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The Co-bondage (Bondage) number of fuzzy graphs and its properties

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Abstract

In this paper, we define the Co-bondage number $b_c(G)$and new type of non-bondage$\{b_{en}$ and $b_{tn}\}$ for any fuzzy graph,and fuzzy strong line graph. A characterization is obtained for fuzzy strong line graphs $L_s(G)$ such that $L_s(G)$ is tree . A necessary condition for a fuzzy double strong line graph of cycle is a fuzzy trees and the exact value of $b_n(G)$ for any graph G is found and exact values of b_c, b_{en} and b_{tn} for some standard graphs are found and some bounds are obtained. Also,find the exact value of $b_{tn}(G)$ for any graph G is found. Moreover we define neighbourhood extension also analysis it properties by using bondage arcs and we also obtained relationships between b_c, $b_{tn}(G)$ and b_t.

Keywords: $\gamma(G)$- Minimum dominating set ,$b_c(G)$- maximum co-bondage number ,b(G)- minimum bondage number , b_{tn}-maximum total non-bondage number, b_{en}- maximum efficient-bondage number ,$L_s(G)$ - strong line graph,$L_s^*(G)$ - double strong line graph.

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1 Introduction

Fuzzy graph theory was introduced by A. Rosenfeld[8] in 1975. Fuzzy graph theory is now finding numerous applications in modern science and technology especially in the fields of neural networks, expert systems, information theory, cluster analysis, medical diagnosis, control theory, etc. Sunil Mathew, Sunitha M.S [10] has obtained the fuzzy graph-theoretic concepts like f- bonds, paths, cycles, trees and connectedness and established some of their properties. V.R. Kulli and B. Janakiram [7] have established the non-bondage number of a graph. First we give the definitions of basic concepts of fuzzy graphs and define the non-bondage and it is properties. All graphs consider here are finite, undirected, distinct labeling with no loop or multi arcs and p nodes and q (fuzzy) arcs. Any undefined term in this paper may be found in Harary[5]. Among the various applications of the theory of domination that have been considered, the one that is perhaps most often discussed concerns a communication network. Such a network consists of existing communication links between a fixed set of sites. The problem is to select a smallest set of sites at which to place transmitters so that every site in the network that does not have a transmitter is joined by a direct communication link to one that does have a transmitter. This problem reduces to that of finding a minimum dominating set in the graph corresponding to the network. This graph has a node representing each site and an arc between two nodes if the corresponding sites have a direct communications link joining them. To minimize the direct communication links in the network, it is non-bondage but in case we want minimum number site to control all other location sites that mean reduce the number of transmitting station. It is Co- bondage,and more

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general graph are connected but need not internal connected so determine say graph connected internal or not is called Neighbourhood Extendable we introduce the following section.

2 Preliminaries

Definition 2.1. *A fuzzy subset of a non-empty set V is a mapping $\sigma : V \to [0,1]$. A fuzzy relation on V is a fuzzy subset of $E(V \times V)$. A fuzzy graph $G = (\sigma, \mu)$ is a pair of function $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$, where $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$for all $u, v \epsilon V$.*

Definition 2.2. *The underlying crisp graph of G=(σ, μ) is denoted by $G^* = (V, E)$, where $V = \{u \epsilon V : \sigma(u) > 0\}$ and $E = \{(u,v) \epsilon V \times v : \mu(u,v) > 0\}$.*

Definition 2.3. *The order $P = \sum_{v \epsilon D} \sigma(v)$. The graph $G = (\sigma, \mu)$ is denoted by G, if unless otherwise mentioned. Let be a fuzzy graph on. The degree of a vertex u is $d_G(u) = \sum\limits_{(u \neq v)} \mu(uv)$and The minimum degree of G is $\delta(G) = \wedge \{d_G(u), u \epsilon V\}$and the maximum degree of G is $\delta(G) = \vee d_G(u), \forall \epsilon V$*

Definition 2.4. *The strength of connectedness between two nodes u and v in a fuzzy graph G is define as the maximum of the strength of all paths between u and v and is denoted by $CONN_G(u,v)$.*

Definition 2.5. *A u-v path P is called a strongest path if its strength equals $CONN_G(u,v)$.*

Definition 2.6. *A fuzzy graph $H = (\tau, \rho)$ is called a fuzzy sub graph of G if $\tau(x) \leq \sigma(x)$for all $x \epsilon V$and $\rho(x,y) \leq \mu(x,y)$ for all$(x,y) \epsilon V$.*

Definition 2.7. *A fuzzy sub graph H =(τ, ρ) is said to be a spanning fuzzy sub graph of G, if $\tau(x) = \sigma(x)$for all x.*

Definition 2.8. *A fuzzy G is said to be connected if there exists a strongest path A path P of length n is a sequence of distinct nodes $u_0 u_1, u_2, u_n$ such that $(u_{i-1}, u_i) > 0$ and degree of membership of a weakest arc is defined as its strength.*

Definition 2.9. *If $u_0 = u_n$ and $n \geq 3$, then P is called a cycle and it is a fuzzy cycle if there is more than one weak arc. Let u be a node in fuzzy graphs G then $N(u) = \{v : (u,v)\}$ is strong arc}is called neighbourhood of u and N[u] = N(u)$\bigcup$ u is called closed neighbourhood of u. Neighbourhood degree of the node is defined by the sum of the weights of the strong neighbour node of u is denoted by $d_s(u) = \sum\limits_{v \epsilon N(u)} \sigma(v)$*

3 Fuzzy dominating set

Definition 3.10. *Let G be a fuzzy graph and u be a node in G then there exist a node v such that (u, v) is a strong arc then u dominates v .*

Definition 3.11. *Let G be a fuzzy graph. A subset D of V is said to be a fuzzy dominating set if for every node $v \epsilon V \setminus D$, there exists $u \epsilon D$ such that u dominates v.*

Definition 3.12. *The domination number of G is the minimum cardinality taken over all dominating sets in G and is denoted by $\gamma(G)$,where $\gamma(G) = \sum\limits_{v \epsilon D} \sigma(v)$. A dominating set with cardinality $\gamma(G)$ is called γ - set of G.But here consider cardinality of γ is total number of elements in set*

Definition 3.13. *Let G be a fuzzy graph without isolated node. A subset D of V is said to be fuzzy total dominating set if for every node $v \epsilon V$, there exists at least one u in D such that u dominates v.*

Definition 3.14. *The domination number of G is the minimum cardinality taken over all dominating sets in G and is denoted by $\gamma_t(G)$,where $\gamma_t(G) = \sum\limits_{v \epsilon v} \sigma(v)$. A dominating set with cardinality $\gamma_t(G)$ is called $\gamma_t(G)$- set of G.But here consider cardinality of γ_t is total number of elements in set*

Definition 3.15. *Let G be a fuzzy graph without isolated node. A subset D of V is said to be fuzzy efficient dominating set if for every node $v \epsilon V - D$, there exists exact one u in D such that u dominates v.*

Definition 3.16. *The domination number of G is the minimum cardinality taken over all dominating sets in G and is denoted by $\gamma_e(G)$,where $\gamma_e(G) = \sum\limits_{v \epsilon v} \sigma(v)$. A dominating set with cardinality $\gamma_e(G)$ is called $\gamma_e(G)$- set of G.But here consider cardinality of γ_e is total number of elements in set*

4 Fuzzy non bondage number

Definition 4.17. *The bondage number b(G) of a fuzzy graph $G(V,E,\sigma,\mu))$ is minimum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that*
$CONN_{G-(x,y)}(u,v) < CONN_G(u,v)$ *for all* $u \epsilon V - \gamma(G)$ *and a* $v\ \epsilon\gamma(G)$. *Here* $\gamma(G)$ *represent minimum dominate set*

Definition 4.18. *The non-bondage number $b_n(G)$ of a fuzzy graph $G(V,E,\sigma,\mu))$ is maximum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that*
$CONN_{G-(x,y)}(u,v) = CONN_G(u,v)$ *for all* $u\epsilon V - \gamma(G)$ *and a* $v\ \epsilon\gamma(G)$.*Here* $\gamma(G)$ *represent minimum dominate set*

Definition 4.19. *The total bondage number $b_t(G)$ of a fuzzy graph $G(V,E,\sigma,\mu))$ is minimum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that*
$CONN_{G-(x,y)}(u,v) < CONN_G(u,v)$ *for all* $u\epsilon V - \gamma_t(G)$ *and a* $v\ \epsilon\gamma_t(G)$.*Here* $\gamma_t(G)$ *represent minimum dominate set*

Definition 4.20. *The total non-bondage number $b_{tn}(G)$of a fuzzy graph $G(V,E,\sigma,\mu))$ is maximum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that*
$CONN_{G-(x,y)}(u,v) = CONN_G(u,v)$ *for all* $u\epsilon V - \gamma_t(G)$ *and a* $v\ \epsilon\gamma_t(G)$.*Here* $\gamma_t(G)$ *represent minimum dominate set*

Definition 4.21. *The efficient bondage number $b_e(G)$ of a fuzzy graph $G(V,E,\sigma,\mu))$ is minimum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that*
$CONN_{G-(x,y)}(u,v) < CONN_G(u,v)$ *for all* $u\epsilon V - \gamma_e(G)$ *and a* $v\ \epsilon\gamma_e(G)$.*Here* $\gamma_e(G)$ *represent minimum dominate set*

Definition 4.22. *The efficient non-bondage number $b_{en}(G)$of a fuzzy graph $G(V,E,\sigma,\mu))$ is maximum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that*
$CONN_{G-(x,y)}(u,v) = CONN_G(u,v)$ *for all* $u\epsilon V - \gamma_e(G)$ *and a* $v\ \epsilon\gamma_e(G)$.*Here* $\gamma_e(G)$ *represent minimum dominate set*

Definition 4.23. *The co - bondage number b_c (G) of fuzzy graph G $(V,E,\sigma,\mu))$ is minimum number of fuzzy arcs required to add graph G such that $\gamma(G+e) < \gamma(G)$ and*
$\mu(e) = Max\{\mu(u,w),\mu(u,x)\}$ *or Max* $\{\mu(w,v),\mu(x,v)\}$, *for any u, v in V.*

5 Co-bondage Number

Theorem 5.1. *For any graph G,*

$$b_c \leq p - 1 - \Delta_n(G) \tag{5.1}$$

where $\Delta_n(G)$ is the total number of strong arcs in Δ of G .

Proof. We know that any graph G has p nodes then p-1 arcs are sufficient to connected all other nodes, but in case G has maximum number strong arcs$\Delta_n(G)$, so we add some arcs to G for connected to all other nodes is called co-bondage and that remain number is $p-1-\Delta_n(G)$. This proves. □

Theorem 5.2. *For any graph G,*

$$b_c(\bar{G}) \leq \delta_n(G)) \tag{5.2}$$

where $\bar{G}$ and $\delta_n(G)$ are the complement and the total number of strong arcs in $\delta$of G

Proof. A node has $q - \delta_n(G)$ strong arcs in complement of any graph, so we need to add $\delta_n(G)$ arcs for connected to all other nodes. Hence proves. □

Proposition 5.1. *For any cycle C_p (all arcs are bondage arc) with $p \geq 4$ nodes.*

$$b_c(C_p) = \begin{cases} 1, & if p = 1(mod 3) \\ 2, & if p = 2(mod 3) \\ 3, & if p = 3 otherwise \end{cases} \tag{5.3}$$

Proof. Let $C_p : v_1 v_2 ... v_p v_1$ denote a cycle on p $\geq$ 4 nodes. We consider the following case.
Case 1: if $p \equiv$ 1(mod 3), then by joining the node $v_{p-1} to v_1$,we obtain a graph G which is a cycle C_{p-1} : $v_1 v_2 ? v_{p-1} v_1$together with a path $v_{p-1} v_p v_1$. This implies that,$\gamma(G) = \gamma(C_{(p-1)}) \leq \gamma((C_p)$.This proves (case 1).
Case 2: if p$\equiv$ 2 (mod 3), then by joining the node $v_{p-1} and v_{p-2} to v_1$,we obtain a graph G which is a cycle $C_{p-2} : v_1 v_2 ... v_{p-2} v_1$ together with a path $v_{p-2} v_{p-1} v_p v_1$. This implies that, $\gamma(G) = \gamma(C_{(p-2)}) \leq \gamma((C_p)$.This proves (case 2)
Case 3: if $p \equiv$ 3 (mod 3), then by joining the node $v_{p-1}, v_{p-2}$and v_{p-3} to v_1 ,we obtain a graph G which is a cycle $C_{p-3} : v_1 v_2 ... v_{p-3} v_1$ together with a path $v_{p-3} v_{p-2} v_{p-1} v_p v_1$. This implies that, $\gamma(G) = \gamma(C_{p-3}) \leq \gamma(C_p)$.This proves (case 3). □

Proposition 5.2. *For any path P_p with $p \leq 4$ nodes.*

$$b_c(P_p) = \begin{cases} 1, & if p = 1(mod 3) \\ 2 & if p = 2(mod 3) \\ 3, & if p = 3 otherwise. \end{cases} \tag{5.4}$$

.

Theorem 5.3. *Let T be a tree with at least two cut nodes such that each cut node is adjacent to an end node then*

$$b_c(T) = r, \tag{5.5}$$

where r is the minimum number of end node adjacent to a cut node

Proof. Let S be the set of all cut node of T. then S is aγ- set for T. Let uϵS be a cut node which is adjacent to minimum number of end nodes $u_1, u_2, ..u_r$. Since there exist a cut node $v \epsilon S$such that v is adjacent to u by joining $u_1, u_2, ..u_r$to v the graph obtain has S-u as a γ- set. This proves. □

Theorem 5.4. *For any graph G,*

$$b_c(G) \leq \Delta_n(G) + 1. \tag{5.6}$$

Furthermore the bound is attained if and only if every γ- set D of G satisfying the following conditions:

- *D is independent*
- *Every node in D is of maximum degree:*
- *Every node in V-D is adjacent to exactly one node in D*

Proof. Let D be a γ- set of G . We consider the following case.
Case 1. Suppose D is not independent. Then there exit two adjacent node u,v ϵD.Let S $\subset$ V-D such that for each node w ϵ S, $N(w) \cap D = \{v\}$. Then by joining each node in S to u , we see that D-{v} is a γ- set of the resulting graph. Thus, $b_c(G) \leq |S| \leq \Delta_n(G) - 1$.
Case 2 . Suppose D is independent Then each vertex v ϵ D is an isolated node in$\langle D \rangle$. Let S be a set define in case 1.Since D has at least two node by joining each in $S \cup \{v\}$ to some node w ϵ D-{v},we obtain a graph which has D-{V} as a γ- set. Hence, $b_c(G) \leq |S \cup \{v\}| \leq \Delta_n(G) + 1$. Other parts of theorem are directly from above case. □

Corollary 5.1. *Fo any graph G*

$$bc(G) \leq min\{p - \Delta_n(G) - 1, \Delta_n(G) + 1\} \tag{5.7}$$

.

Theorem 5.5. *For any non trivial tree T,*

$$b(t) \leq 2 \tag{5.8}$$

.

Theorem 5.6. *Let T be a tree with diam(T) = 5 and has exactly two cut nodes which are adjacent to end nodes and further they have sane degree then,*

$$b_c(T) \geq b(T) + 1 \tag{5.9}$$

, where dia(T) is the diameter of T.

Theorem 5.7. *For any tree T,*

$$b_c(T) \leq 1 + min\{degu\} \tag{5.10}$$

, where u is a cut node adjacent to an end node.

Proof. Since there exist a γ- set containing u and take v other than u , then $N(u) \cap V - \gamma = deg\{u\}$ and add to v with u then $\gamma - u$ as γ -set of G and bc is minimum of such above set .hence proves. □

Theorem 5.8. *let D be a γ -set of G . If there exist a node v in D which is adjacent to every other node in D,then,*

$$b_c \leq p - \gamma(G) - 1 \tag{5.11}$$

.

Proof. we know that $\Delta_n(G) \geq deg(?v) \geq \gamma(G)$ Hence proves. □

6 Non bondage number:

Theorem 6.9. *For any graph G without isolated nodes and $\Delta_n(G) = p - 1$*

$$b_{tn}(G) = q - p + 1. \tag{6.12}$$

Proof. let G be a graph with a node u such that deg u= $\Delta_n(G) = p - 1$.then there exit a node such that e= uv ϵE. Thus u,v is a total dominating set of G. let X be the set of arcs which not strong of u. Then clearly $b_{tn} = |X| = q - \Delta_n(G) = q - p + 1$. □

Theorem 6.10. *For any graph G without isolated nodes,*

$$b_{tn} \leq q - \Delta_n(G). \tag{6.13}$$

Proof. This follow from (12) and that fact that $\Delta_n(G) \leq p - 1$. □

Theorem 6.11. *Let G be a graph without isolated nodes. If H is a sub graph of G , then*

$$b_{tn}(H) \leq b_{tn}(G). \tag{6.14}$$

Proof. Every total non bondage set of H is a total non-bondage set of G. Thus(14)holds. □

Theorem 6.12. *For a complete graph K_p , with p≥ 3 nodes,*

$$b_{tn}(K_P) = \frac{(p-1)(p-2)}{2}. \tag{6.15}$$

Proof. Let K_p be a complete graph with p≥ 3 nodes. The degree of every node of K_p is p-1 by theorem 6.9
$b_{tn}(K_P) = q - p + 1$
=$\frac{(p-1)(p)}{2} - (p-1)$
= $\frac{(p-1)(p-2)}{2}$. □

Theorem 6.13. *For a wheel W_p with $p \geq 4$ nodes,*

$$b_{tn}(W_p) = p - 1. \tag{6.16}$$

Proof. Let W_p be a wheel with $p \geq 4$ nodes. The W_p has a node such that deg u = p-1. By theorem 6.9, $b_{tn}(W_p) = p - 1$. □

Theorem 6.14. *For a complete bipartite graph $K_{m,n}$, 2≤ $m \leq n$, $b_{tn}(K_{mn})$ = mn-m-n-1.*

Theorem 6.15. *For any graph G,*

$$b_{tn}(\bar{G}) + b_{tn}(G) \leq \frac{((p-1)(p-2))}{2} \tag{6.17}$$

Proof. By Theorem 6.10 $b_{tn} \leq q - \Delta_n$,then
$b_{tn}(\bar{G}) + b_{tn}(G) \leq \bar{q} + q - (\Delta_n + \delta_n)$
$= \frac{(p(p-1))}{2} - (\Delta_n + \delta_n)$
$\leq \frac{(p(p-1))}{2} - (p-1)$
$\leq \frac{((p-2)(p-1))}{2}$. □

Theorem 6.16. *For any graph G without isolate nodes,*

$$b_t(G) \leq b_{tn}(G) + 1. \tag{6.18}$$

Proof. Let X be a b_{tn} ? set of G . then for an edge $e\epsilon G - X, X \bigcup \{e\}$is a total bondage set of G hence (18) holds. □

Theorem 6.17. *For any graph G,*

$$b_t(\bar{G}) + b_t(G) \leq \frac{((p-1)(p-2))}{2} + 2 \tag{6.19}$$

Proof. by Theorem 6.16 $b_t(\bar{G}) + b_t(G) \leq b_{tn}(\bar{G}) + b_{tn}(G) + 2 \leq ((p-1)(P-2))/2 + 2$ □

Theorem 6.18. *For any fuzzy graph G,*

$$b_{en}(G) = q - p + \gamma_e(G) \tag{6.20}$$

, where q is total number of fuzzy arcs and p is total number of node.

Proof. Let D be a minimal dominated set of G and its denote by $\gamma(G)$. For each node $v\epsilon V \setminus D$ choose exactly one strong arc which is incident to node v and to a node in D. Let E_1 be the set of all such arcs. The clearly E-E_1 is a $b_{tn}(G)$? set of G. $b_{en}(G) = q - (p - \gamma_e(G))$
$b_{en}(G) = q - p + \gamma_e(G)$ □

Corollary 6.2. *For any graph G without isolate nodes,*

$$b_{en(G)} \leq q - \Delta_n. \tag{6.21}$$

Theorem 6.19. *If K_p is a complete graph with $p \geq 3$ nodes, then*

$$b_{en}(K_p) = \frac{(p-1)(p-2}{2}. \tag{6.22}$$

Proposition 6.3. *If W_p is a wheel with $p \geq 4$ nodes ,then*

$$b_{en}(W_p) = p - 1. \tag{6.23}$$

Proof. Let W_p be a wheel with p $\geq$ 4 nodes then W_p has a node v such that deg v = p-1 $b_{en}(W_p) = p - 1$. □

Proposition 6.4. *If $K_{1,p}$ is a star with p, 1 nodes then*

$$b_{en}(K_{1,p}) = 0. \tag{6.24}$$

Proposition 6.5. *If $K_{m,n}$ is a complete bipartite graph with $2 \leq m \leq n$, then*

$$b_{en}(K_{m,n}) = mn - m - n + 2 \tag{6.25}$$

Proof. Let $K_{m,n}$ be a complete bipartite graph with $2 \leq m \leq n$ then q=mn,p=m+n and $\gamma_e(G)$=2 hence by theorem 6.18, the result □

Theorem 6.20. *For graph G and $\bar{G}$ with no isolated nodes,*

$$b_{en}(\bar{G}) + b_{en}(G) \leq \frac{((p-1)(p-2))}{2} \tag{6.26}$$

7 Relationships between b_n(G) and b

Theorem 7.21. *Let T$\neq P_4$ be a tree with at least two cut nodes .then*

$$b_n(G) \geq b(T). \tag{7.27}$$

Theorem 7.22. *For any fuzzy graph*

$$b(G) \leq b_n(G) + 1 \tag{7.28}$$

Theorem 7.23. *If G be a cycle graph then*

$$b_n(\bar{G}) + b_n(G) \leq \frac{p(p-3)}{2}, \tag{7.29}$$

$p \geq 4$

Proof. by theorem 6 $b_n(G) = q - p + \gamma(G)$
$b_n(\bar{G}) = \bar{q} - p + \gamma(\bar{G})$
$b_n(\bar{G}) + b_n(G) = q - p + \gamma(G) + \bar{q} - p + \gamma(\bar{G})$
$= q + \bar{q} - 2p + ?(\bar{G}) + \gamma(G)$
$= \frac{(p(p-1))}{2} - 2p + \gamma(\bar{G}) + \gamma(G)$
$\leq \frac{(p(p-1))}{2} - 2p + p$
$\leq \frac{(p(p-3))}{2}$. ☐

Theorem 7.24. *For any graph G,*

$$b_n(\bar{G}) + b_n(G) \leq \frac{((p-1)(p-2))}{2} \tag{7.30}$$

Proof. By Theorem 7 $b_n \leq q - \Delta_n$,then
$b_n(\bar{G}) + b_n(G) \leq \bar{q} + q - (\Delta_n + \delta_n)$
$= \frac{(p(p-1))}{2} - (\Delta_n + \delta_n)$
$\leq \frac{(p(p-1))}{2} - (p-1)$
$\leq \frac{((p-2)(p-1))}{2}$. ☐

Theorem 7.25. *If G be a cycle graph then*

$$b(\bar{G}) + b(G) \leq \frac{(p(p-3))}{2} + 2 \tag{7.31}$$

Proof. by Theorem 9
$b(G) \leq b_n(G) + 1$
$b(\bar{G}) + b(G) \leq b_n(\bar{G}) + b_n(G) + 2$
by Theorem 10
$b(\bar{G}) + b(G) \leq \frac{(p(p-3))}{2} + 2$ ☐

Theorem 7.26. *Any graph G,*

$$b(\bar{G}) + b(G) \leq \frac{((p-1)(p-2))}{2} + 2 \tag{7.32}$$

Theorem 7.27. *If G be a tree then*

$$b_n(\bar{G}) + b_n(G) \geq \gamma(\bar{G}) + \gamma(G) - 2, \tag{7.33}$$

if $p \geq 4$

Proof. $b_n(G) \geq \gamma(G) - 1$, then
$b_n(\bar{G}) + b_n(G) \geq \gamma(\bar{G}) + \gamma(G) - 2.$ □

8 Block

Definition 8.24. *A connected fuzzy graph is called block if all nodes are satisfies the condition* $CONN_{G-v}(u, v) = CONN_G(u, v)$ *for every u, v in G..*

9 Strong line graph

Definition 9.25. *Given a fuzzy graph G, its strong line graph* $L_s(G)$ *is a fuzzy graph,* $L_s(G)$ *is a graph G such that Each node of* $L_s(G)$ *represents an arc of G; and ? Two nodes of* $L_s(G)$ *are adjacent if and only if their corresponding arcs are strong and share a common end point in G.*

Definition 9.26. *Given a fuzzy graph G, its double strong line graph* $L_s^*(G)$ *is a fuzzy graph,* $L_s^*(G)$ *is a graph G such that*
Each node of $L_s^*(G)$ *represents a strong arc of G; and Two nodes of* $L_s^*(G)$ *are adjacent if and only if their corresponding arcs are strong and share a common end point in G.*

Theorem 9.28. *For any cycle fuzzy graph, then Ls(G) has one isolate node iff G has a non ? bondage arc.*

Proof. Let $L_s(G)$ has one isolated node, so G has one weakest arc (x, y) by theorem (x, y) non bondage. Conversely, let G has a non-bondage arc (x, y), clearly (x, y) weakest arc and node x and y does not common node for two strong arcs. So corresponding vertex of arc (x, y) in $L_s(G)$ is isolated. □

Theorem 9.29. *Let* $L_s(G)$ *be fuzzy black graph such that* $L_s(G)$ *is a tree. Then a node in* $L_s(G)$ *iff G has non-bondage or a arc adjacent with a non-bondage arc.*

Proof. Given $L_s(G)$ be fuzzy block then $CONN_{G-v}(u, w) = CONNG(u, w)$by definition here node of $L_s(G)$ is a arc of G, so we clearly that $CONN_{G-(x,y)}(u, w) = CONNG(u, w)$, converse true trivially. □

Theorem 9.30. *A complete fuzzy graph of*$L_s(G)$ *is not a complete.*

Proof. Given that G is compete fuzzy graph so there exist a cycle in G and every node of even pair does not adjacent with odd pair so corresponding nodes not adjacent in $L_s(G)$. □

Theorem 9.31. *Let G be a path graph n nodes then* $L_s^*(G)$ *has n-2 arcs.*

Proof. Given G be a path with n nodes then n-1 arcs so $L_s^*(G)$ has a path with n-1 nodes so it has n-2 arcs. □

Theorem 9.32. *Let G be complete graph with n nodes then* $L_s^*(G)$ *is 2(n-2) regular graph.*

Proof. Given G be complete graph so every node has n-1 strong arc then every arc adjacent with 2n-4, so $L_s^*(G)$ has 2(n-2) regular graph. □

10 Neighbourhood Extension

Definition 10.27. *Let G be graph and$S_i \subseteq V$, each S_i is collection of each nodes in G. If $G_E(\tau, \rho)$ said to be Neighbourhood extension, then satisfied following condition.*

- *Each node of G_E represents an strong neighbourhood set of G*
- *Two nodes of G_E are adjacent iff their correspond neighbour set have at least one common node where$\rho(S_i, S_j) = min\{\mu(x, v_i), \mu(v_j, x) / x \epsilon S_i \cap S_j\}$*

Definition 10.28. *Let G^* be Connected graph,if $G_E \cong G^*$ then G said be C- Neighbour Extendable graph also called strong Neighbourhood Extendable otherwise weak Neighbourhood Extendable*

Definition 10.29. *Let G^* be tree graph,if $G_E \cong G^*$ then G said be t- Neighbour Extendable graph also called semi strong Neighbourhood Extendable*

Theorem 10.33. *Let G(not distinct label) be cycle graph then G_E is complement of G.*

Proof. We know that G is connected graph and each node adjacent two nodes since G is cycle,so strong neighbour of each node of G is two.Clearly, if v_i, v_j, are adjacent nodes then $N_s(v_i) \cap N_s(v_j) = \emptyset$ but alternative nodes have some same nodes, so make arcs between them it will be form complement of G. □

Corollary 10.3. *Let G be cycle graph with n nodes then G is strong neighbour extendable if n is odd, otherwise weak neighbour extendable*

Proof. Case 1: If n is odd, by using theorem 10.33 there exist two paths in G_E ie one is $v_1, v_3, ...v_n, v_2$ say P_1 another path say P_2 is $v_2, v_4, ...v_{n-1}, v_1$ so P_1 and P_2 has same nodes say $v_1 and v_2$ clearly G_E is connected graph and G is strong neighbour extendable Case 2: If n is even, by using theorem 10.33 there exist two paths in G_E ie one is $v_1, v_3, ...v_{2n-1}v_1$ say P_1 another path say P_2 is $v_2, v_4, ...v_{2n}, v_2$ so P_1 and P_2 does not have same nodes.Clearly G_E is disconnected graph and G weak neighbour extendable □

Theorem 10.34. *Let G be complete graph with n nodes, then G is strong neighbour extendable*

Proof. We know that G complete graph then every node has n-1 strong arcs so strong neighbour set of every node has n-1 nodes $N_S(v_i) = \{v_1, v_2, ...v_{i-1}, v_{i+1}, ..v_n\} \forall v_i \epsilon G$ then $N_S(v_i) \bigcap N_s(v_j) = \{v_1, v_2, ...v_{i-1}, v_{i+1}, ..v_{j-1}, v_{j+1}, ..v_n\} \forall v_i, v_j \epsilon G$,this implies that G_E is complete graph,so G is strong neighbour extendable □

Theorem 10.35. *Let P_n be path with n nodes then G is not neighbour extendable.*

Proof. LetP_n be path with n nodes then there exist unique path , so every arcs are strong arc and two nodes have one strong neighbour other nodes have two strong neighbours, but their neighbour sets are distinct, so G_E is null graph therefore G is not neighbour extendable. □

Theorem 10.36. *Let G be complete bipartite graph then G is not strong neighbour extendable*

Proof. Given G is complete bipartite graph so node set is partition of two set say V_1 and V_2 then each node of V_1 is strong neighbour of every node in V_2 there two distinct path form in G_E ,One path connect every node in v_1 another path connect every node in V_2 so G_E is disconnect graph therefore G is not strong neighbour extendable. □

Definition 10.30. *(Deficiency Number) Let G be fuzzy graph but G is not strong neighbour extendable then the deficiency number is required number of arcs to make G is strong neighbour extendable .*

11 Conclusion

Above non bondage value ($\neq 0$) is not true for all graphs because $K_{1,n}$ or star graph and P_3 non-bondage value is 0 and also bondage number is equal to 1 for such above graphs and co-bondage of complete graph is not determine,$L^*(G)$ is not tree for all arc are strong or not distinct label .

References

[1] E.J.Cockayne, B.L.Hartnell, S.T. Hedetniemi and R. Laskar , Efficient domination in graphs, Technical Report 558 Clemson University, department of mathematical Sciences (1988)

[2] E.J. Cockayne and S.T Hedetniemi; Towards a theory of domination in graphs, Networks,247-261 (1977)

[3] J.F .Fink, M.S.Jacobson, L.F .Kinch and J.Roberts; The bondage number of graph, Discrete Math., 86, 47-57 (1990).

[4] Gerard Jennhwa Chang , Paul Dorbec ,, Mickael Montassier, Andr Raspaud:Generalized power domination of graphs,Discrete Applied Mathematics 160 (2012) 1691?1698

[5] F. Harary: Graph theory , Addison ? Wesley , Reading, Massachusetts (1963)

[6] R.JahirHussain and R.M. Karthik keyan: Characterization of fuzzy non- bondage number of fuzzy graphs. Mathematical Sciences int., Research Journal vol4 issue 2 ISSN 2278-8697

[7] V.R. Kulli and B. Janakiram; The non-bondage number of a graph , New York Acadency of Science, (1996)

[8] A.Rosenfeld. Fuzzy graphs, InFuzzysets and their applications to Cognitive and Decision. Eds. Academic press, New York (1975)77-95.

[9] K .R. Sandeep Narayan and M.S. Sunitha ; Connectivity in a Fuzzy Graph and its complement ICSRS Publication ,2012

[10] Sunil Mathew, Sunitha M.S; Fuzzy Graphs: Basic Concepts and Applications, Lap Lambert Academic Publication.

[11] G. Suresh Singh and Sunitha Grace Zacharia Some Results on Completely Extendable and Weighted Extendable Graphs Int.journal of Mathematics Archive 2016

[12] H. B.Walikar, B.D. Acharya and E.Sampathkumar , Recent developments in the Theory of Domination in Graphs, MRI Lecture Notes Math. I (1979)

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Siago's K-Fractional Calculus Operators

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Abstract

The aim of present paper is to define a pair of k–Saigo fractional integral and derivative operators involving generalized k–hypergeometric function. The Saigo-k generalized fractional operators involving k–hypergeometric function in the kernel are applied to the generalized k–Mittag-Leffler function and evaluate the formula

$$ {}_2F_{1,k}\left[\begin{array}{cc} (\alpha,k),(\beta,k) & \\ & ;\frac{1}{k} \\ (\gamma,k) & \end{array}\right] = \frac{\Gamma_k(\gamma)\Gamma_k(\gamma-\alpha-\beta)}{\Gamma_k(\gamma-\alpha)\Gamma_k(\gamma-\beta)} $$

using the integral representation for k-hypergeometric function.

Keywords: k-functions and k-fractional calculus.

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1 Introduction

The fractional k-calculus is the k-extension of the classical fractional calculus. The theory of k-calculus operators in recent past have been applied in different and numerous investigations.

Several authors that were dedicated to study such operators and since Diaz et al. [2–4] defined the k-gamma function and the k-symbol. Very recently, Rehman et al. [18] studied the properties of k-beta function. Musbeen and Rehman [14] discuss extension of k-gamma and Pochhamer k-symbol. Musbeen and Habibullah [15] defined k-fractional integration and gave an its application. Musbeen and Habibullah [16] also introduced an integral representation of some generalized confluent k-hypergeometic functions ${}_mF_{m,k}$ and k-hypergeometric functions ${}_{m+1}F_{m,k}$ by using the properties of Pochhammer k- symbols, k-gamma and k-beta functions.

In this paper we evaluate the Saigo k-fractional integral operators and derivatives involving generalized k-hypergeometric function on the k-new generalized Mittag-Leffler function introduced by us [6].

2 Definitions and Preliminaries

In this section, we state some known results and some important definitions which will be used in the sequel.

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Definition 2.1. *Generalized k-Gamma function $\Gamma_k(x)$ defined as [3]*

$$\Gamma_k(x) = \frac{\lim_{n\to\infty} n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k > 0, x \in C \tag{2.1}$$

where $(x)_{n,k}$ is the k-Pochhamer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k), \tag{2.2}$$

$x \in C, k \in R, n \in N^+$.

For $Re(x) > 0$ and $k > 0$, then $\Gamma_k(x)$ defined as the integral

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, \tag{2.3}$$

$$\text{and} \quad \Gamma_k(x+k) = x\Gamma_k(x). \tag{2.4}$$

This give rise to k-beta function defined by

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \quad x > 0,\ y > 0. \tag{2.5}$$

They have also provided some useful and applicable relations

$$B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right) \quad \text{and} \quad B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \tag{2.6}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \tag{2.7}$$

$$(1-kx)^{\frac{\alpha}{k}} = \sum_{n=0}^{\infty}(\alpha)_{n,k}\frac{x^n}{n!}, \tag{2.8}$$

$$(1-x)^{-\frac{\alpha}{k}} = \sum_{n=0}^{\infty}\frac{1}{k^n}(\alpha)_{n,k}\frac{x^n}{n!}. \tag{2.9}$$

Definition 2.2. *k-hypergeometric function F_k define by the series as [15]*

$$F_k((\beta,k);(\gamma,k);x) = \sum_{n=0}^{\infty}\frac{\beta_{n,k}x^n}{(\gamma)_{n,k}n!}, \quad k \in R^+, \beta,\gamma \in C,\ Re(\beta) > 0,\ Re(\gamma) > 0. \tag{2.10}$$

Its integral representation can be determined as follows

$${}_1F_1((\beta,k);(\gamma,k);x) = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt}dt. \tag{2.11}$$

And if $Re(\gamma) > Re(\beta) > 0, k > 0, m \geq 1, m \in Z^+$ and $|x| < 1$, then

$$\begin{aligned} &{}_{m+1}F_{m,k}\left[\begin{matrix} (\alpha,k),(\frac{\beta}{m},k),(\frac{\beta+k}{m},k),\cdots,\left(\frac{\gamma+(m-1)k}{m},k\right) \\ \\ (\frac{\gamma}{m},k),(\frac{\gamma+k}{m},k),\cdots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix} ;x\right] \\ &= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{\frac{-\alpha}{k}}dt. \end{aligned} \tag{2.12}$$

And if $Re(\gamma) > Re(\beta) > 0$ and $|x| < 1$, then

$${}_2F_{1,k}((\alpha,k),(\beta,k);(\gamma,k);x) == \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{\frac{-\alpha}{k}}dt. \tag{2.13}$$

Definition 2.3. *The generalized k-wright function* ${}_p\Psi_q^k(x)$ *defined by [8] for* $k \in R^+; x \in C, a_i, b_j \in C, \alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)$ *and* $(a_i + \alpha_i n), (b_j + \beta_j n) \in C \backslash kZ^-$

$$
{}_p\Psi_q^k(x) = {}_p\Psi_q^k \left[\begin{array}{cc} (\alpha_i, \alpha_i)_{1,p} & \\ & ; x \\ (b_j, \beta_j)_{1,q} & \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k(a_i + \alpha_i n)}{\prod_{j=1}^{q} \Gamma_k(b_j + \beta_j n)} \frac{(x)^n}{n!}, \tag{2.14}
$$

where $\Gamma_k(\cdot)$ denote the k- gamma function and satisfies the condition

$$
\sum_{j=1}^{q} \frac{\beta_j}{k} - \sum_{i=1}^{p} \frac{\alpha_i}{k} > -1. \tag{2.15}
$$

Definition 2.4. *The k-new generalized Mittag-Leffler function* $E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(x)$ *defined by [6] for* $k \in R^+, \alpha, \beta, \gamma, \delta \in C, Re(\alpha) > 0,, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, p, q > 0$ *and* $q \leq Re(\alpha) + p$

$$
E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k} x^n}{\Gamma_k(\alpha n + \beta)(\delta)_{pn,k}}. \tag{2.16}
$$

3 Generalized k-Saigo Fractional Calculus operators

In this section we define the left and right-sided Saigo k-fractional calculus operators. Let $\alpha, \beta, \gamma \in C, K > 0, x \in R^+$, then the generalized k-fractional integration and differentiation operators associated with the k-Gauss hypergeometric function are defined as follows:

$$
(I_{0+,k}^{\alpha,\beta,\gamma} f)(x) = \frac{x^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} {}_2F_{1,k}\left((\alpha+\beta, k), (-\gamma, k); (\alpha, k); (1 - \frac{t}{x}) \right) f(t) dt;
$$
$$
(Re(\alpha) > 0, k > 0), \tag{3.1}
$$

$$
(I_{-,k}^{\alpha,\beta,\gamma} f)(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{\infty} (t-x)^{\frac{\alpha}{k}-1} t^{\frac{-\alpha-\beta}{k}} {}_2F_{1,k}\left((\alpha+\beta, k), (-\gamma, k); (\alpha, k); (1 - \frac{x}{t}) \right) f(t) dt;
$$
$$
(Re(\alpha) > 0, k > 0). \tag{3.2}
$$

Here ${}_2F_{1,k}((\alpha, k), (\beta, k); (\gamma, k); x)$ is the k-Gauss hypergeometric function defined by [16] for $x \in C, |x| < 1, Re(\gamma) > Re(\beta) > 0$

$$
{}_2F_{1,k}((\alpha, k), (\beta, k); (\gamma, k); x) = \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k} (\beta)_{n,k} x^n}{(\gamma)_{n,k} n!}; \tag{3.3}
$$

The corresponding fractional differential operators have their respective forms as

$$
(D_{0+,k}^{\alpha,\beta,\gamma} f)(x) = \left(\frac{d}{dx} \right)^n (I_{0+,k}^{-\alpha+n, -\beta-n, \alpha+\gamma-n} f)(x); \quad Re(\alpha) > 0, k > 0; n = [Re(\alpha) + 1] \tag{3.4}
$$

$$
(D_{0+,k}^{\alpha} f)(x) = \left(\frac{d}{dx} \right)^n \frac{x^{\frac{\alpha+\beta}{k}}}{k\Gamma_k(-\alpha+n)} \int_0^x (x-t)^{-\frac{\alpha}{k}+n-1}
$$
$$
(\times) {}_2F_{1,k}\left((-\alpha-\beta, k), (-\gamma-\alpha+n, k); (-\alpha+n, k); (1 - \frac{t}{x}) \right) f(t) dt;
$$

$$
(D_{-,k}^{\alpha,\beta,\gamma} f)(x) = \left(-\frac{d}{dx} \right)^n (I_{-,k}^{-\alpha+n, -\beta-n, \alpha+\gamma} f)(x); \quad Re(\alpha) > 0, k > 0; n = [Re(\alpha) + 1] \tag{3.5}
$$

$$
(D_{-,k}^{\alpha,\beta,\gamma} f)(x) = \left(-\frac{d}{dx} \right)^n \frac{1}{k\Gamma_k(-\alpha+n)} \int_x^{\infty} (t-x)^{-\frac{\alpha+n}{k}-1} t^{\frac{\alpha+\beta}{k}}
$$
$$
(\times) {}_2F_{1,k}\left((-\alpha-\beta, k), (-\alpha-\gamma, k); (-\alpha+n, k); (1 - \frac{x}{t}) \right) f(t) dt;
$$

where $x > 0, \alpha \in C, Re(\alpha) > 0, k > 0$ and $[Re(\alpha)]$ is the integer part of $Re(\alpha)$.

For $K \to 1$, the operators (3.1) to (3.5) reduce to Saigo's [19] fractional integer and differentiation operators. If we set $\beta = -\alpha$, operators (3.1) to (3.5) reduce to k-Riemann-Liouville operators as follows:

$$(I^{\alpha,-\alpha,\gamma}_{0+,k} f)(x) = (I^{\alpha}_{0+,k} f)(x), \tag{3.6}$$

$$(I^{\alpha,-\alpha,\gamma}_{-,k} f)(x) = (I^{\alpha}_{-,k} f)(x), \tag{3.7}$$

$$(D^{\alpha,-\alpha,\gamma}_{0+,k} f)(x) = (D^{\alpha}_{0+,k} f)(x), \tag{3.8}$$

$$(D^{\alpha,-\alpha,\gamma}_{-,k} f)(x) = (D^{\alpha}_{-,k} f)(x). \tag{3.9}$$

4 Main Result

In this section, we find out the main result.

Theorem 4.1.

$$ {}_2F_{1,k}\left[\begin{matrix} (\alpha,k),(\beta,k) \\ \\ (\gamma,k) \end{matrix} ; \frac{1}{k}\right] = \frac{\Gamma_k(\gamma)(\gamma-\alpha-\beta)}{\Gamma_k(\gamma-\alpha)\Gamma_k(\gamma-\beta)} \tag{4.1}$$

Proof. From equation (2.13), we have the following result

$$ {}_2F_{1,k}\left[\begin{matrix} (\alpha,k),(\beta,k) \\ \\ (\gamma,k) \end{matrix} ; \frac{1}{k}\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{t}-1}(1-xt)^{\frac{-\alpha}{k}} dt.$$

Put $x = 1$ in equation (4.1), we obtain the following

$$ {}_2F_{1,k}\left[\begin{matrix} (\alpha,k),(\beta,k) \\ \\ (\gamma,k) \end{matrix} ; \frac{1}{k}\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\alpha-\beta}{t}-1} dt.$$

On applying the definition of k-beta function, we get the required result

$$ {}_2F_{1,k}\left[\begin{matrix} (\alpha,k),(\beta,k) \\ \\ (\gamma,k) \end{matrix} ; \frac{1}{k}\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} B_k(\beta,\gamma-\alpha-\beta)$$

$$ {}_2F_{1,k}\left[\begin{matrix} (\alpha,k),(\beta,k) \\ \\ (\gamma,k) \end{matrix} ; \frac{1}{k}\right] = \frac{\Gamma_k(\gamma)\Gamma_k(\gamma-\alpha-\beta)}{\Gamma_k(\gamma-\alpha)\Gamma_k(\gamma-\beta)}.$$

□

Lemma 4.1. *Let $\alpha,\beta,\gamma,\rho \in C, Re(\alpha) > 0, k \in R^+(0,\infty)$*

(a) If $Re(\rho) > \max[0, Re(\beta-\gamma)]$, then

$$(I^{\alpha,\beta,\gamma}_{0+,k} t^{\frac{\rho}{k}-1})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)}{\Gamma_k(\rho-\beta)\Gamma_k(\rho+\alpha+\gamma)} x^{\frac{\rho-\beta}{k}-1}. \tag{4.2}$$

(b) If $Re(\rho) > \max[Re(-\beta), Re(-\gamma)]$, then

$$(I^{\alpha,\beta,\gamma}_{-,k} t^{-\frac{\rho}{k}})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho+\beta)\Gamma_k(\rho+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho+\alpha+\beta+\gamma)} x^{\frac{-\rho-\beta}{k}}. \tag{4.3}$$

Proof. **(a):** Taking $f(x) = t^{\frac{\rho}{k}-1}$ in (3.1), we get

$$(I_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \frac{x^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} {}_2F_{1,k}\Big((\alpha+\beta,k),(-\gamma,k);(\alpha,k);(1-\frac{t}{x})\Big) t^{\frac{\rho}{k}-1} dt.$$

We invoke *k*-Gauss hypergeometric series [16] and on changing the order of integration and summation, we have

$$(I_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \frac{x^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_0^x (x-t)^{\frac{\alpha}{k}-1} (1-\frac{t}{x})^n t^{\frac{\rho}{k}-1} dt.$$

On setting $t = xu$, we get

$$(I_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \frac{x^{\frac{\rho-\beta}{k}-1}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_0^1 (1-u)^{\frac{\alpha}{k}+n-1} u^{\frac{\rho}{k}-1} du.$$

On evaluating the inner integral by *k*-beta function and using relation (2.3) and (2.7), we have

$$\begin{aligned} &= x^{\frac{\rho-\beta}{k}-1} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha+\rho)_{n,k} n!} \frac{\Gamma_k(\rho)}{\Gamma_k(\alpha+\rho)} \\ &= x^{\frac{\rho-\beta}{k}-1} \frac{\Gamma_k(\rho)}{\Gamma_k(\alpha+\rho)} \sum_{n=0}^{\infty} k^n {}_2F_{1,k}\Big((\alpha+\beta,k),(-\gamma,k);(\alpha+\rho,k);\frac{1}{k}\Big). \end{aligned} \tag{4.4}$$

Finally use theorem (4.1) and rearrange terms, expression (4.4) yields

$$(I_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho)(\rho-k+\gamma)}{\Gamma_k(\rho-\beta)\Gamma_k(\rho+\alpha+\gamma)} x^{\frac{\rho-\beta}{k}-1}.$$

proof (b): Taking $f(x) = t^{\frac{-\rho}{k}}$ in (3.2), we get

$$(I_{-,k}^{\alpha,\beta,\gamma} t^{\frac{-\rho}{k}})(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{\infty} (t-x)^{\frac{\alpha}{k}-1} t^{\frac{-\alpha-\beta}{k}} {}_2F_{1,k}\Big((\alpha+\beta,k),(-\gamma,k);(\alpha,k);(1-\frac{x}{t})\Big) t^{\frac{-\rho}{k}} dt.$$

On applying *k*-Gauss hypergeometric series [16] and on changing the order of integration and summation, and

$$(I_{-,k}^{\alpha,\beta,\gamma} t^{\frac{-\rho}{k}})(x) = \frac{1}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_x^{\infty} (t-x)^{\frac{\alpha}{k}-1} t^{\frac{-\alpha-\beta}{k}} \left(1-\frac{x}{t}\right)^n t^{\frac{-\rho}{k}} dt.$$

Put $t = \frac{x}{u}$, we have

$$(I_{-,k}^{\alpha,\beta,\gamma} t^{\frac{-\rho}{k}})(x) = \frac{x^{\frac{-\rho-\beta}{k}}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_0^1 (1-u)^{\frac{\alpha}{k}+n-1} u^{\frac{\rho+\beta}{k}-1} du.$$

On evaluating the inner integral by *k*-beta function and using relation (2.3) and (2.7), we have

$$\begin{aligned} &= x^{\frac{-\rho-\beta}{k}} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha+\beta+\rho)_{n,k} n!} \frac{\Gamma_k(\rho+\beta)}{\Gamma_k(\alpha+\beta+\rho)} \\ &= x^{\frac{-\rho-\beta}{k}} \frac{\Gamma_k(\rho+\beta)}{\Gamma_k(\alpha+\beta+\rho)} \sum_{n=0}^{\infty} k^n {}_2F_{1,k}\Big((\alpha+\beta,k),(-\gamma,k);(\alpha+\beta+\rho,k);\frac{1}{k}\Big). \end{aligned}$$

Finally use theorem (4.1) and rearrange terms, expression (4.5) yields

$$(I_{-,k}^{\alpha,\beta,\gamma} t^{-\frac{\rho}{k}})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho+\beta)\Gamma_k(\rho+\gamma)}{\Gamma_k(\rho)\Gamma_k(\alpha+\beta+\rho+\gamma)} x^{\frac{-\rho-\beta}{k}}.$$

□

5 Left side Saigo k-Fractional Integration of the generalized k-Mittag-Leffler function

In this section we have discussed the left-sided Siago k-fractional integration formula of the generalized k-Mittag-Leffler function.

Theorem 5.1. *Let $\alpha,\beta,\gamma,\rho,\delta,\xi \in C$ and $k \in R^+$ be such that $Re(\alpha) > 0, Re(\rho+\gamma-\beta) > 0, p,q > 0, q \leq Re(\nu)+p$. Also, let $c \in R$ and $\nu > 0$. If condition (2.15) is satisfied and $I_{0+,k}^{\alpha,\beta,\gamma}$ be the left sided operator of the generalized k-fractional integration associated with k-Gauss hypergeometric function, then there holds the following relationship*

$$\left(I_{0+,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\rho}{k}-1}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{\frac{\nu}{k}}]\right)\right)(x) = \frac{x^{\frac{\rho-\beta}{k}-1}\Gamma_k(\xi)}{\Gamma_k(\delta)}\,{}_3\Psi_3^k\left[\begin{array}{c}(\rho+\gamma-\beta,\nu),(\delta,qk),(k,k)\\ \\ (\rho-\beta,\nu),(\rho+\alpha+\gamma,\nu),(\xi,pk)\end{array};cx^{\frac{\nu}{k}}\right] \tag{5.1}$$

Proof. Applying (2.16) and (4.2) in the left-side of (5.1), we have

$$\begin{aligned}&\left(I_{0+,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\rho}{k}-1}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{\frac{\nu}{k}}]\right)\right)(x)\\ &= I_{0+,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\rho}{k}-1}\sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}(ct^{\frac{\nu}{k}})^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\right)(x)\\ &= \sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}c^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}I_{0+,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\nu n+\rho}{k}-1}\right)(x)\\ &= x^{\frac{\rho-\beta}{k}-1}\sum_{n=0}^{\infty}\frac{\Gamma_k(\rho+\gamma-\beta+\nu n)\Gamma_k(\delta+qkn)\Gamma_k(\xi)\Gamma_k(1+n)}{\Gamma_k(\delta)\Gamma_k(\xi+pnk)\Gamma_k(\rho+\alpha+\gamma+\nu n)\Gamma_k(\rho-\beta+\nu n)}\frac{(ckx^{\frac{\nu}{k}})^n}{n!}.\end{aligned}$$

On using $\Gamma(n+1) = k^{-n}\Gamma_k(nk+k)$, we get required result

$$\left(I_{0+,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\rho}{k}-1}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{\frac{\nu}{k}}]\right)\right)(x) = \frac{x^{\frac{\rho-\beta}{k}-1}\Gamma_k(\xi)}{\Gamma_k(\delta)}\,{}_3\Psi_3^k\left[\begin{array}{c}(\rho+\gamma-\beta,\nu),(\delta,qk),(k,k)\\ \\ (\rho-\beta,\nu),(\rho+\alpha+\gamma,\nu),(\xi,pk)\end{array};cx^{\frac{\nu}{k}}\right]$$

□

Remark 5.1. *If we put $k = 1$ in equation (5.1), we arrive at the result [6, p.140, Eq.2.1].*

Remark 5.2. *If we set $p = q = k = \xi = 1$ in our formula (5.1), we get the result [1, p.116, Eq.3.1].*

6 Right side Saigo k-Fractional Integration of the generalized k-Mittag-Leffler function

In this section we have discussed the right-sided Siago k-fractional integration formula of the generalized k-Mittag-Leffler function.

Theorem 6.1. *Let $\alpha,\beta,\gamma,\rho,\delta,\xi \in C$ and $k \in R^+$ be such that $Re(\alpha) > 0, Re(\alpha+\rho) > \max[-Re(\beta), -Re(\gamma)]$ with condition $Re(\beta) \neq Re(\gamma), \nu > 0, p,q > 0, q \leq Re(\nu)+p$. Also, let $c \in R, \nu \in R, \nu > 0$ and $I_{-,k}^{\alpha,\beta,\gamma}$ be the right sided operator of the generalized k-fractional integration associated with k-Gauss hypergeometric function, then there holds the formula:*

$$\begin{aligned}&\left(I_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{-\alpha-\rho}{k}}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{-\frac{\nu}{k}}]\right)\right)(x)\\ &= \frac{x^{\frac{-\alpha-\beta-\rho}{k}}\Gamma_k(\xi)}{\Gamma_k(\delta)}\,{}_4\Psi_4^k\left[\begin{array}{c}(\alpha+\beta+\rho,\nu),(\alpha+\gamma+\rho,\nu),(\delta,qk),(k,k)\\ \\ (\alpha+\rho,\nu),(2\alpha+\beta+\rho+\gamma,\nu),(\xi,pk),(\rho,\nu)\end{array};cx^{-\frac{\nu}{k}}\right]\end{aligned} \tag{6.1}$$

Proof. Applying (2.16) and (4.3) in the left-side of (6.1), we have

$$\left(I_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{-\alpha-\rho}{k}}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{-\frac{\nu}{k}}]\right)\right)(x)$$

$$= I_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{-\alpha-\rho}{k}}\sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}(ct^{-\frac{\nu}{k}})^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\right)(x)$$

$$= \sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}c^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}I_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{-(\nu n+\alpha+\rho)}{k}}\right)(x)$$

$$= x^{\frac{-\alpha-\beta-\rho}{k}}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha+\rho+\beta+\nu n)\Gamma_k(\alpha+\rho+\gamma+\nu n)\Gamma_k(\delta+qkn)\Gamma_k(\xi)\Gamma_k(1+n)}{\Gamma_k(\delta)\Gamma_k(\xi+pnk)\Gamma_k(\rho+\nu n)\Gamma_k(\alpha+\rho+\nu n)\Gamma_k(2\alpha+\beta+\rho+\gamma+\nu n)}\frac{(ckx^{-\frac{\nu}{k}})^n}{n!}.$$

On using $\Gamma(n+1)=k^{-n}\Gamma_k(nk+k)$, we get required result

$$\left(I_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{-\alpha-\rho}{k}}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{-\frac{\nu}{k}}]\right)\right)(x)$$

$$=\frac{x^{\frac{-\alpha-\beta-\rho}{k}}\Gamma_k(\xi)}{\Gamma_k(\delta)}\,{}_4\Psi_4^k\left[\begin{matrix}(\alpha+\beta+\rho,\nu),(\alpha+\gamma+\rho,\nu),(\delta,qk),(k,k) \\ (\alpha+\rho,\nu),(2\alpha+\beta+\rho+\gamma,\nu),(\xi,pk),(\rho,\nu)\end{matrix};cx^{-\frac{\nu}{k}}\right]$$

□

Remark 6.1. *If we put $k=1$ in equation (6.1), we can obtain the result [6, p.141, Eq.2.3].*

Remark 6.2. *If we set $p=q=k=\xi=1$ in (6.1), we get the result [1, p.117, Eq.4.1].*

Lemma 6.2. *Let $\alpha,\beta,\gamma,\rho\in C, n=[Re(\alpha)]+1, k\in R^+(0,\infty)$*

(a) If $Re(\rho)>\max[0,Re(-\alpha-\beta-\gamma)]$, then

$$(D_{0+,k}^{\alpha,\beta,\gamma}t^{\frac{\rho}{k}-1})(x)=\frac{\Gamma_k(\rho)\Gamma_k(\rho+\beta+\gamma+\alpha)}{\Gamma_k(\rho+\gamma)\Gamma_k(\rho+\beta+n-nk)}x^{\frac{\rho+\beta+n}{k}-n-1}. \tag{6.2}$$

(b) If $Re(\rho)>\max[Re(-\alpha-\gamma),Re(\beta-nk+n)]$, then

$$(D_{-,k}^{\alpha,\beta,\gamma}t^{-\frac{\rho}{k}})(x)=\frac{\Gamma_k(\rho-\beta-n+nk)\Gamma_k(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)}x^{\frac{-\rho+\beta+n}{k}-n}. \tag{6.3}$$

Proof. **(a):** Taking $f(x)=t^{\frac{\rho}{k}-1}$ in (3.4) and using (4.2), we get

$$(D_{0+,k}^{\alpha,\beta,\gamma}t^{\frac{\rho}{k}-1})(x)=\left(\frac{d}{dx}\right)^n[I_{0+,k}^{-\alpha+n,-\beta-n,\alpha+\gamma-n}t^{\frac{\rho}{k}-1}](x)$$

$$=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\rho)\Gamma_k(\rho+\beta+\alpha+\gamma)}{\Gamma_k(\rho+\beta+n)\Gamma_k(\rho+\gamma)}\left(\frac{d}{dx}\right)^n x^{\frac{\rho+\beta+n}{k}-1}$$

$$=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\rho)\Gamma_k(\rho+\beta+\alpha+\gamma)}{\Gamma_k(\rho+\beta+n)\Gamma_k(\rho+\gamma)}\frac{\Gamma(\frac{\rho+\beta+n}{k})}{\Gamma(\frac{\rho+\beta+n}{k}-n)}x^{\frac{\rho+\beta+n}{k}-n-1}.$$

Applying (2.3) in above equation, we get

$$(D_{0+,k}^{\alpha,\beta,\gamma}t^{\frac{\rho}{k}-1})(x)=\frac{\Gamma_k(\rho)\Gamma_k(\rho+\beta+\gamma+\alpha)}{\Gamma_k(\rho+\gamma)\Gamma_k(\rho+\beta+n-nk)}x^{\frac{\rho+\beta+n}{k}-n-1}.$$

proof (b): Taking $f(x)=t^{-\frac{\rho}{k}}$ in (3.5) and using (4.3), we get

$$(D_{-,k}^{\alpha,\beta,\gamma}t^{-\frac{\rho}{k}})(x)=\left(-\frac{d}{dx}\right)^n[I_{-,k}^{-\alpha+n,-\beta-n,\alpha+\gamma}t^{-\frac{\rho}{k}}](x)$$

$$=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\rho-\beta-n)\Gamma_k(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)}\left(-\frac{d}{dx}\right)^n x^{\frac{-\rho+\beta+n}{k}}$$

$$=\sum_{n=0}^{\infty}(-1)^n k^n\frac{\Gamma_k(\rho-\beta-n)\Gamma_k(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)}\frac{\Gamma(\frac{-\rho+\beta+n+k}{k})}{\Gamma(\frac{-\rho+\beta+n-nk+k}{k})}x^{\frac{-\rho+\beta+n}{k}-n}.$$

On using (2.3), we have

$$= \sum_{n=0}^{\infty} (-1)^n k^n \frac{\Gamma_k(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)} \frac{k^{\frac{\rho-\beta-n}{k}-1}\Gamma\big(\frac{\rho-\beta-n}{k}\big)\Gamma\big(\frac{-\rho+\beta+n}{k}+1\big)}{\Gamma\big(\frac{-\rho+\beta+n-nk+k}{k}\big)} x^{\frac{-\rho+\beta+n}{k}-n}. \tag{6.4}$$

The reflection formula for gamma function see [5]

$$\Gamma(\frac{\rho-\beta-n}{k})\Gamma\left(1-(\frac{\rho-\beta-n}{k})\right) = \frac{\pi}{\sin\left(\frac{\rho-\beta-n}{k}\right)\pi}, \tag{6.5}$$

and

$$\begin{aligned}
\frac{1}{\Gamma\left(1-(\frac{\rho-\beta-n+nk}{k})\right)} &= \frac{\Gamma\left(\frac{\rho-\beta-n+nk}{k}\right)}{\Gamma\left(\frac{\rho-\beta-n+nk}{k}\right)\Gamma\left(1-(\frac{\rho-\beta-n+nk}{k})\right)} \\
&= \Gamma\left(\frac{\rho-\beta-n+nk}{k}\right)\frac{\sin\left(\frac{\rho-\beta-n+nk}{k}\right)\pi}{\pi} \\
&= \Gamma\left(\frac{\rho-\beta-n+nk}{k}\right)\frac{\sin\left(\frac{\rho-\beta-n}{k}\right)\pi\cos n\pi}{\pi} \\
&= k^{1-\left(\frac{\rho-\beta-n+nk}{k}\right)}\Gamma_k(\rho-\beta-n+nk)\frac{\sin\left(\frac{\rho-\beta-n}{k}\right)\pi\cos n\pi}{\pi},
\end{aligned} \tag{6.6}$$

using (6.5) and (6.6) in (6.4), we obtain required result

$$(D^{\alpha,\beta,\gamma}_{-,k}t^{-\frac{\rho}{k}})(x) = \frac{\Gamma_k(\rho-\beta-n+nk)\Gamma_(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)}x^{\frac{-\rho+\beta+n}{k}-n}.$$

□

7 Left and right side Saigo k-Fractional Differentiation of the generalized k-Mittag-Leffler function

In this section we have discussed the left and right sided Siago k-fractional differentiation formula of the generalized k-Mittag-Leffler function.

Theorem 7.1. *Let* $\alpha,\beta,\gamma,\rho,\delta,\xi \in C$ *and* $k \in R^+$ *be such that* $Re(\alpha) > 0, Re(\rho+\beta+\gamma) > 0, \nu > 0, p,q > 0, q \leq Re(\nu)+p, c \in R$ *and* $D^{\alpha,\beta,\gamma}_{0+,k}$ *be the left sided operator of the generalized k-fractional differentiation then there holds the formula:*

$$\begin{aligned}
&\left(D^{\alpha,\beta,\gamma}_{0+,k}\left(t^{\frac{\rho}{k}-1}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{\frac{\nu}{k}}]\right)\right)(x) \\
&= \frac{x^{\frac{\rho+\beta}{k}-1}\Gamma_k(\xi)}{\Gamma_k(\delta)}{}_3\Psi^k_3\left[\begin{matrix}(\rho+\beta+\gamma+\alpha,\nu),(\delta,qk),(k,k) \\ \\ (\rho+\gamma,\nu),(\rho+\beta,\nu+1-k),(\xi,pk)\end{matrix};k^{-1}cx^{\frac{\nu+1-k}{k}}\right]
\end{aligned} \tag{7.1}$$

Proof. Applying (2.16) and (6.2) in the left-side of (7.1), we have

$$\left(D^{\alpha,\beta,\gamma}_{0+,k}\left(t^{\frac{\rho}{k}-1}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{\frac{\nu}{k}}]\right)\right)(x)$$
$$= D^{\alpha,\beta,\gamma}_{0+,k}\left(t^{\frac{\rho}{k}-1}\sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}(ct^{\frac{\nu}{k}})^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\right)(x)$$
$$= \sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}c^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\left(D^{\alpha,\beta,\gamma}_{0+,k}t^{\frac{\nu n+\rho}{k}-1}\right)(x)$$
$$= x^{\frac{\rho+\beta}{k}-1}\frac{\Gamma_k(\xi)}{\Gamma_k(\delta)}\sum_{n=0}^{\infty}c^n\frac{\Gamma_k(\rho+\beta+\alpha+\gamma+\nu n)\Gamma_k(\delta+qkn)\Gamma_k(1+n)}{\Gamma_k(\xi+pnk)\Gamma_k(\rho+\gamma+\nu n)\Gamma_k(\rho+\beta+\nu n+n-nk)n!}.$$

On applying $\Gamma(n+1)=k^{-n}\Gamma_k(nk+k)$, we get required result

$$\left(D^{\alpha,\beta,\gamma}_{0+,k}\left(t^{\frac{\rho}{k}-1}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{\frac{-\nu}{k}}]\right)\right)(x)$$
$$= \frac{x^{\frac{\rho+\beta}{k}-1}\Gamma_k(\xi)}{\Gamma_k(\delta)}{}_3\Psi^k_3\left[\begin{array}{c}(\rho+\beta+\gamma+\alpha,\nu),(\delta,qk),(k,k)\\ \\ (\rho+\gamma,\nu),(\rho+\beta,\nu+1-k),(\xi,pk)\end{array};k^{-1}cx^{\frac{\nu+1-k}{k}}\right]$$

□

Remark 7.1. *If we put $k=1$ in equation (7.1), we get the result [6, p.142, Eq.2.4].*

Remark 7.2. *If we put $p=q=k=\xi=1$ in our formula (7.1), we get the result [1, p.119, Eq.5.1].*

Theorem 7.2. *Let $\alpha,\beta,\gamma,\rho,\delta,\xi\in C$ and $k\in R^+$ be such that $Re(\alpha)>0, Re(\rho)>\max[Re(\alpha+\beta)+n-Re(\gamma)], \nu>0, p,q>0, q\leq Re(\nu)+p$ andc $\in R, Re(\alpha+\beta-\gamma)+n\neq 0$, (where $n=[Re(\alpha)+1]$) and $D^{\alpha,\beta,\gamma}_{-,k}$ be the right sided operator of the generalized k-fractional differentiation then there holds the formula:*

$$\left(D^{\alpha,\beta,\gamma}_{-,k}\left(t^{\frac{\alpha-\rho}{k}}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{\frac{\nu}{k}}]\right)\right)(x)$$
$$= \frac{x^{\frac{\alpha+\beta-\rho}{k}}\Gamma_k(\xi)}{\Gamma_k(\delta)}{}_4\Psi^k_4\left[\begin{array}{c}(\rho-\alpha-\beta,\nu-1+k),(\rho+\gamma,\nu),(\delta,qk),(k,k)\\ \\ (\rho,\nu),(\rho-\alpha,\nu)(\rho+\gamma-\alpha-\beta,\nu),(\xi,pk)\end{array};k^{-1}cx^{\frac{-\nu+1-k}{k}}\right] \quad (7.2)$$

Proof. Applying (2.16) and (6.3) in the left-side of (7.2), we have

$$\left(D^{\alpha,\beta,\gamma}_{-,k}\left(t^{\frac{\alpha-\rho}{k}}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{\frac{-\nu}{k}}]\right)\right)(x) = D^{\alpha,\beta,\gamma}_{-,k}\left(t^{\frac{\alpha-\rho}{k}}\sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}(ct^{-\frac{\nu}{k}})^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\right)(x)$$
$$= \sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}c^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\left(D^{\alpha,\beta,\gamma}_{-,k}t^{-\frac{(\nu n+\rho-\alpha)}{k}}\right)(x).$$

Implying the simplification process used for providing preceding theorems, we obtain

$$\left(D^{\alpha,\beta,\gamma}_{-,k}\left(t^{\frac{\alpha-\rho}{k}}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{\frac{-\nu}{k}}]\right)\right)(x)$$
$$= \frac{x^{\frac{\alpha+\beta-\rho}{k}}\Gamma_k(\xi)}{\Gamma_k(\delta)}{}_4\Psi^k_4\left[\begin{array}{c}(\rho-\alpha-\beta,\nu-1+k),(\rho+\gamma,\nu),(\delta,qk),(k,k)\\ \\ (\rho,\nu),(\rho-\alpha,\nu)(\rho+\gamma-\alpha-\beta,\nu),(\xi,pk)\end{array};k^{-1}cx^{\frac{-\nu+1-k}{k}}\right]$$

□

Remark 7.3. *On taking $k=1$ in equation (7.2), we can produce the result [6, p.143, Eq.(2.5)].*

Remark 7.4. *on setting $p=q=k=\xi=1$ in equation (7.2), we obtained the result [1, p.120, Eq.(6.1)].*

References

[1] V.B.L. Chaurasia and S.C.Pandey, Scientia, On the fractional calculus of generalized Mittag-Leffler function, *Series A: Mathematical Sciences*, 20(2010), 113–122.

[2] R. Diaz and C. Teruel, q, k-Generalized gamma and beta functions, *J. Nonlinear Math. Phys.*, 12(1) (2005), 118-134.

[3] R. Diaz and E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, *Divulg. Mat.*, 15(2) (2007), 179-192.

[4] R. Diaz, C. Ortiz and E. Pariguan, On the k-gamma q-distribution, *Cent. Eur. J. Math.*, 8 (2010), 448-458.

[5] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I. McGraw-Hill, New York-Toronto-London (1953).

[6] A.Gupta and C.L.Parihar, K-new generalized Mittag-Leffler function, *Journal of fractional calculus and applications*, 5(1) (2014), 165-176.

[7] A.Gupta and C.L.Parihar, Fractional differintegral operators of the generalized Mittag-Leffler function, *Journal of fractional calculus and applications*, 5(1) (2014), 165-176.

[8] K.S.Gehlot and J.C.Prajapati, Fractional calculus of generalized k-Wright function, *Journal of fractional calculus and applications*, 4(2) (2013), 283-289.

[9] C. G. Kokologiannaki, Properties and inequalities of generalized k-gamma, beta and zeta functions, *Int. J. Contemp. Math. Sciences*, 5(14) (2010), 653-660.

[10] K.K. Kataria and P.Vellaisamy, The generalized K–Wright function and Marichev-Saigo-Maeda fractional operators, *arXiv: 1408.4762v1 [math.CA]*, 2014, 1-15.

[11] V. Krasniqi, A limit for the k-gamma and k-beta function, *Int. Math. Forum*, 5(33) (2010), 1613-1617

[12] M. Mansour, Determining the k-generalized gamma function $\Gamma_k(x)$ by functional equations, *Int. J. Contemp. Math. Sciences*, 4(21) (2009), 1037-1042.

[13] S. Mubeen, K-analogue of Kummers first formula, *Journal Of inequalities and Special Functions*, 3(3) (2012),41–44.

[14] S. $Mubeen^*$ and A. Rehman, A note on k-gamma function and pochhamer k-symbol, *Journal of inequalities and Mathematical Sciences*, 6(2),(2014), 93-107.

[15] S. Mubeen and G. M. Habibullah, k-Fractional Integrals and Application, *Int. J. Contemp. Math. Science*, 7(2) (2012), 89-94.

[16] S. Mubeen and G. M. Habibullah, An integral representation of some k-hypergeometric functions, *Int. J. Contemp. Math. Science*, 7(4) (2012), 203-207.

[17] L. Romero and R. Cerutti, Fractional Calculus of a k-Wrigth type function, *Int. J. Contemp. Math. Science*, 7(31) (2012), 1547-1557.

[18] A. Rehman, S. $Mubeen^*$, R. Safdar and N. Sadiq, Properties of k-beta function with several variables, *Open Mathematics*, 13(2015), 308-320.

[19] M.Saigo, A Remark on integral operators involving the Gauss hypergeometric functions, *Math .Rep. Kyushu Univ*, No.11 (1978), 135-143.

[20] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon et al. (1993).

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Some Remarks on Semi A-Segal Algebras

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Abstract

In the present paper, we investigate some results on A-Segal algebras. Furthermore, the notion of a semi A-Segal algebra is introduced and some results are given. As an important result, we prove that if B is a finite-dimensional Banach algebra so that $\dim B > 1$, then there is an ideal B' of B such that it is semi B-Segal.

Keywords and Phrases: Semi A-Segal algebra, A-Segal algebra, division algebra.

AMS Subject Classifications (2010): 46H05, 43A20.

1 Introduction

The concept of Segal and abstract Segal algebras were first introduced and studied in [5]. Many authors have considered this notion ever since and investigated several properties of these algebras such as different kinds of amenability, BSE property, Arens regularity and so on (see for instance [1], [3] and [4]). An abstract Segal algebra is a Banach algebra which is perceived as a certain ideal in another Banach algebra. In this paper we focus our attention on a new concept called semi A-Segal algebra and obtain some results on the kind of algebras. Our work is motivated by relative completion of an A-Segal algebra defined by Burnham [2].

2 Main Results

It is known and easy to show that the complex Banach space $\mathbb{C}^n$ is a $\mathbb{R}^{2n}$-Segal algebra. Now, according to Burnham's notation [2]. we have

$$\widetilde{\mathbb{C}^n}^{\mathbb{R}^{2n}} = \bigcup_{\eta>0} \overline{S_{\mathbb{C}^n}(\eta)}^{\mathbb{R}^{2n}}$$

Where $S_{\mathbb{C}^n}(\eta) = \{x = (x_1, \ldots, x_n) \in \mathbb{C}^n : \|x\|_{\mathbb{C}^n} \leq \eta\}$.

Notice that since $\mathbb{C}^n$ is isometrically isomorphic to $\mathbb{R}^{2n}$, thus we have

$$\|\cdot\|_{\mathbb{C}^n} = \|\cdot\|_{\mathbb{R}^{2n}}.$$

Moreover, the following is a norm on $\widetilde{\mathbb{C}^n}^{\mathbb{R}^{2n}}$. We leave verification of the properties to the reader.

$$|||x||| = \inf\left\{t : x \in \overline{S_{\mathbb{C}^n}(t)}^{\mathbb{R}^{2n}}\right\}$$

Now, as an important result, we have:

Theorem 2.1. $\widetilde{\mathbb{C}^n}^{R^{2n}} = \mathbb{R}^{2n}$.

Proof. Since $\bigcup_{\eta>0} \overline{S_{\mathbb{C}^n}(\eta)}^{\mathbb{R}^{2n}} = \mathbb{R}^{2n}$, hence the proof is finished. □

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As a corollary, we obtain the following.

Corollary 2.2. $\widetilde{\mathbb{C}^n}^{\mathbb{R}^{2n}}$ *is a* $\mathbb{R}^{2n}$*-Segal algebra.*

Proof. Since $\mathbb{C}^n$ is a $\mathbb{R}^{2n}$-Segal algebra, so by [2], $\widetilde{\mathbb{C}^n}^{\mathbb{R}^{2n}}$ is also. □

Next, we obtain an interesting result on division algebras.

Theorem 2.3. *Every division algebra is a* $\mathbb{R}^{2n}$*-Segal algebra.*

Proof. Let B be a division algebra. Therefore, by [5] B is a copy of the complex plane $\mathbb{C}$ and in view of proof of the Corollary 2.2 (with $n = 1$), we conclude that it is a $\mathbb{R}^2$-Segal algebra. □

Theorem 2.4. *For any* $x \in \mathbb{C}^n$*, we have*

$$\|x\|_{\mathbb{C}^n} = |||x|||$$

Proof. Since $\mathbb{C}^n$ is a $\mathbb{R}^{2n}$-Segal algebra, thus proof follows from the Theorem6 of [2]. Furthermore, $\|x\|_{\mathbb{R}^{2n}} = |||x|||$. □

Theorem 2.5. $\bigcup_{\eta>0} S_{\mathbb{C}^n}(\eta) = \mathbb{R}^{2n}$.

Proof. According to Theorem 7 of [2]. because of $\mathbb{C}^n$ is a $\mathbb{R}^{2n}$-Segal algebra, we have

$$S_{\mathbb{C}^n}(\eta) = \overline{S_{\mathbb{C}^n}(\eta)}^{\mathbb{R}^{2n}} \cap \mathbb{C}^n$$

Therefore,

$$\bigcup_{\eta>0} S_{\mathbb{C}^n}(\eta) = \left(\bigcup_{\eta>0} \overline{S_{\mathbb{C}^n}(\eta)}^{\mathbb{R}^{2n}} \right) \cap \mathbb{C}^n = \mathbb{R}^{2n} \cap \mathbb{C}^n = \mathbb{R}^{2n}$$

This completes the proof. □

The next result is about the singularity of $\mathbb{C}^n$ as a $\mathbb{R}^{2n}$–Segal algebra.

Theorem 2.6. $\mathbb{C}^n$ *is singular.*

Proof. The proof follows the fact that $\mathbb{C}^n \subseteq \widetilde{\mathbb{C}^n}^{\mathbb{R}^{2n}} = \mathbb{R}^{2n}$ as well as Theorem11 of [2]. □

Here, we wish to introduce a new notion so-called semi A-Segal algebra.

Definition 2.7. *Let* $(A, \|\cdot\|_A)$ *be a Banach algebra. A subalgebra* B *of* A *is called a semi A-Segal algebra with respect to* $\|\cdot\|_B$ *when the following conditions are satisfied:*

(i) B is an ideal(not necessarily dense) in A such that it is a Banach algebra with respect to $\|\cdot\|_B$*;*

(ii) Natural injection from B into A is continuous and the product is a jointly continuous function from $A \times B$ *into B.*

Clearly, every A-Segal algebra is also semi A-Segal. But the converse need not be true; for instance, $\mathbb{R}$ is semi $\mathbb{R}^2$-Segal but not $\mathbb{R}^2$-Segal.

Now, we give some results on semi A-Segal algebras.

Theorem 2.8. *Suppose that m and n are two positive integers with* $n > m$*. Then* $\mathbb{R}^m$ *is a semi* $\mathbb{R}^n$*-Segal algebra.*

Proof. The natural embedding from $\mathbb{R}^m$ into $\mathbb{R}^n$ shows that $\mathbb{R}^m$ is an ideal in $\mathbb{R}^n$. The other conditions of Definition 2.7 are easy to verification. □

Theorem 2.9. *Let B be a finite-dimensional Banach algebra with an even dimension. Then there is an ideal* B' *of B such that it is B-Segal.*

Proof. Suppose that $\dim B = 2k$, for some positive integer k. It is easy to show that B is linearly isomorphic to $\mathbb{R}^k$. Now, by taking $B' = \mathbb{C}^k$, one can arrive at the desired result. □

The following holds for semi A-Segal algebras.

Theorem 2.10. *Let B be a finite-dimensional Banach algebra so that* $\dim B > 1$. *Then there is an ideal* B' *in B such that it is semi B-Segal.*

Proof. Assume that

$$\dim B = k \quad (k \in N, k > 1).$$

As before, B is linearly isomorphic to $\mathbb{R}^k$. We set $B' = \mathbb{R}^{k-1}$. Now, Theorem 2.8 completes the proof. □

References

[1] M.Alaghmandan, Approximate amenability of Segal algebras II, Canad. Math. Bull. 58(1) (2015), 3-6,.

[2] J.T.Burnham, The Relative Completion of an A-Segal Algebra is Closed, Proceeding of the American Mathematical Society, 49(1) (1975), 116-122.

[3] F.Ghahramani and A.T.Lau, Approximate weak amenability, derivations and Arens regularity of Segal algebras, Studia Math, 169(2) (2005), 189-205.

[4] J.Inoue and S.E.Takahasi, Segal algebras in commutative Banach algebras, Rocky Mountain. J. Math, 44(2) (2014), 539-589.

[5] H.Reiter, and J.D.Stegeman, Classical Harmonic Analysis and Locally Compact Groups. London Mathematical Society Monographs: New Series 22. Oxford: Clarendon Press, 2000.

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On Some Decompositions of Continuity via $\delta-$Local Function in Ideal Topological Spaces

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Abstract

We introduce the notions of $\delta^* - pre - continuity$, $\delta^* - B_t - continuity$, and $\delta^* - \beta - continuity$, $\delta^* - B_\beta - continuity$ and to obtain some decompositions of continuity via $\delta - local$ function in ideal topological spaces.

Keywords: $\delta - pre-$open set, $\delta - \beta-$open set, $\beta-$open set, $\beta - I-$open set, $\delta - \alpha^*-$open set, $\delta^* - \alpha-$open set, decomposition of continuity.

2010 MSC: 54C08, 54C10.

1 Introduction and Preliminaries

Ideals in topological spaces have been considered since 1930.This topic has won its importance by Vaidyanathaswamy [13]. Janković and Hamlett investigated further properties of ideal topological space [7]. Recently, in [3] Hatir et al. have introduced and studied $\delta-$local function in ideal topological space. In this paper, we have obtained decompositions of continuity using $\delta-$ local functions in ideal topological spaces.

Throughout this paper, spaces (X, τ) and (Y, τ) (or simply X and Y), always mean topological spaces on which no separation axiom is assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ), respectively.

A subset A of a topological space (X, τ) is said to be regular open (resp. regular closed) [12] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A is called $\delta - open$ [12] if for each $x \epsilon A$, there exists a regular open set G such that $x \in G \subset A$. The complement of a $\delta - open$ set is called $\delta - closed$. A point $x \in X$ is called a $\delta - cluster$ point of A if $Int(Cl(U)) \cap A \neq \phi$ for each open set V containing x. The set of all $\delta - cluster$ points of A is called the $\delta - closure$ of A and is denoted by $\delta Cl(A)$. The $\delta - interior$ of A is the union of all regular open sets of X contained in A and it is denoted by $\delta Int(A)$. A is $\delta - open$ if $\delta Int(A) = A$. $\delta - open$ sets forms a topology τ^δ. τ^δ is the same as the collection of all $\delta - open$ sets of (X, τ) and is denoted by $\delta O(X)$.

An ideal on a topological space (X, τ) is a nonempty collection of subsets of X which is satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$, (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. An ideal topological space is a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^*$: $P(X) \to P(X)$ called a local function [7, 8] of A with respect to τ and I is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X : U \cap A \notin I \; for \; every \; U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$, simply write A^* instead of $A^*(I, \tau)$. For every ideal topological space, there exists a topology $\tau^*(I)$ or briefly τ^*, finer than τ, generated by $\beta(I, \tau) = \{U - W : U \in \tau \text{ and } W \in I\}$, but in general $\beta(I, \tau)$ is not always a topology [7]. Also $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$. If $A \in \tau^*$, $Int^*(A) = A$ and $Int^*(A)$ will denote the τ^* interior of A. If I is an ideal on X then (X, τ, I) is called an ideal topological space.

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Recently, Hatir et al. [3] introduced $\delta-local$ function in ideal topological spaces in the following manner. Let (X,τ,I) be an ideal topological space and A be a subset of X. Then $A^{\delta_*}(I,\tau)=\{x\in X : U\cap A\notin I \; for \; every \; U\in\delta O(X,x)\}$ is called the $\delta-local$ function of I on X with respect to I and τ. We denote simply A^{δ_*} for $A^{\delta_*}(I,\tau)$. Furthermore, $Cl^{\delta_*}(A)=A\cup A^{\delta_*}$ defines a Kuratowski closure operator for $\tau^{\delta_*}(I)$. We will denote τ^{δ_*} the topology generated by Cl^{δ_*}, that is, $\tau^{\delta_*}=\{U\subset X : Cl^{\delta_*}(X-U)=X-U\}$. Therefore, the topology τ^{δ_*} finer than τ^{δ} and also the topology τ^* finer than τ^{δ_*}.

Lemma 1.1. *[3] Let (X,τ,I) be an ideal topological space and A, B subsets of X. Then*
1) If $A\subset B$, then $Cl^{\delta_*}(A)\subset Cl^{\delta_*}(B)$
2) $Cl^{\delta_*}(A\cap B)\subset Cl^{\delta_*}(A)\cap Cl^{\delta_*}(B)$
3) If $U\in\tau^{\delta}$, then $U\cap Cl^{\delta_*}(A)\subset Cl^{\delta_*}(U\cap A)$
4) $Cl^{\delta_*}(\cup_i(A_i))=\cup_i(Cl^{\delta_*}(A_i))$
5) If $I\subset J$, then $Cl^{J\delta_*}(A)\subset Cl^{I\delta_*}(A)$, ($J$ is ideal)

First we shall recall some definitions used in the sequel.

Definition 1.1. *A subset A of an ideal topological space (X,τ,I) is said to be*
1) $\alpha-I-open$ [4] if $A\subset Int(Cl^*(Int(A)))$,
2) $pre-I-open$ [2] if $A\subset Int(Cl^*(A))$,
3) $\beta-I-open$ [4] if $A\subset Cl(Int(Cl^*(A)))$,
4) $\delta^*-\alpha-open$ [6] if $A\subset Int(Cl^{\delta_*}(Int^*(A)))$,
5) $\delta-\alpha^*-open$ [6] if $A\subset Int(\delta Cl(Int^*(A)))$.

Definition 1.2. *A subset A of a topological space (X,τ) is said to be*
1) $\alpha-open$ [10] if $A\subset Int(Cl(Int(A)))$,
2) $pre-open$ [9] if $A\subset Int(Cl(A))$,
3) $\beta-open$ [1] if $A\subset Cl(Int(Cl(A)))$,
4) $\delta-pre-open$ [11] if $A\subset Int(\delta Cl(A))$,
5) $\delta-\beta-open$ [5] if $A\subset Cl(Int(\delta Cl(A)))$.

Definition 1.3. *Let $f:(X,\tau,I)\longrightarrow(Y,\sigma)$ be a function. If for each $V\in\sigma$, $f^{-1}(V)$ is a $\alpha-I-open$ (resp. $pre-I-open$, $\beta-I-open$, $\delta^*-\alpha-open$, $\delta-\alpha^*-open$), then f is said to be $\alpha-I-continuous$ [4] (resp. $pre-I-continuous$ [2], $\beta-I-continuous$ [4], $\delta^*-\alpha-continuous$ [6], $\delta-\alpha^*-continuous$ [6]).*

Definition 1.4. *Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a function. If for each $V\in\sigma$, $f^{-1}(V)$ is an $\alpha-open$ (resp. $pre-open$, $\beta-open$, $\delta-pre-open$, $\delta-\beta-open$), then f is said to be $\alpha-continuous$ [10] (resp. $pre-continuous$ [9], $\beta-continuous$ [1], $\delta-pre-continuous$ [11], $\delta-\beta-continuous$ [5]).*

2 $\delta^*-pre-open$ set and $\delta^*-\beta-open$ set

We give the following generalized open sets to obtain new decompositions of continuity.

Definition 2.5. *A subset A of an ideal topological space (X,τ,I) is said to be*
1) $\delta^*-pre-open$ if $A\subset Int(Cl^{\delta_*}(A))$,
2) $\delta^*-\beta-open$ if $A\subset Cl(Int(Cl^{\delta_*}(A)))$.

Proposition 2.1. *1) Every $\alpha-I-open$ set is $\delta^*-\alpha-open$,*
2) Every $\delta^*-\alpha-open$ set is $\delta-\alpha^*-open$,
3) Every $\delta-\alpha^*-open$ set is $\delta-pre-open$,
4) Every $\delta^*-\alpha-open$ set is $\delta^*-pre-open$,
5) Every $\delta^*-pre-open$ set is $\delta-pre-open$,
6) Every $\delta^*-pre-open$ set is $\delta^*-\beta-open$,
7. Every $\delta^*-\beta-open$ set is $\delta-\beta-open$.

Proof. Straightforward from the definitions of the topologies τ^*, τ^{δ} and τ^{δ_*} and [6]. □

Remark 2.1. *None of them in the proposition 1 is reversible as shown by examples below. Also $\alpha-open$ set and $\delta^*-\alpha-open$ [6], $pre-open$ set and $\delta^*-pre-open$, $\beta-open$ set and $\delta^*-\beta-open$ are independent notions.*

Example 2.1. *Let $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a\},\{b,d\},\{a,c\},\{a,b,d\}\}$ and $I=\{\phi,\{a\}\}$. Then $\tau^*=\{\phi,X,\{a\},\{c\},\{b,d\},\{a,c\},\{a,b,d\},\{b,c,d\}\}$ and $\tau^\delta=\{\phi,X,\{b,d\},\{a,c\}\}$. Take $A=\{b,c,d\}$. Therefore, A is a $\delta^*-pre-open$ set and $\delta^*-\beta-open$ set, but neither $pre-open$ nor $\beta-open$ and not $pre-I-open$ set.*

Example 2.2. *Let $X=\{a,b,c,d,e\}$, $\tau=\{\phi,X,\{a\},\{c\},\{a,c\},\{a,b\},\{a,b,c\}\}$ and $I=\{\phi,\{a\}\}$. Then $\tau^\delta=\{\phi,X,\{c\},\{a,b\},\{a,b,c\}\}$. Take $A=\{a,c,d\}$. Therefore, since $Int(Cl^{\delta_*}(A))=\{a,c\}$ and $Int(Cl(A))=X$, A is $pre-open$ set and $\alpha-open$ set, $\delta-pre-open$ set and also $\delta^*-\beta-open$ set, but neither $\delta^*-pre-open$ nor $\delta^*-\alpha-open$.*

Example 2.3. *Let $X=\{a,b,c,d,e\}$, $\tau=\{\phi,X,\{a\},\{c,e\},\{a,c,e\},\{a,b\},\{a,b,c,e\}\}$ and $I=\{\phi,\{e\}\}$. Then $\tau^\delta=\{\phi,X,\{c,e\},\{a,b\},\{a,b,c,e\}\}$. Take $A=\{a,e\}$. Therefore, A is $\beta-open$ set and also $\delta-\beta-open$ set, but not $\delta^*-\beta-open$ since $\{a,e\}\nsubseteq Cl(Int(Cl^{\delta_*}(A)))=\{a,b,d\}$.*

Proposition 2.2. *The arbitrary union of $\delta^*-pre-open$ sets ($\delta^*-\beta-open$ sets) are $\delta^*-pre-open$ set ($\delta^*-\beta-open$ set).*

Proof. Let A_i be $\delta^*-pre-open$ sets for every i. Then, $A_i\subset Int(Cl^{\delta_*}(A_i))$ for every i. Hence, $\cup_iA_i\subset\cup_i(Int(Cl^{\delta_*}(A_i)))\subset Int(Cl^{\delta_*}(\cup_iA_i))$ by Lemma 1.1(4). Consequently, $\cup_iA_i$ is $\delta^*-pre-open$ set. For $\delta^*-\beta-open$ set, the proof is similar. □

Remark 2.2. *The intersection of two $\delta^*-pre-open$ sets ($\delta^*-\beta-open$ sets) need not be a $\delta^*-pre-open$ set ($\delta^*-\beta-open$ set) as in the following example.*

Example 2.4. *Let $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a\},\{a,c\},\{a,d\},\{a,c,d\}\}$ and $I=\{\phi,\{b\}\}$.Then $\tau^\delta=\{\phi,X\}$ and $\tau^*=\tau$. Take $A=\{b,c\}$ and $B=\{a,b\}$ are $\delta^*-pre-open$ set and $\delta^*-\beta-open$ set , but $A\cap B=\{b\}$ is neither $\delta^*-pre-open$ set nor $\delta^*-\beta-open$ set since $Cl(Int(Cl^{\delta_*}(\{b\})))=\phi$.*

Corollary 2.1. *[3] Let (X,τ,I) be an ideal topological space and $A\subset X$.*
1) If $A\subset A^{\delta_*}$, then $\delta Cl(A)=Cl^{\delta_*}(A)$
2) If $I=\{\phi\}$, then $\delta Cl(A)=Cl^{\delta_*}(A)$.

Proposition 2.3. *Let (X,τ,I) be an ideal topological space and $A\subset X$. If $A\subset A^{\delta_*}$ (If $I=\{\phi\}$), then*
1) $\delta-pre-open$ set and $\delta^*-pre-open$ set are equivalent
2) $\delta-\beta-open$ set and $\delta^*-\beta-open$ set are equivalent.

Proof. By Corollary 2.1, if $A\subset X$, then it $\delta Cl(A)=Cl^{\delta_*}(A)$. Thus we get the result. □

Proposition 2.4. *Let (X,τ,I) be an ideal topological space and $A,B\subset X$. Then the following statements hold:*
1) If $A\in\tau^\delta$ and B is $\delta^*-pre-open$ set, then $A\cap B$ is $\delta^*-pre-open$ set,
2) If $A\in\tau^\delta$ and B is $\delta^*-\beta-open$ set, then $A\cap B$ is $\delta^*-\beta-open$ set.

Proof. 1) Let $A\in\tau^\delta$ and B is $\delta^*-pre-open$ set. Then,

$$\begin{aligned}A\cap B&\subset\delta Int(A)\cap Int(Cl^{\delta_*}(B))=\delta Int(\delta Int(A))\cap Int(Cl^{\delta_*}(B))\\&\subset Int(\delta Int(A)\cap Int(Cl^{\delta_*}(B)))=Int(\delta Int(A)\cap Cl^{\delta_*}(B))\\&\subset Int(Cl^{\delta_*}(A\cap B))\quad\text{(by Lemma 1.1.)}\end{aligned}$$

The proof of (2) are same with the proof of (1). □

Proposition 2.5. *Let I and J be any two ideals on a topological space (X,τ) with $I\subset J$. If a subset A of X is $\delta^*-pre-(J)open$ set ($\delta^*-\beta-(J)open$ set), then it is $\delta^*-pre-(I)open$ set ($\delta^*-\beta-(I)open$ set).*

Proof. Follows from directly Lemma 1.1(5). □

The above discussions are summarized in the following diagram.

Diagram 1

$$\begin{array}{cccccccccccc}
\tau & \rightarrow & \alpha IO & & -\rightarrow & PIO & & --\rightarrow & \beta IO & & & \\
& & | & \searrow & & | & \searrow & & | & \searrow & & \\
& & | & & \alpha O & --\rightarrow & & PO & --\rightarrow & & \beta O & \\
& & \downarrow & & | & \downarrow & & | & \downarrow & & | & \\
& & \delta^*\alpha O & & -\rightarrow & \delta^*PO & & --\rightarrow & \delta^*\beta O & & | & \\
& & & \searrow & \downarrow & & \searrow & \downarrow & & \searrow & \downarrow & \\
& & & & \delta\alpha^*O & --\rightarrow & & \delta PO & --\rightarrow & & \delta\beta O &
\end{array}$$

By αIO,(resp. PIO, βIO, αO, PO, βO, $\delta^*\alpha O$, δ^*PO, $\delta^*\beta O$, $\delta\alpha^*O$, δPO, $\delta\beta O$) in diagram, we denote the family of all $\alpha-I-open$ sets (resp. $pre-I-open$, $\beta-I-open$, $\alpha-open$, $pre-open$, $\beta-open$, $\delta^*-\alpha-open$, $\delta^*-pre-open$, $\delta^*-\beta-open$, $\delta-\alpha^*-open$, $\delta-pre-open$, $\delta-\beta-open$) of a space (X,τ) and (X,τ,I).

Definition 2.6. *A subset A of an ideal topological space (X,τ,I) is called*
1) A $\delta^*-t-set$ if $Int(A)=Int(Cl^{\delta_*}(A))$,
2) A $\delta^*-\beta-set$ if $Int(A)=Cl(Int(Cl^{\delta_*}(A)))$,
3) A $\delta-\alpha^*-set$ [6] if $Int(A)=Int(\delta Cl(Int^*(A)))$,
4) A $\delta^*-\alpha-set$ [6] if $Int(A)=Int(Cl^{\delta_*}(Int^*(A)))$.

Proposition 2.6. *Let A be a subset of an ideal topological space (X,τ,I). The following properties hold:*
1). Every $\delta^*-t-set$ is $\delta^*-\alpha-set$,
2) Every $\delta^*-\beta-set$ is $\delta^*-t-set$,
3) Every $\delta-\alpha^*-set$ is $\delta^*-\alpha-set$ [6].

Proof. Straightforward from the definitions of the topologies τ^{δ} and τ^{δ_*} and [6]. □

Remark 2.3. *None of them in Proposition 2.5 is reversible as shown by examples below. Also the notions of $\delta^*-t-set$ and $\delta-\alpha^*-set$ are independent notions [6].*

Example 2.5. *Let $X=\{a,b,c,d,e\}$, $\tau=\{\phi,X,\{a\},\{c\},\{a,c\},\{a,b\},\{a,b,c\}\}$ and $I=\{\phi,\{c\}\}$. Then $\tau^{\delta}=\{\phi,X,\{c\},\{a,b\},\{a,b,c\}\}$ and $\tau^*=\{\phi,X,\{a\},\{c\},\{a,c\},\{a,b\},\{a,b,c\},\{a,b,d,e\}\}$. Take $A=\{b,c\}$. Therefore A is $\delta^*-\alpha-set$ and $\delta-\alpha^*-set$, but not $\delta^*-\beta-set$ and not $\delta^*-t-set$ since $Cl(Int(Cl^{\delta_*}(\{b,c\})))=X\neq Int(\{b,c\})$ and $\{c\}=Int(\{b,c\})=Int(Cl^{\delta_*}(Int^*(\{b,c\})))=Int(\delta Cl(Int^*(\{b,c\})))=\{c\}$.*

In this example if we take $A=\{c,d\}$, we obtain that A is $\delta^*-t-set$, but not $\delta^*-\beta-set$ since $Int(\{c,d\})\neq Cl(Int(Cl^{\delta_*}(\{c,d\})))=\{c,d,e\}$ and $Int(\{c,d\})=Int(Cl^{\delta_*}(\{c,d\}))=\{c\}$.

Example 2.6. *Let $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a\},\{d\},\{b,d\},\{a,d\},\{a,b,d\}\}$ and $I=\{\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{a,b,d\}\}$. Then $\tau^{\delta}=\{\phi,X,\{a\},\{b,d\},\{a,b,d\}\}$ and $\tau^*=\wp(X)$. if we take $A=\{b,c\}$, then A is $\delta^*-t-set$ and $\delta^*-\alpha-set$, but not $\delta-\alpha^*-set$ since $Int(\{b,c\})=Int(Cl^{\delta_*}(\{b,c\}))=\phi$ and $Int(\{b,c\})=\phi\neq Int(\delta Cl(Int^*(\{b,c\})))=\{b,d\}$.*

Definition 2.7. *Let (X,τ,I) be an ideal topological space. A subset A in X is said to be a δ^*-B_t-set (resp. $\delta^*-B_{\beta}-set$, $\delta^*-B_{\alpha}-set$ [6], $\delta-B\alpha^*-set$ [6]) if there is a $U\in\tau$ and a $\delta^*-t-set$ (resp. $\delta^*-\beta-set$, $\delta-\alpha^*-set$, $\delta^*-\alpha-set$) V in X such that $A=U\cap V$.*

Proposition 2.7. *For a subset A of a space (X,τ,I), the following properties hold:*
1) Every $\delta^*-t-set$ is δ^*-B_t-set,
2) Every $\delta^*-\beta-set$ is $\delta^*-B_{\beta}-set$,
3) Every $\delta-\alpha^*-set$ is $\delta-B\alpha^*-set$ [6],
4) Every $\delta^*-\alpha-set$ is $\delta^*-B_{\alpha}-set$ [6],
5) Every open set is δ^*-B_t-set (resp. $\delta^*-B_{\beta}-set$, $\delta-B\alpha^*-set$, $\delta^*-B_{\alpha}-set$).

Proof. Since $A=A\cap X$ and $X\in\tau$, we get 1-4, also if $A\in\tau$, we get 5. □

Remark 2.4. *None of them in Proposition 2.6 is reversible as shown by example below and [6].*

Example 2.7. *Let* $X = \{a,b,c,d,e\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a,c\}, \{a,b\}, \{a,b,c\}\}$ *and* $I = \{\phi, \{c\}\}$ *and* $\tau^{\delta} = \{\phi, X, \{c\}, \{a,b\}, \{a,b,c\}\}$. *If we take* $A = \{a\}$, *then* A *is* $\delta^* - B_t - set$ ($\delta^* - B_\beta - set$), *but not* $\delta^* - t - set$ ($\delta^* - \beta - set$) *since* $\{a\} \in \tau$ *and* $\{a\} = \{a\} \cap X$ *also* $Int(Cl^{\delta_*}(\{a\})) = \{a,b\} \neq Int(\{a\})$. *In this example,* $\{c,d\}$ *is* $\delta^* - B_t - set$ *and* $\delta^* - B_\beta - set$, *but* $\{c,d\} \notin \tau$.

By Proposition 2.5, we have the following diagram

Diagram 2

$$\begin{array}{ccccc} \delta^* - B_\beta - set & \Longrightarrow & \delta^* - B_t - set & \Longrightarrow & \delta^* - B_\alpha - set \\ & & & & \Uparrow \\ & & & & \delta - B\alpha^* - set \end{array}$$

Theorem 2.1. *Let* A *be a subset of an ideal topological space* (X, τ, I). *Then the following statements are equivalent:*

1) A *is open,*
2) A *is* $\delta^* - pre - open$ *and* $\delta^* - B_t - set$,
3) A *is* $\delta^* - \beta - open$ *and* $\delta^* - B_\beta - set$,
4) A *is* $\delta^* - \alpha - open$ *and* $\delta^* - B_\alpha - set$ *[6],*
5) A *is* $\delta - \alpha^* - open$ *and* $\delta - B\alpha^* - set$ *[6].*

Proof. (1)$\Longrightarrow$(2). This is obvious from diagrams 1-2 and Proposition 2.6 (5).

(2)$\Longrightarrow$(1). Since A is a $\delta^* - B_t - set$, we have $A = U \cap V$, where U is an open set and $Int(V) = Int(Cl^{\delta_*}(V))$. By the hypothesis, A is also $\delta^* - pre - open$, and we have

$$\begin{aligned} A \subset Int(Cl^{\delta_*}(A)) &= Int(Cl^{\delta_*}(U \cap V)) \subset Int(Cl^{\delta_*}(U) \cap Cl^{\delta_*}(V)) \\ &= Int(Cl^{\delta_*}(U)) \cap Int(Cl^{\delta_*}(V)) = Int(Cl^{\delta_*}(U)) \cap Int(V). \end{aligned}$$

Hence

$$\begin{aligned} A &= U \cap V = (U \cap V) \cap U \subset (Int(Cl^{\delta_*}(U)) \cap Int(V)) \cap U \\ &= (Int(Cl^{\delta_*}(U)) \cap U) \cap Int(V) = U \cap Int(V). \end{aligned}$$

Notice $A = U \cap V \supset U \cap Int(V)$. Therefore, we obtain $A = U \cap Int(V)$.

(1) $\Longleftrightarrow$ (3). The proof is same with (1) $\Longleftrightarrow$ (2). □

3 Decompositions of continuity

Definition 3.8. *Let* $f : (X, \tau, I) \longrightarrow (Y, \sigma)$ *be a function. If for each* $V \in \sigma$, $f^{-1}(V)$ *is a* $\delta^* - pre - open$ *set* ($\delta^* - \beta - open$ *set*), *then f is said to be* $\delta^* - pre - continuous$ ($\delta^* - \beta - continuous$).

Definition 3.9. *Let* $f : (X, \tau, I) \longrightarrow (Y, \sigma)$ *be a function. If for each* $V \in \sigma$, $f^{-1}(V)$ *is a* $\delta^* - B_t - set$ (*resp.* $\delta^* - B_\beta - set$, $\delta^* - B_\alpha - set$, $\delta - B\alpha^* - set$), *then f is said to be* $\delta^* - B_t - continuous$ (*resp.* $\delta^* - B_\beta - continuous$, $\delta^* - B_\alpha - continuous$ *[6],* $\delta - B\alpha^* - continuous$ *[6]*).

By Diagrams 1-2, we have the following proposition.

Proposition 3.8. *1) A* $\delta^* - B_\beta - continuous$ *function is* $\delta^* - B_t - continuous$,
2) A $\delta^* - B_t - continuous$ *function is* $\delta^* - B_\alpha - continuous$,
3) A $\delta - B\alpha^* - continuous$ *function is* $\delta^* - B_\alpha - continuous$,
4) A $\delta^* - \alpha - continuous$ *function is* $\delta^* - pre - continuous$,
5) A $\delta^* - pre - continuous$ *function is* $\delta^* - \beta - continuous$,
6) A $\delta^* - \alpha - continuous$ *function is* $\delta - \alpha^* - continuous$,
7) A $\delta - \alpha^* - continuous$ *function is* $\delta - pre - continuous$,
8) A $\delta^* - pre - continuous$ *function is* $\delta - pre - continuous$.

By Theorem 2.1, we have the following main theorem.

Theorem 3.2. *For a function $f : (X, \tau, I) \longrightarrow (Y, \sigma)$, the following properties are equivalent:*
1) f is continuous,
2) f is $\delta^* - pre -$ continuous and $\delta^* - B_t -$ continuous,
3) f is $\delta^* - \beta -$ continuous and $\delta^* - B_\beta -$ continuous.

Remark 3.5. *1) $\delta^* - pre -$ continuous and $\delta^* - B_t -$ continuous are independent of each other,*
2) $\delta^* - \beta -$ continuous and $\delta^* - B_\beta -$ continuous are independent of each other.

Example 3.8. *Let $X = Y = \{a, b, c, d, e\}$, $\tau_1 = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, b, c\}\}$, $I = \{\phi, \{a\}\}$ and then $\tau^\delta = \{\phi, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and also $\tau_2 = \{\phi, Y, \{a, b\}\}$. Define a function $f : (X, \tau_1, I) \longrightarrow (Y, \tau_2)$ as follows: $f(a) = f(c) = a$, $f(b) = c$, $f(d) = b$, $f(e) = d$. Then f is $\delta^* - B_t -$ continuous, but not $\delta^* - pre -$ continuous since $f^{-1}(\{a, b\}) = \{a, c, d\}$ and $\{a, c\} = Int\{a, c, d\} = Int(Cl^{\delta_*}(\{a, c, d\})) = \{a, c\}$, thus $\{a, c, d\}$ is $\delta^* - B_t -$ set, but not $\delta^* - pre - open$.*

Example 3.9. *Let $X = Y = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, d\}, \{a, c, d\}\}$, $I = \{\phi, \{b\}\}$ and then $\tau^\delta = \{\phi, X\}$, $\tau^* = \tau$ and also $\tau_2 = \{\phi, Y, \{b\}\}$. Define an identity function $f : (X, \tau_1, I) \longrightarrow (Y, \tau_2)$. Then f is $\delta^* - B_\beta -$ continuous, but not $\delta^* - \beta -$ continuous, since $f^{-1}(\{b\}) = \{b\}$ and $Int(\{b\}) = \phi = Cl(Int(Cl^{\delta_*}(\{b\})))$, thus $\{b\}$ is $\delta^* - B_\beta -$ set, but not $\delta^* - \beta - open$ set.*

Example 3.10. *Let $X = Y = \{a, b, c, d, e\}$, $\tau_1 = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, b, c\}\}$, $I = \{\phi, \{a\}\}$ and then $\tau^\delta = \{\phi, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_1^* = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, b, c\}, \{a, b, d, e\}\}$. Let $\tau_2 = \{\phi, Y, \{a, b\}\}$. Define a function $f : (X, \tau_1, I) \longrightarrow (Y, \tau_2)$ as follows: $f(a) = c$, $f(b) = a$, $f(c) = b$, $f(d) = d$, $f(e) = d$. Then f is $\delta^* - pre -$ continuous, but not $\delta^* - B_t -$ continuous $f^{-1}(\{a, b\}) = \{b, c\}$ and $\{c\} = Int(\{b, c\}) \neq Int(Cl^{\delta_*}(\{b, c\})) = X$, thus $\{b, c\}$ is $\delta^* - pre - open$, but not $\delta^* - B_t -$ set. In this example, if we take same function, Then f is $\delta^* - \beta -$ continuous, but not $\delta^* - B_\beta -$ continuous since $f^{-1}(\{a, b\}) = \{b, c\}$ and $\{c\} = Int(\{b, c\}) \neq Cl(Int(Cl^{\delta_*}(\{b, c\}))) = X$.*

Corollary 3.2. *For a function $f : (X, \tau) \longrightarrow (Y, \sigma)$, the following properties are equivalent:*
1) f is continuous,
2) f is $\delta - pre -$ continuous and $\delta - B -$ continuous [4].
3) f is $pre -$ continuous and $B -$ continuous [2].

Corollary 3.3. *For a function $f : (X, \tau, I) \longrightarrow (Y, \sigma)$, the following properties are equivalent:*
1) f is continuous,
2. f is $\alpha - I -$ continuous and $C_I -$ continuous [4],
3. f is $pre - I -$ continuous and $B - I -$ continuous [2],
4. f is $\delta^* - \alpha -$ continuous and $\delta^* - B_\alpha -$ continuous [6],
5. f is $\delta - \alpha^* -$ continuous and $\delta - B\alpha^* -$ continuous [6].

References

[1] Abd El-Monsef M. E., El-Deep S. N. and Mahmoud R. A., $\beta - open$ sets and $\beta - continuous$ mappings, *Bull. Fac. Sci. Assiut Univ.*, 12(1)(19839, 77-90.

[2] Dontchev, J., On pre-I-open sets and a decomposition of I continuity, *Banyan Math. J.*, 2(1996).

[3] Hatir E., Al-Omari A. and Jafari S., $\delta - local$ function and its properties, *Fasciculi Mathematici*, 53(2014), 53-64.

[4] Hatir E. and Noiri T., On decompositions of continuity via idealization, *Acta Math. Hungar.*, 96(4)(2002), 341–349

[5] Hatir E. and Noiri T., Decompositions of continuity and complete continuity, *Acta Math. Hungar.*, 113(4)(2006), 281-287.

[6] Hatir E., Another decomposition of continuity via $\delta - local$ function in ideal topological spaces, *Asian J. of Mathematics and Computer Research*, 18(3), 106-112, 2017

[7] Janković D. and Hamlett T. R., New topologies from old via ideals, *Amer. Math. Monthly*, 97(1990), 295-310.

[8] Kuratowski K., Topology I, Warszawa, 1933.

[9] Mashhour A. S., Abd El-Monsef M. E. and El-Deeb S. N., On pre-continuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, 53(1982), 47-53.

[10] Nijastad O., On some classes of nearly open sets, *Pacific J. Math.*, 15(1965), 961-970.

[11] Raychaudhuri S. and Mukherjee M. N., On $\delta - almost$ continuity and $\delta - preopen$ sets, *Bull. Inst. Math. Acad. Sinica* 21(1993), 357-366.

[12] Velićko N. V., H-closed topological spaces, *Amer. Math. Soc. Transl.*, 2(78)(1968), 103-118.

[13] Vaidyanathaswamy R., Set Topology, *Chelsea Publishing Company*, 1960.

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Extremal trees with respect to the first and second reformulated Zagreb index

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Abstract

Let G be a graph with edge set $E(G)$. The first and second reformulated Zagreb indices of G are defined as $EM_1(G) = \sum_{e\in E(G)} deg(e)^2$ and $EM_2(G) = \sum_{e\sim f} deg(e)deg(f)$,respectively, where $deg(e)$ denotes the degree of the edge e, and $e \sim f$ means that the edges e and f are incident. In this paper, the extremal trees with respect to the first and second reformulated Zagreb indices are presented.

Keywords: Tree, first reformulated Zagreb, second reformulated Zagreb, graph operation.

2010 MSC: 05C07, 05C35, 05C76.

1 Basic Definitions

Throughout this paper we consider undirected finite graphs without loops and multiple edges. The vertex and edge sets of a G will be denoted by $V(G)$ and $E(G)$, respectively. For a vertex v in G, the degree of v, $deg(v)$, is the number of edges incident to v and $N[v, G]$ is the set of all vertices adjacent to v. A vertex with degree one is called a pendent and $\Delta = \Delta(G)$ denotes the maximum degree of G. The number of vertices of degree i and the number of edges of G connecting a vertex of degree i with a vertex of degree j are denoted by $n_i = n_i(G)$ and $m_{i,j}(G)$, respectively. One can easily see that $\sum_{i=1}^{\Delta(G)} n_i = |V(G)|$.

Suppose $V(G) = \{v_1, \ldots, v_n\}$ and $d_k = deg(v_k)$, $1 \leq k \leq n$. The sequence $D(G) = (d_1, d_2, \ldots, d_n)$ is called the **degree sequence** of G and for simplicity of our argument, we usually write $D(G) = (x_1^{a_1}, x_2^{a_2}, ..., x_t^{a_t})$, when

$$D(G) = (\overbrace{x_1, ..., x_1}^{a_1\ times}, \overbrace{x_2, ..., x_2}^{a_2\ times}, ..., \overbrace{x_t, ..., x_t}^{a_t\ times}),$$

$x_1 > x_2 > ... > x_t$ and $a_1, ..., a_t$ are positive integers with $a_1 + a_2 + ... + a_t = n$.

Suppose $W \subset V(G)$ and $L \subseteq E(G)$. The notations $G \setminus W$ and $G \setminus L$ stand for the subgraphs of G obtained by deleting the vertices of W and the subgraph obtained by deleting the edges of L, respectively. If $W = \{v\}$ or $L = \{xy\}$, then the subgraphs $G \setminus W$ and $G \setminus L$ will be written as $G - v$ and $G - xy$ for short, respectively. Moreover, for any two nonadjacent vertices x and y of G, let $G + xy$ be the graph obtained from G by adding an edge xy.

A tree is a connected acyclic graph. It is well-known that any tree with at least two vertices has at least two pendent vertices. The set of all $n-$vertex trees will be denoted by $\tau(n)$. We denote the $n-$vertex path, cycle and the star graphs with P_n, C_n and S_n, respectively.

2 Preliminaries

The well-known Zagreb indices are among the oldest and most important degree-based molecular structure-descriptors. The first and second Zagreb indices are defined as the sum of squares of the degrees of

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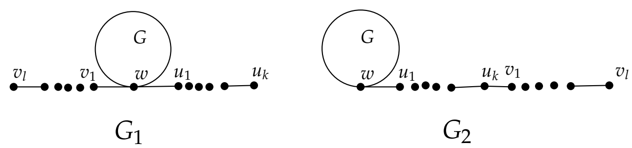

Figure 1: The Graphs G, P, Q, G_1 and G_2 in Transformation A.

the vertices and the sum of product of the degrees of adjacent vertices, respectively. These graph invariant was introduced many years ago by Gutman and Trinajesti/c [3]. We refer to [2, 9], for the history of these graph invariants and their role in QSAR and QSPR studies.

Milićević et al. [8], was introduced the first and second reformulated Zagreb indices of a graph G as edge counterpart of the first and second Zagreb indices, respectively. They are defined as $EM_1(G) = \sum_{e\in E(G)} deg(e)^2$ and $EM_2(G) = \sum_{e\sim f} deg(e)deg(f)$, where $deg(e)$ denotes the degree of the edge e, and $e \sim f$ means that the edges e and f are incident. In a recent paper, Milovanović et al. [7] obtained some relationship between Zagreb and reformulated Zagreb indices.

Zhou and Trinajestić [11], obtained sharp bounds for the reformed Zagreb indices. Ilić and Zhou [4] gave upper and lower bounds for the first reformulated Zagreb index and lower bounds for the second reformulated Zagreb index. They proved that if G is an $n-$vertex unicyclic graph then $EM_i(G) \geq EM_i(C_n)$, $i = 1, 2$, with equality if and only if $G \cong C_n$. Suppose $S_n^\star$ denotes the $n-$vertex unicyclic graph obtained by adding an edge to an $n-$vertex star, connecting two pendent vertices. They also proved that if G is an $n-$vertex unicyclic graph then $EM_i(G) \leq EM_i(S_n^\star)$, $i = 1, 2$, with equality if and only if $G \cong S'_n$.

Ji et al. [6] provided a shorter proof for results given by Ilić and Zhou and characterized the extremal properties of the first reformulated Zagreb index in the class of trees and bicyclic graphs by introducing some graph operations which increase or decrease this invariant. In [5], the authors applied a similar method as those given in [6] to find sharp bound for the first reformulated Zagreb index among all tricyclic graphs.

Su et al. [10], obtained the maximum and minimum of the first reformulated Zagreb index of graphs with connectivity at most k and characterized the corresponding extremal graphs. We encourage the interested readers to consult papers [1, 12] and references therein for more information on Zagreb and reformulated Zagreb indices of simple graphs.

3 Some Graph Transformations

In this section some graph operations are introduced which decreases the first and the second reformulated Zagreb index of graphs.

Transformation A. Suppose G is a graph with a given vertex w such that $deg(w) \geq 1$. We also assume that $P := v_1v_2...v_l$ and $Q := u_1u_2...u_k$ are two paths with l and k vertices, respectively. Define G_1 to be the graph obtained from G, P and Q by attaching vertices v_1w, wu_1, and $G_2 = G_1 - v_1w + u_kv_1$. The above referred graphs have been illustrated in Figure 1.

Lemma 3.1. *Let G_1 and G_2 be two graphs as shown in Figure 1. Then $EM_1(G_2) < EM_1(G_1)$ and $EM_2(G_2) < EM_2(G_1)$.*

Proof. Suppose $deg(w) = x$, $N[w, G] = \{l_1, \ldots l_x\}$ and $deg(l_i) = d_i$, for $i = 1, \ldots, x$. If $k, l \geq 2$ and $x \geq 1$, then

$$\begin{aligned} EM_1(G_1) - EM_1(G_2) &= 2(x+2)^2 + \sum_{i=1}^{x}(d_i + x)^2 \\ &- ((x+1)^2 + 8 + \sum_{i=1}^{x}(d_i + x - 1)^2) > (x+2)^2 - 8 > 0. \end{aligned}$$

If $k = l = 1$, or ($k = 1$ & $l \geq 2$), then a simple calculation shows the validity of $EM_1(G_2) < EM_1(G_1)$, as

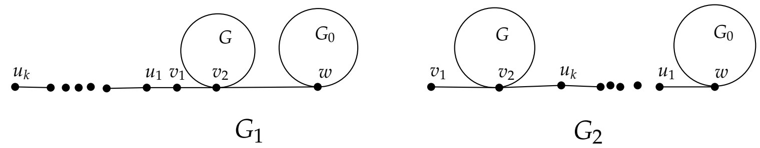

Figure 2: The Graphs G, G_0, P_k, G_1 and G_2 in Transformation B.

desired. Suppose that $k, l \geq 3$. Then,

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) &> 2(x+2) + (x+2)^2 + \sum_{i=1}^{x}(x+2)(x+d_i) + 2 - (4+4+4) \\ &> 0. \end{aligned}$$

If $k, l \in \{1,2\}$, or $(k = 1 \ \& \ l \geq 3)$ or $(k = 2 \ \& \ l \geq 3)$, then a simple calculation shows the validity of $EM_2(G_2) < EM_2(G_1)$, proving the lemma. □

Transformation B. Suppose G and G_0 are two graphs with given vertices $\{v_1, v_2\} \subseteq V(G)$ and $w \in V(G_0)$ such that $d_G(v_1) = 1$, $d_G(v_2) \geq 2$, $d_{G_0}(w) \geq 1$, and $v_1v_2 \in E(G)$. We also assume that $P_k := u_1u_2 \ldots u_k$ is a path with $k \geq 1$ vertices. Construct G_1 as the graph obtained from G, G_0 and P_k by adding the edges v_1u_1, v_2w, and $G_2 = G_1 - \{v_1u_1, v_2w\} + \{v_2u_k, u_1w\}$. The above referred graphs have been illustrated in Figure 2.

Lemma 3.2. *Let G_1 and G_2 be two graphs as shown in Figure 2. Then $EM_1(G_2) < EM_1(G_1)$ and $EM_2(G_2) < EM_2(G_1)$.*

Proof. Suppose $deg(v_2) = x$, $N[v_2, G] = \{l_1 := v_1, \ldots l_x\}$ and $deg(l_i) = d_i$, for $i = 2, \ldots, x$. In addition, suppose $deg(w) = h$, $N[w, G_0] = \{r_1, \ldots r_h\}$ and $deg(r_i) = b_i$, for $i = 1, \ldots, h$. If $x \geq 2$ and $h \geq 1$, then

$$\begin{aligned} EM_1(G_1) - EM_1(G_2) &= 1 + (x+1)^2 + (x+h)^2 - (x^2 + (x+1)^2 + (h+1)^2) \\ &= 2h(x-1) > 0, \end{aligned}$$

as desired. Suppose $k \geq 2$. Then,

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) &= 2 + 2(x+1) + (x+1)(x+h) + \sum_{i=2}^{x}(x+1)(d_i + x - 1) \\ &+ \sum_{i=2}^{x}(x+h)(d_i + x - 1) + \sum_{i=1}^{h}(x+h)(b_i + h - 1) \\ &- \Big(x(x+1) + 2(x+1) + 2(h+1) + \sum_{i=2}^{x} x(d_i + x - 1) \\ &+ \sum_{i=2}^{x}(x+1)(d_i + x - 1) + \sum_{i=1}^{h}(h+1)(b_i + h - 1)\Big) \\ &> 2 + (x+1)(x+h) - x(x+1) - 2(h+1) = h(x-1) > 0. \end{aligned}$$

If $k = 1$, then a simple calculation shows the validity of $EM_2(G_2) < EM_2(G_1)$, proving the lemma. □

Transformation C. Suppose G, G_0 and G' are three graphs with given vertices $w \in V(G_0)$, $\{v_1, v_2\} \subseteq V(G)$ and $y \in V(G')$ such that $d_{G_0}(w) \geq 2$, $d_G(v_1) \geq 2$, $d_G(v_2) = 1$ and $d_{G'}(y) \geq 2$. In addition, we assume that $P_k := u_1u_2 \ldots u_k$ is a path with $k \geq 1$ vertices. Define G_1 to be the graph obtained from G, G_0, G' and P_k by adding the edges wv_1, v_2u_1, u_ky and $G_2 = G_1 - \{wv_1, v_2u_1, u_ky\} + \{wu_1, u_kv_1, v_2y\}$. The above referred graphs have been illustrated in Figure 3.

Lemma 3.3. *Let G_1 and G_2 be two graphs as shown in Figure 3. Then $EM_1(G_2) < EM_1(G_1)$.*

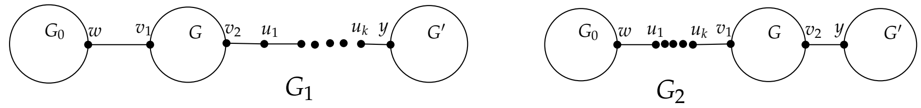

Figure 3: The Graphs G_0, G,G', P_k, G_1 and G_2 in Transformation C.

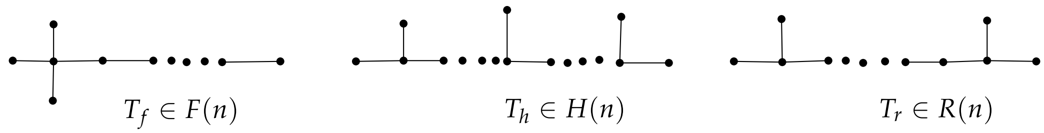

Figure 4: The trees in $F(n)$, $H(n)$ and $R(n)$.

Proof. Suppose that $d_{G_0}(w) = x$, $d_G(v_1) = k$ and $k, x \geq 2$. Then,

$$\begin{aligned} EM_1(G_1) - EM_1(G_2) &= (x+k)^2 + 4 - ((x+1)^2 + (k+1)^2) \\ &= 2kx + 2 - (2(x+k)) > 0, \end{aligned}$$

proving the lemma. □

4 Main Results

The aim of this section is to apply Transformations A-C to obtain an ordering of trees with respect to the first and second reformulated Zagreb indices. For simplicity of our argument, we first introduce a notation. Set

$$F(n) = \{T \in (4^1, 2^{n-5}, 1^4)|\ m_{1,2}(T) = 1,\ m_{1,4}(T) = 3,\ m_{2,4}(T) = 1 \textit{ and } m_{2,2}(T) = n-6\},$$

where $n \geq 7$ is a positive integer. It is easy to see that for each $T \in F(n)$,

$$EM_1(T) = 4n + 20 \quad , \quad EM_2(T) = 4n + 45. \tag{4.1}$$

Theorem 4.1. *Let T' be a tree with $\Delta(T') \geq 4$. If $T' \notin F(n)$, then for each $T \in F(n)$, we have $EM_1(T) < EM_1(T')$ and $EM_2(T) < EM_2(T')$.*

Proof. We first assume that $T' \in (4^1, 2^{n-5}, 1^4)$. Since $T' \notin F(n)$, $m_{1,2}(T) \neq 1$, $m_{1,4}(T) \neq 3$, $m_{2,4}(T) \neq 1$ or $m_{2,2}(T) \neq n-6$. We now apply a repeated application of Transformation B to obtain a tree $Q \in F(n)$. By Lemma 3.2, $EM_1(T) = EM_1(Q) < EM_1(T')$ and $EM_2(T) = EM_2(Q) < EM_2(T')$, as desired.

Next suppose $T' \notin (4^1, 2^{n-5}, 1^4)$. Then by a repeated application of Transformation A, we obtain a tree $G \in (4^1, 2^{n-5}, 1^4)$. If $G \in F(n)$, then by Lemma 3.1, $EM_1(T) = EM_1(G) < EM_1(T')$ and $EM_2(T) = EM_2(G) < EM_2(T')$. In other cases, we obtain the result by replacing T' with G in the first case. □

Suppose $n \geq 10$ and define $H(n) = \{T \in (3^3, 2^{n-8}, 1^5)|m_{1,2}(T) = 0, m_{1,3}(T) = 5, m_{2,3}(T) = 4, m_{3,3}(T) = 0 \textit{ and } m_{2,2}(T) = n-10\}$. It is easy to see that for each $T \in H(n)$,

$$EM_1(T) = 4n + 16. \tag{4.2}$$

Theorem 4.2. *Let T' be a tree with $n \geq 10$ vertices and $\Delta(T') = 3$ such that $n_3(T') \geq 3$. If $T' \notin H(n)$, then for each $T \in H(n)$, $EM_1(T) < EM_1(T')$.*

Proof. Suppose $T' \in (3^3, 2^{n-8}, 1^5)$. Since $T' \notin H(n)$, $m_{1,2}(T) \neq 0$, $m_{1,3}(T) \neq 5$, $m_{2,3}(T) \neq 4$, $m_{3,3}(T) \neq 0$ or $m_{2,2}(T) \neq n-10$. Again a repeated application of Transformations B and C, will result a tree $Q \in H(n)$. Now by Lemmas 3.2 and 3.3, $EM_1(T) = EM_1(Q) < EM_1(T')$. Suppose $n_3(T') \geq 4$. Since $n_3(T') \geq 4$, by a repeated application of Transformation A we obtain a tree $G \in (3^3, 2^{n-8}, 1^5)$. If $G \in H(n)$, then by Lemma 3.1, $EM_1(T) = EM_1(G) < EM_1(T')$. In other cases, we obtain the result by replacing T' with G in the first case. □

Table 1: The Trees with $\Delta \leq 3$ and $n_3 \leq 2$.

Notation	$m_{3,3}$	$m_{2,3}$	$m_{1,2}$	$m_{1,3}$	$m_{2,2}$	EM_1
A_1	0	0	2	0	n-3	4n-10
A_2	0	1	1	2	n-5	4n-2
A_3	0	2	2	1	n-6	4n
A_4	0	3	3	0	n-7	4n+2
A_5	0	2	0	4	n-7	4n+6
A_6	0	3	1	3	n-8	4n+8
A_7	0	4	2	2	n-9	4n+10
A_8	1	1	1	3	n-7	4n+10
A_9	0	5	3	1	n-10	4n+12
A_{10}	1	2	2	2	n-8	4n+12
A_{11}	0	6	4	0	n-11	4n+14
A_{12}	1	3	3	1	n-9	4n+14
A_{13}	1	4	4	0	n-10	4n+16

Suppose $n \geq 8$ and define:
$R(n) = \{T \in (3^2, 2^{n-6}, 1^4)|\ m_{1,2}(T) = 0,\ m_{1,3}(T) = 4,\ m_{2,3}(T) = 2,\ m_{3,3}(T) = 0 \text{ and } m_{2,2}(T) = n-7\}$.
It is easy to see that for each $T \in R(n)$,

$$EM_2(T) = 4n + 12. \tag{4.3}$$

Theorem 4.3. *Let T' be a tree with $n \geq 8$ vertices and $\Delta(T') = 3$ such that $n_3(T') \geq 2$. If $T' \notin R(n)$, then for each $T \in R(n)$, $EM_2(T) < EM_2(T')$.*

Proof. Suppose $T' \in (3^2, 2^{n-6}, 1^4)$. Since $T' \notin R(n)$, $m_{1,2}(T) \neq 0$, $m_{1,3}(T) \neq 4$, $m_{2,3}(T) \neq 2$, $m_{3,3}(T) \neq 0$ or $m_{2,2}(T) \neq n-7$. Again a repeated application of Transformation B, will result a tree $Q \in R(n)$. Now by Lemma 3.2, $EM_2(T) = EM_2(Q) < EM_2(T')$. Suppose $n_3(T') \geq 3$. Since $n_3(T') \geq 3$, by a repeated application of Transformation A we obtain a tree $G \in (3^2, 2^{n-6}, 1^4)$. If $G \in R(n)$, then by Lemma 3.1, $EM_1(T) = EM_1(G) < EM_1(T')$. In other case, we obtain the result by replacing T' with G in the first case. □

Theorem 4.4. *If $n \geq 11$, $T_1 \in A_1$, $T_2 \in A_2$, $T_3 \in A_3$, $T_4 \in A_4$, $T_5 \in A_5$, $T_6 \in A_6$, $T_7 \in A_7$, $T_8 \in A_8$, $T_9 \in A_9$, $T_{10} \in A_{10}$, $T_{11} \in A_{11}$, $T_{12} \in A_{12}$, $T_{13} \in A_{13}$, $T_{14} \in H(n)$ and $T \in \tau(n) \diagdown \{T_1, T_2, ..., T_{14}\}$, then $EM_1(T_1) < EM_1(T_2) < EM_1(T_3) < EM_1(T_4) < EM_1(T_5) < EM_1(T_6) < EM_1(T_7) = EM_1(T_8) < EM_1(T_9) = EM_1(T_{10}) < EM_1(T_{11}) = EM_1(T_{12}) < EM_1(T_{13}) = EM_1(T_{14}) < EM_1(T)$.*

Proof. From Table 1 and Equation 4.2, we have $EM_1(T_1) < EM_1(T_2) < EM_1(T_3) < EM_1(T_4) < EM_1(T_5) < EM_1(T_6) < EM_1(T_7) = EM_1(T_8) < EM_1(T_9) = EM_1(T_{10}) < EM_1(T_{11}) = EM_1(T_{12}) < EM_1(T_{13}) = EM_1(T_{14})$. If $\Delta(T) = 3$ and $n_3(T) \geq 3$ then the proof is completed by applying Theorem 4.2. If $\Delta(T) \geq 4$, then Theorem 4.1 and Equation 4.1, gives the result. Otherwise, $T \in \{T_1, T_2, ..., T_{14}\}$. □

Theorem 4.5. *Suppose that T is a tree with $n(\geq 10)$ vertices, except $T_1', T_2', \ldots, T_8'$, illustrated in Figure 6. Then we have $EM_2(T_1') < EM_2(T_2') < EM_2(T_3') < EM_2(T_4') < EM_2(T_5') < EM_2(T_6') < EM_2(T_7') < EM_2(T_8') < EM_2(T)$.*

Proof. Let $T' \in F(n)$ and $n \geq 10$. We consider the following cases:

Case 1. $\Delta(T) = 3$. If $n_3(T) \geq 2$, then Theorem 4.3 shows that $EM_2(T_8') < M_2(T)$. Now suppose that $n_3(T) = 1$. Clearly, $T_2', T_3', \ldots, T_7'$ are all trees with $n_3(T) = 1$. It is easy to see that $EM_2(T_2') = 4n$, $EM_2(T_3') = 4n+4$, $EM_2(T_4') = 4n+5$, $EM_2(T_5') = 4n+9$, $EM_2(T_6') = 4n+10$ and $EM_2(T_7') = 4n+11$.

Case 2. $\Delta(T) \geq 4$. Then by Theorem 4.1, $EM_2(T') < M_2(T)$. Since $EM_2(T_8') = 4n+12 < 4n+45 = EM_2(T')$, $EM_2(T_8') < EM_2(T)$.

Case 3. $\Delta(T) = 2$. Then $T \cong P_n$ and $M_2(T) = 4n-12$.

This completes the proof. □

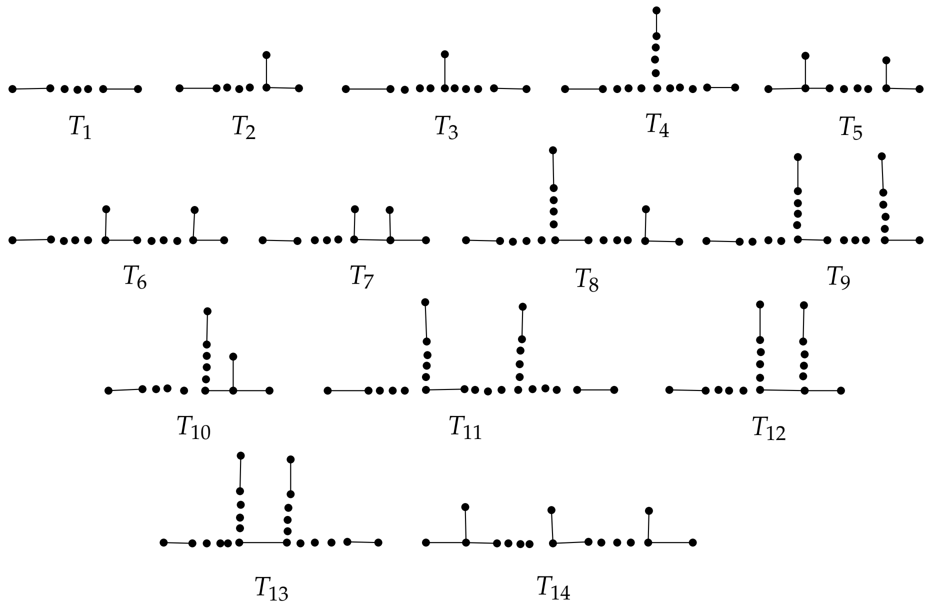

Figure 5: The Trees in Theorem 4.4.

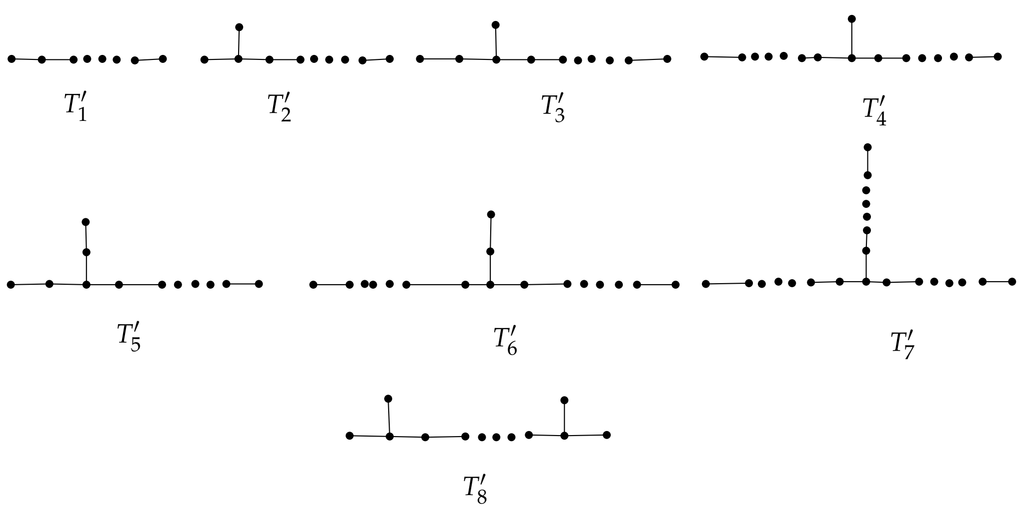

Figure 6: The Trees in Theorem 4.5.

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References

[1] N. De, Some bounds of reformulated Zagreb indices, *Appl. Math. Sci.* (*Ruse*), 6 (101-104) (2012), 5005–5012.

[2] I. Gutman and K.C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.*, 50 (2004), 83–92.

[3] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, 17 (1972), 535–538.

[4] A. Ilić and B. Zhou, On reformulated Zagreb indices, *Discrete Appl. Math.*, 160 (3) (2012), 204–209.

[5] S. Ji, Y. Qu and X. Li, The reformulated Zagreb indices of tricyclic graphs, *Appl. Math. Comput.*, 268 (2015), 590–595.

[6] S. Ji, X. Li and B. Huo, On reformulated Zagreb indices with respect to acyclic, unicyclic and bicyclic graphs, *MATCH Commun. Math. Comput. Chem.*, 72 (3)(2014), 723–732.

[7] E. I. Milovanović, I. Ž. Milovanović, E. Ć. Dolićanin and E. Glogić, A note on the first reformulated Zagreb index, *Appl. Math. Comput.*, 273 (2016), 16–20.

[8] A. Milićević, S. Nikolić and N. Trinajstić, On reformulated Zagreb indices, *Mol. Divers.*, 8 (2004), 393–399.

[9] S. Nikolić, G. Kovaćević, A. Milićević and N. Trinajstić, The Zagreb indices 30 years after, *Croat. Chem. Acta.*, 76 (2003), 113–124.

[10] G. Su, L. Xiong, L. Xu and B. Ma, On the maximum and minimum first reformulated Zagreb index of graphs with connectivity at most k, *Filomat*, 25(4) (2011), 75–83.

[11] B. Zhou and N. Trinajstić, Some properties of the reformulated Zagreb indices, *J. Math. Chem.*, 48(3) (2010), 714–719.

[12] B. Zhou and I. Gutman, Further properties of Zagreb indices, *MATCH Commun. Math. Comput. Chem.*, 54 (2005), 233–239.

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On the total product cordial labeling on the cartesian product of $P_m \times C_n$, $C_m \times C_n$ and the generalized Petersen graph $P(m,n)$

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Abstract

A total product cordial labeling of a graph G is a function $f : V \to \{0,1\}$. For each xy, assign the label $f(x)f(y)$, f is called total product cordial labeling of G if it satisfies the condition that $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the set of vertices and edges which are labeled with $i = 0,1$, respectively. A graph with a total product cordial labeling defined on it is called total product cordial.

In this paper, we determined the total product cordial labeling of the cartesian product of $P_m \times C_n$, $C_m \times C_n$ and the generalized Petersen graph $P(m,n)$.

Keywords: Graph Labeling, Total Product Cordial Labeling.

2010 MSC: 54C05, 54C08, 54C10.

1 Introduction

All graphs considered are finite, simple and undirected. The graph has vertex set $V = V(G)$ and edge set $E = E(G)$ and we let $e = |E|$ and $v = |V|$. A general reference for graph theoretic notions is in [5].

The classic paper of β-valuations by Rosa in 1967 [3] laid the foundations for several graph labeling methods. For a simple graph of order $|V|$ and size $|E|$, Ibrahim Cahit [1] introduced a weaker version of β-valuation or graceful labeling in 1987 and called it cordial labeling. The following notions of product cordial labeling was introduced in 2004 [3].

For a simple graph $G = (V, E)$ and a function $f : V \to \{0,1\}$, assign the label $f(x)f(y)$ for each edge xy. This function f is called a product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the number of vertices and edges labeled with $i = 0,1$. Motivated by this definition, M. Sundaram, R. Ponraj and S. Somasundaram introduce a new type of graph labeling known as total product cordial labeling and investigate the total product cordial behavior of some standard graphs.

A function f is called a total product cordial labeling of G if it satisfies the condition that $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| \leq 1$. A graph with a total product cordial labeling defined on it is called *total product cordial*.

2 Preliminaries

Definition 2.1. *Let $G = (V, E)$ be a simple graph and $f : V \to \{0,1\}$ be a map. For each edge xy, assign the label $f(x)f(y)$, f is called a total product cordial labeling of G if it satisfies the condition that $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the set of vertices and edges which are labeled with $i = 0,1$ respectively. A graph with a total product cordial labeling defined on it is called total product cordial.*

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Definition 2.2. *A Cartesian product, denoted by $G \times H$, of two graphs G and H, is the graph with vertex $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H)$ satisfying the following conditions $(u_1, u_2)(v_1, v_2) \in E(G \times H)$ if and only if either $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$.*

Definition 2.3. *The generalized Petersen graph $P(m, n)$, $m \geq 3$ and $1 \leq n \leq \left\lfloor \frac{m-1}{2} \right\rfloor$, consists of an outer m-cycle $u_0u_1 \ldots u_{m-1}$, a set of m spokes u_iv_i, $0 \leq i \leq m-1$, and m inner edges v_iv_{i+m} with indices taken modulo m.*

Theorem 2.4. [4] C_n is total product cordial if $n \neq 4$.

Remark 2.5. [4] The cycle C_4 is not total product cordial.

3 Total Product Cordial Graphs

This section presents some results of total product cordial labeling on some graphs.

Theorem 3.6. The graph $P_m \times C_n$ is total product cordial graph for all m and n except when $m = 1$ and $n = 4$.

Proof. Let $V(P_m \times C_n) = \{v_{(i,j)} | 1 \leq i \leq m, 1 \leq j \leq n\}$. The order and size of the graph $P_m \times C_n$ are mn and $2mn - n$, respectively. Consider the following cases:
Case 1: m and n are even.
Subcase 1: m is even and $n = 4$
Define the function $f : V(P_m \times C_4) \to \{0, 1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & 1 \leq i \leq m, j = 4 \quad \text{or} \\ & i \text{ is even}, j = 3 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{3m}{2}$ and $v_f(1) = \frac{5m}{2}$. On the other hand, the edges of $P_m \times C_4$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, && 1 \leq i \leq m-1,\, j = 3, 4 \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, && 1 \leq i \leq m,\, j = 3 \quad \text{or} \\ & && 1 \leq i \leq m,\, i \text{ is even},\, j = 2 \\ f(v_{(i,1)}v_{(i,4)}) &= 0, && 1 \leq i \leq m. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{9m-4}{2}$ and $e_f(1) = \frac{7m-4}{2}$.
Hence, $\left| v_f(0) + e_f(0) - v_f(1) - e_f(1) \right| = |6m - 2 - (6m - 2)| = 0$. Thus, the graph $P_m \times C_4$ is total product cordial.
Subase 2: m and n are even, $(n > 4)$
Define the function $f : V(P_m \times C_n) \to \{0, 1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & 1 \leq i \leq m \ \frac{n}{2} + 2 \leq j \leq n \quad \text{or} \\ & i \text{ is even}, j = \frac{n}{2} + 1 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-m}{2}$ and $v_f(1) = \frac{mn+m}{2}$. On the other hand, the edges of $P_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, && 1 \leq i \leq m-1,\, \frac{n}{2} + 1 \leq j \leq n \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, && 1 \leq i \leq m,\, \frac{n}{2} + 1 \leq j \leq n-1 \quad \text{or} \\ & && i \text{ is even},\, j = \frac{n}{2} \\ f(v_{(i,1)}v_{(i,n)}) &= 0, && 1 \leq i \leq m. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn-n+m}{2}$ and $e_f(1) = \frac{2mn-n-m}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn-n}{2} - \frac{3mn-n}{2}\right| = 0$. Thus, the graph $P_m \times C_n$ is total product cordial if m and n is even, $n > 4$.

Case 2: m is even, ($m \geq 2$) and n is odd, ($n \geq 3$).

Define the function $f : V(P_m \times C_n) \rightarrow \{0,1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & 1 \leq i \leq m, \frac{n+3}{2} \leq j \leq n-1 \quad \text{or} \\ & i \text{ is even}, j = \frac{n+1}{2} \quad \text{or} \\ & 1 \leq i \leq \frac{m}{2}, j = n \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-m}{2}$ and $v_f(1) = \frac{mn+m}{2}$. On the other hand, the edges of $P_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) = 0, &\quad 1 \leq i \leq m-1\,,\ \frac{n+1}{2} \leq j \leq n-1 \quad \text{or} \\ &\quad 1 \leq i \leq \frac{m}{2}\,,\ j = n \\ f(v_{(i,j)}v_{(i,j+1)}) = 0, &\quad 1 \leq i \leq m\,,\ \frac{n+1}{2} \leq j \leq n-1 \quad \text{or} \\ &\quad i \text{ is even}\,,\ 1 \leq i \leq m\,,\ j = \frac{n-1}{2} \\ f(v_{(i,1)}v_{(i,n)}) = 0, &\quad \frac{m}{2} + 1 \leq i \leq m. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn-n+m+1}{2}$ and $e_f(1) = \frac{2mn-n-m-1}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn-n+1}{2} - \frac{3mn-n-1}{2}\right| = 1$. Thus, the graph $P_m \times C_n$ is total product cordial if m is even, $m \geq 2$ and n is odd, $n \geq 3$.

Case 3: m and n are odd, ($n \geq 3$).

Subcase 1: If $m = 1$ and n is odd, ($n \geq 3$), the the graph $P_1 \times C_n \cong C_n$, which is total product cordial by Theorem 2.4.

Subcase 2: If $m = 3$ and $n \geq 3$, define the function $f : V(P_3 \times C_n) \rightarrow \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & i = 1, j = 1 \quad \text{or} \\ & i = 1, j = n \quad \text{or} \\ & i = 2, 2 \leq j \leq n-1 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = n$ and $v_f(1) = 2n$. On the other hand, the edges of $P_3 \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) = 0, &\quad i = 1\,,\ 1 \leq j \leq n \quad \text{or} \\ &\quad i = 2\,,\ 2 \leq j \leq n-1 \\ f(v_{(i,j)}v_{(i,j+1)}) = 0, &\quad i = 1\,,\ j = 1 \quad \text{or} \\ &\quad i = 1\,,\ j = n-1 \quad \text{or} \\ &\quad i = 2\,,\ 1 \leq j \leq n-1 \\ f(v_{(i,1)}v_{(i,n)}) = 0, &\quad i = 1. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = 3n$ and $e_f(1) = 2n$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = |4n - 4n| = 0$. Thus, the graph $P_3 \times C_n$ is total product cordial for $n \geq 3$.

Subcase 3: If m and n are odd, ($m, n \geq 5$), define the function $f : V(P_m \times C_n) \rightarrow \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & 2 \leq i \leq \frac{m+1}{2}, 1 \leq j \leq n \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-n}{2}$ and $v_f(1) = \frac{mn+n}{2}$. On the other hand, the edges of $P_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, \qquad 1 \le i \le \frac{m+1}{2}, 1 \le j \le n \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, \qquad 2 \le i \le \frac{m+1}{2}, 1 \le j \le n-1 \\ f(v_{(i,1)}v_{(i,n)}) &= 0, \qquad 2 \le i \le \frac{m+1}{2}. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = mn$ and $e_f(1) = mn - n$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn-n}{2} - \frac{3mn-n}{2}\right| = 0$. Thus the graph $P_m \times C_n$ is total product cordial if m and n is odd, $m, n \ge 5$.

Case 4: m is odd, ($m \ge 5$) and n is even.

Define the function $f : V(P_m \times C_n) \to \{0, 1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & i = \frac{m+1}{2}, 1 \le j \le n \quad \text{or} \\ & \frac{m+5}{2} \le i \le m, 1 \le j \le n \\ 1, & \text{otherwise} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-n}{2}$ and $v_f(1) = \frac{mn+n}{2}$. On the other hand, the edges of $P_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, \qquad \frac{m-1}{2} \le i \le m-1, 1 \le j \le n \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, \qquad i = \frac{m+1}{2}, 1 \le j \le n-1 \quad \text{or} \\ & \qquad\quad \frac{m+5}{2} \le i \le m, 1 \le j \le n-1 \\ f(v_{(i,1)}v_{(i,n)}) &= 0, \qquad i = \frac{m+1}{2} \quad \text{or} \\ & \qquad\quad \frac{m+5}{2} \le i \le m. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = mn$ and $e_f(1) = mn - n$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn-n}{2} - \frac{3mn-n}{2}\right| = 0$. Thus, the graph $P_m \times C_n$ is total product cordial if m is odd and n is even, $n \ge 4$.

On the other hand, if $m = 1$ and $n = 4$, the graph $P_1 \times C_4 \cong C_4$, which is not total product cordial by Remark 2.5. Hence, considering all the cases above, we can say, that the graph $P_m \times C_n$ is total product cordial except if $m = 1$ and $n = 4$. □

Theorem 3.7. The graph $C_m \times C_n$ is total product cordial graph for all $m, n \ge 3$.

Proof. Let $V(C_m \times C_n) = \{v_{(i,j)} | 1 \le i \le m, 1 \le j \le n\}$. The order and size of the graph $C_m \times C_n$ are mn and $2mn$, respectively. To prove the theorem, let us consider the following cases:

Case 1: m is even, n is even.

Subcase 1: If m is even and n is even, ($n > 4$), we will label the vertices of $C_m \times C_n$ using the function defined on Theorem 3.6, Case 2. Accordingly, the number of vertices and edges labeled with 0 and 1 are, $\frac{mn-m}{2}$ and $\frac{mn+m}{2}$, respectively. The additional edge of $C_m \times C_n$ whose label is 0 is $f(v_{(i,j)}v_{(m,j)}) = 0, \quad \frac{n}{2} + 1 \le j \le n$. Thus, the number of edges labeled with 0 and 1 would be $e_f(0) = \frac{2mn+m}{2}$ and $e_f(1) = \frac{2mn-m}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(0) - e_f(0)\right| = \left|\frac{3mn}{2} - \frac{3mn}{2}\right| = 0$. Thus, the graph $C_m \times C_n$ is total product cordial if m and n is even $n > 4$.

Subcase 2: If m is even and $n = 4$, we will label the vertices of $C_m \times C_4$ using the function defined on Theorem 3.6, Case 1. Accordingly, the number of vertices labeled with 0 and 1 are, $\frac{3m}{2}$ and $\frac{5m}{2}$, respectively. The additional edge of $C_m \times C_4$ whose label is 0 is $f(v_{(1,j)}v_{(m,j)}) = 0, \quad j = 3,4$. Thus, the number of edges labeled with 0 and 1 would be $e_f(0) = \frac{9m}{2}$ and $e_f(1) = \frac{7m}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(0) - e_f(0)\right| = |6m - 6m| = 0$. Thus, the graph $C_m \times C_n$ is total product cordial if m and n is even $n = 4$.

Case 2: m is even, $(m \geq 4)$ and n is odd, $(n \geq 3)$.

Subcase 1: If m is even, $m \geq 4$ and $n = 3$, define the function $f : V(C_m \times C_3) \to \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & i \text{ is even}, j = 1,3 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = m$ and $v_f(1) = 2m$. On the other hand, the edges of $C_m \times C_3$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, && 1 \leq i \leq m-1\,,\ j = 1,3 \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, && i \text{ is even}\,,\ j = 1,2 \\ f(v_{(i,1)}v_{(i,3)}) &= 0, && i \text{ is even} \\ f(v_{(1,j)}v_{(m,j)}) &= 0, && j = 2. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{7m}{2}$ and $e_f(1) = \frac{5m}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{9m}{2} - \frac{9m}{2}\right| = 0$. Thus, the graph $C_m \times C_3$ is total product cordial if m is even, $m \geq 4$.

Subcase 2: If m is even, $m \geq 4$ and n is odd, $n \geq 5$, define the function $f : V(C_m \times C_n) \to \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & 1 \leq i \leq m,\ \frac{n+5}{2} \leq j \leq n-1 \quad \text{or} \\ & i \text{ is even},\ j = \frac{n+1}{2} \quad \text{or} \\ & i = 1,3,4,\ldots,m,\ j = \frac{n+3}{2} \quad \text{or} \\ & 1 \leq i \leq \frac{m}{2},\ j = n \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-m-2}{2}$ and $v_f(1) = \frac{mn+m+2}{2}$. On the other hand, the edges of $C_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) = 0, \quad & 1 \leq i \leq m-1\,,\ \frac{n+1}{2} \leq j \leq n-1 \quad \text{or} \\ & 1 \leq i \leq \frac{m}{2}\,,\ j = n \end{aligned}$$

$$\begin{aligned} f(v_{(i,j)}v_{(i,j+1)}) &= 0, && 1 \leq i \leq m\,,\ \frac{n+1}{2} \leq j \leq n-1 \quad \text{or} \\ & && i \text{ is even}\,,\ 1 \leq i \leq m\,,\ j = \frac{n-1}{2} \\ f(v_{(i,1)}v_{(i,n)}) &= 0, && 1 \leq i \leq \frac{m}{2} \\ f(v_{(1,j)}v_{(m,j)}) &= 0, && \frac{n+1}{2} \leq j \leq n. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn+m+2}{2}$ and $e_f(1) = \frac{2mn-m-2}{2}$.

Hence, $\left|v_0 + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3m}{2} - \frac{3m}{2}\right| = 0$. Thus, the graph $C_m \times C_n$ is total product cordial if m is even, $m \geq 4$ and n is odd, $n \geq 5$.

Case 3: m is odd, $(m \geq 3)$ and n is even, $(n \geq 4)$.

Subcase 1: If $m = 3$ and $n \geq 4$, then the graph $C_3 \times C_n \cong C_m \times C_3$, which is total product cordial by Theorem 3.7, Case 2, Subcase 1.

Subcase 2: If $m \geq 5$ and $n \geq 4$, define the function $f: V(C_m \times C_n) \to \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & i = \frac{m+1}{2}, j \text{ is even or} \\ & i = \frac{m+3}{2}, j = 1,3,4,\ldots,n \quad \text{or} \\ & \frac{m+5}{2} \leq i \leq m-1, 1 \leq j \leq n \quad \text{or} \\ & i = m, 1 \leq j \leq \frac{n}{2} \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-n-2}{2}$ and $v_f(1) = \frac{mn+n+2}{2}$. On the other hand, the edges of $C_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) = 0, &\quad \frac{m+1}{2} \leq i \leq m-1, 1 \leq j \leq n \quad \text{or} \\ &\quad i = \frac{m-1}{2}, j \text{ is even} \\ f(v_{(i,j)}v_{(i,j+1)}) = 0, &\quad \frac{m+1}{2} \leq i \leq m-1, 1 \leq j \leq n-1 \quad \text{or} \\ &\quad i = m, 1 \leq j \leq \frac{n}{2} \\ f(v_{(i,1)}v_{(i,n)}) = 0, &\quad \frac{m+1}{2} \leq i \leq m \\ f(v_{(1,j)}v_{(m,j)}) = 0, &\quad \frac{n}{2}+1 \leq j \leq n. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn+n+2}{2}$ and $e_f(1) = \frac{2mn-n-2}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn}{2} - \frac{3mn}{2}\right| = 0$. Thus, the graph $C_m \times C_n$ is total product cordial if m is odd, $m \geq 5$ and n is even, $n \geq 4$.

Case 4: m and n is odd, $m, n \geq 3$

Define the function $f: V(C_m \times C_n) \to \{0,1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & i = \frac{m+1}{2}, j \text{ is odd or} \\ & \frac{m+3}{2} \leq i \leq m-1, 1 \leq j \leq n \quad \text{or} \\ & i = m, 1 \leq j \leq \frac{n-1}{2} \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-n}{2}$ and $v_f(1) = \frac{mn+n}{2}$. On the other hand, the edges of $C_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) = 0, &\quad i = \frac{m-1}{2}, j \text{ is odd or} \\ &\quad \frac{m+1}{2} \leq i \leq m-1, 1 \leq j \leq n \\ f(v_{(i,j)}v_{(i,j+1)}) = 0, &\quad \frac{m+1}{2} \leq i \leq m-1, 1 \leq j \leq n-1 \quad \text{or} \\ &\quad i = m, 1 \leq j \leq \frac{n-1}{2} \\ f(v_{(i,1)}v_{(i,n)}) = 0, &\quad \frac{m+1}{2} \leq i \leq m \\ f(v_{(1,j)}v_{(m,j)}) = 0, &\quad \frac{n+1}{2} \leq j \leq n. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn+n+1}{2}$ and $e_f(1) = \frac{2mn-n-1}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn+1}{2} - \frac{3mn-1}{2}\right| = 1$. Thus, the graph $C_m \times C_n$ is total product cordial if m and n is odd, $m, n \geq 3$.

Considering the cases above, we can say, that the graph $C_m \times C_n$ is total product cordial for all $m, n \geq 3$. □

Theorem 3.8. The generalized Petersen graph $P(m, n)$ is total product cordial graph for all $m \geq 3$.

Proof. Let $V(P(m,n)) = \{v_1, v_2, \ldots, v_{2m}\}$ where v_i, $1 \leq i \leq m$ are vertices of the outer cycle and v_i, $m+1 \leq i \leq 2m$ are the vertices of the inner cycle. The order and size of the generalized Petersen graph $P(m,n)$ are $2m$ and $3m$, respectively. To prove the theorem, let us consider the following cases:
Case 1: m is odd.

Define the function $f : V(P(m,n)) \rightarrow \{0,1\}$ by:

$$f(v_i) = 1, \qquad i = m+1, m+2, \ldots, 2m$$
$$f(v_{2i}) = 0, \qquad i = 1,2,\ldots,\frac{m-1}{2}$$
$$f(v_{2i-1}) = \begin{cases} 0, & \frac{m+3}{4} \leq i \leq \frac{m+1}{2}, m \equiv 1 \pmod 4 \quad \text{or} \\ & \frac{m+5}{4} \leq i \leq \frac{m+1}{2}, m \equiv 3 \pmod 4 \\ 1, & i = 1,2,3,\ldots,\frac{m-1}{4}, m \equiv 1 \pmod 4 \quad \text{or} \\ & i = 1,2,3,\ldots,\frac{m+1}{4}, m \equiv 3 \pmod 4. \end{cases}$$

For m is odd, $m \equiv 1 \pmod 4$, the number of vertices labeled with 0 and 1 would be $v_f(0) = \frac{3m+1}{4}$ and $v_f(1) = \frac{5m-1}{4}$. On the other hand, the edges of the generalized Petersen graph $P(m,n)$ with labels zero are the following:

$$f(v_{2i-1}v_{2i}) = 0, \qquad i = 1,2,\ldots,\frac{m-1}{4}, m \equiv 1 \pmod 4 \quad \text{or}$$
$$\frac{m+3}{4} \leq i \leq \frac{m-1}{2}, m \equiv 1 \pmod 4$$
$$f(v_{2i}v_{2i+1}) = 0, \qquad i = 1,2,\ldots,\frac{m-1}{2}$$
$$f(v_m v_1) = 0$$
$$f(v_{2i}v_{2i+m+1}) = 0, \qquad i = 1,2,\ldots,\frac{m-1}{2}$$
$$f(v_{2i-1}v_{2i+m}) = 0, \qquad \frac{m+3}{4} \leq i \leq \frac{m-1}{2}, m \equiv 1 \pmod 4$$
$$f(v_m v_{m+1}) = 0.$$

In view of the above labeling, we have $e_f(0) = \frac{7m+1}{4}$ and $e_f(1) = \frac{5m-1}{4}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{5m+1}{2} - \frac{5m-1}{2}\right| = 1$. Thus, the generalized Petersen graph $P(m,n)$ is total product cordial if m is odd, $m \equiv 1 \pmod 4$.

Similarly, the generalized Petersen graph $P(m,n)$ is total product cordial if m is odd and $m \equiv 3 \pmod 4$. In view of the vertex labeling defined above, we have $v_f(0) = \frac{3m-1}{4}$ and $v_f(1) = \frac{5m+1}{4}$. On the other hand, the edge labels of the generalized petersen graph $P(m,n)$ are the following;

$$f(v_{2i-1}v_{2i}) = 0, \qquad i = 1,2,\ldots,\frac{m+1}{4}, m \equiv 3 \pmod 4 \quad \text{or}$$
$$\frac{m+5}{4} \leq i \leq \frac{m-1}{2}, m \equiv 3 \pmod 4$$
$$f(v_m v_1) = 0$$
$$f(v_{2i}v_{2i+1}) = 0, \qquad i = 1,2,\ldots,\frac{m-1}{2}$$
$$f(v_{2i}v_{2i+m+1}) = 0, \qquad i = 1,2,\ldots,\frac{m-1}{2}$$
$$f(v_{2i-1}v_{2i+m}) = 0, \qquad \frac{m+5}{4} \leq i \leq \frac{m-1}{2}, m \equiv 3 \pmod 4$$
$$f(v_m v_{m+1}) = 0.$$

In view of the above labeling, we have $e_f(0) = \frac{7m-1}{4}$ and $e_f(1) = \frac{5m+1}{4}$.

Hence, we have $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{5m-1}{2} - \frac{5m+1}{2}\right| = 1$. Thus, the generalized Petersen graph $P(m,n)$ is total product cordial if m is odd, $m \equiv 3 \pmod 4$.

Case 2: m is even

Subcase 1: If $m = 4k, k \in \mathbb{Z}^+$, define the function $f : V(P(m,n)) \to \{0,1\}$ by

$$f(v_i) = 1 \qquad i = m+1, m+2, \ldots, 2m$$

$$f(v_{2i-1}) = \begin{cases} 0, & \frac{m+4}{4} \le i \le \frac{m}{2} \\ 1, & i = 1,2,3,\ldots,\frac{m}{4} \end{cases}$$

$$f(v_{2i}) = 0 \qquad i = 1,2,\ldots,\frac{m}{2}$$

.

In view of the above labeling, we have $v_f(0) = \frac{3m}{4}$ and $v_f(1) = \frac{5m}{4}$. On the other hand, the edges of the generalized Petersen graph $P(m,n)$ with labels zero are the following:

$$\begin{aligned} f(v_{2i-1}v_{2i}) &= 0, && 1 \le i \le \frac{m}{2} \\ f(v_m v_1) &= 0 \\ f(v_{2i}v_{2i+1}) &= 0, && i = 1,2,\ldots,\frac{m-2}{2} \\ f(v_{2i}v_{2i+m+1}) &= 0, && i = 1,2,\ldots,\frac{m}{2}-1 \\ f(v_{2i-1}v_{2i+m}) &= 0, && \frac{m+4}{4} \le i \le \frac{m}{2} \\ f(v_m v_{m+1}) &= 0. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{7m}{4}$ and $e_f(1) = \frac{5m}{4}$.

Hence, $\left| v_f(0) + e_f(0) - v_f(1) - e_f(1) \right| = \left| \frac{5m}{2} - \frac{5m}{2} \right| = 0$. Thus, the generalized Petersen graph $P(m,n)$ is total product cordial if $m = 4k, k \in \mathbb{Z}^+$.

Subcase 2: If $m = 4k+2, k \in \mathbb{Z}^+$, define the function $f : V(P(m,n)) \to \{0,1\}$ by:

$$f(v_i) = \begin{cases} 0, & i = m+1 \quad \text{or} \\ & \frac{m+6}{2} \le i \le m \\ 1, & m+2 \le i \le 2m \end{cases}$$

$$f(v_{2i}) = 0, \qquad i = 1,2,\ldots,\frac{m+2}{4}$$

$$f(v_{2i-1}) = 1, \qquad i = 1,2,\ldots,\frac{m+6}{4}.$$

In view of the above labeling, we have $v_f(0) = \frac{3m-2}{4}$ and $v_f(1) = \frac{5m+2}{4}$. On the other hand, the edges of the generalized Petersen graph $P(m,n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, && \frac{m+6}{2} \le i \le m-1 \\ f(v_{2i-1}v_{2i}) &= 0, && 1 \le i \le \frac{m+6}{4} \\ f(v_m v_1) &= 0 \\ f(v_{2i}v_{2i+1}) &= 0, && i = 1,2,\ldots,\frac{m+2}{4} \\ f(v_{2i}v_{2i+m+1}) &= 0, && i = 1,2,\ldots,\frac{m+2}{4} \\ f(v_m v_{m+1}) &= 0 \\ f(v_i v_{i+m+1}) &= 0, && \frac{m+6}{2} \le i \le m-1. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{7m+2}{4}$ and $e_f(1) = \frac{5m-2}{4}$.

Hence, $\left| v_f(0) + e_f(0) - v_f(1) - e_f(1) \right| = \left| \frac{5m}{2} - \frac{5m}{2} \right| = 0$. Thus, the generalized Petersen graph $P(m,n)$ is total product cordial if $m = 4k+2, k \in \mathbb{Z}^+$.

Considering the cases above, we can say, that the generalized Petersen Graph $P(m,n)$ is total product cordial for all $m \ge 3$. □

References

[1] Cahit, I., "Cordial graphs: A weaker version of graceful and harmonious Graphs", *Ars Combinatoria*, 23 (1987), 201-207.

[2] Gallian, J. A., "A dynamic survey on graph labeling",*The Electronic Journal of Combinatorics*, Sixteenth Edition, (2013).

[3] Sundaram, M., Ponraj, P. and Somasundaram, S., "Product cordial labeling of graphs", *Bulletin of Pure and Applied Sciences*, 23E (1) (2004), 155-162.

[4] Sundaram, M., Ponraj, P. and Somasundaram, S., "Total Product Cordial Labeling of Graphs", *Bulletin of Pure and Applied Sciences*, 23E (1) (2005), 199-203.

[5] West, D. B., An Introduction to Graph Theory, *Prentice Hall*, (1996).

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On $\mathbb{S}$ fuzzy soft sub hemi rings of a hemi ring

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Abstract

In this expose, on endeavors equipped on the way to achieve comprehension of the arithmetical character of $\mathbb{S}$-fuzzy soft sub hemi rings of a hemi ring.

Keywords: Fuzzy soft set, $\mathbb{S}$ fuzzy soft sub hemi ring, anti-$\mathbb{S}$-fuzzy soft sub hemi ring, and pseudo Fuzzy soft co-set.

2010 MSC: 94D05, 05C25.

1 Introduction

A small amount of researchers done their works in near rings and a few kinds of semi rings contain conventional part. Semi rings emerge in a natural approach in a few applications in the theory of automata and formal languages. Soft set premise as a novel mathematical device means and deals with uncertainty which seems to be gratis from the intrinsic difficulties disturbing the obtainable works. The introduction of fuzzy sets as a result of Zadeh. L.A [16], a few scholars developed fuzzy concepts lying on the impression of the concept of fuzzy sets. Dubois.D and Prade. H [8], were urbanized the concept of fuzzy Sets and Systems: Theory and Applications. Aktas. H, CaSman.N [3] were developed by Soft sets and soft groups. In this article, $\mathbb{S}$-Fuzzy soft sub hemi ring of a hemi ring is initiated in addition to the theorems in the company of various example.

2 Preliminaries

Definition 2.1. *Let $\mathbb{R}$ be a hemi ring. A Fuzzy soft sub set (H, C) of $\mathbb{R}$ is supposed to be a $\mathbb{S}$-Fuzzy soft sub hemi ring (SFSHR) of $\mathbb{R}$ if it satisfies the subsequent circumstances:*

(i) $\mu_{(H,C)}(a, b) \geq \mathbb{S}\{\mu_{(H,C)}(a), \mu_{(H,C)}(b)\}$,

(ii) $\mu_{(H,C)}(ab) \geq \mathbb{S}\{\mu_{(H,C)}(a), \mu_{(H,C)}(b)\}$, *in favor of each and every one a and b in $\mathbb{R}$.*

Definition 2.2. *Let $(\mathbb{R}, +, \cdot)$ be a hemi ring. A $\mathbb{S}$-Fuzzy soft sub hemi ring (H, C) of $\mathbb{R}$ is said to be an Fuzzy soft normal sub hemi ring (SFSNSHR) of $\mathbb{R}$ if it satisfies the subsequent conditions:*
(i)$\mu_{(H,C)}(ab) = \mu_{(H,C)}(ba)$ *on behalf of all a and b in $\mathbb{R}$.*

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Definition 2.3. *Let $\mathbb{R}$ and $\mathbb{R}^1$ be some two hemi rings. Assent to f is a mapping from $\mathbb{R}$ to $\mathbb{R}^1$ be any function and let B be a $\mathbb{S}$- Fuzzy soft sub hemi ring in $\mathbb{R}$, V be an $\mathbb{S}$-Fuzzy soft sub hemi ring in $f(\mathbb{R}) = \mathbb{R}^1$, defined by $\mu_{V(b)} = \sup_{(a)\in f^{-1}(b)}(\mu_{(H,C)})(a)$ intended for every a within $\mathbb{R}$ as well as b in $\mathbb{R}^1$. After that B is called a pre image of V under f and it is denoted by $f^{-1}(V)$.*

Definition 2.4. *Let (H, C) be an $\mathbb{S}$-Fuzzy soft sub hemi ring of a hemi ring $(\mathbb{R}, +, \cdot)$ and a in $\mathbb{R}$. Then the pseudo $\mathbb{S}$-Fuzzy soft coset $(x(H, B))^p$ obviously $((x\mu_{(H,C)})^p)(a) = p(x)\mu_{(H,C)}(a)$, for every x in $\mathbb{R}$ and for some p in P.*

3 A few proofs associated by way of $\mathbb{S}$-fuzzy soft sub hemi rings of a hemi ring

Theorem 3.1. *If (H, C) is an $\mathbb{S}$-Fuzzy soft sub hemi ring of a hemi ring $(\mathbb{R}, +, \cdot)$, then (H, C) is an $\mathbb{S}$-Fuzzy soft sub hemi ring of $\mathbb{R}$.*

Proof. Allow (H, C) be an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring $\mathbb{R}$. Think about $(H, C) = \left\{ \langle a, \mu_{(H,C)}, (a)\rangle \right\}$, despite a in $\mathbb{R}$, we obtain $(H, C) = (H, D) = \left\{ \langle a, \mu_{(H,D)}, (a)\rangle \right\}$, somewhere $\mu_{(H,D)}(a) = \mu_{(H,C)}(a)$, visibly, $\mu_{(H,D)}(a + b) \geq \mathbb{S}\{\mu_{(H,D)}(a), \mu_{(H,D)}(b)\}$, in spite of a as well as b in $\mathbb{R}$ in addition to $\mu_{(H,D)}(ab) \geq \mathbb{S}\{\mu_{(H,D)}(a), \mu_{(H,D)}(b)\}$, for all that a moreover b in $\mathbb{R}$. While B is an $\mathbb{S}$-fuzzy soft sub hemi ring of $\mathbb{R}$, we encompass $\mu_{(H,C)}(a + b) \geq \mathbb{S}\{\mu_{(H,C)}(a), \mu_{(H,C)}(b)\}$, for all that a in addition to b in $\mathbb{R}$. Also $\mu_{(H,C)}(ab) \geq \mathbb{S}\{\mu_{(H,C)}(a), \mu_{(H,C)}(b)\}$, every a along with b in $\mathbb{R}$. For this reason $(H, D) = (H, C)$ is an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring $\mathbb{R}$. □

Theorem 3.2. *If (H, C) is an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring $(\mathbb{R}, +, \cdot)$, then (H, C) is an $\mathbb{S}$-fuzzy soft sub hemi ring of $\mathbb{R}$.*

Proof. Conset to (H, C) be an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring $\mathbb{R}$. With the purpose of $(H, C) = \left\{ \langle a, \mu_{(H,C)}(a)\rangle \right\}$, in favor of every one a in $\mathbb{R}$. Let $(H, C) = (H, D) = \left\{ \langle a, \mu_{(H,D)}(a)\rangle \right\}$, designed for the entire a along with b in $\mathbb{R}$. In view of the fact that (H, B) is an $\mathbb{S}$- fuzzy soft sub hemi ring of $\mathbb{R}$, which implies to facilitate $1-\mu_{(H,D)}(ab) \leq \mathbb{S}\{(1-\mu_{(H,D)}(a)), (1-\mu_{(H,D)}(b))\}$, which implies so as to $\mu_{(H,D)}(ab) \geq 1 - \mathbb{S}\{(1 - \mu_{(H,D)}(a)), (1 - \mu_{(H,D)}(b))\} = \mathbb{S}\{\mu_{(H,D)}(a), \mu_{(H,D)}(b)\}$. As a result, $\mu_{(H,D)}(ab) \geq \mathbb{S}\{\mu_{(H,D)}(a), \mu_{(H,D)}(b)\}$, intended for every one of a furthermore b in $\mathbb{R}$. Consequently $(H, D) = (H, C)$ is an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring $\mathbb{R}$. □

Theorem 3.3. *Accede to $(\mathbb{R}, +, \cdot)$ survice a hemi ring and (H, C) be present a non unfilled subset of $\mathbb{R}$. Then (H, C) is a sub hemi ring of $\mathbb{R}$merely if $(H, D) = \left\langle \chi_{(H,C)}, \overline{\chi}_{(H,C)} \right\rangle$ is a $\mathbb{S}$-fuzzy soft sub hemi ring of $\mathbb{R}$, where $\chi_{(H,C)}$ is the characteristic function.*

Proof. Allow $(\mathbb{R}, +, \cdot)$ be a hemi ring in addition to (H, C) be a unbalance subset of $\mathbb{R}$. Primary agree to (H, C) be a sub hemi ring of $\mathbb{R}$. Obtain a with b in $\mathbb{R}$.

Case (i): Condition a furthermore b in (H, C) afterward $a + b, ab$ inside (H, C), given that (H, C) is a sub hemi ring of $\mathbb{R}, \chi_{(H,C)}(a) = \chi_{(H,C)}(b) = \chi_{(H,C)}(a + b) = \chi_{(H,C)}(ab) = 1$ with $\chi_{(H,C)}(a) = \chi_{(H,C)}(b) = \chi_{(H,C)}(a + b) = \chi_{(H,C)}(ab) = 0$. As a result, $\chi_{(H,C)}(a + b) \geq \mathbb{S}\{(a), \mu_{(H,C)}\chi(b)\}$, meant for every one of a also b within $\mathbb{R}$, $\chi_{(H,C)}(ab) \geq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, behalf of all a along with b inside $\mathbb{R}$. Subsequently, $\chi_{(H,C)}(a + b) \leq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, in favor of every part of a in addition to b into $\mathbb{R}, \chi_{(H,C)}(ab) \leq \{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, intended for each and every one a as well as b within $\mathbb{R}$.

Case (ii) Either a or b in (H, C), then $a + b, ab$ may or may not be in (H, C), $\chi_{(H,C)}(a) = 1, \chi_{(H,C)}(b) = 0$ (or) $\chi_{(H,C)}(a) = 0, \chi_{(H,C)}(b) = 1), \chi_{(H,C)}(a + b) = 1, \chi_{(H,C)}(ab) = 1$ (or 0) and $\chi_{(H,C)}(a) = 0, \chi_{(H,C)}(a)(b) = 1$ (or) $\chi_{(H,C)}(a) = 1, \chi_{(H,C)}(b) = 0)\chi_{(H,C)}(a + b) = \chi_{(H,C)}(ab) = 0$ (or 1). Obviously $\chi_{(H,C)}(a + b) \geq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, for all that a along with b in $\mathbb{R}, \chi_{(H,C)}(ab) \geq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, intended for every a and b in $\mathbb{R}$, and $\chi_{(H,C)}(a + b) \leq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, for all a and b in $\mathbb{R}\chi_{(H,C)}(ab) \leq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, for all a and b in $\mathbb{R}$.

Case (iii) If a and b somewhere else (H, C), at that time $a+b, ab$ may well otherwise may not in $(H, C), \chi_{(H,C)}(a) = \chi_{(H,C)}(b) = 0, \chi_{(H,C)}(a+b) = \chi_{(H,C)}(ab) = 1$ or 0 and $\chi_{(H,C)}(a) = \chi_{(H,C)}(b) = 1, \chi_{(H,C)}(a + b) = \chi_{(H,C)}(ab) = 0$ or 1. Evidently $\chi_{(H,C)}(a + b) \geq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, for all a and b in $\mathbb{R}\chi_{(H,C)}(ab) \geq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, for all a and b in $\mathbb{R}$ and $\chi_{(H,C)}(a + b) \leq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, for all a and b in $\mathbb{R}\chi_{(H,C)}(ab) \leq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\}$, for all a and b in $\mathbb{R}$. Subsequently in above conditions we comprise B is a fuzzy soft sub hemi ring of (H, C) hemi ring $\mathbb{R}$. On the contrary, Accede to a and b in (H, C), In view of the fact that (H, C) is non blank subset of $\mathbb{R}$, thus $\chi_{(H,C)}(a) = \chi_{(H,C)}(b) = 1, \chi_{(H,C)}(a) = \chi_{(H,C)}(b) = 0$. While $B = \left\langle \chi_{(H,C)}, \overline{\chi}_{(H,C)} \right\rangle$ is a fuzzy soft sub hemi ring of $\mathbb{R}$, we have $\chi_{(H,C)}(a + b) \geq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\} = \mathbb{S}\{1, 1\} = 1, \chi_{(H,C)}(ab) \geq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(ab)\} = \mathbb{S}\{1, 1\} = 1$. For that reason $\chi_{(H,C)}(a + b) = \chi_{(H,C)}(ab) = 1$. and $\chi_{(H,C)}(a + b) \leq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\} = \max\{0, 0\} = 0, \chi_{(H,C)}(ab) \leq \mathbb{S}\{\chi_{(H,C)}(a), \chi_{(H,C)}(b)\} = \max\{0, 0\} = 0$. Therefore $\chi_{(H,C)}(a + b) = \chi_{(H,C)}(ab) = 0$. Thus $a + b$ as well as ab in (H, C), as a result (H, C) is a sub hemi ring of $\mathbb{R}$. □

Theorem 3.4. *Let (H, C) be an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring H and f is an isomorphism from a hemi ring $\mathbb{R}$ onto H. Then $(H, C) \circ f$ is an $\mathbb{S}$- fuzzy soft sub hemi ring of $\mathbb{R}$.*

Proof. Consent to a and b in $\mathbb{R}$ as well as (H, C) be an fuzzy soft sub hemi ring of a hemi ring H. Subsequently, it is encompassed, $(\mu_{(H,C)} \circ f)(a + b) = \mu_{(H,C)} f(a + b) = \mu_{(H,C)}\{f(a) + f(b)\}$, as f is an isomorphism $\geq \mathbb{S}\{\mu_{(H,C)} f(a), \mu_{(H,C)} f(b)\}, = \mathbb{S}\{(\mu_{(H,C)} \circ f)(a), (\mu_{(H,C)} \circ f)(b)\}$, which implies so to $(\mu_{(H,C)} \circ f)(a + b) \geq \mathbb{S}\{(\mu_{(H,C)} \circ f)(a), (\mu_{(H,C)} \circ f)(b)\}$. And $(\mu_{(H,C)} \circ f)(ab) = \mu_{(H,C)}(f(ab)) = \mu_{(H,C)}(f(a)f(b))$, as f is an isomorphism $\geq \mathbb{S}\{\mu_{(H,C)} f(a), \mu_{(H,C)} f(b)\} = \mathbb{S}\{(\mu_{(H,C)} \circ f)(a), (\mu_{(H,C)} \circ f)(b)\}$, which implies that $(\mu_{(H,C)} \circ f)(ab) \geq \mathbb{S}\{(\mu_{(H,C)} \circ f)(a), (\mu_{(H,C)} \circ f)(b)\}$. For that reason $(H, C) \circ f$ is a $\mathbb{S}$- Fuzzy soft sub hemi ring of a hemi ring $\mathbb{R}$. □

Theorem 3.5. *Let (H, C) be an $\mathbb{S}$-fuzzy soft sub hemi ring of a heim ring h and f is an anti-isomorphism from a hemi ring r onto h. Then $(H, C) \circ f$ is a $\mathbb{S}$-fuzzy soft sub hemi ring of $\mathbb{R}$.*

Proof. Accede to a and b in $\mathbb{R}$ in addition to (H, C) be an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring H. Afterward we have $(\mu_{(H,C)} \circ f)(a + b) = \mu_{(H,C)}(f(a + b)) = \mu_{(H,C)}(f(b) + f(a))$, as f is an anti-isomorphism $\geq \min\{\mu_{(H,C)} f(a), \mu_{(H,C)} f(b)\} = \mathbb{S}\{(\mu_{(H,C)} \circ f)(a), (\mu_{(H,C)} \circ f)(b)\}$, which implies to facilitate $(\mu_{(H,C)} \circ f)(a + b) \geq \mathbb{S}\{(\mu_{(H,C)} \circ f)(a), (\mu_{(H,C)} \circ f)(b)\}$. In addition to, $(\mu_{(H,C)} \circ f)(ab) = \mu_{(H,C)} \circ f(ab) = \mu_{(H,C)} \circ (f(b)f(a))$, as f is an anti-isomorphism $\geq \mathbb{S}\{\mu_{(H,C)}(a), \mu_{(H,C)} f(b)\} = \mathbb{S}\{(\mu_{(H,C)} \circ f)(a), (\mu_{(H,C)} \circ f)(b)\}$, which implies to $(\mu_{(H,C)} \circ f)(ab) \geq \{(\mu_{(H,C)} \circ f)(a), (\mu_{(H,C)} \circ f)(b)\}$. Thus $(G, B) \circ f$ is an $\mathbb{S}$-fuzzy soft sub hemi ring of the hemi ring $\mathbb{R}$. □

Theorem 3.6. *Let (H, C) be an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring $(R, +, \cdot)$, then the pseudo fuzzy soft co-set $(x(H, C))^p$ is an $\mathbb{S}$-fuzzy soft hemi ring of a hemi ring $\mathbb{R}$, for every x in $\mathbb{R}$.*

Proof. Consent to (H, C) be an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring $\mathbb{R}$. In favor of each a and b in $\mathbb{R}$, we have

$$\begin{aligned}((x\mu_{(H,C)})^p)(a+b) &= p(x)\mu_{(H,C)}(a+b) \geq p(x)\mathbb{S}\{(\mu_{(H,C)}(a),(\mu_{(H,C)}(b)\} \\ &= \mathbb{S}\{p(x)(\mu_{(H,C)}(a),(\mu_{(H,C)}(b)\} \\ &= \mathbb{S}\{((x\mu_{(H,C)})^p)(a),((x\mu_{(H,C)})^p)(b)\}.\end{aligned}$$

As a result, $((x\mu_{(H,C)})^p)(a+b) \geq \mathbb{S}\{((x\mu_{(H,C)})^p)(a),((x\mu_{(H,C)})^p)(b)\}$. At this istant

$$\begin{aligned}((x\mu_{(H,C)})^p)(ab) &= p(x)\mu_{(H,C)}(ab) \geq p(x)\mathbb{S}\{\mu_{(H,C)}(a),\mu_{(H,C)}(b)\} \\ &= \mathbb{S}\{p(x)\mu_{(H,C)}(a),p(x)\mu_{(H,C)}(b)\} \\ &= \mathbb{S}\{((x\mu_{(H,C)})^p)(a),((x\mu_{(H,C)})^p)(b)\}.\end{aligned}$$

Consequently, $((x\mu_{(H,C)})^p)(ab) \geq \mathbb{S}\{((x\mu_{(H,C)})^p)(a),((x\mu_{(H,C)})^p)(b)\}$. From now $(x(H,C))^p$ is an $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring $\mathbb{R}$. □

4 Conclusion

In the current work, a novel concept of $\mathbb{S}$-fuzzy soft sub hemi ring of a hemi ring which are defined with some properties and related theorems are studied.

References

[1] M. T. Abu Osman, On some product of fuzzy subgroups, *Fuzzy Sets and Systems*, 24(1987) 79–86.

[2] B. Ahmad and A. Kharal, On fuzzy soft sets, *Advances in Fuzzy Systems*, 2009, 1–6,

[3] H. Aktas and N. CaSman, Soft sets and soft groups, *Inform. Sci.*, 177 (2007), 2726–2735.

[4] M. I. Ali, F. Feng, X. Liu, Min and W. K. Shabir, On some new operations in soft Set Theory, *Computers and Mathematics with Applications,* 57(2009), 1547–1553.

[5] M. I. Ali, M. Shabir and M. Naz, Algebraic structures of soft sets associated with new operations, *Computers and Mathematics with Applications,* 61(2011), 2647–2654.

[6] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems,* 20(1986), 87–96.

[7] B.S. Shieh, Infinite fuzzy relation equations with continuous t-norms, *Information Sciences*, 178(2008) 1961–1967.

[8] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications,* Academic Press, 1980.

[9] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, *Computers and Mathematics with Applications*, 56(2008), 2621–2628.

[10] Y. B. Jun, Soft BCK/BCI-algebra, *Computers and Mathematics with Applications*, 56(2008) 1408–1413.

[11] D. Molodtsov, Soft set theory first result, *Computers and Mathematics with Applications*, 37(1999), 19-31.

[12] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, *Comput. Math. Appl.*, 45(2003), 555–562.

[13] P. K. Maji, R. Biswas and A.R. Roy, Fuzzy soft sets, *The J. Fuzzy Math.*, 9: 589–602 (2011).

[14] Qiu-Mei Sun, Zi-Liong Zhang and Jing Liu, Soft sets and soft modules, *Lecture Notes in Comput. Sci.*, 50(09)(2008), 403-409.

[15] X. Yang, D. Yu, J. Yang and C. Wu, Generalization of Soft Set Theory, *From Crisp to Fuzzy Case, Proceedings of the Second International Conference of Fuzzy Information and Engineering*, (ICFIE-2007), 27-Apr., pp. 345–355.

[16] L. A. Zadeh, Fuzzy sets, *Inform. And Control*, 8(1965), 338–353.

[17] H. J. Zimmermann, *Fuzzy Set Theory and Its Applications*, Kluwer, 1991.

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Asymptotic behavior of solution for a fractional Riemann-Liouville differential equations on time scales

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Abstract

In this paper, we will establish asymptotic behavior of solutions for the fractional order nonlinear dynamic equation on time scales

$$\left(p\left(t\right)_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha}x\left(t\right)\right)^{\Delta}+f\left(t,x^{\sigma}\left(t\right)\right)=0,\qquad \text{for all } t\in\left[t_0,+\infty\right)_{\mathbb{T}},$$

with $\alpha\in[0,1)$, where ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha}x$ is the Riemann-Liouville fractional derivative of order α of x on time scales. We obtain some asymptotic behavior of solutions for the equation by developing a generalized Riccati substitution technique. Our results in this paper some sufficient conditions for asymptotic behavior of all solutions.

Keywords: Oscillation, Dynamic equations, Time scale, Riccati technique, Fractional calculus.

2010 MSC: 34K11, 39A10, 39A99, 34C10, 39A11.

1 Introduction

In this paper, we are concerned with the asymptotic behavior of solutions for the fractional order nonlinear dynamic equation on time scales

$$\left(p\left(t\right)_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha}x\left(t\right)\right)^{\Delta}+f\left(t,x^{\sigma}\left(t\right)\right)=0,\qquad \text{for all } t\in\left[t_0,+\infty\right)_{\mathbb{T}},\tag{1.1}$$

with $\alpha\in[0,1)$, where ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha}x$ is the Riemann-Liouville fractional derivative of order α of x on time scales. Since we are interested in asymptotic behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and is a time scale interval of the form $[t_0,+\infty)_{\mathbb{T}}:=[t_0,+\infty)\cap\mathbb{T}$.

Throughout this paper and without further mention, we formulate the following hypotheses:

(H_1) $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$, is a continuous function verifying

$$xf\left(t,x\right)\geq 0,\qquad \text{for all } t\in\left[t_0,+\infty\right)_{\mathbb{T}},\ x\in\mathbb{R}\backslash\left\{0\right\}.$$

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(H_2) There exist a function $r : \mathbb{T} \to \mathbb{R}$, which is a positive and rd-continuous, such that

$$\frac{f(x)}{x} \geq r(t), \qquad \text{for all } t \in [t_0, +\infty)_{\mathbb{T}},\ x \in \mathbb{R} \setminus \{0\}. \tag{1.2}$$

(H_3) $p : \mathbb{T} \to \mathbb{R}^+$ is a real-valued rd-continuous functions, such that

$$\int_{t_0}^{\infty} \frac{1}{p(t)} \Delta t = \infty. \tag{1.3}$$

By a solution of (1.1) we mean a nontrivial real-valued function $pP_{t_0}^{\mathbb{T}} \mathcal{D}_t^{\alpha} x \in \mathcal{C}_{rd}^1([T_x, +\infty)_{\mathbb{T}}, \mathbb{R})$, where $T_x \in [t_0, +\infty)_{\mathbb{T}}$, which satisfies (1.1) on $[T_x, +\infty)_{\mathbb{T}}$. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration.

A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is non oscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

The theory of dynamic equations on time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D. thesis [1] in order to unify continuous and discrete analysis. The books on the subjects of time scale, that is, measure chain, by Bohner and Peterson [3],[4], summarize and organize much of time scale calculus.

The theory of oscillations is an important branch of the applied theory of dynamic equations related to the study of oscillatory phenomena in technology, natural and social sciences. In recent years, there has been much research activity concerning the oscillation of solutions of various dynamic equations on time scales.

In the last decade, there has been increasing interest in obtaining sufficient conditions for the oscillation and non oscillation of solutions of different classes of dynamic equations on time scales, and we refer the reader to the papers [19], [22].

So far, there are any results on oscillatory of (1.1). Hence the aim of this paper is to give some asymptotic behavior criteria for this equation.

2 Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. (supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. If $\sigma(t) > t$ we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If $\sigma(t) = t$, then t is called right-dense; if $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k := \mathbb{T}$.

The graininess function for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. The Δ-derivative of $f : \mathbb{T} \to \mathbb{R}$ at a right dense point t is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

If t is not right scattered, then the derivative is defined by

$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and at each left-dense point t in the left hand limit at t exists (finite). The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$. We will use the following product and quotient rules for the derivative of the product fg and the quotient $\frac{f}{g}$ where $(g^{\sigma}(t)\, g(t) \neq 0)$ of two differentiable functions f and g,

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta}, \qquad \text{and} \qquad \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g^{\sigma} g}.$$

For $a, b \in \mathbb{T}$, and for a differentiable function f, the Cauchy integral of f^{Δ} is defined

$$\int_a^b f^{\Delta}(t)\,\Delta t = f(b) - f(a),$$

and the improper integrals are defined in the usual way by

$$\int_a^{\infty} f^{\Delta}(t)\,\Delta t = \lim_{b\to\infty}\int_a^{\infty} f^{\Delta}(t)\,\Delta t.$$

For more on the calculus on time scales, we refer the reader to [3, 4].

We introduce the fractional differentiation and fraction integral on time scales is defined [21].

Definition 2.1 (Fractional integral on time scales)**.** *[21] Suppose $\mathbb{T}$ is a time scale, $[a, b]$ is an interval of $\mathbb{T}$, and h is an integrable function on $[a, b]$. Let $0 < \alpha < 1$. Then the (left) fractional integral of order α of h is defined by*

$${}_a^{\mathbb{T}}I_t^{\alpha} h(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,\Delta s.$$

here Γ is the gamma function defined by:

$$\Gamma(t) := \int_0^{\infty} x^{t-1}e^{-x}dx, \qquad \textit{for all } x > 0.$$

Definition 2.2 (Riemann–Liouville fractional derivative on time scales)**.** *[21] Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, and $h : \mathbb{T} \to \mathbb{R}$. The (left) Riemann–Liouville fractional derivative of order α of h is defined by*

$${}_a^{\mathbb{T}}\mathcal{D}_t^{\alpha} h(t) := \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha} h(s)\,\Delta s\right)^{\Delta}.$$

We present some fundamental properties of the fractional operators on time scales.

Theorem 2.1. *[21] Let $\mathbb{T}$ be a time scale with derivative Δ, a, $t \in \mathbb{T}$ and $0 < \alpha < 1$. Then, the following properties hold:*

1. ${}_a^{\mathbb{T}}\mathcal{D}_t^{\alpha} = \Delta \circ {}_a^{\mathbb{T}} I_t^{1-\alpha}$,

2. ${}_a^{\mathbb{T}}\mathcal{D}_t^{\alpha} \circ {}_a^{\mathbb{T}} I_t^{\alpha} = Id$,

3. ${}_a^{\mathbb{T}}\mathcal{D}_t^{n+\alpha} = {}_a^{\mathbb{T}} \mathcal{D}_t^{\alpha} \circ \Delta^n$,

   where $n \in \mathbb{N}$ and Id denotes the identity operator.

3 Main Results

In this section, we establish some sufficient conditions which guarantee that every solution x of (1.1) oscillates on $[t_0, +\infty)_{\mathbb{T}}$ or $\lim_{t\to+\infty} x(t)$ exists (finite).

Lemmas 3.1 give useful information about the behavior of possible non oscillatory solutions of (1.1).

Lemma 3.1. *Let (H_1)-(H_3) holds. Suppose that x is an eventually positive solution of (1.1), then there are only the following two possible cases for $t \in [t_1, +\infty)_{\mathbb{T}}$, where $t_1 \in [t_0, +\infty)_{\mathbb{T}}$ sufficiently large:*

1. $\left(p(t)\,{}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\right)^{\Delta} \le 0$, ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \ge 0$, $x^{\Delta}(t) \ge 0$, ${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \ge 0$,

2. $\left(p(t)\,{}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\right)^{\Delta} \le 0$, ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \ge 0$, $x^{\Delta}(t) \ge 0$, ${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \ge 0$.

Proof. Let x be an eventually positive solution of (1.1). Then there exists a $t_1 \in [t_0, +\infty)_{\mathbb{T}}$ such that $x^{\sigma}(t) > 0$ for $t \in [t_1, +\infty)_{\mathbb{T}}$. From (1.1), we have

$$\left(p(t)\,{}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\right)^{\Delta} \leq 0, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \tag{3.4}$$

Thus, ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)$ is decreasing on $[t_1, +\infty)_{\mathbb{T}}$. We claim that ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \geq 0$, for all $t \in [t_1, +\infty)_{\mathbb{T}}$. If not, then there exist a $t_2 \in [t_1, +\infty)_{\mathbb{T}}$ and a constant $\xi > 0$ such that

$$p(t)\,{}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \leq -\xi, \qquad \text{for all } t \in [t_2, +\infty)_{\mathbb{T}}.$$

By propertied (1) the Theorem 2.1 and (3.4), we obtain

$${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \leq {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t_1) - \xi \int_{t_1}^{t} \frac{1}{p(s)} \Delta s, \qquad \text{for all } t \in [t_2, +\infty)_{\mathbb{T}},$$

hence, we have ${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \to -\infty$ as $t \to +\infty$, which is a contradiction. □

Remark 3.1. *Along the work, we also use the notation*

$$Td(t) := \frac{d^{\sigma}(t)}{d(t)} \qquad \text{and} \qquad \xi_{\alpha} := \Gamma(2-\alpha).$$

Theorem 3.2. *Suppose that* (H_1)-(H_3) *holds and that there exist a positive functions* $\delta, \phi \in \mathcal{C}_{rd}^{1}([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^{+})$ *such that for every sufficiently large* $t_1 \in [t_0, +\infty)_{\mathbb{T}}$,

$$\int_{t_1}^{\infty} \xi_{\alpha} (\sigma(s) - t_0)^{\alpha-1} r(s)\, \delta^{\sigma}(s) - \frac{\left[\delta^{\Delta}(s)\right]^2 p(s)\, T\phi(s)}{4\delta^{\sigma}(s)} \Delta s = \infty, \tag{3.5}$$

and

$$\phi(t) - \xi(t, t_1)\, \phi^{\Delta}(t) \leq 0, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}, \tag{3.6}$$

where

$$\xi(t, t_1) := p(t) \int_{t_1}^{t} \frac{1}{p(\tau)} \Delta\tau, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}.$$

Then the solution x *of* (1.1) *is oscillatory or* $\lim\limits_{t \to +\infty} x(t)$ *exists (finite).*

Proof. Let x be a non oscillatory solution of (1.1). We only consider the case when x is eventually positive, since the case when x is eventually negative is similar. by Lemma 3.1 we see that x satisfies either case (1) or case (2).
Suppose first that x satisfies (1) of lemma 3.1, hen there exists $t_1 \in [t_0, +\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x^{\sigma}(t) > 0$ for all $t \in [t_1, +\infty)_{\mathbb{T}}$.
Define the function w by

$$w(t) := \delta(t) \frac{p(t)\,{}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t)}, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \tag{3.7}$$

Then $w(t) > 0$ for all $t \in [t_1, +\infty)_{\mathbb{T}}$.
Using the product rule and the quotient rule, we get

$$w^{\Delta}(t) = \delta^{\Delta}(t) \frac{p(t)\,{}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t)} + \delta^{\sigma}(t) \frac{\left(p(t)\,{}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\right)^{\Delta}}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} - \delta^{\sigma}(t) \frac{p(t)\,{}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\Delta}(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)}. \tag{3.8}$$

From (1.1), (3.8) and propertied (1) the Theorem 2.1, we obtain

$$w^{\Delta}(t) \leq \delta^{\Delta}(t) \frac{p(t)\, {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t)} - r(t)\,\delta^{\sigma}(t) \frac{x^{\sigma}(t)}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} - \frac{\delta^{\sigma}(t)\, p(t)\, {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\, {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)}. \tag{3.9}$$

Substituting (3.7) in (3.9), we have

$$w^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} w^{\sigma}(t) - r(t)\,\delta^{\sigma}(t) \frac{x^{\sigma}(t)}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} - \frac{\delta^{\sigma}(t)}{\delta^2(t)\, p(t)} \frac{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t)}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} w^2(t). \tag{3.10}$$

Hence, we obtain by (1.1) that ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)$ is decreasing on $[t_1, +\infty)_{\mathbb{T}}$. Then, we obtain

$$\begin{aligned} {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) &\geq \int_{t_1}^{t} \frac{1}{p(\tau)} \left(p(\tau)\, {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(\tau)\right) \Delta\tau \\ &\geq \xi(t, t_1)\, {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t), \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \left(\frac{{}_{a}^{\mathbb{T}}I_t^{1-\alpha} x}{\phi}\right)^{\Delta}(t) &= \frac{{}_{a}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\,\phi(t) - {}_{a}^{\mathbb{T}}I_t^{1-\alpha} x(t)\,\phi^{\Delta}(t)}{\phi(t)\,\phi^{\sigma}(t)} \\ &\leq \frac{{}_{a}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{\phi(t)\,\phi^{\sigma}(t)} \left(\phi(t) - \xi(t, t_1)\,\phi^{\Delta}(t)\right) \leq 0, \quad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \end{aligned}$$

Thus, $\frac{{}_{a}^{\mathbb{T}}I_t^{1-\alpha} x}{\phi}$ is a nondecreasing function on $[t_1, +\infty)_{\mathbb{T}}$, we have

$$\frac{{}_{a}^{\mathbb{T}}I_t^{1-\alpha} x(t)}{\left({}_{a}^{\mathbb{T}}I_t^{1-\alpha} x(t)\right)^{\sigma}} \geq \frac{\phi(t)}{\phi^{\sigma}(t)}, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \tag{3.11}$$

Substituting (3.11) in (3.10), we have

$$w^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} w^{\sigma}(t) - r(t)\,\delta^{\sigma}(t) \frac{x^{\sigma}(t)}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} - \frac{\delta^{\sigma}(t)}{\delta^2(t)} \frac{\phi(t)}{\phi^{\sigma}(t)} w^2(t). \tag{3.12}$$

Since x is a decreasing function, we have

$$\begin{aligned} {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) &= \int_{t_0}^{t} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)\,\Delta s \\ &\leq x(t) \int_{t_0}^{t} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \Delta s \\ &\leq \frac{1}{\xi_{\alpha}} (t-t_0)^{1-\alpha} x(t), \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \end{aligned} \tag{3.13}$$

Substituting (3.13) in (3.12), we get

$$w^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} w^{\sigma}(t) - \xi_{\alpha} (\sigma(t) - t_0)^{\alpha-1} r(t)\,\delta^{\sigma}(t) - \frac{\delta^{\sigma}(t)}{\delta^2(t)\, p(t)} \frac{\phi(t)}{\phi^{\sigma}(t)} w^2(t). \tag{3.14}$$

Using the inequality [22]

$$Bx - Ax^2 \leq \frac{B^2}{4A}, \qquad \text{for all } x \in \mathbb{R}^+,\ A > 0 \text{ and } B \in \mathbb{R}. \tag{3.15}$$

we get

$$w^{\Delta}(t) \leq -\xi_{\alpha}(\sigma(t)-t_0)^{\alpha-1} r(t)\,\delta^{\sigma}(t) + \frac{\left[\delta^{\Delta}(t)\right]^2 p(t)\,T\phi(t)}{4\delta^{\sigma}(t)}.$$

Integrating both sides of the last inequality from t_1 to t, we obtain

$$w(t) - w(t_1) \leq -\int_{t_1}^{t} \xi_{\alpha}(\sigma(s)-t_0)^{\alpha-1} r(s)\,\delta^{\sigma}(s) - \frac{\left[\delta^{\Delta}(s)\right]^2 p(s)\,T\phi(s)}{4\delta^{\sigma}(s)}\Delta s.$$

Since $w(t) > 0$, for all $t \in [t_2, +\infty)_{\mathbb{T}}$, we have

$$\int_{t_1}^{t} \xi_{\alpha}(\sigma(s)-t_0)^{\alpha-1} r(s)\,\delta^{\sigma}(s) - \frac{\left[\delta^{\Delta}(s)\right]^2 p(s)\,T\phi(s)}{4\delta^{\sigma}(s)}\Delta s \leq w(t_1) < \infty,$$

which is a contradiction with (3.5). Hence, case (1) of Lemma 3.1 is not true.
Secondly suppose that x satisfies (2) of lemma 3.1, then clearly $\lim\limits_{t\to+\infty} x(t)$ exists (finite).
Thus, the proof is complete. □

Remark 3.2. *The function ϕ is existent, e.g., by letting*

$$\phi(t) := \int_{t_1}^{t} \frac{1}{p(s)}\Delta s, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}.$$

Remark 3.3. *If we take $\mathbb{T} = \mathbb{R}$, it is clear that $Td(t) = 1$.*

Taking $\delta(t) = 1$ in Theorem 3.2, we have the following the corollary.

Corollary 3.1. *Suppose that (H_1)-(H_3) holds and for every sufficiently large $t_1 \in [t_0, +\infty)_{\mathbb{T}}$,*

$$\int_{t_1}^{\infty} (\sigma(s)-t_0)^{\alpha-1} r(s)\,\Delta s = \infty. \tag{3.16}$$

Then the solution x of (1.1) is oscillatory or $\lim\limits_{t\to+\infty} x(t)$ exists (finite).

Similar to the proof of Theorem 3.2 , we can prove the following theorem.

Theorem 3.3. *Suppose that (H_1)-(H_3) holds and that there exist a positive functions δ, $\phi \in C^1_{rd}([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that (3.6) holds and for every sufficiently large $t_1 \in [t_0, +\infty)_{\mathbb{T}}$,*

$$\lim_{t\to+\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m \left((\sigma(s)-t_0)^{\alpha-1} r(s)\,\delta^{\sigma}(s) - \frac{\left[\delta^{\Delta}(s)\right]^2 T\phi(s)}{4\delta^{\sigma}(s)} \right) \Delta s = \infty,$$

where $m \geq 0$.
Then the solution x of (1.1) is oscillatory or $\lim\limits_{t\to+\infty} x(t)$ exists (finite).

Taking $\delta(t) = 1$ in Theorem 3.3, we have the following the corollary.

Corollary 3.2. *Suppose that (H_1)-(H_3) holds and for every sufficiently large $t_1 \in [t_0, +\infty)_{\mathbb{T}}$,*

$$\lim_{t\to+\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m (\sigma(s)-t_0)^{\alpha-1} r(s)\,\Delta s = \infty, \tag{3.17}$$

where $m \geq 0$.
Then the solution x of (1.1) is oscillatory or $\lim\limits_{t\to+\infty} x(t)$ exists (finite).

4 Example

In the following, we illustrate possible applications with two examples.

Example 4.1. *Consider the fractional order dynamic equation on time scales*

$$\left({}_1^{\mathbb{Z}}\mathcal{D}_t^{\alpha}x\left(t\right)\right)^{\Delta}+\frac{x}{t}=0,\qquad \textit{for all } t\in[1,+\infty)_{\mathbb{Z}}\,. \tag{4.18}$$

Let $r\left(t\right)=\frac{1}{t}$ *and* $f\left(x\right)=x$. *It is easy to see that the conditions* $\left(H_1\right)$, $\left(H_2\right)$ *and* $\left(H_3\right)$ *are satisfied. We will apply Corollary 3.1 and it remains to satisfy the condition* (3.16). *For every sufficiently large* t_1, *since*

$$\int_{t_1}^{\infty}\left(\sigma\left(s\right)-t_0\right)^{\alpha-1}r\left(s\right)\Delta s=\sum_{n=t_1}\frac{\left(n-1\right)^{1-\alpha}}{n}\simeq\sum_{n\geq t_1}\frac{1}{n^{\alpha}}=\infty,$$

which yields that (3.16) *holds.*
Hence, by Corollary 3.1 every solution of (4.18) *is oscillatory or* $\lim_{t\to+\infty}x\left(t\right)$ *exists (finite).*

Example 4.2. *Consider the fractional order dynamic equation on time scales*

$$\left(t_1^{\mathbb{R}}\mathcal{D}_t^{\alpha}x\left(t\right)\right)^{\Delta}+tx\left(t\right)=0,\qquad \textit{for all } t\in[1,+\infty)_{\mathbb{R}}\,. \tag{4.19}$$

Let $p\left(t\right)=t$, $r\left(t\right)=t$ *and* $f\left(x\right)=x$. *It is easy to see that the conditions* $\left(H_1\right)$, $\left(H_2\right)$ *and* $\left(H_3\right)$ *are satisfied. We will apply Corollary 3.1 and it remains to satisfy the condition* (3.16). *For every sufficiently large* t_1, *since*

$$\int_{t_1}^{\infty}\left(t-t_0\right)^{\alpha-1}r\left(t\right)ds=\int_{t_1}^{\infty}\left(s-t_0\right)^{\alpha-1}sds\simeq\int_{t_1}^{\infty}s^{\alpha}ds=\infty,$$

which yields that (3.16) *holds.*
Hence, by Corollary 3.1 every solution of (4.19) *is oscillatory or* $\lim_{t\to+\infty}x\left(t\right)$ *exists (finite).*

References

[1] S. Hilger, *Ein Maßkettenakalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. Thesis, Universität Würzburg, 1988 (in German).

[2] R. Agarwal, M. Bohner, A. Peterson, *Inequalities on time scales*, Math. Inequal. Appl. 4(2001), 535–557.

[3] M. Bohner, A.C.Peterson, *Dynamic Equations on Time Scales. An Introduction with Applications*. Birkäuser Boston, Inc., Boston, MA, 2001.

[4] M. Bohner, A.C. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkäuser Boston, Inc., Boston, MA, 2003.

[5] S. H. Saker, *Oscillation of second-order nonlinear neutral delay dynamic equations on time scales*. J. Comput. App. Mat, 187(2006), 123-141.

[6] L. Erbe, T.S. Hassan, A. Peterson, *Oscillation criteria for nonlinear damped dynamic equations on time scales*. Appl. Math. Comp. 203(2008), 343–357

[7] S. R. Grace, M. Bohner, S. Sun. *Oscillation of fourth-order dynamic equations*. *Hacettepe. J. Math. Statistics*, 39(2010), 545 – 553.

[8] *S.R.Grace, R. P. Agarwal, and Sandra Pinelas*, *On the oscillation of fourth-order superlinear dynamic equations on time Scales,* Dynamic. Systems. Appl, 20(2011), 45-54.

[9] T. Li, E. Thandapani, S. Tang, *Oscillation theorems for fourth-order delay dynamic equations on time scales*. Bulletin. Mat. Anal. Appl, 3(2011), 190-199.

[10] B. Baculíkovà, J. Dzŭurina, *Oscillation theorems for higher order neutral differential equations*. Appl. Mat. Comp. 219(2012), 3769-3778.

[11] S. R. Grace, R. P. Agarwal, P J Y. Wong, A. Zafer, *On the oscillation of fractional differential equations*. Fract. Calc. Appl. Anal. 15(2012), 222-231.

[12] D. X. Chen, *Oscillation criteria of fractional differential equations*. Adv. Differ. Equ. 2012, 1-10.

[13] R.P. Agarwal , M. Bohner, T. Li, C. Zhang. *A new approach in the study of oscillatory behavior of even-order neutral delay differential equations*. Appl. Mat. Comp. 2013, 787–794.

[14] S. R. Grace. *On the Oscillation of nth Order Dynamic Equations on Time-Scales*. Mediterr. J. Math. 10 (2013), 147–156.

[15] C. Zhanga, R. P. Agarwal, M. Bohner, T. Li, *New results for oscillatory behavior of even-order half-linear delay differential equations*. Appl. Mat. Letters, 26 (2013), 179–183.

[16] D. X. Chen, P. X. Qu, Y. H. Lan, *Forced oscillation of certain fractional differential equations,* Adv. Diff. Equ. 2013.

[17] Y. QI, Ji. YU. *Oscillation criteria for fourth-order nonlinear delay dynamic equations*. Elec. J. Diff. Equa, 79(2013),1-17.

[18] Z. Han, Y. Sun, M. Xu, *Oscillation criteria for p-Laplacian delay dynamic equations with damping on time scales*, Adv. Diff. Equa. 2013.

[19] Ya. Shi, Oscillation criteria for nth order nonlinear neutral differential equations. Appl. Mat. Comp, 235 (2014), 423–429.

[20] N. Benkhettou, S. Hassani, D. F.M. Torres, *A conformable fractional calculus on arbitrary time scales*, J. King Saud University-Science (2015).

[21] N. Benkhettou, A. Hammoudi, D. F.M. Torre, *Existence and uniqueness of solution for a fractional Riemann–Liouville initial value problem on time scales*, J. King Saud University–Science (2015).

[22] F. Z. Ladrani, A. Hammoudi, A. Benaissa cherif, *Oscillation theorems for fourth-order nonlinear dynamic equations on time scales*, Elec. J. Mat. Anal. Appl, 3(2015), 46-58.

[23] A. Benaissa cherif, F. Z. Ladrani, A. Hammoudi, *Oscillation theorems for higher order neutral nonlinear dynamic equations on time scales*, Malaya. J. Mat, 4(2016), 599–605.

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Anti S-fuzzy soft subhemirings of a hemiring

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Abstract

In this paper, we have studied the algebraic operations of anti S-fuzzy soft sets to establish their basic properties. We have discussed different algebraic structures of anti S-fuzzy soft sets under the restricted and extended operations of union and intersection in a comprehensive manner. Logical equivalences have also been made in order to give a complete overview of these structures.

Keywords: Fuzzy soft set, anti S fuzzy soft subhemiring, pseudo anti S fuzzy soft coset.

2010 MSC: 03F55, 06D72, 08A72.

1 Introduction

Many fields deal daily with the uncertain data that may not be successfully modeled by the classical mathematics. There are some mathematical tools for dealing with uncertainties; two of them are fuzzy set theory, developed by Zadeh (1965), and soft set theory, introduced by Molodtsov (1999), that are related to this work. At present, work on the soft set theory is progressing rapidly. Maji et al (2003) defined operations of soft sets to make a detailed theoretical study on the soft sets. By using these definitions, the applications of soft set theory have been studied increasingly. Soft decision making (Cagman & Enginoglu 2010a, 2010b, Cagman et al 2010a, 2010b, Chen et al 2005, Feng et al 2010, Herewan & Deris 2009b, Herawan et al 2009, Kong et al 2008, 2009, Maji et al 2002, Xiao et al 2003, Majumdar & Samanta 2008, 2010. Cagman & Enginoglu (2010a) redefine the operations of soft sets to make them more functional for improving several new results. By using these new definitions they also construct a uni-int decision making method which selects a set of optimum elements from the alternatives. Cagman & Enginoglu (2010b) introduced a matrix representation of this work that gives several advantages to compute applications of the soft set theory. Later than the preamble of fuzzy sets as a result of L.A. Zadeh [16], more than a few scholars developed lying on the overview of the notion of fuzzy sets. Ali, M.I., M. Shabir and M. Naz, [5] developed the algebraic structures of soft sets associated with new operations, Fuzzy Sets and Systems: Theory and Applications was urbanized by Dubois, D. and Prade, H.[8], also Maji, P.K., R. Biswas and A.R. Roy, [13] have been produced Fuzzy Soft Sets. In this article, we introduce some properties and theorems in anti S-fuzzy soft subhemirings of a hemiring.

2 Preliminaries

Definition 2.1. *A S-norm is a binary operation* $S : [0,1] \times [0,1] \to [0,1]$ *satisfying the following requirements:*

(i) $0Sx = x, 1Sx = 1$ *(boundary conditions)*

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(ii) $xSy = ySx$ *(commutativity)*

(iii) $xS(ySz) = (xSy)Sz$ *(associativity)*

(iv) *If* $x \in y$ *and* $w \in z$*, then* $xSw \in ySz$ *(monotonicity).*

Definition 2.2. *Let* $(R, +, \cdot)$ *be a hemiring.* (F, A) *-fuzzy subset of R is said to be an anti S-fuzzy soft subhemiring(anti fuzzy soft subhemiring with respect to S-norm) of R if it satisfies the following conditions:*

(i) $\mu_{(F,A)}(x + y) \leq S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$

(ii) $\mu_{(F,A)}(xy) \leq S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$*, for all x and y in R.*

Definition 2.3. *Let* (F, A) *and* (F, B) *be fuzzy subsets of sets G and H, respectively. The anti-product of* (F, A) *and* (F, B)*, denoted by* $(F, A \times B)$ *is defined as* $(F, A \times B) = \{\langle (x, y), \mu_{(F,A\times B)}(x, y)\rangle /$ *for all x in G and y in* $H\}$*, where* $\mu_{(F,A\times B)}(x, y) = \max\{\mu_{(F,A)}(x), \mu_{(F,B)}(y)\}$.

Definition 2.4. *Let (F,A) be a fuzzy subset in a set S, the anti-strongest relation fuzzy relation on S, that is fuzzy relation on (F,A) is (G,V) given by* $\mu_{(G,V)}(x, y) = \max\{\mu_{(F,A)}(x), \mu_{(F,B)}(y)\}$ *for all x and y in S.*

Definition 2.5. *Let* $(R, +, \cdot)$ *and* $(R', +, \cdot)$ *be any two hemirings. Let* $f : R \to R'$ *be any function and* (F, A) *be an anti S fuzzy soft subhemiring in R,* (G, V) *be an anti S- fuzzy soft subhemiring in* $f(R) = R'$*, defined by* $\mu_{(G,V)}(y) = \inf_{x\in f^{-1}(y)} \mu_{(F,A)}(x)$ *for all x in R and y in* R'*. Then* (F, A) *is called a preimage of (G, V) under f and is denoted by* $f^{-1}((G, V))$.

Definition 2.6. *Let* (F, A) *be an anti S fuzzy soft subhemiring of a hemiring* $(R, +, \cdot)$ *and a in R. Then the pseudo anti S fuzzy soft coset* $(a(F, A))^p$ *is defined by* $((a\mu_{(F,A)})^p)(x) = p^{(a)}\mu_{(F,A)}(x)$*, for every x in R, and for some p in P.*

3 Properties of anti S-fuzzy soft subhemiring of hemiring

Theorem 3.1. *Union of any two anti S fuzzy soft subhemiring of a hemiring R is an anti S fuzzy soft subhemiring of R.*

Proof. Let (F, A) and (G, B) be any two S-fuzzy soft subhemirings of a heniring R and x and y in R. Let $(F, A) = \left\{ \left\langle (x), \mu_{(F,A)}(x) \right\rangle /x \in R \right\}$ and $(G, B) = \left\{ \left\langle (x), \mu_{(G,B)}(x) \right\rangle /x \in R \right\}$ and also let $(H, C) = (F, A) \cup (G, B) = \left\{ \left\langle (x), \mu_{(H,C)}(x) \right\rangle /x \in R \right\}$, where $\max\left\{ \mu_{(F,A)}(x), \mu_{(G,B)}(x) \right\} = \mu_{(H,C)}(x)$, now, $\mu_{(H,C)}(x + y, q) \leq \max\left\{ S(\mu_{(F,A)}(x), \mu_{(F,A)}(y)), S(\mu_{(G,B)}(x), \mu_{(G,B)}(y)) \right\} \leq S(\mu_{(H,C)}(x), \mu_{(H,C)}(y))$. Therefore, $\mu_{(H,C)}(x + y) \leq S(\mu_{(H,C)}(x), \mu_{(H,C)}(y))$, for all x and y in R. And, $\mu_{(H,C)}(xy) \leq \max\{S(\mu_{(F,A)}(x), \mu_{(F,A)}(y)), S(\mu_{(G,B)}(x), \mu_{(G,B)}(y))\} \leq S(\mu_{(H,C)}(x), \mu_{(H,C)}(y))$. Therefore, $\mu_{(H,C)}(xy) \leq S(\mu_{(H,C)}(x), \mu_{(H,C)}(y))$, for all x and y in R. Therefore (H, C) is an an anti S-fuzy soft subhemiring of a hemiring R. □

Theorem 3.2. *The Union of a family of anti S-fuzzy soft subhemirings of hemiring R is an anti S-fuzzy soft subhemiring of R.*

Proof. It is trivial. □

Theorem 3.3. *If* (F, A) *and* (F, B) *are two anti S-fuzzy soft subhemirings of the hemirings* R_1 *and* R_2 *respectively, then anti product* $(F, A \times B)$ *is an anti S-fuzzy soft subhemiring of* $R_1 \times R_2$*.*

Proof. Let (F, A) and (G, B) be two anti S-fuzzy subhemirings of the hemirings R_1 and R_2 respectively. Let x_1 and x_2 be in R_1, y_1 and y_2 in R_2. Then (x_1, y_1) and (x_2, y_2) are in $R_1 \times R_2$. Now,

$$\begin{aligned}
&\mu_{(F,A\times B)}[(x_1, y_1) + (x_2, y_2)] \\
&\leq \max\{S(\mu_{(F,A)}(x_1), \mu_{(F,A)}(x_2)), S(\mu_{(F,B)}(y_1), \mu_{(F,B)}(y_2))\} \\
&\leq S(\mu_{(F,A\times B)}(x_1, y_1), \mu_{(F,A\times B)}(x_2, y_2)).
\end{aligned}$$

Therefore,

$$\mu_{(F,A\times B)}[(x_1,y_1)+(x_2,y_2)] \leq S\big(\mu_{(F,A\times B)}(x_1,y_1),\mu_{(F,A\times B)}(x_2,y_2)\big).$$

Also,

$$\begin{aligned}
&\mu_{(F,A\times B)}[(x_1,y_1)(x_2,y_2)]\\
&\leq \max\{S(\mu_{(F,A)}(x_1),\mu_{(F,A)}(x_2)),S(\mu_{(F,B)}(y_1),\mu_{(F,B)}(y_2))\}\\
&\leq S\big(\mu_{(F,A\times B)}(x_1,y_1),\mu_{(F,A\times B)}(x_2,y_2)\big).
\end{aligned}$$

Therefore,

$$\mu_{(F,A\times B)}[(x_1,y_1)(x_2,y_2)] \leq S\big(\mu_{(F,A\times B)}(x_1,y_1),\mu_{(F,A\times B)}(x_2,y_2)\big).$$

Hence $(F, A\times B)$ is an anti S-fuzzy soft subhemiring of a hemiring $R_1\times R_2$. □

Theorem 3.4. *Let (F,A) be a fuzzy soft subset of a hemi ring R and (G,V) be the anti strongest fuzzy relation of R. Then (F,A) is an anti S fuzzy soft subhemiring of R if and only if (G,V) is an anti Sfuzzy soft sub hemi ring of $R\times R$.*

Proof. Suppose that (F,A) is an anti S-fuzzy soft subhemiring of a hemiring R. Then for any $X=(x_1,x_2)$ and $Y=(y_1,y_2)$ are in $R\times R$, We have

$$\begin{aligned}
\mu_{(G,V)} &\leq \max\{S(\mu_{(F,A)}(x_1,y_1),\mu_{(F,A)}(x_2,y_2)\}\\
&\leq \max\{S(\mu_{(F,A)}(x_1),\mu_{(F,A)}(y_1)),S(\mu_{(F,A)}(x_2),\mu_{(F,A)}(y_2)\}\\
&\leq S\big(\mu_{(G,V)}((x_1,x_2)),\mu_{(G,V)}((y_1,y_2))\big)\\
&\leq S\big(\mu_{(G,V)}(X),\mu_{(G,V)}(Y)\big),\text{ for all } X \text{ and } Y \text{ in } R\times R.
\end{aligned}$$

Therefore, $\mu_{(G,V)}(XY)\leq S\big(\mu_{(G,V)}(X),\mu_{(G,V)}(Y)\big)$, for all X and Y in $R\times R$. This proves that (G,V) is an anti S fuzzy soft subhemiring of a hemiring of $R\times R$. Conversely assume that (G,V) is an anti S-fuzzy soft subhemiring of a hemiring of $R\times R$, then for any $X=(x_1,x_2)$ and $Y=(y_1,y_2)$ are in $R\times R$. We have

$$\begin{aligned}
&\max\{S(\mu_{(F,A)}(x_1+y_1),\mu_{(F,A)}(x_2+y_2))\}\\
&=\mu_{(G,V)}(X+Y)\\
&\leq S(\mu_{(G,V)}(X),\mu_{(G,V)}(Y))\\
&=S(\mu_{(G,V)}((x_1,y_1)),\mu_{(G,V)}((y_1,y_2)))\\
&=S\big(\max(\mu_{(F,A)}(x_1),\mu_{(F,A)}(y_1)),\max(\mu_{(F,A)}(x_2),\mu_{(F,A)}(y_2))\big).
\end{aligned}$$

If $X_2=0, y_2=0$, we get $\mu_{(F,A)}(x_1+y_1)\leq S\big(\mu_{(F,A)}(x_1),\mu_{(F,A)}(y_1)\big)$ for all x_1 and y_1 in R. And

$$\begin{aligned}
&\max\{S(\mu_{(F,A)}(x_1y_1),\mu_{(F,A)}(x_2y_2)\}\\
&=\mu_{(G,V)}(x,y)\\
&\leq S(\mu_{(G,V)}(X),\mu_{(G,V)}(Y))\\
&=S(\mu_{(G,V)}((x_1,y_1)),\mu_{(G,V)}((y_1,y_2)))\\
&=S(\max\{\mu_{(F,A)}(x_1),\mu_{(F,A)}(x_2)\},\max\{\mu_{(F,A)}(y_1),\mu_{(F,A)}(y_2)\}).
\end{aligned}$$

If $x_2=0, y_2=0$. We get $\mu_{(F,A)}(x_1y_1)\leq S\big(\mu_{(F,A)}(x_1),\mu_{(F,A)}(y_1)\big)$ for all x_1 and y_1 in R. Therefore (F,A) is an anti S-fuzzy soft subhemiring of R. □

Theorem 3.5. *If (F,A) is an anti S-fuzzy soft subhemiring of a hemiring $(R,+,\cdot)$ if and only if $\mu_{(F,A)}(x+y)\leq S(\mu_{(F,A)}(x),\mu_{(F,A)}(y),\mu_{(F,A)}(xy))\leq S(\mu_{(F,A)}(x),\mu_{(F,A)}(y))$ for all x and y in R.*

Proof. It is trivial. □

Theorem 3.6. *If (F, A) is an anti S-fuzzy soft subhemiring of a hemiring $(R, +, \cdot)$, then $H = \{x/x \in R : \mu_{(F,A)}(x) = 0\}$ is either empty or is a subhemiring of R.*

Proof. It is trivial. □

Theorem 3.7. *If (F, A) is an anti S-fuzzy soft subhemiring of a hemiring $(R, +, \cdot)$. If $\mu_{(F,A)}(x + y) = 1$, then either $\mu_{(F,A)}(x) = 1$ or $\mu_{(F,A)}(y) = 1$, for all x and y in R.*

Proof. It is trivial. □

Theorem 3.8. *If (F, A) is an anti S-fuzzy soft subhemiring of a hemiring $(R, +, \cdot)$, then the pseudo anti S-fuzzy coset $(a(F, A))^p$ is an anti S-soft subhemiring of a hemiring R, for every a in R.*

Proof. Let (F, A) is an anti S-fuzzy subhemiring of a hemiring R. For every x and y in R, we have $((a\mu_{(F,A)})^p)(x + y) \leq p(a)S(\mu_{(F,A)}(x), \mu_{(F,A)}(y)) \in S(p(a)\mu_{(F,A)}(x), p(a)\mu_{(F,A)}(y)) = S((a\mu_{(F,A)}{}^p(x), (a\mu_{(F,A)}{}^p(y))$. Therefore, $((a\mu_{(F,A)})^p)(x + y) \leq S((a\mu_{(F,A)}{}^p(x), (a\mu_{(F,A)}{}^p(y))$. Now, $((a\mu_{(F,A)})^p)(xy) \leq p(a)S(\mu_{(F,A)}(x), \mu_{(F,A)}(y)) \in S(p(a)\mu_{(F,A)}(x), p(a)\mu_{(F,A)}(y)) = S((a\mu_{(F,A)}{}^p(x), (a\mu_{(F,A)}{}^p(y))$. Therefore, $((a\mu_{(F,A)})^p)(xy) \leq S((a\mu_{(F,A)}{}^p(x), (a\mu_{(F,A)}{}^p(y))$. Hence $(a(F, A))^p$ is an anti S-fuzzy soft subhemiring of a hemiring R. □

Theorem 3.9. *Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemi rings. The homomorphic image of an anti S-fuzzy soft subhemiring of R is an anti S-fuzzy soft subhemiring of R'.*

Proof. Let $f : R \to R'$ be a homomorphism. Then $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and y in R. Let $(G, V) = f((F, A))$, where (F, A) is an anti S-fuzzy soft subhemiring of R. Now, for $f(x), f(y)$ in $R', \mu_{(G,V)}((f(x)) + (f(y))) \leq \mu_{(F,A)}(x + y) \leq S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$, which implies that $\mu_{(G,V)}((f(x)) + (f(y))) \leq S(\mu_{(G,V)}((f(x)), \mu_{(G,V)}((f(y))))$. Again, $\mu_{(G,V)}((f(x))(f(y))) \leq \mu_{(G,V)}((f(x))(f(y))) \leq S(\mu_{(G,V)}(x), \mu_{(G,V)}(y))$.

Hence (G, V) is an anti S-fuzzy subhemiring of hemiring R'. □

Theorem 3.10. *Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. The homomorphic preimage of an anti S-fuzzy soft subhemiring of R' is an anti S-fuzzy soft subhemiring of R.*

Proof. Let $f : R \to R'$ be a homomorphism.Then, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and y in R. Let $(G, V) = f((F, A))$, where (G, V) is an anti S- fuzzy soft soft subhemiring of R'. Now, for all x, y in $R, \mu_{(F,A)}((x) + (y)) = \mu_{(G,V)}((f(x)) + (f(y))) \leq S(\mu_{(G,V)}(f(x)), \mu_{(G,V)}(f(y)) = S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$, which implies that $\mu_{(F,A)}((x) + (y)) \leq S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$. Again, $\mu_{(F,A)}((x)(y)) = \mu_{(G,V)}(f(x)f(y)) \leq S(\mu_{(G,V)}(f(x)), \mu_{(G,V)}(f(y))) = S(\mu_{(F,A)}(x), \mu_{(F,A)}(y)$, which implies that $\mu_{(F,A)}((x)(y)) \leq S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$. Hence (F, A) is an anti S fuzzy soft subhemiring of hemiring R. □

Theorem 3.11. *Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. The anti-homomorphic image of an anti S-fuzzy soft subhemiring of R is an anti S-fuzzy soft subhemiring of R'.*

Proof. Let $f : R \to R'$ be a homomorphism.Then, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and y in R. Let $(G, V) = f((F, A))$, where (F, A) is an anti S- fuzzy soft soft subhemiring of R. $\mu_{(G,V)}((f(x)) + (f(y))) \leq \mu_{(F,A)}(y + x) \leq S(\mu_{(F,A)}(y), \mu_{(F,A)}(x)) = S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$, which implies that $\mu_{(G,V)}((f(x)) + f(y))) \leq S(\mu_{(G,V)}(f(x)), \mu_{(G,V)}(f(y)))$. Again, $\mu_{(G,V)}((f(x))(f(y))) \leq \mu_{(F,A)}(yx) \leq S(\mu_{(F,A)}(y), \mu_{(F,A)}(x)) = S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$, which implies that $\mu_{(G,V)}((f(x))(f(y))) \leq S(\mu_{(G,V)}(f(x)), \mu_{(G,V)}(f(y)))$. Hence (G, V) is an anti S fuzzy soft subhemiring of hemiring R'. □

Theorem 3.12. *Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. The anti-homomorphic preimage of an anti S-fuzzy soft subhemiring of R' is an anti S-fuzzy soft subhemiring of R.*

Proof. Let $(G, V) = f((F, A))$ where (G, V) is an anti S fuzzy subhemiring of R'. Let x and y in R. Then $\mu_{(F,A)}((x) + (y)) = \mu_{(G,V)}((f(x)) + (f(y))) \leq S(\mu_{(G,V)}(f(y)), \mu_{(G,V)}(f(x)) = S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$, which implies that, $\mu_{(F,A)}((x) + (y)) \leq S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$. Again $\mu_{(F,A)}((x)(y)) = \mu_{(G,V)}((f(x))(f(y))) \leq S(\mu_{(G,V)}(f(y)), \mu_{(G,V)}(f(x))) = S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$, which implies that $\mu_{(F,A)}((x)(y)) \leq S(\mu_{(F,A)}(x), \mu_{(F,A)}(y))$. Hence (F, A) is an anti S fuzzy soft subhemiring of hemiring R. □

In the following Theorem $\circ$ is the composition operation of functions:

Theorem 3.13. *Let (F, A) be an anti S-fuzzy soft subhemiring of hemiring H and f is an isomorphism from a hemi ring R onto H. Then $(F, A) \circ f$ is an anti S-fuzzy soft subhemiring of R.*

Proof. Let x and y in R. Then we have, $(\mu_{(F,A)} \circ f)((x) + (y)) = \mu_{(F,A)}((f(x)) + (f(y))) \leq S(\mu_{(F,A)}(f(x)), \mu_{(F,A)}(f(y))) \leq S\big((\mu_{(F,A)} \circ f)(x), (\mu_{(F,A)} \circ f)(y)\big)$, which implies that $(\mu_{(F,A)} \circ f)((x) + (y)) \leq S\big((\mu_{(F,A)} \circ f)(x), (\mu_{(F,A)} \circ f)(y)\big)$. And $(\mu_{(F,A)} \circ f)((x)(y)) = \mu_{(F,A)}((f(x))(f(y))) \leq S(\mu_{(F,A)}(f(x)), \mu_{(F,A)}(f(y))) \leq S\big((\mu_{(F,A)} \circ f)(x), (\mu_{(F,A)} \circ f)(y)\big)$, which implies that $(\mu_{(F,A)} \circ f)((x)(y)) \leq S\big((\mu_{(F,A)} \circ f)(x), (\mu_{(F,A)} \circ f)(y)\big)$. Therefore $(F, A) \circ f$ is an anti S fuzzy soft subhemiring of hemiring R.

□

Theorem 3.14. *Let (F, A) be an anti S-fuzzy soft subhemiring of hemiring H and f is an anti-isomorphism from a hemi ring R onto H. Then $(F, A) \circ f$is an anti S-fuzzy soft subhemiring of R.*

Proof. Let x and y in R. Then we have, $(\mu_{(F,A)} \circ f)((x) + (y)) = \mu_{(F,A)}((f(y)) + (f(x))) \leq S(\mu_{(F,A)}(f(x)), \mu_{(F,A)}(f(y))) \leq S\big((\mu_{(F,A)} \circ f)(x), (\mu_{(F,A)} \circ f)(y)\big)$, which implies that $(\mu_{(F,A)} \circ f)((x) + (y)) \leq S\big((\mu_{(F,A)} \circ f)(x), (\mu_{(F,A)} \circ f)(y)\big)$. And $(\mu_{(F,A)} \circ f)((x)(y)) = \mu_{(F,A)}((f(y))(f(x))) \leq S(\mu_{(F,A)}(f(x)), \mu_{(F,A)}(f(y))) \leq \big((\mu_{(F,A)} \circ f)(x), (\mu_{(F,A)} \circ f)(y)\big)$, which implies that $(\mu_{(F,A)} \circ f)((x)(y)) \leq S\big((\mu_{(F,A)} \circ f)(x), (\mu_{(F,A)} \circ f)(y)\big)$. Therefore $(F, A) \circ f$ is an anti S fuzzy soft subhemiring of hemiring R.

□

References

[1] M. T. Abu Osman, On some product of fuzzy subgroups, *Fuzzy Sets and Systems*, 24(1987) 79–86.

[2] B. Ahmad and A. Kharal, On fuzzy soft sets, *Advances in Fuzzy Systems,* 1–6, 2009.

[3] H. Aktas and N. CaSman, Soft sets and soft groups, *Inform. Sci.* 177 (2007) 2726–2735.

[4] M. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set Theory, *Computers and Mathematics with Applications,* 57(2009), 1547–1553.

[5] M. I. Ali, M. Shabir and M. Naz, Algebraic structures of soft sets associated with new operations, *Computers and Mathematics with Applications,* 61(2011), 2647–2654.

[6] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20(1986): 87–96.

[7] Bih-Sheue Shieh, Infinite fuzzy relation equations with continuous t-norms, *Information Sciences*, 178(2008), 1961–1967.

[8] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications,* Academic Press, 1980.

[9] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, *Computers and Mathematics with Applications,* 56(2008) 2621–2628.

[10] Y. B. Jun, Soft BCK/BCI-algebra, *Computers and Mathematics with Applications,* 56(2008) 1408–1413.

[11] D. Molodtsov, Soft set theory first result, *Computers and Mathematics with Applications,* 37 (1999) 19-31

[12] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, *Computers and Mathematics with Applications,* 45 (2003), 555–562.

[13] P. K. Maji, R. Biswas and A.R. Roy, Fuzzy soft sets, *The J. Fuzzy Math.,* 9(2001): 589–602.

[14] Qiu-Mei Sun, Zi-Liong Zhang and Jing Liu, Soft sets and soft modules, *Lecture Notes in Comput. Sci.,* 5009 (2008) 403–409.

[15] X. Yang, D. Yu, J. Yang and C. Wu, Generalization of Soft Set Theory: From Crisp to Fuzzy Case, *Proceedings of the Second International Conference of Fuzzy Information and Engineering (ICFIE-2007),* 27-Apr., pp. 345–355.

[16] L. A. Zadeh, Fuzzy sets, *Inform. And Control,* 8(1965), 338–353.

[17] H. J. Zimmermann, *Fuzzy Set Theory and Its Applications,* Kluwer, 1991.

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Oscillation criteria for nonlinear difference equations with superlinear neutral term

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Abstract

In this paper, the authors obtain sufficient conditions for the oscillation of all solutions of the equation

$$\Delta(a_n\Delta(x_n + p_n x^{\alpha}_{n-k})) + q_n x^{\beta}_{n+1-l} = 0$$

where $\alpha \geq 1$ and $\beta > 0$ are ratio of odd positive integers, and $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are real positive sequences. Examples are provided to illustrate the importance of the main results.

Keywords: Oscillation, nonlinear difference equation, superlinear neutral term.

2010 MSC: 39A11.

1 Introduction

In this paper, we are concerned with the oscillatory behavior of nonlinear neutral difference equation of the form

$$\Delta(a_n\Delta(x_n + p_n x^{\alpha}_{n-k})) + q_n x^{\beta}_{n+1-l} = 0, \ n \geq n_0 \tag{1.1}$$

where n_0 is a nonnegative integer, subject to the following conditions:

(H_1) $\alpha \geq 1$, and β are ratios of odd positive integers;

(H_2) $\{a_n\}$, $\{p_n\}$, and $\{q_n\}$ are positive real sequences for all $n \geq n_0$;

(H_3) k is a positive integer, and l is a nonnegative integer.

Let $\theta = \max\{k, l\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ that satisfies equation (1.1) for all $n \geq n_0$. A solution of equation (1.1) is called oscillatory if its terms are neither eventually positive nor eventually negative, and nonoscillatory otherwise. If all solutions of the difference equation are oscillatory then the equation itself called oscillatory.

As mentioned by Hale [4] and others, neutral equations having a nonlinearity in the neutral term arise in various applications. We choose to investigate the oscillatory behavior of equation (1.1) since similar properties of difference equations with linear neutral term are extensively studied in [1–3, 8, 10].

In particular in [6, 7, 9, 11, 12], the authors considered equation of the type (1.1) when $0 < \alpha \leq 1$ and either

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty, \tag{1.2}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty. \tag{1.3}$$

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In all the results the condition $p_n \to 0$ as $n \to \infty$ is required to apply the theorems. Further in [11], the authors considered equation of the type (1.1) with $\alpha > 1$ and studied the oscillatory behavior under the condition that $\lim_{n\to\infty} \inf q_n > 0$. Motivated by this observation, in this paper we examine the other case $\alpha \geq 1$ and we do not require that either $p_n \to 0$ as $n \to \infty$ or $\lim_{n\to\infty} \inf q_n > 0$. Our method of proof is different from that of in and hence our results are new and complement to that of reported in [6, 7, 9, 11, 12]. Examples are presented to illustrate the importance of the main results.

2 Oscillation Results

In this section, we obtain sufficient conditions for the oscillation of all solutions of the equation (1.1). Define

$$R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s}, \quad A_n = \sum_{s=n}^{\infty} \frac{1}{a_s},$$

$$B_n = \frac{1}{p_{n+k}} \left(1 - \frac{M^{\frac{1}{\alpha}} R_{n+2k}^{\frac{1}{\alpha}}}{R_{n+k}^{\frac{1}{\alpha}} p_{n+2k}^{\frac{1}{\alpha}}} \right) > 0 \text{ for all constants } M > 0,$$

and

$$E_n = \frac{1}{p_{n+k}} \left(1 - \frac{M_1^{\frac{1}{\alpha}-1} A_{n+k}^{\frac{1}{\alpha}-1}}{p_{n+2k}^{\frac{1}{\alpha}}} \right) > 0 \text{ for all constants } M_1 > 0.$$

We set

$$z_n = x_n + p_n x_{n-k}^{\frac{1}{\alpha}}.$$

Due to the from of our equation (1.1), we only need to give proofs for the case of eventually positive solutions since the proofs for the eventually negative solution would be similar.

We begin with the following theorem.

Theorem 2.1. *Assume that* $(H_1) - (H_3)$, *and* (1.2) *hold. If*

$$\sum_{n=n_1}^{\infty} q_n B_{n+1-l}^{\frac{\beta}{\alpha}} = \infty \tag{2.4}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution, say $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. From equation (1.1), we have

$$\Delta(a_n \Delta z_n) = -q_n x_{n+1-l}^{\beta} < 0, \ n \geq n_1. \tag{2.5}$$

In view of condition (1.2), it is easy to see that $\Delta z_n > 0$ for all $n \geq n_1$. Now, it follows from the definition of z_n, one obtains

$$x_n^{\alpha} = \frac{1}{p_{n+k}} (z_{n+k} - x_{n+k}). \tag{2.6}$$

On the other hand $x_{n+k} \leq \frac{z_{n+2k}^{1/\alpha}}{p_{n+2k}^{1/\alpha}}$, and therefore from (2.6), we have

$$x_n^{\alpha} \geq \frac{1}{p_{n+k}} \left(z_{n+k} - \frac{z_{n+2k}^{1/\alpha}}{p_{n+2k}^{\frac{1}{\alpha}}} \right). \tag{2.7}$$

From (2.5), we have $a_n \Delta z_n$ is positive and decreasing and therefore

$$z_n \geq R_n a_n \Delta z_n, \ n \geq n_1, \tag{2.8}$$

and hence

$$\Delta\left(\frac{z_n}{R_n}\right) \le 0, \; n \ge n_1. \tag{2.9}$$

Now (2.7) and (2.9) implies that

$$x_n^\alpha \ge \frac{1}{p_{n+k}}\left(1 - \frac{M^{\frac{1}{\alpha}-1} R_{n+2k}^{\frac{1}{\alpha}}}{R_{n+k}^{\frac{1}{\alpha}} p_{n+2k}^{\frac{1}{\alpha}}}\right) z_{n+k} \tag{2.10}$$

where we have used $z_n \ge M > 0$ for all $n \ge n_1$. In view of (2.5) and (2.10), we obtain

$$\Delta(a_n \Delta z_n) + q_n B_{n+1-l}^{\frac{\beta}{\alpha}} z_{n+1+k-l}^{\frac{\beta}{\alpha}}, \; n \ge n_1. \tag{2.11}$$

Summing the equation (2.11) from n_1 to n, we have

$$\sum_{s=n_1}^{n} q_s B_{s+1-l}^{\frac{\beta}{\alpha}} z_{n+1+k-l}^{\frac{\beta}{\alpha}} \le a_{n_1} \Delta z_{n_1}.$$

Since $z_n \ge M$, it is easy to see from the last inequality that we can obtain a contradiction with (2.4) as $n \to \infty$. This completes the proof. □

Remark 2.1. *In Theorem 2.1, we are not required the conditions $\alpha \ge \beta$ or $\alpha \le \beta$ and $l \ge k$ or $l \le k$.*

Theorem 2.2. *Assume that $(H_1) - (H_3)$, and* (1.2) *hold. If $l > k$, and the first order delay difference equation*

$$\Delta w_n + q_n B_{n+1-l}^{\frac{\beta}{\alpha}} R_{n+1+k-l}^{\frac{\beta}{\alpha}} w_{n+1+k-l}^{\frac{\beta}{\alpha}} = 0 \tag{2.12}$$

is oscillatory, then every solution of equation (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution such that $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \ge n_1 \ge n_0$. Proceeding as in proof of Theorem 2.1, we obtain (2.8) and (2.11). Now combining (2.8) and (2.11), we have

$$\Delta(a_n \Delta z_n) + q_n B_{n+1-l}^{\frac{\beta}{\alpha}} R_{n+1+k-l}^{\frac{\beta}{\alpha}} (a_{n+1+k-l \Delta z_{n+1+k-l}})^{\frac{\beta}{\alpha}} \le 0.$$

Let $w_n = a_n \Delta z_n$. Then $\{w_n\}$ is a positive solution of the inequality

$$\Delta(a_n \Delta z_n) + q_n B_{n+1-l}^{\frac{\beta}{\alpha}} R_{n+1+k-l}^{\frac{\beta}{\alpha}} w_{n+1+k-l}^{\frac{\beta}{\alpha}} \le 0, \; n \ge n_1.$$

But by Lemma 2.7 of [8], the corresponding difference equation (2.12) has positive solution. This contradiction completes the proof. □

Corollary 2.1. *Assume that $(H_1) - (H_3)$ and* (1.2) *hold. If $l > k + 1$, $\alpha = \beta$ and*

$$\liminf_{n\to\infty} \sum_{s=n+1+k-l}^{n-1} q_s B_{s+1-l} R_{s+1+k-l} > \left(\frac{l-k-1}{l-k}\right)^{l-k} \tag{2.13}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 7.5.1 of [3] and Theorem 2.2. □

Corollary 2.2. *Assume that $(H_1) - (H_3)$, and* (1.2) *hold. If $\beta < \alpha$ and $l > k$ and*

$$\sum_{n=n_1}^{\infty} q_n B_{n+1-l}^{\frac{\beta}{\alpha}} R_{n+1+k-l}^{\frac{\beta}{\alpha}} = \infty \tag{2.14}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 1 of [5] and Theorem 2.2. □

Corollary 2.3. *Assume that* $(H_1)-(H_3)$, *and* (1.2) *hold. If* $\beta>\alpha$ *and* $l>k+1$ *and there exists a* $\lambda>\frac{1}{l-k-1}\log\frac{\beta}{\alpha}$ *such that*

$$\lim_{n\to\infty}\inf\left[q_nB_{n+1-l}^{\frac{\beta}{\alpha}}R_{n+1+k-l}^{\frac{\beta}{\alpha}}\exp(-e^{\lambda n})\right]>0$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 2 of [5] and Theorem 2.2. □

Our next results are for the case where (1.3) holds in place of (1.2).

Theorem 2.3. *Let* $\frac{\beta}{\alpha}>1$, $(H_1)-(H_3)$, *and* (1.3) *hold. If condition* (2.4) *holds, and*

$$\lim_{n\to}\sup\sum_{s=n_1}^{n-1}\left[A_{s+1}^{\frac{\beta}{\alpha}}q_sE_{s+1-l}^{\frac{\beta}{\alpha}}-\frac{\frac{\beta}{\alpha}M_1^{1-\frac{\beta}{\alpha}}A_s^{\frac{\beta}{\alpha}-1}}{4a_sA_{s+1}^{\frac{\beta}{\alpha}}}\right]=\infty \tag{2.15}$$

for all constants $M_1>0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution such that $x_n>0$, $x_{n-k}>0$, and $x_{n-l}>0$ for all $n\geq n_1\geq n_0$. From equation (1.1) that (2.5) holds, we then have either $\Delta z_n>0$ or $\Delta z_n<0$ eventually. If $\Delta z_n>0$ holds, then proceeding as Theorem 2.1, we obtain a contradiction to condition (2.4). Next assume that $\Delta z_n<0$ for all $n\geq n_1$. Define

$$u_n=\frac{a_n\Delta z_n}{z_n^{\frac{\beta}{\alpha}}},\ n\geq n_1. \tag{2.16}$$

Then $u_n<0$ for $n\geq n_1$, and from (2.5) we have

$$\Delta z_s\quad\leq\quad\frac{a_n\Delta z_n}{a_s},\ s\geq n.$$

Summing the last inequality from n to j and the letting $j\to\infty$, we obtain

$$\frac{a_n\Delta z_nA_n}{z_n}\geq -1,\ n\geq n_1. \tag{2.17}$$

Thus

$$\frac{-a_n\Delta z_n(-a_n\Delta z_n)^{\frac{\beta}{\alpha}-1}A_n^{\frac{\beta}{\alpha}}}{z_n^{\frac{\beta}{\alpha}}}\leq 1$$

for $n\geq n_1$. Since $-a_n\Delta z_n>0$ and from (2.16), we have

$$-\frac{1}{L^{\frac{\beta}{\alpha}-1}}\leq u_nA_n^{\frac{\beta}{\alpha}}\leq 0, \tag{2.18}$$

where $L=-a_{n_1}\Delta z_{n_1}$. On the other hand from (2.17), one obtains

$$\Delta\left(\frac{z_n}{A_n}\right)\geq 0,\ n\geq n_1. \tag{2.19}$$

From the definition of z_n and (2.19), we have

$$x_n^{\alpha}\geq\frac{1}{p_{n+k}}\left(1-\frac{M_1^{\frac{1}{\alpha}-1}A_{n+k}^{\frac{1}{\alpha}-1}}{p_{n+2k}^{\frac{1}{\alpha}}}\right),\ n\geq n_1, \tag{2.20}$$

where we have used $\frac{z_n}{A_n}\geq M_1$ for all $n\geq n_1$. From (2.5) and (2.20), we obtain

$$\Delta(a_n\Delta z_n)+q_nE_{n+1-l}^{\frac{\beta}{\alpha}}z_{n+1+k-l}^{\frac{\beta}{\alpha}}\leq 0,\ n\geq n_1. \tag{2.21}$$

From (2.16), we have

$$\Delta u_n = \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\frac{\beta}{\alpha}}} - \frac{u_n \Delta z_n^{\frac{\beta}{\alpha}}}{z_{n+1}^{\frac{\beta}{\alpha}}},\ n \geq n_1. \tag{2.22}$$

By Mean value theorem

$$\Delta z_n^{\frac{\beta}{\alpha}} \leq \begin{cases} \frac{\beta}{\alpha} z_{n+1}^{\frac{\beta}{\alpha}-1} \Delta z_n, & \text{if } \frac{\beta}{\alpha} > 1; \\ \frac{\beta}{\alpha} z_n^{\frac{\beta}{\alpha}-1} \Delta z_n, & \text{if } \frac{\beta}{\alpha} < 1, \end{cases} \tag{2.23}$$

and so combining (2.23) with (2.22) and then using the fact that $\Delta z_n < 0$ gives

$$\Delta u_n \quad \leq \quad -q_n E_{n+1-l}^{\frac{\beta}{\alpha}} - \frac{\beta}{\alpha} M_1^{\frac{\beta}{\alpha}-1} A_n^{\frac{\beta}{\alpha}-1} \frac{u_n^2}{a_n} \tag{2.24}$$

since $\frac{z_n}{z_{n+1}} \geq 1$ for all $n \geq n_1$. Multiplying (2.24) by $A_{n+1}^{\frac{\beta}{\alpha}}$ and then summing it from n_1 to $n-1$, we obtain

$$\sum_{s=n_1}^{n-1} A_{s+1}^{\frac{\beta}{\alpha}} \Delta u_s + \sum_{s=n_1}^{n-1} A_{s+1}^{\frac{\beta}{\alpha}} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} + \sum_{s=n_1}^{n-1} \frac{\beta}{\alpha} M_1^{\frac{\beta}{\alpha}-1} A_{s+1}^{\frac{\beta}{\alpha}} A_s^{\frac{\beta}{\alpha}-1} \frac{u_n^2}{a_n} \leq 0. \tag{2.25}$$

Summation by parts formula yields

$$\sum_{s=n_1}^{n-1} A_{s+1}^{\frac{\beta}{\alpha}} \Delta u_s \geq A_n^{\frac{\beta}{\alpha}} u_n - A_{n_1}^{\frac{\beta}{\alpha}} u_{n_1} + \sum_{s=n_1}^{n-1} \frac{\beta}{\alpha} A_s^{\frac{\beta}{\alpha}-1} \frac{u_s}{a_s}. \tag{2.26}$$

Combining (2.25) and (2.26) implies

$$\begin{aligned} A_n^{\frac{\beta}{\alpha}} u_n - A_{n_1}^{\frac{\beta}{\alpha}} u_{n_1} + \sum_{s=n_1}^{n-1} \frac{\beta}{\alpha} A_s^{\frac{\beta}{\alpha}-1} \frac{u_s}{a_s} + \sum_{s=n_1}^{n-1} \frac{\beta}{\alpha} M_1^{\frac{\beta}{\alpha}-1} A_{s+1}^{\frac{\beta}{\alpha}} A_s^{\frac{\beta}{\alpha}-1} \frac{u_n^2}{a_n} \\ + \sum_{s=n_1}^{n-1} A_{s+1}^{\frac{\beta}{\alpha}} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} \leq 0 \end{aligned}$$

which on using completing the square yields

$$\sum_{s=n_1}^{n-1} \left[A_{s+1}^{\frac{\beta}{\alpha}} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} - \frac{\frac{\beta}{\alpha} M_1^{1-\frac{\beta}{\alpha}} A_s^{\frac{\beta}{\alpha}-1}}{4 a_s A_{s+1}^{\frac{\beta}{\alpha}}} \right] \leq \frac{1}{L^{\frac{\beta}{\alpha}-1}} + A_{n_1}^{\frac{\beta}{\alpha}} u_{n_1}$$

when using (2.18). This contradicts (2.15) as $n \to \infty$, and the proof is now completed. □

Theorem 2.4. *Let* $0 < \frac{\beta}{\alpha} < 1$, $(H_1) - (H_3)$, *and* (1.3) *hold. If* $l > k$, *condition* (2.14) *hold, and*

$$\lim_{n \to \infty} \sup \sum_{s=n_1}^{n-1} \left[K^{\frac{\beta}{\alpha}-1} A_{s+1} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} - \frac{1}{4 a_s A_{s+1}} \right] = \infty \tag{2.27}$$

for all constants $K > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.3, we see that $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$, then proceeding as in Corollary 2.2, we obtain a contradiction with condition (2.14). Next, assume that $\Delta z_n < 0$ for all $n \geq n_1$. Proceeding as in the proof of Theorem 2.3 we obtain (2.21). Define

$$u_n = \frac{a_n \Delta z_n}{z_n},\ n \geq n_1. \tag{2.28}$$

Thus $u_n < 0$ for all $n \geq n_1$, and

$$\begin{aligned} \Delta u_n \quad &\leq \quad \frac{\Delta(a_n \Delta z_n)}{z_{n+1}} - \frac{a_n (\Delta z_n)^2}{z_n z_{n+1}} \\ &\leq \quad -q_n E_{n+1-l}^{\frac{\beta}{\alpha}} \frac{z_{n+1+k-l}^{\frac{\beta}{\alpha}}}{z_{n+1}} - \frac{u_n^2}{a_n},\ n \geq n_1. \end{aligned} \tag{2.29}$$

Since $\{z_n\}$ is decreasing there exists a constant $K > 0$ such that $z_n \le K$ for all $n \ge n_1$. Using the last inequality in (2.29), we see that

$$\Delta u_n \le -q_n E_{n+1-l}^{\frac{\beta}{\alpha}} K^{\frac{\beta}{\alpha}-1} - \frac{u_n^2}{a_n}, \; n \ge n_1.$$

Multiplying the last inequality by A_{n+1} and then summing it from n_1 to $n-1$, we have

$$\sum_{s=n_1}^{n-1} A_{s+1}\Delta u_s + \sum_{s=n_1}^{n-1} A_{s+1}\frac{u_s^2}{a_s} + \sum_{s=n_1}^{n-1} K^{\frac{\beta}{\alpha}-1} A_{s+1} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} \le 0. \tag{2.30}$$

Using summation by parts formula in the first term of (2.30) and rearranging, we have

$$A_n u_n - A_{n_1} u_{n_1} + \sum_{s=n_1}^{n-1}\left(\frac{u_s}{a_s} + A_{s+1}\frac{u_s^2}{a_s}\right) + \sum_{s=n_1}^{n-1} K^{\frac{\beta}{\alpha}-1} A_{s+1} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} \le 0$$

which on using completing the square yields

$$\sum_{s=n_1}^{n-1}\left[K^{\frac{\beta}{\alpha}-1} A_{s+1} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} - \frac{1}{4a_s A_{s+1}}\right] \le 1 + A_{n_1} u_{n_1}$$

when using (2.17). This contradicts (2.27) as $n \to \infty$, and the proof is now complete. □

Theorem 2.5. *Let $\alpha = \beta$, $(H_1) - (H_3)$, and* (1.3) *hold. If $l > k+1$, condition* (2.13) *hold, and*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n-1}\left[A_{s+1} q_s E_{s+1-l} - \frac{1}{4a_s A_{s+1}}\right] = \infty \tag{2.31}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Corollary 2.1 and Theorem 2.4, and thus the details are omitted. □

3 Examples

In this section, we present some examples to illustrate the importance of our main results.

Example 3.1. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta\left(x_n + n x_{n-1}^3\right)\right) + n x_{n-2}^{1/3} = 0, \; n \ge 1. \tag{3.32}$$

Here $a_n = \frac{1}{n}$, $p_n = n$, $q_n = n$, $k = 1$, $l = 3$, $\alpha = 3$ and $\beta = \frac{1}{3}$. Simple calculation yields $R_n = \frac{n(n-1)}{2}$ and $B_n = \frac{1}{(n+1)}\left(1 - \frac{M^{-2/3}}{n^{1/3}}\right)$. Now, it is easy to see that all conditions of Theorem 2.1 are satisfied, and hence every solution of equation (3.32) is oscillatory.

Example 3.2. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{2n+1}\Delta\left(x_n + n x_{n-2}^3\right)\right) + 2x_{n-3}^3 = 0, \; n \ge 1. \tag{3.33}$$

Here $a_n = \frac{1}{2n+1}$, $p_n = n$, $q_n = 2$, $k = 2$, $l = 4$ and $\alpha = \beta = 3$. Simple calculation shows that $R_n = n^2 - 1$, $B_n = \frac{1}{(n+2)}\left(1 - \frac{M^{-2/3}(n+5)^{1/3}}{(n+1)^{1/3}(n+4)^{1/3}}\right)$, and it is easy to see that all conditions of Corollary 2.1 are satisfied. Hence every solution of equation (3.33) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (3.33).

Example 3.3. *Consider the neutral difference equation*

$$\Delta\left((n+1)(n+2)\Delta\left(x_n + n^3 x_{n-1}^3\right)\right) + (n+1)^7 x_{n-3}^5 = 0, \; n \ge 1. \tag{3.34}$$

Here $a_n = (n+1)(n+2),\ p_n = n^3,\ q_n = (n+1)^7,\ k = 1,\ l = 4,\ \alpha = 3$ and $\beta = 5$. A simple calculation yields $R_n = \frac{n-1}{n},\ A_n = \frac{1}{n+1},\ B_n = \frac{1}{(n+1)^3}\left(1 - \frac{M^{1/3}(n+1)^{2/3}}{n^{1/3}(n+2)^{2/3}}\right)$ and $E_n = \frac{1}{(n+1)^3}\left(1 - \frac{M^{-2/3}}{(n+2)^{1/3}}\right)$. Now, it is easy to see that all conditions of Theorem 2.3 are satisfied and hence every solution of equation (3.34) is oscillatory.

Example 3.4. *Consider the neutral difference equation*

$$\Delta\left((n+1)(n+2)\Delta\left(x_n + (n-1)x_{n-1}^{5/3}\right)\right) + (n+2)^2 x_{n-2}^{5/3} = 0,\ n \geq 1. \tag{3.35}$$

Here $a_n = (n+1)(n+2),\ p_n = (n-1),\ q_n = (n+2)^2,\ k = 1,\ l = 3,\ \alpha = \beta = 5/3$. A simple calculation shows that $R_n = \frac{n-1}{n}$, $A_n = \frac{1}{n+1}$, $B_n = \frac{1}{n}\left(1 - \frac{M^{-2/5}(n+1)^{3/5}}{(n(n+2))^{3/5}}\right)$, and $E_n = \frac{1}{n}\left(1 - \frac{M_1^{-2/5}(n+2)^{2/5}}{(n+1)^{3/5}}\right)$. It is easy to verify that all conditions of Corollary 2.1 and Theorem 2.5 are satisfied and hence every solution of equation (3.35) is oscillatory.

We conclude this paper with the following remark.

Remark 3.2. *In this paper, we obtain some new oscillation criteria for the equation* (1.1) *using Riccati type transformation and comparison method which involves α and β. Further the results presented in this paper are new and complement to that of in [6, 7, 9, 11, 12].*

References

[1] R.P.Agarwal, S.R.Grace and D.O'Regan, Oscillation Theory For Difference and Functional Differential Equations, Kluwer, The Netherlands, 2000.

[2] R.P.Agarwal, M. Bohner, S.R. Grace and D.O'Regan, Discrete Oscillation Theory, *Hindawi Publ. Corp.*, New York, 2005.

[3] I. Gyori and G. Ladas, Oscillation Theory of Delay differential Equations with Applications, *Clarendan Press, Oxford*, 1991.

[4] J.K.Hale, Theory of Functional Differential Equations, *Springer, New York*, 1987.

[5] X. H. Tang and Y. Liu, Oscillation for nonlinear delay difference equations, *Tamkang J. Math.*, 32(2001), 275-280.

[6] E. Thandapani, Z. Liu, R. Arul and P. S. Raja, Oscillation and asymptotic behavior of second order difference equations with nonlinear neutral term, *Appl.Math. E-Notes.*, 4(2004), 59-67.

[7] E.Thandapani, S.Pandian and R.K.Balasubramanian, Oscillation of solutions of nonlinear neutral difference equations with nonlinear neutral term, *Far East J. Appl. Math.*, 15(2004),47-62.

[8] E. Thandapani and S. Selvarangam, Oscillation theorems of second order quasilinear neutral difference equations, *J. Math. Comput. Sci.*, 2(2012), 866-879.

[9] K.S.Vidhyaa, C.Dharuman J.R.Graef and E.Thandapani, Oscillation of second order difference equations with a sublinear neutral term, *J. Math. Appl.*, (to appear).

[10] J.Yang, X.Guan and W.Liu, Oscillation and asymptotic behavior of second order neutral difference equation, *Ann. Diff. Equ.*, 13(1997), 94-106.

[11] M.K.Yildiz and H.Ogurmez, Oscillation results of higher order nonlinear neutral delay difference equations with a nonlinear neutral term, *Hacettepe J. Math. Stat.*, 43(2014), 809-814.

[12] Z.Zhang, J.Chen and C.Zhang, Oscillation of solutions for second order nonlinear difference equations with nonlinear neutral term, *Comput. Math. Appl.*, 41(2001), 1487-1494.

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Intuitionistic Q-fuzzy ternary subhemiring of a hemiring

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Abstract

In this paper, a generalized intuitionistic Q-fuzzy ternary subhemiring of a hemiring is proposed. Further, some important notions and basic algebraic properties of intuitionistic fuzzy sets are discussed.

Keywords: Q-fuzzy subhemiring, Q-fuzzy ternary subhemiring, intuitionistic fuzzy ternary subhemiring, homomorphism, anti-homomorphism.

2010 MSC: 03F55, 06D72, 08A72, 16Y30, 16Y60.

1 Introduction

An algebra $(R;+;\cdot)$ is said to be a semiring if $(R;+)$ and $(R;\cdot)$ are semigroups satisfying $a.(b+c) = a.b+a.c$ and $(b+c).a = b.a+c.a$ for all a, b and c in R. A Semiring R is said to be additively commutative if $a+b = b+a$ for all a and b in R. Ternary rings are introduced by Lister [9]. And he investigated some of their properties and radical theory of such rings. A Semiring R may have an identity 1, defined by $1.a = a = a.1$ and a zero 0, defined by $0+a = a = a+0$ and $a.0 = 0 = 0.a$ for all a in R. Ternary semirings arise naturally as follows-consider the ring of integers Z which plays a vital role in the theory of ring. The concept of intuitionistic fuzzy subsets (IFS) was presented by K.T.Atanassov [5], as a generalization of the notion of fuzzy set. Solairaju.A and R.Nagarajan, have given a new structure in construction of Q-fuzzy groups [14]. Also Giri.R.D and Chide.B.R [8], given the structure of Prime Radical in Ternary Hemiring. In this paper, we introduce some properties and theorems in intuitionistic Q-fuzzy ternary subhemiring of a hemiring.

2 Preliminaries

Definition 2.1. *Let X be a non-empty set and Q be a non-empty set. A Q-fuzzy subset A of X is function $A : X \times Q \to [0,1]$.*

Definition 2.2. *Let R be a hemiring. A fuzzy subset A of R is said to be a Q-fuzzy ternary subhemiring (FTSHR) of R if it satisfies the following conditions:*

(i) $A(x+y,q) \geq \min\{A(x,q), A(y,q)\}$,

(ii) $A(xyz,q) \geq \min\{A(x,q), A(y,q), A(z,q)\}$, *for all x, y and z in R and q in Q.*

Definition 2.3. *Let R be a hemiring. A Q-fuzzy subset A of R is said to be an anti Q-fuzzy subhemiring (AFTSHR) of R if it satisfies the following conditions:*

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(i) $A(x+y,q) \leq \max\{A(x,q), A(y,q)\}$,

(ii) $A(xyz,q) \leq \max\{A(x,q), A(y,q), A(z,q)\}$, *for all* x, y *and* z *in* R *and* q *in* Q.

Definition 2.4. *An intuitionistic fuzzy subset (IFS) A in X is defined as an object of the form* $A = \{< x, \mu_A(x), \mu_A(x) > / x \in X\}$, *where* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *define the degree of membership and the degree of non-membership of the element* $x \in X$ *respectively and for every* $x \in X$ *satisfying* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.5. *Let R be a hemiring. An intuitionistic Q-fuzzy subset A of R is said to be an intuitionistic Q-fuzzy ternary subhemiring (IFTSHR) of R if it satisfies the following conditions:*

(i) $\mu_A(x+y,q) \geq \min\{\mu_A(x,q), \mu_A(y,q)\}$,

(ii) $\mu_A(xyz,q) \geq \min\{\mu_A(x,q), \mu_A(y,q), \mu_A(z,q)\}$,

(i) $\nu_A(x+y,q) \leq \max\{\nu_A(x,q), \nu_A(y,q)\}$,

(ii) $\nu_A(xyz,q) \leq \max\{\nu_A(x,q), \nu_A(y,q), \nu_A(z,q)\}$, *for all* x, y *and* z *in* R *and* q *in* Q.

Definition 2.6. *Let* $(R, +, \cdot)$ *and* $(R', +, \cdot)$ *be any two hemirings. Then the function* $f : R \to R'$ *is called a homomorphism if* $f(x+y,q) = f(x,q) + f(y,q)$ *and* $f(xyz,q) = f(x,q)f(y,q)f(z,q)$, *for all* x, y *and* z *in* R *and* q *in* Q.

Definition 2.7. *Let* $(R, +, \cdot)$ *and* $(R', +, \cdot)$ *be any two hemirings. Then the function* $f : R \to R'$ *is called an anti-homomorphism if* $f(x+y,q) = f(y,q) + f(x,q)$ *and* $f(xyz,q) = f(z,q)f(y,q)f(x,q)$, *for all* x, y *and* z *in* R *and* q *in* Q.

Definition 2.8. *Let* $(R, +, \cdot)$ *and* $(R', +, \cdot)$ *be any two hemirings. Then the function* $f : R \to R'$ *is called an isomorphism if* f *is bijection.*

Definition 2.9. *Let* $(R, +, \cdot)$ *and* $(R', +, \cdot)$ *be any two hemirings. Then the function* $f : R \to R'$ *is called an anti-isomorphism if* f *is bijection.*

3 Some properties of intuitionistic Q-fuzzy ternary subhemiring of a hemiring

Theorem 3.1. *If A is an intuitionistic Q-fuzzy ternary subhemiring of a hemiring* $(R, +, \cdot)$, *then* $H = \{(x,q)/x \in R : \mu_A(x,q) = 1, \nu_A(x,q) = 0\}$ *is either empty or is a ternary subhemiring of R.*

Proof. If none of the elements satisfies this condition, then H is empty. If (x,q) and (y,q) in H, then $\mu_A(x+y,q) \geq \min\{\mu_A(x,q), \mu_A(y,q)\} = \min\{1,1\} = 1$. Therefore $\mu_A(x+y,q) = 1$, for all (x,q) and (y,q) in H. And $\mu_A(xyz,q) \geq \min\{\mu_A(x,q), \mu_A(y,q), \mu_A(z,q)\} = \min\{1,1,1\} = 1$. Therefore $\mu_A(xyz,q) = 1$, for all $(x,q), (y,q)$ and (z,q) in H. And $\nu_A(x+y,q) \leq \max\{\nu_A(x,q), \nu_A(y,q)\} = \max\{0,0\} = 0$. Therefore $\nu_A(x+y,q) = 0$, for all (x,q) and (y,q) in H. And $\nu_A(xyz,q) \leq \max\{\nu_A(x,q), \nu_A(y,q), \nu_A(z,q)\} = \max\{0,0,0\} = 0$. Therefore $\nu_A(xyz,q) = 0$, for all $(x,q), (y,q)$ and (z,q) in H. Therefore H is a ternary subhemiring of R. Hence H is either empty or is a ternary subhemiring of R. □

Theorem 3.2. *If A is an intuitionistic Q-fuzzy ternary subhemiring of a hemiring* $(R, +, \cdot)$, *then* $H = \{\langle (x,q), \mu_A(x,q) \rangle : 0 < \mu_A(x,q) \leq 1 \, and \, \nu_A(x,q) = 0\}$ *is either empty or is a Q-fuzzy ternary subhemiring of R.*

Proof. By using Theorem 3.1. □

Theorem 3.3. *If A is an intuitionistic Q-fuzzy ternary subhemiring of a hemiring* $(R, +, \cdot)$, *then* $H = \{\langle (x,q), \mu_A(x,q) \rangle : 0 < \mu_A(x,q) \leq 1\}$ *is either empty or is a Q-fuzzy ternary subhemiring of R.*

Proof. By using Theorem.3.2. □

Theorem 3.4. *If A is an intuitionistic Q-fuzzy ternary subhemiring of a hemiring $(R,+,\cdot)$, then A is an intuitionistic Q-fuzzy ternary subhemiring of R.*

Proof. Let A be an intuitionistic Q-fuzzy ternary subhemiring of a hemiring R. Consider $A = \{\langle (x,q),\mu_A(x,q),\nu_A(x,q)\rangle\}$, for all x in R. $A = B = \{\langle (x,q),\mu_B(x,q),\nu_B(x,q)\rangle\}$, where $\mu_B(x,q) = \mu_A(x,q), \nu_B(x,q) = 1-\mu_A(x,q)$. clearly, $\mu_B(x+y,q) \geq \min\{\mu_B(x,q),\mu_B(y,q)\}$ and $\mu_B(xyz,q) \geq \min\{\mu_B(x,q),\mu_B(y,q),\mu_B(z,q)\}$. Since A is an intuitionistic Q-fuzzy ternary subhemiring of R, we have, $\mu_A(x+y,q) \geq \min\{\mu_A(x,q),\mu_A(y,q)\}$ for all x and y in R, which implies that $1-\nu_B(x+y,q) \geq \min\{(1-\nu_B(x,q)),(1-\nu_B(y,q))\}$ which implies that $\nu_B(x+y,q) \leq 1-\min\{(1-\nu_B(x,q)),(1-\nu_B(y,q))\} = \max\{\nu_B(x,q),\nu_B(y,q)\}$. Therefore, $\nu_B(x+y,q) \leq \max\{\nu_B(x,q),\nu_B(y,q)\}$ for all x and y in R and q in Q. And $\mu_A(xyz,q) \geq \min\{\mu_A(x,q),\mu_A(y,q),\mu_A(z,q)\}$ which implies that $1-\nu_B(xyz,q) \geq \min\{(1-\nu_B(x,q)),(1-\nu_B(y,q)),(1-\nu_B(z,q))\}$ which implies that $\nu_B(xyz,q) \leq 1-\min\{(1-\nu_B(x,q)),(1-\nu_B(y,q)),(1-\nu_B(z,q))\} = \max\{\nu_B(x,q),\nu_B(y,q),\nu_B(z,q)\}$. Therefore $\nu_B(xyz,q) \leq \max\{\nu_B(x,q),\nu_B(y,q),\nu_B(z,q)\}$, for all x,y and z in R. Hence $B = A$ is an intuitionistic Q-fuzzy ternary subhemiring of a hemiring R. □

Remark 3.1. *The converse of the above theorem is not true. It is shown by the following example: Consider the hemiring $Z_5 = \{0,1,2,3,4\}$ with addition modulo and multiplicative modulo operation and $Q = \{p\}$. Then $A = \{(\langle 0,0.7,0.2\rangle,p),(\langle 1,0.5,0.1\rangle,p),(\langle 2,0.5,0.4\rangle,p),(\langle 3,0.5,0.1\rangle,p),(\langle 4,0.5,0.4\rangle,p)\}$ is not an intuitionistic Q-fuzzy ternary subhemiring of Z_5, but $A = \{(\langle 0,0.7,0.3\rangle,p),(\langle 1,0.5,0.5\rangle,p),(\langle 2,0.5,0.5\rangle,p),(\langle 3,0.5,0.5\rangle,p),(\langle 4,0.5,0.5\rangle,p)\}$ is an intuitionistic Q-fuzzy ternary subhemiring of Z_5.*

Theorem 3.5. *If A is an intuitionistic Q-fuzzy ternary subhemiring of a hemiring $(R,+,\cdot)$, then A is an intuitionistic Q-fuzzy ternary subhemiring of R.*

Proof. Let A be an intuitionistic Q-fuzzy ternary subhemiring of a hemiring R. That is $A = \{\langle (x,q),\mu_A(x,q)\nu_A(x,q)\rangle\}$, for all x in R and q in Q. Let $A = B = \{\langle (x,q),\mu_B(x,q),\nu_B(x,q)\rangle\}$, where $\mu_B(x,q) = 1-\nu_A(x,q), \nu_B(x,q) = \nu_A(x,q)$. clearly, $\nu_B(x+y,q) \leq \max\{\nu_B(x,q),\nu_B(y,q)\}$ and $\nu_B(xyz,q) \leq \max\{\nu_B(x,q),\nu_B(y,q),\nu_B(z,q)\}$ for all x,y and z in R. Since A is an intuitionistic Q-fuzzy ternary subhemiring of R, we have, $\nu_A(x+y,q) \leq \max\{\nu_A(x,q),\nu_A(y,q)\}$ for all x and y in R, which implies that $1-\mu_B(x+y,q) \leq \max\{(1-\mu_B(x,q)),(1-\mu_B(y,q))\}$ which implies that $\mu_B(x+y,q) \geq 1-\max\{(1-\mu_B(x,q)),(1-\mu_B(y,q))\} = \min\{\mu_B(x,q),\mu_B(y,q)\}$. Therefore, $\mu_B(x+y,q) \geq \min\{(\mu_B(x,q)),(\mu_B(y,q))\}$ for all x and y in R and q in Q. And $\nu_A(xyz,q) \leq \max\{\nu_A(x,q),\nu_A(y,q),\nu_A(z,q)\}$ which implies that $1-\mu_B(xyz,q) \leq \max\{(1-\mu_B(x,q)),(1-\mu_B(y,q)),(1-\nu_B(z,q))\}$ which implies that $\mu_B(xyz,q) \geq 1-\max\{(1-\mu_B(x,q)),(1-\mu_B(y,q)),(1-\mu_B(z,q))\} = \min\{\mu_B(x,q),\mu_B(y,q),\mu_B(z,q)\}$. Therefore $\mu_B(xyz,q) \geq \min\{\mu_B(x,q),\mu_B(y,q),\mu_B(z,q)\}$, for all x,y and z in R. Hence $B = A$ is an intuitionistic Q-fuzzy ternary subhemiring of a hemiring R. □

Remark 3.2. *The converse of the above theorem is not true. It is shown by the following example: Consider the hemiring $Z_5 = \{0,1,2,3,4\}$ with addition modulo and multiplicative modulo operation and $Q = \{p\}$. Then $A = \{(\langle 0,0.5,0.1\rangle,p),(\langle 1,0.6,0.4\rangle,p),(\langle 2,0.5,0.4\rangle,p),(\langle 3,0.6,0.4\rangle,p),(\langle 4,0.5,0.4\rangle,p)\}$ is not an intuitionistic Q-fuzzy ternary subhemiring of Z_5, but $A = \{(\langle 0,0.9,0.1\rangle,p),(\langle 1,0.6,0.4\rangle,p),(\langle 2,0.6,0.4\rangle,p),(\langle 3,0.6,0.4\rangle,p),(\langle 4,0.6,0.4\rangle,p)\}$ is an intuitionistic Q-fuzzy ternary subhemiring of Z_5.*

In The Following Theorem $\circ$ Is The Composition Operation of Functions:

Theorem 3.6. *Let A be an intuitionistic Q-fuzzy ternary subhemiring of a hemiring H and f is an isomorphism from a hemiring R onto H. Then $A \circ f$ is an intuitionistic Q-fuzzy ternary subhemiring of R.*

Proof. Let x and y in R and A be an intuitionistic Q-fuzzy ternary subhemiring H. Then we have $(\mu_A \circ f)(x+y,q) = \mu_A(f(x+y,q)) = \mu_A(f(x,q)+f(y,q)) \geq \min\{\mu_A(f(x,q)),\mu_A(f(y,q))\}$ (as A is an IFTSHR of H) $\geq \min\{(\mu_A \circ f)(x,q),(\mu_A \circ f)(y,q)\}$ which implies that $(\mu_A \circ f)(x+y,q) \geq \{(\mu_A \circ f)(x,q),(\mu_A \circ f)(y,q)\}$, for all x and y in R and q in Q. And $(\mu_A \circ f)(xyz,q) = \mu_A(f(xyz,q)) = \mu_A(f(x,q)f(y,q)f(z,q))$, as f is an isomorphism $\geq \min\{\mu_A(f(x,q)),\mu_A(f(y,q)),\mu_A(f(z,q))\}$, as A is an IFTSHR of $H \geq \min\{(\mu_A \circ f)(x,q),(\mu_A \circ f)(y,q)\}$ which implies that $(\mu_A \circ f)(xyz,q) \geq \min\{(\mu_A \circ$

$f)(x,q), (\mu_A \circ f)(y,q), (\mu_A \circ f)(z,q)\}$, for all x, y and z in R. We have $(\nu_A \circ f)(x+y,q) = \nu_A(f(x+y,q)) = \nu_A(f(x,q)+f(y,q))$, as f is an isomorphism $\leq \max\{\nu_A(f(x,q)), \nu_A(f(y,q))\} \leq \max\{(\nu_A \circ f)(x,q), (\nu_A \circ f)(y,q)\}$ which implies that $(\nu_A \circ f)(x+y,q) \leq \max\{(\nu_A \circ f)(x,q), (\nu_A \circ f)(y,q)\}$, for all x and y in R. And $(\nu_A \circ f)(xyz,q) = \nu_A(f(xyz,q)) = \nu_A(f(x,q)f(y,q)f(z,q))$, as f is an isomorphism $\leq \max\{\nu_A(f(x,q)), \nu_A(f(y,q)), \nu_A(f(z,q))\} \leq \max\{(\nu_A \circ f)(x,q), (\nu_A \circ f)(y,q), (\nu_A \circ f)(z,q)\}$ which implies that $(\nu_A \circ f)(xyz,q) \leq \max\{(\nu_A \circ f)(x,q), (\nu_A \circ f)(y,q), (\nu_A \circ f)(z,q)\}$ for all x, y and z in R. Therefore $(A \circ f)$ is an intuitionistic Q-fuzzy ternary subhemiring of a hemiring R. □

Theorem 3.7. *Let A be an intuitionistic Q-fuzzy ternary subhemiring of a hemiring H and f is an anti-isomorphism from a hemiring R onto H. Then $A \circ f$ is an intuitionistic Q-fuzzy ternary subhemiring of R.*

Proof. Let x and y in R and A be an intuitionistic Q-fuzzy ternary subhemiring H. Then we have $(\mu_A \circ f)(x+y,q) = \mu_A(f(x+y,q)) = \mu_A(f(y,q)+f(x,q))$, as f is an anti-homomorphism $\geq \min\{\mu_A(f(y,q)), \mu_A(f(x,q))\}$ as A is an IFTSHR of $H \geq \min\{(\mu_A \circ f)(x,q), (\mu_A \circ f)(y,q)\}$ which implies that $(\mu_A \circ f)(x+y,q) \geq \min\{(\mu_A \circ f)(x,q), (\mu_A \circ f)(y,q)\}$, for all x and y in R. And $(\mu_A \circ f)(xyz,q) = \mu_A(f(xyz,q)) = \mu_A(f(z,q)f(y,q)f(x,q))$, as f is an anti-isomorphism $\geq \min\{\mu_A(f(z,q)), \mu_A(f(y,q)), \mu_A(f(x,q))\}$, as A is an IFTSHR of $H \geq \min\{(\mu_A \circ f)(x,q), (\mu_A \circ f)(y,q)\}$ which implies that $(\mu_A \circ f)(xyz,q) \geq \min\{(\mu_A \circ f)(x,q), (\mu_A \circ f)(y,q), (\mu_A \circ f)(z,q)\}$ for all x, y and z in R. We have $(\nu_A \circ f)(x+y,q) = \nu_A(f(x+y,q)) = \nu_A(f(y,q)+f(x,q))$, as f is an anti-isomorphism $\leq \max\{\nu_A(f(y,q)), \nu_A(f(x,q))\} \leq \max\{(\nu_A \circ f)(x,q), (\nu_A \circ f)(y,q)\}$ which implies that $(\nu_A \circ f)(x+y,q) \leq \max\{(\nu_A \circ f)(x,q), (\nu_A \circ f)(y,q)\}$, for all x and y in R. And $(\nu_A \circ f)(xyz,q) = \nu_A(f(xyz,q)) = \nu_A(f(z,q)f(y,q)f(x,q))$, as f is an anti-isomorphism $\leq \max\{\nu_A(f(z,q)), \nu_A(f(y,q)), \nu_A(f(x,q))\} \leq \max\{(\nu_A \circ f)(x,q), (\nu_A \circ f)(y,q), (\nu_A \circ f)(z,q)\}$ which implies that $(\nu_A \circ f)(xyz,q) \leq \max\{(\nu_A \circ f)(x,q), (\nu_A \circ f)(y,q), (\nu_A \circ f)(z,q)\}$ for all x, y and z in R and q in Q. Therefore $(A \circ f)$ is an intuitionistic Q-fuzzy ternary subhemiring of a hemiring R. □

References

[1] N. Anitha and K. Arjunan, Homomorphism in Intuitionistic Fuzzy Subhemiring of a Hemiring, *International Journal of Mathematical Science* & *Engg.Appls,* Vol.4(v); 165–172,2010.

[2] N. Anitha, Intuitionistic Q-Fuzzy Subhemiring of a Hemiring, *International Journal on Engineering Technology and Sciences,* Vol.II, 70–73,2015.

[3] N. Anitha, Level set of Intuitionistic Fuzzy Subhemiring of a Hemiring, *International Journal on Engineering Technology and Sciences,* Vol.II, 124–128,2015.

[4] Asok kumer Ray, On Product of Fuzzy Subgroups, *Fuzzy Sets and Systems,* 105; 181–183, 1999.

[5] K. T. Atanassov, Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems,* Vol.20 (1); 87–96,1986.

[6] R. Biswas, Fuzzy Subgroups and Anti-Fuzzy Subgroups, *Fuzzy Sets and Systems,* 35; 121–124, 1990.

[7] T. K. Dutta, S. Kar, On Prime ideals and Prime Radical of Ternary Semiring, *Bull. Calcutta Math. Soc.,* 97, No.5; 445–454, 2005.

[8] R. D. Giri and B. R. Chide, Prime Radical in Ternary Hemirings, *International Journal of Pure and Applied Mathematics,* Vol.95 (5), 631–647, 2014.

[9] W. G. Lister, Ternary Rings, *Trans. Amer. Math. Soc.,* 154; 37–55, 1971.

[10] N. Palaniappan and R. Muthuraj, The Homomorphism, Anti-Homomorphism of a Fuzzy and An Anti-Fuzzy Groups, *Varahmihir Journal of Mathematical Sciences,* Vol.4 (2); 387–399, 2004.

[11] N. Palaniappan and K. Arjunan, The Homomorphism, Anti-Homomorphism of a Fuzzy and An Anti-Fuzzy Ideals, *Varahmihir Journal of Mathematical Sciences,* Vol.6 (1), 181–188, 2006.

[12] N. Palaniappan and K. Arjunan, Some Properties of Intuitionistic Fuzzy Subgroups, *ActaCiencia India,* Vol.XXXIII (2); 321–328, 2007.

[13] Rajesh Kumar, *Fuzzy Algebra,* University of Delhi Publication Division, Volume 1,1993.

[14] A. Solairaju and R. Nagarajan, A New Structure and Construction of Q-Fuzzy Groups, *Advances in Fuzzy Mathematics,* Vol.4 (1); 23–29, 2009.

[15] W. B. Vasantha Kandasamy, *Smarandache Fuzzy Algebra,* American Research Press, Rehoboth, 2003.

[16] L.A. Zadeh, Fuzzy Sets, *Information and Control,* Vol.8; 338–353, 1965.

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Common fixed point of contractive modulus on 2-cone Banach space

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Abstract

In this paper, we have proved the existence of unique common fixed point of four contractive maps on 2-cone Banach space through a contractive modulus and weakly compatible maps.

Keywords: 2-Cone Banach Space, Common Fixed Point, Contractive Modulus, Weakly Compatible.

2010 MSC: 65D30, 65D32.

1 Introduction

In 2007, Huang and Zhang [1] introduced the concept of cone metric spaces and fixed point theorems of contraction mappings ; Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$ has a unique fixed point. In 2009, Karapinar[2] establish Some fixed theorems in cone Banach space. The common fixed point theorems with the assumption of weakly compatible and coincidence point of four maps on an upper semi continuous contractive modulus in cone Banach space are proved by R. Krishnakumar and D.Dhamodharan [5]. Ahmet Sahiner and Tuba Yigit[11] proved 2 -cone Banach spaces and fixed point theorem.

In this paper, we investigate the common fixed point theorems with the assumption of weakly compatible and coincidence point of four maps on an upper semi continuous contractive modulus in 2-cone Banach space

Definition 1.1. *Let E be the real Banach space. A subset P of E is called a cone if and only if:*

i. P is closed, non empty and $P \neq 0$

ii. $ax + by \in P$ *for all* $x, y \in P$ *and non negative real numbers* a, b

iii. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

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Example 1.1. *Let $K > 1.$ be given. Consider the real vector space with*

$$E = \{ax + b : a, b \in R; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. The cone P is regular and so normal.

Definition 1.2. *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

i. $d(x, y) \geq 0$, *and* $d(x, y) = 0$ *if and only if* $x = y$ $\forall x, y \in X$,

ii. $d(x, y) = d(y, x)$, $\forall x, y \in X$,

iii. $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$,

Then (X, d) is called a cone metric space (CMS).

Example 1.2. *Let $E = R^2$*

$$P = \{(x, y) : x, y \geq 0\}$$

$X = R$ and $d : X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.3. *[2] Let X be a vector space over R. Suppose the mapping $\|.\|_c : X \to E$ satisfies*

i. $\|x\|_c \geq 0$ *for all* $x \in X$,

ii. $\|x\|_c = 0$ *if and only if* $x = 0$,

iii. $\|x + y\|_c \leq \|x\|_c + \|y\|_c$ *for all* $x, y \in X$,

iv. $\|kx\|_c = |k|\|x\|_c$ *for all* $k \in R$ *and for all* $x \in X$, *then* $\|.\|_c$ *is called a cone norm on* X, *and the pair* $(X, \|.\|_c)$ *is called a cone normed space* (CNS).

Remark 1.1. *Each Cone normed space is Cone metric space with metric defined by*
$d(x, y) = \|x - y\|_c$

Example 1.3. *Let $X = R^2$, $P = \{(x, y) : x \geq 0, y \geq 0\} \subset R^2$ and $\|(x, y), u\|_c = (a|x|, b|y|), a > 0, b > 0$. Then $(X, \|., u\|_c)$ is a cone normed space over R^2*

Definition 1.4. *Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then $\{x_n\}_{n\geq 0}$ converges to x whenever for every $c \in E$ with $0 \ll E$, there is a natural number $N \in N$ such that $\|x_n - x\|_c \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$*

Definition 1.5. *Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. $\{x_n\}_{n\geq 0}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in N$, such that $\|x_n - x_m\|_c \ll c$ for all $n, m \geq N$*

Definition 1.6. *Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. $(X, \|.\|_c)$ is a complete cone normed space if every Cauchy sequence is convergent. Complete cone normed spaces will be called cone Banach spaces.*

Lemma 1.1. *[2] Let $(X, \|.\|_c)$ be a CNS, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then*

i. *the sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_c \to 0$ as $n \to \infty$,*

ii. *the sequence* $\{x_n\}$ *is Cauchy if and only if* $\|x_n - x_m\|_c \to 0$ *as* $n, m \to \infty$,

iii. *the sequence* $\{x_n\}$ *converges to* x *and the sequence* $\{y_n\}$ *converges to* y, *then* $\|x_n - y_n\|_c \to \|x - y\|_c$.

Definition 1.7. *[11] Let* X *be a linear space over* R *with dimension greater then or equal to 2,* E *be Banach space with the norm* $\|.\|$ *and* $P \subset E$ *be a cone. If the function*

$$\|.,.\| : X \times X \to (E, P, \|.\|)$$

satisfies the following axioms:

1. $\|x, y\|_c \geq 0$ *for every* $x, y \in X$, $\|x, y\|_c = 0$ *if and only if* x *and* y *are linearly dependent,*
2. $\|x, y\|_c = \|y, x\|_c$, *for every* $x, y \in X$
3. $\|\alpha x, y\|_c = |\alpha|\|x, y\|_c$,*for every* $x, y \in X$ *and* $\alpha \in R$
4. $\|x, y + z\|_c \leq \|x, y\|_c + \|y, z\|_c$, *for every* $x, y, z \in X$,

then $(X, \|.,.\|_c)$ *is called a 2 -cone normed space.*

Example 1.4.

If we fix $\{u_1, u_2, ..., u_d\}$ to be a basis for X, we can give the following lemma.

Lemma 1.2. *[11] Let* $(X, \|,.,\|_c)$ *be a 2 -cone normed space. Then a sequence* $\{x_n\}$ *converges to* $x \in X$ *if and only if for each* $c \in E$ *with* $c \gg 0$ *(0 is zero element of E) there exists an* $N = N(c) \in N$ *such that* $n > N$ *implies* $\|x_n - x, u_i\|_c \ll c$ *for every* $i = 1, 2, ..., d$.

Lemma 1.3. *[11] Let* $(X, \|.,.\|_c)$ *be a 2 -cone normed space. Then a sequence* $\{x_n\}$ *converges to* x *in* X *if and only if* $\lim_{n\to\infty} \max \|x_n - x, u_i\|_c = 0$.

Definition 1.8. *[11] A 2 -cone normed space* $(X, \|.,.\|_c)$ *is a 2-cone Banach spaces if any Cauchy sequence in* X *is convergent to an* x *in* X.

Theorem 1.1. *Any 2-cone normed space* X *is a cone normed spaces and its topology agrees with the norm generated by* $\|.,.\|_c^\infty$.

Definition 1.9. *Let* f *and* g *be two self maps defined on a set* X *maps* f *and* g *are said to be commuting of* $fgx = gfx$ *for all* $x \in X$

Definition 1.10. *Let* f *and* g *be two self maps defined on a set* X *maps* f *and* g *are said to be weakly compatible if they commute at coincidence points. that is if* $fx = gx$ *forall* $x \in X$ *then* $fgx = gfx$

Definition 1.11. *Let* f *and* g *be two self maps on set* X. *If* $fx = gx$, *for some* $x \in X$ *then* x *is called coincidence point of* f *and* g

Lemma 1.4. *Let* f *and* g *be weakly compatible self mapping of a set* X. *If* f *and* g *have a unique point of coincidence, that is* $w = fx = gx$ *then* w *is the unique common fixed point of* f *and* g.

2 Main Result

Theorem 2.2. *Let* X *be a 2-cone Banach space (with* $\dim X \geq 2$*). Suppose that the mappings* P, Q, S *and* T *are four self maps of* X *such that* $T(X) \subseteq P(X)$ *and* $S(X) \subseteq Q(X)$ *and satisfying*

$$\|Ty - Sx, u\|_c \leq a\|Px - Qy, u\|_c + b\{\|Px - Sx, u\|_c + \|Qy - Ty, u\|_c\} + c\{\|Px - Ty, u\|_c + \|Qy - Sx, u\|_c\} \tag{2.1}$$

for all $x, y \in X$, *where* $a, b, c \geq 0$ *and* $a + 2b + 2c < 1$. *suppose that the pairs* $\{P, S\}$ *and* $\{Q, T\}$ *are weakly compatible, then* P, Q, S *and* T *have a unique common fixed point.*

Proof. Suppose x_0 is an arbitrary initial point of X and define the sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$

$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (2.1) implies that

$$\begin{aligned}
\|y_{2n+1} - y_{2n}, u\|_c &= \|Tx_{2n+1} - Sx_{2n}, u\|_c \\
&\leq a\|Px_{2n} - Qx_{2n+1}, u\|_c + b\{\|Px_{2n} - Sx_{2n}, u\|_c + \|Qx_{2n} - Tx_{2n+1}, u\|_c\} \\
&\quad + c\{\|Px_{2n} - Tx_{2n+1}, u\|_c + \|Qx_{2n+1} - Sx_{2n}, u\|_c\} \\
&\leq a\|y_{2n-1} - y_{2n}, u\|_c + b\{\|y_{2n-1} - y_{2n}, u\|_c + \|y_{2n} - y_{2n+1}, u\|_c\} \\
&\quad + c\{\|y_{2n-1} - y_{2n+1}, u\|_c + \|y_{2n} - y_{2n}, u\|_c\} \\
&\leq a\|y_{2n-1} - y_{2n}, u\|_c + b\{\|y_{2n-1} - y_{2n}, u\|_c + \|y_{2n} - y_{2n+1}, u\|_c\} \\
&\quad + c\|y_{2n-1} - y_{2n+1}, u\|_c \\
&\leq (a+b+c)\|y_{2n-1} - y_{2n}, u\|_c + (b+c)\|y_{2n} - y_{2n+1}\|c \\
\|y_{2n+1} - y_{2n}, u\|_c &\leq \frac{a+b+c}{1-(b+c)}\|y_{2n} - y_{2n-1}, u\|_c \\
\|y_{2n+1} - y_{2n}, u\|_c &\leq h\|y_{2n} - y_{2n-1}, u\|_c
\end{aligned}$$

where $h = \frac{a+b+c}{1-(b+c)} < 1$ for all $n \in N$

$$\begin{aligned}
\|y_{2n} - y_{2n+1}, u\|_c &\leq h\|y_{2n-1} - y_{2n}, u\|_c \\
&\leq h^2\|y_{2n-2} - y_{2n-1}, u\|_c \\
&\vdots \\
&\leq h^{2n-1}\|y_0 - y_1, u\|_c
\end{aligned}$$

For all $m > n$

$$\begin{aligned}
\|y_n - y_m, u\|_c &\leq \|y_n - y_{n+1}, u\|_c + \|y_{n+1} - y_{n+2}, u\|_c + \cdots + \|y_{m-1} - y_m, u\|_c \\
&\leq (h^n + h^{n+1} + \cdots h^{m-1})\|y_0 - y_1, u\|_c \\
&\leq h^n(1 + h + h^2 + \cdots + h^{m-1-n})\|y_0 - y_1, u\|_c \\
&\leq \frac{h^n}{1-h}\|y_0 - y_1, u\|_c
\end{aligned}$$

$\Rightarrow \ \|y_n - y_m, u\|_c \ll 0$ as $n, m \to \infty$.

Hence $\{y_n\}$ is a Cauchy sequence.

There exists a point l in $(X, \|., u\|_c)$ such that
$\lim_{n\to\infty} \{y_n\} = l$, $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Q_{2n+1} = l$ and $\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = l$

that is,

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Q_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = x^*$$

Since $T(X) \subseteq P(X)$, there exists a point z in X Such that $x^* = Pz$ then by (1)

$$\begin{aligned}
\|Sz - x^*, u\|_c &\leq \|Sz - Tx_{2n-1}, u\|_c + \|Tx_{2n-1} - x^*, u\|_c \\
&\leq a\|Pz - Qx_{2n-1}, u\|_c + b\{\|Pz - Sz, u\|_c + \|Qx_{2n-1} - Tx_{2n-1}, u\|_c\} \\
&\quad + c\{\|Pz - Tx_{2n-1}, u\|_c + \|Qx_{2n-1} - Sz, u\|_c\} + \|Tx_{2n-1} - x^*, u\|_c
\end{aligned}$$

Taking the limit as $n \to \infty$

$$\begin{aligned}\|Sz - x^*, u\|_c &\le a\|x^* - x^*, u\|_c + b\{\|x^* - x^*, u\|_c + \|x^* - Sz, u\|_c\}\\ &\quad + c\{\|x^* - x^*, u\|_c + \|x^* - Sz, u\|_c\} + \|x^* - x^*, u\|_c\\ &\le 0 + b\{\|x^* - Sz, u\|_c + 0\} + c\{0 + \|x^* - Sz, u\|_c\} + 0 + (b+c)\|x^* - Sz, u\|_c\end{aligned}$$

Which is a contraction since $a + 2b + 2c < 1$.

$$\text{therefore} \quad Sz = Pz = x^*$$

Since $S(X) \subseteq Q(X)$ there exists a point $w \in X$ such that $x^* = Qw$.

by (1)

$$\begin{aligned}\|Sz - x^*, u\|_c &\le \|Sz - Tw, u\|_c\\ &\le a\|Pz - Qw, u\|_c + b\{\|Pz - Sz, u\|_c + \|Qw - Tw, u\|_c\} + c\{\|Pz - Tw, u\|_c + \|Qw - Sw, u\|_c\}\\ &\le a\|x^* - x^*, u\|_c + b\{\|x^* - x^*, u\|_c + \|x^* - Tw, u\|_c\} + c\{\|x^* - Tw, u\|_c + \|x^* - x^*, u\|_c\}\\ &\le 0 + +b\{0 + \|x^* - Tw, u\|_c\} + c\{\|x^* - Tw, u\|_c + 0\}\\ \|x^* - Tw, u\|_c &\le (b+c)\|x^* - Tw, u\|_c\end{aligned}$$

which is a contradiction since $a + 2b + 2c < 1$.

$$\text{therefore} \quad Tw = Qw = x^*$$

Thus $Sz = Pz = Tw = Qw = x^*$

Since P and S are weakly compatible maps,

Then $SP(z) = PS(z)$

$Sx^* = Px^*$

To prove that x^* is a fixed point of S

Suppose $Sx^* \neq x^*$ then by (2.1)

$$\begin{aligned}\|Sx^* - x^*, u\|_c &\le \|Sx^* - Tx^*, u\|_c\\ &\le a\|Px^* - Qw, u\|_c + b\{\|Px^* - Sx^*, u\|_c + \|Qw - Tw, u\|_c\} +\\ &\quad + c\{\|Px^* - Tw, u\|_c + \|Qw - Sx^*, u\|_c\}\\ &\le a\|Sx^* - x^*, u\|_c + b\{\|Sx^* - Sx^*, u\|_c + \|x^* - x^*, u\|_c\} +\\ &\quad + c\{\|Sx^* - x^*, u\|_c + \|x^* - Sx^*, u\|_c\}\\ &\le a\|Sx^* - x^*, u\|_c + b\{0 + 0\} + 2c\|Sx^* - x^*, u\|_c\\ \|Sx^* - x^*, u\|_c &\le (a + 2c)\|Sx^* - x^*, u\|_c\end{aligned}$$

Which is a contradiction, Since $a + 2b + 2c < 1$.

$$Sx^* = x^*$$

Hence $Sx^* = Px^* = x^*$ Similarly, Q and T are weakly compatible maps then $TQw = QTw$, that is $Tx^* = Qx^*$

To prove that x^* is a fixed point of T.

Suppose $Tx^* \neq x^*$ by (2.1)

$$\begin{aligned} \|Tx^* - x^*, u\|_c &\leq \|Sx^* - Tx^*, u\|_c \\ &\leq a\|Px^* - Qx^*, u\|_c + b\{\|Px^* - Sx^*, u\|_c + \|Qx^* - Tx^*, u\|_c\} + \\ &\quad + c\{\|Px^* - Tx^*, u\|_c + \|Qx^* - Sx^*, u\|_c\} \\ &\leq a\|x^* - Tx^*, u\|_c + b\{\|x^* - x^*, u\|_c + \|Tx^* - Tx^*, u\|_c\} + \\ &\quad + c\{\|x^* - Tx^*, u\|_c + \|Tx^* - x^*, u\|_c\} \\ &\leq a\|Tx^* - x^*, u\|_c + b\{0 + 0\} + 2c\|Tx^* - x^*, u\|_c \\ \|Tx^* - x^*, u\|_c &\leq (a + 2c)\|Tx^* - x^*, u\|_c \end{aligned}$$

which is a contradiction since $a + 2b + 2c < 1$.

$$Tx^* = x^*.$$

Hence. $Tx^* = Qx^* = x^*$
Thus $Sx^* = Px^* = Tx^* = Qx^* = x^*$
That is, x^* is a common fixed point of P, Q, S and T
To prove that the uniqueness of x^*
Suppose that x^* and y^*, $x^* \neq y^*$ are common fixed points of P, Q, S and T respectively, by (2.1) we have,

$$\begin{aligned} \|x^* - y^*, u\|_c &\leq \|Sx^* - Ty^*, u\|_c \\ &\leq a\|Px^* - Qy^*, u\|_c + b\{\|Px^* - Sx^*, u\|_c + \|Qy^* - Ty^*, u\|_c\} + \\ &\quad + c\{\|Px^* - Ty^*, u\|_c + \|Qy^* - Sx^*, u\|_c\} \\ &\leq a\|x^* - y^*, u\|_c + b\{\|x^* - x^*, u\|_c + \|y^* - y^*, u\|_c\} + c\{\|x^* - y^*, u\|_c + \|y^* - x^*, u\|_c\} \\ &\leq a\|x^* - y^*, u\|_c + b\{0 + 0\} + c\{\|x^* - y^*, u\|_c + \|y^* - x^*, u\|_c\} \\ &\leq (a + 2c)\|x^* - y^*, u\|_c \end{aligned}$$

which is a contradiction. Since $a + 2b + 2c < 1$.

$$\text{therefore} \quad x^* = y^*.$$

Hence x^* is the unique common fixed point of P, Q, S and T respectively. □

Corollary 2.1. *Let X be a 2-cone Banach space (with $\dim X \geq 2$). Suppose that the mappings P, S and T are three self maps of X such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ and satisfying*

$$\|Sx - Ty, u\|_c \leq a\|Px - Py, u\|_c + b\{\|Px - Sy, u\|_c + \|Px - Ty, u\|_c\} + c\{\|Px - Ty, u\|_c + \|Py - Sx, u\|_c\}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + 2b + 2c < 1$. suppose that the pairs $\{P, S\}$ and $\{P, T\}$ are weakly compatible, then P, S and T have a unique common fixed point.

Proof. The proof of the corollary immediate by taking $P = Q$ in the above theorem (2.2). □

Definition 2.12. *A mapping $\Phi : P \cup \{0\} \to P \cup \{0\}$ is said to be contractive modulus if it is continuous and which satisfies*

1. $\Phi(t) = 0$ *if and only if* $t = 0$
2. $\Phi(t) \leq t$ *for* $t \in P$
3. $\Phi(t + s) \leq \Phi(t) + \Phi(s)$ *for* $t, s \in P$

Theorem 2.3. *Let X be a 2-cone Banach space (with $\dim X \geq 2$).. Suppose that the mappings P, Q, S and T are four self maps of X such that $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$ satisfying*

$$\|Sx - Ty, u\|_c \leq \Phi(\lambda(x, y)), \tag{2.2}$$

where Φ is an upper semi continuous contractive modulus and

$$\lambda(x, y) = \max\{\|Px - Qy, u\|_c, \|Px - Sx, u\|_c, \|Qy - Ty, u\|_c, \frac{1}{2}\{\|Px - Ty, u\|_c + \|Qy - Sx, u\|_c\}\}.$$

The pair $\{S, P\}$ and $\{T, Q\}$ are weakly compatible. Then P, Q, S and T have a unique common fixed point.

Proof. Let us take x_0 is an arbitrary point of X and define a sequence $\{y_{2n}\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$

$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (2.2) implies that

$$\begin{aligned}
\|y_{2n} - y_{2n+1}, u\|_c &= \|Sx_{2n} - Tx_{2n+1}, u\|_c \\
&\leq \Phi(\lambda(x_{2n}, x_{2n+1})) \\
&\leq \lambda(x_{2n}, x_{2n+1}) \\
&= \max\{\|Px_{2n} - Qx_{2n+1}, u\|_c, \|Px_{2n} - Sx_{2n}, u\|_c, \|Qx_{2n+1} - Tx_{2n+1}, u\|_c, \\
&\quad \frac{1}{2}\{\|Px_{2n} - Tx_{2n+1}, u\|_c + \|Qx_{2n+1} - Sx_{2n}, u\|_c\}\} \\
&= \max\{\|Tx_{2n-1} - Sx_{2n}, u\|_c, \|Tx_{2n-1} - Sx_{2n}, u\|_c, \|Sx_{2n} - Tx_{2n+1}, u\|_c, \\
&\quad \frac{1}{2}\{\|Tx_{2n-1} - Tx_{2n+1}, u\|_c + \|Sx_{2n} - Sx_{2n}, u\|_c\}\} \\
&= \max\{\|Tx_{2n-1} - Sx_{2n}, u\|_c, \|Tx_{2n-1} - Sx_{2n}, u\|_c, \|Sx_{2n} - Tx_{2n+1}, u\|_c, \\
&\quad \frac{1}{2}\|Tx_{2n-1} - Tx_{2n+1}, u\|_c\} \\
&= \max\{\|y_{2n} - y_{2n-1}, u\|_c, \|y_{2n} - y_{2n+1}, u\|_c, \frac{1}{2}\|y_{2n-1} - y_{2n+1}, u\|_c\} \\
&\leq \max\{\|y_{2n} - y_{2n-1}, u\|_c, \|y_{2n} - y_{2n+1}, u\|_c\}
\end{aligned}$$

Since Φ is an contractive modulus, $\lambda(x_{2n} - x_{2n+1}) = \|y_{2n} - y_{2n+1}, u\|_c$ is not possible. Thus,

$$\|y_{2n} - y_{2n+1}, u\|_c \leq \Phi(\|y_{2n-1} - y_{2n}, u\|_c) \tag{2.3}$$

Since Φ is an upper semi continuous, contractive modulus. Equation (2.3) implies that the sequence $\{\|y_{2n+1} - y_{2n}, u\|_c\}$ is monotonic decreasing and continuous. There exists a real number, say $r \geq 0$ such that

$$\lim_{n\to\infty} \|y_{2n+1} - y_{2n}, u\|_c = r,$$

as $n \to \infty$ equation (2.3) $\Rightarrow$

$$r \leq \Phi(r)$$

which is only possible if $r = 0$ because Φ is a contractive modulus. Thus

$$\lim_{n\to\infty} \|y_{2n+1} - y_{2n}, u\|_c = 0.$$

Claim: $\{y_{2n}\}$ is a Cauchy sequence.
Suppose $\{y_{2n}\}$ is not a Cauchy sequence.
Then there exists an $\epsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$\|y_{m_i} - y_{n_i}, u\|_c \geq \epsilon \quad \text{and} \quad \|y_{m_i} - y_{n_{i-1}}, u\|_c \leq \epsilon \tag{2.4}$$

$$\epsilon \leq \|y_{m_i} - y_{n_i}, u\|_c \leq \|y_{m_i} - y_{n_{i-1}}, u\|_c + \|y_{n_{i-1}} - y_{n_i}, u\|_c$$

$$\text{therefore} \quad \lim_{i\to\infty} \|y_{m_i} - y_{n_i}, u\|_c = \epsilon$$

now

$$\epsilon \leq \|y_{m_{i-1}} - y_{n_{i-1}}, u\|_c \leq \|y_{m_{i-1}} - y_{m_i}, u\|_c + \|y_{m_i} - y_{n_{i-1}}, u\|_c$$

by taking limit $i \to \infty$ we get,

$$\lim_{i\to\infty} \|y_{m_{i-1}} - y_{n_{i-1}}, u\|_c = \epsilon$$

from (2.3) and (2.4)

$$\epsilon \leq \|y_{m_i} - y_{n_i}, u\|_c = \|Sx_{m_i} - Tx_{n_i}, u\|_c \leq \Phi(\lambda(x_{m_i}, x_{n_i}))$$

where implies

$$\epsilon \leq \Phi(\lambda(x_{m_i}, x_{n_i})) \tag{2.5}$$

$$\begin{aligned}
\lambda(x_{m_i}, x_{n_i}) &= \max\{\|Px_{m_i} - Qx_{n_i}, u\|_c, \|Px_{m_i} - Sx_{m_i}, u\|_c, \|Qx_{n_i} - Tx_{n_i}, u\|_c,\\
&\quad \frac{1}{2}(\|Px_{m_i} - Tx_{n_i}, u\|_c + \|Qx_{n_i} - Sx_{m_i}, u\|_c)\}\\
&= \max\{\|Tx_{m_{i-1}} - Sx_{n_{i-1}}, u\|_c, \|Tx_{m_{i-1}} - Sx_{m_i}, u\|_c, \|Sx_{n_{i-1}} - Tx_{n_i}, u\|_c,\\
&\quad \frac{1}{2}(\|Tx_{m_{i-1}} - Tx_{n_i}, u\|_c + \|Sx_{n_{i-1}} - Sx_{m_i}, u\|_c)\}\\
&= \max\{\|y_{m_{i-1}} - y_{n_{i-1}}, u\|_c, \|y_{m_{i-1}} - y_{m_i}, u\|_c, \|y_{n_{i-1}} - y_{n_i}, u\|_c,\\
&\quad \frac{1}{2}(\|y_{m_{i-1}} - y_{n_i}, u\|_c + \|y_{n_{i-1}} - y_{m_i}, u\|_c)\}
\end{aligned}$$

Taking limit as $i \to \infty$, we get

$$\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \max\{\epsilon, 0, 0, \frac{1}{2}(\epsilon, \epsilon)\}$$

$$\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \epsilon$$

Therefore from (2.5) we have, $\epsilon \leq \Phi(\epsilon)$
This is a contraction because $\epsilon > 0$ and Φ is contractive modulus.
Therefore $\{y_{2n}\}$ is Cauchy sequence in X
There exits a point z in X such that $\lim_{n\to\infty} y_{2n} = z$
Thus,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = z \quad \text{and}$$

$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$

$$(i.e) \quad \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$

$T(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$

$$\begin{aligned}
\|Su - z, u\|_c &\leq \|Su - Tx_{2n+1}, u\|_c + \|Tx_{2n+1} - z, u\|_c\\
&\leq \Phi(\lambda(u, x_{2n+1})) + \|Tx_{2n+1} - z, u\|_c
\end{aligned}$$

where

$$\begin{aligned}
\lambda(u, x_{2n+1}) &= \max\{\|Pu - Qx_{2n+1}, u\|_c, \|Pu - Su, u\|_c, \|Qx_{2n+1} - Tx_{2n+1}, u\|_c,\\
&\quad \frac{1}{2}(\|Pu - Tx_{2n+1}, u\|_c + \|Qx_{2n+1} - Su, u\|_c)\}\\
&= \max\{\|z - Sx_{2n}, u\|_c, \|z - Su, u\|_c, \|Sx_{2n} - Tx_{2n+1}, u\|_c,\\
&\quad \frac{1}{2}(\|z - Tx_{2n+1}, u\|_c + \|Sx_{2n} - Su, u\|_c)\}.
\end{aligned}$$

Now taking the limit as $n \to \infty$ we have,

$$\lambda(u, x_{2n+1}) = \max\{\|z - Su, u\|_c, \|z - Su, u\|_c, \|Su - Tu, u\|_c, \frac{1}{2}(\|z - Tu, u\|_c + \|z - Su, u\|_c)\}$$
$$= \max\{\|z - Su, u\|_c, \|z - Su, u\|_c, \|Su - z, u\|_c, \frac{1}{2}(\|z - z, u\|_c + \|z - Su, u\|_c)\}$$
$$= \|z - Su, u\|_c$$

Thus

$$\|Su - z, u\|_c \leq \Phi(\|Su - z, u\|_c) + \|z - z, u\|_c$$
$$= \Phi(\|Su - z, u\|_c)$$

If $Su \neq z$ then $\|Su - z, u\|_c > 0$ and hence as Φ is contracive modulus
$\Phi(\|Su - z, u\|_c) < \|Su - z, u\|_c$ Which is a contradiction, $Su = z$ so, $Pu = Su = z$
So u is a coincidence point if P and S. The pair of maps S and P are weakly compatible $SPu = PSu$ that is $Sz = Pz$.
$S(X) \subseteq Q(X)$, there exists a point $v \in X$ such that $z = Qv$.
Then we have

$$\|z - Tv, u\|_c = \|Su - Tv, u\|_c$$
$$\leq \Phi(\lambda(u, v))$$
$$\leq \lambda(u, v)$$
$$= \max\{\|Pu - Qv, u\|_c, \|Pu - Su, u\|_c, \|Qv - Tv, u\|_c,$$
$$\frac{1}{2}(\|Pu - Tv, u\|_c + \|Qv - Su, u\|_c)\}$$
$$= \max\{\|z - z, u\|_c, \|z - z, u\|_c, \|z - Tv, u\|_c,$$
$$\frac{1}{2}(\|z - Tv, u\|_c + \|z - z, u\|_c)\}$$
$$= \|z - Tv, u\|_c$$

Thus $\|z - Tv, u\|_c \leq \Phi(\|z - Tv, u\|_c)$.
If $Tv \in z$ then $\|z - Tv, u\|_c \geq 0$ and hence as Φ is contractive modulus

$$\Phi(\|z - Tv, u\|_c) < \|z - Tv, u\|_c$$

Therefore $\|z - Tv, u\|_c < \|z - Tv, u\|_c$
which is a contradiction. Therefore $Tv = Qv = z$
So, v is a coincidence point of Q and T.
Since the pair of maps Q and T are weakly compatible, $QTv = TQv$
(i.e) $Qz = Tz$.
Now show that z is a fixed point of S.
We have

$$\|Sz - z\| = \|Sz - Tv, u\|_c$$
$$\leq \Phi(\lambda(z, v))$$
$$\leq \lambda(z, v)$$
$$= \max\{\|Pz - Qv, u\|_c, \|Pz - Sz, u\|_c, \|Qv - Tv, u\|_c, \frac{1}{2}(\|Pz - Tv, u\|_c + \|Qv - Sz, u\|_c)\}$$
$$= \max\{\|Sz - z, u\|_c, \|Sz - Sz, u\|_c, \|z - z, u\|_c, \frac{1}{2}(\|Sz - z, u\|_c + \|z - Sz, u\|_c)\}$$
$$= \|Sz - z, u\|_c$$

Thus $\|Sz - z, u\|_c \leq \Phi(\|Sz - z, u\|_c)$.
If $Sz \neq z$ then $\|Sz - z, u\|_c > 0$ and hence as Φ is contractive modulus $\Phi(\|Sz - z, u\|_c) < \|Sz - z, u\|_c$ which is a contradiction. There exits $Sz = z$. Hence $Sz = Pz = z$
Show that z is a fixed point of T.
We have

$$\begin{aligned}
\|z - Tz, u\|_c &= \|Sz - Tz, u\|_c \\
&\leq \Phi(\lambda(z, z)) \\
&\leq \lambda(z, z) \\
&= \max\{\|Pz - Qz, u\|_c, \|Pz - Sz, u\|_c, \|Qz - Tz, u\|_c, \frac{1}{2}(\|Pz - Tz, u\|_c + \|Qz - Sz, u\|_c)\} \\
&= \max\{\|z - Tz, u\|_c, \|z - z, u\|_c, \|Tz - Tz, u\|_c, \frac{1}{2}(\|z - Tz, u\|_c + \|Tz - z, u\|_c)\} \\
&= \|z - Tz, u\|_c
\end{aligned}$$

Thus $\|z - Tz, u\|_c \leq \Phi(\|z - Tz, u\|_c)$.
If $z \neq Tz$ then $\|z - Tz, u\|_c > 0$ and hence as Φ is contractive modulus

$$\Phi(\|z - Tz, u\|_c) < \|z - Tz, u\|_c.$$

which is a contradiction. Hence $z = Tz$.
Therefore $Tz = Qz = z$.
Therefore $Sz = Pz = Tz = Qz = z$.
That is z is common fixed point of P, Q, S and T.

Uniqueness

Suppose, z and w is $(z \neq w)$ are common fixed point of P, Q, S and T.
we have

$$\begin{aligned}
\|z - w, u\|_c &= \|Sz - Tw, u\|_c \\
&\leq \Phi(\lambda(z, w)) \\
&\leq \lambda(z, w) \\
&= \max\{\|Pz - Qw, u\|_c, \|Pz - Sz, u\|_c, \|Qw - Tw, u\|_c, \frac{1}{2}(\|Pz - Tw, u\|_c + \|Qw - Sz, u\|_c)\} \\
&= \max\{\|z - w, u\|_c, \|z - z, u\|_c, \|w - w, u\|_c, \frac{1}{2}(\|z - w, u\|_c + \|w - z, u\|_c)\} \\
&= \|z - w, u\|_c
\end{aligned}$$

Thus, $\|z - w, u\|_c \leq \Phi(\|z - w, u\|_c)$
Since $z \neq w$, then $\|z - w\| > 0$ and hence as Φ is contractive modulus.

$$\Phi(\|z - w, u\|_c) < \|z - w, u\|_c$$

$$\text{therefore} \quad \|z - w, u\|_c < \|z - w, u\|_c$$

which is a contradiction,

$$\text{therefore} \quad z = w$$

Thus z is the unique common fixed point of P, Q, S and T. □

Corollary 2.2. *Let X be a 2-cone Banach space (with $\dim X \geq 2$). Suppose that the mappings P, S and T are three self maps of X such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ satisfying*

$$\|Sx - Ty, u\|_c \leq \Phi(\lambda(x, y)), \tag{2.6}$$

where Φ *is an upper semi continuous contractive modulus and*

$$\lambda(x,y) = \max\{\|Px-Py,u\|_c, \|Px-Sx,u\|_c, \|Py-Ty,u\|_c, \frac{1}{2}\{\|Px-Ty,u\|_c + \|Py-Sx,u\|_c\}\}.$$

The pair $\{S,P\}$ *and* $\{T,P\}$ *are weakly compatible. Then* P,S *and* T *have a unique common fixed point.*

Proof. The proof of the corollary immediate by taking $P = Q$ in the above theorem (2.3). □

References

[1] Huang, LG, Zhang, *Cone metric spaces and fixed point theorems of contractive mappings* , J. Math. Anal. Appl., 332(2007), 1468–1476.

[2] Karapinar .E, *Fixed point theorems in cone Banach spaces*, Fixed Point Theory and Application, 1-9(2009).

[3] Krishnakumar .R and Marudai .M, *Cone Convex Metric Space and Fixed Point Theorems* , Int. Journal of Math. Analysis, 6(22)(2012), 1087-1093.

[4] Krishnakumar. R and Marudai, M, *Generalization of a Fixed Point Theorem in Cone Metric Spaces*, Int. Journal of Math. Analysis, 5(11)(2011), 507–512.

[5] Krishnakumar .R and Dhamodharan .D, *Common Fixed Point of Four Mapping With Contractive Modulus on Cone Banach Space*, Malaya J. Mat., 5(2)(2017), 310–320.

[6] Krishnakumar .R and Dhamodharan .D, *Some Fixed Point Theorems in Cone Banach Spaces Using* Φ_p *Operator* , International Journal of Mathematics And its Applications, 4(2-B)(2016), 105-112.

[7] Krishnakumar.R and Dhamodharan. D, *Fixed Point Theorems in normal cone metric space*, International J. of Math. Sci. & Engg. Appls. 10(III)(2016), 213–224.

[8] Elvin Rada and Agron Tato, *An Extension of a Fixed Point Result in Cone Banach Space* , Pure and Applied Mathematics Journal, 4(3)(2015), 70–74.

[9] R. K. Gujetiya, Dheeraj Kumari Mali and Mala Hakwadiya, *Common Fixed Point Theorem for Compatible Mapping on Cone Banach Space* ,International Journal of Mathematical Analysis, 8(35)(2014), 1697 –1706.

[10] Thabet Abdeljawad, Erdal Karapinar and Kenan Tas, *Common Fixed Point Theorem in Cone Banach Space* Hacettepe Journal of Mathematics and Statistics, 40 (2)(2011), 211 - 217.

[11] Ahmet Sahiner and Tuba Yigit, *2 -Cone Banach Spaces and Fixed Point Theorem*, Numerical Analysis and Applied Mathematics ICNAAM 2012 AIP Conf. Proc. 1479, 975–978 (2012);

[12] Ahmet Sahiner, *Fixed Point Theorems in symmetric 2 -Cone Banach Space* $(l_p, \|,.,\|_p^c)$, Journal of Nonlinear Analysis and Optimization, Vol. 3, No. 2, (2012), 115–120

[13] B.E Rhoades, *Contraction type mappings on 2 Banach space,* Math. Nachr. 91(1979), 151–155.

[14] H. Gunawan and Mashadi, *On Finite Dimensional 2 -normed spaces*, Soochow J. Math., 27 (3) 2001, 321–329.

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