$

for all $x \in U$. Now replacing x by $\frac{x}{2}$ and multiplying by 4 in (3.20), we obtain

$$\left\| 4 f_2\left(\frac{x}{2}\right) - 4^2 f_2\left(\frac{x}{2^2}\right) \right\| \le 4\Psi\left(\frac{x}{2^2}, \frac{x}{2^2}, \frac{x}{2^2}\right) \tag{3.21}$$

for all $x \in U$. From (3.20)and (3.21), we arrive

$$\begin{aligned} \left\| 4^2 f_2\left(\tfrac{x}{2^2}\right) - f_2(x) \right\| &\le \left\| 4^2 f_2\left(\tfrac{x}{2^2}\right) - 4 f_2\left(\tfrac{x}{2}\right) \right\| + \left\| 4 f_2\left(\tfrac{x}{2}\right) - f_2(x) \right\| \\ &\le 4\Psi\left(\tfrac{x}{2^2}, \tfrac{x}{2^2}, \tfrac{x}{2^2}\right) + \Psi\left(\tfrac{x}{2}, \tfrac{x}{2}, \tfrac{x}{2}\right) \end{aligned} \tag{3.22}$$

for all $x \in U$. Proceeding further and using induction on a positive integer $'n'$, we get

$$\left\| 4^n f_2\left(\frac{x}{2^n}\right) - f_2(x) \right\| \le \frac{1}{4} \sum_{k=1}^{n-1} 4^k \Psi\left(\frac{x}{2^k}, \frac{x}{2^k}, \frac{x}{2^k}\right) \ \le \frac{1}{4} \sum_{k=1}^{\infty} 4^k \Psi\left(\frac{x}{2^k}, \frac{x}{2^k}, \frac{x}{2^k}\right) \tag{3.23}$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{ 4^n f_2\left(\frac{x}{2^n}\right) \right\}$, replace x by $\frac{x}{2^m}$ and multiplying by 4^m in (3.23), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| 4^{m+n} f_2\left(\tfrac{x}{2^{m+n}}\right) - 4^m f_2\left(\tfrac{x}{2^m}\right) \right\| &= 4^m \left\| f_2\left(\tfrac{x}{2^{m+n}}\right) - f_2\left(\tfrac{x}{2^m}\right) \right\| \\ &\le \tfrac{1}{4} \sum_{k=1}^{n-1} 4^{k+m} \Psi\left(\tfrac{x}{2^{k+m}}, \tfrac{x}{2^{k+m}}, \tfrac{x}{2^{k+m}}\right) \\ &\le \tfrac{1}{4} \sum_{k=0}^{\infty} 4^{k+m} \Psi\left(\tfrac{x}{2^{k+m}}, \tfrac{x}{2^{k+m}}, \tfrac{x}{2^{k+m}}\right) \\ &\to 0 \quad as \quad m \to \infty \end{aligned}$$

for all $x \in U$. Hence the sequence $\left\{ 4^n f_2\left(\frac{x}{2^n}\right) \right\}$ is a Cauchy sequence. Since V is complete, there exists a quadratic mapping $Q_2 : U \to V$ such that

$$Q_2(x) = \lim_{n \to \infty} 4^n f_2\left(\frac{x}{2^n}\right), \qquad \forall \ x \in U.$$

The rest of the proof is similar to the case $j = 1$. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.7).

Corollary 3.1. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\| Df(x,y,z) \| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t \right), & t \ne 2; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t \right), & t \ne \frac{2}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + \left(\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t} \right) \right) & t \ne \frac{2}{3}; \end{cases} \tag{3.24}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ such that

$$\| f(2x) - 16 f(x) - Q_2(x) \| \le \begin{cases} 10\kappa_1, \\ \frac{\kappa_2 \|x\|^t}{|4 - 2^t|}, \\ \frac{\kappa_3 \|x\|^{3t}}{|4 - 2^{3t}|}, \\ \frac{\kappa_4 \|x\|^{3t}}{|4 - 2^{3t}|} \end{cases} \tag{3.25}$$

where

$$\begin{aligned} \kappa_1 &= \frac{\rho}{r^4-r^2}, \\ \kappa_2 &= \frac{\rho\left[24+12r^2+12r^2 2^t+12r^t+18\cdot 2^t\right]}{r^4-r^2}, \\ \kappa_3 &= \frac{12\rho 2^t\left[r^2+r^t\right]}{r^4-r^2}, \\ \kappa_4 &= \frac{\rho\left[24+12r^2\left(1+2^t+2^{3t}\right)+18\cdot 2^{3t}+12\cdot r^t\cdot 2^t+12\cdot r^{3t}\right]}{r^4-r^2}. \end{aligned} \tag{3.26}$$

for all $x \in U$.

Theorem 3.2. *Let* $j=\pm 1$. *Let* $\psi : U^3 \to [0,\infty)$ *be a function such that*

$$\sum_{n=0}^{\infty} \frac{\psi\left(2^n x, 2^n y, 2^n z\right)}{16^n} \text{ converges and } \lim_{n\to\infty} \frac{\psi\left(2^n x, 2^n y, 2^n z\right)}{16^n} = 0 \tag{3.27}$$

for all $x, y, z \in U$ *and let* $f : U \to V$ *be an even function which satisfies the inequality*

$$\|Df(x,y,z)\| \le \psi(x,y,z) \tag{3.28}$$

for all $x, y, z \in U$.*Then there exists a unique quartic function* $Q_4 : U \to V$ *which satisfies* (1.7) *and*

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{16^{kj}} \tag{3.29}$$

for all $x \in U$, *where* $\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$ *is defined in* (3.5)*and* $Q_4(x)$*is defined by*

$$Q_4(x) = \lim_{n\to\infty} \frac{1}{16^{nj}} \left\{ f\left(2^{(n+1)j}x\right) - 4f\left(2^{nj}x\right) \right\}, \tag{3.30}$$

for all $x \in U$.

Proof. It follows from (3.12), that

$$\|f(4x) - 20f(2x) + 64f(x)\| \le \Psi(x,x,x), \tag{3.31}$$

where

$$\Psi(x,x,x) = \frac{1}{r^4-r^2}\left[12\left(1-r^2\right)\psi(0,x,0) + 12r^2\psi(x,2x,-x) + 6\psi(0,2x,0) + 12\psi(rx,2x,-x)\right]$$

for all $x \in U$. It is easy to see from (3.31)that

$$\|f(4x) - 4f(2x) - 16\{f(2x) - 4f(x)\}\| \le \Psi(x,x,x) \tag{3.32}$$

for all $x \in U$. Define a mapping $f_4 : U \to V$ by (See Theorem 2.2)

$$f_4(x) = f(2x) - 4f(x) \tag{3.33}$$

for all $x \in U$. Using (3.33)in (3.32), we obtain

$$\|f_4(2x) - 16f_4(x)\| \le \Psi(x,x,x) \tag{3.34}$$

for all $x \in U$. From (3.34), we have

$$\left\| \frac{f_4(2x)}{16} - f_4(x) \right\| \le \frac{\Psi(x,x,x)}{16} \tag{3.35}$$

for all $x \in U$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.2 concerning the stability of (1.7).

Corollary 3.2. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 4; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq \frac{4}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) & t \neq \frac{4}{3}; \end{cases} \tag{3.36}$$

for all $x, y, z \in U$. Then there exists a unique quartic function $Q_4 : U \to V$ such that

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \begin{cases} 2\kappa_1, \\ \frac{\kappa_2\|x\|^t}{|16-2^t|}, \\ \frac{\kappa_3\|x\|^{3t}}{|16-2^{3t}|}, \\ \frac{\kappa_4\|x\|^{3t}}{|16-2^{3t}|} \end{cases} \tag{3.37}$$

for all $x \in U$, where κ_i $(i = 1,2,3,4)$ are defined in (3.26).

Theorem 3.3. *Assume $j = \pm 1$. Let $\psi : U^3 \to [0, \infty)$ be a function satisfying the conditions (3.1)and (3.27)for all $x, y, z \in U$. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{3.38}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ and a unique quartic function $Q_4 : U \to V$ which satisfies (1.7) and

$$\|f(x) - Q_2(x) - Q_4(x)\| \leq \frac{1}{12}\left\{\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{4^{kj}} + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{16^{kj}}\right\} \tag{3.39}$$

for all $x \in U$, where $\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right), Q_2(x)$ and $Q_4(x)$ are defined in (3.5), (3.4)and (3.30) respectively for all $x \in U$.

Proof. By Theorems 3.1 and 3.2, there exists a unique quadratic function $Q_2' : U \to V$ and a unique quartic function $Q_4' : U \to V$ such that

$$\left\|f(2x) - 16f(x) - Q_2'(x)\right\| \leq \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{4^{kj}} \tag{3.40}$$

and

$$\left\|f(2x) - 4f(x) - Q_4'(x)\right\| \leq \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{16^{kj}} \tag{3.41}$$

for all $x \in U$. Now from (3.40)and (3.41), that

$$\begin{aligned} \left\|f(x) + \tfrac{1}{12}Q_2'(x) - \tfrac{1}{12}Q_4'(x)\right\| &= \left\|\left\{-\tfrac{f(2x)}{12} + \tfrac{16f(x)}{12} + \tfrac{Q_2'(x)}{12}\right\} + \left\{\tfrac{f(2x)}{12} - \tfrac{4f(x)}{12} - \tfrac{Q_4'(x)}{12}\right\}\right\| \\ &\leq \tfrac{1}{12}\left\{\left\|f(2x) - 16f(x) - Q_2'(x)\right\| + \left\|f(2x) - 4f(x) - Q_4'(x)\right\|\right\} \\ &\leq \tfrac{1}{12}\left\{\tfrac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{4^{kj}} + \tfrac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{16^{kj}}\right\} \end{aligned}$$

for all $x \in U$. Thus, we obtain (3.39)by defining $Q_2(x) = \frac{-1}{12}Q_2'(x)$ and $Q_4(x) = \frac{1}{12}Q_4'(x)$, where $\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$, $Q_2(x)$ and $Q_4(x)$ are defined in (3.5), (3.4)and (3.30) respectively for all $x \in U$. □

The following corollary is an immediate consequence of Theorem 3.3 concerning the stability of (1.7).

Corollary 3.3. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 2,4; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq \frac{2}{3}, \frac{4}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & t \neq \frac{2}{3}, \frac{4}{3}; \end{cases} \tag{3.42}$$

for all $x,y,z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ and a unique quartic function $Q_4 : U \to V$ such that

$$\|f(x) - Q_2(x) - Q_4(x)\| \leq \begin{cases} \kappa_1, \\ \frac{\kappa_2\|x\|^t}{12}\left\{\frac{1}{|4-2^t|} + \frac{1}{|16-2^t|}\right\}, \\ \frac{\kappa_3\|x\|^{3t}}{12}\left\{\frac{1}{|4-2^{3t}|} + \frac{1}{|16-2^{3t}|}\right\}, \\ \frac{\kappa_4\|x\|^{3t}}{12}\left\{\frac{1}{|4-2^{3t}|} + \frac{1}{|16-2^{3t}|}\right\} \end{cases} \tag{3.43}$$

for all $x \in U$, where κ_i $(i = 1,2,3,4)$ are given in (3.26).

4 STABILITY RESULTS: ODD CASE-DIRECT METHOD

In this section, we discussed the generalized Ulam - Hyers stability of the functional equation (1.7) for odd case.

Theorem 4.4. *Assume $j = \pm 1$. Let $\phi : U^3 \to [0,\infty)$be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{2^n} \text{ converges and } \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{2^n} = 0 \tag{4.1}$$

for all $x,y,z \in U$ and let $f : U \to V$be an odd function which satisfies the inequality

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{4.2}$$

for all $x,y,z \in U$. Then there exists a unique additive function $A : U \to V$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{2^{kj}} \tag{4.3}$$

for all $x \in U$, where $A(x)$and $\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$ are defined by

$$A(x) = \lim_{n\to\infty} \frac{1}{2^{nj}} \left\{f\left(2^{(n+1)j}x\right) - 8f\left(2^{nj}x\right)\right\}, \tag{4.4}$$

$$\begin{aligned} \Phi(2^{kj}x, 2^{kj}x, 2^{kj}x) = \tfrac{1}{r^4-r^2} \Big[& (5-4r^2)\,\phi\left(2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) + 2r^2\phi\left(2^{(k+1)j}x, 2^{(k+1)j}x, -2^{kj}x\right) \\ & + (4-2r^2)\,\phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) + r^2\phi\left(2^{(k+1)j}x, 2^{(k+2)j}x, -2^{(k+1)j}x\right) \\ & + \phi\left(2^{kj}x, 2^{(k+1)j}x, 2^{kj}x\right) + 2\phi\left((1+r)2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) \\ & + 2\phi\left((1-r)2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) + \phi\left((1+2r)2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) \\ & + \phi\left((1-2r)2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) \Big] \end{aligned} \tag{4.5}$$

for all $x \in U$.

Proof. **Case (i):** j=1. Using the oddness of $f$in (4.2), we get

$$\begin{aligned} &\|f(x + r(y+z)) + f(x - r(y+z)) \\ &\quad -r^2 [f(x+y+z) + f(x-y-z)] - 2(1-r^2) f(x)\| \leq \phi(x,y,z) \end{aligned} \tag{4.6}$$

for all $x \in U$. Replacing (x,y,z) by $(x,2x,-x)$ in (4.6), we obtain

$$\left\| f\left((1+r)x\right) + f\left((1-r)x\right) - r^2 f\left(2x\right) - 2\left(1-r^2\right) f\left(x\right) \right\| \le \phi\left(x,2x,-x\right) \tag{4.7}$$

for all $x \in U$. Again replacing x by $2x$ in (4.7), we get

$$\left\| f\left(2(1+r)x\right) + f\left(2(1-r)x\right) - r^2 f\left(4x\right) - 2\left(1-r^2\right) f\left(2x\right) \right\| \le \phi\left(2x,4x,-2x\right) \tag{4.8}$$

for all $x \in U$. Setting (x,y,z) by $(2x,2x,-x)$ in (4.6), we have

$$\left\| f\left((2+r)x\right) + f\left((2-r)x\right) - r^2 f\left(3x\right) - r^2 f\left(x\right) - 2\left(1-r^2\right) f\left(2x\right) \right\| \le \phi\left(2x,2x,-x\right) \tag{4.9}$$

for all $x \in U$. Again setting (x,y,z) by (x,x,x) in (4.6), we obtain

$$\left\| f\left((1+2r)x\right) + f\left((1-2r)x\right) - r^2 f\left(3x\right) - r^2 f\left(x\right) - 2\left(1-r^2\right) f\left(x\right) \right\| \le \phi\left(x,x,x\right) \tag{4.10}$$

for all $x \in U$. Putting (x,y,z) by $(x,2x,x)$ in (4.6), we get

$$\left\| f\left((1+3r)x\right) + f\left((1-3r)x\right) - r^2 f\left(4x\right) - r^2 f\left(2x\right) - 2\left(1-r^2\right) f\left(x\right) \right\| \le \phi\left(x,2x,x\right) \tag{4.11}$$

for all $x \in U$. Again putting (x,y,z) by $((1+r)x,2x,-x)$ in (4.6), we have

$$\left\| f\left((1+2r)x\right) + f\left(x\right) - r^2 f\left((2+r)x\right) - r^2 f\left(rx\right) \; -2\left(1-r^2\right) f\left((1+r)x\right) \right\| \le \phi\left((1+r)x,2x,-x\right) \tag{4.12}$$

for all $x \in U$. Letting (x,y,z) by $((1-r)x,2x,-x)$ in (4.6), we obtain

$$\left\| f\left(x\right) + f\left((1-2r)x\right) - r^2 f\left((2-r)x\right) + r^2 f\left(rx\right) \; -2\left(1-r^2\right) f\left((1-r)x\right) \right\| \le \phi\left((1-r)x,2x,-x\right) \tag{4.13}$$

for all $x \in U$. Adding (4.12)and (4.13), we arrive

$$\begin{aligned} &\Big\| f\left((1+2r)x\right) + f\left((1-2r)x\right) + 2f\left(x\right) - r^2 f\left((2+r)x\right) - r^2 f\left((2-r)x\right) - 2\left(1-r^2\right) f\left((1+r)x\right) \\ &\qquad - 2\left(1-r^2\right) f\left((1-r)x\right) \Big\| \le \phi\left((1+r)x,2x,-x\right) + \phi_a\left((1-r)x,2x,-x\right) \end{aligned} \tag{4.14}$$

for all $x \in U$. Replacing (x,y,z) by $((1+2r)x,2x,-x)$ in (4.6), we get

$$\begin{aligned} &\Big\| f\left((1+3r)x\right) + f\left((1+r)x\right) - r^2 f\left(2(1+r)x\right) - r^2 f\left(2rx\right) \\ &\qquad -2\left(1-r^2\right) f\left((1+2r)x\right) \Big\| \le \phi\left((1+2r)x,2x,-x\right) \end{aligned} \tag{4.15}$$

for all $x \in U$. Again replacing (x,y,z) by $((1-2r)x,2x,-x)$ in (4.6), we obtain

$$\begin{aligned} &\Big\| f\left((1-r)x\right) + f\left((1-3r)x\right) - r^2 f\left(2(1-r)x\right) + r^2 f\left(2rx\right) \\ &\qquad -2\left(1-r^2\right) f\left((1-2r)x\right) \Big\| \le \phi\left((1-2r)x,2x,-x\right) \end{aligned} \tag{4.16}$$

for all $x \in U$. Adding (4.15)and (4.16), we arrive

$$\begin{aligned} &\| f\left((1+3r)x\right) + f\left((1-3r)x\right) + f\left((1+r)x\right) + f\left((1-r)x\right) - r^2 f\left(2(1+r)x\right) \\ &\quad -r^2 f\left(2(1-r)x\right) - 2\left(1-r^2\right) f\left((1+2r)x\right) \; - 2\left(1-r^2\right) f\left((1-2r)x\right) \| \\ &\qquad \le \phi\left((1+2r)x,2x,-x\right) + \phi\left((1-2r)x,2x,-x\right) \end{aligned} \tag{4.17}$$

for all $x \in U$. Now multiplying (4.7)by $2\left(1-r^2\right)$, (4.9)by r^2 and adding (4.10)and (4.14), we have

$$\begin{aligned} &\left(r^4 - r^2\right) \| f\left(3x\right) - 4f\left(2x\right) + 5f\left(x\right) \| \\ &= \| \left\{ 2\left(1-r^2\right) f\left((1+r)x\right) + 2\left(1-r^2\right) f\left((1-r)x\right) - 2r^2\left(1-r^2\right) f\left(2x\right) \; -4\left(1-r^2\right)^2 f\left(x\right) \right\} \\ &\quad + \left\{ r^2 f\left((2+r)x\right) + r^2 f\left((2-r)x\right) - r^4 f\left(3x\right) \; -r^4 f\left(x\right) - 2r^2\left(1-r^2\right) f\left(2x\right) \right\} \\ &\quad + \left\{ -f\left((1+2r)x - f\left((1-2r)x\right) \; + r^2 f\left(3x\right) - r^2 f\left(x\right) + 2\left(1-r^2\right) f\left(x\right) \right\} \right. \\ &\quad + \left\{ f\left((1+2r)x\right) + f\left((1-2r)x\right) \; + 2f\left(x\right) - r^2 f\left((2+r)x\right) - r^2 f\left((2-r)x\right) \right. \\ &\quad \left. -2\left(1-r^2\right) f\left((1+r)x\right) \; -2\left(1-r^2\right) f\left((1-r)x\right) \right\} \| \\ &\le 2\left(1-r^2\right) \phi\left(x,2x,-x\right) + r^2 \phi\left(2x,2x,-x\right) + \phi\left(x,x,x\right) + \phi\left((1+r)x,2x,-x\right) + \phi\left((1-r)x,2x,-x\right) \end{aligned}$$

for all$x \in U$. Hence from the above inequality, we reach

$$\begin{aligned} \|f(3x) - 4f(2x) + 5f(x)\| \le \tfrac{1}{(r^4 - r^2)} \big[& 2\left(1 - r^2\right)\phi(x, 2x, -x) + r^2\phi(2x, 2x, -x) \\ & +\phi(x, x, x) + \phi((1+r)x, 2x, -x) \\ & +\phi((1-r)x, 2x, -x) \big] \end{aligned} \tag{4.18}$$

for all $x \in U$. Also multiplying (4.8)by r^2, (4.10)by $2\left(1 - r^2\right)$ and adding (4.7), (4.11)and (4.17), we have

$$\begin{aligned} & \left(r^4 - r^2\right) \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & = \big\| \{-f((1+r)x) - f((1-r)x) + r^2 f(2x) + 2\left(1 - r^2\right) f(x)\} \\ & \quad + \{r^2 f(2(1+r)x) + r^2 f(2(1-r)x) - r^4 f(4x) - 2r^2\left(1 - r^2\right) f(2x)\} \\ & \quad + \Big\{2\left(1 - r^2\right) f((1+2r)x) + 2\left(1 - r^2\right) f((1-2r)x) - 2r^2\left(1 - r^2\right) f(3x) \\ & \quad + 2r^2\left(1 - r^2\right) f(x) - 4\left(1 - r^2\right)^2 f(x)\Big\} + \{-f((1+3r)x) - f((1-3r)x) \\ & \quad + r^2 f(4x) - r^2 f(2x) + 2\left(1 - r^2\right) f(x)\} + \{f((1+3r)x) + f((1-3r)x) + f((1+r)x) \\ & \quad + f((1-r)x) - r^2 f(2(1+r)x) - r^2 f(2(1-r)x) - 2\left(1 - r^2\right) f((1+2r)x) - 2\left(1 - r^2\right) f((1-2r)x)\} \big\| \\ & \le r^2\phi(2x, 4x, -2x) + 2\left(1 - r^2\right)\phi(x, x, x) + \phi(x, 2x, -x) + \phi(x, 2x, x) \\ & \qquad + \phi((1+2r)x, 2x, -x) + \phi((1-2r)x, 2x, -x) \end{aligned}$$

for all $x \in U$. Hence from the above inequality, we get

$$\begin{aligned} & \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & \quad \le \tfrac{1}{r^4 - r^2} \big[r^2\phi(2x, 4x, -2x) + 2\left(1 - r^2\right)\phi(x, x, x) + \phi(x, 2x, -x) \\ & \qquad + \phi(x, 2x, x) + \phi((1+2r)x, 2x, -x) + \phi((1-2r)x, 2x, -x) \big] \end{aligned} \tag{4.19}$$

for all $x \in U$. Adding (4.18)and (4.19), we arrive

$$\begin{aligned} & \|f(4x) - 10f(2x) + 16f(x)\| \\ & \quad = \|2f(3x) - 8f(2x) + 10f(x) + f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & \quad \le 2\|f(3x) - 4f(2x) + 5f(x)\| + \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & \quad \le \tfrac{1}{r^4 - r^2} \big[\left(5 - 4r^2\right)\phi(x, 2x, -x) + 2r^2\phi(2x, 2x, -x) + \left(4 - 2r^2\right)\phi(x, x, x) \\ & \qquad + r^2\phi(2x, 4x, -2x) + \phi(x, 2x, x) + 2\phi((1+r)x, 2x, -x) \\ & \qquad + 2\phi((1-r)x, 2x, -x) + \phi((1+2r)x, 2x, -x) + \phi((1-2r)x, 2x, -x) \big] \end{aligned} \tag{4.20}$$

for all $x \in U$. From (4.20), we have

$$\|f(4x) - 10f(2x) + 16f(x)\| \le \Phi(x, x, x) \tag{4.21}$$

where

$$\begin{aligned} \Phi(x, x, x) = \tfrac{1}{r^4 - r^2} \big[& \left(5 - 4r^2\right)\phi(x, 2x, -x) + 2r^2\phi(2x, 2x, -x) + \left(4 - 2r^2\right)\phi(x, x, x) \\ & + r^2\phi(2x, 4x, -2x) + \phi(x, 2x, x) + 2\phi((1+r)x, 2x, -x) \\ & + 2\phi((1-r)x, 2x, -x) + \phi((1+2r)x, 2x, -x) + \phi((1-2r)x, 2x, -x) \big] \end{aligned}$$

for all $x \in U$. It follows from (4.21), that

$$\|f(4x) - 8f(2x) - 2\{f(2x) - 8f(x)\}\| \le \Phi(x, x, x) \tag{4.22}$$

for all $x \in U$. Define a mapping $f_1 : U \to V$ by (See Theorem 2.3)

$$f_1(x) = f(2x) - 8f(x) \tag{4.23}$$

for all $x \in U$. Using (4.23)in (4.22), we obtain

$$\|f_1(2x) - 2f_1(x)\| \le \Phi(x, x, x) \tag{4.24}$$

for all $x \in U$. From (4.24), we obtain

$$\left\| \frac{f_1(2x)}{2} - f_1(x) \right\| \le \frac{\Phi(x, x, x)}{2} \tag{4.25}$$

for all $x \in U$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 4.4 concerning the stability of (1.7).

Corollary 4.4. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 1; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq \frac{1}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & t \neq \frac{1}{3}; \end{cases}$$

for all $x,y,z \in U$. Then there exists a unique additive function $A : U \to V$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \begin{cases} \kappa_5, \\ \frac{\kappa_6 \|x\|^t}{|2-2^t|}, \\ \frac{\kappa_7 \|x\|^{3t}}{|2-2^{3t}|}, \\ \frac{\kappa_8 \|x\|^{3t}}{|2-2^{3t}|} \end{cases} \tag{4.26}$$

where

$$\begin{aligned} \kappa_5 &= \frac{\rho(16-3r^2)}{r^4-r^2}, \\ \kappa_6 &= \frac{\rho}{r^4-r^2}\left[30 - 12r^2 + 2(6+r^2)2^t + r^2 2^{2t} + 2(1+r)^t + 2(1-r)^t + (1+2r)^t + (1-2r)^t\right], \\ \kappa_7 &= \frac{\rho}{r^4-r^2}\Big[4 - 2r^2 + 2(3-2r^2)2^t + 2r^2 2^{2t} + r^2 2^{4t} + 2(1+r)^t 2^t \\ &\qquad + 2(1-r)^t 2^t + (1+2r)^t 2^t + (1-2r)^t 2^t\Big], \\ \kappa_8 &= \frac{\rho}{r^4-r^2}\Big[34 - 14r^2 + 2(3-2r^2)2^t + 2(6+r^2)2^{3t} + 2r^2 2^{2t} \\ &\qquad + r^2(2^{4t} + 2^{6t}) + 2(1+r)^t 2^t + 2(1-r)^t 2^t + 2(1+r)^{3t} + 2(1-r)^{3t} \\ &\qquad + (1+2r)^t 2^t + (1-2r)^t 2^t + (1+2r)^{3t} + (1-2r)^{3t}\Big] \end{aligned} \tag{4.27}$$

for all $x \in U$.

Theorem 4.5. *Assume $j = \pm 1$. Let $\phi : U^3 \to [0,\infty)$be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{8^n} \ \textit{converges and} \ \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{8^n} = 0 \tag{4.28}$$

for all $x,y,z \in U$ and let $f : U \to V$ be an odd function which satisfies the inequality

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{4.29}$$

for all $x,y,z \in U$. Then there exists a unique cubic function $C : U \to V$ which satisfies (1.7) and

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{8^{kj}} \tag{4.30}$$

for all $x \in U$, where $\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$is defined in (4.5)and $C(x)$is defined by

$$C(x) = \lim_{n\to\infty} \frac{1}{8^{nj}} \left\{ f\left(2^{(n+1)j}x\right) - 2f\left(2^{nj}x\right) \right\} \tag{4.31}$$

for all $x \in U$.

Proof. It follows from (4.21), that

$$\|f(4x) - 10f(2x) + 16f(x)\| \leq \Phi(x,x,x), \tag{4.32}$$

where

$$\begin{aligned} \Phi(x,x,x) = \frac{1}{r^4-r^2} \big[&(5-4r^2)\phi(x,2x,-x) + 2r^2\phi(2x,2x,-x) + (4-2r^2)\phi(x,x,x) \\ &+ r^2\phi(2x,4x,-2x) + \phi(x,2x,x) + 2\phi((1+r)x,2x,-x) \\ &+ 2\phi((1-r)x,2x,-x) + \phi((1+2r)x,2x,-x) + \phi((1-2r)x,2x,-x) \big] \end{aligned}$$

for all$x \in U$. It is easy to see from (4.32)that

$$\|f(4x) - 2f(2x) - 8\{f(2x) - 2f(x)\}\| \leq \Phi(x,x,x) \tag{4.33}$$

for all$x \in U$. Define a mapping $f_3 : U \to V$by (See Theorem 2.3)

$$f_3(x) = f(2x) - 2f(x) \tag{4.34}$$

for all$x \in U$. Using (4.34)in (4.33), we obtain

$$\|f_3(2x) - 8f_3(x)\| \leq \Phi(x,x,x) \tag{4.35}$$

for all$x \in U$. From (4.35), we have

$$\left\|\frac{f_3(2x)}{8} - f_3(x)\right\| \leq \frac{\Phi(x,x,x)}{8} \tag{4.36}$$

for all $x \in U$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 4.5 concerning the stability of (1.7).

Corollary 4.5. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 3; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq 1; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & t \neq 1; \end{cases} \tag{4.37}$$

for all $x,y,z \in U$. Then there exists a unique cubic function $C : U \to V$ such that

$$\|f(2x) - 2f(x) - C(x)\| \leq \begin{cases} \frac{\kappa_5}{7}, \\ \frac{\kappa_6\|x\|^t}{|8-2^t|}, \\ \frac{\kappa_7\|x\|^{3t}}{|8-2^{3t}|}, \\ \frac{\kappa_8\|x\|^{3t}}{|8-2^{3t}|} \end{cases} \tag{4.38}$$

for all $x \in U$, where κ_i $(i = 5,6,7,8)$ are given in (4.27).

Theorem 4.6. *Assume $j = \pm 1$. Let $\phi : U^3 \to [0,\infty)$ be a function satisfying the conditions (4.1)and (4.28)for all $x,y,z \in U$. Suppose that an odd function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{4.39}$$

for all $x,y,z \in U$. Then there exists a unique additive function $A : U \to V$ and a unique cubic function $C : U \to V$ such that

$$\|f(x) - A(x) - C(x)\| \leq \frac{1}{6}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{2^{kj}} + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{8^{kj}}\right\} \tag{4.40}$$

for all $x \in U$,where $\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$, $A(x)$and $C(x)$ are defined in (4.5), (4.4)and (4.31), respectively for all $x \in U$.

Proof. By Theorems 4.4 and 4.5, there exists a unique additive function $A' : U \to V$ and a unique cubic function $C' : U \to V$ such that

$$\|f(2x) - 8f(x) - A'(x)\| \leq \frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{2^{kj}} \tag{4.41}$$

and

$$\left\| f(2x) - 2f(x) - C'(x) \right\| \leq \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{8^{kj}} \tag{4.42}$$

for all $x \in U$. Now from (4.41)and (4.42), that

$$\begin{aligned} \left\| f(x) + \tfrac{1}{6}A'(x) - \tfrac{1}{6}C'(x) \right\| &= \left\| \left\{ -\tfrac{f(2x)}{6} + \tfrac{8f(x)}{6} + \tfrac{A'(x)}{6} \right\} + \left\{ \tfrac{f(2x)}{6} - \tfrac{2f(x)}{6} - \tfrac{C'(x)}{6} \right\} \right\| \\ &\leq \tfrac{1}{6} \left\{ \| f(2x) - 8f(x) - A'(x) \| + \| f(2x) - 2f(x) - C'(x) \| \right\} \\ &\leq \tfrac{1}{6} \left\{ \tfrac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi(2^{kj}x, 2^{kj}x, 2^{kj}x)}{2^{kj}} + \tfrac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi(2^{kj}x, 2^{kj}x, 2^{kj}x)}{8^{kj}} \right\} \end{aligned}$$

for all $x \in U$. Thus, we obtain (4.40)by defining $A(x) = \frac{-1}{6}A'(x)$ and $C(x) = \frac{1}{6}C'(x)$, where $\Phi\left(2^{k}x, 2^{kj}x, 2^{kj}x\right)$,$A(x)$, and $C(x)$are defined in (4.5),(4.4)and (4.31)respectively for all $x \in U$. □

The following corollary is an immediate consequence of Theorem 4.6 concerning the stability of (1.7).

Corollary 4.6. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ satisfies the inequality*

$$\| Df(x,y,z) \| \leq \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t \right), & t \neq 1, 3; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t \right), & t \neq \frac{1}{3}, 1; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + \left(\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t} \right) \right), & t \neq \frac{1}{3}, 1; \end{cases} \tag{4.43}$$

for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$ and a unique cubic function $C : U \to V$ such that

$$\| f(x) - A(x) - C(x) \| \leq \begin{cases} \frac{4\kappa_5}{21}, \\ \frac{\kappa_6 \|x\|^t}{6} \left\{ \frac{1}{|2-2^t|} + \frac{1}{|8-2^t|} \right\}, \\ \frac{\kappa_7 \|x\|^{3t}}{6} \left\{ \frac{1}{|2-2^{3t}|} + \frac{1}{|8-2^{3t}|} \right\}, \\ \frac{\kappa_8 \|x\|^{3t}}{6} \left\{ \frac{1}{|2-2^{3t}|} + \frac{1}{|8-2^{3t}|} \right\}, \end{cases} \tag{4.44}$$

for all $x \in U$, where κ_i $(i = 5, 6, 7, 8)$ are given in (4.27).

5 STABILITY RESULTS: MIXED CASE

Theorem 5.1. *Let $\psi, \phi : U^3 \to [0, \infty)$ be a function that satisfies (3.1), (3.23), (4.1)and (4.28)for all $x, y, z \in U$. Suppose that a function $f : U \to V$ satisfies the inequalities (3.34)and (4.39)for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$, a unique quadratic function $Q_2 : U \to V$, a unique cubic function $C : U \to V$ and a unique quartic function $Q_4 : U \to V$ such that*

$$\begin{aligned} \| f(x) - A(x) - Q_2(x) - C(x) - Q_4(x) \| \leq \tfrac{1}{2} \Big\{ & \Psi_2\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) + \Psi_4\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) \\ & + \Phi_1\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) + \Phi_3\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) \Big\} \end{aligned} \tag{5.1}$$

for all $x \in U$, where $\Psi_2\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$, $\Psi_4\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$, $\Phi_1\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$ and $\Phi_3\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$ are defined by

$$\Psi_2\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) = \frac{1}{12} \left\{ \frac{1}{4} \left(\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{2^{kj}} + \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(-2^{kj}x, -2^{kj}x, -2^{kj}x\right)}{2^{kj}} \right) \right\} \tag{5.2}$$

$$\Psi_4\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) = \frac{1}{12} \left\{ \frac{1}{16} \left(\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{16^{kj}} + \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(-2^{kj}x, -2^{kj}x, -2^{kj}x\right)}{16^{kj}} \right) \right\} \tag{5.3}$$

$$\Phi_1\left(2^{kj}x,2^{kj}x,2^{kj}x\right)=\frac{1}{6}\left\{\frac{1}{2}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{2^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{2^{kj}}\right)\right\} \tag{5.4}$$

$$\Phi_3\left(2^{kj}x,2^{kj}x,2^{kj}x\right)=\frac{1}{6}\left\{\frac{1}{8}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{8^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{8^{kj}}\right)\right\} \tag{5.5}$$

respectively for all $x \in U$.

Proof. Let $f_e(x)=\frac{1}{2}\{f(x)+f(-x)\}$ for all $x\in U$. Then $f_e(0)=0, f_e(x)=f_e(-x)$. Hence

$$\begin{aligned}\|Df_e(x,y,z)\| &= \tfrac{1}{2}\{\|Df(x,y,z)+Df(-x,-y,-z)\|\}\\ &\le \tfrac{1}{2}\{\|Df(x,y,z)\|+\|Df(-x,-y,-z)\|\}\\ &\le \tfrac{1}{2}\{\psi(x,y,z)+\psi(-x,-y,-z)\}\end{aligned}$$

for all $x\in U$. Hence from Theorem 3.3, there exists a unique quadratic function $Q_2: U\to V$ and a unique quartic function $Q_4: U\to V$such that

$$\begin{aligned}\|f(x)-Q_2(x)-Q_4(x)\| &\le \tfrac{1}{2}\left\{\tfrac{1}{12}\left[\tfrac{1}{4}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{4^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{4^{kj}}\right)\right.\right.\\ &\quad\left.\left.+\tfrac{1}{16}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{16^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{16^{kj}}\right)\right]\right\}\\ &\le \tfrac{1}{2}\left\{\Psi_2\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Psi_4\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\right\},\end{aligned} \tag{5.6}$$

for all $x\in U$. Again $f_o(x)=\frac{1}{2}\{f(x)-f(-x)\}$ for all $x\in U$. Then $f_o(0)=0, f_o(x)=-f_o(-x)$. Hence

$$\begin{aligned}\|Df_o(x,y,z)\| &= \tfrac{1}{2}\{\|Df(x,y,z)-Df(-x,-y,-z)\|\}\\ &\le \tfrac{1}{2}\{\|Df(x,y,z)\|-\|Df(-x,-y,-z)\|\}\\ &\le \tfrac{1}{2}\{\phi(x,y,z)-\phi(-x,-y,-z)\}\end{aligned}$$

for all $x\in U$. Hence from Theorem 4.6, there exists a unique additive function $A: U\to V$ and a unique cubic function $C: U\to V$such that

$$\begin{aligned}\|f(x)-A(x)-C(x)\| &\le \tfrac{1}{2}\left\{\tfrac{1}{6}\left[\tfrac{1}{2}\left(\sum_{k=0}^{\infty}\tfrac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{2^{kj}}+\sum_{k=0}^{\infty}\tfrac{\Phi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{2^{kj}}\right)\right.\right.\\ &\quad\left.\left.+\tfrac{1}{8}\left(\sum_{k=0}^{\infty}\tfrac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{8^{kj}}+\sum_{k=0}^{\infty}\tfrac{\Phi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{8^{kj}}\right)\right]\right\}\\ &\le \tfrac{1}{2}\left\{\Phi_1\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Phi_3\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\right\},\end{aligned} \tag{5.7}$$

for all $x\in U$. Since $f(x)=f_e(x)+f_o(x)$ then it follows from (5.6)and (5.7)that

$$\begin{aligned}\|f(x)-A(x)-Q_2(x)-C(x)-Q_4(x)\| &= \|\{f_e(x)-Q_2(x)-Q_4(x)\}+\{f_o(x)-A(x)-C(x)\}\|\\ &\le \|f_e(x)-Q_2(x)-Q_4(x)\|+\|f_o(x)-A(x)-C(x)\|\\ &\le \tfrac{1}{2}\left\{\Psi_2\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Psi_4\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\right.\\ &\quad\left.+\Phi_1\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Phi_3\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\right\}\end{aligned}$$

for all $x\in U$. Hence the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 5.1 concerning the stability of (1.7).

Corollary 5.1. *Let ρ, t be nonnegative real numbers. Suppose that a function $f: U\to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \begin{cases}\rho, & \\ \rho\left(\|x\|^t+\|y\|^t+\|z\|^t\right), & t\neq 1,2,3,4;\\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right), & t\neq \frac{1}{3},\frac{2}{3},\frac{4}{3},1;\\ \rho\left(\|x\|^t\|y\|^t\|z\|^t+(\|x\|^{3t}+\|y\|^{3t}+\|z\|^{3t})\right), & t\neq\frac{1}{3},\frac{2}{3},\frac{4}{3},1;\end{cases} \tag{5.8}$$

for all $x, y, z \in U$. *Then there exists a unique additive function* $A : U \to V$, *a unique quadratic function* $Q_2 : U \to V$, *a unique cubic function* $C : U \to V$*and a unique quartic function* $Q_4 : U \to V$ *such that*

$$\|f(x) - A(x) - Q_2(x) - C(x) - Q_4(x)\|$$
$$\leq \begin{cases} \frac{1}{2}\left[2\kappa_1 + \frac{8\kappa_5}{21}\right], \\ \frac{1}{2}\left[\frac{\kappa_2}{6}\left\{\frac{1}{4-2^t} + \frac{1}{16-2^t}\right\} + \frac{\kappa_6}{3}\left\{\frac{1}{2-2^t} + \frac{1}{8-2^t}\right\}\right]\|x\|^t, \\ \frac{1}{2}\left[\frac{\kappa_3}{6}\left\{\frac{1}{4-2^{3t}} + \frac{1}{16-2^{3t}}\right\} + \frac{\kappa_7}{3}\left\{\frac{1}{2-2^{3t}} + \frac{1}{8-2^{3t}}\right\}\right]\|x\|^{3t}, \\ \frac{1}{2}\left[\frac{\kappa_4}{6}\left\{\frac{1}{4-2^{3t}} + \frac{1}{16-2^{3t}}\right\} + \frac{\kappa_8}{3}\left\{\frac{1}{2-2^{3t}} + \frac{1}{8-2^{3t}}\right\}\right]\|x\|^{3t} \end{cases} \tag{5.9}$$

for all $x \in U$, *where* κ_i $(i = 1, 2, ..., 8)$ *are respectively, given in* (3.26)*and* (4.27).

6 STABILITY RESULTS FIXED POINT METHOD: EVEN CASE

Theorem 6.1. *Let* $f : U \to V$ *be an even function for which there exists a function* $\psi : U^3 \to [0, \infty)$*with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^{2n}} \psi(\mu_i^n x, \mu_i^n y, \mu_i^n z) = 0 \tag{6.1}$$

with

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x, y, z)\| \leq \psi(x, y, z) \tag{6.2}$$

for all $x, y, z \in U$. *If there exists* $L = L(i) < 1$*such that the function*

$$x \to \Gamma(x, x, x) = \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

has the property

$$\Gamma(x, x, x) = \frac{L}{\mu_i^2} \Gamma(\mu_i x, \mu_i x, \mu_i x) \tag{6.3}$$

for all $x \in U$. *Then there exists a unique quadratic function* $Q_2 : U \to V$*satisfying* (1.7) *and*

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1 - L} \Gamma(x, x, x) \tag{6.4}$$

for all $x \in U$.

Proof. consider the set $X = \{f/f : U \to V, f(0) = 0\}$and introduce the generalized metric on X.

$$d(f, h) = \inf\{M \in (0, \infty) : \|f(x) - h(x)\| \leq M\gamma(x), x \in U\}.$$

It is easy to see that (X, d) is complete. Define $T : X \to X$by $Tf(x) = \frac{1}{\mu_i^2} f(\mu_i x)$
for all $x \in U$. Now for all $f, h \in X$,

$$\begin{aligned} d(f, h) \leq M &= \|f(x) - h(x)\| \leq M\Gamma(x, x, x), x \in U \\ &= \left\|\frac{1}{\mu_i^2} f(\mu_i x) - \frac{1}{\mu_i^2} h(\mu_i x)\right\| \leq \frac{1}{\mu_i^2} M\Gamma(\mu_i x, \mu_i x, \mu_i x), x \in U \\ &= \left\|\frac{1}{\mu_i^2} f(\mu_i x) - \frac{1}{\mu_i^2} h(\mu_i x)\right\| \leq LM\Gamma(x, x, x), x \in U \\ &= \|Tf(x) - Th(x)\| \leq LM\Gamma(x, x, x), x \in U \\ &= d(Tf, Th) \leq LM. \end{aligned}$$

This gives $d(Tf, Th) \leq Ld(f, h)$, for all $f, h \in X$.
i.e., $T$is a strictly contractive mapping on $X$with Lipschitz constant L.

Now, from (3.15)we have

$$\|f_2(2x) - 4f_2(x)\| \le \Psi(x,x,x) \tag{6.5}$$

where $f_2(x) = f(2x) - 16f(x)$ and

$$\Psi(x,x,x) = \frac{1}{r^4 - r^2}\left[12\left(1-r^2\right)\psi(0,x,0) + 12r^2\psi(x,2x,-x) + 6\psi(0,2x,0) + 12\psi(rx,2x,-x)\right]$$

for all $x \in U$. From (6.5), we arrive

$$\left\|\frac{f_2(2x)}{4} - f_2(x)\right\| \le \frac{1}{4}\Psi(x,x,x) = \frac{1}{2^2}\Psi(x,x,x) \tag{6.6}$$

for all $x \in U$. Using (6.3)for the case $i = 0$, it reduces to
$\left\|\frac{f_2(2x)}{4} - f_2(x)\right\| \le L\Gamma(x,x,x)$, for all $x \in U$.
i.e., $d(Tf_2, f_2) \le L \Rightarrow d(Tf_2, f_2) \le L^{1-0} = L^{1-i} < \infty$
Again replacing $x = \frac{x}{2}$, in (6.5), we obtain

$$\left\|f_2(x) - 4f_2\left(\frac{x}{2}\right)\right\| \le \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in U$. Using (6.3)for the case $i = 1$, it reduces to
$\left\|f_2(x) - 4f_2\left(\frac{x}{2}\right)\right\| \le \Gamma(x,x,x)$ for all $x \in U$,
i.e., $d(Tf_2, f_2) \le 1 \Rightarrow d(Tf_2, f_2) \le 1 = L^{1-1} = L^{1-i} < \infty$.
From above two cases, we arrive

$$d(Tf_2, f_2) \le L^{1-i}$$

Therefore $(B_2(i))$ holds.
By$(B_2(ii))$, it follows that there exists a fixed point $Q_2$of Tin X such that

$$Q_2(x) = \lim_{n\to\infty} \frac{1}{\mu_i^{2n}}\left(f(\mu_i^{n+1}x) - 16f(\mu_i^n x)\right) \tag{6.7}$$

for all $x \in U$. In order to prove $Q_2 : U \to V$is quadratic. Replacing (x,y,z) by $(\mu_i^n x, \mu_i^n y, \mu_i^n z)$in (6.2)and divided by μ_i^{2n}, it follows from (6.1)and (6.7), Q_2satisfies (1.7) for all $x,y,z \in U$. i.e., Q_2satisfies the functional equation (1.7) for all $x,y,z \in U$. By $(B_2(iii))$, $Q_2$is the unique fixed point of Tin the set $Y = \{f \in X : d(Tf, Q_2) < \infty\}$, using the fixed point alternative result $Q_2$is the unique function such that
$\|f_2(x) - Q_2(x)\| \le M\Gamma(x,x,x)$ for all $x \in U$and $M > 0$.
Finally, by $(B_2(iv))$, we obtain

$$d(f_2, Q_2) \le \frac{1}{1-L}d(Tf_2, f_2).$$

This implies

$$d(f_2, Q_2) \le \frac{L^{1-i}}{1-L}.$$

Hence we conclude that

$$\|f(2x) - 16f(x) - Q_2(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x)$$

for all $x \in U$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 6.1 concerning the stability of (1.7).

Corollary 6.1. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \ne 2; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \ne \frac{2}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) & t \ne \frac{2}{3}; \end{cases} \tag{6.8}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ such that

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \begin{cases} 10\kappa_1, \\ \frac{\kappa_2\|x\|^t}{|4-2^t|}, \\ \frac{\kappa_3\|x\|^{3t}}{|4-2^{3t}|}, \\ \frac{\kappa_4\|x\|^{3t}}{|4-2^{3t}|} \end{cases} \tag{6.9}$$

where

$$\begin{aligned} \kappa_1 &= \frac{\rho}{r^4-r^2}, \\ \kappa_2 &= \frac{\rho\left[24+12r^2+12r^2 2^t+12r^t+18\cdot 2^t\right]}{r^4-r^2}, \\ \kappa_3 &= \frac{12\rho 2^t\left[r^2+r^t\right]}{r^4-r^2}, \\ \kappa_4 &= \frac{\rho\left[24+12r^2\left(1+2^t+2^{3t}\right)+18\cdot 2^{3t}+12\cdot r^t\cdot 2^t+12\cdot r^{3t}\right]}{r^4-r^2} \end{aligned} \tag{6.10}$$

for all $x \in U$.

Proof. Setting

$$\psi(x,y,z) = \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) \end{cases}$$

for all $x, y, z \in U$. Now $\frac{\Psi(\mu_i^n x, \mu_i^n y, \mu_i^n z)}{\mu_i^{2n}} = \begin{cases} \frac{\rho}{\mu_i^{2n}}, \\ \frac{\rho}{\mu_i^{2n}}\left(\|\mu_i^n x\|^t + \|\mu_i^n y\|^t + \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{2n}}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{2n}}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t + (\|\mu_i^n x\|^{3t} + \|\mu_i^n y\|^{3t} + \|\mu_i^n z\|^{3t})\right) \end{cases} =$

$$\begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

i.e., (6.1)is holds. But we have

$$\Gamma(x,x,x) = \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) = \begin{cases} 30\kappa_1, \\ \frac{\kappa_2\|x\|^t}{2^t}, \\ \frac{\kappa_3\|x\|^{3t}}{2^{3t}}, \\ \frac{\kappa_4\|x\|^{3t}}{2^{3t}}. \end{cases}$$

Also,

$$\frac{1}{\mu_i^2}\Gamma(x,x,x) = \begin{cases} \frac{30\kappa_1}{\mu_i^2}, \\ \frac{\kappa_2\|\mu_i x\|^t}{\mu_i^2 2^t}, \\ \frac{\kappa_3\|\mu_i x\|^{3t}}{\mu_i^2 2^{3t}}, \\ \frac{\kappa_4\|\mu_i x\|^{3t}}{\mu_i^2 2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-2} 30\kappa_1, \\ \mu_i^{t-2}\frac{\kappa_2\|x\|^t}{2^t}, \\ \mu_i^{3t-2}\frac{\kappa_3\|\mu_i x\|^{3t}}{2^{3t}}, \\ \mu_i^{3t-2}\frac{\kappa_4\|\mu_i x\|^{3t}}{2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-2}\Gamma(x,x,x), \\ \mu_i^{t-2}\Gamma(x,x,x), \\ \mu_i^{3t-2}\Gamma(x,x,x), \\ \mu_i^{3t-2}\Gamma(x,x,x). \end{cases}$$

Hence the inequality (6.3)holds either, $L = 2^{-2}$ if $i = 0$and $L = 2^2$ if $i = 1$. Now from (6.4), we prove the following cases for condition (i).

Case: 1 $L = 2^{-2}$ if $i = 0$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{-2})^{1-0}}{1-2^{-2}}30\kappa_1 = 10\kappa_1$$

Case: 2 $L = 2^2$ if $i = 1$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{2})^{1-1}}{1-2^{2}}30\kappa_1 = -10\kappa_1$$

Also, (6.3)holds either,$L = 2^{t-2}$for $t < 2$if $i = 0$and $L = \frac{1}{2^{t-2}}$for $t > 2$if $i = 1$. Now from (6.4), we prove the following cases for condition (ii).
Case: 1 $L = 2^{t-2}$for $t < 2$ if $i = 0$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{t-2})^{1-0}}{1-2^{t-2}}\frac{\kappa_2\|x\|^t}{2^t} = \frac{\kappa_2\|x\|^t}{4-2^t}.$$

Case: 2 $L = \frac{1}{2^{t-2}}$for $t > 2$ if $i = 1$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{t-2}}\right)^{1-1}}{1-\frac{1}{2^{t-2}}}\frac{\kappa_2\|x\|^t}{2^t} = \frac{\kappa_2\|x\|^t}{2^t-4}.$$

Also, (6.3)holds either,$L = 2^{3t-2}$for $3t < 2$if $i = 0$and $L = \frac{1}{2^{3t-2}}$for $3t > 2$if $i = 1$. Now from (6.4), we prove the following cases for condition (iii).
Case: 1 $L = 2^{3t-2}$for $3t < 2$if $i = 0$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-2})^{1-0}}{1-2^{3t-2}}\frac{\kappa_3\|x\|^{3t}}{2^{3t}} = \frac{\kappa_3\|x\|^{3t}}{4-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-2}}$for $3t > 2$ if $i = 1$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-2}}\right)^{1-1}}{1-\frac{1}{2^{3t-2}}}\frac{\kappa_3\|x\|^{3t}}{2^{3t}} = \frac{\kappa_3\|x\|^{3t}}{2^{3t}-4}.$$

Finally, (6.3)holds either,$L = 2^{3t-2}$for $3t < 2$if $i = 0$and $L = \frac{1}{2^{3t-2}}$for $3t > 2$if $i = 1$. Now from (6.4), we prove the following cases for condition (iv).
Case: 1 $L = 2^{3t-2}$for $3t < 2$if $i = 0$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-2})^{1-0}}{1-2^{3t-2}}\frac{\kappa_4\|x\|^{3t}}{2^{3t}} = \frac{\kappa_4\|x\|^{3t}}{4-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-2}}$for $3t > 2$if $i = 1$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-2}}\right)^{1-1}}{1-\frac{1}{2^{3t-2}}}\frac{\kappa_4\|x\|^{3t}}{2^{3t}} = \frac{\kappa_4\|x\|^{3t}}{2^{3t}-4}.$$

Hence the proof of the corollary is complete. □

Theorem 6.2. *Let $f : U \to V$ be an even function for which there exists a function $\psi : U^3 \to [0,\infty)$with the condition*

$$\lim_{n\to\infty}\frac{1}{\mu_i^{4n}}\psi(\mu_i^n x, \mu_i^n y, \mu_i^n z) = 0 \tag{6.11}$$

with

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{6.12}$$

for all $x, y, z \in U$. *If there exists* $L = L(i) < 1$*such that the function*

$$x \to \Gamma(x, x, x) = \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

has the property

$$\Gamma(x, x, x) = \frac{L}{\mu_i^4}\Gamma(\mu_i x, \mu_i x, \mu_i x) \tag{6.13}$$

for all $x \in U$. *Then there exists a unique quartic function* $Q_4 : U \to V$*satisfying* (1.7) *and*

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x, x, x) \tag{6.14}$$

for all $x \in U$.

Proof. consider the set $X = \{f/f : U \to V, f(0) = 0\}$and introduce the generalized metric on X.

$$d(f, h) = \inf\{M \in (0, \infty) : \|f(x) - h(x)\| \leq M\gamma(x), x \in U\}.$$

It is easy to see that (X, d) is complete. Define $T : X \to X$by $Tf(x) = \frac{1}{\mu_i^4}f(\mu_i x)$
for all $x \in U$. Now for all $f, h \in X$,

$$\begin{aligned} d(f, h) \leq M = \|f(x) - h(x)\| &\leq M\Gamma(x, x, x), x \in U \\ = \left\|\frac{1}{\mu_i^4}f(\mu_i x) - \frac{1}{\mu_i^4}h(\mu_i x)\right\| &\leq \frac{1}{\mu_i^4}M\Gamma(\mu_i x, \mu_i x, \mu_i x), x \in U \\ = \left\|\frac{1}{\mu_i^4}f(\mu_i x) - \frac{1}{\mu_i^4}h(\mu_i x)\right\| &\leq LM\Gamma(x, x, x), x \in U \\ = \|Tf(x) - Th(x)\| &\leq LM\Gamma(x, x, x), x \in U \Rightarrow d(Tf, Th) \leq LM. \end{aligned}$$

This gives $d(Tf, Th) \leq Ld(f, h)$, for all $f, h \in X$.
i.e., $T$is a strictly contractive mapping on $X$with Lipschitz constant L.
Now, from (3.30)we have

$$\|f_4(2x) - 16f_4(x)\| \leq \Psi(x, x, x) \tag{6.15}$$

where $f_4(x) = f(2x) - 4f(x)$ and

$$\Psi(x, x, x) = \frac{1}{r^4 - r^2}\left[12\left(1 - r^2\right)\psi(0, x, 0) + 12r^2\psi(x, 2x, -x) + 6\psi(0, 2x, 0) + 12\psi(rx, 2x, -x)\right]$$

for all $x \in U$. From (6.15), we arrive

$$\left\|\frac{f_4(2x)}{16} - f_4(x)\right\| \leq \frac{1}{16}\Psi(x, x, x) = \frac{1}{2^4}\Psi(x, x, x) \tag{6.16}$$

for all $x \in U$. Using (6.13)for the case $i = 0$, it reduces to
$\left\|\frac{f_4(2x)}{16} - f_4(x)\right\| \leq L\Gamma(x, x, x)$, for all $x \in U$.
i.e., $d(Tf_4, f_4) \leq L \Rightarrow d(Tf_4, f_4) \leq L^{1-0} = L^{1-i} < \infty$
Again replacing $x = \frac{x}{2}$, in (6.15), we obtain

$$\left\|f_4(x) - 16f_4\left(\frac{x}{2}\right)\right\| \leq \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in U$. Using (6.13)for the case $i = 1$, it reduces to
$\left\|f_4(x) - 16f_4\left(\frac{x}{2}\right)\right\| \leq \Gamma(x, x, x)$ for all $x \in U$,
i.e., $d(Tf_4, f_4) \leq 1 \Rightarrow d(Tf_4, f_4) \leq 1 = L^{1-1} = L^{1-i} < \infty$.
From above two cases, we arrive

$$d(Tf_4, f_4) \leq L^{1-i}$$

Therefore $(B_2(i))$ holds.

By$(B_2(ii))$, it follows that there exists a fixed point $Q_4$of Tin X such that

$$Q_4\left(x\right)=\lim_{n\to\infty}\frac{1}{\mu_i^{4n}}\left(f(\mu_i^{n+1}x)-4f(\mu_i^nx)\right) \tag{6.17}$$

for all $x\in U$. In order to prove $Q_4:U\to V$is quartic. Replacing $(x,y,z)$by $\left(\mu_i^nx,\mu_i^ny,\mu_i^nz\right)$in (6.12)and divided by μ_i^{4n}. It follows from (6.11)and (6.17), Q_4satisfies (1.7) for all $x,y,z\in U$. i.e., Q_4satisfies the functional equation (1.7) $x,y,z\in U$. By $(B_2(iii))$, $Q_4$is the unique fixed point of Tin the set $Y=\{f\in X:d(Tf,Q_4)<\infty\}$, using the fixed point alternative result $Q_2$is the unique function such that
$\|f_4\left(x\right)-Q_4\left(x\right)\|\leq M\Gamma\left(x,x,x\right)$ for all $x\in U$and $M>0$.
Finally, by $(B_2(iv))$, we obtain

$$d\left(f_4,Q_4\right)\leq\frac{1}{1-L}d\left(Tf_4,f_4\right)$$

This implies

$$d\left(f_4,Q_4\right)\leq\frac{L^{1-i}}{1-L}$$

Hence we conclude that

$$\|f\left(2x\right)-4f\left(x\right)-Q_4\left(x\right)\|\leq\frac{L^{1-i}}{1-L}\Gamma\left(x,x,x\right)$$

for all $x\in U$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 6.2 concerning the stability of (1.7).

Corollary 6.2. *Let ρ,t be nonnegative real numbers. Suppose that an even function $f:U\to V$ satisfies the inequality*

$$\|Df\left(x,y,z\right)\|\leq\begin{cases}\rho,\\ \rho\left(\|x\|^t+\|y\|^t+\|z\|^t\right), & t\neq4;\\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right), & t\neq\frac{4}{3};\\ \rho\left(\|x\|^t\|y\|^t\|z\|^t+(\|x\|^{3t}+\|y\|^{3t}+\|z\|^{3t})\right) & t\neq\frac{4}{3};\end{cases} \tag{6.18}$$

for all $x,y,z\in U$. Then there exists a unique quartic function $Q_4:U\to V$ such that

$$\|f\left(2x\right)-4f\left(x\right)-Q_4\left(x\right)\|\leq\begin{cases}2\kappa_1,\\ \frac{\kappa_2\|x\|^t}{|16-2^t|},\\ \frac{\kappa_3\|x\|^{3t}}{|16-2^{3t}|},\\ \frac{\kappa_4\|x\|^{3t}}{|16-2^{3t}|}\end{cases} \tag{6.19}$$

for all $x\in U$, where κ_i $(i=1,2,3,4)$ are defined in (6.10).

Proof. Setting

$$\psi\left(x,y,z\right)=\begin{cases}\rho,\\ \rho\left(\|x\|^t+\|y\|^t+\|z\|^t\right),\\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right),\\ \rho\left(\|x\|^t\|y\|^t\|z\|^t+(\|x\|^{3t}+\|y\|^{3t}+\|z\|^{3t})\right)\end{cases}$$

for all $x,y,z\in U$. Now $\frac{\psi(\mu_i^nx,\mu_i^ny,\mu_i^nz)}{\mu_i^{4n}}=\begin{cases}\frac{\rho}{\mu_i^{4n}},\\ \frac{\rho}{\mu_i^{4n}}\left(\|\mu_i^nx\|^t+\|\mu_i^ny\|^t+\|\mu_i^nz\|^t\right),\\ \frac{\rho}{\mu_i^{4n}}\left(\|\mu_i^nx\|^t\|\mu_i^ny\|^t\|\mu_i^nz\|^t\right),\\ \frac{\rho}{\mu_i^{4n}}\left(\|\mu_i^nx\|^t\|\mu_i^ny\|^t\|\mu_i^nz\|^t+(\|\mu_i^nx\|^{3t}+\|\mu_i^ny\|^{3t}+\|\mu_i^nz\|^{3t})\right)\end{cases}=$

$$\begin{cases} \to 0 \, as \, n \to \infty, \\ \to 0 \, as \, n \to \infty, \\ \to 0 \, as \, n \to \infty, \\ \to 0 \, as \, n \to \infty. \end{cases}$$ i.e., (6.11)is holds. But we have

$$\Gamma(x,x,x) = \Psi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right) = \begin{cases} 30\kappa_1, \\ \frac{\kappa_2\|x\|^t}{2^t}, \\ \frac{\kappa_3\|x\|^{3t}}{2^{3t}}, \\ \frac{\kappa_4\|x\|^{3t}}{2^{3t}}. \end{cases}$$

Also,

$$\frac{1}{\mu_i^4}\Gamma(x,x,x) = \begin{cases} \frac{30\kappa_1}{\mu_i^4}, \\ \frac{\kappa_2\|\mu_i x\|^t}{\mu_i^4 2^t}, \\ \frac{\kappa_3\|\mu_i x\|^{3t}}{\mu_i^4 2^{3t}}, \\ \frac{\kappa_4\|\mu_i x\|^{3t}}{\mu_i^4 2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-4} 30\kappa_1, \\ \mu_i^{t-4}\frac{\kappa_2\|x\|^t}{2^t}, \\ \mu_i^{3t-4}\frac{\kappa_3\|\mu_i x\|^{3t}}{2^{3t}}, \\ \mu_i^{3t-4}\frac{\kappa_4\|\mu_i x\|^{3t}}{2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-4}\Gamma(x,x,x), \\ \mu_i^{t-4}\Gamma(x,x,x), \\ \mu_i^{3t-4}\Gamma(x,x,x), \\ \mu_i^{3t-4}\Gamma(x,x,x). \end{cases}$$

Hence the inequality (6.13)holds either, $L = 2^{-4}$ if $i = 0$and $L = 2^4$ if $i = 1$. Now from (6.14), we prove the following cases for condition (i).

Case: 1 $L = 2^{-4}$ if $i = 0$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{-4})^{1-0}}{1-2^{-4}}30\kappa_1 = 2\kappa_1$$

Case: 2 $L = 2^4$ if $i = 1$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^4)^{1-1}}{1-2^4}30\kappa_1 = -2\kappa_1$$

Also, (6.13)holds either,$L = 2^{t-4}$for $t < 4$if $i = 0$and $L = \frac{1}{2^{t-4}}$for $t > 4$if $i = 1$. Now from (6.14), we prove the following cases for condition (ii).

Case: 1 $L = 2^{t-4}$ for $t < 4$ if $i = 0$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{t-4})^{1-0}}{1-2^{t-4}}\frac{\kappa_2\|x\|^t}{2^t} = \frac{\kappa_2\|x\|^t}{16-2^t}.$$

Case: 2 $L = \frac{1}{2^{t-4}}$ for $t > 4$ if $i = 1$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{t-4}}\right)^{1-1}}{1-\frac{1}{2^{t-4}}}\frac{\kappa_2\|x\|^t}{2^t} = \frac{\kappa_2\|x\|^t}{2^t-16}.$$

Also, (6.13)holds either,$L = 2^{3t-4}$for $3t < 4$if $i = 0$and $L = \frac{1}{2^{3t-4}}$for $3t > 4$if $i = 1$. Now from (6.14), we prove the following cases for condition (iii).

Case:1 $L = 2^{3t-4}$for $3t < 4$if $i = 0$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-4})^{1-0}}{1-2^{3t-4}}\frac{\kappa_3\|x\|^{3t}}{2^{3t}} = \frac{\kappa_3\|x\|^{3t}}{16-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-4}}$for $3t > 4$if $i = 1$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-4}}\right)^{1-1}}{1-\frac{1}{2^{3t-4}}}\frac{\kappa_3\|x\|^{3t}}{2^{3t}} = \frac{\kappa_3\|x\|^{3t}}{2^{3t}-16}.$$

Finally, (6.13)holds either,$L = 2^{3t-4}$for $3t < 4$if $i = 0$and $L = \frac{1}{2^{3t-4}}$for $3t > 4$if $i = 1$. Now from (6.14), we prove the following cases for condition (iv).

Case:1 $L = 2^{3t-4}$for $3t < 4$if $i = 0$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-4})^{1-0}}{1-2^{3t-4}} \frac{\kappa_4 \|x\|^{3t}}{2^{3t}} = \frac{\kappa_4 \|x\|^{3t}}{16-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-4}}$for $3t > 4$if $i = 1$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-4}}\right)^{1-1}}{1-\frac{1}{2^{3t-4}}} \frac{\kappa_4 \|x\|^{3t}}{2^{3t}} = \frac{\kappa_4 \|x\|^{3t}}{2^{3t}-16}.$$

Hence the proof of the corollary is complete. □

Theorem 6.3. *Let $f : U \to V$ be an even function for which there exists a function $\psi : U^3 \to [0,\infty)$with the condition (6.1)and (6.11) with*

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \le \psi(x,y,z) \tag{6.20}$$

for all $x,y,z \in U$. If there exists $L = L(i) < 1$such that the function

$$x \to \Gamma(x,x,x) = \Psi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right),$$

has the property (6.3)and (6.13), then there exists a unique quadratic function $Q_2 : U \to V$and a unique quartic function $Q_4 : U \to V$satisfying (1.7) and

$$\|f(x) - Q_2(x) - Q_4(x)\| \le \frac{1}{6}\frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.21}$$

for all $x \in U$, where $Q_2(x)$ and $Q_4(x)$ are defined in (6.7)and (6.17) respectively for all $x \in U$.

Proof. By Theorems 6.1 and 6.2, there exists a unique quadratic function $Q_2' : U \to V$ and a unique quartic function $Q_4' : U \to V$ such that

$$\left\|f(2x) - 16f(x) - Q_2'(x)\right\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.22}$$

and

$$\left\|f(2x) - 4f(x) - Q_4'(x)\right\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.23}$$

for all $x \in U$. Now from (6.22)and (6.23), that

$$\begin{aligned} \left\|f(x) + \tfrac{1}{12}Q_2'(x) - \tfrac{1}{12}Q_4'(x)\right\| &= \left\|\left\{-\tfrac{f(2x)}{12} + \tfrac{16f(x)}{12} + \tfrac{Q_2'(x)}{12}\right\} + \left\{\tfrac{f(2x)}{12} - \tfrac{4f(x)}{12} - \tfrac{Q_4'(x)}{12}\right\}\right\| \\ &\le \tfrac{1}{12}\left\{\left\|f(2x) - 16f(x) - Q_2'(x)\right\| + \left\|f(2x) - 4f(x) - Q_4'(x)\right\|\right\} \quad \text{for all } x \in U. \\ &\le \tfrac{1}{12}\left\{\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x) + \tfrac{L^{1-i}}{1-L}\Gamma(x,x,x)\right\} \\ &\le \tfrac{1}{6}\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x) \end{aligned}$$

Thus, we obtain (6.21)by defining $Q_2(x) = \frac{-1}{12}Q_2'(x)$ and $Q_4(x) = \frac{1}{12}Q_4'(x)$, where $Q_2(x)$ and $Q_4(x)$ are defined in (6.7)and (6.17) respectively for all $x \in U$. □

The following corollary is an immediate consequence of Theorem 6.3 concerning the stability of (1.7).

Corollary 6.3. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \ne 2,4; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \ne \frac{2}{3},\frac{4}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & t \ne \frac{2}{3},\frac{4}{3}; \end{cases} \tag{6.24}$$

for all $x,y,z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ and a unique quartic function $Q_4 : U \to V$ such that

$$\|f(x) - Q_2(x) - Q_4(x)\| \leq \begin{cases} 2\kappa_1, \\ \frac{\kappa_2\|x\|^t}{6}\left\{\frac{1}{|4-2^t|} + \frac{1}{|16-2^t|}\right\}, \\ \frac{\kappa_3\|x\|^{3t}}{6}\left\{\frac{1}{|4-2^{3t}|} + \frac{1}{|16-2^{3t}|}\right\}, \\ \frac{\kappa_4\|x\|^{3t}}{6}\left\{\frac{1}{|4-2^{3t}|} + \frac{1}{|16-2^{3t}|}\right\} \end{cases} \tag{6.25}$$

for all $x \in U$, where κ_i $(i = 1,2,3,4)$ are given in (6.10).

STABILITY RESULTS FIXED POINT METHOD: ODD CASE

Theorem 6.1. *Let $f : U \to V$ be an odd function for which there exists a function $\psi : U^3 \to [0,\infty)$with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^n}\phi(\mu_i^n x, \mu_i^n y, \mu_i^n z) = 0 \tag{6.1}$$

with

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{6.2}$$

for all $x,y,z \in U$. If there exists $L = L(i) < 1$such that the function

$$x \to \Gamma(x,x,x) = \Phi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right),$$

has the property

$$\Gamma(x,x,x) = \frac{L}{\mu_i}\Gamma(\mu_i x, \mu_i x, \mu_i x) \tag{6.3}$$

Then there exists a unique additive function $A : U \to V$satisfying (1.7) and

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.4}$$

for all $x \in U$.

Proof. consider the set $X = \{f/f : U \to V, f(0) = 0\}$and introduce the generalized metric on X.

$$d(f,h) = \inf\{M \in (0,\infty) : \|f(x) - h(x)\| \leq M\Gamma(x,x,x), x \in U\}$$

It is easy to see that $(X,d)$is complete. Define $T : X \to X$by $Tf(x) = \frac{1}{\mu_i}f(\mu_i x)$
for all $x \in U$.Now for all $f,h \in X$,

$$\begin{aligned} d(f,h) \leq M &= \|f(x) - h(x)\| \leq M\Gamma(x,x,x), x \in U \\ &= \left\|\frac{1}{\mu_i}f(\mu_i x) - \frac{1}{\mu_i}h(\mu_i x)\right\| \leq \frac{1}{\mu_i}M\Gamma(\mu_i x,\mu_i x,\mu_i x), x \in U \\ &= \left\|\frac{1}{\mu_i}f(\mu_i x) - \frac{1}{\mu_i}h(\mu_i x)\right\| \leq LM\Gamma(x,x,x), x \in U \\ &= \|Tf(x) - Th(x)\| \leq LM\Gamma(x,x,x), x \in U \\ &= d(Tf,Th) \leq LM. \end{aligned}$$

This gives $d(Tf,Th) \leq Ld(f,h)$, for all $f,h \in X$,
i.e., $T$is a strictly contractive mapping on $X$with Lipschitz constant L.
Now, from (4.24)we have

$$\|f_1(2x) - 2f_1(x)\| \leq \Phi(x,x,x) \tag{6.5}$$

where $f_1(x) = f(2x) - 8f(x)$

$$\begin{aligned}\Phi(x,x,x) = \tfrac{1}{r^4-r^2}&\left[\left(5-4r^2\right)\phi(x,2x,-x) + 2r^2\phi(2x,2x,-x) + \left(4-2r^2\right)\phi(x,x,x)\right.\\ &+r^2\phi(2x,4x,-2x) + \phi(x,2x,x) + 2\phi((1+r)x,2x,-x)\\ &\left.+2\phi((1-r)x,2x,-x) + \phi((1+2r)x,2x,-x) + \phi((1-2r)x,2x,-x)\right]\end{aligned}$$

for all $x \in U$. From (6.5), we arrive

$$\left\|\frac{f_1(2x)}{2} - f_1(x)\right\| \leq \frac{1}{2}\Phi(x,x,x) = \frac{1}{2}\Phi(x,x,x) \tag{6.6}$$

for all $x \in U$. Using (6.3)for the case $i = 0$, it reduces to
$\left\|\frac{f_1(2x)}{2} - f_1(x)\right\| \leq L\Gamma(x,x,x)$, for all $x \in U$.
i.e., $d(Tf_1, f_1) \leq L \Rightarrow d(Tf_1, f_1) \leq L^{1-0} = L^{1-i} < \infty$
Again replacing $x = \frac{x}{2}$, in (6.5), we obtain

$$\left\|f_1(x) - 2f_1\left(\frac{x}{2}\right)\right\| \leq \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in U$. Using (6.3)for the case $i = 1$, it reduces to
$\left\|f_1(x) - 2f_1\left(\frac{x}{2}\right)\right\| \leq \Gamma(x,x,x)$ for all $x \in U$,
i.e., $d(Tf_1, f_1) \leq 1 \Rightarrow d(Tf_1, f_1) \leq 1 = L^{1-1} = L^{1-i} < \infty$.
From above two cases, we arrive

$$d(Tf_1, f_1) \leq L^{1-i}$$

Therefore $(B_2(i))$ holds.
By$(B_2(ii))$, it follows that there exists a fixed point $A$of Tin X such that

$$A(x) = \lim_{n\to\infty} \frac{1}{\mu_i^n}\left(f(\mu_i^{n+1}x) - 8f(\mu_i^n x)\right) \tag{6.7}$$

for all $x \in U$. In order to prove $A: U \to V$is additive. Replacing $(x,y,z)$by $\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right)$in (6.2)and divide by μ_i^n. It follows from (6.1)and (6.7), Asatisfies (1.7) for all $x,y,z \in U$. i.e., Asatisfies the functional equation (1.7).
By $(B_2(iii))$, $A$is the unique fixed point of Tin the set $Y = \{f \in X : d(Tf, A) < \infty\}$, using the fixed point alternative result $A$is the unique function such that
$\|f_1(x) - A(x)\| \leq M\Gamma(x,x,x)$ for all $x \in U$and $M > 0$.
Finally, by $(B_2(iv))$, we obtain

$$d(f_1, A) \leq \frac{1}{1-L} d(Tf_1, f_1)$$

This implies

$$d(f_1, A) \leq \frac{L^{1-i}}{1-L}$$

Hence we conclude that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x)$$

for all $x \in U$. This completes the proof of the theorem. □

Corollary 6.1. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f: U \to V$ with$f(0) = 0$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 1; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq \frac{1}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + \left(\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t}\right)\right), & t \neq \frac{1}{3}; \end{cases} \tag{??}$$

for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$ such that

$$\|f(2x) - 8f(x) - A(x)\| \le \begin{cases} \kappa_5, \\ \frac{\kappa_6 \|x\|^t}{|2-2^t|}, \\ \frac{\kappa_7 \|x\|^{3t}}{|2-2^{3t}|}, \\ \frac{\kappa_8 \|x\|^{3t}}{|2-2^{3t}|} \end{cases} \tag{6.8}$$

where

$$\begin{aligned} \kappa_5 &= \frac{\rho\left(16-3r^2\right)}{r^4-r^2}, \\ \kappa_6 &= \frac{\rho}{r^4-r^2}\left[30 - 12r^2 + 2\left(6+r^2\right)2^t + r^2 2^{2t} \right. \\ &\qquad \left. +2(1+r)^t + 2(1-r)^t + (1+2r)^t + (1-2r)^t\right], \\ \kappa_7 &= \frac{\rho}{r^4-r^2}\left[4 - 2r^2 + 2\left(3-2r^2\right)2^t + 2r^2 2^{2t} + r^2 2^{4t} + 2(1+r)^t 2^t \right. \\ &\qquad \left. +2(1-r)^t 2^t + (1+2r)^t 2^t + (1-2r)^t 2^t\right], \\ \kappa_8 &= \frac{\rho}{r^4-r^2}\left[34 - 14r^2 + 2\left(3-2r^2\right)2^t + 2\left(6+r^2\right)2^{3t} + 2r^2 2^{2t} \right. \\ &\qquad +r^2\left(2^{4t}+2^{6t}\right) + 2(1+r)^t 2^t + 2(1-r)^t 2^t + 2(1+r)^{3t} + 2(1-r)^{3t} \\ &\qquad \left. + (1+2r)^t 2^t + (1-2r)^t 2^t + (1+2r)^{3t} + (1-2r)^{3t}\right] \end{aligned} \tag{6.9}$$

for all $x \in U$.

Proof. Setting

$$\psi(x,y,z) = \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + \left(\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t}\right)\right) \end{cases}$$

for all $x, y, z \in U$. Now $\frac{\psi(\mu_i^n x, \mu_i^n y, \mu_i^n z)}{\mu_i^n} = \begin{cases} \frac{\rho}{\mu_i^n}, \\ \frac{\rho}{\mu_i^n}\left(\|\mu_i^n x\|^t + \|\mu_i^n y\|^t + \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^n}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^n}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t + \left(\|\mu_i^n x\|^{3t} + \|\mu_i^n y\|^{3t} + \|\mu_i^n z\|^{3t}\right)\right) \end{cases} =$

$\begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$ i.e., (6.1)is holds. But we have

$$\Gamma(x,x,x) = \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) = \begin{cases} \kappa_5, \\ \frac{\kappa_6 \|x\|^t}{2^t}, \\ \frac{\kappa_7 \|x\|^{3t}}{2^{3t}}, \\ \frac{\kappa_8 \|x\|^{3t}}{2^{3t}} \end{cases}$$

Also,

$$\frac{1}{\mu_i}\Gamma(x,x,x) = \begin{cases} \frac{\kappa_5}{\mu_i}, \\ \frac{\kappa_6 \|\mu_i x\|^t}{\mu_i 2^t}, \\ \frac{\kappa_7 \|\mu_i x\|^{3t}}{\mu_i 2^{3t}}, \\ \frac{\kappa_8 \|\mu_i x\|^{3t}}{\mu_i 2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-1}\kappa_5, \\ \mu_i^{t-1}\frac{\kappa_6 \|x\|^t}{2^t}, \\ \mu_i^{3t-1}\frac{\kappa_7 \|\mu_i x\|^{3t}}{2^{3t}}, \\ \mu_i^{3t-1}\frac{\kappa_8 \|\mu_i x\|^{3t}}{2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-1}\Gamma(x,x,x), \\ \mu_i^{t-1}\Gamma(x,x,x), \\ \mu_i^{3t-1}\Gamma(x,x,x), \\ \mu_i^{3t-1}\Gamma(x,x,x). \end{cases}$$

Hence the inequality (6.3)holds either, $L = 2^{-1}$ if $i = 0$and $L = 2$ if $i = 1$. Now from (6.4), we prove the following cases for condition (i).

Case: 1 $L = 2^{-1}$ if $i = 0$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{-1})^{1-0}}{1-2^{-1}}\kappa_5 = \kappa_5$$

Case: 2 $L = 2$ if $i = 1$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2)^{1-1}}{1-2}\kappa_5 = -\kappa_5$$

Also, (6.3)holds either,$L = 2^{t-1}$for $t < 1$if $i = 0$and $L = \frac{1}{2^{t-1}}$for $t > 1$if $i = 1$. Now from (6.4), we prove the following cases for condition (ii).
Case: 1 $L = 2^{t-1}$for $t < 1$if $i = 0$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{t-1})^{1-0}}{1-2^{t-1}}\frac{\kappa_6\|x\|^t}{2^t} = \frac{\kappa_6\|x\|^t}{2-2^t}.$$

Case: 2 $L = \frac{1}{2^{t-1}}$for $t > 1$if $i = 1$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{t-1}}\right)^{1-1}}{1-\frac{1}{2^{t-1}}}\frac{\kappa_6\|x\|^t}{2^t} = \frac{\kappa_6\|x\|^t}{2^t-2}.$$

Also, (6.3)holds either,$L = 2^{3t-1}$for $3t < 1$if $i = 0$and $L = \frac{1}{2^{3t-1}}$for $3t > 1$if $i = 1$. Now from (6.4), we prove the following cases for condition (iii).
Case:1 $L = 2^{3t-1}$for $3t < 1$if $i = 0$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-1})^{1-0}}{1-2^{3t-1}}\frac{\kappa_7\|x\|^{3t}}{2^{3t}} = \frac{\kappa_7\|x\|^{3t}}{2-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-4}}$for $3t > 1$if $i = 1$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-1}}\right)^{1-1}}{1-\frac{1}{2^{3t-1}}}\frac{\kappa_7\|x\|^{3t}}{2^{3t}} = \frac{\kappa_7\|x\|^{3t}}{2^{3t}-2}.$$

Finally, (6.3)holds either, $L = 2^{3t-1}$for $3t < 1$if $i = 0$and $L = \frac{1}{2^{3t-1}}$for $3t > 1$if $i = 1$. Now from (6.4), we prove the following cases for condition (iv).
Case: 1 $L = 2^{3t-1}$for $3t < 1$if $i = 0$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-1})^{1-0}}{1-2^{3t-1}}\frac{\kappa_8\|x\|^{3t}}{2^{3t}} = \frac{\kappa_8\|x\|^{3t}}{2-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-1}}$for $3t > 1$if $i = 1$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-1}}\right)^{1-1}}{1-\frac{1}{2^{3t-1}}}\frac{\kappa_8\|x\|^{3t}}{2^{3t}} = \frac{\kappa_8\|x\|^{3t}}{2^{3t}-2}.$$

Hence the proof of the corollary is complete. □

Theorem 6.2. *Let $f : U \to V$ be an odd function for which there exists a function $\psi : U^3 \to [0,\infty)$with the condition*

$$\lim_{n\to\infty}\frac{1}{\mu_i^{3n}}\psi(\mu_i^n x, \mu_i^n y, \mu_i^n z) = 0 \tag{6.10}$$

with

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{6.11}$$

for all $x, y, z \in U$. If there exists $L = L(i) < 1$such that the function

$$x \to \Gamma(x, x, x) = \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

has the property

$$\Gamma(x) = \frac{L}{\mu_i^3}\Gamma(\mu_i x, \mu_i x, \mu_i x) \tag{6.12}$$

Then there exists a unique cubic function $C : U \to V$satisfying (1.7) and

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x, x, x) \tag{6.13}$$

for all $x \in U$.

Proof. consider the set $X = \{f/f : U \to V, f(0) = 0\}$and introduce the generalized metric on X.

$$d(f, h) = \inf\{M \in (0, \infty) : \|f(x) - h(x)\| \leq M\Gamma(x, x, x), x \in U\}$$

It is easy to see that $(X, d)$is complete. Define $T : X \to X$by $Tf(x) = \frac{1}{\mu_i}f(\mu_i x)$
for all $x \in U$.Now for all $f, h \in X$,

$$\begin{aligned} d(f, h) \leq M &= \|f(x) - h(x)\| \leq M\Gamma(x, x, x), x \in U \\ &= \left\|\frac{1}{\mu_i^3}f(\mu_i x) - \frac{1}{\mu_i^3}h(\mu_i x)\right\| \leq \frac{1}{\mu_i^3}M\Gamma(\mu_i x, \mu_i x, \mu_i x), x \in U \\ &= \left\|\frac{1}{\mu_i^3}f(\mu_i x) - \frac{1}{\mu_i^3}h(\mu_i x)\right\| \leq LM\Gamma(x, x, x), x \in U \\ &= \|Tf(x) - Th(x)\| \leq LM\Gamma(x, x, x), x \in U \\ &= d(Tf, Th) \leq LM. \end{aligned}$$

This gives $d(Tf, Th) \leq Ld(f, h)$, for all $f, h \in X$,
i.e., $T$is a strictly contractive mapping on $X$with Lipschitz constant L.
Now, from (4.35)we have

$$\|f_3(2x) - 8f_3(x)\| \leq \Phi(x, x, x) \tag{6.14}$$

where $f_3(x) = f(2x) - 2f(x)$

$$\begin{aligned} \Phi(x, x, x) = \frac{1}{r^4 - r^2}\big[&(5 - 4r^2)\phi(x, 2x, -x) + 2r^2\phi(2x, 2x, -x) + (4 - 2r^2)\phi(x, x, x) \\ &+ r^2\phi(2x, 4x, -2x) + \phi(x, 2x, x) + 2\phi((1 + r)x, 2x, -x) \\ &+ 2\phi((1 - r)x, 2x, -x) + \phi((1 + 2r)x, 2x, -x) + \phi((1 - 2r)x, 2x, -x)\big] \end{aligned}$$

for all $x \in U$. From (6.14), we arrive

$$\left\|\frac{f_3(2x)}{8} - f_3(x)\right\| \leq \frac{1}{8}\Phi(x, x, x) = \frac{1}{2^3}\Phi(x, x, x) \tag{6.15}$$

for all $x \in U$. Using (6.12)for the case $i = 0$, it reduces to
$\left\|\frac{f_3(2x)}{8} - f_3(x)\right\| \leq L\Gamma(x, x, x)$, for all $x \in U$.
i.e., $d(Tf_3, f_3) \leq L \Rightarrow d(Tf_3, f_3) \leq L^{1-0} = L^{1-i} < \infty$
Again replacing $x = \frac{x}{2}$, in (6.14), we obtain

$$\left\|f_3(x) - 8f_3\left(\frac{x}{2}\right)\right\| \leq \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in U$. Using (6.3)for the case $i = 1$, it reduces to
$\left\|f_3(x) - 8f_3\left(\frac{x}{2}\right)\right\| \leq \Gamma(x, x, x)$ for all $x \in U$.
i.e., $d(Tf_3, f_3) \leq 1 \Rightarrow d(Tf_3, f_3) \leq 1 = L^{1-1} = L^{1-i} < \infty$.
From above two cases, we arrive

$$d(Tf_3, f_3) \leq L^{1-i}$$

Therefore $(B_2(i))$ holds.

By$(B_2(ii))$, it follows that there exists a fixed point $C$of Tin X such that

$$C(x) = \lim_{n\to\infty} \frac{1}{\mu_i^{3n}} \left(f(\mu_i^{n+1}x) - 2f(\mu_i^n x) \right) \tag{6.16}$$

for all $x \in U$. In order to prove $C : U \to V$is cubic. Replacing $(x, y, z)$by $\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right)$in (6.11)and divide by μ_i^{3n}. It follows from (.711) and (6.16), Csatisfies (1.7) for all $x, y, z \in U$. i.e., Csatisfies the functional equation (1.7). By $(B_2(iii))$, $C$is the unique fixed point of Tin the set $Y = \{f \in X : d(Tf, C) < \infty\}$, using the fixed point alternative result $C$is the unique function such that
$\|f_1(x) - C(x)\| \le M\Gamma(x, x, x)$ for all $x \in U$and $M > 0$.
Finally, by $(B_2(iv))$, we obtain

$$d(f_3, C) \le \frac{1}{1-L} d(Tf_3, f_3)$$

This implies

$$d(f_3, C) \le \frac{L^{1-i}}{1-L}$$

Hence we conclude that

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{L^{1-i}}{1-L} \Gamma(x, x, x)$$

for all $x \in U$. This completes the proof of the theorem. □

Corollary 6.2. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ with$f(0) = 0$ satisfies the inequality*

$$\|Df(x, y, z)\| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 3; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & 3t \neq 3; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & 3t \neq 3; \end{cases} \tag{??}$$

for all $x, y, z \in U$. Then there exists a unique cubic function $C : U \to V$ such that

$$\|f(2x) - 2f(x) - C(x)\| \le \begin{cases} \frac{\kappa_5}{7}, \\ \frac{\kappa_6 \|x\|^t}{|8-2^t|}, \\ \frac{\kappa_7 \|x\|^{3t}}{|8-2^{3t}|}, \\ \frac{\kappa_8 \|x\|^{3t}}{|8-2^{3t}|} \end{cases} \tag{6.17}$$

for all $x \in U$, where κ_i $(i = 5, 6, 7, 8)$ are defined in (6.9).

Proof. Setting

$$\psi(x, y, z) = \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) \end{cases}$$

for all $x, y, z \in U$. Now $\frac{\psi(\mu_i^n x, \mu_i^n y, \mu_i^n z)}{\mu_i^{3n}} = \begin{cases} \frac{\rho}{\mu_i^{3n}}, \\ \frac{\rho}{\mu_i^{3n}} \left(\|\mu_i^n x\|^t + \|\mu_i^n y\|^t + \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{3n}} \left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{3n}} \left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t + (\|\mu_i^n x\|^{3t} + \|\mu_i^n y\|^{3t} + \|\mu_i^n z\|^{3t})\right) \end{cases} =$

$$\begin{cases} \rightarrow 0 \, as \, n \rightarrow \infty, \\ \rightarrow 0 \, as \, n \rightarrow \infty, \\ \rightarrow 0 \, as \, n \rightarrow \infty, \\ \rightarrow 0 \, as \, n \rightarrow \infty. \end{cases}$$ i.e., (.1) is holds. But we have

$$\Gamma(x,x,x) = \Phi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right) = \begin{cases} \kappa_5, \\ \frac{\kappa_6\|x\|^t}{2^t}, \\ \frac{\kappa_7\|x\|^{3t}}{2^{3t}}, \\ \frac{\kappa_8\|x\|^{3t}}{2^{3t}} \end{cases}$$

Also,

$$\frac{1}{\mu_i^3}\Gamma(x,x,x) = \begin{cases} \frac{\kappa_5}{\mu_i^3}, \\ \frac{\kappa_6\|\mu_i x\|^t}{\mu_i^3 2^t}, \\ \frac{\kappa_7\|\mu_i x\|^{3t}}{\mu_i^3 2^{3t}}, \\ \frac{\kappa_8\|\mu_i x\|^{3t}}{\mu_i^3 2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-3}\kappa_5, \\ \mu_i^{t-3}\frac{\kappa_6\|x\|^t}{2^t}, \\ \mu_i^{3t-3}\frac{\kappa_7\|\mu_i x\|^{3t}}{2^{3t}}, \\ \mu_i^{3t-3}\frac{\kappa_8\|\mu_i x\|^{3t}}{2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-3}\Gamma(x,x,x), \\ \mu_i^{t-3}\Gamma(x,x,x), \\ \mu_i^{3t-3}\Gamma(x,x,x), \\ \mu_i^{3t-3}\Gamma(x,x,x). \end{cases}$$

Hence the inequality (6.12)holds either, $L = 2^{-3}$ if $i = 0$and $L = 2^3$ if $i = 1$. Now from (6.13), we prove the following cases for condition (i).
Case: 1 $L = 2^{-3}$ if $i = 0$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{-3})^{1-0}}{1-2^{-3}}\kappa_5 = \frac{\kappa_5}{7}$$

Case: 2 $L = 2^3$ if $i = 1$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^3)^{1-1}}{1-2^3}\kappa_5 = \frac{-\kappa_5}{7}$$

Also, (6.12)holds either,$L = 2^{t-3}$for $t < 3$if $i = 0$and $L = \frac{1}{2^{t-3}}$for $t > 3$if $i = 1$. Now from (6.13), we prove the following cases for condition (ii).
Case:1 $L = 2^{t-3}$ for $t < 3$if $i = 0$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{t-3})^{1-0}}{1-2^{t-3}}\frac{\kappa_6\|x\|^t}{2^t} = \frac{\kappa_6\|x\|^t}{8-2^t}.$$

Case: 2 $L = \frac{1}{2^{t-3}}$ for $t > 3$if $i = 1$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{t-3}}\right)^{1-1}}{1-\frac{1}{2^{t-3}}}\frac{\kappa_6\|x\|^t}{2^t} = \frac{\kappa_6\|x\|^t}{2^t-8}.$$

Also, (6.12)holds either,$L = 2^{3t-3}$for $3t < 3$if $i = 0$and $L = \frac{1}{2^{3t-3}}$for $3t > 3$if $i = 1$. Now from (6.13), we prove the following cases for condition (iii).
Case:1 $L = 2^{3t-3}$ for $3t < 3$ if $i = 0$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-3})^{1-0}}{1-2^{3t-3}}\frac{\kappa_7\|x\|^{3t}}{2^{3t}} = \frac{\kappa_7\|x\|^{3t}}{8-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-3}}$ for $3t > 3$ if $i = 1$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-3}}\right)^{1-1}}{1-\frac{1}{2^{3t-3}}}\frac{\kappa_7\|x\|^{3t}}{2^{3t}} = \frac{\kappa_7\|x\|^{3t}}{2^{3t}-8}.$$

Finally, (6.12)holds either, $L = 2^{3t-3}$for $3t < 3$if $i = 0$and $L = \frac{1}{2^{3t-3}}$for $3t > 3$if $i = 1$. Now from (6.13), we prove the following cases for condition (iv).

Case: 1 $L = 2^{3t-3}$ for $3t < 3$ if $i = 0$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-3})^{1-0}}{1-2^{3t-3}}\frac{\kappa_8\|x\|^{3t}}{2^{3t}} = \frac{\kappa_8\|x\|^{3t}}{8-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-3}}$for $3t > 3$ if $i = 1$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-3}}\right)^{1-1}}{1-\frac{1}{2^{3t-3}}}\frac{\kappa_8\|x\|^{3t}}{2^{3t}} = \frac{\kappa_8\|x\|^{3t}}{2^{3t}-8}.$$

Hence the proof of the corollary is complete. □

Theorem 6.3. *Let $f : U \to V$ be an odd function for which there exists a function $\psi : U^3 \to [0,\infty)$with the condition (6.1)and (6.10) with*

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{6.18}$$

for all $x,y,z \in U$. If there exists $L = L(i) < 1$such that the function

$$x \to \Gamma(x,x,x) = \Phi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right),$$

has the property (6.3)and (6.12), then there exists a unique additive function $A : U \to V$and a unique cubic function $C : U \to V$satisfying (1.7) and

$$\|f(x) - A(x) - C(x)\| \leq \frac{1}{3}\frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.19}$$

for all $x \in U$, where $A(x)$ and $C(x)$ are defined in (6.7)and (6.16) respectively for all $x \in U$.

Proof. By Theorems 6.1 and 6.2, there exists a unique additive function $A' : U \to V$ and a unique cubic function $C' : U \to V$ such that

$$\left\|f(2x) - 8f(x) - A'(x)\right\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.20}$$

and

$$\left\|f(2x) - 2f(x) - C'(x)\right\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.21}$$

for all $x \in U$. Now from (6.20)and (6.21), that

$$\begin{aligned} \left\|f(x) + \tfrac{1}{6}A'(x) - \tfrac{1}{6}C'(x)\right\| &= \left\|\left\{-\tfrac{f(2x)}{6} + \tfrac{8f(x)}{6} + \tfrac{A'(x)}{6}\right\} + \left\{\tfrac{f(2x)}{6} - \tfrac{2f(x)}{6} - \tfrac{C'(x)}{6}\right\}\right\| \\ &\leq \tfrac{1}{6}\left\{\|f(2x) - 8f(x) - A'(x)\| + \|f(2x) - 2f(x) - C'(x)\|\right\} \\ &\leq \tfrac{1}{6}\left\{\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x) + \tfrac{L^{1-i}}{1-L}\Gamma(x,x,x)\right\} = \tfrac{1}{3}\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x) \end{aligned}$$

for all $x \in U$. Thus, we obtain (6.19)by defining $A(x) = \frac{-1}{6}A'(x)$ and $C(x) = \frac{1}{6}C'(x)$, where $A(x)$ and $C(x)$ are defined in (6.7)and (6.16) respectively for all $x \in U$. □

The following corollary is an immediate consequence of Theorem 6.3 concerning the stability of (1.7).

Corollary 6.3. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ with $f(0) = 0$satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 1,3; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq \frac{1}{3}, 1; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + \left(\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t}\right)\right), & t \neq \frac{1}{3}, 1; \end{cases} \tag{6.22}$$

for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$ and a unique cubic function $C : U \to V$ such that

$$\|f(x) - A(x) - C(x)\| \leq \begin{cases} \frac{8\kappa_5}{21}, \\ \frac{\kappa_6\|x\|^t}{3}\left\{\frac{1}{|2-2^t|} + \frac{1}{|8-2^t|}\right\}, \\ \frac{\kappa_7\|x\|^{3t}}{3}\left\{\frac{1}{|2-2^{3t}|} + \frac{1}{|8-2^{3t}|}\right\}, \\ \frac{\kappa_8\|x\|^{3t}}{3}\left\{\frac{1}{|2-2^{3t}|} + \frac{1}{|8-2^{3t}|}\right\}, \end{cases} \tag{6.23}$$

for all $x \in U$, where κ_i $(i = 5, 6, 7, 8)$ are given in (6.9).

7 STABILITY RESULTS: MIXED CASE

Theorem 7.1. *Let $\psi, \phi : U^3 \to [0, \infty)$ be a function that satisfies (6.1), (6.11), (6.1)and (6.10)for all $x, y, z \in U$. Suppose that a function $f : U \to V$ with $f(0) = 0$ satisfies the inequalities (6.20)and (6.18)for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$, a unique quadratic function $Q_2 : U \to V$, a unique cubic function $C : U \to V$ and a unique quartic function $Q_4 : U \to V$ such that*

$$\|f(x) - A(x) - Q_2(x) - C(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\left\{\Gamma_{Q_2Q_4}(x,x,x) + \Gamma_{AC}(x,x,x)\right\} \tag{7.1}$$

for all $x \in U$, where $\Gamma_{Q_2Q_4}(x,x,x)$ and $\Gamma_{AC}(x,x,x)$ are defined by

$$\Gamma_{Q_2Q_4}(x,x,x) = \frac{1}{12}\left[\Gamma(x,x,x) + \Gamma(-x,-x,-x)\right] \tag{7.2}$$

$$\Gamma_{AC}(x,x,x) = \frac{1}{6}\left[\Gamma(x,x,x) + \Gamma(-x,-x,-x)\right] \tag{7.3}$$

respectively for all $x \in U$.

Proof. Let $f_e(x) = \frac{1}{2}\{f(x) + f(-x)\}$ for all $x \in U$. Then $f_e(0) = 0, f_e(x) = f_e(-x)$. Hence

$$\begin{aligned} \|Df_e(x,y,z)\| &= \tfrac{1}{2}\{\|Df(x,y,z) + Df(-x,-y,-z)\|\} \\ &\leq \tfrac{1}{2}\{\|Df(x,y,z)\| + \|Df(-x,-y,-z)\|\} \\ &\leq \tfrac{1}{2}\{\psi(x,y,z) + \psi(-x,-y,-z)\} \end{aligned}$$

for all $x \in U$. Hence from Theorem 6.3, there exists a unique quadratic function $Q_2 : U \to V$ and a unique quartic function $Q_4 : U \to V$such that

$$\begin{aligned} \|f(x) - Q_2(x) - Q_4(x)\| &\leq \tfrac{1}{2}\left\{\tfrac{1}{6}\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x) + \tfrac{1}{6}\tfrac{L^{1-i}}{1-L}\Gamma(-x,-x,-x)\right\} \\ &\leq \tfrac{1}{12}\tfrac{L^{1-i}}{1-L}\{\Gamma(x,x,x) + \Gamma(-x,-x,-x)\}, \end{aligned} \tag{7.4}$$

for all $x \in U$. Again $f_o(x) = \frac{1}{2}\{f(x) - f(-x)\}$ for all $x \in U$. Then $f_o(0) = 0$, $f_o(x) = -f_o(-x)$. Hence

$$\begin{aligned} \|Df_o(x,y,z)\| &= \tfrac{1}{2}\{\|Df(x,y,z) - Df(-x,-y,-z)\|\} \\ &\leq \tfrac{1}{2}\{\|Df(x,y,z)\| - \|Df(-x,-y,-z)\|\} \\ &\leq \tfrac{1}{2}\{\phi(x,y,z) - \phi(-x,-y,-z)\} \end{aligned}$$

for all $x \in U$. Hence from Theorem 6.3, there exists a unique additive function $A : U \to V$ and a unique cubic function $C : U \to V$such that

$$\begin{aligned} \|f(x) - A(x) - C(x)\| &\leq \frac{1}{2}\left\{\frac{1}{3}\frac{L^{1-i}}{1-L}\Gamma(x,x,x) + \frac{1}{3}\frac{L^{1-i}}{1-L}\Gamma(-x,-x,-x)\right\} \\ &\leq \frac{1}{6}\frac{L^{1-i}}{1-L}\{\Gamma(x,x,x) + \Gamma(-x,-x,-x)\}, \end{aligned} \tag{7.5}$$

for all $x \in U$. Since $f(x) = f_e(x) + f_o(x)$ then it follows from (7.4)and (7.5)that

$$\begin{aligned} &\|f(x) - A(x) - Q_2(x) - C(x) - Q_4(x)\| \\ &= \|\{f_e(x) - Q_2(x) - Q_4(x)\} + \{f_o(x) - A(x) - C(x)\}\| \\ &\leq \|f_e(x) - Q_2(x) - Q_4(x)\| + \|f_o(x) - A(x) - C(x)\| \\ &\leq \frac{1}{12}\frac{L^{1-i}}{1-L}\{\Gamma(x,x,x) + \Gamma(-x,-x,-x)\} + \frac{1}{6}\frac{L^{1-i}}{1-L}\{\Gamma(x,x,x) + \Gamma(-x,-x,-x)\} \\ &\leq \frac{L^{1-i}}{1-L}\{\Gamma_{Q_2Q_4}(x,x,x) + \Gamma_{AC}(x,x,x)\} \end{aligned} \quad \text{for all } x \in U.$$

Hence the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 7.1 concerning the stability of (1.7).

Corollary 7.1. *Let ρ, t be nonnegative real numbers. Suppose that a function $f : U \to V$ with $f(0) = 0$satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 1,2,3,4; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right), & t \neq \frac{1}{3},\frac{2}{3},\frac{4}{3},1; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & t \neq \frac{1}{3},\frac{2}{3},\frac{4}{3},1; \end{cases} \tag{7.6}$$

for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$, a unique quadratic function $Q_2 : U \to V$, a unique cubic function $C : U \to V$and a unique quartic function $Q_4 : U \to V$ such that

$$\begin{aligned} &\|f(x) - A(x) - Q_2(x) - C(x) - Q_4(x)\| \\ &\leq \begin{cases} \frac{1}{2}\left[2\kappa_1 + \frac{8\kappa_5}{21}\right], \\ \frac{1}{2}\left[\frac{\kappa_2}{6}\left\{\frac{1}{4-2^t} + \frac{1}{16-2^t}\right\} + \frac{\kappa_6}{3}\left\{\frac{1}{2-2^t} + \frac{1}{8-2^t}\right\}\right]\|x\|^t, \\ \frac{1}{2}\left[\frac{\kappa_3}{6}\left\{\frac{1}{4-2^{3t}} + \frac{1}{16-2^{3t}}\right\} + \frac{\kappa_7}{3}\left\{\frac{1}{2-2^{3t}} + \frac{1}{8-2^{3t}}\right\}\right]\|x\|^{3t}, \\ \frac{1}{2}\left[\frac{\kappa_4}{6}\left\{\frac{1}{4-2^{3t}} + \frac{1}{16-2^{3t}}\right\} + \frac{\kappa_8}{3}\left\{\frac{1}{2-2^{3t}} + \frac{1}{8-2^{3t}}\right\}\right]\|x\|^{3t} \end{cases} \end{aligned} \tag{7.7}$$

for all $x \in U$, where κ_i $(i = 1, 2, ..., 8)$ are respectively, given in (6.10)and (6.9).

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