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On some boundary-value problems of functional integro-differential equations with nonlocal conditions

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Abstract

In this paper, we study the existence of solution for some boundary value problems of functional integro-differential equations with nonlocal boundary conditions.

Keywords: Nonlocal boundary value problems, schauder fixed point theorem, functional integral equation, functional integro-differential equation, lebesgue dominated convergence theorem.

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1 Introduction

Mathematical modelling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro- differential equations, stochastic equations. Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arises in many fields like fluid dynamics, biological models and chemical kinetics integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Consider the following boundary value problems of functional integro-differential equations with the nonlocal boundary conditions.

$$x'(t) = f(t, \int_0^1 k(t,s)x(s)ds), \qquad t \in (0,1) \tag{1.1}$$

$$x(\tau) + \alpha\, x(\xi) = 0, \qquad \tau, \xi \in [0,1], \alpha \neq -1. \tag{1.2}$$

$$x''(t) = f(t, \int_0^1 k(t,s)x'(s)ds), \qquad t \in (0,1) \tag{1.3}$$

$$x(\tau) + \beta\, x(\xi) = 0, \qquad \beta \neq -1 \tag{1.4}$$

$$x'(\tau) + \alpha\, x'(\xi) = 0, \qquad \tau, \xi \in [0,1], \alpha \neq -1. \tag{1.5}$$

Here we study the existence of at least one solution of each of the boundary value problems (1.1)-(1.2) and (1.3)-(1.5).
The existence of exactly one solution of them will be deduced.

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2 Functional integral equation

Here we study the existence of at least one (and exactly one) continuous solution of the functional integral equation.

$$y(t) = f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]ds) \tag{2.6}$$

under the following assumptions

(1) $f: I = [0,1] \times R \to R$ is measurable in $t \in [0,1]$ for all $x \in R$ and continuous in $x \in R$ for all $t \in [0,1]$ and there exists integrable function $a \in L^1[0,1]$ and positive constant $b > 0$ such that

$$|f(t,x)| \leq a(t) + b|x| \qquad t \in I.$$

(2) $a = \sup_t |a(t)|, \qquad t \in [0,1]$

(3) $k : I = [0,1] \times [0,1] \to R$ is continuous $t \in [0,1]$ for every $s \in [0,1]$ and measurable in $s \in [0,1]$ for all $t \in [0,1]$, such that

$$\sup_t \int_0^1 k(t,s)dt \leq M$$

Now for the existence of at least one continuous solution of the functional integral equation (2.6), we have the following theorem.

Theorem 2.1. *Let the assumptions (1)-(3) be satisfied. If* $2Mb < 1$, *then the functional integral equation (2.6) has at least one solution* $y \in C[0,1]$.

Proof. let $C = C[0,1]$ and define the set Q_r by

$$Q_r = \{y \in C : |y| \leq r\} \subset C[0,1]$$

where $r = \frac{a}{1-2bM}$.
Define the operator F associated with the functional integral equation (2.6) by

$$Fy(t) = f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]ds)$$

To show that $F: Q_r \to Q_r$, let $y \in Q_r$, then

$$\begin{aligned}
|Fy(t)| &= |f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]ds)| \\
&\leq |a(t)| + b\,|\int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\,ds| \\
&\leq |a(t)| + b\,[\,|\int_0^1 k(t,s)\int_0^s y(\theta)d\theta ds| + |\int_0^1 k(t,s)[\frac{-1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\,ds|\,] \\
&\leq |a(t)| + b\,[\int_0^1 |k(t,s)|\,|y(s)|\,ds + \int_0^1 |k(t,s)|[\frac{1}{1+\alpha} + \frac{\alpha}{1+\alpha}]\,|y(s)|\,ds\,] \\
&\leq |a(t)| + b\,[\int_0^1 |k(t,s)|\,r\,ds + \int_0^1 |k(t,s)|\,r\,ds\,] \\
&\leq |a(t)| + 2bMr = r. \\
&\leq a + 2bMr = r.
\end{aligned}$$

This proves that $F : Q_r \to Q_r$ and the class of functions $\{F(y)\}$ is uniformly bounded.
Let $t_1, t_2 \in [0,1]$ and $|t_2 - t_1| \le \delta$, then

$$\begin{aligned}
|Fy(t_2) - Fy(t_1)| &= |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_1,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&= |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_1,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&+ f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&\le |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&+ |f(t_1, \int_0^1 k(t_1,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&\le |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&+ L\, |\int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds \\
&- \int_0^1 k(t_1,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds| \\
&\le |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&+ 2L||y||\int_0^1 |k(t_2,s) - k(t_1,s)|ds, \\
&\le |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&+ 2L\, r\int_0^1 |k(t_2,s) - k(t_1,s)|ds.
\end{aligned}$$

This means that the class of functions $F\{y\}$ is equi-continuous on Q_r.
Using Arzela-Ascoli Theorem (see[13]), we find that F is compact.
Now we prove that $F : Q_r \to Q_r$ is continuous.

Let $\{y_n\} \subset Q_r$, and $y_n \to y$, then

$$Fy_n(t) = f(t, \int_0^1 k(t,s)[\int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_n(\theta)d\theta]ds)$$

$$\lim_{n\to\infty} Fy_n(t) = \lim_{n\to\infty} f(t, \int_0^1 k(t,s)[\int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_n(\theta)d\theta]ds)$$

Now

$$\lim_{n\to\infty} f(t, \int_0^1 k(t,s)y_n(s)ds) = f(t, \lim_{n\to\infty}\int_0^1 k(t,s)[\int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_n(\theta)d\theta]ds)$$

then using Lebesgue dominated convergence Theorem (see[13]), we have

$$\begin{aligned}
\lim_{n\to\infty} Fy_n &= \lim_{n\to\infty} f(t, \int_0^1 k(t,s)f(t, \int_0^1 k(t,s)[\int_0^s y_n(\theta)d\theta \\
&- \frac{1}{1+\alpha}\int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_n(\theta)d\theta]ds) \\
&= f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)
\end{aligned}$$

Then $Fy_n(t) \to Fy(t)$.
Which means that the operator F is continuous.
Since all conditions of Schauder fixed point theorem [12] are satisfied, then the operator F has at least one fixed point $y \in C[0,1]$, which completes the proof. ■

Now for the uniqueness of the solution of the functional integral equation (2.6).
Consider following assumptions

(1*) $f : I = [0,1] \times R \to R$ is measurable in $t \in [0,1]$ for all $x \in R$ and satisfies the lipschitz such that

$$|f(t,x) - f(t,y)| \leq b|x-y|, \qquad b > 0 \tag{2.7}$$

(2*) $f(t,0) \in L^1[0,1]$ $\quad \sup_t |f(t,0)| \leq a.$

Theorem 2.2. *Let the assumptions (1*),(2*) and (3) be satisfied. If $2Mb < 1$, then the functional integral equation (2.6) has a unique solution $y \in C[0,1]$.*

Proof. From (2.7) we can obtain

$$|f(t,x)| \leq |f(t,0)| + b\,|x|.$$

This shows that the assumptions of Theorem (2.1) are satisfied
Now let y_1,y_2 be two solution of functional integral equation (2.6)

$$y_1(t) = f(t, \int_0^1 k(t,s)[\int_0^s y_1(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_1(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_1(\theta)d\theta]ds)$$

$$y_2(t) = f(t, \int_0^1 k(t,s)[\int_0^s y_2(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_2(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_2(\theta)d\theta]ds)$$

$$\begin{aligned}
|y_1(t)-y_2(t)| &= |f(t,\int_0^1 k(t,s)[\int_0^s y_1(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_1(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_1(\theta)d\theta]\ ds) \\
&- f(t,\int_0^1 k(t,s)[\int_0^s y_2(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_2(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_2(\theta)d\theta]\ ds)| \\
&\le b\,|\int_0^1 k(t,s)[\int_0^s y_1(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_1(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_1(\theta)d\theta]ds \\
&- \int_0^1 k(t,s)[\int_0^s y_2(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_2(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_2(\theta)d\theta]ds| \\
&\le b\,|\int_0^1 k(t,s)\int_0^s y_1(\theta)d\theta ds - \int_0^1 k(t,s)[\frac{1}{1+\alpha}\int_0^\tau y_1(\theta)d\theta + \frac{\alpha}{1+\alpha}\int_0^\xi y_1(\theta)d\theta]\ ds \\
&- \int_0^1 k(t,s)\int_0^s y_2(\theta)d\theta ds + \int_0^1 k(t,s)[\frac{1}{1+\alpha}\int_0^\tau y_2(\theta)d\theta + \frac{\alpha}{1+\alpha}\int_0^\xi y_2(\theta)d\theta]\ ds| \\
&\le b\,|\int_0^1 k(t,s)[\int_0^s y_1(\theta)d\theta\ -\ \int_0^s y_2(\theta)d\theta]ds| \\
&+ b\,|\int_0^1 k(t,s)[\frac{1}{1+\alpha}\int_0^\tau (y_2(\theta)-y_1(\theta))d\theta\ +\ \frac{\alpha}{1+\alpha}\int_0^\xi (y_2(\theta)-y_1(\theta))d\theta]ds\ | \\
&\le b\,|\int_0^1 k(t,s)\int_0^s (y_1(\theta)-y_2(\theta))d\theta ds| \\
&+ b\,|\int_0^1 k(t,s)[\frac{1}{1+\alpha}||y_2-y_1||\ +\ \frac{\alpha}{1+\alpha}||y_2-y_1||]ds\ | \\
&\le b\ (\ ||y_1-y_2||\int_0^1 |k(t,s)|ds\ +\ ||y_1-y_2||\int_0^1 |k(t,s)|ds) \\
&\le 2bM\ ||y_1-y_2||
\end{aligned}$$

then

$$||y_1-y_2|| \le\ K||y_1-y_2||$$

where $K = 2bM <\ 1$, then

$$||y_1-y_2||(1-k) \le\ 0$$

and

$$||y_1-y_2|| =\ 0$$

which implies that $y_1 = y_2$ then the functional integral equation (2.6) has a unique continuous solution.

3 Nonlocal boundary value problems

Here we study the existence of at least one (and exactly one) solution of each of the functional integro-differential equations (1.1),(1.3).
Consider the functional integro differential equation

$$x'(t)\ =\ f(t,\int_0^1\ k(t,s)\ x(s)\ ds) \qquad t\in(0,1).$$

with the nonlocal boundary value condition

$$x(\tau)+\alpha\ x(\xi)=0. \qquad \tau,\xi\ \in[0,1],\alpha\ \ne\ -1$$

Theorem 3.3. *Let the assumptions of theorem (2.1) be satisfied, then the nonlocal boundary value problem (1.1)-(1.2) has at least one continuous solution* $x\in C[0,1]$.

Proof. Let $x'(t) = y(t)$. Integrating both sides we get

$$x(t) = x(0) + \int_0^t y(s)ds,$$

$$x(\tau) = x(0) + \int_0^\tau y(s)ds$$

and

$$x(\xi) = x(0) + \int_0^\xi y(s)ds$$

Using the nonlocal boundary condition (1.2) we obtain

$$x(0) + \int_0^\tau y(s)ds \; = \; -\alpha\, x(0) - \alpha \int_0^\xi y(s)ds,$$

and

$$x(0) = -\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds,$$

then

$$x(t) = \int_0^t y(s)ds - \frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds \tag{3.8}$$

where y satisfies the functional integral equation

$$y(t) \; = \; f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]ds).$$

This complete the proof of equivalent between the nonlocal problem (1.1)-(1.2) and the functional integral equation (2.6). This implies that there exists at least one solution $x \in C[0,1]$ of the nonlocal problem (1.1)-(1.2).■

Corollary 3.1. *Let the assumptions* (1^*)*,*(2^*) *and (3) be satisfied, then the solution of nonlocal boundary value problem (1.1)-(1.2) has a unique continuous solution* $x \in C[0,1]$.

Consider the functional integro-differential equation

$$x''(t) \; = \; f(t, \int_0^1 k(t,s)x'(s)ds) \qquad\qquad t \in (0,1)$$

with the nonlocal boundary conditions

$$x(\tau) + \beta\, x(\xi) = 0,$$

$$x'(\tau) \; + \; \alpha\, x'(\xi) = 0.$$

Theorem 3.4. *Let the assumptions of theorem (2.1) be satisfied then the boundary value problems (1.3)-(1.5) has at least one continuous solution* $x \in C[0,1]$.

Proof. Let $x''(t) \; = \; y(t)$ integrating both sides, we obtain

$$x'(t) \; = \; x'(0) + \int_0^t y(s)\, ds$$

and

$$x(t) \; = \; x(0) + tx'(0) + \int_0^t (t-s)\, y(s)ds.$$

then

$$x'(\tau) \; = \; x'(0) + \int_0^\tau y(s)\, ds,$$

and

$$x'(\xi) \; = \; x'(0) + \int_0^\xi y(s)\, ds.$$

Using the nonlocal condition (1.5) we obtain

$$x'(0) \;=\; -\frac{1}{1+\alpha}\int_0^{\tau} y(s)ds - \frac{\alpha}{1+\alpha}\int_0^{\xi} y(s)\; ds$$

and

$$x(\tau) \;=\; x(0) + \tau\, x'(0) + \int_0^{\tau} (\tau - s)\, y(s)\; ds,$$

$$x(\xi) \;=\; x(0) + \xi\, x'(0) + \int_0^{\xi} (\xi - s)\, y(s)\; ds,$$

$$x'(0) \;=\; -\frac{1}{1+\alpha}\int_0^{\tau} y(s)ds - \frac{\alpha}{1+\alpha}\int_0^{\xi} y(s)\; ds.$$

Using Boundary condition (1.4) we obtain

$$x(0) = \frac{-\beta\xi - \tau}{1+\beta}x'(0) - \frac{1}{1+\alpha}\int_0^{\tau}(\tau - s)y(s)ds - \frac{1}{1+\beta}\int_0^{\xi}(\xi - s)y(s)ds,$$

$$\begin{aligned} x(t) = {} & \frac{-\beta\xi - \tau}{1+\beta}[-\frac{1}{1+\beta}\int_0^{\tau} y(s)ds - \frac{1}{1+\alpha}\int_0^{\xi} y(s)ds] \\ & - \frac{1}{1+\beta}\int_0^{\tau}(\tau - s)y(s)ds - \frac{1}{1+\beta}\int_0^{\xi}(\xi - s)ds \\ + t[-\frac{1}{1+\alpha}\int_0^{\tau} y(s)ds - {} & \frac{\alpha}{1+\alpha}\int_0^{\xi} y(s)ds] + \int_0^{t}(t - s)y(s)ds, \qquad (3.9) \\ x'(t) = -\frac{1}{1+\alpha}\int_0^{\tau} y(s)ds - {} & \frac{\alpha}{1+\alpha}\int_0^{\xi} y(s)ds + \int_0^{t} y(s)ds, \end{aligned}$$

and y satisfies the functional integral equation

$$y(t) = f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^{\tau} y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^{\xi} y(\theta)d\theta]ds).$$

This complete the proof of equivalent between the nonlocal problem (1.3)-(1.5) and the functional integral equation (2.6). This implies that there exists at least one solution $x \in C[0,1]$ of the nonlocal problem (1.3)-(1.5). ■

Corollary 3.2. *Let the assumptions* (1^*),(2^*) *and (3) be satisfied, then the solution of nonlocal boundary value problem (1.3)-(1.5) has a unique continuous solution* $x \in C[0,1]$.

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Almost Contra Pre Generalized b - Continuous Functions in Topological Spaces

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Abstract

In this paper, the authors introduce a new class of functions called almost contra pre generalized b - continuous function (briefly almost contra pgb-continuous) in topological spaces. Some characterizations and several properties concerning almost contra pgb-continuous functions are obtained.

Keywords: pgb-closed sets, pgb-closed map, pgb-continuous map, contra pgb-continuity.

2010 MSC: 54C05, 54C08, 54C10.

1 Introduction

In 2002, Jafari and Noiri introduced and studied a new form of functions called contra-pre continuous functions. The purpose of this paper is to introduce and study almost contra pgb-continuous functions via the concept of pgb-closed sets. Also, properties of almost contra pgb-continuity are discussed. Moreover, we obtain basic properties and preservation theorems of almost contra pgb-continuous functions and relationships between almost contra pgb-continuity and pgb-regular graphs.

Through out this paper (X,τ) and (Y,σ) represent the non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $A \subseteq X$, the closure of A and interior of A will be denoted by $cl(A)$ and $int(A)$ respectively, union of all pgb-open sets X contained in A is called pgb-interior of A and it is denoted by $pgbint(A)$, the intersection of all pgb-closed sets of X containing A is called pgb-closure of A and it is denoted by $pgbcl(A)$ [9].

2 Preliminaries

Definition 2.1. *Let a subset A of a topological space (X,τ), is called*
1) a pre-open set [8] if $A \subseteq int(cl(A))$.
2) a semi-open set [6] if $A \subseteq cl(int(A))$.
3) a b -open set [3] if $A \subseteq cl(int(A)) \cup int(cl(A))$.
4) a generalized b- closed set (briefly gb- closed) [1] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
5) a generalized αb- closed set (briefly $g\alpha b$- closed) [11] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is α open in X.
6) a regular generalized b- closed set (briefly rgb- closed) [7] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.
7) a pre generalized b- closed set (briefly pgb- closed) [9] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is pre-open in X.

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Definition 2.2. *A function $f : (X, \tau) \to (Y, \sigma)$, is called*
1) almost contra continuous [1] if $f^{-1}(V)$ is closed in (X, τ) for every regular-open set V of (Y, σ).
2) almost contra b-continuous [2] if $f^{-1}(V)$ is b-closed in (X, τ) for every regular-open set V of (Y, σ). 3) almost contra pre-continuous [5] if $f^{-1}(V)$ is pre-closed in (X, τ) for every regular-open set V of (Y, σ).
4) almost contra semi-continuous [4] if $f^{-1}(V)$ is semi-closed in (X, τ) for every regular-open set V of (Y, σ).
5) almost contra rgb-continuous [10] if $f^{-1}(V)$ is rgb-closed in (X, τ) for every regular-open set V of (Y, σ).

3 Almost Contra Pre Generalized b - Continuous Functions

In this section, we introduce almost contra pre generalized b - continuous functions and investigate some of their properties.

Definition 3.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is called almost contra pre generalized b - continuous if $f^{-1}(V)$ is pgb - closed in (X, τ) for every regular open set V in (Y, σ).*

Example 3.1. *Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{b\}, \{c\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Clearly f is almost contra pgb - continuous.*

Theorem 3.1. *If $f : X \to Y$ is contra pgb - continuous then it is almost contra pgb - continuous.*

Proof. Obvious, because every regular open set is open set. □

Remark 3.1. *Converse of the above theorem need not be true in general as seen from the following example.*

Example 3.2. *Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Then f is almost contra pgb - continuous function but not contra pgb - continuous, because for the open set $\{a, c\}$ in Y and $f^{-1}\{a, c\} = \{a, b\}$ is not pgb - closed in X.*

Theorem 3.2. *1) Every almost contra b - continuous function is almost contra pgb - continuous function.*
2) Every almost contra $g\alpha$ - continuous function is almost contra pgb - continuous function.
3) Every almost contra $g\alpha*$ - continuous function is almost contra pgb - continuous function.
4) Every almost contra g - continuous function is almost contra pgb - continuous function.
5) Every almost contra rgb - continuous function is almost contra pgb - continuous function.
6) Every almost contra $g\alpha b$ - continuous function is almost contra pgb - continuous function.

Remark 3.2. *Converse of the above statements is not true as shown in the following example.*

Example 3.3. *i) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{Y, \varphi, \{b\}, \{c\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = c$, $f(c) = b$. Clearly f is almost contra pgb - continuous but f is not almost contra b - continuous. Because $f^{-1}(\{b\}) = \{c\}$ is not b - closed in (X, τ) where $\{b\}$ is regular - open in (Y, σ).*
ii) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Clearly f is almost contra pgb - continuous but f is not almost contra $g\alpha$ - continuous. Because $f^{-1}(\{b\}) = \{a\}$ is not $g\alpha$ - closed in (X, τ) where $\{a\}$ is regular - open in (Y, σ).
iii) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. Clearly f is almost contra pgb - continuous but f is not almost contra $g\alpha*$ - continuous. Because $f^{-1}(\{b\}) = \{b\}$ is not $g\alpha*$ - closed in (X, τ) where $\{b\}$ is regular - open in (Y, σ).
iv) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Clearly f is almost contra pgb - continuous but f is not almost contra g - continuous. Because $f^{-1}(\{b\}) = \{a\}$ is not g - closed in (X, τ) where $\{b\}$ is regular - open in (Y, σ).
v) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{b\}, \{c\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Clearly f is almost contra pgb - continuous but f is not almost contra rgb - continuous. Because $f^{-1}(\{c\}) = \{a\}$ is not rgb - closed in (X, τ) where $\{c\}$ is regular - open in (Y, σ).
vi) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. Clearly f is almost contra pgb - continuous but f is not almost contra $g\alpha b$ - continuous. Because $f^{-1}(\{a\}) = \{a\}$ is not $g\alpha b$ - closed in (X, τ) where $\{b\}$ is regular - open in (Y, σ).

Theorem 3.3. *The following are equivalent for a function $f : X \to Y$,*
(1) f is almost contra pgb - continuous.
(2) for every regular closed set F of Y, $f^{-1}(F)$ is pgb - open set of X.
(3) for each $x \in X$ and each regular closed set F of Y containing $f(x)$, there exists pgb - open U containing x such that $f(U) \subset F$.
(4) for each $x \in X$ and each regular open set V of Y not containing $f(x)$, there exists pgb - closed set K not containing x such that $f^{-1}(V) \subset K$.

Proof. (1) $\Rightarrow$ (2) : Let F be a regular closed set in Y, then $Y - F$ is a regular open set in Y. By (1), $f^{-1}(Y - F) = X - f^{-1}(F)$ is pgb - closed set in X. This implies $f^{-1}(F)$ is pgb - open set in X. Therefore, (2) holds.
(2) $\Rightarrow$ (1) : Let G be a regular open set of Y. Then $Y - G$ is a regular closed set in Y. By (2), $f^{-1}(Y - G)$ is pgb - open set in X. This implies $X - f^{-1}(G)$ is pgb - open set in X, which implies $f^{-1}(G)$ is pgb - closed set in X. Therefore, (1) hold.
(2) $\Rightarrow$ (3) : Let F be a regular closed set in Y containing $f(x)$, which implies $x \in f^{-1}(F)$. By (2), $f^{-1}(F)$ is pgb - open in X containing x. Set $U = f^{-1}(F)$, which implies U is pgb - open in X containing x and $f(U) = f(f^{-1}(F)) \subset F$. Therefore (3) holds.
(3) $\Rightarrow$ (2) : Let F be a regular closed set in Y containing $f(x)$, which implies $x \in f^{-1}(F)$. From (3), there exists pgb - open U_x in X containing x such that $f(U_x) \subset F$. That is $U_x \subset f^{-1}(F)$. Thus $f^{-1}(F) = \{\cup U_x : x \in f^{-1}(F)$, which is union of pgb - open sets. Therefore, $f^{-1}(F)$ is pgb - open set of X.
(3) $\Rightarrow$ (4) : Let V be a regular open set in Y not containing $f(x)$. Then $Y - V$ is a regular closed set in Y containing $f(x)$. From (3), there exists a pgb - open set U in X containing x such that $f(U) \subset Y - V$ .This implies $U \subset f^{-1}(Y - V) = X - f^{-1}(V)$. Hence, $f^{-1}(V) \subset X - U$. Set $K = X - V$, then K is pgb - closed set not containing x in X such that $f^{-1}(V) \subset K$.
(4) $\Rightarrow$ (3) : Let F be a regular closed set in Y containing $f(x)$. Then $Y - F$ is a regular open set in Y not containing $f(x)$. From (4), there exists pgb - closed set K in X not containing x such that $f^{-1}(Y - F) \subset K$. This implies $X - f^{-1}(F) \subset K$. Hence, $X - K \subset f^{-1}(F)$, that is $f(X - K) \subset F$. Set $U = X - K$, then U is pgb - open set containing x in X such that $f(U) \subset F$. □

Theorem 3.4. *The following are equivalent for a function $f : X \to Y$,*
(1) f is almost contra pgb - continuous.
(2) $f^{-1}(Int(Cl(G)))$ is pgb - closed set in X for every open subset G of Y.
(3) $f^{-1}(Cl(Int(F)))$ is pgb - open set in X for every closed subset F of Y.

Proof. (1) $\Rightarrow$ (2) : Let G be an open set in Y. Then $Int(Cl(G))$ is regular open set in Y. By (1), $f^{-1}(Int(Cl(G)) \in pgb - C(X)$.
(2) $\Rightarrow$ (1) : Proof is obvious.
(1) $\Rightarrow$ (3) : Let F be a closed set in Y. Then $Cl(Int(G))$ is regular closed set in Y. By (1), $f^{-1}(Cl(Int(G)) \in pgb - O(X)$.
(3) $\Rightarrow$ (1) : Proof is obvious. □

Definition 3.4. *A function $f : X \to Y$ is said to be R - map if $f^{-1}(V)$ is regular open in X for each regular open set V of Y.*

Definition 3.5. *A function $f : X \to Y$ is said to be perfectly continuous if $f^{-1}(V)$ is clopen in X for each open set V of Y.*

Theorem 3.5. *For two functions $f : X \to Y$ and $g : Y \to Z$, let $g \circ f : X \to Z$ be a composition function. Then, the following properties hold.*
(1) If f is almost contra pgb - continuous and g is an R - map, then $g \circ f$ is almost contra pgb - continuous.
(2) If f is almost contra pgb - continuous and g is perfectly continuous, then $g \circ f$ is contra pgb - continuous.
(3) If f is contra pgb - continuous and g is almost continuous, then $g \circ f$ is almost contra pgb - continuous.

Proof. (1) Let V be any regular open set in Z. Since g is an R - map, $g^{-1}(V)$ is regular open in Y. Since f is almost contra pgb - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is pgb - closed set in X. Therefore $g \circ f$ is almost contra pgb - continuous.
(2) Let V be any regular open set in Z. Since g is perfectly continuous, $g^{-1}(V)$ is clopen in Y. Since f is almost contra pgb - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is pgb - open and pgb - closed set in X. Therefore $g \circ f$

is pgb continuous and contra pgb - continuous.
(3) Let V be any regular open set in Z. Since g is almost continuous, $g^{-1}(V)$ is open in Y. Since f is almost contra pgb - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is pgb - closed set in X. Therefore $g \circ f$ is almost contra pgb - continuous. □

Theorem 3.6. *Let $f : X \to Y$ be a contra pgb - continuous and $g : Y \to Z$ be pgb - continuous. If Y is $Tpgb$ - space, then $g \circ f : X \to Z$ is an almost contra pgb - continuous.*

Proof. Let V be any regular open and hence open set in Z. Since g is pgb - continuous $g^{-1}(V)$ is pgb - open in Y and Y is $Tpgb$ - space implies $g^{-1}(V)$ open in Y. Since f is contra pgb - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is pgb - closed set in X. Therefore, $g \circ f$ is an almost contra pgb - continuous. □

Theorem 3.7. *If $f : X \to Y$ is surjective strongly pgb - open (or strongly pgb - closed) and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is an almost contra pgb - continuous, then g is an almost contra pgb - continuous.*

Proof. Let V be any regular closed (resp. regular open) set in Z. Since $g \circ f$ is an almost contra pgb - continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is pgb - open (resp. pgb - closed) in X. Since f is surjective and strongly pgb - open (or strongly pgb - closed), $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is pgb - open(or pgb - closed). Therefore g is an almost contra pgb - continuous. □

Definition 3.6. *A function $f : X \to Y$ is called weakly pgb - continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in pgb - O(X; x)$ such that $f(U) \subset cl(V)$.*

Theorem 3.8. *If a function $f : X \to Y$ is an almost contra pgb - continuous, then f is weakly pgb - continuous function.*

Proof. Let $x \in X$ and V be an open set in Y containing $f(x)$. Then $cl(V)$ is regular closed in Y containing $f(x)$. Since f is an almost contra pgb - continuous function by Theorem 3.4 (2), $f^{-1}(cl(V))$ is pgb - open set in X containing x. Set $U = f^{-1}(cl(V))$, then $f(U) \subset f(f^{-1}(Cl(V))) \subset cl(V)$. This shows that f is weakly pgb - continuous function. □

Definition 3.7. *A space X is called locally pgb - indiscrete if every pgb - open set is closed in X.*

Theorem 3.9. *If a function $f : X \to Y$ is almost contra pgb - continuous and X is locally pgb - indiscrete space, then f is almost continuous.*

Proof. Let U be a regular open set in Y. Since f is almost contra pgb - continuous $f^{-1}(U)$ is pgb - closed set in X and X is locally pgb - indiscrete space, which implies $f^{-1}(U)$ is an open set in X. Therefore f is almost continuous. □

Lemma 3.1. *Let A and X_0 be subsets of a space X. If $A \in pgb - O(X)$ and $X_0 \in \tau^\alpha$, then $A \cap X_0 \in pgb - O(X_0)$.*

Theorem 3.10. *If $f : X \to Y$ is almost contra pgb - continuous and $X_0 \in \tau^\alpha$ then the restriction $f/X_0 : X_0 \to Y$ is almost contra pgb - continuous.*

Proof. Let V be any regular open set of Y. By Theorem, we have $f^{-1}(V) \in pgb - O(X)$ and hence $(f/X_0)^{-1}(V) = f^{-1}(V) \cap X_0 \in pgb - O(X_0)$. By Lemma 3.1, it follows that f/X_0 is almost contra pgb - continuous. □

Theorem 3.11. *If $f : X \to \prod Y_\lambda$ is almost contra pgb - continuous, then $P_\lambda \circ f : X \to Y_\lambda$ is almost contra pgb - continuous for each $\lambda \in \nabla$, where P_λ is the projection of $\prod Y_\lambda$ onto Y_λ.*

Proof. Let Y_λ be any regular open set of Y. Since P_λ is continuous open, it is an R - map and hence $(P_\lambda)^{-1} \in RO(\prod Y_\lambda)$.
By theorem, $f^{-1}(P_\lambda^{-1}(V)) = (P_\lambda \circ f)^{-1} \in pgb - O(X)$. Hence $P_\lambda \circ f$ is almost contra pgb - continuous. □

4 Pre Generalized b - Regular Graphs and Strongly Contra Pre Generalized b - Closed Graphs

Definition 4.8. *A graph G_f of a function $f : X \to Y$ is said to be pgb - regular (strongly contra pgb - closed) if for each $(x, y) \in (X \times Y) \backslash G_f$, there exist a pgb - closed set U in X containing x and $V \in R - O(Y)$ such that $(U \times V) \cap G_f = \varphi$.*

Theorem 4.12. *If $f : X \to Y$ is almost contra pgb - continuous and Y is T_2, then G_f is pgb - regular in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) \backslash G_f)$. It is obvious that $f(x) \neq y$. Since Y is T_2, there exists $V, W \in RO(Y)$ such that $f(x) \in V$, $y \in W$ and $V \cap W = \varphi$. Since f is almost contra pgb - continuous, $f^{-1}(V)$ is a pgb - closed set in X containing x. If we take $U = f^{-1}(V)$, we have $f(U) \subset V$. Hence, $f(U) \cap W = \varphi$ and G_f is pgb - regular. □

Theorem 4.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. Then f is almost pgb - continuous if and only if g is almost pgb - continuous.*

Proof. **Necessary :** Let $x \in X$ and $V \in pgb - O(Y)$ containing $f(x)$. Then, we have $g(x) = (x, f(x)) \in R - O(X \times Y)$. Since f is almost pgb - continuous, there exists a pgb - open set U of X containing x such that $g(U) \subset X \times Y$. Therefore, we obtain $f(U) \subset V$. Hence f is almost pgb continuous.
Sufficiency : Let $x \in X$ and w be a regular open set of $X \times Y$ containing $g(x)$. There exists $U_1 \in RO(X, \tau)$ and $V \in RO(Y, \sigma)$ such that $(x, f(x)) \in (U_1 \times V) \subset W$. Since f is almost pgb - continuous, there exists $U_2 \in pgb - O(X, \tau)$ such that $x \in U_2$ and $f(U_2) \subset V$. Set $U = U_1 \cap U_2$. We have $x \in U_x \in pgb - O(X, \tau)$ and $g(U) \subset (U_1 \times V) \subset W$. This shows that g is almost pgb - continuous. □

Theorem 4.14. *If a function $f : X \to Y$ be a almost contra pgb - continuous and almost continuous, then f is regular set - connected.*

Proof. Let $V \in RO(Y)$. Since f is almost contra pgb - continuous and almost continuous, $f^{-1}(V)$ is pgb - closed and open. So $f^{-1}(V)$ is clopen. It turns out that f is regular set - connected. □

5 Connectedness

Definition 5.9. *A space X is called pgb - connected if X cannot be written as a disjoint union of two non - empty pgb - open sets.*

Theorem 5.15. *If $f : X \to Y$ is an almost contra pgb - continuous surjection and X is pgb - connected, then Y is connected.*

Proof. Suppose that Y is not a connected space. Then Y can be written as $Y = U_0 \cup V_0$ such that U_0 and V_0 are disjoint non - empty open sets. Let $U = int(cl(U_0))$ and $V = int(cl(V_0))$. Then U and V are disjoint nonempty regular open sets such that $Y = U \cup V$. Since f is almost contra pgb - continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are pgb - open sets of X. We have $X = f^{-1}(U) \cup f^{-1}(V)$ such that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. Since f is surjective, this shows that X is not pgb - connected. Hence Y is connected. □

Theorem 5.16. *The almost contra pgb - continuous image of pgb - connected space is connected.*

Proof. Let $f : X \to Y$ be an almost contra pgb - continuous function of a pgb - connected space X onto a topological space Y. Suppose that Y is not a connected space. There exist non - empty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in Y. Since f is almost contra pgb - continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are pgb - open in X. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are non - empty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not pgb - connected. This is a contradiction and hence Y is connected. □

Definition 5.10. *A topological space X is said to be pgb - ultra connected if every two non - empty pgb - closed subsets of X intersect.*

A topological space X is said to be hyper connected if every open set is dense.

Theorem 5.17. *If X is pgb - ultra connected and $f : X \to Y$ is an almost contra pgb - continuous surjection, then Y is hyper connected.*

Proof. Suppose that Y is not hyperconnected. Then, there exists an open set V such that V is not dense in Y. So, there exist non - empty regular open subsets $B_1 = int(cl(V))$ and $B_2 = Y - cl(V)$ in Y. Since f is almost contra pgb - continuous, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are disjoint pgb - closed. This is contrary to the pgb - ultra - connectedness of X. Therefore, Y is hyperconnected. □

6 Separation axioms

Definition 6.11. *A topological space X is said to be $pgb - T_1$ space if for any pair of distinct points x and y, there exist a pgb - open sets G and H such that $x \in G$, $y \notin G$ and $x \notin H$, $y \in H$.*

Theorem 6.18. *If $f : X \to Y$ is an almost contra pgb - continuous injection and Y is weakly Hausdorff, then X is $pgb - T_1$.*

Proof. Suppose Y is weakly Hausdorff. For any distinct points x and y in X, there exist V and W regular closed sets in Y such that $f(x) \in V$, $f(y) \notin V$, $f(y) \in W$ and $f(x) \notin W$. Since f is almost contra pgb - continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are pgb - open subsets of X such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $y \in f^{-1}(W)$ and $x \notin f^{-1}(W)$. This shows that X is $pgb - T_1$. □

Corollary 6.1. *If $f : X \to Y$ is a contra pgb - continuous injection and Y is weakly Hausdorff, then X is $pgb - T_1$.*

Definition 6.12. *A topological space X is called Ultra Hausdorff space, if for every pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X containing x and y, respectively.*

Definition 6.13. *A topological space X is said to be $pgb - T_2$ space if for any pair of distinct points x and y, there exist disjoint pgb - open sets G and H such that $x \in G$ and $y \in H$.*

Theorem 6.19. *If $f : X \to Y$ is an almost contra pgb - continuous injective function from space X into a Ultra Hausdorff space Y, then X is $pgb - T_2$.*

Proof. Let x and y be any two distinct points in X. Since f is an injective $f(x) \neq f(y)$ and Y is Ultra Hausdorff space, there exist disjoint clopen sets U and V of Y containing $f(x)$ and $f(y)$ respectively. Then $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint pgb - open sets in X. Therefore X is $pgb - T_2$. □

Definition 6.14. *A topological space X is called Ultra normal space, if each pair of disjoint closed sets can be separated by disjoint clopen sets.*

Definition 6.15. *A topological space X is said to be pgb - normal if each pair of disjoint closed sets can be separated by disjoint pgb - open sets.*

Theorem 6.20. *If $f : X \to Y$ is an almost contra pgb - continuous closed injection and Y is ultra normal, then X is pgb - normal.*

Proof. Let E and F be disjoint closed subsets of X. Since f is closed and injective $f(E)$ and $f(F)$ are disjoint closed sets in Y . Since Y is ultra normal there exists disjoint clopen sets U and V in Y such that $f(E) \subset U$ and $f(F) \subset V$. This implies $E \subset f^{-1}(U)$ and $F \subset f^{-1}(V)$. Since f is an almost contra pgb - continuous injection, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint pgb - open sets in X. This shows X is pgb - normal. □

Theorem 6.21. *If $f : X \to Y$ is an almost contra pgb - continuous and Y is semi - regular, then f is pgb - continuous.*

Proof. Let $x \in X$ and V be an open set of Y containing $f(x)$. By definition of semi - regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset V$. Since f is almost contra pgb - continuous, there exists $U \in pgb - O(X, x)$ such that $f(U) \subset G$. Hence we have $f(U) \subset G \subset V$. This shows that f is pgb - continuous function. □

7 Compactness

Definition 7.16. *A space X is said to be:*
(1) pgb - compact if every pgb - open cover of X has a finite subcover.
(2) pgb - closed compact if every pgb - closed cover of X has a finite subcover.
(3) Nearly compact if every regular open cover of X has a finite subcover.
(4) Countably pgb - compact if every countable cover of X by pgb - open sets has a finite subcover.
(5) Countably pgb - closed compact if every countable cover of X by pgb - closed sets has a finite sub cover.
(6) Nearly countably compact if every countable cover of X by regular open sets has a finite sub cover.
(7) pgb - Lindelof if every pgb - open cover of X has a countable sub cover.
(8) pgb - Lindelof if every pgb - closed cover of X has a countable sub cover.
(9) Nearly Lindelof if every regular open cover of X has a countable sub cover.
(10) S - Lindelof if every cover of X by regular closed sets has a countable sub cover.
(11) Countably S - closed if every countable cover of X by regular closed sets has a finite sub - cover.
(12) S - closed if every regular closed cover of x has a finite sub cover.

Theorem 7.22. *Let $f : X \to Y$ be an almost contra pgb - continuous surjection. Then, the following properties hold:*
(1) If X is pgb - closed compact, then Y is nearly compact.
(2) If X is countably pgb - closed compact, then Y is nearly countably compact.
(3) If X is pgb - Lindelof, then Y is nearly Lindelof.

Proof. (1) Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y. Since f is almost contra pgb - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is pgb - closed cover of X. Since X is pgb - closed compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{(V_\alpha) : \alpha \in I_0\}$ which is finite sub cover of Y, therefore Y is nearly compact.
(2) Let $\{V_\alpha : \alpha \in I\}$ be any countable regular open cover of Y . Since f is almost contra pgb - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable pgb - closed cover of X. Since X is countably pgb - closed compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{(V_\alpha) : \alpha \in I_0\}$ is finite subcover for Y . Hence Y is nearly countably compact.
(3) Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y. Since f is almost contra pgb - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is pgb - closed cover of X. Since X is pgb - Lindelof, there exists a countable subset I_0 of I such that $X = \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{(V_\alpha) : \alpha \in I_0\}$ is finite sub cover for Y . Therefore, Y is nearly Lindelof. □

Theorem 7.23. *Let $f : X \to Y$ be an almost contra pgb - continuous surjection. Then, the following properties hold:*
(1) If X is pgb - compact, then Y is S - closed.
(2) If X is countably pgb - closed, then Y is is countably S - closed.
(3) If X is pgb - Lindelof, then Y is S - Lindelof.

Proof. (1) Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y. Since f is almost contra pgb - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is pgb - open cover of X. Since X is pgb - compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{V_\alpha : \alpha \in I_0\}$ is finite sub cover for Y. Therefore, Y is S - closed.
(2) Let $\{V_\alpha : \alpha \in I\}$ be any countable regular closed cover of Y. Since f is almost contra pgb - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable pgb - open cover of X. Since X is countably pgb - compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{V_\alpha : \alpha \in I_0\}$ is finite sub cover for Y. Hence, Y is countably S - closed.
(3) Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y. Since f is almost contra pgb - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is pgb - open cover of X. Since X is pgb - Lindelof, there exists a countable sub - set I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{V_\alpha : \alpha \in I_0\}$ is finite sub cover for Y. Hence, Y is S - Lindelof. □

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General Solution and Two Methods of Generalized Ulam - Hyers Stability of $n-$ Dimensional AQCQ Functional Equation

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Abstract

In this paper, we achieve the general solution and generalized Ulam - Hyers stability of a $n-$ dimensional additive-quadratic-cubic-quartic (AQCQ) functional equation

$$f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) = 4f\left(\sum_{i=1}^{n} v_i\right) + 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) - 6f\left(\sum_{i=1}^{n-1} v_i\right) + f(2v_n) + f(-2v_n) - 4f(v_n) - 4f(-v_n)$$

where n is a positive integer with $n \geq 3$ in Banach Space (**BS**) via direct and fixed point methods. The stability results are discussed in two different ways by assuming n is an odd positive integer and n is an even positive integer.

Keywords: AQCQ functional equation, generalized Ulam - Hyers stability, Banach space, fixed point.

2010 MSC: 39B52, 32B72, 32B82.

1 Introduction

The education of stability problems for functional equations is tied to a question of Ulam [61] regarding the stability of group homomorphisms and certainly answered for a additive functional equation on Banach spaces by Hyers [30] and Aoki [3]. It was further generalized and marvelous outcome obtained by number of authors [24, 44, 53, 58].

The general solution and the generalized Hyers-Ulam-Rassias stability of the generalized mixed type of functional equation

$$f(x+ay) + f(x-ay) = a^2\left[f(x+y) + f(x-y)\right] + 2\left(1-a^2\right)f(x) + \frac{\left(a^4-a^2\right)}{12}\left[f(2y) + f(-2y) - 4f(y) - 4f(-y)\right]. \tag{1.1}$$

for fixed integers a with $a \neq 0, \pm 1$ having solution **additive, quadratic, cubic and quartic** was discussed by K. Ravi et. al., [59] . Its generalized Ulam-Hyers stability in multi-Banach spaces and non-Archimedean normed spaces via fixed point approach was respectively investigated by T.Z. Xu et. al [62, 63].

Very recently, Choonkil Park and Jung Rye Lee [42] proved the Hyers - Ulam stability of the following additive - quadratic - cubic - quartic functional equation

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \tag{1.2}$$

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in paranormed spaces.

During the last seven decades, the stability problems of various functional equations in several spaces have been broadly investigated by number of mathematicians [4] - [18], [20] - [23], [25] - [29], [32] - [34], [39, 40, 43], [48] - [52], [56, 57].

Now, we will recall the fundamental results in fixed point theory [36].

Theorem 1.1. *(Banach's contraction principle) Let (X,d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx,Ty) \leq Ld(x,y)$ for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ \ x \in X$;

(A4) $d(x,x^*) \leq \frac{1}{1-L}\ \ d(x,x^*), \forall\ \ x \in X$.

Theorem 1.2. *Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that
(FP1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T
(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

In this paper, we established the generalized Ulam - Hyers stability of a $n-$ dimensional additive-quadratic-cubic-quartic (AQCQ) functional equation

$$f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) = 4f\left(\sum_{i=1}^{n} v_i\right) + 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) - 6f\left(\sum_{i=1}^{n-1} v_i\right) + f\left(2v_n\right) + f\left(-2v_n\right) - 4f\left(v_n\right) - 4f\left(-v_n\right) \tag{1.3}$$

where n is a positive integer with $n \geq 3$ in Banach Space (**BS**) via direct and fixed point methods. The stability results are discussed in two different ways by assuming n is an odd positive integer and n is an even positive integer.

In section 2, the general solution of (1.3) is present.

In Sections 3 and 4, the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an odd positive integer and n is an even positive integer in Banach space using direct method are discussed, respectively.

In Sections 5 and 6, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an odd positive integer and n is an even positive integer in Banach space using fixed point methods, respectively.

In Section 7, we conclude with the non stable cases for the functional equation (1.3).

2 General Solution

In this section, we provide the general solution of the function equation (1.3). To prove this, let us take $\mathcal{I}$ and $\mathcal{J}$ be real vector spaces.

Lemma 2.1. *If a function $f : \mathcal{I} \to \mathcal{J}$ fulfills (1.3) for all $v_1, \cdots, v_n \in \mathcal{I}$ if and only if $f : \mathcal{I} \to \mathcal{J}$ satisfies (1.2) for all $x, y \in \mathcal{I}$.*

Proof. Let $f : \mathcal{I} \to \mathcal{J}$ be a function fulfills (1.3). Replacing $(v_1, v_2, v_3 \cdots, v_{n-1}, v_n)$ by $(x, 0, 0, \cdots, 0, y)$ in (1.3), we get (1.2) as desired. Conversely, let $f : \mathcal{I} \to \mathcal{J}$ be a function satisfying (1.2). Changing (x, y) by $(v_1 + v_2 + v_3 \cdots + v_{n-1}, v_n)$ in (1.2), we arrive (1.3) as desired. □

Lemma 2.2. *If $f : \mathcal{I} \to \mathcal{J}$ be an odd mapping fulfills (1.3) and let $a : \mathcal{I} \to \mathcal{J}$ be a mapping given by*

$$a(v) = f(2v) - 8f(v) \tag{2.1}$$

for all $v \in \mathcal{I}$ then

$$a(2v) = 2a(v) \tag{2.2}$$

for all $v \in \mathcal{I}$ such that a is additive.

Proof. Using oddness of f in (1.3), we arrive

$$f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) = 4f\left(\sum_{i=1}^{n} v_i\right) + 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) - 6f\left(\sum_{i=1}^{n-1} v_i\right) \tag{2.3}$$

for all $v_1, \cdots, v_n \in \mathcal{I}$. Letting $(v_1, \cdots, v_n)$ by $(0, \cdots, 0)$ in (2.3), we find that

$$f(0) = 0. \tag{2.4}$$

Also, replacing $(v_2, v_3, \cdots, v_{n-1})$ by $(0, 0, \cdots, 0)$ in (2.3), we get

$$f(v_1 + 2v_n) + f(v_1 - 2v_n) = 4f(v_1 + v_n) + 4f(v_1 - v_n) - 6f(v_1) \tag{2.5}$$

for all $v_1, v_n \in \mathcal{I}$. Changing (v_1, v_n) by (v, v) in (2.5), we obtain

$$f(3v) = 4f(2v) - 5f(v) \tag{2.6}$$

for all $v \in \mathcal{I}$. Again changing (v_1, v_n) by $(2v, v)$ in (2.5) and using (2.4), (2.6), we arrive

$$f(4v) = 4f(3v) - 6f(2v) + 4f(v) \tag{2.7}$$

for all $v \in \mathcal{I}$. Using (2.6) in (2.7), we get

$$f(4v) = 10f(2v) - 16f(v) \tag{2.8}$$

for all $v \in \mathcal{I}$. From (2.1), we have

$$a(2v) - 2a(v) = f(4v) - 10f(2v) + 16f(v) \tag{2.9}$$

for all $v \in \mathcal{I}$. Using (2.8) in (2.9), we desired our result. □

Lemma 2.3. *If $f : \mathcal{I} \to \mathcal{J}$ be an odd mapping fulfills (1.3) and let $c : \mathcal{I} \to \mathcal{J}$ be a mapping given by*

$$c(v) = f(2v) - 2f(v) \tag{2.10}$$

for all $v \in \mathcal{I}$ then

$$c(2v) = 8c(v) \tag{2.11}$$

for all $v \in \mathcal{I}$ such that c is cubic.

Proof. It follows from (2.11) that

$$c(2v) - 8c(v) = f(4v) - 10f(2v) + 16f(v) \tag{2.12}$$

for all $u \in \mathcal{I}$. Using (2.8) in (2.12), we desired our result. □

Lemma 2.4. *If $f : \mathcal{I} \to \mathcal{J}$ be an even mapping fulfills (1.3) and let $q_2 : \mathcal{I} \to \mathcal{J}$ be a mapping given by*

$$q_2(v) = f(2v) - 16f(v) \tag{2.13}$$

for all $v \in \mathcal{I}$ then

$$q_2(2v) = 4q_2(v) \tag{2.14}$$

for all $v \in \mathcal{I}$ such that q_2 is quadratic.

Proof. Using evenness of f in (1.3), we get

$$f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) = 4f\left(\sum_{i=1}^{n} v_i\right) + 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) - 6f\left(\sum_{i=1}^{n-1} v_i\right) + 2f\left(2v_n\right) - 8f\left(v_n\right) \tag{2.15}$$

for all $v_1, \cdots, v_n \in \mathcal{I}$. Letting $(v_1, \cdots, v_n)$ by $(0, \cdots, 0)$ in (2.15), we obtain

$$f(0) = 0. \tag{2.16}$$

Replacing $(v_2, v_3, \cdots, v_{n-1})$ by $(0, 0, \cdots, 0)$ in (2.15), we arrive

$$f\left(v_1 + 2v_n\right) + f\left(v_1 - 2v_n\right) = 4f\left(v_1 + v_n\right) + 4f\left(v_1 - v_n\right) - 6f\left(v_1\right) + 2f\left(2v_n\right) - 8f\left(v_n\right) \tag{2.17}$$

for all $v_1, v_n \in \mathcal{I}$. Setting (v_1, v_n) by (v, v) in (2.17), we have

$$f(3v) = 6f(2v) - 15f(v) \tag{2.18}$$

for all $v \in \mathcal{I}$. Again setting (v_1, v_n) by $(2v, v)$ in (2.17) and using (2.16), (2.18), we arrive

$$f(4v) = 4f(3v) - 4f(2v) - 4f(v) \tag{2.19}$$

for all $v \in \mathcal{I}$. Using (2.18) in (2.19), we get

$$f(4v) = 20f(2v) - 64f(v) \tag{2.20}$$

for all $v \in \mathcal{I}$. From (2.13), we establish

$$q_2(2v) - 4q_2(v) = f(4v) - 20f(2v) + 64f(v) \tag{2.21}$$

for all $v \in \mathcal{I}$. Using (2.20) in (2.21), we desired our result. □

Lemma 2.5. *If $f : \mathcal{I} \to \mathcal{J}$ be an even mapping fulfills (1.3) and let $q_4 : \mathcal{I} \to \mathcal{J}$ be a mapping given by*

$$q_4(v) = f(2v) - 4f(v) \tag{2.22}$$

for all $v \in \mathcal{I}$ then

$$q_4(2v) = 16q_4(v) \tag{2.23}$$

for all $v \in \mathcal{I}$ such that q_4 is quartic.

Proof. It follows from (2.23) that

$$q_4(2v) - 4q_4(v) = f(4v) - 20f(2v) + 64f(v) \tag{2.24}$$

for all $v \in \mathcal{I}$. Using (2.20) in (2.24), we desired our result. □

Remark 2.1. *If $f : \mathcal{I} \to \mathcal{J}$ be a mapping fulfills (1.3) then there exists $f_o, f_e : \mathcal{I} \to \mathcal{J}$ and let $a, q_2, c, q_4 : \mathcal{I} \to \mathcal{J}$ be a mapping defined in (2.1), (2.10), (2.13) and (2.22), we have*

$$f_e(v) = \frac{1}{12}(q_4(v) - q_2(v)) \tag{2.25}$$

and

$$f_o(v) = \frac{1}{6}(c(v) - a(v)) \tag{2.26}$$

for all $v \in \mathcal{I}$. Also if we define

$$f(v) = f_e(v) + f_o(v) \tag{2.27}$$

we arrive

$$f(v) = \frac{1}{12}(q_4(v) - q_2(v)) + \frac{1}{6}(c(v) - a(v)) \tag{2.28}$$

for all $v \in \mathcal{I}$.

Throughout this paper, let we consider $\mathcal{Y}$ be a normed space and $\mathcal{Z}$ be a Banach space. Define a mapping $Df_{aqcq}:\mathcal{Y}\to\mathcal{Z}$ by

$$\begin{aligned}Df_{aqcq}(v_1,\cdots,v_n) &= f\left(\sum_{i=1}^{n-1}v_i+2v_n\right)+f\left(\sum_{i=1}^{n-1}v_i-2v_n\right)\\ &\quad -4f\left(\sum_{i=1}^{n}v_i\right)-4f\left(\sum_{i=1}^{n-1}v_i-v_n\right)+6f\left(\sum_{i=1}^{n-1}v_i\right)\\ &\quad -f(2v_n)-f(-2v_n)+4f(v_n)+4f(-v_n)\end{aligned}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$.

3 Stability Results - Direct Method: n Odd Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an odd positive integer in Banach space using direct method.

3.1 f IS AN ODD FUNCTION

Theorem 3.3. *Let $q=\pm1$ and $\omega,\Omega:\mathcal{Y}^n\to[0,\infty)$ be functions such that*

$$\lim_{p\to\infty}\frac{\omega\left(2^{pq}v_1,\cdots,2^{pq}v_n\right)}{2^{pq}}=0 \tag{3.1}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$. Let $Df_{aqcq}:\mathcal{Y}\to\mathcal{Z}$ be an odd function satisfying the inequality

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\|\le\omega\left(v_1,\cdots,v_n\right) \tag{3.2}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$. Then there exists a unique additive function $\mathcal{A}:\mathcal{Y}\to\mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{A}(v)-a(v)\|=\|\mathcal{A}(v)-\{f(2v)-8f(v)\}\|\le\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)}{2^{rq}} \tag{3.3}$$

where $\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)$ and $\mathcal{A}(v)$ are defined by

$$\begin{aligned}&\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)\\ &\quad=4\omega_1(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)+\omega_1(2\cdot2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)\\ &\quad=4\omega\left(2^{rq}v,\ \underbrace{2^{rq}v,2^{rq}v,\cdots,2^{rq}v}_{\frac{n-3}{2}\ times},\underbrace{-2^{rq}v,-2^{rq}v,\cdots,-2^{rq}v}_{\frac{n-3}{2}\ times},0,2^{rq}v\right)\\ &\qquad+\omega\left(2\cdot2^{rq}v,\ \underbrace{2^{rq}v,2^{rq}v,\cdots,2^{rq}v}_{\frac{n-3}{2}\ times},\underbrace{-2^{rq}v,-2^{rq}v,\cdots,-2^{rq}v}_{\frac{n-3}{2}\ times},0,2^{rq}v\right)\end{aligned} \tag{3.4}$$

and

$$\mathcal{A}(v)=\lim_{p\to\infty}\frac{a(2^{pq}v)}{2^{pq}}=\lim_{p\to\infty}\frac{f(2\cdot2^{pq}v)-8f(2^{pq}v)}{2^{pq}} \tag{3.5}$$

for all $v\in\mathcal{Y}$, respectively.

Proof. **Case (i):** Assume $q=1$.

Given, f is an odd function. Using oddness of f in (3.2), we arrive

$$\left\|f\left(\sum_{i=1}^{n-1}v_i+2v_n\right)+f\left(\sum_{i=1}^{n-1}v_i-2v_n\right)-4f\left(\sum_{i=1}^{n}v_i\right)-4f\left(\sum_{i=1}^{n-1}v_i-v_n\right)+6f\left(\sum_{i=1}^{n-1}v_i\right)\right\|\le\omega\left(v_1,\cdots,v_n\right) \tag{3.6}$$

for all $v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n \in \mathcal{Y}$. Replacing

$$(v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n) = \left(v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v \right)$$

in (3.6), we get

$$\begin{aligned} \Big\| f(3v) - 4f(2v) + 5f(v) \Big\| &\le \omega \left(v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v \right) \\ &= \omega_1(v, v, \cdots, -v, 0, v) \end{aligned} \tag{3.7}$$

for all $v \in \mathcal{Y}$. Again replacing

$$(v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n) = \left(2v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v \right)$$

in (3.6), we obtain

$$\begin{aligned} \Big\| f(4v) - 4f(3v) + 6f(2v) - 4f(v) \Big\| &\le \omega \left(2v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v \right) \\ &= \omega_1(2v, v, \cdots, -v, 0, v) \end{aligned} \tag{3.8}$$

for all $v \in \mathcal{Y}$. It follows from (3.7) and (3.8),

$$\begin{aligned} &\Big\| f(4v) - 10f(2v) + 16f(v) \Big\| \\ &\quad = \Big\| f(4v) - 4f(3v) + 4f(3v) + 6f(2v) - 16f(2v) + 20f(v) - 4f(v) \Big\| \\ &\quad \le 4\Big\| f(3v) - 4f(2v) + 5f(v) \Big\| + \Big\| f(4v) - 4f(3v) + 6f(2v) - 4f(v) \Big\| \\ &\quad \le 4\omega_1(v, v, \cdots, -v, 0, v) + \omega_1(2v, v, \cdots, -v, 0, v) \end{aligned} \tag{3.9}$$

for all $v \in \mathcal{Y}$. Define

$$\Omega(v, v, \cdots, -v, 0, v) = 4\omega_1(v, v, \cdots, -v, 0, v) + \omega_1(2v, v, \cdots, -v, 0, v) \tag{3.10}$$

for all $v \in \mathcal{Y}$. Using (3.10) in (3.9), we have

$$\Big\| f(4v) - 10f(2v) + 16f(v) \Big\| \le \Omega(v, v, \cdots, -v, 0, v) \tag{3.11}$$

for all $v \in \mathcal{Y}$. It follows from (3.11), we reach

$$\Big\| \Big\{ f(4v) - 8f(2v) \Big\} - 2\Big\{ f(2v) - 8f(v) \Big\} \Big\| \le \Omega(v, v, \cdots, -v, 0, v) \tag{3.12}$$

for all $v \in \mathcal{Y}$. Using (2.1) in (3.12), we land

$$\Big\| a(2v) - 2a(v) \Big\| \le \Omega(v, v, \cdots, -v, 0, v) \tag{3.13}$$

for all $v \in \mathcal{Y}$. It follows from (3.13) that

$$\Big\| \frac{a(2v)}{2} - a(v) \Big\| \le \frac{\Omega(v, v, \cdots, -v, 0, v)}{2} \tag{3.14}$$

for all $v \in \mathcal{Y}$. Now, replacing v by $2v$ and dividing by 2 in (3.14), we have

$$\left\| \frac{a(2^2 v)}{2^2} - \frac{a(2v)}{2} \right\| \leq \frac{\Omega(2v, 2v, \cdots, -2v, 0, 2v)}{2^2} \tag{3.15}$$

for all $v \in \mathcal{Y}$. From (3.14) and (3.15), we obtain

$$\begin{aligned} \left\| \frac{a(2^2 v)}{2^2} - a(v) \right\| &\leq \left\| \frac{a(2^2 v)}{2^2} - \frac{a(2v)}{2} \right\| + \left\| \frac{a(2v)}{2} - a(v) \right\| \\ &\leq \frac{1}{2} \left[\Omega(v, v, \cdots, -v, 0, v) + \frac{\Omega(2v, 2v, \cdots, -2v, 0, 2v)}{2} \right] \end{aligned} \tag{3.16}$$

for all $v \in \mathcal{Y}$. Generalizing, for a positive integer p, we reach

$$\left\| \frac{a(2^p v)}{2^p} - a(v) \right\| \leq \frac{1}{2} \sum_{r=0}^{p-1} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^r} \tag{3.17}$$

for all $v \in \mathcal{Y}$. Thus, the sequence $\left\{ \frac{a(2^p v)}{2^p} \right\}$ is a Cauchy in $\mathcal{Z}$ and so it converges.

Indeed, to prove the convergence of the sequence $\left\{ \frac{a(2^p v)}{2^p} \right\}$, replacing v by $2^s v$ and dividing by 2^s in (3.17), for any $p, s > 0$, we get

$$\begin{aligned} \left\| \frac{a(2^{p+s} v)}{2^{(p+s)}} - \frac{a(2^s v)}{2^s} \right\| &= \frac{1}{2^s} \left\| \frac{a(2^p \cdot 2^s v)}{2^p} - a(2^s v) \right\| \\ &\leq \frac{1}{2^s} \frac{1}{2} \sum_{r=0}^{p-1} \frac{\Omega(2^r \cdot 2^s v, 2^r \cdot 2^s v, \cdots, -2^r \cdot 2^s v, 0, 2^r \cdot 2^s v)}{2^r} \\ &\leq \frac{1}{2} \sum_{r=0}^{\infty} \frac{\Omega(2^r \cdot 2^s v, 2^r \cdot 2^s v, \cdots, -2^r \cdot 2^s v, 0, 2^r \cdot 2^s v)}{2^r \cdot 2^s} \\ &\to 0 \quad as \;\; s \to \infty \end{aligned}$$

for all $v \in \mathcal{Y}$. Since $\mathcal{Z}$ is complete, we see that a mapping $\mathcal{A}(v) : \mathcal{Y} \to \mathcal{Z}$ defined by

$$\mathcal{A}(v) = \lim_{p \to \infty} \frac{a(2^p v)}{2^p}$$

for all $v \in \mathcal{Y}$. Letting $p \to \infty$ in (3.17), we see that (3.3) holds for all $v \in \mathcal{Y}$. In order to show that $\mathcal{A}$ satisfies (1.3), replacing $(v_1, \cdots, v_n)$ by $(2^p v_1, \cdots, 2^p v_n)$ and dividing by 2^p in (3.2), we have

$$\|\mathcal{A}(v_1, \cdots, v_n)\| = \lim_{p \to \infty} \frac{1}{2} \left\| D f_{aqcq}(2^p v_1, \cdots, 2^p v_n) \right\| \leq \lim_{p \to \infty} \frac{1}{2} \omega(2^p v_1, \cdots, 2^p v_n)$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ and so the mapping $\mathcal{A}$ is additive. Hence, $\mathcal{A}$ satisfies (1.3), for all $v_1, \cdots, v_n \in \mathcal{Y}$.

To prove that $\mathcal{A}$ is unique, we assume now that there is $\mathcal{A}'$ as another additive mapping satisfying (1.3) and the inequality (3.3). Then it follows easily that

$$\mathcal{A}(2^s v) = 2^s \mathcal{A}(v), \qquad \mathcal{A}'(2^s v) = 2^s \mathcal{A}'(v)$$

for all $v \in \mathcal{Y}$ and all $s \in \mathbb{N}$. Thus

$$\begin{aligned} \|\mathcal{A}(v) - \mathcal{A}'(v)\| &= \frac{1}{2^s} \|\mathcal{A}(2^s v) - \mathcal{A}'(2^s v)\| \\ &= \frac{1}{2^s} \left\{ \|\mathcal{A}(2^s v) - a(2^s v) + a(2^s v) - \mathcal{A}'(2^s v)\| \right\} \\ &\leq \frac{1}{2^s} \left\{ \|\mathcal{A}(2^s v) - a(2^s v)\| + \|a(2^s v) - \mathcal{A}'(2^s v)\| \right\} \\ &\leq \sum_{r=0}^{\infty} \frac{\Omega(2^r \cdot 2^s v, 2^r \cdot 2^s v, \cdots, -2^r \cdot 2^s v, 0, 2^r \cdot 2^s v)}{2^{(r+s)}} \end{aligned}$$

for all $v \in \mathcal{Y}$. Letting $s \to \infty$, in the above inequality, we achieve uniqueness of $\mathcal{A}$. Hence the theorem holds for $q = 1$.

Case (ii): Assume $q = -1$.

Now replacing v by $\frac{v}{2}$ in (3.13), we get

$$\left\|a(v) - 2a\left(\frac{v}{2}\right)\right\| \leq \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right) \tag{3.18}$$

for all $v \in \mathcal{Y}$. Now, replacing v by $\frac{v}{2}$ and multiply by 2 in (3.18), we have

$$\left\|2a\left(\frac{v}{2}\right) - 2^2 a\left(\frac{v}{2^2}\right)\right\| \leq 2\Omega\left(\frac{v}{2^2}, \frac{v}{2^2}, \cdots, -\frac{v}{2^2}, 0, \frac{v}{2^2}\right) \tag{3.19}$$

for all $v \in \mathcal{Y}$. From (3.18) and (3.19), we obtain

$$\begin{aligned}\left\|a(v) - 2^2 a\left(\frac{v}{2^2}\right)\right\| &\leq \left\|a(v) - 2a\left(\frac{v}{2}\right)\right\| + \left\|2a\left(\frac{v}{2}\right) - 2^2 a\left(\frac{v}{2^2}\right)\right\| \\ &\leq \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right) + 2\Omega\left(\frac{v}{2^2}, \frac{v}{2^2}, \cdots, -\frac{v}{2^2}, 0, \frac{v}{2^2}\right)\end{aligned} \tag{3.20}$$

for all $v \in \mathcal{Y}$. Generalizing, for a positive integer p, we reach

$$\left\|a(v) - 2^p a\left(\frac{v}{2^p}\right)\right\| \leq \sum_{r=1}^{p-1} 2^r \Omega\left(\frac{v}{2^r}, \frac{v}{2^r}, \cdots, -\frac{v}{2^r}, 0, \frac{v}{2^r}\right) \tag{3.21}$$

for all $v \in \mathcal{Y}$. The rest of the proof is similar to that of case $q = 1$. Hence for $q = -1$ also the theorem holds. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.3 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.1. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\|Df_{aqcq}(v_1, \cdots, v_n)\| \leq \begin{cases} b, & \\ b \sum_{i=1}^{n} ||v_i||^d, & d \neq 1; \\ b\left\{\prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 1; \end{cases} \tag{3.22}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|a(v) - \mathcal{A}(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d(5n - 6 + 2^{rd})}{|2 - 2^d|}, \\ \dfrac{b||v||^{nd}(5n - 6 + 2^{rnd})}{|2 - 2^{nd}|} \end{cases} \tag{3.23}$$

for all $v \in \mathcal{Y}$.

Theorem 3.4. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{3pq}} = 0 \tag{3.24}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality

$$\|Df_{aqcq}(v_1, \cdots, v_n)\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{3.25}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{C}(v) - c(v)\| = \|\mathcal{C}(v) - \{f(2v) - 2f(v)\}\| \leq \frac{1}{8} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} \tag{3.26}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$ is defined in (3.4) and $\mathcal{C}(v)$ is defined by

$$\mathcal{C}(v) = \lim_{p\to\infty} \frac{c(2^{pq}v)}{2^{3pq}} = \lim_{p\to\infty} \frac{f(2 \cdot 2^{pq}v) - 2f(2^{pq}v)}{2^{3pq}} \tag{3.27}$$

for all $v \in \mathcal{Y}$, respectively.

Proof. It follows from (3.11), we reach

$$\left\|\left\{f(4v)-2f(2v)\right\}-8\left\{f(2v)-2f(v)\right\}\right\| \leq \Omega(v,v,\cdots,-v,0,v) \tag{3.28}$$

for all $v \in \mathcal{Y}$. Using (2.10) in (3.28), we land

$$\left\|c(2v)-8c(v)\right\| \leq \Omega(v,v,\cdots,-v,0,v) \tag{3.29}$$

for all $v \in \mathcal{Y}$. The rest of the proof is similar to that of Theorem 3.3 . This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.4 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.2. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\| \leq \begin{cases} b, & \\ b\sum_{i=1}^{n}||v_i||^d, & d \neq 3; \\ b\left\{\prod_{i=1}^{n}||v_i||^d+\sum_{i=1}^{n}||v_i||^{nd}\right\}, & nd \neq 3; \end{cases} \tag{3.30}$$

for all $v_1,\cdots,v_n \in \mathcal{Y}$, then there existss a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|c(v)-\mathcal{C}(v)\| \leq \begin{cases} \frac{5b}{|7|}, \\ \frac{b||v||^d(5n-6+2^{rd})}{|8-2^d|}, \\ \frac{b||v||^{nd}(5n-6+2^{rnd})}{|8-2^{nd}|} \end{cases} \tag{3.31}$$

for all $v \in \mathcal{Y}$.

Theorem 3.5. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0,\infty)$ be functions satisfying (3.1) and (3.24) for all $v_1,\cdots,v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality*

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\| \leq \omega\left(v_1,\cdots,v_n\right) \tag{3.32}$$

for all $v_1,\cdots,v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ and a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v)-\mathcal{A}(v)-\mathcal{C}(v)\| \leq \frac{1}{6}\left\{\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)}{2^{rq}} + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)}{2^{rq}}\right\} \tag{3.33}$$

where $\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)$, $\mathcal{A}(v)$ and $\mathcal{C}(v)$ is defined in (3.4), (3.5) and (3.27) for all $v \in \mathcal{Y}$, respectively.

Proof. **Case (i):** For $q = 1$. Given f is an odd function.

If $f : \mathcal{Y} \to \mathcal{Z}$ satisfies (3.32) then by Theorem 3.3, there exists a unique additive function $\mathcal{A}' : \mathcal{Y} \to \mathcal{Z}$ such that

$$\left\|\mathcal{A}'(v)-(f(2v)-8f(v))\right\| \leq \frac{1}{2}\sum_{r=0}^{\infty}\frac{\Omega(2^rv,2^rv,\cdots,-2^rv,0,2^rv)}{2^r} \tag{3.34}$$

for all $v \in \mathcal{Y}$.

Also, if $f : \mathcal{Y} \to \mathcal{Z}$ satisfies (3.32) then by Theorem 3.4, there exists a unique cubic function $\mathcal{C}' : \mathcal{Y} \to \mathcal{Z}$ such that

$$\left\|\mathcal{C}'(v)-(f(2v)-2f(v))\right\| \leq \frac{1}{8}\sum_{r=0}^{\infty}\frac{\Omega(2^rv,2^rv,\cdots,-2^rv,0,2^rv)}{2^r} \tag{3.35}$$

for all $v \in \mathcal{Y}$. Combining (3.34) and (3.35), we achieve

$$\begin{aligned}
&\left\| \frac{1}{6}\mathcal{A}'(v) - \frac{1}{6}\mathcal{C}'(v) - f(v) \right\| \\
&= \left\| \frac{1}{6}\mathcal{A}'(v) - \frac{1}{6}f(2v) - \frac{8}{6}f(v) - \frac{1}{6}\mathcal{C}'(v) + \frac{1}{6}f(2v) - \frac{2}{6}f(v) \right\| \\
&\leq \left\| \frac{1}{6}\mathcal{A}'(v) - \frac{1}{6}(f(2v) - 8f(v)) \right\| + \left\| \frac{1}{6}\mathcal{C}'(v) - \frac{1}{6}(f(2v) + 2f(v)) \right\| \\
&\leq \frac{1}{6}\left\{ \|\mathcal{A}'(v) - (f(2v) - 8f(v))\| + \|\mathcal{C}'(v) - (f(2v) - 2f(v))\| \right\} \\
&\leq \frac{1}{6}\left\{ \frac{1}{2}\sum_{r=0}^{\infty} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^r} + \frac{1}{8}\sum_{r=0}^{\infty} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^r} \right\}
\end{aligned}$$

for all $v \in \mathcal{Y}$. Defining

$$\mathcal{A}(v) = \frac{1}{6}\mathcal{A}'(v); \qquad \mathcal{C}(v) = \frac{-1}{6}\mathcal{C}(v)$$

we arrive (3.33) as desired. Similarly, we can prove for $j = -1$. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.5 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.3. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\|Df_{aqcq}(v_1, \cdots, v_n)\| \leq \begin{cases} b, & \\ b\sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 1, 3; \\ b\left\{ \prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd} \right\}, & nd \neq 1, 3; \end{cases} \tag{3.36}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ and a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \begin{cases} \frac{5b}{6}\left(1 + \frac{1}{|7|}\right), \\ \frac{b||v||^d(5n - 6 + 2^{rd})}{6}\left(\frac{1}{|2 - 2^d|} + \frac{1}{|8 - 2^d|}\right), \\ \frac{b||v||^{nd}(5n - 6 + 2^{rnd})}{6}\left(\frac{1}{|2 - 2^{nd}|} + \frac{1}{|8 - 2^{nd}|}\right) \end{cases} \tag{3.37}$$

for all $v \in \mathcal{Y}$.

3.2 f IS AN EVEN FUNCTION

Theorem 3.6. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p \to \infty} \frac{\omega\left(2^{pq} v_1, \cdots, 2^{pq} v_n\right)}{2^{2pq}} = 0 \tag{3.38}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality

$$\|Df_{aqcq}(v_1, \cdots, v_n)\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{3.39}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_2(v) - q_2(v)\| = \|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}\| \leq \frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} \tag{3.40}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$ is defined in (3.4) and $\mathcal{Q}_2(v)$ are defined by

$$\mathcal{Q}_2(v) = \lim_{p \to \infty} \frac{q_2(2^{pq}v)}{2^{2pq}} = \lim_{p \to \infty} \frac{f(2 \cdot 2^{pq}v) - 16f(2^{pq}v)}{2^{2pq}} \tag{3.41}$$

for all $v \in \mathcal{Y}$, respectively.

Proof. Given, f is an even function. Using evenness of f in (3.39), we arrive

$$\begin{aligned}\Bigg\| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \\ +6f\left(\sum_{i=1}^{n-1} v_i\right) - 2f(2v_n) + 8f(v_n)\Bigg\| \leq \omega(v_1, \cdots, v_n)\end{aligned} \tag{3.42}$$

for all $v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n \in \mathcal{Y}$. Replacing

$$(v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n) = \left(v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v\right)$$

in (3.42), we get

$$\begin{aligned}\Big\| f(3v) - 6f(2v) + 15f(v)\Big\| &\leq \omega\left(v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v\right) \\ &= \omega_1(v, v, \cdots, -v, 0, v)\end{aligned} \tag{3.43}$$

for all $v \in \mathcal{Y}$. Again replacing

$$(v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n) = \left(2v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v\right)$$

in (3.42), we obtain

$$\begin{aligned}\Big\| f(4v) - 4f(3v) + 4f(2v) + 4f(v)\Big\| &\leq \omega\left(2v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v\right) \\ &= \omega_1(2v, v, \cdots, -v, 0, v)\end{aligned} \tag{3.44}$$

for all $v \in \mathcal{Y}$. It follows from (3.43) and (3.44),

$$\begin{aligned}&\Big\| f(4v) - 20f(2v) + 64f(v)\Big\| \\ &\quad = \Big\| f(4v) - 4f(3v) + 4f(3v) + 4f(2v) - 24f(2v) + 60f(v) + 4f(v)\Big\| \\ &\quad \leq 4\Big\| f(3v) - 6f(2v) + 15f(v)\Big\| + \Big\| f(4v) - 4f(3v) + 4f(2v) + 4f(v)\Big\| \\ &\quad \leq 4\omega_1(v, v, \cdots, -v, 0, v) + \omega_1(2v, v, \cdots, -v, 0, v)\end{aligned} \tag{3.45}$$

for all $v \in \mathcal{Y}$. Define

$$\Omega(v, v, \cdots, -v, 0, v) = 4\omega_1(v, v, \cdots, -v, 0, v) + \omega_1(2v, v, \cdots, -v, 0, v) \tag{3.46}$$

for all $v \in \mathcal{Y}$. Using (3.46) in (3.45), we have

$$\Big\| f(4v) - 20f(2v) + 64f(v)\Big\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.47}$$

for all $v \in \mathcal{Y}$. It follows from (3.47), we reach

$$\Big\| \Big\{ f(4v) - 16f(2v)\Big\} - 4\Big\{ f(2v) - 16f(v)\Big\}\Big\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.48}$$

for all $v \in \mathcal{Y}$. Using (2.13) in (3.48), we land

$$\Big\| q_2(2v) - 4q_2(v)\Big\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.49}$$

for all $v \in \mathcal{Y}$. The rest of the proof is similar to that of Theorem 3.3 . This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.6 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.4. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 2; \\ b\left\{\prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 2; \end{cases} \tag{3.50}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|q_2(v) - \mathcal{Q}_2(v)\| \leq \begin{cases} \dfrac{5b}{|3|}, \\ \dfrac{b||v||^d(5n-6+2^{rd})}{|4-2^d|}, \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|4-2^{nd}|} \end{cases} \tag{3.51}$$

for all $v \in \mathcal{Y}$.

Theorem 3.7. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{4pq}} = 0 \tag{3.52}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{3.53}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_4(v) - q_4(v)\| = \|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}\| \leq \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} \tag{3.54}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$ is defined in (3.4) and $\mathcal{Q}_4(v)$ is defined by

$$\mathcal{Q}_4(v) = \lim_{p\to\infty} \frac{q_4(2^{pq}v)}{2^{4pq}} = \lim_{p\to\infty} \frac{f(2\cdot 2^{pq}v) - 4f(2^{pq}v)}{2^{4pq}} \tag{3.55}$$

for all $v \in \mathcal{Y}$, respectively.

Proof. It follows from (3.47), we reach

$$\left\|\left\{f(4v) - 4f(2v)\right\} - 16\left\{f(2v) - 4f(v)\right\}\right\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.56}$$

for all $v \in \mathcal{Y}$. Using (2.22) in (3.56), we land

$$\left\|q_4(2v) - 16q_4(v)\right\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.57}$$

for all $v \in \mathcal{Y}$. The rest of the proof is similar to that of Theorem 3.3 . This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.7 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.5. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 4; \\ b\left\{\prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 4; \end{cases} \tag{3.58}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$*, then there exists a unique quartic function* $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ *such that*

$$\|q_4(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \frac{5b}{|15|}, \\ \frac{b||v||^d(5n-6+2^{rd})}{|16-2^d|}, \\ \frac{b||v||^{nd}(5n-6+2^{rnd})}{|16-2^{nd}|} \end{cases} \tag{3.59}$$

for all $v \in \mathcal{Y}$.

Theorem 3.8. *Let* $q = \pm 1$ *and* $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ *be functions satisfying (3.38) and (3.52) for all* $v_1, \cdots, v_n \in \mathcal{Y}$. *Let* $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ *be an even function satisfying the inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{3.60}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. *Then there exists a unique quadratic function* $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ *and a unique quartic function* $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ *which satisfies (1.3) and*

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \frac{1}{12}\left\{\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{2rq}} + \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{4rq}}\right\} \tag{3.61}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$, $\mathcal{Q}_2(v)$ *and* $\mathcal{Q}_4(v)$ *is defined in (3.4), (3.41) and (3.55) for all* $v \in \mathcal{Y}$*, respectively.*

Proof. **Case (i):** For $q = 1$. Given f is an even function.

If $f : \mathcal{Y} \to \mathcal{Z}$ satisfies (3.60) then by Theorem 3.6, there exists a unique quadratic function $\mathcal{Q}_2' : \mathcal{Y} \to \mathcal{Z}$ such that

$$\left\|\mathcal{Q}_2'(v) - (f(2v) - 4f(v))\right\| \leq \frac{1}{4}\sum_{r=0}^{\infty}\frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^{2r}} \tag{3.62}$$

for all $v \in \mathcal{Y}$.

Also, if $f : \mathcal{Y} \to \mathcal{Z}$ satisfies (3.60) then by Theorem 3.7, there exists a unique cubic function $\mathcal{Q}_4' : \mathcal{Y} \to \mathcal{Z}$ such that

$$\left\|\mathcal{Q}_4'(v) - (f(2v) - 16f(v))\right\| \leq \frac{1}{16}\sum_{r=0}^{\infty}\frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^{4r}} \tag{3.63}$$

for all $v \in \mathcal{Y}$. Combining (3.62) and (3.63), we achieve

$$\begin{aligned}
&\left\|\frac{1}{12}\mathcal{Q}_2'(v) - \frac{1}{12}\mathcal{Q}_4'(v) - f(v)\right\| \\
&= \left\|\frac{1}{12}\mathcal{Q}_2'(v) - \frac{1}{12}f(2v) - \frac{16}{12}f(v) - \frac{1}{12}\mathcal{Q}_4'(v) + \frac{1}{12}f(2v) - \frac{4}{12}f(v)\right\| \\
&\leq \left\|\frac{1}{12}\mathcal{Q}_2'(v) - \frac{1}{12}(f(2v) - 16f(v))\right\| + \left\|\frac{1}{12}\mathcal{Q}_4'(v) - \frac{1}{12}(f(2v) + 4f(v))\right\| \\
&\leq \frac{1}{12}\left\{\left\|\mathcal{Q}_2'(v) - (f(2v) - 16f(v))\right\| + \left\|\mathcal{Q}_4'(v) - (f(2v) - 4f(v))\right\|\right\} \\
&\leq \frac{1}{12}\left\{\frac{1}{2}\sum_{r=0}^{\infty}\frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^{2r}} + \frac{1}{8}\sum_{r=0}^{\infty}\frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^{4r}}\right\}
\end{aligned}$$

for all $v \in \mathcal{Y}$. Defining

$$\mathcal{Q}_2(v) = \frac{1}{12}\mathcal{Q}_2'(v); \qquad \mathcal{Q}_4(v) = \frac{-1}{12}\mathcal{Q}_4'(v)$$

we arrive (3.61) as desired. Similarly, we can prove for $j = -1$. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.8 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.6. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exist real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 2,4; \\ b\left\{\prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 2,4; \end{cases} \tag{3.64}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \frac{5b}{12}\left(\frac{1}{|3|} + \frac{1}{|15|}\right), \\ \frac{b||v||^d(5n-6+2^{rd})}{12}\left(\frac{1}{|4-2^d|} + \frac{1}{|16-2^d|}\right), \\ \frac{b||v||^{nd}(5n-6+2^{rnd})}{12}\left(\frac{1}{|4-2^{nd}|} + \frac{1}{|16-2^{nd}|}\right) \end{cases} \tag{3.65}$$

for all $v \in \mathcal{Y}$.

3.3 f IS AN ODD - EVEN FUNCTION

Theorem 3.9. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions satisfying (3.32) and (3.60) for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a function satisfying the inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{3.66}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$, a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$, a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\begin{aligned} &\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \\ &\quad \leq \frac{1}{2}\left\{\frac{1}{6}\left[\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega_1(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega_3(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{3rq}}\right]\right. \\ &\qquad \left. + \frac{1}{12}\left[\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega_2(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{2rq}} + \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega_4(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{4rq}}\right]\right\} \end{aligned} \tag{3.67}$$

where

$$\Omega_t(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v) = \Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v) + \Omega(-2^{rq}v, -2^{rq}v, \cdots 2^{rq}v, 0, -2^{rq}v)$$

for $t = 1, 2, 3, 4$ and $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$, $\mathcal{A}(v)$, $\mathcal{Q}_2(v)$, $\mathcal{C}(v)$ and $\mathcal{Q}_4(v)$ is defined in (3.4), (3.5), (3.41), (3.27), and (3.55) for all $v \in \mathcal{Y}$, respectively.

Proof. Let we define

$$f_{ODD}(v) = \frac{f(v) - f(-v)}{2}$$

for all $v \in \mathcal{Y}$. Then $f_{ODD}(0) = 0$ and $f_{ODD}(-v) = -f_{ODD}(v)$ for all $v \in \mathcal{Y}$. Hence

$$\|Df_{ODD}(v_1, \cdots, v_n)\| \leq \frac{\omega(v_1, \cdots, v_n)}{2} + \frac{\omega(-v_1, \cdots, -v_n)}{2} \tag{3.68}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. By Theorem 3.5, we have

$$\begin{aligned}
&\|f_{ODD}(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \\
&\leq \frac{1}{2}\left\{\frac{1}{6}\left(\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} + \frac{\Omega(-2^{rq}v, -2^{rq}v, \cdots 2^{rq}v, 0, -2^{rq}v)}{2^{rq}}\right]\right.\right. \\
&\qquad \left.\left. + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{3rq}} + \frac{\Omega(-2^{rq}v, -2^{rq}v, \cdots 2^{rq}v, 0, -2^{rq}v)}{2^{3rq}}\right]\right)\right\} \\
&\leq \frac{1}{2}\left\{\frac{1}{6}\left(\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega_1(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}}\right] + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega_3(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{3rq}}\right]\right)\right\}
\end{aligned} \tag{3.69}$$

for all $v \in \mathcal{Y}$.

Also, let

$$f_{EVEN}(v) = \frac{f(v) + f(-v)}{2}$$

for all $v \in \mathcal{Y}$. Then $f_{EVEN}(0) = 0$ and $f_{EVEN}(-v) = f_{EVEN}(v)$ for all $v \in \mathcal{Y}$. Hence

$$\|Df_{EVEN}(v_1, \cdots, v_n)\| \leq \frac{\omega(v_1, \cdots, v_n)}{2} + \frac{\omega(-v_1, \cdots, -v_n)}{2} \tag{3.70}$$

for all $v \in \mathcal{Y}$. By Theorem 3.8, we have

$$\begin{aligned}
&\|f_{EVEN}(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \\
&\leq \frac{1}{2}\left\{\frac{1}{12}\left(\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{2rq}} + \frac{\Omega(-2^{rq}v, -2^{rq}v, \cdots 2^{rq}v, 0, -2^{rq}v)}{2^{2rq}}\right]\right.\right. \\
&\qquad \left.\left. + \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{4rq}} + \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{4rq}}\right]\right)\right\} \\
&\leq \frac{1}{2}\left\{\frac{1}{12}\left(\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega_2(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{2rq}}\right] + \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega_4(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{4rq}}\right]\right)\right\}
\end{aligned} \tag{3.71}$$

for all $v \in \mathcal{Y}$. Define

$$f(v) = f_{EVEN}(v) + f_{ODD}(v) \tag{3.72}$$

for all $v \in \mathcal{Y}$. From (3.69),(3.71) and (3.72), we arrive our result. □

The following corollary is an immediate consequence of Theorem 3.9 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.7. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping. If there exists real numbers b and d such that*

$$\|Df_{aqcq}(v_1, \cdots, v_n)\| \leq \begin{cases} b, & \\ b\sum_{i=1}^{n}||v_i||^d, & d \neq 1,2,3,4; \\ b\left\{\prod_{i=1}^{n}||v_i||^d + \sum_{i=1}^{n}||v_i||^{nd}\right\}, & nd \neq 1,2,3,4; \end{cases} \tag{3.73}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there existss a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$, a unique quadratic function

$\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$, a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \frac{5b}{2}\left\{\frac{1}{12}\left[\frac{1}{|3|}+\frac{1}{|15|}\right]+\frac{1}{6}\left[1+\frac{1}{|7|}\right]\right\}, \\ \frac{b||v||^d(5n-6+2^{rd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^d|}+\frac{1}{|16-2^d|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^d|}+\frac{1}{|8-2^d|}\right]\right\}, \\ \frac{b||v||^{nd}(5n-6+2^{rnd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^{nd}|}+\frac{1}{|16-2^{nd}|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^{nd}|}+\frac{1}{|8-2^{nd}|}\right]\right\}, \end{cases} \tag{3.74}$$

for all $v \in \mathcal{Y}$.

4 Stability Results - Banach Space : n is an Even Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an even positive integer in Banach space using direct method.

The proof of the following theorems and corollaries are similar to that of proofs of Section 3. Hence the details of the proofs are omitted.

4.1 f IS AN ODD FUNCTION

Theorem 4.10. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{pq}} = 0 \tag{4.1}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.2}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{A}(v) - a(v)\| = \|\mathcal{A}(v) - \{f(2v) - 8f(v)\}\| \leq \frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{rq}} \tag{4.3}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$ and $\mathcal{A}(v)$ are defined by

$$\begin{aligned} &\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v) \\ &= 4\omega_1(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v) + \omega_1(2\cdot 2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v) \\ &= 4\omega\left(2^{rq}v, \underbrace{2^{rq}v, 2^{rq}v, \cdots, 2^{rq}v}_{\frac{n-2}{2}\ times}, \underbrace{-2^{rq}v, -2^{rq}v, \cdots, -2^{rq}v}_{\frac{n-2}{2}\ times}, 2^{rq}v\right) \\ &\quad + \omega\left(2\cdot 2^{rq}v, \underbrace{2^{rq}v, 2^{rq}v, \cdots, 2^{rq}v}_{\frac{n-2}{2}\ times}, \underbrace{-2^{rq}v, -2^{rq}v, \cdots, -2^{rq}v}_{\frac{n-2}{2}\ times}, 2^{rq}v\right) \end{aligned} \tag{4.4}$$

and

$$\mathcal{A}(v) = \lim_{p\to\infty} \frac{a(2^{pq}v)}{2^{pq}} = \lim_{p\to\infty} \frac{f(2\cdot 2^{pq}v) - 8f(2^{pq}v)}{2^{pq}} \tag{4.5}$$

for all $v \in \mathcal{Y}$, respectively.

Corollary 4.8. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum_{i=1}^{n} ||v_i||^d, & d \neq 1; \\ b\prod_{i=1}^{n} ||v_i||^d, & nd \neq 1; \\ b\left\{\prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 1; \end{cases} \tag{4.6}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there existss a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|a(v) - \mathcal{A}(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{|2-2^d|}, \\ \dfrac{b||v||^d(4+2^{rnd})}{|2-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n-3+2^{rd}+2^{rnd})}{|2-2^{nd}|} \end{cases} \tag{4.7}$$

for all $v \in \mathcal{Y}$.

Theorem 4.11. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{3pq}} = 0 \tag{4.8}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.9}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{C}(v) - c(v)\| = \|\mathcal{C}(v) - \{f(2v) - 2f(v)\}\| \leq \frac{1}{8} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{rq}} \tag{4.10}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$ is defined in (4.4) and $\mathcal{C}(v)$ is defined by

$$\mathcal{C}(v) = \lim_{p\to\infty} \frac{c(2^{pq}v)}{2^{3pq}} = \lim_{p\to\infty} \frac{f(2\cdot 2^{pq}v) - 2f(2^{pq}v)}{2^{3pq}} \tag{4.11}$$

for all $v \in \mathcal{Y}$, respectively.

Corollary 4.9. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, \\ b\sum_{i=1}^{n} ||v_i||^d, & d \neq 3; \\ b\prod_{i=1}^{n} ||v_i||^d, & nd \neq 3; \\ b\left\{\prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 3; \end{cases} \tag{4.12}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|c(v) - \mathcal{C}(v)\| \leq \begin{cases} \dfrac{5b}{|7|}, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{|8-2^d|}, \\ \dfrac{b||v||^d(4+2^{rnd})}{|8-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n-3+2^{rd}+2^{rnd})}{|8-2^{nd}|} \end{cases} \tag{4.13}$$

for all $v \in \mathcal{Y}$.

Theorem 4.12. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions satisfying (4.1) and (4.8) for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.14}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. *Then there exists a unique additive function* $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ *and a unique cubic function* $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ *which satisfies (1.3) and*

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \frac{1}{6}\left\{\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{rq}} + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{3rq}}\right\} \tag{4.15}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$, $\mathcal{A}(v)$ *and* $\mathcal{C}(v)$ *is defined in (4.4), (4.5) and (4.11) for all* $v \in \mathcal{Y}$, *respectively.*

Corollary 4.10. *Let* $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ *be an odd mapping. If there exists real numbers* b *and* d *such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum_{i=1}^{n}||v_i||^d, & d \neq 1, 3; \\ b\prod_{i=1}^{n}||v_i||^d, & nd \neq 1, 3; \\ b\left\{\prod_{i=1}^{n}||v_i||^d + \sum_{i=1}^{n}||v_i||^{nd}\right\}, & nd \neq 1, 3; \end{cases} \tag{4.16}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, *then there exists a unique additive function* $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ *and a unique cubic function* $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ *such that*

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \begin{cases} \frac{5b}{6}\left(1 + \frac{1}{|7|}\right), \\ \frac{b||v||^d(5n - 1 + 2^{rd})}{6}\left(\frac{1}{|2 - 2^d|} + \frac{1}{|8 - 2^d|}\right), \\ \frac{b||v||^d(4 + 2^{rnd})}{6}\left(\frac{1}{|2 - 2^d|} + \frac{1}{|8 - 2^d|}\right), \\ \frac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{6}\left(\frac{1}{|2 - 2^{nd}|} + \frac{1}{|8 - 2^{nd}|}\right) \end{cases} \tag{4.17}$$

for all $v \in \mathcal{Y}$.

4.2 f IS AN EVEN FUNCTION

Theorem 4.13. *Let* $q = \pm 1$ *and* $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ *be function such that*

$$\lim_{p\to\infty}\frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{2pq}} = 0 \tag{4.18}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. *Let* $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ *be an even function satisfying the inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.19}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. *Then there exists a unique quadratic function* $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ *which satisfies (1.3) and*

$$\|\mathcal{Q}_2(v) - q_2(v)\| = \|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}\| \leq \frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{2rq}} \tag{4.20}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$ *is defined in (4.4) and* $\mathcal{Q}_2(v)$ *are defined by*

$$\mathcal{Q}_2(v) = \lim_{p\to\infty}\frac{q_2(2^{pq}v)}{2^{2pq}} = \lim_{p\to\infty}\frac{f(2 \cdot 2^{pq}v) - 16f(2^{pq}v)}{2^{2pq}} \tag{4.21}$$

for all $v \in \mathcal{Y}$, *respectively.*

Corollary 4.11. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum_{i=1}^{n} ||v_i||^d, & d \neq 2; \\ b\prod_{i=1}^{n} ||v_i||^d, & nd \neq 2; \\ b\left\{\prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 2; \end{cases} \tag{4.22}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|a(v) - \mathcal{Q}_{\in}(v)\| \leq \begin{cases} \frac{5b}{|3|}, \\ \frac{b||v||^d(5n-1+2^{rd})}{|4-2^d|}, \\ \frac{b||v||^{nd}(4+2^{rnd})}{|4-2^{nd}|}, \\ \frac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{|4-2^{nd}|} \end{cases} \tag{4.23}$$

for all $v \in \mathcal{Y}$.

Theorem 4.14. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{4pq}} = 0 \tag{4.24}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.25}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_4(v) - q_4(v)\| = \|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}\| \leq \frac{1}{16} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{4rq}} \tag{4.26}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$ is defined in (4.4) and $\mathcal{Q}_4(v)$ is defined by

$$\mathcal{Q}_4(v) = \lim_{p\to\infty} \frac{q_4(2^{pq}v)}{2^{4pq}} = \lim_{p\to\infty} \frac{f(2 \cdot 2^{pq}v) - 4f(2^{pq}v)}{2^{4pq}} \tag{4.27}$$

for all $v \in \mathcal{Y}$, respectively.

Corollary 4.12. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum_{i=1}^{n} ||v_i||^d, & d \neq 4; \\ b\prod_{i=1}^{n} ||v_i||^d, & nd \neq 4; \\ b\left\{\prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 4; \end{cases} \tag{4.28}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|q_4(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \frac{5b}{|15|}, \\ \frac{b||v||^d(5n-1+2^{rd})}{|16-2^d|}, \\ \frac{b||v||^{nd}(4+2^{rnd})}{|16-2^{nd}|}, \\ \frac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{|16-2^{nd}|} \end{cases} \tag{4.29}$$

for all $v \in \mathcal{Y}$.

Theorem 4.15. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0,\infty)$ be functions satisfying (4.18) and (4.24) for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality*

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \le \omega\left(v_1, \cdots, v_n\right) \tag{4.30}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \le \frac{1}{12}\left\{\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{2rq}} + \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{4rq}}\right\} \tag{4.31}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$, $\mathcal{Q}_2(v)$ and $\mathcal{Q}_4(v)$ is defined in (4.4), (4.21) and (4.27) for all $v \in \mathcal{Y}$, respectively.

Corollary 4.13. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \le \begin{cases} b, & \\ b\sum_{i=1}^{n} ||v_i||^d, & d \neq 2,4; \\ b\prod_{i=1}^{n} ||v_i||^d, & nd \neq 2,4; \\ b\left\{\prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 2,4; \end{cases} \tag{4.32}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \le \begin{cases} \frac{5b}{12}\left(\frac{1}{|3|} + \frac{1}{|15|}\right), \\ \frac{b||v||^d(5n - 1 + 2^{rd})}{12}\left(\frac{1}{|4-2^d|} + \frac{1}{|16-2^d|}\right), \\ \frac{b||v||^{nd}(4 + 2^{rnd})}{12}\left(\frac{1}{|4-2^d|} + \frac{1}{|16-2^d|}\right), \\ \frac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{12}\left(\frac{1}{|4-2^{nd}|} + \frac{1}{|16-2^{nd}|}\right) \end{cases} \tag{4.33}$$

for all $v \in \mathcal{Y}$.

4.3 f IS AN ODD - EVEN FUNCTION

Theorem 4.16. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0,\infty)$ be functions satisfying (4.14) and (4.30) for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a function satisfying the inequality*

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \le \omega\left(v_1, \cdots, v_n\right) \tag{4.34}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$, a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$, a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\begin{aligned} &\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \\ &\le \frac{1}{2}\left\{\frac{1}{6}\left[\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega_{11}(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{rq}} + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega_{33}(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{3rq}}\right]\right. \\ &\left. + \frac{1}{12}\left[\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega_{22}(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{2rq}} + \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega_{44}(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{4rq}}\right]\right\} \end{aligned} \tag{4.35}$$

where

$$\Omega_{tt}(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v) = \Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v) + \Omega(-2^{rq}v, -2^{rq}v, \cdots 2^{rq}v, -2^{rq}v)$$

for $tt = 1, 2, 3, 4$ *and* $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$, $\mathcal{A}(v)$, $\mathcal{Q}_2(v)$, $\mathcal{C}(v)$ *and* $\mathcal{Q}_4(v)$ *is defined in (4.4), (4.5), (4.21), (4.11) and (4.27) for all* $v \in \mathcal{Y}$, *respectively.*

Corollary 4.14. *Let* $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ *be a mapping. If there exists real numbers* b *and* d *such that*

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \leq \begin{cases} b, & \\ b \sum_{i=1}^{n} ||v_i||^d, & d \neq 1,2,3,4; \\ b \prod_{i=1}^{n} ||v_i||^d, & nd \neq 1; \\ b \left\{ \prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd} \right\}, & nd \neq 1,2,3,4; \end{cases} \tag{4.36}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, *then there exists a unique additive function* $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$, *a unique quadratic function* $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$, *a unique cubic function* $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ *and a unique quartic function* $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ *such that*

$$\begin{aligned} &\| f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v) \| \\ &\leq \begin{cases} \frac{5b}{2} \left\{ \frac{1}{12} \left[\frac{1}{|3|} + \frac{1}{|15|} \right] + \frac{1}{6} \left[1 + \frac{1}{|7|} \right] \right\}, \\ \frac{b||v||^d(5n-1+2^{rd})}{2} \left\{ \frac{1}{12} \left[\frac{1}{|4-2^d|} + \frac{1}{|16-2^d|} \right] + \frac{1}{6} \left[\frac{1}{|2-2^d|} + \frac{1}{|8-2^d|} \right] \right\}, \\ \frac{b||v||^{nd}(4+2^{rnd})}{2} \left\{ \frac{1}{12} \left[\frac{1}{|4-2^d|} + \frac{1}{|16-2^d|} \right] + \frac{1}{6} \left[\frac{1}{|2-2^d|} + \frac{1}{|8-2^d|} \right] \right\}, \\ \frac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{2} \left\{ \frac{1}{12} \left[\frac{1}{|4-2^{nd}|} + \frac{1}{|16-2^{nd}|} \right] + \frac{1}{6} \left[\frac{1}{|2-2^{nd}|} + \frac{1}{|8-2^{nd}|} \right] \right\}, \end{cases} \end{aligned} \tag{4.37}$$

for all $v \in \mathcal{Y}$.

5 Stability Results - Fixed Point Method: n Odd Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an odd positive integer in Banach space using fixed point method.

5.1 f IS AN ODD FUNCTION

Theorem 5.17. *Let* $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ *be a odd mapping for which there exist a function* $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ *with the condition*

$$\lim_{p \to \infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^p} = 0 \tag{5.1}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ *where*

$$\kappa_j = \begin{cases} 2 & if \quad j = 0; \\ \frac{1}{2} & if \quad j = 1, \end{cases} \tag{5.2}$$

such that the functional inequality

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{5.3}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. *If there exists* $L = L(i)$ *such that the function*

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, 0, v) = \frac{L}{\kappa_j} \Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, 0, \kappa_j v\right). \tag{5.4}$$

for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{A}(v) - a(v)\| = \|\mathcal{A}(v) - \{f(2v) - 8f(v)\}\| \leq \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, 0, v) \tag{5.5}$$

for all $v \in \mathcal{Y}$.

Proof. Consider the set

$$\Gamma = \{p/p : \mathcal{Y} \to \mathcal{Z},\ p(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(p, q) = \inf\{K \in (0, \infty) : \| p(v) - q(v) \| \leq K\psi(v), v \in \mathcal{Y}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon p(v) = \frac{1}{\kappa_j} p(\kappa_j v),$$

for all $v \in \mathcal{Y}$. Now $p, q \in \Gamma$, by [36], we have $d(\Upsilon p, \Upsilon q) \leq Ld(p, q)$, i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

From (3.13), we arrive

$$\left\|a(2v) - 2a(v)\right\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{5.6}$$

for all $v \in \mathcal{Y}$. It follows from (5.6) that

$$\left\|\frac{a(2v)}{2} - a(v)\right\| \leq \frac{\Omega(v, v, \cdots, -v, 0, v)}{2} \tag{5.7}$$

for all $v \in \mathcal{Y}$. Using (5.4) for the case $j = 0$ it reduces to

$$\left\|\frac{a(2v)}{2} - a(v)\right\| \leq L\Psi(v, v, \cdots, -v, 0, v)$$

for all $v \in \mathcal{Y}$,

$$\text{i.e.,}\quad d(\Upsilon a, a) \leq L \Rightarrow d(\Upsilon a, a) \leq L = L^1 < \infty. \tag{5.8}$$

Again replacing $v = \frac{v}{2}$ in (5.6), we get

$$\left\|a(v) - 2a\left(\frac{v}{2}\right)\right\| \leq \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right) \tag{5.9}$$

for all $v \in \mathcal{Y}$. Using (5.4) for the case $j = 1$ it reduces to

$$\left\|a(v) - 2a\left(\frac{v}{2}\right)\right\| \leq \Psi(v, v, \cdots, -v, 0, v)$$

for all $v \in \mathcal{Y}$,

$$\text{i.e.,}\quad d(a, \Upsilon a) \leq 1 \Rightarrow d(a, \Upsilon a) \leq 1 = L^0 < \infty. \tag{5.10}$$

From (5.8) and (5.10), we arrive

$$d(a, \Upsilon a) \leq L^{1-j}.$$

Therefore (FP1) holds.

By (FP2), it follows that there exists a fixed point A of Υ in Γ such that

$$A(v) = \lim_{p\to\infty} \frac{a(\kappa_j^p v)}{\kappa_j^p}, \qquad \forall\ v \in \mathcal{Y}. \tag{5.11}$$

To order to prove A satisfies the functional equation (1.3), the proof is similar to the of Theorem 3.3. By (FP3), A is the unique fixed point of Υ in the set

$$\Delta = \{A \in \Gamma : d(a, A) < \infty\},$$

such that

$$\|a(v) - A(v)\| \leq K\psi(v)$$

for all $v \in \mathcal{Y}$ and $K > 0$. Finally by (FP4), we obtain

$$d(a, A) \leq \frac{1}{1-L} d(a, \Upsilon a)$$

this implies

$$d(a, A) \leq \frac{L^{1-j}}{1-L}$$

which yields

$$\|a(v) - A(v)\| \leq \frac{L^{1-j}}{1-L} \Psi(v, v, \cdots, -v, 0, v)$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 5.17 concerning the stability of (1.3).

Corollary 5.15. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d satisfying (3.22) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|a(v) - \mathcal{A}(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d (5n - 6 + 2^{rd})}{|2 - 2^d|}, \\ \dfrac{b||v||^{nd} (5n - 6 + 2^{rnd})}{|2 - 2^{nd}|} \end{cases} \tag{5.12}$$

for all $v \in \mathcal{Y}$.

Proof. Taking

$$\omega(v_1, \cdots, v_n) = \begin{cases} b, \\ b \sum_{i=1}^{n} ||v_i||^d, \\ b \left\{ \prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd} \right\}, \end{cases}$$

for all $v \in \mathcal{Y}$. Now,

$$\frac{1}{\kappa_j^p} \omega\left(\kappa_j^p v_1, \kappa_j^p \cdots, \kappa_j^p v_n\right) = \begin{cases} \frac{b}{\kappa_j^p}, \\ \frac{b}{\kappa_j^p} \sum_{i=1}^{n} ||\kappa_j^p v_i||^d, \\ \frac{b}{\kappa_j^p} \left\{ \prod_{i=1}^{n} ||\kappa_j^p v_i||^d + \sum_{i=1}^{n} ||\kappa_j^p v_i||^{nd} \right\}, \end{cases} = \begin{cases} \to 0 \text{ as } p \to \infty, \\ \to 0 \text{ as } p \to \infty, \\ \to 0 \text{ as } p \to \infty. \end{cases}$$

Thus, (5.1) is holds. But we have

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots - \frac{v}{2}, 0, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, 0, v) = \frac{L}{\kappa_j} \Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, 0, \kappa_j v\right)$$

for all $v \in \mathcal{Y}$. Hence

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots - \frac{v}{2}, 0, \frac{v}{2}\right) = \begin{cases} 5b, \\ \dfrac{b(5n - 6 + 2^{rd})}{2^d} ||v||^d, \\ \dfrac{b(5n - 6 + 2^{rnd})}{2^d} ||v||^d. \end{cases}$$

Now,

$$\frac{1}{\kappa_j}\Psi\left(\kappa_j v,\kappa_j v,\cdots,-\kappa_j v,0,\kappa_j v\right)=\begin{cases}\dfrac{5b}{\kappa_j}, \\ \dfrac{b(5n-6+2^{rd})}{\kappa_j}||\kappa_j v||^d, \\ \dfrac{b(5n-6+2^{rnd})}{\kappa_j}||\kappa_j v||^{nd},\end{cases}=\begin{cases}\kappa_j^{-1}\Psi(v,v,\cdots,-v,0,v), \\ \kappa_j^{d-1}\Psi(v,v,\cdots,-v,0,v) \\ \kappa_j^{nd-1}\Psi(v,v,\cdots,-v,0,v).\end{cases}$$

Hence the inequality (5.4) holds either, $L=2^{-1}$ if $i=0$ and $L=\frac{1}{2^{-1}}$ if $i=1$. Now from (5.5), we prove the following cases for condition (i).

Case:1 $L=2^{-1}$ if $i=0$

$$\|a(v)-A(v)\|\le\frac{\left(2^{-1}\right)^{1-0}}{1-2^{-1}}\kappa_j^{d-1}\Psi(v,v,\cdots,-v,0,v)=5b.$$

Case:2 $L=\frac{1}{2^{-1}}$ if $i=1$

$$\|a(v)-A(v)\|\le\frac{\left(\frac{1}{2^{-1}}\right)^{1-1}}{1-\frac{1}{2^{-1}}}\kappa_j^{d-1}\Psi(v,v,\cdots,-v,0,v)=-5b.$$

Also the inequality (5.4) holds either, $L=2^{d-1}$ for $d<1$ if $i=0$ and $L=\frac{1}{2^{d-1}}$ for $d>1$ if $i=1$. Now from (5.5), we prove the following cases for condition (ii).

Case:3 $L=2^{d-1}$ for $d<1$ if $i=0$

$$\|a(v)-A(v)\|\le\frac{\left(2^{(d-1)}\right)^{1-0}}{1-2^{(d-1)}}\Psi(v,v,\cdots,-v,0,v)=\frac{b(5n-6+2^{rd})||v||^d}{2-2^d}.$$

Case:4 $L=\frac{1}{2^{d-1}}$ for $d>1$ if $i=1$

$$\|a(v)-A(v)\|\le\frac{\left(\frac{1}{2^{(d-1)}}\right)^{1-1}}{1-\frac{1}{2^{(d-1)}}}\Psi(v,v,\cdots,-v,0,v)=\frac{b(5n-6+2^{rd})||v||^d}{2^d-2}.$$

The proof of condition (iii) is similar to that of condition (ii). Hence the proof is complete. □

The proofs of the subsequent theorems and corollaries are similar to that of Theorem 5.17 and 5.15. Hence the details of the proofs are omitted.

Theorem 5.18. *Let $Df_{aqcq}:\mathcal{Y}\to\mathcal{Z}$ be a odd mapping for which there exists a function $\omega,\Omega:\mathcal{Y}^n\to[0,\infty)$ with the condition*

$$\lim_{p\to\infty}\frac{\omega\left(\kappa_j^p v_1,\cdots,\kappa_j^p v_n\right)}{\kappa_j^{3p}}=0 \tag{5.13}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\|\le\omega\left(v_1,\cdots,v_n\right) \tag{5.14}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$. If there exists $L=L(i)$ such that the function

$$\Psi(v,v,\cdots,-v,0,v)=\Omega\left(\frac{v}{2},\frac{v}{2},\cdots,-\frac{v}{2},0,\frac{v}{2}\right),$$

has the property

$$\Psi(v,v,\cdots,-v,0,v)=\frac{L}{\kappa_j^3}\,\Psi\left(\kappa_j v,\kappa_j v,\cdots,-\kappa_j v,0,\kappa_j v\right). \tag{5.15}$$

for all $v\in\mathcal{Y}$. Then there exists a unique cubic function $\mathcal{C}:\mathcal{Y}\to\mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{C}(v)-c(v)\|=\|\mathcal{C}(v)-\{f(2v)-2f(v)\}\|\le\frac{L^{1-j}}{1-L}\Psi(v,v,\cdots,-v,0,v) \tag{5.16}$$

for all $x\in\mathcal{Y}$.

Corollary 5.16. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d satisfying (3.30) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|c(v) - \mathcal{C}(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d(5n-6+2^{rd})}{|8-2^d|}, \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|8-2^{nd}|} \end{cases} \tag{5.17}$$

for all $v \in \mathcal{Y}$.

Theorem 5.19. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a odd mapping for which there exist a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (5.1) and (5.13) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{5.18}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right),$$

has the properties (5.4) and (5.15) and

$$\Psi(v, v, \cdots, -v, 0, v) = \frac{L}{\kappa_j}\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, 0, \kappa_j v\right). \tag{5.19}$$

for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ and a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \frac{L^{1-j}}{6(1-L)}\Psi(v, v, \cdots, -v, 0, v) \tag{5.20}$$

for all $x \in \mathcal{Y}$.

Corollary 5.17. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d satisfying (3.36) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \begin{cases} \dfrac{5b}{6}\left(1 + \dfrac{1}{|7|}\right), \\ \dfrac{b||v||^d(5n-6+2^{rd})}{6}\left(\dfrac{1}{|2-2^d|} + \dfrac{1}{|8-2^d|}\right), \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{6}\left(\dfrac{1}{|2-2^{nd}|} + \dfrac{1}{|8-2^{nd}|}\right) \end{cases} \tag{5.21}$$

for all $v \in \mathcal{Y}$.

5.2 f IS AN EVEN FUNCTION

Theorem 5.20. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^{2p}} = 0 \tag{5.22}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{5.23}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right),$$

has the property

$$\Psi(v,v,\cdots,-v,0,v)=\frac{L}{\kappa_j^2}\,\Psi\left(\kappa_j v,\kappa_j v,\cdots,-\kappa_j v,0,\kappa_j v\right). \tag{5.24}$$

for all $v\in\mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2:\mathcal{Y}\to\mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_2(v)-q_2(v)\|=\|\mathcal{Q}_2(v)-\{f(2v)+16f(v)\}\|\leq\frac{L^{1-j}}{1-L}\Psi(v,v,\cdots,-v,0,v) \tag{5.25}$$

for all $x\in\mathcal{Y}$.

Corollary 5.18. *Let $Df_{aqcq}:\mathcal{Y}\to\mathcal{Z}$ be an even mapping. If there exists real numbers b and d satisfying (3.50) for all $v_1,\cdots,v_n\in\mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2:\mathcal{Y}\to\mathcal{Z}$ such that*

$$\|q_2(v)-\mathcal{Q}_2(v)\|\leq\begin{cases}\dfrac{5b}{7},\\ \dfrac{b||v||^d(5n-6+2^{rd})}{|4-2^d|},\\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|4-2^{nd}|}\end{cases} \tag{5.26}$$

for all $v\in\mathcal{Y}$.

Theorem 5.21. *Let $Df_{aqcq}:\mathcal{Y}\to\mathcal{Z}$ be a even mapping for which there exists a function $\omega,\Omega:\mathcal{Y}^n\to[0,\infty)$ with the condition*

$$\lim_{p\to\infty}\frac{\omega\left(\kappa_j^p v_1,\cdots,\kappa_j^p v_n\right)}{\kappa_j^{4p}}=0 \tag{5.27}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$ where κ_j is defined (5.2)
such that the functional inequality

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\|\leq\omega\left(v_1,\cdots,v_n\right) \tag{5.28}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$. If there exists $L=L(i)$ such that the function

$$\Psi(v,v,\cdots,-v,0,v)=\Omega\left(\frac{v}{2},\frac{v}{2},\cdots,-\frac{v}{2},0,\frac{v}{2}\right),$$

has the property

$$\Psi(v,v,\cdots,-v,0,v)=\frac{L}{\kappa_j^4}\,\Psi\left(\kappa_j v,\kappa_j v,\cdots,-\kappa_j v,0,\kappa_j v\right). \tag{5.29}$$

for all $v\in\mathcal{Y}$. Then there exists a unique quartic function $\mathcal{Q}_4:\mathcal{Y}\to\mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_4(v)-q_4(v)\|=\|\mathcal{Q}_4(v)-\{f(2v)-4f(v)\}\|\leq\frac{L^{1-j}}{1-L}\Psi(v,v,\cdots,-v,0,v) \tag{5.30}$$

for all $x\in\mathcal{Y}$.

Corollary 5.19. *Let $Df_{aqcq}:\mathcal{Y}\to\mathcal{Z}$ be an even mapping. If there exists real numbers b and d satisfying (3.58) for all $v_1,\cdots,v_n\in\mathcal{Y}$, then there exists a unique quartic function $\mathcal{Q}_4:\mathcal{Y}\to\mathcal{Z}$ such that*

$$\|q_4(v)-\mathcal{Q}_4(v)\|\leq\begin{cases}\dfrac{5b}{15},\\ \dfrac{b||v||^d(5n-6+2^{rd})}{|16-2^d|},\\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|16-2^{nd}|}\end{cases} \tag{5.31}$$

for all $v\in\mathcal{Y}$.

Theorem 5.22. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (5.22) and (5.27) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{5.32}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right),$$

has the properties (5.24) and (5.29) and

$$\Psi(v, v, \cdots, -v, 0, v) = \frac{L}{\kappa_j}\,\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, 0, \kappa_j v\right). \tag{5.33}$$

for all $v \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \frac{L^{1-j}}{12(1-L)}\Psi(v, v, \cdots, -v, 0, v) \tag{5.34}$$

for all $v \in \mathcal{Y}$.

Corollary 5.20. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d satisfying (3.64) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \frac{5b}{12}\left(\frac{1}{|3|} + \frac{1}{|15|}\right), \\ \frac{b||v||^d(5n-1+2^{rd})}{12}\left(\frac{1}{|4-2^d|} + \frac{1}{|16-2^d|}\right), \\ \frac{b||v||^{nd}(5n-1+2^{rnd})}{12}\left(\frac{1}{|4-2^{nd}|} + \frac{1}{|16-2^{nd}|}\right) \end{cases} \tag{5.35}$$

for all $v \in \mathcal{Y}$.

5.3 f IS AN ODD - EVEN FUNCTION

Theorem 5.23. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (5.1), (5.22), (5.13) and (5.27) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{5.36}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right),$$

has the properties (5.4), (5.24), (5.15) and (5.29) and

$$\Psi(v, v, \cdots, -v, 0, v) = \frac{L}{\kappa_j}\,\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, 0, \kappa_j v\right). \tag{5.37}$$

for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \leq \frac{L^{1-j}}{(1-L)}\left(\frac{1}{6} + \frac{1}{12}\right)\Psi(v, v, \cdots, -v, 0, v) \tag{5.38}$$

for all $v \in \mathcal{Y}$.

Corollary 5.21. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping. If there exists real numbers b and d satisfying (3.73) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \frac{5b}{2}\left\{\frac{1}{12}\left[\frac{1}{|3|}+\frac{1}{|15|}\right]+\frac{1}{6}\left[1+\frac{1}{|7|}\right]\right\}, \\ \frac{b||v||^d(5n-1+2^{rd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^d|}+\frac{1}{|16-2^d|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^d|}+\frac{1}{|8-2^d|}\right]\right\}, \\ \frac{b||v||^{nd}(5n-1+2^{rnd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^{nd}|}+\frac{1}{|16-2^{nd}|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^{nd}|}+\frac{1}{|8-2^{nd}|}\right]\right\}, \end{cases} \tag{5.39}$$

for all $v \in \mathcal{Y}$.

6 Stability Results - Fixed Point Method: n Even Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an even positive integer in Banach space using fixed point method.

6.1 f IS AN ODD FUNCTION

Theorem 6.24. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a odd mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^p} = 0 \tag{6.1}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where

$$\kappa_j = \begin{cases} 2 & if \quad j = 0; \\ \frac{1}{2} & if \quad j = 1, \end{cases} \tag{6.2}$$

such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{6.3}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, v) = \frac{L}{\kappa_j}\,\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, \kappa_j v\right). \tag{6.4}$$

for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{A}(v) - a(v)\| = \|\mathcal{A}(v) - \{f(2v) - 8f(v)\}\| \leq \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, v) \tag{6.5}$$

for all $v \in \mathcal{Y}$.

Corollary 6.22. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d fulfilling (4.6) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|a(v) - \mathcal{A}(v)\| \leq \begin{cases} 5b, \\ \frac{b||v||^d(5n-1+2^{rd})}{|2-2^d|}, \\ \frac{b||v||^d(4+2^{rnd})}{|2-2^{nd}|}, \\ \frac{b||v||^{nd}(5n-3+2^{rd}+2^{rnd})}{|2-2^{nd}|} \end{cases} \tag{6.6}$$

for all $v \in \mathcal{Y}$.

Theorem 6.25. *Let $Df_{aqcq}: \mathcal{Y} \to \mathcal{Z}$ be a odd mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0,\infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^{3p}} = 0 \tag{6.7}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{6.8}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there existss $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, v) = \frac{L}{\kappa_j^3}\,\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, \kappa_j v\right). \tag{6.9}$$

for all $v \in \mathcal{Y}$. Then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{C}(v) - c(v)\| = \|\mathcal{C}(v) - \{f(2v) - 2f(v)\}\| \leq \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, v) \tag{6.10}$$

for all $v \in \mathcal{Y}$.

Corollary 6.23. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d fulfilling (4.12) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|c(v) - \mathcal{C}(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d(5n - 1 + 2^{rd})}{|8 - 2^d|}, \\ \dfrac{b||v||^d(4 + 2^{rnd})}{|8 - 2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{|8 - 2^{nd}|} \end{cases} \tag{6.11}$$

for all $v \in \mathcal{Y}$.

Theorem 6.26. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a odd mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0,\infty)$ with the conditions (6.1) and (6.7) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{6.12}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the properties (6.4) and (6.9) for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \frac{L^{1-j}}{6(1-L)}\Psi(v, v, \cdots, -v, v) \tag{6.13}$$

for all $v \in \mathcal{Y}$.

Corollary 6.24. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d fulfilling (4.16) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|a(v) - \mathcal{A}(v)\| \leq \begin{cases} \dfrac{5b}{6} + \dfrac{5b}{7.6}, \\ \dfrac{b||v||^d(5n - 1 + 2^{rd})}{6|2 - 2^d|} + \dfrac{b||v||^d(5n - 1 + 2^{rd})}{6|8 - 2^d|}, \\ \dfrac{b||v||^d(4 + 2^{rnd})}{6|2 - 2^{nd}|} + \dfrac{b||v||^d(4 + 2^{rnd})}{6|8 - 2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{6|2 - 2^{nd}|} + \dfrac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{6|8 - 2^{nd}|} \end{cases} \tag{6.14}$$

for all $v \in \mathcal{Y}$.

6.2 f IS AN EVEN FUNCTION

Theorem 6.27. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^{2p}} = 0 \tag{6.15}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \le \omega\left(v_1, \cdots, v_n\right) \tag{6.16}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, v) = \frac{L}{\kappa_j^2}\,\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, \kappa_j v\right). \tag{6.17}$$

for all $v \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_2(v) - q_2(v)\| = \|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}\| \le \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, v) \tag{6.18}$$

for all $v \in \mathcal{Y}$.

Corollary 6.25. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d fulfilling (4.22) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|q_2(v) - \mathcal{Q}_2(v)\| \le \begin{cases} 5b, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{|4-2^d|}, \\ \dfrac{b||v||^d(4+2^{rnd})}{|4-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{|4-2^{nd}|} \end{cases} \tag{6.19}$$

for all $v \in \mathcal{Y}$.

Theorem 6.28. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^{4p}} = 0 \tag{6.20}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \le \omega\left(v_1, \cdots, v_n\right) \tag{6.21}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, v) = \frac{L}{\kappa_j^4}\,\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, \kappa_j v\right). \tag{6.22}$$

for all $v \in \mathcal{Y}$. Then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_4(v) - q_4(v)\| = \|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}\| \le \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, v) \tag{6.23}$$

for all $v \in \mathcal{Y}$.

Corollary 6.26. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d fulfilling (4.28) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|q_4(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{|16-2^d|}, \\ \dfrac{b||v||^d(4+2^{rnd})}{|16-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{|16-2^{nd}|} \end{cases} \tag{6.24}$$

for all $v \in \mathcal{Y}$.

Theorem 6.29. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (6.15) and (6.20) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{6.25}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there existss $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the properties (6.17) and (6.22) for all $v \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \frac{L^{1-j}}{12(1-L)}\Psi(v, v, \cdots, -v, v) \tag{6.26}$$

for all $v \in \mathcal{Y}$.

Corollary 6.27. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d fulfilling (4.32) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{12|4-2^d|} + \dfrac{b||v||^d(5n-1+2^{rd})}{12|16-2^d|}, \\ \dfrac{b||v||^d(4+2^{rnd})}{12|4-2^{nd}|} + \dfrac{b||v||^d(4+2^{rnd})}{12|16-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{12|4-2^{nd}|} + \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{12|16-2^{nd}|} \end{cases} \tag{6.27}$$

for all $v \in \mathcal{Y}$.

6.3 f IS AN ODD - EVEN FUNCTION

Theorem 6.30. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (6.1), (6.15), (6.7) and (6.20) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{6.28}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the properties (6.4), (6.17), (6.9) and (6.22) for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \leq \frac{L^{1-j}}{(1-L)}\left(\frac{1}{6} + \frac{1}{12}\right)\Psi(v, v, \cdots, -v, v) \tag{6.29}$$

for all $v \in \mathcal{Y}$.

Corollary 6.28. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping and if there exists real numbers b and d fulfilling (4.36) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\|$$
$$\leq \begin{cases} \frac{5b}{2}\left\{\frac{1}{12}\left[\frac{1}{|3|}+\frac{1}{|15|}\right]+\frac{1}{6}\left[1+\frac{1}{|7|}\right]\right\}, \\ \frac{b||v||^d(5n-1+2^{rd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^d|}+\frac{1}{|16-2^d|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^d|}+\frac{1}{|8-2^d|}\right]\right\}, \\ \frac{b||v||^d(4+2^{rnd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^{nd}|}+\frac{1}{|16-2^{nd}|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^{nd}|}+\frac{1}{|8-2^{nd}|}\right]\right\}, \\ \frac{b||v||^{nd}(5n-3+2^{rd}+2^{rnd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^{nd}|}+\frac{1}{|16-2^{nd}|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^{nd}|}+\frac{1}{|8-2^{nd}|}\right]\right\}, \end{cases} \tag{6.30}$$

for all $v \in \mathcal{Y}$.

7 Counter Examples For Non Stable Cases

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = 1$ in condition (ii) of Corollaries 3.1, 4.8, 5.15 and 6.22.

Example 7.1. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & \text{if } |v| < 1 \\ \rho, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$a(v) = f(2v) - 8f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^r}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq 52\rho \sum_{i=1}^{n} |v_i| \tag{7.1}$$

for all $v \in \mathbb{R}$. Then there do not exist a additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{A}(v) - \{f(2v) - 8f(v)\}| \leq \eta |v|, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.2}$$

Proof. Now

$$|a(v)| = |f(2v) - 8f(v)| \leq \sum_{r=0}^{\infty} \frac{|\omega(2^r v)|}{|2^r|} = \sum_{n=0}^{\infty} \frac{\rho}{2^r} = 2\,\rho.$$

Therefore, we see that a is bounded. We are going to prove that a satisfies (7.1).

If $v = 0$ then (7.1) is trivial. If $\sum_{i=1}^{n} |v_i| \geq \frac{1}{2}$ then the left hand side of (7.1) is less than 52ρ. Now suppose that $0 < \sum_{i=1}^{n} |v_i| < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^k} \leq \sum_{i=1}^{n} |v_i| < \frac{1}{2^{k-1}}, \tag{7.3}$$

so that $2^{k-1} v_i (i = 1, 2, \cdots, n) < \frac{1}{2}$ and consequently

$$2^{k-1}\left(\sum_{i=1}^{n-1} v_i + 2v_n\right), 2^{k-1}\left(\sum_{i=1}^{n-1} v_i - 2v_n\right), 2^{k-1}\left(\sum_{i=1}^{n} v_i\right), 2^{k-1}\left(\sum_{i=1}^{n-1} v_i - v_n\right),$$
$$2^{k-1} f\left(\sum_{i=1}^{n-1} v_i\right), 2^{k-1}(2v_n), 2^{k-1}(-2v_n), 2^{k-1}(v_n), 2^{k-1}(-v_n) \in (-1, 1).$$

Therefore for each $r = 0, 1, \ldots, k-1$, we have

$$2^r\left(\sum_{i=1}^{n-1} v_i + 2v_n\right), 2^r\left(\sum_{i=1}^{n-1} v_i - 2v_n\right), 2^r\left(\sum_{i=1}^{n} v_i\right), 2^r\left(\sum_{i=1}^{n-1} v_i - v_n\right),$$
$$2^r f\left(\sum_{i=1}^{n-1} v_i\right), 2^r\left(2v_n\right), 2^r\left(-2v_n\right), 2^r\left(v_n\right), 2^r\left(-v_n\right) \in (-1,1)$$

and

$$\omega\left(2^r\sum_{i=1}^{n-1} v_i + 2^r\cdot 2v_n\right) + \omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r\cdot 2v_n\right) - 4\omega\left(2^r\sum_{i=1}^{n} v_i\right) - 4\omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r v_n\right)$$
$$+ 6\omega\left(2^r\sum_{i=1}^{n-1} v_i\right) - \omega\left(2^r\cdot 2v_n\right) - \omega\left(-2^r\cdot 2v_n\right) + 4\omega\left(2^r v_n\right) + 4\omega f\left(-2^r v_n\right) = 0$$

for $r = 0, 1, \ldots, k-1$. From the definition of a and (7.3), we obtain that

$$\begin{aligned}
&\left|a\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + a\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4a\left(\sum_{i=1}^{n} v_i\right) - 4a\left(\sum_{i=1}^{n-1} v_i - v_n\right)\right.\\
&\qquad\left. +6a\left(\sum_{i=1}^{n-1} v_i\right) - a\left(2v_n\right) - a\left(-2v_n\right) + 4a\left(v_n\right) + 4a\left(-v_n\right)\right|\\
&\le \sum_{r=0}^{\infty}\frac{1}{2^r}\left|\omega\left(2^r\sum_{i=1}^{n-1} v_i + 2^r\cdot 2v_n\right) + \omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r\cdot 2v_n\right) - 4\omega\left(2^r\sum_{i=1}^{n} v_i\right)\right.\\
&\qquad - 4\omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r v_n\right) + 6\omega\left(2^r\sum_{i=1}^{n-1} v_i\right) - \omega\left(2^r\cdot 2v_n\right) - \omega\left(-2^r\cdot 2v_n\right)\\
&\qquad\qquad\left. + 4\omega\left(2^r v_n\right) + 4\omega f\left(-2^r v_n\right)\right|\\
&\le \sum_{r=k}^{\infty}\frac{1}{2^r}\left|\omega\left(2^r\sum_{i=1}^{n-1} v_i + 2^r\cdot 2v_n\right) + \omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r\cdot 2v_n\right) - 4\omega\left(2^r\sum_{i=1}^{n} v_i\right)\right.\\
&\qquad - 4\omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r v_n\right) + 6\omega\left(2^r\sum_{i=1}^{n-1} v_i\right) - \omega\left(2^r\cdot 2v_n\right) - \omega\left(-2^r\cdot 2v_n\right)\\
&\qquad\qquad\left. + 4\omega\left(2^r v_n\right) + 4\omega f\left(-2^r v_n\right)\right|\\
&= \sum_{r=k}^{\infty}\frac{1}{2^r}\times 26\rho = 26\rho\times\frac{2}{2^k} \le 52\rho\sum_{i=1}^{n}|v_i|
\end{aligned}$$

Thus a satisfies (7.1) for all $x \in \mathbb{R}$ with $0 < \sum_{i=1}^{n}|v_i| < \frac{1}{2}$.

We claim that the additive functional equation (1.3) is not stable for $r = 1$ in condition (ii) of Corollary 3.1. Suppose on the contrary that there exist a additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ satisfying (7.2). Since a is bounded and continuous for all $x \in \mathbb{R}$, $\mathcal{A}$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.3, $\mathcal{A}$ must have the form $\mathcal{A}(v) = cv$ for any v in $\mathbb{R}$. Thus, we obtain that

$$|a(v)| \le (\eta + |c|)\,|v|. \tag{7.4}$$

But we can choose a positive integer s with $s\rho > \eta + |c|$.

If $v \in \left(0, \frac{1}{2^{s-1}}\right)$, then $2^r v \in (0,1)$ for all $r = 0, 1, \ldots, s-1$. For this v, we get

$$a(v) = \sum_{n=0}^{\infty}\frac{\omega(2^r v)}{2^r} \ge \sum_{r=0}^{s-1}\frac{\rho(2^r v)}{2^r} = s\rho v > (\eta + |c|)\,v$$

which contradicts (7.4). Therefore the additive functional equation (1.3) is not stable in sense of Ulam, Hyers and Rassias if $r = 1$, assumed in the inequality condition (ii). □

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = \frac{1}{n}$ in condition (iii) of Corollaries 3.1,5.15 and condition (iv) of Corollaries 4.8 and 6.22.

Example 7.2. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & \text{if } |v| < \frac{1}{n} \\ \frac{\rho}{n}, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$a(v) = f(2v) - 8f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^r}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq \frac{52\rho}{n} \sum_{i=1}^{n} |v_i|^{\frac{1}{n}} \tag{7.5}$$

for all $v \in \mathbb{R}$. Then there do not exist a additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{A}(v) - \{f(2v) - 8f(v)\}| \leq \eta |v|^{\frac{1}{n}}, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.6}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = 2$ in condition (ii) of Corollaries 3.1, 4.8, 5.15 and 6.22.

Example 7.3. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v^2, & \text{if } |v| < 1 \\ \rho, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$q_2(v) = f(2v) - 3f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{2r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq \frac{26\rho \times 4^2}{3} \sum_{i=1}^{n} |v_i|^2 \tag{7.7}$$

for all $v \in \mathbb{R}$. Then there do not exist a quadratic mapping $\mathcal{Q}_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}| \leq \eta |v|^2, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.8}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = \frac{2}{n}$ in condition (iii) of Corollaries 3.1,5.15 and condition (iv) of Corollaries 4.8 and 6.22.

Example 7.4. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & \text{if } |v| < \frac{2}{n} \\ \frac{2\rho}{n}, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$q_2(v) = f(2v) - 16f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{2r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) + 6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq \frac{52\rho \times 4^2}{3n} \sum_{i=1}^{n} |v_i|^{\frac{2}{n}} \tag{7.9}$$

for all $v \in \mathbb{R}$. Then there do not exist a quadratic mapping $\mathcal{Q}_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}| \leq \eta |v|^{\frac{2}{n}}, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.10}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = 3$ in condition (ii) of Corollaries 3.1, 4.8, 5.15 and 6.22.

Example 7.5. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v^3, & if\ |v| < 1 \\ \rho, & otherwise \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$c(v) = f(2v) - 3f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{3r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) + 6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq \frac{26\rho \times 8^3}{7} \sum_{i=1}^{n} |v_i|^3 \tag{7.11}$$

for all $v \in \mathbb{R}$. Then there do not exist a cubic mapping $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{C}(v) - \{f(2v) - 2f(v)\}| \leq \eta |v|^3, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.12}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = \frac{3}{n}$ in condition (iii) of Corollaries 3.1,5.15 and condition (iv) of Corollaries 4.8 and 6.22.

Example 7.6. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & if\ |v| < \frac{3}{n} \\ \frac{3\rho}{n}, & otherwise \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$c(v) = f(2v) - 2f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{3r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) + 6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq \frac{78\rho \times 8^3}{7n} \sum_{i=1}^{n} |v_i|^{\frac{3}{n}} \tag{7.13}$$

for all $v \in \mathbb{R}$. Then there do not exist a cubic mapping $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{C}(v) - \{f(2v) - 2f(v)\}| \leq \eta |v|^{\frac{3}{n}}, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.14}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = 2$ in condition (ii) of Corollaries 3.1, 4.8, 5.15 and 6.22.

Example 7.7. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v^4, & \text{if } |v| < 1 \\ \rho, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$q_4(v) = f(2v) - 4f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{4r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq \frac{26\rho \times 4^2}{15} \sum_{i=1}^{n} |v_i|^4 \tag{7.15}$$

for all $v \in \mathbb{R}$. Then there do not exist a quartic mapping $\mathcal{Q}_4 : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}| \leq \eta |v|^4, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.16}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = \frac{4}{n}$ in condition (iii) of Corollaries 3.1,5.15 and condition (iv) of Corollaries 4.8 and 6.22.

Example 7.8. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & \text{if } |v| < \frac{4}{n} \\ \frac{4\rho}{n}, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$q_2(v) = f(2v) - 4f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{4r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq \frac{144\rho \times 4^2}{15n} \sum_{i=1}^{n} |v_i|^{\frac{4}{n}} \tag{7.17}$$

for all $v \in \mathbb{R}$. Then there do not exist a quartic mapping $\mathcal{Q}_4 : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}| \leq \eta |v|^{\frac{4}{n}}, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.18}$$

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Stability of 2-Variable Additive-Quadratic-Cubic-Quartic Functional Equation Using Fixed Point Method

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Abstract

In this paper, the authors proved the generalized Ulam-Hyers stability of 2-variable Additive-Quadratic-Cubic-Quartic functional equation

$$\begin{aligned} f(x+2y,u+2v)+f(x-2y,u-2v) =& 4f(x+y,u+v)-4f(x-y,u-v)-6f(x,u)+f(2y,2v) \\ &+f(-2y,-2v)-4f(y,v)-4f(-y,-v) \end{aligned}$$

using fixed point method.

Keywords: Additive-quadratic-cubic-quartic functional equations, generalized Ulam-Hyers stability, Banach space, fixed point.

2010 MSC: 39B52, 32B72, 32B82.

1 Introduction and Preliminaries

Under what condition is there a homomorphism near an approximately homomorphism between a group and a metric group? This is called the stability problem of functional equations which was first raised by S. M. Ulam [49] in 1940. In next year, D. H. Hyers [24] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by T. Aoki [2] and Th. M. Rassias [44], respectively. The terminology Hyers- Ulam-Rassias stability originates from this historical background. Since then, a great deal of work has been done by a number of authors (for instance, [11, 13, 25, 40–42, 45, 47]).

The stability of mixed type functional equations have been extensively investigated by a number of mathematicians in referenes (see [3–5, 14–22, 28–31, 33–38, 43, 48, 50–52]). In 2003, V. Radu [39] introduced a new method, successively developed in ([7–10]), to obtaining the existence of the exact solutions and the error estimations, based on the fixed point alternative.

Now we will recall the fundamental results in fixed point theory.

Theorem 1.1. *(Banach's contraction principle) Let (X,d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx,Ty) \le Ld(x,y)$ for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

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(A2) $lim_{n\to\infty} T^n x = x^*$, *for any starting point* $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X$;

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X$.

Theorem 1.2. *[12](The alternative of fixed point) Suppose that for a complete generalized metric space* (X, d) *and a strictly contractive mapping* $T : X \to X$ *with Lipschitz constant L. Then, for each given element* $x \in X$, *either*
$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0,$$
or
$(B2)$ *there exists a natural number* n_0 *such that:*
(i) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(ii)*The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T
(iii) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;
(iv) $d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ *for all* $y \in Y$.

Recently, M. Arunkumar et al. introduced and investigated the general solution and generalized Ulam-Hyers stability of 2-Variable AQCQ Functional equation

$$f(x+2y, u+2v) + f(x-2y, u-2v) = 4f(x+y, u+v) - 4f(x-y, u-v) - 6f(x,u) + f(2y, 2v) + f(-2y, -2v) - 4f(y, v) - 4f(-y, -v) \tag{1.1}$$

using direct method.

In this paper, the authors proved the generalized Ulam-Hyers stability of 2-variable Additive-Quadratic-Cubic-Quartic functional equation (1.1) using fixed point method.

Through out this paper, let X be a normed space and Y be a Banach space respectively. Define a mapping $Df : X \to Y$ by

$$Df(x,y,u,v) = f(x+2y, u+2v) + f(x-2y, u-2v) - 4f(x+y, u+v) + 4f(x-y, u-v) + 6f(x,u) - f(2y, 2v) - f(-2y, -2v) + 4f(y, v) + 4f(-y, -v)$$

for all $x, y, u, v \in X$.

2 Stability Results: Odd Case

In this section, the authors presented the generalized Ulam-Hyers stability of the functional equation (1.1) for odd case using fixed point method.

2.1 Additive Stability Results

Theorem 2.1. *Let* $Df : X \to Y$ *be a mapping for which there exist a function* $\alpha : X^4 \to [0, \infty)$ *with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^k} \alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right) = 0 \tag{2.1}$$

where $\mu_i = 2$ *if* $i = 0$ *and* $\mu_i = \frac{1}{2}$ *if* $i = 1$, *such that the functional inequality with*

$$\|D\, f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{2.2}$$

for all $x, y, u, v \in X$. *If there exists* $L = L(i) < 1$ *such that the function*

$$u \to \Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right), \tag{2.3}$$

has the property

$$\Phi(u) = L\frac{1}{\mu_i}\Phi(\mu_i u) \tag{2.4}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ *for all* $x \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying the functional equation (1.1) and*

$$\| f(2u,2u) - 8f(u,u) - A(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{2.5}$$

for all $u \in X$.

Proof. Consider the set

$$\Omega = \{p/p : X \to Y,\ p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = \inf\{K \in (0,\infty) :\| p(u) - q(u) \| \leq K\Phi(u), u \in X\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(u) = \frac{1}{\mu_i} p(\mu_i u),$$

for all $u \in X$. Now $p, q \in \Omega$, we have

$$\begin{aligned} d(p,q) \leq K &\Rightarrow \| p(u) - q(u) \| \leq K\Phi(u), \forall u \in X. \\ &\Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i u) - \frac{1}{\mu_i} q(\mu_i u) \right\| \leq \frac{1}{\mu_i} K\Phi(\mu_i u), \forall u \in X, \\ &\Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i u) - \frac{1}{\mu_i} q(\mu_i u) \right\| \leq LK\Phi(u), \forall u \in X, \\ &\Rightarrow \| Tp(u) - Tq(u) \| \leq LK\Phi(u), \forall u \in X, \\ &\Rightarrow d(p,q) \leq LK. \end{aligned}$$

This implies $d(Tp, Tq) \leq Ld(p,q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

Replacing (x,y,u,v) by (u,u,u,u) in (2.2), we get

$$\|f(3u,3u) - 4f(2u,2u) + 5f(u,u)\| \leq \|\alpha(u,u,u,u)\| \tag{2.6}$$

for all $u \in X$. Replacing (x,y,u,v) by $(2u,u,2u,u)$ in (2.2), we obtain

$$\|f(4u,4u) - 4f(3u,3u) + 6f(2u,2u) - 4f(u,u)\| \leq \|\alpha(2u,u,2u,u)\| \tag{2.7}$$

for all $u \in X$. Now, from (2.6) and (2.7), we have

$$\begin{aligned} &\|f(4u,4u) - 10f(2u,2u) + 16f(u,u)\| \\ &\quad \leq 4\,\|f(3u,3u) - 4f(2u,2u) + 5f(u,u)\| + \|f(4u,4u) - 4f(3u,3u) + 6f(2u,2u) - 4f(u,u)\| \\ &\quad \leq 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u) \end{aligned} \tag{2.8}$$

for all $u \in X$. From (2.8), we arrive

$$\|f(4u,4u) - 10f(2u,2u) + 16f(u,u)\| \leq \beta(u,u,u,u) \tag{2.9}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $u \in X$. It is easy from (2.9) that

$$\|f(4u,4u) - 8f(2u,2u) - 2\,(f(2u,2u) - 8f(u,u))\| \leq \beta(u,u,u,u) \tag{2.10}$$

for all $u \in X$. Let $a : X \to Y$ be a mapping defined by $a(u,u) = f(2u,2u) - 8f(u,u)$. From (2.10), we conclude that

$$\|a(2u,2u) - 2a(u,u)\| \leq \beta(u,u,u,u) \tag{2.11}$$

for all $u \in X$. From (2.11), we arrive

$$\left\| \frac{a(2u,2u)}{2} - a(u,u) \right\| \leq \frac{1}{2}\beta(u,u,u,u) \tag{2.12}$$

for all $u \in X$. Using (2.3) and (2.4) for the case $i = 0$ it reduces to

$$\left\| \frac{a(2u,2u)}{2} - a(u,u) \right\| \leq L\Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f,Tf) \leq L \leq L^1 < \infty.$$

Again replacing $u = \frac{u}{2}$ in (2.11), we get

$$\left\| a(u,u) - 2a\left(\frac{u}{2},\frac{u}{2}\right) \right\| \leq \beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right) \tag{2.13}$$

for all $u \in X$. Using (2.3) and (2.4) for the case $i = 1$ it reduces to

$$\left\| a(u,u) - 2a\left(\frac{u}{2},\frac{u}{2}\right) \right\| \leq \Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f,Tf) \leq 1 \leq L^0 < \infty.$$

In both cases, we arrive

$$d(f,Tf) \leq L^{1-i}.$$

Therefore, $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point A of T in Ω such that

$$A(u,u) = \lim_{k\to\infty} \frac{1}{\mu_i^k}\left(f(\mu_i^{k+1}u,\mu_i^{k+1}u) - 8f(\mu_i^k u,\mu_i^k u)\right) \tag{2.14}$$

for all $u \in X$.

To prove $A : X \to Y$ is additive. Replacing (x,y,u,v) by $\left(\mu_i^k x,\mu_i^k y,\mu_i^k u,\mu_i^k v\right)$ in (2.2) and dividing by μ_i^k, it follows from (2.1) that

$$\begin{aligned}\|A(x,y,u,v)\| &= \lim_{k\to\infty} \frac{\left\| D\, f(\mu_i^k x,\mu_i^k y,\mu_i^k u,\mu_i^k v) \right\|}{\mu_i^k} \\ &\leq \lim_{k\to\infty} \frac{\alpha(\mu_i^k x,\mu_i^k y,\mu_i^k u,\mu_i^k v)}{\mu_i^k} = 0\end{aligned}$$

for all $x,y,u,v \in X$. i.e., A satisfies the functional equation (1.1).

By $(A3)$, A is the unique fixed point of T in the set $\Delta = \{A \in \Omega : d(f,A) < \infty\}$, A is the unique function such that

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq K\Phi(u)$$

for all $u \in X$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f,A) \leq \frac{1}{1-L}d(f,Tf)$$

this implies

$$d(f,A) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(2u,2u) - 8f(u,u) - A(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u)$$

for all $u \in X$. This completes the proof. □

The following corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1.1).

Corollary 2.1. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s+||y||^s+||u||^s+||v||^s\right), & s \neq 1; \\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right), & s \neq \frac{1}{4}; \\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s+||x||^{4s}+||y||^{4s}+||u||^{4s}+||v||^{4s}\right\}, & s \neq \frac{1}{4}; \end{cases} \tag{2.15}$$

for all $x,y,u,v \in X$, then there exists a unique additive function $A : X \to Y$ such that

$$\|f(2u,2u)-8f(u,u)-A(u,u)\| \leq \begin{cases} 5\lambda, \\ \dfrac{\left(18+2^{s+1}\right)\lambda||u||^s}{|2-2^s|}, \\ \dfrac{\left(4+4^s\right)\lambda||u||^{4s}}{|2-2^{4s}|}, \\ \dfrac{\left(22+4^s+2^{4s+1}\right)\lambda||u||^{4s}}{|2-2^{4s}|}. \end{cases} \tag{2.16}$$

for all $u \in X$.

Proof. Setting

$$\alpha(x,y,u,v) = \begin{cases} \lambda; \\ \lambda\left(||x||^s+||y||^s+||u||^s+||v||^s\right); \\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right); \\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s+||x||^{4s}+||y||^{4s}+||u||^{4s}+||v||^{4s}\right\}; \end{cases}$$

for all $x,y,u,v \in X$. Now,

$$\frac{\alpha(\mu_i^k x,\mu_i^k y,\mu_i^k u,\mu_i^k u)}{\mu_i^k} = \begin{cases} \lambda\mu_i^{-k}; \\ \lambda\mu_i^{k(s-1)}\left(||x||^s+||y||^s+||u||^s+||v||^s\right); \\ \lambda\mu_i^{k(4s-1)}\left(||x||^s||y||^s||u||^s||v||^s\right); \\ \lambda\mu_i^{k(4s-1)}\left\{||x||^s||y||^s||u||^s||v||^s+||x||^{4s}+||y||^{4s}+||u||^{4s}+||v||^{4s}\right\}; \end{cases}$$
$$= \begin{cases} \to 0 \text{ as } k\to\infty; \\ \to 0 \text{ as } k\to\infty; \\ \to 0 \text{ as } k\to\infty; \\ \to 0 \text{ as } k\to\infty. \end{cases}$$

Thus, (2.1) is holds.

But we have $\Phi(u) = \beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right)$ has the property $\Phi(u) = L\cdot\frac{1}{\mu_i}\Phi(\mu_i u)$ for all $u \in X$. Hence,

$$\Phi(u) = \beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right) = \begin{cases} 5\lambda, \\ \dfrac{\left(18+2^{s+1}\right)\lambda}{2^s}||u||^s; \\ \dfrac{\left(4+4^s\right)\lambda}{2^{4s}}||u||^{4s}; \\ \dfrac{\left(22+4^s+2^{4s+1}\right)\lambda}{2^{4s}}||u||^{4s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i}\Phi(\mu_i u) = \begin{cases} \dfrac{5\lambda}{\mu_i}; \\ \dfrac{\left(18+2^{s+1}\right)\lambda}{\mu_i 2^s}\left(||\mu_i u||^s\right); \\ \dfrac{\left(4+4^s\right)\lambda}{\mu_i 2^{4s}}\left(||\mu_i u||^{4s}\right); \\ \dfrac{\left(22+4^s+2^{4s+1}\right)\lambda}{\mu_i 2^{4s}}\left(||\mu_i u||^{4s}\right); \end{cases} = \begin{cases} \mu_i^{-1}\Phi(u); \\ \mu_i^{s-1}\Phi(u); \\ \mu_i^{4s-1}\Phi(u); \\ \mu_i^{4s-1}\Phi(u). \end{cases}$$

From (2.5), we prove the following cases:
Case:1 $L = 2^{-1}$ if $i = 0$;

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \lambda \left(\frac{\left(2^{-1}\right)^{1-0}}{1-2^{(-1)}} \right) = 5\lambda.$$

Case:2 $L = 2^1$ if $i = 1$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \lambda \left(\frac{\left(2^{1}\right)^{1-1}}{1-2^{1}} \right) = -5\lambda.$$

Case:3 $L = 2^{s-1}$ for $s < 1$ if $i = 0$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \frac{\left(18+2^{s+1}\right)\lambda}{2^s} \left(\frac{\left(2^{(s-1)}\right)^{1-0}}{1-2^{(s-1)}} \right) ||u||^s = \frac{\left(18+2^{s+1}\right)\lambda}{2-2^s} ||u||^s.$$

Case:4 $L = 2^{1-s}$ for $s > 1$ if $i = 1$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \frac{\left(18+2^{s+1}\right)\lambda}{2^s} \left(\frac{\left(2^{(1-s)}\right)^{1-1}}{1-2^{(1-s)}} \right) ||u||^s = \frac{\left(18+2^{s+1}\right)\lambda}{2^s-2} ||u||^s.$$

Case:5 $L = 2^{4s-1}$ for $s < \frac{1}{4}$ if $i = 0$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \frac{\left(4+4^{s}\right)\lambda}{2^{4s}} \left(\frac{\left(2^{(4s-1)}\right)^{1-0}}{1-2^{(4s-1)}} \right) ||u||^{4s} = \frac{\left(4+4^{s}\right)\lambda}{2-2^{4s}} ||u||^{4s}.$$

Case:6 $L = 2^{1-4s}$ for $s > \frac{1}{4}$ if $i = 1$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \frac{\left(4+4^{s}\right)\lambda}{2^{4s}} \left(\frac{\left(2^{(1-4s)}\right)^{1-0}}{1-2^{(1-4s)}} \right) ||u||^{4s} = \frac{\left(4+4^{s}\right)\lambda}{2^{4s}-2} ||u||^{4s}.$$

This completes the proof. □

2.2 Cubic Stability Results

Theorem 2.2. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^{3k}} \alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right) = 0 \tag{2.17}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$, such that the functional inequality with

$$\|D\, f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{2.18}$$

for all $x,y,u,v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right), \tag{2.19}$$

has the property

$$\Phi(u) = L\frac{1}{\mu_i^3}\Phi\left(\mu_i u\right) \tag{2.20}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $x \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation (1.1) and

$$\|\ f(2u,2u) - 2f(u,u) - C(u,u)\ \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{2.21}$$

for all $u \in X$.

Proof. Consider the set

$$\Omega = \{p/p : X \to Y,\ p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = \inf\{K \in (0,\infty) :\| p(u) - q(u) \|\leq K\Phi(u), u \in X\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(u) = \frac{1}{\mu_i^3} p(\mu_i u),$$

for all $u \in X$. Now $p, q \in \Omega$, we have

$$\begin{aligned} d(p,q) \leq K \Rightarrow & \| p(u) - q(u) \|\leq K\Phi(u), \forall u \in X. \\ & \Rightarrow \left\| \frac{1}{\mu_i^3} p(\mu_i u) - \frac{1}{\mu_i^3} q(\mu_i u) \right\| \leq \frac{1}{\mu_i^3} K\Phi(\mu_i u), \forall u \in X, \\ & \Rightarrow \left\| \frac{1}{\mu_i^3} p(\mu_i u) - \frac{1}{\mu_i^3} q(\mu_i u) \right\| \leq LK\Phi(u), \forall u \in X, \\ & \Rightarrow \| Tp(u) - Tq(u) \|\leq LK\Phi(u), \forall u \in X, \\ & \Rightarrow d(p,q) \leq LK. \end{aligned}$$

This implies $d(Tp, Tq) \leq Ld(p,q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

It is easy from (2.9) that

$$\|f(4u,4u) - 2f(2u,2u) - 8\left(f(2u,2u) - 2f(u,u)\right)\| \leq \beta(u,u,u,u) \tag{2.22}$$

for all $u \in X$. Let $c : X \to Y$ be a mapping defined by $c(u,u) = f(2u,2u) - 2f(u,u)$. From (2.22), we conclude that

$$\|c(2u,2u) - 8c(u,u)\| \leq \beta(u,u,u,u) \tag{2.23}$$

for all $u \in X$. From (2.23), we arrive

$$\left\|\frac{c(2u,2u)}{8} - c(u,u)\right\| \leq \frac{1}{8}\beta(u,u,u,u) \tag{2.24}$$

for all $u \in X$. Using (2.19) and (2.20) for the case $i = 0$ it reduces to

$$\left\|\frac{c(2u,2u)}{8} - c(u,u)\right\| \leq\ L\Phi(u)$$

for all $u \in X$,

$$\text{i.e.,}\quad d(f,Tf) \leq L \leq L^1 < \infty.$$

Again replacing $u = \frac{u}{2}$ in (2.23), we get

$$\left\|c(u,u) - 8c\left(\frac{u}{2},\frac{u}{2}\right)\right\| \leq \beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right) \tag{2.25}$$

for all $u \in X$. Using (2.19) and (2.20) for the case $i = 1$ it reduces to

$$\left\|c(u,u) - 8c\left(\frac{u}{2},\frac{u}{2}\right)\right\| \leq\ \Phi(u)$$

for all $u \in X$,

$$\text{i.e.,}\quad d(f,Tf) \leq 1 \leq L^0 < \infty.$$

In both cases, we arrive

$$d(f,Tf) \leq L^{1-i}.$$

Therefore, $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point C of T in Ω such that

$$C(u,u) = \lim_{k\to\infty} \frac{1}{\mu_i^{3k}} \left(f(\mu_i^{k+1}u, \mu_i^{k+1}u) - 2f(\mu_i^k u, \mu_i^k u) \right) \tag{2.26}$$

for all $u \in X$.

To prove $C : X \to Y$ is cubic. Replacing (x,y,u,v) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right)$ in (2.18) and dividing by μ_i^{3k}, it follows from (2.17) that

$$\begin{aligned} \|C(x,y,u,v)\| &= \lim_{k\to\infty} \frac{\left\|D\, f(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v)\right\|}{\mu_i^{3k}} \\ &\leq \lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v)}{\mu_i^{3k}} = 0 \end{aligned}$$

for all $x,y,u,v \in X$. i.e., C satisfies the functional equation (1.1).

By $(A3)$, C is the unique fixed point of T in the set $\Delta = \{C \in \Omega : d(f,C) < \infty\}$, C is the unique function such that

$$\|f(2u,2u) - 2f(u,u) - C(u,u)\| \leq K\Phi(u)$$

for all $u \in X$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f,C) \leq \frac{1}{1-L} d(f,Tf)$$

this implies

$$d(f,C) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(2u,2u) - 2f(u,u) - C(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u)$$

for all $u \in X$. This completes the proof. □

The following corollary is an immediate consequence of Theorem 2.2 concerning the stability of (1.1).

Corollary 2.2. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 3; \\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right), & s \neq \frac{3}{4}; \\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq \frac{3}{4}; \end{cases} \tag{2.27}$$

for all $x,y,u,v \in X$, then there exists a unique cubic function $A : X \to Y$ such that

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \begin{cases} \dfrac{5\lambda}{7}, \\ \dfrac{\left(18 + 2^{s+1}\right)\lambda||u||^s}{|2^3 - 2^s|}, \\ \dfrac{\left(4 + 4^s\right)\lambda||u||^{4s}}{|2^3 - 2^{4s}|}, \\ \dfrac{\left(22 + 4^s + 2^{4s+1}\right)\lambda||u||^{4s}}{|2^3 - 2^{4s}|}. \end{cases} \tag{2.28}$$

for all $u \in X$.

Proof. Setting

$$\alpha(x,y,u,v) = \begin{cases} \lambda; \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right); \\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases}$$

for all $x, y, u, v \in X$. Now,

$$\frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k u)}{\mu_i^{3k}} = \begin{cases} \lambda \mu_i^{-3k}; \\ \lambda \mu_i^{k(s-3)} \left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda \mu_i^{k(4s-3)} \left(||x||^s ||y||^s ||u||^s ||v||^s\right); \\ \lambda \mu_i^{k(4s-3)} \left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (2.17) is holds.

But we have $\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right)$ has the property $\Phi(u) = L \cdot \frac{1}{\mu_i^3} \Phi(\mu_i u)$ for all $u \in X$. Hence

$$\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right) = \begin{cases} 5\lambda, \\ \frac{\left(18 + 2^{s+1}\right)\lambda}{2^s} ||u||^s; \\ \frac{\left(4 + 4^s\right)\lambda}{2^{4s}} ||u||^{4s}; \\ \frac{\left(22 + 4^s + 2^{4s+1}\right)\lambda}{2^{4s}} ||u||^{4s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i^3}\Phi(\mu_i u) = \begin{cases} \frac{5\lambda}{\mu_i^3}; \\ \frac{\left(18 + 2^{s+1}\right)\lambda}{\mu_i^3 2^s} \left(||\mu_i u||^s\right); \\ \frac{\left(4 + 4^s\right)\lambda}{\mu_i^3 2^{4s}} \left(||\mu_i u||^{4s}\right); \\ \frac{\left(22 + 4^s + 2^{4s+1}\right)\lambda}{\mu_i^3 2^{4s}} \left(||\mu_i u||^{4s}\right); \end{cases} = \begin{cases} \mu_i^{-3}\Phi(u); \\ \mu_i^{s-3}\Phi(u); \\ \mu_i^{4s-3}\Phi(u); \\ \mu_i^{4s-3}\Phi(u). \end{cases}$$

From (2.21), we prove the following cases:

Case:1 $L = 2^{-3}$ if $i = 0$;

$$\|f(2u, 2u) - 2f(u, u) - C(u, u)\| \le 5\lambda \left(\frac{\left(2^{-3}\right)^{1-0}}{1 - 2^{(-3)}}\right) = \frac{5\lambda}{7}.$$

Case:2 $L = 2^3$ if $i = 1$,

$$\|f(2u, 2u) - 2f(u, u) - C(u, u)\| \le 5\lambda \left(\frac{\left(2^3\right)^{1-1}}{1 - 2^3}\right) = \frac{-5\lambda}{7}.$$

Case:3 $L = 2^{s-3}$ for $s < 3$ if $i = 0$,

$$\|f(2u, 2u) - 2f(u, u) - C(u, u)\| \le \frac{\left(18 + 2^{s+1}\right)\lambda}{2^s} \left(\frac{\left(2^{(s-3)}\right)^{1-0}}{1 - 2^{(s-3)}}\right) ||u||^s = \frac{\left(18 + 2^{s+1}\right)\lambda}{2^3 - 2^s} ||u||^s.$$

Case:4 $L = 2^{3-s}$ for $s > 3$ if $i = 1$,

$$\|f(2u, 2u) - 2f(u, u) - C(u, u)\| \le \frac{\left(18 + 2^{s+1}\right)\lambda}{2^s} \left(\frac{\left(2^{(3-s)}\right)^{1-1}}{1 - 2^{(3-s)}}\right) ||u||^s = \frac{\left(18 + 2^{s+1}\right)\lambda}{2^s - 2^3} ||u||^s.$$

Case:5 $L = 2^{4s-3}$ for $s < \frac{3}{4}$ if $i = 0$,

$$\|f(2u,2u) - 2f(u,u) - C(u,u)\| \leq \frac{(4+4^s)\,\lambda}{2^{4s}} \left(\frac{\left(2^{(4s-3)}\right)^{1-0}}{1-2^{(4s-3)}} \right) ||u||^{4s} = \frac{(4+4^s)\,\lambda}{2^3 - 2^{4s}} ||u||^{4s}.$$

Case:6 $L = 2^{3-4s}$ for $s > \frac{3}{4}$ if $i = 1$,

$$\|f(2u,2u) - 2f(u,u) - C(u,u)\| \leq \frac{(4+4^s)\,\lambda}{2^{4s}} \left(\frac{\left(2^{(3-4s)}\right)^{1-0}}{1-2^{(3-4s)}} \right) ||u||^{4s} = \frac{(4+4^s)\,\lambda}{2^{4s} - 2^3} ||u||^{4s}.$$

This finishes the proof. □

2.3 Additive-Cubic Mixed Stability Results

Theorem 2.3. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0,\infty)$ with the condition given in (2.1) and (2.17) respectively, such that the functional inequality*

$$\|D\,f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{2.29}$$

for all $x,y,u,v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right), \tag{2.30}$$

has the property (2.4) and (2.20), then there exists a unique 2-variable additive function $A : X \to Y$ and a unique 2-variable cubic function $C : X \to Y$ satisfying the functional equation (1.1) and

$$\| \; f(u,u) - A(u,u) - C(u,u) \; \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{2.31}$$

for all $u \in X$.

Proof. By Theorems 2.1 and 2.2, there exists a unique 2-variable additive function $A_1 : X \to Y$ and a unique 2-variable cubic function $C_1 : X \to Y$ such that

$$\|f(2u,2u) - 8f(u,u) - A_1(u,u)\| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{2.32}$$

for all $u \in X$ and

$$\|f(2u,2u) - 2f(u,u) - C_1(u,u)\| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{2.33}$$

for all $u \in X$. Now, from (2.32) and (2.33) that

$$\begin{aligned}
\left\| f(u,u) + \frac{1}{6}A_1(u,u) - \frac{1}{6}C_1(u,u) \right\| &= \left\| \left\{ -\frac{f(2u,2u)}{6} + \frac{8}{6}f(u,u) + \frac{1}{6}A_1(u,u) \right\} \right. \\
&\qquad \left. + \left\{ \frac{f(2u,2u)}{6} - \frac{2}{6}f(u,u) - \frac{1}{6}C_1(u,u) \right\} \right\| \\
&\leq \frac{1}{6} \left\| \{-f(2u,2u) + 8f(u,u) + A_1(u,u)\} \right. \\
&\qquad \left. + \{f(2u,2u) - 2f(u,u) - C_1(u,u)\} \right\| \\
&\leq \frac{1}{6} \left\{ \frac{L^{1-i}}{1-L}\Phi(u) + \frac{L^{1-i}}{1-L}\Phi(u) \right\}
\end{aligned}$$

for all $u \in X$. Thus we obtain (2.31) by defining $A(u,u) = \frac{-1}{6}A_1(u,u)$ and $C(u,u) = \frac{1}{6}C_1(u,u)$, where $A(u,u)$ and $C(u,u)$ are defined in (2.14) and (2.26) respectively, for all $u \in X$. □

The following corollary is an immediate consequence of Theorem 2.3 concerning the stability of (1.1).

Corollary 2.3. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 1,3; \\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right), & s \neq \frac{1}{4}, \frac{3}{4}; \\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq \frac{1}{4}, \frac{3}{4}; \end{cases} \tag{2.34}$$

for all $x,y,u,v \in X$, then there exists a unique 2-variable additive function $A : X \to Y$ and a unique 2-variable cubic function $C : X \to Y$ such that

$$\|f(u,u) - A(u,u) - C(u,u)\| \leq \begin{cases} \frac{20\lambda}{21}; \\ \frac{(18+2^{s+1})}{6}\left(\frac{1}{|2-2^s|} + \frac{1}{|2^3-2^s|}\right)\lambda||u||^s; \\ \frac{(4+4^s)}{6}\left(\frac{1}{|2-2^{4s}|} + \frac{1}{|2^3-2^{4s}|}\right)\lambda||u||^{4s}; \\ \frac{(22+4^s+2^{4s+1})}{6}\left(\frac{1}{|2-2^{4s}|} + \frac{1}{|2^3-2^{4s}|}\right)\lambda||u||^{4s}; \end{cases} \tag{2.35}$$

for all $u \in X$.

3 Stability Results: Even Case

In this section, the authors given the generalized Ulam-Hyers stability of the functional equation (1.1) for even case using fixed point method.

3.1 Quadratic Stability Results

Theorem 3.4. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^{2k}} \alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right) = 0 \tag{3.1}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$, such that the functional inequality with

$$\|D\, f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{3.2}$$

for all $x,y,u,v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right), \tag{3.3}$$

has the property

$$\Phi(u) = L\frac{1}{\mu_i^2}\Phi\left(\mu_i u\right) \tag{3.4}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $u \in X$. Then there exists a unique 2-variable quadratic mapping $Q_2 : X \to Y$ satisfying the functional equation (1.1) and

$$\| f(2u,2u) - 16f(u,u) - Q_2(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.5}$$

for all $u \in X$.

Proof. Consider the set

$$\Omega = \{p/p : X \to Y,\ p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = \inf\{K \in (0,\infty) : \| p(u) - q(u) \| \leq K\Phi(u), u \in X\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(u) = \frac{1}{\mu_i^2} p(\mu_i u),$$

for all $u \in X$. Now $p, q \in \Omega$, we have

$$\begin{aligned} d(p,q) \le K \Rightarrow & \| \ p(u) - q(u) \ \| \le K\Phi(u), \forall u \in X. \\ & \Rightarrow \left\| \frac{1}{\mu_i^2} p(\mu_i u) - \frac{1}{\mu_i^2} q(\mu_i u) \right\| \le \frac{1}{\mu_i^2} K\Phi(\mu_i u), \forall u \in X, \\ & \Rightarrow \left\| \frac{1}{\mu_i^2} p(\mu_i u) - \frac{1}{\mu_i^2} q(\mu_i u) \right\| \le LK\Phi(u), \forall u \in X, \\ & \Rightarrow \| \ Tp(u) - Tq(u) \ \| \le LK\Phi(u), \forall u \in X, \\ & \Rightarrow d(p,q) \le LK. \end{aligned}$$

This implies $d(Tp, Tq) \le Ld(p,q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

Replacing (x, y, u, v) by (u, u, u, u) in (3.2) and using evenness of f, we get

$$\|f(3u,3u) - 6f(2u,2u) + 15f(u,u)\| \le \|\alpha(u,u,u,u)\| \tag{3.6}$$

for all $u \in X$. Replacing (x, y, u, v) by $(2u, u, 2u, u)$ in (3.2), we obtain

$$\|f(4u,4u) - 4f(3u,3u) + 4f(2u,2u) + 4f(u,u)\| \le \|\alpha(2u,u,2u,u)\| \tag{3.7}$$

for all $u \in X$. Now, from (3.6) and (3.7), we have

$$\begin{aligned} &\|f(4u,4u) - 20f(2u,2u) + 64f(u,u)\| \\ &\quad \le 4\|f(3u,3u) - 6f(2u,2u) + 15f(u,u)\| + \|f(4u,4u) - 4f(3u,3u) + 4f(2u,2u) + 4f(u,u)\| \\ &\quad \le 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u) \end{aligned} \tag{3.8}$$

for all $u \in X$. From (3.8), we arrive

$$\|f(4u,4u) - 20f(2u,2u) + 64f(u,u)\| \le \beta(u,u,u,u) \tag{3.9}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $u \in X$. It is easy from (3.9) that

$$\|f(4u,4u) - 16f(2u,2u) - 4\left(f(2u,2u) - 16f(u,u)\right)\| \le \beta(u,u,u,u) \tag{3.10}$$

for all $u \in X$. Let $q_2 : X \to Y$ be a mapping defined by $q_2(u,u) = f(2u,2u) - 16f(u,u)$. From (3.10), we conclude that

$$\|q_2(2u,2u) - 4q_2(u,u)\| \le \beta(u,u,u,u) \tag{3.11}$$

for all $u \in X$. From (3.11), we arrive

$$\left\| \frac{q_2(2u,2u)}{4} - q_2(u,u) \right\| \le \frac{1}{4}\beta(u,u,u,u) \tag{3.12}$$

for all $u \in X$. Using (3.3) and (3.4) for the case $i = 0$ it reduces to

$$\left\| \frac{q_2(2u,2u)}{4} - q_2(u,u) \right\| \le \ L\Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f, Tf) \le L \le L^1 < \infty.$$

Again replacing $u = \dfrac{u}{2}$ in (3.11), we get

$$\left\| q_2(u,u) - 4q_2\left(\frac{u}{2}, \frac{u}{2}\right) \right\| \le \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right) \tag{3.13}$$

for all $u \in X$. Using (3.3) and (3.4) for the case $i = 1$ it reduces to

$$\left\| q_2(u,u) - 4q_2\left(\frac{u}{2}, \frac{u}{2}\right) \right\| \leq \Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f, Tf) \leq 1 \leq L^0 < \infty.$$

In both cases, we arrive

$$d(f, Tf) \leq L^{1-i}.$$

Therefore, $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point Q_2 of T in Ω such that

$$Q_2(u,u) = \lim_{k\to\infty} \frac{1}{\mu_i^{2k}} \left(f(\mu_i^{k+1}u, \mu_i^{k+1}u) - 16f(\mu_i^k u, \mu_i^k u) \right) \tag{3.14}$$

for all $u \in X$.

To prove $Q_2 : X \to Y$ is quadratic. Replacing (x,y,u,v) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right)$ in (3.2) and dividing by μ_i^{2k}, it follows from (3.1) that

$$\begin{aligned} \|Q_2(x,y,u,v)\| &= \lim_{k\to\infty} \frac{\left\| D\, f(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v) \right\|}{\mu_i^{2k}} \\ &\leq \lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v)}{\mu_i^{2k}} = 0 \end{aligned}$$

for all $x, y, u, v \in X$. i.e., Q_2 satisfies the functional equation (1.1).

By $(A3)$, Q_2 is the unique fixed point of T in the set $\Delta = \{Q_2 \in \Omega : d(f, Q_2) < \infty\}$, Q_2 is the unique function such that

$$\|f(2u, 2u) - 16f(u,u) - Q_2(u,u)\| \leq K\Phi(u)$$

for all $u \in X$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f, Q_2) \leq \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f, Q_2) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(2u, 2u) - 16f(u,u) - Q_2(u,u) \| \leq \frac{L^{1-i}}{1-L} \Phi(u)$$

for all $u \in X$. This finishes the proof. □

The following corollary is an immediate consequence of Theorem 3.4 concerning the stability of (1.1).

Corollary 3.4. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 2; \\ \lambda\left(||x||^s ||y||^s ||u||^s ||v||^s\right), & s \neq \frac{1}{2}; \\ \lambda\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq \frac{1}{2}; \end{cases} \tag{3.15}$$

for all $x, y, u, v \in X$, then there exists a unique 2-variable quadratic function $Q_2 : X \to Y$ such that

$$\|f(2u,2u) - 16f(u,u) - Q_2(u,u)\| \leq \begin{cases} \dfrac{5\lambda}{3}, \\ \dfrac{\left(18 + 2^{s+1}\right)\lambda||u||^s}{|2^2 - 2^s|}, \\ \dfrac{\left(4 + 4^s\right)\lambda||u||^{4s}}{|2^2 - 2^{4s}|}, \\ \dfrac{\left(22 + 4^s + 2^{4s+1}\right)\lambda||u||^{4s}}{|2^2 - 2^{4s}|}. \end{cases} \tag{3.16}$$

for all $u \in X$.

Proof. Setting

$$\alpha(x,y,u,v)=\begin{cases}\lambda;\\ \lambda\left(||x||^s+||y||^s+||u||^s+||v||^s\right);\\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right);\\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s+||x||^{4s}+||y||^{4s}+||u||^{4s}+||v||^{4s}\right\};\end{cases}$$

for all $x,y,u,v\in X$. Now,

$$\frac{\alpha(\mu_i^k x,\mu_i^k y,\mu_i^k u,\mu_i^k u)}{\mu_i^{2k}}=\begin{cases}\lambda\mu_i^{-2k};\\ \lambda\mu_i^{k(s-2)}\left(||x||^s+||y||^s+||u||^s+||v||^s\right);\\ \lambda\mu_i^{k(4s-2)}\left(||x||^s||y||^s||u||^s||v||^s\right);\\ \lambda\mu_i^{k(4s-2)}\left\{||x||^s||y||^s||u||^s||v||^s+||x||^{4s}+||y||^{4s}+||u||^{4s}+||v||^{4s}\right\};\end{cases}$$
$$=\begin{cases}\to 0 \text{ as } k\to\infty;\\ \to 0 \text{ as } k\to\infty;\\ \to 0 \text{ as } k\to\infty;\\ \to 0 \text{ as } k\to\infty.\end{cases}$$

Thus, (3.1) is holds.

But we have $\Phi(u)=\beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right)$ has the property $\Phi(u)=L\cdot\frac{1}{\mu_i^2}\Phi(\mu_i u)$ for all $u\in X$. Hence,

$$\Phi(u)=\beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right)=\begin{cases}5\lambda,\\ \frac{\left(18+2^{s+1}\right)\lambda}{2^s}||u||^s;\\ \frac{\left(4+4^s\right)\lambda}{2^{4s}}||u||^{4s};\\ \frac{\left(22+4^s+2^{4s+1}\right)\lambda}{2^{4s}}||u||^{4s}.\end{cases}$$

Now,

$$\frac{1}{\mu_i^2}\Phi(\mu_i u)=\begin{cases}\frac{5\lambda}{\mu_i};\\ \frac{\left(18+2^{s+1}\right)\lambda}{\mu_i^2 2^s}\left(||\mu_i u||^s\right);\\ \frac{\left(4+4^s\right)\lambda}{\mu_i^2 2^{4s}}\left(||\mu_i u||^{4s}\right);\\ \frac{\left(22+4^s+2^{4s+1}\right)\lambda}{\mu_i^2 2^{4s}}\left(||\mu_i u||^{4s}\right);\end{cases}=\begin{cases}\mu_i^{-2}\Phi(u);\\ \mu_i^{s-2}\Phi(u);\\ \mu_i^{4s-2}\Phi(u);\\ \mu_i^{4s-2}\Phi(u).\end{cases}$$

From (3.5), we prove the following cases:

Case:1 $L=2^{-2}$ if $i=0$;

$$\|f(2u,2u)-16f(u,u)-Q_2(u,u)\|\leq\lambda\left(\frac{\left(2^{-2}\right)^{1-0}}{1-2^{(-2)}}\right)=\frac{5\lambda}{3}.$$

Case:2 $L=2^1$ if $i=1$,

$$\|f(2u,2u)-16f(u,u)-Q_2(u,u)\|\leq\lambda\left(\frac{\left(2^{2}\right)^{1-1}}{1-2^{2}}\right)=\frac{-5\lambda}{3}.$$

Case:3 $L=2^{s-2}$ for $s<2$ if $i=0$,

$$\|f(2u,2u)-16f(u,u)-Q_2(u,u)\|\leq\frac{\left(18+2^{s+1}\right)\lambda}{2^s}\left(\frac{\left(2^{(s-2)}\right)^{1-0}}{1-2^{(s-2)}}\right)||u||^s=\frac{\left(18+2^{s+1}\right)\lambda}{2^2-2^s}||u||^s.$$

Case:4 $L = 2^{2-s}$ for $s > 2$ if $i = 1$,

$$\|f(2u,2u) - 16f(u,u) - Q_2(u,u)\| \leq \frac{(18+2^{s+1})\,\lambda}{2^s}\left(\frac{\left(2^{(2-s)}\right)^{1-1}}{1-2^{(2-s)}}\right)||u||^s = \frac{(18+2^{s+1})\,\lambda}{2^s-2^2}||u||^s.$$

Case:5 $L = 2^{4s-2}$ for $s < \frac{1}{2}$ if $i = 0$,

$$\|f(2u,2u) - 16f(u,u) - Q_2(u,u)\| \leq \frac{(4+4^s)\,\lambda}{2^{4s}}\left(\frac{\left(2^{(4s-2)}\right)^{1-0}}{1-2^{(4s-2)}}\right)||u||^{4s} = \frac{(4+4^s)\,\lambda}{2^2-2^{4s}}||u||^{4s}.$$

Case:6 $L = 2^{2-4s}$ for $s > \frac{2}{4}$ if $i = 1$,

$$\|f(2u,2u) - 16f(u,u) - Q_2(u,u)\| \leq \frac{(4+4^s)\,\lambda}{2^{4s}}\left(\frac{\left(2^{(2-4s)}\right)^{1-0}}{2-2^{(1-4s)}}\right)||u||^{4s} = \frac{(4+4^s)\,\lambda}{2^{4s}-2^2}||u||^{4s}.$$

This completes the proof. □

3.2 Quartic Stability Results

Theorem 3.5. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^{4k}}\alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right) = 0 \tag{3.17}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$, such that the functional inequality with

$$\|D\,f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{3.18}$$

for all $x,y,u,v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right), \tag{3.19}$$

has the property

$$\Phi(u) = L\frac{1}{\mu_i^4}\Phi(\mu_i u) \tag{3.20}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $x \in X$. Then there exists a unique 2-variable quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1.1) and

$$\| f(2u,2u) - 4f(u,u) - Q_4(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.21}$$

for all $u \in X$.

Proof. Consider the set

$$\Omega = \{p/p : X \to Y,\ p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = \inf\{K \in (0,\infty) : \| p(u) - q(u) \| \leq K\Phi(u), u \in X\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(u) = \frac{1}{\mu_i^4}p(\mu_i u),$$

for all $u \in X$. Now $p, q \in \Omega$, we have

$$\begin{aligned} d(p,q) \le K &\Rightarrow \| p(u) - q(u) \| \le K\Phi(u), \forall u \in X. \\ &\Rightarrow \left\| \frac{1}{\mu_i^4} p(\mu_i u) - \frac{1}{\mu_i^4} q(\mu_i u) \right\| \le \frac{1}{\mu_i^4} K\Phi(\mu_i u), \forall u \in X, \\ &\Rightarrow \left\| \frac{1}{\mu_i^4} p(\mu_i u) - \frac{1}{\mu_i^4} q(\mu_i u) \right\| \le LK\Phi(u), \forall u \in X, \\ &\Rightarrow \| Tp(u) - Tq(u) \| \le LK\Phi(u), \forall u \in X, \\ &\Rightarrow d(p,q) \le LK. \end{aligned}$$

This implies $d(Tp, Tq) \le Ld(p,q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.
It is easy from (3.9) that

$$\|f(4u,4u) - 4f(2u,2u) - 16\,(f(2u,2u) - 4f(u,u))\| \le \beta(u,u,u,u) \tag{3.22}$$

for all $u \in X$. Let $q_4 : X \to Y$ be a mapping defined by $q_4(u,u) = f(2u,2u) - 4f(u,u)$. From (3.22), we conclude that

$$\|q_4(2u,2u) - 16q_4(u,u)\| \le \beta(u,u,u,u) \tag{3.23}$$

for all $u \in X$. From (3.23), we arrive

$$\left\|\frac{q_4(2u,2u)}{16} - q_4(u,u)\right\| \le \frac{1}{16}\beta(u,u,u,u) \tag{3.24}$$

for all $u \in X$. Using (3.19) and (3.20) for the case $i = 0$ it reduces to

$$\left\|\frac{q_4(2u,2u)}{16} - q_4(u,u)\right\| \le \; L\Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f,Tf) \le L \le L^1 < \infty.$$

Again replacing $u = \dfrac{u}{2}$ in (3.23), we get

$$\left\|q_4(u,u) - 16q_4\left(\frac{u}{2},\frac{u}{2}\right)\right\| \le \beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right) \tag{3.25}$$

for all $u \in X$. Using (3.19) and (3.20) for the case $i = 1$ it reduces to

$$\left\|q_4(u,u) - 8q_4\left(\frac{u}{2},\frac{u}{2}\right)\right\| \le \; \Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f,Tf) \le 1 \le L^0 < \infty.$$

In both cases, we arrive

$$d(f,Tf) \le L^{1-i}.$$

Therefore, $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point Q_4 of T in Ω such that

$$Q_4(u,u) = \lim_{k\to\infty} \frac{1}{\mu_i^{4k}} \left(f(\mu_i^{k+1}u, \mu_i^{k+1}u) - 4f(\mu_i^k u, \mu_i^k u)\right) \tag{3.26}$$

for all $u \in X$.

To prove $Q_4 : X \to Y$ is quartic. Replacing (x,y,u,v) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right)$ in (3.18) and dividing by μ_i^{4k}, it follows from (3.17) that

$$\begin{aligned} \|Q_4(x,y,u,v)\| &= \lim_{k\to\infty} \frac{\left\|D\, f(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v)\right\|}{\mu_i^{4k}} \\ &\le \lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v)}{\mu_i^{4k}} = 0 \end{aligned}$$

for all $x, y, u, v \in X$. i.e., Q_4 satisfies the functional equation (1.1).

By $(A3)$, Q_4 is the unique fixed point of T in the set $\Delta = \{Q_4 \in \Omega : d(f, Q_4) < \infty\}, Q_4$ is the unique function such that

$$\|f(2u, 2u) - 4f(u, u) - Q_4(u, u)\| \leq K\Phi(u)$$

for all $u \in X$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f, Q_4) \leq \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f, Q_4) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(2u, 2u) - 4f(u, u) - Q_4(u, u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u)$$

for all $u \in X$. This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.5 concerning the stability of (1.1).

Corollary 3.5. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x, y, u, v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 4; \\ \lambda\left(||x||^s ||y||^s ||u||^s ||v||^s\right), & s \neq 1; \\ \lambda\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq 1; \end{cases} \tag{3.27}$$

for all $x, y, u, v \in X$, then there exists a unique 2-variable quartic function $A : X \to Y$ such that

$$\|f(2u, 2u) - 4f(u, u) - Q_4(u, u)\| \leq \begin{cases} \frac{5\lambda}{15}, \\ \frac{\left(18 + 2^{s+1}\right)\lambda||u||^s}{|2^4 - 2^s|}, \\ \frac{\left(4 + 4^s\right)\lambda||u||^{4s}}{|2^4 - 2^{4s}|}, \\ \frac{\left(22 + 4^s + 2^{4s+1}\right)\lambda||u||^{4s}}{|2^4 - 2^{4s}|}. \end{cases} \tag{3.28}$$

for all $u \in X$.

Proof. Setting

$$\alpha(x, y, u, v) = \begin{cases} \lambda; \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda\left(||x||^s ||y||^s ||u||^s ||v||^s\right); \\ \lambda\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases}$$

for all $x, y, u, v \in X$. Now,

$$\frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k u)}{\mu_i^{4k}} = \begin{cases} \lambda\mu_i^{-4k}; \\ \lambda\mu_i^{k(s-4)}\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda\mu_i^{k(4s-4)}\left(||x||^s ||y||^s ||u||^s ||v||^s\right); \\ \lambda\mu_i^{k(4s-4)}\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (3.17) is holds.

But we have $\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right)$ has the property $\Phi(u) = L \cdot \frac{1}{\mu_i^4}\Phi(\mu_i u)$ for all $u \in X$. Hence

$$\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right) = \begin{cases} 5\lambda, \\ \frac{(18+2^{s+1})\lambda}{2^s}||u||^s; \\ \frac{(4+4^s)\lambda}{2^{4s}}||u||^{4s}; \\ \frac{(22+4^s+2^{4s+1})\lambda}{2^{4s}}||u||^{4s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i^4}\Phi(\mu_i u) = \begin{cases} \frac{5\lambda}{\mu_i^4}; \\ \frac{(18+2^{s+1})\lambda}{\mu_i^4 2^s}(||\mu_i u||^s); \\ \frac{(4+4^s)\lambda}{\mu_i^4 2^{4s}}\left(||\mu_i u||^{4s}\right); \\ \frac{(22+4^s+2^{4s+1})\lambda}{\mu_i^4 2^{4s}}\left(||\mu_i u||^{4s}\right); \end{cases} = \begin{cases} \mu_i^{-4}\Phi(u); \\ \mu_i^{s-4}\Phi(u); \\ \mu_i^{4s-4}\Phi(u); \\ \mu_i^{4s-4}\Phi(u). \end{cases}$$

From (3.21), we prove the following cases:
Case:1 $L = 2^{-4}$ if $i = 0$;

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le 5\lambda\left(\frac{(2^{-4})^{1-0}}{1-2^{(-4)}}\right) = \frac{5\lambda}{16}.$$

Case:2 $L = 2^4$ if $i = 1$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le 5\lambda\left(\frac{(2^4)^{1-1}}{1-2^4}\right) = \frac{-5\lambda}{16}.$$

Case:3 $L = 2^{s-4}$ for $s < 4$ if $i = 0$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le \frac{(18+2^{s+1})\lambda}{2^s}\left(\frac{\left(2^{(s-4)}\right)^{1-0}}{1-2^{(s-4)}}\right)||u||^s = \frac{(18+2^{s+1})\lambda}{2^4-2^s}||u||^s.$$

Case:4 $L = 2^{4-s}$ for $s > 3$ if $i = 1$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le \frac{(18+2^{s+1})\lambda}{2^s}\left(\frac{\left(2^{(4-s)}\right)^{1-1}}{1-2^{(4-s)}}\right)||u||^s = \frac{(18+2^{s+1})\lambda}{2^s-2^4}||u||^s.$$

Case:5 $L = 2^{4s-4}$ for $s < 1$ if $i = 0$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le \frac{(4+4^s)\lambda}{2^{4s}}\left(\frac{\left(2^{(4s-4)}\right)^{1-0}}{1-2^{(4s-4)}}\right)||u||^{4s} = \frac{(4+4^s)\lambda}{2^4-2^{4s}}||u||^{4s}.$$

Case:6 $L = 2^{4-4s}$ for $s > 1$ if $i = 1$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le \frac{(4+4^s)\lambda}{2^{4s}}\left(\frac{\left(2^{(4-4s)}\right)^{1-0}}{1-2^{(4-4s)}}\right)||u||^{4s} = \frac{(4+4^s)\lambda}{2^{4s}-2^4}||u||^{4s}.$$

This finishes the proof. □

3.3 Quadratic-Quartic Mixed Stability Results

Theorem 3.6. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0, \infty)$ with the condition given in (3.1) and (3.17) respectively, such that the functional inequality*

$$\|D f(x, y, u, v)\| \leq \alpha(x, y, u, v) \tag{3.29}$$

for all $x, y, u, v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right), \tag{3.30}$$

has the property (3.4) and (3.20), then there exists a unique 2-variable quadratic function $Q_2 : X \to Y$ and a unique 2-variable quartic function $Q_4 : X \to Y$ satisfying the functional equation (1.1) and

$$\| f(u, u) - Q_2(u, u) - Q_4(u, u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.31}$$

for all $u \in X$.

Proof. By Theorems 3.4 and 3.5, there exists a unique 2-variable quadratic function $Q_{2_1} : X \to Y$ and a unique 2-variable quartic function $Q_{4_1} : X \to Y$ such that

$$\left\|f(2u, 2u) - 16f(u, u) - Q_{2_1}(u, u)\right\| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.32}$$

for all $u \in X$ and

$$\left\|f(2u, 2u) - 4f(u, u) - Q_{4_1}(u, u)\right\| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.33}$$

for all $u \in X$. Now, from (3.32) and (3.33) that

$$\begin{aligned}\left\|f(u, u) + \frac{1}{12}Q_{2_1}(u, u) - \frac{1}{12}Q_{4_1}(u, u)\right\| &= \left\|\left\{-\frac{f(2u, 2u)}{12} + \frac{16}{12}f(u, u) + \frac{1}{12}Q_{2_1}(u, u)\right\}\right. \\ &\qquad \left. + \left\{\frac{f(2u, 2u)}{12} - \frac{4}{12}f(u, u) - \frac{1}{12}Q_{4_1}(u, u)\right\}\right\| \\ &\leq \frac{1}{12}\left\|\{f(2u, 2u) - 16f(u, u) - Q_{2_1}(u, u)\}\right. \\ &\qquad \left. + \{f(2u, 2u) - 4f(u, u) - Q_{4_1}(u, u)\}\right\| \\ &\leq \frac{1}{12}\left\{\frac{L^{1-i}}{1-L}\Phi(u) + \frac{L^{1-i}}{1-L}\Phi(u)\right\}\end{aligned}$$

for all $u \in X$. Thus we obtain (3.31) by defining $Q_2(u, u) = \frac{-1}{12}Q_{2_1}(u, u)$ and $Q_4(u, u) = \frac{1}{12}Q_{4_1}(u, u)$, where $Q_2(u, u)$ and $Q_4(u, u)$ are defined in (3.14) and (3.26) respectively, for all $u \in X$. □

The following corollary is an immediate consequence of Theorem 3.6 concerning the stability of (1.1).

Corollary 3.6. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x, y, u, v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 2, 4; \\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right), & s \neq \frac{1}{2}, 1; \\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq \frac{1}{2}, 1; \end{cases} \tag{3.34}$$

for all $x, y, u, v \in X$, then there exists a unique 2-variable quadratic function $Q_2 : X \to Y$ and a unique 2-variable quartic function $Q_4 : X \to Y$ such that

$$\|f(u, u) - Q_2(u, u) - Q_4(u, u)\| \leq \begin{cases} \frac{\lambda}{18}; \\ \frac{(18 + 2^{s+1})}{6}\left(\frac{1}{|2^2 - 2^s|} + \frac{1}{|2^4 - 2^s|}\right)\lambda||u||^s; \\ \frac{(4 + 4^s)}{12}\left(\frac{1}{|2^2 - 2^{4s}|} + \frac{1}{|2^4 - 2^{4s}|}\right)\lambda||u||^{4s}; \\ \frac{(22 + 4^s + 2^{4s+1})}{12}\left(\frac{1}{|2^2 - 2^{4s}|} + \frac{1}{|2^4 - 2^{4s}|}\right)\lambda||u||^{4s}; \end{cases} \tag{3.35}$$

for all $u \in X$.

4 Additive-Quadratic-Cubic-Quartic Mixed Stability Results

In this section, the authors proved the additive-quadratic-cubic-quartic mixed stability of the functional equation (1.1) using fixed point method.

Theorem 4.7. *Let $j = \pm 1$. Let $Df : X^2 \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0, \infty)$ with the condition given in (2.1), (2.17), (3.1) and (3.17) respectively, such that the functional inequality*

$$\|Df(x, y, u, v)\| \leq \alpha(x, y, u, v) \tag{4.1}$$

for all $x, y, u, v \in X$. Then there exists a unique 2-variable additive mapping $A(u, u) : X^2 \to Y$, a unique 2-variable quadratic mapping $Q_2(u, u) : X^2 \to Y$, a unique 2-variable cubic mapping $C(u, u) : X^2 \to Y$ and a unique 2-variable quartic mapping $Q_4(u, u) : X^2 \to Y$satisfying the functional equation (1.1) and

$$\|g(u, u) - A(u, u) - Q_2(u, u) - C(u, u) - Q_4(u, u)\| \frac{L^{1-i}}{1-L} \left(\Phi_{AC}(u) + \Phi_{Q_2Q_4}(u)\right) \tag{4.2}$$

for all $u \in X$, where $\Phi_{AC}(u)$ and $\Phi_{Q_2Q_4}(u)$ are defined by

$$\Phi_{AC}(u) = \frac{1}{6} [\Phi(u) + \Phi(-u)] \tag{4.3}$$

$$\Phi_{Q_2Q_4}(u) = \frac{1}{12} [\Phi(u) + \Phi(-u)] \tag{4.4}$$

respectively, for all $u \in X$.

Proof. Let $f_o(u, u) = \frac{1}{2} (f(u, u) - f(-u, -u))$ for all $u \in X$. Then $f_o(0, 0) = 0$ and $f_o(-u, -u) = -f_o(u, u)$ for all $u \in X$. Hence

$$\|Df_o(x, y, u, v)\| \leq \frac{1}{2} \{\alpha(x, y, u, v) + \alpha(-x, -y, -u, -v)\} \tag{4.5}$$

for all $x, y, u, v \in X$. By Theorem 2.3, there exists a unique 2-variable additive function $A(u, u) : X^2 \to Y$ and a unique 2-variable cubic function $C(u, u) : X^2 \to Y$ such that

$$\begin{aligned} \|f_o(u, u) - A(u, u) - C(u, u)\| &\leq \frac{1}{2} \left\{ \frac{1}{3} \frac{L^{1-i}}{1-L} \Phi(u) + \frac{1}{3} \frac{L^{1-i}}{1-L} \Phi(-u) \right\} \\ &\leq \frac{1}{6} \frac{L^{1-i}}{1-L} \{\Phi(u) + \Phi(-u)\}, \end{aligned} \tag{4.6}$$

for all $u \in X$. Also, let $f_e(u, u) = \frac{1}{2} (f(u, u) + f(-u, -u))$ for all $u \in X$. Then $f_e(0, 0) = 0$ and $f_e(-u, -u) = f_e(u, u)$ for all $u \in X$. Hence

$$\|Df_e(x, y, u, v)\| \leq \frac{1}{2} \{\alpha(x, y, u, v) + \alpha(-x, -y, -u, -v)\} \tag{4.7}$$

for all $x, y, u, v \in X$. By Theorem 3.6, there exists a unique 2-variable quadratic mapping $Q_2(u, u) : X^2 \to Y$ and a unique 2-variable quartic mapping $Q_4(u, u) : X^2 \to Y$ such that

$$\begin{aligned} \|f_e(u, u) - Q_2(u, u) - Q_4(u, u)\| &\leq\leq \frac{1}{2} \left\{ \frac{1}{6} \frac{L^{1-i}}{1-L} \Phi(u) + \frac{1}{6} \frac{L^{1-i}}{1-L} \Phi(u-) \right\} \\ &\leq \frac{1}{12} \frac{L^{1-i}}{1-L} \{\Phi(u) + \Phi(-u)\}, \end{aligned} \tag{4.8}$$

for all $u \in X$. Define

$$f(u, u) = f_o(u, u) + f_e(u, u) \tag{4.9}$$

for all $u \in X$. Now from (4.6), (4.8) and (4.9)

$$\begin{aligned}
&\|f(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\| \\
&\quad = \|f_o(u,u) + f_e(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\| \\
&\quad \leq \|f_o(u,u) - A(u,u) - C(u,u)\| + \|f_e(u,u) - Q_2(u,u) - Q_4(u,u)\| \\
&\quad \leq \frac{1}{6}\frac{L^{1-i}}{1-L}\{\Phi(u) + \Phi(-u)\} + \frac{1}{12}\frac{L^{1-i}}{1-L}\{\Phi(u) + \Phi(-u)\} \\
&\quad \leq \frac{L^{1-i}}{1-L}\{\Phi_{AC}(u) + \Phi_{Q_2Q_4}(u)\} \qquad (4.10)
\end{aligned}$$

for all $u \in X$. This finishes the proof. □

The following corollary is an immediate consequence of Theorem 4.7, using Corollaries 2.3 and 3.6 concerning stability of (1.1).

Corollary 4.7. *Let $Df : X^2 \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\{||x||^s + ||y||^s + ||u||^s + ||v||^s\}, & s \neq 1,2,3,4; \\ \lambda||x||^s||y||^s||u||^s||v||^s, & s \neq \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1; \\ \lambda\{||x||^s||y||^s||u||^s||v||^s + \{||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\}\}, & s \neq \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1; \end{cases} \qquad (4.11)$$

for all $x, y, u, v \in X$, then there exists a unique 2-variable additive mapping $A(u,u) : X^2 \to Y$, a unique 2-variable quadratic mapping $Q_2(u,u) : X^2 \to Y$, a unique 2-variable cubic mapping $C(u,u) : X^2 \to Y$ and a unique 2-variable quartic mapping $Q_4(u,u) : X^2 \to Y$ such that

$$\|f(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\| \leq \begin{cases} \frac{5\rho}{6}\left(1 + \frac{1}{7} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 15}\right), \\ \frac{(18+2^{s+1})}{6}\left(\frac{1}{|2-2^s|} + \frac{1}{|8-2^s|} + \frac{1}{2|4-2^s|} + \frac{1}{2|16-2^s|}\right)\rho||u||^s, \\ \frac{(4+2^{2s})}{6}\left(\frac{1}{|2-2^{4s}|} + \frac{1}{|8-2^{4s}|} + \frac{1}{2|4-2^{4s}|} + \frac{1}{2|16-2^{4s}|}\right)\rho||u||^{4s} \\ \frac{(22+2^{2s}+2^{4s+1})}{6}\left(\frac{1}{|2-2^{4s}|} + \frac{1}{|8-2^{4s}|} + \frac{1}{2|4-2^{4s}|} + \frac{1}{2|16-2^{4s}|}\right)\rho||u||^{4s} \end{cases} \qquad (4.12)$$

for all $u \in X$.

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On two boundary-value problems of functional integro-differential equations with nonlocal conditions

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Abstract

In this paper we establish the existence of solution for two boundary value problems of Fredholm functional integro-differential equations with nonlocal boundary conditions.

Keywords: Nonlocal boundary value problems, Fredholm functional integral equation, Fredholm functional integro-differential equation, compact in measure.

2010 MSC: 534A08, 47H07, 47H10.

1 Introduction

Mathematical modelling of real-life problems usually results in functional equations, of various types appear in many applications that arise in the fields of mathematical analysis, nonlinear functional analysis, mathematical physics, and engineering. An interesting feature of functional integral equations is their role in the study of many problems of functional integro-differential equations. Several different techniques were proposed to study the existence of solutions of the functional integral equations in appropriate function spaces. Although all of these techniques have the same goal, they differ in the function spaces and the fixed point theorems to be applied. Consider the following boundary value problems of Fredholm functional integro-differential equations.

$$x'(t) = f(t, \int_0^1 k(t,s)x'(s)ds), \qquad a.e. \qquad t \in (0,1) \tag{1.1}$$

with the nonlocal boundary condition

$$x(\tau) + \alpha\, x(\xi) = 0, \qquad \tau, \xi \in [0,1], \alpha \neq -1. \tag{1.2}$$

$$x''(t) = f(t, \int_0^1 k(t,s)x''(s)ds) \qquad a.e. \qquad t \in (0,1) \tag{1.3}$$

with the nonlocal boundary conditions

$$x(\tau) + \beta\, x(\xi) = 0, \qquad \beta \neq -1 \tag{1.4}$$

$$x'(\tau) + \alpha\, x'(\xi) = 0, \qquad \tau, \xi \in [0,1], \alpha \neq -1. \tag{1.5}$$

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Here we study the existence of at least one solution of each of the boundary value problems (1.1)-(1.2) and (1.3)-(1.5).
The existence of exactly one solution of them will be deduced.

2 Functional integral equation

Here we study the existence of at least one (and exactly one) integrable solution of the Fredholm functional integral equation.

$$y(t) = f(t, \int_0^1 k(t,s)y(s)ds) \tag{2.6}$$

under the following assumptions

(1) $f : I = [0,1] \times R \to R$ is measurable in $t \in [0,1]$ for all $x \in R$ and continuous in $x \in R$ for all $t \in [0,1]$ and there exists integrable function $a \in L^1[0,1]$ and positive constant $b > 0$ such that

$$| f(t,x) | \leq a(t) + b|x| \qquad a.e. \quad t \in I.$$

(2)

$$||a|| = \int_0^1 |a(t)|dt, \qquad t \in [0,1]$$

(3) $k : I = [0,1] \times [0,1] \to R$ is continuous $t \in [0,1]$ for every $s \in [0,1]$ and measurable in $s \in [0,1]$ for all $t \in [0,1]$, such that

$$\sup_t \int_0^1 k(t,s)dt \leq M$$

Now for the existence of at least one integrable solution of the functional integral equation (2.6) we have the following theorem.

Theorem 2.1. *Let the assumptions (1)-(3) be satisfied. If $L = Mb < 1$, then the functional integral equation (2.6) has at least one solution $y \in L^1[0,1]$.*

Proof. Let $L^1 = L^1[0,1]$ and define the set B_r by

$$B_r = \{y \in L^1 : ||y|| \leq r\} \subset L^1[0,1],$$

where

$$r = \frac{a}{1 - bM}.$$

Define the operator T associated with the Fredholm functional integral equation (2.6) by

$$Ty(t) = f(t, \int_0^1 k(t,s)y(s)ds).$$

To show that $T : B_r \to B_r$, let $y \in B_r$, then

$$\begin{aligned}
||Ty(t)||_{L^1} &= \int_0^1 |Ty(t)|dt \\
&= \int_0^1 |f(t, \int_0^1 k(t,s)y(s)ds)|dt \\
&\leq \int_0^1 [| a(t) | + b| \int_0^1 k(t,s)y(s)ds|]dt \\
&\leq \int_0^1 | a(t) |dt + b \int_0^1 \int_0^1 |k(t,s)y(s)|dsdt \\
&\leq \int_0^1 | a(t) |dt + bM \int_0^1 |y(s)| ds \\
&\leq || a || + bM ||y|| \\
&\leq ||a|| + bMr = r. \\
&\leq a + bMr = r.
\end{aligned}$$

From this we observe that $T(B_r) \subset B_r$. Then $T : B_r \rightarrow B_r$, Moreover from our assumptions $(1)-(3)$ follows that the operator T is continuous.

To prove that T is a contraction with respect to measure of weak non compactness β on the set B_r.
Let $X \subset B_r$ and let $y \in X$. Futher $\epsilon > 0$ and take a measurable subset $D \subset [0,1]$ such that $\mu(D) \leq \epsilon$, then we get

$$\begin{aligned}
||Ty(t)||_{L^1(D)} &= \int_D |Ty(t)|dt \\
&= \int_D |f(t, \int_0^1 k(t,s)y(s)ds)|dt \\
&\leq \int_D [|a(t)| + b|\int_D k(t,s)y(s)ds|]dt \\
&\leq \int_D |a(t)|dt + b\int_D \int_0^1 |k(t,s)y(s)|dsdt \\
&\leq \int_D |a(t)|dt + bM\int_D |y(s)|ds \\
&\leq ||a||_{L^1(D)} + bM\int_D |y(s)|ds
\end{aligned}$$

for this subset X, the measure of weak non compactness $\beta(X)$ is given by the formula

$$\beta(X) = \lim_{\epsilon \to 0} \{\sup_{y \in X} \{\sup_D \{\int_D |y(t)|dt \, : \, D \subset [0,1] \;\; \mu(D) \leq \epsilon\}\}$$

To value $\beta(TX)$ we notice that

$$\beta(X) = \lim_{\epsilon \to 0} \{\sup_{y \in X} \{\sup_D \{\int_D |a(t)|dt + bM\int_D |y(t)|dt \, : \, D \subset [0,1] \;\; \mu(D) \leq \epsilon\}\}\} = 0$$

Indeed,we have $\beta(TX) \leq bM\beta(X)$.
Since all conditions of Schauder fixed point theorem (see[15]), are satisfied, then the operator T has at least one fixed point $y \in L^1[0,1]$, which completes the proof. ∎

Now for the uniqueness of the solution of the Fredholm functional integral equation (2.6) Consider following assumptions

(1*) $f : I = [0,1] \times R \rightarrow R$ is measurable in $t \in [0,1]$ for all $x \in R$ and satisfies the lipschitz such that

$$|f(t,x) - f(t,y)| \leq b|x-y|, \qquad b > 0 \tag{2.7}$$

(2*) $f(t,0) \in L^1[0,1]$ $\int_0^1 |f(t,0)|dt \leq a$.

Theorem 2.2. *Let the assumptions (1*),(2*) and (3) be satisfied. If $Mb < 1$, then the functional integral equation (2.6) has a unique solution $y \in L^1[0,1]$.*

Proof. From (2.7) we can obtain

$$|f(t,x)| \leq |f(t,0)| + b\,|x|.$$

This shows that the assumptions of Theorem (2.1) are satisfied
Now let y_1,y_2 be two solution of functional integral equation (2.6)

$$y_1(t) = f(t, \int_0^1 k(t,s)y_1(s)ds)$$

$$y_2(t) = f(t, \int_0^1 k(t,s)y_2(s)ds)$$

$$\begin{aligned} ||y_1(t) - y_2(t)||_{L^1} &= \int_0^t |f(t, \int_0^1 k(t,s)y_1\, ds) - f(t, \int_0^1 k(t,s)y_2\, ds)|dt \\ &\leq b \int_0^t | \int_0^1 k(t,s)y_1(s)\, ds - \int_0^1 k(t,s)y_2(s)\, ds|dt \\ &\leq b \int_0^t | \int_0^1 k(t,s)\ (y_1(s) - y_2(s))\, ds|dt \\ &\leq b \int_0^t \int_0^1 |k(t,s)|\ |y_1(s) - y_2(s)|dsdt \\ &\leq bM \int_0^t |y_1(s) - y_2(s)|ds \\ &\leq bM||y_1 - y_2||_{L^1}, \end{aligned}$$

then

$$||y_1 - y_2|| \leq K||y_1 - y_2||$$

where $L = bM < 1$, then

$$||y_1 - y_2||(1-k) \leq 0$$

and

$$||y_1 - y_2|| = 0$$

which implies that $y_1 = y_2$ then the Fredholm functional integral equation (2.6) has a unique integrable solution.

3 Nonlocal boundary value problems

Here we study the existence of at least one (and exactly one) solution of each of the boundary value problems (1.1)-(1.2) and (1.3)-(1.5).
Consider the functional integro differential equation

$$x'(t) = f(t, \int_0^1 k(t,s)\, x'(s)\, ds) \qquad a.e. \qquad t \in (0,1).$$

with the nonlocal boundary value condition

$$x(\tau) + \alpha\, x(\xi) = 0. \qquad \tau, \xi \in [0,1], \alpha \neq -1$$

Theorem 3.3. *Let the assumptions of theorem (2.1) be satisfied, then the nonlocal boundary value problem (1.1)-(1.2) has at least one integrable solution* $x \in L^1[0,1]$.

Proof. Let $x'(t) = y(t)$. Integrating both sides we get

$$x(t) = x(0) + \int_0^t y(s)ds,$$

$$x(\tau) = x(0) + \int_0^\tau y(s)ds$$

and

$$x(\xi) = x(0) + \int_0^\xi y(s)ds$$

Using the nonlocal boundary condition (1.2) we obtain

$$x(0) + \int_0^\tau y(s)ds = -\alpha\, x(0) - \alpha \int_0^\xi y(s)ds,$$

and

$$x(0) = -\frac{1}{1+\alpha} \int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha} \int_0^\xi y(s)ds,$$

then

$$x(t) = \int_0^t y(s)ds - \frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds \tag{3.8}$$

where y satisfies the functional integral equation

$$y(t) \ = \ f(t, \int_0^1 k(t,s)y(s)ds).$$

This complete the proof of equivalent between the nonlocal problem (1.1)-(1.2) and the functional integral equation (2.6). This implies that there exists at least one solution $x \in L^1[0,1]$ of the nonlocal problem (1.1)-(1.2).■

Corollary 3.1. *Let the assumptions (1*),(2*) and (3) be satisfied, then the solution of nonlocal boundary value problem (1.1)-(1.2) has a unique integrable solution* $x \in L^1[0,1]$.

Consider the Fredholm functional integro-differential equation

$$x''(t) \ = \ f(t, \ \int_0^1 k(t,s)x''(s)ds) \qquad a.e. \qquad t \in (0,1)$$

with the nonlocal boundary conditions

$$x(\tau) + \ \beta\, x(\xi) = 0,$$
$$x'(\tau) \ + \ \alpha\, x'(\xi) = 0.$$

Theorem 3.4. *Let the assumptions of theorem (2.1) be satisfied then the boundary value problems (1.3)-(1.5) has at least one integrable solution* $x \ \in \ L^1[0,1]$.

Proof. Let $x''(t) \ = \ y(t)$ integrating both sides we obtain

$$x'(t) \ = \ x'(0) + \int_0^t \ y(s)\ ds$$

and

$$x(t) \ = \ x(0) + tx'(0) + \int_0^t \ (t-s)\ y(s)ds.$$

then

$$x'(\tau) \ = \ x'(0) + \int_0^\tau \ y(s)\ ds,$$

and

$$x'(\xi) \ = \ x'(0) + \int_0^\xi \ y(s)\ ds.$$

Using the nonlocal condition (1.5) we obtain

$$x'(0) \ = \ -\frac{1}{1+\alpha}\int_0^\tau \ y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi \ y(s)\ ds$$

and

$$x(\tau) \ = \ x(0) + \tau\, x'(0) + \int_0^\tau \ (\tau - s)\ y(s)\ ds,$$
$$x(\xi) \ = \ x(0) + \xi\, x'(0) + \int_0^\xi \ (\xi - s)\ y(s)\ ds,$$

$$x'(0) \ = \ -\frac{1}{1+\alpha}\int_0^\tau \ y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi \ y(s)\ ds.$$

Using Boundary condition (1.4) we obtain

$$x(0) = \frac{-\beta\xi - \tau}{1+\beta}x'(0) - \frac{1}{1+\alpha}\int_0^\tau (\tau - s)y(s)ds - \frac{1}{1+\beta}\int_0^\xi (\xi - s)y(s)ds,$$

$$
\begin{aligned}
x(t) &= \frac{-\beta\xi - \tau}{1+\beta}[-\frac{1}{1+\beta}\int_0^{\tau} y(s)ds - \frac{1}{1+\alpha}\int_0^{\xi} y(s)ds] \\
&- \frac{1}{1+\beta}\int_0^{\tau}(\tau - s)y(s)ds - \frac{\beta}{1+\beta}\int_0^{\xi}(\xi - s)ds \\
+ t[&-\frac{1}{1+\alpha}\int_0^{\tau} y(s)ds - \frac{\alpha}{1+\alpha}\int_0^{\xi} y(s)ds] + \int_0^{t}(t-s)y(s)ds, \qquad (3.9)\\
x'(t) &= -\frac{1}{1+\alpha}\int_0^{\tau} y(s)ds - \frac{\alpha}{1+\alpha}\int_0^{\xi} y(s)ds] + \int_0^{t} y(s)ds,
\end{aligned}
$$

and y satisfies the functional integral equation

$$y(t) = f(t, \int_0^1 k(t,s)y(s)ds).$$

This complete the proof of equivalent between the nonlocal problem (1.3)-(1.5) and the functional integral equation (2.6). This implies that there exists at least one solution $x \in L^1[0,1]$ of the nonlocal problem (1.3)-(1.5). ■

Corollary 3.2. *Let the assumptions* (1^*)*,*(2^*) *and (3) be satisfied, then the solution of nonlocal boundary value problem (1.3)-(1.5) has a unique integrable solution* $x \in L^1[0,1]$.

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a_i Type $n-$ Variable Multi $n-$ Dimensional Additive Functional Equation

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Abstract

In this paper, the authors investigated the general solution and generalized Ulam - Hyers stability of a_i type $n-$ variable multi $n-$ dimensional additive functional equation

$$\begin{aligned} 2h\left(\sum_{i=1}^{n} a_i\, x_{1i}, \sum_{i=1}^{n} a_i\, x_{2i}, \ldots\ldots, \sum_{i=1}^{n} a_i\, x_{ni}\right) &= \left(\sum_{i=1}^{n} a_i\right) h\left(\sum_{i=1}^{n} x_{1i}, \sum_{i=1}^{n} x_{2i}, \ldots\ldots, \sum_{i=1}^{n} x_{ni}\right) \\ &\quad + \left(a_1 - \sum_{i=2}^{n} a_i\right) h\left(x_{11} - \sum_{i=2}^{n} x_{1i}, x_{21} - \sum_{i=2}^{n} x_{2i}, \ldots\ldots, x_{n1} - \sum_{i=2}^{n} x_{ni}\right) \end{aligned}$$

where $a_i (i = 1, 2, \ldots n)$ are different integers greater than 1, using two different technique.

Keywords: Additive functional equations, Ulam - Hyers stability, Ulam - Hyers - Rassias stability, Ulam - Gavruta - Rassias stability, Ulam - JRassias stability.

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1 Introduction

During the last seven decades, the perturbation problems of several functional equations have been extensively investigated by number of authors [1, 3, 20, 21, 30, 31, 34, 35]. The terminology generalized Ulam - Hyers stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [8, 18, 22–24].

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of Cauchy (see [24]). The additive function $f(x) = cx$ is the solution of the additive functional equation (1.1).

The solution and stability of various additive functional equations were discussed by D.O. Lee [19], K. Ravi, M. Arunkumar [32], M. Arunkumar [4–6, 8, 9]. W.G. Park, J.H. Bae [16, 27] investigate the general solution and the generalized Hyers-Ulam stability of the multi-additive functional equation and 2- variable

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quadratic functional equation of the forms

$$f(x_1+x_2, y_1+y_2, z_1+z_2) = \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k), \tag{1.2}$$

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x,z) + 2f(y,w). \tag{1.3}$$

The stability of the functional equation (1.3) in fuzzy normed space was proved by M. Arunkumar et., al [7]. Using the ideas in [7], the general solution and generalized Hyers-Ulam-Rassias stability of a 3- variable quadratic functional equation

$$f(x+y, z+w, u+v) + f(x-y, z-w, u-v) = 2f(x,z,u) + 2f(y,w,v). \tag{1.4}$$

was discussed by K. Ravi and M. Arunkumar [33]. Its solution is of the form

$$f(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx. \tag{1.5}$$

Also, M. Arunkumar, S. Hema Latha established the general solution and generalized Ulam - Hyers stability of a 2 - variable Additive Quadratic functional equation

$$f(x+y, u+v) + f(x-y, u-v) = 2f(x,u) + f(y,v) + f(-y,-v) \tag{1.6}$$

having solutions

$$f(x,y) = ax + by \tag{1.7}$$

and

$$f(x,y) = ax^2 + bxy + cy^2 \tag{1.8}$$

in Banach and Non Archimedean Fuzzy spaces respectively. Infact, M. Arunkumar et. al., [11] introduced and discussed a 2 - variable AC - mixed type functional equation

$$f(2x+y, 2z+w) - f(2x-y, 2z-w) = 4[f(x+y, z+w) - f(x-y, z-w)] - 6f(y,w) \tag{1.9}$$

having solutions

$$f(x,y) = ax + by \tag{1.10}$$

and

$$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3. \tag{1.11}$$

Recently, M.Arunkumar et.al., [12] introduced and established the general solution and generalized Ulam - Hyers stability of a 2 - variable Associative functional equation

$$g(x,u) + g(y+z, v+w) = g(x+y, u+v) + g(z,w) \tag{1.12}$$

having solutions

$$g(x,y) = ax + by \tag{1.13}$$

using Banach and Intuitionistic Fuzzy Normed spaces, respectively.

Inspired by the above results in this paper, the authors investigated the general solution generalized Ulam - Hyers stability of a_i type $n-$ variable multi $n-$ dimensional additive functional equation

$$\begin{aligned} 2h\left(\sum_{i=1}^{n} a_i\, x_{1i}, \sum_{i=1}^{n} a_i\, x_{2i}, \ldots\ldots, \sum_{i=1}^{n} a_i\, x_{ni}\right) &= \left(\sum_{i=1}^{n} a_i\right) h\left(\sum_{i=1}^{n} x_{1i}, \sum_{i=1}^{n} x_{2i}, \ldots\ldots, \sum_{i=1}^{n} x_{ni}\right) \\ &+ \left(a_1 - \sum_{i=2}^{n} a_i\right) h\left(x_{11} - \sum_{i=2}^{n} x_{1i}, x_{21} - \sum_{i=2}^{n} x_{2i}, \ldots\ldots, x_{n1} - \sum_{i=2}^{n} x_{ni}\right) \end{aligned} \tag{1.14}$$

having solution

$$h(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} c_i x_i \tag{1.15}$$

where $a_i(i = 1, 2, \dots n)$ are different integers greater than 1, using Hyers direct and Alternative fixed point methods.

In particular, when $n = 1, 2$ in (1.14), we arrive

$$2h\left(a_1\ x_{11}, a_1\ x_{21}, \dots, a_1\ x_{n1}\right) = a_1 h\left(x_{11}, x_{21}, \dots, x_{n1}\right) + a_1 h\left(x_{11}, x_{21} \dots, x_{n1}\right). \tag{1.16}$$

and

$$\begin{aligned} &2h\left(a_1\ x_{11} + a_2\ x_{12}, a_1\ x_{21} + a_1\ x_{22}, \dots, a_1\ x_{n1} + a_1\ x_{n2}\right) \\ &\quad = \left(a_1 + a_2\right) h\left(x_{11} + x_{12}, x_{21} + x_{22}, \dots, x_{n1} + x_{n2}\right) \\ &\qquad + \left(a_1 - a_2\right) h\left(x_{11} - x_{12}, x_{21} - x_{22}, \dots, x_{n1} - x_{n2}\right). \end{aligned} \tag{1.17}$$

2 General Solution

In this section, the general solution of the functional equation (1.14) is given. Through out this section let as assume $\mathcal{A}$ and $\mathcal{B}$ be linear normed spaces.

Lemma 2.1. *If a mapping $h : \mathcal{A}^n \to \mathcal{B}$ satisfies the functional equation (1.14) then h is additive.*

Proof. Assume $h : \mathcal{A}^n \to \mathcal{B}$ be a mapping satisfies the functional equation (1.14). Replacing

$$x_{mi} = 0, \qquad i = 1, 2, \dots n, \qquad m = 1, 2, \cdots n$$

in (1.14), we get

$$h(0, 0, \dots, 0) = 0. \tag{2.1}$$

Again replacing

$$x_{mi} = 0, \qquad i = 2, 3 \dots n, \qquad m = 1, 2, \cdots n$$

in (1.14), we obtain

$$\begin{aligned} 2h(a_1 x_{11}, a_1 x_{21}, \dots, a_1 x_{n1}) &= (a_1 + a_2 + \cdots + a_n) h(x_{11}, x_{21}, \dots, x_{n1}) \\ &\quad + (a_1 - a_2 - \cdots - a_n) h(x_{11}, x_{21}, \dots, x_{n1}) \end{aligned} \tag{2.2}$$

for all $x_{11}, x_{21}, \dots, x_{n1} \in \mathcal{A}$. If we substitute $(x_{11}, x_{21}, \dots, x_{n1})$ by $(x, x \dots, x)$ in (2.2), we reach

$$h(a_1 x, a_1 x, \dots, a_1 x) = a_1\ h(x, x, \dots, x) \tag{2.3}$$

for all $x \in \mathcal{A}$. Putting

$$x_{mi} = 0, \qquad i = 3, 4 \dots n, \qquad m = 1, 2, \cdots n$$

in (1.14), we obtain

$$h(x_{12}, 0, \dots, 0) = -h(-x_{12}, 0, \dots, 0) \tag{2.4}$$

for all $x_{12} \in \mathcal{A}$. So one can show that

$$h(a_1^k x, a_1^k x, \dots, a_1^k x) = a_1^k\ h(x, x, \dots, x) \tag{2.5}$$

for all $x \in \mathcal{A}$ and all $k \in \mathbb{N}$. □

3 Stability Results: Banach Space: Hyers Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.14).

In this section, let we consider $\mathcal{A}$ be a normed space and $\mathcal{B}$ be a Banach space and define a mapping $Dh : \mathcal{A}^n \to \mathcal{B}$ by

$$\begin{aligned} &Dh(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn}) \\ &= 2h\left(\sum_{i=1}^{n} a_i\, x_{1i}, \sum_{i=1}^{n} a_i\, x_{2i}, \dots\dots, \sum_{i=1}^{n} a_i\, x_{ni}\right) - \left(\sum_{i=1}^{n} a_i\right) h\left(\sum_{i=1}^{n} x_{1i}, \sum_{i=1}^{n} x_{2i}, \dots\dots, \sum_{i=1}^{n} x_{ni}\right) \\ &\quad - \left(a_1 - \sum_{i=2}^{n} a_i\right) h\left(x_{11} - \sum_{i=2}^{n} x_{1i}, x_{21} - \sum_{i=2}^{n} x_{2i}, \dots\dots, x_{n1} - \sum_{i=2}^{n} x_{ni}\right) \end{aligned}$$

for all $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn} \in \mathcal{A}$.

Theorem 3.1. *Let $\ell = \pm 1$ and $\vartheta, \Theta : \mathcal{A}^n \to [0, \infty)$ be a function such that*

$$\lim_{s\to\infty} \frac{1}{2^{s\ell}} \vartheta\left(a_1^{s\ell} x_{11}, \dots, a_1^{s\ell} x_{1n}, a_1^{s\ell} x_{21}, \dots, a_1^{s\ell} x_{2n}, a_1^{s\ell} x_{n1}, \dots, a_1^{s\ell} x_{nn}\right) = 0 \tag{3.1}$$

for all $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn} \in \mathcal{A}$. Let $h : \mathcal{A}^n \to \mathcal{B}$ be a function satisfying the inequality

$$\|Dh(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn})\| \le \sum_{j=1}^{n} \vartheta_j\left(x_{j1}, x_{j2}, \dots, x_{jn}\right) \tag{3.2}$$

for all $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn} \in \mathcal{A}$. Then there exists a unique $n-$ variable additive mapping $A : \mathcal{A}^n \to \mathcal{B}$ which satisfies (1.14) and

$$\|h(x, x, \dots, x) - A(x, x, \dots, x)\| \le \frac{1}{a_1} \sum_{t=0}^{\infty} \frac{\Theta(a_1^{t\ell} x)}{a_1^{t\ell}} \tag{3.3}$$

where $\Theta\left(a_1^{t\ell} x\right)$ and $A(x, x, \dots, x)$ are defined by

$$\Theta(a_1^{t\ell} x) = \frac{1}{2} \sum_{j=1}^{n} \vartheta_j\left(a_1^{t\ell} x, \underbrace{0, \dots, 0}_{(n-1)-times}\right) \tag{3.4}$$

and

$$A(x, x, \dots, x) = \lim_{s\to\infty} \frac{1}{a_1^{s\ell}} h(a_1^{s\ell} x, a_1^{s\ell} x, \dots, a_1^{s\ell} x) \tag{3.5}$$

for all $x \in \mathcal{A}$, respectively.

Proof. Given $h : \mathcal{A}^n \to \mathcal{B}$ be a function satisfying the inequality (3.2) for all $x_{11}, \dots, x_{1n}, \dots, x_{n1}, \dots, x_{nn} \in \mathcal{A}$. To establish this theorem, we have to show that

(i) $\left\{\frac{1}{a_1^s}\ h\left(a_1^s x, a_1^s x, \dots, a_1^s x\right)\right\}$ is a Cauchy sequence for every $x \in \mathcal{A}$;

(ii) If

$$A(x, x, \dots, x) = \lim_{s\to\infty} \frac{1}{a_1} h(a_1^s x, a_1^s x, \dots, a_1^s x)$$

then A is additive on $\mathcal{A}$;

(iii) Further A satisfies (3.3), for all $x \in \mathcal{A}$;

(iv) A is unique.

Replacing

$$x_{mi} = 0, \qquad i = 2, 3 \ldots n, \qquad m = 1, 2, \cdots n$$

in (3.2), we get

$$\begin{aligned} \|2h(a_1x_{11}, a_1x_{21}, \ldots, a_1x_{n1}) - (a_1 + a_2 + \cdots + a_n)h(x_{11}, x_{21}, \ldots, x_{n1}) \\ -(a_1 - a_2 - \cdots - a_n)h(x_{11}, x_{21}, \ldots, x_{n1})\| \leq \sum_{j=1}^{n} \vartheta_j \left(x_{j1}, \underbrace{0, \ldots, 0}_{(n-1)-times} \right) \end{aligned} \tag{3.6}$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathcal{A}$. If we substitute

$$x_{m1} = x, \qquad m = 1, 2, \ldots n$$

in (3.7), we arrive

$$\|2h(a_1x, a_1x, \ldots, a_1x) - 2a_1h(x, x, \ldots, x)\| \leq \sum_{j=1}^{n} \vartheta_j \left(x, \underbrace{0, \ldots, 0}_{(n-1)-times} \right) \tag{3.7}$$

for all $x \in \mathcal{A}$. Hence from (3.7), we reach

$$\left\| \frac{1}{a_1} \; h\left(\underbrace{a_1x, a_1x, \ldots, a_1x}_{n-times} \right) - h\left(\underbrace{x, x, \ldots, x}_{n-times} \right) \right\| \leq \frac{1}{2 \times a_1} \sum_{j=1}^{n} \vartheta_j \left(x, \underbrace{0, \ldots, 0}_{(n-1)-times} \right) \tag{3.8}$$

for all $x \in \mathcal{A}$. It follows from (3.8) that

$$\left\| \frac{1}{a_1} \; h\left(\underbrace{a_1x, a_1x, \ldots, a_1x}_{n-times} \right) - h\left(\underbrace{x, x, \ldots, x}_{n-times} \right) \right\| \leq \frac{1}{a_1}\Theta(x) \tag{3.9}$$

where

$$\Theta(x) = \frac{1}{2} \sum_{j=1}^{n} \vartheta_j \left(x, \underbrace{0, \ldots, 0}_{(n-1)-times} \right)$$

for all $x \in \mathcal{A}$. Now replacing x by a_1x and dividing by a_1 in (3.9), we get

$$\left\| \frac{1}{a_1^2} \; h\left(a_1^2x, a_1^2x, \ldots, a_1^2x \right) - \frac{1}{a_1}h\left(a_1x, a_1x, \ldots, a_1x \right) \right\| \leq \frac{1}{a_1^2}\Theta(a_1x) \tag{3.10}$$

for all $x \in \mathcal{A}$. From (3.8) and (3.10), we obtain

$$\left\| \frac{1}{a_1^2} \; h\left(a_1^2x, a_1^2x, \ldots, a_1^2x \right) - h\left(x, x, \ldots, x \right) \right\| \leq \frac{1}{a_1} \left[\Theta(x) + \frac{\Theta(a_1x)}{a_1} \right] \tag{3.11}$$

for all $x \in \mathcal{A}$. Proceeding further and using induction on a positive integer s, we get

$$\left\| \frac{1}{a_1^s} \; h\left(a_1^sx, a_1^sx, \ldots, a_1^sx \right) - h\left(x, x, \ldots, x \right) \right\| \leq \frac{1}{a_1} \sum_{t=0}^{s-1} \frac{\Theta(a_1^tx)}{a_1^t} \tag{3.12}$$

for all $x \in \mathcal{A}$. In order to prove the convergence of the sequence

$$\left\{ \frac{1}{a_1^s} \; h\left(a_1^sx, a_1^sx, \ldots, a_1^sx \right) \right\},$$

replace x by $a_1^r x$ and dividing by a_1^r in (3.12), for any $r, s > 0$, we deduce

$$\begin{aligned}
&\left\| \frac{1}{a_1^{r+s}} \; h\left(a_1^{r+s}x, a_1^{r+s}x, \dots, a_1^{r+s}x\right) - \frac{1}{a_1^r} \; h\left(a_1^r x, a_1^r x, \dots, a_1^r x\right) \right\| \\
&\quad = \frac{1}{a_1^r} \left\| \frac{1}{a_1^s} h\left(a_1^r \cdot a_1^s x, a_1^r \cdot a_1^s x, \dots, a_1^r \cdot a_1^s x\right) - h\left(a_1^r x, a_1^r x, \dots, a_1^r x\right) \right\| \\
&\quad \le \frac{1}{a_1} \sum_{t=0}^{\infty} \frac{\Theta(a_1^{r+s}x)}{a_1^{r+s}} \\
&\quad \to 0 \quad as \;\; r \to \infty
\end{aligned}$$

for all $x \in \mathcal{A}$. Hence the sequence $\left\{ \frac{1}{a_1^s} \; h\left(a_1^s x, a_1^s x, \dots, a_1^s x\right) \right\}$ is a Cauchy sequence. Since $\mathcal{B}$ is complete, there exists a mapping $A : \mathcal{A}^n \to \mathcal{B}$ such that

$$A(x, x, \dots x) = \lim_{s \to \infty} \frac{1}{a_1^s} \; h\left(a_1^s x, a_1^s x, \dots, a_1^s x\right), \quad \forall \;\; x \in \mathcal{A}.$$

Letting $s \to \infty$ in (3.12), we see that (3.3) holds for all $x \in \mathcal{A}$. To prove that A satisfies (1.14), replacing

$$x_{mi} = a_1^s x_{mi}, \qquad i = 1, 2, 3 \dots n, \qquad m = 1, 2, \cdots n$$

and dividing by a_1^s in (3.2), we obtain

$$\begin{aligned}
&\frac{1}{a_1^s} \left\| Dh(a_1^s x_{11}, \dots, a_1^s x_{1n}, a_1^s x_{21}, \dots, a_1^s x_{2n}, a_1^s x_{n1}, \dots, a_1^s x_{nn}) \right\| \\
&\qquad\qquad\qquad \le \frac{1}{a_1^s} \sum_{j=1}^{n} \vartheta_j \left(a_1^s x_{j1}, a_1^s x_{j2}, \dots, a_1^s x_{jn}\right)
\end{aligned}$$

for all $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn} \in \mathcal{A}$. Letting $s \to \infty$ in the above inequality and using the definition of $A(x, x, \dots, x)$, we see that

$$DA(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn}) = 0.$$

Hence A satisfies (1.14) for all $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn} \in \mathcal{A}$. To prove that $A(x, x, \dots x)$ is unique, let $B(x, x, \dots x)$ be another $n-$ variable additive mapping satisfying (1.14) and (3.3), then

$$\begin{aligned}
&\| A(x, x, \dots x) - B(x, x, \dots x) \| \\
&= \frac{1}{a_1^s} \| A(a_1^s x, a_1^s x, \dots a_1^s x) - B(a_1^s x, a_1^s x, \dots a_1^s x) \| \\
&\le \frac{1}{2^n} \{ \| A(a_1^s x, a_1^s x, \dots a_1^s x) - h(a_1^s x, a_1^s x, \dots a_1^s x) \| \\
&\qquad\qquad + \| h(a_1^s x, a_1^s x, \dots a_1^s x) - B(a_1^s x, a_1^s x, \dots a_1^s x) \| \} \\
&\le \frac{2}{a_1} \sum_{t=0}^{\infty} \frac{\Theta(a_1^{t+s}x)}{a_1^{(t+s)}} \\
&\to 0 \quad as \;\; s \to \infty
\end{aligned}$$

for all $x \in \mathcal{A}$. Thus A is unique. Hence for $\ell = 1$ the Theorem holds.

Now, replacing x by $\frac{x}{a_1}$ in (3.7), we reach

$$\left\| 2h(x, x, \dots, x) - 2a_1 h\left(\frac{x}{a_1}, \frac{x}{a_1}, \dots, \frac{x}{a_1}\right) \right\| \le \sum_{j=1}^{n} \vartheta_j \left(\frac{x}{a_1}, \underbrace{0, \dots, 0}_{(n-1)-times} \right) \tag{3.13}$$

for all $x \in \mathcal{A}$. Dividing the above inequality by 2, we obtain

$$\left\| h(x, x, \dots, x) - a_1 h\left(\frac{x}{a_1}, \frac{x}{a_1}, \dots, \frac{x}{a_1} x\right) \right\| \le \Theta\left(\frac{x}{a_1}\right) \tag{3.14}$$

where

$$\Theta\left(\frac{x}{a_1}\right) = \frac{1}{2}\sum_{j=1}^{n}\vartheta_j\left(\frac{x}{a_1}, \underbrace{0,\ldots,0}_{(n-1)-times}\right)$$

for all $x \in \mathcal{A}$. The rest of the proof is similar to that of $\ell = 1$. Hence for $\ell = -1$ also the Theorem holds. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers [21], Ulam-TRassias [31] and Ulam-JMRassias [30] stabilities of (1.14).

Corollary 3.1. *Let ρ and q be nonnegative real numbers. Let $h : \mathcal{A}^n \to \mathcal{B}$ be a function satisfying the inequality*

$$\|Dh(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},x_{n1},\ldots,x_{nn})\| \le \begin{cases} \rho, & \\ \rho\sum\limits_{i=1}^{n}\sum\limits_{m=1}^{n}||x_{mi}||^q, & q \neq 1; \\ \rho\left\{\prod\limits_{i=1}^{n}\prod\limits_{m=1}^{n}||x_{mi}||^q + \sum\limits_{i=1}^{n}\sum\limits_{m=1}^{n}||x_{mi}||^{nq},\right\}, & nq \neq 1; \end{cases} \tag{3.15}$$

for all $x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},x_{n1},\ldots,x_{nn} \in \mathcal{A}$. Then there exists a unique $n-$ variable additive function $A : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x,x\ldots,x) - A(x,x,\ldots x)\| \le \begin{cases} \dfrac{na_1\rho}{2|a_1-1|}, \\ \dfrac{na_1\rho||x||^q}{2|a_1-a_1^q|}, \\ \dfrac{na_1\rho||x||^{nq}}{2|a_1-a_1^{nq}|}, \end{cases} \tag{3.16}$$

for all $x \in \mathcal{A}$.

Now, we will provide an example to illustrate that the functional equation (1.14) is not stable for $q = 1$ in condition (ii) of Corollary 3.1.

Example 3.1. *Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\vartheta(x) = \begin{cases} \mu x, & if\ |x| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x,x\ldots,x) = \sum_{n=0}^{\infty}\frac{\vartheta(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then h satisfies the functional inequality

$$|Dh(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},x_{n1},\ldots,x_{nn})| \le \frac{4\,\mu\,a_1}{(a_1-1)}|x| \tag{3.17}$$

for all $x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},x_{n1},\ldots,x_{nn} \in \mathbb{R}$. Then there do not exist a $n-$ variable additive mapping $A : \mathbb{R}^n \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|h(x,x\ldots,x) - A(x,x,\ldots x)| \le \kappa|x| \qquad for\ all \quad x \in \mathbb{R}. \tag{3.18}$$

Proof. Now

$$|h(x,x\ldots,x)| \le \sum_{n=0}^{\infty}\frac{|\vartheta(a_1^n x)|}{|a_1^n|} = \sum_{n=0}^{\infty}\frac{\mu}{a_1^n} = \frac{a_1\mu}{a_1-1}.$$

Therefore, we see that h is bounded. We are going to prove that h satisfies (3.17).

If $x_{mi} = 0, \quad i = 1,2,\ldots,n, m = 1,2,\ldots,n$ then (3.17) is trivial. If $|x_{mi}| \ge \dfrac{1}{a_1}$ then the left hand side of (3.17) is less than $\dfrac{4\,\mu\,a_1}{a_1-1}$. Now suppose that $0 < |x_{mi}| < \dfrac{1}{a_1}$. Then there exists a positive integer k such that

$$\frac{1}{a_1^k} \le |x_{mi}| < \frac{1}{a_1^{k-1}}, \tag{3.19}$$

so that $a_1^{k-1} x_{mi} < \frac{1}{a_1}$ and consequently

$$a_1^{k-1}(x_{mi}), a_1^{k-1}(-x_{mi}) \in (-1,1).$$

Therefore for each $p = 0, 1, \ldots, k-1$, we have

$$a_1^p(x_{mi}), a_1^p(-x_{mi}) \in (-1,1)$$

and

$$\begin{aligned} 2\vartheta &\left(a_1^p \sum_{i=1}^n a_i\, x_{1i}, a_1^p \sum_{i=1}^n a_i\, x_{2i}, \ldots\ldots, a_1^p \sum_{i=1}^n a_i\, x_{ni} \right) \\ &- \left(\sum_{i=1}^n a_i \right) \vartheta \left(a_1^p \sum_{i=1}^n x_{1i}, a_1^p \sum_{i=1}^n x_{2i}, \ldots\ldots, a_1^p \sum_{i=1}^n x_{ni} \right) \\ &\quad - \left(a_1 - \sum_{i=2}^n a_i \right) \vartheta \left(a_1^p x_{11} - a_1^p \sum_{i=2}^n x_{1i}, a_1^p x_{21} - a_1^p \sum_{i=2}^n x_{2i}, \ldots\ldots, a_1^p x_{n1} - a_1^p \sum_{i=2}^n x_{ni} \right) = 0 \end{aligned}$$

for $p = 0, 1, \ldots, k-1$. From the definition of h and (3.19), we obtain that

$$\begin{aligned} &\Bigg| 2h \left(\sum_{i=1}^n a_i\, x_{1i}, \sum_{i=1}^n a_i\, x_{2i}, \ldots\ldots, \sum_{i=1}^n a_i\, x_{ni} \right) \\ &\quad - \left(\sum_{i=1}^n a_i \right) h \left(\sum_{i=1}^n x_{1i}, \sum_{i=1}^n x_{2i}, \ldots\ldots, \sum_{i=1}^n x_{ni} \right) \\ &\qquad - \left(a_1 - \sum_{i=2}^n a_i \right) h \left(x_{11} - \sum_{i=2}^n x_{1i}, x_{21} - \sum_{i=2}^n x_{2i}, \ldots\ldots, x_{n1} - \sum_{i=2}^n x_{ni} \right) \Bigg| \\ &\leq \sum_{p=0}^{\infty} \frac{1}{a_1^n} \Bigg| 2\vartheta \left(a_1^p \sum_{i=1}^n a_i\, x_{1i}, a_1^p \sum_{i=1}^n a_i\, x_{2i}, \ldots\ldots, a_1^p \sum_{i=1}^n a_i\, x_{ni} \right) \\ &\quad - \left(\sum_{i=1}^n a_i \right) \vartheta \left(a_1^p \sum_{i=1}^n x_{1i}, a_1^p \sum_{i=1}^n x_{2i}, \ldots\ldots, a_1^p \sum_{i=1}^n x_{ni} \right) \\ &\qquad - \left(a_1 - \sum_{i=2}^n a_i \right) \vartheta \left(a_1^p x_{11} - a_1^p \sum_{i=2}^n x_{1i}, a_1^p x_{21} - a_1^p \sum_{i=2}^n x_{2i}, \ldots\ldots, a_1^p x_{n1} - a_1^p \sum_{i=2}^n x_{ni} \right) \Bigg| \\ &\leq \sum_{p=k}^{\infty} \frac{1}{a_1^p} \Bigg| 2\vartheta \left(a_1^p \sum_{i=1}^n a_i\, x_{1i}, a_1^p \sum_{i=1}^n a_i\, x_{2i}, \ldots\ldots, a_1^p \sum_{i=1}^n a_i\, x_{ni} \right) \\ &\quad - \left(\sum_{i=1}^n a_i \right) \vartheta \left(a_1^p \sum_{i=1}^n x_{1i}, a_1^p \sum_{i=1}^n x_{2i}, \ldots\ldots, a_1^p \sum_{i=1}^n x_{ni} \right) \\ &\qquad - \left(a_1 - \sum_{i=2}^n a_i \right) \vartheta \left(a_1^p x_{11} - a_1^p \sum_{i=2}^n x_{1i}, a_1^p x_{21} - a_1^p \sum_{i=2}^n x_{2i}, \ldots\ldots, a_1^p x_{n1} - a_1^p \sum_{i=2}^n x_{ni} \right) \Bigg| \\ &\leq \sum_{p=k}^{\infty} \frac{1}{a_1^p} 4\mu = 4\,\mu \times \frac{a_1}{(a_1 - 1)\, a_1^k} = \frac{4\,\mu\, a_1}{(a_1 - 1)} |x|. \end{aligned}$$

Thus h satisfies (3.17) for all $x_{mi} \in \mathbb{R}$ with $0 < |x_{mi}| < \frac{1}{a_1}$.

We claim that the additive functional equation (1.14) is not stable for $q = 1$ in condition (ii) Corollary 3.1. Indeed, assume the contrary that there exist a additive mapping $A : \mathbb{R}^n \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.18). Since h is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x, x, \ldots, x) = cx$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|h(x, x, \ldots, x)| \leq (\kappa + |c|)\, |x|. \tag{3.20}$$

But, choose a positive integer i with $i\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{i-1}}\right)$, then $2^p x \in (0,1)$ for all $p = 0, 1, \ldots, i-1$. For this x, we get

$$h(x, x, \ldots, x) = \sum_{p=0}^{\infty} \frac{\vartheta(a_1^p x)}{a_1^p} \geq \sum_{p=0}^{i-1} \frac{\mu(2^p x)}{2^p} = i\mu x > (\kappa + |c|)\, x$$

which contradicts (3.20). Therefore the additive functional equation (1.14) is not stable in sense of Ulam, Hyers and Rassias if $q = 1$, assumed in the inequality condition (ii) of (3.16). □

Now, we will provide an example to illustrate that the functional equation (1.14) is not stable for $q = \frac{1}{n}$ in condition (iii) of Corollary 3.1.

Example 3.2. *Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\vartheta(x) = \begin{cases} \mu x, & \text{if } |x| < \frac{1}{n} \\ \frac{\mu}{n}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x, x \ldots, x) = \sum_{n=0}^{\infty} \frac{\vartheta(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then h satisfies the functional inequality

$$|Dh(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn})| \leq \frac{4\,\mu\, a_1}{n(a_1 - 1)} |x| \tag{3.21}$$

for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathbb{R}$. Then there do not exist a $n-$ variable additive mapping $A : \mathbb{R}^n \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|h(x, x \ldots, x) - A(x, x, \ldots x)| \leq \kappa |x| \qquad for\ all \quad x \in \mathbb{R}. \tag{3.22}$$

4 Stability Results: Banach Space: Alternative Fixed Point Method

In this section, we apply a fixed point method for achieving stability of the functional equation (1.14) is present.

Now, first we will recall the fundamental results in fixed point theory.

Theorem 4.2. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1} x), \forall\ n \geq 0, \forall\ x \in X$;

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X$.

Theorem 4.3. *[26] Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that
(FP1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T
(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

In this section, we take let us consider $\mathcal{E}$ and $\mathcal{F}$ to be a normed space and a Banach space, respectively and define a mapping $Dh : \mathcal{E}^n \to \mathcal{F}$ by

$$\begin{aligned} & Dh(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn}) \\ & = 2h\left(\sum_{i=1}^{n} a_i\, x_{1i}, \sum_{i=1}^{n} a_i\, x_{2i}, \ldots\ldots, \sum_{i=1}^{n} a_i\, x_{ni}\right) - \left(\sum_{i=1}^{n} a_i\right) h\left(\sum_{i=1}^{n} x_{1i}, \sum_{i=1}^{n} x_{2i}, \ldots\ldots, \sum_{i=1}^{n} x_{ni}\right) \\ & \quad - \left(a_1 - \sum_{i=2}^{n} a_i\right) h\left(x_{11} - \sum_{i=2}^{n} x_{1i}, x_{21} - \sum_{i=2}^{n} x_{2i}, \ldots\ldots, x_{n1} - \sum_{i=2}^{n} x_{ni}\right) \end{aligned}$$

for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{E}$.

Theorem 4.4. *Let $h : \mathcal{E}^n \to \mathcal{F}$ be a mapping for which there exists a function $\zeta : \mathcal{E}^n \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\tau_i^k} \zeta(\tau_i^k x) = 0 \tag{4.1}$$

where

$$\tau_i = \begin{cases} a_1 & if \quad i = 0; \\ \frac{1}{a_1} & if \quad i = 1, \end{cases} \tag{4.2}$$

such that the functional inequality

$$\|Dh(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn})\| \leq \sum_{j=1}^{n} \vartheta_j\left(x_{j1}, x_{j2}, \ldots, x_{jn}\right) \tag{4.3}$$

for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{E}$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Theta(x) = \frac{1}{2} \sum_{j=1}^{n} \vartheta_j\left(\frac{x}{a_1}, \underbrace{0, \ldots, 0}_{(n-1)-times}\right),$$

has the property

$$\frac{1}{\tau_i} \Theta(\tau_i x) = L\, \Theta(x). \tag{4.4}$$

for all $x \in \mathcal{E}$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.14) and

$$\|h(x, x, \ldots, x) - A(x, x, \ldots, x)\| \leq \frac{L^{1-i}}{1-L} \Theta(x) \tag{4.5}$$

for all $x \in \mathcal{E}$.

Proof. Consider the set

$$\Gamma = \{f / f : \mathcal{E}^n \to \mathcal{F},\ f(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(f, g) = \inf\{K \in (0, \infty) : \| f(x, x, \ldots, x) - g(x, x, \ldots, x) \| \leq K\Theta(x), x \in \mathcal{E}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon f(x, x, \ldots, x) = \frac{1}{\tau_i} f(\tau_i x, \tau_i x, \ldots, \tau_i x),$$

for all $x \in \mathcal{E}$. Now $f, g \in \Gamma$,

$$\begin{aligned} d(f, g) \leq K \Rightarrow & \| f(x, x, \ldots, x) - g(x, x, \ldots, x) \| \leq K\Theta(x), x \in \mathcal{E}. \\ & \Rightarrow \left\| \frac{1}{\tau_i} f(\tau_i x, \tau_i x, \ldots, \tau_i x) - \frac{1}{\tau_i} g(\tau_i x, \tau_i x, \ldots, \tau_i x) \right\| \leq \frac{1}{\tau_i} K\Theta(\tau_i x), x \in \mathcal{E}, \\ & \Rightarrow \left\| \frac{1}{\tau_i} f(\tau_i x, \tau_i x, \ldots, \tau_i x) - \frac{1}{\tau_i} g(\tau_i x, \tau_i x, \ldots, \tau_i x) \right\| \leq LK\Theta(x), x \in \mathcal{E}, \\ & \Rightarrow \| \Upsilon f(x, x, \ldots, x) - \Upsilon g(x, x, \ldots, x) \| \leq LK\Theta(x), x \in \mathcal{E}, \\ & \Rightarrow d(\Upsilon f, \Upsilon g) \leq LK. \end{aligned}$$

This implies $d(\Upsilon f, \Upsilon g) \leq Ld(f,g)$, for all $f,g \in \Gamma$. i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

It follows from, (3.9) that

$$\|2h(a_1x, a_1x, \ldots, a_1x) - 2a_1h(x,x,\ldots,x)\| \leq \sum_{j=1}^{n} \vartheta_j \left(x, \underbrace{0,\ldots,0}_{(n-1)-times} \right) \tag{4.6}$$

for all $x \in \mathcal{E}$. Now, from (4.6), we get

$$\left\| \frac{1}{a_1} \ h\left(a_1x, a_1x, \ldots, a_1x\right) - h\left(x,x,\ldots,x\right) \right\| \leq \frac{1}{2a_1}\Theta(x) \tag{4.7}$$

for all $x \in \mathcal{E}$. Using (4.4) for the case $i = 0$ it reduces to

$$\left\| \frac{1}{a_1} \ h\left(a_1x, a_1x, \ldots, a_1x\right) - h\left(x,x,\ldots,x\right) \right\| \leq \ L\Theta(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(\Upsilon h, h) \leq L \Rightarrow d(\Upsilon h, h) \leq L = L^1 < \infty. \tag{4.8}$$

Again replacing $x = \frac{x}{a_i}$ in (4.6), we get

$$\left\| h(x,x,\ldots,x) - a_1 h\left(\frac{x}{a_i}, \frac{x}{a_i}, \ldots, \frac{x}{a_i}\right) \right\| \leq \frac{1}{2}\sum_{j=1}^{n} \vartheta_j \left(\frac{x}{a_1}, \underbrace{0,\ldots,0}_{(n-1)-times} \right) \tag{4.9}$$

for all $x \in \mathcal{E}$. Using (4.4) for the case $i = 1$ it reduces to

$$\left\| h(x,x,\ldots,x) - a_1 h\left(\frac{x}{a_i}, \frac{x}{a_i}, \ldots, \frac{x}{a_i}\right) \right\| \leq \Theta(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(h, \Upsilon h) \leq 1 \Rightarrow d(h, \Upsilon h) \leq 1 = L^0 < \infty. \tag{4.10}$$

From (4.8) and (4.10), we arrive

$$d(h, \Upsilon h) \leq L^{1-i}.$$

Therefore (FP1) holds.

By (FP2), it follows that there exists a fixed point A of Υ in Γ such that

$$A(x,x,\ldots,x) = \lim_{k\to\infty} \frac{h(\tau_i^k x, \tau_i^k x \ldots, \tau_i^k x)}{\tau_i^k}, \qquad \forall \ x \in \mathcal{E}. \tag{4.11}$$

To order to prove $A : \mathcal{E} \to \mathcal{F}$ satisfies (1.14), replacing

$$x_{mi} = \tau_i^k x_{mi}, \qquad i = 1,2,3\ldots n, \quad m = 1,2,\cdots n$$

in (4.3) and dividing by τ_i^k, it follows from (4.1) that

$$\frac{1}{\tau_i^k} \left\| Dh(\tau_i^k x_{11}, \ldots, \tau_i^k x_{1n}, \tau_i^k x_{21}, \ldots, \tau_i^k x_{2n}, \tau_i^k x_{n1}, \ldots, \tau_i^k x_{nn}) \right\|$$
$$\leq \frac{1}{\tau_i^k} \sum_{j=1}^{n} \vartheta_j \left(\tau_i^k x_{j1}, \tau_i^k x_{j2}, \ldots, \tau_i^k x_{jn} \right)$$

for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{E}$. Letting $k \to \infty$ in the above inequality and using the definition of $A(x,x,\ldots,x)$, we see that

$$DA(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn}) = 0.$$

Hence A satisfies (1.14) for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{A}$.

By (FP3), A is the unique fixed point of Υ in the set

$$\Delta = \{A \in \Gamma : d(h, A) < \infty\},$$

such that

$$\|h(x, x, \ldots, x) - A(x, x, \ldots, x)\| \leq K\Theta(x)$$

for all $x \in \mathcal{E}$ and $K > 0$. Finally by (FP4), we obtain

$$d(h, A) \leq \frac{1}{1-L} d(h, \Upsilon h)$$

this implies

$$d(h, A) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\|h(x, x, \ldots, x) - A(x, x, \ldots, x)\| \leq \frac{L^{1-i}}{1-L}\Theta(x)$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.4 concerning the stability of (1.14).

Corollary 4.2. *Let $h : \mathcal{E} \to \mathcal{F}$ be a mapping and exists real numbers ρ and r such that*

$$\|Dh(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn})\| \leq \begin{cases} \rho, & \\ \rho \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||x_{mi}||^q, & q \neq 1; \\ \rho \left\{ \prod\limits_{i=1}^{n} \prod\limits_{m=1}^{n} ||x_{mi}||^q + \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||x_{mi}||^{nq}, \right\}, & nq \neq 1; \end{cases} \tag{4.12}$$

for all for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{E}$. Then there exists a unique additive function $A : \mathcal{E} \to \mathcal{F}$ such that

$$\|h(x, x \ldots, x) - A(x, x, \ldots x)\| \leq \begin{cases} \dfrac{n\rho}{2|a_1 - 1|}, \\ \dfrac{n\rho||x||^q}{2|a_1 - a_1^q|}, \\ \dfrac{n\rho||x||^{nq}}{2|a_1 - a_1^{nq}|}, \end{cases} \tag{4.13}$$

for all $x \in \mathcal{E}$.

Proof. Setting

$$\vartheta(x) = \begin{cases} \rho, \\ \rho \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||x_{mi}||^q, \\ \rho \left\{ \prod\limits_{i=1}^{n} \prod\limits_{m=1}^{n} ||x_{mi}||^q + \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||x_{mi}||^{nq}, \right\}, \end{cases}$$

for all $x \in \mathcal{E}$. Now,

$$\frac{1}{\tau_i^k}\vartheta(\tau_i^k x) = \begin{cases} \dfrac{\rho}{\tau_i^k}, \\ \dfrac{\rho}{\tau_i^k} \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||\tau_i^k x_i||^q, \\ \dfrac{\rho}{\tau_i^k} \left\{ \prod\limits_{i=1}^{n} \prod\limits_{m=1}^{n} ||\tau_i^k x_{mi}||^q + \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||\tau_i^k x_{mi}||^{nq}, \right\}, \end{cases} = \begin{cases} \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (4.1) is holds. We, already have

$$\Theta(x) = \frac{1}{2} \sum_{j=1}^{n} \vartheta_j \left(\frac{x}{a_1}, \underbrace{0, \ldots, 0}_{(n-1)-times} \right),$$

with the property

$$\frac{1}{\tau_i}\Theta(\tau_i x) = L\,\Theta\,(x)$$

for all $x \in \mathcal{E}$. Hence

$$\Theta(x) = \frac{1}{2}\sum_{j=1}^{n}\vartheta_j\left(\frac{x}{a_1}, \underbrace{0,\ldots,0}_{(n-1)-times}\right) = \begin{cases} \frac{n\rho}{2} \\ \frac{n\rho}{2\cdot a_1^q}||x||^q \\ \frac{n\rho}{2\cdot a_1^{nq}}||x||^{nq}. \end{cases}$$

Also,

$$\frac{1}{\tau_i}\Theta(\tau_i x) = \begin{cases} \frac{n\rho}{2\tau_i} \\ \frac{n\rho}{2\tau_i}||\tau_i x||^q \\ \frac{n\rho}{2\tau_i}||\tau_i x||^{nq}. \end{cases} = \begin{cases} \tau_i^{-1}\frac{n\rho}{2}, \\ \tau_i^{q-1}n\frac{n\rho||x||^q}{2} \\ \tau_i^{nq-1}n\frac{n\rho||x||^{nq}}{2} \end{cases} = \begin{cases} \tau_i^{-1}\Theta(x), \\ \tau_i^{q-1}\Theta(x) \\ \tau_i^{nq-1}\Theta(x). \end{cases}$$

Hence the inequality (4.4) holds either, $L = a_1^{-1}$ if $i = 0$ and $L = \frac{1}{a_1^{-1}}$ if $i = 1$. Now from (4.5), we prove the following cases for condition (i).

Case:1 $L = a_1^{-1}$ if $i = 0$

$$\|h(x) - A(x)\| \leq \frac{\left(a_1^{-1}\right)^{1-0}}{1-a_1^{-1}}\Theta(x) = \frac{n\rho}{2(a_1-1)}.$$

Case:2 $L = \frac{1}{a_1^{-1}}$ or if $i = 1$

$$\|h(x) - A(x)\| \leq \frac{\left(\frac{1}{a_1^{-1}}\right)^{1-1}}{1-\frac{1}{a_1^{-1}}}\Theta(x) = \frac{n\rho}{2(1-a_1)}.$$

Also the inequality (4.4) holds either, $L = a_1^{q-1}$ for $q < 1$ if $i = 0$ and $L = \frac{1}{a_1^{q-1}}$ for $q > 1$ if $i = 1$. Now from (4.5), we prove the following cases for condition (ii).

Case:3 $L = a_1^{q-1}$ for $q < 1$ if $i = 0$

$$\|h(x) - A(x)\| \leq \frac{\left(a_1^{(q-1)}\right)^{1-0}}{1-a_1^{(q-1)}}\Theta(x) = \frac{n\rho||x||^q}{2(a_1 - a_1^q)}$$

Case:4 $L = \frac{1}{a_1^{q-1}}$ for $q > 1$ if $i = 1$

$$\|h(x) - A(x)\| \leq \frac{\left(\frac{1}{a_1^{(q-1)}}\right)^{1-1}}{1-\frac{1}{a_1^{(q-1)}}}\Theta(x) = \frac{n\rho||x||^q}{2(a_1^q - a_1)}.$$

The proof of condition (iii) is similar to that of condition (ii). Hence the proof is complete. □

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Ulam-Hyers Stability of Quadratic Reciprocal Functional Equation in Intuitionistic Random Normed spaces: Various Methods

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Abstract

In this paper, the authors investigated the intuitionistic random stability of a quadratic reciprocal functional equation

$$f(x+2y)+f(2x+y)=\frac{f(x)f(y)\left[5f(x)+5f(y)+8\sqrt{f(x)f(y)}\right]}{\left[2f(x)+2f(y)+5\sqrt{f(x)+f(y)}\right]^2}$$

using direct and fixed point methods.

Keywords: Quadratic reciprocal functional equation, generalized Ulam-Hyers stability, intuitionistic random normed space, fixed point.

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1 Introduction

The study of stability problem for functional equations goes back to a question raised by Ulam [44] concerning the stability of group homomorphisms that affirmatively answered for Banach spaces by Hyers [24]. Hyers Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [37] for linear mappings by considering an unbounded Cauchy difference. The paper by Rassias has provided a lot of influences in the development of what we now call the generalized Hyers-Ulam stability or Hyers- Ulam-Rassias stability of functional equations. J.M. Rassias [35] considered the Cauchy difference controlled by a product of different powers of norm. Afterwards, Găvruţa [21] generalized the Rassas's theorem by using a general control function. In 2008, a special case of Găvruţa's theorem for the unbounded Cauchy difference was obtained by Ravi et al. [38] by considering the summation of both the sum and the product of two p-norms in the sprit of Rassias approach. A large part of proofs in this topic used the direct method (of Hyers): the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution.

In 2003, V. Radu [11] proposed a new method, successively developed in [12–14], to obtaining the existence of the exact solutions and the error estimations, based on the fixed point alternative.

The theory of random normed spaces (RN-spaces) is important as a generalization of the deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also

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provide us with the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. Recently, J.M. Rassias et al. [36] investigated the intuitionistic random stability of the quartic functional equation and C. Park et al. [33] presented the Hyers-Ulam stability of the additive-quadratic functional equation in intuitionistic random normed space.

In 2014, M. Arunkumar and S. Karthikeyan [5] introduced and investigated Hyers-Ulam stability of n-dimensional reciprocal functional equation

$$f\left(\frac{2x}{n}\right)=\sum_{\ell=1}^{n}\left(\frac{f\left(x+\ell y_{\ell}\right)f\left(x-\ell y_{\ell}\right)}{f\left(x+\ell y_{\ell}\right)+f\left(x-\ell y_{\ell}\right)}\right) \tag{1.1}$$

which originates from n-consecutive terms of a harmonic progression in RN-space using direct and fixed point methods.

Recently, Abasalt Bodaghi and Sang Og Kim [1] introduced new 2-dimensional quadratic reciprocal functional equation

$$f(x+2y)+f(2x+y)=\frac{f(x)f(y)\left[5f(x)+5f(y)+8\sqrt{f(x)f(y)}\right]}{\left[2f(x)+2f(y)+5\sqrt{f(x)+f(y)}\right]^{2}}. \tag{1.2}$$

It is easily verified that the quadratic reciprocal function $f(x)=\dfrac{1}{x^2}$ is a solution of the functional equation (1.2).

In this paper, the authors establish intuitionistic random norm stability of a quadratic reciprocal functional equation (1.2) using direct and fixed point methods.

2 Preliminaries of Intuitionistic Random Normed Spaces

In this section, using the idea of intuitionistic random normed spaces introduced by Chang et al. [16], we define the notion of intuitionistic random normed spaces as in [15, 22, 29, 31, 40–42].

Definition 2.1. *A measure distribution function is a function $\mu:\mathbb{R}\to[0,1]$ which is left continuous, non-decreasing on $\mathbb{R}$, $inf_{t\in\mathbb{R}}\mu(t)=0$ and $sup_{t\in\mathbb{R}}\mu(t)=1$.*

We will denote by D the family of all measure distribution functions and by H a special element of D defined by

$$H(t)=\begin{cases}0, & if\ t\leq 0,\\ 1, & if\ t>0.\end{cases} \tag{2.1}$$

If X is a nonempty set, then $\mu: X\to D$ is called a probabilistic measure on X and $\mu(x)$ is denoted by μ_x.

Definition 2.2. *A non-measure distribution function is a function $\nu:\mathbb{R}\to[0,1]$ which is right continuous, non-decreasing on $\mathbb{R}$, $inf_{t\in\mathbb{R}}\nu(t)=0$ and $sup_{t\in\mathbb{R}}\nu(t)=1$.*

We will denote by B the family of all non-measure distribution functions and by G a special element of B defined by

$$G(t)=\begin{cases}1, & if\ t\leq 0,\\ 0, & if\ t>0.\end{cases} \tag{2.2}$$

If X is a nonempty set, then $\nu: X\to B$ is called a probabilistic non-measure on X and $\nu(x)$ is denoted by ν_x.

Lemma 2.1. *[8, 20] Consider the set L^* and operation $\leq_{L^*}$ defined by:*

$$L^*=\left\{(x_1,x_2):(x_1,x_2)\in[0,1]^2\ \ and\ \ x_1+x_2\leq 1\right\},$$

$$(x_1,x_2)\leq_{L^*}(y_1,y_2)\Leftrightarrow x_1\leq y_1, x_2\geq y_2,\quad\forall\,(x_1,x_2)\,,\,(y_1,y_2)\in L^*.$$

Then $(L^*,\leq_{L^*})$ is a complete lattice.

Definition 2.3. *[8] An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universal set U is an object*

$$A_{\zeta,\eta} = \{(\zeta_A(u), \eta_A(u)) \,|u \in U\}$$

for all $u \in U$, $\zeta_A(u) \in [0,1]$ and $\eta_A(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $A_{\zeta,\eta}$ and, furthermore, they satisfy $\zeta_A(u) + \eta_A(u) \leq 1$.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a *triangular norm* $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1,x) = 1 * x = x$ for all $x \in [0,1]$. A *triangular conorm* $S = \diamondsuit$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0,x) = 0 \diamondsuit x = x$ for all $x \in [0,1]$.

Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 2.4. *[8] A triangular norm ($t-$norm) on L^* is a mapping $T : (L^*)^2 \to L^*$ satisfying the following conditions:*

(i) $(\forall \in L^*)\,(T(x, 1_{L^*}) = x)$ *(boundary condition);*

(ii) $\left(\forall (x,y) \in (L^*)^2\right)(T(x,y) = T(y,x))$ *(commutativity);*

(iii) $\left(\forall (x,y,z) \in (L^*)^3\right)(T(x, T(y,z)) = T(T(x,y), z))$ *(associativity);*

(iv) $\left(\forall (x,x',y,y') \in (L^*)^4\right)(x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \Rightarrow T(x,y) \leq_{L^*} T(x',y'))$ *(monotonicity).*

If $(L^*, \leq_{L^*}, T)$ is an Abelian topological monoid with unit 1_{L^*}, then T is said to be a *continuous $t-$norm*.

Definition 2.5. *[8] A continuous $t-$norms T on L^* is said to be continuous $t-$representable if there exist a continuous $t-$norm $*$ and a continuous $t-$conorm $\diamondsuit$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,*

$$T(x,y) = (x_1 * y_1, x_2 \diamondsuit y_2).$$

For example,

$$T(a,b) = (a_1 b_1,\ \min\{a_2 + b_2, 1\})$$

and

$$M(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ are continuous $t-$representable.

Now, we define a sequence T^n recursively by $T^1 = T$ and

$$T^n\left(x^{(1)}, \ldots, x^{(n+1)}\right) = T\left(T^{n-1}\left(x^{(1)}, \ldots, x^{(n)}\right), x^{(n+1)}\right), \quad \forall n \geq 2,\ x^{(i)} \in L^*.$$

Definition 2.6. *[43] A negator on L^* is any decreasing mapping $N : L^* \to L^*$ satisfying $N : (0_{L^*}) = 1_{L^*}$ and $N(1_{L^*}) = 0_{L^*}$. If $N(N(x)) = x$ for all $x \in L^*$, then N is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N : [0,1] \to [0,1]$ satisfying $P_{\mu,\nu}(0) = 1$ and $P_{\mu,\nu}(1) = 0$. N_s denotes the standard negator on $[0,1]$ defined by*

$$N_s(x) = 1 - x, \quad \forall x \in [0,1].$$

Definition 2.7. *[43] Let μ and ν be measure and non-measure distribution functions from $X \times (0, +\infty)$ to $[0,1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and all $t > 0$. The triple $\left(X, P_{\mu,\nu}, T\right)$ is said to be an intuitionistic random normed space (briefly IRN-space) if X is a vector space, T is a continuous $t-$representable and $P_{\mu,\nu}$ is a mapping $X \times (0, +\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,*

(IRN1) $P_{\mu,\nu}(x, 0) = 0_{L^*}$;

(IRN2) $P_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x = 0$;

(IRN3) $P_{\mu,\nu}(\alpha x, t) = P_{\mu,\nu}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;

(IRN4) $P_{\mu,\nu}(x+y,t+s) \geq_{L^*} T(P_{\mu,\nu}(x,t), P_{\mu,\nu}(y,s))$.

In this case, $P_{\mu,\nu}$ *is called an intuitionistic random norm. Here,* $P_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t))$.

Example 2.1. *[43] Let* $(X, \|\cdot\|)$ *be a normed space. Let* $T(a,b) = (a_1 b_1, \min(a_2+b_2, 1))$ *for all* $a = (a_1,a_2), b = (b_1,b_2) \in L^*$ *and* μ,ν *be measure and non-measure distribution functions defined by*

$$P_{\mu,v}(x,t) = (\mu_x(t), v_x(t)) = \left(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\right), \quad \forall t \in \mathbb{R}^+.$$

Then $(X, P_{\mu,\nu}, T)$ *is an IRN-sapce.*

Definition 2.8. *[43] A sequence* $\{x_n\}$ *in an IRN-space* $(X, P_{\mu,\nu}, T)$ *is called a Cauchy sequence if, for any* $\varepsilon > 0$ *and* $t > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$P_{\mu,\nu}(x_n - x_m, t) > L^* \ (N_s(\varepsilon), \varepsilon), \quad \forall n, \ m \geq n_0,$$

where N_s *is the standard negator.*

Definition 2.9. *[43] The sequence* $\{x_n\}$ *is said to be convergent to a point* $x \in X$ *(denoted by* $x_n \xrightarrow{P_{\mu,\nu}} x$*) if* $P_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ *as* $n \to \infty$ *for every* $t > 0$.

Definition 2.10. *[43] An IRN-space* $(X, P_{\mu,\nu}, T)$ *is said to be complete if every Cauchy sequence in X is convergent to a point* $x \in X$.

Now, we use the following notation for a given mapping $\Delta : X \to Y$

$$\Delta(x,y) = f(x+2y) + f(2x+y) - \frac{f(x)f(y)\left[5f(x)+5f(y)+8\sqrt{f(x)f(y)}\right]}{\left[2f(x)+2f(y)+5\sqrt{f(x)+f(y)}\right]^2}.$$

3 Stability Results: Direct Method

In this section, the authors presented the generalized Ulam-Hyers stability of the functional equation (1.2) in intuitionistic random normed spaces using direct method.

Theorem 3.1. *Let X be a linear space and* $(Y, P_{\mu,\nu}, T)$ *be a complete IRN-space. Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *for which there are* $\xi, \zeta : X^2 \to D^+$*,* $\xi(x,y)$ *is denoted by* $\xi_{x,y}$ *and* $\zeta(x,y)$ *is denoted by* $\zeta_{x,y}$*, furthur,*$(\xi_{x,y}(t), \zeta_{x,y}(t))$ *is denoted by* $P'_{\xi,\zeta}(x,y,t)$ *with the property:*

$$P_{\mu,\nu}(\Delta(x,y),t) \geq_{L^*} P'_{\xi,\zeta}(x,y,t) \tag{3.1}$$

for all $x,y \in X$ *and all* $t > 0$*. If*

$$T_{i=1}^{\infty} P'_{\xi,\zeta}\left(\frac{x}{3^{i+n}}, \frac{x}{3^{i+n}}, 3^{i-1+2n}t\right) = 1_{L^*} \tag{3.2}$$

and

$$lim_{n\to\infty} P'_{\xi,\zeta}\left(\frac{x}{3^n}, \frac{x}{3^n}, 3^{2n}t\right) = 1_{L^*} \tag{3.3}$$

for all $x \in X$ *and all* $t > 0$*, then there exists a unique quadratic reciprocal mapping* $R : X \to Y$ *satisfies the inequality*

$$P_{\mu,\nu}(f(x) - R(x), t) \geq_{L^*} T_{i=1}^{\infty} P'_{\xi,\zeta}\left(\frac{x}{3^i}, \frac{x}{3^i}, 3^{i-1}t\right) \tag{3.4}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Replacing (x,y) by (x,x) in (3.1), we get

$$P_{\mu,\nu}\left(f(3x) - \frac{f(x)}{3^2}, t\right) \geq_{L^*} P'_{\xi,\zeta}(x,x,t) \tag{3.5}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{x}{3}$ in (3.5), we obtain

$$P_{\mu,\nu}\left(f(x) - \frac{1}{3^2} f\left(\frac{x}{3}\right), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{x}{3}, \frac{x}{3}, t\right) \tag{3.6}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{x}{3^n}$ in (3.5)and using (IRN3), we have

$$P_{\mu,\nu}\left(\frac{1}{3^{2n}}f\left(\frac{x}{3^n}\right)-\frac{1}{3^{2(n+1)}}f\left(\frac{x}{3^{n+1}}\right),\frac{t}{3^{2n}}\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{x}{3^{n+1}},\frac{x}{3^{n+1}},t\right) \tag{3.7}$$

for all $x \in X$ and all $t > 0$. Using $(IRN3)$ in (3.7), we arrive

$$P_{\mu,\nu}\left(\frac{1}{3^{2n}}f\left(\frac{x}{3^n}\right)-\frac{1}{3^{2(n+1)}}f\left(\frac{x}{3^{n+1}}\right),t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{x}{3^{n+1}},\frac{x}{3^{n+1}},3^{2n}t\right) \tag{3.8}$$

that is,

$$P_{\mu,\nu}\left(\frac{1}{3^{2n}}f\left(\frac{x}{3^n}\right)-\frac{1}{3^{2(n+1)}}f\left(\frac{x}{3^{n+1}}\right),\frac{t}{3^{n}}\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{x}{3^{n+1}},\frac{x}{3^{n+1}},3^{n}t\right) \tag{3.9}$$

for all $n \in \mathbb{N}$ and all $t > 0$. As $3 > 1/3 + 1/3^2 + \ldots + 1/3^k$, by the triangle inequality it follows

$$\begin{aligned} P_{\mu,\nu}\left(f(x)-\frac{1}{3^{2k}}f\left(\frac{x}{3^k}\right),t\right) &\geq_{L^*} T_{n=0}^{k-1}\left\{P'_{\xi,\zeta}\left(\frac{1}{3^{2n}}f\left(\frac{x}{3^n}\right)-\frac{1}{3^{2(n+1)}}f\left(\frac{x}{3^{n+1}}\right),\sum_{n=0}^{k-1}\frac{1}{3^n}t\right)\right\} \\ &\geq_{L^*} T_{i=1}^{k}\left\{P'_{\xi,\zeta}\left(\frac{x}{3^i},\frac{x}{3^i},3^{i-1}t\right)\right\} \end{aligned} \tag{3.10}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\left\{\frac{1}{3^{2n}}f\left(\frac{x}{3^n}\right)\right\}$, replacing x by $\frac{x}{3^m}$ in (3.10), we obtain

$$P_{\mu,\nu}\left(\frac{1}{3^{2m}}f\left(\frac{x}{3^m}\right)-\frac{1}{3^{2(k+m)}}f\left(\frac{x}{3^{k+m}}\right),t\right) \geq_{L^*} T_{i=1}^{k}\left\{P'_{\xi,\zeta}\left(\frac{x}{3^{i+m}},\frac{x}{3^{i+m}},3^{i-1+2m}t\right)\right\} \tag{3.11}$$

for all $x \in X$ and all $t > 0$ and all $k, m \geq 0$. Since the right hand-side of the inequality tends to 1_{L^*} as m tends to infinity, the sequence $\left\{\frac{1}{3^{2n}}f\left(\frac{x}{3^n}\right)\right\}$ is a Cauchy sequence. Therefore, we may define $R(x) = lim_{n\to\infty}\frac{1}{3^{2n}}f\left(\frac{x}{3^n}\right)$ for all $x \in X$.

Now, we prove that R satisfies (1.2). Replacing (x, y) by $\left(\frac{x}{3^n}, \frac{y}{3^n}\right)$ in (3.1), we get

$$P_{\mu,\nu}\left(\frac{1}{3^{2n}}\Delta\left(\frac{x}{3^n},\frac{y}{3^n}\right),t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{x}{3^n},\frac{y}{3^n},3^{2n}t\right) \tag{3.12}$$

for all $x, y \in X$ and $t > 0$. Letting $n \to \infty$ in the above inequality and using the definition of $R(x)$, we see that R satisfies (1.2) for all $x, y \in X$.

Finally, to prove the uniqueness of the quadratic reciprocal function R subject to (3.4), let us assume that there exists another quadratic reciprocal function S which satisfies (3.4). Obviously, we have $R\left(\frac{x}{3^n}\right) = 3^{2n}R(x)$ and $S\left(\frac{x}{3^n}\right) = 3^{2n}S(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence, it follows from (3.4) that

$$\begin{aligned} P_{\mu,\nu}\left(R(x)-S(x),t\right) &\geq_{L^*} P_{\mu,\nu}\left(R\left(\frac{x}{3^n}\right)-S\left(\frac{x}{3^n}\right),3^{2n}t\right) \\ &\geq_{L^*} T\left(P_{\mu,\nu}\left(R\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right),\frac{3^{2n}t}{2}\right),P_{\mu,\nu}\left(f\left(\frac{x}{3^n}\right)-S\left(\frac{x}{3^n}\right),\frac{3^{2n}t}{2}\right)\right) \\ &\geq_{L^*} T\left(T_{i=1}^{\infty}\left(P'_{\xi,\zeta}\left(\frac{x}{3^{i+m}},\frac{x}{3^{i+m}},\frac{3^{i-1+2m}t}{2}\right)\right),T_{i=1}^{\infty}\left(P'_{\xi,\zeta}\left(\frac{x}{3^{i+m}},\frac{x}{3^{i+m}},\frac{3^{i-1+2m}t}{2}\right)\right)\right) \end{aligned}$$

for all $x \in X$ and $t > 0$. By letting $n \to \infty$ in (3.4), we prove the uniqueness of R. This completes the proof. □

From Theorem 3.1, we obtain the following corollary concerning the Hyers-Ulam-Rassias and JMRassias stabilities for the functional equation (1.2).

Corollary 3.1. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$P_{\mu,\nu}\left(\Delta(x,y),t\right) \geq_{L^*} \begin{cases} P'_{\xi,\zeta}\left(\epsilon,t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||x||^s+||y||^s\right),t\right); \\ P'_{\xi,\zeta}\left(\epsilon||x||^s||y||^s,t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||x||^s||y||^s+||x||^{2s}+||y||^{2s}\right),t\right); \end{cases} \tag{3.13}$$

for all $x, y \in X$ and $t > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic reciprocal mapping $R : X \to Y$ such that

$$P_{\mu,\nu}\left(f(x) - R(x), t\right) \geq_{L^*} \begin{cases} P'_{\xi,\zeta}\left(\left|\frac{9}{8}\right|\epsilon, t\right); \\ P'_{\xi,\zeta}\left(\frac{18\epsilon}{\left|3^{s+2}-1\right|}||x||^s, t\right), & s < -2 \quad or \quad s > -2; \\ P'_{\xi,\zeta}\left(\frac{9\epsilon}{\left|3^{2s+2}-1\right|}||x||^{2s}, t\right), & s < -1 \quad or \quad s > -1; \\ P'_{\xi,\zeta}\left(\frac{27\epsilon}{\left|3^{2s+2}-1\right|}||x||^{2s}, t\right), & s < -1 \quad or \quad s > -1; \end{cases} \tag{3.14}$$

for all $x \in X$ and all $t > 0$.

4 Stability Results: Fixed Point Method

In this section, the authors proved the generalized Ulam-Hyers stability of the functional equation (1.2) in intuitionistic random normed spaces using fixed point method.

Now, we will recall the fundamental results in fixed point theory.

Theorem 4.2. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $\Gamma : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(\Gamma x, \Gamma y) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping Γ has one and only fixed point $x^* = \Gamma(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty}\Gamma^n x = x^*$, for any starting point $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(\Gamma^n x, x^*) \leq \frac{1}{1-L}\ d(\Gamma^n x, \Gamma^{n+1}x), \forall\ n \geq 0, \forall\ x \in X$;

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X$.

Theorem 4.3. *[30](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $\Gamma : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
$(B1)$ $\qquad d(\Gamma^n x, \Gamma^{n+1}x) = \infty \quad \forall\ n \geq 0,$
or
$(B2)$ there exists a natural number n_0 such that:
(i) $d(\Gamma^n x, \Gamma^{n+1}x) < \infty$ for all $n \geq n_0$;
(ii)The sequence $(\Gamma^n x)$ is convergent to a fixed point y^* of Γ
(iii) y^* is the unique fixed point of Γ in the set $Y = \{y \in X : d(\Gamma^{n_0}x, y) < \infty\}$;
(iv) $d(y^*, y) \leq \frac{1}{1-L}\ d(y, \Gamma y)$ for all $y \in Y$.

Using above fixed point theorems to prove the stability results, we define the following:
δ_i is a constant such that

$$\delta_i = \begin{cases} 3 & if \quad i = 0; \\ \frac{1}{3} & if \quad i = 1; \end{cases}$$

and Ω is the set such that

$$\Omega = \{g \mid g : X \to Y, g(0) = 0\}.$$

Theorem 4.4. *Let X be a linear space and $\left(Y, P_{\mu,\nu}, T\right)$ be a complete IRN-space. Let $f : X \to Y$ be a mapping for which there exist a function $\xi, \zeta : X^2 \to D^+$ with the condition*

$$T_{i=1}^{\infty} P'_{\xi,\zeta}\left(\frac{x}{3^{i+n}}, \frac{x}{3^{i+n}}, 3^{i-1+2n}t\right) = 1_{L^*} \tag{4.1}$$

and

$$lim_{n\to\infty} P'_{\xi,\zeta}\left(\frac{x}{3^n}, \frac{x}{3^n}, 3^{2n}t\right) = 1_{L^*}, \tag{4.2}$$

and satisfying the functional inequality

$$P_{\mu,\nu}\left(\Delta(x,y),t\right) \geq_{L^*} P'_{\xi,\zeta}\left(x,y,t\right), \ \forall\, x,y \in X, t > 0. \tag{4.3}$$

If there exists L such that the function

$$x \to \beta(x) = \frac{x}{3}, \frac{x}{3} \tag{4.4}$$

has the property

$$P'_{\xi,\zeta}\left(L\delta_i^2\beta(\delta_i x), r\right) = P'_{\xi,\zeta}\left(\beta(x),t\right), \ \forall\, x \in X, t > 0. \tag{4.5}$$

Then there exists a unique quadratic reciprocal function $R : X \to Y$ satisfying the functional equation (1.2) and

$$P_{\mu,\nu}\left(f(x) - R(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{L^{1-i}}{1-L}\beta(x), t\right), \ \forall\, x \in X, t > 0. \tag{4.6}$$

Proof. Let d be a general metric on Ω, such that

$$d(g,h) = inf\left\{K \in (0,\infty) | P_{\mu,\nu}\left(g(x) - h(x), r\right) \geq_{L^*} P'_{\xi,\zeta}\left(K\beta(x),t\right), x \in X, t > 0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $\Gamma : \Omega \to \Omega$ by $\Gamma g(x) = \delta_i^2 g(\delta_i x)$, for all $x \in X$. For $g, h \in \Omega$, we have $d(g,h) \leq K$

$$\begin{aligned}
\Rightarrow \quad & P_{\mu,\nu}\left(g(x) - h(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(K\beta(x), t\right) \\
\Rightarrow \quad & P_{\mu,\nu}\left(\delta_i^2 g(\delta_i x) - \delta_i^2 h(\delta_i x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(K\beta(\delta_i x), \frac{t}{\delta_i^2}\right) \\
\Rightarrow \quad & P_{\mu,\nu}\left(\Gamma g(x) - \Gamma h(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(KL\beta(x), t\right) \\
\Rightarrow \quad & d\left(\Gamma g(x), \Gamma h(x)\right) \leq KL \\
\Rightarrow \quad & d\left(\Gamma g, \Gamma h\right) \leq L d(g,h)
\end{aligned} \tag{4.7}$$

for all $g, h \in \Omega$. Therefore, Γ is strictly contractive mapping on Ω with Lipschitz constant L. Replacing (x, y) by (x, x) in (4.3), we get

$$P_{\mu,\nu}\left(f(3x) - \frac{f(x)}{9}, t\right) \geq_{L^*} P'_{\xi,\zeta}\left(x, x, t\right) \tag{4.8}$$

for all $x \in X, t > 0$. Using (IRN3) in (4.8), we arrive

$$P_{\mu,\nu}\left(9f(3x) - f(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(x, x, \frac{t}{9}\right) \tag{4.9}$$

for all $x \in X, t > 0$, with the help of (4.5) when $i = 0$, it follows from (4.8), we get

$$\begin{aligned}
\Rightarrow \quad & P_{\mu,\nu}\left(9f(3x) - f(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(L\beta(x), t\right) \\
\Rightarrow \quad & d(\Gamma f, f) \leq L = L^1 = L^{1-i}.
\end{aligned} \tag{4.10}$$

Replacing x by $\frac{x}{3}$ in (4.8) and using (IRN3), we obtain

$$P_{\mu,\nu}\left(f(x) - \frac{1}{9}f\left(\frac{x}{3}\right), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{x}{3}, \frac{x}{3}, t\right) \tag{4.11}$$

for all $x \in X, t > 0$, with the help of (4.5) when $i = 1$, it follows from (4.11) we get

$$\begin{aligned}
& P_{\mu,\nu}\left(f(x) - \frac{1}{9}f\left(\frac{x}{3}\right), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\beta(x), t\right) \\
\Rightarrow \quad & d(f, \Gamma f) \leq 1 = L^0 = L^{1-i}
\end{aligned} \tag{4.12}$$

for all $x \in X, t > 0$. Then, from (4.10) and (4.12) we can conclude,

$$d(f, \Gamma f) \leq L^{1-i} < \infty.$$

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point R of Γ in Ω such that

$$\lim_{n\to\infty} P_{\mu,\nu}\left(\delta_i^{2n} f\left(\delta_i^n x\right) - R(x), t\right) \to 1_{L^*}, \qquad \forall x \in X, t > 0. \tag{4.13}$$

Replacing (x, y) by $(\delta_i x, \delta_i y)$ in (4.3), we arrive

$$P_{\mu,\nu}\left(\delta_i^{2n} \Delta(\delta_i x, \delta_i y), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\delta_i x, \delta_i y, \frac{t}{\delta_i^{2n}}\right) \tag{4.14}$$

for all $x, y \in X$ and $t > 0$.

By proceeding the same procedure as in the Theorem 3.1, we can prove the function, $R : X \to Y$ satisfies the functional equation (1.2).

By fixed point alternative, since R is unique fixed point of Γ in the set

$$\nabla = \{f \in \Omega | d(f, Q) < \infty\},$$

therefore, R is a uniqe function such that

$$P_{\mu,\nu}\left(f(x) - R(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(K\beta(x), t\right) \tag{4.15}$$

for all $x \in X, t > 0$ and $K > 0$. Again using the fixed point alternative, we obtain

$$\begin{aligned} & d(f,R) \leq \frac{1}{1-L} d(f, \Gamma f) \\ \Rightarrow \quad & d(f,R) \leq \frac{L^{1-i}}{1-L} \\ \Rightarrow \quad & P_{\mu,\nu}\left(f(x) - R(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{L^{1-i}}{1-L}\beta(x), t\right) \end{aligned} \tag{4.16}$$

for all $x \in X$ and $t > 0$. This completes the proof. □

From Theorem 4.4, we obtain the following corollary concerning the stability for the functional equation (1.2).

Corollary 4.2. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$P_{\mu,\nu}\left(\Delta(x,y), t\right) \geq_{L^*} \begin{cases} P'_{\xi,\zeta}\left(\epsilon, t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||x||^s + ||y||^s\right), t\right); \\ P'_{\xi,\zeta}\left(\epsilon ||x||^s ||y||^s, t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}\right), t\right); \end{cases} \tag{4.17}$$

for all $x, y \in X$ and $t > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic reciprocal mapping $R : X \to Y$ such that

$$P_{\mu,\nu}\left(f(x) - R(x), t\right) \geq_{L^*} \begin{cases} P'_{\xi,\zeta}\left(\left|\frac{9}{8}\right| \epsilon, t\right); \\ P'_{\xi,\zeta}\left(\frac{18\epsilon}{|3^{s+2}-1|} ||x||^s, t\right), & s < -2 \ \ or \ \ s > -2; \\ P'_{\xi,\zeta}\left(\frac{9\epsilon}{|3^{2s+2}-1|} ||x||^{2s}, t\right), & s < -1 \ \ or \ \ s > -1; \\ P'_{\xi,\zeta}\left(\frac{27\epsilon}{|3^{2s+2}-1|} ||x||^{2s}, t\right), & s < -1 \ \ or \ \ s > -1; \end{cases} \tag{4.18}$$

for all $x \in X$ and all $t > 0$.

Proof. Setting

$$P'_{\xi,\zeta}(x,y,t) = \begin{cases} P'_{\xi,\zeta}\left(\epsilon, t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||x||^s + ||y||^s\right), t\right); \\ P'_{\xi,\zeta}\left(\epsilon ||x||^s ||y||^s, t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}\right), t\right); \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then,

$$P'_{\xi,\zeta}\left(\delta_i^k x, \delta_i^k y, \frac{t}{\delta_i^{2k}}\right) = \begin{cases} P'_{\xi,\zeta}\left(\epsilon, \frac{t}{\delta_i^{2k}}\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||\delta_i^k x||^s + ||\delta_i^k y||^s\right), \frac{t}{\delta_i^{2k}}\right); \\ P'_{\xi,\zeta}\left(\epsilon||\delta_i^k x||^s||\delta_i^k y||^s, \frac{t}{\delta_i^{2k}}\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||\delta_i^k x||^s||\delta_i^k y||^s + ||\delta_i^k x||^{2s} + ||\delta_i^k y||^{2s}\right), \frac{t}{\delta_i^{2k}}\right); \end{cases}$$

$$= \begin{cases} P'_{\xi,\zeta}\left(\epsilon, \delta_i^{-2k} t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||x||^s + ||y||^s\right), \delta_i^{-(2+s)k} t\right); \\ P'_{\xi,\zeta}\left(\epsilon||x||^s||y||^s, \delta_i^{-(2+2s)k} t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(||x||^s||y||^s + ||x||^{2s} + ||y||^{2s}\right), \delta_i^{-(2+2s)k} t\right); \end{cases}$$

$$= \begin{cases} \to 1_{L^*} \text{ as } k \to \infty; \\ \to 1_{L^*} \text{ as } k \to \infty; \\ \to 1_{L^*} \text{ as } k \to \infty; \\ \to 1_{L^*} \text{ as } k \to \infty. \end{cases}$$

Thus, (4.1) is holds. But we have $\beta(x) = \left(\frac{x}{3}, \frac{x}{3}\right)$ has the property

$$P'_{\xi,\zeta}\left(\delta_i^2 \beta(\delta_i x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\beta(x), t\right), \ \forall\, x \in X, t > 0.$$

Hence,

$$P'_{\xi,\zeta}\left(\beta(x), t\right) = P'_{\xi,\zeta}\left(\frac{x}{3}, \frac{x}{3}, t\right) = \begin{cases} P'_{\xi,\zeta}\left(\epsilon, t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(\left\|\frac{x}{3}\right\|^s + \left\|\frac{x}{3}\right\|^s\right), t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left\|\frac{x}{3}\right\|^s\left\|\frac{x}{3}\right\|^s, t\right); \\ P'_{\xi,\zeta}\left(\epsilon\left(\left\|\frac{x}{3}\right\|^s\left\|\frac{x}{3}\right\|^s + \left\|\frac{x}{3}\right\|^{2s} + \left\|\frac{x}{3}\right\|^{2s}\right), t\right); \end{cases}$$

$$= \begin{cases} P'_{\xi,\zeta}\left(\epsilon, t\right); \\ P'_{\xi,\zeta}\left(\frac{2\epsilon}{3^s}||x||^s, t\right); \\ P'_{\xi,\zeta}\left(\frac{\epsilon}{3^{2s}}||x||^{2s}, t\right); \\ P'_{\xi,\zeta}\left(\frac{3\epsilon}{3^{2s}}||x||^{2s}, t\right). \end{cases}$$

for all $x \in X$ and $t > 0$. Now,

$$P'_{\xi,\zeta}\left(\delta_i^2 \beta(\delta_i x), t\right) = \begin{cases} P'_{\xi,\zeta}\left(\delta_i^2 \epsilon, t\right); \\ P'_{\xi,\zeta}\left(\frac{2\epsilon}{3^s}\delta_i^2||\delta_i x||^s, t\right); \\ P'_{\xi,\zeta}\left(\frac{\epsilon}{3^{2s}}\delta_i^2||\delta_i x||^{2s}, t\right); \\ P'_{\xi,\zeta}\left(\frac{3\epsilon}{3^{2s}}\delta_i^2||\delta_i x||^{2s}, t\right); \end{cases} = \begin{cases} P'_{\xi,\zeta}\left(\delta_i^2 \epsilon, t\right); \\ P'_{\xi,\zeta}\left(\frac{2\epsilon}{3^s}\delta_i^{2+s}||x||^s, t\right); \\ P'_{\xi,\zeta}\left(\frac{\epsilon}{3^{2s}}\delta_i^{2+2s}||x||^{2s}, t\right); \\ P'_{\xi,\zeta}\left(\frac{3\epsilon}{3^{2s}}\delta_i^{2+2s}||x||^{2s}, t\right); \end{cases}$$

$$= \begin{cases} P'_{\xi,\zeta}\left(\delta_i^2 \beta(x), t\right); \\ P'_{\xi,\zeta}\left(\delta_i^{2+s} \beta(x), t\right); \\ P'_{\xi,\zeta}\left(\delta_i^{2+2s} \beta(x), t\right); \\ P'_{\xi,\zeta}\left(\delta_i^{2+2s} \beta(x), t\right), \end{cases}$$

for all $x \in X$ and $t > 0$. Now, from (4.6), we prove the following cases:
Case:1 $L = 3^2$ if $i = 0$;

$$P_{\mu,\nu}\left(f(x) - R(x), r\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{L}{1-L}\beta(x), t\right) = P'_{\xi,\zeta}\left(\frac{-9}{8}\epsilon, t\right).$$

Case:2 $L = 3^{-2}$ if $i = 1$;

$$P_{\mu,\nu}\left(f(x) - R(x), r\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{1}{1-L}\beta(x), 2t\right) = P'_{\xi,\zeta}\left(\frac{9}{8}\epsilon, t\right).$$

Case:3 $L = 3^{s+2}$ for $s < -2$ if $i = 0$;

$$P_{\mu,\nu}\left(f(x) - R(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\frac{3^{s+2}\epsilon}{1-3^{s+2}}\beta(x)||x||^s, t\right) = P'_{\xi,\zeta}\left(\frac{18\epsilon}{1-3^{s+2}}||x||^s, t\right).$$

Case:4 $L = 3^{-s-2}$ for $s > -2$ if $i = 1$;

$$P_{\mu,\nu}\left(f(x) - R(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\left(\frac{\epsilon}{1-3^{-s-2}}\right)\beta(x)||x||^s, t\right) = P'_{\xi,\zeta}\left(\frac{18\epsilon}{3^{s+2}-1}||x||^s, t\right).$$

Case:5 $L = 3^{2s+2}$ for $s < -1$ if $i = 0$;

$$P_{\mu,\nu}\left(f(x) - R(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\left(\frac{3^{2s+2}\epsilon}{1-3^{2s+2}}\right)\beta(x)||x||^{2s}, t\right) = P'_{\xi,\zeta}\left(\frac{9\epsilon}{1-3^{2s+2}}||x||^{2s}, t\right).$$

Case:6 $L = 3^{-2s-2}$ for $s > -1$ if $i = 1$;

$$P_{\mu,\nu}\left(f(x) - R(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\left(\frac{\epsilon}{1-3^{-2s-2}}\right)\beta(x)||x||^{2s}, t\right) = P'_{\xi,\zeta}\left(\frac{9\epsilon}{3^{2s+2}-1}||x||^{2s}, t\right).$$

Hence complete the proof. □

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On a new subclass of bi-univalent functions of Sakaguchi type satisfying subordinate conditions

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Abstract

In this paper, we introduce and investigate a new subclass of the function class Σ of bi-univalent functions defined in the open unit disk. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass.

Keywords: Bi-univalent functions; Sakaguchi functions; coefficient bounds; subordination.

2010 MSC: 30C45, 30C50.

1 Introduction and Definitions

Let A denote the class of analytic functions in the unit disc

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The Koebe one-quarter theorem [5] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z\,, \ \ (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w\,, \ \left(|w| < r_0(f)\ , \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as

$$f(z) \prec g(z)\,, \qquad (z \in U)$$

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if there exists a Schwarz function $w(z)$, analytic in U, with

$$w(0) = 0 \text{ and } |w(z)| < 1, \qquad (z \in U)$$

such that

$$f(z) = g(w(z)) \qquad (z \in U).$$

Let Σ denote the class of bi-univalent functions defined in the unit disc U. For a brief history and interesting examples in the class Σ, (see [14]). The research into Σ was started by Lewin ([10]). It focused on problems connected with coefficients and obtained the bound for the second coefficient. Several authors have subsequently studied similar problems in this direction (see [4], [12]). Recently, Srivastava et al. [14] introduced and investigated subclasses of the bi-univalent functions and obtained bounds for the initial coefficients; it was followed by such works as those by Frasin and Aouf [6] and others (see, for example, [1], [3], [9], [11], [15]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [7], [8]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, ...\}$) is still an open problem.

Motivated by the earlier work of Sakaguchi [13] on the class of starlike functions with respect to symmetric points denoted by S_S consisting of functions $f \in A$ satisfy the condition $Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0$, $(z \in U)$, we introduce a new subclass of the function class Σ of bi-univalent functions, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass.

2 Coefficient Estimates

In the following, let ϕ be an analytic function with positive real part in U, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{2.2}$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \;\; v(w) = c_1 w + \sum_{n=2}^{\infty} c_n w^n \quad (|z| < 1, |w| < 1). \tag{2.3}$$

It is well known that

$$|b_1| \leq 1, \;\; |b_2| \leq 1 - |b_1|^2, \;\; |c_1| \leq 1, \;\; |c_2| \leq 1 - |c_1|^2. \tag{2.4}$$

Next, the equations (2.2) and (2.3) lead to

$$\phi(u(z)) = 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + \cdots, \;\; |z| < 1 \tag{2.5}$$

and

$$\phi(v(w)) = 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + \cdots, \;\; |w| < 1. \tag{2.6}$$

Definition 2.1. *A function $f \in \Sigma$ is said to be in the class $S_\Sigma(\phi, s, t)$, if the following subordination hold*

$$\frac{(s-t)zf'(z)}{f(sz) - f(tz)} \prec \phi(z)$$

and

$$\frac{(s-t)wg'(w)}{g(sw)-g(tw)} \prec \phi(w)$$

where $g(w)=f^{-1}(w)$, $s,t\in\mathbb{C}$ *with* $s\neq t$, $|t|\leq 1$.

Theorem 2.1. *Let* f *given by (1.1) be in the class* $S_{\Sigma}(\phi,s,t)$. *Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|(3-2s-2t+st)B_1^2-(2-s-t)^2B_2\right|+|2-s-t|^2B_1}} \tag{2.7}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{|3-s^2-t^2-st|}; & if \;\; B_1 \leq \dfrac{|2-s-t|^2}{|3-s^2-t^2-st|} \\ \dfrac{\left|(3-2s-2t+st)B_1^2-(2-s-t)^2B_2\right|B_1+\left|3-s^2-t^2-st\right|B_1^3}{\left|3-s^2-t^2-st\right|\left[\left|(3-2s-2t+st)B_1^2-(2-s-t)^2B_2\right|+|2-s-t|^2B_1\right]}; & . \\ & if \;\; B_1 > \dfrac{|2-s-t|^2}{|3-s^2-t^2-st|} \end{cases} \tag{2.8}$$

Proof. Let $f\in S_{\Sigma}(\phi,s,t)$. Then, there are analytic functions $u,v:U\to U$ given by (2.3) such that

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = \phi(u(z)) \tag{2.9}$$

and

$$\frac{(s-t)wg'(w)}{g(sw)-g(tw)} = \phi(v(w)) \tag{2.10}$$

where $g(w)=f^{-1}(w)$. Since

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} =$$

$$1+(2-s-t)a_2z+\left[\left(3-s^2-t^2-st\right)a_3-\left(2s+2t-s^2-t^2-2st\right)a_2^2\right]z^2+\cdots$$

and

$$\frac{(s-t)wg'(w)}{g(sw)-g(tw)} =$$

$$1-(2-s-t)a_2w+\left[\left(6-s^2-t^2-2s-2t\right)a_2^2-\left(3-s^2-t^2-st\right)a_3\right]w^2+\cdots,$$

it follows from (2.5), (2.6), (2.9) and (2.10) that

$$(2-s-t)a_2 = B_1b_1, \tag{2.11}$$

$$\left(3-s^2-t^2-st\right)a_3-\left(2s+2t-s^2-t^2-2st\right)a_2^2 = B_1b_2+B_2b_1^2, \tag{2.12}$$

and

$$-(2-s-t)a_2 = B_1c_1, \tag{2.13}$$

$$\left(6-s^2-t^2-2s-2t\right)a_2^2-\left(3-s^2-t^2-st\right)a_3 = B_1c_2+B_2c_1^2. \tag{2.14}$$

From (2.11) and (2.13) we obtain

$$c_1 = -b_1. \tag{2.15}$$

By adding (2.14) to (2.12), further computations using (2.11) to (2.15) lead to

$$\left[2\left(3-2s-2t+st\right)B_1^2-2\left(2-s-t\right)^2B_2\right]a_2^2=B_1^3\left(b_2+c_2\right). \tag{2.16}$$

(2.15) and (2.16), together with (2.4), we find that

$$\left|\left(3-2s-2t+st\right)B_1^2-\left(2-s-t\right)^2B_2\right|\left|a_2\right|^2\leq B_1^3\left(1-\left|b_1\right|^2\right). \tag{2.17}$$

which gives us the desired estimate on $|a_2|$ as asserted in (2.7).

Next, in order to find the bound on $|a_3|$, by subtracting (2.14) from (2.12), we obtain

$$2\left(3-s^2-t^2-st\right)a_3-2\left(3-s^2-t^2-st\right)a_2^2=B_1\left(b_2-c_2\right)+B_2\left(b_1^2-c_1^2\right). \tag{2.18}$$

Then, in view of (2.4) and (2.15) , we have

$$\left|3-s^2-t^2-st\right|B_1\left|a_3\right|\leq\left[\left|3-s^2-t^2-st\right|B_1-\left|2-s-t\right|\right]\left|a_2\right|^2+B_1^2.$$

Notice that (2.7), we get the desired estimate on $|a_3|$ as asserted in (2.8). □

Corollary 1. *If we let*

$$\phi\left(z\right)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2z^2+...\quad\left(0<\alpha\leq1\right),$$

then inequalities (2.7) and (2.8) become

$$\left|a_2\right|\leq\frac{2\alpha}{\sqrt{\left|2\left(3-2s-2t+st\right)-\left(2-s-t\right)^2\right|\alpha+\left|2-s-t\right|^2}}$$

and

$$\left|a_3\right|\leq\begin{cases}\dfrac{2\alpha}{\left|3-s^2-t^2-st\right|}; & if\ \ 0<\alpha\leq\dfrac{\left|2-s-t\right|^2}{2\left|3-s^2-t^2-st\right|}\\[2ex] \dfrac{2\left[\left|2(3-2s-2t+st)-(2-s-t)^2\right|+2\left|3-s^2-t^2-st\right|\right]\alpha^2}{\left|3-s^2-t^2-st\right|\left[\left|2(3-2s-2t+st)-(2-s-t)^2\right|\alpha+\left|2-s-t\right|^2\right]}; & \\[2ex] & if\ \ \dfrac{\left|2-s-t\right|^2}{2\left|3-s^2-t^2-st\right|}<\alpha\leq1.\end{cases}.$$

Corollary 2. *If we let*

$$\phi\left(z\right)=\frac{1+\left(1-2\alpha\right)z}{1-z}=1+2\left(1-\alpha\right)z+2\left(1-\alpha\right)z^2+\cdots\quad\left(0\leq\alpha<1\right),$$

then inequalities (2.7) and (2.8) become

$$\left|a_2\right|\leq\frac{2\left(1-\alpha\right)}{\sqrt{\left|2\left(3-2s-2t+st\right)\left(1-\alpha\right)-\left(2-s-t\right)^2\right|+\left|2-s-t\right|^2}}$$

and

$$\left|a_3\right|\leq\begin{cases}\dfrac{2\left(1-\alpha\right)}{\left|3-s^2-t^2-st\right|}; & if\ \ \dfrac{2\left|3-s^2-t^2-st\right|-\left|2-s-t\right|^2}{2\left|3-s^2-t^2-st\right|}\leq\alpha<1\\[2ex] \dfrac{2\left[\left|2(3-2s-2t+st)(1-\alpha)-(2-s-t)^2\right|+2\left|3-s^2-t^2-st\right|(1-\alpha)\right](1-\alpha)}{\left|3-s^2-t^2-st\right|\left[\left|2(3-2s-2t+st)(1-\alpha)-(2-s-t)^2\right|+\left|2-s-t\right|^2\right]}; & \\[2ex] & if\ \ 0\leq\alpha<\dfrac{2\left|3-s^2-t^2-st\right|-\left|2-s-t\right|^2}{2\left|3-s^2-t^2-st\right|}\end{cases}.$$

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Common Fixed Point of Four Mapping With Contractive Modulus on Cone Banach Space

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Abstract

In this paper, we have proved the existence of unique common fixed point of four contractive maps on cone Banach space through an upper semi continuous contractive modulus and weakly compatible maps.

Keywords: Cone Banach Space, Common Fixed Point, Contractive Modulus, Weakly Compatible.

2010 MSC: 65D30, 65D32.

1 Introduction

The notion of cone metric space is initiated by Huang and Zhang [3] and also they discussed some properties of the convergence of sequences and proved the fixed point theorems of a contraction mapping for cone metric spaces; Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$ has a unique fixed point. Some fixed theorems in cone Banach space are proved by Karapinar[5].

In this paper, we investigate the common fixed point theorems with the assumption of weakly compatible and coincidence point of four maps on an upper semi continuous contractive modulus in cone Banach space

Definition 1.1. *Let E be the real Banach space. A subset P of E is called a cone if and only if:*

i. *P is closed, non empty and $P \neq 0$*

ii. *$ax + by \in P$ for all $x, y \in P$ and non negative real numbers a, b*

iii. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

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Example 1.1. *[12] Let $K > 1$. be given. Consider the real vector space with*

$$E = \{ax + b : a, b \in R; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. The cone P is regular and so normal.

Definition 1.2. *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

i. $d(x,y) \geq 0$, *and* $d(x,y) = 0$ *if and only if* $x = y$ $\forall x, y \in X$,

ii. $d(x,y) = d(y,x)$, $\forall x, y \in X$,

iii. $d(x,y) \leq d(x,z) + d(z,y)$, $\forall x, y, z \in X$,

Then (X, d) is called a cone metric space (CMS).

Example 1.2. *Let $E = R^2$*

$$P = \{(x,y) : x, y \geq 0\}$$

$X = R$ and $d : X \times X \to E$ such that

$$d(x,y) = (|x - y|, \alpha|x - y|)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.3. *[5] Let X be a vector space over R. Suppose the mapping $\|.\|_c : X \to E$ satisfies*

i. $\|x\|_c \geq 0$ *for all* $x \in X$,

ii. $\|x\|_c = 0$ *if and only if* $x = 0$,

iii. $\|x + y\|_c \leq \|x\|_c + \|y\|_c$ *for all* $x, y \in X$,

iv. $\|kx\|_c = |k|\|x\|_c$ *for all $k \in R$ and for all $x \in X$, then $\|.\|_c$ is called a cone norm on X, and the pair $(X, \|.\|_c)$ is called a cone normed space* (CNS).

Remark 1.1. *Each Cone normed space is Cone metric space with metric defined by*
$d(x,y) = \|x - y\|_c$

Example 1.3. *Let $X = R^2$, $P = \{(x,y) : x \geq 0, y \geq 0\} \subset R^2$ and $\|(x,y)\|_c = (a|x|, b|y|), a > 0, b > 0$. Then $(X, \|.\|_c)$ is a cone normed space over R^2*

Definition 1.4. *Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n \geq 0}$ be a sequence in X. Then $\{x_n\}_{n \geq 0}$ converges to x whenever for every $c \in E$ with $0 \ll E$, there is a natural number $N \in N$ such that $\|x_n - x\|_c \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$*

Definition 1.5. *Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n \geq 0}$ be a sequence in X. $\{x_n\}_{n \geq 0}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in N$, such that $\|x_n - x_m\|_c \ll c$ for all $n, m \geq N$*

Definition 1.6. *Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n \geq 0}$ be a sequence in X. $(X, \|.\|_c)$ is a complete cone normed space if every Cauchy sequence is convergent. Complete cone normed spaces will be called cone Banach spaces.*

Lemma 1.1. *[5] Let $(X, \|.\|_c)$ be a CNS, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then*

i. the sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_c \to 0$ as $n \to \infty$,

ii. *the sequence* $\{x_n\}$ *is Cauchy if and only if* $\|x_n - x_m\|_c \to 0$ *as* $n, m \to \infty$,

iii. *the sequence* $\{x_n\}$ *converges to* x *and the sequence* $\{y_n\}$ *converges to* y, *then* $\|x_n - y_n\|_c \to \|x - y\|_c$.

Definition 1.7. *Let* f *and* g *be two self maps defined on a set* X *maps* f *and* g *are said to be commuting of* $fgx = gfx$ *for all* $x \in X$

Definition 1.8. *Let* f *and* g *be two self maps defined on a set* X *maps* f *and* g *are said to be weakly compatible if they commute at coincidence points. that is if* $fx = gx$ *forall* $x \in X$ *then* $fgx = gfx$

Definition 1.9. *Let* f *and* g *be two self maps on set* X. *If* $fx = gx$, *for some* $x \in X$ *then* x *is called coincidence point of* f *and* g

Lemma 1.2. *Let* f *and* g *be weakly compatible self mapping of a set* X. *If* f *and* g *have a unique point of coincidence, that is* $w = fx = gx$ *then* w *is the unique common fixed point of* f *and* g.

2 Main Result

Theorem 2.1. *Let* $(X, \|.\|_c)$ *be a Cone Banch space with the norm* $d(x, 0) = \|x\|_c$. *Suppose that the mappings* P, Q, S *and* T *are four self maps of* $(X, \|.\|_c)$ *such that* $T(X) \subseteq P(X)$ *and* $S(X) \subseteq Q(X)$ *and satisfying*

$$\|Ty - Sx\|_c \leq a\|Px - Qy\|_c + b\{\|Px - Sx\|_c + \|Qy - Ty\|_c\} + c\{\|Px - Ty\|_c + \|Qy - Sx\|_c\} \tag{2.1}$$

for all $x, y \in X$, *where* $a, b, c \geq 0$ *and* $a + 2b + 2c < 1$. *suppose that the pairs* $\{P, S\}$ *and* $\{Q, T\}$ *are weakly compatible, then* P, Q, S *and* T *have a unique common fixed point.*

Proof. Suppose x_0 is an arbitrary initial point of X and define the sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$

$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (2.1) implies that

$$\begin{aligned}
\|y_{2n+1} - y_{2n}\|_c &= \|Tx_{2n+1} - Sx_{2n}\|_c \\
&\leq a\|Px_{2n} - Qx_{2n+1}\|_c + b\{\|Px_{2n} - Sx_{2n}\|_c + \|Qx_{2n} - Tx_{2n+1}\|_c\} \\
&\quad + c\{\|Px_{2n} - Tx_{2n+1}\|_c + \|Qx_{2n+1} - Sx_{2n}\|_c\} \\
&\leq a\|y_{2n-1} - y_{2n}\|_c + b\{\|y_{2n-1} - y_{2n}\|_c + \|y_{2n} - y_{2n+1}\|_c\} \\
&\quad + c\{\|y_{2n-1} - y_{2n+1}\|_c + \|y_{2n} - y_{2n}\|_c\} \\
&\leq a\|y_{2n-1} - y_{2n}\|_c + b\{\|y_{2n-1} - y_{2n}\|_c + \|y_{2n} - y_{2n+1}\|_c\} \\
&\quad + c\|y_{2n-1} - y_{2n+1}\|_c \\
&\leq (a + b + c)\|y_{2n-1} - y_{2n}\|_c + (b + c)\|y_{2n} - y_{2n+1}\|c \\
\|y_{2n+1} - y_{2n}\|_c &\leq \frac{a + b + c}{1 - (b + c)}\|y_{2n} - y_{2n-1}\|_c \\
\|y_{2n+1} - y_{2n}\|_c &\leq h\|y_{2n} - y_{2n-1}\|_c
\end{aligned}$$

where $h = \frac{a+b+c}{1-(b+c)} < 1$ for all $n \in N$

$$\begin{aligned}
\|y_{2n} - y_{2n+1}\|_c &\leq h\|y_{2n-1} - y_{2n}\|_c \\
&\leq h^2\|y_{2n-2} - y_{2n-1}\|_c \\
&\vdots \\
&\leq h^{2n-1}\|y_0 - y_1\|_c
\end{aligned}$$

For all $m > n$

$$\begin{aligned}\|y_n - y_m\|_c &\le \|y_n - y_{n+1}\|_c + \|y_{n+1} - y_{n+2}\|_c + \cdots + \|y_{m-1} - y_m\|_c \\ &\le (h^n + h^{n+1} + \cdots h^{m-1})\|y_0 - y_1\|_c \\ &\le h^n(1 + h + h^2 + \cdots + h^{m-1-n})\|y_0 - y_1\|_c \\ &\le \frac{h^n}{1-h}\|y_0 - y_1\|_c\end{aligned}$$

$\Rightarrow \ \|y_n - y_m\|_c \ll 0$ as $n, m \to \infty$.

Hence $\{y_n\}$ is a Cauchy sequence.

There exists a point l in $(X, \|.\|_c)$ such that
$\lim_{n\to\infty}\{y_n\} = l$, $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Q_{2n+1} = l$ and $\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = l$

that is,

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Q_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = x^*$$

Since $T(X) \subseteq P(X)$, there exists a point z in X Such that $x^* = Pz$ then by (1)

$$\begin{aligned}\|Sz - x^*\|_c &\le \|Sz - Tx_{2n-1}\|_c + \|Tx_{2n-1} - x^*\|_c \\ &\le a\|Pz - Qx_{2n-1}\|_c + b\{\|Pz - Sz\|_c + \|Qx_{2n-1} - Tx_{2n-1}\|_c\} \\ &\quad + c\{\|Pz - Tx_{2n-1}\|_c + \|Qx_{2n-1} - Sz\|_c\} + \|Tx_{2n-1} - x^*\|_c\end{aligned}$$

Taking the limit as $n \to \infty$

$$\begin{aligned}\|Sz - x^*\|_c &\le a\|x^* - x^*\|_c + b\{\|x^* - x^*\|_c + \|x^* - Sz\|_c\} \\ &\quad + c\{\|x^* - x^*\|_c + \|x^* - Sz\|_c\} + \|x^* - x^*\|_c \\ &\le 0 + b\{\|x^* - Sz\|_c + 0\} + c\{0 + \|x^* - Sz\|_c\} + 0 + (b+c)\|x^* - Sz\|_c\end{aligned}$$

Which is a contraction since $a + 2b + 2c < 1$.

$$\text{therefore} \quad Sz = Pz = x^*$$

Since $S(X) \subseteq Q(X)$ there exists a point $w \in X$ such that $x^* = Qw$.
by (1)

$$\begin{aligned}\|Sz - x^*\|_c &\le \|Sz - Tw\|_c \\ &\le a\|Pz - Qw\|_c + b\{\|Pz - Sz\|_c + \|Qw - Tw\|_c\} + c\{\|Pz - Tw\|_c + \|Qw - Sw\|_c\} \\ &\le a\|x^* - x^*\|_c + b\{\|x^* - x^*\|_c + \|x^* - Tw\|_c\} + c\{\|x^* - Tw\|_c + \|x^* - x^*\|_c\} \\ &\le 0 + +b\{0 + \|x^* - Tw\|_c\} + c\{\|x^* - Tw\|_c + 0\} \\ \|x^* - Tw\|_c &\le (b+c)\|x^* - Tw\|_c\end{aligned}$$

which is a contradiction since $a + 2b + 2c < 1$.

$$\text{therefore} \quad Tw = Qw = x^*$$

Thus $Sz = Pz = Tw = Qw = x^*$
Since P and S are weakly compatible maps,
Then $SP(z) = PS(z)$
$Sx^* = Px^*$

To prove that x^* is a fixed point of S

Suppose $Sx^* \neq x^*$ then by (2.1)

$$\begin{aligned}\|Sx^* - x^*\|_c &\leq \|Sx^* - Tx^*\|_c \\ &\leq a\|Px^* - Qw\|_c + b\{\|Px^* - Sx^*\|_c + \|Qw - Tw\|_c\} + \\ &\quad + c\{\|Px^* - Tw\|_c + \|Qw - Sx^*\|_c\} \\ &\leq a\|Sx^* - x^*\|_c + b\{\|Sx^* - Sx^*\|_c + \|x^* - x^*\|_c\} + \\ &\quad + c\{\|Sx^* - x^*\|_c + \|x^* - Sx^*\|_c\} \\ &\leq a\|Sx^* - x^*\|_c + b\{0+0\} + 2c\|Sx^* - x^*\|_c \\ \|Sx^* - x^*\|_c &\leq (a+2c)\|Sx^* - x^*\|_c\end{aligned}$$

Which is a contradiction, Since $a + 2b + 2c < 1$.

$$Sx^* = x^*$$

Hence $Sx^* = Px^* = x^*$ Similarly, Q and T are weakly compatible maps then $TQw = QTw$, that is $Tx^* = Qx^*$

To prove that x^* is a fixed point of T.

Suppose $Tx^* \neq x^*$ by (2.1)

$$\begin{aligned}\|Tx^* - x^*\|_c &\leq \|Sx^* - Tx^*\|_c \\ &\leq a\|Px^* - Qx^*\|_c + b\{\|Px^* - Sx^*\|_c + \|Qx^* - Tx^*\|_c\} + \\ &\quad + c\{\|Px^* - Tx^*\|_c + \|Qx^* - Sx^*\|_c\} \\ &\leq a\|x^* - Tx^*\|_c + b\{\|x^* - x^*\|_c + \|Tx^* - Tx^*\|_c\} + \\ &\quad + c\{\|x^* - Tx^*\|_c + \|Tx^* - x^*\|_c\} \\ &\leq a\|Tx^* - x^*\|_c + b\{0+0\} + 2c\|Tx^* - x^*\|_c \\ \|Tx^* - x^*\|_c &\leq (a+2c)\|Tx^* - x^*\|_c\end{aligned}$$

which is a contradiction since $a + 2b + 2c < 1$.

$$Tx^* = x^*.$$

Hence. $Tx^* = Qx^* = x^*$

Thus $Sx^* = Px^* = Tx^* = Qx^* = x^*$

That is, x^* is a common fixed point of P, Q, S and T

To prove that the uniqueness of x^*

Suppose that x^* and y^*, $x^* \neq y^*$ are common fixed points of P, Q, S and T respectively, by (2.1) we have,

$$\begin{aligned}\|x^* - y^*\|_c &\leq \|Sx^* - Ty^*\|_c \\ &\leq a\|Px^* - Qy^*\|_c + b\{\|Px^* - Sx^*\|_c + \|Qy^* - Ty^*\|_c\} + \\ &\quad + c\{\|Px^* - Ty^*\|_c + \|Qy^* - Sx^*\|_c\} \\ &\leq a\|x^* - y^*\|_c + b\{\|x^* - x^*\|_c + \|y^* - y^*\|_c\} + c\{\|x^* - y^*\|_c + \|y^* - x^*\|_c\} \\ &\leq a\|x^* - y^*\|_c + b\{0+0\} + c\{\|x^* - y^*\|_c + \|y^* - x^*\|_c\} \\ &\leq (a+2c)\|x^* - y^*\|_c\end{aligned}$$

which is a contradiction. Since $a + 2b + 2c < 1$.

$$\text{therefore } \; x^* = y^*.$$

Hence x^* is the unique common fixed point of P, Q, S and T respectively. □

Corollary 2.1. *Let $(X, \|.\|_c)$ be a Cone Banach space with the norm $d(x, 0) = \|x\|_c$.. Suppose that the mappings P, S and T are three self maps of $(X, \|.\|_c)$ such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ and satisfying*

$$\|Sx - Ty\|_c \leq a\|Px - Py\|_c + b\{\|Px - Sy\|_c + \|Px - Ty\|_c\} + c\{\|Px - Ty\|_c + \|Py - Sx\|_c\}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + 2b + 2c < 1$. suppose that the pairs $\{P, S\}$ and $\{P, T\}$ are weakly compatible, then P, S and T have a unique common fixed point.

Proof. The proof of the corollary immediate by taking $P = Q$ in the above theorem (2.1). □

Definition 2.10. *A function $\Phi : [0, \infty) \to [0, \infty)$ is said to be contractive modulus if $\Phi : [0, \infty) \to [0, \infty)$ and $\Phi(t) < t$ for $t > 0$*

Definition 2.11. *A real valued function Φ defined on $X \subseteq R$ is said to be upper semi continuous if $\lim_{n\to\infty} \sup \Phi(t_n) \leq \Phi(t)$, for every sequence $\{t_n\} \in X$ with $t_n \to t$ as $n \to \infty$.*

Remark 2.2. *It is clear that every continuous function is upper semi continuous but converse may not true.*

Theorem 2.2. *Let $(X, \|.\|_c)$ be a Cone Banch space with the norm $d(x, 0) = \|x\|_c$.. Suppose that the mappings P, Q, S and T are four self maps of $(X, \|.\|_c)$ such that $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$ satisfying*

$$\|Sx - Ty\|_c \leq \Phi(\lambda(x, y)), \tag{2.2}$$

where Φ is an upper semi continuous contractive modulus and

$$\lambda(x, y) = \max\{\|Px - Qy\|_c, \|Px - Sx\|_c, \|Qy - Ty\|_c, \frac{1}{2}\{\|Px - Ty\|_c + \|Qy - Sx\|_c\}\}.$$

The pair $\{S, P\}$ and $\{T, Q\}$ are weakly compatible. Then P, Q, S and T have a unique common fixed point.

Proof. Let us take x_0 is an arbitrary point of X and define a sequence $\{y_{2n}\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$

$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (2.2) implies that

$$\begin{aligned}
\|y_{2n} - y_{2n+1}\|_c &= \|Sx_{2n} - Tx_{2n+1}\|_c \\
&\leq \Phi(\lambda(x_{2n}, x_{2n+1})) \\
&\leq \lambda(x_{2n}, x_{2n+1}) \\
&= \max\{\|Px_{2n} - Qx_{2n+1}\|_c, \|Px_{2n} - Sx_{2n}\|_c, \|Qx_{2n+1} - Tx_{2n+1}\|_c, \\
&\quad \frac{1}{2}\{\|Px_{2n} - Tx_{2n+1}\|_c + \|Qx_{2n+1} - Sx_{2n}\|_c\}\} \\
&= \max\{\|Tx_{2n-1} - Sx_{2n}\|_c, \|Tx_{2n-1} - Sx_{2n}\|_c, \|Sx_{2n} - Tx_{2n+1}\|_c, \\
&\quad \frac{1}{2}\{\|Tx_{2n-1} - Tx_{2n+1}\|_c + \|Sx_{2n} - Sx_{2n}\|_c\}\} \\
&= \max\{\|Tx_{2n-1} - Sx_{2n}\|_c, \|Tx_{2n-1} - Sx_{2n}\|_c, \|Sx_{2n} - Tx_{2n+1}\|_c, \\
&\quad \frac{1}{2}\|Tx_{2n-1} - Tx_{2n+1}\|_c\} \\
&= \max\{\|y_{2n} - y_{2n-1}\|_c, \|y_{2n} - y_{2n+1}\|_c, \frac{1}{2}\|y_{2n-1} - y_{2n+1}\|_c\} \\
&\leq \max\{\|y_{2n} - y_{2n-1}\|_c, \|y_{2n} - y_{2n+1}\|_c\}
\end{aligned}$$

Since Φ is an contractive modulus, $\lambda(x_{2n}-x_{2n+1}) = \|y_{2n}-y_{2n+1}\|_c$ is not possible. Thus,

$$\|y_{2n}-y_{2n+1}\|_c \leq \Phi(\|y_{2n-1}-y_{2n}\|_c) \tag{2.3}$$

Since Φ is an upper semi continuous, contractive modulus. Equation (2.3) implies that the sequence $\{\|y_{2n+1}-y_{2n}\|_c\}$ is monotonic decreasing and continuous. There exists a real number, say $r \geq 0$ such that

$$\lim_{n\to\infty} \|y_{2n+1}-y_{2n}\|_c = r,$$

as $n \to \infty$ equation (2.3) $\Rightarrow$

$$r \leq \Phi(r)$$

which is only possible if $r = 0$ because Φ is a contractive modulus. Thus

$$\lim_{n\to\infty} \|y_{2n+1}-y_{2n}\|_c = 0.$$

Claim: $\{y_{2n}\}$ is a Cauchy sequence.
Suppose $\{y_{2n}\}$ is not a Cauchy sequence.
Then there exists an $\epsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$\|y_{m_i}-y_{n_i}\|_c \geq \epsilon \quad \text{and} \quad \|y_{m_i}-y_{n_{i-1}}\|_c \leq \epsilon \tag{2.4}$$

$$\epsilon \leq \|y_{m_i}-y_{n_i}\|_c \leq \|y_{m_i}-y_{n_{i-1}}\|_c + \|y_{n_{i-1}}-y_{n_i}\|_c$$

$$\text{therefore} \quad \lim_{i\to\infty} \|y_{m_i}-y_{n_i}\|_c = \epsilon$$

now

$$\epsilon \leq \|y_{m_{i-1}}-y_{n_{i-1}}\|_c \leq \|y_{m_{i-1}}-y_{m_i}\|_c + \|y_{m_i}-y_{n_{i-1}}\|_c$$

by taking limit $i \to \infty$ we get,

$$\lim_{i\to\infty} \|y_{m_{i-1}}-y_{n_{i-1}}\|_c = \epsilon$$

from (2.3) and (2.4)

$$\epsilon \leq \|y_{m_i}-y_{n_i}\|_c = \|Sx_{m_i}-Tx_{n_i}\|_c \leq \Phi(\lambda(x_{m_i},x_{n_i}))$$

where implies

$$\epsilon \leq \Phi(\lambda(x_{m_i},x_{n_i})) \tag{2.5}$$

$$\begin{aligned}
\lambda(x_{m_i},x_{n_i}) &= \max\{\|Px_{m_i}-Qx_{n_i}\|_c, \|Px_{m_i}-Sx_{m_i}\|_c, \|Qx_{n_i}-Tx_{n_i}\|_c, \\
&\quad \frac{1}{2}(\|Px_{m_i}-Tx_{n_i}\|_c + \|Qx_{n_i}-Sx_{m_i}\|_c)\} \\
&= \max\{\|Tx_{m_{i-1}}-Sx_{n_{i-1}}\|_c, \|Tx_{m_{i-1}}-Sx_{m_i}\|_c, \|Sx_{n_{i-1}}-Tx_{n_i}\|_c, \\
&\quad \frac{1}{2}(\|Tx_{m_{i-1}}-Tx_{n_i}\|_c + \|Sx_{n_{i-1}}-Sx_{m_i}\|_c)\} \\
&= \max\{\|y_{m_{i-1}}-y_{n_{i-1}}\|_c, \|y_{m_{i-1}}-y_{m_i}\|_c, \|y_{n_{i-1}}-y_{n_i}\|_c, \\
&\quad \frac{1}{2}(\|y_{m_{i-1}}-y_{n_i}\|_c + \|y_{n_{i-1}}-y_{m_i}\|_c)\}
\end{aligned}$$

Taking limit as $i \to \infty$, we get

$$\lim_{i\to\infty} \lambda(x_{m_i},x_{n_i}) = \max\{\epsilon, 0, 0, \frac{1}{2}(\epsilon,\epsilon)\}$$

$$\lim_{i\to\infty} \lambda(x_{m_i},x_{n_i}) = \epsilon$$

Therefore from (2.5) we have, $\epsilon \leq \Phi(\epsilon)$
This is a contraction because $\epsilon > 0$ and Φ is contractive modulus.

Therefore $\{y_{2n}\}$ is Cauchy sequence in X
There exits a point z in X such that $\lim_{n\to\infty} y_{2n} = z$
Thus,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = z \quad \text{and}$$
$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$
$$(i.e) \quad \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$

$T(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$

$$\begin{aligned}\|Su - z\|_c &\le \|Su - Tx_{2n+1}\|_c + \|Tx_{2n+1} - z\|_c \\ &\le \Phi(\lambda(u, x_{2n+1})) + \|Tx_{2n+1} - z\|_c\end{aligned}$$

where

$$\begin{aligned}\lambda(u, x_{2n+1}) &= \max\{\|Pu - Qx_{2n+1}\|_c, \|Pu - Su\|_c, \|Qx_{2n+1} - Tx_{2n+1}\|_c, \\ &\quad \frac{1}{2}(\|Pu - Tx_{2n+1}\|_c + \|Qx_{2n+1} - Su\|_c)\} \\ &= \max\{\|z - Sx_{2n}\|_c, \|z - Su\|_c, \|Sx_{2n} - Tx_{2n+1}\|_c, \\ &\quad \frac{1}{2}(\|z - Tx_{2n+1}\|_c + \|Sx_{2n} - Su\|_c)\}.\end{aligned}$$

Now taking the limit as $n \to \infty$ we have,

$$\begin{aligned}\lambda(u, x_{2n+1}) &= \max\{\|z - Su\|_c, \|z - Su\|_c, \|Su - Tu\|_c, \frac{1}{2}(\|z - Tu\|_c + \|z - Su\|_c)\} \\ &= \max\{\|z - Su\|_c, \|z - Su\|_c, \|Su - z\|_c, \frac{1}{2}(\|z - z\|_c + \|z - Su\|_c)\} \\ &= \|z - Su\|_c\end{aligned}$$

Thus

$$\begin{aligned}\|Su - z\|_c &\le \Phi(\|Su - z\|_c) + \|z - z\|_c \\ &= \Phi(\|Su - z\|_c)\end{aligned}$$

If $Su \neq z$ then $\|Su - z\|_c > 0$ and hence as Φ is contracive modulus
$\Phi(\|Su - z\|_c) < \|Su - z\|_c$ Which is a contradiction, $Su = z$ so, $Pu = Su = z$
So u is a coincidence point if P and S. The pair of maps S and P are weakly compatible $SPu = PSu$ that is $Sz = Pz$.
$S(X) \subseteq Q(X)$, there exists a point $v \in X$ such that $z = Qv$.
Then we have

$$\begin{aligned}\|z - Tv\|_c &= \|Su - Tv\|_c \\ &\le \Phi(\lambda(u, v)) \\ &\le \lambda(u, v) \\ &= \max\{\|Pu - Qv\|_c, \|Pu - Su\|_c, \|Qv - Tv\|_c, \\ &\quad \frac{1}{2}(\|Pu - Tv\|_c + \|Qv - Su\|_c)\} \\ &= \max\{\|z - z\|_c, \|z - z\|_c, \|z - Tv\|_c, \\ &\quad \frac{1}{2}(\|z - Tv\|_c + \|z - z\|_c)\} \\ &= \|z - Tv\|_c\end{aligned}$$

Thus $\|z - Tv\|_c \leq \Phi(\|z - Tv\|_c)$.
If $Tv \in z$ then $\|z - Tv\|_c \geq 0$ and hence as Φ is contractive modulus

$$\Phi(\|z - Tv\|_c) < \|z - Tv\|_c$$

Therefore $\|z - Tv\|_c < \|z - Tv\|_c$
which is a contradiction. Therefore $Tv = Qv = z$
So, v is a coincidence point of Q and T.
Since the pair of maps Q and T are weakly compatible, $QTv = TQv$
(i.e) $Qz = Tz$.
Now show that z is a fixed point of S.
We have

$$\begin{aligned}
\|Sz - z\| &= \|Sz - Tv\|_c \\
&\leq \Phi(\lambda(z, v)) \\
&\leq \lambda(z, v) \\
&= \max\{\|Pz - Qv\|_c, \|Pz - Sz\|_c, \|Qv - Tv\|_c, \frac{1}{2}(\|Pz - Tv\|_c + \|Qv - Sz\|_c)\} \\
&= \max\{\|Sz - z\|_c, \|Sz - Sz\|_c, \|z - z\|_c, \frac{1}{2}(\|Sz - z\|_c + \|z - Sz\|_c)\} \\
&= \|Sz - z\|_c
\end{aligned}$$

Thus $\|Sz - z\|_c \leq \Phi(\|Sz - z\|_c)$.
If $Sz \neq z$ then $\|Sz - z\|_c > 0$ and hence as Φ is contractive modulus $\Phi(\|Sz - z\|_c) < \|Sz - z\|_c$
which is a contradiction. There exits $Sz = z$. Hence $Sz = Pz = z$
Show that z is a fixed point of T.
We have

$$\begin{aligned}
\|z - Tz\|_c &= \|Sz - Tz\|_c \\
&\leq \Phi(\lambda(z, z)) \\
&\leq \lambda(z, z) \\
&= \max\{\|Pz - Qz\|_c, \|Pz - Sz\|_c, \|Qz - Tz\|_c, \frac{1}{2}(\|Pz - Tz\|_c + \|Qz - Sz\|_c)\} \\
&= \max\{\|z - Tz\|_c, \|z - z\|_c, \|Tz - Tz\|_c, \frac{1}{2}(\|z - Tz\|_c + \|Tz - z\|_c)\} \\
&= \|z - Tz\|_c
\end{aligned}$$

Thus $\|z - Tz\|_c \leq \Phi(\|z - Tz\|_c)$.
If $z \neq Tz$ then $\|z - Tz\|_c > 0$ and hence as Φ is contractive modulus

$$\Phi(\|z - Tz\|_c) < \|z - Tz\|_c.$$

which is a contradiction. Hence $z = Tz$.
Therefore $Tz = Qz = z$.
Therefore $Sz = Pz = Tz = Qz = z$.
That is z is common fixed point of P, Q, S and T.
Uniqueness
Suppose, z and w is $(z \neq w)$ are common fixed point of P, Q, S and T.

we have

$$\begin{aligned}
\|z-w\|_c &= \|Sz-Tw\|_c \\
&\leq \Phi(\lambda(z,w)) \\
&\leq \lambda(z,w) \\
&= \max\{\|Pz-Qw\|_c, \|Pz-Sz\|_c, \|Qw-Tw\|_c, \frac{1}{2}(\|Pz-Tw\|_c+\|Qw-Sz\|_c)\} \\
&= \max\{\|z-w\|_c, \|z-z\|_c, \|w-w\|_c, \frac{1}{2}(\|z-w\|_c+\|w-z\|_c)\} \\
&= \|z-w\|_c
\end{aligned}$$

Thus, $\|z-w\|_c \leq \Phi(\|z-w\|_c)$
Since $z \neq w$, then $\|z-w\| > 0$ and hence as Φ is contractive modulus.

$$\Phi(\|z-w\|_c) < \|z-w\|_c$$

$$\text{therefore} \quad \|z-w\|_c < \|z-w\|_c$$

which is a contradiction,

$$\text{therefore} \quad z = w$$

Thus z is the unique common fixed point of P, Q, S and T. □

Corollary 2.2. *Let $(X, \|.\|_c)$ be a Cone Banch space with the norm $d(x,0) = \|x\|_c$. Suppose that the mappings P, S and T are three self maps of $(X, \|.\|_c)$ such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ satisfying*

$$\|Sx-Ty\|_c \leq \Phi(\lambda(x,y)), \tag{2.6}$$

where Φ is an upper semi continuous contractive modulus and

$$\lambda(x,y) = \max\{\|Px-Py\|_c, \|Px-Sx\|_c, \|Py-Ty\|_c, \frac{1}{2}\{\|Px-Ty\|_c+\|Py-Sx\|_c\}\}.$$

The pair $\{S,P\}$ and $\{T,P\}$ are weakly compatible. Then P, S and T have a unique common fixed point.

Proof. The proof of the corollary immediate by taking $P = Q$ in the above theorem (2.2). □

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A comparative study of ASM and NWCR method in transportation problem

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Abstract

The transportation model is a special class of the linear programming problem. It deals with the situation in which commodity is shipped from sources to destinations. The objective is to minimize the total shipping cost while satisfying both the supply limit and the demand requirements. In this paper, a new method named ASM-method for finding an optimal solution for a transportation problem. The most attractive feature of this method is that it requires very simple arithmetical and logical calculation. So it is very easy to understand and use.

Keywords: Transportation problem, optimal solution , ASM (Assigning Shortest Minimax)-method, NWCR method.

2010 MSC: 90B06, 90C08, 68T27, 68T37.

1 Introduction

A transportation problem is one of the earliest and most important applications of linear programming problem. Description of a classical transportation problem can be given as follows. A certain amount of homogeneous commodity is available at number of sources and a fixed amount is required to meet the demand at each number of destinations. A balanced condition (i.e. Total demand is equal to total supply) is assumed. It deals with the situation in which a commodity is shipped from sources to destinations. The objective is to be determined the amounts shipped from each source to each destination that minimize the total shipping cost while satisfying both the supply limit and the demand requirements. Nowadays transportation problem has become a standard application for industrial organizations having several manufacturing units, warehouses and distribution centers [1–9].

2 Definitions

A set of non-negative values $x_{ij}, i = 1, 2, 3, \cdots ,$ and $j = 1, 2, 3, \cdots , n$ that satisfies the constraints is called a feasible solution to the transportation problem.

A feasible solution is said to be optimal if it minimizes the total transportation cost.

Optimality test can be performed if the number of allocation cells in an initial basic feasible solution = $m + n - 1\{$ No. of rows + No. of coloums - 1$\}$. Otherwise optimality test cannot be performed.

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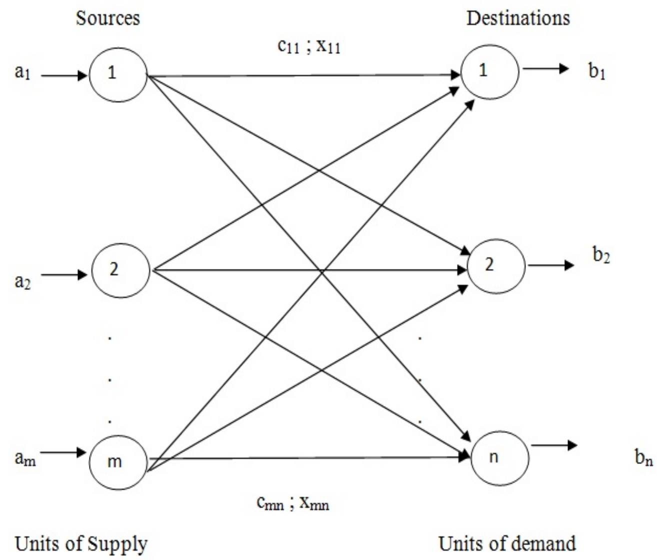
3 General Formation of Transportation Problem



4 Different Methods to Finding Optimal Solution

For finding an optimal solution for transportation problems it was required to solve the problem into two stages.

(1) In first stage Initial basic feasible solution (IBFS) was obtained by opting any of the available methods such as North West Corner, Matrix Minima, Least Cost Method, Row Minima, Column Minima and Vogels Approximation Method etc.

(2) Next and last stage MODI (Modified Distribution) method was adopted to get an optimal solution.

Here a much easier heuristic approach is proposed (ASM-Method) for finding an optimal solution directly with lesser number of iterations and very easy computations. The stepwise procedure of proposed method is carried out as follows.

4.1 ASM Method

Step 1:
Construct the transportation table from given transportation problem.
Step 2:
Subtract each row entries of the transportation table from the respective row minimum and then subtract each column entries of the resulting transportation table from respective column minimum.
Step 3:
Now there will be at least one zero in each row and in each column in the reduced cost matrix. Select the first zero (row-wise) occurring in the cost matrix. Suppose $(i,j)^{th}$ zero is selected, count the total number of zeros (excluding the selected one) in the i^{th} row and j^{th} column. Now select the next zero and count the total number of zeros in the corresponding row and column in the same manner. Continue it for all zeros in the cost matrix.
Step 4:
Now choose a zero for which the number of zeros counted in step 3 is minimum and supply maximum possible amount to that cell. If tie occurs for some zeros in step 3 then choose a $(k.l)^{th}$ zero breaking tie such that the total sum of all the elements in the k^{th} row and l^{th} column is maximum. Allocate maximum possible amount to that cell.

Step 5:

After performing step 4, delete the row or column for further calculation where the supply from a given source is depleted or the demand for a given destination is satisfied.

Step 6:

Check whether the resultant matrix possesses at least one zero in each row and in each column. If not, repeat step 2, otherwise go to step 7.

Step 7:

Repeat step 3 to step 6 until and unless all the demands are satisfied and all the supplies are exhausted.

4.2 Numerical Example

ASM Method

	1	2	3	4	Supply
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	

Solution:

Row reduced matrix

	1	2	3	4	Supply
A	0	2	6	3	250
B	6	8	4	0	300
C	11	14	3	0	400
Demand	200	225	275	250	

Column reduced matrix.

	1	2	3	4	Supply
A	0	0	3	3	250
B	6	6	1	0	300
C	11	12	0	0	400
Demand	200	225	275	250	

Using ASM method and final table

	1	2	3	4	Supply
A	200 11	50 13	17	14	250
B	16	175 18	14	125 10	300
C	21	24	275 13	125 10	400
Demand	200	225	275	250	

Transportation Cost

$$= (11 * 200) + (13 * 50) + (18 * 175) + (10 * 125) + (13 * 275) + (10 * 125)$$
$$= 2200 + 650 + 3150 + 1250 + 3575 + 1250$$
$$= Rs.12075$$

4.3 Optimality Check

To find initial basic feasible solution for the above example North West Corner Rule (NWCR) Mmethod is used and allocations are obtained as follows:

	1	2	3	4	Supply
A	200 11	50 13	17	14	250
B	16	175 18	125 14	10	300
C	21	24	150 13	250 10	400
Demand	200	225	275	250	

Transportation cost

$$= (11 * 200) + (13 * 50) + (18 * 175) + (10 * 125) + (14 * 125) + (13 * 150)$$
$$= 2200 + 650 + 3150 + 1750 + 1950 + 2500$$
$$= Rs.12200$$

By applying NWCR (North West Corner Rule) the optimal solution is Rs.12200.

To finding the optimal solution by using NWCR Rs.12200 and ASM method Rs.12075 for transportation problem. From these two methods ASM method provides the minimum transportation cost. Thus the ASM method is optimal.

Problem 2

	1	2	3	Supply
A	5	1	7	10
B	6	4	6	80
C	3	2	5	15
Demand	75	20	50	

Solution:

	1	2	3	Supply
A	5	1	7	10
B	6	4	6	80
C	3	2	5	15
Demand	75	20	50	

Supply $= 10 + 80 + 15 = 105$
Demand $= 75 + 20 + 50 = 145$
Supply $\neq$ Demand
$=>$ Unbalanced transportation problem.

Step:1

Introducing a dummy row with demand 40 units and cost 0.

	1	2	3	
A	5	1	7	10
B	6	4	6	80
C	3	2	5	15
D	0	0	0	40
	75	20	50	

Supply $= 10 + 80 + 15 + 40 = 145$
Demand $= 75 + 20 + 50 = 145$
Supply = Demand
=> Balanced transportation problem.

ASM Method

Row reduced matrix.

	1	2	3	
A	4	0	6	10
B	2	0	2	80
C	1	0	3	15
D	0	0	0	40
	75	20	50	

Column reduced matrix.

	1	2	3	
A	4	0	6	10
B	2	0	2	80
C	1	0	3	15
D	0	0	0	40
	75	20	50	

Using ASM and Final Table

	1	2	3	
A	5	1 (10)	7	10
B	6 (60)	4 (10)	6 (10)	80
C	3 (15)	2	5	15
D	0	0	0 (40)	40
	75	20	50	

Transportation Cost

$$\begin{aligned} &= (1*10) + (6*60) + (4*10) + (6*10) + (3*15) + (0*40) \\ &= 10 + 360 + 40 + 60 + 45 + 0 \\ &= Rs.515. \end{aligned}$$

Optimality Check

	1	2	3	
A	5 [10]	1	7	10
B	6 [65]	4 [15]	6	80
C	3	2 [5]	5 [10]	15
D	0	0	0 [40]	40
	75	20	50	

Transportation Cost

$$\begin{aligned} &= (5*10) + (6*65) + (4*15) + (2*5) + (5*10) + (0*40) \\ &= 50 + 390 + 60 + 10 + 50 + 0 \\ &= Rs.560. \end{aligned}$$

By applying NWCR (North West Corner Rule) the optimal solution is Rs.560.

To finding the optimal solution by using NWCR Rs.560 and ASM method Rs.515 for transportation problem. From these two methods ASM method provides the minimum transportation cost. Thus the ASM method is optimal.

5 Conclusion

In this paper ASM method provides an optimal solution with less iteration for transportation problem. This method provides less time and make easy to understand. So it will be helpful for decision makers who are dealing with this problem.

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Common fixed point theorem for two weakly compatible self mappings in $b-$metric spaces

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Abstract

We prove some fixed and common fixed point theorems for two weakly compatible self mappings in complete $b-$metric spaces. Our results improve and generalize several known results from the current literature and its extension.

Keywords: common fixed point, coincidence point, b-metric space, $g-\alpha-$admissible mapping, $\alpha-$regular, triangular $\alpha-$admissible, weakly compatible self mappings.

2010 MSC: 47H10, 54H25

1 Introduction

It is well known that the Banach contraction principle has been improved in different directions at different spaces by mathematicians over the years. In [9, 10], S. Czerwik introduced the notion of a b-metric space which is a generalization of usual metric space and generalized the Banach contraction principle in the context of complete b-metric spaces. In the sequel, several papers have been published on the fixed point theory in b-metric spaces (see, e.g., [2–7, 12–14, 18, 26]). On the other hand, more recently, Samet et al. in [24] introduced the concept of $\alpha-\psi$-contractive type mappings and α-admissible mappings in metric spaces. Then, Karapinar and Samet [16] introduced the concept of generalized $\alpha-\psi$-contractive type, which was inspired by the notion of $\alpha-\psi$-contractive mappings. Furthermore, they [16] obtained various fixed point theorems for this generalized class of contractive mappings. Also, It should be noted that the study of common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity (see[1, 19–22]). In this paper, we prove coincidence fixed point and some common fixed point theorems for two weakly compatible self mappings in complete $b-$metric spaces.

Definition 1.1. *[9] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a b-metric space iff for all $x, y, z \in X$, the following conditions are satisfied:*

(i) $d(x,y) = 0$ *iff* $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a b-metric space with the parameter s.
It is obvious that a $b-$metric space with base $s = 1$ is a metric space. There are examples of $b-$metric spaces which are not metric spaces (see, e.g., Singh and Prasad [26]).
The notions of a Cauchy sequence and a convergent sequence in $b-$metric spaces are defined by Boriceanu[8]. As usual, a $b-$metric space is said to be complete if and only if each Cauchy sequence in this space is convergent. Note that a $b-$metric, in the general case, is not continuous [2].

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Definition 1.2. *[15] Let X be a non-empty set and $T, g : X \to X$ are given self-mappings on X. The pair $\{T, g\}$ is said to be weakly compatible if $Tgx = gTx$, whenever $Tx = gx$ for some x in X.*

Samet et al. [24] defined the notion of $\alpha-$admissible mappings as follows.

Definition 1.3. *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be $\alpha-$admissible if*

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Recently, Rosa et al. [23] introduced the following new notions of $g-\alpha-$admissible mapping.

Definition 1.4. *Let $T, g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. The mapping T is $g-\alpha-$admissible if, for all $x, y \in X$ such that $\alpha(gx, gy) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$. If g is the identity mapping, then T is called $\alpha-$admissible.*

Definition 1.5. *[17] An $\alpha-$admissible map T is said to be triangular $\alpha-$admissible if*

$$\alpha(x, z) \geq 1 \ \text{and} \ \alpha(z, y) \geq 1 \ \Longrightarrow \alpha(x, y) \geq 1.$$

2 Main Results

Let Φ denote the family of all real functions $\varphi : \mathbb{R}_+^5 \to \mathbb{R}$ with the following conditions:

(1) φ is upper-semicontinuous and non-decreasing in each coordinate variable;

(2) $\max\{\varphi(0, 0, t, t, 0), \varphi(t, 0, 0, t, t), \varphi(t, t, t, t, 0)\} < t$ for each $t > 0$.

The above family Φ is considered by Ding [11]. It is motivated by Singh and Meade [25].
In this section, we prove some common fixed point results for two self-mappings.

Definition 2.6. *Let (X, d) be a b-metric space, $g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. X is $\alpha-$regular with respect to g, if for every sequence $\{x_n\} \subseteq X$ such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx \in gX$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that for all $k \in \mathbb{N}$, $\alpha(gx_{n(k)}, gx) \geq 1$. If g is the identity mapping, then T is called $\alpha-$regular.*

Our first result is the following.

Lemma 2.1. *Let $T, g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. Suppose T be a $g-\alpha-$admissible and triangular $\alpha-$admissible. Assume that there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$. Then*

$$\alpha(gx_m, gx_n) \geq 1 \ for\ all\ m, n \in \mathbb{N}\ with\ m < n,$$

where

$$gx_{n+1} = Tx_n.$$

Proof. Since there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$ and T is a $g-\alpha-$admissible, we deduce that

$$\alpha(gx_0, gx_1) = \alpha(gx_0, Tx_0) \geq 1 \Longrightarrow \alpha(gx_1, gx_2) = \alpha(Tx_0, Tx_1) \geq 1,$$
$$\alpha(gx_1, gx_2) \geq 1 \Longrightarrow \alpha(gx_2, gx_3) = \alpha(Tx_1, Tx_2) \geq 1.$$

By continuing this process, we get

$$\alpha(gx_n, gx_{n+1}) \geq 1, \ n = 0, 1, 2, \cdots.$$

Suppose that $m < n$. Since $\alpha(gx_m, gx_{m+1}) \geq 1$, $\alpha(gx_{m+1}, gx_{m+2}) \geq 1$ and T is triangular $\alpha-$admissible, we have $\alpha(gx_m, gx_{m+2}) \geq 1$. Again, since $\alpha(gx_m, gx_{m+2}) \geq 1$ and $\alpha(gx_{m+2}, gx_{m+3}) \geq 1$, we have $\alpha(gx_m, gx_{m+3}) \geq 1$. Continuing this process inductively, we obtain

$$\alpha(gx_m, gx_n) \geq 1.$$

□

Theorem 2.1. *Let (X,d) be a complete b-metric space, $T,g: X \to X$ be such that $TX \subseteq gX$ and $\alpha: X \times X \to \mathbb{R}$. Assume that gX is closed that the following condition holds:*

$$\alpha(x,y)s^3 d(Tx,Ty) \leq \varphi(d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{1}{2s}d(gx,Ty), \frac{1}{2s}d(gy,Tx)), \tag{2.1}$$

for $x,y \in X$ and $\varphi \in \Phi$. Assume also that the following conditions hold:

(i) T is $g-\alpha-$admissible and triangular $\alpha-$admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) X is $\alpha-$regular with respect to g.

Then T and g have a coincidence point.
Moreover, if the following conditions hold:

(a) The pair $\{T,g\}$ is weakly compatible;

(b) either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(gx_0, Tx_0) \geq 1$ (using the condition (ii)). Since $TX \subseteq gX$ we can choose a point $x_1 \in X$ such that $Tx_0 = gx_1$. Also, there exists $x_2 \in X$ such that $Tx_1 = gx_2$, this can be done through the reality $TX \subseteq gX$. Continuing this process having chosen $x_1, x_2, ..., x_n \in X$, we have $x_{n+1} \in X$ such that

$$gx_{n+1} = Tx_n, \; n = 0,1,2,\cdots. \tag{2.2}$$

By Lemma 2.1, we have

$$\alpha(gx_n, gx_{n+1}) \geq 1, \; n = 0,1,2,\cdots. \tag{2.3}$$

If $Tx_{n_0} = Tx_{n_0+1}$ for some n_0, then by (2.2), we get

$$gx_{n_0} = Tx_{n_0+1} = Tx_{n_0},$$

that is, T and g have a coincidence point at $x = x_{n_0}$, and so the proof is completed. So, we suppose that for all $n \in \mathbb{N}$, $Tx_n \neq Tx_{n+1}$. Now, for all $n \in \mathbb{N}$ by (2.1) and (2.3), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq s^3 d(gx_n, gx_{n+1}) = s^3 d(Tx_{n-1}, Tx_n) \\ &\leq \alpha(gx_{n-1}, gx_n) s^3 d(Tx_{n-1}, Tx_n) \\ &\leq \varphi(d(gx_{n-1}, gx_n), d(gx_{n-1}, Tx_{n-1}), d(gx_n, Tx_n), \frac{1}{2s}d(gx_{n-1}, Tx_n), \frac{1}{2s}d(gx_n, Tx_{n-1})) \\ &= \varphi(d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{1}{2s}d(gx_{n-1}, gx_{n+1}), \frac{1}{2s}d(gx_n, gx_n)) \\ &= \varphi(d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{1}{2s}d(gx_{n-1}, gx_{n+1}), 0). \end{aligned} \tag{2.4}$$

If $d(gx_{n-1}, gx_n) \leq d(gx_n, gx_{n+1})$, from (2.4),

$$\frac{d(gx_{n-1}, gx_{n+1})}{2s} \leq \frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{2},$$

and using the properties of the function φ, we get

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \varphi(d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{2}, 0) \\ &\leq \varphi(d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), 0) \\ &< d(gx_n, gx_{n+1}), \end{aligned}$$

which is a contradiction. So $d(gx_n, gx_{n+1}) < d(gx_{n-1}, gx_n)$ for all $n \in \mathbb{N}$, that is, the sequence of nonnegative numbers $\{d(gx_n, gx_{n+1})\}$ is decreasing. Hence, it converges to a nonnegative number, say $\delta \geq 0$. If $\delta > 0$, then letting $n \to \infty$ in (2.4) and since φ is continuous, then we obtain

$$\delta \leq \varphi(\delta, \delta, \delta, \delta, 0) < \delta,$$

which is a contraction. Therefore

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{2.5}$$

Now, we claim that

$$\lim_{n,m\longrightarrow\infty} d(gx_n, gx_m) = 0. \tag{2.6}$$

Assume on the contrary that there exists $\epsilon > 0$ and subsequences $\{gx_{m(k)}\}$, $\{gx_{n(k)}\}$ of $\{gx_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(gx_{m(k)}, gx_{n(k)}) \geq \epsilon. \tag{2.7}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.7) and $n(k) > m(k) \geq k$. Thus,

$$d(gx_{m(k)}, gx_{n(k)-1}) < \epsilon. \tag{2.8}$$

Using the triangle inequality in $b-$metric space and (2.7) and (2.8) we obtain that

$$\begin{aligned}\epsilon \leq d(gx_{n(k)}, gx_{m(k)}) &\leq sd(gx_{n(k)}, gx_{n(k)-1}) + sd(gx_{n(k)-1}, gx_{m(k)})\\ &< sd(gx_{n(k)}, gx_{n(k)-1}) + s\epsilon.\end{aligned}$$

Taking the the upper limit as $k \longrightarrow \infty$ and using (2.5) we obtain

$$\epsilon \leq \limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)}) \leq s\epsilon. \tag{2.9}$$

Also

$$\begin{aligned}\epsilon &\leq d(gx_{m(k)}, gx_{n(k)}) \leq sd(gx_{m(k)}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)})\\ &\leq s^2 d(gx_{m(k)}, gx_{n(k)}) + s^2 d(gx_{n(k)}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)})\\ &\leq s^2 d(gx_{m(k)}, gx_{n(k)}) + (s^2 + s)d(gx_{n(k)}, gx_{n(k)+1}).\end{aligned}$$

So from (2.5) and (2.9), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)+1}) \leq s^2\epsilon. \tag{2.10}$$

Also

$$\begin{aligned}\epsilon &\leq d(gx_{n(k)}, gx_{m(k)}) \leq sd(gx_{n(k)}, gx_{m(k)+1}) + sd(gx_{m(k)+1}, gx_{m(k)})\\ &\leq s^2 d(gx_{n(k)}, gx_{m(k)}) + s^2 d(gx_{m(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})\\ &\leq s^2 d(gx_{n(k)}, gx_{m(k)}) + (s^2 + s)d(gx_{m(k)}, gx_{m(k)+1}).\end{aligned}$$

So from (2.5) and (2.9), we get

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)+1}) \leq s^2\epsilon. \tag{2.11}$$

Also

$$d(gx_{m(k)+1}, gx_{n(k)}) \leq sd(gx_{m(k)+1}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)}),$$

so from (2.5) and (2.11), we have

$$\frac{\epsilon}{s^2} \leq \limsup_{k\longrightarrow\infty} d(gx_{n(k)+1}, gx_{m(k)+1}). \tag{2.12}$$

Now using inequality (2.1) and Lemma 2.1, we have

$$\begin{aligned} s^3 d(gx_{m(k)+1}, gx_{n(k)+1}) &= s^3 d(gx_{m(k)+1}, gx_{n(k)+1}) = s^3\, d(Tx_{m(k)}, Tx_{n(k)}) \\ &\leq \alpha(gx_{m(k)}, gx_{n(k)}) s^3\, d(Tx_{m(k)}, Tx_{n(k)}) \\ &\leq \varphi(d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)}, gx_{m(k)+1}), d(gx_{n(k)+1}, gx_{n(k)}), \\ &\quad \frac{1}{2s} d(gx_{m(k)}, gx_{n(k)+1}), \frac{1}{2s} d(gx_{n(k)}, gx_{m(k)+1})). \end{aligned}$$

Since φ is upper-semicontinuous, by (2.5),(2.10),(2.11) and (2.12)

$$\begin{aligned} s\epsilon = s^3.\frac{\epsilon}{s^2} &\leq s^3 \limsup_{k\to\infty} d(gx_{m(k)+1}, gx_{n(k)+1}) \\ &\leq \varphi(s\epsilon, 0, 0, \frac{s\epsilon}{2}, \frac{s\epsilon}{2}) \\ &\leq \varphi(s\epsilon, 0, 0, s\epsilon, s\epsilon) \\ &< s\epsilon. \end{aligned}$$

which is a contradiction. So, we conclude that $\{gx_n\}$ is a Cauchy sequence in (X, d). By virtue of (2.2) we get $\{Tx_n\} = \{gx_{n+1}\} \subseteq gX$ and gX is closed, there exists $x \in X$ such that

$$\lim_{n\to\infty} gx_n = gx. \tag{2.13}$$

Now, we claim that x is a coincidence point of T and g. On the contrary, assume that $d(Tx, gx) > 0$. Since X is $\alpha-$regular with respect to g and (2.13), we have

$$\alpha(gx_{n(k)+1}, gx) \geq 1 \; for\ all\ k \in \mathbb{N}. \tag{2.14}$$

Also by the use of triangle inequality in $b-$metric space, we have

$$\begin{aligned} d(gx, Tx) &\leq sd(gx, gx_{n(k)+1}) + sd(gx_{n(k)+1}, Tx) \\ &= sd(gx, gx_{n(k)+1}) + sd(Tx_{n(k)}, Tx). \end{aligned}$$

In the above inequality, if k tends to infinity, then, we have

$$d(gx, Tx) \leq \lim_{k\to\infty} sd(Tx_{n(k)}, Tx). \tag{2.15}$$

By property of φ, (2.14) and (2.15), we have

$$\begin{aligned} s^2 d(gx, Tx) &\leq \lim_{k\to\infty} s^3 d(Tx_{n(k)}, Tx) \leq \lim_{k\to\infty} \alpha(gx_{n(k)+1}, gx) s^3 d(Tx_{n(k)}, Tx) \\ &\leq \lim_{k\to\infty} [\varphi(d(gx_{n(k)}, gx), d(gx_{n(k)}, Tx_{n(k)}), d(gx, Tx), \frac{1}{2s} d(gx_{n(k)}, Tx), \frac{1}{2s} d(gx, Tx_{n(k)})] \\ &= \lim_{k\to\infty} [\varphi(d(gx_{n(k)}, gx), d(gx_{n(k)}, gx_{n(k)+1}), d(gx, Tx), \frac{1}{2s} d(gx_{n(k)}, Tx), \frac{1}{2s} d(gx, gx_{n(k)+1})] \\ &\leq \varphi(0, 0, d(gx, Tx), \frac{1}{2s} d(gx, Tx), 0) \\ &\leq \varphi(0, 0, d(gx, Tx), d(gx, Tx), 0) \\ &< d(gx, Tx), \end{aligned}$$

which is a contradiction. Hence, $d(gx, Tx) = 0$, that is, $gx = Tx$ and x is a coincidence point of T and g. We claim that, if $Tu = gu$ and $Tv = gv$, then $gu = gv$. By hypotheses, $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$. Suppose that $\alpha(u, v) \geq 1$, then

$$\begin{aligned} s^3 d(gu, gv) = s^3 d(Tu, Tv) &\leq \alpha(u, v) s^3 d(Tu, Tv) \\ &\leq \varphi(d(gu, gv), d(gu, Tu), d(gv, Tv), \frac{1}{2s} d(gu, Tv), \frac{1}{2s} d(gv, Tu)) \\ &= \varphi(d(gu, gv), d(gu, gu), d(gv, gv), \frac{1}{2s} d(gu, gv), \frac{1}{2s} d(gv, gu)) \\ &\leq \varphi(d(gu, gv), 0, 0, d(gu, gv), d(gv, gu)) \\ &< d(gu, gv), \end{aligned}$$

which is a contradiction. Thus we deduce that $gu = gv$. Similarly, if $\alpha(v, u) \geq 1$ we can prove that $gu = gv$. Now, we show that T and g have a common fixed point. Indeed, if $w = Tu = gu$, owing to the weakly compatible of T and g, we get $Tw = T(gu) = g(Tu) = gw$. Thus w is a coincidence point of T and g, then $gu = gw = w = Tw$. Therefore, w is a common fixed point of T and g. The uniqueness of common fixed point of T and g is a consequence of the conditions (2.1) and (b), and so we omit the details. □

Example 2.1. *Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |x(t)| < \infty$. Define $d : X \times X \longrightarrow [0, \infty)$ by*

$$d(x,y) = \Big(\int_0^1 |x(t) - y(t)| dt \Big)^2.$$

Then d is a b-metric on X, with $s = 2$.
The operator $T : X \longrightarrow X$ defined by

$$Tx(t) = \frac{1}{\sqrt{8}} \ln \big(|x(t)| + 1\big),$$

and the operator $g : X \longrightarrow X$ defined by

$$gx(t) = e^{\sqrt{8}|x(t)|} - 1.$$

Now, we prove that T and g have a unique common fixed point. For all $x, y \in X$, we have

$$\begin{aligned}
2^3 d(Tx, Ty) &= 2^3 \Big(\int_0^1 |Tx(t) - Ty(t)| dt \Big)^2 \leq 8 \Big(\int_0^1 |\frac{1}{\sqrt{8}} \ln(|x(t)| + 1) - \frac{1}{\sqrt{8}} \ln(|y(t)| + 1)| dt \Big)^2 \\
&\leq \Big(\int_0^1 |(\ln(|x(t)| + 1) - \ln(|y(t)| + 1))| dt \Big)^2 \leq \Big(\int_0^1 \ln(\frac{|x(t)| + 1}{|y(t)| + 1}) dt \Big)^2 \\
&\leq \Big(\int_0^1 \ln(1 + \frac{|x(t) - y(t)|}{|y(t)| + 1}) dt \Big)^2 \leq \Big(\ln(1 + \int_0^1 |x(t) - y(t)| dt) \Big)^2 \\
&\leq \Big(\ln(1 + \int_0^1 |e^{\frac{4}{\sqrt{2}}|x(t)|} - e^{\frac{4}{\sqrt{2}}|y(t)|}| dt) \Big)^2 \leq \Big(\ln(1 + \sqrt{(\int_0^1 |e^{\frac{4}{\sqrt{2}}|x(t)|} - e^{\frac{4}{\sqrt{2}}|y(t)|}| dt)^2}) \Big)^2 \\
&\leq \Big(\ln(1 + \sqrt{d(gx, gy)}) \Big)^2 \\
&\leq \varphi(d(gx, gy), d(gx, Tx), d(gy, Ty), \frac{1}{2s} d(gx, Ty), \frac{1}{2s} d(gy, Tx)).
\end{aligned}$$

Now, if we define $x_0 = 0$, $\alpha(x, y) = 1$ and $\varphi(t) = \ln^2(1 + \sqrt{t})$ for all $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}^2$, where $t = \max\{t_1, t_2, t_3, t_4, t_5\}$. Thus, by using Theorem 2.1 we obtain that T and g have a unique common fixed point.

From Theorem 2.1, if we choose $g = I_X$ the identity mapping on X, we deduce the following corollary.

Corollary 2.1. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a self-mapping on X. If there exist $\alpha : X \times X \to \mathbb{R}$ and $\varphi \in \Phi$ such that for all $x, y \in X$,*

$$\alpha(x, y) s^3 d(Tx, Ty) \leq \varphi(d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2s} d(x, Ty), \frac{1}{2s} d(y, Tx)).$$

Also that the following conditions hold:

(i) T is $\alpha-$admissible and triangular $\alpha-$admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) X is $\alpha-$regular;

(iv) either $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$ whenever $Tu = u$ and $Tv = v$.

Then T has a unique fixed point.

Example 2.2. *Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |x(t)| < \infty$. Define $d : X \times X \longrightarrow [0, \infty)$ by*

$$d(x,y) = \Big(\int_0^1 |x(t) - y(t)| dt \Big)^2.$$

Then d is a b-metric on X, with $s = 2$.
The operator $T : X \longrightarrow X$ defined by

$$Tx(t) = \frac{1}{\sqrt{8}} \ln\big(|x(t)| + 1\big).$$

Now, we prove that Thas a unique fixed point. For all $x, y \in X$, we have

$$\begin{aligned}
2^3 d(Tx, Ty) &= 2^3\Big(\int_0^1 |Tx(t) - Ty(t)|dt\Big)^2 \leq 8\Big(\int_0^1 |\frac{1}{\sqrt{8}}\ln(|x(t)|+1) - \frac{1}{\sqrt{8}}\ln(|y(t)|+1)|dt\Big)^2 \\
&\leq \Big(\int_0^1 |(\ln(|x(t)|+1) - \ln(|y(t)|+1))|dt\Big)^2 \leq \Big(\int_0^1 \ln(\frac{|x(t)|+1}{|y(t)|+1})dt\Big)^2 \\
&\leq \Big(\int_0^1 \ln(1 + \frac{|x(t)-y(t)|}{|y(t)|+1})dt\Big)^2 \leq \Big(\ln(1 + \int_0^1 |x(t)-y(t)|dt)\Big)^2 \\
&\leq \Big(\ln(1 + \sqrt{(\int_0^1 |x(t)-y(t)|dt)^2})\Big)^2 \\
&\leq \Big(\ln(1 + \sqrt{d(x,y)})\Big)^2 \\
&\leq \varphi(d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s}d(x,Ty), \frac{1}{2s}d(y,Tx)).
\end{aligned}$$

Now, if we define $x_0 = 0$, $\alpha(x,y) = 1$ and $\varphi(t) = \ln^2(1+\sqrt{t})$ for all $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}^2$, where $t = \max\{t_1, t_2, t_3, t_4, t_5\}$. Thus, by using Corollary 2.1 we obtain that T has a unique fixed point.

From Theorem 2.1, if the function $\alpha : X \times X \to \mathbb{R}$ is such that $\alpha(x,y) = 1$ for all $x, y \in X$, we deduce the following theorem.

Theorem 2.2. *Let (X,d) be a complete b-metric space, $T, g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed such that for all $x, y \in X$,*

$$s^3 d(Tx,Ty) \leq \varphi(d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{1}{2s}d(gx,Ty), \frac{1}{2s}d(gy,Tx)),$$

where $\varphi \in \Phi$. Then T and g have a coincidence point. Moreover, if T and g are weakly compatible, then T and g have a unique common fixed point.

In Theorem 2.1, if we put

$$\varphi(t_1, t_2, t_3, t_4, t_5) = k\ \max\{t_1, t_2, t_3, t_4 + t_5\}$$

for all $t_i \in \mathbb{R}_+ (i = 1, 2, 3, 4, 5)$, we deduce the following theorem.

Theorem 2.3. *Let (X,d) be a complete b-metric space, $T, g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and there exist $\alpha : X \times X \to \mathbb{R}$ and $0 < k < \frac{1}{2}$ such that for all $x, y \in X$,*

$$\alpha(x,y)s^3 d(Tx,Ty) \leq k\ \max\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty) + d(gy,Tx)}{2s}\}.$$

Assume also that the following conditions hold:

(i) T is $g - \alpha-$admissible and triangular $\alpha-$admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) X is $\alpha-$regular with respect to g.

Then T and g have a coincidence point.
Moreover, if the following conditions hold:

(a) The pair $\{T, g\}$ is weakly compatible;

(b) either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

Example 2.3. *Let $X = [0, \infty)$ be endowed with b-metric $d(x, y) = (|x - y|)^2 = (x - y)^2$, where $s = 2$. Define $T, g : X \longrightarrow X$ by*

$$T(x) = \begin{cases} \frac{1}{8}x, & 0 \leq x \leq \frac{4}{3}, \\ x - \dfrac{2}{3}, & x > \frac{4}{3}. \end{cases}$$

and

$$g(x) = \frac{3}{4}x \ \forall x \in X.$$

Now, we define the mapping $\alpha : X \times X \rightarrow \mathbb{R}_+$ by

$$\alpha(x, y) = \begin{cases} 1, & \textit{if } (x, y) \in [0, 1], \\ 0, & \textit{otherwise}. \end{cases}$$

It is easily seen that the pair $\{T, g\}$ is weakly compatible, $T(X) \subset g(X)$ and $g(X)$ is closed. For all $x, y \in X$, we have

$$\begin{aligned} \alpha(x, y)s^3 d(Tx, Ty) &= 1.8.|\frac{1}{8}x - \frac{1}{8}y|^2 = \frac{2}{9}|\frac{3}{4}x - \frac{3}{4}y|^2 \\ &= \frac{2}{9}d(gx, gy) \\ &\leq \frac{1}{3} \max\{d(gx, gy), d(gx, Tx), d(gy, Ty), \frac{d(gx, Ty) + d(gy, Tx)}{2s}\}. \end{aligned}$$

Moreover, there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$. Indeed, for $x_0 = 1$, we have $\alpha(g(1), T(1)) = \alpha(\frac{3}{4}, \frac{1}{8}) = 1$. Let $x, y \in X$ such that $\alpha(gx, gy) \geq 1$, that is, $gx, gy \in [0, 1]$ and by the definition of g, we have $x, y \in [0, \frac{4}{3}]$. So, by definition of T and α, we have $T(x) = \frac{1}{8}x \in [0, 1], T(y) = \frac{1}{8}y \in [0, 1]$ and $\alpha(Tx, Ty) = 1$. Thus, T is $g - \alpha-$admissible and hence (i) is satisfied.
Finally, it remains to show that X is $\alpha-$regular with respect to g. In so doing, let $\{x_n\} \subseteq X$ such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \rightarrow gx \in gX$ as $n \rightarrow \infty$. Since $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, by the definition of α, we have $gx_n \in [0, 1]$ for all $n \in \mathbb{N}$ and $gx \in [0, 1]$. Then, $\alpha(gx_n, gx) \geq 1$. Now, all the hypotheses of Theorem 2.3 are satisfied. Consequently, 0 is the unique common fixed point of T and g.

Remark 2.1. *Since a $b-$metric space is a metric space when $s = 1$, so our results can be viewed as the generalization an the extension of several comparable results.*

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Initial value problems for fractional differential equations involving Riemann-Liouville derivative

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Abstract

Existence results are obtained for fractional differential equations with C_p continuity of functions. Monotone method for nonlinear initial value problem is developed by introducing the notion of coupled lower and upper solutions. As an application of the method existence and uniqueness results are obtained.

Keywords: Fractional derivative, initial value problem, coupled lower and upper solutions, existence and uniqueness.

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1 Introduction

The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, and in many other fields, like theory of fractals. Analytical as well as numerical methods are available for studying fractional differential equations such as compositional method, transform method, Adomain methods and power series method etc. (see details in [4, 23] and references therein). Monotone method [5] coupled with method of lower and upper solutions is an effective mechanism that offers constructive procedure to obtain existence results in a closed set. Basic theory of fractional differential equations with Riemann-Liouville fractional derivative is well developed in [2, 7, 9]. Lakshamikantham and Vatsala [1, 6, 8] obtained the local and global existence of solution of Riemann-Liouville fractional differential equation and uniqueness of solution. In the year 2009, McRae developed monotone method for Riemann-Liouvile fractional differential equation with initial conditions and studied the qualitative properties of solutions of initial value problem [10]. Nanware and Dhaigude [11, 13, 14, 16–22] developed monotone method for system of fractional differential equations with various conditions and successfully applied to study qualitative properties of solutions. Nanware obtained existence results for the solution of fractional differential equations involving Caputo derivative with boundary conditions [12, 15]. In 2012, Yaker and Koksal have studied initial value problem (1.1) − (1.2) for Riemann- Liouville fractional differential equations. They have proved existence results by using concept of lower and upper solutions and local existence results under the strong hypothesis that the functions are locally Holder continuous.
In this paper, we develop monotone method without such strong hypothesis for the following nonlinear Riemann-Liouville fractional differential equation with initial condition

$$D^q u(t) = f(t, u(t)) + g(t, u(t)), \qquad t \in [t_0, T] \tag{1.1}$$

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$$u^0 = u(t)(t-t_0)^{1-q}\}_{t=t_0} \tag{1.2}$$

where $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $J = [t_0, T]$, $f(t,u)$ is nondecreasing in u , $g(t,u)$ is nonincreasing in u for each t and D^q denotes the Riemann-Liouville fractional derivative with respect to t of order $q(0 < q < 1)$. This is called initial value problem(IVP). We develop monotone method coupled with lower and upper solutions for the IVP $(1.1) - (1.2)$. The method is applied to obtain existence and uniqueness of solution of the IVP $(1.1) - (1.2)$.

The paper is organized in the following manner : In section 2, we consider some definitions and lemmas required in next section and obtained result for nonstrict inequalities. In section 3, we improve the existence results due to Yaker and Koksal. In section 4, we develop monotone method and apply it to obtain existence and uniqueness results for Riemann-Liouville fractional differential equation with initial condition when nonlinear function on the right hand side is considered as sum of nondecreasing and nonincreasing functions.

2 Preliminaries

In this section, we discuss some basic definitions and results which are required for the development of monotone method for fractional differential equation with initial condition involving Riemann-Liouville derivative when nonlinear function on the right hand side is considered as sum of nondecreasing and nonincreasing functions.

The Riemann-Liouville fractional derivative of order $q(0 < q < 1)$ [23] is defined as

$$D_a^q u(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-q-1} u(\tau) d\tau, \quad \text{for } a \le t \le b.$$

Lemma 2.1. *[2] Let $m \in C_p([t_0, T], \mathbb{R})$ and for any $t_1 \in (t_0, T]$ we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \le t \le t_1$. Then it follows that $D^q m(t_1) \ge 0$.*

Lemma 2.2. *[6] Let $\{u_\epsilon(t)\}$ be a family of continuous functions on $[t_0, T]$, for each $\epsilon > 0$ where $D^q u_\epsilon(t) = f(t, u_\epsilon(t))$, $u_\epsilon(t_0) = u_\epsilon(t)(t-t_0)^{1-q}\}_{t=t_0}$ and $|f(t, u_\epsilon(t))| \le M$ for $t_0 \le t \le T$. Then the family $\{u_\epsilon(t)\}$ is equicontinuous on $[t_0, T]$.*

Now, we introduce the notion of lower and upper solutions for the initial value problem $(1.1) - (1.2)$.

Definition 2.1. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are said to be lower and upper solutions of the IVP* $(1.1) - (1.2)$ *if*

$$D^q v(t) \le f(t, v(t)) + g(t, v(t)), \qquad v^0 \le u^0$$
$$D^q w(t) \ge f(t, w(t)) + g(t, w(t)), \qquad w^0 \ge u^0.$$

Definition 2.2. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are said to be lower and upper solutions of type I of IVP* $(1.1) - (1.2)$ *if*

$$D^q v(t) \le f(t, v(t)) + g(t, w(t)), \qquad v^0 \le u^0$$
$$D^q w(t) \ge f(t, w(t)) + g(t, v(t)), \qquad w^0 \ge u^0.$$

Definition 2.3. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are said to be lower and upper solutions of type II of IVP* $(1.1) - (1.2)$ *if*

$$D^q v(t) \le f(t, w(t)) + g(t, v(t)), \qquad v^0 \le u^0$$
$$D^q w(t) \ge f(t, v(t)) + g(t, w(t)), \qquad w^0 \ge u^0.$$

Definition 2.4. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are said to be lower and upper solutions of type III of IVP* $(1.1) - (1.2)$ *if*

$$D^q v(t) \le f(t, w(t)) + g(t, w(t)), \qquad v^0 \le u^0$$
$$D^q w(t) \ge f(t, v(t)) + g(t, v(t)), \qquad w^0 \ge u^0.$$

3 Existence Results

In this section, we improve the existence results due to Yaker and Koksal [24] for IVP (1.1) − (1.2). We now state and prove the following existence results.

Theorem 3.1. *Suppose that:*

(i) *$v(t)$ and $w(t)$ in $C_p(J,\mathbb{R})$ are coupled lower and upper solutions of type I of IVP* (1.1)-(1.2) *with $v(t) \le w(t)$ on J.*

(ii) *$f(t,u), g(t,u) \in C[\Omega,\mathbb{R}]$ and $g(t,u(t))$ is nonincreasing in u for each t on J.*

Then there exist a solution $u(t)$ of IVP (1.1)-(1.2) *satisfying $v(t) \le u \le w(t)$ on J.*

Proof. Let $P : J \times \mathbb{R} \to \mathbb{R}$ be defined by

$$P(t,u) = min\{w(t), max(u(t), v(t))\}$$

Then $f(t,P(t,u(t)) + g(t,P(t,u(t)))$ defines a continuous extension of $f+g$ to $J\times\mathbb{R}$ which is bounded, since $f+g$ is uniformly bounded on Ω. By Lemma 2.2, it follows that the family $P_\epsilon(t,u(t))$ is equicontinuous on J. By Ascoli-Arzela theorem the sequences $\{P_\epsilon(t,u(t))\}$ has convergent subsequences $\{P_{\epsilon_n}(t,u_1)\}$ which converges uniformly to $P(t,u)$. Since $f+g$ is uniformly continuous, we obtain that $f(t,P_{\epsilon_n}(t,u)) + g(t,P_{\epsilon_n}(t,u))$ tends uniformly to $f(t,P(t,u)) + g(t,P(t,u))$ as $n\to\infty$. Hence $P(t,u(t))$ is the solution of

$$D^q u(t) = f(t,P(t,u)) + g(t,P(t,u)), \quad u(t) = u(t_0)(t-t_0)^{1-q}\}_{t=t_0} = u^0. \tag{3.3}$$

It follows that the equation (3.3) has a solution on the interval J.
We wish to prove that $v(t) \le u(t) \le w(t)$ on J. For $\epsilon > 0$, consider $w_\epsilon(t) = w(t) + \epsilon\gamma(t)$ and $v_{i\epsilon}(t) = v_i(t) - \epsilon\gamma(t)$, where $\gamma(t) = (t-t_0)^{q-1}E_{q,q}((t-t_0)^q)$ Then we have $w^0_\epsilon = w^0 + \epsilon\gamma^0, \quad v^0_\epsilon = v^0 - \epsilon\gamma^0$, where $\gamma^0 > 0$. This shows that $v^0_\epsilon < u^0 < w^0_\epsilon$. Next we show that $u < w_\epsilon, \quad t_0 \le t \le T$. On the contrary, suppose that $v_\epsilon \ge u \ge w_\epsilon$. Then there exists $t_1 \in (t_0,T]$ such that $u(t_1) = w_\epsilon(t_1)$ and $v_\epsilon > u > w_\epsilon, \quad t_0 \le t < t_1$. Thus $u(t_1) > w(t_1)$ and hence $P(t_1,u(t_1)) = w(t_1)$.
Set $m(t) = u(t) - w_\epsilon(t)$ we have $m(t_1) = 0$ and $m(t) \le 0, \quad t_0 \le t \le t_1$. By Lemma 2.1, we have $D^q u(t_1) \ge D^q w_\epsilon(t_1)$ which gives a contradiction

$$\begin{aligned} f(t_1,w(t_1)) + g(t_1,w(t_1)) &= f(t_1,P(t_1,u(t_1)) + g(t_1,P(t_1,u(t_1))) \\ &= D^q u(t_1) \\ &\ge D^q w_\epsilon(t_1) \\ &= D^q w(t_1) + \epsilon\gamma(t_1) \\ &> D^q w(t_1) \\ &\ge f(t_1,w(t_1)) + g(t_1,v(t_1)) \end{aligned}$$

Similarly, we prove $v_\epsilon < u, \quad t_0 \le t \le T$. For this, suppose there exists $t_1 \in (t_0,T]$ such that $v_\epsilon(t_1) = u(t_1)$ and $v_\epsilon(t) > u(t), \quad t_0 \le t < t_1$. Thus $u(t_1) < v(t_1 \le w(t_1)$ and hence $P(t_1,u(t_1)) = v(t_1)$.
Set $m(t) = v_\epsilon(t) - u(t)$ we have $m(t_1) = 0$ and $m(t) \le 0, \quad t_0 \le t \le t_1$. Applying Lemma 2.1, we have $D^q u(t_1) \ge D^q w_\epsilon(t_1)$. Since $g(t,u)$ is nonincreasing in u for each t and $\gamma(t) > 0$, we get a contradiction

$$\begin{aligned} f(t_1,v(t_1)) + g(t_1,v(t_1)) &= f(t_1,P(t_1,u(t_1)) + g(t_1,P(t_1,u(t_1))) \\ &= D^q u(t_1) \\ &\le D^q v_\epsilon(t_1) \\ &= D^q v(t_1) - \epsilon\gamma(t_1) \\ &< D^q v(t_1) \\ &\le f(t_1,v(t_1)) + g(t_1,w(t_1)) \end{aligned}$$

Consequently, we get $v_\epsilon(t) < u(t) < w_\epsilon(t)$ on J. In the limiting case $\epsilon\to 0$ we get $v(t) \le u(t) \le w(t)$ on J. □

Theorem 3.2. *Suppose that:*

(i) *$v(t)$ and $w(t)$ in $C_p(J,\mathbb{R})$ are coupled lower and upper solutions of type II of IVP* (1.1)-(1.2) *with $v(t) \le w(t)$ on J.*

(ii) *$f(t,u), g(t,u) \in C[\Omega,\mathbb{R}]$ and $f(t,u)$ is nonincreasing in u for each t on J.*

Then there exists a solution $u(t)$ of IVP (1.1)-(1.2) *satisfying $v(t) \le u \le w(t)$ on J.*

Proof. Proof can be given on the same line as in Theorem 3.1. □

Theorem 3.3. *Suppose that:*

(i) *$v(t)$ and $w(t)$ in $C_p(J,\mathbb{R})$ are coupled lower and upper solutions of type III of IVP* (1.1)-(1.2) *with $v(t) \le w(t)$ on J.*

(ii) *$f(t,u(t)), g(t,u(t)) \in C[\Omega,\mathbb{R}]$ are both nonin creasing in u for each t on J.*

Then there exists a solution $u(t)$ of IVP (1.1)-(1.2) *satisfying $v(t) \le u \le w(t)$ on J.*

Proof. Proof can be given on the same line as in Theorem 3.1. □

4 Monotone Method

In this section we develop monotone method for Riemann-liouville fractional differential equations with initial conditions for all types of coupled lower and upper solutions defined in section 2 and we apply the method to obtain extremal solutions and uniqueness of solution of the IVP (1.1)-(1.2).

Theorem 4.4. *Assume that:*

(i) *$f(t,u(t))$ and $g(t,u(t))$ in $C[\Omega,R^2]$ and $f(t,u(t))$ nonincreasing in u for each $t \in [t_0,T]$,*

(ii) *$v_0(t)$ and $w_0(t)$ in $C(J,R)$ are coupled lower and upper solutions of type I of IVP* (1.1)-(1.2) *such that $v_0(t_0) \le w_0(t_0)$ on J.*

(iii) *$f(t,u(t)), g(t,u(t))$ satisfies one-sided Lipschitz condition,*

$$f(t,u(t)) - f(t,\overline{u}(t)) \ge -M(u-\overline{u}), M > 0, \overline{u} \ge u,$$
$$g(t,u(t)) - g(t,\overline{u}(t)) \ge -N(u-\overline{u}), N > 0, \overline{u} \ge u$$

Then there exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ such that

$$\{v_n(t)\} \to v(t) \quad and \quad \{w_n(t)\} \to w(t) as\ n \to \infty$$

and $v(t)$ and $w(t))$ are minimal and maximal solutions of the IVP (1.1)-(1.2).

Proof. For any η in $C(J,R)$ such that for $v_0 \le \eta$ on J, we consider the following linear fractional differential equation

$$D^q u(t) = f(t,\eta(t)) + g(t,\eta(t)) - M(u-\eta) - N(u-\eta), \quad u(t)(t-t_0)^{1-q}\}_{t=t_0} = u^0 \tag{4.4}$$

Since the right hand side of equation (4.4) is known, it is clear that for every η there exists a unique solution $u(t)$ of IVP (4.4) on J.

For each η and μ in $C(J,R)$ such that $v_0 \le \eta$ and $w_0 \le \mu$, define a mapping A by $A[\eta,\mu] = u(t)$ where $u(t)$ is the unique solution of IVP (4.4). This mapping defines the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$. Firstly, we prove

(I) $v_0 \le A[v_0,w_0], \quad , w_0 \ge A[w_0,v_0]$

(II) A possesses the monotone property on the segment
$[v_0, w_0] \in C(J, R^2) : v_0 \leq u \leq w_0\}$

Set $A[v_0, w_0] = v_1(t)$, where $v_1(t)$ is the unique solution of IVP (4.4) with $\eta(t) = v_0(t)$ and v_0 is lower solution of IVP (1.1)-(1.2).
Consider $p(t) = v_0(t) - v_1(t)$ so that, we have

$$\begin{aligned}
D^q p(t) &= D^q v_0(t) - D^q v_1(t) \\
&\leq f(t, v_0(t)) + g(t, v_0(t)) - f(t, v_0)) - g(t, v_0) + M(v_1 - v_0) \\
&\leq -Mp(t) \\
\text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\
\text{and} \quad p(t)(t - t_0)^{1-q}\}_{t=t_0} &\leq 0
\end{aligned}$$

By Lemma 2.1, we have $p(t) \leq 0$ on $t_0 \leq t \leq T$.This implies $v_0(t) \leq v_1(t)$. Thus $v_0 \leq A[v_0, w_0]$. Similarly we can prove $w_0 \geq A[w_0, v_0]$.
Let $\eta(t)$ and $\mu(t)$ in $[v_0, w_0]$ be such that $\eta(t) \leq \mu(t)$. Suppose that $A[\eta, \mu] = u(t)$ and $A[\eta, \mu] = v(t)$
Consider $p(t) = u(t) - v(t)$ we find by Lipschitz condition that

$$\begin{aligned}
D^q p(t) &= D^q u(t) - D^q v(t) \\
&= f(t, \eta(t)) + g(t, \eta(t)) - f(t, \eta(t)) - g(t, \eta(t)) + M(u - \eta) \\
&\leq -M(u - v) \\
&\leq -Mp(t) \\
\text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\
\text{and} \quad p(t)(t - t_0)^{1-q}\}_{t=t_0} &\leq 0
\end{aligned}$$

As before in (I),we have $A[\eta, \mu] \leq A[\eta, \mu]$. This shows that operator A possesses monotone property on $[v_0, w_0]$. Now in view of (I) and (II), define the sequences

$$v_n(t) = A[v_{n-1}, w_{n-1}], \quad w_n(t) = A[v_{n-1}, w_{n-1}] \qquad \text{on the segment} \quad [v_0, w_0].$$

It follows that

$$v_0(t) \leq v_1(t) \leq v_2(t) \leq ...v_n(t) \leq w_n(t) \leq w_{n-1}(t) \leq ... \leq w_1(t) \leq w_0(t). \tag{4.5}$$

Obviously the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are monotonic and bounded hence they are uniformly bounded on J. By Lemma 2.2 it follows that the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are equicontinuous on J and by Ascoli-Arzela Theorem, there exists subsequences $\{v_{n_k}(t)\}$ and $\{w_{n_k}(t)\}$ that converge uniformly on J. By (4.5) it follows that the sequences $\{v_{n_k}(t)\}$ and $\{w_{n_k}(t)\}$ converge uniformly and monotonically to $v(t)$ and $w(t)$ where

$$\lim_{n\to\infty} v_n(t) = v(t) \quad \lim_{n\to\infty} w_n(t) = w(t) \quad \text{on} \quad [t_0, T]$$

Using following fractional Volterra integral equations

$$\begin{aligned}
v_{n+1}(t) &= v_0^0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \{f(s, v_n) + g(s, v_n) - M(v_n - \eta) - N(v_n - \eta)\} ds \\
w_{n+1}(t) &= w_0^0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \{f(s, w_n) + g(s, w_n) - M(w_n - \mu) - N(w_n - \mu)\} ds
\end{aligned} \tag{4.6}$$

it follows that $v(t)$ and $w(t)$ are solutions of IVP (1.1)-(1.2).

To prove that $v(t)$ and $w(t)$ are the minimal and maximal solutions of IVP (1.1)-(1.2), we need to prove that if $u(t)$ is any solution of IVP (1.1)-(1.2) such that $v_0 \leq u \leq w_0$ on $[t_0, T]$ then $v_0 \leq v \leq u \leq w_0$ on J. Suppose that for some n, $v_n(t) \leq u(t) \leq w_n(t)$ on J. Firstly, we prove $v_{n+1}(t) \leq u(t)$ on $[t_0, T]$. Set $p(t) = v_{n+1}(t) - u(t)$

so that by Lipschitz condition we have

$$\begin{aligned} D^q p(t) &= D^q v_{n+1}(t) - D^q u(t) \\ &= f(t,v_n) + g(t,v_n) - M(v_{n+1} - v_n) - f(t,u) - g(t,u) \\ &\leq -M(v_n - u) - M_i(v_n - u) - M(v_{n+1} - v_n) \\ &\leq -Mp(t) \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t)(t-t_0)^{1-q}\}_{t=t_0} &\leq 0 \end{aligned}$$

By Lemma 2.1, it follows that $p(t) \leq 0$. This implies $v_{n+1}(t) \leq u(t)$ on J.
Secondly, we prove that $u(t) \leq w_{n+1}(t)$. Consider $p(t) = u(t) - w_{n+1}(t)$. By Lipschitz condition we have

$$\begin{aligned} D^q p(t) &= D^q u(t) - D^q w_{n+1}(t) \\ &= f(t,u) + g(t,u) - M(w_{n+1} - w_n) - f(t,w_{n+1}) - g(t,w_{n+1}) \\ &\leq -M(u - w_{n+1}) \\ &\leq -Mp(t) \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t)(t-t_0)^{1-q}\}_{t=t_0} &\leq 0 \end{aligned}$$

Using Lemma 2.1,we get $p(t) \leq 0$. It follows that $u(t) \leq w_{n+1}(t)$. Since $v_0 \leq u \leq w_0$ on J, by induction we have $v_n(t) \leq u(t) \leq w_n(t)$ for all n. In limiting case as $n \to \infty$, it follows that $v(t) \leq u(t) \leq w(t)$ on J. □

Lastly,we prove the uniqueness of solution of IVP (1.1)-(1.2) in the following

Theorem 4.5. *Assume that (i)-(ii) of Theorem 4.1 hold and if*

$$|f(t,u(t)) - f(t,\overline{u})| \leq M|(u - \overline{u})|, \quad v_0 \leq \overline{u} \leq u \leq w_0, \quad M > 0$$

then $v(t) = w(t) = u(t)$ is the unique solution of IVP (1.1) − (1.2).

Proof. We need to prove only $v(t) \geq w(t)$. Set $p(t) = w(t) - v(t)$, we find by Lipschitz condition that

$$\begin{aligned} D^q p(t) &= D^q w(t) - D^q v(t) \\ &= f(t,w(t)) + g(t,w(t)) - f(t,v(t)) - g(t,v(t)) \\ &\leq Mp(t) \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t)(t-t_0)^{1-q}\}_{t=t_0} &\leq 0 \end{aligned}$$

Hence by Lemma 2.1, we have $v(t) \geq w(t)$. This shows that $v(t) = w(t) = u(t)$ is the unique solution of IVP (1.1)-(1.2). □

Theorem 4.6. *Assume that:*

(i) *$f(t,u(t))$ and $g(t,u(t))$ in $C[\Omega, R^2]$ and $f(t,u(t))$ nonincreasing in u for each $t \in [t_0, T]$,*

(ii) *$v_0(t)$ and $w_0(t)$ in $C(J,R)$ are coupled lower and upper solutions of type II of IVP* (1.1)-(1.2) *such that $v_0(t_0) \leq w_0(t_0)$ on J*

(iii) *$f(t,u(t)), g(t,u(t))$ satisfies one-sided Lipschitz condition,*

$$\begin{aligned} f(t,u(t)) - f(t,\overline{u}(t)) \geq -M(u - \overline{u}), M > 0, \overline{u} \geq u, \\ g(t,u(t)) - g(t,\overline{u}(t)) \geq -N(u - \overline{u}), N > 0, \overline{u} \geq u \end{aligned}$$

Then there exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ such that

$$\{v_n(t)\} \to v(t) \quad \text{and} \quad \{w_n(t)\} \to w(t) \text{as } n \to \infty$$

and $v(t)$ and $w(t))$ are minimal and maximal solutions of the IVP (1.1) − (1.2).

Proof. Proof can be given on the same line as in Theorem 4.1 □

Theorem 4.7. *Assume that (i)-(ii) of Theorem 4.3 hold and if*

$$|f(t,u(t)) - f(t,\overline{u})| \leq M|(u-\overline{u})|, \quad v_0 \leq \overline{u} \leq u \leq w_0, \quad M > 0$$

then $v(t) = w(t) = u(t)$ *is the unique solution of IVP* (1.1) − (1.2).

Proof. Proof can be given on the same line as in Theorem 4.2. □

Theorem 4.8. *Assume that:*

(i) $f(t,u(t))$ *and* $g(t,u(t))$ *in* $C[\Omega, R^2]$ *and* $f(t,u(t))$ *nonincreasing in* u *for each* $t \in [t_0, T]$,

(ii) $v_0(t)$ *and* $w_0(t)$ *in* $C(J,R)$ *are coupled lower and upper solutions of type III of IVP* (1.1)-(1.2) *such that* $v_0(t_0) \leq w_0(t_0)$ *on* J

(iii) $f(t,u(t)), g(t,u(t))$ *satisfies one-sided Lipschitz condition,*

$$f(t,u(t)) - f(t,\overline{u}(t)) \geq -M(u-\overline{u}), M > 0, \overline{u} \geq u,$$

$$g(t,u(t)) - g(t,\overline{u}(t)) \geq -N(u-\overline{u}), N > 0, \overline{u} \geq u$$

Then there exist monotone sequences $\{v_n(t)\}$ *and* $\{w_n(t)\}$ *such that*

$$\{v_n(t)\} \to v(t) \quad \text{and} \quad \{w_n(t)\} \to w(t) \text{as } n \to \infty$$

and $v(t)$ *and* $w(t))$ *are minimal and maximal solutions of the IVP* (1.1)-(1.2).

Proof. Proof can be given on the same line as in Theorem 4.1 □

Theorem 4.9. *Assume that (i)-(ii) of Theorem 4.5 hold and if*

$$|f(t,u(t)) - f(t,\overline{u})| \leq M|(u-\overline{u})|, \quad v_0 \leq \overline{u} \leq u \leq w_0, \quad M > 0$$

then $v(t) = w(t) = u(t)$ *is the unique solution of IVP* (1.1) − (1.2).

Proof. Proof can be given on the same line as in Theorem 4.2 □

5 Conclusion

Existence results obtained by Yaker and Koksal are improved for the class of continuous functions. Monotone method coupled with lower and upper solutions is developed for the initial value problem (1.1) − (1.2) when the function on the right hand side is sum of nondecreasing and nonincreasing functions. The method developed is successfully applied to obtain existence and uniqueness of solutions of the IVP (1.1) − (1.2).

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Boundary Value Problems for Fractional Differential Equations and Inclusions in Banach Spaces

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Abstract

In this paper, we are concerned with the existence of solutions for boundary value problems, first for a class of fractional differential equations and second for a class of fractional differential inclusions. The methods include techniques associated with measure of noncompactness in conjunction with fixed point theorems of Mönch type.

Keywords: Fractional differential equation, fractional differential inclusion, boundary value problem, measure of noncompactness, fixed point.

2010 MSC: 26A33, 34A08, 34A60, 34B15.

1 Boundary Value Problems for Fractional Differential Equations in Banach Spaces

1.1 Introduction

In this section, we are concerned with the existence of solutions for boundary value problems (BVP for short), for a class of fractional order differential equations, when we apply the method associated with the technique of measure of noncompactness and a fixed point theorem of Mönch type. This technique was mainly initiated in the monograph of Banas and Goebel [11] and subsequently developed and used in many papers; see, for example, Banas and Sadarangani [12], Guo *et al.* [25], Lakshimikantham and Leela [38], Mönch [42], and Szufla [47].

1.2 Preliminaries

We introduce notations, definitions, and preliminary facts, many of which will be used throughout the remainder of this paper.

Let $C(J, E)$ be the Banach space of all continuous functions from J into E with the norm

$$\|y\| = \sup\{|y(t)| : 0 \leq t \leq T\},$$

and we let $L^1(J, E)$ denote the Banach space of functions $y : J \longrightarrow E$ which are Bochner integrable with norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

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Let $L^\infty(J,E)$ be the Banach space of functions $y : J \to E$ which are bounded and equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 \,:\, \|y(t)\| \le c : \text{a.e. } t \in J\}.$$

Let $AC^1(J,E)$ is the space of functions $y : J \to E$, which are absolutely continuous whose first derivative y' is absolutely continuous.

For a given set V of functions $v : J \to E$, we define

$$V(t) = \{\vartheta(t) \,:\, \vartheta \in V\},\ t \in J,$$

$$V(J) = \{\vartheta(t) \,:\, \vartheta \in V,\ t \in J\}.$$

Definition 1.2.1. *([37, 45]). The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $r \in \mathbb{R}_+$ is defined by*

$$I_a^r h(t) = \int_a^t \frac{(t-s)^{r-1}}{\Gamma(r)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^r h(t) = h(t) * \varphi_r(t)$, where $\varphi_r(t) = \frac{t^{r-1}}{\Gamma(r)}$ for $t > 0$, and $\varphi_r(t) = 0$ for $t \le 0$, and $\varphi_r \to \delta(t)$ as $r \to 0$, where δ is the delta function.

Definition 1.2.2. *([37, 45]). For a function h given on the interval $[a,b]$, the r Riemann-Liouville fractional-order derivative of h, is defined by*

$$(D_{a+}^r h)(t) \quad = \quad \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-r-1} h(s)ds.$$

Here $n = [r] + 1$ and $[r]$ denotes the integer part of r.

For convenience, we first recall the definition of the Kuratowski measure of noncompactness, and summarize the main properties of this measure.

Definition 1.2.3. *([6, 11]) Let E be a Banach space and let Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha \,:\, \Omega_E \to [0,\infty)$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0, : B \subset \bigcup_{j=1}^m B_j \text{ and } diam(B_j) \le \epsilon\}\,; \text{here } B \in \Omega_E.$$

Properties:

(1) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(2) $\alpha(B) = \alpha(\overline{B})$.

(3) $A \subset B \Rightarrow \alpha(A) \le \alpha(B)$.

(4) $\alpha(A+B) \le \alpha(A) + \alpha(B)$.

(5) $\alpha(cB) = c\alpha(B)\,; c \in \mathbb{R}$.

(6) $\alpha(conB) = \alpha(B)$.

Here $\overline{B}$ and $conB$ denote the closure and the convex hull of the bounded set B, respectively.

The details of α and its properties can be found in [6, 11].

Definition 1.2.4. *A multivalued map $F \,:\, J \times E \to E$ is said to be Carathéodory if*

(1) $t \to F(t,u)$ is measurable for each $u \in E$.

(2) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in J$.

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 1.2.1. *([42],[5]) Let D be a bounded, closed and convex subset of a Banach space E such that $0 \in D$, and let N be a continuous mapping of D into itself. if the implication,*

$$V = \overline{co}N(V) \text{ or} V = N(V) \cup \{0\} \Longrightarrow \alpha(V) = 0, \tag{1.1}$$

holds for every subset V of D, then N has a fixed point.

Lemma 1.2.1. *([47]) Let D be a bounded, closed and convex subset of a Banach space $C(J, E)$, G be a continuous function on $J \times J$, and $f : J \times E \to E$ be a function satisfying the Carathéodory conditions, and suppose there exists $p \in L^1(J, \mathbb{R}_+)$ such that for each $t \in J$ and each bounded set $B \subset E$ one has*

$$\lim_{k \to 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t)\alpha(B); \text{ where } J_{t,k} = [t-k, t] \cap J. \tag{1.2}$$

If V is an equicontinuous subset of D, then

$$\alpha(\{\int_J G(s,t)f(s,y(s))ds \; : y \in V\}) \leq \int_J \|G(t,s)\| p(s)\alpha(V(s))ds. \tag{1.3}$$

1.3 Boundary Value Problems of Order $r \in (1, 2]$

We consider the boundary value problem with nonlocal conditions

$$D^r y(t) = f(t, y(t)), \text{ for a.e. } t \in J = [0, T], \tag{1.4}$$

$$y(0) = 0, \; y(T) = g(y), \tag{1.5}$$

where $1 < r \leq 2$, D^r is the Riemann-Liouville fractional derivative, $f : J \times E \to E$ is a continuous function, $g : E \to E$ is a continuous function and $(E, |\cdot|)$ denotes a Banach space.

Later we will study another boundary value problem for another fractional differential equation with nonlocal conditions. In particular, we will consider the boundary value problem with nonlocal conditions,

$$D^r y(t) = f(t, y(t)), \text{ for a.e. } t \in J = [0, T], \tag{1.6}$$

$$y(0) = 0, \; \beta y(\eta) = y(T), \tag{1.7}$$

where $1 < r \leq 2$, $0 < \beta\eta^{r-1} < 1$, $0 < \eta < 1$, and D^r, f, $(E, |\cdot|)$ are as in (1.4)-(1.5).

1.3.1 Main Results for (1.4)-(1.5) and (1.6)-(1.7)

Let us start by defining what we mean by a solution of the problem (1.4)-(1.5).

Definition 1.3.5. *A function $y \in C([0, T], E)$ is said to be a solution of (1.4)-(1.5) if y satisfies the equation $D^r y(t) = f(t, y(t))$ on J, and the conditions $y(0) = 0, \; y(T) = g(y)$.*

For the existence of solutions for the problem (1.4)-(1.5), we need the following auxiliary lemma.

Lemma 1.3.2. *[10] Let $r > 0$, and $h \in C(0, T) \cap L(0, T)$ then*

$$I^\alpha D^\alpha h(t) = h(t) + c_1 t^{r-1} + c_2 t^{r-2} + \ldots + c_n t^{r-n}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where n is the smallest integer greater than or equal to r.

Lemma 1.3.3. *Let $1 < \alpha \leq 2$ and let $h : [0, T] \to \mathbb{R}$ be continuous. A function $y \in C([0, T], E)$ is a solution of the fractional integral equation*

$$\begin{array}{rcl} y(t) & = & \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds \\ & + & \frac{t^{r-1}}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} h(s) ds - \frac{t^{r-1}}{T^{r-1}} g(y) \end{array} \tag{1.8}$$

if and only if y is a solution of the fractional BVP

$$D^r y(t) = h(t), \;\; t \in [0, T], \tag{1.9}$$

$$y(0) = 0, \; y(T) = g(y). \tag{1.10}$$

Proof: Assume y satisfies (1.9). Then Lemma 1.3.2 implies that

$$y(t) = c_1 t^{r-1} + c_2 t^{r-2} + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds.$$

From (1.10), a simple calculation gives

$$c_2 = 0,$$

and

$$c_1 = \frac{1}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} h(s) ds + \frac{1}{T^{r-1}} g(y).$$

Hence we get equation (1.8). Conversely, it is clear that if y satisfies the integral equation (1.8), then equations (1.9)-(1.10) hold. □

Theorem 1.3.2. *Assume the following hypotheses hold:*

(H1) *The function $f : J \times E \longrightarrow E$ satisfies the Carathéodory conditions.*

(H2) *There exists $p \in L^\infty(J, \mathbb{R}_+)$, such that*

$$\|f(t,y)\| \leq p(t)\|y\| \text{ for a.e. } t \in J \text{ and each } y \in E.$$

(H3) *There exists constant $k^* > 0$ such that*

$$\|g(y)\|| \leq k^*\|y\| \text{ for each } y \in E.$$

(H4) *For almost each $t \in J$ and each bounded set $B \subset E$ we have*

$$\lim_{k \to 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t)\alpha(B).$$

(H5) *For almost each bounded set $B \subset E$ we have*

$$\alpha(g(B)) \leq k^*\alpha(B).$$

Then the BVP (1.4)-(1.5) has at least one solution on $C(J, B)$, provided that

$$\frac{T^r + T^{2r}}{\Gamma(r)}\|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} < 1. \tag{1.11}$$

Proof. Transform the problem (1.4)-(1.5) into a fixed point problem. Consider the operator

$$\begin{aligned}(Ny)(t) \quad = \quad & \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s, y(s)) ds \\ & + \tfrac{t^{r-1}}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} f(s, y(s)) ds - \tfrac{t^{r-1}}{T^{r-1}} g(y).\end{aligned}$$

Clearly, from Lemma 1.3.3, the fixed points of N are solutions to (1.4)-(1.5).

Now, let $R > 0$ and consider the set

$$D_R = \{y \in C(J, E) : \|y\|_\infty \leq R\}.$$

We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, E)$. Then, for each $t \in J$,

$$\begin{aligned}|(Ny_n)(t) - (Ny)(t)| \quad \leq \quad & \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y_n(s) - f(s, y(s))\| ds \\ & + \tfrac{t^{r-1}}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} \|f(s, y_n(s) - f(s, y(s))\| ds.\end{aligned}$$

Let $\rho > 0$ be such that

$$\|y_n\|_\infty \leq \rho, \ \|y\|_\infty \leq \rho.$$

By (H2)-(H3) we have

$$\|f(s, y_n(s) - f(s, y(s))\| \leq 2\rho p(s) := \sigma(s); \ \sigma \in L^1(J, \mathbb{R}_+).$$

Since f is a Carathéodory function, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: *N maps D_R into itself.*

For each $y \in D_R$, by (H2) and (1.11) we have for each $t \in J$

$$\begin{array}{rcl} \|N(y)(t)\| & \leq & \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y(s))\| ds \\ & + & \frac{t^{r-1}}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} \|f(s, y(s))\| ds + \frac{t^{r-1}}{T^{r-1}} \|g(y)\| \\ & \leq & \frac{T^r + T^{2r}}{\Gamma(r)} \|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} \\ & \leq & R. \end{array}$$

Step 3: *$N(D_R)$ is bounded and equicontinuous.*

By Step 2, it is obvious that $N(D_R) \subset C(J, E)$ is bounded.

For the equicontinuity of $N(D_R)$, let $t_1, t_2 \in J$, $t_1 < t_2$, and $y \in D_R$. We have

$$\begin{array}{rcl} |(Ny)(t_2) - (Ny)(t_1)| & = & \Big| \frac{1}{\Gamma(r)} \int_0^{t_1} [(t_2 - s)^{r-1} - (t_1 - s)^{r-1}] f(s, y(s)) ds \\ & & + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (t_2 - s)^{r-1} f(s, y(s)) ds \\ & & + \frac{(t_1 - t_2)^{r-1}}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} |f(s, y(s))| ds \\ & & + \frac{(t_1 - t_2)^{r-1}}{T^{r-1}} g(y) \Big| \\ & \leq & \frac{p(t)}{\Gamma(r)} \int_0^{t_1} [(t_1 - s)^{r-1} - (t_2 - s)^{r-1}] ds \\ & & + \frac{p(t)}{\Gamma(r)} \int_{t_1}^{t_2} (t_2 - s)^{r-1} ds \\ & & + \frac{p(t)(t_2 - t_1)^{r-1}}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} ds + \frac{k^*(t_1 - t_2)^{r-1}}{T^{r-1}} \\ & \leq & \frac{p(t)}{\Gamma(r+1)} [(t_2 - t_1)^r + t_1^r - t_2^r] + \frac{p(t)}{\Gamma(r+1)} (t_2 - t_1)^r \\ & & + \frac{p(t)(t_2 - t_1)^{r-1}}{T^{r-1}\Gamma(r)} + \frac{k^*(t_1 - t_2)^{r-1}}{T^{r-1}} \\ & \leq & \frac{p(t)}{\Gamma(r+1)} (t_2 - t_1)^r + \frac{p(t)}{\Gamma(r+1)} (t_1^r - t_2^r) \\ & + & \frac{p(t)(t_2 - t_1)^{r-1}}{T^{r-1}\Gamma(r)} + \frac{k^*(t_1 - t_2)^{r-1}}{T^{r-1}}. \end{array}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_R such that $V \subset \overline{co}(N(V) \cup \{0\})$. V is bounded and equicontinuous, and therefore the function $\vartheta \to \vartheta = \alpha(V(t))$ is continuous on J. By (H3), Lemma1.2.1, and the properties of the measure α, we have for each $t \in J$

$$\begin{array}{rcl} \vartheta(t) & \leq & \alpha(N(V)(t) \cup \{0\}) \\ & \leq & \alpha(N(V)(t)) \\ & \leq & \int_0^t \frac{T^r + T^{2r}}{\Gamma(r)} p(s)\alpha(V(s))ds + \frac{k^* T^r}{\Gamma(r)} \alpha(V(t)) \\ & \leq & \|\vartheta\|_{L^\infty} \left[\frac{T^r + T^{2r}}{\Gamma(r)} \|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} \right]. \end{array}$$

This means that

$$\|\vartheta\|_{L^\infty} \left(1 - \frac{T^r + T^{2r}}{\Gamma(r)} \|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} \right) \leq 1.$$

By (1.11) it follows that $\|\vartheta\|_\infty = 0$, that is, $\vartheta = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying now Theorem 1.2.1 we conclude that N has a fixed point which is a solution of the problem (1.4)-(1.5). □

Example. As an application of Theorem 1.3.2, we consider the fractional differential equation

$$D^r y(t) = \frac{2}{19 + e^t} |y(t)|, \text{ for a.e. } t \in J = [0,1], \quad 1 < r \leq 2, \tag{1.12}$$

$$y(0) = 0, \ y(1) = \sum_{i=1}^{n} c_i y(t_i), \tag{1.13}$$

where $0 < t_1 < t_2 < \cdots < t_n < 1$, $c_i, \ i = 1, \ldots, n$, are given positive constants with $\sum_{i=1}^n c_i < \frac{4}{5}$.

Set

$$f(t,x) = \frac{2}{19 + e^t} x, \ (t,x) \in J \times [0, \infty),$$

Clearly, conditions (H1) and (H2) hold with

$$p(t) = \frac{2}{19 + e^t}.$$

Condition (1.11) is satisfied with $T = 1$ and $k^* = \frac{4}{5}$. Indeed

$$\frac{T^r + T^{2r}}{\Gamma(r)} \|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} \quad \leq \quad \frac{2}{25\Gamma(r)} < 1,$$

which is satisfied for each $r \in (1,2]$. Then by Theorem 1.2.1 (namely, Theorem 1.3.2), the problem (1.12)-(1.13) has a solution on $[0,1]$.

Now we study the fractional boundary value problem (1.6)-(1.7).

Definition 1.3.6. *A function $y \in AC([0,T], E)$ is said to be a solution of (1.6)-(1.7) if y satisfies the equation $D^r y(t) = f(t, y(t))$ on J, and the conditions (1.7).*

For the existence of solutions for the problem (1.6)-(1.7), we need the following auxiliary lemma.

Lemma 1.3.4. *Let $1 < r \leq 2$ and let $h : [0,T] \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$\begin{array}{rcl} y(t) & = & -\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds + \frac{t^{r-1}}{T^{r-1} - \beta\eta^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} h(s) ds \\ & & -\frac{\beta t^{r-1}}{T^{r-1} - \beta\eta^{r-1}\Gamma(r)} \int_0^\eta (\eta - s)^{r-1} h(s) ds \end{array} \tag{1.14}$$

if and only if y is a solution of the fractional BVP

$$D^r y(t) = h(t), \ t \in [0,T], \tag{1.15}$$

$$y(0) = 0, \ \beta y(\eta) = y(T). \tag{1.16}$$

Proof: Assume y satisfies (1.15), then Lemma 1.3.3 implies that

$$y(t) = -I^r h(t) + c_1 t^{r-1} + c_2 t^{r-2},$$

for $c_1, c_2 \in \mathbb{R}$. Consequently, the general solution is

$$y(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{r-1} h(s) ds + c_1 t^{r-1} + c_2 t^{r-2}.$$

From $y(0) = 0$, a simple calculation gives

$$c_2 = 0,$$

and from $\beta y(\eta) = y(T)$ combined with

$$y(T) = -\frac{1}{\Gamma(r)} \int_0^T (T-s)^{r-1} h(s) ds + c_1 T^{r-1},$$

$$y(\eta) = -\frac{1}{\Gamma(r)} \int_0^\eta (\eta-s)^{r-1} h(s) ds + c_1 \eta^{r-1},$$

we have

$$c_1 = \frac{1}{T^{r-1} - \beta\eta^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} h(s) ds - \frac{\beta}{T^{r-1} - \beta\eta^{r-1}\Gamma(r)} \int_0^\eta (\eta-s)^{r-1} h(s) ds.$$

Hence we get equation (1.14). Conversely, it is clear that if y satisfies equation (1.14), then equations (1.15)-(1.16) hold. □

Remark 1.3.1. *The problem (1.15)-(1.16) is equivalent to*

$$y(t) = \int_0^T G(t,s) h(s) ds. \tag{1.17}$$

where

$$G(t,s) = \begin{cases} \frac{[t(T-s)]^{r-1} - \beta t^{r-1}(\eta-s)^{r-1}}{T^{r-1} - \beta\eta^{r-1}\Gamma(r)} - \frac{(t-s)^{r-1}T^{r-1} - \beta\eta^{r-1}}{T^{r-1} - \beta\eta^{r-1}\Gamma(r)}, & 0 \le s \le t \le T, s \le \eta, \\ \frac{[t(T-s)]^{r-1} - (t-s)^{r-1}T^{r-1} - \beta\eta^{r-1}}{T^{r-1} - \beta\eta^{r-1}\Gamma(r)}, & 0 \le \eta \le s \le t \le T, \\ \frac{[t(T-s)]^{r-1} - \beta t^{r-1}(\eta-s)^{r-1}}{T^{r-1} - \beta\eta^{r-1}\Gamma(r)}, & 0 \le t \le s \le \eta \le T, \\ \frac{[t(T-s)]^{r-1}}{T^{r-1} - \beta\eta^{r-1}\Gamma(r)}, & 0 \le t \le s \le T, \eta \le s. \end{cases} \tag{1.18}$$

Remark 1.3.2. *The function $t \to \int_0^T |G(t,s)| ds$ is continuous on $[0,T]$, and hence is bounded.*

Theorem 1.3.3. *Assume (H1),(H2) and the following hypothesis:*

(H6) *For almost each $t \in J$ and each bounded set $B_1 \subset E$ we have*

$$\lim_{l \to 0^+} \alpha(f(J_{t,l} \times B_1)) \le p(t) \alpha(B_1).$$

Then the BVP (1.6)-(1.7) has at least one solution on $C(J,E)$, provided that

$$G^* T \|p\|_{L^\infty} < 1. \tag{1.19}$$

Proof: Transform the problem (1.6)-(1.7) into a fixed point problem. Consider the operator

$$(N_1 y)(t) = \int_0^T G(t,s) f(s, y(s)) ds \tag{1.20}$$

where the function $G(t,s)$ is given by (1.18).

(We remark that, from Lemma 1.3.4, the fixed points of N_1 are solutions to (1.6)-(1.7).)
Let $R_1 > 0$ and consider the set

$$D_{R_1} = \{y \in C(J,E) \, : \|y\|_\infty \leq R_1\}.$$

We shall show that N_1 satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *N_1 is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,E)$. Then, for each $t \in J$,

$$\begin{aligned} |(N_1y_n)(t) - (N_1y)(t)| &\leq \int_0^T |G(t,s)| \|f(s,y_n(s) - f(s,y(s))\| ds \\ &\leq \int_0^T \sup_{(t,s)\in J\times J} |G(t,s)| \|f(s,y_n(s) - f(s,y(s))\| ds. \\ &\leq \int_0^T G^* \|f(s,y_n(s) - f(s,y(s))\| ds, \end{aligned}$$

where

$$G^* = \sup_{(t,s)\in J\times J} |G(t,s)|.$$

Let $\rho > 0$ be such that

$$\|y_n\|_\infty \leq \rho \text{ and } \|y\|_\infty \leq \rho.$$

By (H1)-(H2) we have

$$\|f(s,y_n(s) - f(s,y(s))\| \leq 2\rho G^* p(s) := \sigma(s); \;\; \sigma \in L^1(J, \mathbb{R}_+).$$

Since f is a Carathéodory function, the Lebesgue dominated convergence theorem implies that

$$\|N_1(y_n) - N_1(y)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: *N_1 maps D_{R_1} into itself.* For each $y \in D_{R_1}$, by (H2) and (1.19) we have for each $t \in J$,

$$\begin{aligned} \|N_1(y)(t)\| &\leq \int_0^T |G(t,s)| \|f(s,y(s))\| ds \\ &\leq T\|p\|_{L^\infty} G^* \|p^*\|_{L^\infty} \\ &\leq R_1. \end{aligned}$$

Step 3: *$N_1(D_{R_1})$ is bounded and equicontinuous.*

By Step 2, it is obvious that $N_1(D_{R_1}) \subset C(J,E)$ is bounded.
For the equicontinuity of $N_1(D_{R_1})$. Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and $y \in D_{R_1}$. We have

$$\begin{aligned} |N_1(y)(\tau_2) - N_2(y)(\tau_1)| &= \int_0^T |G(\tau_2,s) - G(\tau_1,s)||f(s,y(s))| ds \\ &\leq \int_0^T p(s) R_1 |G(\tau_2,s) - G(\tau_1,s)| ds \end{aligned} \quad (1.21)$$

As $\tau_1 \longrightarrow \tau_2$, the right-hand side of the above inequality tends to zero.
Now let V be a subset of D_{R_1} such that $V \subset \overline{co}(N_1(V) \cup \{0\})$.
V is bounded and equicontinuous, and therefore the function $\vartheta \to \vartheta = \alpha(V(t))$ is continuous on J. By (H6), Lemma 1.2.1, and the properties of the measure α, we have for each $t \in J$,

$$\begin{aligned} \vartheta(t) &\leq \alpha(N_1(V)(t) \cup \{0\}) \\ &\leq \alpha(N_1(V)(t)) \\ &\leq \int_0^T p(s)|G(t,s)|\alpha(V(s)) ds \\ &\leq \vartheta\|_{L^\infty}[TG^*\|p\|_{L^\infty}]. \end{aligned}$$

This implies that

$$\|\vartheta\|_{L^\infty}(1 - [TG^*\|p\|_{L^\infty}]) \leq 1.$$

By (1.19) it follows that $\|\vartheta\|_\infty = 0$, that is, $\vartheta = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_{R_1}. Applying now Theorem 1.2.1 we conclude that N_1 has a fixed point which is a solution of the problem (1.6)-(1.7). □

1.4 Boundary Value Problems of Order $r \in (2,3]$

In this section, we will consider the boundary value problem

$$D^r y(t) = f(t, y(t)), \text{ for a.e. } t \in J = [0, T], \tag{1.22}$$

$$y(0) = 0, \ y'(T) = 0, \ y''(0) = 0, \tag{1.23}$$

where $2 < r \leq 3$, D^r is the Riemann-Liouville fractional derivative and f and $(E, |\cdot|)$ are as in (1.4)-(1.5). We will make use of some of the hypotheses (H1) - (H5) of Theorem 1.3.2 in this section.

1.4.1 Main Results for (1.22)-(1.23)

Let us start by defining what we mean by a solution of the problem (1.22)–(1.23).

Definition 1.4.7. *A function $y \in AC^2([0,T], E)$ is said to be a solution of (1.22)-(1.23) if it satisfies $D^r y(t) = f(t, y(t))$ on J, and the conditions $y(0) = 0, \ y'(T) = 0 \ y''(0) = 0$.*

For the existence of solutions for the problem (1.22)-(1.23), we need the following auxiliary lemma.

Lemma 1.4.5. *Let $2 < r \leq 3$ and let $h : [0,T] \to E$ be continuous. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds \\ &\quad - \tfrac{t^{r-1}}{\Gamma(r) T^{r-2}} \int_0^T (T-s)^{r-2} h(s) ds. \end{aligned} \tag{1.24}$$

if and only if y is a solution of the fractional BVP

$$D^r y(t) = h(t), \ t \in [0, T], \tag{1.25}$$

$$y(0) = 0, \ y'(T) = 0, \ y''(0) = 0. \tag{1.26}$$

Proof: Assume y satisfies (1.25), then Lemma 1.3.2 implies that

$$y(t) = c_1 t^{r-1} + c_2 t^{r-2} + c_3 t^{r-3} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{r-1} h(s) ds.$$

From (1.26), a simple calculation gives

$$c_2 = 0, \, c_3 = 0$$

and

$$c_1 = -\frac{1}{\Gamma(r) T^{r-2}} \int_0^T (T-s)^{r-2} h(s) ds.$$

Hence we get equation (1.24). Conversely, it is clear that if y satisfies equation (1.24), then equations (1.25)-(1.26) hold. □

Theorem 1.4.4. *Assume (H1), (H2) and the following hypothesis:*

(H7) For almost each $t \in J$ and each bounded set $B_2 \subset E$ we have

$$\lim_{k \to 0^+} \alpha(f(J_{t,k} \times B_2)) \leq p(t) \alpha(B_2).$$

Then the BVP (1.22)-(1.23) has at least one solution on $C(J, B)$, provided that

$$\left[\frac{T^r}{\Gamma(r+1)} + \frac{T^{2r}}{(r-1)\Gamma(r)} \right] \|p\|_{L^\infty} < 1. \tag{1.27}$$

Proof. Transform the problem (1.22)-(1.23) into a fixed point problem. Consider the operator

$$\begin{aligned} (N_2 y)(t) &= \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s, y(s)) ds \\ &\quad - \tfrac{t^{r-1}}{T^{r-2}\Gamma(r)} \int_0^T (T-s)^{r-2} f(s, y(s)) ds. \end{aligned}$$

Remark 1.4.3. *Clearly, from Lemma 1.4.5, the fixed points of N_2 are solutions to (1.22)-(1.23).*

Let $R_2 > 0$ and consider the set

$$D_{R_2} = \{y \in C(J, E) \, : \|y\|_\infty \leq R_2\}.$$

We shall show that N_2 satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *N_2 is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, E)$. Then, for each $t \in J$,

$$\begin{aligned} |(N_2y_n)(t) - (N_2y)(t)| \quad &\leq \quad \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y_n(s) - f(s, y(s))\| ds \\ &\quad - \tfrac{t^{r-1}}{T^{r-2}\Gamma(r)} \int_0^T (T-s)^{r-2} \|f(s, y_n(s) - f(s, y(s))\| ds. \end{aligned}$$

Let $\rho > 0$ be such that

$$\|y_n\|_\infty \leq \rho, \; \|y\|_\infty \leq \rho.$$

By (H2) we have

$$\|f(s, y_n(s) - f(s, y(s))\| \leq 2\rho p(s) := \sigma(s); \; \sigma \in L^1(J, \mathbb{R}_+).$$

Since f is a Carathéodory function, the Lebesgue dominated convergence theorem implies that

$$\|N_2(y_n) - N_2(y)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: *N_2 maps D_{R_2} into itself.* For each $y \in D_{R_2}$, by (H2) and (1.27) we have for each $t \in J$

$$\begin{aligned} \|N_2(y)(t)\| \quad &\leq \quad \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y(s))\| ds \\ &\quad - \quad \tfrac{t^{r-1}}{T^{r-2}\Gamma(r)} \int_0^T (T-s)^{r-2} \|f(s, y(s))\| ds \\ &\leq \quad R_2 \left[\left[\tfrac{T^r}{\Gamma(r+1)} + \tfrac{T^{2r}}{(r-1)\Gamma(r)} \right] \|p\|_{L^\infty} \right] \\ &\leq \quad R_2. \end{aligned}$$

Step 3: *$N_2(D_{R_2})$ is bounded and equicontinuous.*

By Step 2, it is obvious that $N_2(D_{R_2}) \subset C(J, E)$ is bounded.
For the equicontinuity of $N_2(D_{R_2})$. Let $t_1, \, t_2 \in J, \, t_1 < t_2$, and $y \in D_{R_2}$. we have

$$\begin{aligned} |(N_2y)(t_2) - (N_2y)(t_1)| \quad &= \quad \left\| \frac{1}{\Gamma(r)} \int_0^{t_1} [(t_2-s)^{r-1} - (t_1-s)^{r-1}] f(s, y(s)) ds \right. \\ &\quad + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (t_2-s)^{r-1} f(s, y(s)) ds \\ &\quad \left. + \frac{(t_2-t_1)^{r-1}}{T^{r-2}\Gamma(r)} \int_0^T (T-s)^{r-2} |f(s, y(s))| ds \right\| \\ &\leq \quad \frac{p(t)}{\Gamma(r)} \int_0^{t_1} [(t_1-s)^{r-1} - (t_2-s)^{r-1}] ds \\ &\quad + \frac{p(t)}{\Gamma(r)} \int_{t_1}^{t_2} (t_2-s)^{r-1} ds \\ &\quad + \frac{p(t)(t_2-t_1)^{r-1}}{T^{r-2}\Gamma(r-1)} \int_0^T (T-s)^{r-2} ds \\ &\leq \quad \frac{p(t)}{\Gamma(r+1)} [(t_2-t_1)^r + t_1^r - t_2^r] + \frac{p(t)}{\Gamma(r+1)} (t_2-t_1)^r \\ &\quad + \frac{p(t)(t_2-t_1)^{r-1}}{T^{2r}\Gamma(r)} \\ &\leq \quad \frac{p(t)}{\Gamma(r+1)} (t_2-t_1)^r + \frac{p(t)}{\Gamma(r+1)} (t_1^r - t_2^r) \\ &\quad + \frac{T^{2r} p(t)(t_2-t_1)^{r-1}}{\Gamma(r)}. \end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

Now let V_2 be a subset of D_{R_2} such that $V_2 \subset \overline{co}(N_2(V_2) \cup \{0\})$.
V_2 is bounded and equicontinuous, and therefore the function $\vartheta \to \vartheta = \alpha(V_2(t))$ is continuous on J. By (H3), Lemma 1.2.1, and the properties of the measure α, we have for each $t \in J$

$$\begin{array}{rcl}\vartheta(t) & \leq & \alpha(N_2(V_2)(t) \cup \{0\}) \\ & \leq & \alpha(N_2(V_2)(t)) \\ & \leq & \int_0^t \left[\frac{T^r}{\Gamma(r+1)} + \frac{T^{2r}}{\Gamma(r)}\right] p(s)\alpha(V_2(s))ds \\ & \leq & \|\vartheta\|_{L^\infty} \left[\left[\frac{T^r}{\Gamma(r+1)} + \frac{T^{2r}}{\Gamma(r)}\right] \|p\|_{L^\infty}\right].\end{array}$$

This means that

$$\|\vartheta\|_{L^\infty} \left(1 - \left[\frac{T^r}{\Gamma(r+1)} + \frac{T^{2r}}{\Gamma(r)}\right] \|p\|_{L^\infty}\right) \leq 1.$$

By (1.27) it follows that $\|\vartheta\|_\infty = 0$, that is, $\vartheta = 0$ for each $t \in J$, and then $V_2(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V_2 is relatively compact in D_{R_2}. Applying now Theorem 1.2.1, we conclude that N_2 has a fixed point which is a solution of the problem (1.22)–(1.23). ☐

2 Boundary Value Problems for Fractional Differential Inclusions in Banach Spaces

2.1 Introduction

In this section, we are concerned with the existence of solutions for boundary value problems, for a class of fractional order differential inclusions, when the right hand side is convex valued. This result relies on the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. Recently, this has proved to be a valuable tool in solving fractional differential equations and inclusions in Banach spaces; for details, see the papers of Lasota *et al.* [39], Agarwal *et al.* [4] and Benchohra *et al.* [18], [19], [20]. This result extends to the multivalued case some previous results in the literature, and constitutes an interesting contribution to this emerging field.

2.2 Preliminaries

We introduce notations, definitions, and preliminary facts that will be used in the remainder of this section.

Let $(E, \|\cdot\|)$ be a Banach space. Let $P_{cl}(E) = \{A \in \mathcal{P}(E) : A \text{ closed}\}$, $P_c(E) = \{A \in \mathcal{P}(E) : A \text{ convex}\}$, $P_{cp,c}(E) = \{A \in \mathcal{P}(E) : A \text{ compact and convex}\}$. A multivalued mapping $G : E \to \mathcal{P}(E)$ has a fixed point if there is $x \in E$ such that $x \in G(E)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Let X, Y be two sets, and $N : X \to \mathcal{P}(Y)$ be a set-valued map. We define

$$graph(N) = \{(x, y) \ : x \in X,\, y \in N(X)\}.$$

For more details on multi-valued maps see the books of Deimling [23], Aubin *et al.* [7, 8] and Hu and Papageorgiou [34].

Let $R > 0$, and let

$$B = \{x \in E \ : |x| \leq R\},$$

and

$$U = \{x \in C(J, E) \ : \|x\| \leq R\},$$

Clearly $\overline{U} = C(J, B)$.

For each $y \in C(J, E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J, E) \, : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}.$$

Theorem 2.2.5. *([32]) Let E be a Banach space and $C \subset L^1(J, E)$ be countable with $|u(t)| \leq h(t)$ for a.e. $t \in J$, and every $u \in C$, where $h \in L^1(J, \mathbb{R}_+)$. Then the function $\phi(t) = \alpha(C(t))$ belongs to $L^1(J, \mathbb{R}_+)$ and satisfies*

$$\alpha\left(\left\{\int_0^T u(s)ds \, : u \in C\right\}\right) \leq 2\int_0^T \alpha(C(s))ds,$$

where α is the Kuratowski measure of non compactness.

Let us now recall the set-valued analog of Mönch's fixed point theorem.

Theorem 2.2.6. *([44]) Let K be a closed, convex subset of a Banach space E, U a relatively open subset of K, and $N : \overline{U} \to \mathcal{P}_c(K)$. Assume that $graph(N)$ is closed, that N maps compact sets into relatively compact sets, and that, for some $x_0 \in U$, the following two conditions are satisfied :*

$$\begin{cases} M \subset \overline{U}, M \subset conv(x_0 \cup N(M)) \\ and\, \overline{M} = \overline{U} \; with\, C \subset M countable \end{cases} \Rightarrow \overline{M} compact, \tag{2.28}$$

$$x \in (1-\lambda)x_0 + \lambda N(x) \; for \; all x \in \overline{U} \setminus U, \; \lambda \in (0,1). \tag{2.29}$$

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

Lemma 2.2.6. *([39]) Let I be a compact real interval. Let F be a Carathéodory multivalued map and let Θ be a linear continuous map from $L^1(I, E) \to C(I, E)$. Then the operator*

$$\Theta \circ S_{F,y} \, : C(I, E) \to \mathcal{P}_{cp,c}(C(I, E)), \, y \mapsto (\Theta \circ S_{F,y})(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C(I, E) \times C(I, E)$.

2.3 Boundary Value Problems of Order $r \in (1, 2]$

We consider the boundary value problem

$$D^r y(t) \in F(t, y(t)), \; \text{for a.e. } t \in J = [0, T], \tag{2.30}$$

$$y(0) = 0, \; \beta y(\eta) = y(T), \tag{2.31}$$

where $1 < r \leq 2$, $0 < \beta\eta^{\alpha-1} < 1$, $0 < \eta < 1$, D^r is as in (1.6)-(1.7), $(E, || \cdot ||)$ is a Banach space, $F : J \times E \to \mathcal{P}(E)$ is a multivalued map, and $\mathcal{P}(E)$ is the family of all nonempty subsets of E.

2.3.1 Main Results for (2.30)-(2.31)

Let us start by defining what we mean by a solution of the problem (2.30)-(2.31).

Definition 2.3.8. *A function $y \in AC^1([0, T], E)$ is said to be a solution of (2.30)-(2.31) if there exists a function $v \in L^1(J, E)$ with $v(t) \in F(t, y(t))$, for a.e. $t \in J$, such that $D^r y(t) = v(t)$ on J, and the condition (2.31) is satisfied.*

For the existence of solutions for the problem (2.30)-(2.31), we make use of the auxiliary Lemma 1.3.2 and Lemma 1.3.4.

Theorem 2.3.7. *Assume the following hypotheses hold:*

(H'1) $\quad F : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carathéodory multi-valued map.*

(H'2) *For each $R > 0$, there exists a function $p \in L^1(J, E)$ and such that*

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v|,\ v(t) \in F(t,y)\} \le p(t)$$

for each $(t,y) \in J \times E$ with $|y| \le R$, and

$$\lim_{R\to+\infty} \inf \frac{\int_0^T p(t)dt}{R} = \delta < \infty.$$

(H'3) *There exists a Carathéodory function $\psi,: J \times [0,2R] \to \mathbb{R}_+$ such that*

$$\alpha(F(t,M)) \le \psi(t, \alpha(M)), \text{ a.e. } t \in J, \text{ and each } M \subset B,$$

and the unique solution $\varphi \in C(J,[0,2R])$ of the inequality,

$$\phi(t) \le 2\left[\int_0^t G(t,s)\varphi(s,\phi(s))ds\right] \tag{2.32}$$

is $\phi \equiv 0$.

Then the BVP (2.30)-(2.31) has at least one solution on $C(J,B)$, provided that

$$TG^*\|p\|_{L^\infty} < 1, \tag{2.33}$$

where

$$G^* = \sup_{(t,s)\in J\times J} |G(t,s)|.$$

Proof. Transform the problem (2.30)-(2.31) into a fixed point problem. Consider the multivalued operator

$$Q(y) = \left\{h \in C(J,E):\ h(t) = \int_0^t G(t,s)v(s)ds,\ \ v \in S_{F,y}\right\}.$$

We shall show that Q satisfies the assumptions of the set-valued analog of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *$Q(y)$ is convex for each $y \in C(J,E)$.*

Indeed, if $h_1,\ h_2$ belong to $Q(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = \int_0^t G(t,s)v_i(s)ds,\ i = 1,2.$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = \int_0^t G(t,s)[dv_1(s) + (1-d)v_2(s)]ds.$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in Q(y).$$

Step 2: *$Q(M)$ is relatively compact for each compact $M \subset \overline{U}$.*

Let $M \subset \overline{U}$ be a compact set and let $\{h_n\}$ by any sequence of elements of $Q(M)$. We show that $\{h_n\}$ has a convergent subsequence by using the Ascoli-Arzela criterion of compactness in $C(J,E)$. Since $h_n \in Q(M)$ there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$h_n(t) = \int_0^t G(t,s)v_n(s)ds.$$

Using Theorem 2.2.5 and the properties of the measure of noncompactness of Kuratowski, α, we have

$$\alpha(\{h_n(t)\}) \le 2\left[\int_0^t G^*\alpha(\{(v_n(s)\})ds\right]. \tag{2.34}$$

On the other hand, since $M(s)$ is compact in E, the set $\{v_n(s)\,;n\geq 1\}$ is compact. Consequently, $\alpha(\{v_n(s)\,;n\geq 1\})=0$ for a.e. $s\in J$.
Now (2.32) implies that$\{h_n(t)\,;n\geq 1\}$ is relatively compact in E, for each $t\in J$. In addition, for each $\tau_1,\ \tau_2\in J,\ \tau_1<\tau_2$, and $y\in D_R$, we have

$$\begin{aligned}|Q(y)(\tau_2)-Q(y)(\tau_1)| &= \int_0^T|G(\tau_2,s)-G(\tau_1,s)||v(s)|ds\\ &\leq \int_0^T p(s)|G(\tau_2,s)-G(\tau_1,s)|ds\\ &\leq \int_0^T p(s)ds|G(\tau_2,s)-G(\tau_1,s)|ds\end{aligned}\tag{2.35}$$

As $\tau_1\longrightarrow\tau_2$, the right-hand side of the above inequality tends to zero.
This shows that $\{h_n\,;n\geq 1\}$ is equicontinuous. Consequently, $\{h_n\,;n\geq 1\}$ is relatively compact in $C(J,E)$.

Step 3: *Q has a closed graph.*

Let $(y_n,h_n)\in graph(Q),\ n\geq 1,$ with $\|y_n-y\|,\ \|h_n-h\|\to 0$, as $n\to\infty$. We must show that $(y,h)\in graph(Q)$.

$(y_n,h_n)\in graph(Q)$ means that $h_n\in Q(y_n)$, which means that there exists $v_n\in S_{F,y_n}$, such that for each $t\in J$,

$$h_n(t)=\int_0^t G(t,s)v_n(s)ds.$$

Consider the continuous linear operator

$$\Theta\,:L^1(J,E)\to C(J,E),$$

$$\Theta(v)(t)\mapsto h_n(t)=\int_0^t G(t,s)v_n(s)ds$$

Clearly,

$$\|h_n(t)-h(t)\|\to 0,\ \text{as},\ n\to\infty.$$

From Lemma 2.2.6 it follows that $\Theta\circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t)\in\Theta(S_{F,y_n}).$$

Since $y_n\to y,$ Lemma 2.2.6 implies that

$$h(t)=\int_0^t G(t,s)v(s)ds$$

for some $v\in S_{F,y}$.

Step 4: Suppose $M\subset\overline{U},\ M\subset conv(\{0\}\cup Q(M)),$ and $\overline{M}=\overline{C}$ for some countable set $C\subset M$. Using an estimation of type (2.35), we see that $Q(M)$ is equicontinuous. Then from $M\subset conv(\{0\}\cup Q(M))$, We deduce that M is equicontinuous, too. In order to apply the Ascoli-Arzela theorem, it remains to show that $M(t)$ is relatively compact in E for each $t\in J$. Since

$$C\subset M\subset conv(\{0\}\cup Q(M))\ \text{and } C \text{ is countable,}$$

we can find a countable set $H=\{h_n\,:n\geq 1\}\subset Q(M)$ with $C\subset conv(\{0\}\cup H)$. Then, there exist $y_n\in M$ and $v_n\in S_{F,y_n}$ such that

$$h_n(t)=\int_0^t G(t,s)v_n(s)ds$$

From $M\subset\overline{C}\subset\overline{conv}(\{0\}\cup H))$, and according to Theorem 2.2.5, we have

$$\alpha(M(t))\leq(\alpha(\overline{C}(t))\leq\alpha(H(t))=\alpha(\{h_n((t)\,:n\geq 1\}).$$

Using(2.34), we obtain

$$\alpha(M(t))\leq 2\left[\int_0^t G^*\alpha(\{v_n(s)\})ds\right].$$

Now, since $v_n(s) \in M(s)$, we have

$$\alpha(M(t)) \leq 2\left[\int_0^t G^*\alpha(\{v_n(s);\, n \geq 1\})ds\right]$$

Also, since $v_n(s) \in M(s)$, we have

$$\alpha(\{v_n(s);\, n \geq 1\}) = \alpha(M(s))$$

It follows that

$$\begin{aligned}\alpha(M(t)) &\leq 2\left[\int_0^t G^*\alpha(M(s))ds\right] \\ &\leq 2\left[\int_0^t G^*\psi(s,\alpha(M(s)))ds\right].\end{aligned}$$

Also, the function ϕ given by $\phi(t) = \alpha(M(t))$ belongs to $C(J,[0,2R])$. Consequently by (H'3), $\phi \equiv 0$; that is, $\alpha(M(t)) = 0$ for all $t \in J$.
Now, by the Ascoli-Arzela theorem, M is relatively compact in $C(J,E)$.

Step 5: Let $h \in Q(y)$ with $y \in \overline{U}$. Since $|y(s)| \leq R$ and by (H'2), we have $Q(\overline{U}) \subseteq \overline{U}$, because if it were not true, then there exists a function $y \in \overline{U}$, but $\|Q(y)\|_{\mathcal{P}} > R$ and

$$h(t) = \int_0^t G(t,s)v(s)ds$$

for some $v \in S_{F,y}$. On the other hand, we have

$$\begin{aligned}R &\leq \|Q(y)\|_{\mathcal{P}} \\ &\leq \int_0^t |G(t,s)||v(s)|ds \\ &\leq TG^*\|p\|_{L^\infty}.\end{aligned}$$

Dividing both sides by R and taking the lower limits as $R \to \infty$, we conclude that $[TG^*\|p\|_{L^\infty}]\delta \geq 1$ which contradicts (2.33). Hence $Q(\overline{U}) \subseteq \overline{U}$.
As a consequence of Steps 1-5 together with Theorem 2.2.6, we can conclude that Q has a fixed point $y \in C(J,B)$ which is a solution of the problem (2.30)-(2.31). □

2.4 Boundary Value Problems of Order $r \in (2,3]$

In this section,we are concerned with the existence of solutions for the boundary value problem for a fractional differential inclusion,

$$D^r y(t) \in F(t,y(t)), \text{ for a.e. } t \in J = [0,T], \tag{2.36}$$

$$y(0) = 0,\ y'(0) = 0,\ y''(T) = 0, \tag{2.37}$$

where $2 < r \leq 3$, D^r is the Riemann-Liouville fractional derivative, F and $(E, ||\cdot||)$ as are in (2.30)-(2.31).

2.4.1 Main Results for (2.36)-(2.37)

Let us start by defining what we mean by a solution of the problem (2.36)–(2.37).

Definition 2.4.9. *A function $y \in AC^2([0,T],E)$ is said to be a solution of (2.36)-(2.37) if there exist a function $v \in L^1(J,E)$ with $v(t) \in F(t,y(t))$, for a.e. $t \in J$, such that $D^r y(t) = v(t)$ on J, and the condition $y(0) = 0,\ y'(0) = 0\ y''(T) = 0$.*

For the existence of solutions for the problem (2.36)-(2.37), we will make use of the auxiliary Lemma 1.3.2 and Lemma 1.4.5.

Theorem 2.4.8. *Assume (H'1)-(H'2) and the following hypothesis hold:*

(H'4) *There exists a Carathéodory function* $\psi : J \times [0, 2R] \to \mathbb{R}_+$ *such that*

$$\alpha(F(t, M)) \leq \psi(t, \alpha(M)), \text{ a.e. } t \in J, \text{ and each } M \subset B,$$

and the unique solution $\varphi \in C(J, [0, 2R])$ *of the inequality*

$$\begin{aligned} \phi(t) \quad \leq \quad & 2[\tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \varphi(s, \phi(s)) ds \\ & - \tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} \varphi(s, \phi(s)) ds], t \in J \end{aligned} \tag{2.38}$$

is $\phi \equiv 0$.

Then the BVP (2.36)-(2.37) has at least one solution on $C(J, B)$*, provided that*

$$\delta < \left[\frac{T}{\Gamma(r+1)} + \frac{T^2}{(r-1)(r-2)\Gamma(r-1)} \right]. \tag{2.39}$$

Proof. Transform the problem (2.36)-(2.37) into a fixed point problem. Consider the multivalued operator

$$Q_2(y) = \left\{ h \in C(J, E) : \begin{array}{rcl} h(t) & = & \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v(s) ds - \\ & & \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \times \\ & & \int_0^T (T-s)^{r-3} v(s) ds \end{array}, \ v \in S_{F,y} \right\}.$$

We shall show that Q_2 satisfies the assumptions of the set-valued analog of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *$Q_2(y)$ is convex for each $y \in C(J, E)$.*

Indeed, if $h_1, \ h_2$ belong to $Q_2(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) \quad = \quad & \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v_i(s) ds \\ & - \tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} v_i(s) ds, \ i = 1, 2. \end{aligned}$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) \quad = \quad & \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} [dv_1(s) + (1-d)v_2(s)] ds + \\ & \tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \times \\ & \int_0^T (T-s)^{r-3} [dv_1(s) + (1-d)v_2(s)] ds. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *$Q_2(M)$ is relatively compact for each compact $M \subset \overline{U}$.*

Let $M \subset \overline{U}$ be a compact set and let $\{h_n\}$ by any sequence of elements of $Q_2(M)$. We show that$\{h_n\}$ has a convergent subsequence by using the Ascoli-Arzela criterion of compactness in $C(J, E)$. Since $\{h_n\} \subset Q_2(M)$ there exist $\{y_n\} \in M$ and $v_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) \quad = \quad & \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v_n(s) ds \\ & - \tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} v_n(s) ds. \end{aligned}$$

Using Theorem 2.2.5 and the properties of the measure of Kuratowski α, we have

$$\begin{aligned} \alpha(\{h_n(t)\}) \quad \leq \quad & 2\Big[\tfrac{1}{\Gamma(r)} \int_0^t \alpha(\{(t-s)^{r-1} v_n(s)\}) ds \\ & - \tfrac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T \alpha(\{(T-s)^{r-3} v_n(s)\}) ds \Big]. \end{aligned} \tag{2.40}$$

On the other hand, since $M(s)$ is compact in E, the set $\{v_n(s)\,;n\geq 1\}$ is compact. Consequently, $\alpha(\{v_n(s)\,;n\geq 1\})=0$ for a.e. $s\in J$. Furthermore

$$\alpha(\{(t-s)^{r-1}v_n(s)\})=(t-s)^{r-1}\alpha(\{v_n(s)\,;n\geq 1\})=0$$

$$\alpha(\{(T-s)^{r-1}v_n(s)\})=(T-s)^{r-1}\alpha(\{v_n(s)\,;n\geq 1\})=0$$

for a.e. $t,\ s\in J$. Now (2.40) implies that$\{h_n(t)\,;n\geq 1\}$ is relatively compact in E, for each $t\in J$.

In addition for each t_1 and t_2 from J, $t_1<t_2$, we have

$$\begin{aligned}|h_n(t_2)-h_n(t_1)| &= \Big|\tfrac{1}{\Gamma(r)}\int_0^{t_1}[(t_2-s)^{r-1}-(t_1-s)^{r-1}]v_n(s)ds\\ &\quad+\tfrac{1}{\Gamma(r)}\int_{t_1}^{t_2}(t_2-s)^{r-1}v_n(s)ds\Big|\\ &\quad+\tfrac{(t_2-t_1)^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T(T-s)^{r-3}|v_n(s)|ds\\ &\leq \tfrac{p(t)}{\Gamma(r)}\int_0^{t_1}[(t_1-s)^{r-1}-(t_2-s)^{r-1}]ds\\ &\quad+\tfrac{p(t)}{\Gamma(r)}\int_{t_1}^{t_2}(t_2-s)^{r-1}ds\\ &\quad+\tfrac{p(t)(t_2-t_1)^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T(T-s)^{r-3}ds\\ &\leq \tfrac{p(t)}{\Gamma(r+1)}[(t_2-t_1)^r+t_1^r-t_2^r]+\tfrac{p(t)}{\Gamma(r+1)}(t_2-t_1)^r\\ &\quad+\tfrac{p(t)(t_2-t_1)^{r-1}}{\Gamma(r-1)}\\ &\leq \tfrac{p(t)}{\Gamma(r+1)}(t_2-t_1)^r+\tfrac{p(t)}{\Gamma(r+1)}(t_1^r-t_2^r)\\ &\quad+\tfrac{T^rp(t)(t_2-t_1)^{r-1}}{\Gamma(r-1)}.\end{aligned}\tag{2.41}$$

As $t_1\longrightarrow t_2$, the right-hand side of the above inequality tends to zero.This shows that $\{h_n\,;n\geq 1\}$ is equicontinuous. Consequently,$\{h_n\,;n\geq 1\}$ is relatively compact in $C(J,E)$.

Step 3: *Q_2 has a closed graph.*

Let $(y_n,h_n)\in graph(Q_2)$, $n\geq 1$, with $\|y_n-y\|$, $\|h_n-h\|\to 0$, as $n\to\infty$. We must show that $(y,h)\in graph(Q_2)$.
$(y_n,h_n)\in graph(Q_2)$ means that $h_n\in Q_2(y_n)$, which means that there exists $v_n\in S_{F,y_n}$, such that for each $t\in J$,

$$\begin{aligned}h_n(t) &= \tfrac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}v_n(s)ds\\ &\quad-\tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T(T-s)^{r-3}v_n(s)ds.\end{aligned}$$

Consider the continuous linear operator

$$\Theta\,:L^1(J,E)\to C(J,E)$$

$$\begin{aligned}\Theta(v)(t)\mapsto h_n(t) &= \tfrac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}v_n(s)ds\\ &\quad-\tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T(T-s)^{r-3}v_n(s)ds.\end{aligned}$$

Clearly,

$$\|h_n(t)-h(t)\|\to 0,\ \text{as } n\to\infty.$$

From Lemma 2.2.6 it follows that $\Theta\circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t)\in\Theta(S_{F,y_n}).$$

Since $y_n\to y$, Lemma 2.2.6 implies that

$$\begin{aligned}h(t) &= \tfrac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}v(s)ds\\ &\quad-\tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T(T-s)^{r-3}v(s)ds,\end{aligned}$$

for some $v\in S_{F,y}$.

Step 4: Suppose $M\subset\overline{U}$, $M\subset conv(\{0\}\cup Q_2(M))$, and $\overline{M}=\overline{C}$ for some countable set $C\subset M$. Using an estimation of type (2.41), we see that $Q_2(M)$ is equicontinuous. Then from $M\subset conv(\{0\}\cup Q_2(M))$, we deduce that M is equicontinuous, too. In order to apply the Ascoli-Arzela theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since

$$C \subset M \subset conv(\{0\} \cup N(M)) \text{ and } C \text{ is countable,}$$

we can find a countable set $H = \{h_n : n \geq 1\} \subset Q_2(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exists $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) &= \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v_n(s)ds \\ &\quad - \tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} v_n(s)ds. \end{aligned}$$

From $M \subset \overline{C} \subset \overline{conv}(\{0\} \cup H))$, and according to Theorem 2.2.5, we have

$$\alpha(M(t)) \leq (\alpha(\overline{c}(t)) \leq \alpha(H(t)) = \alpha(\{h_n((t) : n \geq 1\}).$$

Using (2.40), we obtain

$$\begin{aligned} \alpha(M(t)) &\leq 2[\tfrac{1}{\Gamma(r)} \int_0^t \alpha(\{(t-s)^{r-1} v_n(s)\})ds \\ &\quad - \tfrac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T \alpha(\{(T-s)^{r-3} v_n(s)\})ds]. \end{aligned}$$

Now, since $v_n \in M(s)$ we have

$$\begin{aligned} \alpha(M(t)) &\leq 2[\tfrac{1}{\Gamma(r)} \int_0^t \alpha(\{(t-s)^{r-1} v_n(s);\, n \geq 1\})ds \\ &\quad - \tfrac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T \alpha(\{(T-s)^{r-3} v_n(s);\, n \geq 1\})ds]. \end{aligned}$$

Also, since $v_n(s) \in M(s)$ we have

$$\alpha(\{(t-s)^{r-1} v_n(s);\, n \geq 1\}) = (t-s)^{r-1} \alpha(M(s))$$

and

$$\alpha(\{(T-s)^{r-1} v_n(s);\, n \geq 1\}) = (T-s)^{r-1} \alpha(M(s)).$$

It follows that

$$\begin{aligned} \alpha(M(t)) &\leq 2[\tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha(M(s))ds \\ &\quad - \tfrac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} \alpha(M(s))ds] \\ &\leq 2[\tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \psi(s, \alpha(M(s)))ds \\ &\quad - \tfrac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} \psi(s, \alpha(M(s)))ds]. \end{aligned}$$

Also, the function ϕ given by $\phi(t) = \alpha(M(t))$ belong to $C(J, [0, 2R])$. Consequently by (H'4), $\phi \equiv 0$, that is $\alpha(M(t)) = 0$ for all $t \in J$.
Now, by the Ascoli-Arzela theorem, M is relatively compact in $C(J, E)$.

Step 5: Let $h \in Q_2(y)$ with $y \in \overline{U}$. Since $|y(s)| \leq R$ and (H'2), we have $N(\overline{U}) \subseteq \overline{U}$, because if it is not true, then there exists a function $y \in \overline{U}$ but $\|Q_2(y)\|_{\mathcal{P}} > R$ and

$$\begin{aligned} h(t) &= \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v(s)ds \\ &\quad - \tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} v(s)ds, \end{aligned}$$

for some $v \in S_{F,y}$. On the other hand we have

$$\begin{aligned} R &\leq \|Q_2(y)\|_{\mathcal{P}} \\ &\leq \tfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} |v(s)|ds \\ &\quad - \tfrac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} |v(s)|ds \\ &\leq \tfrac{T}{\Gamma(r+1)} \int_0^t p(s)ds \\ &\quad - \tfrac{T^2}{(r-1)(r-2)\Gamma(r-1)} \int_0^T p(s)ds \\ &\leq \left[\tfrac{T}{\Gamma(r+1)} + \tfrac{T^2}{(r-1)(r-2)\Gamma(r-1)}\right] \int_0^T p(s)ds. \end{aligned}$$

Dividing both sides by R and taking the lower limits as $R \to \infty$, we conclude that $\left[\frac{T}{\Gamma(r+1)} + \frac{T^2}{(r-1)(r-2)\Gamma(r-1)}\right]\delta \geq 1$ which contradicts (2.39). Hence $Q_2(\overline{U}) \subseteq \overline{U}$.
As a consequence of Steps 1-5 together with Theorem 2.2.6, we can conclude that Q_2 has a fixed point $y \in C(J, B)$ which is a solution of the problem (2.36)-(2.37).

□

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Conditions for Oscillation and Convergence of Solutions to Second Order Neutral Delay Difference Equations with Variable Coefficients

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Abstract

In this paper, we deals with the second order neutral functional difference equation of the form

$$\Delta\left(r(n)\Delta(x(n)-p(n)x(n-\tau))\right)+q(n)f(x(n-\sigma))=0; \quad n\geq n_0 \quad (*)$$

where $\{r(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are sequences of real numbers, τ and σ are positive integers and $f:R\to R$ is a real valued function. We determine sufficient conditions under which every solutions of $(*)$ is either oscillatory or tends to zero.

Keywords: Oscillation, nonoscillation, second order, neutral, delay difference equations.

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1 Introduction

In this paper we deals with the second order neutral functional difference equation of the form

$$\Delta\left(r(n)\Delta(x(n)-p(n)x(n-\tau))\right)+q(n)f(x(n-\sigma))=0; \quad n\geq n_0 \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, τ and σ are positive integers, $\{r(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are sequences of real numbers, and $f:R\to R$ is a continuous function.

Throughout this paper we assume the following conditions to be hold:

(i) $\{q(n)\}$ is a sequence of nonnegative real numbers and $\{q(n)\}$ is not identically zero for sufficiently large values of n;

(ii) $\{p(n)\}$ is a sequence of nonnegative real numbers and there exist a constant p such that $0\leq p(n)\leq p<1$;

(iii) $\{r(n)\}$ is a sequence of positive real numbers;

(iv) there exist a constant k such that $\frac{f(u)}{u}\geq k>0$ for all $u\neq 0$.

Let $\{x(n)\}$ be a real sequences. We will also define a companion or associated sequence $\{z(n)\}$ of it by

$$z(n)=x(n)-p(n)x(n-\tau), \quad n\geq n_0. \tag{1.2}$$

Let $\theta=\max\{\tau,\sigma\}$. For any real sequence $\{\phi(n)\}$ defined in $n_0-\theta\leq n\leq n_0-1$, the equation (1.1) has a solution $\{x(n)\}$ defined for $n\geq n_0$ and satisfying the initial condition $x(n)=\phi(n)$ for $n_0-\theta\leq n\leq n_0-1$. A

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solution $\{x(n)\}$ of equation (1.1) is oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In this paper we establish sufficient conditions for the oscillation of solutions to (1.1) under the following two cases:

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} = \infty \tag{1.3}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} < \infty. \tag{1.4}$$

Recently, there has been much interest in studying the oscillatory and asymptotic behaviour of difference equations; see, for example, [3-10] and the references cited therein. For the general theory of difference equations one can refer to [1,2].

In [7], sternal et al. established sufficient conditions for every bounded solutions of (1.1) is either oscillatory are tends to zero as $n \to \infty$ under the conditions (1.3) and

$$\sum_{n=n_0}^{\infty} q(n) = \infty.$$

Rath et al. in [6] established sufficient conditions under which every solution of (1.1) is oscillatory are tends to zero as $n \to \infty$.

In [5] we established sufficient conditions for oscillation of all solutions of the equation (1.1) where $\{p(n)\}$ is a nonnegative real sequence.

In this paper our aim is to determine sufficient conditions under which every solution of (1.1) is oscillatory or tends to zero as $n \to \infty$. Our established results are discrete analogues of some well-known results due to [8].

In the sequel, for our convenience, when we write a fractional inequality without mentioning its domain of validity are assume that it holds for all sufficiently large values of n.

2 Some Useful Lemmas

In this section, we state and prove the following lemmas which are useful in proving our main results of this paper.

Lemma 2.1. [3] *Let $\{x(n)\}$ be an eventually positive solutions of (1.1) and $\{z(n)\}$ be its associated sequence defined by (1.2). If $\{\Delta z(n)\}$ is eventually negative or $\limsup_{n\to\infty} x(n) > 0$, then $z(n) > 0$, eventually.*

Lemma 2.2. *Assume that (1.3) holds. Let $\{x(n)\}$ be an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$. Then its associated sequence $\{z(n)\}$ defined by (1.2) satisfies $z(n) > 0, r(n)\Delta z(n) > 0$ and $\Delta(r(n)\Delta z(n)) < 0$ eventually.*

Proof. Assume that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$. Then it follows from (1.1) that $\Delta(r(n)\Delta z(n)) = -q(n)x(\sigma(n)) < 0$. Consequently $\{r(n)\Delta z(n)\}$ is decreasing and thus either $\Delta z(n) > 0$ or $\Delta z(n) < 0$, eventually. If we let $\Delta z(n) < 0$, then by Lemma 2.1, $z(n) > 0$ eventually. Then also $r(n)\Delta z(n) < -c < 0$ and summing this from n_1 to $n-1$, we have

$$z(n) \le z(n_1) - c\sum_{s=n_1}^{n-1} \frac{1}{r(s)} \to -\infty \quad as \quad n \to \infty.$$

This contradicts the positivity of $\{z(n)\}$ and hence $\Delta z(n) > 0$. Since $\limsup_{n\to\infty} x(n) > 0$, by Lemma 2.1 we have $z(n) > 0$ eventually and the proof is complete. □

3 Main Results

In this section, we derive sufficient conditions for oscillation of all solutions of (1.1). For the sake of convenience we use the following notations.

$$\begin{aligned} Q(n): &= \min\{q(n), q(n-\tau)\} \\ (\Delta\eta(n))_+ &= \max\{0, \Delta\eta(n)\} \\ R(n) &= \sum_{s=n_0}^{n-1} \frac{1}{r(s)}, \end{aligned}$$

and

$$\beta(n) = \sum_{s=n}^{\infty} \frac{1}{r(s)}.$$

Theorem 3.1. *Assume that (1.3) holds and $\sigma > \tau$. Suppose that there exist a positive real valued sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[k\eta(s)Q(s) - \frac{(1+p)r(s-\sigma)\left((\Delta\eta(s))_+\right)^2}{4\eta(s)} \right] = \infty. \tag{3.1}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$ and let $\{z(n)\}$ be its associated sequence defined by (1.2). Then there exists an integer $n_1 \geq n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$, $x(n-\sigma) > 0$ and $z(n) > 0$ for all $n \geq n_1$. Then from (1.1), we have

$$\Delta(r(n)\Delta z(n)) \leq -kq(n)x(n-\sigma) \leq 0, \quad n \geq n_1. \tag{3.2}$$

This shows that $\{r(n)\Delta z(n)\}$ is a decreasing sequence. Then by Lemma 2.2, $z(n) > 0$ and $\Delta z(n) > 0$, eventually. Now from (3.2), we have

$$\Delta(r(n)\Delta z(n)) + p\Delta(r(n-\tau)\Delta z(n-\tau)) + kq(n)x(n-\sigma) + pkq(n-\tau)x(n-\tau-\sigma) \leq 0.$$

or

$$\Delta(r(n)\Delta z(n)) + p\Delta(r(n-\tau)\Delta z(n-\tau)) + kQ(n)z(n-\sigma) \leq 0. \tag{3.3}$$

Define a sequence $\{u(n)\}$ by

$$u(n) = \eta(n)\frac{r(n)\Delta z(n)}{z(n-\sigma)}, \quad n \geq n_1. \tag{3.4}$$

Clearly $u(n) > 0$. Taking difference on both sides of (3.4) and using the fact, from (3.2) that $\Delta z(n-\sigma) \geq \frac{r(n+1)\Delta z(n+1)}{r(n-\sigma)}$, we have

$$\begin{aligned} \Delta u(n) &\leq \eta(n)\frac{\Delta(r(n)\Delta z(n))}{z(n-\sigma)} - \frac{\eta(n)}{\eta^2(n+1)}\frac{u^2(n+1)}{r(n-\sigma)} + \frac{u(n+1}{\eta(n+1)}\Delta\eta(n) \\ &\leq \eta(n)\frac{\Delta(r(n)\Delta z(n))}{z(n-\sigma)} - \frac{\eta(n)u^2(n+1)}{\eta^2(n+1)r(n-\sigma)} + \frac{u(n+1)}{\eta(n+1)}(\Delta\eta(n))_+. \end{aligned} \tag{3.5}$$

Similarly we introduce another sequence $\{v(n)\}$ defined by

$$v(n) = \eta(n)\frac{r(n-\tau)\Delta z(n-\tau)}{z(n-\sigma)}, \quad n \geq n_1. \tag{3.6}$$

Then $v(n) > 0$. Taking difference on both sides of (3.6), by (3.2) and $\sigma > \tau$, we see that

$$\Delta z(n-\sigma) \geq \frac{r(n-\tau+1)\Delta z(n-\tau+1)}{r(n-\sigma)},$$

and

$$\Delta v(n) \leq \eta(n)\frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n-\sigma)} - \frac{\eta(n)}{\eta^2(n+1)}\frac{v^2(n+1)}{r(n-\sigma)} + \frac{v(n+1)}{\eta(n+1)}\Delta\eta(n)$$

$$\leq \eta(n)\frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n-\sigma)} - \frac{\eta(n)}{\eta^2(n+1)}\frac{v^2(n+1)}{r(n-\sigma)} + \frac{v(n+1)}{\eta(n+1)}(\Delta\eta(n))_+. \tag{3.7}$$

From (3.5) and (3.7) we have

$$\Delta u(n) + p\Delta v(n) \leq \eta(n)\frac{\Delta(r(n)\Delta z(n))}{z(n-\sigma)} + p\eta(n)\frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n-\sigma)}$$

$$+\frac{u(n+1)}{\eta(n+1)}(\Delta\eta(n))_+ - \frac{\eta(n)}{\eta^2(n+1)}\frac{u^2(n+1)}{r(n-\sigma)} + p\frac{v(n+1)}{\eta(n+1)}(\Delta\eta(n))_+$$

$$- p\frac{\eta(n)}{\eta^2(n+1)}\frac{v^2(n+1)}{r(n-\sigma)}. \tag{3.8}$$

In view of (3.3) and the above inequality, we have

$$\Delta u(n) + p\Delta v(n) \leq -kQ(n)\eta(n) + \frac{u(n+1)(\Delta\eta(n))_+}{\eta(n+1)} - \frac{\eta(n)u^2(n+1)}{\eta^2(n+1)r(n-\sigma)}$$

$$+\frac{p(\Delta\eta(n))_+}{\eta(n+1)}v(n+1) - p\frac{\eta(n)v^2(n+1)}{\eta^2(n+1)r(n-\sigma)}$$

$$\leq -k\eta(n)Q(n) + (1+p)\frac{r(n-\sigma)((\Delta\eta(n))_+)^2}{4\eta(n)}.$$

Summing the above inequality from n_1 to $n-1$, we get

$$u(n) + pv(n) \leq u(n_1) + pv(n_1) - \sum_{s=n_1}^{n-1}\left[k\eta(s)Q(s) - \frac{(1+p)r(s-\sigma((\Delta\eta(n))_+)^2}{4\eta(s)}\right]$$

which implies that

$$\sum_{s=n_1}^{n-1}\left[k\eta(s)Q(s) - \frac{(1+p)r(s-\sigma)((\Delta\eta(s))_+)^2}{4\eta(s)}\right] \leq u(n_1) + pv(n_1)$$

which contradicts (3.1). This completes the proof. □

Choosing $\eta(n) = R(n-\sigma+1)$. By Theorem 3.1, we have the following results.

Corollary 3.2. *Assume that (1.3) holds and $\sigma > \tau$. If*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[kR(s-\sigma+1)Q(s) - \frac{(1+p)}{4r(s-\sigma)R(s-\sigma+1)}\right] = \infty. \tag{3.9}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Corollary 3.3. *Assume that (1.3) holds and $\sigma > \tau$. If*

$$\liminf_{n\to\infty}\frac{1}{\ln R(n-\sigma)}\sum_{s=n_0}^{n-1}R(s-\sigma+1)Q(s) > \frac{1+p}{4k}, \tag{3.10}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Proof. We can easily prove that (3.10) yields that there exists a constant $\epsilon > 0$ such that for all large n,

$$\frac{1}{\ln R(n-\sigma)} \sum_{s=n_0}^{n-1} R(s-\sigma+1)Q(s) > \frac{1+p}{4k} + \epsilon$$

which follows that

$$\sum_{s=n_0}^{n-1} R(s-\sigma+1)Q(s) - \left(\frac{1+p}{4k}\right) \ln R(n-\sigma) \geq \epsilon \ln R(n-\sigma),$$

that is

$$\sum_{s=n_0}^{n-1} \left[R(s-\sigma+1)Q(s) - \frac{1+p}{4kr(s-\sigma)R(s-\sigma+1)}\right] \geq \epsilon \ln R(n-\sigma) - \frac{(1+p)}{4k} \ln R(n_0-\sigma). \tag{3.11}$$

Now it is clear that (3.11) implies (3.9) and the assertion of Corollary 3.3 follows from Corollary 3.2. □

Corollary 3.4. *Assume that (1.3) holds and $\sigma > \tau$. If*

$$\liminf_{n\to\infty} \left[Q(n)R^2(n-\sigma+1)r(n-\sigma)\right] > \frac{1+p}{4k}, \tag{3.12}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Proof. It is easy to verify that (3.12) yields the existence of $\epsilon > 0$ such that for all large n,

$$Q(n)R^2(n-\sigma+1)r(n-\sigma) \geq \frac{1+p}{4k} + \epsilon.$$

Dividing the above inequality by $R(n-\sigma+1)r(n-\sigma)$, we have

$$Q(n)R(n-\sigma+1) - \frac{1+p}{4kR(n-\sigma+1)r(n-\sigma)} \geq \frac{\epsilon}{R(n-\sigma+1)r(n-\sigma)},$$

which implies that (3.9) holds. Therefore by Corollary 3.2, every solution of (1.1) is either oscillatory or tends to zero. □

Next, choosing $\eta(n) = n$. By Theorem 3.1, we have the following result.

Corollary 3.5. *Assume that (1.3) holds and $\sigma > \tau$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[ksQ(s) - \frac{1+p}{4s}\right] = \infty, \tag{3.13}$$

then every solution of (3.13) is either oscillatory or tends to zero.

Theorem 3.6. *Assume that (1.4) holds and $\sigma > \tau$. Suppose that there exists a positive real valued sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that (3.1) holds and*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[kQ(s)\beta(s+1) - \frac{1+p}{4r(s)\beta(s+1)}\right] = \infty. \tag{3.14}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$. Let $\{z(n)\}$ be the sequence defined by (1.2). Then by Lemma 2.1, $z(n) > 0$, eventually. Then there exists an integer $n_1 \geq n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$, $x(n-\sigma) > 0$ and $z(n) > 0$, for all $n \geq n_1$.

Clearly we can see that $\{r(n)\Delta z(n)\}$ is nonincreasing sequence eventually. Consequently, it is easy to conclude that there exist two possible cases of sign of $\{\Delta z(n)\}$, that is, $\Delta z(n) > 0$ or $\Delta z(n) < 0$ for $n \geq n_2 \geq n_1$. If $\Delta z(n) > 0$, then we are back to the case of Theorem 3.1, and we can get a contradiction to (3.1). If $\Delta z(n) < 0$, then we define the sequence $\{u(n)\}$ by

$$u(n) = \frac{r(n)\Delta z(n)}{z(n)}, \quad n \geq n_2. \tag{3.15}$$

Clearly $u(n) < 0$, Noting that $\{r(n)\Delta z(n)\}$ is nonincreasing, we have

$$r(s)\Delta z(s) \leq r(n)\Delta z(n), \quad s \geq n \geq n_2.$$

Dividing the above inequality by $r(s)$ and summing from n to $l-1$, we get

$$z(l) \leq z(n) + r(n)\Delta z(n) \sum_{s=n}^{l-1} \frac{1}{r(s)}, \quad l \geq n \geq n_2.$$

Letting $l \to \infty$ in the above inequality, we have

$$0 \leq z(n) + r(n)\Delta z(n)\beta(n), \quad n \geq n_2.$$

Therefore

$$\frac{r(n)\Delta z(n)}{z(n)}\beta(n) \geq -1, \quad n \geq n_2.$$

From (3.15), we have

$$-1 \leq u(n)\beta(n) \leq 0. \tag{3.16}$$

Similarly, we introduce another sequence $\{v(n)\}$ by

$$v(n) = \frac{r(n-\tau)\Delta z(n-\tau)}{z(n)}, \quad n \geq n_2. \tag{3.17}$$

Clearly $v(n) < 0$. Noting that $\{r(n)\Delta z(n)\}$ is nonincreasing, we have $r(n-\tau)\Delta z(n-\tau) \geq r(n)\Delta z(n)$. Then $v(n) > u(n)$. From (3.16), we obtain

$$-1 \leq v(n)\beta(n) \leq 0, \quad n \geq n_2. \tag{3.18}$$

Taking difference on both sides of (3.15), we have

$$\Delta u(n) = \frac{\Delta(r(n)\Delta z(n))}{z(n)} - \frac{\Delta u^2(n)}{r(n)}. \tag{3.19}$$

Again, taking difference on both sides of (3.17), we obtain

$$\Delta v(n) \leq \frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n)} - \frac{v^2(n)}{r(n)}. \tag{3.20}$$

From (3.19) and (3.20), we can obtain

$$\Delta u(n) + p\Delta v(n) \leq \frac{\Delta(r(n)\Delta z(n))}{z(n)} + p\frac{(r(n-\tau)\Delta z(n-\tau))}{z(n)} - \frac{u^2(n)}{r(n)} - p\frac{v^2(n)}{r(n)}. \tag{3.21}$$

On the other hand, proceed as in the proof of Theorem 3.1, we have that (3.3) holds. Therefore by (3.3) and (3.21), we get

$$\Delta u(n) + p\Delta v(n) \leq -kQ(n) - \frac{u^2(n)}{r(n)} - p\frac{v^2(n)}{r(n)}. \tag{3.22}$$

Multiplying by $\beta(n+1)$ on (3.22) and summing from n_2 to $n-1$, we have

$$[\beta(n)u(n) - \beta(n_2)u(n_2)] - \sum_{s=n_2+1}^{n-1} u(s)\Delta\beta(s) + p\,[\beta(n)v(n) - \beta(n_2)v(n_2)]$$

$$-p\sum_{s=n_2+1}^{n-1} v(s)\Delta\beta(s) + k\sum_{s=n_2}^{n-1} Q(s)\beta(s+1) + \sum_{s=n_2}^{n-1} \frac{u^2(s)}{r(s)}\beta(s+1)$$

$$+p\sum_{s=n_2}^{n-1} \frac{v^2(s)\beta(s+1)}{r(s)} \leq 0$$

or

$$[\beta(n)u(n)-\beta(n_2)u(n_2)]+\sum_{s=n_2+1}^{n-1}\frac{u(s)}{r(s)}+p\left[\beta(n)v(n)-\beta(n_2)v(n_2)\right]$$
$$+p\sum_{s=n_2+1}^{n-1}\frac{v(s)}{r(s)}+k\sum_{s=n_2+1}^{n-1}Q(s)\beta(s+1)+\sum_{s=n_2+1}^{n-1}\frac{u^2(s)\beta(s+1)}{r(s)}$$
$$+p\sum_{s=n_2+1}^{n-1}\frac{v^2(s)\beta(s+1)}{r(s)}\leq 0$$

or

$$[\beta(n)u(n)+p\beta(n)v(n)]+\sum_{s=n_2+1}^{n-1}\left[\frac{u(s)}{r(s)}+\frac{u^2(s)\beta(s+1)}{r(s)}\right]$$
$$+p\sum_{s=n_2+1}^{n-1}\left[\frac{v(s)}{r(s)}+\frac{v^2(s)\beta(s+1)}{r(s)}\right]+k\sum_{s=n_2+1}^{n-1}Q(s)\beta(s+1)$$
$$\leq \beta(n_2)u(n_2)+p\beta(n_2)v(n_2)$$

or

$$[\beta(n)u(n)+p\beta(n)v(n)]+\sum_{s=n_2+1}^{n-1}\left[kQ(s)\beta(s+1)-\frac{1+p}{4r(s)\beta(s+1)}\right]$$
$$\leq \beta(n_2)u(n_2)+p\beta(n_2)v(n_2).$$

By (3.16) and (3.18) we obtain a contradiction with (3.14). This completes the proof. □

Corollary 3.7. *Assume that (1.4) holds and $\sigma>\tau$. Furthermore assume that one of conditions (3.9), (3.10), (3.12) and (3.13) holds, and one has (3.14). Then every solution of (1.1) is either oscillatory or tends to zero.*

Theorem 3.8. *Assume that (1.4) holds and $\sigma>\tau$. Suppose that there exists a positive real sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that (3.1) holds, and*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\beta^2(s+1)Q(s)=\infty. \tag{3.23}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) and let $\{z(n)\}$ be its associated sequence defined by (1.2). Then by Lemma 2.1, $z(n)>0$, eventually. Then there exists an integer $n_1\geq n_0$ such that $x(n)>0$, $x(n-\tau)>0$, $x(n-\sigma)>0$ and $z(n)>0$ for all $n\geq n_1$. Also we see that $\{r(n)\Delta z(n)\}$ is nonincreasing eventually. Consequently, it is easy to see that there exist two possible cases of the sign of $\{\Delta z(n)\}$, that is, $\Delta z(n)>0$ or $\Delta z(n)<0$ for $n\geq n_2\geq n_1$. If $\Delta z(n)>0$, then we have back to the case of Theorem 3.1 and we can get a contradiction to (3.1). If $\Delta z(n)<0$, then we define the sequences $\{u(n)\}$ and $\{v(n)\}$ as in Theorem 3.6. Then proceed as in the proof of Theorem 3.6, we obtain (3.16), (3.18) and (3.22).

Multiplying (3.22) by $\beta^2(n+1)$ and summing from n_2 to $n-1$ yields,

$$\beta^2(n)u(n)-\beta^2(n_2)u(n_2)+2\sum_{s=n_2+1}^{n-1}\frac{u(s)\beta(s)}{r(s)}+\sum_{s=n_2}^{n-1}\frac{u^2(s)\beta^2(s+1)}{r(s)}$$
$$+p\beta^2(n)v(n)-p\beta^2(n_2)v(n_2)+2p\sum_{s=n_2+1}^{n-1}\frac{v(s)\beta(s)}{r(s)}+p\sum_{s=n_2}^{n-1}\frac{v^2(s)\beta^2(s+1)}{r(s)}$$
$$+k\sum_{s=n_2}^{n-1}\beta^2(s+1)Q(s)\leq 0 \tag{3.24}$$

If follows from (1.4) and (3.16) that

$$\begin{aligned}\left|\sum_{s=n_2+1}^{\infty}\frac{u(s)\beta(s+1)}{r(s)}\right| &\leq \sum_{s=n_2+1}^{\infty}\frac{|u(s)\beta(s)|}{r(s)} \leq \sum_{s=n_2+1}^{\infty}\frac{1}{r(s)} < \infty,\\ \sum_{s=n_2}^{n-1}\frac{u^2(s)\beta^2(s+1)}{r(s)} &\leq \sum_{s=n_2}^{n-1}\frac{u^2(s)\beta^2(s)}{r(s)} < \sum_{s=n_2}^{\infty}\frac{1}{r(s)} < \infty.\end{aligned}$$

In view of (3.18), we get

$$\begin{aligned}\left|\sum_{s=n_2+1}^{\infty}\frac{v(s)\beta(s)}{r(s)}\right| &\leq \sum_{s=n_2+1}^{\infty}\frac{|v(s)\beta(s)|}{r(s)} \leq \sum_{s=n_2+1}^{\infty}\frac{1}{r(s)} < \infty,\\ \sum_{s=n_2}^{\infty}\frac{v^2(s)\beta^2(s+1)}{r(s)} &\leq \sum_{s=n_2}^{\infty}\frac{v^2(s)\beta^2(s)}{r(s)} \leq \sum_{s=n_2}^{\infty}\frac{1}{r(s)} < \infty.\end{aligned}$$

From (3.24), we have

$$\limsup_{n\to\infty}\sum_{s=n_2}^{n-1}\beta^2(s+1)Q(s) < \infty,$$

which is a contradiction with (3.23). This completes the proof. □

Corollary 3.9. *Assume that (1.4) holds and $\sigma > \tau$. Suppose also that one of conditions (3.9), (3.10), (3.12) and (3.13) holds and one has (3.23). Then every solution of (1.1) is either oscillatory or tends to zero.*

In the following, we give some new oscillation results for (1.1) when $\sigma \leq \tau$.

Theorem 3.10. *Assume that (1.3) holds and $\sigma \leq \tau$. Moreover, suppose that there exists a positive real valued sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[k\eta(s)Q(s) - \frac{(1+p)r(s-\tau)((\Delta\eta(s))_+)^2}{4\eta(s)}\right] = \infty. \tag{3.25}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$ and let $\{z(n)\}$ be its associated sequence defined by (1.2). Then by Lemma 2.2 $z(n) > 0$ and $\Delta z(n) > 0$ eventually. Then there exists an integer $n_1 \geq n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$, $x(n-\sigma) > 0$ and $z(n) > 0$ for all $n \geq n_1$. Similar to the proof of Theorem 3.1, there exists an integer $n_2 \geq n_1$ such that (3.3) hold for $n \geq n_2$. Define a sequence $\{u(n)\}$ by

$$u(n) = \eta(n)\frac{r(n)\Delta z(n)}{z(n-\tau)}, \quad n \geq n_2. \tag{3.26}$$

Then $u(n) > 0$. Taking difference on both sides of (3.26), by (3.2), we get

$$\Delta z(n-\tau) \geq \frac{r(n)\Delta z(n)}{r(n-\tau)},$$

and

$$\begin{aligned}\Delta u(n) &\leq \frac{\eta(n)\Delta(r(n)\Delta z(n))}{z(n-\tau)} - \frac{\eta(n)u^2(n+1)}{\beta^2(n+1)r(n-\tau)} + \frac{u(n+1)}{\eta(n+1)}\Delta\eta(n)\\ &\leq \frac{\eta(n)\Delta(r(n)\Delta z(n))}{z(n-\tau)} - \frac{\eta(n)u^2(n+1)}{\eta^2(n+1)r(n-\tau)} + \frac{u(n+1)}{\eta(n+1)}(\Delta\eta(n))_+.\end{aligned} \tag{3.27}$$

Also we define an another sequence $\{v(n)\}$ by

$$v(n) = \eta(n)\frac{r(n-\tau)\Delta z(n-\tau)}{z(n-\tau)}, \quad n \geq n_2. \tag{3.28}$$

Note that $\sigma \leq \tau$. The rest of the proof is similar to that of the Theorem 3.1 and so is omitted. This completes the proof. □

Corollary 3.11. *Assume that (1.3) holds and $\sigma \leq \tau$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[kR(s-\tau+1)Q(s) - \frac{1+p}{4r(s-\tau)R(s-\tau+1)} \right] = \infty, \tag{3.29}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Corollary 3.12. *Assume that (1.3) hold and $\sigma \leq \tau$. If*

$$\liminf_{n\to\infty} \frac{1}{\ln R(n-\tau)} \sum_{s=n_0}^{n-1} R(s-\tau+1)Q(s) > \frac{1+p}{4k}, \tag{3.30}$$

then every solution of (1.1) is oscillatory or tends to zero.

Proof. By Corollary 3.11, the proof is similar to that of Corollary 3.3, we omit the details. □

Corollary 3.13. *Assume that (1.3) holds and $\sigma \leq \tau$. If*

$$\liminf_{n\to\infty} \Big(Q(n)R^2(n-\tau+1)r(n-\tau) \Big) > \frac{1+p}{4k}, \tag{3.31}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Proof. By Corollary 3.11, the proof is similar to that of Corollary 3.4 and so is omitted. □

Next, choosing $\eta(n) = n$. From Theorem 3.8 we have the following result.

Corollary 3.14. *Assume that (1.3) holds and $\sigma \leq \tau$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[ksQ(s) - (1+p)\frac{r(s-\tau)}{4s} \right] = \infty, \tag{3.32}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Theorem 3.15. *Assume that (1.4) hold and $\sigma \leq \tau$. Further suppose that there exists a positive real valued sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that (3.25) holds. Suppose also that one of (3.14) and (3.23) holds. Then every solution of (1.1) is either oscillatory or tends to zero.*

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$ and let $\{z(n)\}$ be its associated sequence defined by (1.2).

Then by Lemma 2.1, $z(n) > 0$, eventually. Then there exists an integer $n_1 \geq n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$, $x(n-\sigma) > 0$ and $z(n) > 0$ for all $n \geq n_1$. In view of (3.2), $\{r(n)\Delta z(n)\}$ is nonincreasing eventually. Consequently, it is easy to conclude that there exists two possible cases of the sign of $\{\Delta z(n)\}$. That is, $\Delta z(n) > 0$ or $\Delta z(n) < 0$ for $n \geq n_2 \geq n_1$. If $\Delta z(n) > 0$, then we are base to the case of Theorem 3.10. If $\Delta z(n) < 0$, then by the proof of Theorem 3.6 or Theorem 3.8, we can obtain a contradiction to (3.14) or (3.23) respectively. The proof is complete. □

Corollary 3.16. *Assume that (1.4) holds and $\sigma \geq \tau$. Suppose that one of conditions (3.29), (3.30) and (3.32) holds, and one has (3.14) or (3.23). Then every solution of (1.1) is either oscillatory or tends to zero.*

4 Some Example

In this section we give some examples to illustrate our results.

Example 4.1. *Consider the following second order neutral delay difference equation*

$$\Delta\left[(n+2\sigma)\Delta(x(n)-p(n)x(n-\tau))\right] + \frac{\lambda}{n+\sigma} f(x(n-\sigma)) = 0; \quad n = 0,1,2,.... \tag{4.1}$$

where $0 \leq p(n) \leq p < 1$, τ *and* σ *are positive integers with* $\sigma > \tau$, $r(n) = n + 2\sigma$, $q(n) = \frac{\lambda}{n+\sigma}$, $\lambda > 0$ *and* $f(x) = x(1 + x^2)$. *Take* $\eta(n) = n + \sigma$, *we have* $k = 1$, $Q(n) = \frac{\lambda}{n+\tau}$. *Now*

$$\limsup_{n\to\infty} \sum_{s=0}^{n-1} \left[k\eta(s)Q(s) - \frac{(1+p)r(s-\tau)((\Delta\eta(s))_+)^2}{4\eta(s)} \right]$$
$$= \sum_{s=0}^{n-1} \left[\lambda - \left(\frac{1+p}{4} \right) \right] = \infty,$$

for $\lambda > \frac{1+p}{4}$. *Hence by Theorem 3.1, every solution of (1.1) is either oscillatory or tends to zero.*

Example 4.2. *Consider the following second order neutral delay difference equation*

$$\Delta \left[(n+\tau)\Delta(x(n) - p(n)x(n-\tau)) \right] + \frac{\lambda}{n} \Big((x(n-\sigma)(1 + x^2(n-\sigma)) \Big) = 0, \quad n = 1, 2, ... \tag{4.2}$$

where $0 \leq p(n) \leq p < 1$, τ *and* σ *are positive integers with* $\sigma \leq \tau$, $r(n) = n + \tau$, $q(n) = \frac{\lambda}{n}$, $\lambda > 0$ *and* $k = 1$. *Clearly,* $Q(n) = \frac{\lambda}{n}$. *Now*

$$\limsup_{n\to\infty} \sum_{s=1}^{n-1} \left[ksQ(s) - \left(\frac{1+p}{4} \right) \frac{r(s-\tau)}{s} \right]$$
$$= \limsup_{n\to\infty} \sum_{s=1}^{n-1} \left[\lambda - \left(\frac{1+p}{4} \right) \right]$$
$$= \infty$$

for $\lambda > \frac{1+p}{4}$. *Hence by Corollary 3.14, every solution of (4.2) is either oscillatory or tends to zero.*

Example 4.3. *Consider the following second order neutral delay difference equation*

$$\Delta \left[e^n \Delta(x(n) - p(n)x(n-1)) \right] + e^{2n} x(n-2)(1 + x^2(n-2)) = 0, \quad n = 0, 1, ... \tag{4.3}$$

where $0 \leq p(n) \leq p < 1$, τ *and* σ *are positive integers with* $\sigma > \tau$, $r(n) = e^n$, $q(n) = e^{2n}$ *and* $k = 1$. *We have,* $Q(n) = e^{2n-2}$, $\beta(n+1) = \frac{1}{e^n(e-1)}$. *Clearly,* $\sum_{n=0}^{\infty} \frac{1}{r(n)} < \infty$. *Also*

$$\limsup_{n\to\infty} \sum_{s=0}^{n-1} \beta^2(s+1)Q(s)$$
$$= \limsup_{n\to\infty} \sum_{s=0}^{n-1} \frac{1}{e^{2s}(s-1)^2} e^{2s-2}$$
$$= \infty$$

Also,

$$\liminf_{n\to\infty} \left[Q(n)R^2(n-\sigma+1)r(n-\sigma) \right]$$
$$= \liminf_{n\to\infty} \left[e^{2n-2}R^2(n-1)r(n-2) \right]$$
$$= \liminf_{n\to\infty} \left[e^{2n-2} \left(\frac{e^n - 1}{e^{n-1}(e-1)} \right)^2 e^{n-2} \right]$$
$$= \liminf_{n\to\infty} \left[\frac{(e^n-1)^2}{(e-1)^2} e^{n-2} \right]$$
$$> \frac{1+p}{\epsilon}$$

Then by Corollary 3.9, every solution of (1.1) is either oscillatory or tends to zero.

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Oscillation conditions for first order neutral difference equations with positive and negative variable co-efficients

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Abstract

In this article, we analysis the oscillatory properties of first order neutral difference equations with positive and negative variable coefficients of the forms

$$\Delta[x(n)+p(n)x(n-\tau)]+\sum_{i=1}^{m}q_i(n)x(n-\sigma_i)-\sum_{j=1}^{k}r_j(n)x(n-\rho_j)=0; \quad n=0,1,2,..., \qquad (*)$$

and

$$\Delta[x(n)+p(n)x(n+\tau)]+\sum_{i=1}^{m}q_i(n)x(n+\sigma_i)-\sum_{j=1}^{k}r_j(n)x(n+\rho_j)=0; \quad n=0,1,2,..., \qquad (**)$$

where $\{p(n)\}$ is a sequence of real numbers, $\{q_i(n)\}$ and $\{r_j(n)\}$ are sequences of positive real numbers, τ is a positive integer, σ_i and ρ_j are nonnegative integers, for $i=1,2,...,m$ and $j=1,2,...,k$. We established sufficient conditions for oscillation of solutions to $(*)$ and $(**)$.

Keywords and Phrases: Oscillatory properties, neutral, delay, advanced, difference equation, positive and negative coefficients.

AMS Subject Classifications (2010): 39A10, 39A12.

1 Introduction

In this article, we analysis the oscillatory properties of the first order neutral delay and advanced difference equations with several positive and negative coefficients of the forms

$$\Delta[x(n)+p(n)x(n-\tau)]+\sum_{i=1}^{m}q_i(n)x(n-\sigma_i)-\sum_{j=1}^{k}r_j(n)x(n-\rho_j)=0; \quad n=0,1,2,..., \qquad (1.1)$$

and

$$\Delta[x(n)+p(n)x(n+\tau)]+\sum_{i=1}^{m}q_i(n)x(n+\sigma_i)-\sum_{j=1}^{k}r_j(n)x(n+\rho_j)=0; \quad n=0,1,2,..., \qquad (1.2)$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, $\{p(n)\}$ is a sequence of real numbers, $\{q_i(n)\}$ and $\{r_j(n)\}$ are sequences of positive real numbers, τ is a positive integer, and σ_i and ρ_j are nonnegative integers for $i=1,2,...,m$ and $j=1,2,...,k$.

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Throughout the paper we assumed that there exist a constant p such that $-1 < p \leq p(n) \leq 0$; eventually and $\{p(n)\}$ is monotonically.

In the last many years there has been an improving curiosity in the work of the oscillation concept of neutral difference and differential equations. The oscillation and asymptotic properties of these equations has been used in many areas of applied mathematics, such as population dynamics [4], stability theory [12,13], circuit theory [3], bifurcation analysis [2], dynamical behavior of delayed network systems [14] and so on.

In [11], *Öğü*nmez et al. established sufficient conditions for oscillation of all solutions of (1.1) and (1.2) when $p \equiv 0$, $m = k$, $q_i(n) = q_i$ and $r_j(n) \equiv r_j$. In [8], we derived sufficient conditions for oscillation of all solutions of the equations (1.1) and (1.2) for the cases $-1 < p < 0$, $m = k$, $q_i(n) = q_i$ and $r_j(n) = r_j$. The results obtained in [8] improves the results in [11]; In [9], we derived sufficient conditions for oscillation of all solutions of the equations (1.1) and (1.2) for the cases $p(n) \equiv p$ with $-1 < p < 0$.

For the general background of difference equations, one can refer to the books [1,5] and the papers [2-4, 6-14] and reference cited therein. Our main aim in this paper is to obtain the sufficient conditions for the oscillation of all solutions of equations (1.1) and (1.2).

Let $n^* = \max\{\tau, \sigma_i, \rho_j\}$ for $i = 1, 2, ..., m$ and $j = 1, 2..., k$. A solution of (1.1) on $N(n_0) = \{n_0, n_0 + 1, ...\}$ is defined as a real sequence $\{x(n)\}$ defined for $n \geq n_0 - n^*$ and which satisfies (1.1) for $n \in N(n_0)$. A solution $\{x(n)\}$ of (1.1) on $N(n_0)$ is said to be oscillatory if for every positive integers $N_0 > n_0$, there exists $n \geq N_0$ such that $x(n)x(n+1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory.

Furthermore, unless otherwise stated, when we write a functional inequality it indicates that it holds for all sufficiently large values of n.

2 Some Useful Lemmas

The following lemmas are very useful to prove our main results.

Lemma 2.1. *Let $\{x(n)\}$ be an eventually positive solution of the delay difference equation*

$$\Delta[x(n) + p(n)x(n-\tau)] + \sum_{i=1}^{m} q_i(n)x(n-\sigma_i) = 0. \tag{2.1}$$

Set

$$z(n) = x(n) + p(n)x(n-\tau). \tag{2.2}$$

Then $z(n) > 0$ and $\Delta z(n) < 0$ eventually.

Proof. From (2.1) and (2.2), we obtain

$$\Delta z(n) = -\sum_{i=1}^{m} q_i(n)x(n-\sigma_i) \leq 0. \tag{2.3}$$

This shows that $\{z(n)\}$ is a decreasing sequence.

Then either $z(n) > 0$ or $z(n) < 0$ eventually. If $z(n) < 0$, then

$$x(n) \leq -p(n)x(n-\tau) \leq -px(n-\tau)$$

or

$$x(n + k\tau) \leq (-p)^k x(n),$$

which implies that $x(n) \to 0$ as $n \to \infty$. Since $\{p(n)\}$ is bounded, we have $z(n) \to 0$ as $n \to \infty$ and consequently $z(n) > 0$, eventually.

This completes the proof. □

Lemma 2.2. [6] *Assume that*

$$\frac{(\bar{m}+1)^{\bar{m}+1}}{\bar{m}^{\bar{m}}} \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) > 1, \tag{2.4}$$

where $\alpha_i(n) \geq 0$, $1 \leq i \leq r$ and $\bar{m} = \min_{1 \leq i \leq r} m_i$. Then the delay difference inequality

$$\Delta x(n) + \sum_{i=1}^{r} \alpha_i(n)x(n-m_i) \leq 0; \quad n = 0, 1, 2, ..., \tag{2.5}$$

has no eventually positive solution.

Lemma 2.3. *Let $\{x(n)\}$ be an eventually positive solution of the neutral advanced difference equation*

$$\Delta[x(n)+p(n)x(n+\tau)]-\sum_{i=1}^{m}q_i(n)x(n+\sigma_i)=0; \quad n\geq n_0. \tag{2.6}$$

Set

$$z(n)=x(n)+p(n)x(n+\tau). \tag{2.7}$$

If

$$\sum_{n=n_0}^{\infty}\sum_{i=1}^{m}q_i(n)=+\infty, \tag{2.8}$$

then $z(n)>0$ and $\Delta z(n)>0$ eventually.

Proof. From (2.6) and (2.7), we have

$$\Delta z(n)=\sum_{i=1}^{m}q_i(n)x(n+\sigma_i)\geq 0. \tag{2.9}$$

This shows that $\{z(n)\}$ is an increasing sequence. Then either $z(n)>0$ or $z(n)<0$, eventually.

If $z(n)<0$, then

$$x(n)<-p(n)x(n+\tau)<x(n+\tau).$$

This shows that $\{x(n)\}$ is bounded from below by a positive constant, say M.

From (2.9), we have

$$\Delta z(n)\geq M\sum_{i=1}^{m}q_i(n), \tag{2.10}$$

which, in view of (2.8), implies that $z(n)\to+\infty$ as $n\to+\infty$. This is a contradiction and this completes the proof. □

Lemma 2.4. [8] *Consider the advanced difference inequality*

$$\Delta x(n)-\sum_{i=1}^{m}q_i(n)x(n+\sigma_i)\geq 0; \quad n\geq n_0. \tag{2.11}$$

If

$$\frac{\sigma^{\sigma}}{(\sigma-1)^{\sigma-1}}\sum_{i=1}^{m}\liminf_{n\to\infty}q_i(n)>1, \tag{2.12}$$

where $\sigma=\min_{1\leq i\leq m}\sigma_i$, then (2.11) cannot have an eventually positive solution.

3 Sufficient Conditions for Oscillations of Equation (1.1)

In this section, we establish sufficient conditions for the oscillation of all solutions of the neutral delay difference equation (1.1).

Theorem 3.1. *Assume that $\Delta p(n)\leq 0$ and $m=k$. Suppose that for $i=1,2,...,m$, $\sigma_i=\rho_i$, $\sigma_i>\tau$, $q_i(n)-r_i(n)\geq 0$ and not identically zero and*

$$q_i(n)-r_i(n)\geq q_i(n-\tau)-r_i(n-\tau). \tag{3.1}$$

Suppose that for $i=1,2,...,m$,

$$\frac{(\sigma'-\tau+1)^{\sigma'-\tau+1}}{(\sigma'-\tau)^{\sigma'-\tau}}\sum_{i=1}^{m}\liminf_{n\to\infty}\left(\frac{q_i(n)-r_i(n)}{1+p(n-\sigma+\tau-\sigma_i)}\right)>1, \tag{3.2}$$

where

$$\sigma'=\min_{1\leq i\leq m}\sigma_i \quad and \quad \sigma=\max_{1\leq i\leq m}\sigma_i.$$

Then every solution of (1.1) is oscillatory.

Proof. Assume the contrary. Without loss of generality, we suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) and let $\{z(n)\}$ be its associated sequence defined by (2.2). Then by Lemma 2.1, $z(n) > 0$ and $\Delta z(n) < 0$, eventually.

Then the equation (1.1) becomes,

$$\Delta z(n) = \sum_{i=1}^{m}(r_i(n) - q_i(n))x(n-\sigma_i). \tag{3.3}$$

Set

$$y(n) = z(n) + p(n-\sigma)z(n-\tau). \tag{3.4}$$

Then

$$\begin{aligned}\Delta y(n) &\leq \sum_{i=1}^{m}(r_i(n) - q_i(n))x(n-\sigma_i)\\ &+ p(n-\sigma)\sum_{i=1}^{m}(r_i(n-\tau) - q_i(n-\tau))x(n-\sigma_i-\tau)\\ &\leq \sum_{i=1}^{m}(r_i(n) - q_i(n))(x(n-\sigma_i) + p(n-\sigma_i)x(n-\sigma_i-\tau))\end{aligned}$$

$$= \sum_{i=1}^{m}(r_i(n) - q_i(n))z(n-\sigma_i) \leq 0. \tag{3.5}$$

This shows that $\{y(n)\}$ is a decreasing sequence. By applying the procedure used in Lemma 2.1, we can easily show that $y(n) > 0$, eventually.

Now, from (3.4), we have

$$\frac{y(n)}{1+p(n-\sigma)} \leq z(n-\tau), \tag{3.6}$$

or

$$\frac{y(n+\tau-\sigma_i)}{1+p(n-\sigma+\tau-\sigma_i)} \leq z(n-\sigma_i). \tag{3.7}$$

Using (3.7) in (3.5), we have

$$\Delta y(n) + \sum_{i=1}^{m}\left(\frac{q_i(n) - r_i(n)}{1+p(n-\sigma+\tau-\sigma_i)}\right)y(n-(\sigma_i-\tau)) \leq 0. \tag{3.8}$$

In view of (3.2) and Lemma 2.2, the delay difference inequality (3.8) has no eventually positive solution, which contradicts the fact that $y(n) > 0$, eventually.

This completes the proof. □

Theorem 3.2. *Assume that $\Delta p(n) \leq 0$ and $m = k$. Suppose that*

(i) *there exists a partition of the set $\{1, 2, ..., m\}$ into two disjoint subsets I and J such that $i \in I$ implies $\sigma_i > \rho_i$ and $j \in J$ implies $\sigma_j = \rho_j$;*

(ii) *$g_i(n) = q_i(n) - r_i(n-\sigma_i+\rho_i) \geq 0$ and not identically zero for $i = 1, 2, ..., m$;*

and

(iii) *$g_i(n) \geq g_i(n-\tau)$ and $\sigma_i > \tau$ for $i = 1, 2, ..., m$.*

Suppose further that

$$\sum_{i\in I}\sum_{s=n-\sigma_i+\rho_i}^{n-1} r_i(s) \leq 1 + p(n) \tag{3.9}$$

and

$$\frac{(\sigma' - \tau + 1)^{\sigma'-\tau+1}}{(\sigma' - \tau)^{\sigma'-\tau}}\sum_{i=1}^{m}\liminf_{n\to\infty}\left(\frac{g_i(n)}{1+p(n+\tau-\sigma-\sigma_i)}\right) > 1, \tag{3.10}$$

where $\sigma' = \min_{1\leq i\leq m}\sigma_i$ and $\sigma = \max_{1\leq i\leq m}\sigma_i$.

Then every solution of (1.1) is oscillatory.

Proof. On the contrary, we assume without loss of generality that $\{x(n)\}$ is an eventually positive solution of (1.1). Set

$$z(n) = x(n) + p(n)x(n-\tau) - \sum_{i\in I}\sum_{s=n-\sigma_i+\rho_i}^{n-1} r_i(s)x(s-\rho_i). \tag{3.11}$$

Then by Lemma 2.1 in [10], $z(n) > 0$ and $\Delta z(n) \le 0$ eventually.

Now,

$$\begin{aligned}\Delta z(n) &= -\sum_{i=1}^{m} q_i(n)x(n-\sigma_i) + \sum_{i=1}^{m} r_i(n)x(n-\rho_i)\\ &- \sum_{i\in I} r_i(n)x(n-\rho_i) + \sum_{i\in I} r_i(n-\sigma_i+\rho_i)x(n-\sigma_i)\\ &= -\sum_{i=1}^{m} q_i(n)x(n-\sigma_i) + \sum_{i=1}^{m} r_i(n-\sigma_i+\rho_i)x(n-\sigma_i)\end{aligned}$$

$$\Delta z(n) = -\sum_{i=1}^{m} g_i(n)x(n-\sigma_i). \tag{3.12}$$

Set

$$y(n) = z(n) + p(n-\sigma)z(n-\tau) \tag{3.13}$$

where $\sigma = \max_{1\le i\le m}\sigma_i$. Then

$$\begin{aligned}\Delta y(n) &\le \Delta z(n) + p(n-\sigma)\Delta z(n-\tau)\\ &\le -\sum_{i=1}^{m} g_i(n)x(n-\sigma_i) - p(n-\sigma)\sum_{i=1}^{m} g_i(n-\tau)x(n-\tau-\sigma_i)\\ &\le -\sum_{i=1}^{m} g_i(n)[x(n-\sigma_i) + p(n-\sigma)x(n-\tau-\sigma_i)]\\ &\le -\sum_{i=1}^{m} g_i(n)[x(n-\sigma_i) + p(n-\sigma_i)x(n-\tau-\sigma_i)]\end{aligned}$$

$$= -\sum_{i=1}^{m} g_i(n)z(n-\sigma_i) \le 0. \tag{3.14}$$

This shows that $\{y(n)\}$ is a nonincreasing sequence. We claim that $y(n) > 0$, eventually.

Otherwise $y(n) < 0$. This implies that

$$z(n) < -p(n-\sigma)z(n-\tau) \le -pz(n-\tau)$$

and hence we have $z(n) \to 0$ as $n \to \infty$. Since $\{p(n)\}$ is bounded, we have $y(n) \to 0$ as $n \to \infty$, which is a contradiction.

From (3.13), we get

$$\frac{y(n)}{1+p(n-\sigma)} \le z(n-\tau)$$

or

$$\frac{y(n+\tau-\sigma_i)}{1+p(n+\tau-\sigma-\sigma_i)} \le z(n-\sigma_i) \tag{3.15}$$

Using (3.15) is (3.14), we have

$$\Delta y(n) + \sum_{i=1}^{m}\left[\frac{g_i(n)}{1+p(n+\tau-\sigma-\sigma_i)}\right] y(n-(\sigma_i-\tau)) \le 0. \tag{3.16}$$

By Lemma 2.2 and (3.10), the delay difference inequality (3.16) has no eventually positive solution, which leads to a contradiction.

This completes the proof.

□

Theorem 3.3. *Assume that $\Delta p(n) \le 0$. Suppose that*

(*i*) *there exists a positive integer $l \le m$ and a partition of the set $\{1, 2, ..., k\}$ into l disjoint subsets $J_1, J_2, ..., J_l$ such that $j \in J_i$ implies $\rho_j < \sigma_i$;*

(*ii*) *$g_i(n) = q_i(n) - \sum_{u \in J_i} r_u(n - \sigma_i + \rho_u) \ge 0$ and are not identically zero for $i = 1, 2, ..., l$, $g_i(n) = q_i(n)$ for $i = l + 1, ..., m$;*

and

(*iii*) *$g_i(n) \ge g_i(n - \tau)$ and $\sigma_i > \tau$ for $i = 1, 2, ..., m$.*

Suppose further that

$$\sum_{i=1}^{l} \sum_{j \in J_i} \sum_{s=n-\sigma_i+\rho_j}^{n-1} r_j(s) \le 1 + p(n), \quad \text{eventually} \tag{3.17}$$

and

$$\frac{(\sigma' - \tau + 1)^{\sigma' - \tau + 1}}{(\sigma' - \tau)^{\sigma' - \tau}} \sum_{i=1}^{m} \liminf_{n \to \infty} \left(\frac{g_i(n)}{1 + p(n + \tau - \sigma - \sigma_i)} \right) > 1, \tag{3.18}$$

where $\sigma' = \min_{1 \le i \le m} \sigma_i$ and $\sigma = \max_{1 \le i \le m} \sigma_i$.

Then every solution of (1.1) is oscillatory.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1). Set

$$z(n) = x(n) + p(n)x(n - \tau) - \sum_{i=1}^{l} \sum_{u \in J_i} \sum_{s=n-\sigma_i+\rho_u}^{n-1} r_u(s)x(s - \rho_u). \tag{3.19}$$

Then

$$\begin{aligned} \Delta z(n) &= \Delta[x(n) + p(n)x(n - \tau)] \\ &- \sum_{i=1}^{l} \sum_{u \in J_i} \left[\sum_{s=n+1-\sigma_i+\rho_u}^{n} r_u(s)x(s - \rho_u) - \sum_{s=n-\sigma_i+\rho_u}^{n-1} r_u(s)x(s - \rho_u) \right] \\ &= -\sum_{i=1}^{m} q_i(n)x(n - \sigma_i) + \sum_{j=1}^{k} r_j(n)x(n - \rho_j) \\ &- \sum_{i=1}^{l} \sum_{u \in J_i} [r_u(n)x(n - \rho_u) - r_u(n - \sigma_i + \rho_u)x(n - \sigma_i)] \\ &= -\sum_{i=1}^{m} q_i(n)x(n - \sigma_i) + \sum_{i=1}^{l} \sum_{u \in J_i} r_u(n - \sigma_i + \rho_u)x(n - \sigma_i) \end{aligned}$$

or

$$\Delta z(n) = -\sum_{i=1}^{m} g_i(n)x(n - \sigma_i) \le 0. \tag{3.20}$$

This shows that $\{z(n)\}$ is nonincreasing sequence. By Lemma 2.1 in [10], we can show that $z(n) > 0$, eventually.

Set

$$y(n) = z(n) + p(n - \tau)z(n - \tau), \tag{3.21}$$

where $\sigma = \max_{1 \le i \le m} \sigma_i$.

Then

$$\begin{aligned} \Delta y(n) &\le \Delta z(n) + p(n - \sigma)\Delta z(n - \tau) \\ &= -\sum_{i=1}^{m} g_i(n)x(n - \sigma_i) - p(n - \sigma) \sum_{i=1}^{m} g_i(n - \tau)x(n - \tau - \sigma_i) \end{aligned}$$

$$\leq -\sum_{i=1}^{m} g_i(n)z(n-\sigma_i) \leq 0. \tag{3.22}$$

Clearly $\{y(n)\}$ is a nonincreasing sequence. By applying the procedure in Theorem 3.2, we can easily show that $y(n) > 0$, eventually.

Again from (3.21)

$$y(n) \leq (1 + p(n-\sigma))z(n-\tau)$$

or

$$\frac{y(n+\tau-\sigma_i)}{1+p(n+\tau-\sigma-\sigma_i)} \leq z(n-\sigma_i). \tag{3.23}$$

Using (3.23) in (3.22), we obtain

$$\Delta y(n) + \sum_{i=1}^{m}\left(\frac{g_i(n)}{1+p(n+\tau-\sigma-\sigma_i)}\right) y(n-(\sigma_i-\tau)) \leq 0. \tag{3.24}$$

But in view of Lemma 2.2 and (3.18), the delay difference inequality (3.24) has no eventually positive solution. This contradiction compelets the proof. □

4 Sufficient Conditions for Oscillation of Equation (1.2)

Theorem 4.1. *Assume that $\Delta p(n) \geq 0$ and $m = k$. Suppose that for $i = 1, 2, ..., m$, $\sigma_i = \rho_i$, $\rho_i > \tau$, $h_i(n) = r_i(n) - q_i(n) \geq 0$ and are not identically zero, and $h_i(n) \geq h_i(n+\tau)$.*

Suppose further that

$$\sum_{n=0}^{\infty}\sum_{i=1}^{m} h_i(n) = +\infty \tag{4.1}$$

and

$$\frac{(\rho'-\tau)^{\rho'-\tau}}{(\rho'-\tau-1)^{\rho'-\tau-1}} \sum_{i=1}^{m} \liminf_{n\to\infty}\left(\frac{h_i(n)}{1+p(n+\rho-\tau+\rho_i)}\right) > 1, \tag{4.2}$$

where $\rho' = \min_{1\leq i\leq m}\rho_i$ and $\rho = \max_{1\leq i\leq m}\rho_i$.

Then every solution of (1.2) is oscillatory.

Proof. For the sake of contradiction, without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.2).

Set

$$z(n) = x(n) + p(n)x(n+\tau). \tag{4.3}$$

Then from (1.2) and (4.3), we obtain

$$\Delta z(n) = \sum_{i=1}^{m} h_i(n)x(n+\rho_i) \geq 0. \tag{4.4}$$

This shows that $\{z(n)\}$ is an eventually increasing sequence. Then by Lemma 2.3, the sequence $\{z(n)\}$ is an eventually positive.

Set

$$y(n) = z(n) + p(n+\rho)z(n+\tau) \tag{4.5}$$

where $\rho = \max_{1\leq i\leq m}\rho_i$. Then

$$\begin{aligned}
\Delta y(n) &\geq \Delta z(n) + p(n+\rho)\Delta z(n+\tau) \\
&= \sum_{i=1}^{m} h_i(n)x(n+\rho_i) + p(n+\rho)\sum_{i=1}^{m} h_i(n+\tau)x(n+\rho_i+\tau) \\
&\geq \sum_{i=1}^{m} h_i(n)\left[x(n+\rho_i) + p(n+\rho_i)x(n+\rho_i+\tau)\right]
\end{aligned}$$

$$= \sum_{i=1}^{m} h_i(n) z(n+\rho_i) \geq 0. \tag{4.6}$$

This shows that $\{y(n)\}$ is an increasing sequence. But in view of (4.1) and Lemma 2.3, we get $y(n) > 0$, eventually.

From(4.5), we have

$$\frac{y(n)}{1+p(n+\rho)} \leq z(n+\tau) \tag{4.7}$$

or

$$\frac{y(n-\tau+\rho_i)}{1+p(n+\rho-\tau+\rho_i)} \leq z(n+\rho_i). \tag{4.8}$$

Using (4.8) in (4.6), we obtain

$$\Delta y(n) - \sum_{i=1}^{m} \left(\frac{h_i(n)}{1+p(n+\rho-\tau+\rho_i)} \right) y(n+\rho_i-\tau) \geq 0. \tag{4.9}$$

But in view of (4.2) and Lemma 2.4, the advanced difference inequality (4.9) cannot have an eventually positive solution. This is a contradiction and this completes the proof. □

Theorem 4.2. *Assume that $\Delta p(n) \geq 0$ and $m = k$. Suppose that*

(*i*) *there exist a partition of the set $\{1, 2, ..., m\}$ into two disjoint subsets I and J such that $i \in I$ implies $\rho_i > \sigma_i$ and $j \in J$ implies $\rho_j = \sigma_j$;*

(*ii*) *$h_i(n) = r_i(n) - q_i(n+\rho_i-\sigma_i) \geq 0$ and are not identically zero for $i = 1, 2, ..., m$;*

(*iii*) *$h_i(n) \geq h_i(n+\tau)$ and $\rho_i > \tau$ for $i = 1, 2, ...m$.*

Suppose further that

$$\sum_{n=0}^{\infty} \sum_{i=1}^{m} h_i(n) = +\infty \tag{4.10}$$

and

$$\frac{(\rho' - \tau)^{\rho'-\tau}}{(\rho'-\tau-1)^{\rho'-\tau-1}} \sum_{i=1}^{m} \liminf_{n\to\infty} \left(\frac{h_i(n)}{1+p(n+\rho-\tau+\rho_i)} \right) > 1, \tag{4.11}$$

where $\rho' = \min_{1\leq i\leq m} \rho_i$ and $\rho = \max_{1\leq i\leq m} \rho_i$.

Then every solution $\{x(n)\}$ of (1.2) is either oscillatory or $\liminf_{n\to\infty} x(n) = 0$.

Proof. On the contrary we may assume, without loss of generality that $\{x(n)\}$ is an eventually positive solution such that

$$\liminf_{n\to\infty} x(n) > 0. \tag{4.12}$$

Set

$$z(n) = x(n) + p(n)x(n+\tau) - \sum_{i\in I} \sum_{s=n}^{n+\rho_i-\sigma_i-1} q_i(s)x(s+\sigma_i). \tag{4.13}$$

Then from (1.2) and (4.13), we have

$$\begin{aligned} \Delta z(n) &= -\sum_{i=1}^{m} q_i(n)x(n+\sigma_i) + \sum_{i=1}^{m} r_i(n)x(n+\rho_i) \\ &- \sum_{i\in I} q_i(n+\rho_i-\sigma_i)x(n+\rho_i) + \sum_{i\in I} q_i(n)x(n+\sigma_i) \\ &= -\sum_{i=1}^{m} q_i(n+\rho_i-\sigma_i)x(n+\rho_i) + \sum_{i=1}^{m} r_i(n)x(n+\rho_i) \end{aligned}$$

$$= \sum_{i=1}^{m} h_i(n)x(n+\rho_i) \geq 0. \tag{4.14}$$

This shows that $\{z(n)\}$ is a nondecreasing sequence.

Then either

$$\lim_{n\to\infty} z(n) = +\infty \tag{4.15}$$

or

$$\lim_{n\to\infty} z(n) = L \in R. \tag{4.16}$$

Assume that (4.16) holds. But in view of (4.10) and (4.12), and from (4.14), we have

$$\lim_{n\to\infty} z(n) = +\infty,$$

which is a contradiction to the assumption and so (4.15) holds. Thus we have $z(n) > 0$. eventually. Set

$$y(n) = z(n) + p(n+\rho)z(n+\tau) \tag{4.17}$$

where $\rho = \max_{1\le i\le m} \rho_i$. Then

$$\Delta y(n) = \sum_{i=1}^{m} h_i(n)x(n+\rho_i) + p(n+\rho)\sum_{i=1}^{m} h_i(n+\tau)x(n+\tau+\rho_i)$$

$$\ge \sum_{i=1}^{m} h_i(n)z(n+\rho_i) \ge 0. \tag{4.18}$$

This shows that $\{y(n)\}$ is an increasing sequence. By repeating the steps followed in the Theorem 4.1, we can easily show that $y(n) > 0$, eventually.

Again from (4.17), we have

$$\frac{y(n)}{1+p(n+\rho)} \le z(n+\tau)$$

or

$$\frac{y(n-\tau+\rho_i)}{1+p(n-\tau+\rho+\rho_i)} \le z(n+\rho_i). \tag{4.19}$$

Using (4.19) in (4.18), we obtain

$$\Delta y(n) - \sum_{i=1}^{m}\left(\frac{h_i(n)}{1+p(n-\tau+\rho+\rho_i)}\right) y(n-\tau+\rho_i) \ge 0; \tag{4.20}$$

But in view of (4.11) and the Lemma 2.4, the advanced difference inequality (4.20) cannot have an eventually positive solution. This is a contradiction and this completes the proof. □

Theorem 4.3. *Assume that $\Delta p(n) \ge 0$. Suppose that*

(i) there exist a positive integer $l \le k$ and a partition of the set $\{1,2,...,m\}$ into l disjoint subsets $I_1, I_2, ..., I_l$ such that $i \in I_j$ implies $\rho_j > \sigma_j$;

(ii) $a_j(n) = r_j(n) - \sum_{i\in I_j} q_i(n+\rho_j-\sigma_i) \ge 0$ for $j = 1,2,...,l$ and are not identically zero and $a_j(n) = r_j(n)$ for $j = l+1,...,k$;

(iii) $\rho_j > \tau$ for $j = 1,2,...k$;

(iv) $a_j(n) \ge a_j(n+\tau)$ for $j = 1,2,...,k$.

Suppose further that

$$\sum_{n=0}^{\infty}\sum_{j=1}^{k} \liminf_{n\to\infty} a_j(n) > 1 \tag{4.21}$$

and

$$\frac{(\rho'-\tau)^{\rho'-\tau}}{(\rho'-\tau-1)^{\rho'-\tau-1}} \sum_{j=1}^{k} \liminf_{n\to\infty}\left(\frac{a_j(n)}{1+p(n-\tau+\rho-\rho_j)}\right) > 1 \tag{4.22}$$

where $\rho' = \min_{1\le j\le k} \rho_j$ and $\rho = \max_{1\le j\le k} \rho_j$.

Then every solution $\{x(n)\}$ of (1.2) is either oscillatory or $\liminf_{n\to\infty} x(n) = 0$.

Proof. On the contrary, without loss of generality that we may suppose that $\{x(n)\}$ is an eventually positive solution such that

$$\liminf_{n\to\infty} x(n) > 0. \tag{4.23}$$

Set

$$z(n) = x(n) + p(n)x(n-\tau) - \sum_{j=1}^{l}\sum_{i\in I_j}\sum_{s=n}^{n+\rho_j-\sigma_i-1} q_i(s)x(s+\sigma_i). \tag{4.24}$$

Then from (1.2) and (4.24), we have

$$\begin{aligned}\Delta z(n) &= -\sum_{i=1}^{m} q_i(n)x(n+\sigma_i) + \sum_{j=1}^{k} r_j(n)x(n+\rho_j)\\ &- \sum_{j=1}^{l}\sum_{i\in I_j}\left(q_i(n+\rho_j-\sigma_i)x(n+\rho_j) - q_i(n)x(n+\sigma_i)\right)\\ &= \sum_{j=1}^{l} r_j(n)x(n+\rho_j) - \sum_{j=1}^{l}\sum_{i\in I_j} q_i(n+\rho_j-\sigma_i)x(n+\rho_j)\\ &+ \sum_{j=l+1}^{k} r_j(n)x(n+\rho_j)\end{aligned}$$

or

$$\Delta z(n) = \sum_{j=1}^{k} a_j(n)x(n+\rho_j) \geq 0. \tag{4.25}$$

This shows that $\{z(n)\}$ is an increasing sequence. In view of (4.21) and (4.23) and from (4.25), we obtain $z(n) \to +\infty$ as $n \to \infty$. Since $\{z(n)\}$ increases to $+\infty$. We have $z(n) > 0$, eventually.

Set

$$y(n) = z(n) + p(n+\rho)z(n+\tau), \tag{4.26}$$

where $\rho = \max_{1\leq j\leq k} \rho_j$. Then

$$\begin{aligned}\Delta y(n) &\geq \Delta z(n) + p(n+\rho)\Delta z(n+\tau)\\ &= \sum_{j=1}^{k} a_j(n)x(n+\rho_j) + p(n+\rho)\sum_{j=1}^{k} a_j(n+\tau)x(n+\tau+\rho_j)\end{aligned}$$

or

$$\Delta y(n) \geq \sum_{j=1}^{k} a_j(n)z(n+\rho_j) \geq 0. \tag{4.27}$$

Since $\{y(n)\}$ is increasing, $z(n) \to \infty$ as $n \to \infty$ and $z(n) > 0$ eventually, we can easily show from (4.27), that $y(n) \to \infty$ as $n \to \infty$ and consequently $y(n) > 0$, eventually.

From (4.26), we have

$$\frac{y(n)}{1+p(n+\rho)} \leq z(n+\tau)$$

or

$$\frac{y(n-\tau+\rho_j)}{1+p(n+\rho-\tau+\rho_j)} \leq z(n+\rho_j). \tag{4.28}$$

Using (4.28) in (4.27), we have

$$\Delta y(n) - \sum_{j=1}^{k}\left(\frac{a_j(n)}{1+p(n-\tau+\rho_j+\rho)}\right) y(n+\rho_j-\tau) \geq 0. \tag{4.29}$$

This shows that the difference inequality (4.29) has an eventually positive solution $\{y(n)\}$. On the other hand, in view of (4.22) and Lemma 2.4, the advanced difference inequality (4.29) cannot have an eventually positive solution, which leads to a contradiction. This completes the proof. □

Conclusion: We presents sufficient conditions for oscillation of all solutions of first order neutral delay and advanced difference equations with positive and negative variable coefficients. Our results improves the earlier results in the literature.

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Numerical Investigation of the Nonlinear Integro-Differential Equations using He′s Homotopy Perturbation Method

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Abstract

In this paper, He's Homotopy Perturbation Method (HHPM), by construction, produces approximate solutions of nonlinear integro-differential equations [2]. The purpose of this paper is to extend the He's Homotopy Perturbation method to the nonlinear integro-differential equations. Efficient error estimation for the He's Homotopy Perturbation method is also introduced. Details of this method are presented and compared with Single-Term Haar Wavelet Series (STHWS) method [2] numerical results along with estimated errors are given to clarify the method and its error estimator.

Keywords: Integro-Differential Equations, Nonlinear integro-differential equations, Single-term Haar wavelet series, He's Homotopy Perturbation Method.

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1 Introduction

Mathematical modelling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro- differential equations, stochastic equations. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in many fields like fluid dynamics, biological models and chemical kinetics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution [6].

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models, [1, 8, 9].

In this article we developed numerical methods for nonlinear IDEs to get discrete solutions via He's Homotopy Perturbation method which was studied by S. Sekar et al. [3, 4]. The subject of this paper is to try to find numerical solutions of nonlinear integro-differential equations using He's Homotopy Perturbation method and compare the discrete results with the single-term Haar wavelet series method (STHWS) which is presented previously by Sekar et al. [2]. Finally, we show the method to achieve the desired accuracy. Details of the structure of the present method are explained in sections. We apply He's Homotopy Perturbation method and STHWS methods for nonlinear IDEs. In Section 4, it's proved the efficiency of the He's Homotopy Perturbation method. Finally, Section 5 contains some conclusions and directions for future expectations and researches.

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2 He's Homotopy Perturbation Method

In this section, we briefly review the main points of the powerful method, known as the He's homotopy perturbation method [2]. To illustrate the basic ideas of this method, we consider the following differential equation:

$$A(u) - f(t) = 0, u(0) = u_0, t \in \Omega \tag{2.1}$$

where A is a general differential operator, u_0 is an initial approximation of Eq. (2.1), and $f(t)$ is a known analytical function on the domain of Ω. The operator A can be divided into two parts, which are L and N, where L is a linear operator, but N is nonlinear. Eq. (2.1) can be, therefore, rewritten as follows:

$$L(u) + N(u) - f(t) = 0$$

By the homotopy technique, we construct a homotopy $U(t,p) : \Omega \times [0,1] \to \Re$, which satisfies:

$$H(U,p) = (1-p)[LU(t) - Lu_0(t)] + p[AU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \tag{2.2}$$

or

$$H(U,p) = LU(t) - Lu_0(t) + pLu_0(t) + p[NU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \tag{2.3}$$

where $p \in [0,1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously, from Eqs. (2.2) or (2.3) we will have $H(U,0) = LU(t) - Lu_0(t) = 0, H(U,1) = AU(t) - f(t) = 0$.

The changing process of p from zero to unity is just that of $U(t,p)$ from $u_0(t)$ to $u(t)$. In topology, this is called homotopy. According to the He's Homotopy Perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (2.2) or (2.3) can be written as a power series in p :

$$U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2U_2 + p^3U_3 + ... \tag{2.4}$$

Setting $p = 1$, results in the approximate solution of Eq.(2.1)

$$U(t) = \lim_{p\to 1} U = U_0 + U_1 + U_2 + U_3 + ...$$

Applying the inverse operator $L^{-1} = \int_0^t (.)dt$ to both sides of Eq. (2.3), we obtain

$$U(t) = U(0) + \int_0^t Lu_0(t)dt - p\int_0^t Lu_0(t)dt - p[\int_0^t (NU(t) - f(t))dt] \tag{2.5}$$

where $U(0) = u_0$.

Now, suppose that the initial approximations to the solutions, $Lu_0(t)$, have the form

$$Lu_0(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t) \tag{2.6}$$

where α_n are unknown coefficients, and $P_0(t), P_1(t), P_2(t), ...$ are specific functions. Substituting (2.4) and (2.6) into (2.5) and equating the coefficients of p with the same power leads to

$$\begin{cases} p^0 : U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \\ p^1 : U_1(t) = -\sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt - \int_0^t (NU_0(t) - f(t))dt \\ p^2 : U_2(t) = -\int_0^t NU_1(t)dt \\ \vdots \\ p^j : U_j(t) = -\int_0^t NU_{j-1}(t)dt \end{cases} \tag{2.7}$$

Now, if these equations are solved in such a way that $U_1(t) = 0$, then Eq. (2.7) results in $U_1(t) = U_2(t) = U_3(t) = \ldots = 0$ and therefore the exact solution can be obtained by using

$$U(t) = U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \tag{2.8}$$

It is worth noting that, if $U(t)$ is analytic at $t = t_0$, then their Taylor series

$$U(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

can be used in Eq. (2.8), where $a_0, a_1, a_2, ...$ are known coefficients and α_n are unknown ones, which must be computed.

3 General format for nonlinear integro-differential equations

The equation is of the form [3]

$$\frac{\partial}{\partial t} u(x,t) + \int_{n=0}^{t} Ru(x,s)ds = g(x,t) \tag{3.9}$$

is an example of general nonlinear integro-differential equations defined on a Hilbert space. In the equation R is a nonlinear operator that contains partial derivatives with respect to x and g is an inhomogeneous term. On particular interest is the following special case (3.9) becomes.

$$\frac{\partial}{\partial t} u(x,t) - \int_{n=0}^{t} a(t-s) \frac{\partial}{\partial x} \sigma(\frac{\partial}{\partial x} u(x,s))ds = g(x,t), 0 < x < 1, 0 < t < T \tag{3.10}$$

with the initial condition

$$u(x,0) = f(x) \tag{3.11}$$

The problem arises in the theory of one-dimensional viscoelasticity [3]. It is also a special model for one dimensional heat flow in materials with memory [3].

A numerical solution to the nonlinear problem given by (3.10) and (3.11) was obtained using Galerkin's method [7]. In this paper, the STHWS method and He's Homotopy Perturbation method are described and applied to compute numerical solutions to (3.10) and (3.11). It will be shown that the algorithms are efficient and accurate with only two or three iterations.

4 Numerical Experiments

Different forms of the kernel $a(.)$ and the nonlinear function $\sigma(.)$ [7] in (3.10) are considered. The inhomogeneous term $g(x,t)$ and initial condition $f(x)$ in (3.11) are also chosen appropriately so that exact solutions are available. The exact solutions are then compared with the numerical solutions derived through the STHWS method and He's Homotopy Perturbation method.

4.1 Example

In this example [5], $a(\zeta) = e^{-\zeta}$, $\sigma(\zeta) = \zeta^2$, $g(x,t) = e^{-(x+t)} + 2e^{-2x}(e^{-t} - e^{-2t})$ and the initial condition $u(x,0) = e^{-x}$. With these choices, (3.10) and (3.11) become

$$\frac{\partial}{\partial t} u(x,t) - \int_0^t e^{-(t-s)} \frac{\partial}{\partial x} [(\frac{\partial}{\partial x} u(x,s))^2]ds = e^{-(x+t)} + 2e^{-2x}(e^{-t} - e^{-2t}), u(x,0) = e^{-x}$$

The exact solution for this problem is $u(x,t) = e^{-(x+t)}$

4.2 Example

In this example [5], $a(\zeta) = e^{-2\zeta}$, $\sigma(\zeta) = \zeta^2$, $g(x,t) = cos(x+t) + \frac{1}{4}[sin2(x+t) - cos2(x+t) - e^{-2t}(sin2x - cos2x)]$ and the initial condition $u(x,0) = sinx$.

The exact solution for this problem is $u(x,t) = sin(x+t)$

Table 1 shows the errors between the exact solution and numerical solutions. The above examples 4.1 and 4.2 has been solved numerically using the STHWS method [2] and He's Homotopy Perturbation method. The obtained results (with step size $x = 0.2$ and $t = 0.01$) along with exact solutions of the examples 4.1 and 4.2, absolute errors between them are calculated and are presented in Table 1. A graphical representation is given for the nonlinear integro-differential equations in Figures 1 and 2, using three-dimensional effect to highlight the efficiency of the He's Homotopy Perturbation method.

Figure 1: Error estimation of the Example 4.1

Figure 2: Error estimation of the Exaample 4.2

Table 1: Numerical results for the Examples 4.1 and 4.2

	Exact Solution		STHWS Error		HHPM Error	
t	Example 4.1	Example 4.2	Example 4.1	Example 4.2	Example 4.1	Example 4.2
0.0	0.99005	0.29552	1.63E-04	1.00E-05	1.63E-06	1.00E-07
0.2	0.81058	0.47943	2.77E-04	2.62E-04	2.77E-06	2.62E-06
0.4	0.66365	0.64422	3.43E-04	3.48E-04	3.43E-06	3.48E-06
0.6	0.54335	0.78333	4.63E-04	4.18E-04	4.63E-06	4.18E-06
0.8	0.44486	0.89121	5.48E-05	5.42E-04	5.48E-07	5.42E-06
1.0	0.36422	0.96356	6.11E-05	6.62E-04	6.11E-07	6.62E-06

5 Conclusion

The obtained results (approximate solutions) of the nonlinear integro-differential equation [2] show that the He's Homotopy Perturbation method works well for finding the solution. The efficiency and the accuracy of the He's Homotopy Perturbation method have been illustrated by suitable examples. From the Table 1, it can be observed that for most of the time intervals, the absolute error is less in the He's Homotopy Perturbation method when compared to the single term Haar wavelet series method [2], which yields a small error, along with the exact solutions. From the Figures 1 - 2, it can be predicted that the He's Homotopy Perturbation method solution match well to the problem when compared to the single term Haar wavelet series method [2]. Hence the He's Homotopy Perturbation method is more suitable for studying the nonlinear integro-differential equation.

The researcher has successfully introduced He's Homotopy Perturbation method which has been exclusively developed for solving nonlinear integro-differential equation. Finally, in this paper, it is concluded that from the Table and Figures, which indicate the error to be almost, less with the nonlinear integro-differential equation using He's Homotopy Perturbation method.

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Soft Almost Semi-Continuous Mappings

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Abstract

In the present paper the concept of soft almost semi-continuous mappings and soft almost semi-open mappings in soft topological spaces have been introduced and studied.

Keywords: Soft regular open set, Soft semi open set, Soft almost continuous mappings, Soft semi-continuous mappings, Soft almost semi-continuous mappings and Soft almost semi-open mappings.

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1 Introduction

The theory of soft set was proposed by Molodtsov in 1999 [10]. It is a method for handling uncertain data. In 2011 Shabir and Naz [11] initiated the study of soft topological spaces. Many researchers worked on the findings of structures of soft set theory ,soft topology and applied to many problems having uncertainties.Theoretical study of soft sets and soft topological spaces have been by some authors in [1, 3, 5–7, 10, 11, 13–15].In 2013, Chen [2] introduced the concept of soft semi-open sets and soft-semi-closed sets in soft topological spaces.The section 2, of this paper gives the basic concept of soft set theory and soft topology.In section 3, we define the concepts of soft almost continuous mappings.It is shown that every soft almost continuous mapping is soft almost semi continuous and the example shows that the converse may not be true.Several characterization and properties of soft almost continuous mappings in soft topological spaces have been studies in this section. Section 4, introduces and studied soft almost open mappings.Last section give the conclusion of this paper.

2 Preliminaries

Let U is an initial universe set , E be a set of parameters , P(U) be the power set of U and $A \subseteq E$.

Definition 2.1. *[10] A pair (F, A) is called a soft set over U, where F is a mapping given by F: $A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For all $e \in A$, F(e) may be considered as the set of e-approximate elements of the soft set (F, A).*

Definition 2.2. *[6] For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B), denoted by (F, A) $\subseteq$ (G,B), if:*

(a) *$A \subseteq B$ and*

(b) *$F(e) \subseteq G(e)$ for all $e \in E$.*

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Definition 2.3. *[6] Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal denoted by (F, A) = (G, B) If (F, A) $\subseteq$ (G, B) and (G, B) $\subseteq$ (F, A).*

Definition 2.4. *[7] The complement of a soft set (F, A), denoted by $(F, A)^c$, is defined by $(F, A)^c$ = (F^c, A), where F^c : A $\rightarrow$ P(U) is a mapping given by $F^c(e) = U - F(e)$, for all e $\in$ E.*

Definition 2.5. *[6] Let a soft set (F, A) over U.*

(a) *Null soft set denoted by ϕ if for all e $\in$ A, F (e) = ϕ.*

(b) *Absolute soft set denoted by $\widetilde{U}$, if for each e $\in$ A, F(e) = U.*

Clearly, $\widetilde{U}^c = \phi$ and $\phi^c = \widetilde{U}$.

Definition 2.6. *[1] Union of two sets (F, A) and (G, B) over the common universe U is the soft (H, C), where C = A $\cup$ B, and for all e $\in$ C,*

$$H(e) = \begin{cases} F(e), & if e \in A - B \\ G(e), & if e \in B - A \\ F(e) \cup G(e), & \text{if } e \in A \cap B \end{cases}$$

Definition 2.7. *[1] Intersection of two soft sets (F, A) and (G, B) over a common universe U, is the soft set (H, C) where C = A $\cap$ B and H(e) = F(e) $\cap$ G(e) for each e $\in$ E.*

Let X and Y be an initial universe sets and E and K be the non empty sets of parameters, S(X, E) denotes the family of all soft sets over X and S(Y, K) denotes the family of all soft sets over Y.

Definition 2.8. *[11] A subfamily τ of S(X , E) is called a soft topology on X if:*

1. *$\widetilde{\phi}$, $\widetilde{X}$ belong to τ.*
2. *The union of any number of soft sets in τ belongs to τ.*
3. *The intersection of any two soft sets in τ belongs to τ.*

The triplet (X, τ, E) is called a soft topological space over X. The members of τ are called soft open sets in X and their complements called soft closed sets in X.

Definition 2.9. *If (X ,τ, E) is soft topological space and a soft set (F, E) over X.*

(a) *The soft closure of (F, E) is denoted by Cl(F,E) is defined as the intersection of all soft closed super sets of (F, E) [11].*

(b) *The soft interior of (F, E) is denoted by Int(F,E) is defined as the soft union of all soft open subsets of (F, E) [14].*

Definition 2.10. *[14] The soft set (F, E)$\in$ S(X, E) is called a soft point if there exist x $\in$ X and e $\in$ E such that F(e) = $\{x\}$ and F(e′) = ϕ for each e′ $\in$ E − $\{e\}$, and the soft point (F, E) is denoted by $(x_e)_E$.*

Definition 2.11. *[14] The soft point $(x_e)_E$ is said to be in the soft set (G, E), denoted by $(x_e)_E \in$ (G, E) if $(x_e)_E \subset$ (G,E).*

Definition 2.12. *[2, 13] A soft set (F,E) in a soft topological space (X,τ,E) is said to be :*
(a) Soft regular open if (F,E)= Int(Cl(F,E)).
(b) Soft regular closed if its complement is soft regular open.
(c) Soft semi-open if (F,E) $\subseteq$ Cl(Int(F,E)).
(d) Soft semi-closed if its complement is soft semi-open.

Remark 2.13. *[4, 13] Every soft regular open (resp. soft regular closed) set is soft open (resp. closed) and every soft open (resp. closed) set is soft semi-open (resp. semi-closed) but the converses may not be true.*

Definition 2.14. *[2] Let (F,E) be a soft set in a soft topological space (X ,τ,E).*

(a) The soft semi-closure of (F, E) is denoted by sCl(F,E) is defined as the smallest soft semi-closed set over which contains (F, E).

(b) The soft semi-interior of (F, E) is denoted by sInt(F,E) is defined as the largest soft semi-open set over which is contained in (F, E).

Definition 2.15. *[5] Let S(X,E) and S(Y,K) be families of soft sets. Let u: $X \to Y$ and p: $E \to K$ be mappings. Then a mapping f_{pu}: S(X, E) $\to$ S(Y, K) is defined as :*

(i)Let (F, A) be a soft set in S(X, E). The image of (F, A) under f_{pu}, written as f_{pu} (F, A) = (f_{pu}(F), p(A)), is a soft set in S(Y,K) such that

$$f_{pu}(F)(k) = \begin{cases} \bigcup_{e \in p^{-1}(k) \bigcap A} u(F(e)), & p^-1(k) \bigcap A \neq \phi \\ \phi, & p^{-1}(k) \bigcap A = \phi \end{cases}$$

For all k $\in$ K.

(ii) Let (G , B) be a soft set in S(Y , K). The inverse image of (G , B) under f_{pu}, written as

$$f_{pu}^{-1}(G)(e) = \begin{cases} u^{-1}G(p(e)), & p(e) \in B \\ \phi, & otherwise \end{cases}$$

For all e $\in$ E.

Definition 2.16. *[8, 12] Let (X,τ,E) and (Y,υ,K) be a soft topological spaces. A soft mapping f_{pu} : (X,τ,E)$\to$(Y,υ,K) is said to be :*

(a) Soft almost continuous if f_{pu}^{-1}(G, K) is soft open in X , for all soft regular open set (G,K) in Y.

(b) Soft almost open if f_{pu} (F, E) is soft open in Y, for all soft regular open set (F, E) in X.

(c) Soft semi-continuous mapping if f_{pu}^{-1}(G, K) is soft semi-open in X , for all soft open set (G,K) in Y.

(d) Soft semi-open if f_{pu} (F, E) is soft semi-open in Y , for all soft open set (F, E) in X.

(e) Soft semi-irresolute if f_{pu}^{-1}(G, K) is soft semi-open in X , for all soft semi-open set (G, K) in Y.

3 Soft Almost Semi-Continuous Mappings

Definition 3.1. *A soft mapping f_{pu} : (X ,τ,E) $\to$ (Y,ϑ,K) is said to be soft almost semi-continuous if the inverse image of every soft regular open set over Y is soft semi-open over X.*

Remark 3.2. *Every soft almost continuous mapping is soft almost semi-continuous but converse may not be true.*

Example 3.3. *Let X = $\{x_1, x_2\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1, y_2\}$, K = $\{k_1, k_2\}$. The soft sets (F_1,E), (F_2,E), (F_3,E), (G_1,K),(G_2,K) are defined as follows :*

$F_1(e_1) = \phi$, $F_1(e_2) = \{x_1\}$,

$F_2(e_1) = \{x_1\}$, $F_2(e_2) = \phi$,

$F_3(e_1) = \{x_1\}$, $F_3(e_2) = \{x_1\}$,

$G_1(k_1) = \{y_1\}$, $G_1(k_2) = \{y_2\}$,

$G_2(k_1) = \{y_2\}$, $G_2(k_2) = \{y_1\}$

Let $\tau = \{\phi$, (F_1,E),(F_2,E), (F_3,E), $\tilde{X}$ } and $\upsilon = \{\phi$,(G_1,K),(G_2,K), $\tilde{Y}$ } are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\to$ (Y,υ,K) defined by u(x_1) = y_1, u(x_2) = y_2 and p(e_1) = k_1, p(e_2) = k_2 is soft almost semi-continuous mapping not soft almost continuous.

Remark 3.4. *Every soft semi-continuous mapping is soft almost semi-continuous but converse may not be true.*

Example 3.5. *Let X = $\{x_1, x_2, x_3\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1, y_2, y_3\}$, K = $\{k_1, k_2\}$. The soft set (G,K) is defined as follows :*

G(k_1) = $\{y_1\}$, G(k_2) = ϕ

Let $\tau = \{\phi$, $\tilde{X}$ } , and $\upsilon = \{\phi$,(G,K), $\tilde{Y}$ } are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\to$ (Y,υ,K) defined by u(x_1) = y_1 , u(x_2) = y_2 , u(x_3) = y_3 and p(e_1)= k_1, p(e_2) = k_2 is soft almost semi-continuous but not soft semi-continuous.

Theorem 3.6. *Let f_{pu} : (X ,τ,E) $\to$ (Y,ϑ,K) be a soft mapping. Then the following conditions are equivalent:*

(a) f_{pu} is soft almost semi-continuous.

(b) f_{pu}^{-1} (G,K) is soft semi-closed set in X for every soft regular closed set (G,K) in Y.

(c) f_{pu}^{-1} (A,K) $\subset$ sInt(f_{pu}^{-1}(Int(Cl(A,K)))) for every soft open set (A,K) in Y.

(d) sCl(f_{pu}^{-1}(Cl(Int(G,K)))) $\subset$ f_{pu}^{-1}(G,K) for every soft closed set (G,K) in Y.

(e) For each soft point $(x_e)_E$ over X and each soft regular open set (G,K) over Y containing f_{pu} ($(x_e)_E$),there exists a soft semi-open set (F,E) over X such that $(x_e)_E \in$ (F,E) and (F,E) $\subset$ f_{pu}^{-1}(G,K).

(f) For each soft point $(x_e)_E$ over X and each soft regular open set (G,K) over Y containing $f_{pu}((x_e)_E)$,there exists a soft semi-open set (F,E) over X such that $(x_e)_E \in$ (F,E) and f_{pu}(F,E) $\subset$ (G,K).

Proof: (a)⇔(b) Since f_{pu}^{-1} ($(G,K)^C$) = $(f_{pu}^{-1}(G,K))^C$ for every soft set (G,K) over Y.

(a)⇒(c) Since (A,K) is soft open set over Y, (A,K) $\subset$ Int(Cl(A,K)) and hence, f_{pu}^{-1} (A,K) $\subset$ f_{pu}^{-1}(Int(Cl(A,K))).Now Int (Cl(A,K)) is a soft regular open set over Y. By (a), f_{pu}^{-1} (Int(Cl(A,K))) is soft semi-open set over X. Thus, f_{pu}^{-1} (A,K) $\subset$ f_{pu}^{-1} (Int(Cl(A,K))) = sInt(f_{pu}^{-1}(Int(Cl(A,K)))).

(c)⇒(a) Let (A,K) be a soft regular open set over Y, then we have f_{pu}^{-1} (A,K) $\subset$ sInt(f_{pu}^{-1}(Int(Cl(A,K)))) = sInt(f_{pu}^{-1}(A,K)).Thus, f_{pu}^{-1} (A,K) = sInt(f_{pu}^{-1}(A,K)) shows that f_{pu}^{-1} (A,K) is a soft semi-open set over X.

(b)⇒(d) Since (G,K) is soft closed set over Y, Cl(Int(G,K)) $\subset$ (G,K) and f_{pu}^{-1}(Cl(Int (G,K))) $\subset$ f_{pu}^{-1} (G,K). Cl(Int(G,K)) is soft regular closed set over Y. Hence, f_{pu}^{-1}(Cl(Int(G,K is soft semi-closed set over X.Thus, sCl(f_{pu}^{-1}(Cl(Int(G,K)))) = f_{pu}^{-1}(Cl(Int(G,K))) $\subset$ f_{pu}^{-1} (G,K).

(d)⇒(b) Let (G,K) be a soft regular closed set over Y,then we have sCl(f_{pu}^{-1}(G,K)) = sCl(f_{pu}^{-1}(Cl(Int(G,K)))) $\subset$ f_{pu}^{-1} (G,K) .Thus, sCl(f_{pu}^{-1}(G,K)) $\subset$ f_{pu}^{-1} (G,K), shows that f_{pu}^{-1} (G,K) is soft semi-closed set over X.

(a)⇒(e) Let $(x_e)_E$be a soft point over X and (G,K) be a soft regular open set over Y such that $f_{pu}((x_e)_E) \in$ (G,K), Put (F,E) = f_{pu}^{-1}(G,K) .Then by (a), (F,E) is soft semi-open set, $(x_e)_E\in$ (F,E) and (F,E) $\subset$ f_{pu}^{-1}(G,K).

(e)⇒(f) Let $(x_e)_E$ be a soft point over X and (G,K) be a soft regular open set over Y such that $f_{pu}((x_e)_E)$. By (e) there exists a soft semi-open set (F,E) such that $(x_e)_E \in$ (F,E), (F,E) $\subset$ f_{pu}^{-1}(G,K). And so , we have $(x_e)_E \in$ (F,E), f_{pu}(F,E) $\subset$ f_{pu}(f_{pu}^{-1}(G,K)) $\subset$ (G,K).

(f)⇒(a) Let (G,K) be a soft regular open set over Y and $(x_e)_E$ be a soft point over X such that $(x_e)_E\in$ f_{pu}^{-1}(G,K).Then $f_{pu}((x_e)_E) \in$ f_{pu}(f_{pu}^{-1}(G,K)) $\subset$(G,K). By (f) ,there exists a soft semi-open set (F,E) such that $(x_e)_E \in$ (F,E) and f_{pu}(F,E) $\subset$(G,K) .This shows that $(x_e)_E \in$ (F,E) $\subset$ f_{pu}^{-1}(G,K). it follows that f_{pu}^{-1}(G,K) is soft semi-open set and hence f_{pu}^{-1} is soft almost semi-continuous.

Definition 3.7. *A soft topological space (X ,τ,E) is said to be soft semiregular if for each soft open set (F,E) and each soft point $(x_e)_E \in$ (F,E), there exists a soft open set (G,E) such that $(x_e)_E \in$ (G,E) and (G,E) $\subset$ Int(Cl(G,E)) $\subset$ (F,E).*

Theorem 3.8. *Let f_{pu} : (X ,τ,E) → (Y,ϑ,K) be a soft mapping from a soft topological space (X ,τ,E) to a soft semiregular space (Y,ϑ,K).Then f_{pu} is soft almost semi-continuous if and only if f_{pu} is soft semi-continuous.*

Proof: Necessity: Let $(x_e)_E$ be a soft point in X and (F,K) be a soft open set in Y such that f_{pu}($(x_e)_E) \in$ (F,K) .Since (Y,ϑ,K) is soft semiregular there exists a soft open set (G,K) in Y such that $f_{pu}((x_e)_E) \in$ (G,K) and (G,K) $\subset$ Int(Cl(G,K) $\subset$ (F,K). Since Int(Cl(G,K)) is soft regular open in Y and f_{pu} is soft almost semi-continuous,by theorem 3.6 (f) there exists a soft semi-open set (A,E) in X such that $(x_e)_E \in$ (A,E) and f_{pu}(A,E) Int(Cl(G,K)).Thus, (A,E) is soft semi-open set such that $(x_e)_E \in$ (A,E) and f_{pu}(A,E) $\subset$ (F,K). Hence by theorem [26] [2], f_{pu} is soft semi-continuous.

Sufficiency : Obvious.

Lemma 3.9. *If f_{pu} : (X ,τ,E) → (Y,ϑ,K) be a soft mapping and f_{pu} is a soft open and soft continuous mapping then f_{pu}^{-1}(G,K) is soft semi-open in X for every (G,K) is soft semi-open in Y.*

Proof: Let (G,K) is soft semi-open in Y.Then, (G,K) $\subseteq$ Int(Cl(Int(G,K))).Since f_{pu} is soft continuous we have,f_{pu}^{-1}(G,K) $\subseteq$ f_{pu}^{-1}(Int(Cl(Int(G,K)))) $\subseteq$ Int(f_{pu}^{-1}(Cl(Int(G,K)))).By the openness of f_{pu}, we have f_{pu}^{-1}(Cl(Int(G,K))) $\subseteq$ Cl(f_{pu}^{-1}(Int(G,K))). Again f_{pu} is soft continuous f_{pu}^{-1}(Int(G,K)) $\subseteq$ Int(f_{pu}^{-1}(G,K)).Thus, f_{pu}^{-1}(G,K)$\subseteq$ Int(Cl(Int(f_{pu}^{-1}(G,K)))).

Consequently, f_{pu}^{-1}(G,K) is soft semi-open in X.

Theorem 3.10. *If soft mapping $f_{p_1u_1}$: (X ,τ,E) → (Y,ϑ,K) is soft open soft continuous and soft mapping $g_{p_2u_2}$:(Y,ϑ,K) → (Z,η,T) is soft almost semi-continuous, then $g_{p_2u_2}$ o$f_{p_1u_1}$: (X ,τ,E) → (Z,η,T) is soft almost semi-continuous.*

Proof : Suppose (U,T) is a soft regular open set in Z. Then $g_{p_2u_2}^{-1}$(U,T) is a soft semi-open set in Y because $g_{p_2u_2}$ is soft almost semi-continuous. Since $f_{p_1u_1}$ being soft open and continuous.By lemma 3.9 ($f_{p_1u_1}^{-1}$ ($g_{p_2u_2}^{-1}$(U,T)) is soft semi-open in X. Consequently, $g_{p_2u_2}$ o$f_{p_1u_1}$: (X ,τ,E) → (Z,η,T) is soft almost semi-continuous.

Lemma 3.11. *If (A,E) be a soft semi-open set over X and (Y,E) is soft open in a soft topological space (X ,τ,E). Then (A,E) $\cap$ (Y,E) is soft semi-open in (Y,E).*

Proof: Obvious.

Theorem 3.12. *Let f_{pu} : (X ,τ,E) → (Y,ϑ,K) be a soft almost semi-continuous mapping and (A,E) is soft open set in X, Then f_{pu}/ (A,E) is soft almost semi-continuous.*

Proof : Let (G,K) be a soft regular open set in Y then f_{pu}^{-1} (G,K) is soft semi-open in X. Since (A,E) is soft open in X, By lemma 3.11 (A,E) $\cap$ f_{pu}^{-1} (G,K) = $[f_{pu}/(A,E)]^{-1}$ (G,K) is soft semi-open in (A,E). Therefore, f_{pu}/ (A,E) is soft almost semi-continuous.

4 Soft Almost Semi-Open Mappings

Definition 4.1. *A soft mapping f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) is said to be soft almost semi-open if for each soft regular open set (F,E) in X, f_{pu}(F,E) is soft semi-open in Y.*

Remark 4.2. *Every soft almost open is soft almost semi-open but converse may not be true.*

Example 4.3. *Let X = $\{x_1 , x_2 \}$, E = $\{e_1 , e_2 \}$ and Y = $\{y_1 , y_2\}$, K = $\{k_1 , k_2 \}$. The soft sets (F_1,E) ,(F_2,E) ,(G,K) are defined as follows :*

$F_1(e_1) = \{x_1\}$, $F_1(e_2) = \{x_2\}$,
$F_2(e_1) = \{x_2\}$, $F_2(e_2) = \{x_1\}$,
$G_1(k_1) = \phi$ $G_1(k_2) = \{y_1\}$,
$G_2(k_1) = \{y_1\}$ $G_2(k_2) = \phi$,
$G_3(k_1) = \{y_1\}$ $G_3(k_2) = \{y_1\}$

Let $\tau = \{\phi$, (F_1,E) ,(F_2,E), $\tilde{X}$ $\}$, and $\upsilon = \{\phi$,(G_1,K),(G_2,K),(G_3,K) $\tilde{Y}$ $\}$ are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = y_1 , u(x_2)= y_2 and p(e_1)= k_1 , p(e_2) = k_2 is soft almost semi-open mapping but not soft almost open .

Remark 4.4. *Every soft semi-open mappings is soft almost semi-open but converse may not be true.*

Example 4.5. *Let X = $\{x_1 , x_2, x_3 \}$, E = $\{e_1 , e_2 \}$ and Y = $\{y_1 , y_2, y_3\}$, K = $\{k_1 , k_2 \}$. The soft sets (F,E) is defined as follows :*

$F(e_1) = \{x_1\}$, $F(e_2) = \phi$,

Let $\tau = \{\phi$, (F,E), $\tilde{X}$ $\}$ and $\upsilon = \{\phi$, $\tilde{Y}$ $\}$ are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = y_1 , u(x_2)= y_2 and p(e_1)= k_1 , p(e_2) = k_2 is soft almost semi-open mapping but not soft semi-open.

Theorem 4.6. *Let $f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Y,ϑ,K) and $g_{p_2u_2}$:(Y,ϑ,K) $\rightarrow$ (Z,η,T) be two soft mappings, If $f_{p_1u_1}$ is soft almost open and $g_{p_2u_2}$ is soft semi-open. Then the soft mapping $g_{p_2u_2}\, o f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Z,η,T) is soft almost semi-open.*

Proof : Let (F,E) be soft regular open in X. Then $f_{p_1u_1}$(F,E) is soft open in Y because $f_{p_1u_1}$ is soft almost open.Therefore, $g_{p_2u_2}$ ($f_{p_1u_1}$)(F,E) is soft semi-open in Z. Because $g_{p_2u_2}$ is soft semi-open. Since ($g_{p_2u_2}\, o f_{p_1u_1}$)(F,E) = ($g_{p_2u_2}$($f_{p_1u_1}$ (F,E)), it follows that the soft mapping ($g_{p_2u_2}\, o f_{p_1u_1}$) is soft almost semi-open.

Definition 4.7. *A soft mapping f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) is said to be soft semi-irresolute if the inverse image of soft semi-open set of Y is soft semi-open set in X.*

Theorem 4.8. *Let $f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Y,ϑ,K) and $g_{p_2u_2}$:(Y,ϑ,K) $\rightarrow$ (Z,η,T) be two soft mappings, such that $g_{p_2u_2}\, o f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Z,η,T) is soft almost semi-open and $g_{p_2u_2}$ is soft semi-irresolute and injective then $f_{p_1u_1}$ is soft almost semi-open.*

Proof : Suppose (F,E) is soft regular open set in X. Then $g_{p_2u_2}\, o f_{p_1u_1}$(F,E) is soft semi-open in Z because $g_{p_2u_2}\, o f_{p_1u_1}$ is soft almost semi-open. Since $g_{p_2u_2}$ is injective, we have ($g^{-1}_{p_2u_2}$ ($g_{p_2u_2}\, o f_{p_1u_1}$)(F,E)) = $f_{p_1u_1}$ (F,E). Therefore $f_{p_1u_1}$ (F, E) is soft semi-open in Y, because $g_{p_2u_2}$ is soft semi-irresolute. This implies $f_{p_1u_1}$ is soft almost semi-open.

Theorem 4.9. *Let soft mapping f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) be soft almost semi-open mapping. If (G,K) is soft set of Y and (F,E) is soft regular closed set of X containing f^{-1}_{pu} (G,K) then there is a soft semi-closed set (A,K) of Y containing (G,K) such that f^{-1}_{pu} (A,K) $\subset$ (F,E).*

Proof: Let (A, K) = $(f_{pu}(F, E)^C)^C$. Since f^{-1}_{pu} (G,K) $\subset$ (F,E) we have $f_{pu}(F, E)^C \subset$ (G,K). Since f_{pu} is soft almost semi-open then (A,K) is soft semi-closed set of Y and f^{-1}_{pu} (A,K) = (f^{-1}_{pu} $(f_{pu}(F, E)^C)^C \subset ((F, E)^C)^C$ = (F,E). Thus, f^{-1}_{pu} (A,K) $\subset$ (F,E).

5 Conclusion

Continuity of soft mappings played very important role in the development of soft topology.In this paper we have introduced soft almost semi-continuous(resp. soft almost semi-open) mappings and it is shown by the examples that the class of soft almost semi-continuous(resp. soft almost semi-open)mappings properly contains the class of all soft almost continuous(resp. soft almost open) mappings. Various properties and characterization of these soft mappings have been studied. The class of all soft almost mappings introduced in this paper will be useful to study various strong and weak forms of soft separation axioms,soft connectedness and soft compactness in soft topology.

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On generalized b star - closed set in Topological Spaces

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Abstract

In this paper, we introduce a new class of sets called generalized b star - closed sets in topological spaces (briefly gb^*- closed set). Also we discuss some of their properties and investigate the relations between the associated topology.

Keywords: gb^* - closed set, b - closed set, gb closed set.

2010 MSC: 54A05.

1 Introduction

In 1970, Levine introduced the concept of generalized closed set and discussed the properties of sets, closed and open maps, compactness, normal and separation axioms. Later in 1996 Andrjivic gave a new type of generalized closed set in topological space called b closed sets. The investigation on generalization of closed set has lead to significant contribution to the theory of separation axiom, generalization of continuity and covering properties. A.A.Omari and M.S.M. Noorani made an analytical study and gave the concepts of generalized b closed sets in topological spaces.

In this paper, a new class of closed set called generalized b star - closed set is introduced to prove that the class forms a topology. The notion of generalized b star - closed set and its different characterizations are given in this paper. Throughout this paper (X, τ) and (Y, σ) represent the non - empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

Let $A \subseteq X$, the closure of A and interior of A will be denoted by $cl(A)$ and $int(A)$ respectively, union of all b - open sets X contained in A is called b - interior of A and it is denoted by $bint(A)$, the intersection of all b - closed sets of X containing A is called b - closure of A and it is denoted by $bcl(A)$.

2 Preliminaries

Definition 2.1. *Let A subset A of a topological space (X, τ), is called*

1) a pre-open set [13] if $A \subseteq int(cl(A))$.

2) a semi-open set [?] if $A \subseteq cl(int(A))$.

3) a α -open set [9] if $A \subseteq int(cl(int(A)))$.

4) a b -open set [2] if $A \subseteq cl(int(A)) \cup int(cl(A))$.

5) a generalized $*$ closed set (briefly g^*-closed)[8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$ open in X.

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6) *a generalized b -closed set (briefly gb- closed) [1] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

7) *a* α *generalized star -closed set (briefly* αg^* *- closed) [12] if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* α*-open in* X.

8) *a generalized star semi -closed set (briefly* g^*s*- closed) [14] if* $scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is gs-open in* X.

9) *a regular generalized b-closed set (briefly rgb- closed) [11] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.

3 Generalized b star - closed set

In this section, we introduce generalized b star - closed set and investigate some of their properties.

Definition 3.2. *A subset* A *of a topological space* (X, τ)*, is called generalized b star - closed set (briefly* gb^* *- closed set) if* $bcl(A) \subset U$ *whenever* $A \subset U$ *and* U *is* g^* *-open in* X.

Theorem 3.1. *Every closed set is* gb^* *- closed.*

Proof. Let A be any closed set in X such that $A \subset U$, where U is g^* open. Since $bcl(A) \subset cl(A) = A$. Therefore $bcl(A) \subset U$. Hence A is gb^* - closed set in X. □

The converse of above theorem need not be true as seen from the following example.

Example 3.1. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$*. The set* $\{b\}$ *is* gb^* *- closed set but not a closed set.*

Theorem 3.2. *Every pre-closed set is* gb^* *- closed set.*

Proof. Let A be pre-closed set in X such that $A \subseteq U$ where U is g^* open. Since A is pre closed $bcl(A) \subseteq pcl(A) \subseteq A$. Therefore $bcl(A) \subseteq U$. Hence A is gb^* -closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.2. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$*. The set* $\{a, c\}$ *is* gb^* *-closed set but not a pre-closed set.*

Theorem 3.3. *Every semi-closed set is* gb^* *- closed set.*

Proof. Let A be any semi-closed set in X such that $A \subseteq U$ where U is g^* open. Since A is semi closed set, $bcl(A) \subseteq scl(A) \subseteq U$. Therefore $bcl(A) \subseteq U$. Hence A is gb^* closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.3. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a, b\}\}$*. The set* $\{a, c\}$ *is* gb^* *- closed set but not a semi-closed set.*

Theorem 3.4. *Every* αg^* *- closed set is* gb^* *- closed set.*

Proof. Let A be any αg^* -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is αg^* -closed set, $bcl(A) \subseteq \alpha cl(A) \subseteq U$. Therefore $bcl(A) \subseteq U$. Hence A is gb^* -closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.4. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$*. The set* $\{b\}$ *is* gb^* *- closed set but not a* αg^* *-closed set.*

Theorem 3.5. *Every b -closed set is* gb^* *- closed set.*

Proof. Let A be any b -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is b-closed, $bcl(A) = A$. Therefore $bcl(A) \subset U$. Hence A is gb^* - closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.5. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$*. The set* $\{a, c\}$ *is* gb^* *-closed set but not a b-closed set.*

Theorem 3.6. *Every* g^* *- closed set is* gb^* *-closed set.*

Proof. Let A be any g^* -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is g^* -closed, $bcl(A) \subseteq cl(A) \subseteq U$. Therefore $bcl(A) \subseteq U$. Hence A is gb^*- closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.6. *Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a, c\}\}$. The set $\{a, b\}$ is gb^* -closed set but not a g^* -closed set.*

Theorem 3.7. *Every g^*s -closed set is gb^* -closed set.*

Proof. Let A be g^*s -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is g^*s closed set, $bcl(A) \subseteq scl(A) \subseteq U$. Hence A is gb^* -closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.7. *Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. The set $\{a, c\}$ is gb^* - closed set but not a g^*s -closed set.*

Theorem 3.8. *Every gb^* -closed set is rgb -closed set.*

Proof. Let A be any gb^* -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is gb^* closed, $bcl(A) \subseteq pcl(A) \subseteq U$. Hence A is rgb -closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.8. *Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$. The set $\{a, b\}$ is rgb -closed set but not a gb^* - closed set.*

4 Characteristics of gb^*-closed set

Theorem 4.9. *If A and B are gb^*-closed sets in X then $A \cup B$ is gb^*-closed set in X.*

Proof. Let A and B are gb^*-closed sets in X and U be any g^* open set containing A and B. Therefore $bcl(A) \subseteq U$, $bcl(B) \subseteq U$. Since $A \subseteq U, B \subseteq U$ then $A \cup B \subseteq U$. Hence $bcl(A \cup B) = bcl(A) \cup bcl(B) \subseteq U$. Therefore $A \cup B$ is gb^*-closed set in X. □

Theorem 4.10. *If a set A is gb^* - closed set if and only if $bcl(A) - A$ contains no non empty g^* -closed set.*

Proof. Necessary: Let F be a g^* closed set in X such that $F \subseteq bcl(A) - A$. Then $A \subseteq XF$. Since A is gb^* closed set and $X - F$ is g^* open then $bcl(A) \subseteq X - F$. (i.e.) $F \subseteq X - bcl(A)$. So $F \subseteq (X - bcl(A)) \cap (bcl(A) - A)$. Therefore $F = \varphi$.

Sufficiency: Let us assume that $bcl(A) - A$ contains no non empty g^* closed set. Let $A \subseteq U$, U is g^* open. Suppose that $bcl(A)$ is not contained in U, $bcl(A) \cap U^c$ is a non-empty g^* closed set of $bcl(A) - A$ which is contradiction. Therefore $bcl(A) \subseteq U$. Hence A is gb^*-closed. □

Theorem 4.11. *If A is gb^*-closed set in X and $A \subseteq B \subseteq bcl(A)$, Then B is gb^*-closed set in X.*

Proof. Since $B \subseteq bcl(A)$, we have $bcl(B) \subseteq bcl(A)$ then $bcl(B) - B \subseteq bcl(A) - A$. By theorem 4.10, $bcl(A) - A$ contains no non empty g^* closed set. Hence $bcl(B) - B$ contains no non empty g^* closed set. Therefore B is gb^*-closed set in X. □

Theorem 4.12. *If $A \subseteq Y \subseteq X$ and suppose that A is gb^* closed set in X then A is gb^*-closed set relative to Y.*

Proof. Given that $A \subseteq Y \subseteq X$ and A is gb^*-closed set in X. To prove that A is gb^*- closed set relative to Y. Let us assume that $A \subseteq Y \cap U$, where U is g^* open in X. Since A is gb^*-closed set, $A \subseteq U$ implies $bcl(A) \subseteq U$. It follows that $Y \cap bcl(A) \subseteq Y \cap U$. That is A is gb^*-closed set relative to Y. □

Theorem 4.13. *If A is both g^* open and gb^*-closed set in X, then A is g^* closed set.*

Proof. Since A is g^* open and gb^* closed in X, $bcl(A) \subseteq U$. But $A \subseteq bcl(A)$. Therefore $A = bcl(A)$. Hence A is g^* closed set. □

Theorem 4.14. *For $xinX$, then the set $X - \{x\}$ is a gb^* -closed set or g^* -open.*

Proof. Suppose that $X - \{x\}$ is not g^* open, then X is the only g^* open set containing $X - \{x\}$. (i.e.) $bcl(X - \{x\}) \subseteq X$. Then $X - \{x\}$ is gb^* - closed in X. □

5 Generalized b star - open set and generalized b star - neighbourhoods

In this section, we introduce generalized b star - open sets (briefly gb^* - open) and generalized b star - neighbourhoods (briefly gb^* - neighbourhood) in topological spaces by using the notions of gb^* - open set and study some of their properties.

Definition 5.3. *A subset A of a topological space (X, τ), is called semi generalized b^* - open set (briefly gb^* - open set) if A^c is gb^* - closed in X. We denote the family of all gb^* - open sets in X by gb^* - $O(X)$.*

Theorem 5.15. *If A and B are gb^* - open sets in a space X. Then $A \cap B$ is also gb^* - open set in X.*

Proof. If A and B are gb^* - open sets in a space X. Then A^c and B^c are gb^* - closed sets in a space X. By Theorem 4.13 $A^c \cup B^c$ is also gb^* - closed set in X. (i.e.) $A^c \cup B^c = (A \cap B)^c$ is a gb^* - closed set in X. Therefore $A \cap B$ gb^* - open set in X. □

Remark 5.1. *The union of two gb^*-open sets in X is generally not a gb^*-open set in X.*

Example 5.9. *Let $X = \{a, b, c\}$ with $\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}\}$. If $A = \{b\}$, $B = \{c\}$ are pgb-open sets in X, then $A \cup B = \{b, c\}$ is not gb^* open set in X.*

Theorem 5.16. *If $int(B) \subseteq B \subseteq A$ and if A is gb^* -open in X, then B is gb^* - open in X.*

Proof. Suppose that $int(B) \subseteq B \subseteq A$ and A is gb^* -open in X then $A^c \subseteq B^c \subseteq cl(A^c)$. Since A^c is gb^* - closed in X, by Theorem 5.15 B is gb^* - open in X. □

Definition 5.4. *Let x be a point in a topological space X and let $x \in X$. A subset N of X is said to be a gb^* - neighbourhood of x iff there exists a gb^* - open set G such that $x \in G \subset N$.*

Definition 5.5. *A subset N of Space X is called a gb^* - neighbourhood of $A \subset X$ iff there exists a gb^* - open set G such that $A \subset G \subset N$.*

Theorem 5.17. *Every neighbourhood N of $x \in X$ is a gb^* - neighbourhood of x.*

Proof. Let N be a neighbourhood of point $x \in X$. To prove that N is a gb^* - neighbourhood of x. By Definition of neighbourhood, there exists an open set G such that $x \in G \subset N$. Hence N is a sg^*b - neighbourhood of x. □

Remark 5.2. *In general, a gb^* - neighbourhood of $x \in X$ need not be a neighbourhood of x in X as seen from the following example.*

Example 5.10. *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{c\}, \{a, c\}\}$. Then gb^* - $O(X) = \{X, \varphi, \{c\}, \{a, c\}, \{b, c\}\}$. The set $\{b, c\}$ is gb^* - neighbourhood of point c, since the gb^* - open sets $\{c\}$ is such that $c \in \{c\} \subset \{b, c\}$. However, the set $\{b, c\}$ is not a neighbourhood of the point c, since no open set G exists such that $c \in G \subset \{b, c\}$.*

Remark 5.3. *The gb^* - neighbourhood N of $x \in X$ need not be a gb^* - open in X.*

Theorem 5.18. *If a subset N of a space X is gb^* - open, then N is gb^* - neighbourhood of each of its points.*

Proof. Suppose N is gb^* - open. Let $x \in N$. We claim that N is gb^* - neighbourhood of x. For N is a gb^* - open set such that $x \in N \subset N$. Since x is an arbitrary point of N, it follows that N is a gb^* - neighbourhood of each of its points. □

Remark 5.4. *In general, a gb^* - neighbourhood of $x \in X$ need not be a neighbourhood of x in X as seen from the following example.*

Example 5.11. *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \varphi, \{a, b\}\}$. Then $gb^* - O(X) = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{b, c\}$ is gb^*-neighbourhood of point b, since the gb^*-open sets $\{b\}$ is such that $b \in \{b\} \subset \{b, c\}$. Also the set $\{b, c\}$ is gb^*-neighbourhood of point $\{b\}$. Since the gb^*-open set $\{a, b\}$ is such that $b \in \{b\} \subset \{a, b\}$. (i.e.) $\{b, c\}$ is gb^*-neighbourhood of each of its points. However, the set $\{b, c\}$ is not a gb^*-open set in X.*

Theorem 5.19. *Let X be a topological space. If F is gb^* - closed subset of X and $x \in F^c$. Prove that there exists a gb^* - neighbourhood N of x such that $N \cap F = \varphi$.*

Proof. Let F be gb^* - closed subset of X and $x \in F^c$. Then F^c is gb^* - open set of X. So by Theorem 5.18 F^c contains a gb^* - neighbourhood of each of its points. Hence there exists a gb^* - neighbourhood N of x such that $N \subset F^c$. (i.e.) $N \cap F = \varphi$. □

Definition 5.6. *Let x be a point in a topological space X. The set of all gb^* - neighbourhood of x is called the gb^* - neighbourhood system at x, and is denoted by gb^* - $N(x)$.*

Theorem 5.20. *Let a gb^* - neighbourhood N of X be a topological space and each $x \in X$, Let gb^* - $N(X, \tau)$ be the collection of all gb^* - neighbourhood of x. Then we have the following results.*

(i) For all $x \in X$, $gb^* - N(x) \neq \phi$.

(ii) $N \in gb^* - N(x) \Rightarrow x \in N$.

(iii) $N \in gb^* - N(x), M \supset N \Rightarrow M \in gb^* - N(x)$.

(iv) $N \in gb^* - N(x), M \in gb^* - N(x) \Rightarrow N \cap M \in gb^* - N(x)$. *if finite intersection of gb^* open set is gb^* open.*

(v) $N \in gb^* - N(x) \Rightarrow$ *there exists* $M \in gb^* - N(x)$ *such that* $M \subset N$ *and* $M \in gb^* - N(y)$ *for every* $y \in M$.

Proof. 1. Since X is gb^* - open set, it is a gb^* - neighbourhood of every $x \in X$. Hence there exists at least one gb^* - neighbourhood (namely - X) for each $x \in X$. Therefore $gb^* - N(x) \neq \phi$ for every $x \in X$.

2. If $N \in gb^* - N(x)$, then N is gb^* - neighbourhood of x. By Definition of gb^* - neighbourhood, $x \in N$.

3. Let $N \in gb^* - N(x)$ and $M \supset N$. Then there is a gb^* - open set G such that $x \in G \subset N$. Since $N \subset M$, $x \in G \subset M$ and so M is gb^* - neighbourhood of x. Hence $M \in gb^* - N(x)$.

4. Let $N \in gb^* - N(x)$, $M \in gb^* - N(x)$. Then by Definition of gb^* - neighbourhood, there exists gb^* - open sets G_1 and G_2 such that $x \in G_1 \subset N$ and $x \in G_2 \subset M$. Hence

$$x \in G_1 \cap G_2 \subset N \cap M \tag{5.1}$$

Since $G_1 \cap G_2$ is a gb^* - open set, it follows from (5.1) that $N \cap M$ is a gb^* - neighbourhood of x. Hence $N \cap M \in gb^* - N(x)$.

5. Let $N \in gb^* - N(x)$, Then there is a gb^* - open set M such that $x \in M \subset N$. Since M is gb^* - open set, it is gb^* - neighbourhood of each of its points.
Therefore $M \in gb^* - N(y)$ for every $y \in M$.

□

Theorem 5.21. *Let X be a nonempty set, and for each $x \in X$, let $gb^* - N(x)$ be a nonempty collection of subsets of X satisfying following conditions.*

(i) $N \in gb^* - N(x) \Rightarrow x \in N$.

(ii) $N \in gb^* - N(x), M \in gb^* - N(x) \Rightarrow N \cap M \in gb^* - N(x)$.

Let τ consists of the empty set and all those non-empty subsets of G of X having the property that $x \in G$ implies that there exists an $N \in gb^* - N(x)$ such that $x \in N \subset G$, Then τ is a topology for X.

Proof. 1. $\varphi \in \tau$ By definition. We have to show that $x \in \tau$. Let x be any arbitrary element of X. Since $gb^* - N(x)$ is non-empty, there is an $N \in gb^* - N(x)$ and so $x \in N$ by (i). Since N is a subset of X, we have $x \in N \subset X$. Hence $x \in \tau$.

2. Let $G_1 \in \tau$ and $G_2 \in \tau$. If $x \in G_1 \cap G_2$ then $x \in G_1$ and $x \in G_2$. Since $G_1 \in \tau$ and $G_2 \in \tau$ there exists $N \in gb^* - N(x)$ and $M \in gb^* - N(x)$, such that $x \in N \subset G_1$ and $x \in M \subset G_2$. Then $x \in N \cap M \subset G_1 \cap G_2$. But $N \cap M \in gb^* - N(x)$ by (2). Hence $G_1 \cap G_2 \in \tau$.

□

6 Conclusion

The classes of generalized b star -closed sets defined using g^* open sets form a topology. The gb^*-closed sets can be used to derive a new decomposition of continuity, closed maps and open maps, contra continuous function, almost contra continuous function, closure and interior. This idea can be extended to fuzzy topological space and fuzzy intuistic topological spaces.

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On the Natural Lift Curves for the Involute Spherical Indicatrices in Minkowski 3-space

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Abstract

This study presents some new conditions of being integral curve for the geodesic spray of the natural lift curves of the spherical indicatrices of the involutes of a given spacelike curve with a timelike binormal in Minkowski 3-space. Furthermore, depending on these conditions some interesting results about the spacelike evolute curve were obtained. Additionally we illustrate an example of our main results.

Keywords: Minkowski space, involute-evolute curve couple, geodesic spray, natural lift curve, spherical indicatrix.

2010 MSC: 51B20, 53A35.

1 Introduction

One of the most significant curve is an involute of a given curve. The concept of involute was first studied by Huygens when he was considering clocks without pendula for use on ships at sea. An involute of a given curve is some other curve that always remains perpendicular to the tangent lines to that given curve. This can also be thought as the process of winding or unwinding a string tautly around a curve. The original curve is called an evolute. In addition to this, involute-evolute curve couple is a well known concept in the classical differential geometry, see [8, 11, 13]. The basic local theory of space curve are mainly developed by the Frenet-Serret theorem which expresses the derivative of a geometrically chosen basis of $\mathbf{E}^3$ by the aid of itself is proved. Then it is observed that by the solution of some of special ordinary differential equations, further classical topics, for instance spherical curves, Bertrand curves, involutes and evolutes are investigated, see for the details [10].

In differential geometry, especially the theory of space curves, the Darboux vector is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. In terms of the Frenet-Serret apparatus, the Darboux vector ω can be expressed as $\omega = \tau t + \kappa b$. In addition to this, the concepts of the natural lift and the geodesic sprays have been given by Thorpe in 1979 [16]. Çalışkan et al. [9] have studied the natural lift curves and the geodesic sprays in the Euclidean 3-space $\mathbf{E}^3$. Then Bilici et al. [5] have proposed the natural lift curves and the geodesic sprays for the spherical indicatrices of the the involute-evolute curve couple in $\mathbf{E}^3$.

Spherical images (indicatrices) are a well known concept in classical differential geometry of curves [10]. Kula and Yaylı [19] have studied spherical images of the tangent indicatrix and binormal indicatrix of a slant helix and they have shown that the spherical images are spherical helices. In recent years some of the classical differential geometry topics have been extended to Lorentzian geometry. In [20] Süha at all investigated tangent and trinormal spherical images of timelike curve lying on the pseudo hyperbolic space in Minkowski

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space-time. İyigün [21] defined the tangent spherical image of a unit speed timelike curve lying on the on the pseudo hyperbolic space in H_0^2. In [6] author adapted this problem for the spherical indicatrices of the involutes of a timelike curve in Minkowski 3-space $\mathbf{E}_1^3$. However, this problem is not solved in other cases of the space curve.

In the present paper, the natural lift curves for the spherical indicatrices of the involutes of a given spacelike curve with a timelike binormal have been investigated in Minkowski 3-space $\mathbf{E}_1^3$. With this aim we translate tangents of the involutes of a spacelike curve with a timelike binormal curve to the center of the unit hypersphere S_1^2 we obtain a spacelike curve $\alpha_{t^*}^* = t^*$ on the unit hypersphere . This curve is called the first spherical indicatrix or tangent indicatrix of α^*. One consider the principal normal indicatrix $\overline{\alpha^*}_{n^*} = n^*$ and the binormal indicatrix $\overline{\alpha^*}_{b^*} = b^*$ on the unit hypersphere H_0^2. Then the natural lift curves of the spherical indicatrices of the involutes of a given spacelike curve α with a timelike binormal are investigated in Minkowski 3-space $\mathbf{E}_1^3$ and some new results were obtained. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2 Preliminaries

Let M be a hypersurface in $\mathbf{E}_1^3$ equipped with a metric g, where the metric g means a symmetric non-degenerate $(0,2)$ tensör field on M with constant signature. For a hypersurface M, let TM be the set $\cup\left\{T_p(M) : p \in M\right\}$ of all tangent vectors to M. A technicality: For each $p \in M$ replace $0 \in T_p(M)$ by 0_p (other-wise the zero tangent vector is in every tangent space). Then each $v \in TM$ is in a unique $T_p(M)$, and the projection $\pi : TM \to M$ sends v to p. Thus $\pi^{-1}(p) = T_p(M)$. There is a natural way to make TM a manifold, called the *tangent bundle* of M.

A vector field $X \in \chi(M)$ is exactly a smooth section of TM, that is, a smooth function $X : M \to TM$ such that $\pi \circ X = I \ (identity)$. Let M be a hypersurface in $\mathbf{E}_1^3$. A curve $\alpha : I \to TM$ is an integral curve of $X \in \chi(M)$ provided $\alpha' = X_\alpha$; that is,

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \text{ for all } s \in I, \text{ [14]},$$

For any parametrized curve $\alpha : I \to TM$, the parametrized curve given by $\overline{\alpha} : I \to TM$

$$s \to \overline{\alpha}(s) = \left(\alpha(s), \alpha'(s)\right) = \alpha'(s)\mid_{\alpha(s)}$$

is called the *natural lift* of α on TM. Thus, we can write

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}\left(\alpha'(s)\mid_{\alpha(s)}\right) = D_{\alpha'(s)}\alpha'(s), \tag{2.1}$$

where D is the standard connection on $\mathbf{E}_1^3$.

For $v \in TM$, the smooth vector field $X \in \chi(M)$ defined by

$$X(v) = \varepsilon g(v, S(v))\, \xi \mid_{\alpha(s)},\ \varepsilon = g(\xi, \xi) \tag{2.2}$$

is called the *geodesic spray* on the manifold TM, where ξ is the unit normal vector field of M and S is the shape operator of M.

The Minkowski three-dimensional space $\mathbf{E}_1^3$ is the real vector space $\mathbb{R}^3$ endowed with the standard flat Lorentzian metric given by [2]

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}_1^3$. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are arbitrary vectors in $\mathbf{E}_1^3$ then we define the Lorentzain vector product of u and v as the following:

$$u \times v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Since g is an indefinite metric, recall that a vector $v \in \mathbf{E}_1^3$ can have one of three Lorentzian characters: it can be space-like if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$ and null if $g(v, v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbf{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors α' are respectively spacelike, timelike or null (lightlike), for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in \mathbf{E}_1^3$ is given by $\|a\| = \sqrt{|g(a,a)|}$. α is called an unit speed curve if velocity vector σ of α satisfies $\|\sigma\| = 1$. For vectors $v, w \in \mathbf{E}_1^3$ it is said to be orthogonal if and only if $g(v,w) = 0$.

Denote by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the moving Frenet frame along the curve α in the space $\mathbf{E}_1^3$. For an arbitrary curve α with first and second curvature, κ and τ in the space $\mathbf{E}_1^3$, the following Frenet formulae are given in [12]: If α is a spacelike curve with a timelike binormal vector $\mathbf{b}$, then the Frenet formulae read

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}, \tag{2.3}$$

where $g(\mathbf{t},\mathbf{t}) = 1,\ g(\mathbf{n},\mathbf{n}) = 1,\ g(\mathbf{b},\mathbf{b}) = -1,\ g(\mathbf{t},\mathbf{n}) = g(\mathbf{t},\mathbf{b}) = g(\mathbf{n},\mathbf{b}) = 0$.

The angle between two vectors in Minkowski space is defined by [15]:

Definition 2.1. Let X and Y be spacelike vectors in $\mathbf{E}_1^3$ that span a spacelike vector subspace, then we have $|g(X,Y)| \leq \|X\|\|Y\|$ and hence, there is a unique positive real number θ such that

$$|g(X,Y)| = \|X\|\|Y\|\cos\theta.$$

The real number θ is called the Lorentzian spacelike angle between X and Y.

Definition 2.2. Let X and Y be spacelike vectors in $\mathbf{E}_1^3$ that span a timelike vector subspace, then we have $|g(X,Y)| > \|X\|\|Y\|$ and hence, there is a unique positive real number θ such that

$$|g(X,Y)| = \|X\|\|Y\|\cosh\theta.$$

The real number θ is called the Lorentzian timelike angle between X and Y.

Definition 2.3. Let X be a spacelike vector and Y a positive timelike vector in $\mathbf{E}_1^3$, then there is a unique non-negative real number θ such that

$$|g(X,Y)| = \|X\|\|Y\|\sinh\theta.$$

The real number θ is called the Lorentzian timelike angle between X and Y.

Definition 2.4. Let X and Y be positive (negative) timelike vectors in $\mathbf{E}_1^3$, then there is a unique non-negative real number θ such that

$$g(X,Y) = \|X\|\|Y\|\cosh\theta.$$

The real number θ is called the Lorentzian timelike angle between X and Y.

The Darboux vector for the spacelike curve with a timelike binormal is defined by [17]:

$$\omega = \tau t - \kappa b.$$

There are two cases corresponding to the causal characteristic of Darboux vector ω

Case 1. If $|\kappa| < |\tau|$, then ω is a spacelike vector. In this situation, we can write

$$\kappa = \|\omega\| \sin h\theta,\ \tau = \|\omega\| \cos h\theta,\ g(\omega,\omega) = \|\omega\|^2 = \tau^2 - \kappa^2$$

and the unit vector c of direction ω is

$$c = \frac{1}{\|\omega\|}\omega = \cos h\theta t - \sin h\theta b,$$

where θ is the Lorentzian timelike angle between $-b$ and timelike unit vector c' Lorentz orthogonal to the normalisation of the Darboux vector c as Fig. 1.

Case 2. If $|\kappa| > |\tau|$, then ω is a timelike vector. In this situation, we have

$$\kappa = \|\omega\| \cos h\theta,\ \tau = \|\omega\| \sin h\theta,\ g(\omega,\omega) = -\|\omega\|^2 = \kappa^2 - \tau^2$$

and the unit vector c of direction ω is

$$c = \frac{1}{\|\omega\|}\omega = \sin h\theta t - \cos h\theta b,$$

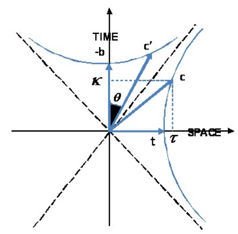

Figure 1: Lorentzian timelike angle θ

Proposition 2.5. Let α be a spacelike (or timelike) curve with curvatures κ and τ. The curve is a general helix if and only if $\frac{\tau}{\kappa}$ is constant, [3].

Remark 2.6 . We can easily see from Lemma 3.2, 3.3, and 3.4 in [1] that: $\frac{\tau(s)}{\kappa(s)} = \cot\theta$, $\frac{\tau(s)}{\kappa(s)} = \coth\theta$ or $\frac{\tau(s)}{\kappa(s)} = \tanh\theta$, if θ =constant then α is a general helix.

Lemma 2.7. *The natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on M* [6].

Remark 2.8. Let α be a spacelike curve with a timelike binormal. In this situation its involute curve α^* must be a spacelike curve with a spacelike or timelike binormal. $(\alpha,\ \alpha^*)$ being the involute-evolute curve couple, the following lemma was given by [4].

Lemma 2.9. *Let $(\alpha,\ \alpha^*)$ be the involute-evolute curve couple. The relations between the Frenet vectors of the curve couple as follow.*

I. If ω is a spacelike vector $(|\kappa| < |\tau|)$, then

$$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{n}^* \\ \mathbf{b}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh\theta & 0 & -\cosh\theta \\ -\cosh\theta & \tau & \sinh\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}.$$

II. If ω is a timelike vector $(|\kappa| > |\tau|)$, then

$$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{n}^* \\ \mathbf{b}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh\theta & 0 & \sinh\theta \\ -\sinh\theta & \tau & \cosh\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}.$$

Remark 2.10. In this situation I., the causal characteristics of the Frenet frame of the involute curve α^* is $\{t^*\ spacelike,\ n^*\ timelike,\ b^*\ spacelike\}$. If α is a spacelike curve with timelike ω, then the causal characteristics of the Frenet frame of the curve α^* must be of he form $\{t^*\ spacelike,\ n^*\ spacelike,\ b^*\ timelike\}$.

Definition 2.10. Let S_1^2 and H_0^2 be hyperspheres in $\mathbf{E}_1^3$. The Lorentzian sphere and hyperbolic sphere of radius 1 in are given by

$$S_1^2 = \left\{a = (a_1, a_2, a_3)\epsilon\mathbf{E}_1^3 : g(a,a) = 1\right\}$$

and

$$H_0^2 = \left\{a = (a_1, a_2, a_3)\epsilon\mathbf{E}_1^3 : g(a,a) = -1\right\}$$

respectively, [14].

3 The Natural Lift Curves for the Spherical Indicatrices of the Involutes of a Spacelike Curve with a Timelike Binormal

3.1 The natural lift of tangent indicatrix of the curve α^*

Let α be a spacelike curve with timelike binormal and spacelike ω $(|\kappa| < |\tau|)$. We will investigate how evolute curve α must be a curve satisfying the condition that the natural lift curve $\overline{\alpha^*}_{t^*}$ is an integral curve of geodesic spray, where $\alpha^*_{t^*}$ is the spherical indicatrix of tangent vector of involute curve α^*.

If the natural lift curve $\overline{\alpha^*}_{t^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.1.

$$\overline{D}_{\alpha'^*_{t^*}}\alpha'^*_{t^*}=0, \tag{3.4}$$

where $\overline{D}$ is the connection on the Lorentzian sphere S_1^2 and the equation of the spherical indicatrix of tangent vector of the involute curve α^* is $\alpha^*_{t^*}=t^*$. Thus from Lemma 2.2.I and the last equation we obtain

$$-\frac{\theta'}{\|\omega\|}\cosh\theta+\frac{\theta'}{\|\omega\|}\sinh\theta=0.$$

Because of $\{t,\ n,\ b\}$ are linear independent, we can easily see that

$$\theta=\text{cons}\tan\text{t},$$

according to Remark 2.1, we have

$$\frac{\tau}{\kappa}=\coth\theta=\text{cons}\tan\text{t}.$$

Result 3.1.1. If the curve α is a general helix, then the spherical indicatrix $\alpha^*_{t^*}$ of the involute curve α^* is a geodesic on the Lorentzian sphere S_1^2. In this case, from the Lemma 2.1 the natural lift $\overline{\alpha^*}_{t^*}$ of $\alpha^*_{t^*}$ is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$. In the case of a spacelike curve with timelike binormal and timelike ω, similar result can be easily obtained in following same procedure.

Remak 3.1.2. From the classification of all W-curves (i.e. a curves for which a curvature and a torsion are constants) in [1, 18], Case 1. and Case 2. we have following results with relation to curve α.

Result 3.1.3. If the curve α with κ =constant $>0,\ \tau$ =constant$\neq 0$ and $\kappa<|\tau|$ then α is a part of a spacelike hyperbolic helix,

$$\alpha(s)=\frac{1}{\|\omega\|^2}\left(\kappa\sinh\left[\|\omega\|s\right],\ \kappa\cosh\left[\|\omega\|s\right],\ \tau\|\omega\|s\right).$$

Result 3.1.4. Let α be a spacelike curve with timelike binormal and timelike ω. If the curve α with κ =constant $>0,\ \tau$ =constant$\neq 0$ and $\kappa>|\tau|$ then α is a part of a spacelike circular helix,

$$\alpha(s)=\frac{1}{\|\omega\|^2}\left(\tau\|\omega\|s,\ \kappa\cos\left[\|\omega\|s\right],\ \kappa\sin\left[\|\omega\|s\right]\right).$$

Result 3.1.5. Let α be a spacelike curve with timelike binormal and timelike ω. If the curve α with κ =constant $>0,\ \tau=0$ then α is a part of a circle.

From Lemma 3.1 in [7], we can write the following result:

Result 3.1.6. There is no spacelike W-curve with timelike binormal with condition $|\tau|=|\kappa|$.

Example 3.1.7. Let $\alpha(s)=\left(\sinh s,\ \cosh s,\ \sqrt{2}s\right)$ be a unit speed spacelike hyperbolic helix with timelike binormal and spacelike ω such that

$$\begin{aligned} t &= \left(\cosh s,\ \sinh s,\ \sqrt{2}\right) \\ n &= (\sinh s,\ \cosh s,\ 0) \\ b &= \left(\sqrt{2}\cosh s,\ \sqrt{2}\sinh s,\ 1\right),\ \kappa=1 \text{ and } \tau=\sqrt{2}. \end{aligned}$$

If α is a spacelike curve then its involute curve is a spacelike. In this situation, the involutes of the curve α can be given by the equation

$$\alpha^*(s)=\left(\sinh s+(c-s)\cosh s,\ \cosh s+(c-s)\sinh s,\ c\sqrt{2}\right),$$

where $c\in\mathbb{R}$. One can see a special example of such a curve α as Fig. 2. and its involute curve α^* as Fig. 3. when $s=[-5,\ 5]$ and $c=2$.

The short calculations give the following equation of the spherical indicatrices of the involute curve α^* and its natural lifts.

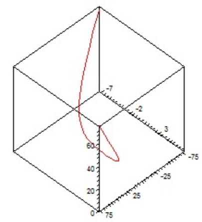

Figure 2: Spacelike curve α

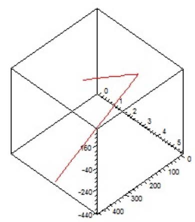

Figure 3: Involute curve α

$$\begin{aligned} \alpha^*_{t^*} &= t^* = (\sinh s,\ \cosh s,\ 0) \qquad & \overline{\alpha^*}_{t^*} &= (\cosh s,\ \sinh s,\ 0) \\ \alpha^*_{n^*} &= n^* = (\cosh s,\ \sinh s,\ 0) & \overline{\alpha^*}_{n^*} &= (\sinh s,\ \cosh s,\ 0) \\ \alpha^*_{b^*} &= b^* = (0,\ 0,\ 1) & \overline{\alpha^*}_{b^*} &= (0,\ 0,\ 0) \end{aligned}$$

Since

$$g\left(\alpha^{*\prime}_{t^*},\ \alpha^{*\prime}_{t^*}\right) = 1 > 0$$

$\alpha^*_{t^*}$ is spacelike. For being $\alpha^*_{t^*}$ is a spacelike curve, its spherical image is geodesic which lies on the Lorentzian unit sphere S^2_1 as Fig. 4. and natural lift curve of the tangent indicatrix as Fig. 5. One consider the principal normal indicatrix is a geodesic which lies on H^2_0 as Fig. 6 and its natural lift as Fig. 7.

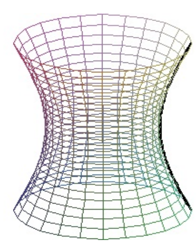

Figure 4: Spherical image of tangent indicatrix of the involute curve α^*

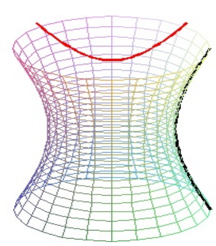

Figure 5: Tangent indicatrix of the involute curve α^* and its natural lift

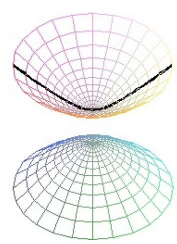

Figure 6: Spherical image of principal normal indicatrix of the involute curve α^*

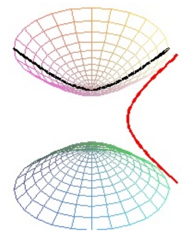

Figure 7: Principal normal indicatrix of the involute curve α^* and its natural lift

3.2 The natural lift of principal normal indicatrix of the curve α^*

Let α be a spacelike curve with timelike binormal and spacelike ω $(|\kappa| < |\tau|)$. In this section, we will investigate how α must be a curve satisfying the condition that the natural lift curve $\overline{\alpha^*}_{n^*}$ of $\alpha^*_{n^*}$ is an integral curve of geodesic spray, where $\alpha^*_{n^*}$ is the spherical indicatrix of principal normal vector of α^*. If the natural lift curve $\overline{\alpha^*}_{n^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.1. we have

$$\overline{\overline{D}}_{\alpha'^*_{n^*}} \alpha'^*_{n^*} = 0, \tag{3.5}$$

and from the Lemma 2.2. I. and the equation (5) we get,

$$\left[\left(\sigma' \cosh\theta + \theta'\sigma \sinh\theta - \frac{\kappa}{k_n}\right) t + \left(-\frac{k_n'}{k_n^2}\right) n + \left(\frac{\tau}{k_n} - \sigma' \sinh\theta - \theta'\sigma\cosh\theta\right) b\right] \frac{1}{\|\omega\|\, k_n} = 0,$$

where $\sigma = \frac{\gamma_n}{k_n}$ ($\gamma_n = \frac{\theta'}{\|\omega\|}$ and $k_n = \frac{1}{\|\omega\|}\sqrt{\theta'^2 + \|\omega\|^2}$ are the geodesic curvatures of the curve α with respect to S_1^2 and $\mathbf{E}_1^3$, respectively.) and $\overline{\overline{D}}$ is the connection of hyperbolic sphere H_0^2. Since $\{t,\ n,\ b\}$ are linear independent, we get

$$\begin{aligned} \sigma' \cosh\theta + \theta'\sigma \sinh\theta - \frac{\kappa}{k_n} &= 0 \\ \frac{k_n'}{k_n^2} &= 0 \\ \frac{\tau}{k_n} - \sigma' \sinh\theta - \theta'\sigma\cosh\theta &= 0, \end{aligned}$$

and we obtain

$$\gamma_n = \text{cons}\tan\text{t},\ k_n = \text{cons}\tan\text{t}.$$

Therefore, we can write the following result.

Result 3.2.1. If the geodesic curvatures of the evolute curve α with respect to S_1^2 and $\mathbf{E}_1^3$ are constant, then the spherical indicatrix $\alpha^*_{n^*}$ is a geodesic on the hyperbolic sphere H_0^2. In this case, the natural lift $\overline{\alpha^*}_{n^*}$ of $\alpha^*_{n^*}$ is an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$. In particular, if the evolute curve α is a spacelike curve with timelike binormal and timelike ω $(|\kappa| > |\tau|)$, then the similar result can be easily obtained by taking S_1^2 instead of H_0^2 in following same procedure.

3.3 The natural lift of binormal indicatrix of the curve α^*

Let α be a spacelike curve with timelike binormal and spacelike ω $(|\kappa| < |\tau|)$. We will investigate how α must be a curve satisfying the condition that the natural lift curve $\overline{\alpha^*}_{b^*}$ is an integral curve of geodesic spray, where $\alpha^*_{b^*}$ is the spherical indicatrix of binormal vector of α^* and $\overline{\alpha^*}_{b^*}$ is the natural lift of the curve $\alpha^*_{b^*}$. If the natural lift curve $\overline{\alpha^*}_{b^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.1. we have

$$\overline{D}_{\alpha'^*_{b^*}} \alpha'^*_{b^*} = 0, \tag{3.6}$$

from the Lemma 2.2. I. and the equation (6) we have,

$$\frac{\|\omega\|}{\theta'} n = 0.$$

Since $\{t,\ n,\ b\}$ are linear independent, we obtain

$$\kappa = 0,\ \tau = 0.$$

Thus, we can give the following result.

Result 3.3.1. The spherical indicatrix $\alpha^*_{b^*}$ of the involute curve α^* can not be a geodesic line on the Lorentzian sphere S^2_1, because, the evolute curve α whose curvature and torsion are equal to 0 is a straight line. In this case $(\alpha,\ \alpha^*)$ can not occur the involute-evolute curve couple. Therefore, the natural lift $\overline{\alpha^*}_{b^*}$ of the curve $\alpha^*_{b^*}$ can never be an integral curve of the geodesic spray on the tangent bundle $T\left(S^2_1\right)$. If the evolute curve α is a spacelike curve with timelike binormal and timelike ω , then the similar result can be easily obtained by taking S^2_1 instead of H^2_0 in following same procedure.

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The General Solution and Stability of Nonadecic Functional Equation in Matrix Normed Spaces

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Abstract

In this paper, we present the general solution of a nonadecic functional equation and establish the Ulam-Hyers stability of nonadecic functional equation in matrix normed spaces by using the fixed point method.

Keywords: Hyers-Ulam stability, fixed point, nonadecic functional equation, matrix normed spaces.

2010 MSC: 39B52, 39B82, 46L07, 47H10, 47L25.

1 Introduction

In 1940, an interesting talk presented by S. M. Ulam [27] triggered the study of stability problems for various functional equations. He raised a question concerning the stability of homomorphism. In the following year, 1941, D. H. Hyers [5] was able to give a partial solution to Ulam's question. The result of Hyers was generalized by Aoki [1] for additive mappings. In 1978, Th. M. Rassias [14] succeeded in extending the result of Hyers theorem by weakening the condition for the Cauchy difference.

The stability phenomenon that was presented by Th. M. Rassias is called the generalized Hyers-Ulam stability. This concept actually means that if one is studying a Hyers-Ulam stable system, one need not have to reach the exact solution, which usually is quite difficult or time consuming. This is quite useful in many applications for example optimization, numerical analysis, biology, life sciences, economics etc., where finding the exact solution is quite difficult.

From 1982-1994, J. M. Rassias (see [16]- [23]) solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings, by involving a product of different powers of norms. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [4] by replacing the unbounded Cauchy difference by a general control function. A further generalization of the Hyers-Ulam stability for a large class of mapping was obtained by Isac and Th. M. Rassias [6]. They also presented some applications in non-linear analysis, especially in fixed point theory. This terminology may also be applied to the cases of other functional equations [2, 3, 13, 15, 26, 29]. Also, the generalized Hyers-Ulam stability of functional equations and inequalities in matrix normed spaces has been studied by number of authors [7–10, 12, 28].

K. Ravi and B. V. Senthil Kumar [24] discussed the general solution of undecic functional equation

$$f(x+6y)-11f(x+5y)+55f(x+4y)-165f(x+3y)+330f(x+2y)$$
$$-462f(x+y)-462f(x)-330f(x-y)+165f(x-2y)$$
$$-55f(x-3y)+11f(x-4y)-f(x-5y)=39916800f(y)$$

and proved the stability of this functional equation in quasi β - normed spaces by applying the fixed point method.

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Very recently, K. Ravi et. al., [25] discussed the general solution of quattuordecic functional equation

$$f(x+7y)-14f(x+6y)+91f(x+5y)-364f(x+4y)+1001f(x+3y)-2002f(x+2y)$$
$$-3003f(x+y)-3432f(x)+3003f(x-y)-2002f(x-2y)+1001f(x-3y)$$
$$-364f(x-4y)+91f(x-5y)-14f(x-6y)+f(x-7y)=87178291200f(y)$$

and its stability in quasi β - normed spaces.

In this paper, we introduce the following new functional equation

$$f(x+10y)-19f(x+9y)+171f(x+8y)-969f(x+7y)+3876f(x+6y)-11628f(x+5y)$$
$$+27132f(x+4y)-50388f(x+3y)+75582f(x+2y)-92378f(x+y)$$
$$+92378f(x)-75582f(x-y)+50388f(x-2y)-27132f(x-3y)$$
$$+11628f(x-4y)-3876f(x-5y)+969f(x-6y)-171f(x-7y)$$
$$+19f(x-8y)-f(x-9y)=19!f(y) \tag{1.1}$$

where $19! = 121645100400000000$, is said to be nonadecic functional equation since the function $f(x) = cx^{19}$ is its solution. In this paper, we determine the general solution of the functional equation (1.1) and we also prove the Ulam-Hyers stability of the functional equation (1.1) in matrix normed spaces by using fixed point approach.

2 General Solution of Nonadecic Functional Equation (1.1)

In this section, we present the general solution of nonadecic functional equation (1.1). For this, let us consider $\mathcal{A}$ and $\mathcal{B}$ be real vector spaces.

Theorem 2.1. *If $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying (1.1) for all $x, y \in \mathcal{A}$, then f is nonadecic.*

Proof. Letting $x = y = 0$ in (1.1), one gets $f(0) = 0$. Replacing $x = 0$, $y = x$ and $x = x$, $y = -x$ in (1.1) and adding the two resulting equations, we get

$$f(-x) = -f(x)$$

Hence, f is an odd mapping. Replacing $x = 0$, $y = 2x$ and $x = 10x$, $y = x$ in (1.1) and subtracting the two resulting equations, we get

$$19f(19x)-189f(18x)+969f(17x)-3724f(16x)+11628f(15x)-27930f(14x)$$
$$+50388f(13x)-72675f(12x)+92378f(11x)-100130f(10x)$$
$$+75582f(9x)-34884f(8x)+27132f(7x)-34884f(6x)+3876f(5x)$$
$$+24225f(4x)+171f(3x)-(16815+19!)f(2x)+19!f(x)=0 \tag{2.2}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(9x, x)$ in (1.1), we obtain that

$$f(19x)-19f(18x)+171f(17x)-969f(16x)+3876f(15x)-11628f(14x)$$
$$+27132f(13x)-50388f(12x)+75582f(11x)-92378f(10x)$$
$$+92378f(9x)-75582f(8x)+50388f(7x)-27132f(6x)+11628f(5x)$$
$$-3876f(4x)+969f(3x)-171f(2x)+(19-19!)f(x)=0 \tag{2.3}$$

for all $x \in \mathcal{A}$. Multiplying (2.3) by 19, and then subtracting (2.2) from the resulting equation, we get

$$172f(18x)-2280f(17x)+14687f(16x)-62016f(15x)+193002f(14x)-18240f(3x)$$
$$-465120f(13x)+884697f(12x)-1343680f(11x)+1655052f(10x)$$
$$-1679600f(9x)+1401174f(8x)-930240f(7x)+480624f(6x)$$
$$-217056f(5x)+97869f(4x)+(13566-19!)f(2x)+20(19!)f(x)=0 \tag{2.4}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(8x, x)$ in (1.1), we have

$$f(18x)-19f(17x)+171f(16x)-969f(15x)+3876f(14x)-11628f(13x)$$
$$+27132f(12x)-50388f(11x)+75582f(10x)-92378f(9x)$$
$$+92378f(8x)-75582f(7x)+50388f(6x)-27132f(5x)+11628f(4x)$$
$$-3876f(3x)+969f(2x)-(170+19!)f(x)=0 \tag{2.5}$$

for all $x \in \mathcal{A}$. Multiplying (2.5) by 172, and then subtracting (2.4) from the resulting equation , we get

$$\begin{aligned}988f(17x) - 14725f(16x) + 104652f(15x) - 473670f(14x) + 1534896f(13x)\\ +7323056f(11x) - 11345052f(10x) + 14209416f(9x) - 14487842f(8x)\\ +12069864f(7x) - 8186112f(6x) + 4449648f(5x) - 1902147f(4x)\\ + 648432f(3x) - 3782007f(12x) - (180234 + 19!)f(2x) + 192(19!)f(x) = 0\end{aligned} \tag{2.6}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(7x, x)$ in (1.1), it follows that

$$\begin{aligned}f(17x) - 19f(16x) + 171f(15x) - 969f(14x) + 3876f(13x) - 11628f(12x)\\ +27132f(11x) - 50388f(10x) + 75582f(9x) - 92378f(8x)\\ +92378f(7x) - 75582f(6x) + 50388f(5x) - 27132f(4x) + 11628f(3x)\\ - 3875f(2x) + (950 - 19!)f(x) = 0\end{aligned} \tag{2.7}$$

for all $x \in \mathcal{A}$. Multiplying (2.7) by 988, and then subtracting (2.6) from the resulting equation, we get

$$\begin{aligned}4047f(16x) - 64296f(15x) + 483702f(14x) - 2294592f(13x) + 7706457f(12x)\\ +38438292f(10x) - 60465600f(9x) + 76781622f(8x) - 79199600f(7x)\\ +66488904f(6x) - 45333696f(5x) + 24904269f(4x) - 10840032f(3x)\\ - 19483360f(11x) + (3648266 - 19!)f(2x) + 1180(19!)f(x) = 0\end{aligned} \tag{2.8}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(6x, x)$ in (1.1), we have

$$\begin{aligned}f(16x) - 19f(15x) + 171f(14x) - 969f(13x) + 3876f(12x) - 11628f(11x)\\ +27132f(10x) - 50388f(9x) + 75582f(8x) - 92378f(7x) + 92378f(6x)\\ - 75582f(5x) + 50388f(4x) - 27131f(3x) + 11609f(2x) - (3705 + 19!)f(x) = 0\end{aligned} \tag{2.9}$$

for all $x \in \mathcal{A}$. Multiplying (2.9) by 4047, and then subtracting (2.8) from the resulting equation, we arrive at

$$\begin{aligned}12597f(15x) - 208335f(14x) + 1626951f(13x) - 7979715f(12x) + 27575156f(11x)\\ +143454636f(9x) - 229098732f(8x) + 294654166f(7x) - 307364862f(6x)\\ +260546658f(5x) - 179015967f(4x) + 98959125f(3x)\\ - 71364912f(10x) - (43333357 + 19!)f(2x) + 5227(19!)f(x) = 0\end{aligned} \tag{2.10}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(5x, x)$ in (1.1), we obtain

$$\begin{aligned}f(15x) - 19f(14x) + 171f(13x) - 969f(12x) + 3876f(11x) - 11628f(10x)\\ +27132f(9x) - 50388f(8x) + 75582f(7x) - 92378f(6x) + 92378f(5x)\\ - 75581f(4x) + 50369f(3x) - 26961f(2x) + (10659 - 19!)f(x) = 0\end{aligned} \tag{2.11}$$

for all $x \in \mathcal{A}$. Multiplying (2.11) by 12597, and then subtracting (2.10) from the resulting equation, we arrive at

$$\begin{aligned}31008f(14x) - 527136f(13x) + 4226778f(12x) - 21250816f(11x)\\ +75113004f(10x) - 198327168f(9x) + 405638904f(8x) - 657452288f(7x)\\ +856320804f(6x) - 903139008f(5x) + 773077890f(4x) - 535539168f(3x)\\ + (296294360 - 19!)f(2x) + 17824(19!)f(x) = 0\end{aligned} \tag{2.12}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(4x, x)$ in (1.1), we get

$$\begin{aligned}f(14x) - 19f(13x) + 171f(12x) - 969f(11x) + 3876f(10x) - 11628f(9x)\\ +27132f(8x) - 50388f(7x) + 75582f(6x) - 92377f(5x)\\ + 92359f(4x) - 75411f(3x) + 49419f(2x) - (23256 + 19!)f(x) = 0\end{aligned} \tag{2.13}$$

for all $x \in \mathcal{A}$. Multiplying (2.13) by 31008, and then subtracting (2.12) from the resulting equation, we obtain

$$\begin{aligned}62016f(13x) - 1075590f(12x) + 8795936f(11x) - 45074004f(10x)\\ +162233856f(9x) - 435670152f(8x) + 904978816f(7x) + 1802805120f(3x)\\ -1487325852f(6x) + 1961287008f(5x) - 2090789982f(4x) + 48832(19!)f(x)\\ - (1236089992 + 19!)f(2x) = 0\end{aligned} \tag{2.14}$$

for all $x \in \mathcal{A}$. Replacing (x,y) by $(3x,x)$ in (1.1), we have

$$\begin{aligned} &f(13x) - 19f(12x) + 171f(11x) - 969f(10x) + 3876f(9x) - 11628f(8x) \\ &\quad +27132f(7x) - 50387f(6x) + 75563f(5x) - 92207f(4x) \\ &\quad + 91409f(3x) - 71706f(2x) + (38760 - 19!)f(x) = 0 \end{aligned} \tag{2.15}$$

for all $x \in \mathcal{A}$. Multiplying (2.15) by 62016, and then subtracting (2.14) from the resulting equation, we obtain

$$\begin{aligned} &102714f(12x) - 1808800f(11x) + 15019500f(10x) - 78140160f(9x) + 285451896f(8x) \\ &\quad -777639296f(7x) + 1637474340f(6x) - 2724828000f(5x) + 3627519330f(4x) \\ &\quad - 3866015424f(3x) + (3210829304 - 19!)f(2x) + 110848(19!)f(x) = 0 \end{aligned} \tag{2.16}$$

for all $x \in \mathcal{A}$. Replacing (x,y) by $(2x,x)$ in (1.1), it follows that

$$\begin{aligned} &f(12x) - 19f(11x) + 171f(10x) - 969f(9x) + 3876f(8x) - 11627f(7x) + 27113f(6x) \\ &\quad - 50217f(5x) + 74613f(4x) - 88502f(3x) + 80750f(2x) - (48450 - 19!)f(x) = 0 \end{aligned} \tag{2.17}$$

for all $x \in \mathcal{A}$. Multiplying (2.17) by 102714, and then subtracting (2.16) from the resulting equation, we obtain

$$\begin{aligned} &142766f(11x) - 2544594f(10x) + 21389706f(9x) - 112667568f(8x) + 416616382f(7x) \\ &\quad -1147410342f(6x) + 2433160938f(5x) - 4036280352f(4x) + 5224379004f(3x) \\ &\quad - (5083326196 + 19!)f(2x) + 213562(19!)f(x) = 0 \end{aligned} \tag{2.18}$$

for all $x \in \mathcal{A}$. Replacing (x,y) by (x,x) in (1.1), we get

$$\begin{aligned} &f(11x) - 19f(10x) + 171f(9x) - 968f(8x) + 3857f(7x) - 11457f(6x) + 26163f(5x) \\ &\quad - 46512f(4x) + 63954f(3x) - 65246f(2x) + (41990 - 19!)f(x) = 0 \end{aligned} \tag{2.19}$$

for all $x \in \mathcal{A}$. Multiplying (2.19) by 142766, and then subtracting (2.18) from the resulting equation, we obtain

$$\begin{aligned} &167960f(10x) - 3023280f(9x) + 25529920f(8x) - 134032080f(7x) \\ &\quad +488259720f(6x) - 1302025920f(5x) + 2604051840f(4x) \\ &\quad - 3906077760f(3x) + (4231584240 - 19!)f(2x) + 356328(19!)f(x) = 0 \end{aligned} \tag{2.20}$$

for all $x \in \mathcal{A}$. Replacing (x,y) by $(0,x)$ in (1.1), we obtain that

$$\begin{aligned} &f(10x) - 18f(9x) + 152f(8x) - 798f(7x) + 2907f(6x) - 7752f(5x) \\ &\quad + 15504f(4x) - 23256f(3x) + 25194f(2x) - (16796 + 19!)f(x) = 0 \end{aligned} \tag{2.21}$$

for all $x \in \mathcal{A}$. Multiplying (2.21) by 167960, and then subtracting (2.20) from the resulting equation, we can obtain that

$$f(2x) = 2^{19}f(x)$$

for all $x \in \mathcal{A}$. Hence $f : \mathcal{A} \to \mathcal{B}$ is a nonadecic mapping. This completes the proof. □

3 Ulam-Hyers Stability of Nonadecic Functional Equation (1.1)

In this section, We will investigate the Ulam-Hyers stability for the functional equation (1.1) in matrix normed spaces by using the fixed point method.

Throughout this section, let us consider $(X, \|.\|_n)$ be a matrix normed space, $(Y, \|.\|_n)$ be a matrix Banach space and let n be a fixed non-negative integer.

For a mapping $f : X \to Y$, define $\mathcal{G}f : X^2 \to Y$ and $\mathcal{G}f_n : M_n(X^2) \to M_n(Y)$ by,

$$\begin{aligned} \mathcal{G}f(a,b) = {} & f(a+10b) - 19f(a+9b) + 171f(a+8b) - 969f(a+7b) + 3876f(a+6b) \\ & - 11628f(a+5b) + 27132f(a+4b) - 50388f(a+3b) - f(a-9b) \\ & + 75582f(a+2b) - 92378f(a+b) + 92378f(a) - 75582f(a-b) \\ & + 50388f(a-2b) - 27132f(a-3b) + 11628f(a-4b) - 19!f(b) \\ & - 3876f(a-5b) + 969f(a-6b) - 171f(a-7b) + 19f(a-8b), \end{aligned}$$

$$\begin{aligned}\mathcal{G}f_n(x_{ij},y_{ij}) = & f_n(x_{ij}+10y_{ij}) - 19f_n(x_{ij}+9y_{ij}) + 171f_n(x_{ij}+8y_{ij}) - 969f_n(x_{ij}+7y_{ij})\\ & + 3876f_n(x_{ij}+6y_{ij}) - 11628f_n(x_{ij}+5y_{ij}) + 27132f_n(x_{ij}+4y_{ij})\\ & - 50388f_n(x_{ij}+3y_{ij}) + 75582f_n(x_{ij}+2y_{ij}) - 92378f_n(x_{ij}+y_{ij})\\ & + 92378f_n(x_{ij}) - 75582f_n(x_{ij}-y_{ij}) + 50388f_n(x_{ij}-2y_{ij})\\ & - 27132f_n(x_{ij}-3y_{ij}) + 11628f_n(x_{ij}-4y_{ij}) - 3876f_n(x_{ij}-5y_{ij})\\ & + 969f_n(x_{ij}-6y_{ij}) - 171f_n(x_{ij}-7y_{ij}) + 19f_n(x_{ij}-8y_{ij})\\ & - f_n(x_{ij}-9y_{ij}) - 19!f_n(y_{ij})\end{aligned}$$

for all $a,b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 3.2. *Let $l = \pm 1$ be fixed and $\psi : X^2 \to [0,\infty)$ be a function such that there exists a $\eta < 19$ with*

$$\psi(a,b) \le 2^{19l}\eta\psi(\frac{a}{2^l},\frac{b}{2^l}) \qquad \forall\, a,b \in X. \tag{3.22}$$

Let $f : X \to Y$ be a mapping satisfying

$$\left\|\mathcal{G}f_n([x_{ij}],[y_{ij}])\right\| \le \sum_{i,j=1}^{n} \psi(x_{ij},y_{ij}) \qquad \forall\, x = [x_{ij}], y = [y_{ij}] \in M_n(X). \tag{3.23}$$

Then there exists a unique nonadecic mapping $\mathcal{N}_{\mathcal{D}} : X \to Y$ such that

$$\left\|f_n([x_{ij}]) - \mathcal{N}_{\mathcal{D}n}([y_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \frac{\eta^{\frac{1-l}{2}}}{2^{19}(1-\eta)}\overline{\psi}(x_{ij}) \qquad \forall\, x = [x_{ij}] \in M_n(X), \tag{3.24}$$

where $$\begin{aligned}\overline{\psi}(x_{ij}) = \frac{1}{19!}[&\psi(0,2x_{ij}) + \psi(10x_{ij},x_{ij}) + 19\psi(9x_{ij},x_{ij}) + 172\psi(8x_{ij},x_{ij})\\ &+988\psi(7x_{ij},x_{ij}) + 4047\psi(6x_{ij},x_{ij}) + 12597\psi(5x_{ij},x_{ij})\\ &+31008\psi(4x_{ij},x_{ij}) + 62016\psi(3x_{ij},x_{ij}) + 102714\psi(2x_{ij},x_{ij})\\ &+142766\psi(x_{ij},x_{ij}) + 167960\psi(0,x_{ij})]\end{aligned}$$

Proof. Substituting $n = 1$ in (3.23), we obtain

$$\|\mathcal{G}f(a,b)\| \le \psi(a,b) \tag{3.25}$$

Replacing (a,b) by $(0,2a)$ in (3.25), we get

$$\begin{aligned}&\|f(20a) - 18f(18a) + 152f(16a) - 798f(14a) + 2907f(12a) - 7752f(10a)\\ &+15504f(8a) - 23256f(6a) + 25194f(4a) - (16796 + 19!)f(2a)\| \le \psi(0,2a)\end{aligned} \tag{3.26}$$

for all $a \in X$. Replacing (a,b) by $(10a,a)$ in (3.25), we obtain

$$\begin{aligned}&\|f(20a) - 19f(19a) + 171f(18a) - 969f(17a) + 3876f(16a) - 11628f(15a)\\ &+27132f(14a) - 50388f(13a) + 75582f(12a) - 92378f(11a)\\ &+92378f(10a) - 75582f(9a) + 50388f(8a) - 27132f(7a) + 11628f(6a)\\ &-3876f(5a) + 969f(4a) - 171f(3a) + 19f(2a) - (1 + 19!)f(a)\| \le \psi(10a,a)\end{aligned} \tag{3.27}$$

for all $a \in X$. Combining (3.26) and (3.27), we arrive at

$$\begin{aligned}&\|19f(19a) - 189f(18a) + 969f(17a) - 3724f(16a) + 11628f(15a) - 27930f(14a)\\ &+50388f(13a) - 72675f(12a) + 92378f(11a) - 100130f(10a)\\ &+75582f(9a) - 34884f(8a) + 27132f(7a) - 34884f(6a) + 3876f(5a)\\ &+24225f(4a) + 171f(3a) - (16815 + 19!)f(2a) + 19!f(a)\| \le \psi(0,2a) + \psi(10a,a)\end{aligned} \tag{3.28}$$

for all $a \in X$. Replacing (a,b) by $(9a,a)$ in (3.25), we obtain

$$\begin{aligned}&\|f(19a) - 19f(18a) + 171f(17a) - 969f(16a) + 3876f(15a) - 11628f(14a)\\ &+27132f(13a) - 50388f(12a) + 75582f(11a) - 92378f(10a)\\ &+92378f(9a) - 75582f(8a) + 50388f(7a) - 27132f(6a) + 11628f(5a)\\ &-3876f(4a) + 969f(3a) - 171f(2a) + (19 - 19!)f(a)\| \le \psi(9a,a)\end{aligned} \tag{3.29}$$

$\forall\ a \in X$. Multiplying (3.29) by 19, and combining the resulting inequality with (3.28), we get

$$\begin{aligned}\|172f(18a) - 2280f(17a) + 14687f(16a) - 62016f(15a) + 193002f(14a)\\ -465120f(13a) + 884697f(12a) - 1343680f(11a) + 1655052f(10a)\\ -1679600f(9a) + 1401174f(8a) - 930240f(7a) + 480624f(6a)\\ -217056f(5a) + 97869f(4a) - 18240f(3a) + (13566 - 19!)f(2a)\\ +20(19!)f(a)\| \leq \psi(0,2a) + \psi(10a,a) + 19\psi(9a,a)\end{aligned} \tag{3.30}$$

for all $a \in X$. Replacing (a, b) by $(8a, a)$ in (3.25), we obtain

$$\begin{aligned}\|f(18a) - 19f(17a) + 171f(16a) - 969f(15a) + 3876f(14a) - 11628f(13a)\\ +27132f(12a) - 50388f(11a) + 75582f(10a) - 92378f(9a)\\ +92378f(8a) - 75582f(7a) + 50388f(6a) - 27132f(5a) + 11628f(4a)\\ -3876f(3a) + 969f(2a) - (170 + 19!)f(a)\| \leq \psi(8a,a)\end{aligned} \tag{3.31}$$

$\forall\ a \in X$. Multiplying (3.31) by 172, and combining the resulting inequality with (3.30), we get

$$\begin{aligned}\|988f(17a) - 14725f(16a) + 104652f(15a) - 473670f(14a) + 1534896f(13a)\\ +7323056f(11a) - 11345052f(10a) + 14209416f(9a) - 14487842f(8a)\\ +12069864f(7a) - 8186112f(6a) + 4449648f(5a) - 1902147f(4a)\\ +648432f(3a) - 3782007f(12a) - (180234 + 19!)f(2a)\\ +192(19!)f(a)\| \leq \psi(0,2a) + \psi(10a,a) + 19\psi(9a,a) + 172\psi(8a,a)\end{aligned} \tag{3.32}$$

for all $a \in X$. Replacing (a, b) by $(7a, a)$ in (3.25), we get

$$\begin{aligned}\|f(17a) - 19f(16a) + 171f(15a) - 969f(14a) + 3876f(13a) - 11628f(12a)\\ +27132f(11a) - 50388f(10a) + 75582f(9a) - 92378f(8a)\\ +92378f(7a) - 75582f(6a) + 50388f(5a) - 27132f(4a) + 11628f(3a)\\ -3875f(2a) + (950 - 19!)f(a)\| \leq \psi(7a,a)\end{aligned} \tag{3.33}$$

$\forall\ a \in X$. Multiplying (3.33) by 988, and combining the resulting inequality with (3.32), we get

$$\begin{aligned}\|4047f(16a) - 64296f(15a) + 483702f(14a) - 2294592f(13a) + 7706457f(12a)\\ +38438292f(10a) - 60465600f(9a) + 76781622f(8a) - 79199600f(7a)\\ +66488904f(6a) - 45333696f(5a) + 24904269f(4a) - 10840032f(3a)\\ -19483360f(11a) + (3648266 - 19!)f(2a)\\ +1180(19!)f(a)\| \leq \psi(0,2a) + \psi(10a,a) + 19\psi(9a,a) + 172\psi(8a,a) + 988\psi(7a,a)\end{aligned} \tag{3.34}$$

for all $a \in X$. Replacing (a, b) by $(6a, a)$ in (3.25), we get

$$\begin{aligned}\|f(16a) - 19f(15a) + 171f(14a) - 969f(13a) + 3876f(12a) - 11628f(11a)\\ +27132f(10a) - 50388f(9a) + 75582f(8a) - 92378f(7a)\\ +92378f(6a) - 75582f(5a) + 50388f(4a) - 27131f(3a) + 11609f(2a)\\ -(3705 + 19!)f(a)\| \leq \psi(6a,a)\end{aligned} \tag{3.35}$$

$\forall\ a \in X$. Multiplying (3.35) by 4047, and combining the resulting inequality with (3.34), we arrive at

$$\begin{aligned}\|12597f(15a) - 208335f(14a) + 1626951f(13a) - 7979715f(12a) + 27575156f(11a)\\ +143454636f(9a) - 229098732f(8a) + 294654166f(7a) - 307364862f(6a)\\ +260546658f(5a) - 179015967f(4a) + 98959125f(3a)\\ -71364912f(10a) - (43333357 + 19!)f(2a) + 5227(19!)f(a)\|\\ \leq \psi(0,2a) + \psi(10a,a) + 19\psi(9a,a) + 172\psi(8a,a) + 988\psi(7a,a) + 4047\psi(6a,a)\end{aligned} \tag{3.36}$$

for all $a \in X$. Replacing (a, b) by $(5a, a)$ in (3.25), we obtain

$$\begin{aligned}\|f(15a) - 19f(14a) + 171f(13a) - 969f(12a) + 3876f(11a) - 11628f(10a)\\ +27132f(9a) - 50388f(8a) + 75582f(7a) - 92378f(6a)\\ +92378f(5a) - 75581f(4a) + 50369f(3a) - 26961f(2a)\\ +(10659 - 19!)f(a)\| \leq \psi(5a,a)\end{aligned} \tag{3.37}$$

$\forall\ a \in X$. Multiplying (3.37) by 12597, and combining the resulting inequality with (3.36), we arrive at

$$\|31008f(14a) - 527136f(13a) + 4226778f(12a) - 21250816f(11a) \\ +75113004f(10a) - 198327168f(9a) + 405638904f(8a) - 657452288f(7a) \\ +856320804f(6a) - 903139008f(5a) + 773077890f(4a) - 535539168f(3a) \\ +(296294360 - 19!)f(2a) + 17824(19!)f(a)\| \leq \psi(0,2a) + \psi(10a,a) \\ + 19\psi(9a,a) + 172\psi(8a,a) + 988\psi(7a,a) + 4047\psi(6a,a) + 12597\psi(5a,a) \quad (3.38)$$

for all $a \in X$. Replacing (a,b) by $(4a,a)$ in (3.25), we get

$$\|f(14a) - 19f(13a) + 171f(12a) - 969f(11a) + 3876f(10a) - 11628f(9a) \\ +27132f(8a) - 50388f(7a) + 75582f(6a) - 92377f(5a) \\ +92359f(4a) - 75411f(3a) + 49419f(2a) - (23256 + 19!)f(a)\| \leq \psi(4a,a) \quad (3.39)$$

$\forall\ a \in X$. Multiplying (3.39) by 31008, and combining the resulting inequality with (3.38), we obtain

$$\|62016f(13a) - 1075590f(12a) + 8795936f(11a) - 45074004f(10a) \\ +162233856f(9a) - 435670152f(8a) + 904978816f(7a) + 1802805120f(3a) \\ -1487325852f(6a) + 1961287008f(5a) - 2090789982f(4a) + 48832(19!)f(a) \\ -(1236089992 + 19!)f(2a)\| \leq \psi(0,2a) + \psi(10a,a) + 19\psi(9a,a) + 172\psi(8a,a) \\ + 988\psi(7a,a) + 4047\psi(6a,a) + 12597\psi(5a,a) + 31008\psi(4a,a) \quad (3.40)$$

for all $a \in X$. Replacing (a,b) by $(3a,a)$ in (3.25), we get

$$\|f(13a) - 19f(12a) + 171f(11a) - 969f(10a) + 3876f(9a) - 11628f(8a) \\ +27132f(7a) - 50387f(6a) + 75563f(5a) - 92207f(4a) \\ +91409f(3a) - 71706f(2a) + (38760 - 19!)f(a)\| \leq \psi(3a,a) \quad (3.41)$$

$\forall\ a \in X$. Multiplying (3.53) by 62016, and combining the resulting inequality with (3.40), we obtain

$$\|102714f(12a) - 1808800f(11a) + 15019500f(10a) - 78140160f(9a) \\ +285451896f(8a) - 2724828000f(5a) + 3627519330f(4a) - 3866015424f(3a) \\ -777639296f(7a) + 1637474340f(6a) + (3210829304 - 19!)f(2a) \\ +110848(19!)f(a)\| \leq \psi(0,2a) + \psi(10a,a) + 19\psi(9a,a) + 12597\psi(5a,a) \\ + 172\psi(8a,a) + 988\psi(7a,a) + 4047\psi(6a,a) + 31008\psi(4a,a) + 62016\psi(3a,a) \quad (3.42)$$

for all $a \in X$. Replacing (a,b) by $(2a,a)$ in (3.25), we get

$$\|f(12a) - 19f(11a) + 171f(10a) - 969f(9a) + 3876f(8a) - 11627f(7a) \\ +27113f(6a) - 50217f(5a) + 74613f(4a) - 88502f(3a) \\ +80750f(2a) - (48450 - 19!)f(a)\| \leq \psi(2a,a) \quad (3.43)$$

for all $a \in X$. Multiplying (3.43) by 102714, and combining the resulting inequality with (3.42), we obtain

$$\|142766f(11a) - 2544594f(10a) + 21389706f(9a) - 112667568f(8a) \\ +416616382f(7a) - 1147410342f(6a) + 2433160938f(5a) \\ -4036280352f(4a) + 5224379004f(3a) - (5083326196 + 19!)f(2a) \\ +213562(19!)f(a)\| \leq \psi(0,2a) + \psi(10a,a) + 19\psi(9a,a) + 12597\psi(5a,a) \\ +172\psi(8a,a) + 988\psi(7a,a) + 4047\psi(6a,a) \\ + 31008\psi(4a,a) + 62016\psi(3a,a) + 102714\psi(2a,a) \quad (3.44)$$

for all $a \in X$. Replacing (a,b) by (a,a) in (3.25), we get

$$\|f(11a) - 19f(10a) + 171f(9a) - 968f(8a) + 3857f(7a) - 11457f(6a) \\ +26163f(5a) - 46512f(4a) + 63954f(3a) - 65246f(2a) \\ +(41990 - 19!)f(a)\| \leq \psi(a,a) \quad (3.45)$$

for all $a \in X$. Multiplying (3.45) by 142766, and combining the resulting inequality with (3.44), we get

$$\begin{aligned}&\|167960f(10a) - 3023280f(9a) + 25529920f(8a) - 134032080f(7a)\\ &\quad +488259720f(6a) - 1302025920f(5a) + 2604051840f(4a)\\ &\quad -3906077760f(3a) + (4231584240 - 19!)f(2a) + 356328(19!)f(a)\|\\ &\quad \leq \psi(0,2a) + \psi(10a,a) + 19\psi(9a,a) + 12597\psi(5a,a) + 172\psi(8a,a) + 988\psi(7a,a)\\ &\qquad + 4047\psi(6a,a) + 31008\psi(4a,a) + 62016\psi(3a,a) + 102714\psi(2a,a) + 142766\psi(a,a) \end{aligned} \tag{3.46}$$

for all $a \in X$. Replacing (a,b) by $(0,a)$ in (3.25), we get

$$\begin{aligned}&\|f(10a) - 18f(9a) + 152f(8a) - 798f(7a) + 2907f(6a) - 7752f(5a)\\ &\quad +15504f(4a) - 23256f(3a) + 25194f(2a) - (16796 + 19!)f(a)\| \leq \psi(0,a)\end{aligned} \tag{3.47}$$

for all $a \in X$. Multiplying (3.47) by 167960, and combining the resulting inequality with (3.46), we obtain

$$\begin{aligned}&\|-19!f(2a) + 524288(19!)f(a)\| \leq \psi(0,2a) + \psi(10a,a) + 19\psi(9a,a) + 12597\psi(5a,a)\\ &\quad +172\psi(8a,a) + 988\psi(7a,a) + 4047\psi(6a,a) + 31008\psi(4a,a)\\ &\quad + 62016\psi(3a,a) + 102714\psi(2a,a) + 142766\psi(a,a) + 167960\psi(0,a)\end{aligned} \tag{3.48}$$

for all $a \in X$. From (3.48), we can obtain

$$\begin{aligned}\left\|-f(2a) + 2^{19}f(a)\right\| \leq \frac{1}{19!}&[\psi(0,2a) + \psi(10a,a) + 19\psi(9a,a) + 172\psi(8a,a)\\ &+988\psi(7a,a) + 4047\psi(6a,a) + 12597\psi(5a,a)\\ &+31008\psi(4a,a) + 62016\psi(3a,a) + 102714\psi(2a,a)\\ &+142766\psi(a,a) + 167960\psi(0,a)]\end{aligned} \tag{3.49}$$

Therefore,

$$\left\|f(2a) - 2^{19}f(a)\right\| \leq \overline{\psi}(a) \qquad \forall\, a \in X. \tag{3.50}$$

Thus

$$\left\|f(a) - \frac{1}{2^{19l}}f(2^l a)\right\| \leq \frac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{19}}\overline{\psi}(a) \qquad \forall\, a \in X. \tag{3.51}$$

We consider the set $\mathcal{M} = \{f : X \to Y\}$ and introduce the generalized metric ρ on $\mathcal{M}$ as follows:

$$\rho(f,g) = \inf\left\{\mu \in \mathbb{R}_+ : \|f(a) - g(a)\| \leq \mu\overline{\psi}(a), \forall a \in X\right\},$$

It is easy to check that $(\mathcal{M}, \rho)$ is a complete generalized metric (see also [11]). Define the mapping $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{P}f(a) = \frac{1}{2^{19l}}f(2^l a) \qquad \forall\, f \in \mathcal{M} \text{ and } a \in X.$$

Let $f, g \in \mathcal{M}$ and ν be an arbitrary constant with $\rho(f,g) = \nu$. Then

$$\|f(a) - g(a)\| \leq \nu\overline{\psi}(a) \qquad \text{for all } a \in X.$$

Utilizing (3.22), we find that

$$\|\mathcal{P}f(a) - \mathcal{P}g(a)\| = \left\|\frac{1}{2^{19l}}f(2^l a) - \frac{1}{2^{19l}}g(2^l a)\right\| \leq \eta\nu\overline{\psi}(a) \quad \text{for all } a \in X.$$

Hence it holds that $\rho(\mathcal{P}f, \mathcal{P}g) \leq \eta\nu$, that is, $\rho(\mathcal{P}f, \mathcal{P}g) \leq \eta\rho(f,g)$ for all $f, g \in \mathcal{M}$.

It follows from (3.51) that $\rho(f, \mathcal{P}f) \leq \dfrac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{19}}$.

Therefore according to Theorem 2.2 in [3], there exists a mapping $\mathcal{N}_{\mathcal{D}} : X \to Y$ which satisfying:

1. $\mathcal{N}_{\mathcal{D}}$ is a unique fixed point of $\mathcal{P}$ in the set $\mathcal{S} = \{g \in \mathcal{M} : \rho(f,g) < \infty\}$, which is satisfied

$$\mathcal{N}_{\mathcal{D}}(2^l a) = 2^{19l}\mathcal{N}_{\mathcal{D}}(a) \qquad \forall\, a \in X. \tag{3.52}$$

   In other words, there exists a μ satisfying

$$\|f(a) - g(a)\| \leq \mu\overline{\psi}(a) \forall\, a \in X.$$

2. $\rho(\mathcal{P}^k f, \mathcal{N}_{\mathcal{D}}) \to 0$ as $k \to \infty$. This implies that

$$\lim_{k\to\infty} \frac{1}{2^{19kl}} f(2^{kl}a) = \mathcal{N}_{\mathcal{D}}(a) \qquad \forall\, a \in X.$$

3. $\rho(f, \mathcal{N}_{\mathcal{D}}) \leq \frac{1}{1-\eta}\rho(f, \mathcal{P}f)$, which implies the inequality $\rho(f, \mathcal{N}_{\mathcal{D}}) \leq \frac{\eta^{\frac{1-l}{2}}}{2^{19}(1-\eta)}$.

$$\text{So} \quad \|f(a) - \mathcal{N}_{\mathcal{D}}(a)\| \leq \frac{\eta^{\frac{1-l}{2}}}{2^{19}(1-\eta)} \overline{\psi}(a) \qquad \forall\, a \in X. \tag{3.53}$$

It follows from (3.22) and (3.23) that

$$\begin{aligned}\|\mathcal{G}\mathcal{N}_{\mathcal{D}}(a,b)\| &= \lim_{k\to\infty} \frac{1}{2^{19kl}} \left\|\mathcal{G}f(2^{kl}a, 2^{kl}b)\right\| \\ &\leq \lim_{k\to\infty} \frac{1}{2^{19kl}} \psi(2^{kl}a, 2^{kl}b) \leq \lim_{k\to\infty} \frac{2^{kl}\eta^k}{2^{19kl}} \psi(a,b) = 0\end{aligned}$$

for all $a, b \in X$. Hence

$$\begin{aligned}&\mathcal{N}_{\mathcal{D}}(a+10b) - 19\mathcal{N}_{\mathcal{D}}(a+9b) + 171\mathcal{N}_{\mathcal{D}}(a+8b) - 969\mathcal{N}_{\mathcal{D}}(a+7b) + 3876\mathcal{N}_{\mathcal{D}}(a+6b) \\ &\quad + 27132\mathcal{N}_{\mathcal{D}}(a+4b) - 50388\mathcal{N}_{\mathcal{D}}(a+3b) + 75582\mathcal{N}_{\mathcal{D}}(a+2b) - 92378\mathcal{N}_{\mathcal{D}}(a+b) \\ &\quad + 92378\mathcal{N}_{\mathcal{D}}(a) - 75582\mathcal{N}_{\mathcal{D}}(a-b) + 50388\mathcal{N}_{\mathcal{D}}(a-2b) - 27132\mathcal{N}_{\mathcal{D}}(a-3b) \\ &\quad + 11628\mathcal{N}_{\mathcal{D}}(a-4b) - 3876\mathcal{N}_{\mathcal{D}}(a-5b) + 969\mathcal{N}_{\mathcal{D}}(a-6b) - 171\mathcal{N}_{\mathcal{D}}(a-7b) \\ &\qquad\qquad - 11628\mathcal{N}_{\mathcal{D}}(a+5b) + 19\mathcal{N}_{\mathcal{D}}(a-8b) - \mathcal{N}_{\mathcal{D}}(a-9b) = 19!\mathcal{N}_{\mathcal{D}}(b)\end{aligned}$$

Therefore, the mapping $\mathcal{N}_{\mathcal{D}} : X \to Y$ is nonadecic mapping.
By Lemma 2.1 in [9] and (3.53),

$$\begin{aligned}\left\|f_n([x_{ij}]) - \mathcal{N}_{\mathcal{D}n}([x_{ij}])\right\| &\leq \sum_{i,j=1}^{n} \left\|f(x_{ij}) - \mathcal{N}_{\mathcal{D}}(x_{ij})\right\| \\ &\leq \sum_{i,j=1}^{n} \frac{\eta^{\frac{1-l}{2}}}{2^{19}(1-\eta)} \overline{\psi}(x_{ij}) \qquad \forall\, x = [x_{ij}] \in M_n(X),\end{aligned}$$

where $\overline{\psi}(x_{ij}) = \frac{1}{19!}[\psi(0, 2x_{ij}) + \psi(10x_{ij}, x_{ij}) + 19\psi(9x_{ij}, x_{ij}) + 172\psi(8x_{ij}, x_{ij})$
$+988\psi(7x_{ij}, x_{ij}) + 4047\psi(6x_{ij}, x_{ij}) + 12597\psi(5x_{ij}, x_{ij})$
$+31008\psi(4x_{ij}, x_{ij}) + 62016\psi(3x_{ij}, x_{ij}) + 102714\psi(2x_{ij}, x_{ij})$
$+142766\psi(x_{ij}, x_{ij}) + 167960\psi(0, x_{ij})]$,

Thus $\mathcal{N}_{\mathcal{D}} : X \to Y$ is a unique nonadecic mapping satisfying (3.24). □

Corollary 3.1. *Let $l = \pm 1$ be fixed and let t, ϵ be positive real numbers with $t \neq 19$. Let $f : X \to Y$ be a mapping such that*

$$\left\|\mathcal{G}f_n([x_{ij}], [y_{ij}])\right\|_n \leq \sum_{i,j=1}^{n} \epsilon(\left\|x_{ij}\right\|^t + \left\|y_{ij}\right\|^t) \qquad \forall\, x = [x_{ij}], y = [y_{ij}] \in M_n(X). \tag{3.54}$$

Then there exists a unique nonadecic mapping $\mathcal{N}_{\mathcal{D}} : X \to Y$ such that

$$\left\|f_n([x_{ij}]) - \mathcal{N}_{\mathcal{D}n}([x_{ij}])\right\|_n \leq \sum_{i,j=1}^{n} \frac{\epsilon_s}{l(2^{19} - 2^t)} \left\|x_{ij}\right\|^t \qquad \forall\, x = [x_{ij}] \in M_n(X),$$

where $\epsilon_s = \frac{\epsilon}{19!}[667054 + 102715(2^t) + 62016(3^t) + 31008(4^t) + 12597(5^t)$
$+ 4047(6^t) + 988(7^t) + 172(8^t) + 19(9^t) + 10^t]$

Proof. The proof follows from Theorem 3.2 by taking $\psi(a,b) = \epsilon(\|a\|^t + \|b\|^t)$ for all $a, b \in X$. Then we can choose $\eta = 2^{l(t-19)}$, and we can obtain the required result. □

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $t = 19$ in corollary 3.1.

Example 3.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \epsilon x^{19}, & if\ |x| < 1 \\ \epsilon, & otherwise \end{cases}$$

where $\epsilon > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^{19n}}$$

for all $x \in \mathbb{R}$. Then f satisfies the inequality

$$\begin{aligned} \| f(x+10y) &- 19f(x+9y) + 171f(x+8y) - 969f(x+7y) + 3876f(x+6y) \\ &+27132f(x+4y) - 50388f(x+3y) + 75582f(x+2y) - 92378f(x+y) \\ &+92378f(x) - 75582f(x-y) + 50388f(x-2y) - 27132f(x-3y) \\ &+11628f(x-4y) - 3876f(x-5y) + 969f(x-6y) - 171f(x-7y) \\ &\qquad -11628f(x+5y) + 19f(x-8y) - f(x-9y) - 19!f(y)\| \end{aligned}$$

$$\leq \frac{(121645100400000000)}{524287}(524288)^2 \epsilon (|x|^{19} + |y|^{19}) \tag{3.55}$$

for all $x, y \in \mathbb{R}$. Then there does not exist a nonadecic mapping $\mathcal{N}_{\mathcal{D}} : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ such that

$$|f(x) - \mathcal{N}_{\mathcal{D}}(x)| \leq \lambda\, |x|^{19} \qquad \forall\, x \in \mathbb{R}. \tag{3.56}$$

Proof. It is easy to see that f is bounded by $\dfrac{524288\epsilon}{524287}$ on $\mathbb{R}$.
If $|x|^{19} + |y|^{19} = 0$, then (3.55) is trivial.
If $|x|^{19} + |y|^{19} \geq \frac{1}{2^{19}}$, then L.H.S of (3.55) is less than $\dfrac{(121645100400000000)(524288)\epsilon}{524287}$.
Suppose that $0 < |x|^{19} + |y|^{19} < \frac{1}{2^{19}}$, then there exists a non-negative integer k such that

$$\frac{1}{2^{19(k+1)}} \leq |x|^{19} + |y|^{19} < \frac{1}{2^{19k}}, \tag{3.57}$$

so that $2^{19(k-1)}\, |x|^{19} < \frac{1}{2^{19}}, 2^{19(k-1)}\, |y|^{19} < \frac{1}{2^{19}}$, and

$$\begin{aligned} &2^n(x), 2^n(y), 2^n(x+10y), 2^n(x+9y), 2^n(x+8y), 2^n(x+7y), \\ &2^n(x+6y), 2^n(x+5y), 2^n(x+4y), 2^n(x+3y), 2^n(x+7y), 2^n(x+2y), \\ &2^n(x+y), 2^n(x-y), 2^n(x-2y), 2^n(x-3y), 2^n(x-4y), 2^n(x-5y), \\ &\qquad 2^n(x-6y), 2^n(x-7y), 2^n(x-8y), 2^n(x-9y) \in (-1,1) \end{aligned}$$

for all $n = 0, 1, 2, ..., k-1$. Hence

$$\begin{aligned} \psi(2^n(x+10y)) &- 19\psi(2^n(x+9y)) + 171\psi(2^n(x+8y)) - 969\psi(2^n(x+7y)) \\ &+ 3876\psi(2^n(x+6y)) - 11628\psi(2^n(x+5y)) + 27132\psi(2^n(x+4y)) \\ &- 50388\psi(2^n(x+3y)) + 75582\psi(2^n(x+2y)) - 92378\psi(2^n(x+y)) \\ &+ 92378\psi(2^n x) - 75582\psi(2^n(x-y)) + 50388\psi(2^n(x-2y)) \\ &- 27132\psi(2^n(x-3y)) + 11628\psi(2^n(x-4y)) - 3876\psi(2^n(x-5y)) \\ &+ 969\psi(2^n(x-6y)) - 171\psi(2^n(x-7y)) + 19\psi(2^n(x-8y)) \\ &\qquad - \psi(2^n(x-9y)) - 19!\psi(2^n y) = 0 \end{aligned}$$

for $n = 0, 1, 2, ..., k-1$. From the definition of f and (3.57), we obtain that

$$\begin{aligned} | f(x+10y) &- 19f(x+9y) + 171f(x+8y) - 969f(x+7y) + 3876f(x+6y) \\ &+27132f(x+4y) - 50388f(x+3y) + 75582f(x+2y) - 92378f(x+y) \\ &+92378f(x) - 75582f(x-y) + 50388f(x-2y) - 27132f(x-3y) \\ &+11628f(x-4y) - 3876f(x-5y) + 969f(x-6y) - 171f(x-7y) \\ &\qquad -11628f(x+5y) + 19f(x-8y) - f(x-9y) - 19!f(y)| \end{aligned}$$

$$\begin{aligned} \leq \textstyle\sum_{n=0}^{\infty} \frac{1}{2^{19n}}\, &|\psi(2^n(x+10y)) - 19\psi(2^n(x+9y)) + 171\psi(2^n(x+8y)) \\ &+3876\psi(2^n(x+6y)) - 11628\psi(2^n(x+5y)) + 27132\psi(2^n(x+4y)) \\ &-50388\psi(2^n(x+3y)) + 75582\psi(2^n(x+2y)) - 92378\psi(2^n(x+y)) \end{aligned}$$

$$+92378\psi(2^n x) - 75582\psi(2^n(x-y)) + 50388\psi(2^n(x-2y))$$
$$-27132\psi(2^n(x-3y)) + 11628\psi(2^n(x-4y)) - 3876\psi(2^n(x-5y))$$
$$+969\psi(2^n(x-6y)) - 171\psi(2^n(x-7y)) + 19\psi(2^n(x-8y))$$
$$-969\psi(2^n(x+7y)) - \psi(2^n(x-9y)) - 19!\psi(2^n y)|$$
$$\leq \sum_{n=k}^{\infty} \frac{(121645100400000000)\epsilon}{2^{19n}} = \frac{(524288)(121645100400000000)\epsilon}{2^{19k}524287}$$
$$\leq \frac{(121645100400000000)}{524287}(524288)^2\epsilon(|x|^{19} + |y|^{19}).$$

Therefore, f satisfies (3.55) for all $x, y \in \mathbb{R}$. Now, we claim that functional equation (1.1) is not stable for $t = 19$ in corollary 3.1. Suppose on the contrary that there exists a nonadecic mapping $\mathcal{N}_{\mathcal{D}} : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying (3.56). Then there exists a constant $c \in \mathbb{R}$ such that $\mathcal{N}_{\mathcal{D}}(x) = cx^{19}$ for any $x \in \mathbb{R}$. Thus we obtain the following inequality

$$|f(x)| \leq (\lambda + |c|)\,|x|^{19} \tag{3.58}$$

Let $m \in \mathbb{N}$ with $m\epsilon > \lambda + |c|$. If $x \in (0, \frac{1}{2^{m-1}})$, then $2^n x \in (0,1)$ for all $n = 0, 1, 2, ..., m-1$, and for this case we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^{19n}} \geq \sum_{n=0}^{m-1} \frac{\epsilon(2^n x)^{19}}{2^{19n}} = m\epsilon x^{19} > (\lambda + |c|)\,|x|^{19}$$

which is a contradiction to (3.58). Therefore the nonadecic functional equation (1.1) is not stable for $t = 19$. □

4 Conclusion

In this investigation, we identified a general solution of nonadecic functional equation and establised the generalized Ulam -Hyers stability of this functional equation in matrix normed spaces by using the fixed point method and also provided an example for non-stability.

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Asymptotic behavior of the oscillatory solutions of first order neutral delay difference equations

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Abstract

In this article, the asymptotic behavior of oscillatory solutions of a class of first order neutral delay difference equations with variable co-efficients and constant delays is investigated. We established a sufficient conditions of the equations under consideration approach zero as the independent variable tends to infinity.

Keywords: Oscillatory solutions, asymptotic behavior, neutral, delay difference equation.

2010 MSC: 39A10, 39A12.

1 Introduction

We consider the first order neutral delay difference equation with variable co-efficients of the form

$$\Delta[x(n) - p(n)x(n-\tau)] - q(n)x(n-\sigma) = 0; \quad n \geq n_0; \tag{1}$$

where $\{p(n)\}$, $\{q(n)\}$ are sequences of real numbers, τ and σ are positive integers with $\tau > \sigma$ and Δ is the forward difference operator defined by the equation

$$\Delta x(n) = x(n+1) - x(n).$$

In the oscillation theory of difference equations one of the important problems is to find sufficient conditions in order that all oscillatory solutions of (1) tends to zero as $n \to \infty$. Considerably less is known about the behavior of oscillatory solutions to first order neutral delay difference equations with variable co-efficients. We choose to refer to the papers [9,10,13].

By a solutions of equation (1), we mean a real sequence $\{x(n)\}$ which is defined for $n \geq n_0 - \max\{\tau, \sigma\}$ and satisfies equation (1) for all $n \in N(n_0) = \{n_0, n_0+1, n_0+2, ...\}$. A non trivial solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

Philos et al. [7] consider the first order neutral delay differential equation

$$[x(t) - p(t)x(t-\sigma)]' = \phi(t)x(t-\tau), \quad t \geq t_0 \tag{1'}$$

and obtained sufficient conditions for all solutions of the equation $(1')$ to tend to zero as $t \to \infty$.

The purpose of the present paper is to obtain sufficient conditions for all oscillatory solutions of (1) tend to zero as $n \to \infty$. Our obtained results are discrete analogues of some well known results due to [7]. With

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respect to the oscillation and asymptotic behavior of difference equation, reader can refer to [3-6, 8-14]. For the several background in difference equation, one can refer to [1,2].

Throughout this paper, we define $N(a) = \{a, a+1, a+2, ...\}$ and $N(a,b) = \{a, a+1, a+2, ...b\}$ where a and b are integers with $a \leq b$.

The following conditions are assumed to be hold throughout the paper.

(C_1) $\{p(n)\}$ is a sequence of nonnegative real numbers,

(C_2) $\{q(n)\}$ is a sequence of positive real numbers,

(C_3) τ and σ are positive integers such that $\tau > \sigma$.

In section 2, we shall state and prove some lemmas, which play a crucial role in proving our theorem.

2 Some Lemmas

Lemma 2.1. *Assume that $\{p(n)\}$ is a sequence of nonnegative real numbers and $0 \leq p(n) \leq p < 1$. Assume also that $\{q(n)\}$ is a sequence of positive real numbers. Then every oscillatory solution of the neutral delay difference equation (1) which is eventually of one sign (ie, it is either eventually nonnegative or eventually non positive), tends to zero at ∞.*

Proof. Without loss of generality, we suppose that $\{x(n)\}$ is an oscillatory solution of (1) which is eventually nonnegative. We observe that, if $\{x(n)\}$ is eventually identically zero, then it tends to zero at ∞. So, we assume that $\{x(n)\}$ is not eventually identically zero. Set

$$z(n) = x(n) - p(n)x(n-z). \tag{2}$$

By taking into account (2) and the fact that $\{x(n)\}$ is nonnegative, from (1) we conclude that $\{\Delta z(n)\}$ is eventually nonnegative and $\{\Delta z(n)\}$ is not eventually identically zero. They $\{z(n)\}$ is increasing on $N(n_1)$ where $n_1 \geq n_0$ such that $x(n) \geq 0$, $n \geq n_1 - \tau$ and it is not eventually identically zero. This guarantees that $\{z(n)\}$ is either negative eventually positive or eventually negative. Assume that $\{z(n)\}$ is eventually positive i.e. $\{z(n)\}$ is positive on $N(n_2)$ when $n_2 \geq n_1$. Since $\{x(n)\}$ is oscillatory, there exists an integer $\xi \geq n_2$ with $x(\xi) = 0$ then

$$\begin{aligned} 0 < z(\xi) &= x(\xi) - p(\xi)x(\xi - \tau) \\ &= -p(\xi)x(\xi - z) \end{aligned} \tag{3}$$

consequently $p(\xi)x(\xi - z) < 0$.

Hence given $\{p(n)\}$ is assume to nonnegative on $N(n_0)$, it follows immediately that $x(\xi - z) < 0$. This contradicts the fact that $\{x(n)\}$ is nonnegative on $N(n_1)$. This contradiction establishes that $\{z(n)\}$ is always eventually negative on $N(n_1)$.

Therefore

$$z(n) = x(n) - p(n)x(n-\tau) < 0, \quad n \geq n_1$$

and so we have

$$x(n) < p(n)x(n-\tau). \tag{4}$$

Let us suppose that $\{x(n)\}$ is unbounded. Then as $\{x(n)\}$ is nonnegative on $N(n_1 - \tau)$. We can consider a sequence of integers $\{m_k\}$ with $n_1 \leq m_0 < m_1 < m_2 < ...$ and $\lim_{k\to\infty} m_k = \infty$ $k = 0, 1, ...$ such that

$$\max_{n \in N(n_1 - \tau, m_k)} x(n) = x(m_k) > 0 \quad (k = 0, 1, 2, 3, ...)$$

and $\lim_{k\to\infty} x(m_k) = \infty$.

Then by taking into account that $\{p(n)\}$ is nonnegative on $N(n_0)$ and using (4) and $0 \leq p(n) \leq p < 1$, we obtain

$$0 < x(m_0) < p(m_0)x(m_0) \leq px(m_0).$$

That is, $0 < x(m_0) < px(m_0)$. As $0 \le m < 1$, this is a contradiction, which shows that $\{x(n)\}$ is necessary bounded on $N(n_1 - \tau)$. Hence there exists a positive real constant k such that

$$0 \le x(n) < K \quad for \quad all \quad n \in N(n_1 - \tau). \tag{5}$$

Now, we take into account the hypothesis that $\{p(n)\}$ is nonnegative on $N(n_0)$ and we use (4) and (5) to obtain to $n \ge n_1$.

$$0 \le x(n) < p(n)x(n-\tau) \le pK, \quad for \quad all \quad n \in N(n_1).$$

Finally, by an easily induction, we can prove that

$$0 \le x(n) < p^i K \quad for \quad all \quad n \in N(n_1 + (i-1)\tau), \quad (i = 0, 1, 2, 3, ...) \tag{6}$$

But, as $0 \le p < 1$ we have

$$\lim_{i\to\infty} p^i = 0$$

Hence it follows easily from (6) that

$$\lim_{n\to\infty} x(n) = 0$$

The proof of the lemma is finished. □

Lemma 2.2. *Assume that $\{p(n)\}$ is a sequence of nonnegative real numbers on $N(n_0)$ and $\{q(n)\}$ is a sequence of positive real numbers on $N(n_0)$. Let $\{x(n)\}$ be an oscillatory solution of the neutral delay difference equation (1) and let $\bar{n}$ be an integer with $\bar{n} > n_0$. If*

$$x(\bar{n}) > 0 \quad and \quad z(\bar{n}) > 0, \tag{7}$$

then either $x(\xi) \le 0$ or $z(\xi) \le 0$ for at least one $\xi \in N(\bar{n}+1, \bar{n}+\tau-1)$.

Proof. First of all, we will prove the following claim.

Claim: Let $n_1 \ge n_0$. If both $x(n)$ and $z(n)$ are positive on $N(n_1, n_1+\tau)$, then $x(n)$ and $z(n)$ are also positive on $N(n_1+\tau, n_1+\tau+\sigma)$. In order to establish our claim, we assume that $x(n) > 0$ and $z(n) > 0$ for all $n \in N(n_1, n_1+\tau)$. So, we have $x(n-\sigma) > 0$, for all $n \in N(n_1+\sigma, n_1+\tau+\sigma)$. From this and (3), we concluded that $\Delta z(n) > 0$ for all $n \in N(n_1+\sigma, n_1+\tau+\sigma)$. This guarantees that $\{z(n)\}$ is increasing on $N(n_1+\sigma, n_1+\tau+\sigma)$ which together with the facts that $N(n_1+\tau, n_1+\tau+\sigma) \subset N(n_1+\sigma, n_1+\tau+\sigma)$ and $z(n_1+\sigma) > 0$ implies that $\{z(n)\}$ is always positve on $N(n_1+\tau, n_1+\tau+\sigma)$. We see that $x(n-\tau) > 0$ on $N(n_1+\tau, n_1+\tau+\sigma)$. Hence, by taking into account the fact that $\{z(n)\}$ is positive on $N(n_1+\tau, n_1+\tau+\sigma)$ and using the assumption that $\{p(n)\}$ is nonnegative on $N(n_0)$, we obtain, for every $n \in N(n_1+\tau, n_1+\tau+\sigma)$

$$x(n) = z(N) + p(n)x(n-\tau) > p(n)x(n-\tau) \ge 0.$$

This implies that $\{x(n)\}$ is always positive on $N(n_1+\tau, n_1+\tau+\sigma)$ and completes the proof of our claim. □

Now, let us suppose that (7) holds.

We will show that either

$$x(\xi) \le 0 \quad or \quad z(\xi) \le 0 \quad for \quad atleast \quad one \quad \xi \in N(\bar{n}+1, \bar{n}+\tau-1) \tag{8}$$

If (8) is not true, then $x(n) > 0$ and $z(n) > 0$ for every $n \in N(\bar{n}+1, \bar{n}+\tau-1)$. So, because of (6), both $\{x(n)\}$ and $\{z(n)\}$ must be positive on $N(\bar{n}, \bar{n}+\tau-1)$. By our claim, $\{x(n)\}$ and $\{z(n)\}$ are also positive on $N(\bar{n}+\tau-1, \bar{n}+\tau+\sigma-1)$. Consequently, $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\bar{n}, \bar{n}+\tau+\sigma-1)$. By using again our claim (with $n_1 = \bar{n}+\sigma-1$), we see that $\{x(n)\}$ and $\{z(n)\}$ are also positive on $N(\bar{n}+\tau+\sigma-1, \bar{n}+\tau+2\sigma-1)$. Thus, $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\bar{n}, \bar{n}+\tau+2\sigma-1)$. Following the procedure, we can conclude that, for any nonnegative integer k, both $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\bar{n}, \bar{n}+\tau+k\sigma-1)$. This guarantees that $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\bar{n})$. But, the fact that $\{x(n)\}$ is positive on $N(\bar{n})$ contradictory the oscillatory character of $\{x(n)\}$. So (8) has been proved.

Lemma 2.3. *Let $0 \le p(n) \le p < 1$ and $\{f(n)\}$ be an unbounded sequence of real numbers on $N(n_0 - \tau)$. We define*

$$g(n) = f(n) - p(n)f(n-\tau), \quad n \ge n_0.$$

Then the sequence $\{g(n)\}$ is also unbounded. Moreover, there exists $m_0 \geq n_0$, such that for any $m \geq m_0$, the following statement is true:

If

$$|g(n)| \leq |g(m)|, \quad for \quad every \quad n \in N(m_0, m), \tag{9}$$

then

$$|f(n)| \leq \frac{1}{1-p} |g(m)| \quad for \quad all \quad n \in N(n_0 - \tau, m) \tag{10}$$

Proof. The hypothesis that $\{f(n)\}$ is unbounded guarantees the existence of a sequence of integer $\{m_k\}_{k=0,1,...}$ with $n_0 \leq m_0 < m_1 < ...$ and $\lim_{k\to\infty} m_k = \infty$ such that

$$\max_{n\in N(n_0-\tau,m_k)} |f(n)| = |f(m_k)|, \quad (k = 0, 1, 2, 3, ...) \tag{11}$$

and

$$\lim_{k\to\infty} |m_k| = \infty. \tag{12}$$

By taking into account, the assumption that $\{p(n)\}$ is nonnegative on $N(n_0)$ and using $0 \leq p(n) \leq p < 1$ and (11) we obtain for $k = 0, 1, 2, ...$

$$\begin{aligned} |g(m_k)| &= |f(m_k) - p(m_k)f(m_k - \tau)| \\ &\geq |f(m_k)| - p(m_k) |f(m_k - \tau)| \\ &\geq |f(m_k)| - p |f(m_k)| \end{aligned}$$

Hence, we have

$$|g(m_k)| \geq (1-p) |f(m_k)|, \quad (k = 0, 1, 2, ...)$$

So in view of (12) and because of the fact that $1 - p > 0$, it follows that

$$\lim_{k\to\infty} |g(m_k)| = \infty.$$

This guarantees that $|g(n)|$ is necessary unbounded.

Now, let m be an arbitrary point with $m \geq m_0$, and assume that (9) is satisfied. As $\{p(n)\}$ is assume to be nonnegative on $N(n_0)$, we can use (9) and $0 \leq p(n) \leq p < 1$ to obtain, for $n \in N(m_0, m)$,

$$\begin{aligned} |g(m)| &\geq |g(n)| = |f(n) - p(n)f(n - \tau)| \\ &\geq |f(n)| - p |f(n - \tau)| \\ &\geq |f(n)| - p \max_{s\in N(n_0-\tau,m)} |f(s)|. \end{aligned}$$

Thus,

$$|g(m)| \geq \max_{n\in N(m_0,m)} |f(n)| - p \max_{n\in N(n_0-\tau,m)} |f(n)|, n \in N(n_0 - \tau, m). \tag{13}$$

On the otherhand, by using (11) with $k = 0$, we can immediately see that

$$\max_{n\in N(m_0,m)} |f(n)| = \max_{n\in N(n_0-\tau,m)} |f(n)|,$$

Hence (13), yields

$$|g(m)| \geq (1-p) \max_{n\in N(n_0-\tau,m)} |f(n)|.$$

So since $1 - p > 0$, we have

$$\max_{n\in N(n_0-\tau,m)} |f(n)| \leq \frac{1}{1-p} |g(m)|.$$

The proof of the lemma is now complete. □

3 Main Results

Theorem 3.1. *Assume that*

$$0 \leq p(n) \leq p < \frac{1}{2}. \tag{14}$$

If

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) < 2(1-2p) \tag{15}$$

then every oscillatory solution of equation (1) tends to zero as $n \to \infty$.

Proof. Let $\{x(n)\}$ be an oscillatory solution of (1). First it will be shown that the solution $\{x(n)\}$ is bounded. Next, by the use of the boundedness of $\{x(n)\}$, we shall prove that the solution $\{x(n)\}$ tend to zero as $n \to \infty$.

Suppose, that the sake of contradiction, that the solution $\{x(n)\}$ is unbounded. We see that condition (15) implies, in particular, that

$$\sum_{s=n}^{n+\tau-1} q(s) < 2(1-2p) \quad for \quad all \quad n$$

and consequently there exists an integer $n_1 \geq n_0$ such that

$$\sum_{s=n}^{n+\tau-1} q(n) < 2(1-2p) \quad for \quad every \quad n \geq n_1. \tag{16}$$

By taking into account the fact that $\{p(n)\}$ is nonnegative on $N(n_0)$ and that (14) holds and using the fact that $\{x(n)\}$ is unbounded, we can apply Lemma 2.3 to conclude that the sequence $\{z(n)\}$ is also unbounded, where $z(n)$ is defined by (2). Moreover, there exists a $m_0 \geq n_0$ such that, for any $m \geq m_0$, the following statements is true.

If

$$|z(n)| \leq |z(m)| \quad for \quad every \quad n \in N(m_0, m), \tag{17}$$

then

$$|x(n)| \leq \frac{1}{1-p}|z(m)| \quad for \quad all \quad n \in N(n_0-\tau, m). \tag{18}$$

Also, as $\{x(n)\}$ is unbounded, it is obvious that $\{x(n)\}$ does not tend to zero at $n \to \infty$.

In view of Lemma 2.1, $\{x(n)\}$ cannot be eventually of one sign, i.e., it is neither eventually nonnegative nor eventually nonpositive. This means that $\{x(n)\}$ changes sign for arbitrarly large values of n. So, in view of (1) and (2), the sequence $\{\Delta z(n)\}$ changes sign for arbitrarly large values of n and consequently $\{z(n)\}$ cannot be eventually monotone. From this fact and the unboundness of $\{z(n)\}$ we conclude that there exists an integer $m \geq \max\{n_1+\sigma, m_0, n_0+\tau\}$ with $z(m) \neq 0$ such that

$$z(m)\Delta z(m) \leq 0 \tag{19}$$

and

$$|z(n)| \leq |z(m)| \quad for \quad every \quad n \in N(n_0, m). \tag{20}$$

We observe that $m \geq m_0$ and that (20) implies (17). Hence (18) holds true. Furthermore, we see that $\{-x(n)\}$ is also an oscillatory solution of (1), which is unbounded, and that

$$-z(n) = -x(n) + p(n)x(n-\tau) \quad for \quad n \geq n_0.$$

Thus, as $z(m) \neq 0$, we may (and do) assume that

$$z(m) > 0. \tag{21}$$

So (18) becomes

$$|x(n)| \leq \frac{1}{1-p}z(m) \quad for \quad all \quad n \in N(n_0-\tau, m). \tag{22}$$

Now, we will show that

$$x(m) > 0.$$

Assume, for the sake of contradiction, that $x(m) \leq 0$. As $m \geq n_0 + \tau$, we have $n_0 \leq m - \tau < m$. Consequently, (22) susures that

$$|x(m-\tau)| \leq \frac{1}{1-p} z(m).$$

By using this inequality as well as (21) and taking into account the fact that $\{p(n)\}$ is nonnegative on $N(n_0)$, we obtain

$$\begin{aligned} 0 < z(m) &= x(m) - p(m)x(m-\tau) \\ &\leq -p(m)x(m-\tau) \\ &\leq p(m)\,|x(m-\tau)| \\ &\leq p\frac{1}{1-p}z(m) \end{aligned}$$

and consequently

$$1 \leq \frac{p}{1-p}, \quad i.e. \quad p \geq \frac{1}{2}.$$

This contradiction proves that $x(m) \leq 0$. In view of (19) and (21), we have

$$\Delta z(m) \leq 0.$$

Prove this and (1), we have $x(m-\sigma) \leq 0$. Note that $m - \sigma \geq n_1$. Let us denote the integer ξ_1 less then m such that $x(\xi_1)z(\xi_1) \leq 0$ and

$$x(n) > 0 \quad and \quad z(n) > 0 \quad for \quad every \quad n \in N(\xi_1 + 1, m).$$

It is obvious that $m - \sigma \leq \xi_1 \leq m - 1$. Since $\{x(n)\}$ is oscillatory, then there exists an integers $\xi_2 > m$ such that

$$x(\xi_2)z(\xi_2) \leq 0$$

and

$$x(n) > 0 \quad and \quad z(n) > 0, \quad for \quad every \quad n \in N(m, \xi_2 - 1).$$

We note that $x(n) > 0$ and $z(n) > 0$ as $N(\xi_1 + 1, \xi_2 - 1)$.

We shall establish the following inequality

$$2z(m) \leq \left\{ 2p + \sum_{n=\xi_1}^{\xi_1+\tau-1} q(n) \right\} \max_{n \in N(m-2\tau, m-1)} |x(n)|. \tag{23}$$

Inequality (23) is an immediate consequence of the next inequalities:

$$z(m) \leq \left\{ p + \sum_{n=\xi_1}^{m-1} q(n) \right\} \max_{n \in N(m-2\tau, m-1)} |x(n)| \tag{24}$$

and

$$z(m) \leq \left\{ p + \sum_{n=m}^{\xi_1+\tau-1} q(n) \right\} \max_{n \in N(n-2\tau, m-1)} |x(n)|. \tag{25}$$

So, we will prove that (24) and (25) hold.

Proof of inequality (24). We see that (1) and (2) gives

$$z(m) = z(\xi_1) + \sum_{n=\xi_1}^{m-1} q(n)x(n-\sigma) \tag{26}$$

First, let us assume that $z(\xi_1) \leq 0$. Then from (26) we obtain

$$z(m) \leq \sum_{n=\xi_1}^{m-1} q(n)x(n-\sigma) \leq \sum_{n=\xi_1}^{m-1} q(n)\,|x(n-\sigma)|.$$

As $m-\sigma \le \xi_1 \le m-1$, we have

$$m-2\tau \le m-2\sigma \le \xi_1-\sigma \le m-\sigma-1 < m-1.$$

Hence $n-\sigma \in N(m-2\tau, m-1)$ whenever $n \in N(\xi_1, m-1)$. So, we get

$$z(m) \le \left\{\sum_{n=\xi_1}^{m-1} q(n)\right\} \max_{n\in N(n-2\tau,m-1)} |x(n)|$$

which, as $p \ge 0$, implies (24), Next, we assume that $x(\xi_1) \le 0$. Then from (26), we obtain

$$\begin{aligned} z(m) &= x(\xi_1) - p(\xi_1)x(\xi_1-\tau) + \sum_{n=\xi_1}^{m-1} q(n)x(n-\sigma) \\ &\le -p(\xi_1)x(\xi_1-\tau) + \sum_{n=\xi_1}^{m-1} q(n)x(n-\sigma) \\ &\le p(\xi_1)\left|x(\xi_1-\tau\right| + \sum_{n=\xi_1}^{m-1} q(n)\left|x(n-\sigma)\right|. \end{aligned}$$

But, as $m-\sigma \le \xi_1 \le m-1$, we have

$$m-2\tau \le m-\tau-\sigma \le \xi_1-\tau \le \xi_1-\sigma \le m-1-\sigma < m-1$$

or

$$m-2\tau \le m-\sigma-\tau \le \xi_1-\tau \le \xi_1-\sigma \le m-1-\sigma < m-1.$$

Thus,

$$|x(\xi_1-\tau)| \le \max_{n\in N(m-2\tau,m-1)} |x(n)|.$$

Also as we have previously seen,

$$n-\sigma \in N(m-2\tau, m-1) \quad \textit{whenever} \quad n \in N(\xi_1, m-1).$$

Thus the last inequality becomes

$$z(m) \le p \max_{n\in N(m-2\tau,m-1)} |x(n)| + \left\{\sum_{n=\xi_1}^{m-1} q(n)\right\} \max_{n\in N(m-2\tau,m-1)} |x(n)|.$$

Consequently (24) is fulfilled. The proof of (24) is finished. □

Proof of inequality (25)

We distinguish between two cases: Either $\xi_2 > \xi_1+\tau$, or $\xi_2 \le \xi_1+\tau$.

Case 1: $\xi_2 > \xi_1+\tau$. Then there is an integer $\bar{n}$ with $\xi_1 < \bar{n} < \xi_2-\tau$ such that $x(\bar{n}) > 0$ and $z(\bar{n}) > 0$. So by Lemma 2, either $x(\xi) \le 0$ or $z(\xi) \le 0$ for atleast one $\xi \in N(\bar{n}+1, \bar{n}+\tau-1)$. Since $\xi_1 < \bar{n} < \xi < \bar{n}+\tau-k < \xi_2$. This is a contradiction to the fact that both $x(n) > 0$ and $z(n) > 0$ on $N(\xi_1+1, \xi_2-1)$

Case 2: $\xi_2 \le \xi_1+\tau$. From (1) and (2), we have

$$z(m) = z(\xi_2) - \sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma). \tag{27}$$

We examine the two subcases, where either $\xi_2 \le m+\sigma$ or $\xi_2 > m+\sigma$.

Subcase 2.1 $\xi_2 \le m+\sigma$. Suppose first that $z(\xi_2) \le 0$ Then from (26),

$$\begin{aligned} z(m) &= -\sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma) \\ &\le \sum_{n=m}^{\xi_2-1} q(n)\left|x(n-\sigma)\right|. \end{aligned}$$

We observe that

$$m-\tau \leq m-\sigma \leq n-\sigma \leq \xi_2-1-\sigma \leq m-1.$$

So, $n-\sigma \in N(m-\tau, m-1)$ whenever $n \in N(m, \xi_2-1)$. Hence from the above inequality, we obtain

$$\begin{aligned} z(m) &\leq \left\{\sum_{n=m}^{\xi_2-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)| \\ &\leq \left\{\sum_{n=m}^{\xi_1+\tau-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)|. \end{aligned}$$

Consequently, as $p \geq 0$ inequality (25) is always fulfilled. Next, let us suppose that $x(\xi_2) \leq 0$. Then from (1) and (2), we have

$$\begin{aligned} z(m) &= [x(\xi_2)-p(\xi_2)x(\xi_2-\tau)] - \sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma) \\ &\leq -p(\xi_2)x(\xi_2-\tau) - \sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma) \\ &\leq p(\xi_2)\,|x(\xi_2-\tau)| + \sum_{n=m}^{\xi_2-1} q(n)\,|x(n-\sigma)| \\ &\leq p\,|x(\xi_2-\tau)| + \sum_{n=m}^{\xi_2-1} q(n)\,|x(n-\sigma)|. \end{aligned}$$

But, $m-\tau < \xi_2-\tau \leq (m+\sigma)-(\sigma+1) = m-1$. Also, as we have seen above, we have $n-\sigma \in N(m-\tau, m-1)$ whenever $n \in N(m, \xi_2-1)$. From these, we obtain

$$\begin{aligned} z(m) &\leq p \max_{n\in N(m-\tau,m-1)} |x(n)| + \left(\sum_{n=m}^{\xi_2-1} q(n)\right) \max_{n\in N(m-\tau,m-1)} |x(n)| \\ &\leq p \max_{n\in N(m-\tau,m-1)} |x(n)| + \left[\sum_{n=m}^{\xi_1+\tau-1} q(n)\right] \max_{n\in N(m-\tau,m-1)} |x(n)| \\ &\leq \left\{p + \sum_{n=m}^{\xi_1+\tau-1} q(n)\right\} \max_{n\in N(m-2\tau,m-1)} |x(n)|. \end{aligned}$$

So (25) holds true

Subcase 2.2: $\xi_2 > m+\sigma$. First, let $z(\xi_2) \leq 0$. Then (27) is written as

$$z(m) \leq -\sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma). \tag{28}$$

If $n \in N(m+\sigma, \xi_2-1)$, then $\xi_1 < m \leq n-\sigma \leq \xi_2-1-\sigma \leq \xi_2-1$. Consequently, $n-\sigma \in N(\xi_1+1, \xi_2-1)$. So we have $x(n-\sigma) > 0$ for every $n \in N(m+\sigma, \xi_2-1)$. Hence, it follows from (26), that

$$\begin{aligned} z(m) &\leq -\sum_{n=m}^{m+\sigma-1} q(n)x(n-\sigma) - \sum_{n=m+\sigma}^{\xi_2-1} q(n)x(n-\sigma) \\ &< -\sum_{n=m}^{m+\sigma-1} q(n)x(n-\sigma) \\ &\leq \sum_{n=m}^{m+\sigma-1} q(n)\,|x(n-\sigma)|. \end{aligned}$$

But, for any $n \in N(m, m+\sigma-1)$, it holds $m-\tau \leq m-\sigma-1 \leq n-\sigma-1 \leq m-1$. Thus, we derive

$$\begin{aligned} z(m) &\leq \left\{\sum_{n=m}^{m+\sigma-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)| \\ &\leq \left\{\sum_{n=m}^{\xi_2-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)| \\ &\leq \left\{\sum_{n=m}^{\xi_1+\tau-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)|, \end{aligned}$$

which, as $p \geq 0$, guarantees that (25) holds true. Next, let $x(\xi_2) \leq 0$. Then (27) becomes,

$$z(m) \leq -p(\xi_2)x(\xi_2-\tau) - \sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma). \tag{29}$$

As above, $n-\sigma \in N(\xi_1+1, \xi_2-1)$ for every $N \in N(m+\sigma, \xi_2-1)$. Consequently $x(n-\sigma) > 0$ for each $n \in N(m-\sigma, \xi_2-1)$. We notice that $\xi_2-\tau \leq \xi_1 < m < \xi_2-\sigma$. If $n \in N(m, \xi_2-\sigma)$, then from (29), we get

$$\begin{aligned} z(m) &\leq -p(\xi_2)x(\xi_2-\tau) - \sum_{n=m}^{m+\sigma-1} q(n)x(n-\sigma) - \sum_{n=m+\sigma}^{\xi_2-1} q(n)x(n-\sigma) \\ &\leq -p(\xi_2)x(\xi_2-\tau) - \sum_{n=m}^{m+\sigma-1} q(n)x(n-\sigma) \\ z(m) &\leq p\,|x(\xi_2-\tau)| + \sum_{n=m}^{m+\sigma-1} q(n)\,|x(n-\sigma)| \end{aligned}$$

But $m-\tau < \xi_2-\tau \leq \xi_1 < m$. Moreover, as before, we have $[n-\sigma \in N(m-\tau, m-1)]$, for every $n \in N(m, m+\sigma-1)$. Thus, we obtain

$$\begin{aligned} z(m) &\leq p \max_{n\in N(m-\tau,m-1)} |x(n)| + \left\{\sum_{n=m}^{\xi_2-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)| \\ &\leq \left\{p + \sum_{n=m}^{\xi_1+\tau-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)|. \end{aligned}$$

So inequality (25) is always satisfied.

Now, we will make use of inequality (23), which has been already established, to arrive at a contradiction. Since m is choose so that $m \geq n_0+\tau$, we have $m-2\tau \geq n_0-\tau$ and consequently from (22) in particular that

$$|x(n)| \leq \frac{1}{1-p} z(m) \quad for \quad all \quad n \in N(m-2\tau, m).$$

This can equivalently be written as

$$\max_{n\in N(m-2\tau,m)} |x(n)| \leq \frac{1}{1-p} z(m)$$

and so inequality (23) yields

$$2z(m) \leq \left\{2p + \sum_{n=\xi_1}^{\xi_1+\tau-1} q(n)\right\} \frac{1}{1-p} |z(m)|.$$

Thus, in view of (21), we have

$$2 \leq \left\{2p + \sum_{n=\xi_1}^{\xi_1+\tau-1} q(n)\right\} \frac{1}{1-p}$$

i.e.,

$$\sum_{n=\xi_1}^{\xi_1+\tau-1} q(n) \geq 2(1-2p)$$

As $\xi_1 \geq m - \sigma \geq n_1$ the last inequality contradicts (16). This contradiction finishes the proof of the fact that the solution $\{x(n)\}$ is bounded.

The proof of the theorem will be accomplished by proving that the solution $\{x(n)\}$ tends to zero as $n \to \infty$. To this end, we will make use of the fact that the solution $\{x(n)\}$ is always bounded.

Suppose, for the sake of contradiction, that $\{x(n)\}$ does not tend to zero as $n \to \infty$, and define

$$\mu = \limsup_{n \to \infty} |x(n)|.$$

It is obvious that $0 < \mu < \infty$. Moreover, we put

$$\lambda = \limsup_{n \to \infty} |z(n)|.$$

Now, for $n \geq n_0$

$$\begin{aligned} |z(n)| &= |x(n) - p(n)x(n-\tau)| \\ &\leq |x(n)| + p\,|x(n-\tau)| \\ &\leq |x(n)| + m\,|x(n-\tau)|. \end{aligned}$$

So, as $\{x(n)\}$ is bounded, it follows that $\{z(n)\}$ is also bounded. Consequently λ must be finite. Furthermore, it holds.

$$\lambda \geq \mu(1-p), \tag{30}$$

which guarantees, in particular that $\lambda > 0$. In fact, let $\in$ be an arbitrary positive real number. From the definition of μ it follows that, for some point $n_\in \geq n_0 - \tau$, we have

$$|x(n)| \leq \mu + \in \quad for \quad every \quad n \geq n_\in. \tag{31}$$

Hence by using (31) we obtain for each $n \geq n_\in + \tau$

$$\begin{aligned} |z(n)| &= |x(n) - p(n)x(n-\tau)| \\ &\geq |x(n)| - p\,|x(n-\tau)| \\ &\geq |x(n)| - m(\mu + \in). \end{aligned}$$

Consequently,

$$\limsup_{n \to \infty} |z(n)| \geq \limsup_{n \to \infty} |x(n)| - p(\mu + \in)$$

i.e.,

$$\lambda \geq \mu - m(\mu + \in).$$

This inequality holds true for all real numbers $\in > 0$ and so (30) is always satisfied.

Since the solution $\{x(n)\}$ is not eventually of one sign, i.e., it changes sign for arbitrarly large values of n. Thus the sequence $\{\Delta z(n)\}$ changes sign for arbitrarly large values of n, which ensures that $\{z(n)\}$ is not monotone. By this fact and fact that $\lambda > 0$. we conclude that there exists a sequence of integers $\{m_k\}_{k=1}^{\infty}$ with $n_0 \leq m_1 < m_2 < ...$ and $\lim_{k \to \infty} m_k = \infty$ such that $z(m_k) \neq 0$ $(k = 1, 2, ...)$ and

$$z(m_k)\Delta z(m_k) \leq 0 \tag{32}$$

and

$$\lim_{n \to \infty} |z(m_k)| = \lambda. \tag{33}$$

We remark that the sequence $\{m_k\}_{k=1}^{\infty}$ can be chosen so that either $z(m_k) > 0$ for all $k = 1, 2, ...$ or $z(m_k) < 0$ for all $n = 1, 2, 3, ...$ We see that

$$-z(n) = -x(n) + p(n)x(n-\tau) \quad for \quad n \geq n_0$$

and that

$$\limsup_{n \to \infty} |-z(n)| = \lambda.$$

Also, it is obvious that $\{-x(n)\}$ is a bounded oscillatory solution of (1), which does not tend to zero as $n \to \infty$. After there observations, we may (and do) restrict ourselves only to the case where

$$z(m_k) > 0 \quad (k = 1, 2, 3, ...) \tag{34}$$

In view of (34), equality (33) becomes

$$\lim_{n\to\infty} z(m_k) = \lambda. \tag{35}$$

It is clear that we have either $x(m_k) = 0$ for infinitely many $k \in \{1, 2, 3, ...\}$ or $x(m_k) \neq 0$. So, we examine separately the following two cases:

Case I: $x(m_k) \leq 0$ for infinitely many $k \in \{1, 2, 3, ...\}$. Let $\{m_{k_i}\}_{i=1}^{\infty}$ be a sub sequence of $\{m_k\}_{k=1}^{\infty}$ such that

$$x(m_{k_i}) \leq 0 \quad (i = 1, 2, 3, ...). \tag{36}$$

Clearly, $\lim_{n\to\infty} m_{k_i} = \infty$. It follows from (32) (34) and (35) that

$$\Delta z(m_{k_i}) \leq 0 \quad (i = 1, 2, 3, ...), \tag{37}$$

$$z(m_{k_i}) > 0 \quad (i = 1, 2, 3, ...) \tag{38}$$

and

$$\lim_{i\to\infty} z(m_{k_i}) = \lambda, \quad \textit{respectively}. \tag{39}$$

By (37), (1) and (2), we have

$$q(m_{k_i})x(m_{k_i} - \sigma) \leq 0 \quad (i = 1, 2, 3, ...).$$

Consequently, we get

$$x(m_{k_i} - \sigma) \leq 0 \quad (i = 1, 2, 3, ...). \tag{40}$$

Using (2), (36) and (38) we obtain for $i = 1, 2, 3, ...$

$$\begin{aligned} 0 < z(m_{k_i}) &= x(m_{k_i}) - p(m_{k_i})x(m_{k_i} - \tau) \\ &\leq -p(m_{k_i})x(m_{k_i} - \tau) \\ &\leq p\left|x(m_{k_i} - \tau)\right| \\ &\leq p \max_{n\in N(m_{k_i}-\tau, m_{k_i}-1)} |x(n)|. \end{aligned}$$

We consider an integer $j \in \{1, 2, 3, ...\}$ such that $m_{k_i} \geq \tau$. Then we obviously have $m_{k_i} \geq \tau$ for all $i \geq j$, so, it holds

$$z(m_{k_i}) \leq m \max_{n\in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)|, \quad for \quad i \geq j. \tag{41}$$

Next using (1), (2) and (40) we obtain, for $i \geq j$,

$$\begin{aligned} z(m_{k_i}) &= z(m_{k_i} - \tau) + \sum_{n=m_k-\tau}^{m_{k_i}-1} q(n)x(n-\sigma) \\ &= \left[x(m_{k_i} - \tau) - p(m_{k_i} - \tau)x(m_{k_i} - 2\tau)\right] + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n)x(n-\sigma) \\ &\leq -p(m_{k_i} - \tau)x(m_{k_i} - 2\tau) + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n)x(n-\sigma) \\ &\leq p(m_{k_i} - \tau)\left|x(m_{k_i} - 2\tau)\right| + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n)\left|x(n-\sigma)\right| \\ &\leq p \max_{n\in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)| + \left[\sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n)\right] \max_{n\in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)|. \end{aligned}$$

Therefore,

$$z(m_{k_i}) \leq \left\{ p + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n) \right\} \max_{n\in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)| \tag{42}$$

A combinations of (41) and (42) leads to

$$2z(m_{k_i}) \leq \left\{ 2p + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n) \right\} \max_{n\in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)| \quad for \quad i \geq j. \tag{43}$$

Let $\in < 0$ be an arbitrary real numbers. In view of definition of μ, there exists an integer $n_\in \geq n_0 - \tau$ so that (31) holds. Choose an integer $l \geq j$ such that $m_{k_l} \geq n_\in + 2\tau$. It is obvious that $m_{k_i} \geq n_\in + 2\tau$ for all integers $i \geq l$. It follows from (31) that

$$\max_{n\in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)| \leq \mu + \in \quad for \quad i \geq l.$$

Hence, from (43) we get

$$2z(m_{k_i}) \leq (\mu + \in) \left\{ 2p + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n) \right\} \quad for \quad every \quad i \geq l,$$

which gives

$$\begin{aligned} 2\lim_{i\to\infty} z(m_{k_i}) &\leq (\mu + \in) \left[2p + \limsup_{i\to\infty} \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n) \right] \\ &\leq (\mu + \in) \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n-\tau}^{n-1} q(s) \right\} \\ &= (\mu + \in) \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \right\}. \end{aligned}$$

So because of (39), we have

$$2\lambda \leq (\mu + \in) \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \right\}. \tag{44}$$

By combining (28) and (42), we obtain

$$2\mu(1-p) \leq (\mu + \in) \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \right\}.$$

As this inequality is satisfied for every real number $\in > 0$, we always have

$$2\mu(1-p) \leq \mu \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \right\}.$$

Thus, since $\mu > 0$, it holds

$$2(1-p) \leq 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau_1} q(s).$$

i.e.,

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \geq 2(1-2p). \tag{45}$$

Inequality (45) contradicts condition (15).

Case II: $x(m_k) > 0$ for all large n. This means that there exists an integer $r \in \{1, 2, 3, ...\}$ such that

$$x(m_k) > 0 \quad for \quad all \quad k \geq r. \tag{46}$$

It is clear that the integer r can be chosen to be arbitrary large; so it will be considered that $m_r \geq n_0 + \tau$. Then we have $m_k \geq n_0 + \tau$ for all $k \geq r$.

Let as consider an arbitrary large k with $k \geq r$. We observe that in view of (34) and (46) it holds $x(m_k) > 0$ and $z(m_k) > 0$.

Furthermore by (32) and (34), it holds $\Delta z(m_k) \leq 0$. From this and (1), we have $x(m_k - \sigma) \leq 0$, where $m_k - \sigma \geq n_0$. Let ξ_k^1 be the integer with $m_k - \sigma \leq \xi_k^1 \leq m_k$ such that either $x(\xi_k^1) \leq 0$ or $z(\xi_k^1) \leq 0$ and $x(n) > 0$ and $z(n) > 0$ on $N(\xi_k^1 + 1, m_k)$.

On the otherhand, by the oscillatory character of $\{x(n)\}$ we may find an integer ξ_k^2 with $m_k < \xi_k^2$ such that either $x(\xi_k^2) \leq 0$ of $z(\xi_k^2) \leq 0$ and $x(n) > 0$ and $z(n) > 0$ on $N(m_k, \xi_k^2 - 1)$. It follows that both $x(n) > 0$ and $z(n) > 0$ on $N(\xi_k^1, \xi_k^2)$. Thus we have defined two sequence $\{\xi_k^1\}$ and $\{\xi_k^2\}$ of integers such that $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\xi_k^1 + 1, \xi_k^2 - 1)$. Since $\xi_k^1 \geq m_k - \sigma$ for $k \geq r$, we always have $\lim_{k\to\infty} \xi_k^1 = \infty$. Following the same procedure as when establishing (23), we can prove that

$$2z(m_k) \leq \left\{ 2p + \sum_{n=\xi_k^1}^{\xi_k^1+\tau-1} q(n) \right\} \max_{n \in N(m_k - 2\tau, m_k - 1)} |x(n)| \quad for \quad k \leq r. \tag{47}$$

Consider an arbitrary real number $\in > 0$ and let $n_\in \geq n_0 - \tau$ be an integer such that (31) is satisfied. Moreover, let $l \geq r$, be an integer such that $m_l \geq n_\in + 2\tau$. Then we obviously have $m_k \geq n_\in + 2\tau$ for every $k \geq l$. So (31) guarantees that

$$\max_{n \in N(m_k - 2\tau, m_k - 1)} |x(n)| \leq \mu + \in \quad for \quad k \geq l$$

Thus, from (47)we obtain

$$2z(m_k) \leq (\mu + \in) \left\{ 2p + \sum_{n=\xi_k^1}^{\xi_k^1+\tau-1} q(n) \right\} \quad for \quad all \quad k \geq l.$$

Therefore,

$$\begin{aligned} 2 \lim_{k\to\infty} z(m_k) &\leq (\mu + \in) \left\{ 2p + \limsup_{k\to\infty} \sum_{n=\xi_k}^{\xi_k^1+\tau-1} q(n) \right\} \\ &\leq (\mu + \in) \left\{ 2p + \limsup_{k\to\infty} \sum_{s=n}^{n+\tau-1} q(n) \right\} \end{aligned}$$

which because of (33), leads to (44). By the method used previously we can see that (45) is always satisfied. But (45) contradicts condition (15).

In both Cases I and II we have arrived at a contradiction. This contradiction shows that the solution $\{x(n)\}$ tend to zero as $n \to \infty$.

The proof of the theorem is complete.

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Existence of strongly continuous solutions for a functional integral inclusion

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Abstract

In this paper we are concerned with the existence of strongly continuous solution $x \in C[I,E]$ of the nonlinear functional integral inclusion

$$x(t) \in F(t, \int_0^t g(s, x(m(s)))ds), \quad t \in [0,T]$$

under the assumption that the set-valued function F has Lipschitz selection in the Banach space E.

Keywords: Set-valued function, continuous solutions, Functional integral inclusions, selections of the set-valued function, Lipschitz selections.

2010 MSC: 47B38, 47H30.

1 Introduction

Let E be a Banach space, $I = [0,T]$ and let $L^1(I)$ be the class of all Lebesgue integrable functions defined on the interval I.
Denote by $C[I,E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.

$$\|x\|_C = \sup \|x\|_E.$$

Consider the functional integral inclusion

$$x(t) \in F(t, \int_0^t g(s, x(m(s)))ds), \quad t \in [0,T] \tag{1.1}$$

where $F : I \times E \to P(E)$ is a nonlinear set-valued mapping, and $P(E)$ denote the family of nonempty subsets of the Banach space E.
Indeed a set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2], [9]-[13]), and a functional integral inclusion was studied by B.C. Dhage and D. O'Regan (see [3], [4] and [14]).
Here we study the existence of strongly continuous solution $x \in C[I,E]$ of the functional integral inclusion (1.1) in the Banach space E under a set of several suitable assumptions on the set-valued function F.
Our study is based on the selections of the set-valued function F, on which we have a functional integral equation, such a type has been studied in several papers (see [1], [7]-[8] and [15]).

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2 Preliminaries

We present some definitions and results that will be used in this work.
Let E be a Banach space and let $x : I \to E$.

Definition 2.1. *[6] A set-valued map F from $I \times E$ to the family of all nonempty closed subsets of E is called Lipschitzian if there exists $L > 0$ such that for all $t_1, t_2 \in I$ and all $x_1, x_2 \in E$, we have*

$$H(F(t_1, x_1), F(t_2, x_2)) \leq L(|t_1 - t_2| + \|x_1 - x_2\|),$$

where $H(A, B)$ is the Hausdorff metric between the two subsets $A, B \in I \times E$.

Denote $S_F = Lip(I, E)$ be the set of all Lipschitz selections of the set-valued function F with values in the Banach space E.

Let $E = R^n$. The following theorem assures the existence of Lipschitzian selection.

Theorem 2.1. *[6] Let M be a metric space and F be Lipschitzian set-valued function from M into the nonempty compact convex subsets of R^n. Assume, moreover, that for some $\lambda > 0, F(x) \subset \lambda B$ for all $x \in M$ where B is the unit ball of R^n. Then there exists a constant C and a single-valued function $f : M \to R^n, f(x) \in F(x)$ for $x \in M$; this function is Lipschitzian with constant l.*

Denote $S_F^* = Lip(M, R^n)$ to be the set of all Lipschitz selections of the set-valued function F with values in the space R^n.

Theorem 2.2. *[5] "Schauder fixed point theorem".*
Let Q be a convex subset of a Banach space X, $T : Q \to Q$ be a compact, continuous map. Then T has at least one fixed point in Q.

3 Existence of solution in E

In this section, we present our main result by proving the existence of strongly continuous solution $x \in C[I, E]$ of the functional integral inclusion (1.1) in the Banach space E, under the assumption that the set-valued function F has Lipschitz selection in E.
Consider now the functional integral inclusion (1.1) under the following assumptions
(H1) The set $F(t, x)$ is compact and convex for all $(t, x) \in I \times E$.
(H2) The set-valued map F is Lipschitzian with a Lipschitz constant $L > 0$.
(H3) The set of all Lipschitz selections S_F is nonempty.
(H4) The function $g : [0, T] \times E \to E$ satisfies Caratheodory condition i.e. $g(t, .)$ is continuous in $x \in E$ for each $t \in I$ and $g(., x)$ is measurable in $t \in I$ for each $x \in E$.
(H5) There exists an integrable function $a \in L^1[I, E]$ and a positive constant $b > 0$ such that

$$\|g(t, x)\| \leq \|a(t)\| + b\|x\|, \quad \forall\, t \in I, \;\; x \in E.$$

(H6) $m : [0, T] \to [0, T]$ is continuous.

Remark 3.1. *From assumptions (H1) and (H3), there exists $f \in S_F$ such that*

$$\|f(t_2, x) - f(t_1, y)\|_C \leq L(|t_2 - t_1| + \|x - y\|_C),$$

and

$$x(t) = f(t, \int_0^t g(s, x(m(s)))ds, \quad t \in [0, T] \tag{3.2}$$

Then the solution of the functional integral equation (3.2), if it exists, is a solution of the functional integral inclusion (1.1).

Definition 3.2. *By a solution of the functional integral inclusion (1.1) we mean the function* $x(.) \in C[I, E]$ *satisfying (1.1).*

For the existence of strongly continuous solution $x \in C[I, E]$ of the functional integral inclusion (1.1) we have the following theorem.

Theorem 3.3. *Let the assumptions (H1)-(H6) be satisfied. Then there exists a strongly continuous solution* $x \in C[I, E]$ *of the functional integral inclusion (1.1).*

Proof. Let the set-valued function F satisfy the assumptions (H1)-(H3), then there exists a selection $f \in S_F,\ f : I \times E \to E$, such that

$$\|f(t_2, x) - f(t_1, y)\|_C \leq L(|t_2 - t_1| + \|x - y\|_C),$$

for every $t_1, t_2 \in I\ \ and\ \ x, y \in E$.
And f satisfy the functional integral equation (3.2).
Define the operator A by

$$Ax(t) = f(t, \int_0^t g(s, x(m(s)))ds, \quad t \in [0, T]$$

Let the set Q_r be defined as

$$Q_r = \{x \in C[I, E], \|x\|_C \leq r\};\ \ r = \frac{LK + M}{1 - LbT}.$$

Then, it is clear that it is nonempty, bounded, closed and convex set.
Let $x \in Q_r$ be an arbitrary element, then

$$\begin{aligned}
\|Ax(t)\| &= \|f(t, \int_0^t g(s, x(m(s)))ds\| \\
&\leq L\|\int_0^t g(s, x(m(s)))ds\| + \|f(t, 0)\| \\
&\leq L\|\int_0^t g(s, x(m(s)))ds\| + \sup|f(t, 0)| \\
&\leq L\int_0^t \|g(s, x(m(s)))\|ds + \sup|f(t, 0)| \\
&\leq L\int_0^t \{\|a(s)\| + b\|x(m(s))\|\}ds + \sup|f(t, 0)| \\
&\leq L\int_0^t \|a(s)\|ds + Lb\int_0^t \|x(m(s))\|ds + \sup|f(t, 0)| \\
&\leq L\int_0^t \|a(s)\|ds + Lb\int_0^t \sup_{m(s)\in[0,T]} \|x(m(s))\|ds + \sup|f(t, 0)| \\
&\leq L\int_0^t \|a(s)\|ds + Lb\int_0^t \sup_{s\in[0,T]} \|x(m(s))\|ds + \sup|f(t, 0)| \\
&\leq LK + Lb\|x\|T + M,
\end{aligned}$$

where $K = \int_0^t \|a(s)\|ds$, and $M = \sup|f(t, 0)|$.
Then
$\|Ax(t)\| \leq LK + LbrT + M = r$, where $r = \frac{LK+M}{1-LbT}$
Hence

$$\|Ax\|_C \leq r.$$

Which proves that $AQ_r \subset Q_r,\ \ i.e.\ \ A : Q_r \to Q_r$.
Finally, we will show that A is compact.
Let $x \in Q_r$, since $Ax \in Q_r,\ \|Ax\| \leq r$, then Ax is bounded $\forall\ x \in Q_r$.
Therefore, A is bounded and the class $\{Ax\}$ is uniformly bounded.

Now, let $t_1, t_2 \in [0, T]$, then $\forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$, such that $|t_2 - t_1| < \delta$, whenever

$$\begin{aligned}
\|Ax(t_2) - Ax(t_1)\| &= \|f(t_2, \int_0^{t_2} g(s, x(m(s)))ds - f(t_1, \int_0^{t_1} g(s, x(m(s)))ds\| \\
&\leq L\{|t_2 - t_1| + \| \int_0^{t_2} g(s, x(m(s)))ds - \int_0^{t_1} g(s, x(m(s)))ds\|\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|g(s, x(m(s)))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \{\|a(s)\| + b\|x(m(s))\|\}ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\int_{t_1}^{t_2} \|x(m(s))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\int_{t_1}^{t_2} \sup_{m(s)\in[0,T]} \|x(m(s))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\int_{0}^{t_2} \sup_{s\in[0,T]} \|x(m(s))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\|x\||t_2 - t_1|\} \\
&\leq L\{\delta + \int_{t_1}^{t_2} \|a(s)\|ds + br\delta\} = \varepsilon.
\end{aligned}$$

Then

$$\|Ax(t_2) - Ax(t_1)\| \leq \varepsilon.$$

Hence the class $\{Ax\}$ is equicontinuous, $x \in Q_r$, and by Arzela theorem, A is compact.
Then by Schauder fixed point theorem, there exists at least one fixed point, and then there exists at least one strongly continuous solution $x \in C[I, E]$ for the functional integral equation (3.2).
Consequently, there exists a strongly continuous solution $x \in C[I, E]$ for the functional integral inclusion (1.1). □

4 Existence of solution in R^n

In this section, we present the existence of strongly continuous solution $x \in C[I, R^n]$ of the functional integral inclusion (1.1) in the space R^n, under the assumption that the set-valued function F has Lipschitz selection in R^n.
Consider now the functional integral inclusion (1.1) under the following assumptions
(I) The set $F(t, x)$ is compact and convex for all $(t, x) \in I \times R^n$.
(II) The set-valued map F is Lipschitzian with a Lipschitz constant $L > 0$

$$\|F(t_2, x) - F(t_1, y)\|_C \leq L(|t_2 - t_1| + \|x - y\|_C),$$

for every $t_1, t_2 \in I$ *and* $x, y \in R^n$.
(III) The function $g : [0, T] \times R^n \to R^n$ satisfies Caratheodory condition i.e. $g(t, .)$ is continuous in $x \in R^n$ for each $t \in I$ and $g(., x)$ is measurable in $t \in I$ for each $x \in R^n$.
(IV) There exists an integrable function $a \in L^1[I, R^n]$ and a positive constant $b > 0$ such that

$$\|g(t, x)\| \leq \|a(t)\| + b\|x\|, \quad \forall\, t \in I, \quad x \in R^n.$$

(V) $m : [0, T] \to [0, T]$ is continuous.

Definition 4.3. *By a solution of the functional integral inclusion (1.1) we mean the function $x(.) \in C[I, R^n]$ satisfying (1.1).*

Now for the existence of strongly continuous solution $x \in C[I, R^n]$ of the functional integral inclusion (1.1) we have the following theorem.

Theorem 4.4. *Let the assumptions (I)-(V) be satisfied. Then there exists a strongly continuous solution* $x \in C[I, R^n]$ *of the functional integral inclusion (1.1).*

Proof. Let the set-valued function F satisfy the assumptions (I)-(II), then from Theorem (2.1) with $M = I \times R^n$, we deduce that there exists a selection $f \in F$, which satisfies:
(i) $f : I \times R^n \to R^n$ is continuous
(ii) f satisfy Lipschitz condition with a Lipschitz constant $L > 0$

$$\|f(t_2, x) - f(t_1, y)\|_C \leq L(|t_2 - t_1| + \|x - y\|_C),$$

for every $t_1, t_2 \in I \ \ and \ \ x, y \in R^n$.
And f satisfy the functional integral equation (3.2).
Define the operator A by

$$Ax(t) = f(t, \int_0^t g(s, x(m(s)))ds), \quad t \in [0, T]$$

Let the set Q_r be defined as

$$Q_r = \{x \in C[I, R^n], \|x\|_C \leq r\}; \ \ r = \frac{LK + M}{1 - LbT}.$$

Then, it is clear that it is nonempty, bounded, closed and convex set.
Let $x \in Q_r$ be an arbitrary element, then

$$\begin{aligned} \|Ax(t)\| &= \|f(t, \int_0^t g(s, x(m(s)))ds\| \\ &\leq L\| \int_0^t g(s, x(m(s)))ds\| + \|f(t, 0)\| \\ &\leq L\| \int_0^t g(s, x(m(s)))ds\| + \sup |f(t, 0)| \\ &\leq L \int_0^t \|g(s, x(m(s)))\|ds + \sup |f(t, 0)| \\ &\leq L \int_0^t \{\|a(s)\| + b\|x(m(s))\|\}ds + \sup |f(t, 0)| \\ &\leq L \int_0^t \|a(s)\|ds + Lb \int_0^t \|x(m(s))\|ds + \sup |f(t, 0)| \\ &\leq L \int_0^t \|a(s)\|ds + Lb \int_0^t \sup_{m(s) \in [0,T]} \|x(m(s))\|ds + \sup |f(t, 0)| \\ &\leq L \int_0^t \|a(s)\|ds + Lb \int_0^t \sup_{s \in [0,T]} \|x(m(s))\|ds + \sup |f(t, 0)| \\ &\leq LK + Lb\|x\|T + M, \end{aligned}$$

where $K = \int_0^t \|a(s)\|ds$, and $M = \sup |f(t, 0)|$.
Then
$\|Ax(t)\| \leq LK + LbrT + M = r$, where $r = \frac{LK+M}{1-LbT}$
Hence

$$\|Ax\|_C \leq r.$$

Which proves that $AQ_r \subset Q_r, \ \ i.e. \ \ A : Q_r \to Q_r$.
Finally, we will show that A is compact.
Let $x \in Q_r$, since $Ax \in Q_r, \ \|Ax\| \leq r$, then Ax is bounded $\forall \ x \in Q_r$.
Therefore, A is bounded and the class $\{Ax\}$ is uniformly bounded.

Now, let $t_1, t_2 \in [0,T]$, then $\forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$, such that $|t_2 - t_1| < \delta$, whenever

$$\begin{aligned}
\|Ax(t_2) - Ax(t_1)\| &= \|f(t_2, \int_0^{t_2} g(s, x(m(s)))ds - f(t_1, \int_0^{t_1} g(s, x(m(s)))ds\| \\
&\leq L\{|t_2 - t_1| + \|\int_0^{t_2} g(s, x(m(s)))ds - \int_0^{t_1} g(s, x(m(s)))ds\|\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|g(s, x(m(s)))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \{\|a(s)\| + b\|x(m(s))\|\}ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\int_{t_1}^{t_2} \|x(m(s))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\int_{t_1}^{t_2} \sup_{m(s)\in[0,T]} \|x(m(s))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\int_{0}^{t_2} \sup_{s\in[0,T]} \|x(m(s))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\|x\||t_2 - t_1|\} \\
&\leq L\{\delta + \int_{t_1}^{t_2} \|a(s)\|ds + br\delta\} = \varepsilon.
\end{aligned}$$

Then

$$\|Ax(t_2) - Ax(t_1)\| \leq \varepsilon.$$

Hence the class $\{Ax\}$ is equicontinuous, $x \in Q_r$, and by Arzela theorem, A is compact.
Then by Schauder fixed point theorem, there exists at least one fixed point, and then there exists at least one strongly continuous solution $x \in C[I, R^n]$ for the functional integral equation (3.2).
Consequently, there exists a strongly continuous solution $x \in C[I, R^n]$ for the functional integral inclusion (1.1). □

Corollary 4.1. *Let $n = 1$. If F satisfy the assumptions (I)-(II), then from Theorem (2.1) with $M = I \times R$, we deduce that there exists a selection $f \in F$, which satisfies (i)-(ii), and f satisfy the functional integral equation (3.2). Hence there exists a strongly continuous solution $x \in C[I, R]$ for the functional integral inclusion (1.1).*

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$g^*\omega\alpha$-Separation Axioms in Topological Spaces

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Abstract

In this paper, we introduce and study the new separation axioms called g$^*\omega\alpha$-T_i (i= 0,1,2) and weaker forms of regular and normal spaces called g$^*\omega\alpha$-normal and g$^*\omega\alpha$-regular spaces using g$^*\omega\alpha$-closed sets in topological spaces.

Keywords: $g^*\omega\alpha$-closed sets, $g^*\omega\alpha$-T_0 spaces, $g^*\omega\alpha$-T_1 spaces, $g^*\omega\alpha$-T_2 spaces, g$^*\omega\alpha$-regular spaces, g$^*\omega\alpha$-normal spaces.

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1 Introduction

General Topology plays an important role in many fields of applied sciences as well as branches of mathematics. More importantly, generalized closed sets suggest some new separation axioms which have been found to be very useful in the study of certain objects of digital topology.

Maheshwari and Prasad [7] introduced the new class of spaces called s-normal spaces using semi open sets [4]. It was further studied by Noiri and Popa [6], Dorsett [2] and Arya [1]. Munshi [8] and R. Devi [3] introduced g-regular and g-normal spaces and their properties in topological spaces. Recently, Patil P. G. et. al. [9],[11] introduced and studied the concepts of g$^*\omega\alpha$-closed sets and g$^*\omega\alpha$-continuous functions in topological spaces.

In this paper, we introduce new weaker forms of separation axioms called g$^*\omega\alpha$-T_0, g$^*\omega\alpha$-T_1, g$^*\omega\alpha$-T_2 spaces and new class of spaces namely g$^*\omega\alpha$-regular and g$^*\omega\alpha$-normal spaces and their characterizations are obtained.

2 Preliminary

Throughout this paper space (X,τ) and (Y,σ) (or simply X and Y) always denote topological spaces on which no separation axioms are assumed unless explicitly stated.

For a subset A of a space X, the closure (resp. α-closure [5]) and interior (resp. α-interior) of A is denoted by cl(A) (resp. α-cl(A)) and int(A) (resp. α-int(A)).

Definition 2.1. *[9] A subset A of a topological space X is said to be a generalized star $\omega\alpha$-closed (briefly $g^*\omega\alpha$-closed) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\omega\alpha$-open in X.*
The family of all $g^*\omega\alpha$-closed (resp.$g^*\omega\alpha$-open) subsets of a space X is denoted by $G^*\omega\alpha C(X)$ (resp.$G^*\omega\alpha O(X)$).

Definition 2.2. *[10] The intersection of all $g^*\omega\alpha$-closed sets containing a subset A of X is called $g^*\omega\alpha$-closure of A and is denoted by $g^*\omega\alpha$-cl(A).*
A set A is $g^*\omega\alpha$-closed if and only if $g^*\omega\alpha$-cl(A) = A.

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Definition 2.3. *[10] The union of all $g^*\omega\alpha$-open sets contained in a subset A of X is called $g^*\omega\alpha$-interior of A and it is denoted by $g^*\omega\alpha$-int(A).*
A set A is called $g^*\omega\alpha$-open if and only if $g^*\omega\alpha$-int(A) = A.

Definition 2.4. *A function f : $X \to Y$ is called a*
(i) $g^*\omega\alpha$-continuous[11] if $f^{-1}(V)$ is $g^*\omega\alpha$-closed in X for every closed set V in Y.
(ii) $g^*\omega\alpha$-irresolute[11] if $f^{-1}(V)$ is $g^*\omega\alpha$-closed in X for every $g^*\omega\alpha$-closed set V in Y.
(iii) $g^*\omega\alpha$-open[11] if f(V) is $g^*\omega\alpha$-open in Y for every open set V in X.
(iv) pre $g^*\omega\alpha$-open[11] if f(V) is $g^*\omega\alpha$-open set in Y for every $g^*\omega\alpha$-open set V in X.

Definition 2.5. *[10] A topological space X is said to be a $T_{g^*\omega\alpha}$-space if every $g^*\omega\alpha$-closed set is closed.*

3 $g^*\omega\alpha$-Separation Axioms

In this section, we introduce weaker forms of separation axioms such as g$^*\omega\alpha$-T_0, g$^*\omega\alpha$-T_1 and g$^*\omega\alpha$-T_2 spaces and obtain their properties.

Definition 3.1. *A topological space X is said to be a $g^*\omega\alpha$-T_0 if for each pair of distinct points in X, there exists a $g^*\omega\alpha$-open set containing one point but not other.*

Example 3.2. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Then the space (X, τ) is $g^*\omega\alpha$-T_0 space.*

Theorem 3.3. *A space X is $g^*\omega\alpha$-T_0 if and only if $g^*\omega\alpha$-closures of distinct points are distinct.*
Proof: *Let x, $y \in X$ with $x \neq y$ and X be $g^*\omega\alpha$-T_0 space. Since, X is $g^*\omega\alpha$-T_0, there exists $g^*\omega\alpha$-open set G such that $x \in G$ but $y \notin G$. Also $x \notin$ X-G and $y \in$ X-G where X-G is $g^*\omega\alpha$-closed in X. Since $g^*\omega\alpha$-cl($\{y\}$) is the intersection of all $g^*\omega\alpha$-closed sets which contains y and hence $y \in g^*\omega\alpha$-cl($\{y\}$). But $x \notin g^*\omega\alpha$-cl($\{y\}$) as $x \notin$ X-G. Therefore $g^*\omega\alpha$-cl($\{x\}$) $\neq g^*\omega\alpha$-cl($\{$ y $\}$).*
Conversely, suppose for any pair of distinct points x, $y \in X$, $g^*\omega\alpha$-cl($\{x\}$) $\neq g^*\omega\alpha$-cl($\{y\}$). Then, there exists at least one point $z \in X$ such that $z \in g^*\omega\alpha$-cl($\{x\}$) but $z \notin g^*\omega\alpha$-cl($\{y\}$). We claim that $x \notin g^*\omega\alpha$-cl($\{y\}$). If $x \in g^*\omega\alpha$-cl($\{y\}$), then $g^*\omega\alpha$-cl($\{x\}$) $\subseteq g^*\omega\alpha$-cl($\{y\}$), so $z \in g^*\omega\alpha$-cl($\{y\}$) which is contradiction. Hence $x \notin g^*\omega\alpha$-cl($\{y\}$) implies $x \in X$ - $g^*\omega\alpha$-cl($\{y\}$), which is $g^*\omega\alpha$-open set in X containing x but not y. Hence X is $g^*\omega\alpha$-T_0-space.

Theorem 3.4. *Every subspace of a $g^*\omega\alpha$-T_0 space is $g^*\omega\alpha$-T_0 space.*
Proof: *Let y_1, y_2 be two distinct points of Y then y_1 and y_2 are also distinct points of X. Since X is $g^*\omega\alpha$-T_0, there exists $g^*\omega\alpha$-open set G such that $y_1 \in G$, $y_2 \notin G$. Then $G \cap Y$ is $g^*\omega\alpha$-open set in Y containing y_1 but not y_2. Hence Y is $g^*\omega\alpha$-T_0-space.*

Definition 3.5. *[11] A mapping f : $X \to Y$ is said to be a pre $g^*\omega\alpha$-open if the image of every $g^*\omega\alpha$-open set of X is $g^*\omega\alpha$-open in Y.*

Lemma 3.6. *The property of a space being $g^*\omega\alpha$-T_0 space is preserved under bijective and pre $g^*\omega\alpha$-open.*
Proof: *Let X be a $g^*\omega\alpha$-T_0-space and f : $X \to Y$ be bijective, pre $g^*\omega\alpha$-open. Let y_1, $y_2 \in Y$ with $y_1 \neq y_2$. Since f is bijective, there exist x_1, $x_2 \in X$ such that f(x_1) = y_1 and f(x_2) = y_2. Also, since X is $g^*\omega\alpha$-T_0, there exists $g^*\omega\alpha$-open set G in X such that $x_1 \in G$ but $x_2 \notin G$. Then f(G) is $g^*\omega\alpha$-open set containing f(x_1) but not f(x_2) as X is $g^*\omega\alpha$-open. Thus, there exists $g^*\omega\alpha$-open set f(G) in Y such that $y_1 \in$ f(G) and $y_2 \notin$ f(G). Hence Y is $g^*\omega\alpha$-T_0 space.*

Theorem 3.7. *If $G^*\omega\alpha O(X)$ is open under arbitrary union for a topological space X, then each of the following properties are equivalent:*
(a) X is $g^*\omega\alpha$-T_0
(b) each one point set is $g^*\omega\alpha$-closed in X
(c) each subset of X is the intersection of all $g^*\omega\alpha$-open set containing it
(d) the intersection of all $g^*\omega\alpha$-open set containing the point $x \in X$ is the set $\{x\}$.
Proof: *(a) $\Rightarrow$ (b): Let $x \in X$ and X be $g^*\omega\alpha$-T_0 space. Then for any $y \in X$ such that $y \neq x$, then there exists $g^*\omega\alpha$-open set G_y containing y but not x. Therefore $y \in G_y \subseteq \{x\}^c$. Now varying y over $\{x\}^c$, we get $\{x\}^c = \cup\{ G_y : y \in \{x\}^c\}$, $\{x\}^c$ is union of $g^*\omega\alpha$-open set. That is $\{x\}$ is $g^*\omega\alpha$-closed in X.*
(b) $\Rightarrow$ (c): Let us assume that each one point set is $g^*\omega\alpha$-closed in X. If $A \subseteq X$, then for each point $y \notin A$, there exists $\{y\}^c$ such that $A \subseteq \{y\}^c$ and each of these sets $\{y\}^c$ is $g^*\omega\alpha$-open. Therefore A = Y $\cap\{\{y\}^c : y \in A^c\}$. Thus the

intersection of all $g^*\omega\alpha$-open sets containing A is the set A itself.
(c) $\Rightarrow$ (d): Obvious.
(d) $\Rightarrow$ (a): Let us assume that the intersection of all $g^*\omega\alpha$-open set containing the point $x \in X$ is $\{x\}$. Let $x, y \in X$ with $x \neq y$. By hypothesis, there exists $g^*\omega\alpha$-open set G_x such that $x \in G_x$ and $y \notin G_x$. That is, X is $g^*\omega\alpha$-T_0 space.

Theorem 3.8. *If X is $g^*\omega\alpha$-T_0, $T_{g^*\omega\alpha}$-space and Y is $g^*\omega\alpha$-closed subspace of X, then Y is $g^*\omega\alpha$-T_0-space.*

Theorem 3.9. *If $f : X \to Y$ is bijective, pre $g^*\omega\alpha$-open and X is $g^*\omega\alpha$-T_0 space, then Y is also $g^*\omega\alpha$-T_0 space.*
Proof: *Let y_1 and y_2 be two distinct points of Y. Then there exist x_1 and x_2 of X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. As X is $g^*\omega\alpha$-T_0, there exists $g^*\omega\alpha$-open set G such that $x_1 \in G$ and $x_2 \notin G$. Therefore, $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \notin f(G)$. Then f(G) is $g^*\omega\alpha$-open in Y. Thus, there exists $g^*\omega\alpha$-open set f(G) in Y such that $y_1 \in f(G)$ and $y_2 \notin f(G)$. Therefore Y is $g^*\omega\alpha$-T_0 space.*

Definition 3.10. *A topological space X is said to be a $g^*\omega\alpha$-T_1 if for each pair of distinct points x, y in X, there exist a pair of $g^*\omega\alpha$-open sets, one containing x but not y and the other containing y but not x.*

Remark 3.11. *Every T_1-space is $g^*\omega\alpha$-T_1-space.*

Example 3.12. $X = \{a, b, c\}$ *and* $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. *Then (X, τ) is $g^*\omega\alpha$-T_1 space but not T_1-space.*

Remark 3.13. *Every $g^*\omega\alpha - T_1$ space is $g^*\omega\alpha - T_0$ space.*

Example 3.14. *Let $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$. Then the space X is $g^*\omega\alpha$-T_0 but not $g^*\omega\alpha$-T_1 space.*

Theorem 3.15. *A space X is $g^*\omega\alpha$-T_1 if and only if every singleton subset $\{x\}$ of X is $g^*\omega\alpha$-closed in X.*
Proof: *Let x, y be two distinct points of X such that $\{x\}$ and $\{y\}$ are $g^*\omega\alpha$-closed. Then $\{x\}^c$ and $\{y\}^c$ are $g^*\omega\alpha$-open in X such that $y \in \{x\}^c$ but $x \notin \{x\}^c$ and $x \in \{y\}^c$ but $y \notin \{y\}^c$. Hence X is $g^*\omega\alpha$-T_1-space.*
Conversely, let x be any arbitrary point of X. If $y \in \{x\}^c$, then $y \neq x$. Now the space being $g^*\omega\alpha$-T_1 and y is different from x, there must exists $g^*\omega\alpha$-open set G_y such that $y \in G_y$ but $x \notin G_y$. Thus for each $y \in \{x\}^c$, there exists a $g^*\omega\alpha$-open set G_y such that $y \in G_y \subseteq \{x\}^c$. Therefore $\cup\{y : y \neq x\} \subseteq \cup\{G_y : y \neq x\} \subseteq \{x\}^c$ which implies that $\{x\}^c \subseteq \cup\{G_y : y \neq x\} \subseteq \{x\}^c$. Therefore $\{x\}^c = \cup\{G_y : y \neq x\}$. Since, G_y is $g^*\omega\alpha$-open set in X and the union of $g^*\omega\alpha$-open set is again $g^*\omega\alpha$-open in X, so $\{x\}^c$ is $g^*\omega\alpha$-open in X. Hence $\{x\}$ is $g^*\omega\alpha$-closed in X.

Corollary 3.16. *A space X is $g^*\omega\alpha$-T_1 if and only if every finite subset of X is $g^*\omega\alpha$-closed.*

Theorem 3.17. *Let $f : X \to Y$ be bijective and $g^*\omega\alpha$-open. If X is $g^*\omega\alpha$-T_1 and $T_{g^*\omega\alpha}$-space then, Y is $g^*\omega\alpha$-T_1-space.*
Proof: *Let y_1 and y_2 be any two distinct points of Y. Since f is bijective, then there exist distinct points x_1 and x_2 of X such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then there exist $g^*\omega\alpha$-open sets G and H such that $x_1 \in G$, $x_2 \notin G$ and $x_1 \notin H$, $x_2 \in H$. Therefore $y_1 = f(x_1) \in f(G)$ but $y_2 = f(x_2) \notin f(G)$ and $y_2 = f(x_2) \in f(H)$ and $y_1 = f(x_1) \notin f(H)$. As X is $T_{g^*\omega\alpha}$-space, G and H are open sets in X and as f is $g^*\omega\alpha$-open, f(G) and f(H) are $g^*\omega\alpha$-open subsets of Y. Thus, there exist $g^*\omega\alpha$-open sets such that $y_1 \in f(G)$, $y_2 \notin f(G)$ and $y_2 \in f(H)$, $y_1 \notin f(H)$. Hence Y is $g^*\omega\alpha$-T_1-space.*

Theorem 3.18. *Let $f : X \to Y$ be $g^*\omega\alpha$-irresolute and injective. If Y is $g^*\omega\alpha$-T_1 then X is $g^*\omega\alpha$-T_1.*
Proof: *Let $x, y \in Y$ such that $x \neq y$. Then there exist $g^*\omega\alpha$-open sets U and V in Y such that $f(x) \in U$, $f(y) \in V$ and $f(x) \notin V$, $f(y) \notin U$. Then $x \in f^{-1}(U)$, $y \in f^{-1}(V)$ and $x \notin f^{-1}(V)$, $y \notin f^{-1}(U)$, since f is $g^*\omega\alpha$-irresolute. Hence X is $g^*\omega\alpha$-T_1 space.*

Theorem 3.19. *If $f : X \to Y$ is $g^*\omega\alpha$-continuous, injective and Y is T_1 then, X is $g^*\omega\alpha$-T_1 space.*
Proof: *For any two distinct points x_1 and x_2 in X there exist disjoint points y_1 and y_2 of Y such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since Y is T_1, there exist open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$. That is, $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Again, since f is $g^*\omega\alpha$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $g^*\omega\alpha$-open sets in X. Thus, for two distinct points x_1 and x_2 of X, there exist $g^*\omega\alpha$-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Therefore X is $g^*\omega\alpha$-T_1 space.*

Definition 3.20. *A space X is said to be $g^*\omega\alpha$-T_2 if for each pair of distinct points x, y of X, there exist disjoint $g^*\omega\alpha$-open sets U and V such that $x \in U$ and $y \in V$.*

Example 3.21. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the space (X, τ) is $g^*\omega\alpha$-T_2 space, but not $g^*\omega\alpha$-T_1 and $g^*\omega\alpha$-T_0 space.*

Theorem 3.22. *Let X be a topological space. Then X is $g^*\omega\alpha$-T_2 if and only if the intersection of all $g^*\omega\alpha$-closed neighborhood of each point of X is singleton set.*

Proof: *Let x and y be any two distinct points of X. Since, X is $g^*\omega\alpha$-T_2 there exist $g^*\omega\alpha$-open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \phi$. Since, $G \cap H = \phi$, $x \in G \subseteq$ X-H, so X-H is $g^*\omega\alpha$-closed neighborhood of x which does not contains y. Thus y does not belong to the intersection of all $g^*\omega\alpha$-closed neighborhood of x. Since y is arbitrary, the intersection of all $g^*\omega\alpha$-closed neighborhood of x is the singleton $\{x\}$.*

Conversely, let $\{x\}$ be the intersection of all $g^*\omega\alpha$-closed neighborhood of an arbitrary point $x \in X$ and y be a point of X different from x. Since y does not belong to the intersection, there exists $g^*\omega\alpha$-closed neighborhood N of x, such that $y \notin N$. Since, N is $g^*\omega\alpha$ neighborhood of x there exists $g^*\omega\alpha$-open set G such that $x \in G \subseteq N$. Thus G and X-N are $g^*\omega\alpha$-open sets such that $x \in G$, $y \in$ X-N and $G \cap$ (X-N) = ϕ. Hence X is $g^*\omega\alpha$-T_2 space.

Theorem 3.23. *If $f : X \to Y$ is an injective, $g^*\omega\alpha$-irresolute and Y is $g^*\omega\alpha$-T_2 then, X is $g^*\omega\alpha$-T_2.*

Proof: *Let x_1 and x_2 be any two distinct points in X. So, $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$ as f is bijective. Then y_1 and $y_2 \in$ Y such that $y_1 \neq y_2$. Since, Y is $g^*\omega\alpha$-T_2, there exist $g^*\omega\alpha$-open sets G and H such that $y_1 \in G$, $y_2 \in H$ and $G \cap H = \phi$. Then $f^{-1}(G)$ and $f^{-1}(H)$ are $g^*\omega\alpha$-open sets of X as f is $g^*\omega\alpha$-irresolute. Now $f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$. Then $y_1 \in G$ implies $f^{-1}(y_1) \in f^{-1}(G)$ and $x_1 \in f^{-1}(G)$, $y_2 \in H$ that is, $f^{-1}(y_2) \in f^{-1}(H)$ so $x_2 \in f^{-1}(H)$. Thus for every pair of distinct points x_1 and x_2 of X, there exist disjoint $g^*\omega\alpha$-open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$, $x_2 \in f^{-1}(H)$. Hence X is $g^*\omega\alpha$-T_2 space.*

Theorem 3.24. *If $f : X \to Y$ is $g^*\omega\alpha$-continuous, injective and Y is T_2 then X is $g^*\omega\alpha$-T_2 space.*

Proof: *For any two distinct points x_1 and x_2 of X, there exist disjoint points y_1 and y_2 of Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is T_2, there exist disjoint open sets U and V in Y such that $y_1 \in U$ and $y_2 \in V$, that is $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Again, since f is $g^*\omega\alpha$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $g^*\omega\alpha$-open sets in X. Further $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\phi) = \phi$. Thus for two disjoint points x_1 and x_2 of X, there exist disjoint $g^*\omega\alpha$-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Therefore X is $g^*\omega\alpha$-T_2 space.*

Theorem 3.25. *The following properties are equivalent for any topological space X:*
(a) $g^*\omega\alpha$-T_2 space
(b) for each $x \neq y$, there exists $g^*\omega\alpha$-open set U such that $x \in U$ and $y \notin g^*\omega\alpha$-cl(U)
(c) for each $x \in X$, $\{x\} = \cap\{g^*\omega\alpha\text{-cl}(U)$: U is $g^*\omega\alpha$-open in X and $x \in U\}$.

Proof: *$(a) \Rightarrow (b)$: Let $x \in X$ and $x \neq y$, then there exist disjoint $g^*\omega\alpha$-open sets U and V such that $x \in U$ and $y \in V$. Then $X - V$ is $g^*\omega\alpha$-closed. Since $U \cap V = \phi$, $U \subseteq X - V$. Therefore $g^*\omega\alpha$-cl(U) $\subseteq g^*\omega\alpha\text{-}cl(X - V) = X - V$. Now $y \notin X - V$ implies that $y \notin g^*\omega\alpha$-cl(U).*
$(b) \Rightarrow (c)$: For each $x \neq y$, there exists $g^*\omega\alpha$-open set U such that $x \in U$ and $y \notin g^*\omega\alpha$-cl(U). So $y \notin \cap\{g^*\omega\alpha$-cl(U): U is $g^*\omega\alpha$-open in X, $x \in U\} = \{x\}$.
$(c) \Rightarrow (a)$: Let $x, y \in X$ and $x \neq y$. Then by hypothesis, there exists $g^*\omega\alpha$-open set U such that $x \in U$ and $y \notin g^*\omega\alpha$-cl(U). This implies that, there exists $g^*\omega\alpha$-closed set V such that $y \notin V$. Therefore $y \in X - V$ and $X - V$ is $g^*\omega\alpha$-open set. Thus, there exist two disjoint $g^*\omega\alpha$-open sets U and $X - V$ such that $x \in U$ and $y \in X - V$. Therefore X is $g^*\omega\alpha$-T_2 space.

4 $g^*\omega\alpha$-Normal Spaces

In this section, the concept of $g^*\omega\alpha$-normal spaces are introduced and obtained their characterizations.

Definition 4.1. *A space X is said to be a $g^*\omega\alpha$-normal if for any pair of disjoint $g^*\omega\alpha$-closed sets A and B in X, there exist disjoint open sets U and V in X such that $A \subseteq U$, $B \subseteq V$.*

Remark 4.2. *Every $g^*\omega\alpha$-normal space normal.*
However, the converse is not true in general as seen from the following example.

Example 4.3. *Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the space (X, τ) is normal but not $g^*\omega\alpha$-normal.*

Remark 4.4. *If X is normal and $T_{g^*\omega\alpha}$-space then X is $g^*\omega\alpha$-normal.*

Theorem 4.5. *The following are equivalent for a space X:*
(a) X is normal
(b) for any disjoint closed sets A and B, there exist disjoint $g^*\omega\alpha$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$

(c) for any closed set A and any open set V containing A, there exists $g^*\omega\alpha$-open set U in X such that $A \subseteq U \subseteq cl(U) \subseteq V$.

Proof: *(a) $\Rightarrow$ (b): Follows from [9].*

(b) $\Rightarrow$ (c): Let A be a closed and V be an open set containing A. Then A and X-V are disjoint closed sets in X. Then there exist $g^*\omega\alpha$-open sets U and W such that $A \subseteq U$ and X-V $\subseteq$ W. Since X-V is closed, X-V is $g^*\omega\alpha$-closed [9]. We have, X-V $\subseteq$ int(W) and U $\cap$ int(W) = ϕ and so, cl(U) $\cap$ int(W) = ϕ and hence $A \subseteq U \subseteq cl(U) \subseteq$ X-int(W) $\subseteq$ V.

(c) $\Rightarrow$ (a): Let A, B be disjoint closed sets in X. Then A $\subseteq$ X-B and X-B is open. Then there exists $g^*\omega\alpha$-open set G of X such that A $\subseteq$ G $\subseteq$ cl(G) $\subseteq$ X-B. Then A is $g^*\omega\alpha$-closed by [9]. We have A $\subseteq$ int(G), put U = int(G) and V = int(X-G). Then U and V are disjoint open sets of X such that A $\subseteq$ U and B $\subseteq$ V. Therefore X is normal.

Theorem 4.6. *The following statements are equivalent for a topological space X:*

(a) X is $g^*\omega\alpha$-normal

(b) for each closed set A and for each open set U containing A, there exists $g^*\omega\alpha$-open set V containing A such that $g^*\omega\alpha$-cl(V) $\subseteq$ U

(c) for each pair of disjoint closed sets A and B there exists $g^*\omega\alpha$-open set U containing A such that $g^*\omega\alpha$-cl(U) $\cap B = \phi$.

Proof *$(a) \Rightarrow (b)$: Let A be closed and U be an open set containing A. Then $A \cap (X \setminus U) = \phi$ and therefore disjoint closed sets in X. Since X is $g^*\omega\alpha$-normal, there exist disjoint $g^*\omega\alpha$-open sets V and W such that $A \subseteq U$, $X - U \subseteq W$, that is $X - W \subseteq U$. Now $V \cap W = \phi$, implies $V \subseteq X - W$. Therefore $g^*\omega\alpha\text{-}cl(V) \subseteq g^*\omega\alpha\text{-}cl(X - W) = X - W$ since $X - W$ is $g^*\omega\alpha$-closed. Thus, $A \subseteq V \subseteq g^*\omega\alpha\text{-}cl(V) \subseteq X - W \subseteq U$. That is $A \subseteq V \subseteq g^*\omega\alpha\text{-}cl(V) \subseteq U$.*

$(b) \Rightarrow (c)$: Let A and B be disjoint closed sets in X then $A \subseteq X - B$ and $X - B$ is an open set containing A. Then there exists $g^*\omega\alpha$-open set U such that $A \subseteq U$ and $g^*\omega\alpha\text{-}cl(U) \subseteq X - B$, which implies $g^*\omega\alpha\text{-}cl(U) \cap B = \phi$.

$(c) \Rightarrow (a)$: Let A and B be disjoint closed sets in X. Then there exists $g^*\omega\alpha$-open set U such that $A \subseteq U$ and $g^*\omega\alpha\text{-}cl(U) \cap B = \phi$ or $B \subset X - g^*\omega\alpha\text{-}cl(U)$. Now U and $X - g^*\omega\alpha\text{-}cl(U)$ are disjoint $g^*\omega\alpha$-open sets of X such that $A \subseteq U$ and $B \subseteq X - g^*\omega\alpha\text{-}cl(U)$. Hence X is $g^*\omega\alpha$-normal.

Theorem 4.7. *If X is normal and F $\cap$ A = ϕ where F is $\omega\alpha$-closed and A is $g^*\omega\alpha$-closed then there exist open sets U and V such that F $\subseteq$ U and A $\subseteq$ V.*

Proof: *Let X be a normal and $F \cap A = \phi$. Since, F is $\omega\alpha$-closed and A is $g^*\omega\alpha$-closed such that $A \subseteq X - F$ and $X - F$ is $\omega\alpha$-open. Therefore $cl(A) \subseteq X - F$ implies that $cl(A) \cap F = \phi$. Now F is closed, so F and cl(A) are disjoint closed sets in X. As X is a normal, there exist disjoint open sets U and V of X such that $F \subseteq U$ and $cl(A) \subseteq V$.*

Theorem 4.8. *If X is $g^*\omega\alpha$-normal and Y is $g^*\omega\alpha$-closed subset of X then, the subspace Y is also $g^*\omega\alpha$-normal.*

Proof: *Let A and B be any two disjoint $g^*\omega\alpha$-closed sets in Y, then A and B are $g^*\omega\alpha$-closed sets in X by [9]. Since X is $g^*\omega\alpha$-normal, there exist disjoint open sets U and V in X such that A $\subseteq$ U, B $\subseteq$ V. Therefore U$\cap$Y and V$\cap$Y are disjoint open subsets of the subspace Y such that A $\subseteq$ U$\cap$Y and B $\subseteq$ V$\cap$Y. Hence the subspace Y is $g^*\omega\alpha$-normal.*

Remark 4.9. *The property of being $g^*\omega\alpha$-normal is closed hereditary.*

Theorem 4.10. *If f : X $\to$ Y is pre $g^*\omega\alpha$-closed, continuous injective and Y is $g^*\omega\alpha$-normal then, X is $g^*\omega\alpha$-normal.*

Proof: *Let A and B be disjoint $g^*\omega\alpha$-closed sets in X. Since, f is pre $g^*\omega\alpha$-closed, f(A) and f(B) are disjoint $g^*\omega\alpha$-closed sets in Y. Again, since Y is $g^*\omega\alpha$-normal there exist disjoint open sets U and V such that f(A) $\subseteq$ U, f(B) $\subseteq$ V. Thus A $\subseteq$ f^{-1}(U), B $\subseteq$ f^{-1}(V) and f^{-1}(U) $\cap$ f^{-1}(V) = ϕ. Then f^{-1}(U) and f^{-1}(V) are open sets in X as f is continuous. Hence X is $g^*\omega\alpha$-normal.*

Theorem 4.11. *If $f : X \to Y$ is $g^*\omega\alpha$-irresolute, bijective, open map from a $g^*\omega\alpha$-normal space X on to a space Y then Y is $g^*\omega\alpha$-normal.*

Proof: *Let A and B be two disjoint $g^*\omega\alpha$-closed sets in Y. Since, f is $g^*\omega\alpha$-irresolute and bijective, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $g^*\omega\alpha$-closed sets in X. As X is $g^*\omega\alpha$-normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$, that is $A \subseteq f(U)$ and $B \subseteq f(V)$. Then $f(U)$ and $f(V)$ are open sets in Y and $f(U) \cap f(V) = \phi$. Thus Y is $g^*\omega\alpha$-normal.*

5 $g^*\omega\alpha$-Regular Spaces

The concept of $g^*\omega\alpha$-regular spaces and their properties are studied in this section.

Definition 5.1. *A topological space X is said to be a $g^*\omega\alpha$-regular if for each $g^*\omega\alpha$-closed set F and each point $x \notin F$ there exist disjoint open sets U and V in X such that $x \in U$ and $F \subseteq V$.*

Remark 5.2. *Every $g^*\omega\alpha$-regular space is regular.*
However, the converse need not be true as seen from the following example.

Example 5.3. *From Example 4.3, the space (X, τ) is regular but not $g^*\omega\alpha$-regular.*

Theorem 5.4. *Every $g^*\omega\alpha$-regular T_0-space is $g^*\omega\alpha$-T_2.*
Proof: *Let x and y be any two points in X such that $x \neq y$. Let V be an open set which contains x but not y. Then, $X - V$ is a closed set containing y but not x. Then there exist disjoint open sets U and W such that $x \in U$ and $X - V \subset W$. Since $y \in X - V$, $y \in W$. Thus for $x, y \in X$ with $x \neq y$ there exist disjoint $g^*\omega\alpha$-open sets U and W such that $x \in U$ and $y \in W$. Hence X is $g^*\omega\alpha$-T_2 space.*

Theorem 5.5. *In a topological spaces X, the following properties are equivalent:*
(a) X is $g^*\omega\alpha$-regular space
(b) for each point $x \in X$ and each $g^*\omega\alpha$-open neighborhood A of X, there exists open neighborhood V of X such that cl(V) $\subseteq$ A.
Proof: *(a) $\Rightarrow$ (b): Suppose X is $g^*\omega\alpha$-open neighborhood of x. Then there exists $g^*\omega\alpha$-open set G such that $x \in G \subseteq A$. Since $X - G$ is $g^*\omega\alpha$-closed and $x \notin X - G$. By hypothesis there exist open sets U and V such that $X - G \subseteq U$, $x \in V$ and $U \cap V = \phi$ and so $V \subseteq X - U$. Now cl(V) $\subseteq$ cl($X - U$) $= X - U$ and $X - G \subseteq U$ implies $X - U \subseteq G \subseteq A$. Therefore cl(V) $\subseteq$ A.*
(b) $\Rightarrow$ (a): Let F be a closed set in X with $x \notin F$. Then $x \in X - F$ and $X - F$ is $g^*\omega\alpha$-open and so $X - F$ is $g^*\omega\alpha$-neighborhood of X. By hypothesis, there exists open neighborhood V of X such that $x \in V$ and cl(V) $\subseteq X - F$, which implies $F \subseteq X - cl(V)$. Then $X - cl(V)$ is an open set containing F and $V \cap (X - cl(V)) = \phi$. Therefore X is $g^*\omega\alpha$-regular.

Theorem 5.6. *If X is $g^*\omega\alpha$-regular and Y is open, $g^*\omega\alpha$-closed subspace of X, then the subspace Y is $g^*\omega\alpha$-regular.*
Proof: *Let A be $g^*\omega\alpha$-closed subspace of Y and $y \notin A$ then A is $g^*\omega\alpha$-closed in X. Since X is $g^*\omega\alpha$-regular there exist open sets U and V in X such that $y \in U$ and $A \subseteq V$. Therefore U$\cap$Y and V$\cap$Y are disjoint open sets of the subspace Y, such that $y \in U\cap Y$ and $A \subseteq V\cap Y$. Hence Y is $g^*\omega\alpha$-regular.*

Theorem 5.7. *Let $f : X \to Y$ be bijective, $g^*\omega\alpha$-irresolute and open. If X is $g^*\omega\alpha$-regular then Y is also $g^*\omega\alpha$-regular.*
Proof: *Let F be $g^*\omega\alpha$-closed set of Y and $y \notin F$. Since f is $g^*\omega\alpha$-irresolute, f^{-1}(F) is $g^*\omega\alpha$-closed in X. Let f(x) = y, so x = f^{-1}(y) and $x \notin f^{-1}$(F). Again, X is $g^*\omega\alpha$-regular there exist open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq V$, U$\cap$V = ϕ. Since, f is open and bijective, so $y \in$ f(U), F $\subseteq$ f(V) and f(U) $\cap$ f(V) = f(U $\cap$ V) = f(ϕ) = ϕ. Hence Y is $g^*\omega\alpha$-regular.*

Theorem 5.8. *If $f : X \to Y$ is bijective, pre $g^*\omega\alpha$-closed and open map from a space X in to a $g^*\omega\alpha$-regular space Y. If X is $T_{g^*\omega\alpha}$ space then X is $g^*\omega\alpha$-regular.*
Proof: *Let $x \in X$ and F be a $g^*\omega\alpha$-closed set in X with $x \notin F$. Since X is $T_{g^*\omega\alpha}$ space so, F is closed in X. Then f(F) is $g^*\omega\alpha$-closed with f(x) $\notin$ f(F) in Y as f is pre $g^*\omega\alpha$-closed. Again, since Y is $g^*\omega\alpha$-regular there exist open sets U and V such that f(x) $\in$ U and f(F) $\subseteq$ V. Therefore $x \in f^{-1}$(U) and F $\subseteq f^{-1}$(V). Hence X is $g^*\omega\alpha$-regular space.*

Theorem 5.9. *Every subspace of a $g^*\omega\alpha$-regular space is $g^*\omega\alpha$-regular.*
Proof: *Let Y be subspace of a $g^*\omega\alpha$-regular space X. Let $x \in Y$ and F be a $g^*\omega\alpha$-closed set in Y such that $x \notin F$. Then there exists $g^*\omega\alpha$-closed set A of X with F = Y $\cap$ A and $x \notin A$. Therefore, we have $x \in X$, A is $g^*\omega\alpha$-closed in X such that $x \notin A$. Since, X is $g^*\omega\alpha$-regular, there exist open sets G and H such that $x \in G$, $A \subseteq H$ and G $\cap$ H = ϕ. Note that Y $\cap$ G and Y $\cap$ H are open sets in Y. Also $x \in G$ and $x \in Y$ which implies $x \in Y \cap G$ and $A \subseteq H$ implies $Y \cap G \subseteq Y \cap H$, $F \subseteq Y \cap H$. Also (Y $\cap$ G) $\cap$ (Y $\cap$ H) = ϕ. Hence Y is $g^*\omega\alpha$-regular space.*

Theorem 5.10. *Let $f : X \to Y$ be continuous, $g^*\omega\alpha$-closed, surjective and open map. If X is regular then Y is also regular.*
Proof: *Let $y \in Y$ and V be an open set containing y in Y. Let x be a point of X such that y = f(x). Since, X is regular and f is continuous there exists open set U such that $x \in U \subseteq$ cl(U) $\subseteq f^{-1}$(V). Hence $y \in$ f(U) $\subseteq$ f(cl(U)) $\subseteq$ V. Again, since f is $g^*\omega\alpha$-closed map, f(cl(U)) is $g^*\omega\alpha$-closed set contained in the open set V. Hence cl(f(cl(U))) $\subseteq$ V. Therefore $y \in$ f(U) $\subseteq$ f(cl(U)) $\subseteq$ cl(f(cl(U))) $\subseteq$ V. This implies $y \in$ f(U) $\subseteq$ cl(f(U)) $\subseteq$ V and f(U) is open. Hence Y is regular.*

Theorem 5.11. *If $f : X \to Y$ is $g^*\omega\alpha$-irresolute, open, bijective and X is $g^*\omega\alpha$-regular then, Y is $g^*\omega\alpha$-regular.*
Proof: *Let F be a $g^*\omega\alpha$-closed set in Y and $y \notin F$. Take $y = f(x)$ for some $x \in X$. Since, f is $g^*\omega\alpha$-irresolute, $f^{-1}(F)$ is $g^*\omega\alpha$-closed in X and $x \neq f^{-1}(F)$. Then there exist disjoint open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq V$, that is $y = f(x) \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Therefore Y is $g^*\omega\alpha$-regular.*

Theorem 5.12. *If $f: X \to Y$ be pre $g^*\omega\alpha$-open, closed, injective and Y is $g^*\omega\alpha$-regular then, X is $g^*\omega\alpha$-regular.*
Proof: *Let F be a $g^*\omega\alpha$-closed set in X and $x \notin F$. Since, f is pre $g^*\omega\alpha$-closed, $f(F)$ is $g^*\omega\alpha$-closed in Y such that $f(x) \notin f(F)$. Now Y is $g^*\omega\alpha$-regular, there exist open sets G and H such that $f(x) \in G$ and $f(H) \subseteq H$. his implies that $x \in f^{-1}(G)$ and $F \subseteq f^{-1}(H)$. Further $f^{-1}(G) \cap f^{-1}(H) = \phi$. Hence X is $g^*\omega\alpha$-regular.*

6 Conclusion

The research in topology over last two decades has reached a high level in many directions. By researching generalizations of closed sets, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, $g^*\omega\alpha$-separation axioms are defined by using $g^*\omega\alpha$-closed sets will have many possibilities of applications in digital topology and computer graphics.

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Some Existence results for implicit fractional differential equations with impulsive conditions

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Abstract

In this paper, we investigate the existence of solutions for implicit impulsive fractional order differential equations with non-local conditions. An example is included to prove the applicability of the results.

Keywords: Existence, Implicit Impulsive Fractional Differential Equations, Non-local Condition.

2010 MSC: 26A23, 34G20.

1 Introduction

The theory of fractional differential equations is a new branch of mathematics by valuable tools in the modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc.(For details [1–4, 6, 11–14, 18–21]).

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Benchohra *et al* [7], Lakshmikantham *et al* [17], and Samoilenko and Perestyuk [22], K.Balachandran and J.Y.Park [5] and the references therein [15, 16].

Benchohra et al. studied the following Fractional Differential Equations Caputo's derivative:

In [8], u is bounded on J, $t \in J = [0, \infty)$ and $1 < \alpha \leq 2$.

$$^{c}D^{\alpha}u(t) = f(t, u(t), ^{c}D^{\alpha-1}u(t)),\ u(0) = u_0$$

In [9], The existence results for nonlinear implicit fractional-order differential equations given by

$$^{c}D^{\alpha}y(t) = f(t, y(t), ^{c}D^{\alpha}y(t)), y(0) = y_0, t \in J = [0, T],\ 0 < \alpha \leq 1.$$

Inspiration by the above works, we study the existence of solutions for the implicit fractional order differential equations with impulsive and nonlocal conditions of the form

$$^{c}D^{\alpha}y(t) = f(t, y(t), ^{c}D^{\alpha}y(t)), \quad t \in J' := J \backslash \{t_1, ..., t_m\}, J = [0, T], 0 < \alpha \leq 1. \tag{1}$$

$$y(t_k^{+}) = y(t_k^{-}) + y_k,\ k = 1, 2, ..., m \quad y_k \in X \tag{2}$$

$$y(0) = y_0 - \eta(t), \tag{3}$$

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where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is a given function, $\eta : C \to \mathbb{X}$ is continuous, and $y_0 \in \mathbb{X}$ and t_k satisfy $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$.

In this paper is planned as shadows. Section 2 has definitions and elementary results of the fractional calculus. In section 3, implicit impulsive fractional differential equations is attained and proved the theorems on the existence and uniqueness of a solution to the problem (1.1 - 1.3). In section 4, an illustrative example is provided in support of the results of a problem (1.1 - 1.3).

2 Preliminaries

In this section, we introduce notations, definition and preliminary facts. We introduce the Banach space $PC(J, X) = \{x : J \to X : x \in C(t_k, t_{k+1}], X\}, k = 0, 1, 2, ..., m$ and their exist $x(t_k^-)$ and $x(t_k^+), k = 0, 1, 2, .., m$ with $x(t_k^-) = x(t_k)$ with the norm $||x||_{PC} := sup\{||x(t)|| : t \in J\}$.

Definition 2.1. *The fractional order integral of the function $h \in L^1([0, T], \mathbb{X}_+)$ of order $\alpha \in \mathbb{X}_+$ is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function.

Definition 2.2. *For a function h given on the interval $[0, T]$, the Caputo fractional order derivative of order α of h, is defined by*

$$({}^cD^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = |\alpha| + 1$, $|\alpha|$ denoted the integral part of real number α, provided $h^{(n)}(t)$ exists.

Lemma 2.1. *Let a function $f(t, u, v) : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ be continuous. Then the problem (1.1)-(1.3) is equivalent to the problem:*

$$y(t) = \begin{cases} y_0 - \eta(t) + I^\alpha g(t), \ for\ t \in [0, t_1] \\ y_0 - \eta(t) + y_1 + I^\alpha g(t), \ for\ t \in (t_1, t_2] \\ y_0 - \eta(t) + y_1 + y_2 + I^\alpha g(t), \ for\ t \in (t_2, t_3] \\ \vdots \\ y_0 - \eta(t) + \sum_{i=1}^{m} y_i + I^\alpha g(t), \ for\ t \in (t_m, T] \end{cases} \tag{2.1}$$

where $g \in C(J, \mathbb{X})$ satisfies the functional equation

$$g(t) = \begin{cases} f(t, y_0 - \eta(t) + I^\alpha g(t), g(t)), \ for\ t \in [0, t_1] \\ f(t, y_0 - \eta(t) + y_1 + I^\alpha g(t), g(t)), \ for\ t \in (t_1, t_2] \\ f(t, y_0 - \eta(t) + y_1 + y_2 + I^\alpha g(t), g(t)), \ for\ t \in (t_2, t_3] \\ \vdots \\ f(t, y_0 - \eta(t) + \sum_{i=1}^{m} y_i + I^\alpha g(t), g(t)), \ for\ t \in (t_m, T] \end{cases}$$

Proof. If

$${}^cD^\alpha y(t) = g(t)$$

then

$$I^\alpha\, {}^cD^\alpha y(t) = I^\alpha g(t).$$

So we obtain for $t \in [t_0, t_1]$,

$$y(t) = y(0) - \eta(t) + I^\alpha g(t),$$
$$y(t) = y_0 - \eta(t) + I^\alpha g(t), \ for\ t \in [0, t_1]$$

For $t \in (t_1, t_2]$ we have

$$
\begin{aligned}
y(t) &= y(t_1^+) - \eta(t) - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(t, u(t), {}^c D^\alpha u(t)) du + I^\alpha g(t), \\
&= y(t_1) - \eta(t) + y_1 + I^\alpha g(t), \\
y(t) &= y_0 - \eta(t) + y_1 + I^\alpha g(t), \ for \ t \in (t_1, t_2]
\end{aligned}
$$

Similarly $t \in (t_m, T]$ we get

$$
y(t) = y_0 - \eta(t) + \sum_{i=1}^{m} y_i + I^\alpha g(t), \ for \ t \in (t_m, T]
$$

Therefore, we have

$$
y(t) = \begin{cases}
y_0 - \eta(t) + I^\alpha g(t), \ for \ t \in [0, t_1] \\
y_0 - \eta(t) + y_1 + I^\alpha g(t), \ for \ t \in (t_1, t_2] \\
y_0 - \eta(t) + y_1 + y_2 + I^\alpha g(t), \ for \ t \in (t_2, t_3] \\
\vdots \\
y_0 - \eta(t) + \sum_{i=1}^{m} y_i + I^\alpha g(t), \ for \ t \in (t_m, T]
\end{cases}
$$

The proof is completed. □

Lemma 2.2. *[10] Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact map. Then either (i) F has a fixed point in $\overline{U}$, or (ii) there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.*

Theorem 2.1. (Krasnoselkii) *Let M be a closed convex and nonempty subset of a Banach space $\mathbb{X}$. Let A and B be two operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

3 Main Results

To prove the main result we need the following assumptions :

(A_1) The function $f : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, w are continuous.

(A_2) There exist constants $K_1 > 0$ and $0 < K_2 < 1$ such that

$$
||f(t, u_1, v_1) - f(t, u_2, v_2)|| \leq K_1 ||u_1 - u_2|| + K_2 ||v_1 - v_2||
$$

for any $u_1, u_2, v_1 \& v_2 \in \mathbb{X}$ and $t \in J$.

(A_3) η is continuous, and there exists a constant $b < 1$ such that

$$
||\eta(y_1) - \eta(y_2)|| \leq b ||y_1 - y_2||
$$

$$
\forall \ y_1, y_2 \in \mathbb{X},
$$

(A_4) The function $f : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, w are continuous and $\eta : C \to \mathbb{X}$ is continuous.

Theorem 3.2. *Assume that the assumptions $(A_1) - (A_3)$ holds. If*

$$
\sum_{i=1}^{m} x_i + \frac{\eta K_1 T^\alpha}{(1 - K_2)\Gamma(\alpha + 1)} < 1 \tag{3.1}
$$

then there exists a unique solution for $(1.1) - (1.3)$ on J

Proof. Define the operator $M : C(J, \mathbb{X}) \to C(J, \mathbb{X})$ by

The formula of solutions for equation $(1.1) - (1.3)$ should be

$$M(y)(t) = \begin{cases} y_0 - \eta(t) + I^\alpha g(t), \ for\ t \in [0, t_1] \\ y_0 - \eta(t) + y_1 + I^\alpha g(t), \ for\ t \in (t_1, t_2] \\ y_0 - \eta(t) + y_1 + y_2 + I^\alpha g(t), \ for\ t \in (t_2, t_3] \\ \vdots \\ y_0 - \eta(t) + \sum_{i=0}^{m} y_i + I^\alpha g(t), \ for\ t \in (t_m, T] \end{cases} \tag{3.2}$$

$$\text{where} \quad g(t) = f(t, y(t), g(t)), \quad g \in C(J, \mathbb{X}).$$

In general case $t \in (t_m, T])$:

$$M(y)(t) = y_0 - \eta(t) + \sum_{i=1}^{m} y_i + I^\alpha g(t), \quad \text{for} \quad t \in (t_m, T]) \tag{3.3}$$

where

$$g(t) = f(t, y(t), g(t)), \quad g \in C(J, \mathbb{X}).$$

Clearly, the fixed points of operation M are solutions of problem $(1.1) - (1.3)$.

Let $y_1, y_2 \in C(J, \mathbb{X})$. Then for $t \in J$, we have

$$(My_1)(t) - (My_2)(t) = \eta(y_1) - \eta(y_2) + \sum_{i=1}^{m} y_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(g(s) - h(s))ds,$$

where $g, h \in C(J, \mathbb{X})$ be such that

$$g(t) = f(t, y_1(t), g(t)),$$
$$h(t) = f(t, y_2(t), h(t)),$$

Then, for $t \in J$

$$||(My_1)(t) - (My_2)(t)|| = ||\eta(y_1) - \eta(y_2)|| + \sum_{i=1}^{m} ||y_i|| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}||(g(s) - h(s))||ds \tag{3.4}$$

By (A_2) we have

$$\begin{aligned} ||g(t) - h(t)|| &= ||f(t, y_1(t), g(t)) - f(t, y_2(t), h(t))|| \\ &\le K_1||y_1(t) - y_2(t)|| + K_2||y_1(t) - y_2(t)|| \\ &\le \frac{K_1}{1 - K_2}||y_1(t) - y_2(t)|| \end{aligned}$$

Therefore (3.4)

$$\begin{aligned} ||(My_1)(t) - (My_2)(t)|| &\le \sum_{i=1}^{m} ||y_i|| + \frac{b||y_1 - y_2||K_1}{(1 - K_2)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}||y_1(t) - y_2(t)||ds, \\ &\le \sum_{i=1}^{m} ||y_i|| + \frac{bK_1 T^\alpha}{(1 - K_2)\Gamma(\alpha + 1)}||y_1 - y_2||_\infty. \end{aligned}$$

Thus

$$||My_1 - My_2||_\infty \le \sum_{i=1}^{m} ||y_i|| + \frac{bK_1 T^\alpha}{(1 - K_2)\Gamma(\alpha + 1)}||y_1 - y_2||_\infty.$$

By (3.2), the operator M is a continuous. Hence, by Banach's contraction principle, M has a unique fixed point which is a unique solution of the problem $(1.1) - (1.3)$. The proof is completed. □

Theorem 3.3. *Assume the $(A_1)-(A_3)$. Then the problem (1.1)-(1.3) has at least one solution on $[0,T]$.*

Proof. Choose

$$\frac{||y||}{y_0-\eta(t)+\sum_{i=1}^{m}||y_i||+\frac{\varphi(||x||)||p||_{L_1}}{\Gamma(\alpha+1)}T^{\alpha}}\leq 1.$$

Case: (i) M maps bounded sets (balls) into bounded sets in $C([0,T],\mathbb{X})$.

For a positive number r, let $B_r=\{x\in C([0,T],\mathbb{X}):||x||\leq r\}$ be a bounded ball in $C([0,T],\mathbb{X})$. Then for $t\in(t_m,T]$ we have

$$\begin{aligned}||M(y)(t)||&\leq y_0-\eta(t)+\sum_{i=1}^{m}||y_i||+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}||g(s)||ds,\\&\leq y_0-\eta(t)+\sum_{i=1}^{m}||y_i||+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}||f(t,y(t),g(t))||ds,\\&\leq y_0-\eta(t)+\sum_{i=1}^{m}||y_i||+\frac{\varphi(||x||)||p||_{L_1}}{\Gamma(\alpha+1)}T^{\alpha}\end{aligned}$$

Consequently

$$||M(y)||\leq y_0-\eta(t)+\sum_{i=1}^{m}||y_i||+\frac{\varphi(r)||p||_{L_1}}{\Gamma(\alpha+1)}T^{\alpha}.$$

Case: (ii) M maps bounded sets (balls) into equicontinuous sets in $C([0,T],\mathbb{X})$.

Let $\sup_{(t,x)\in[0,T]\times B_r}||f(t,u,v)||=f^*<\infty,\quad \mu_1,\mu_2\in[0,T]$ with $\mu_1,\mu_2\in(t_m,T]$ and $x\in B_r$. Then we have

$$\begin{aligned}||M(y)(\mu_1)-M(y)(\mu_2)||&=y_0-\eta(t)+\sum_{i=1}^{m}||y_i||+\frac{1}{\Gamma(\alpha)}\int_0^{\mu_1}(\mu_1-s)^{\alpha-1}||f(t,y(t),g(t))||ds\\&+\frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}(\mu_2-s)^{\alpha-1}||f(t,y(t),g(t))||ds,\\&\leq y_0-\eta(t)+\sum_{i=1}^{m}||y_i||+\frac{f^*}{\Gamma(\alpha+1)}||\mu_2^{\alpha}-\mu_1^{\alpha}||\end{aligned}$$

Obvisously the right-hand side of the above inequality tends to zero independently of $x\in B_r$ as $\mu_2-\mu_1\to 0$. As M satisfies the above assumptions, therefore it follows by the Arzela-Ascoli theorem that $M:C([0,T],\mathbb{X})\to C([0,T],\mathbb{X})$ is completely continuous. Let y be a solution. Then, for $t\in[0,T]$ and following the similar computations as in the first step, we have

$$||y||\leq y_0-\eta(t)+\sum_{i=1}^{m}||y_i||+\frac{\varphi(||x||)||p||_{L_1}}{\Gamma(\alpha+1)}T^{\alpha}.$$

Consequently, we have

$$\frac{||y||}{y_0-\eta(t)+\sum_{i=1}^{m}||y_i||+\frac{\varphi(||x||)||p||_{L_1}}{\Gamma(\alpha+1)}T^{\alpha}}\leq 1.$$

There exist N^* such that $||x||\neq N^*$. Let us set c

$$U=\{x\in C([0,T],\mathbb{X}):||x||<N^*\}.$$

Note that the operator $M:\overline{U}\to C([0,T],\mathbb{X})$ is continuous and completely continuous. Consequently, by the nonlinear alternative of Lerary-Schauder type, we deduce that M has fixed point $y\in\overline{U}$ which is a solution of the problem $(1.1)-(1.3)$. The proof is completed. □

Theorem 3.4. *(Existence results via Krasnoselskii's fixed point theorem) Assume that* $|f(t,u,v)| \leq \mu||u-v||, \quad \mu \in C([0,T],\mathbb{X}^+)$. *Then the problem (1.1-(1.3) has at least one solution on* $[0,T]$ *if*

$$L\sum_{i=1}^{m}||y_i|| < 1 \tag{3.5}$$

Proof. Choose a suitable constant r as

$$r \geq \frac{(\mu-b)T^\alpha}{\Gamma(\alpha+1)}||u-v|| + y_0 + \sum_{i=1}^{m}||y_i||$$

Define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_r = \{y \in C([0,T],\mathbb{X}) : ||y|| \leq r\}$ as

$$(\mathcal{P}y)(t) = I^\alpha g(t)$$
$$(\mathcal{Q}y)(t) = y_0 - \eta(t) + \sum_{i=1}^{m} y_i$$

For $u, v \in B_r$, we obtain

$$\begin{aligned} ||\mathcal{P}y + \mathcal{Q}y|| &\leq \frac{\mu T^\alpha}{\Gamma(\alpha+1)}||u-v|| + y_0 - b||u-v|| + \sum_{i=1}^{m}||y_i|| \\ &\leq \frac{(\mu-b)T^\alpha}{\Gamma(\alpha+1)}||u-v|| + y_0 + \sum_{i=1}^{m}||y_i|| \\ &\leq r \end{aligned}$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_r$. It follows from the assumption to gether with (3.5) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on B_r as

$$||\mathcal{P}x|| \leq \frac{(\mu-b)T^\alpha}{\Gamma(\alpha+1)}||u-v||$$

Now we prove the compactness of the operator $\mathcal{P}$.

Let $\sup_{(t,x)\in[0,T]\times B_r} ||f(t,u,v)|| = f^* < \infty, \quad \mu_1, \mu_2 \in [0,T]$ with $\mu_1, \mu_2 \in (t_m, T]$ and $x \in B_r$. Then we have

$$\begin{aligned} ||\mathcal{P}(y)(\mu_1) - \mathcal{P}(y)(\mu_2)|| &= \frac{1}{\Gamma(\alpha)}\int_0^{\mu_1}(\mu_1-s)^{\alpha-1}||f(t,y(t),g(t))||ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}(\mu_2-s)^{\alpha-1}||f(t,y(t),g(t))||ds, \\ &\leq \frac{f^*}{\Gamma(\alpha+1)}||\mu_2^\alpha - \mu_1^\alpha|| \end{aligned}$$

which is independent of y and tends to zero as $\mu_2 - \mu_1 \to 0$. Thus, $\mathcal{P}$ is equicontinuous. So $\mathcal{P}$ is relatively compact on B_r. Hence, by the Arzela-Ascoli theorem, $\mathcal{P}$ is compact on B_r. Thus all the assumations of Theorem 1 are satisfied. So the conclusion of Theorem 1 implies that the impulsive implicit fractional non-local problem (1.1)-(1.3) has at least one solution on $[0,T]$. The proof is completed. □

4 Example

Consider the following Implicit fractional differential equation with nonlocal impulsive condition of the form

$${}^cD^\alpha y(t) = \frac{1}{(t+2)^2}\left[\frac{|y(t)|}{1+|y(t)|} - \frac{|D^\alpha y(t)|}{1+|D^\alpha y(t)|}\right] \tag{4.1}$$

$$y(t_k^+) = y(t_k^-) + \frac{1}{4}, \tag{4.2}$$

$$y(0) = y_0 - \sum_{i=1}^{m} c_i y(t_i) \tag{4.3}$$

Take $J = [0,1]$. Set

$$f(t, y(t),^c D^\alpha y(t)) = \frac{1}{(t+2)^2}\left[\frac{|y(t)|}{1+|y(t)|} - \frac{|D^\alpha y(t)|}{1+|D^\alpha y(t)|}\right], t \in J^{'}.x \in X$$

Let $y_1, y_2 \in X$ and $t \in J^{'}$. Then we have

$$||f(t, y_1(t),^c D^\alpha y_1(t)) - f(t, y_2(t),^c D^\alpha y_2(t))|| \leq \frac{K_1}{4(1-K_2)}||y_1 - y_2||$$

Hence the condition $(A_1) - (A_3)$ hold. Note that $K_1 = \frac{1}{4}$ and $K_2 = \frac{1}{8}$. Then by Theorem 2, the problem equations $(1.1) - (1.3)$ has an unique solution on $[0,1]$ for the values of α satisfying equation (4.1).

5 Conclusion

We have proven an existence result for implicit fractional differential equations with impulsive condition. In the future, we will extend the results to other fractional derivatives and boundary value problems.

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Order divisor graphs of finite groups

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Abstract

For each finite group G we associate a simple undirected graph $OD(G)$, order divisor graph. We investigate the interconnection between the group theoretic properties of G and the graph theoretic properties of order divisor graph $OD(G)$. For a finite group G, we obtain the density, the girth and the diameter of $OD(G)$. Further, we obtain the relation $G \cong G'$ if and only if $OD(G) \cong OD(G')$, for every distinct finite groups G and G', and $Auto(G) \subseteq Auto(OD(G))$.

Keywords: Finite group, finite subgroups, isomorphism, automorphism, order divisor graph.

2010 MSC: 14LXX; 20D20; 20LXX.

1 Introduction

Graph theory is a branch of mathematics started by Euler [1] as early as 1736. In recent years, graph theory has found many applications in engineering and applied science, and many books have been published on graph theory and applied graph theory such as those by Chen [2], Thulasiraman and Swamy [3], Wilson and Beineke [4], Mayeda [5], and Deo citedeo. Further, the applications of graph theory are much extensive and powerful in the context of engineering science.

Algebraic graph theory is a specific branch of modern mathematics in which algebraic methods are applied to problems about graphs (Biggs [7]). It is the application of abstract algebra to graph theory. For this reason, group theory is the crowning glory of algebraic graph theory.

The concept of finite groups plays a fundamental role in theory of algebraic graphs. Few decades back the algebraic graph theory was not applicable to ordinary human activities. Now it has been used successfully for information transmission, protecting, coding and decoding with high security through public communication networks. For further studying of algebraic graph theory see [8].

The idea of a divisor graph of a finite set of positive integers was introduced by Sing and Santhosh [9]. According to these authors, a divisor graph X is an ordered pair (V, E) where V is a subset of positive integers N and $ab \in E$ if and only if either a divides b or b divides a for all $a \neq b$. They were studied basic properties of divisor graphs. In [10], Chartrand, Muntean, Saenpholophant and Zhang were studied further properties of divisor graphs. Moreover, the author Yu-Ping Tsau [11] introduced another notation $D[n]$ for a divisor graph of the set $[n] = \{1, 2, \cdots, n\}$. He studied several specific properties of $D[n]$ such as the vertex-chromatic number, the clique number, and the independence number.

Rajkumar and Devi [12] defined an undirected Co-prime graph of subgroups, denoted by $P(G)$ having all the proper subgroups of G as its vertices and two distinct vertices H and K are adjacent in $P(G)$ if and only if $|H|$ and $|K|$ are relatively prime. These authors proved that $P(G)$ is weakly χ-perfect and every simple graph is an induced subgraph of $P(Z_n)$, for some n.

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Subgroup of a group is the shadow that precedes everything in this paper, and we are using subgroups of a finite group as vertices of an order divisor graph. Orders of subgroups of a finite group play an important role in this paper, and they motivated us to define order divisor graph $OD(G)$, where G is a finite group. We hope that this order divisor graph will be a foundation for a new construction in graph theory and algebraic graph theory.

Let G be a finite group with identity e and let $S(G)$ be its set of subgroups. We associate simple undirected graph $OD(G)$ to G with vertices $S(G)$, and for distinct $H, K \in S(G)$, the vertices H and K are adjacent in $OD(G)$ if and only if either $|H|$ divides $|K|$ or $|K|$ divides $|H|$. Thus $OD(G)$ is the empty graph if and only if $|G| = 1$, and $OD(G)$ in the nonempty graph if and only if $|G| \neq 1$.

The main aim of this paper is to study the interplay of group theoretic properties of G with graph theoretic properties of $OD(G)$. This study helps illuminate the structure of $S(G)$ through the structure of $OD(G)$. For $H, K \in S(G)$, define $H \sim K$ if either $|H|$ divides $|K|$ or $|K|$ divides $|H|$. So, the relation $\sim$ is always not reflexive, not symmetric but transitive because $OD(G)$ is undirected simple graph having without multiple edges. Further the relation $\sim$ is transitive if and only of $OD(G)$ is complete.

In this paper, some properties of the order divisor graph $OD(G)$ are studied, the number of vertices in each order divisor graph, the density, the girth and diameter are found. Complete characterizations, in terms of $|G| \neq p$, are given of the cases in which the graph $OD(G)$ is never Eulerian, never a path, never a bipartite, never a star, or never a complete bipartite. Further we verify that the results $G \cong G'$ if and only if $OD(G) \cong OD(G')$ and $Auto(G) \subseteq Auto(OD(G))$ with few examples. This study investigates compositions between finite group theory, number theory and graph theory via studying properties of order division graph $OD(G)$ of a finite group G.

2 Basic Definitions and Notations

In this paper basic definitions and concepts of graph theory are briefly presented. A graph X consist of a nonempty set $V(X)$ of vertices and a set $E(X)$ of elements called edges together with a relation of a incidence which associates with each member a pair of vertices, called its ends. A graph with no loops and no multiple edges is called a simple graph whose order and size are $|V(X)|$ and $|E(X)|$ respectively.

For any vertex x in a graph X, $deg(x)$ be the number of edges with the vertex x as an end point. A graph in which all vertices have the same degree is called a regular graph. A graph X is called connected if there is a path between any two distinct vertices in X. A graph X is complete if every two distinct vertices in X are adjacent. A complete graph with n vertices is denoted by K_n.

A graph X is called planar if it can drawn in the plane so that its edges intersect only at their ends. Also, a connected graph X is called Eulerian if their exist a closed trial congaing every edge of X. A path of length n is called an n-path and is denoted by P_n. A cycle of length n is called n-cycle and is denoted by C_n. A complete bipartite graph denoted by $K_{m,n}$ and the graph $K_{1,n}$ is called star graph. For further definitions and proofs of graph theory the reader may refer to Pirzada [13] and West [14].

Let G be a finite set. Then a group G is called finite. So, the number of elements in G is the order of G and is denoted by $|G|$. Unless mentioned otherwise, all groups considered in this paper are finite. A nonempty subset H of G is called subgroup of G if H is itself a group under the same binary operation on G. Every group G has at least two subgroups, G itself and the set $\{e\}$ consisting of the identity element alone, called trivial subgroups of G, otherwise subgroups of G are called proper. Throughout the paper, we consider $S(G)$ as a set of subgroups of G and $|S(G)|$ denote cardinality of $S(G)$.

A subgroup of a given group G can always be constructed by choosing any element a in G and forming the set of all its powers a^n, $n = 0, \pm 1, \pm 2, \cdots$ this is called the cyclic subgroup generated by an element a and is denoted by $C_n = \{a, a^2, \cdots, a^{n-1}, a^n = e\} = \langle a \rangle$.

Usually, $Z, Z_n, U_n, S_n, A_n, Q_8, D_4$ and V_4 denotes by the group of integers, integers modulo n, non zero integers modulo n, permutations, even permutations, Quaternions, Dihedral and Kelians respectively. Basic definitions for group theory see [15, 16].

Theorem 2.1. *(Lagranges theorem) [15] Let H be a subgroup of a finite group G. Then the order of H divides the order of G.*

Theorem 2.2. *Let a be any element of a group G. Then $\langle a \rangle$ is a cyclic subgroup of G.*

Let $n \geq 1$ be a positive integer. Then the cardinality of the set $D(n) = \{d : d|n\}$ is called divisor function of n and denoted by $d(n)$. In particular, $|D(n)| = d(n), n \geq 1$ an integer. If m and n are positive integers, then $gcd(m,n)$ is the greatest common divisor and $lcm[m,n]$ is the least common multiple of m and n. However, $gcd(m,n) = 1$ if and only m and n are relatively primes, which play an important role in the algebraic graph theory. For further definitions of number theory, the reader may refer to Rose [17].

Theorem 2.3. *If G is a finite cyclic group, then $|S(G)| = d(|G|)$, and if G is a finite non cyclic group, then $|S(G)| > d(|G|)$.*

3 Properties of Order Divisor graph

In this section, we show that $OD(G)$ is always connected and has small density, girth and diameter, and we determine a necessary and sufficient condition for $OD(G)$ is complete.

Definition 3.1. *Let $S(G) = \{H : H \text{ is a subgroup of } G\}$. An undirected simple graph $OD(G)$ is called an order divisor graph of subgroups of a finite group G whose vertex set is $S(G)$ and two distinct vertices $H, K \in S(G)$ are adjacent in $OD(G)$ if and only if either $|H|\,|\,|K|$ or $|K|\,|\,|H|$, where $|H|$, $|K|$ denotes the order of H and K respectively.*

Before studying properties of the order divisor graph of a group we give an example to illustrates their usefulness.

Example 3.1. *The graphs shown in Figure 1 are the order divisor graphs of groups Z_6 and S_6 respectively.*

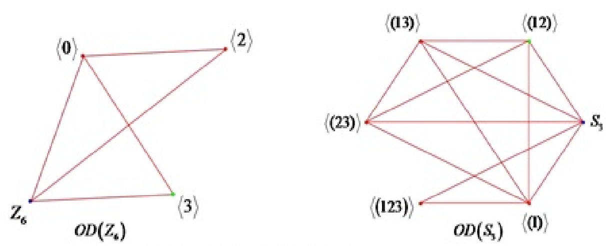

Figure1 order divisor grpahs of Z_6 and S_3

Theorem 3.4. *For any finite group G, the order divisor graph $OD(G)$ is connected.*

Proof. Let G be a finite group with identity element e. Then the vertex $\langle e \rangle$ is adjacent to all the remaining vertices of the order divisor graph $OD(G)$, since $|\langle e \rangle|\,|\,|H|$ for every vertex $H \neq \langle e \rangle$ in $OD(G)$. This implies that there exist a path between any two vertices in $OD(G)$, and hence $OD(G)$ is connected. □

Theorem 3.5. *Let order of a finite group G is not a power of single prime. Then $OD(G)$ is never complete.*

Proof. Consider the group G, whose order is $|G| = p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_r^{\alpha_r}, r > 1$, a prime decomposition. Suppose the graph $OD(G)$ is complete. Then any two vertices H_i and H_j are adjacent in $OD(G), i \neq j$. That is, either $|H_i|\,|\,|H_j|$ or $|H_j|\,|\,|H_i|$ for $i \neq j$. So without loss of generality, we may assume that $|H_i| = p_i$ and $|H_j| = p_j$. It is clear that $|H_i| \not| \, |H_j|$ and $|H_j| \not| \, |H_i|$ in $OD(G)$, since $p_i \not| \, p_j$ and $p_j \not| \, p_i$ for each $i \neq j$. It turns out that $OD(G)$ is never complete. □

Theorem 3.6. *Let G be a group of Composite order. Then the order divisor graph of G must contain a cycle of length* 3.

Proof. Suppose $|G| \neq p$, a prime. Then there is at least one proper subgroup H of G. By the Lagranges Theorem [2.1], $|H|\,|\,|G|$, $|\langle e \rangle|\,|\,|H|$ and $|\langle e \rangle|\,|\,|G|$. It is clear that the unordered pairs $(\langle e \rangle, H)$, (H, G) and $(G, \langle e \rangle)$ form a cycle $C_3 = (\langle e \rangle, H, G, \langle e \rangle)$ of length 3 in $OD(G)$. □

Theorem 3.7. *[15] Let H and K be two subgroups of a finite group G. Then $|HK| = \dfrac{|H||K|}{|H \cap K|}$. In particular, $|HK| = |H||K|$ if either $H \cap K = \{e\}$ or $gcd(|H|, |K|) = 1$.*

Theorem 3.8. *[16] Let C_p and C_q be cyclic groups with respect to multiplication of prime orders p and q respectively. Then $C_p \times C_q \cong C_{pq}$.*

Theorem 3.9. *(Syllows Third Theorem)[16] Let G be a group of order $p^n m$, where p is a prime, $n \geq 1$ and $gcd(p,m) = 1$. Then the number of Sylow p-subgroups of G by n_p is of the form $1 + kp, k \geq 0$ and $1 + kp$ divides m.*

Theorem 3.10. *[13] A graph is non-planar if and only if it has a subgraph homomorphic to either K_5 or $K_{3,3}$.*

Theorem 3.11. *[13] Let X be simple planar graph having $|V(X)| \leq 3$ vertices and $|E(X)|$ edges. Then $|E(X)| \leq 3|V(X)| - 6$.*

Theorem 3.12. *[13] For a maximal planar graph X of order $|V(X)| \geq 3, |E(X)| = 3|V(X)| - 6$.*

Theorem 3.13. *Let p and q be distinct prime with $p < q$. If G is a group of order pq, then the order divisor graph of G is either planar or maximal planar.*

Proof. Suppose $|G| = pq$, where p and q are distinct primes with $p < q$. Then, by the third Sylow Theorem [3.6], there is a unique q-Sylow subgroup, say Q of order q. So there exist two cases on p and q.

Case:1 If $p \not| \, q-1$, then there is a normal p-subgroup, say P of G such that $P \cap Q = \{e\}$ and $|PQ| = |P||Q| = pq = |G|$. Thus $G = P \times Q \cong C_p \times C_q \cong C_{pq}$, by Theorem [3.5]. So, in this case the vertex set of the order divisor graph $OD(G)$ is $S(G) = \big(\langle e \rangle, H, K, G\big)$ where H and K are proper subgroup of C_{pq} with H is not adjoint to K, since $|H| \not| \, |K|$ and $|K| \not| \, |H|$. It is clear that the order and side of $OD(G)$ are 4 and 5 respectively. Therefore by the Theorem [3.10], $OD(G) \ncong K_5$ or $K_{3,3}$, and hence $OD(G)$ is planar.

Case:2 If $p | q-1$, then the group G isomorphic to either of the following groups:
$\langle a, b : a^q = b^p = 1, ab = ba \rangle$ and $\langle a, b : a^q = b^p = 1, ab = a^{\alpha} \rangle$ where $\alpha \neq 1$.

(i) Suppose $G \cong \langle a, b : a^q = b^p = 1, ab = ba \rangle$. Then
$G \cong \langle a : a^q = 1 \rangle \times \langle b : b^p = 1 \rangle \Rightarrow G \cong C_p \times C_q \cong C_{pq}$ this is similar to case(1).

(ii) Suppose $G \cong \big\langle a, b : a^q = b^p = 1, b^{-1}ab = a^{\alpha} \big\rangle$. Then $G \neq C_{pq}$. This implies that G is a non abelian group of order pq generate by a and b. Thus the vertex set of the graph $OD(G)$ is $S(G) = \{\langle e \rangle, \langle a \rangle, \langle b \rangle, \langle ab \rangle, \langle a^2 b \rangle, G\}$ where $\langle a^2 b \rangle = \langle ab^2 \rangle$. It is clear that $|\langle e \rangle| = 1, |\langle a \rangle| = q |\langle b \rangle| = |\langle ab \rangle| || \langle a^2 b \rangle| = p$ and $|G| = pq$. Consequently, order and size of the graph $OD(G)$ are 6 and 12 respectively. We shall now show that $OD(G)$ is maximal planar. Suppose $OD(G)$ is not maximal planar. Then it must satisfy the relation $|E\big(OD(G)\big)| \neq 3|V\big(OD(G)\big)| - 6$, by the Theorem [3.9]. This implies that that $12 \neq 3 \times 6 - 6 \Rightarrow 12 \neq 12$, which is not true. So our assumption is wrong, and hence $OD(G)$ is a maximal planar grpah.

□

Lemma 3.1. *Let H and K be two subgraphs of a finite non-cyclic group G. Then $|H \cap K| \neq gcd\big(|H|, |K|\big)$.*

Proof. Since $H \cap K$ is a subgroup of both H and K in G. So, by the Lagranges Theorem [2.1], $|H \cap K| | |H|$ and $|H \cap K| | |K|$. Hence $|H \cap K| | gcd\big(|H|, |K|\big)$. Further, suppose $gcd\big(|H|, |K|\big) | |H \cap K|$, then there exists two positive integers q and q' such that $q gcd\big(|H|, |K|\big) = q'|H \cap K| = |G|$, since $gcd\big(|H|, |K|\big)$ and $|H \cap K|$ both divides $|G|$. Hence $q \neq q'$. Therefore, if $q' | q$ then $q = nq'$ for some positive integer n. Then $|H \cap K| = n gcd\big(|H|, |K|\big)$, a contradiction since $H \cap K$ is the subgroup of both H and K. Hence $|H \cap K| \neq gcd\big(|H|, |K|\big)$. □

Our next example turns out to be a very useful one for abelian groups and another for non-abelian groups.

Example 3.2. *Consider the subgroups $H = \langle a \rangle = \{e, a\}$ and $K = \langle b \rangle = \{e, b\}$ of the finite abelian group $V_4 = \{e, a, b, c : a^2 = b^2 = c^2 = e\}$. Therefore, $|H| = 2, |K| = 2, |H \cap K| = 1$. Hence $gcd\big(|H|, |K|\big) = 2 \neq 1 = |H \cap K|$.*

Next, $H = \langle (12)(345) \rangle, K = \langle (123)(45) \rangle$, are both cyclic subgroups of order 6 and $H \cap K = \{I\}$ in the finite non-abelain group S_5. Therefore $gcd\big(|H|, |K|\big) = 6 \neq 1 = |H \cap K|$.

The Example [3.2] tells us that the result $|HK| \neq lcm\big[|H|, |K|\big]$ for finite non-cyclic groups, but it must be true for finite cyclic groups. This illustrates the following lemmas.

Lemma 3.2. *Let H and K be two subgraphs of a finite non-cyclic group G. Then $|HK| \neq lcm\big[|H|, |K|\big]$.*

Proof. Follows from Lemma [3.1] and Theorem [3.7]. □

Lemma 3.3. *Let H and K be two subgraphs of a finite cyclic group G. Then* $|HK| = lcm\big[|H|, |K|\big]$.

Proof. Let $d = gcd\big(|H|, |K|\big)$.Then d divides both H and K. Also, by the Lagranges Theorem [2.1], $d\big||G|$, so there exists a unique subgroup, say L such that $|L| = d$, since G is cyclic. But H and K must have a subgroup of this order d and there is only one subgroup in G. Therefore L is a subgroup of $H \cap K$. But $|H \cap K|$ divides both H and K. It is clear that $|H \cap K|\big|gcd\big(|H|, |K|\big) \Rightarrow |H \cap K|\big|d \Rightarrow |H \cap K|\big||L|$. Hence $L = H \cap K$ and $|L| = |H \cap K| = d = gcd\big(|H|, |K|\big)$. Applying Theorem [3.7] yields $|HK| = \dfrac{|H||K|}{gcd\big(|H|, |K|\big)} = lcm\big[|H|, |K|\big]$. □

Theorem 3.14. *Let $|G|$ be a composite number. Then $deg(H) \geq 2$, for every vertex H in $OD(G)$.*

Proof. Let $|G| \neq p$, a prime. Then G has at least one proper subgroup, say H. It is clear that $deg\big(\langle e \rangle\big) \geq 2$ and $deg(G) \geq 2$ for $\langle e \rangle, G \in S(G)$, since $\langle e \rangle - H - G - \langle e \rangle$ is a cycle of length 3 in the graph $OD(G)$.

Now we show that $deg(H) \geq 2$, for every vertex H in $OD(G)$. First suppose $S(G) = \{\langle e \rangle, H, G\}$ be the vertex set of $OD(G)$. Then, by the Lagranges Theorem [2.1], vertex H is adjacent to both the vertices $\langle e \rangle$ and G in $OD(G)$. Therefore, $deg(H) = 2$. Further, if K is another vertex of $OD(G)$ such that $K \neq H, \langle e \rangle, G$. Now consider two cases on the group G.

Case (1): If G is a finite cyclic group then we have the following subcases.

Subcase (1): Suppose H is adjacent to K in $OD(G)$.Then trivially $deg(H) > 2$.

Subcase (2): Suppose H is not adjacent to K in $OD(G)$. Then $|H|\big||K|$ and $|K|\big||H|$.

$$\begin{aligned} &\Rightarrow |H|\big|lcm\big[|H|, |K|\big] \\ &\Rightarrow |H|\big||HK|, \quad \text{by the Lemma [3.3]} \\ &\Rightarrow H \text{ is adjoint to another vertex } HK \text{ in } OD(G). \end{aligned}$$

This shows that $deg(H) > 2$.

Case (2): If G is a finite non-cyclic group, then the edges in $OD(G)$ has any one of the following possibilities:

Subcase (1): Suppose either $|H|\big||K|$ or $|K|\big||H|$. Then H is adjacent to K, and hence $deg(H) > 2$.

Subcase (2): Suppose $|H| \not| \, |K|$ and $|K| \not| \, |H|$. Then either $gcd\big(|H|, |K|\big) = 1$ or $gcd\big(|H|, |K|\big) = d, d > 1$. If $gcd\big(|H|, |K|\big) = 1$, then $|H|\big||H|\big||K|$

$$\begin{aligned} &\Rightarrow |H|\big||HK|, \quad \text{by the Theorem [3.4]} \\ &\Rightarrow H \text{ is adjoint to } HK \text{ and thus } deg(H) > 2. \end{aligned}$$

Otherwise, if $gcd\big(|H|, |K|\big) = d, d > 1$, then $gcd\left(\dfrac{|H|}{d}, \dfrac{|K|}{d}\right) = 1$ this implies that the vertex H is adjacent to another new vertex whose order is $\dfrac{|H|}{d}\dfrac{|K|}{d}$ in $OD(G)$. Therefore, $deg(H) > 2$.

Summarizing the results of the two cases we find $deg(H) \geq 2$ for every vertex H in $OD(G)$. □

Now we are going to study the useful consequences of the Theorem [3.14].

Corollary 3.1. *For any finite group G, the order divisor graph is never a path of length 2.*

Proof. Suppose $OD(G)$ is the path $P_n : H_0 - H_1 - \cdots - H_{n-1} - H_n$ of length $n > 2$, where H_0 and H_n are the initial and terminal vertices of P_n. Then, by the definition of the path, we have $deg(H_0) = deg(H_n) = 1$. This violates result of the Theorem [3.14]. Hence $OD(G)$ is never a path of length $n > 2$. □

Corollary 3.2. *If G is a group of composite order, then $OD(G)$ is never a star graph.*

Proof. Let $|G| \neq p$, a prime. Assume $OD(G)$ is a star graph of order $|S(G)|$. If $H_1, H_2, \cdots, H_{|S(G)|-1}, H_{|S(G)|}$ are the vertices of the $OD(G)$, where $|S(G)| > 2$, then the graph $OD(G)$ contains $|S(G)| - 1$ pendent vertices. That is, $deg(H_i) = 1$, for every $1 \leq i \leq |S(G)| - 1$. This is a contradiction to the Theorem [3.14] . So our assumption is wrong, and hence $OD(G)$ is never a star graph. □

Theorem 3.15. *[14] A simple graph is Eulerian if and only if degree of each vertex is even.*

Corollary 3.3. *Let $|G| \neq P^{2n}, n \geq 1$. Then $OD(G)$ is never Eulerian.*

Proof. suppose that the graph $OD(G)$ is Eulerian. Then the degree of each vertex is even. From the Theorem [3.14], degree of each vertex in $OD(G)$ is at least 2, that is $deg(H_i) \geq 2$, for every $H_i \in S(G)$ and $1 \leq i \leq |S(G)|$. So without loss of generality, we may assume that $deg(H_1) = 2, deg(H_2) = 3, deg(H_3) = 4$, and so on. Then, we found that degree of each vertex can not be even. This is a contradiction to the result of the Theorem [3.15]. Thus, by contraposition, the result follows. □

Before going to the study of further properties of order divisor graph we shall prove that the following consequences of the Theorem [3.14]. First we study the completeness of the order divisor graph $OD(G)$. In particular, we give a necessary and sufficient condition for $OD(G)$ to be completed.

Theorem 3.16. *The order divisor graph of a group G is complete if and only if no two proper divisors of $|G|$ are relatively prime.*

Proof. **Necessity:** Suppose $OD(G)$ is a complete graph of order $|S(G)|$. Then any two vertices $H_i, H_j \in S(G)$ are adjacent in $OD(G), i \neq j$. Therefore, $|H_i| \big| |H_j|$ or $|H_j| \big| |H_i|$ for each $i \neq j. \Rightarrow gcd(|H_i|, |H_j|) \neq 1$, for each $i \neq j$. This implies that $|H_i|$ and $|H_j|$ are two proper divisors of $|G|$ which are not relatively prime. So no two proper divisors of $|G|$ are relatively prime.

Sufficient: Suppose no two proper divisor of $|G|$ are relatively prime. That is, $gcd(|H_i|, |H_j|) \neq 1$, for every two proper subgroups H_i and H_j of G. We shall now show that $OD(G)$ is a complete graph. Suppose that $OD(G)$ is not complete. Then there exists two vertices H_i and H_j in $S(G)$ such that $|H_i| \nmid |H_j|$ and $|H_j| \nmid |H_i|$. That is, either $gcd(|H_i|, |H_j|) = 1$ or $gcd(|H_i|, |H_j|) = d, d > 1$. But, by hypothesis $gcd(|H_i|, |H_j|) \neq 1$, for proper divisors $|H_i|$ and $|H_j|$ of $|G|, i \neq j$. Therefore, $gcd(|H_i|, |H_j|) = d, d > 1 \Rightarrow gcd\left(\frac{|H_i|}{d}, \frac{|H_j|}{d}\right)$ and $\frac{|H_i|}{d}\bigg||G|$ and $\frac{|H_j|}{d}\bigg||G| \Rightarrow \frac{|H_i|}{d}$ and $\frac{|H_j|}{d}$ are both proper divisors of $|G|$ and which are relatively prime, a contradiction to our hypothesis. Hence $OD(G)$ is a complete graph. □

The following results are immediate consequences of the Theorem [3.16].

Theorem 3.17. *Let G be a finite group. Then the following statements are equivalent.*

(a) The graph $OD(G)$ is complete.

(b) $|G| = P^n, n \geq 1$ an integer.

Proof. For (a) implies (b), first assume that $OD(G)$ is a complete graph of a graph G. There are two possibilities: either G is cyclic or not. If G is cyclic group and $|G|$ is a composite number not divisible by p^{n+1} for any prime p, since in this case $\langle p \rangle$ is not adjacent to $\langle p^{n+1} \rangle$ in $OD(G)$. Moreover, $|G|$ is not divisible by square free integer $P_1, p_2, \cdots, p_n, p_i$'s are distinct primes, since in this case $\langle p_i \rangle, \langle p_j \rangle \in S(G), i \neq j$, are not adjacent in $OD(G)$ because $(|\langle p_i \rangle|, |\langle p_j \rangle|) = 1$. Finally, $q^n \nmid p^n$, since if q is prime, then $\langle q \rangle$ is not adjacent to $\langle p \rangle$ in $OD(G)$. So, $\langle G \rangle = p^n$. If G is not a cyclic group, then we have to prove that $\langle G \rangle = p^n$. Assume that $\langle G \rangle \neq p^n$. In view of the Theorem [3.16], $OD(G)$ is never complete. Thus $\langle G \rangle = p^n$.

For (b) implies (a), we assume that $\langle G \rangle = p^n$. Then we shall now show that $OD(G)$ is a complete graph. We consider the following two cases:

Case (1): If G is a cyclic group, then the order of $OD(G)$ is $|S(G)| = d(p^n) = n + 1$, which are $1, p, p^2, \cdots, p^{n-1}, p^n$. It is clear that for any two vertices $H, K \in S(G), |H|, |K| \in \{1, p, p^2, \cdots, p^{n-1}, p^n\}$. This implies that $|H| \big| |K|$ or $|K| \big| |H|$, since $p^i | p^j$ or $p^j | p^i$ for $i \neq j$. That is, no two proper divisors of $|G|$ are relatively prime, hence $OD(G)$ is complete.

Case (2): If G is not a cyclic graph, then $|S(G)| > d(|G|)$. But by Lagranges Theorem [2.1], order of each and every subgroup divides $|G|$, that is $|H| \big| p^n$, for every $H \in S(G)$. Therefore $|H| \in \{1, p, p^2, \cdots, p^{n-1}, p^n\}$. This implies that no two proper divisors in $|G|$ are relatively prime, so in this case also $OD(G)$ is complete. □

The following example show how the result in the Theorem [3.17] can be used to study the structures of order divisor graphs of abelian and nonabelian groups.

Example 3.3. *Figure 2 illustrates the Theorem [3.14].*

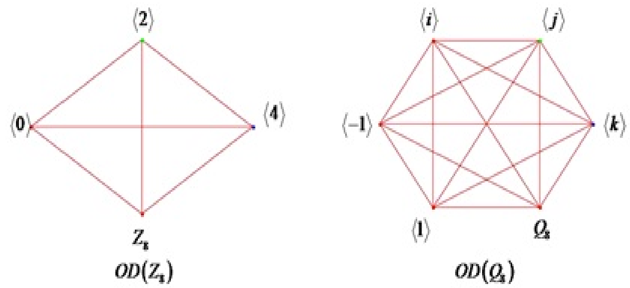

Figure2 order divisor grpahs of Z_8 and Q_8.

Remark 3.1. *Theorem [3.14] tells us that the following properties, if* $|G| = p^n, n > 1$ *then* $OD(G)$ *is* $(|S(G)| - 1)$-*regular graph and the size of* $OD(G)$ *is* $\frac{(|S(G)|)(|S(G)|-1)}{2}$.

(1) *If G is a cyclic group of order* p^n*, then complete order divisor graph* $OD(G)$ *is n-regular.*

(2) *If G is not a cyclic group of order* p^n*, then the order divisor graph* $OD(G)$ *is also complete but not n-regular. This point is illustrated as follows.*

Example 3.4. (1) *The order divisor graph of a cyclic group* Z_8 *is complete 3-regular.*

(2) *The order divisor graph of a non-cyclic group* Q_8 *is complete but 5-regular.*

(3) *If is an abelian but not cyclic group of order* p^n*, then the order divisor graph of G is also complete but never n-regular.*

For example, the Figure 3 shows that Lattice of subgroups of Klein 4 group V_4 *and its order divisor graph.*

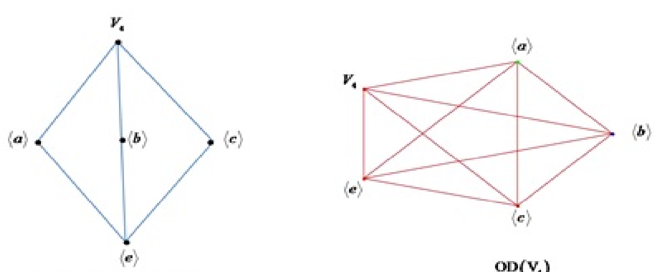

Figure3 order divisor graph of V_4

Corollary 3.4. *Let G be a finite group. Then the order divisor graph* $OD(G)$ *is isomorphic to* K_2 *if and only if G is isomorphic to one of the groups:* $Z_p, C_p, \frac{S_n}{A_n}, Aut(Z), \frac{A_4}{V_4}$.

Proof. It is obvious, since $OD(G) \cong K_2$ if and only if $|G| = p$, a prime. □

Corollary 3.5. *The order divisor graph of a group G is complete if and only if G is isomorphic to one of the groups:*

(a) p-group

(b) $Diag(n, Z)$.

Proof. (a) Since G is a p-group if and only if $|G| = p^n, n \geq 1$. Hence $OD(G)$ is complete.

(b) Since $Diag(n, Z)$ is an abelian group of all $n \times n$ diagonal matrices over the set of integers whose diagonal elements are ± 1. So, $Diag(n, Z) = 2^n$. Hence $OD(Diag(n, Z))$ is complete.

□

Definition 3.2. *The density of a simple graph is the ratio of order and size of the graph respectively.*

Theorem 3.18. *Let $|G| = p^n, n \geq 1$, Then $density(OD(G)) = \dfrac{2}{|S(G)| - 1}$.*

Proof. We have, $density(OD(G)) = \dfrac{|S(G)|}{\frac{1}{2}|S(G)|(|S(G)| - 1)} = \dfrac{2}{|S(G)| - 1}$.

□

Corollary 3.6. *If G is a cyclic group of order $p^n, n \geq 1$ then $density(OD(G)) = \dfrac{2}{n}$.*

Proof. The number of subgroups of a cyclic group of order $p^n, n \geq 1$ is $|S(G)| = d(|G|) = d(p^n) = n + 1$. Therefore $density(OD(G)) = \dfrac{2}{(n+1) - 1} = \dfrac{2}{n}$.

The girth of a simple graph X, denoted by $gir(X)$ is the length of a shortest cycle in X. If X is acyclic graph, then $gir(X) = \infty$. Let m and n be two distinct vertices of a simple graph X. Then the diameter of X, denoted by $diam(X)$, is given by $diam(X) = \sup\{d(m, n) : m, n \text{ distinct vertices of } X\}$, where $d(m, n)$ is the length of the shortest path between m and n. □

Theorem 3.19. *For $|G| > 1$, the girth of order divisor graph of a group G is given by*

$$gir(OD(G)) = \begin{cases} \infty & \text{if} \quad |G| = p \\ 3 & \text{if} \quad |G| \neq p. \end{cases}$$

Proof. If $|G| = p$, a prime, then $OD(G) \cong K_2$, an acyclic graph. It is clear that the girth of $OD(G)$ is infinite. If $|G| \neq p$, then there are two possibilities on $|G|$: either $|G| = p^n, n > 1$, or $|G| \neq p^n, n \geq 1$. Suppose $|G| = p^n$. Then, in view of Theorem [3.17], $OD(G)$ is complete graph with three or more than three vertices and so $gir(OD(G)) = 3$. On the other hand, if $|G| \neq p^n$, in view of Theorem [3.5], the order divisor graph $OD(G)$ always have a three cycle $C_3 = (\langle e \rangle, H, G, \langle e \rangle)$ which is smallest for any proper subgroup H of G. Hence $gir(OD(G)) = 3$. □

Corollary 3.7. *Let G be a group of composite order. Then the graph $OD(G)$ is never complete bipartite.*

Proof. Follows directly from Theorem [3.19], since the girth of complete bipartite graph is 4. □

Theorem 3.20. *Let G be a finite group. Then $diam(OD(G)) \leq 2$.*

Proof. Let p be a prime and $n \geq 1$ be a positive integer. Then we consider the following two cases on $|G|$, G is a finite group.

Case (1) Suppose $|G| = p^n, n \geq 1$. Then the diameter of $OD(G)$ is 1, since this is possible from the Theorem [3.14] and the diameter of a compete graph is 1.

Case (2) Suppose $|G| \neq p^n, n \geq 1$. Then $OD(G)$ is never complete graph by the Theorem [3.5]. Therefore $diam(OD(G)) \geq 1$. But we have to prove that $diam(OD(G)) \leq 2$. For this let H and K be any two distinct vertices of $OD(G)$. If H is adjacent to K, then obviously $d(H, K) = 1$ because $H - K$ is a path of length 1. Otherwise if H is not adjacent to K, then H and K must be proper subgroups of G. So there exist a path of length 2 in $OD(G)$, which is either of the following:

(1) $H - \langle e \rangle - K$

(2) $H - G - K$

(3) $H - H \cap K - K$

(4) $H - HK - K$. Therefore, $d(H, K) = 2$.

From the above two cases we conclude that $diam(OD(G)) \leq 2$. □

4 Isomorphisms of Order Divisor Graphs

This section describes the necessary and sufficient condition for two isomorphic groups and their order divisor graphs. Further we study $Auto(OD(G))$, the group of graph automorphisms of $OD(G)$, and we show that $Auto(G) \subseteq Auto(OD(G))$.

We know that a graph isomorphism f of a graph X to a graph Y is a bijection $f : X \to Y$ which preserves adjacency. The set $Auto(X)$ of all graph automorphisms of X forms a group under the usual composition of functions.

Theorem 4.21. *Let G and G' be any two distinct finite groups. Then G is isomorphic to G' if and only if $|G| = |G'|$ and $|S(G)| = |S(G')|$.*

Proof. Let $S(G)$ and $S(G')$ be the set of subgroups of finite groups G and G' respectively.

Sufficient: Suppose $G \cong G'$. Then there exist an isomorphism φ from G onto G'.

(i) Since φ is bijetive, G and G' have the same cardinality, that is, $|G| = |G'|$.

(ii) Let $H \in S(G)$. Then $\varphi(H) = \{\varphi(x) : x \in H\}$ is a subgroup of G'. Now define a map $f : S(G) \to S(G')$ by the relation $f(H) = \varphi(H)$, for every $H \in S(G)$.

f **is one-to-one:** Suppose that $f(H) = f(K)$. Then $\varphi(H) = \varphi(K) \Rightarrow H = K$, since φ is bijection.

f **is onto:** Let H' belongs to $S(G')$. We must find a subgroup H in $S(G)$ such that $f(H) = H'$. If such a subgroup H is to exist, it must have the property that $\varphi(H) = H'$. For we can solve for H to obtain $H = \varphi^{-1}(H') = \{g \in G : \varphi(g) \in H'\}$ and verify that $\varphi(\varphi^{-1}(H')) = H'$. It is clear that f is onto.

Therefore f is a bijection from $S(G)$ onto $S(G')$, hence $|S(G)| = |S(G')|$.

Necessity: Let $|G| = |G'|$ and $|S(G)| = |S(G')|$. Then we shall show that $G \cong G'$.

For this we define a map $\psi : G \to G'$ by $\psi(a) = a'$, for every $a \in G$. Put $a' = \psi(a)$ and $b' = \psi(b)$ for $a, b \in G$, then a bijection $\psi : G \to G'$ satisfying $\psi(ab) = a'b' = \psi(a)\psi(b)$. Then we say that G and G' are isomorphic under the corresponding group elements. Further we shall show that G and G' are isomorphic under corresponding subgroups. Let $a \in G$. Then we define a map $g : G \to G'$ by the relation $g(\langle a \rangle) = \langle \psi(a) \rangle$ where $\langle a \rangle \in S(G)$ and $\langle \psi(a) \rangle \in S(G')$.

g **is one-to-one:** For this let $a, b \in G$, then $g(\langle a \rangle) = g(\langle b \rangle) \Rightarrow \langle \psi(a) \rangle = \langle \psi(b) \rangle$
$\Rightarrow \psi(\langle a \rangle) = \psi(\langle b \rangle) \Rightarrow \langle a \rangle = \langle b \rangle$, since ψ is a bijection.

g **is onto:** By the way of construction of map g, for every subgroup $\langle \psi(a) \rangle$ of G', there exist a subgroup $\langle a \rangle$ in G such that $g(\langle a \rangle) = \langle \psi(a) \rangle$. Therefore g is onto.

g **is a homomorphism** Let $\langle a \rangle, \langle b \rangle \in S(G)$. Then

$$\begin{aligned} g(\langle a \rangle \langle b \rangle) &= g(\langle ab \rangle) \\ &= \langle \psi(ab) \rangle = \psi(\langle ab \rangle) \\ &= \psi(\langle a \rangle \langle b \rangle) = \psi(\langle a \rangle)\psi(\langle b \rangle) \\ &= \langle \psi(a) \rangle \langle \psi(b) \rangle = g(\langle a \rangle)(\langle b \rangle). \end{aligned}$$

Therefore g preserves subgroups from G onto G'. Hence $G \cong G'$. □

Example 4.5. *(1) The symmetric group S_3 is isomorphic to the Dihedral group D_3 because $|S_3| = |D_3| = 6$ and $|S(S_3)| = |S(D_3)| = 6$.*

(2) The symmetric group S_3 is not isomorphic to Z_6 since $|S_3| = |Z_6| = 6$ but $|S(Z_6)| = 4$ and $|S(Z_6)| = 4$.

(3) The Klein-4 group V_4 is not isomorphic to Z_4 since $|V_4| = |Z_4| = 4$ but $|S(V_4)| = 5$ and $|S(Z_4)| = 3$.

Theorem 4.22. *[16] Let φ be an isomorphism from a group G onto G'. Then $|a| = |\varphi(a)|$, for every $a \in G$. Moreover, $|H| = |\varphi(H)|$, for every $H \in S(G)$. In particular, a group isomorphism preserves the order of elements and the order of subgroups respectively.*

The next theorem provides a necessary and sufficient condition for order divisor graphs are isomorphic.

Theorem 4.23. *Let G and G' be two finite groups. Then $G \cong G'$ if and only if $OD(G) \cong OD(G')$.*

Proof. **Sufficient:** Suppose $G \cong G'$. Then there exist an isomorphism f from G onto G'. Now to show that $OD(G) \cong OD(G')$. For this we define a map $\varphi : OD(G) \to OD(G')$ by the relation $\phi(H) = f(H)$, for every $H \in S(G)$. It is clear that ϕ is well defined bijective map, and further we show that ϕ preserves adjacency. To do this, let (H, K) be an edge of the graph $OD(G)$ with end vertices H and K. Then by the definition of order divisor graph, either $|H|||K|$ or $|K|||H|$. By the Theorem [4.22], this implies that $|f(H)|||f(K)|$ or $|f(K)|||f(H)| \Rightarrow |\phi(H)|||\phi(K)|$ or $|\phi(K)|||\phi(H)|$ is adjacent to $\phi(K)$ in $OD(G')$, it follows that ϕ preserves adjacency. Hence $OD(G) \cong OD(G')$.

Necessity: Suppose $OD(G) \cong OD(G')$. Then there exist an isomorphism ϕ from a graph $OD(G)$ to a graph $OD(G')$ is a bijection that maps $V(OD(G))$ to $V(OD(G'))$ and $E(OD(G))$ to $E(OD(G'))$ such that each edge of $OD(G)$ with end vertices H and K is mapped to an edge with end vertices $\phi(H)$ and $\phi(K)$. Therefore $|V(OD(G))| = |V(OD(G'))|$ and $|E(OD(G))| = |E(OD(G'))|$. This shows that $|G| = |G'|$ and $|S(G)| = |S(G')|$. Applying Theorem [4.21] yields $G \cong G'$.

□

Example 4.6. *Figure 4 shows that the relation $Z_2 \times Z_3 \cong Z_6 \Leftrightarrow OD(Z_2 \times Z_3) \cong OD(Z_6)$ is true.*

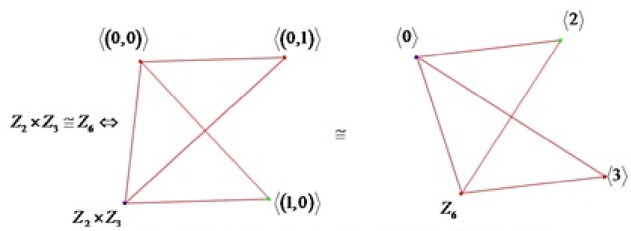

Figure 4 the relation $Z_2 \times Z_3 \cong Z_6 \Leftrightarrow OD(Z_2 \times Z_3) \cong OD(Z_6)$

The following remarks, which are the main results of this section contains the complete description for isomorphic and non-isomorphic finite groups with their corresponding order divisor graphs.

Remark 4.2. *(1) $OD(Z_m \times Z_n) \cong OD(Z_{m,n}) \Leftrightarrow gcd(m, n) = 1$.*

(2) $OD(U_m \times U_n) \cong OD(U_{m,n}) \Leftrightarrow gcd(m, n) = 1$.

Remark 4.3. *Let G and G' be two finite groups. Then $G \ncong G' \Leftrightarrow OD(G) \ncong OD(G')$. Below are the order divisor graphs of groups V_4 and Z_4. This example shows that non-isomorphic groups may have the non-isomorphic order divisor graphs.*

Theorem 4.24. *Let G be a finite group. Then $Auto(G) \subseteq Auto(OD(G))$.*

Proof. Let G be a finite group. Then $Auto(G)$ and $Auto(OD(G))$ are both finite groups. Now we show that $Auto(G) \subseteq Auto(OD(G))$. For this we consider $\varphi \in Auto(G)$, then φ is an isomorphism of G onto G. Suppose two vertices $H, K \in S(G)$ are adjacent in $OD(G)$. Then either $|H|||K|$ or $|K|||H|$. This implies that either $|\varphi(H)|||\varphi(K)|$ or $|\varphi(K)|||\varphi(H)|$, since $|H| = |\varphi(H)|$, for every $H \in S(G)$.

$$\Rightarrow \varphi(H) \text{ and } \varphi(K) \text{ are adjacent in } OD(G).$$
$$\Rightarrow \varphi \text{ is an isomorphism from } OD(G) \text{ to } OD(G).$$
$$\Rightarrow \varphi \in Auto(OD(G)).$$

Hence $Auto(G) \subseteq Auto(OD(G))$. □

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Numerical investigation of the hybrid fuzzy differential equations using He′s homotopy perturbation method

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Abstract

This paper presents an efficient method namely He's Homotopy Perturbation Method (HHPM) is introduced for solving hybrid fuzzy differential equations based on Seikkala derivative with initial value problem [2]. The proposed method is tested on hybrid fuzzy differential equations. The discrete solutions obtained through He's Homotopy Perturbation Method are compared with Leapfrog method [13]. The applicability of the He's Homotopy Perturbation Method is more suitable to solve the hybrid fuzzy differential equations. Error graphs are presented to highlight the efficiency of the He's Homotopy Perturbation Method.

Keywords: Fuzzy differential equations, Fuzzy initial value problems, Hybrid Fuzzy Differential Equations, Leapfrog method, He's Homotopy Perturbation Method.

2010 MSC: 65L80, 65L05.

1 Introduction

Hybrid systems are devoted to modelling, design, and validation of interactive systems of computer programs and continuous systems. That is, control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics can be modelled by hybrid system. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named hybrid fuzzy differential systems. For analytical results on stability properties and comparison theorems we refer reader to [V. Lakshmikantham and X. Z. Liu [9]; V.Lakshmikantham and R. N. Mohapatra [8]; M. Sambandham [11]].

Hyunsoo Kim and Rathinasamy Sakthivel [7] obtained the numerical solution of hybrid fuzzy differential equations using improved predictorcorrector method. T.Jayakumar and K. Kanakarajan [2] obtained numerical solution for hybrid fuzzy system by improved Euler method. T. Jayakumar and K. Kanagarajan [4] derived the numerical solution for hybrid fuzzy system by Runge-Kutta method of order five, T. Jayakumar and K. Kanakarajan [2] claimed that the numerical solution for hybrid fuzzy system by improved Euler method. K. Kanagarajan and M. Sambath [6] stated the numerical solution hybrid fuzzy differential equations by improved predictor- corrector method. K. Kanagarajan and S. Muthukumar [5] extended Runge-Kutta method of order four for hybrid fuzzy differential equations. S. Pederson and M.Sambandham [10] proposed the numerical solution to hybrid fuzzy systems.

Recently, T.Jayakumar and K. Kanagarajan [3] solved the hybrid fuzzy differential equations using Adams Fifth Order Predictor-Corrector Method. S. Sekar and K. Prabhavathi [13] solved the same hybrid fuzzy differential equations using Leapfrog method. The objective of this paper is to use the He's Homotopy

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Perturbation Method (discussed by Sekar et al. [14–16]) to solve the hybrid fuzzy differential equations (discussed by T.Jayakumar and K. Kanagarajan [3] and S. Sekar and K. PRabhavathi [13]).

2 He's Homotopy Perturbation Method

In this section, we briefly review the main points of the powerful method, known as the He's homotopy perturbation method [14–16]. To illustrate the basic ideas of this method, we consider the following differential equation:

$$A(u) - f(t) = 0, u(0) = u_0, t \in \Omega \tag{2.1}$$

where A is a general differential operator, u_0 is an initial approximation of Eq. (2.1), and $f(t)$ is a known analytical function on the domain of Ω. The operator A can be divided into two parts, which are L and N, where L is a linear operator, but N is nonlinear. Eq. (2.1) can be, therefore, rewritten as follows:

$$L(u) + N(u) - f(t) = 0$$

By the homotopy technique, we construct a homotopy $U(t,p) : \Omega \times [0,1] \to \Re$, which satisfies:

$$H(U,p) = (1-p)[LU(t) - Lu_0(t)] + p[AU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \tag{2.2}$$

or

$$H(U,p) = LU(t) - Lu_0(t) + pLu_0(t) + p[NU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \tag{2.3}$$

where $p \in [0,1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously, from Eqs. (2.2) or (2.3) we will have $H(U,0) = LU(t) - Lu_0(t) = 0, H(U,1) = AU(t) - f(t) = 0$.

The changing process of p from zero to unity is just that of $U(t,p)$ from $u_0(t)$ to $u(t)$. In topology, this is called homotopy. According to the He's Homotopy Perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (2.2) or (2.3) can be written as a power series in p :

$$U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2 U_2 + p^3 U_3 + ... \tag{2.4}$$

Setting $p = 1$, results in the approximate solution of Eq.(2.1)

$$U(t) = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + ...$$

Applying the inverse operator $L^{-1} = \int_0^t (.)dt$ to both sides of Eq. (2.3), we obtain

$$U(t) = U(0) + \int_0^t Lu_0(t)dt - p\int_0^t Lu_0(t)dt - p[\int_0^t (NU(t) - f(t))dt] \tag{2.5}$$

where $U(0) = u_0$.

Now, suppose that the initial approximations to the solutions, $Lu_0(t)$, have the form

$$Lu_0(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t) \tag{2.6}$$

where α_n are unknown coefficients, and $P_0(t), P_1(t), P_2(t), ...$ are specific functions. Substituting (2.4) and (2.6) into (2.5) and equating the coefficients of p with the same power leads to

$$\begin{cases} p^0 : U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \\ p^1 : U_1(t) = -\sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt - \int_0^t (NU_0(t) - f(t))dt \\ p^2 : U_2(t) = -\int_0^t NU_1(t)dt \\ \vdots \\ p^j : U_j(t) = -\int_0^t NU_{j-1}(t)dt \end{cases} \tag{2.7}$$

Now, if these equations are solved in such a way that $U_1(t) = 0$, then Eq. (2.7) results in $U_1(t) = U_2(t) = U_3(t) = \ldots = 0$ and therefore the exact solution can be obtained by using

$$U(t) = U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \tag{2.8}$$

It is worth noting that, if $U(t)$ is analytic at $t = t_0$, then their Taylor series

$$U(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

can be used in Eq. (2.8), where $a_0, a_1, a_2, ...$ are known coefficients and α_n are unknown ones, which must be computed.

3 Some basic results on hybrid fuzzy differential equations

Denote by E^1 the set of all functions $u : R \to [0, 1]$ such that

(i) u is normal, that is, there exist an $x_0 \in R$ such that $u(x_0) = 1$,

(ii) u is a fuzzy convex, that is, for $x, y \in R$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1\lambda)y) \geq min\{u(x), u(y)\}$

(iii) u is upper semicontinuous, and

(iv) $[u]^0 = \{\overline{x \in R : u(x) > 0}\}$ is compact. For $0 < \alpha \leq 1$, we define $[u]^\alpha = \{\overline{x \in R : u(x) \geq \alpha}\}$.

An example of a $u \in E^1$ is given by

$$u(x) = \begin{cases} 4x - 3, & \text{if } x \in (0.75, 1], \\ -2x + 3, & \text{if } x \in (1, 1.5), \\ 0, & \text{if } x \notin (0.75, 1.5). \end{cases}$$

The α-level sets of u in (6.1) are given by $[u]^\alpha = [0.75 + 0.25\alpha, 1.50.5\alpha]$. For later purpose, we define $\hat{o} \in E^1$ as $\hat{o}(x) = 1$ if $x = 0$ and $\hat{o}(x) = 0$ if $x \neq 0$.

Next we review the Seikkala derivative [12] of $x : I \to E^1$ where $I \subset R$ is an interval. If $[x(t)^a] = [\underline{x}^a(t), \bar{x}^a(t)]$ for all $t \in I$ and $a \in [0, 1]$, then $[x'(t)^a] = [\underline{x}'(t)^a, (\bar{x}^a)'(t)]$ if $x'(t) \in E^1$. Next consider the initial value problem (IVP)

$$u(x) = \begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x_0 \end{cases} \tag{3.9}$$

where $f : [0, \infty) \times R \to R$ is continuous. We would like to interpret (3.9) using the Seikkala derivative and $x_0 \in E^1$. Let $[x_0]^a = [\underline{x}_0^a, \bar{x}_0^a]$ and $[x(t)]^a = [\underline{x}^a(t), \bar{x}^a(t)]$. By the Zadeh extension principle we get $f : [0, \infty) \times E^1 \to E^1$ where $[f(t, x)]^a = minf(t, u) : u \in [\underline{x}^a(t), \bar{x}^a(t)], maxf(t, u) : u \in [\underline{x}^a(t), \bar{x}^a(t)]$. Then $x : [0, \infty) \to E^1$ is a solution of (6.1) using the Seikkala derivative and $x_0 \in E^1$ if

$(\underline{x}^a)'(t) = minf(t, u) : u \in [\underline{x}^a(t), \bar{x}^a(t)], \underline{x}^a(0) = \underline{x}_0^a$,

$(\bar{x}^a)'(t) = maxf(t, u) : u \in [\underline{x}^a(t), \bar{x}^a(t)], \bar{x}^a(0) = \bar{x}_0^a$, for all $t \in [0, \infty)$ and $a \in [0, 1]$. Lastly consider an $f : [0, \infty) \times R \times R \to R$ which is continuous and the IVP

$$\begin{cases} x'(t) = f(t, x(t), k), \\ x(0) = x_0 \end{cases} \tag{3.10}$$

As in (3.9), to interpret (3.10) using the Seikkala derivative and $x_0, k \in E^1$, by the Zadeh extension principle we use $f : [0, \infty) \times E^1 \times E^1 \to E^1$ where

$[f(t, x, k)]^a = [minf(t, u, u_k) : u \in [\underline{x}^a(t), \bar{x}^a(t)], u_k \in [\underline{k}^a, \bar{k}^a]$,

$maxf(t, u, u_k) : u \in [\underline{x}^a(t), \bar{x}^a(t)], u_k \in [\underline{k}^a, \bar{k}^a]]$,

where $k^a = [\underline{k}^a, \bar{k}^a]]$. Then $x : [0, \infty) \to E^1$ is a solution of (6.2) using the Seikkala derivative and $x_0, k \in E^1$ if

$(\underline{x}^a)'(t) = minf(t, u, u_k) : u \in [\underline{x}^a(t), \bar{x}^a(t)], u_k \in [\underline{k}^a, \bar{k}^a], \underline{x}^a(0) = \underline{x}_0^a$,

$(\bar{x}^a)'(t) = maxf(t, u, u_k) : u \in [\underline{x}^a(t), \bar{x}^a(t)], u_k \in [\underline{k}^a, \bar{k}^a], \underline{x}^a(0) = \underline{x}_0^a$,

for all $t \in [0, \infty)$ and $a \in [0, 1]$.

4 The hybrid fuzzy differential systems

In this section, we study the fuzzy initial value problem for a hybrid fuzzy differential systems.

$$x'(t) = f(t, x(t), \lambda_k x(t_k)), t \in [t_k, t_{k+1}], x(t_k) = x_{t_k} \tag{4.11}$$

where x' denotes Seikkala differentiation, $0 \le t_0 < t_1 < ... < t_k < ..., t_k \to \infty$, $f \in C[R^+ \times E^1 \times E^1, E^1], \lambda_k \in C[E^1, E^1]$. To be specific the system look like

$$x'(t) = \begin{cases} x_0'(t) = f(t, x_0(t), \lambda_0 x(t_0)), x_0(t_0) = x_0, t_0 \le t \le t_1, \\ x_1'(t) = f(t, x_1(t), \lambda_1 x(t_1)), x_1(t_1) = x_1, t_1 \le t \le t_2, \\ ... \\ x_k'(t) = f(t, x_k(t), \lambda_k x(t_k)), x_k(t_k) = x_k, t_k \le t \le t_{k+1}, \\ ... \end{cases} \tag{4.12}$$

Assuming that the existence and uniqueness of solution of (4.11) hold for each $[t_k, t_{k+1}]$, by the solution of (4.12) we mean the following function:

$$x(t) = x(t, t_0, x_0) \begin{cases} x_0(t), t_0 \le t \le t_1, \\ x_1(t), t_1 \le t \le t_2, \\ ... \\ x_k(t), t_k \le t \le t_{k+1}, \\ ... \end{cases} \tag{4.13}$$

We note that the solution of (4.13) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in E^1$ and $k = 0, 1, 2, ...$

Using a representation of fuzzy numbers studied by Goestschel and Woxman [1] and Wu and Ma [17], we may represent $x \in E^1$ by a pair of functions $(\underline{x}(r), \bar{x}(r)), 0 \le r \le 1$, such that

(i) $(\underline{x}(r)$, is bounded, left continuous, and non decreasing,

(ii) $\bar{x}(r)$ is bounded, left continuous, and non increasing, and

(iii) $(\underline{x}(r) \le \bar{x}(r)), 0 \le r \le 1$.

For example, $u \in E^1$ given in (1) is represented by $(\underline{u}(r), \bar{u}r) = (0.75 + 0.25r, 1.5 - 0.5r), 0 \le r \le 1$, which is similar to $[u]^a$ given by (3.10).

Therefore we may replace (4.13) by an equivalent system

$$\begin{cases} \underline{x}'(t) = \underline{f}(t, x, \lambda_k x(t_k)) \equiv F_k(t, \underline{x}, \bar{x}), (\underline{x}(t_k) = \bar{x}_k), \\ \underline{x}'(t) = \bar{f}(t, x, \lambda_k x(t_k)) \equiv G_k(t, \underline{x}, \bar{x}), (\underline{x}(t_k) = \bar{x}_k), \end{cases}$$

which possesses a unique solution $(\underline{x}, \bar{x})$ which is a fuzzy function. That is for each t, the pair $[\underline{x}(t;r), \bar{x}(t;r)]$ is a fuzzy number, where $\underline{x}(t;r), \bar{x}(t;r)$ are respectively the solutions of the parametric form given by

$$\begin{cases} \underline{x}'(t) = F_k(t, \underline{x}(t;r), \bar{x}(t,r)), \underline{x}(t_k;r) = \underline{x}_k(r), \\ \underline{x}'(t) = G_k(t, \underline{x}(t;r), \bar{x}(t,r)), \underline{x}(t_k;r) = \underline{x}_k(r), \end{cases}$$

for $r \in [0, 1]$.

5 Numerical Experiments

In this section, the exact solutions and approximated solutions obtained by Leapfrog method and He's Homotopy Perturbation Method. To show the efficiency of the He's Homotopy Perturbation Method, we have considered the following problem taken from [13], with step size $r = 0.1$ along with the exact solutions.

The discrete solutions obtained by the two methods, Leapfrog method and He's Homotopy Perturbation Method. The absolute errors between them are tabulated and are presented in Table 1. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of $'r'$ and are presented in Figures 1 – 2 for the following problem, using three dimensional effects.

5.1 Example

Consider the following hybrid fuzzy IVP, [13]

$$\left.\begin{aligned} x'(t) = x(t) + m(t)\lambda_k x(t_k), t \in [t_k, t_{k+1}], t_k = k, k = 0, 1, 2, 3, ... \\ x(t,r) = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t], 0 \le r \le 1, \end{aligned}\right\} \quad (5.14)$$

Where

$$m(t) = \begin{cases} 2(t(mod1)) & \text{if } t(mod1) \le 0.5 \\ 2(1 - t(mod1)) & \text{if } t(mod1) > 0.5 \end{cases}$$

$$\lambda_k(\mu) = \begin{cases} \hat{0} & \text{if } k = 0 \\ \mu & \text{if } k = 1, 2, ... \end{cases}$$

The hybrid fuzzy IVP (5.14) is equivalent to the following systems of fuzzy IVPs:
$x_0'(t) = x_0(t), t \in [0, 1]$,
$x(0; r) = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t], 0 \le r \le 1$,
$x_i'(t) = x_i(t) + m(t)x_{i-1}(t), t \in [t_i, t_{i+1}], x_i(t) = x_{i-1}(t_i), i = 1, 2, ...$
In (5.14) $x(t) + m(t)\lambda_k x(t_k)$ is continuous function of t, x and $\lambda_k x(t_k)$. Therefore by Example 5.1 of Kaleva [?], for each $k = 0, 1, 2, ...$ the fuzzy IVP

$$\left.\begin{aligned} x'(t) = x(t) + m(t)\lambda_k x(t_k), t \in [t_k, t_{k+1}], t_k = k, \\ x(t_k) = x_{t_K} \end{aligned}\right\} \quad (5.15)$$

has a unique solution $[t_k, t_{k+1}]$. To numerically solve the hybrid fuzzy IVP (5.15) we applied the He's Homotopy Perturbation Method for hybrid fuzzy differential equation with N = 2 to obtain $y_{1,2}(r)$ approximating $x(2.0; r)$. The Exact and Approximate solutions by Leapfrog method and He's Homotopy Perturbation Method are compared and the absolute error were shown in Table 1. From the Table 1, shows that He's Homotopy Perturbation Method approximate solutions have less error compare to Leapfrog method solutions [?] in the all the stages.

Table 1: Error calculations

	Leapfrog Error		HHPM Error	
t	$Y_1(t_i; r)$	$Y_2(t_i; r)$	$Y_1(t_i; r)$	$Y_2(t_i; r)$
0.1	1.01E-09	1.11E-09	1.01E-11	1.11E-11
0.2	2.01E-09	2.11E-09	2.01E-11	2.11E-11
0.3	3.01E-09	3.11E-09	3.01E-11	3.11E-11
0.4	4.01E-09	4.11E-09	4.01E-11	4.11E-11
0.5	5.01E-09	5.11E-09	5.01E-11	5.11E-11
0.6	6.01E-09	6.11E-09	6.01E-11	6.11E-11
0.7	7.01E-09	7.11E-09	7.01E-11	7.11E-11
0.8	8.01E-09	8.11E-09	8.01E-11	8.11E-11
0.9	9.01E-09	9.11E-09	9.01E-11	9.11E-11
1.0	1.01E-08	1.11E-08	1.01E-10	1.11E-10

6 Conclusion

The obtained results of the fuzzy hybrid differential equation show that the He's Homotopy Perturbation method works well for finding the solutions. From the Table 1, it can be observed that for most of the time intervals, the absolute error is less in He's Homotopy Perturbation method when compared to the Leapfrog method [13], which yields a little error, along with the exact solutions of the problem.

From the results shown in the Figures 1 – 2, it can be said that the error is very less in He's Homotopy Perturbation method when compared to the Leapfrog method [S. Sekar and K. Prabhavathi [13]]. Moreover,

Figure 1: Error estimation of Example 5.1 at $Y_1(t_i; r)$

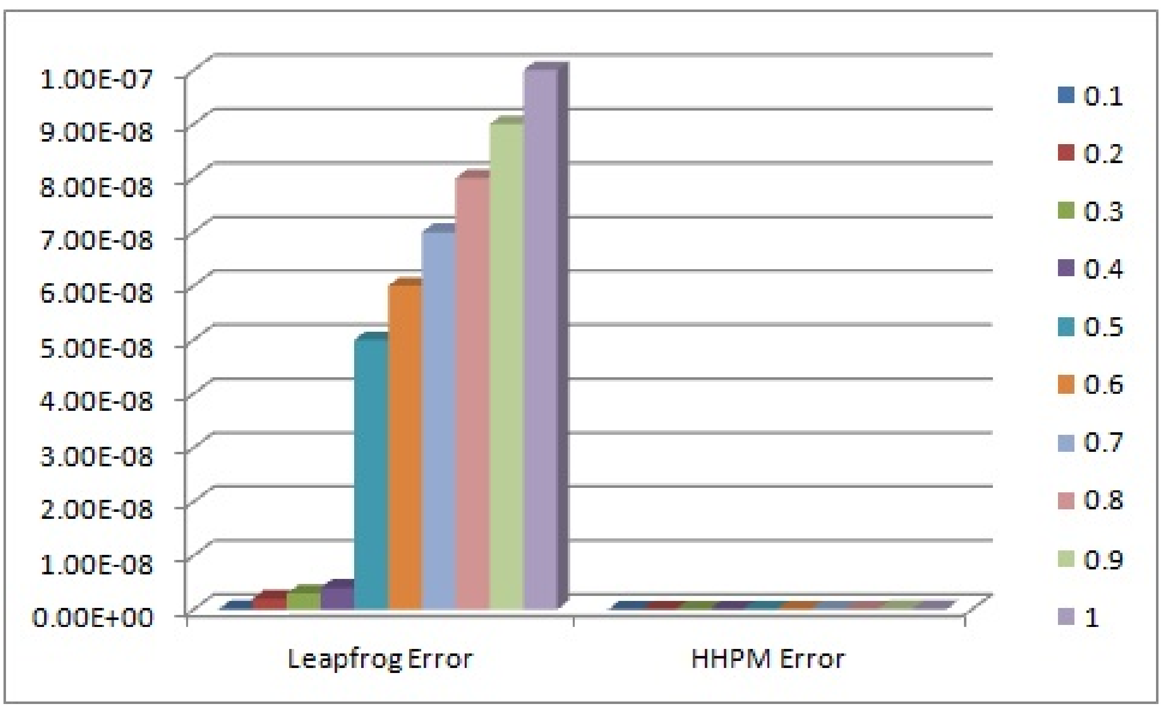

Figure 2: Error estimation of Example 5.1 at $Y_2(t_i; r)$

the He's Homotopy Perturbation method is highly stable because it is based on the Perturbation method and hence one can get the results for any length of time.

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