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Existence and monotonicity of positive solutions for hybrid Caputo-Hadamard fractional integro-differential equations

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Abstract. The purpose of this paper is to present new results on the existence, uniqueness and monotonicity of positive solutions for hybrid Caputo-Hadamard fractional integro-differential equations. Our results are based on the method of upper and lower solutions, and the Dhage and Banach fixed point theorems. Two examples are given to illustrate our obtained results.

AMS Subject Classifications: 40A05, 40A99, 46A70, 46A99.

Keywords: Fractional integro-differential equations, fixed point theorems, existence, positivity, monotonicity.

Contents

1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, physics, chemistry, biology, medicine, etc. In particular, problems concerning qualitative analysis of fractional differential equations with and without delay have received the attention of many authors, see [1]–[14], [16]–[22] and the references therein.

Hybrid Fractional differential equations arise from a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [2, 3, 13, 14, 21, 22].

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Let $J = [t_0, T]$. Haoues et al. [18] investigated the existence, uniqueness and monotonicity of positive solutions for the following hybrid fractional integro-differential equation

$$\begin{cases} {}^{C}D_{t_0}^{\alpha}\left(\dfrac{x(t)}{p(t)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}g(s,x(s))\,ds}\right)=f(t,x(t)),\ t\in J,\\ x(t_0)=p(t_0)\theta\geq 0,\end{cases}$$

where ${}^{C}D_{t_0}^{\alpha}$ is the Caputo fractional derivative of order $0<\alpha\leq 1$, $0<\beta\leq 1$, $0\leq t_0<T$, $f,g: J\times\mathbb{R}\to\mathbb{R}$ and $p: J\to\mathbb{R}$ are given continuous functions. By using the method of the upper and lower solutions and the Dhage and Banach fixed point theorems, the authors obtained the existence, uniqueness and monotonicity of a positive solution.

In this paper, we extend the results in [18] by proving the existence, uniqueness and monotonicity of positive solutions for the following hybrid nonlinear Caputo-Hadamard fractional integro-differential equation

$$\begin{cases} {}^{CH}D_{t_0}^{\alpha}\left(\dfrac{x(t)}{p(t)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}g(s,x(s))\,\frac{ds}{s}}\right)=f(t,x(t)),\ t\in J,\\ x(t_0)=p(t_0)\theta\geq 0,\end{cases} \tag{1.1}$$

where ${}^{CH}D_{t_0}^{\alpha}$ is the Caputo-Hadamard fractional derivative of order $0<\alpha\leq 1$, $0<\beta\leq 1$, $1\leq t_0<T$, $f,g: J\times\mathbb{R}\to\mathbb{R}$ and $p: J\to\mathbb{R}$ are given continuous functions. To prove the existence, uniqueness and monotonicity of positive solutions, we transform (1.1) into an integral equation and then by the method of upper and lower solutions and use Dhage and Banach fixed point theorems.

2. Preliminaries

Let $X = C(J)$ be the Banach space of all real-valued continuous functions defined on the compact interval J, endowed with the maximum norm. Define the subset $\mathcal{C}_\theta = \{x\in X : x(t)\geq p(t_0)\theta,\ t\in J\}$ of X.

Definition 2.1 ([19]). *The Hadamard fractional integral of order $\alpha>0$ for a continuous function $x:[t_0,+\infty)\to\mathbb{R}$ is defined as*

$${}^{H}I_{t_0}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}.$$

Definition 2.2 ([19]). *The Caputo-Hadamard fractional derivative of order $\alpha>0$ for a continuous function $x:[t_0,+\infty)\to\mathbb{R}$ is defined as*

$${}^{CH}D_{t_0}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^{n}(x)(s)\frac{ds}{s},$$

where $\delta^n=\left(t\frac{d}{dt}\right)^n$ and $n=[\alpha]+1$.

Lemma 2.3 ([19]). *Let $\alpha>0$ and $x\in C^{n-1}[t_0,+\infty)$ and $\delta^n(x)$ exists almost everywhere on any bounded interval of $[t_0,+\infty)$. Then*

$$\left({}^{H}I_{t_0}^{\alpha\,CH}D_{t_0}^{\alpha}x\right)(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(t_0)}{k!}(\log t)^{k}.$$

In particular, when $0<\alpha\leq 1$, $\left({}^{H}I_{t_0}^{\alpha\,CH}D_{t_0}^{\alpha}x\right)(t)=x(t)-x(t_0)$.

Definition 2.4. *For any $x \in [a,b] \subset \mathbb{R}^{+}$, we define the upper-control function by*

$$U(t,x) = \sup\{f(t,s) : a \le s \le x\},$$

and the lower-control function by

$$L(t,x) = \inf\{f(t,s) : x \le s \le b\}.$$

It is obvious that these functions are non-decreasing on $[a,b]$, i.e.

$$L(t,x) \le f(t,x) \le U(t,x),\ t \in J.$$

Definition 2.5. *A function $x \in X$ is positive bounded below if $x \in \mathcal{C}_\theta$. In particular, we call x as nonnegative function if $p(t_0)\theta = 0$ and positive function if $p(t_0)\theta > 0$.*

The following fixed point theorem due to Dhage [15] is essential tool for the proof of the first result.

Theorem 2.6 ([15])**.** *Let S be a nonempty bounded closed convex subset of a Banach algebra X. Let $\mathcal{B} : S \to X$ and $\mathcal{A} : S \to X$ be two operators such that*
i) $\mathcal{A}$ is Lipschiz with a Lipschitz constant σ,
ii) $\mathcal{B}$ is completely continuous,
iii) $\mathcal{A}x\mathcal{B}y \in S$ for all $x, y \in S$.
Then the product operator equation

$$\mathcal{A}x\mathcal{B}x = x,$$

has a solution, whenever $\sigma M < 1$, where $M = \sup\{\|\mathcal{B}x\| : x \in S\}$.

3. Existence of positive solutions

In this section, we will discuss the existence of positive solutions for (1.1). We introduce the following conditions

($\mathbf{H1}$) For $t \in J$ and $x \in X$, we have

$$p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} g(s, x(s)) \frac{ds}{s} > 0,$$

and

$$\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \ge 0.$$

($\mathbf{H2}$) Let $x^*, x_* \in \mathcal{C}_\theta$, such that $x_*(t_0) = x^*(t_0) = p(t_0)\theta$ and $p(t_0)\theta \le x_*(t) \le x^*(t) \le b$, $t \in J$. Moreover,

$$\begin{cases} {}^{CH}D_{t_0}^{\alpha}\left(\dfrac{x^*(t)}{p(t) + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s, x^*(s))\frac{ds}{s}}\right) \ge U(t, x^*(t)), \\ {}^{CH}D_{t_0}^{\alpha}\left(\dfrac{x_*(t)}{p(t) + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s, x_*(s))\frac{ds}{s}}\right) \le L(t, x_*(t)), \end{cases} \tag{3.1}$$

for any $t \in J$.

($\mathbf{H3}$) Let g be monotonic non-decreasing with respect to x and there exists $L_g > 0$ such that

$$|g(t,x) - g(t,y)| \le L_g |x - y|,\ t \in J,\ x, y \in \mathbb{R},$$

where

$$0 < L_g \frac{\left(\log \frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)} \left(|\theta| + \frac{c_f \left(\log \frac{T}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right) < 1.$$

$(\mathbf{H4})$ There exists $L_f > 0$ such that

$$|f(t,x) - f(t,y)| \leq L_f |x-y|, \ t \in J, \ x,y \in \mathbb{R}.$$

The functions x^* and x_* are respectively called the pair of upper and lower solutions for (1.1).

From Lemma 2.3, we deduce the following lemma.

Lemma 3.1. *Suppose that* $\frac{x}{h}$ *is differentiable on* J*. Then the equation*

$$\begin{cases} {}^{CH}D_{t_0}^{\alpha}\left(\frac{x(t)}{h(t)}\right) = f(t,x(t)), \ t \in J, \\ x(t_0) = p(t_0)\,\theta, \end{cases} \tag{3.2}$$

is equivalent to

$$x(t) = h(t)\left(\theta + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\right), \ t \in J. \tag{3.3}$$

By the previous lemma, (1.1) is equivalent to

$$\begin{aligned} x(t) = &\left(p(t) + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s,x(s))\frac{ds}{s}\right) \\ &\times\left(\theta + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\right), \ t \in J. \end{aligned}$$

Hence, according to the Dhage fixed point theorem 2.6, we define the operators $\mathcal{A}, \mathcal{B} : \mathcal{C}_\theta \to \mathcal{C}_\theta$ by

$$(\mathcal{A}x)(t) = p(t) + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s,x(s))\frac{ds}{s}, \tag{3.4}$$

and

$$(\mathcal{B}x)(t) = \theta + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}, \tag{3.5}$$

for $t \in J$.

Theorem 3.2. *Suppose that* $(\mathbf{H1}) - (\mathbf{H3})$ *are satisfied, then the problem (1.1) has at last one positive bounded below solution* $x \in \mathcal{C}_\theta$ *satisfying* $x_*(t) \leq x(t) \leq x^*(t)$*,* $t \in J$*.*

Proof. Let $S = \{x \in \mathcal{C}_\theta : x_*(t) \leq x(t) \leq x^*(t), \ t \in J\}$, endowed with the norm $\|x\| = \max_{t\in J}|x(t)|$, then for any $x \in S$, we have $\|x\| \leq b$. Hence, S is a convex, bounded and closed subset of $\mathcal{C}_\theta$. Moreover, the continuity of g and f implies the continuity of the operators $\mathcal{A}$ and $\mathcal{B}$ on S. Now, if $x \in S$ there exists a positive constant c_f such that $\max\{|f(t,x(t))| : (t,x) \in J \times S\} \leq c_f$. Then

$$\begin{aligned} |(\mathcal{B}x)(t)| &= \left| \theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s} \right)^{\alpha-1} f(s,x(s)) \frac{ds}{s} \right| \\ &\leq |\theta| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s} \right)^{\alpha-1} |f(s,x(s))| \frac{ds}{s} \\ &\leq |\theta| + \frac{c_f}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{ds}{s} \\ &\leq |\theta| + \frac{c_f \left(\log \frac{t}{t_0} \right)^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

So,

$$\|\mathcal{B}x\| \leq |\theta| + \frac{c_f \left(\log \frac{T}{t_0} \right)^{\alpha}}{\Gamma(\alpha+1)}.$$

Hence, $\mathcal{B}(S)$ is uniformly bounded. Next, we show the equicontinuity of $\mathcal{B}$. Let $x \in S$, then for any $t_1, t_2 \in J$, $t_2 > t_1$, we get

$$\begin{aligned} &|(\mathcal{B}x)(t_2) - (\mathcal{B}x)(t_1)| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_{t_0}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} f(s,x(s)) \frac{ds}{s} - \int_{t_0}^{t_1} \left(\log \frac{t_1}{s} \right)^{\alpha-1} f(s,x(s)) \frac{ds}{s} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left(\left(\log \frac{t_1}{s} \right)^{\alpha-1} - \left(\log \frac{t_2}{s} \right)^{\alpha-1} \right) |f(s,x(s))| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} |f(s,x(s))| \frac{ds}{s} \\ &\leq \frac{c_f}{\Gamma(\alpha)} \left(\int_{t_0}^{t_1} \left(\log \frac{t_1}{s} \right)^{\alpha-1} - \left(\log \frac{t_2}{s} \right)^{\alpha-1} \frac{ds}{s} + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} \frac{ds}{s} \right) \\ &\leq \frac{c_f}{\Gamma(\alpha+1)} \left(\left(\log \frac{t_1}{t_0} \right)^{\alpha} - \left(\log \frac{t_2}{t_0} \right)^{\alpha} + 2 \left(\log \frac{t_2}{t_1} \right)^{\alpha} \right) \\ &\leq \frac{2c_f}{\Gamma(\alpha+1)} \left(\log \frac{t_2}{t_1} \right)^{\alpha}. \end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the previous inequality is independent of x and tends to zero. Therefore, $\mathcal{B}(S)$ is equicontinuous. The Arzela-Ascoli theorem implies that $\mathcal{B}$ is compact. Hence $\mathcal{B}$ is completely continuous.

By hypothesis $(\mathbf{H3})$, for any $x, y \in S$, we get

$$
\begin{aligned}
&\left|(\mathcal{A}x)(t)-(\mathcal{A}y)(t)\right| \\
&=\left|\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s,x(s))\frac{ds}{s}-\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s,y(s))\frac{ds}{s}\right| \\
&\leq \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}\left|g(s,x(s))-g(s,y(s))\right|\frac{ds}{s} \\
&\leq \frac{L_g}{\Gamma(\beta)}\left(\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}\frac{ds}{s}\right)\|x-y\| \\
&\leq \frac{L_g\left(\log\frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)}\|x-y\|.
\end{aligned}
$$

Then $\mathcal{A}$ is Lipschiz mapping with Lipschitz constant $\sigma = L_g\frac{\left(\log\frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)}$, that satisfying $\sigma \sup\{\|\mathcal{B}x\| : x \in S\} < 1$.

We need to show that $\mathcal{A}x\mathcal{B}y \in S$ for all $x, y \in S$. Indeed, by Definition 2.4 and the hypothesis $(\mathbf{H3})$, we obtain

$$
\begin{aligned}
&(\mathcal{A}x)(t)(\mathcal{B}y)(t) \\
&=\left(p(t)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s,x(s))\frac{ds}{s}\right)\left(\theta+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,y(s))\frac{ds}{s}\right) \\
&\leq\left(p(t)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s,x(s))\frac{ds}{s}\right)\left(\theta+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} U(s,y(s))\frac{ds}{s}\right) \\
&\leq\left(p(t)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s,x^*(s))\frac{ds}{s}\right)\left(\theta+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} U(s,x^*(s))\frac{ds}{s}\right) \\
&\leq x^*(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathcal{A}x)(t)(\mathcal{B}y)(t) \\
&\geq\left(p(t)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s,x(s))\frac{ds}{s}\right)\left(\theta+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} L(s,y(s))\frac{ds}{s}\right) \\
&\geq\left(p(t)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\beta-1} g(s,x_*(s))\frac{ds}{s}\right)\left(\theta+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} L(s,x_*(s))\frac{ds}{s}\right) \\
&\geq x_*(t).
\end{aligned}
$$

Hence, $x_*(t) \leq \mathcal{A}x(t)\mathcal{B}y(t) \leq x^*(t)$, $t \in J$, that is $(\mathcal{A}x\mathcal{B}y)(S) \subseteq S$. According to the Dhage fixed point theorem, the operator equation $\mathcal{A}x\mathcal{B}x = x$ has at last one fixed point $x \in S$. Therefore, the problem (1.1) has at last one positive bounded below solution $x \in \mathcal{C}_\theta$. ∎

Next, we consider many particular cases of the previous theorem.

Corollary 3.3. *Suppose that* $(\mathbf{H3})$ *holds and there exist* $k_1, k_2, k_3, k_4 \in X$*, such that*

$$
k_1(t) \leq g(t,x(t)) \leq k_2(t), \tag{3.6}
$$

and

$$k_3(t) \leq f(t, x(t)) \leq k_4(t). \tag{3.7}$$

If

$$p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_1(s) \frac{ds}{s} > 0, \tag{3.8}$$

and

$$\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} k_3(s) \frac{ds}{s} \geq 0, \tag{3.9}$$

then the problem (1.1) has at least one positive bounded below solution. Moreover

$$\begin{aligned} &\left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_1(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} k_3(s) \frac{ds}{s}\right) \\ &\leq x(t) \\ &\leq \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_2(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} k_4(s) \frac{ds}{s}\right). \end{aligned}$$

Proof. By the assumption (3.7) and the definition of control functions, we have

$$k_3(t) \leq L(t, x(t)) \leq U(t, x(t)) \leq k_4(t),$$

for any $t \in J$. Now, we consider the fractional differential equations

$${}^{CH}D_{t_0}^{\alpha} \left(\frac{x(t)}{p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_1(s) \frac{ds}{s}} \right) = k_3(t), \ x(t_0) = p(t_0)\theta, \tag{3.10}$$

and

$${}^{CH}D_{t_0}^{\alpha} \left(\frac{x(t)}{p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_2(s) \frac{ds}{s}} \right) = k_4(t), \ x(t_0) = p(t_0)\theta. \tag{3.11}$$

In accordance of Lemma 3.1, the solutions of (3.10) and (3.11) are given respectively by

$$x(t) = \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_1(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} k_3(s) \frac{ds}{s}\right),$$

and

$$x(t) = \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_2(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} k_4(s) \frac{ds}{s}\right).$$

Therefore,

$$\begin{aligned} x(t) &= \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_1(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} k_3(s) \frac{ds}{s}\right) \\ &\leq \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_1(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} L(s, x(s)) \frac{ds}{s}\right), \end{aligned}$$

and

$$x(t) = \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_2(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} k_4(s) \frac{ds}{s}\right)$$
$$\geq \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_2(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} U(s, x(s)) \frac{ds}{s}\right).$$

One can define the upper and lower solutions as

$$x^*(t) = \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_2(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} U(s, x^*(s)) \frac{ds}{s}\right),$$

and

$$x_*(t) = \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_1(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} L(s, x_*(s)) \frac{ds}{s}\right).$$

Hence by Theorem 3.2, the problem (1.1) has a positive bounded below solution $x \in \mathcal{C}_\theta$. ■

Corollary 3.4. *Let $k \in X$ and $\varphi \in \mathbb{R}_+$ such that $\varphi < k(t) = \lim_{x\to\infty} f(t,x) < \infty$ for $t \in J$. If $(\mathbf{H3})$, (3.6) and (3.8) hold and $\theta \in \mathbb{R}_+$, then the problem (1.1) has at least one positive bounded below solution. Moreover for $0 < \omega < \varphi$,*

$$\left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_1(s) \frac{ds}{s}\right) \left(\theta + \frac{(\varphi - \omega)\left(\log \frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right)$$
$$\leq x(t)$$
$$\leq \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_2(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} k(s) \frac{ds}{s} + \frac{\omega\left(\log \frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right).$$

Corollary 3.5. *Suppose that $(\mathbf{H3})$, (3.6) and (3.8) hold, and*

$$\lim_{x\to\theta} \frac{f(t,x)}{x} = \gamma(t),$$

where $\gamma \in X$, $t \in J$. Then there exists a positive bounded below solution of the problem (1.1).

Corollary 3.6. *Let μ, ν and ξ are real positive numbers such that $\mu \leq f(t, x(t)) \leq \nu x(t) + \xi$, for $t \in J$. If (3.6), (3.8) and $(\mathbf{H3})$ hold and $\theta \in \mathbb{R}_+$, then the problem (1.1) has at least one positive bounded below solution. Moreover*

$$\left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_1(s) \frac{ds}{s}\right) \left(\theta + \frac{\mu\left(\log \frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right)$$
$$\leq x(t)$$
$$\leq \left(p(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} k_2(s) \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} (\nu x(s) + \xi) \frac{ds}{s}\right).$$

4. Uniqueness of positive solutions

In this portion, we will prove the uniqueness of the bounded below positive solution of (1.1) using the Banach contraction mapping principle.

Theorem 4.1. *Suppose that* $(\mathbf{H1})-(\mathbf{H4})$ *hold. If*

$$L_g \frac{\left(\log \frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)}\left(\theta + c_f \frac{\left(\log \frac{T}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right) + \left(p_m + c_g \frac{\left(\log \frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)}\right) L_f \frac{\left(\log \frac{T}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{4.1}$$

then the problem (1.1) has a unique positive bounded below solution.

Proof. Let c_f and c_g are positive real numbers such that,

$$|f(t,x(t))| \le c_f,\ |g(t,x(t))| \le c_g,$$

for any $t \in J$ and $x, y \in \mathcal{C}_\theta$. According to Theorem 3.2, the problem (1.1) has at least one positive bounded below solution in S. Now, we need only to prove that the product operator $\mathcal{A}x\mathcal{B}x$ is a contraction mapping on X, where $\mathcal{A}$ and $\mathcal{B}$ are defined as in (3.4) and (3.5). Indeed, for any $x, y \in \mathcal{C}_\theta$ and $t \in J$, we get

$$\begin{aligned}
&|(\mathcal{A}x\mathcal{B}x)(t) - (\mathcal{A}y\mathcal{B}y)(t)| \\
&\le \left(\frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} |g(s,x(s)) - g(s,y(s))| \frac{ds}{s}\right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s,x(s))| \frac{ds}{s}\right) \\
&+ \left(|p(t)| + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\beta-1} |g(s,y(s))| \frac{ds}{s}\right) \\
&\times \left(\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s,x(x)) - f(s,y(s))| \frac{ds}{s}\right) \\
&\le \left(L_g \frac{\left(\log \frac{t}{t_0}\right)^{\beta}}{\Gamma(\beta+1)} \|x-y\|\right) \left(\theta + c_f \frac{\left(\log \frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right) \\
&+ \left(|p(t)| + c_g \frac{\left(\log \frac{t}{t_0}\right)^{\beta}}{\Gamma(\beta+1)}\right) \left(L_f \frac{\left(\log \frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} \|x-y\|\right) \\
&\le \left(L_g \frac{\left(\log \frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)} \left(\theta + c_f \frac{\left(\log \frac{T}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right) + \left(p_m + c_g \frac{\left(\log \frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)}\right) L_f \frac{\left(\log \frac{T}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right) \|x-y\|,
\end{aligned}$$

where $p_m = \max_{t \in J} |p(t)|$. Hence, by (4.1) the product operator $\mathcal{A}x\mathcal{B}x$ is a contraction mapping. Therefore, the problem (1.1) has a unique positive bounded below solution $x \in \mathcal{C}_\theta$. ■

5. Monotonicity of positive solutions

Theorem 5.1. *Let* p, g *and* f *be non-decreasing functions with respect to both variables,* $f(t_0, x(t_0)) \ge 0$ *and* $g(t_0, x(t_0)) \ge 0$*. Moreover, let* $(\mathbf{H1})-(\mathbf{H4})$ *hold, then there is a monotonic non-decreasing positive bounded below solution of the problem (1.1).*

Proof. Define a subset $R=\{x\in S: x \text{ is nondecreasing on } J\}$, then R is a closed and convex subset of S. The operator $\mathcal{B}$ is uniformly bounded and completely continuous and the operator $\mathcal{A}$ is Lipschitzian with Lipschitz constant σ, and satisfying $\sigma\sup\{\|\mathcal{B}x\|: x\in R\}\leq 1$. It remains for applying the Dhage theorem that $\mathcal{A}x\mathcal{B}x\in R$ whenever $x,y\in R$. To this end, it suffices to consider $x,y\in R$, $t_1,t_2\in J$ with $t_1<t_2$. It follows that

$$\begin{aligned}
&\mathcal{A}x(t_2)\mathcal{B}y(t_2)-\mathcal{A}x(t_1)\mathcal{B}y(t_1)\\
&=\mathcal{A}x(t_2)(\mathcal{B}y(t_2)-\mathcal{B}y(t_1))+(\mathcal{A}x(t_2)-\mathcal{A}x(t_1))\mathcal{B}y(t_1)\\
&=\frac{1}{\Gamma(\alpha)}\left(p(t_2)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t_2}\left(\log\frac{t_2}{s}\right)^{\beta-1}g(s,x(s))\frac{ds}{s}\right)\\
&\times\left(\int_{t_0}^{t_1}\left(\left(\log\frac{t_2}{s}\right)^{\alpha-1}-\left(\log\frac{t_1}{s}\right)^{\alpha-1}\right)f(s,y(s))\frac{ds}{s}+\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}\right)\\
&+\left(p(t_2)-p(t_1)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t_1}\left(\left(\log\frac{t_2}{s}\right)^{\beta-1}-\left(\log\frac{t_1}{s}\right)^{\beta-1}\right)g(s,x(s))\frac{ds}{s}\right.\\
&\left.+\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\beta-1}g(s,x(s))\frac{ds}{s}\right)\left(\theta+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}\right).
\end{aligned}$$

Since $\left(\log\frac{t_2}{s}\right)^{\alpha-1}-\left(\log\frac{t_1}{s}\right)^{\alpha-1}<0$ and $\left(\log\frac{t_2}{s}\right)^{\beta-1}-\left(\log\frac{t_1}{s}\right)^{\beta-1}<0$, then

$$\begin{aligned}
&\mathcal{A}x(t_2)\mathcal{B}y(t_2)-\mathcal{A}x(t_1)\mathcal{B}y(t_1)\\
&\geq\frac{f(t_1,y(t_1))}{\Gamma(\alpha)}\left(p(t_2)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t_2}\left(\log\frac{t_2}{s}\right)^{\beta-1}g(s,x(s))\frac{ds}{s}\right)\\
&\times\left(\int_{t_0}^{t_1}\left(\left(\log\frac{t_2}{s}\right)^{\alpha-1}-\left(\log\frac{t_1}{s}\right)^{\alpha-1}\right)\frac{ds}{s}+\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\\
&+\left(p(t_2)-p(t_1)+\frac{g(t_1,x(t_1))}{\Gamma(\beta)}\left(\int_{t_0}^{t_1}\left(\left(\log\frac{t_2}{s}\right)^{\alpha-1}-\left(\log\frac{t_1}{s}\right)^{\alpha-1}\right)\frac{ds}{s}\right.\right.\\
&\left.\left.+\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\right)\left(\theta+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}\right)\\
&\geq\frac{f(t_1,y(t_1))}{\Gamma(\alpha+1)}\left(p(t_2)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t_2}\left(\log\frac{t_2}{s}\right)^{\beta-1}g(s,x(s))\frac{ds}{s}\right)\left(\left(\log\frac{t_2}{t_0}\right)^{\alpha}-\left(\log\frac{t_1}{t_0}\right)^{\alpha}\right)\\
&+\left(p(t_2)-p(t_1)+\frac{g(t_1,x(t_1))}{\Gamma(\beta+1)}\left(\left(\log\frac{t_2}{t_0}\right)^{\beta}-\left(\log\frac{t_1}{t_0}\right)^{\beta}\right)\right)\\
&\times\left(\theta+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}\right)\\
&\geq 0.
\end{aligned}$$

Therefore, with the Dhage fixed point theorem the product operator $\mathcal{AB}: R\to R$ has a fixed point with the positivity and monotonicity nondecreasing properties which is a solution of the problem (1.1). ■

Remark 5.2. *The results of Theorem 5.1 are valid in Corollaries 3.3-3.6 if the assumptions of Theorem 5.1 are added to Corollaries 3.3-3.6.*

6. Examples

Example 6.1. *Consider the Caputo-Hadamard fractional integro-differential equation*

$$\begin{cases} {}^{C}D^{\frac{2}{3}}\left(\dfrac{x(t)}{\frac{5+3t}{8}+\frac{1}{\Gamma\left(\frac{1}{4}\right)}\int_1^t\left(\log\frac{t}{s}\right)^{-\frac{3}{4}}\left(\frac{x(s)+2}{x(s)+3}\right)\frac{ds}{s}}\right)=\frac{1}{5+t}\left(\frac{tx(t)}{x(t)+4}+5\right),\ t\in(1,e], \\ x(1)=0, \end{cases} \tag{6.1}$$

where $\alpha=2/3$, $\beta=1/4$, $\theta=0$, $p(t)=\frac{5+3t}{8}$, $f(t,x)=\frac{1}{5+t}\left(\frac{tx}{x+4}+5\right)$ *and* $g(t,x)=\frac{x+2}{x+3}$. *Since* g *is nondecreasing on* x,

$$\frac{2}{3}\leq g(t,x)\leq 1 \text{ and } \frac{5}{5+e}\leq f(t,x)\leq 1 \text{ for } t\in[1,e],\ x\in[0,+\infty),$$

and

$$L_g\frac{\left(\log\frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)}\left(|\theta|+c_f\frac{\left(\log\frac{T}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right)\simeq 0.136<1.$$

Then, by Corollary 3.3, (6.1) has a positive solution which verifies $x_*(t)\leq x(t)\leq x^*(t)$ *where*

$$x_*(t)=\left(\frac{5+3t}{8}+\frac{2(\log t)^{\frac{1}{4}}}{3\Gamma\left(\frac{5}{4}\right)}\right)\frac{5}{5+e}\frac{(\log t)^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)},$$

and

$$x^*(t)=\left(\frac{5+3t}{8}+\frac{(\log t)^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)}\right)\frac{(\log t)^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)}.$$

This positive solution is unique due to the condition (4.1) is satisfied since

$$\begin{aligned} &L_g\frac{\left(\log\frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)}\left(|\theta|+c_f\frac{\left(\log\frac{T}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)}\right)+\left(p_m+c_g\frac{\left(\log\frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)}\right)L_f\frac{\left(\log\frac{T}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} \\ &\simeq 0.404<1. \end{aligned}$$

The property of non-decreasing of this solution is not valid in spite of f.

Example 6.2. *Consider the Caputo-Hadamard fractional integro-differential equation*

$$\begin{cases} {}^{C}D^{\frac{1}{3}}\left(\dfrac{x(t)}{\frac{t}{3}+1+\frac{1}{\Gamma\left(\frac{2}{5}\right)}\int_1^t\left(\log\frac{t}{s}\right)^{-\frac{3}{5}}(2-\cos(x(s)))\frac{ds}{s}}\right)=\frac{t}{6}\sin(x(t)),\ t\in(1,e], \\ x(1)=4, \end{cases} \tag{6.2}$$

where $\alpha=1/3$, $\beta=2/5$, $\theta=1/3$, $p(t)=\frac{t}{3}+1$, $f(t,x)=\frac{t}{6}\sin x$ *and* $g(t,x)=2-\cos x$ *for* $t\in[1,e]$, $x\in\left[0,\frac{\pi}{2}\right]$. *Hence,* g *is nondecreasing on* x *and*

$$0\leq f(t,x)\leq\frac{e}{6},\ 1\leq g(t,x)\leq 2.$$

Since

$$L_g \frac{\left(\log \frac{T}{t_0}\right)^{\beta}}{\Gamma(\beta+1)} \left(|\theta| + \frac{c_f \left(\log \frac{T}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} \right) \simeq 0.947 < 1.$$

Then, by Corollary 3.3, (6.2) has a positive solution which verifies $x_*(t) \leq x(t) \leq x^*(t)$ *where*

$$x_*(t) = \frac{t}{9} + \frac{1}{3} + \frac{(\log t)^{\frac{2}{5}}}{3\Gamma\left(\frac{7}{5}\right)},$$

and

$$x^*(t) = \left(\frac{t}{3} + 1 + \frac{2(\log t)^{\frac{2}{5}}}{\Gamma\left(\frac{7}{5}\right)} \right) \left(\frac{1}{3} + \frac{e(\log t)^{\frac{1}{3}}}{6\Gamma\left(\frac{4}{3}\right)} \right).$$

We could not guarantee this positive solution is unique due to the condition (4.1) is not satisfied. The property of non-decreasing of this positive solution is valid since f *and* g *are increasing on* $\left[0, \frac{\pi}{2}\right]$ *and* p *is increasing on* $[1, e]$.

7. Conclusion

The hybrid nonlinear Caputo-Hadamard fractional integro-differential equation is considered. So, we have studied the existence, uniqueness and monotonicity of positive solutions. The main tool of this work is the method of upper and lower solutions and the Dhage and Banach fixed point theorems. However, by introducing a new fixed mapping, we obtain new positivity conditions.

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On rectifying curves in Minkowski 3-space

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Abstract. In this study, we investigate a rectifying curve by using a dilation of a unit speed curve on pseudo-sphere or pseudo-hyperbolic space and its centrode. Firstly, considering a causal character of any curve, we study the connection between Serret-Frenet apparatus of the curve on pseudo-sphere or pseudo-hyperbolic space and its dilation. Then, we extend necessary conditions when the centrode is a rectifying curve. Also, we examine some properties of centrode which is a rectifying curve.

AMS Subject Classifications: 53C40, 53C50.

Keywords: Rectifying curve, Minkowski 3−space, Frenet equations, centrode, helix.

Contents

1. Introduction

Characterization of a curve allows to classify curves according to some properties. Thus, instead of doing analysis for each curve, working on these classes is appeared as a more convenient way. One example for the characterized curves is a rectifying curve. The curve was put forward by B. Y. Chen in [1]. In three dimensional Euclidean space $\mathbb{E}^3$, if $\alpha : I \subseteq \mathbb{R} \to \mathbb{E}^3$ be a unit speed regular curve with Frenet frame $\{T, N, B\}$ where T is a tangent vector field, N is a normal vector field and B is a binormal vector field, then an osculating plane is a plane spanned by the vector fields $\{T, N\}$, a rectifying plane is a plane spanned by the vector fields $\{T, B\}$ and a normal plane is a plane spanned by the vector fields $\{N, B\}$. The rectifying curve is a curve whose position vector field is located on the rectifying plane. Thus, Chen showed that a rectifying curve α is denoted by the equation

$$\alpha(s) = \eta(s)\, T(s) + \xi(s)\, B(s)$$

where the functions η and ξ are differentiable. Also, it is known that the rectifying curve is characterized as the curve whose ratio of torsion and curvature is a linear fuction of arc parameter in $\mathbb{E}^3$ [1]. By analyzing the characterization of rectifying curve, it is interpreted kinematically because the position vector field of the curve

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states the axis of instantaneous rotation at each point. In addition to that, the connection between rectifying curves and their centrodes is used to study in general mechanics. Firstly, Chen [2] established the connection between rectifying curves and their centrodes in Euclidean 3-space. Since these curves are important, many studies have been conducted on this subject. In this section, we just mention some of them. One of them was made by İlarslan et. al [5]. According to that, they gave some characterizations of timelike, spacelike, null rectifying curves. Moreover, these curves in dual space were examined in [9]. So, they were explored by taking advantage of the relationship between the curve on a unit dual sphere and the surface theory. Additionally, the modified Darboux vector of curve was described and it was shown that this vector is a rectifying curve [8]. Deshmukh et.al [3] developed the necessary circumstances for the centrode of a curve to be a rectifying curve in Euclidean 3-space and also, they presented the results about the dilation for rectifying curves and centrodes. This study has been the main motivation of our study.

In this study, we research on the rectifying curves in the Minkowski 3-space. First of all, we use that the dilation of $u(t)$ is written as $\alpha(t) = f(t)\, u(t)$ where $u(t)$ is a curve on the unit sphere S^2 centered at the origin and the fuction $f(t)$ is positive differentiable. If the curve $\alpha(t)$ is the rectifying curve, then the dilation factor $f(t)$ is given by $f(t) = a\sec(t + t_0)$. Here, $a > 0$ and t_0 are constants [1]. However, this dilation factor in Minkowski 3-space is defined as different ways for the curves $\alpha(t)$ and $u(t)$ [5]. Considering this difference, Frenet-Serret apparatus of α are given in terms of the curve $u(t)$. It was also shown in [4] that the centrode of $\alpha(t)$ is rectifying curve in Minkowski 3-space when it has non-zero constant curvature and non-constant torsion. Here, we generalize this result. For this, we show that the centrode of any helix is not a rectifying curve in Minkowski 3-space. Then, using this feature and considering the causal characters of Frenet-Serret vectors, we obtain that the centrode of non-helix curve is the rectifying curve if and only if it satisfies the condition $a\kappa - b\tau = c$ where a, b, c are constants. Finally, there are many cases from the choices of plane and curve, so we consider all the cases and give some notations for the centrode of $\alpha(t)$. We find the relations between Frenet-Serret apparatus of the centrode, which is a rectifying curve, and the Frenet-Serret apparatus of α.

2. Preliminaries

Let $\mathbb{E}_1^3$ be a space with the metric g denoted by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 \tag{2.1}$$

where $x = (x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbb{E}_1^3$ and also, it is called the three-dimensional Minkowski space. Pseudo-sphere of radius 1 centered at origin is a hyperquadric in $\mathbb{E}_1^3$ and is given by

$$S_1^2(1) = \left\{ v \in \mathbb{E}_1^3 \mid g(v, v) = 1 \right\} \tag{2.2}$$

and pseudo-hyperbolic space of radius 1 centered at origin is defined by

$$H_0^2(1) = \left\{ v \in \mathbb{E}_1^3 \mid g(v, v) = -1 \right\}. \tag{2.3}$$

Let $\alpha(s)$ be a curve with an arc-length parameter s. It is the non-null curve that satisfies the property $g(\alpha'(s), \alpha'(s)) = \pm 1$ where $\prime$ is the derivation of α [7]. For the Frenet frame $\{T, N, B\}$ of a unit speed non-null curve $\alpha(s)$ in $\mathbb{E}_1^3$, the Frenet formulas are given in [6] by

$$\begin{aligned} T'(s) &= \varepsilon_1 \kappa(s) N(s), \\ N'(s) &= -\varepsilon_0 \kappa(s) T(s) + \varepsilon_2 \tau(s) B(s), \\ B'(s) &= -\varepsilon_1 \tau(s) N(s) \end{aligned} \tag{2.4}$$

for $g(T, T) = \varepsilon_0 = \pm 1$, $g(N, N) = \varepsilon_1 = \pm 1$ and $g(B, B) = -\varepsilon_0 \varepsilon_1 = \varepsilon_2$ where T, N, B are known as the tangent vector field, the principal normal vector field, the binormal vector field, respectively. Assume that for

Frenet frame of α, there exists the vector product as

$$B = T \times N, -\varepsilon_1 T = N \times B, -\varepsilon_0 N = B \times T. \tag{2.5}$$

The centrode of $\alpha : I \to \mathbb{E}_1^3$ is defined by

$$d = -\varepsilon_0 \varepsilon_1 \tau T - \varepsilon_0 \varepsilon_1 \kappa B \tag{2.6}$$

which is the angular velocity vector of the motion of a mass particle along the curve α and it obeys the laws of motion [4]

$$T' = d \times T, N' = d \times N, B' = d \times B. \tag{2.7}$$

In addition, the rectifying curves are formed by expanding a unit speed curve on a unit sphere with a special factor f. To characterize such rectifying curves in $\mathbb{E}_1^3$, İlarslan et.al [5] gave a theorem. According to this theorem, a unit speed non-null curve $\alpha = \alpha(s)$ is written as the dilation of the curve $u(t)$ to be rectifying curve. So, it is a rectifying curve with a spacelike rectifying plane if and only if it satisfies the condition

$$\alpha(t) = u(t) \frac{l}{\cos t} \tag{2.8}$$

where $l \in \mathbb{R}_0^+$ and $u(t)$ is a unit speed spacelike curve on $S_1^2(1)$. Also, it is a spacelike (or timelike) rectifying curve with a spacelike (or timelike) position vector if and only if it satisfies the condition

$$\alpha(t) = u(t) \frac{l}{\sinh t} \tag{2.9}$$

where $u(t)$ is a unit speed timelike (or spacelike) curve on $S_1^2(1)$. Finally, it is a spacelike (or timelike) rectifying curve with a spacelike (or timelike) position vector if and only if it satisfies the condition

$$\alpha(t) = u(t) \frac{l}{\cosh t} \tag{2.10}$$

where $u(t)$ is a unit speed spacelike (or timelike) curve on $H_0^2(1)$ (or $S_1^2(1)$).

3. On the Rectifying Curves and the Dilation of Curves

In Minkowski 3-space, let $u(t)$ be a unit speed spacelike or timelike curve lying on $S_1^2(1)$ or $H_1^2(1)$ and $\alpha(t)$ be a dilated rectifying curve of $u(t)$. Now, we give the connections between the Frenet-Serret apparatus of the curves $\alpha(t)$ and $u(t)$ in Minkowski 3-space. For this, the different results are obtained by considering some theorems in [5], because the dilation $\alpha(t)$ of $u(t)$ is defined according to the causal characters of the curves $\alpha(t)$ and $u(t)$.

We assume that α is a spacelike rectifying curve with spacelike position vector on timelike rectifying plane and $u(t)$ is a unit speed timelike curve on pseudo-sphere $S_1^2(1)$. Then, there exists the dilation $\alpha(t) = \frac{a}{\sinh t} u(t)$. In this case, since the rectifying plane is timelike, clearly its normal vector N_α is spacelike, so $g(N_\alpha, N_\alpha) = 1$. If the curve α is spacelike, the tangent of α is also spacelike. Furthermore, its position vector is spacelike, so we have $g(\alpha, \alpha) > 0$, $g(u', u') = -1$ and $g(u, u) = 1$.

We know that $T_u = u'$ and $\{u, u', u \times u'\}$ is an orthonormal frame of $\mathbb{E}_1^3$. We easily write

$$u'' = T_u' = au + bT_u + cu \times T_u.$$

When we use timelike curve u and $\langle u', u \rangle = 0$, we find $a = 1$. Similarly, $b = 0$ is found. Also, we assume $c = \langle u'', u \times u' \rangle = r$. Then, we get

$$u'' = u + ru \times u'. \tag{3.1}$$

From (3.1) and using Frenet equations for $u(t)$, there exists

$$\|u''\| = \sqrt{1+r^2} = \kappa_u \tag{3.2}$$

where κ_u is a curvature of the unit speed timelike curve $u(t)$ and we find

$$T_u = u', N_u = \frac{1}{\kappa_u}u + \frac{r}{\kappa_u}u \times u'. \tag{3.3}$$

Theorem 3.1. *Let $u(t)$ be a unit speed timelike curve on the unit sphere $S_1^2(1)$ in $\mathbb{E}_1^3$, $\alpha(t)$ be a spacelike rectifying curve on a timelike rectifying plane and its position vector be a spacelike vector defined by $\alpha(t) = \dfrac{a}{\sinh(t+t_0)}u(t)$, $a \in \mathbb{R}^+$. Then, the relation between the Frenet-Serret apparatus of $\alpha(t)$ and $u(t)$ is as follows:*

$$\begin{aligned}
T_\alpha &= \sinh(t+t_0)\,u' - \cosh(t+t_0)\,u, \\
N_\alpha &= u \times u', \\
B_\alpha &= \sinh(t+t_0)\,u - \cosh(t+t_0)\,u', \\
\kappa_\alpha &= \frac{1}{a}\sqrt{\kappa_u^2 - 1}\sinh^3(t+t_0), \\
\tau_\alpha &= \frac{1}{a}\sqrt{\kappa_u^2 - 1}\cosh(t+t_0)\sinh^2(t+t_0)
\end{aligned}$$

where $T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha$ are the tangent vector, normal vector, binormal vector, curvature, torsion of $\alpha(t)$, respectively, and κ_u is the curvature of $u(t)$.

Proof. Let α is given by $\alpha(t) = \dfrac{a}{\sinh(t+t_0)}u(t)$ and $\{u, u', u \times u'\}$ is an orthonormal frame of $\mathbb{E}_1^3$. Thus, we get $\|\alpha'(t)\| = \dfrac{a}{\sinh^2(t+t_0)} = \upsilon_\alpha$ and

$$T_\alpha = \sinh(t+t_0)\,u' - \cosh(t+t_0)\,u. \tag{3.4}$$

Then differentiating T_α, we obtain

$$\begin{aligned}
T'_\alpha &= \cosh(t+t_0)\,u' + \sinh(t+t_0)\,u'' - \sinh(t+t_0)\,u - \cosh(t+t_0)\,u' \\
&= r\sinh(t+t_0)\,u \times u' \\
&= \varepsilon_1 \upsilon_\alpha \kappa_\alpha N_\alpha.
\end{aligned}$$

From (3.2) we write $r = \sqrt{\kappa_u^2 - 1}$. Furthermore, for the left side of (3.2) we say that $u \times u'$ is equal to N_α because both of them are unit vectors. So, if we write $\left[\dfrac{r\sinh(t+t_0)}{a}\sinh^2(t+t_0)\right] u \times u' = a\kappa_\alpha N_\alpha$, then we get $N_\alpha = u \times u'$ and also

$$\kappa_\alpha = \frac{1}{a}\sqrt{\kappa_u^2 - 1}\sinh^3(t+t_0). \tag{3.5}$$

We know that $B_\alpha = T_\alpha \times N_\alpha$. Then

$$B_\alpha = \sinh(t+t_0)\,u - \cosh(t+t_0)\,u'. \tag{3.6}$$

After differentiating (3.6), we find

$$\begin{aligned}
B'_\alpha &= \cosh(t+t_0)\,u + \sinh(t+t_0)\,u' - \sinh(t+t_0)\,u' - \cosh(t+t_0)\,u'' \\
&= -r\cosh(t+t_0)\,u \times u' \\
&= -\varepsilon_1 \upsilon_\alpha \tau_\alpha N_\alpha
\end{aligned}$$

which gives $[r\cosh(t+t_0)]\, u\times u' = \upsilon_\alpha \tau_\alpha N_\alpha$ that is,

$$\tau_\alpha = \frac{1}{a}\sqrt{\kappa_u^2-1}\cosh(t+t_0)\sinh^2(t+t_0)\,. \tag{3.7}$$

Thus, we establish the desired relationships. ■

Every curves, which is constituted by $\alpha(t) = \dfrac{a}{sinh(t+t_0)}u(t)$, do not have to be a rectifying curve. This result is valid for the curves which are not arc of the great circle. To show that, let $u(t)$ be an arc of the great circle on $S_1^2(1)$ given by $u(t) = (\sinh t, 0, \cosh t)$. We write $\alpha(t) = a\left(\dfrac{\sinh t}{\sinh(t+t_0)}, 0, \dfrac{\cosh t}{\sinh(t+t_0)}\right)$. Hence, we have $\|\alpha'(t)\| = \dfrac{a}{\sinh^2(t+t_0)}$ and $T_\alpha = (\sinh t_0, 0, -\cosh t_0)$. Thus, $\kappa_\alpha = 0$ and finally we say that $\alpha(t)$ is not a rectifying curve.

Theorem 3.2. *Let $u(t)$ be a unit speed spacelike curve on the unit sphere $S_1^2(1)$ in $\mathbb{E}_1^3$, $\alpha(t)$ be a rectifying curve on a spacelike rectifying plane defined by $\alpha(t) = \dfrac{a}{\cos(t+t_0)}u(t)\,, a\in\mathbb{R}^+$. Then the relation between the Frenet-Serret apparatus of $\alpha(t)$ and $u(t)$ is as follows:*

$$\begin{aligned}
T_\alpha &= \cos(t+t_0)\,u' + \sin(t+t_0)\,u,\\
N_\alpha &= u\times u',\\
B_\alpha &= -\cos(t+t_0)\,u + \sin(t+t_0)\,u',\\
\kappa_\alpha &= \frac{1}{a}\sqrt{1-\kappa_u^2}\cos^3(t+t_0)\,,\\
\tau_\alpha &= \frac{-1}{a}\sqrt{1-\kappa_u^2}\sin(t+t_0)\cos^2(t+t_0)
\end{aligned}$$

where $T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha$ are the tangent vector, normal vector, binormal vector, curvature, torsion of $\alpha(t)$, respectively, and κ_u is the curvature of $u(t)$.

Theorem 3.3. *Let $u(t)$ be a unit speed spacelike curve on the unit sphere $S_1^2(1)$ in $\mathbb{E}_1^3$, $\alpha(t)$ be a timelike rectifying curve on a timelike rectifying plane and its position vector be a timelike vector defined by $\alpha(t) = \dfrac{a}{\sinh(t+t_0)}u(t)\,,\ a\in\mathbb{R}^+$. Then the relation between the Frenet-Serret apparatus of $\alpha(t)$ and $u(t)$ is as follows:*

$$\begin{aligned}
T_\alpha =&\frac{\sinh(t+t_0)}{\sqrt{\cosh 2(t+t_0)}}u' - \frac{\cosh(t+t_0)}{\sqrt{\cosh 2(t+t_0)}}u,\\
N_\alpha =&\frac{-2\sinh(t+t_0)}{f(t)\sqrt{\cosh 2(t+t_0)}}\left(\cosh(t+t_0)u' + \sinh(t+t_0)u\right)\\
&-\frac{1}{f(t)}\sqrt{(1-\kappa_u^2)\cosh 2(t+t_0)}u\times u',\\
B_\alpha =&\frac{\sqrt{1-\kappa_u^2}}{f(t)}\{\sinh(t+t_0)\,u + \cosh(t+t_0)\,u'\} + \frac{2\sinh(t+t_0)}{f(t)}u\times u',\\
\kappa_\alpha =&\frac{\sinh^3(t+t_0)}{a\cosh^{3/2}2(t+t_0)}\{4\sinh^2(t+t_0) - (1-\kappa_u)^2\cosh 2(t+t_0)\}^{1/2},\\
\tau_\alpha =&\frac{\sqrt{\cosh 2(t+t_0)}}{f^2(t)}\Big[-2\kappa_u\sinh(t+t_0) - \sqrt{1-\kappa_u^2}\cosh(t+t_0)\left(1+\kappa_u^2\right)\\
&-2\frac{f'(t)}{f(t)}\sqrt{1-\kappa_u^2}\sinh(t+t_0)\left(1+\sqrt{1-\kappa_u^2}\right)\Big]
\end{aligned}$$

where $T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha$ are the tangent vector, normal vector, binormal vector, curvature, torsion of $\alpha(t)$, respectively, and κ_u is the curvature of u. Also, $f(t)$ is the function such that $f(t) = \{(1 - cosh2(t+t_0))^2 - (1-\kappa_u^2)\, cosh2(t+t_0)\sinh^2(t+t_0)\}$.

Theorem 3.4. *Let $u(t)$ be a unit speed spacelike curve on the unit sphere $H_0^2(1)$ in $\mathbb{E}_1^3$, $\alpha(t)$ be a spacelike rectifying curve on a timelike rectifying plane and its position vector be a timelike vector defined by $\alpha(t) = \dfrac{a}{\cosh(t+t_0)} u(t)$. Then the relation between the Frenet apparatus of $\alpha(t)$ and $u(t)$ is as follows:*

$$\begin{aligned}
T_\alpha &= \cosh(t+t_0)\, u' - \sinh(t+t_0)\, u, \\
N_\alpha &= u \times u', \\
B_\alpha &= -\cosh(t+t_0)\, u + \sinh(t+t_0)\, u', \\
\kappa_\alpha &= \frac{1}{a}\sqrt{1+\kappa_u^2}\cosh^3(t+t_0), \\
\tau_\alpha &= -\frac{1}{a}\sqrt{1+\kappa_u^2}\sinh(t+t_0)\cosh^2(t+t_0)
\end{aligned}$$

where $T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha$ are the tangent vector, normal vector, binormal vector, curvature, torsion of $\alpha(t)$, respectively, and κ_u is the curvature of $u(t)$.

Theorem 3.5. *Let $u(t)$ be a unit speed timelike curve on the unit sphere $S_1^2(1)$ in $\mathbb{E}_1^3$, $\alpha(t)$ be a timelike rectifying curve on a timelike rectifying plane and its position vector be a spacelike vector defined by $\alpha(t) = \dfrac{a}{\cosh(t+t_0)} u(t)$. Then the relation between the Frenet-Serret apparatus of $\alpha(t)$ and $u(t)$ is as follows:*

$$\begin{aligned}
T_\alpha &= \cosh(t+t_0)\, u' - \sinh(t+t_0)\, u, \\
N_\alpha &= u \times u', \\
B_\alpha &= \cosh(t+t_0)\, u - \sinh(t+t_0)\, u', \\
\kappa_\alpha &= \frac{1}{a}\sqrt{\kappa_u^2 - 1}\cosh^3(t+t_0), \\
\tau_\alpha &= \frac{1}{a}\sqrt{\kappa_u^2 - 1}\sinh(t+t_0)\cosh^2(t+t_0)
\end{aligned}$$

where $T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha$ are the tangent vector, normal vector, binormal vector, curvature, torsion of $\alpha(t)$, respectively, and κ_u is the curvature of $u(t)$.

4. Centrodes as Rectifying Curves

In this section, considering the Frenet vectors of the curve α in $\mathbb{E}_1^3$ and their causal characters, we give a proposition that if the curve α is the helix, then its centrode is a line segment. Then using this proposition, we examine the features, which should be provided by the curves whose centrodes are the rectifying curves except the helix. In the previous studies [1, 4], it was shown for the curves with the constant (or non-constant) curvature κ and the non-constant (or constant) torsion τ. Here, this result has been expanded.

Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed curve with Frenet frame $\{T_\alpha, N_\alpha, B_\alpha\}$ and also,

$$g(T_\alpha, T_\alpha) = \varepsilon_0, g(N_\alpha, N_\alpha) = \varepsilon_1, g(B_\alpha, B_\alpha) = \varepsilon_2. \tag{4.1}$$

Morever, let $d : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be the centrode of α with its Frenet frame $\{T_d, N_d, B_d\}$. Then we have $T'_\alpha = d \times T_\alpha$, $N'_\alpha = d \times N_\alpha$ and $B'_\alpha = d \times B_\alpha$. For all the cases of the unit speed curve α, the centrode d is written by

$$d = -\varepsilon_0\varepsilon_1\tau_\alpha T_\alpha - \varepsilon_0\varepsilon_1\kappa_\alpha B_\alpha. \tag{4.2}$$

Now, we find the Serret-Frenet apparatus of d. For this, we use $d' = -\varepsilon_0\varepsilon_1\tau'_\alpha T_\alpha - \varepsilon_0\varepsilon_1\kappa'_\alpha B_\alpha$. Using d' we obtain $\|d'\|^2 = (\tau'_\alpha)^2\varepsilon_0 + \varepsilon_2(\kappa'_\alpha)^2$, then the speed of centrode d is given by $\upsilon_d = \sqrt{\varepsilon_0(\tau'_\alpha)^2 + \varepsilon_2(\kappa'_\alpha)^2}$. Additionally,

$$T_d = -\frac{\varepsilon_0\varepsilon_1\tau'_\alpha T_\alpha}{\upsilon_d} - \frac{\varepsilon_0\varepsilon_1\kappa'_\alpha B_\alpha}{\upsilon_d}. \tag{4.3}$$

Differentiating (4.3) we find

$$T'_d = \left(-\frac{\varepsilon_0\varepsilon_1\tau'_\alpha}{\upsilon_d}\right)' T_\alpha - \frac{\varepsilon_0\varepsilon_1\tau'_\alpha}{\upsilon_d}T'_\alpha + \left(-\frac{\varepsilon_0\varepsilon_2\kappa'_\alpha}{\upsilon_d}\right)' B_\alpha - \frac{\varepsilon_0\varepsilon_2\kappa'_\alpha}{\upsilon_d}B'_\alpha.$$

The centrode d does not have to be unit speed, then we get

$$\begin{aligned}\varepsilon_1^d\kappa_d\upsilon_d N_d &= \left(-\frac{\varepsilon_0\varepsilon_1\tau'_\alpha}{\upsilon_d}\right)' T_\alpha - \frac{\varepsilon_0\tau'_\alpha}{\upsilon_d}\kappa_\alpha N_\alpha - \left(\frac{\varepsilon_0\varepsilon_1\kappa'_\alpha}{\upsilon_d}\right)' B_\alpha + \frac{\varepsilon_0\kappa'_\alpha}{\upsilon_d}\tau_\alpha N_\alpha \\ &= -\left(\frac{\varepsilon_0\varepsilon_1\tau'_\alpha}{\upsilon_d}\right)' T_\alpha - \frac{\varepsilon_0\tau'_\alpha\kappa_\alpha - \varepsilon_0\kappa'_\alpha\tau_\alpha}{\upsilon_d}N_\alpha - \left(\frac{\varepsilon_0\varepsilon_1\kappa'_\alpha}{\upsilon_d}\right)' B_\alpha. \end{aligned} \tag{4.4}$$

Using this equation we give the following proposition about helices.

Proposition 4.1. *Let $\alpha : I \to \mathbb{E}_1^3$ be a unit speed curve whose curvature κ_α and torsion τ_α satisfy $\kappa_\alpha > 0$ and $\varepsilon_0(\tau'_\alpha)^2 + \varepsilon_2(\kappa'_\alpha)^2 \neq 0$. Then α is a helix if and only if its centrode d is a line segment.*

Proof. We assume that the centrode of α is a line segment, then its curvature is zero, namely $\kappa_d = 0$. In (4.4), the coefficients of $T_\alpha, N_\alpha, B_\alpha$ are also zero since $T_\alpha, N_\alpha, B_\alpha$ vectors are linearly independent. Then, we write the following:

(i) $\left(\frac{\varepsilon_0\varepsilon_1\tau'_\alpha}{\upsilon_d}\right)' = 0$,

(ii) $\frac{\varepsilon_0\tau'_\alpha\kappa_\alpha - \varepsilon_0\kappa'_\alpha\tau_\alpha}{\upsilon_d} = 0$,

(iii) $\left(\frac{\varepsilon_0\varepsilon_1\kappa'_\alpha}{\upsilon_d}\right)' = 0$.

For (ii), from $\upsilon_d \neq 0$ there is $\varepsilon_0\tau'_\alpha\kappa_\alpha - \varepsilon_0\kappa'_\alpha\tau_\alpha = 0$. We know that $\varepsilon_0 \neq 0$. If we divide both sides of the equation to ε_0, then we find $\tau'_\alpha\kappa_\alpha - \kappa'_\alpha\tau_\alpha = 0$. Thus, we write that $\frac{\tau'_\alpha}{\kappa'_\alpha} = \frac{\tau_\alpha}{\kappa_\alpha}$ and

$$\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = \frac{\tau'_\alpha\kappa_\alpha - \tau_\alpha\kappa'_\alpha}{\kappa_\alpha^2} = 0.$$

Consequently, $\frac{\tau_\alpha}{\kappa_\alpha}$ is constant and α is a helix. Conversely, if α is a helix, then we write $\tau_\alpha = c\kappa_\alpha$, $c \neq 0$. Using (4.4),

$$\begin{aligned}\varepsilon_1^d\kappa_d\upsilon_d N_d &= -\left(\frac{\varepsilon_0\varepsilon_1 c\kappa'_\alpha}{\upsilon_d}\right)' T_\alpha - \frac{\varepsilon_0 c\kappa'_\alpha\kappa_\alpha - \varepsilon_0\kappa'_\alpha c\kappa_\alpha}{\upsilon_d}N_\alpha - \left(\frac{\varepsilon_0\varepsilon_1\kappa'_\alpha}{\upsilon_d}\right)' B_\alpha \\ &= \frac{(-\varepsilon_0\kappa''_\alpha\upsilon_d + \varepsilon_0\kappa'_\alpha\upsilon'_d)}{\upsilon_d^2}(\varepsilon_1 cT_\alpha + \varepsilon_1 B_\alpha).\end{aligned}$$

Now, we investigate the cases in Minkowski 3-space. Firstly, we obtain

$$\kappa_d^2\upsilon_d^2 g(N_d, N_d) = \left(\varepsilon_0 c^2 + \varepsilon_2\right)\left[\left(\frac{\kappa'_\alpha}{\upsilon_d}\right)'\right]^2$$

and also

$$\varepsilon_0^d = \varepsilon_0 \left(\frac{c\kappa'_\alpha}{\upsilon_d}\right)^2 + \varepsilon_2 \left(\frac{\kappa'_\alpha}{\upsilon_d}\right)^2$$

where ε_0^d is the signature of T_d, that is $g(T_d, T_d) = \varepsilon_0^d$. Here, differentiating both sides, we write

$$2\varepsilon_0 \frac{c\kappa'_\alpha}{\upsilon_d}\left(\frac{c\kappa'_\alpha}{\upsilon_d}\right)' + 2\varepsilon_2 \frac{\kappa'_\alpha}{\upsilon_d}\left(\frac{\kappa'_\alpha}{\upsilon_d}\right)' = 0.$$

Hence, it is $\varepsilon_0 c^2 + \varepsilon_2 = 0$, $\frac{\kappa'_\alpha}{\upsilon_d} = 0$ or $\left(\frac{\kappa'_\alpha}{\upsilon_d}\right)' = 0$.

(1) For $\varepsilon_0 c^2 + \varepsilon_2 = 0$, there exists $\kappa_d^2 \upsilon_d^2 g\left(N_d, N_d\right) = 0$. From $g\left(N_d, N_d\right) \neq 0$ and $\upsilon_d^2 \neq 0$, then $\kappa_d = 0$.

(2) For $\frac{\kappa'_\alpha}{\upsilon_d} = 0$, clearly $\kappa_d = 0$.

(3) For $\left(\frac{\kappa'_\alpha}{\upsilon_d}\right)'$ 0, clearly $\kappa_d = 0$.

Therefore, if α is a helix, then the centrode of α is a line segment. ■

Theorem 4.2. *Let α be a unit speed spacelike (timelike) curve in $\mathbb{E}_1^3$ with a timelike(spacelike) binormal, $\kappa_\alpha, \tau_\alpha \neq 0$ and $\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2 \neq 0$. If α is not a helix, then the centrode d of α is a rectifying curve if and only if κ_α and τ_α satisfy the equation $a\kappa_\alpha - b\tau_\alpha = c$ where a, b, c are constants and they provide the conditions $c \neq 0, a^2 - b^2 \neq 0$.*

Proof. Let $\alpha = \alpha\left(t\right)$ be a unit speed curve in $\mathbb{E}_1^3$ and $\kappa_\alpha, \tau_\alpha \neq 0$, $\varepsilon_0\left(\tau'_\alpha\right)^2 + \varepsilon_2\left(\kappa'_\alpha\right)^2 \neq 0$, namely α is not a null curve. If α is not a helix, then we use (4.4) and (4.2). So, we get

$$\varepsilon_1^d \kappa_d \upsilon_d g(N_d\left(t\right), d\left(t\right)) = \varepsilon_0 \tau_\alpha \left(\frac{\tau'_\alpha}{\upsilon_d}\right)' + \varepsilon_2 \kappa_\alpha \left(\frac{\kappa'_\alpha}{\upsilon_d}\right)'. \tag{4.5}$$

If the centrode d of α is a rectifying curve, the multiplication of the position vector field of d and N_d is zero. Since α is not a helix, d is not a line segment and $\kappa_d \neq 0$. From $\kappa_d > 0$ and $\upsilon_d \neq 0$, we write $g(N_d, d) = 0$. Then we find

$$\varepsilon_0 \tau_\alpha \left(\frac{\tau'_\alpha}{\upsilon_d}\right)' + \varepsilon_2 \kappa_\alpha \left(\frac{\kappa'_\alpha}{\upsilon_d}\right)' = 0. \tag{4.6}$$

This equation shows that, if the centrode d of α is rectifying curve, then it satisfies (4.2). Now, we use (4.6) and try to obtain better notation. Let α be a unit speed curve in $\mathbb{E}_1^3$ with a timelike binormal vector. We have

$$\tau_\alpha \left(\frac{\tau'_\alpha}{\upsilon_d}\right)' - \kappa_\alpha \left(\frac{\kappa'_\alpha}{\upsilon_d}\right)' = 0$$

and from the hypotesis of the theorem, we take $\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2 \neq 0$. Then, it is $\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2 > 0$ or $\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2 < 0$.

$Case\ 1$: We assume that $\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2 > 0$. In this situation, if τ'_α is zero, then the condition $\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2 > 0$ is not satisfied. So $\tau'_\alpha \neq 0$. Then, let $\theta_1(t)$ be a function defined by $sin^{-1}\left(\frac{\kappa'_\alpha}{\tau'_\alpha}\right)$. Using the equations

$$\sin\theta_1\left(t\right) = \frac{\kappa'_\alpha}{\tau'_\alpha}, \cos\theta_1\left(t\right) = \frac{\sqrt{\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2}}{\tau'_\alpha},$$

$$\tan\theta_1\left(t\right) = \frac{\kappa'_\alpha}{\sqrt{\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2}}, \sec\theta_1\left(t\right) = \frac{\tau'_\alpha}{\sqrt{\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2}},$$

then we get

$$\tau_\alpha \left(\sec\theta_1(t)\right)' - \kappa_\alpha \left(\tan\theta_1(t)\right)' = 0,$$
$$\theta_1'(t)\sec\theta_1(t)\left[\tau_\alpha \tan\theta_1(t) - \kappa_\alpha \sec\theta_1(t)\right] = 0.$$

For $\tau_\alpha \tan\theta_1(t) - \kappa_\alpha \sec\theta_1(t) = 0$, we find easily that $\tau_\alpha \tan\theta_1(t) = \kappa_\alpha \sec\theta_1(t)$. Then

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{\sec\theta_1(t)}{\tan\theta_1(t)} = \frac{1}{\sin\theta_1(t)} = \csc\theta_1(t) = \frac{\tau_\alpha'}{\kappa_\alpha'}.$$

So, α is a helix. But, it is contrast to the hypotesis. For $\sec\theta_1(t) = 0$, we write $\dfrac{\tau_\alpha'}{\sqrt{\left(\tau_\alpha'\right)^2 - \left(\kappa_\alpha'\right)^2}} = 0$, namely, $\tau_\alpha' = 0$ and this is contrast to the hypotesis, too. For $\theta_1'(t) = 0$, then it is clear that $\theta_1(t) = constant$. Furthermore $\sin\theta_1(t) = \dfrac{\kappa_\alpha'}{\tau_\alpha'} = c_1$ and c_1 is a constant. If we choose $\dfrac{b}{a} = c_1$, then there is $a\kappa_\alpha' - b\tau_\alpha' = 0$. From here, $a\kappa_\alpha - b\tau_\alpha = c$.

Case 2 : We assume that $\left(\tau_\alpha'\right)^2 - \left(\kappa_\alpha'\right)^2 < 0$. In this situation, $\kappa_\alpha' \neq 0$. Let $\theta_2(t)$ be a function defined by $sin^{-1}\left(\dfrac{\tau_\alpha'}{\kappa_\alpha'}\right)$. Using the equations

$$\sin\theta_2(t) = \frac{\tau_\alpha'}{\kappa_\alpha'}, \cos\theta_2(t) = \frac{\sqrt{\left(\kappa_\alpha'\right)^2 - \left(\tau_\alpha'\right)^2}}{\kappa_\alpha'},$$
$$\tan\theta_2(t) = \frac{\tau_\alpha'}{\sqrt{\left(\kappa_\alpha'\right)^2 - \left(\tau_\alpha'\right)^2}}, \sec\theta_2(t) = \frac{\kappa_\alpha'}{\sqrt{\left(\kappa_\alpha'\right)^2 - \left(\tau_\alpha'\right)^2}},$$

we get

$$\theta_2'(t)\sec\theta_2(t)\left[-\tau_\alpha \sec\theta_2(t) + \kappa_\alpha \tan\theta_2(t)\right] = 0.$$

Thus, we obtain $a\kappa_\alpha - b\tau_\alpha = c$. Conversely, if κ_α and τ_α provides the equation $a\kappa_\alpha - b\tau_\alpha = c$, then we find $\tau_\alpha\left(\dfrac{\tau_\alpha'}{\upsilon_d}\right)' - \kappa_\alpha\left(\dfrac{\kappa_\alpha'}{\upsilon_d}\right)' = 0$ from $\left(\dfrac{\kappa_\alpha'}{\upsilon_d}\right)' = 0$ and $\left(\dfrac{\tau_\alpha'}{\upsilon_d}\right)' = 0$. Also, the proof is done for timelike curve with spacelike binormal vector, similarly. ■

Theorem 4.3. *Let α be a unit speed spacelike curve in $\mathbb{E}_1^3$ with a spacelike binormal vector, $\kappa_\alpha, \tau_\alpha \neq 0$ and $\left(\tau_\alpha'\right)^2 + \left(\kappa_\alpha'\right)^2 \neq 0$. If α is not a helix, then the centrode d of α is a rectifying curve if and only if κ and τ satisfy the equation $a\kappa_\alpha - b\tau_\alpha = c$ where a, b, c are constants and they provide the conditions $c \neq 0, a^2 + b^2 \neq 0$.*

Proof. We assume that $\varepsilon_0 = 1$ and $\varepsilon_2 = 1$. Then, there exists

$$\tau_\alpha\left(\frac{\tau_\alpha'}{\upsilon_d}\right)' + \kappa_\alpha\left(\frac{\kappa_\alpha'}{\upsilon_d}\right)' = 0$$

and $\left(\tau_\alpha'\right)^2 + \left(\kappa_\alpha'\right)^2 \neq 0$. Thus, we get $\kappa_\alpha' \neq 0$ or $\tau_\alpha' \neq 0$.

Case 1 : We assume that $\kappa_\alpha' \neq 0$. Let $\gamma_1(t)$ be a function given by $\gamma_1(t) = \tan^{-1}\left(\dfrac{\tau_\alpha'}{\kappa_\alpha'}\right)$. Then, using

$$\sin\left(\gamma_1(t)\right) = \frac{\tau_\alpha'}{\sqrt{\left(\tau_\alpha'\right)^2 + \left(\kappa_\alpha'\right)^2}}, \cos\left(\gamma_1(t)\right) = \frac{\kappa_\alpha'}{\sqrt{\left(\tau_\alpha'\right)^2 + \left(\kappa_\alpha'\right)^2}},$$

we get

$$\tau_\alpha \cos(\gamma_1(t))\gamma_1'(t) - \kappa_\alpha \sin(\gamma_1(t))\gamma_1'(t) = 0.$$

Thus, we write $\gamma_1'(t) = 0$ and it is a constant. So, we obtain $a\kappa - b\tau = c$. Since $\frac{\tau_\alpha}{\kappa_\alpha}$ is non-constant, it is $c \neq 0$ and $a^2 + b^2 \neq 0$.

$Case\ 2$: We assume that $\tau_\alpha' \neq 0$. Let $\gamma_2(t)$ is a function given by $\gamma_2(t) = \tan^{-1}\left(\frac{\kappa_\alpha'}{\tau_\alpha'}\right)$. Then, $\tan(\gamma_2(t)) = \frac{\kappa_\alpha'}{\tau_\alpha'}$ and using

$$\sin(\gamma_2(t)) = \frac{\kappa_\alpha'}{\sqrt{(\tau_\alpha')^2 + (\kappa_\alpha')^2}}, \cos(\gamma_2(t)) = \frac{\tau_\alpha'}{\sqrt{(\tau_\alpha')^2 + (\kappa_\alpha')^2}},$$

we get

$$(\kappa_\alpha \cos(\gamma_2(t)) - \tau_\alpha \sin(\gamma_2(t)))\gamma_2'(t) = 0.$$

It is easy to show that $a\kappa_\alpha - b\tau_\alpha = c$. ■

Now, we find the relations between the Frenet-Serret apparatus of the centrode d, which is a rectifying curve, and the Frenet-Serret apparatus of α.

Theorem 4.4. *Let α be a unit speed spacelike curve in $\mathbb{E}_1^3$ with a timelike binormal vector and its Serret-Frenet apparatus be $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$. The centrode $d(t)$ of $\alpha(t)$ is a rectifying curve.*

(1) If $d(t)$ is a spacelike curve, then the Serret-Frenet apparatus $\{T_d, N_d, B_d, \kappa_d, \tau_d\}$ of centrode is given by

$$\begin{aligned} T_d &= \frac{-\delta}{\sqrt{1-\hat{c}^2}} T_\alpha - \frac{\delta \hat{c}}{\sqrt{1-\hat{c}^2}} B_\alpha, \\ N_d &= N_\alpha, \\ B_d &= \frac{-\delta}{\sqrt{1-\hat{c}^2}} B_\alpha - \frac{\delta \hat{c}}{\sqrt{1-\hat{c}^2}} T_\alpha, \\ \kappa_d &= \frac{\hat{c}\tau_\alpha - \kappa_\alpha}{\tau_\alpha'(1-\hat{c}^2)}, \\ \tau_d &= \frac{\hat{c}\kappa_\alpha - \tau_\alpha}{\tau_\alpha'(1-\hat{c}^2)} \end{aligned}$$

where δ is the signature of τ_α', $\hat{c} = \frac{b}{a}$ and a, b is defined as in Theorem 4.2.

(2) If $d(t)$ is a timelike curve, then the Serret-Frenet apparatus $\{T_d, N_d, B_d, \kappa_d, \tau_d\}$ of centrode is given by

$$\begin{aligned} T_d &= \frac{-\delta\bar{c}}{\sqrt{1-\bar{c}^2}} T_\alpha - \frac{\delta}{\sqrt{1-\bar{c}^2}} B_\alpha, \\ N_d &= N_\alpha, \\ B_d &= \frac{-\delta\bar{c}}{\sqrt{1-\bar{c}^2}} B_\alpha - \frac{\delta}{\sqrt{1-\bar{c}c^2}} T_\alpha, \\ \kappa_d &= \frac{\tau_\alpha - \bar{c}\kappa_\alpha}{\kappa_\alpha'(1-\bar{c}^2)}, \\ \tau_d &= \frac{\kappa_\alpha - \bar{c}\tau_\alpha}{\kappa_\alpha'(1-\bar{c}^2)} \end{aligned}$$

where δ is the signature of κ_α', $\bar{c} = \frac{a}{b}$ and a, b is defined as in Theorem 4.2.

Proof. From $a\kappa'_\alpha - b\tau'_\alpha = 0$, since $d(t)$ is a spacelike rectifying curve and $\tau'_\alpha \neq 0$, we write $\kappa'_\alpha = \frac{b}{a}\tau'_\alpha$. For υ_d, we write

$$\upsilon_d = \sqrt{\left|\left(\tau'_\alpha\right)^2 - \left(\kappa'_\alpha\right)^2\right|} = |\tau'_\alpha|\sqrt{1-\hat{c}^2}$$

where $\hat{c} = \frac{b}{a}$. For the Frenet-Serret apparatus, we give the following:

$$\begin{aligned} T_d &= -\frac{\varepsilon_0\varepsilon_1\tau'_\alpha}{\upsilon_d}T_\alpha - \frac{\varepsilon_0\varepsilon_1\kappa'_\alpha}{\upsilon_d}B_\alpha = -\frac{\tau'_\alpha}{|\tau'_\alpha|\sqrt{1-\hat{c}^2}}T_\alpha - \frac{\hat{c}\tau'_\alpha}{|\tau'_\alpha|\sqrt{1-\hat{c}^2}}B_\alpha, \\ \varepsilon_1^d\kappa_d\upsilon_d N_d &= \left(-\frac{\tau'_\alpha}{\upsilon_d}\right)'T_\alpha - \frac{\tau'_\alpha\kappa_\alpha - \kappa'_\alpha\tau_\alpha}{\upsilon_d}N_\alpha - \left(\frac{\kappa'_\alpha}{\upsilon_d}\right)'B_\alpha. \end{aligned}$$

Then we get

$$\begin{aligned} N_d &= N_\alpha, \\ \kappa_d &= \frac{-\tau'_\alpha\kappa_\alpha + \kappa'_\alpha\tau_\alpha}{\left(|\tau'_\alpha|\sqrt{1-\hat{c}^2}\right)^2} = \frac{\hat{c}\tau_\alpha - \kappa_\alpha}{\tau'_\alpha\left(1-\hat{c}^2\right)}. \end{aligned}$$

Finally, we see easily that

$$\begin{aligned} B_d = T_d \times N_d &= \frac{-\delta}{\sqrt{1-\hat{c}^2}}B_\alpha - \frac{\hat{c}\delta}{\sqrt{1-\hat{c}^2}}T_\alpha, \\ -\tau_d\varepsilon_1^d\upsilon_d N_d &= \frac{-\delta}{\sqrt{1-\hat{c}^2}}\left(-\varepsilon_1\tau_\alpha N_\alpha\right) - \frac{\hat{c}\delta}{\sqrt{1-\hat{c}^2}}\varepsilon_1\kappa_\alpha N_\alpha, \end{aligned}$$

then we have $\tau_d = \frac{\hat{c}\kappa_\alpha - \tau_\alpha}{\tau'_\alpha\left(1-\hat{c}^2\right)}$. ■

Theorem 4.5. *Let α be a unit speed timelike curve in $\mathbb{E}^3$ with a spacelike binormal vector. The centrode $d(t)$ of $\alpha(t)$ is a rectifying curve.*

(1) If $d(t)$ is a spacelike curve, then the Serret-Frenet apparatus $\{T_d, N_d, B_d, \kappa_d, \tau_d\}$ of centrode is given by

$$\begin{aligned} T_d &= \frac{\delta\overline{c}}{\sqrt{1-\overline{c}^2}}T_\alpha + \frac{\delta}{\sqrt{1-\overline{c}^2}}B_\alpha, \\ N_d &= N_\alpha, \\ B_d &= \frac{\delta\overline{c}}{\sqrt{1-\overline{c}^2}}B_\alpha + \frac{\delta}{\sqrt{1-\overline{c}^2}}T_\alpha, \\ \kappa_d &= \frac{\overline{c}\kappa_\alpha - \tau_\alpha}{\kappa'_\alpha\left(1-\overline{c}^2\right)}, \\ \tau_d &= \frac{\overline{c}\tau_\alpha - \kappa_\alpha}{\kappa'_\alpha\left(1-\overline{c}^2\right)} \end{aligned}$$

where δ is the signature of κ'_α, $\overline{c} = \frac{a}{b}$ and a, b is defined as in Theorem 4.2.

(2) If $d(t)$ is a timelike curve, then the Serret-Frenet apparatus $\{T_d, N_d, B_d, \kappa_d, \tau_d\}$ of centrode is given by

$$
\begin{aligned}
T_d &= \frac{\delta}{\sqrt{1-\hat{c}^2}}T_\alpha + \frac{\delta\hat{c}}{\sqrt{1-\hat{c}^2}}B_\alpha,\\
N_d &= N_\alpha,\\
B_d &= \frac{\delta}{\sqrt{1-\hat{c}^2}}B_\alpha + \frac{\delta\hat{c}}{\sqrt{1-\hat{c}^2}}T_\alpha,\\
\kappa_d &= \frac{\kappa_\alpha - \hat{c}\tau_\alpha}{\tau'_\alpha\left(1-\hat{c}^2\right)},\\
\tau_d &= \frac{\tau_\alpha - \hat{c}\kappa_\alpha}{\tau'_\alpha\left(1-\hat{c}^2\right)}
\end{aligned}
$$

where δ is the signature of τ'_α, $\hat{c} = \dfrac{b}{a}$ and a, b is defined as in Theorem 4.2.

Theorem 4.6. *Let α be a unit speed spacelike curve in $\mathbb{E}_1^3$ with a spacelike binormal and the centrode $d(t)$ of $\alpha(t)$ be a rectifying curve.*

(1) If $\tau'_\alpha \neq 0$, then the Frenet-Serret apparatus of centrode is given by

$$
\begin{aligned}
T_d &= \frac{\delta}{\sqrt{1+\hat{c}^2}}T_\alpha + \frac{\delta\hat{c}}{\sqrt{1+\hat{c}^2}}B_\alpha,\\
N_d &= N_\alpha,\\
B_d &= \frac{\delta}{\sqrt{1+\hat{c}^2}}B_\alpha - \frac{\delta\hat{c}}{\sqrt{1+\hat{c}^2}}T_\alpha,\\
\kappa_d &= \frac{\kappa_\alpha - \hat{c}\tau_\alpha}{\tau'_\alpha\left(1+\hat{c}^2\right)},\\
\tau_d &= \frac{\tau_\alpha + \kappa_\alpha\hat{c}}{\tau'_\alpha\left(1+\hat{c}^2\right)}
\end{aligned}
$$

where δ is the signature of τ'_α, $\hat{c} = \dfrac{b}{a}$ and a, b is defined as in Theorem 4.3.

(2) If $\kappa'_\alpha \neq 0$, then the Frenet-Serret apparatus of centrode is given by

$$
\begin{aligned}
T_d &= \frac{\delta\overline{c}}{\sqrt{1+\overline{c}^2}}T_\alpha + \frac{\delta}{\sqrt{1+\overline{c}^2}}B_\alpha,\\
N_d &= N_\alpha,\\
B_d &= \frac{\delta\overline{c}}{\sqrt{1+\overline{c}^2}}B_\alpha - \frac{\delta}{\sqrt{1+\overline{c}^2}}T_\alpha,\\
\kappa_d &= \frac{\overline{c}\kappa_\alpha - \tau_\alpha}{\kappa'_\alpha\left(1+\overline{c}^2\right)},\\
\tau_d &= \frac{\overline{c}\tau_\alpha + \kappa_\alpha}{\kappa'_\alpha\left(1+\overline{c}^2\right)}
\end{aligned}
$$

where δ is the signature of τ'_α, $\overline{c} = \dfrac{a}{b}$ and a, b is defined as in Theorem 4.3.

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Disjunctive total domination in some tree networks

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Abstract. Networking has been an essential field of multidisciplinary study, including computational theory, mathematics, social sciences, computer science, and other theoretical and applied sciences. The vulnerability determines the network's resistance to interruption of information flow after the breakdown of particular stations or transmission connections. Recently, new vulnerability parameter namely the disjunctive total domination number has been defined by Henning and Naicker [14]. This measure finds the critical vertices with an important position in the graph. In this context, we consider and compute exact formulae for the disjunctive total domination number in some tree networks.

AMS Subject Classifications: 05C69, 68M10, 68R10.

Keywords: Network Design and Communication, Complex Networks, Disjunctive total domination number, Trees.

Contents

1. Introduction

The connected graph can model the network, the vertex in the graph represents a network node, and the edge represents a contact connection between the two nodes [7, 16]. There are many parameters in graph theory for network analysis and to determine some properties of the network. Graph theory techniques facilitate representation and analysis during a vulnerability assessment of complex networks. The theory is based on a set of measurements that evaluate networks and include graph vulnerability parameters. The proposed solutions for the network's vulnerability were rooted in the graph theoretical principles, especially the concepts of domination [19].

Theory of domination is one of the most important branches of graph vulnerability, which has wide application in network designings. It has a wide variety of uses in many areas, such as computer science, communication networks, transportation networks, biological and social networks, operations research, chemistry, economics, engineering, and applied mathematics; the principle of domination has recently become the center of graph theory research activity. This is largely due to a variety of new parameters that can be developed from the basic definition of domination [10, 14, 15]. Disjunctive total domination is the new domination parameter defined recently. Henning and Naicker [14] defined the disjunctive total domination as a relaxation of total domination.

A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G and is denoted

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by $\gamma(G)$ [3, 10]. A *total dominating set*, abbreviated a TD-set, of a graph G, with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G [3, 15]. Let $S \subseteq V(G)$. The set S can be the *disjunctive total dominating set* of the graph G if and only if it satisfies any of the following properties. For every vertex $v \in V(G)$;

(i) v is adjacent to a vertex of S,

(ii) at least two vertices at a distance of 2 from the vertex v must be in the set S.

The *disjunctive total dominating set* of the graph G is briefly called DDT-set. The *disjunctive total domination number* of G is the minimum cardinality of a DTD-set of G and denoted by $\gamma_t^d(G)$. A DTD-set of cardinality $\gamma_t^d(G)$ is called a $\gamma_t^d(G)$-set. Clearly, every TD-set is a DTD-set, furthermore the result $\gamma_t^d(G) \leq \gamma_t(G)$ is obtained in [12–14]. This parameter is studied on grids, trees, permutation graphs, claw-free graphs, shadow distance graph of some special graphs and it is applied on some graph modifications such as bondage and subdivision [1, 2, 12–14].

We consider the disjunctive total domination number as a metric for network vulnerability. In this model, we find the critical vertices with an important position in the graph. Since disjunctive total domination number is considered to be a reasonable measure for the vulnerability of graphs, it is of particular interest to evaluate the disjunctive total domination number of different classes of graphs. Suppose one can break a more complex network into smaller networks, then under some conditions. In that case, the optimization problem's solutions on the smaller networks can be combined to solve the optimization problem on the larger network. Thus, calculation of the disjunctive total domination number for simple graph types is important.

For notation and graph theory terminology, we in general follow [10, 19]. Specifically, let $G = (V(G), E(G))$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The set of all adjacent vertices to vertex $v \in V(G)$ in G is called neighborhood and denoted by $N_G(v)$ or $N(v)$. The close neighborhood of this vertex is defined as $N_G(v) \cup \{v\}$ and denoted by $N_G[v]$ or $N[v]$. The other basic parameter for graphs is the degree of vertex $v \in V(G)$, which is defined as the number of vertices in $N_G(v)$ and denoted by $deg(v)$. Assume that the vertices u and v belong to the graph G. For these vertices, $d(u, v)$ is defined as the distance of the shortest path between these vertices.Furthermore, $diam(G)$ is defined as the *diameter* of G, and it is the highest distance value within the vertices of G. $\Delta(G) = max\{deg(v)|v \in V(G)\}$ and $\delta(G) = min\{deg(v)|v \in V(G)\}$ represent the maximum and minimum degree, respectively. The vertex with $deg(v) = 1$ is said to be a pendant vertex or leaf vertex. The vertex adjacent to the pendant vertex is called the support vertex.

Now, we make use of the following known theorems in our results.

Theorem 1.1. *[14] For $n \geq 3$, $\gamma_t^d(C_n) = \frac{2n}{5}$ if $n \equiv 0(mod5)$; and $\gamma_t^d(C_n) = \lceil \frac{2(n+1)}{5} \rceil$, otherwise.*

Theorem 1.2. *[14] For $n \geq 3$, $\gamma_t^d(P_n) = \lceil \frac{2(n+1)}{5} \rceil + 1$ if $n \equiv 1(mod5)$; and $\gamma_t^d(P_n) = \lceil \frac{2(n+1)}{5} \rceil$, otherwise.*

Lemma 1.3. *[14] If v is a support vertex in a graph G with exactly one neighbor w that is not a leaf, then there is a $\gamma_t^d(G)$-set that contains v. Further if $deg(w) = 2$, then there is a $\gamma_t^d(G)$-set that contains both v and w.*

2. Disjunctive Total Domination Numbers of Some Trees

In this section, the distinctive total dominance numbers of certain tree-type networks such as the double comet graph, the double star graph, the comet graph, the generalized caterpillars, the comb graph, the thorn graph P_n^*, the binomial tree and the complete k-ary tree are computed and exact formulae are presented.

Definition 2.1. *[8] The double star graph $S(x, y)$, where $x, y \geq 0$, is the graph consisting of the union of two star graphs $K_{1,x}$ and $K_{1,y}$ together with an edge joining their centers.*

Theorem 2.2. *If $G \cong S(x, y)$ of order $x + y$, where $x, y \geq 0$, then, $\gamma_t^d(G) = 2$.*

Proof. Note that $V(G) = V(K_{1,x}) \cup V(K_{1,y})$. Furthermore, let u_1 and u_2 be the central vertices of $K_{1,x}$ and $K_{1,y}$, respectively. It is easily seen that $deg(x_i) = deg(y_i) = 1$ for every vertices x_i and y_i, where $x_i \in V(K_{1,x})$ and $y_i \in V(K_{1,y})$, $deg(u_1) = x + 1$ and $deg(u_2) = y + 1$. If a DTD-set of G is considered S, then taking u_1 and u_2 to the set S yields $\gamma_t^d(G) \leq 2$. Furthermore, we have $\gamma_t^d(G) \geq 2$ for any graph G by the definition of disjunctive total domination number. So, $\gamma_t^d(G) \geq 2$. As a consequence, by combining the lower and upper bounds, we obtain $\gamma_t^d(G) = 2$. ■

Definition 2.3. *[4] The comet graph $C(t, r)$ is the graph obtained by identifying one end of the path P_t with the center of the star graph $K_{1,r}$. This graph is illustrated in Figure 1.*

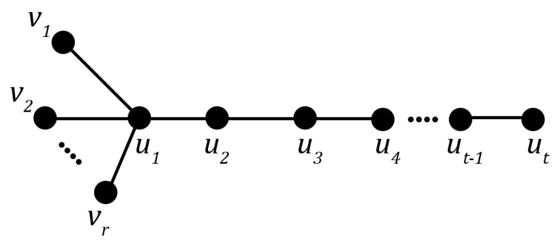

Figure 1: The comet graph $C(t, r)$.

Theorem 2.4. *If $G \cong C(t, r)$ of order $t + r$, then* $\gamma_t^d(G) = \begin{cases} \left\lceil \frac{2(t+1)}{5} \right\rceil + 1 , \text{ if } t \equiv 0, 1, 4(mod5); \\ \left\lceil \frac{2(t+1)}{5} \right\rceil , \text{ otherwise.} \end{cases}$

Proof. Note that $V(G) = V(P_{t-1}) \cup V(K_{1,r})$. Furthermore, $V(P_{t-1}) = \{u_2, u_3, ..., u_t\}$ and $V(K_{1,r}) = \{u_1, v_1, ..., v_r\}$, where u_1 is the center vertex. Suppose S is a DTD-set in G. By Lemma 1.3, u_1 must be in S. Thus, all vertices v_i and u_2 are disjunctively totally dominated by the vertex u_1. The disjunctive total undominated vertices by S are the vertices of the path graph with $(t-1)$ vertices. So, the rest of the proof has to be made similar to the proof of Theorem 1.2. In this case, if $t \equiv 0, 1, 4(mod5)$, then $\gamma_t^d(G) = |S| = \lceil 2(t+1)/5 \rceil + 1$, while if $t \equiv 2, 3(mod5)$,then $\gamma_t^d(G) = |S| = \lceil 2(t + 1)/5 \rceil$ are obtained. ■

Definition 2.5. *[5] The graph obtained by adding x and y vertices, which are pendant, to the end vertices of the path graph with $n - x - y$ vertices is called double comet $DC(n, x, y)$. For $x, y \geq 1$ and $n \geq x + y + 2$ the double comet $DC(n, x, y)$ is one of the tree graphs. $DC(n, x, y)$ is a graph with n vertices, $x + y$ of which is leaves. This graph is illustrated in Figure 2.*

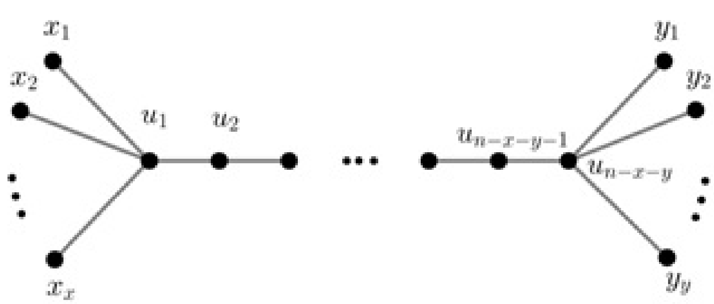

Figure 2: The double comet graph $DC(n, x, y)$.

Theorem 2.6. *If $G \cong DC(n,x,y)$ with $x,y \geq 2$ and $n \geq x+y+3$, then*

$$\gamma_t^d(G) = \begin{cases} \left\lceil \frac{2(n-x-y+1)}{5} \right\rceil + 2 \text{ , if } (n-x-y) \equiv 4(mod5); \\ \left\lceil \frac{2(n-x-y+1)}{5} \right\rceil \text{ , if } (n-x-y) \equiv 2(mod5); \\ \left\lceil \frac{2(n-x-y+1)}{5} \right\rceil + 1 \text{ , otherwise.} \end{cases}$$

Proof. Note that $V(G) = V(P_{n-x-y-2}) \cup V(K_{1,x}) \cup V(K_{1,y})$ in which $V(P_{n-x-y-2}) = \{u_2, u_3, ..., u_{n-x-y-1}\}$, $V(K_{1,x}) = \{u_1, x_1, x_2..., x_x\}$ and $V(K_{1,y}) = \{u_{n-x-y}, y_1, y_2..., y_y\}$. Suppose S is a DTD-set in G. By Lemma 1.3, the vertices u_1 and u_{n-x-y} must be taken to the set S. Thus the vertices not disjunctively totally dominated by the set S form the path graph. As in the proof of Theorem 1.2, the construction of the set S is continued. In this case, if $(n-x-y) \equiv 4(mod5)$, then $\gamma_t^d(G) = |S| = \lceil (2(n-x-y+1)/5) \rceil + 2$; if $(n-x-y) \equiv 2(mod5)$, then $\gamma_t^d(G) = |S| = \lceil (2(n-x-y+1)/5) \rceil$ and otherwise $\gamma_t^d(G) = |S| = \lceil (2(n-x-y+1)/5) \rceil + 1$ are obtained. ■

Definition 2.7. *[17] The graph obtained by joining a pendant edge at each vertex of a path P_n is called a comb graph and is denoted by $P_n \Box K_1$. The graph $P_5 \Box K_1$ is illustrated in Figure 3.*

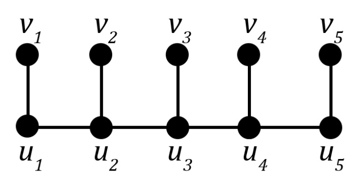

Figure 3: The comb graph $P_5 \Box K_1$.

Theorem 2.8. *If $G \cong P_n \Box K_1$ of order $2n$, then $\gamma_t^d(G) = 4 + \lfloor (n-4)/2 \rfloor$.*

Proof. Note that $V(G) = \{u_i, v_i \mid 1 \leq i \leq n\}$ and $E(G) = \{u_i v_i \mid 1 \leq i \leq n\} \cup \{u_i u_{i+1} \mid 1 \leq i \leq n-1\}$. It is obvious that $|V(G)| = 2n$, $|E(G)| = 2n-1$, $deg(v_i) = 1$ where $i \in \{1, ..., n\}$, $deg(u_1) = deg(u_n) = 2$ and $deg(u_i) = 3$ where $i \in \{2, ..., n-1\}$. We set the upper limits to the disjunctive total domination number of G, first. Suppose D is a DTD-set in G. According to degree of vertices of G, some u_i-vertices $(i \in \{2, ..., n-1\})$ must be taken to the set D. To disjunctively totally dominate the vertices v_1 and v_n, it must be $\{u_1, u_n\} \subseteq D$. Since $N_G(u_1) = \{v_1, u_2\}$ and $N_G(u_n) = \{v_n, u_{n-1}\}$, $\{u_2, u_{n-1}\} \subset D$ should be to disjunctively totally dominate the vertices u_1 and u_n. Thus, the set D is as follows:

$$D = \bigcup_{i=0}^{\lfloor \frac{n-4}{2} \rfloor + 1} \{u_{2i+4}\} \cup \{u_1, u_2, u_{n-1}, u_n\}.$$

Clearly, the set D is a DTD-set for every $n \geq 5$. Furthermore, we get $|D| = 4 + \lfloor (n-4)/2 \rfloor$, also is an upper bound. Thus, $\gamma_t^d(G) \leq 4 + \lfloor (n-4)/2 \rfloor$ is obtained.
To prove the inverse of equality, let the set T be a $\gamma_t^d(G)$-set of G. Assume that the two vertices are adjacent in T. Furthermore, the set S is as follows:

$$S = \bigcup_{i=0}^{\lfloor \frac{n}{3} \rfloor - 1} \{u_{3i+1}, u_{3i+2}\}$$

, where $S \subseteq T$. If $n \equiv 1(mod 3)$, we have $T = S \cup \{v_{n-1}, v_n\}$; if $n \equiv 0(mod 3)$, we have $T = S \cup \{v_n\}$ and if $n \equiv 2(mod 3)$, we have $T = S$. Thus, we obtain $|T| = 2\lfloor n/3 \rfloor + 2$ for $n \equiv 1(mod 3)$, $|T| = 2\lfloor n/3 \rfloor + 1$ for $n \equiv 0(mod 3)$ and $|T| = 2\lfloor n/3 \rfloor$ for $n \equiv 2(mod 3)$. These results contradict the previous upper bound for $n \geq 7$.

Furthermore, we get $\{u_1, u_2, u_{n-1}, u_n\} \subset T$. However, apart from these vertices, no two vertices in T should be adjacent to each other. If the distance between the two vertices is at least three, all vertices in G cannot be disjunctively totally dominated. So, the distance between two vertices must be exactly 2. Thus, it is easy to see that $\gamma_t^d(G) \geq 4 + \lfloor (n-4)/2 \rfloor$, also we have $\gamma_t^d(G) = 4 + \lfloor (n-4)/2 \rfloor$.

■

Corollary 2.9. *If $G \cong P_n \square K_1$ of order $2n$, then $\gamma_t^d(G) = 4 + \gamma(P_{n-4})$.*

Definition 2.10. *[9] Let $p_1, p_2, \ldots, p_n$ be non-negative integers and the graph G be such a graph, where $|V(G)| = n$. The thorn graph of the graph G with parameters $p_1, p_2, \ldots, p_n$ is obtained by attaching p_i new vertices of degree one to the vertex u_i of the graph G, where $i = \overline{1, n}$. The thorn graph of the graph G will be denoted by G^* or if the respective parameters need to be specified, by $G^*(p_1, p_2, \ldots, p_n)$. The graph $P_7^*(2, 1, 1, 3, 2, 1, 4)$ is illustrated in Figure 4.*

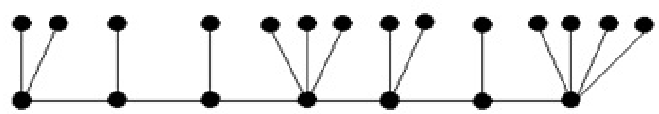

Figure 4: The thorn graph $P_7^*(2, 1, 1, 3, 2, 1, 4)$.

Theorem 2.11. *If $G \cong P_n^*$ is a thorn graph of P_n with $p_i \geq 2$, then $\gamma_t^d(G) = \gamma_t^d(P_n \square K_1)$.*

Proof. The proof is quite close to that of Theorem 2.8, so we omit it.

■

Definition 2.12. *[18] $C_{(t,0)}P_n$ is a generalized Caterpillar obtained from the path graph P_n by attaching t vertices of degree one to each vertex of degree two of P_n. The tree $C_{(t,1)}P_n$ is a generalized Caterpillar obtained from the path graph P_n by attaching m vertices of degree two to each vertex of degree two of P_n. The graph $C_{(3,0)}P_7$ and $C_{(3,1)}P_7$ are illustrated in Figure 5.*

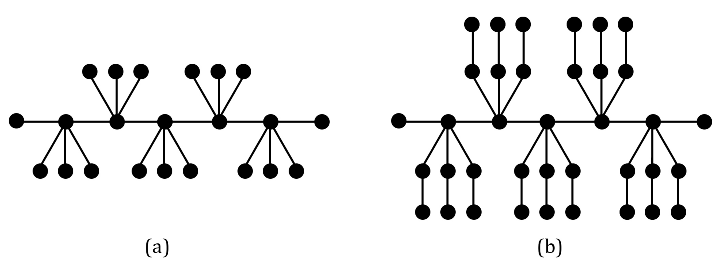

Figure 5: **(a)** The graph $C_{(3,0)}P_7$ and **(b)** the graph $C_{(3,1)}P_7$.

Theorem 2.13. *Let $G \cong C_{(t,0)}P_n$ be generalized caterpillar graph with $(n + t(n-2))$-vertices. Then, for $t \geq 2$ and $p_i = t$, $\gamma_t^d(G) = \gamma_t^d(P_{n-2}^*)$.*

Proof. The proof is quite close to that of Theorem 2.8, so we omit it. ■

Theorem 2.14. *Let $G \cong C_{(t,1)}P_n$ be generalized caterpillar graph with $(n+t(n-2))$-vertices. Then, for $t \geq 3$, $\gamma_t^d(G) = t(n-2)$.*

Proof. The graph G has $(n+t(n-2))$-vertices. Let $V(G) = V_1 \cup V_2 \cup V_3$, where $V_1 = \{u_i \in V(P_n) \mid 1 \leq i \leq n\}$, $V_2 = \{v_i \in (V(G) - V(P_n)) \mid deg(v_i) = 2$ and $1 \leq i \leq t(n-2)\}$ and $V_3 = \{w_i \in (V(G) - V(P_n)) \mid deg(w_i) = 1$ and $1 \leq i \leq t(n-2)\}$. Clearly, we have $deg(u_1) = deg(u_n) = 1$ and $deg(u_i) = 2$ for the vertices of V_1, where $i \in \{2, ..., n-1\}$. Suppose S is a $\gamma_t^d(G)$-set of G. Since $deg(w_i) = 1$ for each vertex $w_i \in V_3$, the all vertices of V_2 must be taken to the set S. Therefore, each vertex of V_1 is disjunctively totally dominated by the set S. Thus, all vertices in G are disjunctively totally dominated by S. It is easily seen that the set S is unique and there is no other set of $\gamma_t^d(G)$-set. Note that $|V_2| = t(n-2)$. Then, $|S| = t(n-2)$. Hence, we get $\gamma_t^d(G) = t(n-2)$. Thus, the proof holds. ■

Definition 2.15. *[6] The binomial tree B_n with root R is the tree defines as follows:*

i. *If $n = 0$, then $B_n = B_0 = R$, i.e., the binomial tree of order zero consists of a single root R.*

ii. *If $n > 0$, then $B_n = R, B_0, B_1, \ldots, B_{n-1}$, i.e., the binomial tree of order $n > 0$ comprises the root R and n binomial subtrees $B_0, B_1, \ldots, B_{n-1}$.*

In Figures 6 and 7, the binomial trees B_4 and B_5 are illustrated.

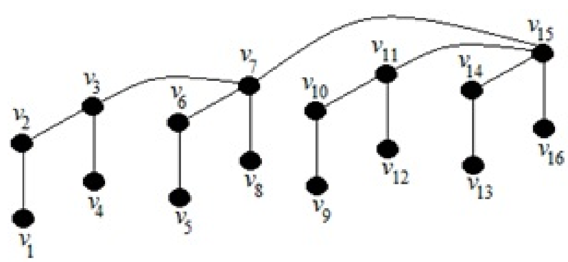

Figure 6: The binomial tree B_4.

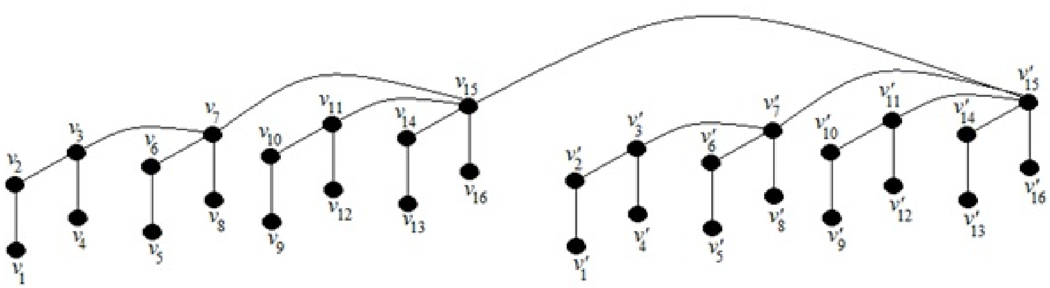

Figure 7: The binomial tree B_5.

Theorem 2.16. *If $G \cong B_n$ of order 2^n with $n \geq 5$, then $\gamma_t^d(G) = 7(2^{n-4})$.*

Proof. The binomial tree B_n has 2^n-vertices. It is clear that $\gamma_t^d(B_0) = 1$, $\gamma_t^d(B_1) = \gamma_t^d(B_2) = 2$, and $\gamma_t^d(B_3) = 4$ for $n \leq 3$. Let $n = 4$, and let S_1 be a $\gamma_t^d(G)$-set of B_4. It is easily seen that $S_1 = \{v_2, v_3, v_6, v_{10}, v_{11}, v_{14}, v_{15}\}$ (see Figure 6). Then, we get $\gamma_t^d(B_4) = 7$.

Suppose S is a $\gamma_t^d(G)$-set of B_5. Since B_5 has two copies of B_4, the vertices $\{v_2, v_3, v_6, v_{10}, v_{11}, v_{14}, v_{15}\}$ and $\{v'_2, v'_3, v'_6, v'_{10}, v'_{11}, v'_{14}, v'_{15}\}$ in the first and second copies of B_4, respectively, must be taken to the set S

(see Figure 7). Thus, we have $S = \{v_2, v_3, v_6, v_{10}, v_{11}, v_{14}, v_{15}, v'_2, v'_3, v'_6, v'_{10}, v'_{11}, v'_{14}, v'_{15}\}$. Then, we obtain $\gamma_t^d(B_5) = 2\gamma_t^d(B_4) = 14$. With the same thought, $\gamma_t^d(B_6) = 2\gamma_t^d(B_5) = 28$ and $\gamma_t^d(B_7) = 2\gamma_t^d(B_6) = 56$ are obtained. If this continue for $n \geq 5$, we get the following recurrence formula:

$$\gamma_t^d(B_n) = 2\gamma_t^d(B_{n-1}) \text{ for } n \geq 5. \quad \textbf{(1)}$$

From this formula, we have:

$$\gamma_t^d(B_n) = 2\gamma_t^d(B_{n-1}) = 2(2\gamma_t^d(B_{n-2})) = 2^2\gamma_t^d(B_{n-1}) = ... = 2^{n-4}\gamma_t^d(B_4).$$

Furthermore, we obtain

$$\gamma_t^d(B_n) = 2^i\gamma_t^d(B_{n-i}), i \in \{1, 2, n-1\}. \quad \textbf{(2)}$$

This equality can be seen by the induction method.
Let $i = 1$.
Thus, we have $\gamma_t^d(B_n) = 2\gamma_t^d(B_{n-1})$, also is clear by Eq. **(1)**. We prove this statement with induction on i. When $i = 1$, we have $\gamma_t^d(B_n) = 2^i\gamma_t^d(B_{n-i})$ and via the Eq. **(1)**, this is valid. We suppose that the result is true for $i = k$ and prove it for $i = k+1$. By induction hypothesis and Eq. **(1)**, we get

$$\gamma_t^d(B_n) = 2^k\gamma_t^d(B_{n-k}) = 2^k(2\gamma_t^d(B_{n-k-1})) = 2^{k+1}\gamma_t^d(B_{n-k-1}).$$

This means the claim is valid where $i = k+1$. Hence, we get
$\gamma_t^d(B_n) = 2^i\gamma_t^d(B_{n-i})$, where $1 \leq i \leq n-4$,
$\gamma_t^d(B_n) = 7$.
Since the first case obtained when $i = n-4$ is $n = 4$, the following is obtained from Eq. **(2)**.
$\gamma_t^d(B_n) = 2^{n-4}(B_{n-(n-4)}) = 2^{n-4}(\gamma_t^d(B_4)) = 7(2^{n-4})$.

■

Definition 2.17. *[6] The complete k-ary tree of height h, T_h^k , is a rooted tree with each leaf having the same depth and each vertex except the leaves in degree k. T_h^k in which every non-leaf vertex has exactly k-children and the distance from the root to each leaf is exactly h. The complete k-ary tree for $k \geq 2$ has $\dfrac{k^{h+1}-1}{k-1}$ vertices and $\dfrac{k^{h+1}-1}{k-1} - 1 = \dfrac{k^{h+1}-k}{k-1}$ edges. The complete binary tree is the complete k-ary tree with $k = 2$. Figure 8 shows an example of a complete k-ary tree T_4^2 ($k = 2$ and $h = 4$). In Figure 8, the value of L expresses the depth of each vertex.*

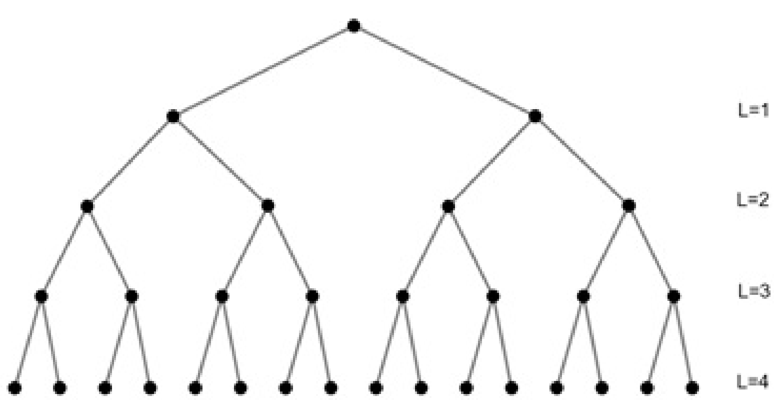

Figure 8: The complete 2-ary tree T_4^2 ($k = 2$ and $h = 4$).

Theorem 2.18. *If $G \cong T_h^k$ of order $\dfrac{k^{h+1}-1}{k-1}$, where $h \geq 4$ and $k \geq 3$, then*

$$\gamma_t^d(G) = \begin{cases} \left(\displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} k^{4i+3}\right) + 1 \ , h \equiv 0(mod4); \\ \left(\displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} k^{4i+4}\right) + 2 \ , h \equiv 1(mod4); \\ \left(\displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} k^{4i+1}\right) \qquad , h \equiv 2(mod4); \\ \left(\displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} k^{4i+2}\right) \qquad , h \equiv 3(mod4). \end{cases}$$

Proof. By the definition of complete k-ary tree T_h^k, we know that there are $k^0, k^1, ..., k^h$ vertices in the levels $0^{th}, 1^{th}, ..., h^{th}$, respectively. Suppose S is a DTD-set in T_h^k for $h \geq 4$ and $k \geq 3$. We know that the vertices that in the level h^{th} are called leaf vertices. Since the degree of each leaf vertex is 1, the set S must include support vertices in the level $(h-1)^{th}$. Furthermore, the distance between each vertex in S and at least two vertices in S is two. Thus, the vertices which are in the levels h^{th}, $(h-2)^{th}$ and $(h-3)^{th}$ are disjunctively totally dominated by the set S. Then, the vertices that in the level $(h-5)^{th}$ must be added to the set S. It is easy to see that distance of the any two vertices which are in distinct levels is four. With the same thought, the set will be occurred. But, we have four cases according to h.

Case 1. $h \equiv 0(mod4)$.
Let $L = \displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} \{4i+3\}$, where the elements of L is the levels of tree T_h^k. Let the set S includes the vertices which are in the levels in L. Thus, we get $|S| = \displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} k^{4i+3}$. But the root vertex of T_h^k is not disjunctively totally dominated by S. If any vertex in the first level is added to S, then root vertex is disjunctively totally dominated. So, $\gamma_t^d(T_h^k) = |S| = \left(\displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} k^{4i+3}\right) + 1$ is obtained.

Case 2. $h \equiv 1(mod4)$.
Let $L = \displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} \{4i+4\}$, where the elements of L is the levels of tree T_h^k. Let the set S includes the vertices which are in the levels in L. Thus, we get $|S| = \displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} k^{4i+4}$. But the root vertex and the vertices which are in first level of T_h^k are not disjunctively totally dominated by S. If any vertex in the first level and the root vertex are added to S, then all vertices of T_h^k are disjunctively totally dominated by S. So, $\gamma_t^d(T_h^k) = |S| = \left(\displaystyle\sum_{i=0}^{\lceil h/4\rceil -1} k^{4i+4}\right) + 2$ is obtained.

Case 3. $h \equiv 2(mod4)$.

Let $L = \sum_{i=0}^{\lceil h/4 \rceil -1} \{4i+1\}$, where the elements of L is the levels of tree T_h^k. Let the set S includes the vertices which are in the levels in L. Thus, we get $|S| = \sum_{i=0}^{\lceil h/4 \rceil -1} k^{4i+1}$. Clearly, all vertices of T_h^k are disjunctively totally dominated by S. So, $\gamma_t^d(T_h^k) = |S| = \sum_{i=0}^{\lceil h/4 \rceil -1} k^{4i+1}$ is obtained.

Case 4. $h \equiv 3(mod4)$.

Let $L = \sum_{i=0}^{\lceil h/4 \rceil -1} \{4i+2\}$, where the elements of L is the levels of tree T_h^k. This case is similar to the *Case* 3. So, $\gamma_t^d(T_h^k) = |S| = \sum_{i=0}^{\lceil h/4 \rceil -1} k^{4i+2}$ is obtained.

Thus, the proof of theorem is completed by the *Cases* 1, 2, 3 and 4.

■

Theorem 2.19. *If $G \cong T_h^k$ of order $\frac{k^{h+1}-1}{k-1}$, where $h \leq 3$ and $k \geq 3$, then $\gamma_t^d(G) = k^{h-1}$.*

Proof. Suppose S is a DTD-set in T_h^k. Due to leaf vertices of T_h^k, the all support vertices must be taken to the set S. It is clear that all vertices of T_h^k are disjunctively totally dominated by S. So, we get $\gamma_t^d(T_h^k) = k^{h-1}$.

■

Theorem 2.20. *If $G \cong T_h^2$ is a complete 2-ary tree of order $2^{h+1}-1$, where $h \geq 4$, then*

$$\gamma_t^d(G) = \begin{cases} \frac{2(2^h-1)}{3} & \text{, if } t \equiv 0(mod2); \\ 2^{h-1} + 2^{h-3} & \text{, otherwise.} \end{cases}$$

Proof. By the definition of complete k-ary tree T_h^k, we know that T_h^2 consists of 2 copies of T_{h-1}^2 , also T_{h-1}^2 consists of 2 copies of T_{h-2}^2 , etc. Clearly, we get $T_1^2 \cong S_{1,2}$, where $S_{1,2}$ is a star graph. It is easily seen that $\gamma_t^d(T_1^2) = 2$. Let D be a $\gamma_t^d(G)$-set of T_h^2 for $h \leq 3$. To be disjunctively totally dominated each vertex of T_h^2, all vertices which are in $(h-1)^{th}$ level must be added to the set D. Thus, all vertices except the vertices in D are disjunctively totally dominated. Therefore, the vertex which in level zero must be added to D. So, all vertices of T_h^2 are disjunctively totally dominated by the set D. Clearly, $|D| = 2^{h-1}+1$, also the set D is the minimum DTD-set. As a result, $\gamma_t^d(T_h^2) = 2^{h-1}+1$ for $h \leq 3$.

Let $h \geq 4$, and let S_1 be a $\gamma_t^d(G)$-set of T_4^2. Clearly, the set S_1 has the vertices of v_1, v_2, v_i, where $i \in \{7, 8, ..., 14\}$ in the Figure 8. So, all vertices in T_4^2 and S_1 are disjunctively totally dominated by S_1. It is easy to see that $|S_1| = 2^3 + 2 = 10$. Hence, $\gamma_t^d(T_4^2) = 10$. Let S be $\gamma_t^d(G)$-set of T_5^2. Since the tree T_5^2 has k-copies of T_4^2, all vertices which are correspond to vertices of S_1 in all copies of T_4^2 must be added to the set S. That is, $S = \bigcup_{j=1}^{2} S_1$. Thus, we get $|S| = 2|S_1|$. So, $\gamma_t^d(T_5^2) = 2(\gamma_t^d(T_4^2)) = 2^4 + 2^2$ is obtained.

With the same thought, we consider the tree T_6^2. If the vertices of $\gamma_t^d(G)$-set of the tree T_5^2 which is copy of T_6^2, all vertices of T_6^2 are not disjunctively totally dominated. Because, the vertices which are in the DTD-set are not dominated. To disjunctively total dominate of these vertices, the vertices which are in 1-level must be taken in to the $\gamma_t^d(G)$-set. Thus, we obtain $\gamma_t^d(T_6^2) = 2(\gamma_t^d(T_5^2)) + 2$. Furthermore, we get following recursive formulas for $\gamma_t^d(T_h^2)$, where $h \geq 5$:

$$\gamma_t^d(T_h^2) = \begin{cases} 2\gamma_t^d(T_{h-1}^2) + 2 \text{ , } \mathit{if}\ h \equiv 0(mod2); \\ \\ \gamma_t^d(T_{h-1}^2) \qquad \text{ , } \mathit{otherwise}. \end{cases}$$

Let $h = 2k + 1$, where $k \in Z^+$. Then we have

$$\gamma_t^d(T_h^2) = 2\gamma_t^d(T_{h-1}^2) = 2(2\gamma_t^d(T_{h-2}^2)) = 2^2\gamma_t^d(T_{h-2}^2) = ... = 2^{n-4}(T_4^2).$$

Let $h = 2k$, where $k \in Z^+$. Then we have

$$\begin{aligned} \gamma_t^d(T_h^2) &= 2\gamma_t^d(T_{h-1}^2) + 2 \\ &= 2(2\gamma_t^d(T_{h-2}^2) + 2) = 2^2(\gamma_t^d(T_{h-2}^2)) + 2 \\ &= 2^2(2\gamma_t^d(T_{h-3}^2) + 2 = 2^3(\gamma_t^d(T_{h-3}^2)) + 2^3 + 2 \\ &. \\ &. \\ &. \\ &= 2^{h-4}\gamma_t^d(T_4^2) + 2^{h-5} + 2^{h-7} + ... + 2^{h-(h-3)} + 2^{h-(h-1)}. \end{aligned}$$

Clearly, we get following result for $1 \leq i \leq h - 4$.

$$\gamma_t^d(T_h^2) = 2^i\gamma_t^d(T_{h-i}^2), \text{ if } h = 2k + 1;$$

$$\gamma_t^d(T_h^2) = 2^i\gamma_t^d(T_{h-i}^2) + \sum_{i=0}^{\frac{h-4}{2}-1} 2^{2i+1}, \text{ if } h = 2k.$$

If we use geometric series for $h = 2k$, then we have $\gamma_t^d(T_h^2) = 2^i\gamma_t^d(T_{h-i}^2) + 2\left(\frac{2^i-1}{2^2-1}\right)$.

These equalities can be proved by induction method, also remaining of the proof is similar to the proof of Theorem 2.16. So, the remaining of proof is omitted. Thus, the proof holds.

■

3. Conclusion

Various measures to determine the network robustness were suggested in the literature, and a number of graph-theoretical parameters were used to assess network reliability. We have discussed the disjunctive total domination number for some tree networks in this work. Suppose one can break a more complex network into smaller networks, then under some conditions. In that case, the optimization problem's solutions on the smaller networks can be combined to solve the optimization problem on the larger network. Thus, calculation of the disjunctive total domination number for simple graph types is important.

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Application of Simpson's method for solving singular Volterra integral equation

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Abstract. In this paper, the numerical scheme for solving singular Volterra integral equation is obtained by considering non-variable subinterval and the function under the integrals were approximated by the Simpson's rule. The error bound for the numerical scheme is established where the scheme derived has convergence of order 3. The scheme obtained is compared with exact solution of the tested problems which shows that the scheme is effective.

Keywords: Singular Volterra integral equation, convergence order, Simpson's rule, exact solution, error bound.

1. Introduction and Background

Singular Volterra integral equation of this form

$$u(t) = \int_0^t \frac{s^{\mu-1}}{t^{\mu}} u(s)ds + g(t), \quad t \in (0, T], \tag{1.1}$$

with $\mu > 0$, $g(t) \in C[0, T]$ is a given function and the kernel is weakly type has been considered by many authors. Diogo $et\ al.$[3], investigated the application of product integration method for the numerical solutions base on graded meshes by Trapezoidal method. Diogo $et\ al.$ [1], utilized the analytic results for the existence and uniqueness solution of (1.1). Further more, Euler's and Trapezoidal methods were used to develop new schemes, comparison between them was made and error bound analysis were developed. Diogo $et\ al.$ [2], used a class of singular Volterra integral equation of the form (1.1) and obtained the numerical schemes which uses Euler method and Trapezoidal rules. The numerical approximation base on the product Euler scheme converges to the smooth solution but with poor order of convergence. However, Diogo and Lima [4], analyzed discrete supperconvergence properties of spline collocation results and for a certain choice of parameter the attainable convergence order of (1.1) was considered. Diogo and Lima [5], proved that a higher order attained at the meshes points by special choice of the collocation methods. Also Diogo [6], utilized the iterated methods on the collocation results.

In this article we consider the work in Diogo $et\ al.$ (2006) which we used the Simpson's method in the case of when $0 < \mu \leq 1$, Eqn (1.1) has a family of solutions in the space $C[0, T]$. The work has been organized as follows; In section 2, we derived the scheme by applying the Simpson's method. In section 3, we estimates the error bound analysis for the convergence results of the propose scheme. Also, in section 4 we tested the scheme by means of some examples and finally in section 5 the conclusion was given.

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2. Derivation of the Scheme by Simpson's rule Approach

2.1. Definitions of the basic concepts

We start by presenting some definitions, theorems and lemmas;

Definition 2.1. *A kernel is called separable if it can be expressed as the outer product of two variables (vectors). For examples*

$$u(t) = \int_0^t \frac{s^{\mu-1}}{t^{\mu}} u(s)ds + g(t), \quad t \in [0,T],$$

where $k(t,s) = \frac{s^{\mu-1}}{t^{\mu}}$ *that can be expressed as* $k(t,s) = \frac{1}{t^{\mu}} s^{\mu-1}$
otherwise, it is nonseparable.

Theorem 2.2. *Mean Value Theorem: Let* $u(x)$ *be a function which is continuous on the closed interval* $[a,b]$ *and which is differentiable at every point of (a,b). Then there is a point* $c \in (a,b)$ *such that*

$$u'(c) = \frac{u(b) - u(a)}{b-a},$$

Lemma 2.3. *Special Gronwall lemma: Let* (e_n) *and* (e_j) *be nonnegative sequences and* C *a nonnegative constant if*

$$u_n \leq C + \sum_{k=0}^{n-1} g_k u_k \quad for \quad n \geq 0,$$

then

$$u_n \leq C \prod_{j=0}^{n-1} (1+g_j) \leq \quad C \exp^{\left(\sum_{j=0}^{n-1} g_j\right)} \quad for \quad n \geq 0.$$

Lemma 2.4. *(i) If* $0 < \mu \leq 1$ *and* $g \in C^1[0,T]$ *(with* $g(0) = 0$ *if* $\mu = 1$*) then equation(1) has a family of solutions* $u \in C[0,T]$ *given by the formula*

$$u(t) = c_0 t^{1-\mu} + g(t) + \gamma + t^{1-\mu} \int_0^t s^{\mu-2}(g(s) - g(0))ds, \tag{2.1}$$

where

$$\gamma := \begin{cases} \frac{1}{\mu-1} g(0) & if \quad \mu < 1, \\ 0 & if \quad \mu = 1, \end{cases}$$

and c_0 *is an arbitrary constant. Out of this family of solutions there is one particular solution* $u \in C^1[0,T]$.

(ii) If $\mu \leq 1$ *and* $g \in C^m[0,T], m \geq 0$ *then the unique solution* $u \in C^m[0,T]$ *of (1) is given by*

$$u(t) = g(t) + t^{1-\mu} \int_0^t s^{\mu-2} g(s)ds.,$$

We note that Eqn (2.4) can be obtained from Eqn (2.1) with $c_0 = 0$. *Indeed; it follows from Eqn (2.1) that*

$$c_0 = \lim_{t \to 0^+} t^{\mu-1} u(t),$$

and this limit is zero when $\mu > 1$.

2.2. Derivation of the scheme

Let us reformulate (1.1) into a new form by choosing some fixed real number $\alpha > 0$. Substituting t by $t+\alpha$ in (1.1) we have

$$u(t+\alpha) = \int_0^{t+\alpha} \frac{s^{\mu-1}}{(t+\alpha)^{\mu}} u(s)ds + g(t+\alpha), \quad t \in [0,T], \tag{2.2}$$

by splitting of the interval we have

$$u(t+\alpha) = \frac{1}{(t+\alpha)^{\mu}} \int_0^{\alpha} s^{\mu-1} u(s)ds + \int_{\alpha}^{t+\alpha} \frac{s^{\mu-1}}{(t+\alpha)^{\mu}} u(s)ds + g(t+\alpha), \quad t \in [\alpha,T], \tag{2.3}$$

or, equivalently,

$$u(t+\alpha) = \frac{I_{\alpha}}{(t+\alpha)^{\mu}} + \int_0^{t} \frac{(s+\alpha)^{\mu-1}}{(t+\alpha)^{\mu}} u(s+\alpha)ds + g(t+\alpha), \tag{2.4}$$

where

$$I_{\alpha} = \int_0^{\alpha} s^{\mu-1} u(s)ds. \tag{2.5}$$

Since I_{α} is known exactly for a chosen the exact solution by using the solution formula then we can apply the numerical method in (2.4) and obtain the approximation.
Now, let us define a uniform grid X_h with stepsize $h = \frac{t}{n}$

$$X_h := \{t_i = ih + \alpha, \quad 0 \le i \le N\}.$$

Setting $t = nh$ in (2.4) we have

$$u(t_n) = \frac{I_{\alpha}}{t_n^{\mu}} + \frac{1}{t_n^{\mu}} \int_0^{nh} (s+\alpha)^{\mu-1} u(s+\alpha)ds + g(t_n). \tag{2.6}$$

In the Simpson's method, we approximates the integral on the right-hand side of (2.6) by considering each subinterval using:

$$u(s+\alpha) \approx \frac{1}{6}\left[u(t_{j+1})(s-jh) + 4u\left(\frac{t_j + t_{j+1}}{2}\right)(t_{j+1}-t_j) + u(t_j)((j+1)h - s)\right], \tag{2.7}$$

on each subinterval $s \in [jh, (j+1)h]$. Defining the following

$$D_j^1 := \int_{jh}^{(j+1)h} (s+\alpha)^{\mu-1}(s-jh)ds$$
$$D_j^2 := \int_{jh}^{(j+1)h} (s+\alpha)^{\mu-1}ds$$
$$D_j^3 := \int_{jh}^{(j+1)h} (s+\alpha)^{\mu-1}((j+1)h - s)ds$$

which can be obtain analytically.
Hence the scheme:

$$u(t_n)_n^h = \frac{I_{\alpha}}{t_n^{\mu}} + \frac{h}{t_n^{\mu}} \sum_{j=0}^{n-1} \frac{(D_j^1 u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3 u_j^h)}{6} + g(t_n), \quad n = 1,2,...,N. \tag{2.8}$$

2.3. Algorithm: Simpson's rule approach

Step1: Given $n = 1$, $\epsilon = 10^{-3}$, $t \in [0, T]$, $\mu \in (0, 1]$, $\alpha > 0$, $u(t)$, $g(t)$, I_α.
Step2: Set $h = \frac{t}{n}$
Step3: Compute

$$\begin{aligned} t_n &= nh + \alpha \\ t_n^\mu &= (nh + \alpha)^\mu \\ u_n^h &= \frac{I_\alpha}{t_n^\mu} + \frac{h}{t_n^\mu} \sum_{j=0}^{n-1} \frac{(D1_j u_{j+1}^h + D2_j u_{j/2}^h + D3_j u_j^h)}{6} + g(t_n) \end{aligned}$$

where
$D1_j := \frac{(jh+\alpha)^{\mu+1} - ((j+1)h+\alpha)^{\mu+1} + h((j+1)h+\alpha)^\mu(\mu+1)}{(\mu+1)\mu}$,
$D2_j := \frac{h(((j+1)h+\alpha)^\mu - (jh+\alpha)^\mu)}{\mu}$,
$D3_j := \frac{((j+1)h+\alpha)^{\mu+1} - (jh+\alpha)^{\mu+1} - h(jh+\alpha)^\mu(\mu+1)}{(\mu+1)\mu}$,
$u_{j+1}^h := u\left((j+1)h + \alpha\right)$,
$u_{j/2}^h := 4u\left(\frac{(jh+\alpha)+((j+1)h+\alpha)}{2}\right)$
$u_j^h := u\left(jh + \alpha\right)$
If $|u(t) - u_n^h| \leq 10^{-2}$ stop, else
Step4: set $n = n + 1$ and go to Step3.

3. Error Bound of the Scheme in Simpson's Rule Approach

In this section we present the error bound for the convergence of the scheme.

Theorem 3.1. *Consider (1.1) with $0 < \mu \leq 1$ and $u \in C^1[0, T]$. Let $\alpha \neq 0$ be fixed in the equivalent (2.4) and assume the integral I_α is known exactly for a chosen particular solution (corresponding to a certain value of the parameter c_0). Then the approximate solution obtained by the product Simpson's method converges with order 3 to the particular exact solution.*

Proof. The solution u of the exact solution satisfies

$$u(t_n)_n^h = \frac{I_\alpha}{t_n^\mu} + \frac{h}{t_n^\mu} \sum_{j=0}^{n-1} \frac{(D_j^1 u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3 u_j^h)}{6} + g(t_n) + \eta(h, t_n), \quad n \geq 1, \tag{3.1}$$

where $\eta(h, t_n)$ is the consistency error given by

$$\eta(h, t_n) = \int_0^{t_n} \frac{s^{\mu-1}}{t_n^\mu} u(s) ds - \frac{h}{t_n^\mu} \sum_{j=0}^{n-1} \frac{(D_j^1 u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3 u_j^h)}{6}, \tag{3.2}$$

but the exact solution is

$$u(t_n) = \frac{I_\alpha}{t_n^\mu} + \frac{1}{t_n^\mu} \int_\alpha^T s^{\mu-1} u(s) ds + g(t_n). \tag{3.3}$$

Setting $e_n = u(t_n) - u(t_n)^h$ for $n \geq 1$ and by utilizing (3.3) and (3.1) this gives

$$\begin{aligned} e_n &= \frac{1}{t_n^\mu}\int_\alpha^T s^{\mu-1}u(s)ds - \frac{h}{t_n^\mu}\sum_{j=0}^{n-1}\frac{(D_j^1u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3u_j^h)}{6} + \eta(h,t_n) \\ &= \frac{1}{t_n^\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} s^{\mu-1}u(t_j)ds - \frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} s^{\mu-1}\left(\frac{u(t_{j+1})^h + u(t_{j/2})^h + u(t_j)^h}{6}\right)ds \\ &+ \eta(h,t_n) \\ &= \frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left(u(t_j) - Su(t_j)^h\right)s^{\mu-1}ds + \eta(h,t_n) \end{aligned}$$

let $Su(t_j)^h = \frac{u(t_{j+1})^h + u(t_{j/2})^h + u(t_j)^h}{6}$ and defining $e_j^s := Se_j = \left(u(t_j) - Su(t_j)^h\right)$ yield

$$e_n = \frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1} e_j^s \int_{t_j}^{t_{j+1}} s^{\mu-1}ds + \eta(h,t_n), \quad n \geq 1, \tag{3.4}$$

but

$$\begin{aligned} \frac{1}{t_n^\mu}\int_{t_j}^{t_{j+1}} s^{\mu-1}ds &\leq \frac{t_j^{\mu-1}}{t_n^\mu}\int_{t_j}^{t_{j+1}} ds \\ &= h\frac{t_j^{\mu-1}}{t_n^\mu} \\ &\leq h\left(\frac{t_j}{t_n}\right)^\mu \frac{1}{t_j} \\ &\leq \frac{h}{\alpha}. \end{aligned} \tag{3.5}$$

Since $\alpha \neq 0$ and for $\alpha > 0$ choose $\alpha \leq t_j\left(\frac{t_n}{t_j}\right)^\mu$. By utilizing (3.5) in (3.4) we have

$$e_n \leq \frac{h^3}{\alpha}\sum_{j=0}^{n-1} e_j^s + \eta(h,t_n), \quad n \geq 1. \tag{3.6}$$

Taking the modulus in (3.6) we have

$$|e_n| \leq \frac{h^3}{\alpha}\sum_{j=0}^{n-1}\left|e_j^s\right| + |\eta(h,t_n)|, \quad n \geq 1 \tag{3.7}$$

On the other hand from equation (3.2) we have

$$\begin{aligned} |\eta(h,t_n)| &= \left|\int_0^{t_n}\frac{s^{\mu-1}}{t_n^\mu}u(s)ds - \frac{h}{t_n^\mu}\sum_{j=0}^{n-1}\frac{(D_j^1u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3u_j^h)}{6}\right| \\ &= \left|\frac{1}{t_n^\mu}\sum_{j=0}^{n-1} D_j u(t_j) - \frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1} D_j\left(\frac{u(t_{j+1})^h + u(t_{j/2})^h + u(t_j)^h}{6}\right)\right| \\ &= \left|\frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1} D_j\left(u(s) - Su(t_j)^h\right)\right| \end{aligned}$$

but

$$D_j := \int_{t_j}^{t_{j+1}} s^{\mu-1} ds$$

Therefore,

$$|\eta(h, t_n)| \leq \frac{h^2}{t_n^\mu} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} s^{\mu-1} \left| \left(u(s) - Su(t_j)^h\right) \right| ds \tag{3.8}$$

by applying the mean value theorem in (3.8), we have

$$|\eta(h, t_n)| \leq \frac{h^3}{t_n^\mu} \max_{s\in[\alpha,T]} |u'(s)| \int_\alpha^{t_n} s^{\mu-1} ds \tag{3.9}$$

Defining $M(\alpha) := \max_{s\in[\alpha,T]} |u'(s)|$

$$\begin{aligned}|\eta(h, t_n)| &\leq \frac{M(\alpha)h^3}{t_n^\mu} \int_\alpha^{t_n} s^{\mu-1} ds \\ &= \frac{M(\alpha)h^3}{\mu} \left(\frac{t_n^\mu - \alpha^\mu}{t_n^\mu}\right) \\ &= \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right) \frac{M(\alpha)h^3}{\mu}\end{aligned}$$

we obtained the following bound

$$|\eta(h, t_n)| \leq \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right) \frac{M(\alpha)h^3}{\mu} \tag{3.10}$$

substitute (3.10) into (3.7) we have

$$|e_n| \leq \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right) \frac{M(\alpha)h^3}{\mu} + \frac{h^3}{\alpha} \sum_{j=0}^{n-1} |e_j^s| \tag{3.11}$$

by applying the special Gronwall lemma for the discrete in (3.11) we have

$$|e_n| \leq \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right) \frac{M(\alpha)h^3}{\mu} \prod_{j=0}^{n-1} \left(1 + \frac{n-1}{\alpha}\right)$$

we obtained the error bound as

$$|e_n| \leq \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right) \frac{M(\alpha)h^3}{\mu} \exp\left(\frac{T-1}{\alpha}\right)$$

Hence, a third order convergence follows.

4. Main Results

In this section we tested the scheme using Maple13 version 10 with the stoping rule as $|u_n^h - u(t)| \leq 10^{-3}$.

Problem 4.1. *Given $g(t) = 1 + t + t^2$ and $0 < \mu \leq 1$ in (1.1), then using (2.1) we obtained the general form of its family of solutions:*

$$u(t) = c_0 t^{1-\mu} + \frac{\mu}{\mu - 1} + \frac{\mu + 1}{\mu} t + \frac{\mu + 2}{\mu + 1} t^2 \tag{4.1}$$

where c_0 is an arbitrary constant. The exact solution (4.1) when $t = 1.02$ is compared with numerical solution (2.8) and errors are presented in Table 1

Table 1: The results obtained by the numerical scheme (2.8) on problem1.

n	u_n^h Eqn (2.8)	$\lvert u(t)-u_n^h\rvert$
80	2.4543	$6.664E-1$
82	2.5122	$6.085E-1$
84	2.5707	$5.500E-1$
86	2.6299	$4.908E-1$
88	2.6899	$4.308E-1$
90	2.7506	$3.701E-1$
92	2.1190	$3.088E-1$
94	2.8741	$2.466E-1$
96	2.9369	$1.838E-1$
98	3.0004	$1.203E-1$
100	3.0647	$5.600E-2$
102	3.1297	$9.000E-3$

Table (1) shows that the numerical results of problem 4.1 obtained from scheme (2.8) which has exact solution of $u(t) = 3.1207$ and the best result is obtained when $n = 102$ with corresponding to an error of $9.000E-3$.

Problem 4.2. *Given $g(t) = 1 + t$ and $0 < \mu \leq 1$ in (1.1), then using (2.1) we obtained the general form of its family of solutions:*

$$u(t) = c_0 t^{1-\mu} + \frac{\mu}{\mu - 1} + \frac{\mu + 1}{\mu} t \tag{4.2}$$

where c_0 is an arbitrary constant. The exact solution (4.2) when $t = 1.02$ is compared with numerical solution (2.8) and errors are presented in Table (2)

Table 2: The results obtained by the numerical scheme (2.8) on problem2.

n	u_n^h Eqn (2.8)	$\lvert u(t)-u_n^h\rvert$
80	1.7946	$2.652E-1$
82	1.8195	$2.403E-1$
84	1.8442	$2.156E-1$
86	1.8689	$1.909E-1$
88	1.8934	$1.664E-1$
90	1.9179	$1.419E-1$
92	1.9423	$1.175E-1$
94	1.9666	$9.320E-2$
96	1.9908	$6.900E-2$
98	2.0149	$4.490E-2$
100	2.0389	$2.090E-2$
102	2.0663	$6.500E-3$

Table (2) shows that the numerical results of problem 4.2 obtained from scheme (2.8) which has exact solution of $u(t) = 2.0598$ and the best result is obtained when $n = 102$ with corresponding to an error of $6.50E - 3$. The error decreases when the number of iterations are increased. The results is an improvement when compared with the work of [5] which uses Euler's method with number of iterations up to 1600 corresponding to an error of $4.82E - 2$.

Problem 4.3. *Given $g(t) = 1 + t + t^3$ and $0 < \mu < 1$ in (1.1), then using (2.1) we obtained the general form of its family of solutions:*

$$u(t) = c_0 t^{1-\mu} + \frac{\mu}{\mu - 1} + \frac{\mu + 1}{\mu} t + \frac{\mu + 3}{\mu + 2} t^3 \tag{4.3}$$

where c_0 is an arbitrary constant. The exact solution (4.3) when $t = 1.02$ is compared with numerical solution (2.8) and errors are presented in Table (3)

Table 3: The results obtained by the numerical scheme (2.8) on problem3.

n	u_n^h Eqn (2.8)	$\|u(t) - u_n^h\|$
80	2.3278	$8.247E - 1$
82	2.3932	$7.593E - 1$
84	2.4604	$6.921E - 1$
86	2.5296	$6.229E - 1$
88	2.6008	$5.517E - 1$
90	2.6739	$4.786E - 1$
92	2.7493	$4.032E - 1$
94	2.8268	$3.257E - 1$
96	2.9065	$2.460E - 1$
98	2.9884	$1.641E - 1$
100	3.0727	$7.980E - 2$
102	3.1593	$6.800E - 3$

Table (3) shows that the numerical results of problem (4.3) obtained from scheme (2.8) which has exact solution of $u(t) = 3.1525$ and the best result is obtained when $n = 102$ with corresponding to an error of $6.800E-3$. The error decreases when the number of iterations are increased. The results is an improvement when compared with the work of [5] which uses Euler's method with number of iterations up to 1600 corresponding to an error of $4.82E - 2$.

4.1. The comparison of the numerical schemes

Here we presented the scheme (2.8) derived from Midpoint's rule when compared with Euler's method in [5].

Table 4: The comparison of scheme (2.8) and Euler's methods in [5] using errors of problem 1 and 2.

n	scheme (2.8) Errors1	scheme (2.8) Errors2	Euler's in [5] Errors
80	$5.759E-1$	$2.321E-1$	$3.919E-1$
82	$5.183E-1$	$2.072E-1$	$4.173E-1$
84	$4.599E-1$	$1.825E-1$	$4.423E-1$
86	$4.009E-1$	$1.578E-1$	$4.671E-1$
88	$3.412E-1$	$1.333E-1$	$4.817E-1$
90	$2.807E-1$	$1.089E-1$	$5.159E-1$
92	$2.196E-1$	$8.460E-2$	$5.400E-1$
94	$1.578E-1$	$6.030E-2$	$5.638E-1$
96	$9.520E-2$	$3.620E-2$	$5.874E-1$
98	$3.190E-2$	$1.210E-2$	$6.108E-1$
99	$1.000E-4$	$1.000E-4$	$6.224E-1$

Table (4) Shows that the errors obtained from the scheme (2.8) is an improvement when compared with the work of [5] which uses Euler's method, since the error decreases when the number of iterations are increased. This shows that the scheme obtained has a better result when compared with the Euler's method with number of iterations up to 1600 corresponding to an error of $4.82E-2$.

5. Conclusion

The function under the integrals were approximated base on the concepts of Simpson's rule. We used error bound estimates for the convergence of the scheme obtained. The numerical results were obtained by means of some examples so as to test the efficiency, accuracy and effectiveness of the new scheme derived. The new approach of the numerical scheme obtained from Simpson's rule was compared with exact solutions.

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(e)-Convergence for double sequences

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Abstract. We define the notion of (e)-summability of double sequences and series of complex numbers. We also obtain a criteria for this summability method with regards to Berezin symbols of an diagonal operator, and show regularity of (e)-summability method for double sequences.

AMS Subject Classifications: 40H05.

Keywords: Berezin symbol, diagonal operator, double sequence, (e)-summability.

Contents

1. Introduction and Background

A double sequence $\{a_{mn}\}_{m,n=0}^{\infty}$ is called the convergent in Pringsheim's sense [14] provided that there exists a number a such that a_{mn} converges to a as both m and n approach to infinity independently of one another

$$\lim_{m,n\to\infty} a_{mn} = a,$$

that is if for every $\varepsilon > 0$ there exists $K = K(\varepsilon) \in \mathbb{N}$ such that $|a_{mn} - a| < \varepsilon$ for every $m, n \geq K$ and also a is said to the Pringsheim's limit of a_{mn}. It is obvious that $\{a_{mn}\}$ is convergent in Pringsheim's sense if and only if for every $\varepsilon > 0$ there exists an integer $K = K(\varepsilon) \in \mathbb{N}$ such that $|a_{mn} - a_{ij}| < \varepsilon$ for $\min\{m, n, i, j\} \geq K$. A double sequence $\{a_{mn}\}$ is bounded provided that there exists a positive number N such that $|a_{mn}| \leq N$ for every m and n, i.e., $\sup_{m,n} |a_{mn}| < \infty$.

A double sequence $\{a_{mn}\}$ is said to be convergent regularly provided that it is convergent in Pringsheim's sense and the following limits hold:

$$\lim_{m,n\to\infty} a_{mn} = x_m \;\; (m = 1, 2, ...),$$
$$\lim_{m,n\to\infty} a_{mn} = x_n \;\; (n = 1, 2, ...).$$

It is well known that a convergent double sequence in Pringsheim's sense fails in general to be bounded. The concept of regular convergence, which was introduced by Hardy in [7], lacks this advantage. Moreover, the regular convergence requires the convergence of rows and columns of a double sequence. (For more information about several type convergence for double sequences, see [18] and its references.)

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A reproducing kernel Hilbert space (shorty, RKHS) $\mathcal{H} = \mathcal{H}(\Omega)$ on some set Ω is a Hilbert space of functions on Ω such that for every $\lambda \in \Omega$ the linear functional (evaluation functional) $f \to f(\lambda)$ is bounded on $\mathcal{H}$. If $\mathcal{H}$ is RKHS on set Ω, then by the classical Riesz Representation Theorem for every $\lambda \in \Omega$ there is a unique element $k_\lambda \in \mathcal{H}$ for which $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The function k_λ is said to be reproducing kernel at λ. We know that (see, Aronzajn [1] and Saitoh [15]) provided that $(e_j)_{j\in J}$ is an orthonormal basis for the RKHS $\mathcal{H}$,

$$k_\lambda(z) = \sum_{j\in J}^{\infty} \overline{e_j(\lambda)} e_j(z), \quad z \in \Omega.$$

The function

$$\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}} = \frac{1}{\left(\sum_{j\in J} |e_j(\lambda)|^2\right)^{1/2}} \sum_{j\in J} \overline{e_j(\lambda)} e_j(z)$$

is called the normalized reproducing kernel at λ.

Berezin [2, 3] introduced the concept of contravariant and covariant symbols of an operator. The contravariant symbol of a Toeplitz operator, which is the so-called Berezin symbol, was firstly used by Berger and Coburn in [4, 5].

Let A be a bounded operator on reproducing kernel Hilbert spaces. Then the function

$$\widetilde{A}(\lambda) := < A\widehat{k}_\lambda, \widehat{k}_\lambda >, \ \lambda \in \Omega,$$

is called the Berezin symbol, which is a bounded function by the norm of the operator (see [2]). On the reproducing kernel Hilbert spaces, $\widetilde{A}_1(\lambda) = \widetilde{A}_2(\lambda)$ for all λ implies $A_1 = A_2$, that is, the Berezin symbol uniquely determines the operator. Therefore, the Berezin symbol includes many information about the operator that induces it. Prosperous applications of the Berezin symbol are up to now commonly in the study of operator theory, such as Toeplitz and Hankel operators [19]. The Berezin symbol technique is motivated by its connections with quantum physics (see, for example, [2, 3]). Readers can found more informations about Berezin symbols and its applications, for instance in [9, 13, 19].

A RKHS $\mathcal{H}(\Omega)$ is standard provided that the underlying set Ω is a subset of a topological space and the boundary of Ω is non-empty and has the property that $(k_{\mathcal{H},\lambda_n})_n$ converges weakly to 0 whenever $(\lambda_n)_n$ is a sequence in Ω that converges to a point in $\partial\Omega$. It is obvious that $\lim_{n\to\infty} \widetilde{K}(\lambda_n) = 0$ for any compact operator K on the standard RKHS $\mathcal{H}$ whenever $(\lambda_n)_n \subset \Omega$ converges to a point of $\partial\Omega$. In this case, the Berezin symbol of a compact operator on a standard RKHS vanishes on the boundary (see [13]).

Karaev [11] introduced (e)-convergent for single sequences and series of complex numbers. Later, he [12] gave a Tauberian-type therom for (e)-convergent sequences. Using the Berezin symbol technique, new proofs for (L)-convergence and Abel convergence were given in [8, 16].

2. Main Results

In this section, we define a concept of (e)-convergence for double sequences and series of complex numbers. We obtain a criteria for this summability method with regards to Berezin symbols of an diagonal operator, and show regularity of (e)-summability method for double sequences.

Recall that a method is called the regular provided that it sums each convergent sequence to its ordinary limit. For instance, Abel, Cezaro and Borel methods are regular (see [6]).

Let $\mathcal{H} = \mathcal{H}(\Omega \times \Omega)$ be a reproducing kernel Hilbert space on some set $\Omega \times \Omega$, $\{e_{mn}\}_{m,n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ and

$$k_{\lambda,\mu}(z, w) := \sum_{m,n\geq 0} \overline{e_{mn}(\lambda, \mu)} e_{mn}(z, w),$$

be a reproducing kernel of $\mathcal{H} = \mathcal{H}(\Omega \times \Omega)$.

Definition 2.1. *Let $\{a_{mn}\}_{m,n\geq 0}$ be a double sequence of complex numbers.*

(a) The sequence $\{a_{mn}\}_{m,n\geq 0}$ is (e)-convergent to L provided that

$$\sum_{m,n=0}^{\infty} a_{mn} \left| e_{mn}(\lambda,\mu) \right|^2$$

converges for all $(\lambda,\mu) \in \Omega \times \Omega$ and

$$\lim_{(\lambda,\mu)\to(\zeta,\xi)} \frac{1}{\sum_{m,n=0}^{\infty} \left| e_{mn}(\lambda,\mu) \right|^2} \sum_{m,n=0}^{\infty} a_{mn} \left| e_{mn}(\lambda,\mu) \right|^2 = L,$$

for every $(\zeta,\xi) \in \partial\Omega \times \partial\Omega$.

(b) The series $\sum_{m,n=0}^{\infty} a_{mn}$ is (e)-summable to L provided that

$$\sum_{m,n=0}^{\infty} a_{mn} \left| e_{mn}(\lambda,\mu) \right|^2$$

converges for each $(\lambda,\mu) \in \Omega \times \Omega$ and

$$\lim_{(\lambda,\mu)\to(\zeta,\xi)} \sum_{m,n=0}^{\infty} a_{mn} \left| e_{mn}(\lambda,\mu) \right|^2 = L$$

for each $(\zeta,\xi) \in \partial\Omega \times \partial\Omega$.

It was shown that Abel and Borel summability for double sequences coincide with concept of (e)-summability for Hardy space and Fock space, respectively (see [10, 17]).

Let $\{a_{mn}\}_{m,n\geq 0}$ be a double sequence of complex numbers. Diagonal operator $D_{\{a_{mn}\}}$ on $\mathcal{H}$ is defined by

$$D_{\{a_{mn}\}} e_{mn}(\lambda,z) = a_{mn} e_{mn}(\lambda,z), \; m,n = 0,1,2,...,$$

with respect to the orthonormal basis $e = \{e_{mn}(\lambda,z)\}_{m,n\geq 0}$ of $\mathcal{H}$.

The following result is main theorem of this section.

Theorem 2.2. *Let $\{a_{mn}\}_{m,n\geq 0}$ be a bounded double sequence of complex numbers.*
(a) *The sequence $\{a_{mn}\}_{m,n\geq 0}$ is (e)-convergent to L if and only if*

$$\lim_{(\lambda,\mu)\to(\zeta,\xi)} \widetilde{D}_{\{a_{mn}\}}(\lambda,\mu) = L$$

for every $(\zeta,\xi) \in \partial\Omega \times \partial\Omega$.

(b) *The series $\sum_{m,n=0}^{\infty} a_{mn}$ is (e)-summable to L if and only if*

$$\lim_{(\lambda,\mu)\to(\zeta,\xi)} \left(\sum_{m,n=0}^{\infty} \left| e_{mn}(\lambda,\mu) \right|^2 \right) \widetilde{D}_{\{a_{mn}\}}(\lambda,\mu) = L$$

for every $(\zeta,\xi) \in \partial\Omega \times \partial\Omega$.

(c) *(e)-summability method for double sequences is regular provided that $\mathcal{H}$ is a standard functional Hilbert space.*

Proof. As $\{a_{mn}\}_{m,n\geq 0}$ is a bounded double sequence, $D_{\{a_{mn}\}}$ is a bounded operator on $\mathcal{H}$. Calculating the Berezin symbol of diagonal operator, we have

$$\begin{aligned}\widetilde{D}_{\{a_{mn}\}}(\lambda,\mu) &= \left\langle D_{\{a_{mn}\}}\widehat{k}_{\lambda,\mu},\widehat{k}_{\lambda,\mu}\right\rangle \\ &= \frac{1}{\|k_{\lambda,\mu}\|^2}\left\langle D_{\{a_{mn}\}}\sum_{m,n=0}^{\infty}\overline{e_{mn}(\lambda,\mu)}e_n(z,w), k_{\lambda,\mu}\right\rangle \\ &= \frac{1}{\sum_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\left\langle \sum_{m,n=0}^{\infty}\overline{e_{mn}(\lambda,\mu)}a_{mn}e_{mn}(z,w), k_{\lambda,\mu}\right\rangle \\ &= \frac{1}{\sum_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}a_{m,n}|e_{mn}(\lambda,\mu)|^2\end{aligned}$$

for all $(\lambda,\mu)\in\Omega\times\Omega$. Therefore

$$\widetilde{D}_{\{a_{mn}\}}(\lambda,\mu) = \frac{1}{\sum_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}a_{mn}|e_{mn}(\lambda,\mu)|^2,\ (\lambda,\mu)\in\Omega\times\Omega. \tag{1}$$

As $\sup\limits_{(\lambda,\mu)\in\Omega\times\Omega}\left|\widetilde{D}_{\{a_{mn}\}}(\lambda,\mu)\right| \leq \left\|\widetilde{D}_{\{a_{mn}\}}\right\| = \sup\limits_{m,n\geq 0}|a_{mn}| < \infty$, formula (1) immediately implies the claims (a) and (b) of the theorem.

Let us show the claim (c). Let $\{a_{mn}\}_{m,n=0}^{\infty}$ converges to L. Then $D_{\{a_{mn}-L\}}$ is a compact operator, and hence $\widetilde{D}_{\{a_{mn}-L\}}$ vanishes on the boundary of $\Omega\times\Omega$ (since $\mathcal{H}$ is a standard reproducing kernel Hilbert space), that is, $\widetilde{D}_{\{a_{mn}-L\}}(\lambda,\mu)\to 0$ as $(\lambda,\mu)\to(\zeta,\xi)\in\partial\Omega\times\partial\Omega$. Taking into consideration this and formula (1), we get

$$\begin{aligned}\lim_{(\lambda,\mu)\to(\zeta,\xi)}\widetilde{D}_{\{a_{mn}\}}(\lambda,\mu) &= \lim_{(\lambda,\mu)\to(\zeta,\xi)}\frac{1}{\sum\limits_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}a_{mn}|e_{mn}(\lambda,\mu)|^2 \\ &= \lim_{(\lambda,\mu)\to(\zeta,\xi)}\frac{1}{\sum\limits_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}(a_{mn}-L+L)|e_{mn}(\lambda,\mu)|^2 \\ &= \lim_{(\lambda,\mu)\to(\zeta,\xi)}\frac{1}{\sum\limits_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}(a_{mn}-L)|e_{mn}(\lambda,\mu)|^2 + L \\ &= \lim_{(\lambda,\mu)\to(\zeta,\xi)}\widetilde{D}_{\{a_{mn}-L\}} + L,\end{aligned}$$

which gives that (e)-$\lim_{m,n} a_{mn} = L$. So, the proof is completed. ■

We can obtain the following result from Theorem 1 by putting $\mathcal{H}=\mathcal{D}\left(\mathbb{D}^2\right)$ and $\mathcal{H}=\mathcal{F}\left(\mathbb{C}^2\right)$.

Corollary 2.3. *Let $\{a_{mn}\}_{m,n\geq 0}$ be a bounded double sequence of complex numbers.*
(a) If $D_{\{a_{mn}\}}$ is a diagonal operator on the Dirichlet space $\mathcal{D}\left(\mathbb{D}^2\right)$ with diagonal elements $a_{mn},\ m,n\geq 0$, with respect to the orthonormal basis of $\mathcal{D}$, then the double sequence $\{a_{mn}\}_{m,n\geq 0}$ is (L)-convergent (logarithmic convergent) to L if and only if

$$\lim_{\lambda,\mu\to 1^-}\widetilde{D}_{\{a_{mn}\}}\left(\sqrt{x},\sqrt{y}\right) = L,$$

where $x=|\lambda|^2$ and $y=|\mu|^2$.
(b) If $D_{\{a_{mn}\}}$ is a diagonal operator on the Fock space $\mathcal{F}\left(\mathbb{C}^2\right)$ with diagonal elements $a_{mn},\ m,n\geq 0$, with

respect to the orthonormal basis of $\mathcal{D}$, then the double sequence $\{a_{mn}\}_{m,n\geq 0}$ is Borel convergent to L if and only if

$$\lim_{\lambda,\mu\to\infty} \widetilde{D}_{\{a_{mn}\}}\left(\sqrt{2x},\sqrt{2y}\right) = L,$$

where $x = \dfrac{|\lambda|^2}{2}$ and $y = \dfrac{|\mu|^2}{2}$.

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Common fixed point theorem satisfying rational contraction in complex valued dislocated metric space

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Abstract. In this article we introduce a notion of fixed point theorem satisfying rational contraction in complex valued dislocated metric space and also support our main theorem to provide an example.

AMS Subject Classifications: 47H10, 54H25.

Keywords: Complex valued metric spaces, dislocated metric space, fixed point.

Contents

1. Introduction

In Mathematical analysis, general topology and functional analysis the fixed point theory play a very important role. Many applications of fixed point theory in computer science, engineering field, image processing and mathematics etc. Banach contraction mapping principle play a crucial role in the fixed poin theory. The concept of dislocated metric space was first introduced by Hitzler in 2001. He generalized the Banach contraction mapping principle in the dislocated metric space. The beauty of dislocated metric space that the self distance between two points need not be necessarily zero. The logical programming, topology, electronic engineering and computer science etc. these are the fields which the dislocated metric space play a very vital role. Azam et al. introduced the complex valued metric spaces and proved Banach contraction mapping principle. So many researchers proved many contraction principle by this complex valued metric spaces. Ozgur edge and Ismet karaca introduced the complex valued disclocated metric spaces. Now we are going to prove the complex valued dislocated metric spaces in the fixed point theorem satisfying rational contraction mapping. Before entering into our main results we shall recall some basic definition and results which are needful.

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2. Preliminaries

We recollect some basic definitions and notions which is useful for proving our main results.
Let C be the set of complex numbers and $v_1, v_2 \in C$. Define a partial order $\leq$ on C as follows:
$v_1 \leq v_2$ if and only if $Re(v_1) \leq Re(v_2)$, $Im(v_1) \leq Im(v_2)$.
Consequently, one can infer that $v_1 \leq v_2$ if one of the following conditions is satisfied:
(i) $Re(v_1) = Re(v_2)$, $Im(v_1) < Im(v_2)$,
(ii)$Re(v_1) < Re(v_2)$, $Im(v_1) = Im(v_2)$,
(iii)$Re(v_1) < Re(v_2)$, $Im(v_1) < Im(v_2)$,
(iv)$Re(v_1) = Re(v_2)$, $Im(v_1) = Im(v_2)$.
In particular,we write $v_1 \lneq v_2$ if $v_1 \neq v_2$ and one of $(i), (ii)$ and (iii) is satisfied and we write $v_1 < v_2$ if only (iii) is satisfied. Notice that
(a) If $0 \leq v_1 \lneq v_2$, then $|v_1| < |v_2|$,
(b) If $v_1 \leq v_2$ and $v_2 < v_3$ then $v_1 < v_3$,
(c) If $p, q \in R$ and $p \leq q$ then $pv \leq qv$ for all $v \in C$.
Now we define a complex valued dislocated metric space

Definition 2.1. *Consider γ_d be a non void set and define a function $\gamma_d : H \times H \to C$ satisfies the following conditions such that for all $u, r, w \in H$*
(1)$\gamma_d(u, r) = \gamma_d(r, u)$
(2)$\gamma_d(u, r) = \gamma_d(r, u) = 0$ if and only if $u = r$
(3)$\gamma_d(u, r) \leq \gamma_d(u, w) + \gamma_d(w, r)$
Then γ_d is said to be complex valued dislocated metric space and call (H, γ_d) is a complex valued dislocated metric space.

Example 2.2. *Consider the function that $\gamma_d : H \times H \to C$ be defined by $\gamma_d(u, r) = max\{u, r\}$ where $H = C$ then it is called as complex valued dislocated metric space.*

Remark 2.3. *Every complex valued metric space is a complex valued dislocated metric space but converse need not be true.*

Definition 2.4. *Consider (H, γ_d) be a complex valued dislocated metric space and define a sequence $\{u_n\}$ in H for each $u \in H$*
(i)let the sequence $\{u_n\}$ be convergent to u in (H, γ_d) is said to be complex valued dislocated metric space then for each $\epsilon > 0$ we can find $n_0 \in N$ such that $\gamma_d(u_n, u) < \epsilon$ for each $n > n_0$ which is denoted by $u_n \to u$
(ii)Consider the sequence $\{u_n\}$ be cauchy sequence in (H, γ_d) is called complex valued dislocated metric space if $lim_{n\to\infty}\gamma_d(u_n, u_{n+b}) = 0$ for each $b > 0$
(iii)Let (H,γ_d) be a complex valued complete dislocated metric space if every complex valued cauchy sequence in H converges to some $u \in H$.

We state the two lemmas which are useful to prove our main theorem

Lemma 2.5. *Consider (H,γ_d) be a complex valued dislocated metric space.Let $\{u_n\}$ be sequence in H. Then $\{u_n\}$ converges to u if and only if $|\gamma_d(u_n, u)| \to 0$ as $n \to \infty$*

Lemma 2.6. *Consider (H,γ_d) be a complex valued dislocated metric space.Let $\{u_n\}$ be sequence in H. Then $\{u_n\}$ is a complex valued dislocated metric cauchy sequence if and only if $|\gamma_d(u_n, u_{n+b})| \to 0$ as $n \to \infty$*

3. Main Results

In this section, we prove the theorem by using new rational contraction mapping in complex valued dislocated metric space.
Now we first define the rational contraction mapping in complex valued dislocated metric space

Definition 3.1. *Let (H, γ_d) be a complete complex valued dislocated metric space. Consider the function $G, T : H \to H$ which satisfies the rational contraction conditions*
$\gamma_d(Gu, Tr) \leq a[\gamma_d(u, r)] + \frac{3b[\gamma_d(u,Tr)]^2}{1+\gamma_d(u,r)+\gamma_d(r,Tr)} + c[\gamma_d(u, Tr) + \gamma_d(u, Gu)]$ *for each $u, r \in H$ and the non negativity constants are a, b, c*

Theorem 3.2. *Let (H, γ_d) be a complete complex valued dislocated metric space. Consider the function $G, T : H \to H$ which satisfies the rational contraction conditions of (3.1) with $2a + 6b + 3c < 1$ .Then G has unique common fixed point.*

Proof. Let u_0 be the arbitrary point in H. Now define $u_{k+1} = Gu_k$,$u_{k+2} = Tu_{k+1}$, for each $k \in Z^+$ Therefore,

$$\begin{aligned}
\gamma_d(u_{k+1}, u_{k+2}) &= \gamma_d(Gu_k, Tu_{k+1}) \\
&\leq a[\gamma_d(u_k, u_{k+1})] + \frac{3b[\gamma_d(u_k,Tu_{k+1})]^2}{1+\gamma_d(u_k,u_{k+1})+\gamma_d(u_{k+1},Tu_{k+1})}+ \\
&\quad c[\gamma_d(u_k, Tu_{k+1}) + \gamma_d(u_k, Gu_k)] \\
&\leq a[\gamma_d(u_k, u_{k+1})] + \frac{3b[\gamma_d(u_k,u_{k+2})]^2}{1+\gamma_d(u_k,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})}+ \\
&\quad c[\gamma_d(u_k, u_{k+2}) + \gamma_d(u_k, u_{k+1})] \\
&\leq a[\gamma_d(u_k, u_{k+1})] + \frac{3b[\gamma_d(u_k,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})]^2}{1+\gamma_d(u_k,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})}+ \\
&\quad c[\gamma_d(u_k, u_{k+1}) + \gamma_d(u_{k+1}, u_{k+2}) + \gamma_d(u_k, u_{k+1})] \\
|\gamma_d(u_{k+1}, u_{k+2})| &\leq a|\gamma_d(u_k, u_{k+1})| + 3b|\gamma_d(u_k, u_{k+1}) + \gamma_d(u_{k+1}, u_{k+2})|+ \\
&\quad c|2\gamma_d(u_k, u_{k+1}) + \gamma_d(u_{k+1}, u_{k+2})|
\end{aligned}$$

Since
$|1 + d(u_k, u_{k+1}) + d(u_{k+1}, u_{k+2})| > |d(u_k, u_{k+1}) + d(u_{k+1}, u_{k+2})|$
Now

$$\begin{aligned}
|\gamma_d(u_{k+1}, u_{k+2})| &\leq a|\gamma_d(u_k, u_{k+1})| + 3b|\gamma_d(u_k, u_{k+1})| + 3b|\gamma_d(u_{k+1}, u_{k+2})|+ \\
&\quad 2c|\gamma_d(u_k, u_{k+1})| + c|\gamma_d(u_{k+1}, u_{k+2})|
\end{aligned}$$

Therefore $|\gamma_d(u_{k+1}, u_{k+2})| \leq \frac{a+3b+2c}{1-(3b+c)}|\gamma_d(u_k, u_{k+1})|$
Similarly,

$$\begin{aligned}
\gamma_d(u_{k+2}, u_{k+3}) &= \gamma_d(Gu_{k+1}, Tu_{k+2}) \\
&\leq a[\gamma_d(u_{k+1}, u_{k+2})] + \frac{3b[\gamma_d(u_{k+1},Tu_{k+2})]^2}{1+\gamma_d(u_{k+1},u_{k+2})+\gamma_d(u_{k+2},Tu_{k+2})}+ \\
&\quad c[\gamma_d(u_{k+1}, Tu_{k+2}) + \gamma_d(u_{k+1}, Gu_{k+1})] \\
&\leq a[\gamma_d(u_{k+1}, u_{k+2})] + \frac{3b[\gamma_d(u_{k+1},u_{k+3})]^2}{1+\gamma_d(u_{k+1},u_{k+2})+\gamma_d(u_{k+2},u_{k+3})}+ \\
&\quad c[\gamma_d(u_{k+1}, u_{k+3}) + \gamma_d(u_{k+1}, u_{k+2})] \\
&\leq a[\gamma_d(u_{k+1}, u_{k+2})] + \frac{3b[\gamma_d(u_{k+1},u_{k+2})+\gamma_d(u_{k+2},u_{k+3})]^2}{1+\gamma_d(u_{k+1},u_{k+2})+\gamma_d(u_{k+2},u_{k+3})}+ \\
&\quad c[\gamma_d(u_{k+1}, u_{k+2}) + \gamma_d(u_{k+2}, u_{k+3}) + \gamma_d(u_{k+1}, u_{k+2})]
\end{aligned}$$

$$\begin{aligned}
|\gamma_d(u_{k+2}, u_{k+3})| &\leq a|\gamma_d(u_{k+1}, u_{k+2})| + 3b|\gamma_d(u_{k+1}, u_{k+2}) + \gamma_d(u_{k+2}, u_{k+3})|+ \\
&\quad c|2\gamma_d(u_{k+1}, u_{k+2}) + \gamma_d(u_{k+2}, u_{k+3})|
\end{aligned}$$

Since
$|1 + d(u_{k+1}, u_{k+2}) + d(u_{k+2}, u_{k+3})| > |d(u_{k+1}, u_{k+2}) + d(u_{k+2}, u_{k+3})|$
Now

$$\begin{aligned}
|\gamma_d(u_{k+2}, u_{k+3})| &\leq a|\gamma_d(u_{k+1}, u_{k+2})| + 3b|\gamma_d(u_{k+1}, u_{k+2})| + 3b|\gamma_d(u_{k+2}, u_{k+3})|+ \\
&\quad 2c|\gamma_d(u_{k+1}, u_{k+2})| + c|\gamma_d(u_{k+2}, u_{k+3})|
\end{aligned}$$

Therefore
$|\gamma_d(u_{k+2}, u_{k+3})| \leq \frac{a+3b+2c}{1-(3b+c)}|\gamma_d(u_{k+1}, u_{k+2})|$

Since $a+3b+2c<1$ therefore $\alpha=\frac{a+3b+2c}{1-(3b+c)}<1$
Then, we have
$|\gamma_d(u_{n+1},u_{n+2})|\leq\alpha|\gamma_d(u_n,u_{n+1})|\leq ....\leq\alpha^{n+1}|\gamma_d(u_0,u_1)|$
Therefore for every $m>n$ we have
$|\gamma_d(u_n,u_m)|\leq|\gamma_d(u_n,u_{n+1})|+|\gamma_d(u_{n+1},u_{n+2})|+........+|\gamma_d(u_{m-1},u_m)|$
$|\gamma_d(u_n,u_m)|\leq[\alpha^n+\alpha^{n+1}+........+\alpha^{m-1}]|\gamma_d(u_0,u_1)|$
$\leq\frac{\alpha^n}{1-\alpha}|\gamma_d(u_0,u_1)|$
$\gamma_d(u_n,u_m)|\leq\frac{\alpha^n}{1-\alpha}|\gamma_d(u_0,u_1)|\to 0$ as $n,m\to\infty$
Hence $\{u_n\}$ is a cauchy sequence. Since H is complete there must exist $x\in H$ such that $\{u_n\}\to x$ as $n\to\infty$
Suppose on contrary that $x\neq Gx$ so $\gamma_d(x,Gx)=y$
Now
$y\leq\gamma_d(x,x_{k+2})+\gamma_d(u_{k+2},Gx)$
$\leq\gamma_d(x,u_{k+2})+\gamma_d(Tu_{k+1},Gx)$
$\leq a[\gamma_d(x,u_{k+1})]+\frac{3b[\gamma_d(x,Tu_{k+1})]^2}{1+\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},Tu_{k+1})}+c[\gamma_d(x,Tu_{k+1})+\gamma_d(x,Gx)]$
$\leq a[\gamma_d(x,u_{k+1})]+\frac{3b[\gamma_d(x,u_{k+2})]^2}{1+\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})}+c[\gamma_d(x,u_{k+2})+\gamma_d(x,Gx)]$
$|y|\leq a|\gamma_d(x,u_{k+1})|+3b|\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})|+c|\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})+\gamma_d(x,Gx)|$
Since
$|1+\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})|>|\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})|$
Therefore
$|y|\leq a|\gamma_d(x,u_{k+1})|+3b|\gamma_d(x,u_{k+1})|+3b|\gamma_d(u_{k+1},u_{k+2})|+c|\gamma_d(x,u_{k+1})|+$
$c|\gamma_d(u_{k+1},u_{k+2})|+c|\gamma_d(x,Gx)|$
Letting $n\to\infty$ we have
$|\gamma_d(x,Gx)|\leq\frac{a+6b+2c}{1-c}|\gamma_d(x,x)|$ Since $a+6b+2c<1$
Therefore, we have $|\gamma_d(x,Gx)|\to 0$ which is the contradiction.
Hence $Gx=x$ similarly we prove that $Tx=x$
To prove the uniqueness of common fixed point of G and T, let $d\in H$ be the another common fixed point of G and T, we have
$\gamma_d(x,d)=\gamma_d(Gx,Td)\leq a[\gamma_d(x,d)]+\frac{3b[\gamma_d(x,Td)]^2}{1+\gamma_d(x,d)+\gamma_d(d,Td)}+c[\gamma_d(x,Td)+\gamma_d(x,Gx)]$

$\leq a[\gamma_d(x,d)]+\frac{3b[\gamma_d(x,d)]^2}{1+\gamma_d(x,d)+\gamma_d(d,d)}+c[\gamma_d(x,d)+\gamma_d(x,x)]$

$\leq a[\gamma_d(x,d)]+\frac{3b[\gamma_d(x,d)+\gamma_d(d,d)]^2}{1+\gamma_d(x,d)+\gamma_d(d,d)}+c[\gamma_d(x,d)+\gamma_d(x,x)]$
$|\gamma_d(x,d)|\leq a|\gamma_d(x,d)|+3b|\gamma_d(x,d)+\gamma_d(d,d)|+c|\gamma_d(x,d)+\gamma_d(x,x)|$
Since $|1+\gamma_d(x,d)+\gamma_d(d,d)|>|\gamma_d(x,d)+\gamma_d(d,d)|$
Now,
$|\gamma_d(x,d)|\leq a|\gamma_d(x,d)|+3b|\gamma_d(x,d)|+3b|\gamma_d(d,d)|+c|\gamma_d(x,d)|+c|\gamma_d(x,x)|$
$|\gamma_d(x,d)|\leq\frac{3b}{1-(a+3b+c)}|\gamma_d(d,d)|+\frac{c}{1-(a+3b+c)}|\gamma_d(x,x)|$
Since $a+6b+3c<1$ therefore we have $x=d$ which shows the uniqueness of common fixed point. ■

Corollary 3.3. *Let (H,γ_d) be a complete complex valued dislocated metric space. Consider the function $G,T:H\to H$ which satisfies the rational contraction conditions $\gamma_d(Gu,Tr)\leq a[\gamma_d(u,r)]+c[\gamma_d(u,Tr)+\gamma_d(u,Gu)]$ for each $u,r\in H$ and the non negativity constants are a,c with $2a+3c<1$ .Then G has unique common fixed point.*

Corollary 3.4. *Let (H,γ_d) be a complete complex valued dislocated metric space. Consider the function $G:H\to H$ which satisfies the contraction conditions $\gamma_d(Gu^n,Gr^n)\leq a[\gamma_d(u,r)]$ for each $u,r\in H$ and the non negativity constant a with $a<1$ .Then G has unique fixed point.*

Example 3.5. *Let X= C be set of complex numbers.Define $f:C\times C\to C$as follows where $z_1=x_1+iy_1$*

$z_2 = x_2 + iy_2$ *.Then* (C, f) *is a complete complex valued dislocated metric space.*
Define $G : C \to C$ *as*

$$G(x) = \begin{cases} 0, & \text{if } x, y \in Q. \\ 1+2i, & \text{if } x, y \in Q^c \\ 2 & \text{if } x \in Q^c, y \in Q \\ 5i & \text{if } x \in Q, y \in Q^c \end{cases}$$

Let us consider $x = \sqrt{3}$ *and* $y = 0$ *we obtain,*
$f(G(\sqrt{3}), L(0)) = f(3, 0) = 3 \preccurlyeq \alpha f(\sqrt{3}, 0) = \alpha\sqrt{3}$
Therefore ,$\alpha \succcurlyeq \sqrt{3}$*, which is a contradiction as* $0 \preccurlyeq \alpha \prec 1$
We notice that $G^2 z = 0$ *so that* $0 = f(G^2 z_1, G^2 z_2) \preccurlyeq \alpha f(z_1, z_2)$ *which shows that* G^2 *satisfies the requirement of Bryant theorem and z=0 is the unique fixed point of T.*

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On solutions of the Diophantine equation $L_n + L_m = 3^a$

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Abstract. Let $(L_n)_{n\geq 0}$ be the Lucas sequence given by $L_0 = 2, L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. In this paper, we are interested in finding all powers of three which are sums of two Lucas numbers, i.e., we study the exponential Diophantine equation $L_n + L_m = 3^a$ in nonnegative integers n, m, and a. The proof of our main theorem uses lower bounds for linear forms in logarithms, properties of continued fractions, and a version of the Baker-Davenport reduction method in Diophantine approximation.

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Contents

1. Introduction

The determination of perfect powers of Lucas and Fibonacci sequences does not date from today. The real contribution of determination of perfect powers of Lucas and Fibonacci sequences began in 2006. By classical and modular approaches of Diophantine equations, Bugeaud, Mignotte, and Siksek [5] defined all perfect powers of Lucas and Fibonacci sequences by solving the equations $F_n = y^p$ and $L_n = y^p$ respectively. From there, many researchers tackled similar problems. It is in the same thought that, others have determined the powers of 2 of the sum/difference of two Lucas numbers [3], powers of 2 of the sum/difference of Fibonacci numbers [4], powers of 2 and of 3 of the product of Pell numbers and Fibonacci numbers.

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We move our interest on the powers of 3 as a sum of two Lucas numbers. This paper follows the following steps : We first give the generalities on binary linear recurrence, then we demonstrate an important inequality on Lucas numbers and finally determine and reduce a coarse bound by section 3. The section 4 is devoted to the reduction of the obtained bound in section 3 and discussion of possible different cases. We know from Bravo and Lucas [3] that the only solutions of the Diophantine equation $F_n + F_m = 2^a$ in positive integers n, m and a with $n \geq m$ are given by

$$2F_1 = 2, \quad 2F_2 = 2, \quad 2F_3 = 4, \quad 2F_6 = 16,$$

and

$$F_2 + F_1 = 2, \quad F_4 + F_1 = F_4 + F_2 = 4, \quad F_5 + F_4 = 8, \quad F_7 + F_4 = 16.$$

and in [4] that all solutions of the Diophantine equation $L_n + L_m = 2^a$ in nonnegative integers $n \geq m$ and a, are

$$2L_0 = 4, \quad 2L_1 = 2, \quad 2L_3 = 8, \quad L_2 + L_1 = 4, \quad L_4 + L_1 = 8, \quad \text{and} \quad L_7 + L_2 = 32.$$

Here in this paper, we determine all the solutions of the following Diophantine equation:

$$L_n + L_m = 3^a \tag{1.1}$$

in nonnegative integers $n \geq m$ and a.

We are interested in finding all powers of three which are sums of two Lucas numbers, i.e., we study the exponential Diophantine equation $L_n + L_m = 3^a$ in nonnegative integers n, m, and a. The proof of our main theorem uses lower bounds for linear forms in logarithms, properties of continued fractions, and a version of the Baker-Davenport reduction method in Diophantine approximation.

We notice that many authors have already tackled this type of problems.

2. Preliminaries

2.1. Generalities

Definition 2.1. *Let $k \geq 1$. The sequence $\{H_n\}_{n\geq 0} \subseteq \mathbb{C}$ is called a recurrent linear sequence of order k if the sequence satisfies*

$$H_{n+k} = a_1 H_{n+k-1} + a_2 H_{n+k-2} + \cdots + a_k H_n$$

for all $n \geq 0$ with $a_1, \ldots, a_k \in \mathbb{C}$, fixed.

We suppose that $a_k \neq 0$ (otherwise, the sequence $\{H_n\}_{n\geq 0}$ satisfies a recurrence of order less than k). If $a_1, \ldots, a_k \in \mathbb{Z}$ and $H_0, \ldots, H_{k-1} \in \mathbb{Z}$, then we can easily prove by induction on n that H_n is an integer for all $n \geq 0$. The polynomial

$$f(X) = X^k - a_1 X^{k-1} - a_2 X^{k-2} - \cdots - a_k \in \mathbb{C},$$

is called the characteristic polynomial of $(H_n)_{n\geq 0}$. We suppose that

$$f(X) = \prod_{i=1}^{m} (X - \alpha_i)^{\sigma_i},$$

where $\alpha_1, \ldots, \alpha_m$ are distinct roots of $f(X)$ with respectively $\sigma_1, \ldots, \sigma_m$ their multiplicities.

Definition 2.2. *We define the sequences $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ for all positive integers $\mathbb{N}$ by*

$$\begin{cases} A_{n+2} = aA_{n+1} + A_n, \; A_0 = 0, \; A_1 = 1 \\ B_{n+2} = aB_{n+1} + B_n, \; B_0 = 2, \; B_1 = a. \end{cases}$$

For $a = 1$, $(A_n)_{n\geq 0} = (F_n)_{n\geq 0}$ and $(B_n)_{n\geq 0} = (L_n)_{n\geq 0}$, which are Fibonacci and Lucas sequences respectively, defined above.

Remark 2.3. *If $k = 2$, the sequence $(H_n)_{n\geq 0}$ is called a binary recurrent sequence. In this case, the characteristic polynomial is of the form*

$$f(X) = X^2 - a_1 X - a_2 = (X - \alpha_1)(X - \alpha_2).$$

Suppose that $\alpha_1 \neq \alpha_2$, then $H_n = c_1\alpha_1^n + c_2\alpha_2^n$ for all $n \geq 0$.

Definition 2.4. *The binary recurrent sequence $\{H_n\}_{n\geq 0}$ is said to be non degenerated if $c_1c_2\alpha_1\alpha_2 \neq 0$ and α_1/α_2 is not a root of unity.*

Binet's formula for the general term of Fibonacci and Lucas sequences is obtained using standard methods for solving recurrent sequences, which are given by :

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

where $(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ are the zeros of the characteristic polynomial $X^2 - X - 1$.

Definition 2.5. *For all algebraic numbers γ, we define its measure by the following identity :*

$$\mathrm{M}(\gamma) = |a_d| \prod_{i=1}^{d} \max\{1, |\gamma_i|\},$$

where γ_i are the roots of $f(x) = a_d \prod_{i=1}^{d}(x - \gamma_i)$ is the minimal polynomial of γ.

Let us define now another height, deduced from the last one, called the absolute logarithmic height. It is the most used one.

Definition 2.6. *(Absolute logarithmic height)*

For a non-zero algebraic number of degree d on $\mathbb{Q}$ where the minimal polynomial on $\mathbb{Z}$ is $f(x) = a_d \prod_{i=1}^{d}(x - \gamma_i)$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log|a_d| + \sum_{i=1}^{d} \log\max\{1, |\gamma_i|\}\right) = \frac{1}{d}\log \mathrm{M}(\gamma).$$

the usual logarithmic absolute height of γ.

The following properties of the logarithmic height, will also be used in the next section:

- $h(\gamma \pm \eta) \leq h(\gamma) + h(\eta) + \log 2$.
- $h(\gamma\eta^{\pm 1}) \leq h(\gamma) + h(\eta)$.
- $h(\gamma^s) = |s| h(\gamma)$.

2.2. Inequalities involving the Lucas numbers

In this section, we state and prove important inequalities associated with the Lucas numbers that will be used in solving the equation (1.1).

Proposition 2.7. *For $n \geq 2$, we have*

$$0.94\,\alpha^n < (1 - \alpha^{-6})\alpha^n \leq L_n \leq (1 + \alpha^{-4})\alpha^n < 1.15\,\alpha^n \tag{2.1}$$

Proof. This follows directly from the formula $L_n = \alpha^n + (-1)^n\alpha^{-n}$. ∎

2.3. Linear forms in logarithms and continued fractions

In order to prove our main result, we have to use a Baker-type lower bound several times for a non-zero linear forms of logarithms in algebraic numbers. There are many of these methods in the literature like that of Baker and Wüstholz in [1]. We recall the result of Bugeaud, Mignotte, and Siksek which is a modified version of the result of Matveev [8]. With the notation of section 2, Laurent, Mignotte, and Nesterenko [7] proved the following theorem:

Theorem 2.8. *Let γ_1, γ_2 be two non-zero algebraic numbers, and let $\log\gamma_1$ and $\log\gamma_2$ be any determination of their logarithms. Put $D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]/[\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}]$, and*

$$\Gamma := b_2 \log\gamma_2 - b_1 \log\gamma_1,$$

where b_1 and b_2 are positive integers. Further, let A_1, A_2 be real numbers > 1 such that

$$\log A_i \geq \max\left\{h(\gamma_i), \frac{|\log\gamma_i|}{D}, \frac{1}{D}\right\}, \quad (i = 1, 2).$$

Then, assuming that γ_1 and γ_2 are mutiplicatively independent, we have

$$\log|\Gamma| > -30.9 \cdot D^4 \left(\max\left\{\log b', \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log A_1 \cdot \log A_2,$$

where

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

We shall also need the following theorem due to Matveev, Lemma due to Dujella and Pethő and Lemma due to Legendre [6, 8].

Theorem 2.9. *(Matveev [8])*

Let $n \geq 1$ an integer. Let $\mathbb{L}$ a field of algebraic number of degree D. Let $\eta_1, \ldots, \eta_l$ non-zero elements of $\mathbb{L}$ and let $b_1, b_2, \ldots, b_l$ integers,

$$B := \max\{|b_1|, ..., |b_l|\},$$

and

$$\Lambda := \eta_1^{b_1} \cdots \eta_l^{b_l} - 1 = \left(\prod_{i=1}^{l} \eta_i^{b_i}\right) - 1.$$

Let $A_1, \ldots, A_l$ reals numbers such that

$$A_j \geq \max\{Dh(\eta_j), |\log(\eta_j)|, 0.16\}, 1 \leq j \leq l.$$

Assume that $\Lambda \neq 0$, So we have

$$\log|\Lambda| > -3 \times 30^{l+4} \times (l+1)^{5.5} \times d^2 \times A_1 ... A_l(1 + \log D)(1 + \log nB)$$

Further, if $\mathbb{L}$ is real, then

$$\log|\Lambda| > -1.4 \times 30^{l+3} \times (l)^{4.5} \times d^2 \times A_1 ... A_l(1 + \log D)(1 + \log B).$$

During our calculations, we get upper bounds on our variables which are too large, so we have to reduce them. To do this, we use some results from the theory of continued fractions. In particular, for a non-homogeneous linear form with two integer variables, we use a slight variation of a result due to Dujella and Pethő, (1998) which is in itself a generalization of the result of Baker and Davemport [2].

For a real number X, we write $\|X\| := \min\{| X - n |: n \in \mathbb{Z}\}$ for the distance of X to the nearest integer.

Lemma 2.10. *(Dujella and Pethő, [6])*

Let M a positive integer, let p/q the convergent of the continued fraction expansion of κ such that $q > 6M$ and let A, B, μ real numbers such that $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\,\|\kappa q\|$.
If $\varepsilon > 0$ then there is no solution of the inequality

$$0 < m\kappa - n + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leqslant m \leqslant M.$$

Lemma 2.11. *(Legendre)*

Let τ real number such that x, y are integers such that

$$\left|\tau - \frac{x}{y}\right| < \frac{1}{2y^2}.$$

then $\frac{x}{y} = \frac{p_k}{q_k}$ is the convergence of τ.

Further,

$$\left|\tau - \frac{x}{y}\right| > \frac{1}{(q_{k+1}+2)y^2}.$$

3. Main Results

Our main result can be stated in the following theorem.

Theorem 3.1. *The only solutions (n, m, a) of the exponential Diophantine equation*
$L_n + L_m = 3^a$ in nonnegative integers $n \geq m$ and a, are : $(1, 0, 1)$ and $(4, 0, 2)$

$$\textit{i.e}\quad L_1 + L_0 = 3, \quad \textit{and} \quad L_4 + L_0 = 9.$$

Proof. First, we study the case $n = m$, next we assume $n > m$ and study the case $n \leq 200$ with *SageMath* in the range $0 \leq m < n \leq 200$ and finally we study the case $n > 200$. Assume throughout that equation (1.1) holds. First of all, observe that if $n = m$, then the original equation (1.1) becomes

$$L_n = \frac{3^a}{2}.$$

This equation has no solution because, $\forall n > 0, L_n \in \mathbb{Z}$. So from now, we assume $n > m$.
If $n \leq 200$, the search with *SageMath* in the range $0 \leq m < n \leq 200$ gives the solutions $(n, m, a) \in \{(1, 0, 1), (4, 0, 2)\}$. Now for the rest of the paper, we assume that $n > 200$. Let first get a relation between a and n which is important for our purpose. Combining (1.1) and the right inequality of (2.1), we get:

$$3^a = L_n + L_m \leq 2\alpha^n + 2\alpha^m < 2^{n+1} + 2^{m+1} = 2^{n+1}(1 + 2^{n-m}) \leq 2^{n+1}(1 + 1/2) < 2^{n+2}.$$

Taking log both sides, we obtain

$$a \log 3 \leq (n+2)\log 2 \Longrightarrow a \leq (n+2)c_1 \quad \text{where} \quad c_1 = \frac{\log 2}{\log 3}.$$

Rewriting equation (1.1) as:

$$L_n + L_m = \alpha^n + \beta^n + L_m = 3^a \Longrightarrow \alpha^n - 3^a = -\beta^n - L_m.$$

Taking absolute value both sides, we get

$$|\alpha^n - 3^a| = |\beta^n + L_m| \leq |\beta|^n + L_m < \frac{1}{2} + 2\alpha^m \quad \because |\beta|^n < \frac{1}{2}, \quad \text{and} \quad L_m < 2\alpha^m.$$

Dividing both sides by α^n and considering that $n > m$, we get:

$$\left|1 - \alpha^{-n} \cdot 3^a\right| < \frac{\alpha^{-n}}{2} + 2\alpha^{m-n} < \frac{1}{\alpha^{n-m}} + \frac{2}{\alpha^{n-m}} \quad \because \frac{1}{2\alpha^n} < \frac{1}{\alpha^{n-m}}; \quad n > m$$

Hence

$$\left|1 - \alpha^{-n} \cdot 3^a\right| < \frac{3}{\alpha^{n-m}} \tag{3.1}$$

Let's take

$$\gamma_1 := \alpha, \quad \gamma_2 := 3, \quad b_1 := n, \quad b_2 := a, \quad \Gamma := a \log 3 - n \log \alpha$$

in order to apply Theorem 2.8. Therefore equation (3.1) can be rewritten as:

$$\left|1 - e^{\Gamma}\right| < \frac{3}{\alpha^{n-m}} \quad \text{where} \quad e^{\Gamma} = \alpha^{-n} 3^a. \tag{3.2}$$

Since $\mathbb{Q}(\sqrt{5})$ is the algebraic number field containing γ_1, γ_2; so we can take $D := 2$. Using equation (1.1) and Binet formula for Lucas sequence, we have :

$$\alpha^n = L_n - \beta^n < L_n + 1 \leq L_n + L_m = 3^a$$

which implies $1 < 3^a \alpha^{-n}$ and so $\Gamma > 0$. Combining this with (3.2), we get

$$0 < \Gamma < \frac{3}{\alpha^{n-m}} \tag{3.3}$$

where we used the fact that $x \leq e^x - 1, \quad \forall x \in \mathbb{R}$. Applying log on right and left hand side of (3.3), we get

$$\log \Gamma < \log 3 - (n - m) \log \alpha. \tag{3.4}$$

Logarithm height of γ_1 and γ_2 are:

$h(\gamma_1) = \frac{1}{2} \log \alpha = 0.2406 \cdots$, $h(\gamma_2) = \log 3 = 1.09862 \cdots$, thus we can choose

$$\log A_1 := 0.5 \quad \text{and} \quad \log A_2 := 1.1.$$

Finally, by recalling that $a \leq (n + 2)c_1; \quad c_1 = 0.63093$, we get :

$$b' := \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1} = \frac{n}{2.2} + a = 0.45n + a < 0.45n + (n + 2)c_1 < 2n.$$

It is easy to see that α and 3 are multiplicatively independent. Then by Theorem 2.8, we have

$$\log \Gamma \geq -30.9 \cdot 2^4 \left(\max \left\{ \log(2n), \frac{21}{2}, \frac{1}{2} \right\} \right)^2 \cdot 0.5 \cdot 1.1$$

$$\log \Gamma > -272 \left(\max \left\{ \log(2n), \frac{21}{2}, \frac{1}{2} \right\} \right)^2. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain the following important result

$$(n - m) \log \alpha < 276 \left(\max \left\{ \log(2n), \frac{21}{2}, \frac{1}{2} \right\} \right)^2. \tag{3.6}$$

Let us find a second linear form in logarithm. For this, we rewrite (1.1) as follows:

$$\alpha^n(1+\alpha^{n-m}) - 3^a = -\beta^n - \beta^m.$$

Taking absolute values in the above relation, we get

$$|\alpha^n(1+\alpha^{m-n}) - 3^a| < 2, \quad \beta = (1-\sqrt{5})/2, \quad |\beta|^n < 1 \quad \text{and} \quad |\beta|^m < 1; \forall n > 200, \quad m \geq 0.$$

Dividing both sides of the above inequality by $\alpha^n(1+\alpha^{m-n})$, we obtain

$$\left|1 - 3^a\alpha^{-n}(1+\alpha^{m-n})^{-1}\right| < \frac{2}{\alpha^n} \quad \text{i.e} \quad |\Lambda| < \frac{2}{\alpha^n}. \tag{3.7}$$

All the conditions are now met to apply a Matveev's theorem (Theorem 2.9).

- Data:

$$t := 3; \quad \gamma_1 := 3; \quad \gamma_2 := \alpha; \quad \gamma_3 := 1 + \alpha^{m-n}$$

$$b_1 := a; \qquad b_2 := -n, \quad b_3 = -1.$$

  As before, $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ contains $\gamma_1, \gamma_2, \gamma_3$ and has $D := [\mathbb{K} : \mathbb{Q}] = 2$. Before continuing with the calculations, let's check whether $\Lambda \neq 0$.

  $\Lambda \neq 0$ comes from the fact that if it was zero, we would have

$$3^a = \alpha^n + \alpha^m \tag{3.8}$$

  Taking the conjugate of the above relation in $\mathbb{Q}(\sqrt{5})$, we get :

$$3^a = \beta^n + \beta^m. \tag{3.9}$$

  Combining (3.8) and (3.9), we get :

$$\alpha^n < \alpha^n + \alpha^m = |\beta^n + \beta^m| \leq |\beta|^n + |\beta|^m < 2.$$

  Recall that $n > 200$. This relation is impossible for $n > 200$. Hence $\Lambda \neq 0$.

- **Calculation of** $h(\gamma_3)$

  Let us now estimate $h(\gamma_3)$ where $\gamma_3 = 1 + \alpha^{m-n}$

$$\gamma_3 = 1 + \alpha^{m-n} < 2 \quad \text{and} \quad \gamma^{-1} = \frac{1}{1+\alpha^{m-n}} < 1$$

  so $|\log \gamma_3| < 1$. Notice that

$$h(\gamma_3) \leq |m-n| \left(\frac{\log \alpha}{2}\right) + \log 2 = \log 2 + (n-m)\left(\frac{\log \alpha}{2}\right).$$

- The calculation of A_1 and A_2 gives :

$$A_1 := 2.2$$

  and

$$A_2 := 0.5$$

  and we can take

$$A_3 := 2 + (n-m)\log\alpha \quad \text{since} \quad h(\gamma_3) := \log 2 + (n-m)\left(\frac{\log\alpha}{2}\right)$$

- **Calculation of B**

  Since $a < (n+2)c_1$, it follows that, $B = \max\{1, n, a\}$. Thus we can take $B = n+1$.

The Matveev's theorem gives the lower bound on the left hand side of (3.7) by replacing the data. We get :

$$\exp\left(-C(1+\log(n+1))\cdot 2.2\cdot 0.5\cdot(2+(n-m)\log\alpha)\right)$$

where

$$C := 1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2) < 9.7\times 10^{11}.$$

Replacing in equation (3.7), we get:

$$\exp\left(-C(1+\log(n+1))\cdot 2.2\cdot 0.5\cdot(2+(n-m)\log\alpha)\right) < |\Lambda| < \frac{2}{\alpha^n}$$

which leads to

$$n\log\alpha - \log 2 < C((1+\log(n+1))\cdot 1.1\cdot(2+(n-m)\log\alpha) < 2C\log n\cdot 1.1\cdot(2+(n-m)\log\alpha)$$

then

$$n\log\alpha - \log 2 < 1.26\times 10^{12}\log n\cdot(2+(n-m)\log\alpha) \tag{3.10}$$

where we used inequality $1+\log(n+1) < 2\log n$, which holds for $n > 200$.

Now, using (3.6) in the right term of the above inequality (3.10) and doing the related calculations, we get

$$n < 7.3\times 10^{14}\log n\left(\max\left\{\log(2n), \frac{21}{2}\right\}\right)^2. \tag{3.11}$$

If $\max\{\log(2n), 21/2\} = 21/2$, it follows from (3.11) that $n < 8.04825\cdot 10^{16}\log n \Longrightarrow n < 3.5\cdot 10^{18}$. On the other hand, if $\max\{\log(2n), 21/2\} = \log(2n)$, then from (3.11), we get $n < 7.3\cdot 10^{14}\log n\log^2(2n)$ and so $n < 7.2\cdot 10^{19}$. We can easily see that for the two possible values of $\max\{\log(2n), 21/2\}$, $n < 7.2\cdot 10^{19}$.

All the calculations done so far can be summarized in the following lemma.

Lemma 3.2. *If (n, m, a) is a solution in positive integers of (1.1) with conditions $n > m$ and $n > 200$, then inequalities*

$$a \leq n+2 < 1.2\times 10^{20} \quad \textit{hold.}$$

4. Reducing of the bound on n

Dividing across inequality (3.3) : $0 < a\log 3 - n\log\alpha < \dfrac{3}{\alpha^{n-m}}$ by $\log\alpha$, we get

$$0 < a\gamma - n < \frac{7}{\alpha^{n-m}}; \quad \text{where} \quad \gamma := \frac{\log 3}{\log\alpha}. \tag{4.1}$$

The continued fraction of the irrational number γ is :

$$[a_0, a_1, a_2, ...] = [1, 2, 3, 1, 1, 2, 3, 2, 4, 2, 1, 11, 2, 1, 11, ...]$$

and let denote p_k/q_k its convergent. An inspection using *SageMath* gives the following inequality

$$49778965253620415 75 = q_{41} < 1.2 \times 10^{20} < q_{42} = 805929983250536127817.$$

Furthermore, $a_M := \max\{a_i | i = 0, 1, ..., 42\} = 161$ Now applying Lemma 2.11 and properties of continued fractions, we obtain

$$|a\gamma - n| > \frac{1}{(a_M + 2)a}. \tag{4.2}$$

Combining equation (4.1) and (4.2), we get

$$\frac{1}{(a_M+2)a} < |a\gamma - n| < \frac{7}{\alpha^{n-m}} \Longrightarrow \frac{1}{(a_M+2)a} < \frac{7}{\alpha^{n-m}} \Longrightarrow \alpha^{n-m} < 7\cdot(161+2)a < 1.3692\cdot 10^{23}.$$

Applying $\log$ above and divide by $\log\alpha$, we get :

$$(n-m) \le \frac{\log(7\cdot 163\cdot a)}{\log\alpha} < 111.$$

To improve the upper bound on n, let consider

$$z := a\log 3 - n\log\alpha - \log\rho(u) \quad \text{where} \quad \rho = 1 + \alpha^{-u}. \tag{4.3}$$

From (3.7), we have

$$|1 - e^z| < \frac{2}{\alpha^n}. \tag{4.4}$$

Since $\Lambda \neq 0$, then $z \neq 0$. Two cases arise : $z < 0$ and $z > 0$. For each case, we will apply Lemma 2.10.

- **Case 1 :** $z > 0$

  From (4.4), we obtain $0 < z \le e^z - 1 < \dfrac{2}{\alpha^n}$. Replacing (4.3) in the above inequality, we get:

  $$0 < a\log 3 - n\log\alpha - \log\rho(n-m) \le 3^a\alpha^{-n}\rho(n-m)^{-1} - 1 < 2\alpha^{-n}$$

  hence

  $$0 < a\log 3 - n\log\alpha - \log\rho(n-m) < 2\alpha^{-n}$$

  and by diving above inequality by $\log\alpha$

  $$0 < a\left(\frac{\log 3}{\log\alpha}\right) - n - \frac{\log\rho(n-m)}{\log\alpha} < 5\cdot\alpha^{-n}. \tag{4.5}$$

  Taking, $\gamma := \dfrac{\log 3}{\log\alpha}, \quad \mu := -\dfrac{\log\rho(n-m)}{\log\alpha}, \quad A := 5, \quad B := \alpha$, inequality (4.5) becomes

  $$0 < a\gamma - n + \mu < AB^{-n}.$$

Since γ is irrational, we are now ready to apply lemma 2.10 of Dujella and Pethö on (4.5) for $n - m \in \{1, ..., 111\}$.
Since $a \leq 1.2 \times 10^{20}$ from lemma 3.2, we can take $M = 1.2 \times 10^{20}$, and we get

$$n < \frac{\log(Aq/\varepsilon)}{\log B} \quad \text{where} \quad q > 6M$$

and q is the denominator of the convergent of the irrational number γ such that $\varepsilon := ||\mu q|| - M||\gamma q|| > 0$. With the help of *SageMath*, with conditions $z > 0$, and (n, m, a) a possible zero of (1.1), we get $n < 112$ which contradicts our assumption $n > 200$. Then it is false.

- **Case 2 :** $z < 0$

  Since $n > 200$, then $\frac{2}{\alpha^n} < \frac{1}{2}$. Hence (4.4) implies that $|1 - e^{|z|}| < 2$. Also, since $z < 0$, we have

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^{|z|} - 1| < \frac{4}{\alpha^n}.$$

  Replacing (4.3) in the above inequality and dividing by $\log 3$, we get:

$$0 < n\left(\frac{\log \alpha}{\log 3}\right) - a + \frac{\rho(n-m)}{\log 3} < \frac{4}{\log 3} \cdot \alpha^{-n} < 4 \cdot \alpha^{-n} \tag{4.6}$$

In order to apply lemma 3.2 on (4.6) for $n - m \in \{1, 2, ..., 111\}$, let's take again $M = 1.2 \times 10^{20}$. With the help of *SageMath*, with conditions $z < 0$, and (n, m, a) a possible zero of (1.1), we get $n < 111$ which contradicts our assumption $n > 200$. Then it is false.

This completes the proof of our main result (Theorem 3.1). ■

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A characterization of involutes of a given curve in $\mathbb{E}^3$ via directional q-frame

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Abstract. The orthogonal trajectories of the first tangents of the curve are called the involutes of α. In the present study, we obtain a characterization of involute curves of order k of the given curve α using directional q-frame. In virtue of the formulas, some results are obtained.

AMS Subject Classifications: Primary 53a04; Secondary 53C26.

Keywords: Frenet curve, Frenet frame, involute curve, directional q-frame.

Contents

1. Introduction

In differential geometry, there are many significant results and properties of curves. In the light of numerous studies authors introduce new works by using frame fields. The directional q- frame field is known as one of the frame field of the differential geometry. The q-frame has some useful advantages comparing to the other well-known frames Frenet and Bishop. One can define and calculate this frame even along a line ($\kappa = 0$). Dede et al. offered the directional q-frame along a space curve to built a tubular surface. They obtained a parametric representation of a directional tubular surface using the q-frame $[1]$.

Involutes of a curve is another attractive research subject among geometers. The idea of a string involute is due to C. Huygens (1658), who is also known as on optician. He discovered involutes trying to build a more accurate clock $[2]$. There are many brillant works on involutes of a given curve in different aspects. For instance, Frenet frame of involute-evolute couple in the space $\mathbb{E}^3$ were given in $[3]$. T. Soyfidan and M. A. Güngör studied a quaternionic curve Euclidean 4-space $\mathbb{E}^4$ and gave the on the quaternionic involute-evolute curves for quaternionic curve $[4]$. Another is As and Sarıoğlugil study's. They obtained on the Bishop curvatures of involute-evolute curve couple in $\mathbb{E}^3$ $[5]$.

In this paper, the characterization of involutes of the 1 st. and 2 nd. order of a curve are given and proved in .$\mathbb{E}^3$ by the help of directional q-frame.

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2. Preliminaries

There are a number of different adapted frames along a space curve, like the parallel transport frame [6, 7] and the Frenet frame [8] . The Frenet frame is the most well-known frame along a space curve. Let $\alpha(s)$ be a space curve with a non-vanishing second derivative. The Frenet frame is described as follows:

$$t = \frac{\alpha^{'}}{\|\alpha'\|}, \quad b = \frac{\alpha^{'} \wedge \alpha^{''}}{\|\alpha' \wedge \alpha''\|}, \quad n = b \wedge t$$

The curvature κ and the torsion τ are obtain by;

$$\kappa = \frac{\left\|\alpha^{'} \wedge \alpha^{''}\right\|}{\|\alpha'\|^{3}}, \quad \tau = \frac{\det\left(\alpha^{'}, \alpha^{''}, \alpha^{'''}\right)}{\|\alpha' \wedge \alpha''\|^{2}}$$

The well-known Frenet formulas are obtain by;

$$\begin{bmatrix} t^{'} \\ n^{'} \\ b^{'} \end{bmatrix} = \varphi \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$$

where $\varphi = \left\|\alpha^{'}(s)\right\|$.

As an alternative to the Frenet frame they define a new adapted frame along a space curve, the q-frame [1] . Dede et al. defined the directional q-frame along a space curve [9] . The directional q-frame offers two key advantages over the Frenet Frame [10, 11] : a) it is well defined even if the curve has vanishing second derivative [12] , b) it avoid the redundant twist around the tangent.

The directional q-frame of a regular curve $\alpha(s)$ is obtained by;

$$t = \frac{\alpha^{'}}{\|\alpha'\|}, \quad n_q = \frac{t \wedge k}{\|t \wedge k\|}, \quad b_q = t \wedge n_q \tag{1}$$

where k is the projection vector.

The varitation equations of the directional q-frame is obtained by;

$$\begin{bmatrix} t^{'} \\ n_q^{'} \\ b_q^{'} \end{bmatrix} = \left\|\alpha^{'}\right\| \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \tag{2}$$

where the q-curvatures are expressed as follows:

$$k_1 = \frac{\left\langle t^{'}, n_q \right\rangle}{\|\alpha'\|}, \quad k_2 = \frac{\left\langle t^{'}, b_q \right\rangle}{\|\alpha'\|}, \quad k_3 = -\frac{\left\langle n_q, b_q^{'} \right\rangle}{\|\alpha'\|}. \tag{3}$$

[9] .

3. Involutes of order 1 st. and order 2 nd. in $\mathbb{E}^3$ according to projection vector

As is well known q-frame is defined by the help of the projection vector k. For simplicity firstly we have choosen the projection vector $k = (0; 0; 1)$. For the cases t and k are parallel, the projection vector can be chosen as

$k=(0;1;0)$, $k=(1;0;0)$ (see [9]) . This part we classified the q-frame into three types: z axis directional q-frames identified with the projection vector $k=(0;0;1)$ (see Theorem 3.1 and 3.2) , y axis directional q-frames identified with the projection vector $k=(0;1;0)$ (see Theorem 3.3 and 3.4) and x axis directional q-frames identified with the projection vector $k=(1;0;0)$ (see Theorem 3.5 and 3.6) .

Definition 3.1. Let α= $\alpha(s)$ be a regular generic curve in $\mathbb{E}^n$ given with the arclength parameter s (i.e., $\left\|\alpha^{'}(s)\right\|=1$). Then the curves which are orthogonal to the system of k-dimensional osculating hyperplanes of α, are called the involutes of order k [13] of the curve α. For simplicity, we call the involutes of order 1, simply the involutes of the given curve [14] .

The theorems below are given by taking $k=(0;0;1)$.

Theorem 3.1. Let $\alpha=\alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be first order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1},\overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1}=-\sqrt{k_1^2+k_2^2},\quad \overline{k_2}=\frac{\left[k_1^{'}k_2-k_2^{'}k_1\right]-\left\|\alpha^{'}\right\|k_3\left[k_1^2+k_2^2\right]}{\left\|\alpha^{'}\right\|\left[k_1^2+k_2^2\right]},$$
$$\overline{k_3}=0$$

Proof:

$$s\rightarrow\alpha(s)\rightarrow\sum_{i=1}^{3}\alpha_i(s)\,e_i\quad(1\leq i\leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s)=\alpha(s)+\lambda(s)\,t(s)$$

we by using (2) , differentiate this equation respect to s, we obtain

$$\overline{\alpha}^{'}(s)=\alpha^{'}(s)+\lambda^{'}(s)\,t(s)+\lambda(s)\left\|\alpha^{'}\right\|\left[k_1n_q+k_2b_q\right]$$

Since

$$\left\langle\overline{\alpha}^{'}(s),t(s)\right\rangle=0$$

and

$$\alpha^{'}(s)=\left\|\alpha^{'}\right\|t(s)$$

we write

$$\lambda(s)=c-\|\alpha\|$$

So, we get

$$\overline{\alpha}^{'}(s)=\alpha^{'}(s)-\left\|\alpha^{'}\right\|t(s)+(c-\|\alpha\|)\left\|\alpha^{'}\right\|\left[k_1n_q+k_2b_q\right]$$

$$=(c-\|\alpha\|)\left\|\alpha^{'}\right\|\left[k_1n_q+k_2b_q\right]\tag{4}$$

Using norm of the equation (4), we get

$$\left\|\overline{\alpha}^{'}(s)\right\|=(c-\|\alpha\|)\sqrt{k_1^2+k_2^2}\left\|\alpha^{'}\right\|\tag{5}$$

and by using the equations (1), (4) and (5), we get

$$\bar{t}(s)=\frac{\left[k_1n_q+k_2b_q\right]}{\sqrt{k_1^2+k_2^2}}\tag{6}$$

if we have chosen the projection vector $k = (0; 0; 1)$

$$\bar{t} \wedge k = \frac{k_1 t}{\sqrt{k_1^2 + k_2^2}} \tag{7}$$

Hence, by taking norm of equation (7), we get

$$\left\| \bar{t} \wedge k \right\| = \sqrt{\frac{k_1^2}{\left(\sqrt{k_1^2 + k_2^2}\right)^2}} \tag{8}$$

Moreover, using the equations (1), (7) and (8), we have

$$\overline{n_q}(s) = t \tag{9}$$

In addition, using the equations (6),and (9)

$$\bar{t} \wedge \overline{n_q} = \frac{k_2 n_q - k_1 b_q}{\sqrt{k_1^2 + k_2^2}} \tag{10}$$

Therefore, from (1) and (10) , we get

$$\overline{b_q}(s) = \frac{k_2 n_q - k_1 b_q}{\sqrt{k_1^2 + k_2^2}} \tag{11}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1} = -\sqrt{k_1^2 + k_2^2} \tag{12}$$

$$\overline{k_2} = \frac{\left[k_1' k_2 - k_2' k_1\right] - \left\|\alpha'\right\| k_3 \left[k_1^2 + k_2^2\right]}{\|\alpha'\| \left[k_1^2 + k_2^2\right]} \tag{13}$$

$$\overline{k_3} = 0 \tag{14}$$

This completes the proof.

Theorem 3.2. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be second order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1}, \overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are vanishes.

$$\overline{k_1} = 0, \quad \overline{k_2} = 0, \quad \overline{k_3} = 0$$

Proof:

$$s \rightarrow \alpha(s) \rightarrow \sum_{i=1}^{3} \alpha_i(s) e_i \quad (1 \leq i \leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s) = \alpha(s) + \lambda_1(s) t(s) + \lambda_2(s) n_q(s)$$

we by using (2) , differentiate this equation respect to s, we obtain

$$\begin{aligned} \overline{\alpha}'(s) &= \alpha'(s) + \lambda_1'(s) t(s) + \lambda_1(s) \left\|\alpha'\right\| [k_1 n_q + k_2 b_q] \\ &\quad + \lambda_2'(s) n_q(s) - \lambda_2(s) \left\|\alpha'\right\| k_1 t + \lambda_2(s) \left\|\alpha'\right\| k_3 b_q \end{aligned}$$

Since

$$\langle \overline{\alpha}^{'}(s), t(s)\rangle = 0, \quad \langle \overline{\alpha}^{'}(s), n_q(s)\rangle = 0$$

and

$$\alpha^{'}(s) = \left\|\alpha^{'}\right\| t(s)$$

So, we get

$$\overline{\alpha}^{'}(s) = \left\|\alpha^{'}\right\| [\lambda_1 k_2 + \lambda_2 k_3] b_q$$

if we take

$$\lambda_1 k_2 = \theta(s), \quad \lambda_2 k_3 = \varphi(s)$$

we obtain

$$\overline{\alpha}^{'}(s) = \left\|\alpha^{'}\right\| [\theta(s) + \varphi(s)] b_q \tag{15}$$

Using norm of the equation (15), we get

$$\left\|\overline{\alpha}^{'}(s)\right\| = \sqrt{\|\alpha^{'}\| [\theta(s) + \varphi(s)]^2} \tag{16}$$

and by using the equations (1), (15) and (16), we attain

$$\bar{t}(s) = \frac{\left\|\alpha^{'}\right\| [\theta(s) + \varphi(s)] b_q}{\sqrt{\|\alpha^{'}\| [\theta(s) + \varphi(s)]^2}} = b_q \tag{17}$$

if we have chosen the projection vector $k = (0; 0; 1)$

$$\bar{t} \wedge k = 0 \tag{18}$$

Hence, by taking norm of equation (18), we get

$$\left\|\bar{t} \wedge k\right\| = 0 \tag{19}$$

Moreover, using the equations (1), (18) and (19), we have

$$\overline{n_q}(s) = 0 \tag{20}$$

In addition, using the equations (17),and (20)

$$\bar{t} \wedge \overline{n_q} = 0 \tag{21}$$

Therefore, from (1) and (21) , we get

$$\overline{b_q}(s) = 0 \tag{22}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1} = 0 \tag{23}$$

$$\overline{k_2} = 0 \tag{24}$$

$$\overline{k_3} = 0 \tag{25}$$

This completes the proof.

The theorems below are given by taking $k=(0;1;0)$.

Theorem 3.3. Let $\alpha=\alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be first order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1}, \overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1}=\sqrt{k_1^2+k_2^2}, \quad \overline{k_2}=\frac{\left[k_2^{'}k_1-k_2k_1^{'}\right]+\left\|\alpha^{'}\right\|k_3\left[k_1^2+k_2^2\right]}{\|\alpha'\|\left[k_1^2+k_2^2\right]},$$
$$\overline{k_3}=0$$

Proof:

$$s\to\alpha(s)\to\sum_{i=1}^{3}\alpha_i(s)e_i \quad (1\leq i\leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s)=\alpha(s)+\lambda(s)t(s)$$

we by using (2), differentiate this equation respect to s, we obtain

$$\overline{\alpha}^{'}(s)=\alpha^{'}(s)+\lambda^{'}(s)t(s)+\lambda(s)\left\|\alpha^{'}\right\|\left[k_1n_q+k_2b_q\right]$$

Since

$$\left\langle\overline{\alpha}^{'}(s),t(s)\right\rangle=0$$

and

$$\alpha^{'}(s)=\left\|\alpha^{'}\right\|t(s)$$

we write

$$\lambda(s)=c-\|\alpha\|$$

So, we get

$$\overline{\alpha}^{'}(s)=\alpha^{'}(s)-\left\|\alpha^{'}\right\|t(s)+(c-\|\alpha\|)\left\|\alpha^{'}\right\|\left[k_1n_q+k_2b_q\right]$$

$$=(c-\|\alpha\|)\left\|\alpha^{'}\right\|\left[k_1n_q+k_2b_q\right] \tag{26}$$

Using norm of the equation (26), we get

$$\left\|\overline{\alpha}^{'}(s)\right\|=(c-\|\alpha\|)\sqrt{k_1^2+k_2^2}\left\|\alpha^{'}\right\| \tag{27}$$

and by using the equations (1), (26) and (27), we get

$$\bar{t}(s)=\frac{\left[k_1n_q+k_2b_q\right]}{\sqrt{k_1^2+k_2^2}} \tag{28}$$

if we have chosen the projection vector $k=(0;1;0)$

$$\bar{t}\wedge k=\frac{-k_2t}{\sqrt{k_1^2+k_2^2}} \tag{29}$$

Hence, by taking norm of equation (29), we get

$$\left\|\bar{t}\wedge k\right\|=\sqrt{\frac{k_2^2}{\left(\sqrt{k_1^2+k_2^2}\right)^2}} \tag{30}$$

Moreover, using the equations (1), (29) and (30), we have

$$\overline{n_q}(s) = -t \tag{31}$$

In addition, using the equations (28),and (31)

$$\bar{t} \wedge \overline{n_q} = \frac{-k_2 n_q + k_1 b_q}{\sqrt{k_1^2 + k_2^2}} \tag{32}$$

Therefore, from (1) and (32) , we get

$$\overline{b_q}(s) = \frac{-k_2 n_q + k_1 b_q}{\sqrt{k_1^2 + k_2^2}} \tag{33}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1} = \sqrt{k_1^2 + k_2^2} \tag{34}$$

$$\overline{k_2} = \frac{\left[k_2' k_1 - k_2 k_1'\right] + \left\|\alpha'\right\| k_3 \left[k_1^2 + k_2^2\right]}{\|\alpha'\| \left[k_1^2 + k_2^2\right]} \tag{35}$$

$$\overline{k_3} = 0 \tag{36}$$

This completes the proof.

Theorem 3.4. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be second order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1}, \overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1} = k_2, \quad \overline{k_2} = k_3, \quad \overline{k_3} = k_1$$

Proof:

$$s \to \alpha(s) \to \sum_{i=1}^{3} \alpha_i(s) e_i \quad (1 \leq i \leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s) = \alpha(s) + \lambda_1(s) t(s) + \lambda_2(s) n_q(s)$$

we by using (2) , differentiate this equation respect to s, we obtain

$$\begin{aligned} \overline{\alpha}'(s) &= \alpha'(s) + \lambda_1'(s) t(s) + \lambda_1(s) \left\|\alpha'\right\| [k_1 n_q + k_2 b_q] \\ &+ \lambda_2'(s) n_q(s) - \lambda_2(s) \left\|\alpha'\right\| k_1 t + \lambda_2(s) \left\|\alpha'\right\| k_3 b_q \end{aligned}$$

Since

$$\langle \overline{\alpha}'(s), t(s) \rangle = 0, \quad \langle \overline{\alpha}'(s), n_q(s) \rangle = 0$$

and

$$\alpha'(s) = \left\|\alpha'\right\| t(s)$$

So, we get

$$\overline{\alpha}'(s) = \left\|\alpha'\right\| [\lambda_1 k_2 + \lambda_2 k_3] b_q$$

if we take

$$\lambda_1 k_2 = \theta(s), \quad \lambda_2 k_3 = \varphi(s)$$

we obtain

$$\overline{\alpha}^{'}(s)=\left\|\alpha^{'}\right\|[\theta(s)+\varphi(s)]\,b_q \tag{37}$$

Using norm of the equation (37), we get

$$\left\|\overline{\alpha}^{'}(s)\right\|=\sqrt{\|\alpha^{'}\|\,[\theta(s)+\varphi(s)]^2} \tag{38}$$

and by using the equations (1), (37) and (38), we attain

$$\bar{t}(s)=\frac{\left\|\alpha^{'}\right\|[\theta(s)+\varphi(s)]\,b_q}{\sqrt{\|\alpha^{'}\|\,[\theta(s)+\varphi(s)]^2}}=b_q \tag{39}$$

if we have chosen the projection vector $k=(0;1;0)$

$$\bar{t}\wedge k=-t \tag{40}$$

Hence, by taking norm of equation (40), we get

$$\left\|\bar{t}\wedge k\right\|=1 \tag{41}$$

Moreover, using the equations (1), (40) and (41), we have

$$\overline{n_q}(s)=-t \tag{42}$$

In addition, using the equations (39),and (42)

$$\bar{t}\wedge\overline{n_q}=-n_q \tag{43}$$

Therefore, from (1) and (43) , we get

$$\overline{b_q}(s)=-n_q \tag{44}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1}=k_2 \tag{45}$$

$$\overline{k_2}=k_3 \tag{46}$$

$$\overline{k_3}=k_1 \tag{47}$$

This completes the proof.

The theorems below are given by taking $k=(1;0;0)$.

Theorem 3.5. Let $\alpha=\alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be first order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1},\overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1}=\frac{\left[k_1^{'}k_2-k_1k_2^{'}\right]-\left\|\alpha^{'}\right\|k_3\left[k_1^2+k_2^2\right]}{\|\alpha^{'}\|\left[k_1^2+k_2^2\right]},\quad \overline{k_2}=\sqrt{k_1^2+k_2^2},$$
$$\overline{k_3}=0$$

Proof:

$$s\to\alpha(s)\to\sum_{i=1}^{3}\alpha_i(s)\,e_i\quad(1\leq i\leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s) = \alpha(s) + \lambda(s)\, t(s)$$

we by using (2), differentiate this equation respect to s, we obtain

$$\overline{\alpha}^{'}(s) = \alpha^{'}(s) + \lambda^{'}(s)\, t(s) + \lambda(s) \left\|\alpha^{'}\right\| [k_1 n_q + k_2 b_q]$$

Since

$$\langle \overline{\alpha}^{'}(s), t(s) \rangle = 0$$

and

$$\alpha^{'}(s) = \left\|\alpha^{'}\right\| t(s)$$

we write

$$\lambda(s) = c - \|\alpha\|$$

So, we get

$$\begin{aligned} \overline{\alpha}^{'}(s) &= \alpha^{'}(s) - \left\|\alpha^{'}\right\| t(s) + (c - \|\alpha\|) \left\|\alpha^{'}\right\| [k_1 n_q + k_2 b_q] \\ &= (c - \|\alpha\|) \left\|\alpha^{'}\right\| [k_1 n_q + k_2 b_q] \end{aligned} \tag{48}$$

Using norm of the equation (48), we get

$$\left\|\overline{\alpha}^{'}(s)\right\| = (c - \|\alpha\|) \sqrt{k_1^2 + k_2^2} \left\|\alpha^{'}\right\| \tag{49}$$

and by using the equations (1), (48) and (49), we get

$$\bar{t}(s) = \frac{[k_1 n_q + k_2 b_q]}{\sqrt{k_1^2 + k_2^2}} \tag{50}$$

if we have chosen the projection vector $k = (1; 0; 0)$

$$\bar{t} \wedge k = \frac{[k_2 n_q - k_1 b_q]}{\sqrt{k_1^2 + k_2^2}} \tag{51}$$

Hence, by taking norm of equation (51), we get

$$\left\|\bar{t} \wedge k\right\| = \sqrt{\frac{k_1^2}{\left(\sqrt{k_1^2 + k_2^2}\right)^2}} \tag{52}$$

Moreover, using the equations (1), (51) and (52), we have

$$\overline{n_q}(s) = \frac{[k_2 n_q - k_1 b_q]}{\sqrt{k_1^2 + k_2^2}} \tag{53}$$

In addition, using the equations (50),and (53)

$$\bar{t} \wedge \overline{n_q} = t \tag{54}$$

Therefore, from (1) and (54), we get

$$\overline{b_q}(s) = t \tag{55}$$

Consequently, by using the equations (3), we obtain

$$\overline{k_1} = \frac{\left[k_1^{'} k_2 - k_1 k_2^{'}\right] - \left\|\alpha^{'}\right\| k_3 \left[k_1^2 + k_2^2\right]}{\|\alpha^{'}\| \left[k_1^2 + k_2^2\right]} \tag{56}$$

$$\overline{k_2} = \sqrt{k_1^2 + k_2^2} \tag{57}$$

$$\overline{k_3} = 0 \tag{58}$$

This completes the proof

Theorem 3.6. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be second order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1}, \overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1} = -k_3, \quad \overline{k_2} = k_2, \quad \overline{k_3} = k_1$$

Proof:

$$s \to \alpha(s) \to \sum_{i=1}^{3} \alpha_i(s) e_i \quad (1 \leq i \leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s) = \alpha(s) + \lambda_1(s) t(s) + \lambda_2(s) n_q(s)$$

we by using (2), differentiate this equation respect to s, we obtain

$$\begin{aligned}\overline{\alpha}^{'}(s) = \alpha^{'}(s) + \lambda_1^{'}(s) t(s) + \lambda_1(s) \left\|\alpha^{'}\right\| [k_1 n_q + k_2 b_q] \\ + \lambda_2^{'}(s) n_q(s) - \lambda_2(s) \left\|\alpha^{'}\right\| k_1 t + \lambda_2(s) \left\|\alpha^{'}\right\| k_3 b_q\end{aligned}$$

Since

$$\langle \overline{\alpha}^{'}(s), t(s) \rangle = 0, \quad \langle \overline{\alpha}^{'}(s), n_q(s) \rangle = 0$$

and

$$\alpha^{'}(s) = \left\|\alpha^{'}\right\| t(s)$$

So, we get

$$\overline{\alpha}^{'}(s) = \left\|\alpha^{'}\right\| [\lambda_1 k_2 + \lambda_2 k_3] b_q$$

if we take

$$\lambda_1 k_2 = \theta(s), \quad \lambda_2 k_3 = \varphi(s)$$

we obtain

$$\overline{\alpha}^{'}(s) = \left\|\alpha^{'}\right\| [\theta(s) + \varphi(s)] b_q \tag{59}$$

Using norm of the equation (59), we get

$$\left\|\overline{\alpha}^{'}(s)\right\| = \sqrt{\|\alpha'\| [\theta(s) + \varphi(s)]^2} \tag{60}$$

and by using the equations $(1), (59)$ and (60), we attain

$$\bar{t}(s) = \frac{\left\|\alpha^{'}\right\| [\theta(s) + \varphi(s)] b_q}{\sqrt{\|\alpha'\| [\theta(s) + \varphi(s)]^2}} = b_q \tag{61}$$

if we have chosen the projection vector $k = (1; 0; 0)$

$$\bar{t} \wedge k = n_q \tag{62}$$

Hence, by taking norm of equation (62), we get

$$\left\|\bar{t} \wedge k\right\| = 1 \tag{63}$$

Moreover, using the equations (1), (62) and (63), we have

$$\overline{n_q}\,(s) = n_q \tag{64}$$

In addition, using the equations (61),and (64)

$$\bar{t} \wedge \overline{n_q} = -t \tag{65}$$

Therefore, from (1) and (65) , we get

$$\overline{b_q}\,(s) = -t \tag{66}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1} = -k_3 \tag{67}$$

$$\overline{k_2} = k_2 \tag{68}$$

$$\overline{k_3} = k_1 \tag{69}$$

This completes the proof.

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Nadaraya-Watson estimation of a nonparametric autoregressive model

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Abstract. We investigate the asymptotic behavior of the Nadaraya-Watson (NW) estimator of the regression function of a $\tau-$mixing process. We prove the strong consistency and the asymptotic normality of this estimator and we illustrate these two properties using simulated data.

AMS Subject Classifications: $62E20$, $62G05$, $62G08$, $62G20$.

Keywords: Nonparametric autoregression, Nonparametric estimation, Asymptotic normality, Nadaraya-Watson estimator, $\tau-$mixing.

Contents

1. Introduction

From the seminal works by Rosenblatt [20], nonparametric function estimation has been widely investigated. Parzen [19] proposed a family of kernels for nonparametric density function estimation. He obtained the same result as Rosenblatt [20]. These different works allowed Nadaraya [17] and Watson [22] to independently propose a nonparametric estimator of the regression function. This is the Nadaraya-Watson (NW) estimator. Theoretical and practical aspects of this estimator have been studied. Interesting properties have been obtained. For an overview on the question, we refer to Bercu et al. [2], Li et al. [15] and the references therein. The NW estimation method was initially restricted to independent and identically distributed data (see, for example, [16, 18, 21] and the references therein). Then, it has been adapted by several studies to the $\alpha-$, $\beta-$ and $\phi-$mixing processes (see, for example, [5, 7, 12] and the references therein). There are very few studies suitable for $\tau-$mixing processes. This paper presents itself as one of the few contributions on the estimation of the regression function of $\tau-$mixing process. We refer the reader to Dedecker and Prieur [6] for the definition of a $\tau-$mixing process.
More recently, Hong and Linton [13] proposed an infinite dimensional NW type estimator for the regression

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function of an $\alpha-$dependent process. In our paper, we use a NW estimator, as Hong and Linton [13] , to estimate the regression function of a $p-$Markov process. These processes are generally $\beta-$dependant. However, there are some that are neither $\alpha-$dependent nor $\beta-$dependent (but $\tau-$dependent) (see [1]). Among these, we can mention some nonparametric autoregressive (NAR) processes. According to Fan and Yao [10] (p. 19), a sequence $(X_t)_{t\in\mathbb{Z}}$ is a NAR process if it is a solution of (2.1). In our study, we show the strong consistency and the asymptotic normality of the NW estimator of the regression function of NAR process under the assumption of a $\tau-$mixing condition on the sample. Our results go further than those of Hong and Linton ([13] , Theorem 1) since we get the strong consistency.
The remainder of this paper is organized as follows. Section 2 discusses the model and the assumptions. Section 3 contains the main results and their proof. Section 4 is devoted to a small simulation.

2. Notations and Assumptions

In this paper, we shall use the following notations : $\|z\| := \sup_{1\leqslant i\leqslant p} |z_i|$, for any $z = (z_1, z_2, \dots, z_p)^{'} \in \mathbb{R}^p$ where $Z^{'}$ denotes the transpose of Z. For any $v \in \mathbb{R}$, $[v]$ denotes the largest integer close to v;
Let $(X_t)_{t\in\mathbb{Z}}$ be a stochastic process satisfying :

$$X_t = f(Y_t) + \xi_t,\ t \in \mathbb{Z}; \tag{2.1}$$

where $X_t \in \mathbb{R}$, $Y_t = (X_{t-1}, X_{t-2}, ..., X_{t-p})^{'} \in \mathbb{R}^p$, $(\xi_t)_{t\in\mathbb{Z}}$ is a sequence of independent identically distributed random variables with $\mathbb{E}(\xi_t) = 0$ and $\sigma^2(\xi_t) > 0$, $t \in \mathbb{Z}$. The random variable ξ_t is independent of X_i, for $i < t$ and $f(.) : \mathbb{R}^p \to \mathbb{R}$, $z \longmapsto \mathbb{E}(X_t|Y_t = z)$, $t \in \mathbb{Z}$, is an unknown measurable function.
Let $x \in \mathbb{R}^p$, we observe $(X_1, Y_1), ..., (X_T, Y_T)$ and estimate $f(x)$ by :

$$\widehat{f}_T(x) = \begin{cases} \dfrac{\sum_{t=1}^{T} K_t(x) X_t}{\sum_{t=1}^{T} K_t(x)}; \text{ if } \sum_{t=1}^{T} K_t(x) \neq 0 \\ 0, \qquad \text{otherwise}; \end{cases} \tag{2.2}$$

where $K_t(x) = K\left(\left\|h_T^{-1}(x - Y_t)\right\|\right)$, $t = 1, ..., T$; $K(.)$ denotes the kernel function and $h_T > 0$.
Our goal is to establish the consistency and the asymptotic normality of $\widehat{f}_T(x)$. Zhu and Politis [23] have done this for nonparametric functional autoregression models. Hong and Linton [13] also proved it for $\alpha-$dependent processes.
The assumptions needed for the theoretical results are stated below.

(A_1) : There exists an Orlicz function $\Phi(.)$ such that :

$$\Phi(uv) \leqslant \Phi(u)\Phi(v),\ \text{for all } u, v \in \mathbb{R}_+;$$

and for all $y, z \in \mathbb{R}^p$,

$$|f(y) - f(z)| \leqslant \sum_{j=1}^{p} \varpi_j |y_j - z_j|,$$

where $(\varpi_j)_{1\leqslant j\leqslant p}$ is a sequence of nonnegative real numbers such that $\varpi = \sum_{j=1}^{p} \varpi_j < 1$, $|f(0, 0, ..., 0)| + \|\xi_1\|_\Phi < \infty$ and $\|.\|_\Phi$ denotes the Orlicz norm associated with $\Phi(.)$ (see [9] for the definition of the Orlicz norm).

(A_2) : The kernel $K : [0, +\infty[\longrightarrow [0, +\infty[$ is bounded and has compact support, that is, there exists $\lambda > 0$ such that $K(\upsilon) = 0$ for all $\upsilon > \lambda$. There exists two real constants $0 < C_1 < C_2 < \infty$ such that $C_1 \leqslant K(\upsilon) \leqslant C_2$, $\upsilon \in [0, \lambda]$ and $\int_{\mathbb{R}} K(\nu)d\nu = 1$.

(A_3) : For $t = 1, ..., T$, $\varphi_x(\lambda h_T) := \mathbb{P}(\|h_T^{-1}(Y_t - x)\| \leqslant \lambda) > 0$ (λ is defined in Assumption (A_2)) and $h_T \longrightarrow 0$ as $T \longrightarrow \infty$.

From Assumption (A_1), Doukhan and Wintenberger [9] show the existence of a strongly stationary and $\tau-$dependent solution of (2.1) such that $\tau(i) = O(a^i)$, $0 < a < 1$ (see Corollary 3.1 of [9]). According to Remark 3.1 of Doukhan and Wintenberger [9], this solution is an ergodic process. So $(Y_t)_{t\in\mathbb{Z}}$ and $(X_t, Y_t)_{t\in\mathbb{Z}}$ are strongly stationary and ergodic processes (see Theorem 36.4 of [3]). Assumption (A_1) also reflects the continuity of the application $f(.)$. Assumption (A_2) was borrowed from Hong and Linton [13] (Assumption $B3$). Assumption (A_3) expresses the possibility of observing the sample in a neighbourhood of x. This is a classic assumption in the nonparametric framework. It naturally extends the hypothesis of the strictly positive density of the explanatory variable.

3. Main Results

Theorem 3.1. *Under Assumptions (A_1), (A_2) and (A_3), for T big enough,*

$$\widehat{f}_T(x) = f(x) + o(1) \ \ \textit{almost surely } (a.s.) \tag{3.1}$$

Proof. According to Assumption (A_3); we have, for $t = 1, ..., T$, $\mathbb{P}\left(\frac{\|x-Y_t\|}{h_T} \leqslant \lambda\right) > 0$, so $\mathbb{E}\bigg(K_t(x)\bigg) > 0$.
Let :

$$\widehat{f}_{1,T}(x) = \frac{\frac{1}{T}\sum_{t=1}^{T} K_t(x)X_t}{\mathbb{E}(K_1(x))} \quad \text{and} \quad \widehat{f}_{2,T}(x) = \frac{\frac{1}{T}\sum_{t=1}^{T} K_t(x)}{\mathbb{E}(K_1(x))}. \tag{3.2}$$

According to Equation (20) of Hong and Linton [13], we can write :

$$\widehat{f}_T(x) - f(x) = \frac{\mathbb{E}\left(\widehat{f}_{1,T}(x)\right) - f(x)}{\widehat{f}_{2,T}(x)} + \frac{\widehat{f}_{1,T}(x) - \mathbb{E}\left(\widehat{f}_{1,T}(x)\right)}{\widehat{f}_{2,T}(x)} - \frac{f(x)\bigg(\widehat{f}_{2,T}(x) - 1\bigg)}{\widehat{f}_{2,T}(x)}. \tag{3.3}$$

Let us study the asymptotic behavior of $\widehat{f}_T(x) - f(x)$. To do it, we shall study the asymptotic behaviors of $\widehat{f}_{2,T}(x)$, $\mathbb{E}\left(\widehat{f}_{1,T}(x)\right) - f(x)$ and $\widehat{f}_{1,T}(x) - \mathbb{E}\left(\widehat{f}_{1,T}(x)\right)$.
We start with the asymptotic behavior of $\widehat{f}_{2,T}(x)$.
According to Assumption (A_1), $(X_t)_{t\in\mathbb{Z}}$ is strongly stationary and ergodic. Since $K_t(x)$ is a measurable transformation of $(X_{t-1}, ..., X_{t-p})'$ and $\mathbb{E}(K_1(x)) < +\infty$ (see Assumption (A_2)), we have by Krengel [14], for T big enough,

$$\frac{1}{T}\sum_{t=1}^{T} K_t(x) \longrightarrow \mathbb{E}(K_1(x)) \ a.s.$$

So, we have for T big enough :

$$\widehat{f}_{2,T}(x) \longrightarrow 1 \ a.s. \tag{3.4}$$

According to Assumptions (A_1) and (A_2), $|\mathbb{E}(K_1(x)X_1)| < \infty$. And $K_t(x)X_t$ is a measurable transformation of $(X_t, X_{t-1}, ..., X_{t-p})'$. Therefore, we show as in (3.4), for T big enough :

$$\frac{1}{T}\sum_{t=1}^{T} K_t(x)X_t \longrightarrow \mathbb{E}(K_1(x)X_1)\ a.s.$$

Therefore, for T big enough :

$$\widehat{f}_{1,T}(x) - \mathbb{E}\left(\widehat{f}_{1,T}(x)\right) \longrightarrow 0\ a.s. \tag{3.5}$$

Using the same reasoning as the proof of Equation (53) in Hong and Liton [13] (see also the proof of Lemma 6.2 of [11]), we show, for T big enough :

$$\mathbb{E}\left(\widehat{f}_{1,T}(x)\right) - f(x) \longrightarrow 0. \tag{3.6}$$

Gathering (3.3), (3.4), (3.5) and (3.6), we get (3.1). ■

Theorem 3.2. *Under Assumptions (A_1), (A_2) and (A_3), for T big enough,*

$$\varsigma^2 := \lim_{T\to\infty} \frac{1}{T} Var\left(\sum_{t=1}^{T} X_t\right) \ < \ +\infty. \tag{3.7}$$

And

$$\sqrt{T}\mathbb{E}(K_1(x))\Big(\widehat{f}_T(x) - f(x) + o(1)\Big) + o\Big(\sqrt{\ln\ln(T)}\Big) \xrightarrow{d} N(0, \varsigma^2), \tag{3.8}$$

where $\xrightarrow{d}$ denotes convergence in distribution.

Proof. According to (3.3), (3.4) and (3.6), we have *a.s.*, for T big enough :

$$\begin{aligned}
\widehat{f}_T(x) - f(x) &= \widehat{f}_{1,T}(x) - \mathbb{E}\left(\widehat{f}_{1,T}(x)\right) + o(1) \\
&= \frac{1}{T\mathbb{E}(K_1(x))}\sum_{t=1}^{T}\Big(K_t(x)X_t - \mathbb{E}(K_1(x)X_1)\Big) + o(1) \\
&= \frac{1}{T\mathbb{E}(K_1(x))}\sum_{t=1}^{T}\Big\{(K_t(x)-1)X_t - \mathbb{E}\Big((K_1(x)-1)X_1\Big)\Big\} \\
&+ \frac{1}{T\mathbb{E}(K_1(x))}\sum_{t=1}^{T}\Big(X_t - \mathbb{E}(X_1)\Big) + o(1).
\end{aligned} \tag{3.9}$$

Since $(K_t(x) - 1)X_t$ is a measurable transformation of $(X_t, X_{t-1}, ..., X_{t-p})'$, so we have, for T big enough :

$$\frac{1}{T}\sum_{t=1}^{T}\Big\{(K_t(x)-1)X_t - \mathbb{E}\Big((K_1(x)-1)X_1\Big)\Big\} \longrightarrow 0\ a.s.$$

we have *a.s.*, for T big enough :

$$\widehat{f}_T(x) - f(x) = \frac{1}{T\mathbb{E}(K_1(x))}\sum_{t=1}^{T}\Big(X_t - \mathbb{E}(X_1)\Big) + o(1). \tag{3.10}$$

The function $s \longmapsto |s|^2 \ln(1+|s|)$ is measurable. So $(|X_t - \mathbb{E}(X_1)|^2 \ln(1+|X_t - \mathbb{E}(X_1)|))_t$ is stationary because $(X_t)_t$ is strongly stationary and ergodic. Therefore $\mathbb{E}(|X_t - \mathbb{E}(X_1)|^2 \ln(1+|X_t - \mathbb{E}(X_1)|)) < \infty$.
According to the Hypothesis (A_1), the mixing coefficient $\tau(.)$ of the process $(X_t)_{t\in\mathbb{Z}}$ is such that $\tau(i) = O(a^i)$, $0 < a < 1$.
From item 3 of Corollary 2 of Dedecker and Prieur [6], we have (3.7) and there exists a sequence $(Z_t)_{1\leqslant t\leqslant T}$ of independent $N(0;\varsigma^2)-$distributed random variables such that :

$$\sum_{t=1}^{T}\Big(X_t - \mathbb{E}(X_1)\Big) = \sum_{t=1}^{T} Z_t + o\Big(\sqrt{T\ln\ln(T)}\Big)\ a.s.; \tag{3.11}$$

where ς^2 is defined in (3.7).
According to (3.10) and (3.11), we have, for T big enough :

$$\sqrt{T}\mathbb{E}(K_1(x))\Big(\widehat{f}_T(x) - f(x) + o(1)\Big) + o\Big(\sqrt{\ln\ln(T)}\Big) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} Z_t\ \ a.s. \tag{3.12}$$

From the Central Limit Theorem, we have for T big enough :

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} Z_t \xrightarrow{d} N(0,\varsigma^2).$$

Back to (3.12), we get (3.8). ■

4. Simulation study

In this section we present some results of our simulation study. We first (Section 4.1) focus on the strong consistency of estimator of regression function defined in (2.2). And we verify numerically the asymptotic normality of this estimator in Section 4.2. The simulation study was performed using R software and the results presented in these simulations correspond to 200 replications. Here, the Orlicz space is $L^1(\mathbb{R})$ and we use the absolute value function as Orlicz function.
Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by :

$$f : x \longmapsto 0.2x. \tag{4.1}$$

We consider :

$$X_t = f(X_{t-1}) + \xi_t,\ t = 1, ..., T; \tag{4.2}$$

where $X_0 = 0$ and $(\xi_t)_t$ is a sequence of independent identically uniformly distributed on $[-0.3, 0.3]$.

We choose the uniform kernel on $[0, 1]$; for the bandwidth, we choose $h_T = T^{-1/6}$. We numerically verify (3.1) and (3.8) at point 0.

4.1. Simulation of strong consistency of $\widehat{f}_T(0)$

The samples are taken with size which varies between 100 and 500 observations. Table 4.1 reports the root mean square error (RMSE). The RMSE is calculated from the following formula :

$$RMSE = \sqrt{\frac{1}{r}\sum_{i=1}^{r}(\widehat{f}_{T,r}(0) - f(0))^2},$$

where r denotes the number of replications (here $r = 200$) and $\widehat{f}_{T,r}(0)$, the value of $\widehat{f}_T(0)$ at the r^{th} replication (see (2.2) for the definition of $\widehat{f}_T(0)$).
As it can be seen in Table 4.1, the RMSE decreases when the sample size increases. This corroborates the convergence of estimator.

T	*RMSE*
100	0.018896
200	0.011016
500	0.007764

Table 4.1 : *RMSE values*

4.2. Simulation of asymptotic normality of $\widehat{f}_T(0)$

The purpose of this subsection is to illustrate the asymptotic normality of estimator $\widehat{f}_T(0)$ (see (3.8)). To this purpose, we randomly generate samples of size $T \in \{100, 300, 500\}$ of $\widehat{f}_T(0)$. Figure 4.1 shows the histogram and the $Q - Q$ plot of the estimator $\widehat{f}_{500}(0)$. In addition to these graphical representations, we performed a Shapiro-Wilk normality test. The results of the test are presented in Table 4.2 where W refers to the test statistic.

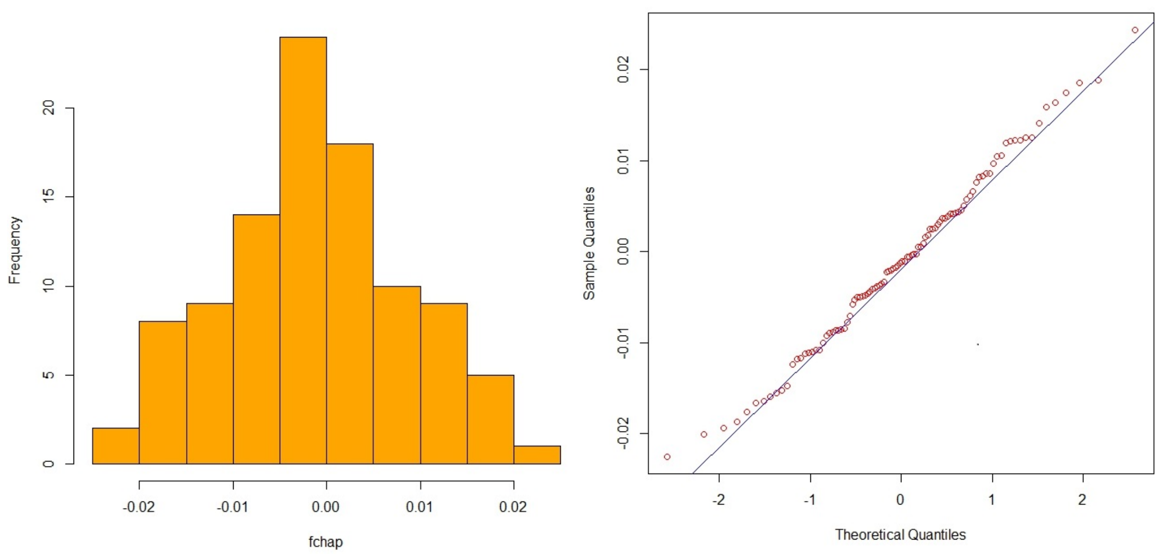

Figure 4.1 : Graphical illustration of the normality of $\widehat{f}_{500}(0)$.

Figure 4.1 is composed of two sub-figures: an histogram (on the left) and a $Q - Q$ plot (on the right). On the left side of Figure 4.1, we have plotted the histogram of $\widehat{f}_{500}(0)$ (orange colour). The shape of the histogram reminds us of the graphical representation of the density of normal distribution. This presumption is accentuated with the quantile cloud of dots. On the right side of Figure 4.1, we have plotted $Q-Q$ plot in red and Henry's line in blue. Most of the points seem to line up with Henry's line. And the extremities of the cloud seem to move away from it. Figure 4.1 therefore shows a presumption of normality of the sample. To confirm the normality of sample, we have performed the Shapiro-Wilk test. The test results show high values of $p - value$. This value increases when the sample size increases. In view of results, we can confirm the normality of these samples .

T	W	*p-value*
100	0.98869	0.5604
300	0.98961	0.6334
500	0.99248	0.8547

Table 4.2 : Shapiro-Wilk normality test on $\widehat{f}_T(0)$, $T \in \{100, 300, 500\}$.

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