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Existence results on nonautonomous partial functional differential equations with state-dependent infinite delay

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Abstract. The aim of this work is to establish the existence of mild solutions for some nondensely nonautonomous partial functional differential equations with state-dependent infinite delay in Banach space. We assume that, the linear part is not necessarily densely defined and generates an evolution family under the hyperbolique conditions. We use the classic Shauder Fixed Point Theorem, the Nonlinear Alternative Leray-Schauder Fixed Point Theorem and the theory of evolution family, we show the existence of mild solutions. Secondly, we obtain the existence of mild solution in a maximal interval using Banach's Fixed Point Theorem which may blow up at the finite time, we show that this solution depends continuously on the initial data under the global Lipschitz condition on the second argument of F and we get the existence of global mild solution. We propose some model arising in dynamic population for the application of our results.

AMS Subject Classifications: 34K43, 35R10, 47J35.

Keywords: Nondensely nonautonomous equations, evolution family, hyperbolique conditions, mild solutions, state-dependent infinite delay.

Contents

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1. Introduction

Partial differential equations play a crucial role in providing mathematical answers to natural phenomena and they continue to be an indispensable tool in scientific investigations of real-world problems. The future behaviors of many phenomenas are therefore supposed to be described by the solutions of an ordinary or partial differential equations. These have long played important roles in the history of mathematical modeling and will undoubtedly continue to serve as indispensable tools in future investigations. They are encountered in a variety of problems in physics, chemistry, biology, medicine, economics, engineering, climate and disease modeling and many others.

In this work, we study the existence of mild solutions for the following partial functional differential equation with state-dependent infinite delay

$$\begin{cases} \dot{x}(t) = A(t)x(t) + F(t, x_{\rho(t,x_t)}); \quad t \in J := [0, b], \\ \\ x_0 = \varphi \in \mathcal{B} \end{cases} \tag{1.1}$$

in a Banach space $(X, \|\cdot\|)$. Here $(A(t))_{t\geq 0}$ is a given family of closed linear operators in X with non necessarily dense domain and satisfying the hyperbolic conditions ($\mathbf{A}_1$) through ($\mathbf{A}_3$) introduce by Tanaka in [45, 46] which will be specified later. The phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X satisfying some Axioms which will be described in the sequel. $F : J \times \mathcal{B}$ is continuous and $\rho : J \times \mathcal{B} \to (-\infty, b]$ are appropriate functions. The history x_t ($t \geq 0$), represents the mapping defined from $(-\infty, 0]$ into X by

$$x_t(\theta) = x(t + \theta) \qquad \text{for } \theta \in (\text{-}\infty, 0].$$

For the nonautonomous dynamical systems, the basic law of evolution is not static in the sense that the environment change with time. Parameters in real-world situations and particularly in the life sciences are rarely constant over time. The theory of nonautonomous dynamical systems is a well-developed and successful mathematical framework to describe time-varying phenomena. Its applications in the life sciences range from simple predator-prey models to complicated signal traduction pathways in biological cells, in physics from the motion of a pendulum to complex climate models, and beyond that to further fields as diverse as chemistry (reaction kinetics), economics, engineering, sociology, demography, and biosciences. Nonautonomous differential equations has received the great attention see for instance the works [22, 26, 28, 40, 42, 47, 51] and some recent works [9, 37–39]. For some applications, we refer the reader to the handbook by Peter E. Kloeden and Christian Pötzsche [44]. Note that when $A(t) := A$ is independent of t, the theory of partial functional differential equations was studied by several authors. Hernández et al. [34] studied the existence of mild solutions of Equation (1.1) by using the classical C_0-semigroup theory. Later on, Belmekki et al. [12] obtained the existence results of the following partial functional differential equations with state-dependent delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t - \tau(x(t)))) \quad \text{for} \quad t \in [a, b]; \\ \\ x_0 = \varphi \in C([-r, 0]; X) \end{cases} \tag{1.2}$$

where the operator A satisfies the usual Hille-Yosida condition except the density of D(A) in X. They obtained their results by using the variation of constants formula which is given in terms of integrated semigroups. In the autonomous case where $\rho(t, x_t) = t$, we refer the reader to Adimy et al [2], K. Ezzinbi et al [23, 24], Hale and Lunel [30], G. F. Webb. [48, 49], Wu [50], and the papers [2, 3, 13, 14, 16–18, 18, 36].

The literature related to partial nonautonomous functional differential equations with delay for which $\rho(t, \psi) = t$ is very extensive and we refer the reader to the papers in [9, 13, 25, 37, 38, 40, 47] concerning this case. Recently Kpoumié et al in [9], investigate several results on the existence of solutions of the following nonautonomous equation :

$$\begin{cases} \dot{x}(t) = A(t)x(t) + F(t, x_t) \quad \text{for} \quad t \geq 0, \\ \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.3}$$

where $(A(t))_{t\geq 0}$ is a given family of closed linear operators on a Banach space $(X, \|\cdot\|)$ not necessarily densely defined satisfying the hyperbolic conditions, $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ to X satisfying some Axioms and F a continuous function defined on $[0, +\infty) \times \mathcal{B}$ with values in X. In this context, they have studied the local existence of the mild solutions which may blow up at the finite time, the global existence of mild solutions are given and under sufficient conditions, the existence of the strict solutions have been obtained.

Functional differential equations with state-dependent delay appear frequently in applications as models of equations and for this reason the study of this type of equation has attracted attention in recent years and more than ten years ago we refer the reader to the handbook by Cañada et al. [5], the book [19], the papers [6, 8, 11, 12, 20, 26, 27, 31, 32] and the references therein. In [39], we investigated the existence of mild solutions of the following nonautonomous equation:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + F(t, x(t - \rho(x(t)))) \quad \text{for} \quad t \in [0, a] \\ \\ x_0 = \varphi \in C([-r, 0], X), \end{cases} \tag{1.4}$$

where $(A(t))_{t\geq 0}$ is a given family of closed linear operators on a Banach space $(X, \|\cdot\|)$ not necessarily densely defined and satisfying the hyperbolic conditions ($\mathbf{A}_1$) through ($\mathbf{A}_3$) introduced by Tanaka in [46] which will be specified in Section 2. F is a given function defined on $[0, +\infty) \times X$ with values in X, the initial data $\rho : [-r; 0] \to X$ is a continuous function, ρ is a positive bounded continuous function on X and r is the maximal delay defined by

$$r = \sup_{x \in X} \rho(x)$$

.

In this paper, we study the existence of at least one mild solutions where the family of closed linear operators on a Banach space is not necessarily densely defined. Note that there are many examples where evolution equations are not densely defined. One can refer to [1, 4, 21] for references and discussion on this subject. Our work is motivated by [9, 34]. The results obtained is a continuation of work done by Hernãndez et al in [34], Belmekki et al. [12] and Kpoumié et al in [39].

In the whole of this work we employ an axiomatic definition for the phase space $\mathcal{B}$ due to Hale and Kato [29]. We assume that $\mathcal{B}$ is a normed linear space of functions mapping $(-\infty, 0]$ to X endowed with a normed $|\cdot|_{\mathcal{B}}$ and satisfying the following Axioms:

($\mathbf{B}_1$) There exist a positive constant H and functions $K(\cdot); M(\cdot) : [0, +\infty) \to [0; +\infty)$, with K continuous and M locally bounded, and the are independent of x, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if x is a function mapping $(-\infty, \sigma + a[$ into X, $a > 0$, such that $x_\sigma \in \mathcal{B}$, and $x(\cdot)$ is continuous on $[\sigma, \sigma + a[$, then for every t in $[\sigma, \sigma + a[$ the following conditions hold :

- (i) $x_t \in \mathcal{B}$,
- (ii) $\|x(t)\|_X \leq H\|x_t\|_{\mathcal{B}}$ which is equivalent to
- $(ii)'$ $\|\varphi(0)\|_X \leq H\|\varphi\|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$.
- (iii) $\|x_t\|_{\mathcal{B}} \leq M(t - \sigma)\|x_\sigma\|_{\mathcal{B}} + K(t - \sigma) \sup_{\sigma \leq s \leq t} \|x(s)\|_X$

($\mathbf{B}_2$) For the function $x(\cdot)$ in ($\mathbf{B}_1$), $t \mapsto x_t$ is a $\mathcal{B}-$valued continuous function for $t \in [\sigma; \sigma + a[$.

(B) The space $\mathcal{B}$ is complete.

For examples and more details on phase space, see the book by Y. Hino, S. Murakami and T. Naito [35].

The organization of this work is as follows: in Section 2, we recall some results on nonautonomous evolution family with nondensely domain theory that will be used to develop our main results. In Section 3, we use the variant of Shauder's Fixed Point Theorem and the nonlinear alternative of Leray-Schauder's to prove the existence of at least one mild solution. In Section 4, we propose an application to some models with state dependent delay.

2. Nonautonomous evolution family with nondense domain

In this section, we recall some notations, definitions and preliminary facts concerning our work. Throughout this paper we used the results which are detailed in [43, 45, 46]. We assume that $\mathcal{B}(X)$ is the Banach space of all bounded linear operators from X to itself. In this work, we assume the following hyperbolic assumptions:

(A$_1$) $D(A(t)) := D$ independent of t and not necessarily densely defined.

(A$_2$) The family $(A(t))_{t\geq 0}$ is stable that means there are constants $M \geq 1$ and $w \in \mathbb{R}$ such that:

$$(w, +\infty) \subset \rho(A(t)) \quad \text{and} \quad \left\| \prod_{j=1}^{k} R(\lambda, A(t_j)) \right\| \leq M(\lambda - w)^{-k}$$

for $t \geq 0$, $\lambda > w$ and for very finite sequence $\{t_j\}_{j=1}^{k}$ with $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k < +\infty$ and $k = 1, 2, \ldots$, where $\rho(A(t))$ is the resolvent set of $A(t)$ and $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$.

(A$_3$) The mapping $t \longmapsto A(t)x$ is continuously differentiable in X for all $x \in D$.

We recall here the classical result which gives us the existence and explicit formula of the evolution family generated by $(A(t))_{t\geq 0}$ due to Oka and Tanaka [43] and Tanaka [46].

Theorem 2.1. *(Oka and Tanaka [43]; Tanaka [46]) Assume that $(A(t))_{t\geq 0}$ satisfies conditions (**A$_1$**) -(**A$_3$**). Then the limit*

$$U(t, s)x = \lim_{\lambda \to 0^+} \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(i\lambda))^{-1} x$$

exists for $x \in \overline{D}$ and $t \geq s \geq 0$, where the convergence is uniform on $\Gamma := \{(t, s) : t \geq s \geq 0\}$. Moreover, the family $\{U(t, s) : (t, s) \in \Gamma\}$ satisfies the following properties:

i) *$U(t, s) : \overline{D} \to \overline{D}$ for $(t, s) \in \Gamma$;*

ii) *$U(t, t)x = x$ and $U(t, s)x = U(t, r)U(r, s)x$ for $x \in \overline{D}$ and $t \geq r \geq s \geq 0$;*

iii) *the mapping $(t, s) \mapsto U(t, s)x$ is continuous on Γ for any $x \in \overline{D}$;*

iv) *$\|U(t, s)x\| \leq Me^{w(t-s)}\|x\|$ for $x \in \overline{D}$ and $(t, s) \in \Gamma$;*

v) *$U(t, s)D(s) \subset D(t)$ for all $t \geq s \geq 0$ where $D(t) := \{x \in D : A(t)x \in \overline{D}\}$;*

vi) *for all $x \in D(s)$ and $t \geq s \geq 0$, the function $t \mapsto U(t, s)x$ is continuously differentiable with: $\frac{\partial}{\partial t}U(t, s)x = A(t)U(t, s)x$ and $\frac{\partial^+}{\partial s}U(t, s)x = -U(t, s)A(s)x$.*

Let $\lambda > 0$, $t \geq s \geq 0$ and $x \in X$. We define $U_\lambda(t, s)$ by:

$$U_\lambda(t, s)x = \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(i\lambda))^{-1} x$$

Remark 2.1. *For $x \in X$, $\lambda > 0$ and $t \geq r \geq s \geq 0$ one can see that*

$$U_\lambda(t,t)x = x \quad \textit{and} \quad U_\lambda(t,s)x = U_\lambda(t,r)U_\lambda(r,s)x.$$

We consider the following nonautonomous linear evolution equation:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t) \ \text{ for } \ t \in [0,a], \\ x(0) = x_0 \in X \end{cases} \tag{2.1}$$

where $f : [0,a] \to X$ is a function.

Theorem 2.2. *(Tanaka [46]) Assume that ($\mathbf{A}_1$)-($\mathbf{A}_3$) hold. Let $x_0 \in \overline{D}$ and $f \in L^1([0,a],X)$. Then the limit*

$$x(t) := U(t,0)x_0 + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,r)f(r)dr \tag{2.2}$$

exists uniformly for $t \in [0,a]$ and x is a continuous function on $[0,a]$.

Definition 2.1. *(Tanaka [46]) For $x_0 \in \overline{D}$, a continuous function $x : [0,a] \to X$ is called a mild solution of the initial value of Equation (2.1) if x satisfies the following equation:*

$$x(t) = U(t,0)x_0 + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,r)f(r)dr. \tag{2.3}$$

Lemma 2.1. *(Ezzinbi, Békollè and Kpoumiè [37]) Assume $f \in L^1([0,a],X)$. If x is the mild solution of Equation (2.1), then*

$$\|x(t)\| \leq Me^{wt}\|x_0\| + \int_0^t Me^{\omega(t-s)}\|f(s)\|ds.$$

Definition 2.2. *(Kpoumiè, Ezzinbi and Békollè [38]) For $\varphi(0) \in \overline{D}$, a continuous function $x : (-\infty,b] \to X$ is a mild solution of Equation (1.3) if x satisfies the following equation*

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,s)F(s,x_s)ds & \textit{for } 0 \leq t \leq b, \\ \varphi(t) & \textit{for } -\infty \leq t \leq 0. \end{cases} \tag{2.4}$$

In the whole of this work, we assume that ($\mathbf{A}_1$) - ($\mathbf{A}_3$) are true and $w > 0$.

3. Existence of mild solutions

In this section, we use some Fixed Point Theorems and the Kuratowski's measure of noncompactness to establish the existence of mild solutions of Equation (1.1). In this work, we always assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous.

Definition 3.1. *Let $\varphi(0) \in \overline{D}$. We say that a continuous function $x : (-\infty,b] \to X$ is a mild solution of Equation (1.1) if x satisfies the following equation*

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,s)F(s,x_{\rho(s,x_s)})ds & \textit{for } 0 \leq t \leq b, \\ \varphi(t) & \textit{for } -\infty \leq t \leq 0. \end{cases} \tag{3.1}$$

We introduce the Kuratowski's measure of noncompactness $\alpha(\cdot)$ of bounded sets K on a Banach space Y which is defined by:

$$\alpha(K) = \inf\{\varepsilon > 0 : \text{K has a finite cover of ball with diameter} < \varepsilon\}.$$

Some basic properties of $\alpha(\cdot)$ are given in the following Lemma.

Lemma 3.1. *(Akhmerov et al. in [7])*

(i) $\alpha(A_1) \leq diaA_1$, *where* $dia(A_1) = \sup_{x,y\in A_1} |x-y|$,

(ii) $\alpha(A_1) = 0$ *if and only if* A_1 *is relatively compact in* X,

(iii) $\alpha(A_1 \cup A_2) = max(\alpha(A_1), \alpha(A_2))$,

(iv) if $A_1 \subset A_2$, *then* $\alpha(A_1) \leq \alpha(A_2)$,

(v) $\alpha(A_1 + A_2) \leq \alpha(A_1) + \alpha(A_2)$,

(vi) $\alpha(B(0,\varepsilon)) = 2\varepsilon$ *if* $dimX = +\infty$.

The terminology and notations employed in this work coincide with those generally used in functional analysis. In particular, for Banach spaces $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, the notation $L(X, Y)$ stands for the Banach space of bounded linear operators from X into Y, and we abbreviate this notation to $L(X)$ when $X = Y$. Moreover $B_r(z, X)$ denotes the open ball with center at z and radius $r > 0$ in X and for a bounded function $x : J \to X$ and $0 \leq t \leq b$ we employ the notation $\|x\|_{X,t}$ for $\|x\|_{X,t} := \sup\limits_{\theta\in[0,t]} \|x(\theta)\|$. We will simply write $\|x\|_t$ when no confusion arises.

To prove our main result we will use the following variant of Schauder's Theorem see Radu Precup [41] and the Nonlinear Alternative of Leray-Schauder see A. Granas [27] or W. Arendt [10].

Theorem 3.1. *(Schauder) Let* X *be a Banach space,* $D \subset X$ *a nonempty convex bounded closed set and let* $\mathcal{T} : D \to D$ *be a completely continuous operator. Then* $\mathcal{T}$ *has at least one fixed point.*

Theorem 3.2. *(Leray-Schauder) Let* $\mathcal{W}$ *be a convex subset of a Banach space* X *and assume that* $0 \in \mathcal{W}$. *Let* $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ *be a completely continuous map. Then either*

(i) $\mathcal{F}$ *has a fixed point in* $\mathcal{W}$, *or*

(ii) the set $\{x \in \mathcal{W} : \ x = \alpha\mathcal{F}(x), \ 0 < \alpha < 1\}$ *is unbounded.*

Theorem 3.3 (Banach's Fixed Point Theorem)**.** *Let (E, d) be a non empty complete metric space and a mapping* $T : E \to E$ *such that* T^p *is a strict contraction (*$p \in \mathbb{N}^\star$*). Then T admits a unique fixed point* $\bar{x}$ *in E (i.e.* $T(\bar{x}) = \bar{x}$*) and the sequence* $(x_n)_n$ *define by* $x_n = T(x_{n-1})$ *with* $x_0 \in E$, *converges to* $\bar{x}$.

Lemma 3.2. *(Lemma Bellman-Gronwall) Let* f, g *the continuous positives fonctions from* $[a, b]$ *to* $\mathbb{R}_+$.
If Ψ *is constant, then from*

$$g(t) \leq \Psi + \int_a^t f(s)g(s)ds \ \textit{ for all } t \in [a, b],$$

it follows that

$$g(t) \leq \Psi \exp\Big(\int_a^t f(s)ds\Big) \ \textit{ for all } t \in [a, b].$$

Let us consider the following assumptions:

$(\mathbf{C}_1)$**.** $U(t, s)_{t>s}$ is compact on D for $t > s$.

$(\mathbf{C}_2)$. The function $F : J \times \mathcal{B} \to X$ satisfies the following properties.

(a) The function $F(\cdot,\psi) : J \to X$ is strongly measurable for every $\psi \in \mathcal{B}$.

(b) The function $F(t,\cdot) : \mathcal{B} \to X$ is continuous for each t $\in$ J.

(c) Let $L^1(J,[0,+\infty))$ be the space of integrable functions from J to $[0,+\infty)$. There exist $p \in L^1(J,[0,+\infty))$ and a continuous non-decreasing function $V : (0,+\infty) \to (0,+\infty)$ such that

$$\|F(t,\psi)\| \leq p(t)V(\|\psi\|_{\mathcal{B}}) \text{ for all } (t,\psi) \in J \times \mathcal{B}.$$

$(\mathbf{C}_3)$. Let $\varphi \in \mathcal{B}$ such that $x_0 = \varphi$ and $t \longmapsto \varphi_t$ is a $\mathcal{B}$-valued well defined continuous function on ρ^- where $\rho^- = \{\rho(s,\psi) : (s,\psi) \in J \times \mathcal{B}, \rho(s,\psi) \leq 0\}$, and there exists a continuous and bounded function $\eta : \rho^- \to (0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq \eta(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \rho^-$.

Remark 3.1. *For $\varphi \in \mathcal{B}$ such that $\varphi_t \in \mathcal{B}$ and $\varphi = x_0$ we can see that for all $t < 0$, $\varphi_t = x_t$. In fact if for all $t < 0, \varphi_t \neq x_t$, then for all $\theta \in (-\infty,0]$, $\varphi_t(\theta) \neq x_t(\theta)$ hence $\varphi(t+\theta) \neq x(t+\theta)$ thus for all $t \in (-\infty,0]$, $\varphi(t) \neq x(t)$ which is absurd because $\varphi = x_0$ that means for all $t \in (-\infty;0]$, $\varphi(t) = x(t)$. Therefore for all $t < 0$, $\varphi_t = x_t$.*

To continue with the next step we need the following Lemma due to E. Hernández.

Lemma 3.3. *(Hernández et al. [33]) Let $\varphi \in \mathcal{B}$ such that $\varphi_t \in \mathcal{B}$ for every $t \in \rho^-$. Assume that there exists a locally bounded function $\eta : \rho^- \to [0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq \eta(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \rho^-$ and $\zeta = \sup\{\eta(s) : s \in \rho^-\}$. If $x : (-\infty,b] \to X$ is continuous on J and $x_0 = \varphi$, then*

$$\|x_s\|_{\mathcal{B}} \leq (M_b + \zeta)\,\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\leq\theta\leq s}\|x(\theta)\|, \qquad s \in \rho^- \cup J$$

$$\textit{Where } K_b = \sup_{t\in J} K(t) \qquad , \qquad M_b = \sup_{t\in J} M(t)$$

.

In the sequel, we prove the existence of mild solution of equation (1.1).

Theorem 3.4. *Let Ω be a nonempty open subset of $\mathcal{B}$ and the function $F : [0,b] \times \mathcal{B} \to X$ is Carathéodory mapping. Assume that $(\boldsymbol{C}_1) - (\boldsymbol{C}_3)$ and $(\boldsymbol{A}_1) - (\boldsymbol{A}_3)$ hold. Let $\varphi \in \Omega$ be such that $\varphi(0) \in \overline{D}$. Then, Equation (1.1) has at least one mild solution $x(\cdot,\varphi)$ define on $]-\infty,a] \to X$, for some $a \in]0,b]$.*

Proof. We use the classic Schauder's Fixed Point Theorem.

Step 1. Let $\varphi \in \Omega$ be such that $\varphi(0) \in \overline{D}$. Then, there exists a constants $r > 0$, $r < b$ such that $\overline{B}_X(\varphi,r) = \{\psi \in \mathcal{B}$ such that $\|\psi - \varphi\|_{\mathcal{B}} \leq r\} \subset \Omega$ and $\|F(s,\psi)\| \leq \|p\|_{L^1}V(\|\psi\|)$ for all $s \in [0,r]$ and $\psi \in \overline{B}_X(\varphi,r)$.
Define the function $y : (-\infty,b] \longrightarrow X$ defined by:

$$y(t) = \begin{cases} U(t,0)\varphi(0) & \text{for } t \in J, \\ \\ \varphi(t) & \text{for } -\infty \leq t \leq 0. \end{cases}$$

By virtue of Axioms $(\mathbf{B}_1) - (i)$ and $(\mathbf{B}_2)$, $y_t \in \mathcal{B}$ and $t \mapsto y_t$ is a continuous function. Then for $\gamma \in (0,r)$ there exists $b_1 \in (0,r]$ such that $\|y_t - \varphi\|_{\mathcal{B}} \leq \gamma$ for all $t \in [0,b_1]$.
Set $K_b := \sup_{t\in[0,b]} K(t)$. Let a be a constant such that:

$$0 < a \leq \min\left\{b_1, \frac{r-\gamma}{Me^{wa}K_b\|p\|_{L^1}V(l)}\right\}$$

where $l = (M_b + \zeta + K_b + K_b H)\|\varphi\|_{\mathcal{B}} + K_b H r$. For $u \in C([0,a];X)$ such that $u(0) = \varphi(0)$, we define its extension on $(\infty, a]$ by :

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for } t \in [0,a], \\ \\ \varphi(t) & \text{for } -\infty \leq t \leq 0. \end{cases}$$

Let us introduce the following space:

$$\mathbb{F}_a := \Big\{ u : [0,a] \to X \text{ continuous such that } \; u_0 = \varphi \text{ and } \sup_{0 \leq t \leq a} \|\tilde{u}_t - \varphi\|_{\mathcal{B}} \leq r \Big\}$$

endowed with the uniform norm topology. $\|.\|_{\mathbb{F}_a}$ defined by:

$$\|u\|_{\mathbb{F}_a} := \|u_0\|_{\mathcal{B}} + \sup_{0 \leq s \leq a} \|u(s)\|$$

The restriction of y to $(\infty, a]$ is an element of $\mathbb{F}_a$. In fact $\|y_t - \varphi\|_{\mathcal{B}} \leq \gamma$ for all $t \in [0, b_1]$ whereas $\gamma < r$ then $\|\tilde{y}_t - \varphi\|_{\mathcal{B}} \leq r$ for all $t \in [0,a]$ thus $y \in \mathbb{F}_a$. Therefore $\mathbb{F}_a$ is nonempty.

For all $u \in \mathbb{F}_a$, we have

$$\begin{aligned} \|u\|_{\mathbb{F}_a} &= \|u_0\|_{\mathcal{B}} + \sup_{0 \leq s \leq a} \|u(s)\| \\ &\leq \|u_0 - \varphi\|_{\mathcal{B}} + \|\varphi\|_{\mathcal{B}} + \sup_{0 \leq s \leq a} H\|u_s\| \text{ by } (\mathbf{B}_1) - (iii) \\ &\leq \|\varphi\|_{\mathcal{B}} + H \sup_{0 \leq s \leq a} \{\|(u_s - \varphi) + \varphi\|_{\mathcal{B}}\} \text{ since } u_0 = \varphi \\ &\leq \|\varphi\|_{\mathcal{B}} + H\{\sup_{0 \leq s \leq a} \|(u_s - \varphi)\|_{\mathcal{B}} + \|\varphi\|_{\mathcal{B}}\} \\ &\leq \|\varphi\|_{\mathcal{B}} + H(r + \|\varphi\|_{\mathcal{B}}). \end{aligned} \tag{3.2}$$

Then $\mathbb{F}_a$ is bounded.

By using the triangular inequality in $\mathcal{B}$ it is clear that $\lambda p + (1-\lambda)q \in \mathbb{F}_a$ for any $p, q \in \mathbb{F}_a$, with $\lambda \in [0,1]$. Indeed

$$\begin{aligned} \|\lambda\tilde{p}_t + (1-\lambda)\tilde{q}_t - \varphi\|_{\mathcal{B}} &= \|\lambda\tilde{p}_t + (1-\lambda)\tilde{q}_t - (1-\lambda)\varphi + (1-\lambda)\varphi - \varphi\|_{\mathcal{B}} \\ &= \|\lambda\tilde{p}_t + (1-\lambda)(\tilde{q}_t - \varphi) + \varphi - \lambda\varphi - \varphi\|_{\mathcal{B}} \\ &= \|\lambda(\tilde{p}_t - \varphi) + (1-\lambda)(\tilde{q}_t - \varphi)\|_{\mathcal{B}} \\ &\leq \lambda\|(\tilde{p}_t - \varphi)\| + (1-\lambda)\|(\tilde{q}_t - \varphi)\|_{\mathcal{B}} \\ &\leq \lambda r + (1-\lambda)r \\ &= r. \end{aligned}$$

Then $\mathbb{F}_a$ is convex.

Now we prove that $\mathbb{F}_a$ is closed. To prove that, consider a convergent sequence $(\tilde{u}_t^n)_{n \in \mathbb{N}}$ of $\mathbb{F}_a$ which converges to $\tilde{u}_t$. We want to show that $\tilde{u}_t \in \mathbb{F}_a$.

$$\begin{aligned} \|\tilde{u}_t - \varphi\|_{\mathcal{B}} &= \|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} + \|\tilde{u}_t^n - \varphi\|_{\mathcal{B}} \\ &\leq \|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} + r, \text{ since } \tilde{u}_t^n \in \mathbb{F}_a. \end{aligned}$$

whereas

$$\|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} \leq K_b \sup_{0\leq s\leq t} \|\tilde{u}(s) - \tilde{u}^n(s)\| + M_b\|\tilde{u}_0 - \tilde{u_0^n}\|_{\mathcal{B}}$$

$$\leq Max(K_b, M_b) \sup_{0\leq s\leq t} \|\tilde{u}(s) - \tilde{u}^n(s)\| + Max(K_b, M_b)\|\tilde{u}_0 - \tilde{u}_0^n\|_{\mathcal{B}}$$

$$\leq Max(K_b, M_b)\|\tilde{u} - \tilde{u}^n\|_{\mathbb{F}_a}.$$

Thus $\|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} \leq Max(K_b, M_b)\|\tilde{u} - \tilde{u}^n\|_{\mathbb{F}_a}$ as $\|\tilde{u} - \tilde{u}^n\|_{\mathbb{F}_a} \to 0$ with $n \to +\infty$ hence $\|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} \to 0$ with $n \to +\infty$ then $\|\tilde{u}_t - \varphi\| \leq r$, hence $\tilde{u}_t \in \mathbb{F}_a$ thus $\mathbb{F}_a$ is closed.

To continue our proof, we need the following Lemma.

Lemma 3.4. *Let $\varphi \in \mathcal{B}$ such that $\varphi_t \in \mathcal{B}$ for every $t \in \rho^-$. Assume that there exists a locally bounded function $\eta : \rho^- \to [0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq \eta(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \rho^-$ and $\zeta = \sup\{\eta(s) : s \in \rho^-\}$. If $u \in \mathbb{F}_a$, then*

$$\|\tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} \leq l < +\infty$$

Where $l = M_b + \zeta + K_b + K_bH)\|\varphi\|_{\mathcal{B}} + K_bHr$.

Proof.

$$\begin{aligned}\|\tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} &\leq (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\leq\theta\leq\rho(s,\tilde{u}_s)} \|\tilde{u}(\theta)\|, \text{ by the Lemma 3.3}\\ &\leq (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b(\|\varphi\|_{\mathcal{B}} + \sup_{0\leq\theta\leq a} \|u(\theta)\|)\\ &\leq (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b\|u\|_{\mathbb{F}_a} \text{ since } \|u\|_{\mathbb{F}_a} = \|\varphi\|_{\mathcal{B}} + \sup_{0\leq\theta\leq a} \|u(\theta)\|\\ &\leq (M_b + \zeta + K_b + K_bH)\|\varphi\|_{\mathcal{B}} + K_bHr. \text{ By relation (3.2)}\end{aligned}$$

■

Consider the mapping $\mathcal{K}$ defined on $\mathbb{F}_a$ by:

$$\begin{cases}(\mathcal{K}x)(t) = U(t,0)\varphi(0) + \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,s)F(s, \tilde{x}_{\rho(s,\tilde{x}_s)})ds \quad \text{for} \quad t \in [0,a],\\ \varphi(t) \qquad \text{for} \quad -\infty < t \leq 0.\end{cases} \tag{3.3}$$

From definition (3.1), theorem (2.1) and the assumptions on φ, we infer that $(\mathcal{K}x)(\cdot)$ is well defined. We claim that $\mathcal{K}(\mathbb{F}_a) \subset \mathbb{F}_a$. In fact, Axiom $(\mathbf{C}_2)$ implies that for every $x \in \mathbb{F}_a$, the mapping $s \mapsto F(s, \tilde{x}_{\rho(s,\tilde{x})})$ is continuous on $[0, a]$. Hence this mapping $v := \mathcal{K}x$ is continuous on $[0, a]$. In the other hand, One has

$$\|\tilde{v}_t - \varphi\|_{\mathcal{B}} \leq \|\tilde{v}_t - y_t\|_{\mathcal{B}} + \|y_t - \varphi\|_{\mathcal{B}}$$

$$\leq \|\tilde{v}_t - y_t\|_{\mathcal{B}} + \gamma$$

On one hand, by Axiom $(\mathbf{B}_1) - (iii)$, we have for any $t \in [0, a]$,

$$\|\tilde{v}_t - y_t\|_{\mathcal{B}} \leq K_b \sup_{0\leq s\leq t} \|v(s) - y(s)\|$$

For any $t \in [0,a]$

$$\begin{aligned}
\|v(t)-y(t)\| &= \left\|U(t,0)\varphi(0) - \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)F(s,\tilde{v}_{\rho(s,\tilde{v_s})})ds - U(t,0)\varphi(0)\right\| \\
&= \left\|\lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)F(s,\tilde{v}_{\rho(s,\tilde{v_s})})ds\right\| \\
&\leq \int_0^t Me^{w(t-s)}\left\|F(s,\tilde{v}_{\rho(s,\tilde{v_s})})\right\|ds \\
&\leq Me^{wa}\int_0^t \|p\|_{L^1}V(\|\tilde{v}_{\rho(s,\tilde{v_s})}\|_{\mathcal{B}})ds \\
&\leq Me^{wa}a\|p\|_{L^1}V(l), \text{ by Lemma } 3.4 \\
&\leq \tfrac{r-\gamma}{K_b}.
\end{aligned}$$

hence $K_b \sup\limits_{0\leq s\leq t}\|v(s)-y(s)\| \leq r-\gamma$ then $\|\tilde{v}_t - y_t\|_{\mathcal{B}} \leq r-\gamma$ we have $\|\tilde{v}_t - \varphi\|_{\mathcal{B}} \leq r$, for any $t \in [0,a]$. Therefore $v \in \mathbb{F}_a$. We have proved that $\mathbb{F}_a$ is a nonempty, bounded, convex and closed subset of $\mathbb{F}_a$:

Now we want to prove that $\mathcal{K}$ is a completely continuous operator.

Step 2. The continuity of $\mathcal{K}$. Let $(u^n)_{n\in\mathbb{N}^*}$ be a sequence in $\mathbb{F}_a$ such that $\lim\limits_{n\to\infty} u^n = u$. For $t \in [0,a]$, we have by Axiom ($\mathbf{B}_1 - iii$) :

$$\begin{aligned}
\|\tilde{u}_t - \tilde{u}^n_t\|_{\mathcal{B}} &\leq K_b \sup_{0\leq s\leq t}\|\tilde{u}(s)-\tilde{u}^n(s)\| + M_b\|\tilde{u}_0 - \tilde{u}^n_0\|_{\mathcal{B}} \\
&\leq Max(K_b,M_b)\sup_{0\leq s\leq t}\|\tilde{u}(s)-\tilde{u}^n(s)\| + Max(K_b,M_b)\|\tilde{u}_0 - \tilde{u}^n_0\|_{\mathcal{B}} \\
&\leq Max(K_b,M_b)\|\tilde{u}-\tilde{u}^n\|_{\mathbb{F}_a}.
\end{aligned}$$

then $\lim\limits_{n\to\infty}\tilde{u}^n_s = \tilde{u_s}$. we recall that $\rho : [0,a]\times\mathcal{B} \to (-\infty,a]$ is continuous then $\lim\limits_{n\to\infty}\rho(s,\tilde{u}^n)_s = \rho(s,\tilde{u_s})$.

Let us study therefore the convergence of the sequence $(\tilde{u}^n_{\rho(s,\tilde{u}^n_s)})_{n\in\mathbb{N}}$ for $s \in [0,a]$. At first, if $s \in [0,a]$ such that $\rho(s,\tilde{u}_s) > 0$, and there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > N$ $\rho(s,\tilde{u}^n_s) > 0$.

In this case one has

$$\begin{aligned}
\|\tilde{u}^n_{\rho(s,\tilde{u}^n_s)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} &\leq \|\tilde{u}^n_{\rho(s,\tilde{u}^n_s)} - \tilde{u}_{\rho(s,\tilde{u}^n_s)}\|_{\mathcal{B}} + \|\tilde{u}_{\rho(s,\tilde{u}^n_s)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} \\
&\leq K_b \sup_{0\leq\theta\leq\rho(s,\tilde{u}^n_s)}\|u^n(\theta)-u(\theta)\| + M_b\|\varphi-\varphi\| + \|\tilde{u}_{\rho(s,\tilde{u}^n_s)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} \\
&\quad \text{by } (\mathbf{B}_1 - iii) \\
&\leq K_b\|u^n-u\|_a + \|\tilde{u}_{\rho(s,\tilde{u}^n_s)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}}
\end{aligned}$$

whereas

$$\lim_{n\to\infty} u^n = u \text{ then } \|u^n-u\|_a \to 0 \text{ for } n \to +\infty,$$

$$\|\tilde{u}_{\rho(s,\tilde{u}^n_s)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} \to 0 \text{ for } n \to +\infty \text{ by } (\mathbf{B}_1 - iii),$$

which proves that $\tilde{u}^n_{\rho(s,\tilde{u}^n_s)} \to \tilde{u}_{\rho(s,\tilde{u}_s)}$ in $\mathcal{B}$ as $n \to +\infty$ for every $s \in [0,a]$ such that $\rho(s,\tilde{u}_s) > 0$. Similar, if $s \in [0,a]$ such that $\rho(s,\tilde{u}_s) < 0$, and there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > N$ $\rho(s,\tilde{u}^n_s) < 0$.

In this case one has

$$\begin{aligned}\|\tilde{u}^n_{\rho(s,\tilde{u}^n_s)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} &= \|\varphi_{\rho(s,\tilde{u}^n_s)} - \varphi_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}}, \text{ by Remark 3.1}\\ &\leq \eta(t)\|\varphi - \varphi\|, \text{ by } (\mathbf{C}_3) \text{ with } t<0.\end{aligned}$$

Which proves that $\tilde{u}^n_{\rho(s,\tilde{u}^n_s)} \to \tilde{u}_{\rho(s,\tilde{u}_s)}$ in $\mathcal{B}$ as $n \to +\infty$ for every $s \in [0,a]$ such that $\rho(s,\tilde{u}_s) < 0$. Then

$$\lim_{n\to\infty} \tilde{u}^n_{\rho(s,\tilde{u}^n_s)} = \tilde{u}_{\rho(s,\tilde{u}_s)}.$$

For $t \in [0,b]$, we have :

$$\begin{aligned}\|(\mathcal{K}u^n)(t) - (\mathcal{K}u)(t)\| &= \Big\| \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)F(s,\tilde{u}^n_{\rho(s,\tilde{u}^n_s)})ds - \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds\Big\|\\ &= \Big\| \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)(F(s,\tilde{u}^n_{\rho(s,\tilde{u}^n_s)}) - F(s,\tilde{u}_{\rho(s,\tilde{u}_s)}))ds\Big\|\\ &\leq Me^{\omega b}\int_0^t \Big\|F(s,\tilde{u}^n_{\rho(s,\tilde{u}^n_s)}) - F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})\Big\|ds.\end{aligned}$$

As $\lim_{n\to\infty} \tilde{u}^n_{\rho(s,\tilde{u}^n_s)} = \tilde{u}_{\rho(s,\tilde{u}_s)}$, $F(s,\cdot)$ is continuous from assumption $(\mathbf{C}_2)-(b)$, then $\Big(F(s,\tilde{u}^n_{\rho(s,\tilde{u}^n_s)})\Big)_{n\in\mathbb{N}}$ converges to $F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})$, and from assumption $(\mathbf{C}_2)-(c)$ we can conclude by the Lebesgue Dominated Convergence Theorem that $\mathcal{K}u^n \to \mathcal{K}u$.

Next, we will show now that the range of $\mathcal{K}$; $Range(\mathcal{K}) := \{\mathcal{K}u, u \in \mathbb{F}_a\}$, is relatively compact in $\mathbb{F}_a$. By the Arzela–Ascoli theorem, it suffices to prove that $Range(\mathcal{K})(t)$ is relatively compact in X for each $t \in [0,a]$, and $Range(\mathcal{K})$ is equicontinuous on $[0,a]$.

Step 3. The set of fonctions $Range(\mathcal{K})(t)$ of is relatively compact on $\mathbb{F}_a$. To prove this assertion, it is sufficient to show that the set $\Big\{(\mathcal{K}u)(t) - U(t,0)\varphi(0):\ u \in \mathbb{F}_a\Big\}$ is relatively compact.

Let $0 < \epsilon < t \leq a$. Then

$$\begin{aligned}\mathcal{K}u(t) - U(t,0)\varphi(0) &= \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds\\ &= \lim_{\lambda\to 0^+}\int_0^{t-\epsilon} U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds + \lim_{\lambda\to 0^+}\int_{t-\epsilon}^t U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds\\ &= U(t,t-\epsilon)\lim_{\lambda\to 0^+}\int_0^{t-\epsilon} U_\lambda(t-\epsilon,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds + \lim_{\lambda\to 0^+}\int_{t-\epsilon}^t U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds.\end{aligned}$$

We claim that

$$\Big\{\lim_{\lambda\to 0^+}\int_0^{t-\epsilon} U_\lambda(t-\epsilon,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds:\ \ u \in \mathbb{F}_a\Big\}$$

is a bounded. In fact, for $u \in \mathbb{F}_a$:

$$\Big\| \lim_{\lambda \to 0^+} \int_0^{t-\epsilon} U_\lambda(t-\epsilon, s) F(s, \tilde{u}_{\rho(s,\tilde{u}_s)}) ds \Big\| \leq M e^{\omega a} \int_0^{t-\epsilon} p(s) V(\|\tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}}) ds$$
$$\leq M e^{\omega a} V(l) \int_0^{t-\epsilon} p(s) ds \text{ by Lemma 3.4.}$$

Where $l = M_b + \zeta + K_b + K_b H)\|\varphi\|_{\mathcal{B}} + K_b H r$. Since $U(t, t-\epsilon)$ is a compact operator for $0 < \epsilon < t$, the set

$$U(t, t-\epsilon)\Big\{ \lim_{\lambda \to 0^+} \int_0^{t-\epsilon} U_\lambda(t-\epsilon, s) F(s, \tilde{u}_{\rho(s,\tilde{u}_s)}) ds : \;\; u \in \mathbb{F}_a \Big\}$$

is relatively compact in X for every ϵ, $0 < \epsilon < t$. We know that,

$$\Big\| \lim_{\lambda \to 0^+} \int_{t-\epsilon}^{t} U_\lambda(t, s) F(s, \tilde{u}_{\rho(s,\tilde{u}_s)}) ds \Big\| \leq M e^{\omega a} \int_{t-\epsilon}^{t} p(s) V(\|\tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}}) ds$$
$$\leq M e^{\omega a} V(l) \int_{t-\epsilon}^{t} p(s) ds \text{ by Lemma 3.4.}$$

Where $l = M_b + \zeta + K_b + K_b H)\|\varphi\|_{\mathcal{B}} + K_b H r$. Thus

$$\lim_{\lambda \to 0^+} \int_{t-\epsilon}^{t} U_\lambda(t, s) F(s, \tilde{u}_{\rho(s,\tilde{u}_s)}) ds \in B\Big(0, M e^{\omega a} V(l) \int_{t-\epsilon}^{t} p(s) ds\Big).$$

By Lemma 2.1 it follows that

$$\alpha\Big(B\Big(0, M e^{\omega a} V(l) \int_{t-\epsilon}^{t} p(s) ds\Big)\Big) = 2 M e^{\omega a} V(l) \int_{t-\epsilon}^{t} p(s) ds. \tag{3.4}$$

where $\alpha(\cdot)$ is Kuratowski's measure of noncompactness of sets in X. Letting ϵ tends to 0 , we obtain in relation (3.4) that $\alpha\Big(B\Big(0, M e^{\omega a} V(l) \int_{t-\epsilon}^{t} p(s) ds\Big)\Big) = 0$. By Lemma 2.1,

$$\Big\{ \lim_{\lambda \to 0^+} \int_{t-\epsilon}^{t} U_\lambda(t, s) F(s, \tilde{u}_{\rho(s,\tilde{u}_s)}) ds : \; u \in \mathbb{F}_a \Big\}$$

is relatively compact. Then

$$\Big\{ (\mathcal{K}u)(t) - U(t, 0)\varphi(0) : \; u \in \mathbb{F}_a \Big\}$$

is relatively compact. Hence, $Range(\mathcal{K})(t)$ is relatively compact in X for each $t \in J$.

Step 4. The set of fonctions $Range(\mathcal{K})$ is equicontinuous on $[0, a]$. For every $0 \leq t_0 \leq t \leq a$, one has:

$$\begin{aligned}
(\mathcal{K}u)(t) - (\mathcal{K}u)(t_0) &= \Big(U(t,0) - U(t_0,0)\Big)\varphi(0) + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,s) F(s, \tilde{u}_{\rho(s,\tilde{u}_s)}) ds \\
&\quad - \lim_{\lambda \to 0^+} \int_0^{t_0} U_\lambda(t_0,s) F(s, \tilde{u}_{\rho(s,\tilde{u}_s)}) ds \\
&= \Big(U(t,0) - U(t_0,0)\Big)\varphi(0) + \lim_{\lambda \to 0^+} \int_{t_0}^t U_\lambda(t,s) F(s, \tilde{u}_{\rho(s,\tilde{u}_s)}) ds \\
&\quad + \Big(U(t,t_0) - I\Big) \lim_{\lambda \to 0^+} \int_0^{t_0} U_\lambda(t_0,s) F(s, \tilde{u}_{\rho(s,\tilde{u}_s)}) ds.
\end{aligned}$$

This implies that

$$\|(\mathcal{K}u)(t)-(\mathcal{K}u)(t_0)\| \leq \Big\|\Big(U(t,0)-U(t_0,0)\Big)\varphi(0)\Big\| + Me^{\omega b}V(l)\int_{t_0}^{t} p(s)ds + \Big\|\Big(U(t,t_0)-I\Big)\lim_{\lambda\to 0^+}\int_0^{t_0} U_\lambda(t_0,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds\Big\|.$$

Since $Range(\mathcal{K})(t_0)$ is relatively compact and

$$\Big\{\lim_{\lambda\to 0^+}\int_0^{t_0} U_\lambda(t_0,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds: \quad u\in\mathbb{F}_a\Big\} \subseteq Range(\mathcal{K})(t_0)$$

. There exists a compact set G such that:

$$\Big\{\lim_{\lambda\to 0^+}\int_0^{t_0} U_\lambda(t_0,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds: \quad u\in\mathbb{F}_a\Big\} \subseteq G.$$

Then

$$\lim_{\substack{t\to t_0\\ t>t_0}}\sup_{u\in G}\Big\|\Big(U(t,t_0)-I\Big)u\Big\| = 0.$$

Thus, we get

$$\lim_{\substack{t\to t_0\\ t>t_0}}\|(\mathcal{K}u)(t)-(\mathcal{K}u)(t_0)\| = 0 \text{ for all } u\in\mathbb{F}_a.$$

Using similar argument for $0\le t\le t_0\le b$, we conclude that $Range(\mathcal{K})$ is equicontinuous. Then by Arzelá-Ascoli's Theorem, $Range(\mathcal{K})$ is retlatively compact. Since $\mathcal{K}$ is continuous by Step 2, we can conclude that $\mathcal{K}$ is a completely continuous operator. The existence of at least one a mild solution for Equation (1.1) is now a consequence of the variant of Schauder's Fixed Point Theorem. □

Theorem 3.5. *Let $(\boldsymbol{C}_1)-(\boldsymbol{C}_3)$ be satisfied. If $\rho(t,\psi)\le t$ for every $(t,\psi)\in J\times\mathcal{B}$ and*

$$MK_be^{\omega b}\|p\|_{L^1} < \int_N^\infty \frac{ds}{V(s)} \tag{3.5}$$

where $N=(M_b+\zeta)\|\varphi\|_{\mathcal{B}}+K_b\|\varphi(0)\|_X$ with $K_b=\sup_{t\in J}k(t)$, $M_b=\sup_{t\in J}M(t)$, $\zeta:=\sup\{\eta(s): s\in\rho^-\}$. Then there exists a mild solution of Equation (1.1).

Proof. Let $E=C(J,X)$ and $\mathcal{K}:E\to E$ be the operator defined by (3.3). In order to use Leray Schauder Alternative Theorem. We claim that the set

$$\xi := \Big\{x\in C(J,X): \quad x=\mu\mathcal{K}(x), \quad 0<\mu<1\Big\} \text{ is bounded. Indeed}$$

$$\begin{aligned}
\|x\| &\le Me^{\omega t}\|\varphi(0)\| + \int_0^t Me^{\omega(t-s)}\|F(s,\tilde{x}_{\rho(s,\tilde{x_s})})\|ds\\
&\le Me^{\omega t}H\|\varphi\|_{\mathcal{B}} + M\int_0^t e^{\omega(t-s)}p(s)V(\|\tilde{x}_{\rho(s,\tilde{x_s})}\|_{\mathcal{B}})ds \quad \text{by } (\mathbf{B}_1)-(ii)' \text{ and } (\mathbf{C}_2)\\
&\le Me^{\omega t}H\|\varphi\|_{\mathcal{B}} + Me^{\omega b}\int_0^t p(s)V\Big((M_b+\zeta)\|\varphi\|_{\mathcal{B}} + K_b\sup_{0\le\theta\le\rho(s,\tilde{x_s})}\|\tilde{x}(\theta)\|\Big)ds \quad \text{by Lemma 3.3}\\
&\le Me^{\omega t}H\|\varphi\|_{\mathcal{B}} + Me^{\omega b}\int_0^t p(s)V\Big((M_b+\zeta)\|\varphi\|_{\mathcal{B}} + K_b\|x\|_{\rho(s,\tilde{x_s})}\Big)ds\\
&\le Me^{\omega t}H\|\varphi\|_{\mathcal{B}} + Me^{\omega b}\int_0^t p(s)V\Big((M_b+\zeta)\|\varphi\|_{\mathcal{B}} + K_b\|x\|_s\Big)ds
\end{aligned}$$

since $\rho(t,\tilde{x_t}) \leq t$ for every $t \in J$. If

$$\vartheta(t) := (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b\|x\|_t,$$

we obtain that

$$\vartheta(t) \leq (M_b + \zeta + K_b M e^{\omega b} H)\|\varphi\|_{\mathcal{B}} + MK_b e^{\omega b}\int_0^t p(s)V(\vartheta(s))ds$$

since $\|x\|_t \leqslant \|x\|$ for all $t \in J$. Setting

$$\nu(t) := (M_b + \zeta + K_b M e^{\omega b} H)\|\varphi\|_{\mathcal{B}} + MK_b e^{\omega b}\int_0^t p(s)V(\vartheta(s))ds$$

and using the nondecreasing character of V, we have :

$$\nu(t) \leq (M_b + \zeta + K_b M e^{\omega b} H)\|\varphi\|_{\mathcal{B}} + MK_b e^{\omega b}\int_0^t p(s)V(\nu(s))ds$$

since $\vartheta(t) \leqslant \nu(t)$ for every $t \in J$. Since ν is defferentiable, we have

$$\nu'(t) \leq MK_b e^{\omega b} p(t)V(\nu(t)) \text{ for every } t \in J.$$

Thus

$$\int_{\nu(0)=N}^{\nu(t)} \frac{ds}{V(s)} \leq MK_b e^{\omega b}\int_0^t p(s)ds.$$

Hence

$$\int_{\nu(0)=N}^{\nu(t)} \frac{ds}{V(s)} \leq MK_b e^{\omega b}\|p\|_{L^1}.$$

Using relation (3.5), we get

$$\int_{\nu(0)=N}^{\nu(t)} \frac{ds}{V(s)} < \int_N^{+\infty} \frac{ds}{V(s)}.$$

This implies that, the set of functions $\{\nu(\cdot) : 0 < \mu < 1\}$ is bounded in $C(J : X)$. Thus the set $\{x(\cdot) : 0 < \mu < 1\}$ is also bounded in $C(J : X)$ since

$$(M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b\|x\|_t \leq \nu(t) \text{ for all } t \in J.$$

We obtain the completely continuous property of $\mathcal{K}$ by proceeding as in the proof of Theorem 3.4. Since E is convex and $0 \in E$, then the Nonlinear Alternative Leray-Schauder's Fixed Point Theorem guaranties the existence of at least one mild solution for Equation (1.1).

Arguing as in the proof of Theorem 3.2 we can prove that $\mathcal{K}$ is completely continuous. Then by the Nonlinear Alternative Leray-Schauder's Fixed Point Theorem the exists at least one mild solution for Equation (1.1). □

4. Global existence of mild solutions and Blowing up phenomena

Let us give the following local Lipschitz condition on the nonlinear part F of Equation (1.1):

(**C**$_4$) For each $\alpha > 0$ there exists a positive constant $r_0(\alpha)$ such that for $\varphi, \psi \in \mathcal{B}$ with $|\varphi|_{\mathcal{B}}, |\psi|_{\mathcal{B}} \leq \alpha$, we have:

$$\|F(t,\varphi) - F(t,\psi)\| \leq r_0(\alpha)|\varphi - \psi|_{\mathcal{B}} \text{ for } t \geq 0.$$

Contrarily to the previous results, if we replace conditions ((**C**$_2$) by condition (**C**$_4$), the following local existence results hold.

Theorem 4.1. *Assume that (C_1), (C_3) and (C_4) hold. Then, for $\varphi \in \mathcal{B}$ such that $\varphi(0) \in \overline{D}$, Equation (1.1) has a mild solution $x(.,\varphi)$ in a maximal interval $(-\infty, a_{max})$ and either*

$$a_{max} = +\infty \quad or \quad \limsup_{t \to a_{max}^-} \|x(t,\varphi)\| = +\infty.$$

Moreover, $x(.,\varphi)$ depends continuously on the initial data φ in the sense that, if $\varphi \in \mathcal{B}$, $\varphi(0) \in \overline{D}$ and $t \in [0, a_{max})$, then there exist positive constants k and $\varepsilon > 0$ such that, for $\psi \in \mathcal{B}$ and $|\varphi - \psi|_{\mathcal{B}} \leq \varepsilon$, we have

$$\|x(s,\varphi) - x(s,\psi)\| \leq k|\varphi - \psi|_{\mathcal{B}} \text{ for } s \in]-\infty, a].$$

Proof. Let $x(.,\varphi)$ be a mild solution of Equation (1.1) in $(-\infty, b]$. We know that, $x(t) \in \overline{D}$ for all $t \in [0,a]$. Repeating the procedure used in the local existence result, this yields existence of $a > a_1$ and a function $x(.,x_a(.,\varphi)) : (-\infty, a_1] \to X$ which satisfies for $t \in [a, a_1]$:

$$x(.,x_a(.,\varphi)) = U(t,0)x(a,\varphi) + \lim_{\lambda \to 0^+} \int_a^t U_\lambda(t,s) F(s, \tilde{x}_{\rho(s,\tilde{x}_s)}(.,x_a(.,\varphi)))ds.$$

Proceeding inductively, we obtain the maximal interval of existence $(-\infty, a_{\max})$ of the solution $x(.,\varphi)$. Assume that $a_{\max} < +\infty$ and $\lim_{t \to a_{\max}^-} \sup \|x(t,\varphi)\| < M$. We claim that $x(.,\varphi)$ is uniformly continuous and consequently $\lim_{t \to a_{\max}^-} x(.,\varphi)$ exists in X, which contradicts the maximality of $[0, a_{max}[$. In the following, we show uniform continuity of $x(.,\varphi)$. Let $t, t+h \in [0, a_{max})$ with $h > 0$. Then,

$$\begin{aligned}
&\|x(t+h,\varphi) - x(t,\varphi)\| \\
&\leq \|U(t+h,0)\varphi(0) - U(t,0)\varphi(0)\| \\
&+ \Big\| \lim_{\lambda \to 0^+} \int_0^{t+h} U_\lambda(t+h,\tau) F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)}) d\tau - \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,\tau) F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)}) d\tau \Big\| \\
&\leq \|U(t+h,0)\varphi(0) - U(t,0)\varphi(0)\| + \Big\| U_\lambda(t+h,t) \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,\tau) F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)}) d\tau \\
&+ \lim_{\lambda \to 0^+} \int_t^{t+h} U_\lambda(t+h,\tau) F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)}) d\tau - \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,\tau) F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)}) d\tau \Big\| \\
&\leq \|(U(t+h,0) - U(t,0))\varphi(0)\| + \Big\| (U(t+h,t) - I) \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,\tau) F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)}) d\tau \Big\| \\
&+ \lim_{\lambda \to 0^+} \Big\| \int_t^{t+h} U_\lambda(t+h,\tau) F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)}) d\tau \Big\|.
\end{aligned}$$

Since $\mathcal{W} := \Big\{ \int_0^t U_\lambda(t,\tau) F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)}) d\tau : \ x \in \mathbb{F}_a \Big\} \subseteq G$ with G compact. We obtain that

$$\lim_{\substack{h \to 0 \\ t+h > t}} \| (U(t+h,t) - I) x \| = 0 \quad \text{for } x \in \mathcal{W}.$$

Since

$$\lim_{\lambda \to 0^+} \Big\| \int_t^{t+h} U_\lambda(t+h,\tau) F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)}) d\tau \Big\| \leq M e^{wh} \|p\|_{L^1} V(l) h,$$

then

$$\lim_{\substack{h \to 0 \\ t+h > t}} \|x(t+h,\varphi) - x(h,\varphi)\| = 0.$$

Similarly, we show that

$$\lim_{\substack{h \to 0 \\ t+h < t}} \|x(t+h,\varphi) - x(t,\varphi)\| = 0.$$

Then $x(.,\varphi)$ is uniformly continuous on $[0,a_{\max})$ and therefore, $\lim_{t\to a_{\max}} x(.,\varphi)$ exists. If we define $x(a_{\max},\varphi) := \lim_{t\to a_{\max}} x(.,\varphi)$, we can extend $x(.,\varphi)$ beyond $a_{\max}$ which contradict the maximality of $]-\infty,a_{\max})$.

We prove now that $\mathcal{K}$ is strict contraction in $\mathbb{F}_a(\varphi)$ and for this end, we consider $x,z\in\mathbb{F}_a(\varphi)$. For $t\in[0,a]$, we have

$$\|(\mathcal{K}x)-(\mathcal{K}z)\|_{\mathbb{F}_a} = \sup_{0\le t\le b}\|(\mathcal{K}x)(t)-(\mathcal{K}z)(t)\|$$

and

$$\begin{aligned}
\|(\mathcal{K}x)-(\mathcal{K}z)\|_{\mathbb{F}_a} &= \Big\|\lim_{\lambda\to0^+}\int_0^t U_\lambda(t,s)F(s,\tilde{x}_{\rho(s,\tilde{x}_s)})ds - \lim_{\lambda\to0^+}\int_0^t U_\lambda(t,s)F(s,\tilde{z}_{\rho(s,\tilde{z}_s)})ds\Big\| \\
&\le \int_0^t Me^{w(t-s)}\|F(s,\tilde{x}_{\rho(s,\tilde{x}_s)})-F(s,\tilde{z}_{\rho(s,\tilde{z}_s)})\|ds \\
&\le Me^{wb}r_0(\alpha)\int_0^t \|\tilde{x}_{\rho(s,\tilde{x}_s)}-\tilde{z}_{\rho(s,\tilde{z}_s)}\|_{\mathcal{B}}ds \\
&\le K_bMe^{wb}r_0(\alpha)\int_0^t \sup_{0\le\theta\le\rho(s,\tilde{x}_s)}\|x(\theta)-z(\theta)\|_X ds \\
&\le K_bMe^{wb}r_0(\alpha)a\|x-z\|_{\mathbb{F}_a}.
\end{aligned}$$

Following the same reasoning, we can see that

$$\begin{aligned}
\|(\mathcal{K}^2x)(t)-(\mathcal{K}^2z)(t)\|_{\mathbb{F}_a} &\le K_bMe^{wb}r_0(\alpha)\int_0^t \sup_{0\le\theta\le\rho(s,\tilde{x}_s)}\|(\mathcal{K}x)(\theta)-(\mathcal{K}z)(\theta)\|_X ds \\
&\le (K_bMe^{wb}r_0(\alpha))^2\int_0^t \sup_{0\le\theta\le s}\int_0^\theta \sup_{0\le\xi\le p}\|x(\xi)-z(\xi)\|_X dpds \\
&\le (K_bMe^{wb}r_0(\alpha))^2\int_0^t\int_0^s \|x-z\|_{\mathbb{F}_a}dpds \\
&\le \frac{(K_bMe^{wb}r_0(\alpha))^2a^2}{2}\|x-z\|_{\mathbb{F}_a}.
\end{aligned}$$

We can repeat the previous argument, and we obtain

$$\|(\mathcal{K}^nx)(t)-(\mathcal{K}^nz)(t)\|_{\mathbb{F}_a} \le \frac{(K_bMe^{wb}r_0(\alpha))^na^n}{n!}\|x-z\|_{\mathbb{F}_a}.$$

Since $\frac{(K_bMe^{wb}r_0(\alpha))^na^n}{n!}\to0$ as $n\to+\infty$ then $\exists n\in\mathbb{N}$ such that $\frac{(K_bMe^{wb}r_0(\alpha))^na^n}{n!}<1$. It follows that $\mathcal{K}^n$ is strict contraction and by the Banach fixe point theorem, we deduce there $\exists! x\in\mathbb{F}_a$ such that $\mathcal{K}^nx=x$. Thus $\mathcal{K}^nx=x$ implies that $\mathcal{K}^{n+1}x=\mathcal{K}x$ on the other hand $\mathcal{K}^n(\mathcal{K}x)=\mathcal{K}(x)$ it follows that $\mathcal{K}(x)$ is a fixed point of $\mathcal{K}^n$ and since fixed point is unique then we get $\mathcal{K}(x)=x$. Equation (1.1) has a unique mild solution $x(.,\varphi)$ which is defined on the interval $(-\infty,a]$. This is true for all $a>0$, then $x(.,\varphi)$ is a global solution of Equation (1.1) on $\mathbb{R}$.

Next, we prove that the solution depends continuously on initial data. Let $\varphi\in\mathcal{B}$ and $t\in[0,a[$ be fixed. Show that $x(\cdot,\varphi)$ is continuous in the sense of φ. $\forall\varepsilon>0$, look for $k(a)>0$ such that for $\psi\in\mathcal{B}$ and $|\varphi-\psi|_{\mathcal{B}}\le\varepsilon$ implies that

$$\|x(\iota,\varphi)-x(\iota,\psi)\|\le k(a)|\varphi-\psi|_{\mathcal{B}} \text{ for } \iota\in]-\infty,a].$$

We have by Lemme 3.3

$$\begin{aligned}|x_s(\cdot,\varphi)-x_s(\cdot,\psi)|_{\mathcal{B}} &\leq (M_b+\zeta)\,|\varphi-\psi|_{\mathcal{B}}+K_b\sup_{0\leq\theta\leq s}\|x(\theta,\varphi)-x_s(\theta,\psi)\|_X, \qquad s\in\rho^-\cup J\\ &\leq (M_b+\zeta)\,|\varphi-\psi|_{\mathcal{B}}+K_b\sup_{0\leq\theta\leq s}\|U(\theta,0)(\varphi-\psi)\|_X\\ &+K_b\sup_{0\leq\theta\leq s}\lim_{\lambda\to 0^+}\int_0^\theta\|U(\theta,\tau)F(\tau,\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\varphi))-F(\tau,\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\psi))\|d\tau\\ &\leq (M_b+\zeta)\,|\varphi-\psi|_{\mathcal{B}}+K_b\sup_{0\leq\theta\leq s}\|U(\theta,0)(\varphi-\psi)\|_X\\ &+K_b\sup_{0\leq\theta\leq s}\lim_{\lambda\to 0^+}\int_0^\theta Me^{w(\theta-\tau)}\|F(\tau,\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\varphi))-F(\tau,\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\psi))\|d\tau\\ &\leq (M_b+\zeta+HK_bMe^{wa})\,|\varphi-\psi|_{\mathcal{B}}\\ &+K_bMe^{wa}r_0(\alpha)\int_0^\theta\|\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\varphi)-\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\psi)\|d\tau\end{aligned}$$

using the Bellman-Gronwall Lemma it follows that

$$\leq (M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}|\varphi-\psi|_{\mathcal{B}}.$$

Hence we can write

$$|x_s(\vartheta,\varphi)-x_s(\vartheta,\psi)|_{\mathcal{B}}\leq (M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}|\varphi-\psi|_{\mathcal{B}} \text{ for } \vartheta\in]-\infty,0]$$

thus

$$|x(s+\vartheta,\varphi)-x(s+\vartheta,\psi)|_{\mathcal{B}}\leq (M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}\|\varphi-\psi\|_{\mathcal{B}} \text{ for } \vartheta\in]-\infty,0]$$

therefore

$$\|x(\iota,\varphi)-x(\iota,\psi)\|_{\mathcal{B}}\leq (M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}\|\varphi-\psi\|_{\mathcal{B}} \text{ for } \iota\in]-\infty,a]$$

It is clear that $(M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}>0$ hence, we deduce the continuous dependence on the initial data. □

Corollary 4.1. *Assume that (**C**$_4$) holds. Let q_1 and q_2 be continuous fonctions from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that*

$$\|F(t,\phi)\|\leq q_1(t)|\phi|_{\mathcal{B}}+q_2(t) \quad \textit{for } t\in\mathbb{R}^+ \textit{ and } \phi\in\mathcal{B}.$$

Then, for $\phi\in\mathcal{B}$ such that $\phi(0)\in\overline{D}$, Equation (1.1) has a unique mild solution which is defined on $\mathbb{R}$.

Proof. Let $x(\cdot,\phi)$ the solution of Equation (1.1) defined on a maximal interval $(-\infty,a_{max})$. Then by the Theorem 4.1

$$a_{max}=+\infty \;\text{ or }\; \limsup_{t\to a_{max}^-}\|x(t,\varphi)\|=+\infty.$$

We assume that $a_{max}<+\infty$ and $\limsup_{t\to a_{max}^-}\|x(t,\varphi)\|=+\infty$.

For all $t\in[0,a_{max}[$:

On has

$$
\begin{aligned}
\|x(t,\phi)\| &\leq \|U(t,0)\|\|\phi(0)\| + \lim_{\lambda\to 0^+}\int_0^t \|U_\lambda(t,s)\|\|F(s,\tilde{x}_{\rho(s,\tilde{x_s})})\|ds \\
&\leq Me^{\omega t}\|\phi(0)\| + \int_0^t Me^{\omega t}\Big(q_1(t)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}} + q_2(t)\Big)ds \\
&\leq Me^{\omega a_{max}}\Big(\|\phi(0)\| + \int_0^t q_2(\theta)d\theta\Big) + Me^{\omega a_{max}}\int_0^t q_1(t)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}}ds
\end{aligned}
$$

By Lemma 3.3

$$
|\tilde{x}_{\rho(s,\tilde{x_s})}(\cdot,\phi)|_{\mathcal{B}} \leq (M_b+\zeta)\,\|\phi\|_{\mathcal{B}} + K_b \sup_{0\leq\theta\leq_{\rho(s,\tilde{x_s})}}\|x(\theta,\phi)\|
$$

.

Thus

$$
\begin{aligned}
|\tilde{x}_{\rho(s,\tilde{x_s})}(\cdot,\phi)|_{\mathcal{B}} &\leq (M_b+\zeta)\,|\phi|_{\mathcal{B}} + K_b \sup_{0\leq\theta\leq_{\rho(s,\tilde{x_s})}}\Big[M_b e^{\omega a_{max}}\Big(\|\phi(0)\| + \int_0^t q_2(\theta)d\theta\Big) \\
&\quad + M_b e^{\omega a_{max}}\int_0^t q_1(t)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}}ds\Big] \\
&\leq (M_b+\zeta)\,|\phi|_{\mathcal{B}} + K_b M_b e^{\omega a_{max}}\Big(\|\phi(0)\| + \int_0^{\rho(s,\tilde{x_s})} q_2(\theta)d\theta\Big) \\
&\quad + M_b e^{\omega a_{max}}\int_0^{\rho(s,\tilde{x_s})} q_1(\theta)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}}ds \\
&= P_1 + P_1\int_0^{\rho(s,\tilde{x_s})} q_1(\theta)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}}ds.
\end{aligned}
$$

With $P_1 = (M_b+\zeta)\,|\phi|_{\mathcal{B}} + K_b M_b e^{\omega a_{max}}\Big(\|\phi(0)\| + \int_0^{\rho(s,\tilde{x_s})} q_2(\theta)d\theta\Big)$ and $P_1 = M_b e^{\omega a_{max}}$.
By Gronwall's Lemma, we deduce that

$$
|\tilde{x}_{\rho(s,\tilde{x_s})}(\cdot,\phi)|_{\mathcal{B}} \leq P_1 e^{a_{max}P_2\int_0^{\rho(s,\tilde{x_s})} q_1(\theta)d\theta}.
$$

Hence $\limsup\limits_{t\to a_{max}^-}\|x(t,\varphi)\| < +\infty$. Therefore $a_{max} = +\infty$

■

5. Application

For illustration of our previous result, we propose to study the following model.

$$
\begin{cases}
\dfrac{\partial}{\partial t}v(t,x) = \delta(t)\dfrac{\partial^2}{\partial x^2}v(t,x) + \beta(t)\displaystyle\int_{-\infty}^0 g\Big(\theta, v\Big(\theta + t - \rho_1(t)\rho_2\Big(\int_0^\pi w(s)|v(t,\theta)|^2 ds\Big), x\Big)\Big)d\theta \\
\quad \text{for} \quad 0\leq t\leq b \text{ and } x\in[0,\pi], \\
v(t,0) = v(t,\pi) = 0 \quad \text{for} \quad 0\leq t\leq b, \\
v(\theta,x) = v_0(\theta,x) \quad \text{for} \quad \theta\leq 0, 0\leq x\leq\pi,
\end{cases}
\tag{5.1}
$$

where $\delta(\cdot)$ is a positive function in $C^1(\mathbb{R}^+,\mathbb{R}^+)$ with $\delta_0 := \inf_{t\geq 0}\delta(t) > 0$ and $\beta : [0,b] \to \mathbb{R}^+$ with $\beta \in L^1(J;[0,+\infty))$. $g : \mathbb{R}^- \times \mathcal{B} \to \mathbb{R}$ and $v_0 : (-\infty,0]\times[0,\pi] \to \mathbb{R}$ are functions. The functions $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$ are continuous and $w : \mathbb{R} \to \mathbb{R}$ is a positive continuous function. To rewrite Eq. (5.1) in the abstract form, we introduce the space $X := C([0,\pi],\mathbb{R})$ of continuous functions from $[0,\pi]$ to $\mathbb{R}$ equipped with the uniform norm topology and we consider the operator $A : D \subset X \to X$ defined by:

$$\begin{cases} D = \{z \in C^2([0,\pi]) : z(0) = z(\pi) = 0\} \\ \\ Az(t,x) = \Delta z(t,x) \ \text{ with } \Delta := \dfrac{\partial^2}{\partial x^2}; \ \ t \in [0,b] \ \text{ and } \ x \in [0,\pi]. \end{cases}$$

Then it is well know that

$$\begin{cases} \overline{D} = \{z \in C([0,\pi]:\mathbb{R}) : z(0) = z(\pi) = 0\} \neq X, \\ \\ (0,+\infty) \subset \rho(A) \quad \text{and} \quad \|R(\lambda,A)\| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0. \end{cases} \tag{5.2}$$

We choose the space of bounded uniformly continuous functions from $\mathbb{R}^-$ to X denoted by $BUC(\mathbb{R}^-,X)$ as a phase space $\mathcal{B} := BUC(\mathbb{R}^-,X)$ endowed with the following norm:

$$\|\psi\|_{\mathcal{B}} := \sup_{\theta\leq 0}\|\psi(\theta)\|.$$

Then, $\mathcal{B}$ satisfies Axioms $(\mathbf{B}_1) - (\mathbf{B})$.

By defining the operators $F : I \times \mathcal{B} \to X$ and $\tau : I \times \mathcal{B} \to \mathbb{R}$ by:

$y(t)(x) := v(t,x).$

$\varphi(\theta)(x) := v_0(\theta,x)$ for $\theta \leq 0$.

$F(t,\phi)(x) := \beta(t)\displaystyle\int_{-\infty}^{0} g\Big(\theta,\phi(\theta)(x)\Big)d\theta.$

$\tau(t,\phi) := t - \rho_1(t)\rho_2\Big(\displaystyle\int_0^{\pi} w(s)|\phi(0)(x)|^2 ds\Big).$

Suppose that $\phi\mathcal{B}$ and let $(A(t))_{t\geq 0}$ be the family of operators defined by $A(t) := \delta(t)\dfrac{\partial^2}{\partial x^2}$. Then, Equation (5.1) takes the following abstract form :

$$\begin{cases} \dot{y}(t) = A(t)y(t) + F(t, y_{\tau(t,y_t)}) \quad \text{for} \quad t \in [0,b], \\ \\ y_0 = \varphi \in \mathcal{B}, \end{cases} \tag{5.3}$$

We have $D(A(t)) = D$ independent of t and for $\lambda > 0$,

$$\begin{aligned} R(\lambda, A(t)) &= (\lambda I - \delta(t)A)^{-1} \\ &= \frac{1}{\delta(t)}R\Big(\frac{\lambda}{\delta(t)}, A\Big). \end{aligned} \tag{5.4}$$

Using (5.2) and (5.4), we have for every $\lambda > 0$, $\lambda \in \rho(A(t))$ and $\|R(\lambda,A(t))\| \leq \frac{1}{\lambda}$.
Then $(0,+\infty) \subset \rho(A(t))$ and

$$\Big\|\prod_{i=1}^{k} R(\lambda, A(t_i))\Big\| \leq \frac{1}{\lambda^k}, \ \ 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k < +\infty.$$

Hence, the family of linear operators $(A(t))_{t\geq 0}$ on X satisfies the assumptions ($\mathbf{A}_1$) - ($\mathbf{A}_3$).
It is known from [10] that, the part Δ_0 of $\Delta = \dfrac{\partial^2}{\partial x^2}$ in $\overline{D(\Delta)}$ given by

$$\begin{cases} D(\Delta_0) = \Big\{ z \in D(\Delta) : \Delta z \in \overline{D(\Delta)} \Big\} \\ \\ \Delta_0 z = \Delta z, \end{cases} \tag{5.5}$$

generates a compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(\Delta)}$ such that

$$\|T_0(t)\| \leq e^{-t} \quad \text{for} \quad t \geq 0. \tag{5.6}$$

Thus, the part $A_0(.)$ of $A(.)$ in $\overline{D}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ on $\overline{D}$ given by

$$U(t,s) = T_0\left(\int_s^t \delta(\tau) d\tau\right)$$

which is compact for $t > s$. By (5.6), one has

$$\|U(t,s)\| \leq e^{-\delta_0(t-s)}.$$

Hence ($\mathbf{C}_1$) is satisfies. We assume that:

1) $g : \mathbb{R}^- \times \mathcal{B} \to \mathbb{R}^+$ is nondecreasing integrable function which satisfies : $g(\theta, 0) = 0$ for $\theta \leq 0$.

2) v_0 is uniformly continuous and bounded with respect to $\theta \in \mathbb{R}^-$, uniformly with respect to $x[0,\pi]$.

Under the above conditions, we claim that $\varphi \in \mathcal{B}$. In fact,

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\leq 0} \|\varphi(\theta)\| = \sup_{\substack{\theta\leq 0 \\ x\in[0,\pi]}} \|v_0(\theta, x)\| < +\infty.$$

and

$$\begin{aligned} \|\varphi(\theta) - \varphi(\theta')\| &= \sup_{x\in[0,\pi]} \|\varphi(\theta)(x) - \varphi(\theta')(x)\| \\ &= \sup_{x\in[0,\pi]} \|v_0(\theta, x) - v_0(\theta', x)\| \to 0 \text{ as } \|\theta - \theta'\| \to 0. \end{aligned}$$

Therefore, $\varphi \in \mathcal{B}$ with $\varphi(0) \in \overline{D}$.

On the other hand, we have:

$$\|F(t, \phi)\| \leq \beta(t) \int_{-\infty}^0 \Big\| g\Big(\theta, \phi(\theta)\Big) \Big\| d\theta$$

for $\phi \in \mathcal{B}$. F satisfies ($\mathbf{C}_2$) with $p(t) = \beta(t)$ and $V(\|\phi\|_{\mathcal{B}}) = \displaystyle\int_{-\infty}^0 \Big\| g\Big(\theta, \phi(\theta)\Big) \Big\| d\theta.$

3) Let $\varphi \in \mathcal{B}$ such that $x_0 = \varphi$ and $t \longmapsto \varphi_t$ is a $\mathcal{B}$-valued. We assume that $\|\varphi_t\|_{\mathcal{B}} \leq \eta(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \tau^-$ where $\eta : \tau^- \to (0, \infty)$ is a continuous and bounded function with

$$\tau^- = \{\tau(s, \psi) : (s, \psi) \in J \times \mathcal{B}, \tau(s, \psi) \leq 0\}.$$

Hence ($\mathbf{C}_3$) is satisfies.

Then, the existence of mild solutions can be deduced from a direct application of Theorem 3.5 and we have the following result.

Theorem 5.1. *Assume* $\varphi(0) \in \overline{D}$ *and*

$$K_b e^{\delta_0 b}\|p\|_{L^1} < \int_N^\infty \frac{ds}{V(s)} \tag{5.7}$$

where $M = 1$, $\omega = \delta_0$, $N = (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b\|\varphi(0)\|$. *Then there exists at least one mild solution of Equation (5.1).*

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Oscillation condition for first order linear dynamic equations on time scales

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Abstract. In this paper, we deal with the first-order dynamic equations with nonmonotone arguments

$$y^{\Delta}(\zeta)+\sum_{i=1}^{m} r_i(\zeta) y\left(\psi_i(\zeta)\right)=0,\ \zeta \in [\zeta_0,\infty)_{\mathbb{T}}$$

where $r_i \in C_{rd}\left([\zeta_0,\infty)_{\mathbb{T}},\mathbb{R}^+\right)$, $\psi_i \in C_{rd}\left([\zeta_0,\infty)_{\mathbb{T}},\mathbb{T}\right)$ and $\psi_i(\zeta) \leq \zeta,\ \lim_{\zeta\to\infty}\psi_i(\zeta)=\infty$ for $1 \leq i \leq m.$ Also, we present a new sufficient condition for the oscillation of delay dynamic equations on time scales. Finally, we give an example illustrating the result.

AMS Subject Classifications: 39A12, 39A21, 34C10, 34N05.

Keywords: Dynamic equation, nonmonotone delays, oscillatory solution, time scales.

Contents

1. Introduction and Background

We consider the delay dynamic equation with several delays which are not necessarily monotone

$$y^{\Delta}(\zeta)+\sum_{i=1}^{m} r_i(\zeta) y\left(\psi_i(\zeta)\right)=0,\ \ \zeta \in [\zeta_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale unbounded above with $\zeta_0 \in \mathbb{T}$, $r_i \in C_{rd}([\zeta_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$, $\psi_i \in C_{rd}([\zeta_0,\infty)_{\mathbb{T}},\mathbb{T})$ do not have to be monotone for $1 \leq i \leq m$ such that

$$\psi_i(\zeta) \leq \zeta \ \text{ for all } \zeta \in \mathbb{T},\ \ \lim_{\zeta\to\infty}\psi_i(\zeta)=\infty. \tag{1.2}$$

First of all, we would like to remind some basic concepts about time scales calculus. A function $r:\mathbb{T}\to\mathbb{R}$ is said to be positively regressive (it means that $r\in\mathcal{R}^+$) if it is rd-continuous and satisfies $1+\mu(\zeta)r(\zeta)>0$ for all $\zeta\in\mathbb{T}$, where $\mu:\mathbb{T}\to\mathbb{R}_0^+$ is the graininess function defined by $\mu(\zeta):=\sigma(\zeta)-\zeta$ with the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ defined with the help of $\sigma:=\inf\{s\in\mathbb{T}: s>\zeta\}$ for $\zeta\in\mathbb{T}$. If $\sigma(\zeta)=\zeta$ or $\mu(\zeta)=0$, a point $\zeta\in\mathbb{T}$ is said to be right-dense, otherwise it is right-scattered.

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A function $y:\mathbb{T}\to\mathbb{R}$ is called a solution of (1.1), if $y(\zeta)$ is delta differentiable for $\zeta\in\mathbb{T}^{\kappa}$ and satisfies (1.1) for $\zeta\in\mathbb{T}^{\kappa}$. It is called that a solution y of (1.1) has a generalized zero at ζ if $y(\zeta)=0$ or if $\mu(\zeta)>0$ and $y(\zeta)y(\sigma(\zeta))<0$. Let $\sup\mathbb{T}=\infty$ and then a nontrivial solution y of (1.1) is called oscillatory on $[\zeta,\infty)$ if it has arbitrarily large generalized zeros in $[\zeta,\infty)$. Also, we refer to book of Bohner and Peterson [2,3] for more detailed information.

For $m=1$, we have the following equation which is the form of (1.1) with single delay.

$$y^{\Delta}(\zeta)+r(\zeta)y\left(\psi(\zeta)\right)=0,\ \ \zeta\in[\zeta_0,\infty)_{\mathbb{T}}. \tag{1.3}$$

Recently, there has been remarkable interest for the oscillatory solutions of this equation. See [1-21] and the references cited therein. Concerning Eq. (1.3) which have monotone arguments, see also Zhang and Deng [20], Bohner [4], Zhang et al. [21], Şahiner and Stavroulakis [19], Agarwal and Bohner [1], and Karpuz and Öcalan [9]. As you seen, many articles have been dedicated to the equations which have monotone terms, but a few is related with the more general case of nonmonotone delay terms. Now, we mention the results which contain delay arguments which are not necessarily monotone.

Suppose that $\psi(\zeta)$ does not have to be monotone and

$$\vartheta(\zeta)=\sup_{s\le\zeta}\psi(s),\ \zeta\in\mathbb{T},\ \zeta\ge 0. \tag{1.4}$$

Obviously, $\vartheta(\zeta)$ is nondecreasing and $\psi(\zeta)\le\vartheta(\zeta)$ for all $\zeta\ge 0$.

In 2017, Öcalan et al. [16] found out the result given below. If

$$\limsup_{\zeta\to\infty}\int_{\vartheta(\zeta)}^{\sigma(\zeta)} r(s)\Delta s>1, \tag{1.5}$$

where $\vartheta(\zeta)$ is defined by (1.4), then all solutions of (1.3) oscillate. In 2020, Öcalan [17] obtained the following criteria. If $-r\in\mathcal{R}^{+}$ and

$$\limsup_{\zeta\to\infty}\int_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r(s)}{e_{-r}\left(\vartheta(\zeta),\psi(s)\right)}\Delta s>1 \tag{1.6}$$

or

$$\liminf_{\zeta\to\infty}\int_{\psi(\zeta)}^{\zeta} \frac{r(s)}{e_{-r}\left(\vartheta(s),\psi(s)\right)}\Delta s>\frac{1}{e}, \tag{1.7}$$

where $\vartheta(\zeta)$ is defined by (1.4),

$$e_{-\lambda r}\left(\zeta,\psi(\zeta)\right)=\exp\left\{\int_{\psi(\zeta)}^{\zeta}\xi_{\mu(s)}(-\lambda r(s))\Delta s\right\}$$

and

$$\xi_h(z)=\begin{cases}\frac{\mathrm{Log}(1+hz)}{h} & \text{, if } h\neq 0\\ z & \text{, if } h=0\end{cases},$$

then all solutions of equation (1.3) oscillate. Also, (1.7) implies that the following condition. If

$$\liminf_{\zeta\to\infty}\int_{\psi(\zeta)}^{\zeta} r(s)\Delta s>\frac{1}{e}, \tag{1.8}$$

where $\vartheta(\zeta)$ is given with (1.4), then all solutions of (1.3) oscillate.
Eventually, Öcalan [18] presented the following result.
Let

$$\bar{\alpha} := \liminf_{\zeta\to\infty} \int_{\psi(\zeta)}^{\zeta} r(s)\Delta s. \tag{1.9}$$

If $-r \in \mathcal{R}^+$, $0 \leq \bar{\alpha} \leq \frac{1}{e}$ and

$$\limsup_{t\to\infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r(s)}{e_{-r}\left(\vartheta(\zeta), \psi(s)\right)}\Delta s > 1 - \frac{1-\bar{\alpha}-\sqrt{1-2\bar{\alpha}-\left(\bar{\alpha}\right)^2}}{2}, \tag{1.10}$$

where $\vartheta(\zeta)$ and $\bar{\alpha}$ are defined by (1.4) and (1.9), resp., then every solution of (1.3) is oscillatory.
Kılıç and Öcalan [12] obtained the following criteria which are the first results for (1.1) with several nonmonotone arguments.
Set $\psi_i(\zeta)$ are not necessarily monotone for $1 \leq i \leq m$ and

$$\vartheta_i(\zeta) = \sup_{s\leq\zeta}\left\{\psi_i(s)\right\} \text{ and } \vartheta(\zeta) = \max_{1\leq i\leq m}\left\{\vartheta_i(\zeta)\right\},\ \zeta \in \mathbb{T},\ \zeta \geq 0. \tag{1.11}$$

Obviously, $\vartheta_i(\zeta)$ are nondecreasing and $\psi_i(\zeta) \leq \vartheta_i(\zeta) \leq \vartheta(\zeta)$ for all $\zeta \geq 0$ and $1 \leq i \leq m$.
Teorem A: Suppose that $-\sum_{i=1}^{m} r_i \in \mathcal{R}^+$ and (1.11) holds. If

$$\limsup_{\zeta\to\infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} r_i(s)\Delta s > 1 \tag{1.12}$$

or

$$\liminf_{\zeta\to\infty} \int_{\psi(\zeta)}^{\zeta} \sum_{i=1}^{m} r_i(s)\Delta s > \frac{1}{e}, \tag{1.13}$$

where $\psi(\zeta) = \max_{1\leq i\leq m}\left\{\psi_i(\zeta)\right\}$, then every solution of (1.1) oscillates.
Further assume that

$$\alpha := \liminf_{\zeta\to\infty} \int_{\psi(\zeta)}^{\zeta} \sum_{i=1}^{m} r_i(s)\Delta s. \tag{1.14}$$

Teorem B: Suppose that $-\sum_{i=1}^{m} r_i \in \mathcal{R}^+$ and $0 \leq \alpha \leq \frac{1}{e}$. If

$$\limsup_{\zeta\to\infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} r_i(s)\Delta s > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.15}$$

then every solution of (1.1) oscillate.
Lately, Öcalan and Kılıç [13] established the following results for (1.1).

Teorem C: Suppose that $-\sum_{i=1}^{m} r_i \in \mathcal{R}^+$, (1.2) and (1.11) hold. If

$$\limsup_{\zeta\to\infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))} \Delta s > 1 \tag{1.16}$$

or

$$\liminf_{\zeta\to\infty} \int_{\psi(\zeta)}^{\zeta} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(s), \psi_i(s))} \Delta s > \frac{1}{e}, \tag{1.17}$$

where $\psi(\zeta) = \max_{1\le i\le m} \{\psi_i(\zeta)\}$, then all solutions of (1.1) are oscillatory.

Although dynamic equations with several arguments are more comprehensive than dynamic equations with one delay, there are not many studies on this subject. So, in this article, we are interested in studying the oscillatory behavior of first order dynamic equations with several delays on time scale. We present one criterion to check the oscillation of (1.1). Our result is an extension and complement to some results published in the literature.

2. Main Results

In this section, we introduce a new sufficient condition for the oscillatory solutions of (1.1) when the arguments $\psi_i(\zeta)$ do not have to be monotone for $1 \le i \le m$ and $0 < \alpha \le \frac{1}{e}$. The following lemmas will be useful to obtain our main result.

The lemma given below can be easily obtained from [4].

Lemma 2.1. *Let* $-\sum_{i=1}^{m} r_i \in \mathcal{R}^+$. *If*

$$y^{\Delta}(\zeta) + y(\zeta)\sum_{i=1}^{m} r_i(\zeta) \le 0,$$

then

$$y(\zeta) \le e_{-\sum_{j=1}^{m} r_j}(\zeta, s)\, y(s) \;\; \textit{for all}\; \zeta \ge s,\; s, \zeta \in \mathbb{T}. \tag{2.1}$$

The result given below can be easily produced by applying a nearly same procedure to [21, Lemma 2.4] when the case $\psi_i(\zeta)$ do not have to be monotone for $1 \le i \le m$. Therefore, the proof of this lemma is not presented here.

Lemma 2.2. *Suppose that $\psi_i(\zeta)$ are not necessarily monotone for $1 \le i \le m$. Let $0 \le \alpha \le \frac{1}{e}$ and $y(\zeta)$ be an eventually positive solution of (1.1). Then, we have*

$$\liminf_{\zeta\to\infty} \frac{y(\sigma(\zeta))}{y(\vartheta(\zeta))} \ge \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.2}$$

where $\psi(\zeta) = \max_{1\le i\le m} \{\psi_i(\zeta)\}$, $\vartheta(\zeta)$ and α are defined by (1.11) and (1.14) respectively.

Theorem 2.3. *Assume that $-\sum_{i=1}^{m} r_i \in \mathcal{R}^+$, (1.2) holds and*

$$\limsup_{\zeta\to\infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))} \Delta s > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.3}$$

where $\vartheta(\zeta)$ *and* α *are defined by* (1.11) *and* (1.14) *resp., then all solutions of* (1.1) *oscillatory.*

Proof. Assume, for the sake of contradiction, that there exists an eventually positive solution $y(\zeta)$ of (1.1). If $y(\zeta)$ is an eventually negative solution of (1.1), the proof of the theorem can be done similarly. Then there exists $\zeta_1 > \zeta_0$ such that $y(\zeta)$, $y\left(\psi_i(\zeta)\right), y\left(\psi(\zeta)\right), y\left(\vartheta(\zeta)\right) > 0$ for all $\zeta \geq \zeta_1$ and $1 \leq i \leq m$. So, using (1.1) we obtain

$$y^{\Delta}(\zeta) = -\sum_{i=1}^{m} r_i(\zeta) y(\psi_i(\zeta)) \leq 0 \text{ for all } \zeta \geq \zeta_1,$$

which implies that $y(\zeta)$ is an eventually nonincreasing function. From this fact and taking into account that $\psi_i(\zeta) \leq \vartheta_i(\zeta) \leq \zeta$ for $1 \leq i \leq m$, (1.1) gives

$$y^{\Delta}(\zeta) + y\left(\zeta\right) \sum_{i=1}^{m} r_i(\zeta) \leq 0,\ \zeta \geq \zeta_1 \tag{2.4}$$

and then, we obtain the below expression from Lemma 2.1.

$$y(\vartheta(\zeta)) \leq e_{-\sum\limits_{j=1}^{m} r_j}\left(\vartheta(\zeta), \psi_i(s)\right) y(\psi_i(s)) \text{ for all } \vartheta(\zeta) \geq \psi_i(s). \tag{2.5}$$

On the other hand, integrating (1.1) from $\vartheta(\zeta)$ to $\sigma(\zeta)$ and with the help of (2.5), we have

$$y(\sigma(\zeta)) - y\left(\vartheta(\zeta)\right) + \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} r_i(s) y(\psi_i(s)) \Delta s = 0$$

$$y(\sigma(\zeta)) - y\left(\vartheta(\zeta)\right) + \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} r_i(s) \frac{y\left(\vartheta(\zeta)\right)}{e_{-\sum\limits_{j=1}^{m} r_j}\left(\vartheta(\zeta), \psi_i(s)\right)} \Delta s \leq 0$$

$$y(\sigma(\zeta)) - y\left(\vartheta(\zeta)\right) + y\left(\vartheta(\zeta)\right) \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}\left(\vartheta(\zeta), \psi_i(s)\right)} \Delta s \leq 0$$

or

$$\int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}\left(\vartheta(\zeta), \psi_i(s)\right)} \Delta s \leq 1 - \frac{y(\sigma(\zeta))}{y\left(\vartheta(\zeta)\right)}. \tag{2.6}$$

Consequently, from (2.6) we obtain

$$\limsup_{\zeta \to \infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}\left(\vartheta(\zeta), \psi_i(s)\right)} \Delta s \leq 1 - \liminf_{\zeta \to \infty} \frac{y\left(\sigma(\zeta)\right)}{y\left(\vartheta(\zeta)\right)} \tag{2.7}$$

and from (2.2) the last inequality turns into

$$\limsup_{\zeta \to \infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}\left(\vartheta(\zeta), \psi_i(s)\right)} \Delta s \leq 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2},$$

which contradicts to (2.3) and this completes the proof. ∎

Example 2.4. *Let $m = 2$ and $\mathbb{T} = 3\mathbb{Z} = \{3k : k \in \mathbb{Z}\}$. Then, we get for $\zeta \in \mathbb{T}$*

$$\sigma(\zeta) = \zeta + 3, \;\; \mu(\zeta) = 3 \;\; \textit{and} \;\; y^{\Delta}(\zeta) = \frac{y(\zeta+3) - y(\zeta)}{3}.$$

So, (1.1) becomes

$$\frac{y(\zeta+3) - y(\zeta)}{3} + r_1(\zeta) y\left(\psi_1(\zeta)\right) + r_2(\zeta) y\left(\psi_2(\zeta)\right) = 0, \;\; \zeta \in \{3k : k \in \mathbb{Z}\}.$$

Let $\psi_1(\zeta) = \zeta - 3$, $\psi_2(\zeta) = \zeta - 6$, then $\psi(\zeta) = \max_{1 \le i \le m} \{\psi_i(\zeta)\} = \psi_1(\zeta) = \zeta - 3$. Since $1 \le i \le m$, $r_i(\zeta) \in \{3k : k \in \mathbb{Z}\}$, we suppose that

$$r_1(3\zeta) = 0.09 \text{ , } r_1(3\zeta + 3) = 0.07 \textit{ and } r_2(\zeta) = 0.03 \;\; \zeta = 0, 3, 6, \ldots.$$

If $\mathbb{T} = h\mathbb{Z}$, from Theorem 1.79 [2], we know the formula given below.

$$\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h \;\; \textit{for } a < b. \tag{2.8}$$

Then, by using (2.8) we obtain that for $1 \le i \le m$, $r_i(\zeta)$, $\psi(\zeta) \in \{3k : k \in \mathbb{Z}\}$

$$\begin{aligned}
\alpha \; : &= \liminf_{\zeta\to\infty} \int_{\psi(\zeta)}^{\zeta} \sum_{i=1}^{m} r_i(s)\Delta s = \liminf_{\zeta\to\infty} \sum_{j=\frac{\zeta-3}{3}}^{\frac{\zeta}{3}-1} \sum_{i=1}^{2} 3r_i(3j) \\
&= \liminf_{\zeta\to\infty} \sum_{j=\frac{\zeta-3}{3}}^{\frac{\zeta}{3}-1} 3(r_1(3j) + r_2(3j)) = \liminf_{\zeta\to\infty} 3(r_1(\zeta-3) + r_2(\zeta-3)) \\
&= 0.36 < \frac{1}{e}
\end{aligned}$$

and

$$\begin{aligned}
M \; : &= \limsup_{\zeta\to\infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))} \Delta s = \limsup_{\zeta\to\infty} \sum_{k=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} \sum_{i=1}^{2} \frac{3r_i(3k)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(3k))} \\
&= \limsup_{\zeta\to\infty} \sum_{k=\frac{\zeta-3}{3}}^{\frac{\zeta+3}{3}-1} \left[\frac{3r_1(3k)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_1(3k))} + \frac{3r_2(3k)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_2(3k))} \right].
\end{aligned}$$

Now, we obtain that

$$\begin{aligned} e_{-r}\left(\vartheta(\zeta),\psi_1(3k)\right) &= \exp\left\{\int_{\psi_1(3k)}^{\vartheta(\zeta)} \xi_{\mu(u)}\sum_{j=1}^{m}(-r_j(u))\Delta u\right\} = \exp\left\{\int_{\psi_1(3k)}^{\vartheta(\zeta)} \xi_{\mu(u)}(-(r_1(u)+r_2(u))\Delta u\right\} \\ &= \exp\left\{\sum_{i=\frac{\psi_1(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \frac{3\log\left(1-\mu(3i)(r_1(3i)+r_2(3i))\right)}{\mu(3i)}\right\} \\ &= \exp\left\{\sum_{i=\frac{\psi_1(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \log\left(1-3(r_1(3i)+r_2(3i)\right)\right\} \\ &= \exp\left\{\log\prod_{i=\frac{\psi_1(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1}\left(1-3(r_1(3i)+r_2(3i)\right)\right\} = \prod_{i=\frac{\psi_1(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1}\left(1-3(r_1(3i)+r_2(3i)\right) \end{aligned}$$

and

$$\begin{aligned} e_{-r}\left(\vartheta(\zeta),\psi_2(3k)\right) &= \exp\left\{\int_{\psi_2(3k)}^{\vartheta(\zeta)} \xi_{\mu(u)}\sum_{j=1}^{m}(-r_j(u))\Delta u\right\} = \exp\left\{\int_{\psi_2(3k)}^{\vartheta(\zeta)} \xi_{\mu(u)}(-(r_1(u)+r_2(u))\Delta u\right\} \\ &= \exp\left\{\sum_{i=\frac{\psi_2(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \frac{3\log\left(1-\mu(3i)(r_1(3i)+r_2(3i))\right)}{\mu(3i)}\right\} \\ &= \exp\left\{\sum_{i=\frac{\psi_2(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \log\left(1-3(r_1(3i)+r_2(3i)\right)\right\} \\ &= \exp\left\{\log\prod_{i=\frac{\psi_2(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1}\left(1-3(r_1(3i)+r_2(3i)\right)\right\} = \prod_{i=\frac{\psi_2(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1}\left(1-3(r_1(3i)+r_2(3i)\right). \end{aligned}$$

Then,

$$\begin{aligned} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r_1(s)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_1(s))}\Delta s &= \sum_{k=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} \frac{3r_1(3k)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_1(3k))} \\ &= \sum_{j=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} 3r_1(3j)\prod_{i=\frac{\psi_1(3j)}{3}}^{\frac{\vartheta(\zeta)}{3}-1}\frac{1}{(1-3(r_1(3i)+r_2(3i))} \\ &= \sum_{j=\frac{\zeta-3}{3}}^{\frac{\zeta+3}{3}-1} 3r_1(3j)\prod_{i=j-1}^{\frac{\zeta-3}{3}-1}\frac{1}{(1-3(r_1(3i)+r_2(3i))} \\ &= 3r_1(\zeta-3)\frac{1}{(1-3(r_1(\zeta-6)+r_2(\zeta-6))} + 3r_1(\zeta) \\ &\cong 0.42187 + 0.21 \end{aligned}$$

and

$$\int_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r_2(s)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_2(s))}\Delta s = \sum_{k=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} \frac{3r_2(3k)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_2(3k))}$$

$$= \sum_{j=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} 3r_2(3j) \prod_{i=\frac{\psi_2(3j)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \frac{1}{(1-3(r_1(3i)+r_2(3i))}$$

$$= \sum_{j=\frac{\zeta-3}{3}}^{\frac{\zeta+3}{3}-1} 3r_2(3j) \prod_{i=j-2}^{\frac{\zeta-3}{3}-1} \frac{1}{(1-3(r_1(3i)+r_2(3i))}$$

$$= 3r_2(\zeta-3)\frac{1}{(1-3(r_1(\zeta-9)+r_2(\zeta-9))}\frac{1}{(1-3(r_1(\zeta-6)+r_2(\zeta-6))}$$

$$+3r_2(\zeta)\frac{1}{(1-3(r_1(\zeta-6)+r_2(\zeta-6))}$$

$$\cong 0.219726 + 0.14062$$

Thus, we have

$$M \ : \ = \limsup_{\zeta\to\infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_i(s))}\Delta s$$

$$= \limsup_{t\to\infty} \left[\int_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r_1(s)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_1(s))} + \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r_2(s)}{e_{-\sum_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_2(s))} \right]\Delta s$$

and

$$M \cong 0,992216 \not> 1$$

implies that (1.16) fails. However, since

$$M \cong 0.992216 > 1 - \frac{1-0.36-\sqrt{1-2(0.36)-(0.36)^2}}{2} \cong 0.87391,$$

which means that all solutions of this equation oscillate by Theorem 2.3. As you can see above, all results which are obtained in literature before can't hold. But, our new oscillation condition holds.

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Plancherel formula for the Shehu transform

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Abstract. We discuss some existence conditions of the Shehu transform, provide a Plancherel formula and also relate the Shehu equicontinuity to exponential L^2-equivanishing.

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Keywords: Shehu transform, Plancherel formula, exponential L^2-equivanishing, Shehu equicontinuity.

Contents

1. Introduction

Integral transforms have many applications in various fields of mathematical sciences and engineering such as physics, mechanics, chemestry, acoustic, etc. For example, integral transforms such as the Fourier transform and the Laplace transform are highly efficient in signal processing and solving differential equations.
In [9] the authors introduced a Laplace-type integral which they called the *Shehu transform*. The Shehu transform of a function $f : \mathfrak{R}_+ \to \mathbb{C}$ is defined by

$$\mathbb{S}\{f\}(s,u) = \int_0^\infty e^{-\frac{s}{u}t} f(t)dt,\ s \geq 0, u > 0. \tag{1.1}$$

provided that this integral exists, the symbol $\mathfrak{R}_+$ stands for the set of nonnegative real numbers. This integral transform generalizes both the Laplace transform [5] and the Yang transform [10]. Many authors used the Shehu transform to solve partial or ordinary differential equations related to real life problems [4], [1], [2], [3], [7]. Authors in [6] extended the Shehu transform to distributions and measures.
This paper is mainly devoted to search a Plancherel formula for the Shehu transform. The remainder of the paper is structured as follows. In Section 2 we discuss some existence conditions after replacing the first variable of the Shehu transform of a function with a complex variable and in Section 3, a Plancherel formula is given and Shehu equicontinuity and exponential L^2-equivanishing are related.

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2. Extension to complex variables and existence conditions

In order to obtain the Plancherel theorem in the next section for the Shehu transform, we want the first variable of $\mathbb{S}\{f\}(s,u)$ to be a complex variable. This is why we consider the Shehu transform of the function $f:\mathfrak{R}_+\to\mathbb{C}$ in the form

$$\mathbb{S}\{f\}(z,u)=\int_0^\infty e^{-\frac{z}{u}t}f(t)dt,\ z\in\mathbb{C},u>0. \tag{2.1}$$

In what follows, we discuss some existence conditions. The symbol $\mathscr{R}_e(z)$ denotes the real part of the complex number z. The complex vector spaces $L^1(\mathfrak{R}_+)$ and $L^2(\mathfrak{R}_+)$ are

$$L^1(\mathfrak{R}_+)=\left\{f:\mathfrak{R}_+\to\mathbb{C}:\int_0^\infty|f(t)|dt<\infty\right\} \tag{2.2}$$

and

$$L^2(\mathfrak{R}_+)=\left\{f:\mathfrak{R}_+\to\mathbb{C}:\int_0^\infty|f(t)|^2dt<\infty\right\}. \tag{2.3}$$

Theorem 2.1. *Consider the function $f:\mathfrak{R}_+\to\mathbb{C}$. If $f\in L^1(\mathfrak{R}_+)$ and $\mathscr{R}_e(z)\geq 0$ then $\mathbb{S}\{f\}(z,u)$ exists.*

Proof. Assume $f\in L^1(\mathfrak{R}_+)$. Set $z=x+iy$ with $\mathscr{R}_e(z)=x\geq 0$. Then

$$\begin{aligned}|e^{-\frac{z}{u}t}f(t)| &=|e^{-\frac{x}{u}t}e^{-i\frac{y}{u}t}f(t)|\\ &=e^{-\frac{x}{u}t}|f(t)|\\ &\leq|f(t)| \text{ because } e^{-\frac{x}{u}t}\leq 1.\end{aligned}$$

Finally $\int_0^\infty|f(t)|dt<\infty\Rightarrow\int_0^\infty|e^{-\frac{z}{u}t}f(t)|dt<\infty$. Thus $\mathbb{S}\{f\}(z,u)$ exists. ■

Theorem 2.2. *Consider the function $f:\mathfrak{R}_+\to\mathbb{C}$. If $f\in L^2(\mathfrak{R}_+)$ and $\mathscr{R}_e(z)>0$ then $\mathbb{S}\{f\}(z,u)$ exists.*

Proof. Assume $f\in L^2(\mathfrak{R}_+)$. Set $z=x+iy$ with $\mathscr{R}_e(z)=x>0$. Then

$$\begin{aligned}|e^{-\frac{z}{u}t}f(t)| &=|e^{-\frac{x}{u}t}e^{-i\frac{y}{u}t}f(t)|\\ &=e^{-\frac{x}{u}t}|f(t)|.\end{aligned}$$

Now, $\int_0^\infty e^{-\frac{2x}{u}t}dt=\left[-\frac{u}{2x}e^{-\frac{2x}{u}t}\right]_0^\infty=\frac{u}{2x}<\infty$. Thus the function $t\mapsto e^{-\frac{x}{u}t}$ is in $L^2(\mathfrak{R}_+)$. Since f is assumed to be in $L^2(\mathfrak{R}_+)$ we see that the product $t\mapsto e^{-\frac{x}{u}t}f(t)$ is integrable (use the Hölder inequality). We have $\int_0^\infty e^{-\frac{x}{u}t}|f(t)|dt<\infty$. Therefore $\int_0^\infty|e^{-\frac{z}{u}t}f(t)|dt<\infty$. Thus $\mathbb{S}\{f\}(z,u)$ exists. ■

Let $\alpha\geq 0$. Consider the function $f:\mathfrak{R}_+\to\mathbb{C}$ and set

$$f_\alpha(t)=f(t)e^{-\alpha t},\ t\in\mathfrak{R}_+. \tag{2.4}$$

Theorem 2.3. *If $f_{\frac{\alpha}{u}}\in L^1(\mathfrak{R}_+)$ and $\mathscr{R}_e(z)\geq\alpha$ then $\mathbb{S}\{f\}(z,u)$ exists.*

Proof. Assume $f_{\frac{\alpha}{u}}\in L^1(\mathfrak{R}_+)$. Set $z=x+iy\in\mathbb{C}$ with $\mathscr{R}_e(z)=x\geq\alpha$. We have

$$|e^{-\frac{z}{u}t}f(t)|=|e^{-\frac{x}{u}t}e^{-\frac{iy}{u}t}f(t)|=e^{-\frac{x}{u}t}|f(t)|.$$

$$\begin{aligned}\text{Now, } x\geq\alpha &\Rightarrow e^{-\frac{x}{u}t}|f(t)|\leq e^{-\frac{\alpha}{u}t}|f(t)|\\ &\Rightarrow\int_0^\infty e^{-\frac{x}{u}t}|f(t)|\leq\int_0^\infty e^{-\frac{\alpha}{u}t}|f(t)|<\infty.\end{aligned}$$

Therefore $\int_0^\infty|e^{-\frac{z}{u}t}f(t)|dt<\infty$. Thus $\mathbb{S}\{f\}(z,u)$ exists. ■

3. Plancherel formula and applications

In harmonic analysis, the Plancherel theorem holds for the Fourier transform. it states that the L^2-norms of a function in the time domain and that of its Fourier transform in the Fourier domain are equal. In other words it expresses conservation of energy for signals in the time domain and the Fourier domain. We would like to obtain the analogue of this result for the Shehu transform. To achieve our gaol we apply the Plancherel formula for the Laplace transform proved in [8]. We start with the following definition.

Definition 3.1. *[8] The function f is called a* Laplace-Pego function *of order α if $f_\alpha \in L^1(\Re_+) \cap L^2(\Re_+)$.*

Theorem 3.2. *[8] If f is a Laplace-Pego function of order $x \geq 0$, then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathcal{L}\{f\}(x+iy)|^2dy = \int_0^{\infty} e^{-2xt}|f(t)|^2dt. \tag{3.1}$$

Hereafter is the analogue of the Plancherel formula for the Shehu transform.

Theorem 3.3. *If f is a Laplace-Pego function of order $\dfrac{x}{u}$ then*

$$\frac{1}{2\pi u}\int_{-\infty}^{\infty}|\mathbb{S}\{f\}(x+iy,u)|^2dy = \int_0^{\infty} e^{-\frac{2x}{u}t}|f(t)|^2dt. \tag{3.2}$$

Proof. Assume that f is a Laplace-Pego function of order $\dfrac{x}{u}$. From Theorem 3.2, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathcal{L}\{f\}(x+iy)|^2dy = \int_0^{\infty} e^{-2xt}|f(t)|^2dt.$$

Replacing x and y by $\dfrac{x}{u}$ and $\dfrac{y}{u}$ respectively we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathcal{L}\{f\}(\frac{x}{u}+i\frac{y}{u})|^2d\frac{y}{u} &= \int_0^{\infty} e^{-2\frac{x}{u}t}|f(t)|^2dt;\\ \Rightarrow \frac{1}{2\pi u}\int_{-\infty}^{\infty}|\mathcal{L}(f)(\frac{x+iy}{u})|^2dy &= \int_0^{\infty} e^{-2\frac{x}{u}t}|f(t)|^2dt\end{aligned}$$

$$\Rightarrow \frac{1}{2\pi u}\int_{-\infty}^{\infty}|\mathbb{S}(f)(x+iy,u)|^2dy = \int_0^{\infty} e^{-\frac{2x}{u}t}|f(t)|^2dt.$$

■

Definition 3.4. *[8] A family $\mathcal{A}$ of Laplace-Pego functions of common order x is said to be* exponentially L^2-equivanishing *at x if for any given strictly positive number ε, we can find a strictly positive number T that depends only on ε and the order x, such that for any function f in $\mathcal{A}$, we have*

$$\int_T^{\infty} e^{-2xt}|f(t)|^2dt < \varepsilon. \tag{3.3}$$

The author in [8] related the concept of exponential L^2-equivanishing to the notion of Laplace equicontinuity. We obtain the analogue result for Shehu equicontinuity as an application of the Plancherel formula.

Definition 3.5. *A family $\mathcal{A}$ of functions is said to be* Shehu equicontinuous *at x if for any given strictly positive number ε, we can find a strictly positive number η that depends only on ε, such that for any strictly positive number u and any function f in $\mathcal{A}$, we have*

$$\frac{1}{2\pi u}\int_{-\infty}^{\infty}|\mathbb{S}\{f\}(x+iy+\eta,u)-\mathbb{S}\{f\}(x+iy,u)|^2dy < \varepsilon. \tag{3.4}$$

If $\mathcal{A}$ is a family of Lapalce-Pego functions of common order α, we set

$$\mathcal{A}_\alpha = \{f_\alpha \,:\, f \in \mathcal{A}\}.$$

We recall that $f_\alpha(t) = f(t)e^{-\alpha t}$, $t \geq 0$.

Theorem 3.6. *Let $\mathcal{A}$ be a family of Laplace-Pego functions with common order $\frac{x}{u} \geq 0$. If $\mathcal{A}$ is Shehu equicontinuous at x then it is exponentially L^2-equivanishing at $\frac{x}{u}$. Moreover, if $\mathcal{A}_{\frac{x}{u}}$ is L^2-bounded, then the inverse is also true.*

Proof. We follow the great lines of the proof of [8, Theorem 6]. Let $\varepsilon > 0$. Let η be such that (3.4) holds. There exists $T > 0$ such that $|e^{-\frac{\eta}{u}T} - 1| \geq \frac{1}{2}$. Then

$$\begin{aligned} &\frac{1}{2\pi u}\int_{-\infty}^{\infty} |\mathbb{S}\{f\}(x+iy+\eta,u) - \mathbb{S}\{f\}(x+iy,u)|^2 dy < \varepsilon \\ \Rightarrow &\frac{1}{2\pi u}\int_{-\infty}^{\infty} \left|\int_0^\infty \left[e^{-\frac{x+iy}{u}t}e^{-\frac{\eta}{u}t}f(t) - e^{-\frac{x+iy}{u}t}f(t)\right]dt\right|^2 dy < \varepsilon \\ \Rightarrow &\frac{1}{2\pi u}\int_{-\infty}^{\infty} \left|\int_0^\infty e^{-\frac{x+iy}{u}t}(e^{-\frac{\eta}{u}t} - 1)f(t)dt\right|^2 dy < \varepsilon \end{aligned}$$

Now using Theorem 3.3, we obtain

$$\begin{aligned} &\int_0^\infty e^{\frac{-2x}{u}t}\left|f(t)(e^{-\frac{\eta}{u}t} - 1)\right|^2 dt < \varepsilon, \\ \Rightarrow &\int_T^\infty e^{\frac{-2x}{u}t}\left|f(t)(e^{-\frac{\eta}{u}t} - 1)\right|^2 dt < \varepsilon, \\ \Rightarrow &\frac{1}{2}\int_T^\infty e^{\frac{-2x}{u}t}\left|f(t)\right|^2 dt < \varepsilon, \\ \Rightarrow &\int_T^\infty e^{\frac{-2x}{u}t}\left|f(t)\right|^2 dt < 2\varepsilon. \end{aligned}$$

Thus $\mathcal{A}$ is exponentially L^2-equivanishing at $\frac{x}{u}$.

Now assume that $\mathcal{A}_{\frac{x}{u}}$ is L^2-bounded and that $\mathcal{A}$ is exponentially L^2-equivanishing at $\frac{x}{u}$. Let $\varepsilon > 0$ and choose T such that (3.3) holds for $\frac{x}{u}$ instead of x. As $\mathcal{A}_{\frac{x}{u}}$ is L^2-bounded, there exists a constant $M > 0$ such that for any $f \in \mathcal{A}$ we have $\int_0^\infty e^{\frac{-2x}{u}t}\left|f(t)\right|^2 \, dt < M.$

Let $\eta > 0$ be such that $\left|e^{-\frac{\eta}{u}T} - 1\right|^2 M < \varepsilon$. Set

$$B = \frac{1}{2\pi u}\int_{-\infty}^{\infty} |\mathbb{S}\{f\}(x+iy+\eta,u) - \mathbb{S}\{f\}(x+iy,u)|^2 dy.$$

We have

$$B = \int_{-\infty}^{\infty} |\mathbb{S}\{g\}(x+iy,u)|^2 dy$$

where $g(t) = (e^{-\frac{\eta}{u}t} - 1)f(t)$. Now, using Theorem 3.3 we have

$$B=\int_0^T e^{-\frac{2x}{u}t}|f(t)|^2\left|e^{-\frac{\eta}{u}t}-1\right|^2dt+\int_T^\infty e^{-\frac{2x}{u}t}|f(t)|^2\left|e^{-\frac{\eta}{u}t}-1\right|^2dt.$$

For $t\le T$, $|e^{-\frac{\eta}{u}t}-1|^2\le|e^{-\frac{\eta}{u}T}-1|^2$. Therefore,

$$\int_0^T e^{-\frac{2x}{u}t}|f(t)|^2\left|e^{-\frac{\eta}{u}t}-1\right|^2dt<\frac{\varepsilon}{M}\int_0^T e^{-\frac{2x}{u}t}|f(t)|^2dt<\frac{\varepsilon}{M}\int_0^\infty e^{-\frac{2x}{u}t}|f(t)|^2dt<\varepsilon.$$

On the other hand, note that $\left|e^{-\frac{\eta}{u}t}-1\right|^2<1$. It follows that

$$\int_T^\infty e^{-\frac{2x}{u}t}|f(t)|^2\left|e^{-\frac{\eta}{u}t}-1\right|^2dt<\int_T^\infty e^{-\frac{2x}{u}t}|f(t)|^2dt<\varepsilon.$$

Therefore, $B<2\varepsilon$, and hence $\mathcal{A}$ is Shehu equicontinuous at x.

■

4. Conclusion

In this paper, some existence conditions of the Shehu integral transform of a function have been discussed, a Plancherel formula provided and Shehu equicontinuity and exponential L^2-equivanishing are related.

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Integrity and vertex neighbor integrity of some graphs

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Abstract. The integrity $I(G)$ of a noncomplete connected graph G is a measure of network vulnerability and is defined by $I(G) = \min_{S\subset V(G)}\{|S| + m(G-S)\}$, where S and $m(G-S)$ denote the subset of V and the order of the largest component of $G-S$, respectively. The vertex neigbor integrity denoted as $VNI(G)$ is the concept of the integrity of a connected graph G and is defined by $VNI(G) = \min_{S\subset V(G)}\{|S| + m(G-S)\}$, where S is any vertex subversion strategy of G and $m(G-S)$ is the number of vertices in the largest component of $G-S$. If a network is modelled as a graph, then the integrity number shows not only the difficulty to break down the network but also the damage that has been caused. This article includes several results on the integrity of the $k-ary\ tree\ H_n^k$, the diamond-necklace N_k, the diamond-chain L_k and the thorn graph of the cycle graph and the vertex neighbor integrity of the H_n^2, H_n^3.

AMS Subject Classifications: 03D20, 05C07, 05C69, 68M10.

Keywords: Integrity, vertex neighbor integrity, vulnerability.

Contents

1. Introduction

Let $G = (V(G), E(G))$ be a graph. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. The number of the neighbor vertices of the vertex v is called degree of v and denoted by $deg_G(v)$. The minimum and maximum degrees of a vertex of G are denoted by $\delta(G)$ and $\Delta(G)$. A vertex v is said to be pendant vertex if $deg_G(v) = 1$. A vertex u is called support if u is adjacent to a pendant vertex [6]. The complement $\overline{G}$ of a graph G is a graph whose vertex set is $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are nonadjacent in G [6].
Let G be a graph and $S \subseteq V(G)$. We denote by $< S >$ the subgraph of G induced by S. A set S is said to be an *independent set* of G, if no pair of vertices of S are adjacent in G. The *independence number* of G, denoted by $\beta(G)$, is the cardinality of a maximum independent set of G. We denote by $\Omega(G)$ the set of all maximum independent sets of G. A vertex and an edge are said to *cover* each other if they are incident. A set of vertices

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which cover all the edges of a graph G is called a *vertex cover* for G, while a set of edges which covers all the vertices is an *edge cover*. The smallest number of vertices in any vertex cover for G is called its *vertex covering number* and is denoted by $\alpha(G)$ [6]. For any graph G of order n, $\alpha(G) + \beta(G) = n$. For a graph G, we denote the minimum number of colors necessary to color G by $\chi(G)$, the chromatic number of the graph G. The connectivity $\kappa = \kappa(G)$ of a graph G is the minimum number of points whose removal results in a disconnected or trivial graph.

The vulnerability of a communication network gives us an idea of the resilience and robustness of the network after some centers in the network have failed. Vulnerability can be measured by certain parameters. In the analysis of the vulnerability of a communication network to disruption, attention is paid to the number of non-working elements and the size of the largest remaining group in it. Especially in a hostile relationship where mutual communication still takes place, it is desirable that the adversary's network be such that the two quantities can be made small at the same time. [1]

One of the parameters used to measure the vulnerability of the graph is the integrity value. Formally, the vertex integrity (frequintly called just the integrity) is

$$I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}$$

where $m(G - S)$ denotes the order of a largest component of $G - S$. This concept was introduced by Barefoot, Entringer and Swart [2], who discovered many of the early results on the subject. If G is a graph of order n, then $1 \le I(G) \le n$ and if H is any subgraph of G, then $I(H) \le I(G)$. The integrity of the binomial trees was given in [8].

The concept of the vertex neighbor-integrity was introduced as a measure of graph vulnerability by M.B.Cozzens and Shu-Shih Y.Wu [4]. Let u be a vertex of a graph $G = (V, E)$. Then $N(u) = \{v \in V(G), v \; and \; u \; are \; adjacent\}$ is the open neighborhood of u, and $N[u] = \{u\} \cup N(u)$ denotes the closed neighborhood of u. A vertex u of a graph G is said to be subverted if the closed neighborhood $N[u]$ is deleted from G. A set of vertices $S = \{u_1, u_2, ..., u_m\}$ is called a vertex subversion strategy of G if each of the vertices in S has been subverted from G. Let $G - S$ be the survival subgraph when S has been a vertex subversion strategy of G. The vertex neighbor integrity of a graph G, $VNI(G)$, is defined to be

$$VNI(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}$$

where S is any vertex subversion strategy of G and $m(G - S)$ is the number of vertices in the largest component of $G - S$. The set S is called the $VNI - set$ of a graph G, which gives its neighbor integrity. The neighbor integrity for total graphs was given in [9]

In this article, we give a recursive formula on the integrity of H_n^k and we calculate the integrity of the diamond-necklace N_k, the diamond-chain L_k and the thorn graph G^* of a cycle C_n. Then, we present a result on the vertex neighbor-integrity of H_n^2 and H_n^3 .

2. Basic Results on Integrity and Vertex Neighbor Integrity

Theorem 2.1. *[1] Define the comet $C_{t,r}$ to be the graph obtained by identifying one end of the path P_t with the center of the star $K_{1,r}$. Then, $I(C_{t,r}) \le I(C_{t+1,r-1}) \le I(C_{t,r}) + 1$.*

Theorem 2.2. *[1] The integrity of*
***(a)** the complete graph K_p is p;*
***(b)** the null graph $\overline{K_p}$ is 1;*
***(c)** the star $K_{1,n}$ is 2;*
***(d)** the path P_n is $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2.$;*

(e) *the cycle* C_n *is* $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$.*;*
(f) *the comet* $C_{p-r,r}$ *is* $I(P_p)$*, if* $r \leq \sqrt{p+1} - \frac{5}{4}$*;* $\lceil 2\sqrt{p-r} \rceil - 1$*, otherwise;*
(g) *the complete bipartite graph* $K_{m,n}$ *is* $1 + minm, n$*;*
(h) *any complete multipartite graph of order* p *and largest partite set of order* r *is* $p - r + 1$.

Theorem 2.3. *[1] Let* G *be a graph of order* p.
(a) $I(G) = 1$ *if and only if* G *is null.*
(b) $I(G) = 2$ *if and only if all nontrivial components of* G *are edges or the only nontrivial component is a star.*
(c) $I(G) = p - 1$ *if and only if* G *is not complete and* $\overline{G}$ *has girth at least 5.*
(d) $I(G) = p$ *if and only if* G *is complete.*

Theorem 2.4. *[1] If in graph* G*,* v *is a vertex for which* $deg(v) \geq I(G - v)$*, then* $I(G) = 1 + I(G - v)$.

Parameters that will be discussed here include the following:

- δ, the minimum vertex degree;
- κ, the connectivity;
- α, the covering number;
- β, the independence number;
- χ, the chromatic number.

Theorem 2.5. *For any graph* G*,*
(a) $I(G) \leq \alpha + 1$.
(b) $I(G) \geq \delta + 1$.
(c) $I(G) \geq \chi$.
(d) $I(G) \geq (p - \kappa(G))/\beta(G) + \kappa(G)$.
(e) $I(G) = \kappa(G) + 1$ *if and only if* $\kappa(G) = \alpha(G)$*;*
(f) $I(G) = \alpha(G) + 1$ *if and only if* G *does not contain* $2K_2$ *as an induced subgraph;*
(g) $I(G) = \delta(G) + 1$ *if and only if* $G \cong rK_n$ *or* $G \cong rK_n + F$ *for some graph* F *satisfying* $\delta(F) \geq |G| - (2r - 1)n - 1$.

Theorem 2.6. *[4] Let* P_n *be the path on* n *vertices. Then we have*

$$VNI(P_n) = \begin{cases} \lceil 2\sqrt{n+3} \rceil - 4, & if \ n \geq 2 \\ 1, & if \ n = 1 \end{cases}$$

Theorem 2.7. *[4] Let* C_n *be the* $n - cycle$*, where* $n \geq 3$*. Then*

$$VNI(C_n) = \begin{cases} \lceil 2\sqrt{n} \rceil - 3, & if \ n > 4, \\ 2, & if \ n = 4, \\ 1, & if \ n = 3. \end{cases}$$

3. Integrity of H_n^k, N_k, L_k and the Thorn Graph

Integrity of G is defined to be $I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}$, where $m(G - S)$ denotes the order of a largest component of $G - S$. In this section we calculated a recursive formula about the integrity of H_n^3 and we give a common result for the integrity of H_n^k.

Definition 3.1. *[3] The complete* $k-ary\ tree$ H_n^k *of depth* n *is the rooted tree in which all vertices at level* $n-1$ *or less have exactly* k *children, and all vertices at level* n *are leaves. A* $2-ary\ tree$ *,* H_4^2 *is illustrated in Figure 1.*

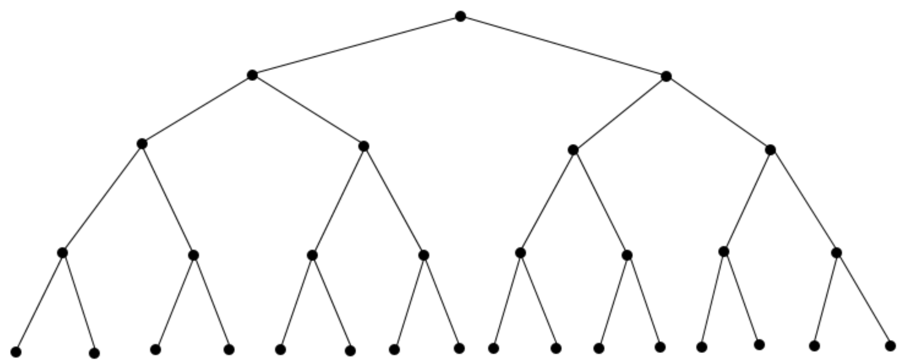

Figure 1: 2-ary tree H_4^2

Theorem 3.2. *The integrity of a complete* $3-ary\ tree$ H_n^3 *is given by*

$$I(H_n^3) = \begin{cases} 3^{\frac{n}{2}} + I(H_{n-2}^3), & if\ n \equiv 0(mod\ 2) \\ 3^{\lfloor \frac{n}{2} \rfloor} + I(H_{n-1}^3), & if\ n \equiv 1(mod\ 2) \end{cases}$$

Proof. **Case 1.** $n \equiv 0(\text{mod } 2)$
In this case S consists of the all vertices at the level $n/2$. Hence, $m(H_n^3 - S) = \sum_{i=0}^{(n/2)-1} 3^i$, $|S| = 3^{n/2}$ and $I(H_n^3) = \sum_{i=0}^{n/2} 3^i = \frac{3^{(n/2)+1}-1}{2}$. To express this function as a recursive function we use induction. Certainly $I(H_4^3) = 3^2 + I(H_2^3) = 9 + 4 = 13$. Assume $I(H_{n-2}^3) = \frac{3^{\frac{n-2}{2}+1}-1}{2}$, then we need to show that $I(H_n^3) = \frac{3^{(n/2)+1}-1}{2}$.

$$I(H_n^3) = 3^{n/2} + I(H_{n-2}^3) \tag{3.1}$$

$$= 3^{n/2} + \frac{3^{\frac{n-2}{2}+1} - 1}{2} \tag{3.2}$$

$$= 3^{n/2} + \frac{3^{\frac{n}{2}} - 1}{2} \tag{3.3}$$

$$= \frac{3^{\frac{n}{2}+1} - 1}{2} \tag{3.4}$$

Case 2. $n \equiv 1(\text{mod } 2)$
In this case S consists of the all vertices at the level $\lfloor n/2 \rfloor$. Hence, $m(H_n^3 - S) = \sum_{i=0}^{\lfloor (n/2) \rfloor} 3^i$, $|S| = 3^{\lfloor n/2 \rfloor}$ and $I(H_n^3) = \frac{3^{\lfloor (n/2) \rfloor+1}-1}{2} + 3^{\lfloor n/2 \rfloor}$. To express this function as a recursive function we use induction. Certainly $I(H_3^3) = 3 + I(H_2^3) = 3 + 4 = 7$. Assume $I(H_{n-1}^3) = \frac{3^{\lfloor \frac{n}{2} \rfloor -1+1}-1}{2} + 3^{\lfloor \frac{n}{2} \rfloor}$, then we need to show that $I(H_n^3) = \frac{3^{\lfloor (n/2) \rfloor+1}-1}{2} + 3^{\lfloor \frac{n}{2} \rfloor}$.

$$I(H_n^3) = 3^{\lfloor n/2 \rfloor} + I(H_{n-1}^3) \tag{3.5}$$

$$= 3^{\lfloor n/2 \rfloor} + \frac{3^{\lfloor \frac{n}{2} \rfloor} - 1}{2} + 3^{\lfloor n/2 \rfloor} \tag{3.6}$$

$$= \frac{3.3^{\lfloor n/2 \rfloor} - 1}{2} + 3^{\lfloor n/2 \rfloor} \tag{3.7}$$

$$= \frac{3^{\lfloor (n/2) \rfloor + 1} - 1}{2} + 3^{\lfloor \frac{n}{2} \rfloor} \tag{3.8}$$

The proof is completed. ∎

Corollary 3.3. *The integrity of a complete $k - ary\ tree$ H_n^k, $k \geq 3$ is given by*

$$I(H_n^k) = \begin{cases} k^{\frac{n}{2}} + I(H_{n-2}^k), & if\ n \equiv 0(mod\ 2) \\ k^{\lfloor \frac{n}{2} \rfloor} + I(H_{n-1}^k), & if\ n \equiv 1(mod\ 2) \end{cases}$$

Definition 3.4. *[7] For $k \geq 2$ an integer, let $D_1, D_2, ..., D_k$ be k disjoint copies of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and where $a_i b_i$ is the missing edge in D_i. N_k is obtained from the disjoint union of these k diamonds by adding the edges $\{a_i b_{i+1} | i = 1, 2, ..., k-1\}$ and adding the edge $a_k b_1$. We call N_k a diamond-necklace with k diamonds.*

Definition 3.5. *[7] For $k \geq 1$, we define a diamond-chain L_k with k diamonds as follows. Let L_k be obtained from a diamond-necklace N_{k+1} with $k+1$ diamonds $D_1, D_2, ..., D_{k+1}$ by removing the diamond D_{k+1} and adding two disjoint triangles T_1 and T_2 and adding an edge joining b_1 to a vertex of T_1 and adding an edge joining a_k to a vertex of T_2. A diamond-necklace, N_6, with six diamonds and a diamond-chain, L_2, with two diamonds is illustrated in Figure 2.*

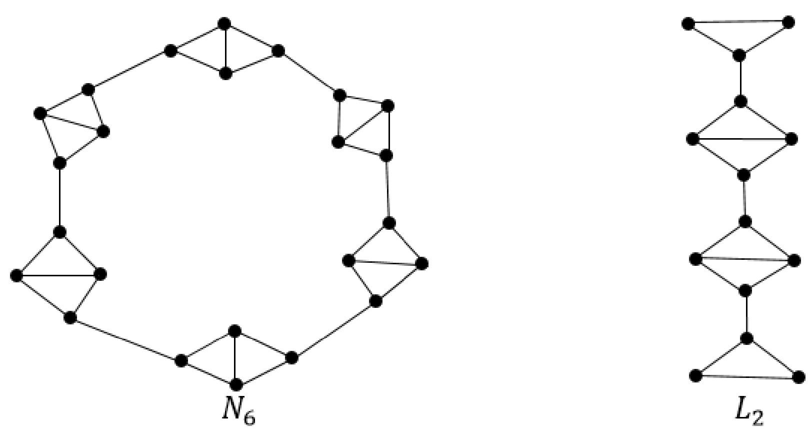

Figure 2: A diamond-necklace N_6 and a diamond-chain L_2

Theorem 3.6. *Let N_k be a diamond-necklace with k vertices.The integrity of N_k is $I(N_k) = \lceil 4\sqrt{k} - 1 \rceil$.*

Proof. For the set S, we choose the vertices among a_i, b_i , where $i \in \{\overline{1,k}\}$ to minimalized the value $|S| + m(N_k - S)$. So, if we remove r vertices from N_k, then we have r components. Number of vertices of a largest component $m(N_k - S) \geq \frac{4k-r}{r}$.

$$I(N_k) = \min_{S \subset V(N_k)} \{|S| + m(N_k - S)\} \tag{3.9}$$

$$\geq \min_{r} \{r + \frac{4k-r}{r}\} \tag{3.10}$$

The function $f(r) = r + \frac{4k-r}{r}$ takes its minimum value at $r = 2\sqrt{k}$. We substitute the minimum value in the function $f(r)$. As the integrity is integer valued, we round this up to get a lower bound and obtain $I(N_k) = \lceil 4\sqrt{k} - 1 \rceil$. ■

Theorem 3.7. *Let L_k be a diamond-chain with $4k + 6$ vertices.The integrity of L_k is $I(L_k) = \lceil \frac{(-1+\sqrt{4k+7})^2+4k+6}{\sqrt{4k+7}} \rceil$.*

Proof. For the set S, we choose the vertices among a_i, b_i and the vertex of order 3 in T_2, where $i \in \{\overline{1,k}\}$ to minimalized the value $|S| + m(L_k - S)$. So, if we remove r vertices from L_k, then we have $r + 1$ components. Number of vertices of a largest component $m(L_k - S) \geq \frac{4k+6-r}{r+1}$.

$$I(L_k) = \min_{S \subset V(L_k)} \{|S| + m(L_k - S)\} \tag{3.11}$$

$$\geq \min_{r} \{r + \frac{4k+6-r}{r+1}\} \tag{3.12}$$

The function $f(r) = \frac{r^2+4k+6}{r+1}$ takes its minimum value at $r = -1 + \sqrt{4k+7}$. We substitute the minimum value in the function $f(r)$. As the integrity is integer valued, we round this up to get a lower bound and obtain $I(L_k) = \lceil \frac{(-1+\sqrt{4k+7})^2+4k+6}{\sqrt{4k+7}} \rceil$. ■

Definition 3.8. *[5] Let $p_1, p_2, ..., p_n$ be non-negative integers and G be such a graph, $V(G) = n$. The thorn graph of the graph G, with parameters $p_1, p_2, ..., p_n$, is obtained by attaching p_i new vertices of degree 1 to the vertex u_i of the graph G, $i = 1, 2, ..., n$. The thorn graph of the graph G will be denoted by G^* or by $G^*(p_1, p_2, ..., p_n)$, if the respective parameters need to be specified. The thorn graph G^* of the cycle C_6 is illustrated in Figure 3.*

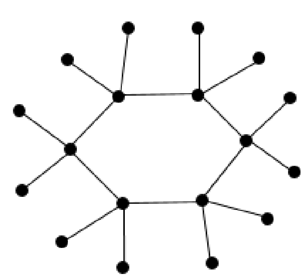

Figure 3: The thorn graph G^*

Theorem 3.9. *If G^* is a thorn graph of C_n with $p_1 = p_2 = ... = p_n = p$, then $I(G^*) = 2\lceil \sqrt{n}\sqrt{p+1} \rceil - p - 1$.*

Proof. For the set S, we choose the vertices of order $p+2$ in C_n to minimalized the value $|S| + m(G^* - S)$. So, if we remove r vertices from C_n, then we have r components. Number of vertices of a largest component $m(C_n - S) \geq \frac{(n-r)p+n-r}{r}$.

$$I(N_k) = \min_{S \subset V(G^*)} \{|S| + m(G^* - S)\} \tag{3.13}$$

$$\geq \min_r \{r + \frac{(n-r)p+n-r}{r}\} \tag{3.14}$$

The function $f(r) = r + \frac{(n-r)p+n-r}{r}$ takes its minimum value at $r = \sqrt{np+n}$. Hence if we substitute the minimum value in the function $f(r)$, we have $I(C_n) = 2\sqrt{n}\sqrt{p+1} - p - 1$. Since the integrity is integer valued, we round this up to get a lower bound and obtain $I(C_n) = 2\lceil \sqrt{n}\sqrt{p+1} \rceil - p - 1$. ■

4. Vertex Neighbor Integrity of H_n^2 and H_n^3

The vertex neighbor integrity of G is defined to be $VNI(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}$, where $m(G - S)$ is the largest connected component in the graph $G - S$. Here, we calculated the $VNI(H_n^2)$ and $VNI(H_n^3)$.

Theorem 4.1. *The vertex neighbor integrity of a complete* $2-ary\ tree$ H_n^2 *is given by*

$$VNI(H_n^2) = \lfloor \frac{\sqrt{2^{n+3}-3}-1}{2} \rfloor$$

Proof. We have $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ vertices in H_n^2. If we remove $|S| = r$ vertices from H_n^2, then we have $4r + 1$ components in $H_n^2 - S$. Hence, $m(H_n^2 - S) \geq \frac{2^{n+1}-1-r}{4r+1}$.

$$VNI(H_n^2) = \min_{S \subset V(H_n^2)} \{|S| + m(H_n^2 - S)\} \tag{4.1}$$

$$\geq \min_r \{r + \frac{2^{n+1} - 1 - r}{4r+1}\} \tag{4.2}$$

The function $f(r) = r + \frac{2^{n+1}-1-r}{4r+1} = \frac{2^{n+1}+4r^2-1}{4r+1}$ takes its minimum value at $r = \frac{\sqrt{2^{n+3}-3}-1}{4}$. Hence if we substitute the minumum value in the function $f(r)$, we have $VNI(H_n^2) = \lfloor \frac{\sqrt{2^{n+3}-3}-1}{2} \rfloor$.

The proof is completed. ■

Theorem 4.2. *The vertex neighbor integrity of* H_n^3*, is given by*

$$VNI(H_n^3) = \begin{cases} 3^{\frac{n}{2}} + VNI(H_{n-2}^3), & if\ n \equiv 1(mod\ 2) \\ 3^{\lfloor \frac{n}{2} \rfloor} + VNI(H_{n-3}^3), & if\ n \equiv 0(mod\ 2) \end{cases}$$

Proof. The proof is similar to the proof of Theorem 3.2. ■

The following table gives the vertex neighbor integrity values of H_n^2 and H_n^3 for $n = 2, ..., 10$.

Table 1: The vertex neighbor integrity of H_n^2 and H_n^3

n	2	3	4	5	6	7	8	9	10
H_n^2	2	3	5	7	11	15	23	31	47
H_n^3	3	4	7	13	31	40	94	121	283

5. Acknowledgement

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A mathematical reason to wear a face mask during a COVID-19 like pandemic

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Abstract. The new coronavirus called COVID-19 started spreading in China since the end of 2019, and shortly, it became a serious matter over the entire world, infecting millions of people and killing many of them. That virus lead researchers to looking for ways to eradicate it, the first thing being to prevent people from getting contaminated. One way someone can protect himself and others is to wear a face mask as recommended by the World Health Organization. In this paper, we give a simple mathematical model showing why everyone should wear a face mask during a COVID-19 like pandemic. In order to illustrate the situation, we carry out a short simulation work, showing how various populations can be affected. We also show the number of contamination rounds needed to contaminate the whole population if nothing is done to stop the contamination process.

AMS Subject Classifications: 92C99; 03C65.

Keywords: Coronavirus, face mask, mathematical model, epidemiology, simulation.

Contents

1. Introduction

The new coronavirus disease called COVID-19 is caused by the virus severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) and started spreading in China since the end of 2019. This has become a serious worldwide concern, especially in the most developed countries where the spread has taken most governments by surprise. Very quickly, variants of this virus also started circulating in different countries. The dominant ones are: Alpha (α) for B.1.1.7 (United Kingdom variant), Beta (β) for B.1.351 (South Africa), Gamma (γ) for P.1 (Brazil), Delta (δ) for B.1.617.2 (India), etc. Since then, many methods have been used in order to flatten the curve of

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the expansion of the virus. Washing hands, respecting a certain social distance between people and wearing face masks became famous on media and social networks.

As for the researchers, many models have been developed to appreciate and predict the spread of the virus. We can mention, e.g., Li et al. [3], who studied the transmission process of the virus and realized forward prediction and backward inference of the epidemic situation. On the other hand, Roy and Bhattacharya [7] discussed a mathematical model regarding the spread of COVID-19 in India, aiming at finding the nature of time dependence of the number of symptomatic patients, officially recorded in the country, during a certain period. They based their study on a differential equation that has been formed to find how the number of asymptomatic patients increases with time, and they discussed the impact of imposition of a countrywide lockdown and its withdrawal. In order to investigate the spread and mitigation of the COVID-19 virus in the UK, Hritonenko et al. [2] consider an integral model with finite memory using a realistic infection distribution. They construct and justify an efficient regularization algorithm for finding the transmission rate.

The research on the international epidemics and the future development trend has become a hot topic of current research, and many teams have studied the transmission law and preventive measures of the COVID-19, many leading to interesting results (see, e.g., Mizumoto and Chowell [5]; Riou and Althaus [6]; Shao and Wu [8]). However, many models that work for some areas do not work in others, and this has been the case for epidemiological models for decades.

The aim of this paper is to give a simple mathematical model showing why everyone should wear a face mask when pandemic such as COVID-19 rises. In fact, if the face masks proved to be effectively protecting from getting the virus, then wearing them is worth it, although they are uncomfortable. See Dhaene et al. [1], on which this work is based.

The layout of this work is as follows. In the next section we discuss the model when no policy is involved. In Section 3, the model involving the ”wear-a-mask” (WAM) policy is discussed. In Section 4, we give results of a simulation work showing the impact of contamination on populations of various sizes and the number of rounds needed to fully contaminate those populations if nothing is done to stop the contamination process. Some concluding remarks are given in Section 5.

2. The model without any policy

Let us denote by R_0, the basic reproduction number, that is, the average number of persons infected by a person carrying the virus in an homogeneous population where everyone is susceptible to be infected. An R_0 value greater than 1 means that the epidemic will grow, otherwise, it will reverse. Moreover, the higher the R_0 value, more contagious the disease is. Figure 1 shows the basic reproduction number, R_0, of some of the SARS-CoV-2 variants in comparison to those of other diseases. From this figure, it appears that the basic reproduction number of Delta variant of SARS-CoV-2 is within the range 5 and 8, meaning that each infected person can infect 5 to 8 other persons. However, another recent study by Liu and Rocklöv [4] shows that the basic reproductive number, R_0, of Delta variant varies from 3.2 to 8, with a mean of 5.08. In both cases, the Delta variant is more contagious than the original SARS-CoV-2 virus. More details can be found on UNSW Newsroom (see link in the references) [9].

Figure 1: The basic reproduction number, R_0, of some of the SARS-CoV-2 variants in comparison to those of other diseases. Source: Imperial College London, Lancet, Australian Government.

With a reproduction number of R_0, an infected person can infect on average R_0 persons, making a total of $1 + R_0$ infected persons. Each one of the R_0 infected persons will infect on average R_0 other persons, yielding $1 + R_0 + R_0^2$ infected persons. The new infected ones will further infect on average R_0 others each, making in total $1 + R_0 + R_0^2 + R_0^3$, and so on.

Suppose that the above process where infected people are infecting others happens n times. Then after n infection rounds, the total number of infected people, which we denote by $M(R_0, n)$, will be equal to $1 + R_0 + R_0^2 + R_0^3 + ... + R_0^n$. By remarking that the terms of this expression follow a geometric progression of common ratio R_0 and first term 1, $M(R_0, n)$ becomes:

$$M(R_0, n) = \frac{1 - R_0^{n+1}}{1 - R_0}. \tag{2.1}$$

Since $M(R_0, n)$ can be decimal, we will consider the integer part where necessary to account for the fact that we deal with a discrete variable (number of persons).

We can derive the number of infection rounds needed to contaminate a popilation of size N as:

$$n = \frac{ln\left(\frac{1-N+R_0N}{R_0}\right)}{ln(R_0)}. \tag{2.2}$$

As it can be seen from equation (2.1), the number of contaminations depends on two parameters: the reproduction number, R_0, and the number, n, of infection rounds. However, we can note that, for a fixed value of the infection rounds, a small change in R_0 values will produce a huge impact on the number of contaminations. In order to illustrate this idea, let us compare the number of contaminations for a disease with $R_0 = 1.9$ with that of a disease with $R_0 = 2$ after $n = 20$ rounds. We get $M(1.9, 20) = 793564$ and $M(2, 20) = 2097151$. We can see that although epidemics with basic reproduction numbers 1.9 and 2 might look similar to each other, the latter is vastly a much more severe disease.

3. The model with "wear-a-mask" (WAM) policy

Let us now assume that there is a policy where everybody in the population wears a mask in public areas. Suppose this policy reduces the basic reproduction number from R_0 to $R_0 \times p$, where $p \in (0,1)$. Then after n rounds of contaminations, the total number of contaminated persons is:

$$M_p(R_0,n) = \frac{1-(R_0 \times p)^{n+1}}{1-(R_0 \times p)}. \tag{3.1}$$

The number of rounds needed to contaminate a population of size N is therefore:

$$n = \frac{ln\left(\frac{1-N+R_0 \times p \times N}{R_0 \times p}\right)}{ln(R_0 \times p)}. \tag{3.2}$$

Obviously, since $p < 1$, we have $M_p(R_0,n) < M(R_0,n)$, meaning that the policy will reduce the number of contaminations.

Table 1 to Table 4 show the impact of the reproduction number, R_0, on the number of contaminations $M(R_0,n)$ and $M_p(R_0,n)$ and the comparison of these numbers for various values of R_0 (ranging from 4 to 8), p ($p = 0.5,\ 0.7,\ 0.8,\ 0.9$) and fixed values of n ($n = 5,\ 10,\ 15,\ 20$). From these tables, we conclude that a relatively small difference in the basic reproduction number, R_0, will have a huge impact on the total number of infections, $M(R_0,n)$ and $M_p(R_0,n)$, after a sufficient number of transmission of the virus. Particularly, from $n = 15$ we observe an exponential increase on $M(R_0,n)$ and $M_p(R_0,n)$.

Table 1: Impact of R_0 ($R_0 = 4$ to 8) on $M(R_0,n)$ and $M_p(R_0,n)$ and comparison of these numbers of contaminations for $p = 0.5$ and fixed n ($n = 5,\ 10,\ 15,\ 20$).

			$n=5$	$n=10$	$n=15$	$n=20$
$p=0.5$	$R_0=4$	$M(R_0,n)$	1365	1398101	1.431656e+09	1.466016e+12
		$M_p(R_0,n)$	63	2047	65535	2097151
		M_p/M	4.62%	0.15%	0%	0%
	$R_0=5$	$M(R_0,n)$	3906	12207031	3.814697e+10	1.192093e+14
		$M_p(R_0,n)$	162	15894	1552204	151582450
		M_p/M	4.15%	0.13%	0%	0%
	$R_0=6$	$M(R_0,n)$	9331	72559411	5.642220e+11	4.387390e+15
		$M_p(R_0,n)$	364	88573	21523360	5230176601
		M_p/M	3.90%	0.12%	0%	0%
	$R_0=7$	$M(R_0,n)$	19608	329554457	5.538822e+12	9.309098e+16
		$M_p(R_0,n)$	734	386196	202837711	106534169022
		M_p/M	3.74%	0.12%	0%	0%
	$R_0=8$	$M(R_0,n)$	37449	1227133513	4.021071e+13	1.317625e+18
		$M_p(R_0,n)$	1365	1398101	1431655765	1466015503701
		M_p/M	3.64%	0.11%	0%	0%

Table 2: Impact of R_0 ($R_0 = 4$ to 8) on $M(R_0, n)$ and $M_p(R_0, n)$ and comparison of these numbers of contaminations for $p = 0.7$ and fixed n ($n = 5,\ 10,\ 15,\ 20$).

			$n=5$	$n=10$	$n=15$	$n=20$
$p=0.7$	$R_0=4$	$M(R_0,n)$	1365	1398101	1.431656e+09	1.466016e+12
		$M_p(R_0,n)$	267	46074	7929686	1364728160
		M_p/M	19.56%	3.30%	0.55%	0.09%
	$R_0=5$	$M(R_0,n)$	3906	12207031	3.814697e+10	1.192093e+14
		$M_p(R_0,n)$	735	386196	202837711	1,06534e+11
		M_p/M	18.79%	3.16%	0.53%	0.09%
	$R_0=6$	$M(R_0,n)$	9331	72559411	5.642220e+11	4.387390e+15
		$M_p(R_0,n)$	1715	2241776	2929804677	3,829e+12
		M_p/M	18.38%	3.09%	0.52%	0.09%
	$R_0=7$	$M(R_0,n)$	19608	329554457	5.538822e+12	9.309098e+16
		$M_p(R_0,n)$	3549	10025182	28318658314	7,99932e+13
		M_p/M	18.09%	3.04%	0.51%	0.09%
	$R_0=8$	$M(R_0,n)$	37449	1227133513	4.021071e+13	1.317625e+18
		$M_p(R_0,n)$	6704	36924146	203353e+11	1,11993e+15
		M_p/M	17.90%	3.01%	0.51%	0.08%

Table 3: Impact of R_0 ($R_0 = 4$ to 8) on $M(R_0, n)$ and $M_p(R_0, n)$ and comparison of these numbers of contaminations for $p = 0.8$ and fixed n ($n = 5,\ 10,\ 15,\ 20$).

			$n=5$	$n=10$	$n=15$	$n=20$
$p=0.8$	$R_0=4$	$M(R_0,n)$	1365	1398101	1.431656e+09	1.466016e+12
		$M_p(R_0,n)$	487	163766	5.495117e+07	1.843855e+10
		M_p/M	35.68%	11.71%	3.84%	1.26%
	$R_0=5$	$M(R_0,n)$	3906	12207031	3.814697e+10	1.192093e+14
		$M_p(R_0,n)$	1365	1398101	1.431656e+09	1.466016e+12
		M_p/M	34.95%	11.45%	3.75%	1.23%
	$R_0=6$	$M(R_0,n)$	9331	72559411	5.642220e+11	4.387390e+15
		$M_p(R_0,n)$	3218	8201060	2.089663e+10	5.324544e+13
		M_p/M	34.49%	11.30%	3.70%	1.21%
	$R_0=7$	$M(R_0,n)$	19608	329554457	5.538822e+12	9.309098e+16
		$M_p(R_0,n)$	6704	36924146	2.033530e+11	1.119930e+15
		M_p/M	34.19%	11.20%	3.67%	1.20%
	$R_0=8$	$M(R_0,n)$	37449	1227133513	4.021071e+13	1.317625e+18
		$M_p(R_0,n)$	12725	136642548	1.467188e+12	1.575381e+16
		M_p/M	33.98%	11.14%	3.65%	1.20%

Table 4: Impact of R_0 ($R_0 = 4$ to 8) on $M(R_0, n)$ and $M_p(R_0, n)$ and comparison of these numbers of contaminations for $p = 0.9$ and fixed n ($n = 5,\ 10,\ 15,\ 20$).

			$n = 5$	$n = 10$	$n = 15$	$n = 20$
$p = 0.9$	$R_0 = 4$	$M(R_0, n)$	1365	1398101	1.431656e+09	1.466016e+12
		$M_p(R_0, n)$	837	506237	306102350	1,85088e+11
		M_p/M	61.25%	36.21%	21.38%	12.63%
	$R_0 = 5$	$M(R_0, n)$	3906	12207031	3.814697e+10	1.192093e+14
		$M_p(R_0, n)$	2372	4377938	8078526911	1,49072e+13
		M_p/M	60.73%	35.86%	21.18%	12.51%
	$R_0 = 6$	$M(R_0, n)$	9331	72559411	5.642220e+11	4.387390e+15
		$M_p(R_0, n)$	5635	25874900	1,18808e+11	5,45527e+14
		M_p/M	60.38%	35.66%	21.06%	12.43%
	$R_0 = 7$	$M(R_0, n)$	19608	329554457	5.538822e+12	9.309098e+16
		$M_p(R_0, n)$	11797	117076619	1,16191e+12	1,15312e+16
		M_p/M	60.16%	35.53%	20.98%	12.39%
	$R_0 = 8$	$M(R_0, n)$	37449	1227133513	4.021071e+13	1.317625e+18
		$M_p(R_0, n)$	22470	434776209	8,41256e+12	1,62776e+17
		M_p/M	60.00%	35.43%	20.92%	12.35%

Another aspect that is worth highlighting is that these tables clearly indicate the positive impact of the policy on the contamination rate. As an example (see Dhaene et al. [1]), let us suppose that the factor p is equal to 90%. This means that wearing face masks reduces the basic reproduction number by 10%. At step 20 of a chain of 20 rounds of infections, the number of infected people under the wear-a-mask policy is only about 12% of the number of people that would have been infected without introducing that policy (Table 4).

Instead of 90%, let us now look at what happens if the factor p equals 80%, which means that wearing masks reduces the reproduction number by 20%. In this case we find that after 20 steps, the number of infected people is reduced to 1.2% of the original number of infected people (Table 3).

4. Simulations

In this section, we carry out a short simulation work to show how various populations can be affected by the virus. We also show the number of infection rounds needed to contaminate the whole population if nothing is done to stop the contamination process. In order to achieve this, we have considered populations of sizes $N = 100$, 1000, 5000, 10000, 100000, 1000000, 3000000, and 5000000. For the basic reproduction number, we take $R_0 = 4$. The results are given in Tables 5.

Table 5: Number of rounds needed to contaminate the full population for different values of p (0.3, 0.5, 0.7, 0.9, 1) and $R_0 = 4$.

Population size	Rounds with No policy	Rounds with WAM policy			
	$p = 1$	$p = 0.9$	$p = 0.7$	$p = 0.5$	$p = 0.3$
100	3.12	3.34	4.05	5.66	15.70
1000	4.78	5.14	6.28	8.97	28.09
5000	5.94	6.40	7.84	11.29	36.89
10000	6.44	6.94	8.52	12.29	40.69
100000	8.10	8.73	10.75	15.61	53.32
1000000	9.76	10.53	12.99	18.93	65.95
3000000	10.55	11.39	14.06	20.52	71.97
5000000	10.92	11.79	14.55	21.25	74.78

It is clear from this table that, for fixed value of p, the number of rounds needed to contaminate the full population increases as the population size increases. Moreover, for fixed population size, the number of rounds required to contaminate that population decreases as the value of p increases.

5. Concluding remarks

This work clearly illustrate the importance of wearing masks to fight the COVID-19 and similar diseases. We have seen that the reproduction number can be considerably reduced if a ”wear-a-mask” policy is made and especially if people respect it. Simulation results show that the infection process is slowed down with the WAM policy, and this could allow more time to researchers to find cure or vaccines to completely eradicate the virus.

The models considered are, of course, too simple to explain and take into account all effects of wearing masks or not. For example, mandating mask protection could make people become less cautious about social distancing, and this could reduce the positive effects of wearing masks. Therefore, wearing a face mask with the intention to decrease the infection rate will only be fully effective if it is surrounded by a sufficient educative support and combined with other regulations. Further, the models should not be applied for any number of infection rounds because when a high proportion of the population becomes immune, the reproduction number will automatically go down.

While we focused on the WAM policy, it is clear that similar observations as the ones we made also hold for any other strategy that adapts the number of infected people. We suggest that people wear masks all the time, even if they have been infected and recovered or if they have been vaccinated. In fact, although approaches of solutions are being obtained, different variants of the virus are springing out, and the available solutions may not be efficient against them.

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On solutions of the Diophantine equation $L_n \pm L_m = p^a$

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Abstract. Lucas sequence is one of the most studied binary recurrence sequence defined by the relation $L_{n+2} = L_{n+1} + L_n;\ L_0 = 2, L_1 = 1$. In this paper, we investigate all the sums and differences of two Lucas numbers that are powers of a odd prime p satisfying $p < 10^3$.

AMS Subject Classifications: 11B39, 11D72, 11J86.

Keywords: Fibonacci numbers, Lucas numbers, linear forms in logarithms, Baker's method, reduction method.

Contents

1. Introduction

The Fibonacci and Lucas numbers are always being studied by many researchers whether as solutions of Diophantine equations or their existence in the nature. The Fibanacci (F_n) and Lucas (L_n) numbers are the most common binary recurrence sequences definded by the relations:

$$F_{n+2} = F_{n+1} + F_n;\ L_{n+2} = L_{n+1} + L_n$$

with the initial values $F_0 = 0, L_0 = 2, F_1 = L_1 = 1$. Both the sequences have the characteristic equation $x^2 - x - 1 = 0$ with the characteristic roots $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The closed form or the binet form of these numbers are given by:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n; \quad n \geq 0. \tag{1.1}$$

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Bugeaud et al. [5] investigated the Diophantine equations $F_n = y^p$ and $L_n = y^p$ and determined all perfect powers in Fibonacci and Lucas sequences. Similar Diophantine equations have been tackled by many researcher involving powers of $2, 3, 5$ and the recurrence sequences such as Fibonacci, Lucas, Pell and k-Fibonacci numbers (see [3, 4, 8, 11–13]) .

In this paper, we explore the solutions of the Diophantine equation:

$$L_n \pm L_m = p^a, \tag{1.2}$$

where p is any odd prime and n, m, a are nonnegative integers satisfying $n \geq m$.

2. Preliminaries

This section deals with the basic concepts of algebraic numbers, some results concerning the bounds of linear forms in logarithms and reduction methods from the theory of continued fractions, which plays a vital role during the proof of our main result.

Let γ be an algebraic number of degree d having the minimal polynomial

$$a_0 \prod_{i=1}^{d} (x - \gamma_i) \in \mathbb{Z}[x],$$

where γ^i are conjugates of γ and $a_0 > 0$. If $\gamma \neq 0$, then its absolute logarithmic height is defined as

$$h(\gamma) = \frac{1}{d}(\log |a_0| + \sum_{i=1}^{d} \log max\{1, |\gamma_i|\}).$$

The following properties of the logarithmic height holds, which will be used in the forthcoming sections as and when necessary with or without any further references:

- $h(\gamma \pm \eta) \leq h(\gamma) + h(\eta) + \log 2$
- $h(\gamma\eta^{\pm 1}) \leq h(\gamma) + h(\eta)$
- $h(\gamma^s) = |s|h(\gamma); \quad s \in \mathbb{Z}.$

2.1. Inequalities involving the Lucas numbers

Inequalities involving the Lucas numbers In this section, we state and prove important inequalities associated with the Lucas numbers that will be used in solving the equation 1.2

Proposition 2.1 (P. Tiebekabe and I. Diouf [12])**.**
For $n \geq 2$, we have

$$0.94\alpha^n < (1 - \alpha^{-6})\alpha^n \leq L_n \leq (1 + \alpha^{-4})\alpha^n < 1.15\alpha^n \tag{2.1}$$

Proof.
This follows directly from the formula $L_n = \alpha^n + (-1)^n\alpha^{-n}$. ■

Proposition 2.2. *[5]*
The only prime powers in Fibonacci and Lucas sequences are

$$F_1 = F_2 = 1,\ F_6 = 2^3,\ L_1 = 1, L_3 = 2^2.$$

2.2. Linear forms in logarithms and continued fractions

In order to prove our main result, we have to use a Baker-type lower bound several times for a non-zero linear forms of logarithms in algebraic numbers. There are many of these methods in the literature like that of Baker and Wüstholz in [1]. We recall the result of Bugeaud, Mignotte, and Siksek which is a modified version of the result of Matveev [10]. With the notation of section 2, Laurent, Mignotte, and Nesterenko [9] proved the following theorem:

Theorem 2.3.
Let γ_1, γ_2 be two non-zero algebraic numbers, and let $\log\gamma_1$ and $\log\gamma_2$ be any determination of their logarithms. Put $D = [\mathbb{Q}(\gamma_1,\gamma_2):\mathbb{Q}]/[\mathbb{R}(\gamma_1,\gamma_2):\mathbb{R}]$, and

$$\Gamma := b_2\log\gamma_2 - b_1\log\gamma_1,$$

where b_1 and b_2 are positive integers. Further, let A_1, A_2 be real numbers > 1 such that

$$\log A_i \geq max\left\{h\left(\gamma_i\right), \frac{|\log\gamma_i|}{D}, \frac{1}{D}\right\}, (i=1,2).$$

Then assuming that γ_1 and γ_2 are multiplicatively independent, we have

$$\log|\Gamma| > -30.9\cdot D^4\left(max\{\log b', \frac{21}{D}, \frac{1}{2}\}\right)^2\log A_1\cdot\log A_2,$$

where

$$b^{'} = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

We shall also need the following theorem due to Mantveev, Lemma due to Dujella and Pethő and Lemma due to Legendre [7, 10].

Theorem 2.4 (Matveev [10])**.**
Let $n \geq 1$ an integer. Let $\mathbb{L}$ a field of algebraic number of degree D. Let $\eta_1, \eta_2, ..., \eta_l$ non-zero elements of $\mathbb{L}$ and let $b_1, b_2, ..., b_l$ integers,

$$B := max\left\{|b_1|, |b_2|, ..., |b_l|\right\},$$

and

$$\Lambda := \eta_1^{b_1}...\eta_l^{b_l} - 1 = \left(\prod_{i=1}^{l}\eta_i^{b_i}\right) - 1.$$

Let $A_1, A_2, ..., A_l$ reals numbers such that

$$A_j \geq max\left\{Dh(\eta_j), |\log(\eta_j)|, 0.16\right\}, 1 \leq j \leq l.$$

Assume that $\Lambda \neq 0$, so we have

$$\log|\Lambda| > -3\times 30^{l+4}\times(l+1)^{5.5}\times d^2\times A_1...A_l(1+\log D)(1+\log nB)$$

Further, if $\mathbb{L}$ is real, then

$$\log|\Lambda| > -1.4\times 30^{l+3}\times(l)^{4.5}\times d^2\times A_1...A_l(1+\log D)(1+\log B).$$

During our calculations, we get upper bounds on our variables which are too large, so we have to reduce them. To do this, we use some results from the theory of continued fractions. In particular, for a non-homogeneous linear form with two integer variables, we use a slight variation of a result due to Dujella and Pethő, (1998) which is in itself a generalization of the result of Baker and Davemport [2].

For a real number X, we write $||X|| := min\left\{|X-n| : n \in \mathbb{Z}\right\}$ for the distance of X to the nearest integer.

Lemma 2.5 (Dujella and Pethő, [7]).
Let M a positive integer, let p/q the convergent of the continued fraction expansion of k such that $q > 6M$ and let A, B, μ real numbers such that $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\kappa q||$. If $\epsilon > 0$ then there is no solution of the inequality

$$0 < m\kappa - n + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log (Aq/\epsilon)}{\log B} \leq m \leq M.$$

Lemma 2.6 (Legendre).
Let τ real number such that x, y are integers such that

$$|\tau - \frac{x}{y}| < \frac{1}{2y^2}.$$

then $\frac{x}{y} = \frac{p_k}{q_k}$ is the convergence of τ.

Further

$$|\tau - \frac{x}{y}| < \frac{1}{(q_{k+1} + 2)y^2}.$$

3. Main Results

This section deals with the main findings of the following Diophantine equation.

Theorem 3.1.
The only solutions (n, m, a) of the exponential Diophantine equation (1.2) in non negative integers n, m, a and odd prime p are listed in Table 1 and Table 2.

p	(n, m, a)
3	$(3,1,1), (4,3,1)(5,0,2)$
5	$(4,0,1), (7,3,2), (8,6,2)$
7	$(5,3,1), (6,5,1)$
11	$(6,4,1), (7,6,1)$
29	$(8,6,1), (9,8,1)$
47	$(9,7,1), (10,9,1)$
199	$(12,10,1), (13,12,1)$
521	$(14,12,1), (15,14,1)$

Table 1: $L_n - L_m = p^a$.

p	(n, m, a)
3	$(1,0,1), (4,0,2)$
5	$(2,0,1), (3,1,1), (6,4,2)$
7	$(3,2,1)$
11	$(4,3,1)$
29	$(6,5,1)$
47	$(7,6,1)$
199	$(10,9,1)$
521	$(12,11,1)$

Table 2: $L_n + L_m = p^a$.

Corollary 3.1.
The only solutions (p, a) of the double exponential Diophantine equations $L_u - L_v = L_s + L_t$ in non negative intergers $u,\ v,\ s$ and t with $u > v$ are liste in Table 3.

(p,a)	$L_n \pm L_m$
$(3,1)$	$L_1 + L_0,\ L_3 - L_1,\ L_4 - L_3$
$(5,1)$	$L_2 + L_0,\ L_3 + L_1,\ L_4 - L_0$
$(7,1)$	$L_3 + L_2,\ L_5 - L_3,\ L_6 - L5$
$(3,2)$	$L_4 + L_0,\ L_5 - L_0$
$(11,1)$	$L_4 + L_3,\ L_6 - L_4,\ L_7 - L_6$
$(5,2)$	$L_6 + L_4,\ L_7 - L_3$
$(29,1)$	$L_6 + L_5,\ L_8 - L_6,\ L_9 - L_8$
$(47,1)$	$L_7 + L_6,\ L_9 - L_7,\ L_{10} - L_9$
$(199,1)$	$L_{10} + L_9,\ L_{12} - L_{10}, L_{13} - L_{12}$
$(521,1)$	$L_{12} + L_{11},\ L_{14} - L_{12},\ L_{15} - L_{14}$

Table 3: $L_n \pm L_m = p^a$.

Solutions in Table 3 are intersections of those in Table 1 and Tabler 2.

Proof of theorem 3.1.
It is obvious that, the case $n = m$ is not possible. Therefore, we assume that $n > m$. A computation using *SageMath* in the range $0 \le m < n \le 200$ reveals that there does not exist any solution of (1.2) other than the solutions listed in Table 1. Furthermore, it is easy to observe that when $1 \le (n - m) \le 3$, $L_n \pm L_m$ results either in $L_k, 2L_k$ or $5F_k$ for some values of k and hence, using Proposition 2.2 we obtain the solutions of (1.2). So from now on, we assume that $n > 200$ and $(n - m) \ge 4$.

Combining (1.1), (1.2) and (2.1) we get:

$$p^a = L_n \pm L_m \le L_n + L_m \le \alpha^{n+1} + \alpha^{m+1} < 2\alpha^{n+1} < 2^{n+2}.$$

Applying logarithms on both sides of the above inequality, we obtain

$$a \log p \le (n+2) \log 2 \implies a \le (n+2) \frac{\log 2}{\log p}.$$

It is easy to observe that for any prime p, $0 < \frac{\log 2}{\log p} < 4/5$ and hence, $a \le n + 1$. Indeed, for all $n > 200$ and any prime p, $a < n$. Using (1.1) in (1.2) we can obtain the inequality:

$$L_n \pm L_m = \alpha^n + \beta^n \pm L_m = p^a \implies \alpha^n - p^a = -\beta^n \pm L_m.$$

Taking absolute value both sides, we get

$$|\alpha^n - p^a| = |\beta^n \pm L_m| \le |\beta|^n + L_m < \frac{1}{2} + 2\alpha^m$$

$\because |\beta|^n < \frac{1}{2}$, and $L_m < 2\alpha^m$. Dividing both sides by α^n and considering that $n > m$, we get:

$$|1 - \alpha^{-n} \cdot p^a| < \frac{\alpha^{-n}}{2} + 2\alpha^{m-n} < \frac{1}{\alpha^{n-m}} + \frac{2}{\alpha^{n-m}} \because \frac{1}{2\alpha^n} < \frac{1}{\alpha^{n-m}}; n > m.$$

Hence

$$|1 - \alpha^{-n} \cdot p^a| < \frac{3}{\alpha^{n-m}} \tag{3.1}$$

To apply Theorem 2.4, we take $\Gamma := \alpha^{-n} \cdot p^a - 1$ with $\eta_1 = \alpha, \eta_2 = p, b_1 = -n, b^2 = a$. The logarithm heights of η_1 and η_2 are:
$h(\eta_1) = \frac{1}{2}\log\alpha = 0.2406\ldots, h(\eta_2) = \log p$, thus we can choose

$$A_1 := 0.5 \text{ and } A_2 := 2\log p, B := \max\{1, n, a\} = n$$

Using Theorem 2.4, we have

$$\log|\Gamma| > -1.4 \times 30^{2+3} \times 2^{4.5} \times 2^2 \times 0.5 \cdot 2\log p \cdot (1 + \log 2)(1 + \log n),$$

which when combined with (3.1) gives

$$(n-m)\log\alpha < 6.23 \cdot 10^9 \log p \cdot (1 + \log n). \tag{3.2}$$

We define a second linear form in logarithm by rewriting (1.2) as follows:

$$\alpha^n \left(1 \pm \alpha^{m-n}\right) - p^a = -\beta^n \mp \beta^m.$$

Taking absolute values in the above relation with the fact that $|\beta| < 1$, we get

$$|\alpha^n \left(1 \pm \alpha^{m-n}\right) - p^a| < 2, \quad \forall n > 200, m \geq 0.$$

Dividing both sides of the above inequality by $\alpha^n \left(1 + \alpha^{m-n}\right)$, we obtain

$$|1 - p^a\alpha^{-n}\left(1 \pm \alpha^{m-n}\right)^{-1}| < \frac{2}{\alpha^n}\ . \tag{3.3}$$

We define

$$\Lambda := p^a\alpha^{-n}\left(1 \pm \alpha^{m-n}\right)^{-1} - 1$$

and take

$$t := 3, \gamma_1 := p, \gamma_2 := \alpha, \gamma_3 := 1 + \alpha^{m-n}, b_1 := a; b_2 := -n, b_3 = -1.$$

As before, $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ contains $\gamma_1, \gamma_2, \gamma_3$ and has $D := [\mathbb{K} : \mathbb{Q}] = 2$. If $\Lambda = 0$, then

$$p^a = \alpha^n \pm \alpha^m,$$

which is not possible for $n > m$. Therefore $\Lambda \neq 0$.

Let us now estimate $h(\gamma_3)$ where $\gamma_3 = 1 \pm \alpha^{m-n}$

$$\gamma_3 = 1 \pm \alpha^{m-n} < 2 \ \ and \ \ \gamma_3^{-1} = \frac{1}{1 + \alpha^{m-n}} < \frac{5}{2}$$

so $|\log\gamma_3| < 1$. Notice that

$$h(\gamma_3) \leq |m - n|\left(\frac{\log\alpha}{2}\right) + \log 2 = \log 2 + (n - m)\left(\frac{\log\alpha}{2}\right)$$

Proceeding as before, we take

$$A_1 := 2\log p, \quad A_2 := 0.5$$

and we can take

$$A_3 := 2 + (n-m)\log\alpha \text{ since } h(\gamma_3) := \log 2 + (n-m)\left(\tfrac{\log\alpha}{2}\right)$$

Recalling, $a < (n+2)\frac{\log 2}{\log p} < n$, it follows that, $B = \max\{1, n, a\}$. Thus we can take $B = n$. The Matveev's theorem gives the lower bound on the left hand side of (3.3) by replacing the data. We get:

$$\exp\left(-C\left(1+\log n\right)\cdot 2\log p\cdot 0.5\cdot\left(2+(n-m)\log\alpha\right)\right) < |\Lambda| < \frac{2}{\alpha^n}$$

which leads to

$$n\log\alpha - \log 2 < C((1+\log n)\cdot\log p\cdot(2+(n-m)\log\alpha) < 2C\log n\cdot\log p\cdot(2+(n-m)\log\alpha),$$

where $C := 1.4\times 30^{3+3}\times 3^{4.5}\times 2^2(1+\log 2) < 9.7\times 10^{11}$. Then

$$n\log\alpha - \log 2 < 1.94\times 10^{12}\log n\log p\cdot(2+(n-m)\log\alpha) \tag{3.4}$$

where we used inequality $1+\log n < 2\log n$, which holds for $n > 200$. Now, using (3.2) in the right term of the above inequality (3.4) and doing the related calculations, we get

$$n < 5.05\times 10^{22}\log^2 n\log^2 p. \tag{3.5}$$

Hence,

$$n < 2.1\times 10^{26}\log^2 p.$$

All the calculations done so far can be summarized in the following lemma.

Lemma 3.2.
If (n, m, p, a) *is a solution in positive integers of* (1.2) *with conditions* $n > m$ *and* $n > 200$*, then inequalities*

$$a \leq n+2 < 2.11\times 10^{26}\log^2 p$$

hold.

4. Reducing of the bound on n

Rewriting (3.1) as

$$|1 - e^{a\log p - n\log\alpha}| < \frac{3}{\alpha^{n-m}}$$

and using the fact that $|\Lambda| < 2|e^{\Lambda}-1|$ whenever $|e^{\Lambda}-1| < \frac{1}{2}$, we obtain the inequality

$$0 < |a\log p - n\log\alpha| < \frac{3}{\alpha^{n-m}}$$

for all $(n-m)\geq 4$. Dividing the above inequality by $\log\alpha$, we get

$$0 < |a\gamma_p - n| < \frac{7}{\alpha^{n-m}};\ \text{where}\ \gamma_p := \frac{\log p}{\log\alpha} \tag{4.1}$$

We run a computer program to find the continued fraction $[a_0, a_1, a_2, ...]$ of the irrational number γ_p. Let p_k/q_k denotes the k^{th} convergent of γ_p. For each prime p, we compute the denominators $q_k(p)$ and $q_{k+1}(p)$ of the convergents of γ_p such that $q_k(p) < 2.11\times 10^{26}\log^2 p < q_{k+1}(p)$ and find $a_M(p) := \max\{a_i | i = 0, 1, ..., k+1\}$. Therefore, taking a_M to be the maximum of all $a_M(p)$, we get $a_M = 130620$.

Now applying Lemma 2.6 and properties of continued fractions, we obtain

$$|a\gamma_p - n| > \frac{1}{(a_M+2)a}. \tag{4.2}$$

Combining equation (4.1) and (4.2), we get

$$\frac{1}{(a_M+2)a} < |a\gamma_p - n| < \frac{7}{\alpha^{n-m}} \implies \frac{1}{(a_M+2)a} < \frac{7}{\alpha^{n-m}}$$
$$\implies \alpha^{n-m} < 7 \cdot (a_M+2)a < 1.93 \times 10^{32} \log^2 p < 9.21 \times 10^{33}.$$

Applying $\log$ above and divide by $\log \alpha$, we get:

$$(n-m) \leq \frac{\log(9.21 \times 10^{33})}{\log \alpha} < 163.$$

To improve the upper bound on n, let consider

$$z := a\log p - n\log\alpha - \log\rho(u) \text{ where } \rho = 1 \pm \alpha^{-u}. \tag{4.3}$$

From (3.3), we have

$$|1-e^z| < \frac{2}{\alpha^n}. \tag{4.4}$$

Since $\Lambda \neq 0$, then $z \neq 0$. Two cases arise: $z < 0$ and $z > 0$. for each case, we will apply Lemma 2.5.

- Case 1: $z > 0$ From (4.4) , we obtain $0 < z \leq e^z - 1 < \frac{2}{\alpha^n}$. Replacing (4.3) in the above inequality, we get :

  $$0 < a\log p - n\log\alpha - \log\rho(n-m) \leq p^a\alpha^{-n}\rho(n-m)^{-1} - 1 < 2\alpha^{-n}$$

  hence

  $$0 < a\log 3 - n\log\alpha - \log\rho(n-m) < 2\alpha^{-n}$$

  and by dividing above inequality by $\log\alpha$

  $$0 < a\left(\frac{\log p}{\log\alpha}\right) - n - \frac{\log\rho(n-m)}{\log\alpha} < 5 \cdot \alpha^{-n}. \tag{4.5}$$

  Taking, $\gamma_p := \frac{\log p}{\log\alpha}$, $\mu := -\frac{\log\rho(n-m)}{\log\alpha}$, $A := 5$, $B := \alpha$, inequality (4.5) becomes

  $$0 < a\gamma_p - n + \mu < AB^{-n}.$$

  Since γ_p is irrational, we are now ready to apply Lemma 2.5 of Dujella and Petho on (4.5) for $n - m \in \{4, 5, ..., 163\}$. Since $a \leq 2.11 \times 10^{26} \log^2 p$ from Lemma 3.2, we can take $M = 2.55 \times 10^{27}$, and we get

  $$n < \frac{\log(Aq_p/\epsilon)}{\log B}$$

  where $q_p > 6M$ and q_p is the denominator of the convergent of the irrational number γ_p such that $\epsilon_p := ||\mu q_p|| - M||\gamma q_p|| > 0$.

  With the help of *SageMath*, with conditions $z > 0$, and (n, m, a) a possible zero of (1.2) , we get $n < 143$ which contradicts our assumption $n > 200$. Then it is false.

- Case 2: $z < 0$ Since $n > 200$, then $\frac{2}{\alpha^n} < \frac{1}{2}$. Hence (4.4) implies that $|1 - e^{|z|}| < 2$. Also, since $z < 0$, we have

  $$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^{|z|} - 1| < \frac{4}{\alpha^n}.$$

  Replacing (4.3) in the above inequality and dividing by $\log p$, we get:

  $$0 < n\left(\frac{\log\alpha}{\log p}\right) - a + \frac{\rho(n-m)}{\log p} < \frac{4}{\log p} \cdot \alpha^{-n} < 4 \cdot \alpha^{-n} \tag{4.6}$$

In order to apply Lemma 3.2 on (4.6) for $n - m \in \{4, 5, ..., 111\}$, we take $M = 2.55 \times 10^{27}$. With the help of *SageMath*, with conditions $z < 0$, and (n, m, a) a possible zero of (1.2), we get $n < 143$ which contradicts our assumption $n > 200$. Then it is false.

This completes the proof of our main result. ■

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θ_f-Approximations via fuzzy proximity relations: Semigroups in digital images

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Abstract. This article introduces θ_f-approximations of sets in fuzzy proximal relator space where $\theta \in [0,1)$. θ_f-approximation provides a more sensitive approach for the upper approximations of subsets or subimages. θ_f-approximation of a subimage are given with an example in digital images. Furthermore, θ_f-approximately groupoid and semigroup in fuzzy proximal relator space are introduced.

AMS Subject Classifications: 08A05, 68Q32, 54E05.

Keywords: Proximity space, relator space, descriptive approximation, approximately semigroup.

Contents

1. Introduction

Efremovič discovered the proximity spaces in 1951 [2]. He defined proximity space using with proximity relation for proximity of arbitrary subsets of a set. In [10], one can find a list of publications on proximity spaces. A proximity measure is a measure of the closeness or nearness between two nonempty sets.

A relator is a set of binary relations on a nonempty set X that is denoted by $\mathcal{R}$. A relator space is defined as the pair $(X, \mathcal{R})$. In 2016, Peters introduced the concept of proximal relator space $(X, \mathcal{R}_\delta)$ where $\mathcal{R}_\delta$ is a family of proximity relations on X [14].

Zadeh defined fuzzy sets in 1965, which he interpreted as a generalization of set. A fuzzy set A in a universe X is a mapping $A : X \to [0,1]$ [23]. For some applications of fuzzy sets please see [16, 17, 19]. Fuzzy similarity measure between fuzzy sets are given in [22]. Fuzzy similarity measure between sets using with fuzzy proximity relation $\mu_{\mathcal{R}}$ and fuzzy proximal relator space $(X, \mu_{\mathcal{R}})$ are introduced in [11]. Studies in the field of algebraic topology were also discussed with a different perspective on these issues, and semitopological δ-groups were published in 2023 [7].

Fuzzy similarity measures and fuzzy proximity relations are useful tools for applications in the applied sciences such as digital image processing and computer vision. A digital image endowed with fuzzy proximity

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relation $\mu_{\mathcal{R}}$ is fuzzy proximal relator space. Therefore one can working on pixels in digital images to obtain classifications or approximations.

Let $A \subseteq X$. A descriptively upper approximation of A is defined with

$$\Phi^* A = \{x \in X \mid x\delta_\Phi A\}.$$

It means, descriptively upper approximation of A consists of all elements in X that only have exactly the same properties as elements in A. But sometimes more sensitive calculations may be needed.

For more sensitive approach to compute upper approximations of subsets, we can also consider elements with somewhat similar properties, even if they do not have the same properties. To do this, the concept of fuzzy set is one of the most effective mathematical tool. Therefore, θ_f-approximations of sets in fuzzy proximal relator space are developed. Main advantages of this study is that it effectively uses the concepts of fuzzy sets, proximity relations and upper approximation of sets together.

In section 2, definitions of Efremovič proximity, set description, descriptively near sets, descriptively upper approximation of sets, fuzzy proximity relation and fuzzy proximal relator space are given.

In section 3, θ_f-approximations of sets in fuzzy proximal relator space are introduced, where $\theta \in [0, 1)$. θ_f-approximation provides a more sensitive approach for the upper approximations of subsets or subimages. θ_f-approximation of a subimage are given with an example in digital images. Furthermore, θ_f-approximately groupoid and semigroup in fuzzy proximal relator space are introduced.

2. Preliminaries

Definition 2.1. *[2, 3] Let X be a nonempty set and δ be a relation on $P(X)$. δ is called an Efremovič proximity that satisfy following axioms:*

(A_1) *$A\ \delta\ B$ implies $B\ \delta\ A$,*

(A_2) *$A\ \delta\ B$ implies $A \neq \emptyset$ and $B \neq \emptyset$,*

(A_3) *$A \cap B \neq \emptyset$ implies $A\ \delta\ B$,*

(A_4) *$A\ \delta\ (B \cup C)$ iff $A\ \delta\ B$ or $A\ \delta\ C$,*

(A_5) *$\{x\}\ \delta\ \{y\}$ iff $x = y$,*

(A_6) *$A\ \underline{\delta}\ B$ implies $\exists E \subseteq X$ such that $A\ \underline{\delta}\ E$ and $E^c\ \underline{\delta}\ B$*

for all $A, B, C \in P(X)$ and all $x, y \in X$. Efremovič proximity relation is denoted by δ_E.

Definition 2.2. *[9] Let X be a nonempty set and δ be a relation on $P(X)$. δ is called a Lodato proximity that satisfy the axioms $(A_1) - (A_5)$ and*

(A_7) $A\ \delta\ B$ and $\{b\}\ \delta\ C$ $(\forall b \in B)$ implies $A\ \delta\ C$ for all $A, B, C \in P(X)$. Lodato proximity relation is denoted by $\delta_{\mathcal{L}}$.

Let X be a nonempty set and $\mathcal{R}$ be a set of relations on X. $\mathcal{R}$ and $(X, \mathcal{R})$ is called a relator and a relator space, respectively [20]. Let $\mathcal{R}_\delta$ be a family of proximity relations on X. Then $(X, \mathcal{R}_\delta)$ is a proximal relator space. As in [14], $\mathcal{R}_\delta$ contains proximity relations such as basic proximity $\delta_{\mathcal{B}}$ [18], Efremovič proximity δ_E [2, 3], Lodato proximity $\delta_{\mathcal{L}}$ [9], Wallman proximity δ_ω [21], descriptive proximity δ_Φ [12, 15].

In a discrete space, a non-abstract point has a location and features. Features can be measured using probe functions [8]. Let X be a nonempty set of non-abstract points in a proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$.

In this space, a function $\Phi : X \to \mathbb{R}^n$, $\Phi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$ is an object description represents a feature vector of $x \in X$ where each $\varphi_i : X \to \mathbb{R}$ is a probe function $(1 \leq i \leq n)$ that describes feature of a non-abstract point such as pixel in a digital image.

Throughout this work, nonempty set of non-abstract points X was considered. Efremovič proximity δ_E [3] and descriptive proximity δ_Φ in defining a descriptive proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$ were considered. Also, instead of the notions proximal relator space and fuzzy proximal relator space, the terms PR-space and FPR-space were used briefly, respectively.

Definition 2.3. *[10] Let Φ be an object description and $A \subseteq X$. Then the set description of A is defined as*

$$\mathcal{Q}(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2.4. *[10, 13] Let $A, B \subseteq X$. Then the descriptive (set) intersection of A and B is defined as*

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

Definition 2.5. *[12] Let $\delta_\Phi \in \mathcal{R}_{\delta_\Phi}$ and $A, B \subseteq X$. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset$, then A is called a descriptively near B and denoted by $A\delta_\Phi B$. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) = \emptyset$, then $A \underline{\delta}_\Phi B$ reads A is descriptively far from B.*

Definition 2.6. *[22] Let X be an universal set and $\mathcal{F}(X)$ be a class of all fuzzy sets of X. A function $\mu : \mathcal{F}(X) \times \mathcal{F}(X) \longrightarrow [0,1]$ is called a fuzzy similarity measure if the following axioms satisfy:*
(μ_1) $\mu(A, \emptyset) = 0$ and $\mu(A, A) = 1$,
(μ_2) $\mu(A, B) = \mu(B, A)$,
(μ_3) $A \subseteq B \subseteq C$ implies $\mu(A, B) \geq \mu(A, C)$ and $\mu(B, C) \geq \mu(A, C)$
for all $A, B, C \in \mathcal{F}(X)$.

Definition 2.7. *[11] Let $(X, \mathcal{R})$ be a PR-space,*

$$\begin{array}{rll} \mu_{\mathcal{R}}: & \mathcal{P}(X) \times \mathcal{P}(X) & \longrightarrow [0,1] \\ & (A, B) & \longmapsto \mu_{\mathcal{R}}(A, B) \end{array}$$

be a fuzzy relation and $A, B \subseteq X$. Then $\mu_{\mathcal{R}}$ is called a fuzzy proximity relation if it satisfies the following axioms:
$(\mu_{\mathcal{R}})_1$ $\mu_{\mathcal{R}}(A, \emptyset) = 0$,
$(\mu_{\mathcal{R}})_2$ $\mu_{\mathcal{R}}(A, B) = \mu_{\mathcal{R}}(B, A)$,
$(\mu_{\mathcal{R}})_3$ $\mu_{\mathcal{R}}(A, B) \neq 0$ implies A is fuzzy proximal to B,
$(\mu_{\mathcal{R}})_4$ $\mu_{\mathcal{R}}(A, B \cup C) \neq 0$ implies $\mu_{\mathcal{R}}(A, B) \neq 0$ or $\mu_{\mathcal{R}}(A, C) \neq 0$
for all $A, B, C \in \mathcal{P}(X)$.
The set of all fuzzy proximity relations on $\mathcal{P}(X)$ is denoted by $\mathcal{P}_{\mu_{\mathcal{R}}}(X)$. Therefore $\mu_{\mathcal{R}}(A, B)$ is called a fuzzy proximity measure of A with B.
If $\mu_{\mathcal{R}}(A, B) > 0$, then A is fuzzy proximal to B. Also, if $\mu_{\mathcal{R}}(A, B) > \theta$, then A is θ-fuzzy proximal to B for $\theta \in (0, 1)$.

Definition 2.8. *[11] Let $(X, \mathcal{R})$ be a PR-space and $\mu_{\mathcal{R}}$ be a fuzzy proximity relation. Then $(X, \mathcal{R}, \mu_{\mathcal{R}})$ is called a FPR-space and shortly denoted by $(X, \mu_{\mathcal{R}})$.*

3. θ_f-Approximations and θ_f-Approximately Semigroups

Definition 3.1. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space and $A \subseteq X$. A θ_f-approximation of A is determined with*

$$A^{\theta}_{\mu_{\mathcal{R}}} = \bigcup_{\mu_{\mathcal{R}}(A,B) > \theta} B,$$

where $B \in \mathcal{P}(X)$ and $\theta \in [0, 1)$.

For clarify the mechanism of θ_f-approximation please see Example 3.3.

Example 3.2. *Let X be a digital image and x, y be pixels of X. Probe functions $\varphi(x) = (R_x, G_x, B_x)$ and $\varphi_y = (R_y, G_y, B_y)$ are represent the RGB codes of pixels x, y. Let*

$$\begin{array}{rl} \mu_{\mathcal{R}} : X \times X & \longrightarrow [0,1] \\ (x, y) & \longmapsto \mu_{\mathcal{R}}(x, y) = \frac{|765 - D_{x,y}|}{765} \end{array}$$

be a fuzzy relation where

$$D_{x,y} = \sqrt{2\,\Delta R^2 + 4\,\Delta G^2 + 3\,\Delta B^2}$$

is a weighted Euclidean distance of pixels with respect to RGB such that $\Delta R = R_x - R_y$, $\Delta G = G_x - G_y$ *and* $\Delta B = B_x - B_y$. *In the definition of* $\mu_{\mathcal{R}}$, 765 *is the maximum value of* $D_{x,y}$.

Furthermore, fuzzy relationship between x *and* $y \cup z$ *means that*

$$\mu_{\mathcal{R}}(x, y \cup z) = \frac{|765 - \min\{D_{x,y}, D_{x,z}\}|}{765}$$

for all $x, y, z \in X$.

Now lets show that $\mu_{\mathcal{R}}$ *is a fuzzy proximity relation.*

$(\mu_{\mathcal{R}})_1$ *Since there is no similarity between* $x \in X$ *and* $\emptyset$, *it is clear that* $\mu_{\mathcal{R}}(x, \emptyset) = 0$.

$(\mu_{\mathcal{R}})_2$ $\mu_{\mathcal{R}}(x, y) = \mu_{\mathcal{R}}(y, x)$ *by* $D_{x,y} = D_{y,x}$ *for all* $x, y \in X$.

$(\mu_{\mathcal{R}})_3$ *Obviously* $\mu_{\mathcal{R}}(x, y) \neq 0$ *implies* A *is fuzzy proximal to* B.

$(\mu_{\mathcal{R}})_4$ *Let* $\mu_{\mathcal{R}}(x, y) = 0$ *and* $\mu_{\mathcal{R}}(x, z) = 0$ *for all* $x, y, z \in X$. *Then* $\mu_{\mathcal{R}}(x, y) = \frac{|765 - D_{x,y}|}{765} = 0$, *that is,* $D_{x,y} = 765$. *Similarly* $D_{x,z} = 765$. *Hence* $D_{x,y} = D_{x,z} = 765$ *and so* $\min\{D_{x,y}, D_{x,z}\} = 765$. *Thus* $\mu_{\mathcal{R}}(x, y \cup z) = \frac{|765 - \min\{D_{x,y}, D_{x,z}\}|}{765} = 0$. *Therefore* $\mu_{\mathcal{R}}(x, y \cup z) \neq 0$ *implies* $\mu_{\mathcal{R}}(x, y) \neq 0$ *or* $\mu_{\mathcal{R}}(x, z) \neq 0$ *for all* $x, y, z \in X$.

Consequently, $\mu_{\mathcal{R}}$ *is a fuzzy proximity relation from Definition 2.7.*

Example 3.3. *Let* X *be a digital image consists of* 16 *pixels as in Fig. 1. Also, digital image* X *endowed with fuzzy proximity relation* $\mu_{\mathcal{R}}$ *from Example 3.2 is a* FPR*-space by Definition 2.8.*

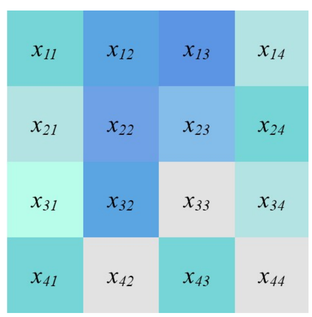

Figure 1: Digital image X

In digital image X, *a pixel* x_{ij} *is an element at position* (i, j) *(row and column). Table 1 lists the RGB codes for each pixel.*

Table 1. RGB codes for each pixel in X.

	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}
Red	117	91	91	180	180	110	132	117
Green	213	165	149	227	227	161	188	213
Blue	215	227	227	228	228	230	234	215

	x_{31}	x_{32}	x_{33}	x_{34}	x_{41}	x_{42}	x_{43}	x_{44}
Red	183	91	226	180	117	226	117	226
Green	253	165	226	227	213	226	213	226
Blue	233	227	226	228	215	226	215	226

From Example 3.2, values of fuzzy proximity relation $\mu_{\mathcal{R}}$ *are given in Table 2.*

Table 2. Values of fuzzy proximity relation $\mu_{\mathcal{R}}$.

$\mu_{\mathcal{R}}$	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	x_{41}	x_{42}	x_{43}	x_{44}
x_{11}	1	0.863	0.823	0.874	0.874	0.859	0.916	1	0.834	0.863	0.794	0.874	1	0.794	1	0.794
x_{12}	0.863	1	0.958	0.769	0.769	0.963	0.902	0.862	0.714	1	0.704	0.769	0.863	0.704	0.863	0.704
x_{13}	0.823	0.958	1	0.738	0.738	0.952	0.872	0.824	0.679	0.958	0.679	0.738	0.874	0.915	0.874	0.915
x_{14}	0.874	0.769	0.738	1	1	0.784	0.864	0.874	0.931	0.769	0.915	1	0.874	0.915	0.874	0.915
x_{21}	0.874	0.769	0.738	1	1	0.784	0.864	0.874	0.931	0.769	0.915	1	0.874	0.915	0.874	0.915
x_{22}	0.859	0.963	0.952	0.784	0.784	1	0.918	0.859	0.724	0.963	0.726	0.784	0.859	0.726	0.859	0.726
x_{23}	0.916	0.902	0.872	0.864	0.864	0.918	1	0.917	0.806	0.902	0.799	0.864	0.917	0.799	0.917	0.799
x_{24}	1	0.862	0.824	0.874	0.874	0.859	0.917	1	0.834	0.863	0.794	0.874	1	0.794	1	0.794
x_{31}	0.834	0.714	0.679	0.931	0.931	0.724	0.806	0.834	1	0.714	0.893	0.931	0.834	0.893	0.834	0.893
x_{32}	0.863	1	0.958	0.769	0.769	0.963	0.902	0.863	0.714	1	0.704	0.769	0.863	0.703	0.863	0.704
x_{33}	0.794	0.704	0.679	0.915	0.915	0.726	0.799	0.794	0.893	0.704	1	0.915	0.778	1	0.794	1
x_{34}	0.874	0.769	0.738	1	1	0.784	0.864	0.874	0.931	0.769	0.915	1	0.874	0.915	0.874	0.915
x_{41}	1	0.863	0.824	0.874	0.874	0.859	0.917	1	0.834	0.863	0.778	0.874	1	0.905	0.884	0.794
x_{42}	0.794	0.704	0.679	0.915	0.915	0.726	0.799	0.794	0.893	0.703	1	0.915	0.905	1	0.794	1
x_{43}	1	0.863	0.824	0.874	0.874	0.859	0.917	1	0.834	0.863	0.794	0.874	0.884	0.794	1	0.794
x_{44}	0.794	0.704	0.679	0.915	0.915	0.726	0.799	0.794	0.893	0.704	1	0.915	0.794	1	0.794	1

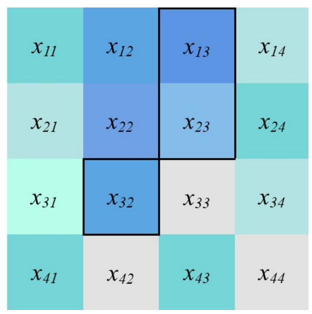

Figure 2: Subimage S

Let $S = \{x_{13}, x_{23}, x_{32}\}$ be a subimage of X as in Fig. 2 and $\theta = 0.92$. From Definition 3.1, θ_f-approximation of S is

$$S^{\theta}_{\mu_{\mathcal{R}}} = \bigcup_{\mu_{\mathcal{R}}(S,x)>\theta} x = \{x_{13}, x_{23}, x_{32}, x_{12}, x_{22}\}$$

where $x \in X$. Hence θ_f-approximation of subimage S consists of θ-fuzzy proximal pixels with S as in Fig. 3.

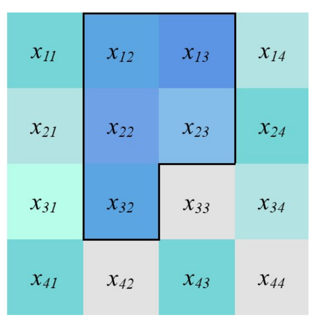

Figure 3: θ_f-approximation of subimage S

Lemma 3.4. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A \subseteq X$ and $\theta \in [0, 1)$. Then the following statements hold:*
(i) $\emptyset^{\theta}_{\mu_{\mathcal{R}}} = \emptyset$,
(ii) $A \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$,
(iii) $X^{\theta}_{\mu_{\mathcal{R}}} = X$.

Proof. It is straightforward. ■

Theorem 3.5. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A, B \subseteq X$ and $\theta \in [0,1)$. Then the following statements hold:*
(i) If $A \subseteq B$, then $B^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$,
(ii) $\left(A^{\theta}_{\mu_{\mathcal{R}}}\right)^{\theta}_{\mu_{\mathcal{R}}} = A^{\theta}_{\mu_{\mathcal{R}}}$,
(iii) $A^{\theta}_{\mu_{\mathcal{R}}} \cap B^{\theta}_{\mu_{\mathcal{R}}} \subseteq (A \cap B)^{\theta}_{\mu_{\mathcal{R}}}$,
(iv) $(A \cup B)^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}} \cup B^{\theta}_{\mu_{\mathcal{R}}}$.

Proof. (i) Let $A \subseteq B$ and $x \in B^{\theta}_{\mu_{\mathcal{R}}}$ where $\theta \in [0,1)$. Then $\mu_{\mathcal{R}}(B,x) > \theta$ and hence $\mu_{\mathcal{R}}(A,x) > \theta$ since $A \subseteq B$. Thus $x \in A^{\theta}_{\mu_{\mathcal{R}}}$. Therefore $B^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$.

(ii) It is clear that $\left(A^{\theta}_{\mu_{\mathcal{R}}}\right)^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$ from (i). Also, $A^{\theta}_{\mu_{\mathcal{R}}} \subseteq \left(A^{\theta}_{\mu_{\mathcal{R}}}\right)^{\theta}_{\mu_{\mathcal{R}}}$ by Lemma 3.4 (ii) and so $\left(A^{\theta}_{\mu_{\mathcal{R}}}\right)^{\theta}_{\mu_{\mathcal{R}}} = A^{\theta}_{\mu_{\mathcal{R}}}$.

(iii) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, from (i) $A^{\theta}_{\mu_{\mathcal{R}}} \subseteq (A \cap B)^{\theta}_{\mu_{\mathcal{R}}}$ and $B^{\theta}_{\mu_{\mathcal{R}}} \subseteq (A \cap B)^{\theta}_{\mu_{\mathcal{R}}}$. Thus $A^{\theta}_{\mu_{\mathcal{R}}} \cap B^{\theta}_{\mu_{\mathcal{R}}} \subseteq (A \cap B)^{\theta}_{\mu_{\mathcal{R}}}$.

(iv) Because of $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $(A \cup B)^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$ and $(A \cup B)^{\theta}_{\mu_{\mathcal{R}}} \subseteq B^{\theta}_{\mu_{\mathcal{R}}}$ from (i). Therefore $(A \cup B)^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}} \cup B^{\theta}_{\mu_{\mathcal{R}}}$. ■

Theorem 3.6. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A_i \subseteq X$ $(i = 1, 2, \cdots, n)$, $n \in \mathbb{N}$ and $\theta \in [0,1)$. Then the following statements hold:*
(i) $\bigcap\limits_i (A_i)^{\theta}_{\mu_{\mathcal{R}}} \subseteq \left(\bigcap\limits_i A_i\right)^{\theta}_{\mu_{\mathcal{R}}}$,
(ii) $\left(\bigcup\limits_i A_i\right)^{\theta}_{\mu_{\mathcal{R}}} \subseteq \bigcup\limits_i (A_i)^{\theta}_{\mu_{\mathcal{R}}}$.

Theorem 3.7. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A \subseteq X$ and $\theta_i \in [0,1)$ $(i = 1, 2, \cdots, n)$, $n \in \mathbb{N}$. Then*
(i) If $\theta_1 > \theta_2$, then $A^{\theta_1}_{\mu_{\mathcal{R}}} \subseteq A^{\theta_2}_{\mu_{\mathcal{R}}}$,
(ii) If $\theta_1 > \theta_2 > \cdots > \theta_n$, then $A^{\theta_1}_{\mu_{\mathcal{R}}} \subseteq A^{\theta_2}_{\mu_{\mathcal{R}}} \subseteq \cdots \subseteq A^{\theta_n}_{\mu_{\mathcal{R}}}$.

Proof. (i) Let $\theta_1 > \theta_2$ and $x \in A^{\theta_1}_{\mu_{\mathcal{R}}}$. Then $\mu_{\mathcal{R}}(A,x) > \theta_1$ and so $\mu_{\mathcal{R}}(A,x) > \theta_2$ since $\theta_1 > \theta_2$. Hence $x \in A^{\theta_2}_{\mu_{\mathcal{R}}}$. Consequently, $A^{\theta_1}_{\mu_{\mathcal{R}}} \subseteq A^{\theta_2}_{\mu_{\mathcal{R}}}$.

(ii) It is easily obtained from (i). ■

Theorem 3.8. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A \subseteq X$, $\theta_i \in [0,1)$, $n \in \mathbb{N}$ and $\bigwedge\limits_i \theta_i = \alpha$, $\bigvee\limits_i \theta_i = \beta$. Then*
(i) $\bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} = A^{\alpha}_{\mu_{\mathcal{R}}}$,
(ii) $\bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} = A^{\beta}_{\mu_{\mathcal{R}}}$.

Proof. (i) Let $x \in \bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}}$. Then $x \in A^{\theta_i}_{\mu_{\mathcal{R}}}$ and so $\mu_{\mathcal{R}}(A,x) > \theta_i$ for at least i. Hence $\mu_{\mathcal{R}}(A,x) > \alpha$ from $\bigwedge\limits_i \theta_i = \alpha$. Thus $x \in A^{\alpha}_{\mu_{\mathcal{R}}}$. Therefore $\bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} \subseteq A^{\alpha}_{\mu_{\mathcal{R}}}$. Similarly, we can show that $A^{\alpha}_{\mu_{\mathcal{R}}} \subseteq \bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}}$. As a results, $\bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} = A^{\alpha}_{\mu_{\mathcal{R}}}$ for all i.

(ii) Let $x \in \bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}}$. Then $x \in A^{\theta_i}_{\mu_{\mathcal{R}}}$ and so $\mu_{\mathcal{R}}(A,x) > \theta_i$ for all i. Hence $\mu_{\mathcal{R}}(A,x) > \beta$ from $\bigvee\limits_i \theta_i = \beta$. Thus $x \in A^{\beta}_{\mu_{\mathcal{R}}}$. Therefore $\bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} \subseteq A^{\beta}_{\mu_{\mathcal{R}}}$. Similarly, we can show that $A^{\beta}_{\mu_{\mathcal{R}}} \subseteq \bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}}$. Consequently, $\bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} = A^{\beta}_{\mu_{\mathcal{R}}}$ for all i. ■

Definition 3.9. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space and let "$\cdot$" be a binary operation defined on X. $G \subseteq X$ is called a θ_f-approximately groupoid in FPR-space if $x \cdot y \in G^{\theta}_{\mu_{\mathcal{R}}}$ for all $x, y \in G$.*

Let we consider G is a θ_f-approximately groupoid with the operation "$\cdot$" in $(X, \mu_{\mathcal{R}})$, $g \in G$ and $A, B \subseteq G$. The subsets $g \cdot A, A \cdot g, A \cdot B \subseteq G^{\theta}_{\mu_{\mathcal{R}}} \subseteq X$ are described as follows:

$$g \cdot A = gA = \{ga | a \in A\},$$

$$A \cdot g = Ag = \{ag | a \in A\},$$

$$A \cdot B = AB = \{ab | a \in A, b \in B\}.$$

Definition 3.10. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, "$\cdot$" be a binary operation on X and $S \subseteq X$. S is named a θ_f-approximately semigroup in FPR-space if the conditions mentioned below are obtained:*
(1) $x \cdot y \in S^{\theta}_{\mu_{\mathcal{R}}}$,
(2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *property satisfy on* $S^{\theta}_{\mu_{\mathcal{R}}}$
for all $x, y, z \in S$.

If θ_f-approximately semigroup has approximately identity element $e_\theta \in S^{\theta}_{\mu_{\mathcal{R}}}$ such that $x \cdot e_\theta = e_\theta \cdot x = x$ for all $x \in S$, then S is called a θ_f-approximately monoid in FPR-space. If $x \cdot y = y \cdot x$ property holds in $S^{\theta}_{\mu_{\mathcal{R}}}$ for all $x, y \in S$, then S is commutative θ_f-approximately semigroup in FPR-space.

Example 3.11. *Assume X is a 16 pixel digital image, as shown in Fig. 1 and $S = \{x_{13}, x_{23}, x_{32}\}$ be a subimage of X. From Example 3.3, θ_f-approximation of S is*

$$S^{\theta}_{\mu_{\mathcal{R}}} = \{x_{13}, x_{23}, x_{32}, x_{12}, x_{22}\}$$

where $\theta = 0.92$.
Let

$$\begin{aligned} \cdot : X \times X &\longrightarrow X \\ (x_{ij}, x_{kl}) &\longmapsto x_{ij} \cdot x_{kl} = x_{pr} \end{aligned}$$

be a binary operation on X such that $p = \min\{i, k\}$ and $r = \min\{j, l\}$.
By Definition 3.10, since
(1) $x_{ij} \cdot x_{kl} \in S^{\theta}_{\mu_{\mathcal{R}}}$,
(2) $(x_{ij} \cdot x_{kl}) \cdot x_{mn} = x_{ij} \cdot (x_{kl} \cdot x_{mn})$ *property satisfy on $S^{\theta}_{\mu_{\mathcal{R}}}$ for all $x_{ij}, x_{kl}, x_{mn} \in S$*
are satisfied, S is indeed a θ_f-approximately semigroup in FPR-space $(X, \mu_{\mathcal{R}})$ with " $\cdot$ ".
Also, since $x_{ij} \cdot x_{kl} = x_{kl} \cdot x_{ij}$ for all $x_{ij}, x_{kl} \in S$ property satisfies in $S^{\theta}_{\mu_{\mathcal{R}}}$, S is a commutative θ_f-approximately semigroup.

Definition 3.12. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $S \subseteq X$ be a θ_f-approximately semigroup and $T \subseteq S$ ($T \neq \emptyset$). T is called a θ_f-approximately subsemigroup if T is a θ_f-approximately semigroup with the operation in S.*

Theorem 3.13. *Let S be a θ_f-approximately semigroup and $T \subseteq S$ ($T \neq \emptyset$). If $T^{\theta}_{\mu_{\mathcal{R}}}$ is a θ_f-approximately groupoid and $T^{\theta}_{\mu_{\mathcal{R}}} \subseteq S^{\theta}_{\mu_{\mathcal{R}}}$, then T is a θ_f-approximately subsemigroup of S .*

Proof. Since $T^{\theta}_{\mu_{\mathcal{R}}}$ is a θ_f-approximately groupoid, thus $x \cdot y \in T^{\theta}_{\mu_{\mathcal{R}}}$ for all $x, y \in T$. Furthermore, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property satisfies on $T^{\theta}_{\mu_{\mathcal{R}}}$ for all $x, y, z \in T$, since S is a θ_f-approximately semigroup and $T^{\theta}_{\mu_{\mathcal{R}}} \subseteq S^{\theta}_{\mu_{\mathcal{R}}}$. Consequently, T is a θ_f-approximately subsemigroup of S. ■

Definition 3.14. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $S \subseteq X$ be a θ_f-approximately semigroup and $I \subseteq S$.*
(1) I is called a θ_f-approximately left ideal of S if $I^{\theta}_{\mu_{\mathcal{R}}}$ is a left ideal of S, i.e., $S(I^{\theta}_{\mu_{\mathcal{R}}}) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$.
(2) I is called a θ_f-approximately right ideal of S if $I^{\theta}_{\mu_{\mathcal{R}}}$ is a right ideal of S, i.e., $(I^{\theta}_{\mu_{\mathcal{R}}})S \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$.
(3) I is called a θ_f-approximately bi-ideal of S if $I^{\theta}_{\mu_{\mathcal{R}}}$ is a bi-ideal of S, i.e., $(I^{\theta}_{\mu_{\mathcal{R}}})S(I^{\theta}_{\mu_{\mathcal{R}}}) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$.

Example 3.15. *In Example 3.11, let we use θ_f-approximately semigroup $S = \{x_{13}, x_{23}, x_{32}\}$. From Definition 3.14, obviously $S \subseteq S$ is a θ_f-approximately left ideal, θ_f-approximately right ideal and also θ_f-approximately bi-ideal of S.*

Theorem 3.16. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space and $S \subseteq X$. If S is a semigroup in X, then S is a θ_f-approximately semigroup in FPR-space.*

Proof. Assume that $S \subseteq X$ be a semigroup. Using Lemma 3.4 (ii), $S \subseteq S^{\theta}_{\mu_{\mathcal{R}}}$ is obtained. Hence $x \cdot y \in S^{\theta}_{\mu_{\mathcal{R}}}$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ condition is also accurate in $S^{\theta}_{\mu_{\mathcal{R}}}$ for all $x, y, z \in S$. After that S is a θ_f-approximately semigroup in FPR-space. ■

The Theorem 3.16 shows that θ_f-approximately semigroup is a generalization of a semigroup.

Theorem 3.17. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space and $S \subseteq X$. If I is a left (right) ideal of θ_f-approximately semigroup S and $\left(S^{\theta}_{\mu_{\mathcal{R}}}\right)\left(I^{\theta}_{\mu_{\mathcal{R}}}\right) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$ ($\left(I^{\theta}_{\mu_{\mathcal{R}}}\right)\left(S^{\theta}_{\mu_{\mathcal{R}}}\right) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$), then I is a θ_f-approximately left (right) ideal of S.*

Proof. Let we consider I be a left ideal of θ_f-approximately semigroup S, that is, $SI \subseteq I$. We know that $S \subseteq S^{\theta}_{\mu_{\mathcal{R}}}$. Hence, from the hypothesis $\left(S^{\theta}_{\mu_{\mathcal{R}}}\right)\left(I^{\theta}_{\mu_{\mathcal{R}}}\right) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$,

$$\begin{aligned} S\left(I^{\theta}_{\mu_{\mathcal{R}}}\right) & \subseteq \left(S^{\theta}_{\mu_{\mathcal{R}}}\right)\left(I^{\theta}_{\mu_{\mathcal{R}}}\right) \\ & \subseteq I^{\theta}_{\mu_{\mathcal{R}}}. \end{aligned}$$

As a results, $I^{\theta}_{\mu_{\mathcal{R}}}$ is a left ideal of S and so I is a θ_f-approximately left ideal of S. Also, It is obviously if I is a right ideal of θ_f*-approximately semigroup* S and $\left(I^{\theta}_{\mu_{\mathcal{R}}}\right)\left(S^{\theta}_{\mu_{\mathcal{R}}}\right) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$, I is a θ_f-approximately right ideal of S. ■

4. Conclusions

This work proposed θ_f-approximations of sets in fuzzy proximal relator space to provide more sensitive approach for approximations or clustering. From the examples and results, it was verified that θ_f-approximation is able to classify the pixels in digital images more precisely according to the selected $\theta \in [0, 1)$. Other results about θ_f-approximately algebraic structures provides a theoretical basis for further studies. Future studies should investigate the performance of this theory with experimental studies in any applied fields.

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The pantograph equation with nonlocal conditions via Katugampola fractional derivative

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Abstract. We study a Pantograph-type equation with Katugampola fractional derivatives. Under nonlocal conditions, we establish some existence and uniqueness results for the problem. Then, some other main results are proved by introducing new definitions related to ULAM stability.

AMS Subject Classifications: 30C45, 39B72, 39B82.

Keywords: Caputo derivative, Lane Emden equation, existence of solutions, Ulam-Hyers stability.

Contents

1. Introduction

It's seen now that technology is a very important matter basis for peoples life, governments systems, specially with the COVID-19 global pandemic happening. As the technology grow faster the need of mathematical modeling grow bigger.

Nowadays, the fractional calculus theory has proven it important use as a tool in modeling many real life problems as energy-saving, national economics growth, Image processing, engineering, biology, physics and fluid dynamics and many other researches area see [9, 12, 20, 26]. The fractional calculus theory is based on the study of partial and ordinary differential equations, where the derivation or the integration operator is of non-integer order α or complex with $Re(\alpha) > 0$. The most three known approaches of operators of fractional calculus theory were given by Gr$\ddot{u}$nwald-Letnikov in 1867; 1868, Riemann-Liouville in 1832; 1847 and Caputo 1967 [15]. The treatment of a fractional differential equation mostly involve the study of the exitance and uniqueness of the solution or only the existence of the solutions also the stability of this solutions is implicated, many scholars has given a widely amount of interesting results in such researches see [2, 4, 6, 8, 11, 16, 22, 28].

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In 1971 Ockendon and Taylor [21] did the research on the way in which the electric current is collected by the pantograph of an electric locomotive using a delay equation

$$\begin{cases} w'(t) = aw(t) + bw(\epsilon t) & 0 \leq t \leq T, 0 < \epsilon < 1, \\ w(0) = w_0, \end{cases}$$

which is now called the Pantograph equation. Since that time many researchers studied and used it in different mathematical and scientific areas as number theory, probability, electrodynamics, medicine, see [21, 25, 27] and the bibliography therein.

A lot of researches have been done on the fractional pantograph equations due to their importance to many areas of research, such as [24] in which K. Balachandran and S. Kiruthika treated the existence of solutions for the following nonlinear fractional pantograph equation:

$$\begin{aligned} &D^{\alpha}u(t) = f(t, u(t), u(\lambda t)), \;\; t \in [0, T] \\ &u(0) = u_0. \end{aligned}$$

Also in [23] Y. Jalilian and M. Ghasemi considered the following fractional integro-differential equation of Pantograph type connected with appropriate initial condition

$$\begin{cases} {}_cD^{\alpha}u(t) = f(t, u(t), u(pt)) + \displaystyle\int_0^{qt} g_1(t, s, u(s))ds \\ + \displaystyle\int_0^t g_2(t, s, u(s))ds, \;\; t \in [0, T] \\ u(0) = u_0. \end{cases}$$

where ${}_cD^{\alpha}$ is the derivative in the sense of Caputo of order $\alpha \in (0, 1]$.
In this paper, we shall study the following nonlinear fractional pantograph problem

$$\begin{cases} {}_cD^{\alpha,\rho}y(t) = f(t, y(t), y(pt)) + g(t, y(t), y((1-p)t)) \\ \\ y(0) - I^{\beta}y(\xi) = 0, \quad 0 < \xi < T, \;\; \alpha \in (0, 1] \; t \in [0, T] \\ \end{cases} \tag{1.1}$$

where ${}_cD^{\alpha,\rho}$ is the Katugampola-type fractional derivative in Caputo sense of order α, $0 < p < 1$, $\rho > 0$, and I^{β} is the integral of order $\beta > 0$, and $f, g : [0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are two given functions.
To the best of our knowledge, this is the first time where such problem is studied.

2. Preliminaries

We recall some definitions and lemmas that will be used later. For more details we refer to $[17 - -19]$.

Definition 2.1. *Let $\alpha > 0$, and $f : [a, b] \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral of order α of f is defined by:*

$$I_a^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau)d\tau,$$

where $\Gamma(\alpha) := \displaystyle\int_0^{\infty} e^{-u}u^{\alpha-1}du.$
In particular when $a = 0$ we denote simply

$$I^{\alpha}f(t) = I_0^{\alpha}f(t)$$

Definition 2.2. *For a function $f \in C^n([a,b],\mathbb{R})$ and $n-1<\alpha\leq n$, the Caputo fractional derivative of f is defined by:*

$$
\begin{aligned}
{}_cD^{\alpha}f(t) &= I_a^{n-\alpha}f^{(n)}(t)\\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds.
\end{aligned}
$$

Definition 2.3. *Let $f:[a,b]\longmapsto\mathbb{R}$ be an integrable function, $\alpha\in(0,1]$ and $\gamma>0$. The Katugampola integral of order α of f is given by*

$$
{}^{\gamma}I_a^{\alpha}f(t)=\frac{\gamma^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^{\gamma}-s^{\gamma})^{-(1-\alpha)}s^{\gamma-1}f(s)ds. \tag{2.1}
$$

When $a=0$ we denote simply

$$
{}^{\gamma}I^{\alpha}f(t)={}^{\gamma}I_0^{\alpha}f(t)
$$

Lemma 2.4. *Let $\alpha>0$, $\beta>0$ such that $\alpha+\beta\leq 1$. Then,*

$$
{}^{\gamma}I_a^{\alpha}\,{}^{\gamma}I_a^{\beta}={}^{\gamma}I_a^{\alpha+\beta} \tag{2.2}
$$

Proof. Let $f:[a,b]\to\mathbb{R}$ be a continues function then for all $\alpha>0,\beta>0$ we have

$$
\begin{aligned}
I_a^{\alpha}\left[I_a^{\beta}[f(t)]\right] &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}I_a^{\beta}[f(s)]ds\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[(t-s)^{\alpha-1}\int_a^s (s-x)^{\beta-1}f(x)dx\right]ds\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\alpha-1}ds\int_a^s (s-x)^{\beta-1}f(x)dx\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)dx\int_x^t (t-s)^{\alpha-1}(s-x)^{\beta-1}ds.
\end{aligned} \tag{2.3}
$$

By changing the variables$s=x+(t-x)\varrho$ and using Beta function we get

$$
\begin{aligned}
I_a^{\alpha}\left[I_a^{\beta}[f(t)]\right] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)dx\int_0^1 (t-x-(t-x)\varrho)^{\alpha-1}\\
&\quad * (x+(t-x)\varrho-x)^{\beta-1}(t-x)d\varrho.\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)(t-x)^{\alpha+\beta-1}dx\int_0^1 (1-\varrho^{\alpha-1})\varrho^{\beta-1}d\varrho\\
&= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)(t-x)^{\alpha+\beta-1}dx\\
&= \frac{1}{\Gamma(\alpha)+\beta)}\int_a^t f(x)(t-x)^{\alpha+\beta-1}dx.\\
&= I_a^{\alpha+\beta}[f(t)].
\end{aligned} \tag{2.4}
$$

■

Definition 2.5. *Let $f:[a,b]\longmapsto\mathbb{R}$ be an integrable function, $\alpha\in(0,1)$ and $\gamma>0$. The Katugampola fractional derivatives of order α of $f(t)$ is defined by*

$$
\begin{aligned}
D_a^{\alpha,\gamma}f(t) &= t^{1-\gamma}\frac{d}{dt}\left({}^{\gamma}I_a^{1-\alpha}f\right)(t)\\
&= \frac{\gamma^{\alpha}}{\Gamma(1-\alpha)}t^{1-\gamma}\frac{d}{dt}\int_a^t (t^{\gamma}-s^{\gamma})^{-\alpha}s^{\gamma-1}f(s)ds.
\end{aligned} \tag{2.5}
$$

In particular when $a = 0$ we denote simply

$$D^{\alpha,\gamma} f(t) = D_0^{\alpha,\gamma} f(t)$$

Definition 2.6. *The Caputo-Katugampola fractional derivatives of order α is defined by*

$$\begin{aligned} {}_cD_a^{\alpha,\gamma} f(t) &= D_a^{\alpha,\gamma}\big[f(t) - f(a)\big] \\ &= \frac{\gamma^\alpha}{\Gamma(1-\alpha)} t^{1-\gamma} \frac{d}{dt} \int_a^t (t^\gamma - s^\gamma)^{-\alpha} s^{\gamma-1} \big[f(s) - f(a)\big] ds. \end{aligned} \tag{2.6}$$

In particular when $a = 0$ we denote simply

$${}_cD^{\alpha,\gamma} f(t) = {}_cD_0^{\alpha,\gamma} f(t)$$

To study (1.1) we need the following lemma

Lemma 2.7. *Let $f \in C^1([a,b])$. Then,*

$${}_cD_a^{\alpha,\gamma} f(t) = \frac{\gamma^\alpha}{\Gamma(1-\alpha)} \int_a^t (t^\gamma - s^\gamma)^{-\alpha} f'(s) ds. \tag{2.7}$$

Proof. If we set for a fixed t,
$u_t(s) = -\frac{1}{\gamma(1-\alpha)}(t^\gamma - s^\gamma)^{1-\alpha}$ and $v(s) = f(s) - f(a)$, then we have $u_t'(s) = s^{\gamma-1}(t^\gamma - s^\gamma)^{-\alpha}$ and $v'(s) = f'(s)$.
Thus, we can write:

$${}_cD_a^{\alpha,\gamma} f(t) = \frac{\gamma^\alpha}{\Gamma(1-\alpha)} t^{1-\gamma} \frac{d}{dt} \int_a^t u_t'(s) v(s) ds,$$

and, by an integration by parts, we have

$${}_cD_a^{\alpha,\gamma} f(t) = -\frac{\gamma^\alpha}{\Gamma(1-\alpha)} t^{1-\gamma} \frac{d}{dt} \int_a^t u_t(s) f'(s) ds,$$

and since $u_t(t) = 0$, we get

$${}_cD_a^{\alpha,\gamma} f(t) = -\frac{\gamma^\alpha}{\Gamma(1-\alpha)} t^{1-\gamma} \int_a^t \frac{\partial}{\partial t}(u_t(s)) f'(s) ds,$$

that corresponds exactly to (2.7). ■

Remark 2.8. *Note that we can rewrite (2.7) in the form*

$${}_cD_a^{\alpha,\gamma} f(t) = {}^\gamma I_a^{1-\alpha}\big(t^{1-\gamma} f'(t)\big). \tag{2.8}$$

Now we have

Lemma 2.9. *Given $f \in C^1([a,b])$, then*

$${}^\gamma I_a^\alpha \, {}_cD_a^{\alpha,\gamma} f(t) = f(t) - f(a).$$

Proof. Indeed, using the formula (2.8), we can write

$${}^\gamma I_a^\alpha \, {}_cD_a^{\alpha,\gamma} f(t) = {}^\gamma I_a^\alpha \; {}^\gamma I_a^{1-\alpha}\big(t^{1-\gamma} f'(t)\big). \tag{2.9}$$

But, thanks to Lemma 2.4, ${}^{\gamma}I_a^{\alpha}\,{}^{\gamma}I_a^{1-\alpha} = {}^{\gamma}I_a^{1}$. Thus,

$$\begin{aligned}{}^{\gamma}I_a^{\alpha}\,{}_cD_a^{\alpha,\gamma}f(t) &= {}^{\gamma}I_a^{1}\big(t^{1-\gamma}f'(t)\big)\\ &= \int_a^t s^{\gamma-1}s^{1-\gamma}f'(s)ds \\ &= f(t)-f(a).\end{aligned} \tag{2.10}$$

■

Let us introduce now the following Lemma:

Lemma 2.10. *Let $F \in C([0,1])$. Then, the problem*

$$\begin{cases} {}_cD^{\alpha,\rho}y(t) = F(t) \;\; \alpha \in \;(0,1]\; t \in [0,T] \\ \\ y(0) - I^{\beta}y(\xi) = 0, \quad 0<\xi<T, \end{cases} \tag{2.11}$$

admits as a solution the function:

$$\begin{aligned} y(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}F(s)ds + \frac{\Gamma(\beta+1)\rho^{1-\alpha}}{\Gamma(\alpha)\Gamma(\beta)(\Gamma(\beta+1)-\xi^{\beta})} \\ \times \int_0^{\xi}(\xi-u)^{\beta-1}\bigg(\int_0^u (u^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}F(s)ds\bigg)du, \end{aligned} \tag{2.12}$$

provided that $T^{\beta} < \Gamma(\beta+1)$.

Proof. Using Lemma 9, we obtain

$$y(t) = {}^{\rho}I^{\alpha}F(t) + y(0). \tag{2.13}$$

Using the boundary condition we get

$$\begin{aligned} y(0) &= I^{\beta}({}^{\rho}I^{\alpha}F(\xi) + y(0)) \\ &= I^{\beta}y(0) + I^{\beta}\,{}^{\rho}I^{\alpha}F(\xi) \\ &= y(0)\frac{\xi^{\beta}}{\beta\Gamma(\beta)} + I^{\beta}\,{}^{\rho}I^{\alpha}F(\xi) \\ &= y(0)\frac{\xi^{\beta}}{\Gamma(\beta+1)} + I^{\beta}\,{}^{\rho}I^{\alpha}F(\xi). \end{aligned} \tag{2.14}$$

Thus,

$$\begin{aligned} y(0) &= \frac{\Gamma(\beta+1)}{(\Gamma(\beta+1)-\xi^{\beta})}I^{\beta}\,{}^{\rho}I^{\alpha}F(\xi) \\ &= \frac{\Gamma(\beta+1)\rho^{1-\alpha}}{\Gamma(\alpha)\Gamma(\beta)(\Gamma(\beta+1)-\xi^{\beta})}\int_0^{\xi}(\xi-u)^{\beta-1} \\ &* \bigg(\int_0^u (u^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}F(s)ds\bigg)du. \end{aligned} \tag{2.15}$$

Finally, inducting (2.15) in (2.13) we obtain (2.12). ■

In the following section we will study of the existence as well as the existence and uniqueness of the solution ([1, 5, 13, 14]), and examine the Ulam-Hyers stability ([3, 7, 10]) for the introduced problem (1)

3. Main Results

We consider the following hypotheses:

$(P1)$: $f, g : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, are continuous.

$(P2)$: There are nonnegative constants L_f and L_g, such that for all t $\in J$, $x_i, x_i^* \in \mathbb{R},\ i = 1, 2$

$$|f(t, x_1, x_2) - f(t, x_1{}^*, x_2{}^*)| \leq L_f \sum_{i=1}^{2} |x_i - x_i{}^*|,$$

$$|g(t, x_1, x_2) - g(t, x_1{}^*, x_2{}^*)| \leq L_g \sum_{i=1}^{2} |x_i - x_i{}^*|.$$

$(P3)$: There exist positive constants λ, δ, that satisfy for all $t \in [0,T]$, and for all $x, x^* \in \mathbb{R}$

$$|f(t, x, x^*)| \leq \lambda, \quad \text{and} \quad |g(t, x, x^*)| \leq \delta.$$

Also, we consider the quantities:

$$A_1 = \frac{2\Gamma(\beta+1)\big(L_f + L_g\big)T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)|\Gamma(\beta+1) - T^\beta|}$$

$$A_2 = \frac{2\big(L_f + L_g\big)T^{\rho\alpha+\beta}}{\rho^\alpha \Gamma(\alpha+1)|\Gamma(\beta+1) - T^\beta|}.$$

3.1. Existence of a unique solution

The first main result deals with the existence of a unique solution for (1.1). We have:

Theorem 3.1. *Assume that* $(P2)$ *is satisfied. Then, the problem* (1.1) *has a unique solution, provided that* $A_1 < 1$ *and* $\Gamma(\beta+1) > T^\beta$.

Proof. Let us introduce the Banach space

$$E := C([0,T], \mathbb{R}), \text{with the norm: } \|x\|_E = \sup_{t \in [0,T]} |x(t)|.$$

Then, we define the nonlinear operator $H : E \to E$ as follows:

$$\begin{aligned} Hy(t) = & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(f(s, y(s), y(ps)) \\ & + g(s, y(s), y((1-p)s)) \Big) ds + \frac{\beta\, \rho^{1-\alpha}}{\Gamma(\alpha)(\Gamma(\beta+1) - \xi^\beta)} \\ & \times \int_0^\xi (\xi - u)^{\beta-1} \Big(\int_0^u (u^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \\ & \times \Big(f(s, y(s), y(ps)) + g(s, y(s), y((1-p)s)) \Big) ds \Big) du. \end{aligned} \tag{3.1}$$

We shall prove that H is a contraction mapping in E.

For $y, x \in E$ and for each $t \in [0, T]$, we have

$$\begin{aligned} & \big|Hy(t) - Hx(t)\big| = \Big| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(f(s, y(s), y(ps)) \\ & f(s, x(s), x(ps) \Big) ds + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(g(s, x(s), x((1-p)s)) \\ & - g(s, y(s), y((1-p)s)) \Big) ds + \frac{\beta\, \rho^{1-\alpha}}{\Gamma(\alpha)|\Gamma(\beta+1) - \xi^\beta|} \int_0^\xi (\xi - u)^{\beta-1} \\ & \times \Big[\int_0^u (u^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(f(s, y(s), y(ps)) - f(s, x(s), x(ps) \\ & + g(s, y(s), y((1-p)s)) - g(s, x(s), x((1-p)s)) \Big) ds \Big] du. \end{aligned} \tag{3.2}$$

Then,

$$\begin{aligned} & \big|Hy(t) - Hx(t)\big| \le (L_f + L_g) \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(|y(s) - x(s)| \\ & + |y(ps) - x(ps)| \Big) ds + \frac{(L_f + L_g)\beta\, \rho^{1-\alpha}}{\Gamma(\alpha)|\Gamma(\beta+1) - T^\beta|} \int_0^\xi (\xi - u)^{\beta-1} \\ & \Big[\int_0^u (u^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(|y(s) - x(s)| + |y(ps) - x(ps)| \Big) ds \Big] du. \end{aligned} \tag{3.3}$$

Hence, a straightforward computation gives

$$\begin{aligned} \big\|Hy - Hx\big\|_E & \le \Big[\frac{2\big(L_f + L_g\big) T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + \frac{2\big(L_f + L_g\big) T^{\rho\alpha+\beta}}{\rho^\alpha \Gamma(\alpha+1)|\Gamma(\beta+1) - T^\beta|} \Big] \big\|y - x\big\| \\ & \le \frac{2\big(L_f + L_g\big) T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \Big(1 + \frac{T^\beta}{|\Gamma(\beta+1) - T^\beta|} \Big) \big\|y - x\big\| \\ & \le \frac{2\big(L_f + L_g\big) \Gamma(\beta+1) T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)|\Gamma(\beta+1) - T^\beta|} \big\|y - x\big\| \end{aligned} \tag{3.4}$$

Consequently,

$$\big\|Hy - Hx\big\|_E \le A_1 \big\|y - x\big\|_E.$$

■

3.2. Existence of at least one solution

The second main result deals with the existence of at least one solution.

Theorem 3.2. *Assume that hypotheses* $(P1), (P2)$ *and* $(P3)$ *are satisfied with* $A_2 < 1$. *Then, the problem* (1.1) *has at least one solution provided that* $\Gamma(\beta+1) > T^{\beta}$.

Proof. We put

$$r \geq \frac{(\lambda+\delta)\Gamma(\beta+1)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)(\Gamma(\beta+1)-T^{\beta})}$$

and consider the ball $B_r := \{x \in E,\ \|x\|_E \leq r\}$.
Then, we define the operators M and N on B_r as:

$$\begin{aligned}(My)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}\Big(f(s,y(s),y(ps)) \\ + g(s,y(s),y((1-p)s))\Big)ds\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}(Ny)(t) = \frac{\beta\,\rho^{1-\alpha}}{\Gamma(\alpha)(\Gamma(\beta+1)-\xi^{\beta})}\int_0^{\xi}(\xi-u)^{\beta-1} \\ \left[\int_0^u (u^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}\Big(f(s,y(s),y(ps)) \right. \\ \left. + g(s,y(s),y((1-p)s))\Big)ds\right]du.\end{aligned} \tag{3.6}$$

For $y, x \in B_r$, we find that

$$\begin{aligned}\|Mx+Ny\|_E &\leq \frac{(\lambda+\delta)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{(\lambda+\delta)T^{\rho\alpha+\beta}}{\rho^{\alpha}\Gamma(\alpha+1)(\Gamma(\beta+1)-T^{\beta})} \\ &\leq \frac{(\lambda+\delta)\Gamma(\beta+1)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)(\Gamma(\beta+1)-T^{\beta})}\end{aligned} \tag{3.7}$$

Then, we can write $\|Mx+Ny\|_E \leq r$. Thus, $\quad Mx+Ny \in B_r$.
Furthermore, for $x, y \in B_r$, we obtain

$$\big\|Nx-Ny\big\|_E \leq A_2\|x-y\|. \tag{3.8}$$

That is to say that N is contractive on B_r.
Now we prove that M is a compact operator on B_r.
We have

$$\begin{aligned}\big\|(My_n)-(My)\big\|_E \leq \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\|f(s,y_n(s),y_n(ps)) \\ - f(s,y(s),y(ps))\| + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \\ \times\|g(s,y_n(s),y_n(1-p)(s)) - g(s,y(s),y(1-p)(s))\|.\end{aligned}$$

Thanks to $(P1)$, and since $s \mapsto y(s)$ is bounded on $[0,T]$, and $\|y_n-y\|_E \to 0$, we reduce the continuity of f and g to a compact set of $[0,T]\times\mathbb{R}^2$, so that we obtain $\|My_n - My\|_E \to 0$.

Also, for $y \in B_r$, we get

$$\|My\|_E \leq \frac{(\lambda+\delta)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < \infty. \tag{3.9}$$

Consequently, M is uniformly bounded on B_r.
Now, we prove that M is equicontinuous. Let $t_1, t_2 \in [0,T]$,
$t_1 < t_2$. Then for $y \in B_r$, we have

$$\begin{aligned}\big|My(t_1) - My(t_2)\big| \leq \bigg|&\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}({t_1}^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}\Big(f(s,y(s),y(ps))\\ &+g(s,y(s),y((1-p)s))\Big)ds - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_2}({t_2}^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}\\ &\times\Big(f(s,y(s),y(ps))+g(s,y(s),y((1-p)s))\Big)ds\bigg|.\end{aligned} \tag{3.10}$$

Hence,

$$\begin{aligned}\big|My(t_1) - My(t_2)\big| \leq \bigg|&\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}({t_1}^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}\Big(f(s,y(s),y(ps))\\ &+g(s,y(s),y((1-p)s))\Big)ds - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}({t_2}^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}\\ &\times\Big(f(s,y(s),y(ps))+g(s,y(s),y((1-p)s))\Big)ds\bigg| * \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\\ &\times\int_{t_1}^{t_2}({t_2}^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}\Big|f(s,y(s),y(ps))+g(s,y(s),y((1-p)s))\Big|ds.\\ &\leq \frac{\rho^{1-\alpha}(\lambda+\delta)}{\Gamma(\alpha)}\int_0^{t_1}\big(({t_2}^{\rho}-s^{\rho})^{\alpha-1}-({t_1}^{\rho}-s^{\rho})^{\alpha-1}\big)s^{\rho-1}\,ds\\ &+\frac{\rho^{1-\alpha}(\lambda+\delta)}{\Gamma(\alpha)}\int_{t_1}^{t_2}({t_2}^{\rho}-s^{\rho})^{\alpha-1}\,s^{\rho-1}ds\end{aligned} \tag{3.11}$$

Then, we get

$$\big|My(t_2) - My(t_1)\big| \leq \frac{(\lambda+\delta)\big({t_2}^{\rho\alpha}-{t_1}^{\rho\alpha}\big)}{\rho^{\alpha}\Gamma(\alpha+1)}. \tag{3.12}$$

The right hand side of (3.12) tends to zero independently of y as $t_1 \to t_2$.
This implies that M is relatively compact, and by the Arzela-Ascoli theorem, we conclude that M is compact on B_r.
Hence, the existence of the solution of the (1.1) holds by Krasnoselskii fixed point theorem. ■

3.3. UH-Stability

Definition 3.3. *The equation* (1.1) *has the* UH *stability if there exists a real number* $k > 0$*, such that for each* $\varepsilon > 0$*, for any* $t \in [0,T]$*, and for each* $x \in E$ *that verify*

$$\Big|{}_cD^{\alpha,\rho}x(t) - f(t,x(t),x(pt)) - g(t,x(t),x((1-p)t))\Big| \leq \varepsilon \tag{3.13}$$

there exists a solution $y \in E$ *of* (1.1)*; that is*

$${}_cD^{\alpha,\rho}y(t) = f(t,y(t),y(pt)) + g(t,y(t),y((1-p)t)) \tag{3.14}$$

such that,

$$\|x-y\|_E \leq k\,\varepsilon.$$

Definition 3.4. *The problem* (1.1) *has the* UH *stability in the generalized sense if there exists* $\phi \in C(\mathbb{R}^+, \mathbb{R}^+)$*, such that* $\phi(0) = 0$ *: for each* $\varepsilon > 0$*, and for any* $x \in E$ *satisfying* (3.13)*, there exists a solution* $y \in E$ *of equation* (1.1)*, such that*

$$\|x - y\|_E < \phi(\varepsilon).$$

Theorem 3.5. *Let the assumptions of Theorem (3.1) hold and* $L'_f + L'_g < 1$*. If the inequality*

$$\begin{aligned}\left\|{}_C D^{\rho,\alpha} x(t)\right\|_E \geq & \frac{\left[2\left(L'_f + L'_g\right)r + \lambda + \delta\right]T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \\ & + \left(\frac{2\Gamma(\beta+1)\left(L'_f + L'_g\right)r + \lambda + \delta}{\rho^\alpha \Gamma(\alpha+1)(\Gamma(\beta+1)+T)}\right) \\ & \times \frac{\Gamma(\rho\alpha+1)T^{\rho\alpha+\beta}}{\Gamma(\rho\alpha+\beta+1)} \end{aligned} \tag{3.15}$$

is valid, then problem (1.1) *has the UH stability.*

Proof. Let $\varepsilon > 0$ and let $x \in E$ be a function which satisfies (3.13) and let $y \in E$ be the unique solution of the equation (1.1). We have:

$$\begin{aligned}\|x\|_E \leq & \frac{\left[2\left(L'_f + L'_g\right)r + \gamma + \delta\right]T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \\ & + \left(\frac{2\Gamma(\beta+1)\left(L'_f + L'_g\right)r + \gamma + \delta}{\rho^\alpha \Gamma(\alpha+1)(\Gamma(\beta+1)+T)}\right) \\ & \times \frac{\Gamma(\rho\alpha+1)T^{\rho\alpha+\beta}}{\Gamma(\rho\alpha+\beta+1)} \end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16), we obtain

$$\|x\|_E \leq \|{}_c D^{\rho,\alpha} x(t)\|_E \tag{3.17}$$

Therefore, we get

$$\begin{aligned}\|x - y\| \leq & \|{}_c D^{\rho,\alpha}\left(x - y\right)\| \\ \leq & \sup_{t \in J} \Big|{}_c D^{\rho,\alpha} x(t) - {}_c D^{\rho,\alpha} y(t) - f(t, x(t), x(pt)) \\ & + g(t, x(t), x((1-p)t)) - f(t, y(t), y(pt)) \\ & + g(t, y(t), y((1-p)t)) + f(t, x(t), x(pt)) \\ & - g(t, x(t), x((1-p)t)) + f(t, y(t), y(pt)) \\ & - g(t, y(t), y((1-p)t))\Big|. \end{aligned} \tag{3.18}$$

Thanks to (3.13) and (3.14), we get

$$\|x - y\| \leq \varepsilon + \left(L'_f + L'_g\right)\|x - y\| \tag{3.19}$$

But since,

$$L'_f + L'_g < 1$$

then, we can write

$$\|x - y\|_E \leq \frac{\varepsilon}{1 - (L'_f + L'_g)} = \varepsilon k. \tag{3.20}$$

Consequently, (1.1) has the UH stability.

Taking $\phi(\varepsilon) = \varepsilon k$, we can state that the equation (1.1) has the generalized UH stability. ■

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The sequence of the hyperbolic k-Padovan quaternions

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Abstract. This work introduces the hyperbolic k-Padovan quaternion sequence, performing the process of complexification of linear and recurrent sequences, more specifically of the generalized Padovan sequence. In this sense, there is the study of some properties around this sequence, deepening the investigative mathematical study of these numbers.

AMS Subject Classifications: 11B37, 11B39.

Keywords: hyperbolic numbers, quaternions, k-Padovan sequence.

Contents

1. Introduction and Background

Studies of recursive linear sequences have been noticed in the mathematical literature. Based on this, there is the concern to carry out an investigative study on the process of complexification of certain sequences. So soon, in this work, the hyperbolic quaternion k-Padovan sequence is introduced, presenting algebraic properties around these numbers.

The Padovan sequence is a linear and recurrent third-order sequence, named after the Italian architect Richard Padovan. Thus, its recurrence is given by: $P_n = P_{n-2} + P_{n-3}$, $n \geq 3$ and being $P_0 = P_1 = P_2 = 1$ your initial conditions [13–16].

The quaternions were developed by Willian Rowan Hamilton (1805-1865), arose from the attempt to generalize complex numbers in the form $z = a + bi$ in three dimensions [10]. Thus are presented as formal sums of scalars with usual vectors of three-dimensional space, existing four dimensions. Second Halici (2012) [8], a quaternion is a hyper-complex number and is described by:

$$q = a + bi + cj + dk$$

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where a, b, c are real numbers or scalar and i, j, k the orthogonal part at the base $\mathbb{R}^3$. The quaternionic product being $i^2 = j^2 = k^2 = ijk = -1, ij = k = -ji, jk = i = -kj$ and $ki = j = -ik$.

Being $q_1 = a_1 + b_1i + c_1j + d_1k$ and $q_2 = a_2 + b_2i + c_2j + d_2k$ two distinct quaternions. The addition, equality and multiplication scalar operations between them are:

$$q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k.$$

$q_1 = q_2$ only if $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$. And for $\alpha \in \mathbb{R}$, we have $\alpha q_1 = \alpha a_1 + \alpha b_1 i + \alpha c_1 j + \alpha d_1 k$. The conjugate of the quaternion is denoted by $\overline{q} = a - bi - cj - dk$.

There are also other works, such as [3, 6, 7, 9] that address the quaternions in the scope of numerical sequences, which are also used as a basis for this research.

As for hyperbolic numbers, the set of these numbers $\mathbb{H}$ can be described as:

$$\mathbb{H} = \left\{ z = x + hy | h \notin \mathbb{R}, h^2 = 1, x, y \in \mathbb{R} \right\}.$$

The addition and multiplication of two of these hyperbolic numbers n_1 e n_2,are given by [12]:

$$\begin{aligned} n_1 \pm n_2 &= (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2) \\ n_1 n_2 &= (x_1 + hy_1)(x_2 + hy_2) = (x_1x_2) + h(y_1y_2) + h(x_1y_2 + x_2y_1) \end{aligned}$$

In this sense, there are works on hyperbolic numbers and the quaternion sequence, used as a basis for this investigative process [1, 2, 4, 5, 11].

2. The hyperbolic k-Padovan quaternions

The sequence of k-Padovan is defined by $P_{k,n} = P_{k,n-2} + kP_{k,n-3}, n \geq 3, k \geqslant 1$ with initial values $P_{k,0} = P_{k,1} = P_{k,2} = 1$. In turn, we have the characteristic polynomial of this sequence as being $x^3 - x - k = 0$.

Definition 2.1. *The hyperbolic k-Padovan quaternions are given by:*

$$\mathbb{H}P_{k,n} = P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + kP_{k,n+3},$$

where $i^2 = j^2 = k^2 = -1$,$ij = k = -ji$,$jk = i = -kj$,$ki = j = -ik$.

According to the definitions presented, a study is carried out on the operations of addition, subtraction, and multiplication of hyperbolic k-Padovan quaternions.

$$\begin{aligned} \mathbb{H}P_{k,n} \pm \mathbb{H}P_{k,m} = {} & (P_{k,n} \pm P_{k,m}) + i(P_{k,n+1} \pm P_{k,m+1}) + j(P_{k,n+2} \pm P_{k,m+2}) \\ & + k(P_{k,n+3} \pm P_{k,m+3}), \end{aligned}$$

$$\begin{aligned} \mathbb{H}P_{k,n}\mathbb{H}P_{k,m} = {} & (P_{k,n}P_{k,m} + P_{k,n+1}P_{k,m+1} + P_{k,n+2}P_{k,m+2} + P_{k,n+3}P_{k,m+3}) \\ & + i(P_{k,n}P_{k,m+1} + P_{k,n+1}P_{k,m} + P_{k,n+2}P_{k,m+3} - P_{k,n+3}P_{k,m+2}) \\ & + j(P_{k,n}P_{k,m+2} + P_{k,n+2}P_{k,m} - P_{k,n+1}P_{k,m+3} + P_{k,n+3}P_{k,m+1}) \\ & + k(P_{k,n}P_{k,m+3} + P_{k,n+3}P_{k,m} + P_{k,n+1}P_{k,m+2} - P_{k,n+2}P_{k,m+1}) \\ & \neq \mathbb{H}P_{k,m}\mathbb{H}P_{k,n} \end{aligned}$$

The conjugate of the hyperbolic k-Padovan quaternary numbers is represented by:

$$\overline{\mathbb{H}P}_{k,n} = P_{k,n} - iP_{k,n+1} - jP_{k,n+2} - kP_{k,n+3}.$$

Theorem 2.2. *Let $P_{k,n}$ be the nth term of the k-Padovan sequence and $\mathbb{H}P_{k,n}$ the nth term of the quaternionic k-Padovan sequence hyperbolic, for $n \geqslant 1$ the following relations are given:*

$$(i)\mathbb{H}P_{k,n+3} = \mathbb{H}P_{k,n+1} + k\mathbb{H}P_{k,n};$$
$$(ii)\mathbb{H}P_{k,n} - i\mathbb{H}P_{k,n+1} + j\mathbb{H}P_{k,n+2} - k\mathbb{H}P_{k,n+3} = P_{k,n} + P_{k,n+2} + P_{k,n+4} + P_{k,n+6}.$$

Proof. (i) Based on Definition 2.1, we have:

$$\begin{aligned}
\mathbb{H}P_{k,n+1} + k\mathbb{H}P_{k,n} &= P_{k,n+1} + iP_{k,n+2} + jP_{k,n+3} + kP_{k,n+4} \\
&+ k(P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + kP_{k,n+3}) \\
&= (P_{k,n+1} + kP_{k,n}) + i(P_{k,n+2} + kP_{k,n+1}) + j(P_{k,n+3} + kP_{k,n+2}) \\
&+ k(P_{k,n+4} + kP_{k,n+3}) \\
&= P_{k,n+3} + iP_{k,n+4} + jP_{k,n+5} + kP_{k,n+6} \\
&= \mathbb{H}P_{k,n+3}
\end{aligned}$$

For (ii), we have:

$$\begin{aligned}
\mathbb{H}P_{k,n} - i\mathbb{H}P_{k,n+1} + j\mathbb{H}P_{k,n+2} - k\mathbb{H}P_{k,n+3} &= P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + kP_{k,n+3} \\
&- i(P_{k,n+1} + iP_{k,n+2} + jP_{k,n+3} + kP_{k,n+4}) \\
&- j(P_{k,n+2} + iP_{k,n+3} + jP_{k,n+4} + kP_{k,n+5}) \\
&- k(P_{k,n+3} + iP_{k,n+4} + jP_{k,n+5} + kP_{k,n+6}) \\
&= P_{k,n} + P_{k,n+2} - kP_{k,n+3} + jP_{k,n+4} + kP_{k,n+3} \\
&+ P_{k,n+4} - iP_{k,n+5} - jP_{k,n+4} + iP_{k,n+5} + P_{k,n+6} \\
&= P_{k,n} + P_{k,n+2} + P_{k,n+4} + P_{k,n+6}
\end{aligned}$$

■

Theorem 2.3. *Let $\overline{\mathbb{H}P}_{k,n}$ be the quaternionic conjugate of hyperbolic k-Padovan, then:*

$$\mathbb{H}P_{k,n} + \overline{\mathbb{H}P}_{k,n} = 2P_{k,n}$$

Proof. According to Definition 2.1, we have:

$$\begin{aligned}
\mathbb{H}P_{k,n} + \overline{\mathbb{H}P}_{k,n} &= P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + kP_{k,n+3} \\
&+ P_{k,n} - iP_{k,n+1} - jP_{k,n+2} - kP_{k,n+3} \\
&= 2P_{k,n}
\end{aligned}$$

■

3. Some properties

Hereinafter, some properties of the hyperbolic quaternion k-Padovan sequence are studied, based on the definitions discussed in the previous section.

Theorem 3.1. *The generating function of the hyperbolic k-Padovan quaternions is given by:*

$$g(\mathbb{H}P_{k,n}, x) = \frac{\mathbb{H}P_{k,0} + \mathbb{H}P_{k,1}x + (\mathbb{H}P_{k,2} - \mathbb{H}P_{k,0})x^2}{1 - x^2 - kx^3}.$$

Proof. Performing the multiplication of the function by x^2, kx^3 in the equations below, we have:

$$g(\mathbb{H}P_{k,n},x)=\sum_{n=0}^{\infty}\mathbb{H}P_{k,n}x^n=\mathbb{H}P_{k,0}+\mathbb{H}P_{k,1}x+\mathbb{H}P_{k,2}x^2+\ldots+\mathbb{H}P_{k,n}x^n+\ldots \tag{3.1}$$

$$x^2g(\mathbb{H}P_{k,n},x)=\mathbb{H}P_{k,0}x^2+\mathbb{H}P_{k,1}x^3+\mathbb{H}P_{k,2}x^4+\ldots+\mathbb{H}P_{k,n-2}x^n+\ldots \tag{3.2}$$

$$kx^3g(\mathbb{H}P_{k,n},x)=\mathbb{H}P_{k,0}kx^3+\mathbb{H}P_{k,1}kx^4+\mathbb{H}P_{k,2}kx^5+\ldots+\mathbb{H}P_{k,n-3}kx^n+\ldots \tag{3.3}$$

Based on the Equation (3.1-3.2+3.3), we have:

$$\begin{aligned}(1-x^2-kx^3)g(\mathbb{H}P_{k,n},x)=\mathbb{H}P_{k,0}+\mathbb{H}P_{k,1}x+(\mathbb{H}P_{k,2}-\mathbb{H}P_{k,0})x^2+(\mathbb{H}P_{k,3}-\mathbb{H}P_{k,1}-\mathbb{H}P_{k,0})x^3\\+\ldots+(\mathbb{H}P_{k,n}-\mathbb{H}P_{k,n-2}-\mathbb{H}P_{k,n-3})x^n+\ldots\end{aligned}$$

Thus:

$$\begin{aligned}(1-x^2-kx^3)g(\mathbb{H}P_{k,n},x)&=\mathbb{H}P_{k,0}+\mathbb{H}P_{k,1}x+(\mathbb{H}P_{k,2}-\mathbb{H}P_{k,0})x^2\\ g(\mathbb{H}P_{k,n},x)&=\frac{\mathbb{H}P_{k,0}+\mathbb{H}P_{k,1}x+(\mathbb{H}P_{k,2}-\mathbb{H}P_{k,0})x^2}{1-x^2-kx^3}.\end{aligned}$$

■

Theorem 3.2. *For $n\in\mathbb{N}$, the Binet formula of the hyperbolic k-Padovan quaternions is expressed by:*

$$Q_{k,n}^{(n)}=C_1r_1^n+C_2r_2^n+C_3r_3^n,$$

where C_1,C_2,C_3 are the coefficients of the Binet formula of the sequence and r_1,r_2,r_3 the roots of the characteristic polynomial ($x^3-x-k=0$).

Proof. Based on the k-Padovan sequence recurrence formula, its respective defined initial values and its characteristic polynomial whose roots are r_1,r_2,r_3, it is possible to obtain, by solving the linear system of equations, the values of coefficients C_1,C_2,C_3.

The discriminant $\Delta=\frac{(-k)^2}{4}-\frac{1}{27}$, referring to the 3rd degree polynomial, determines how the roots of the polynomial will be. Thus, when $\Delta\neq 0$ all roots will be distinct, concluding that $k^2\neq\frac{64}{27}$. Note also that $r_1r_2r_3=k$, $r_1+r_2+r_3=0$ and that when $k\neq 0$, there will be at least one root equal to zero, there being no Binet formula for this case. ■

Theorem 3.3. *For $n\ geqslant 2$ and $n\ in\ mathbbN$, the matrix form of the hyperbolic k-Padovan quaternions is given by:*

$$\begin{bmatrix}0&1&k\\1&0&0\\0&1&0\end{bmatrix}^n\begin{bmatrix}Q_{k,2}&Q_{k,1}&Q_{k,0}\\Q_{k,1}&Q_{k,0}&Q_{k,-1}\\Q_{k,0}&Q_{k,-1}&Q_{k,-2}\end{bmatrix}=\begin{bmatrix}\mathbb{H}_{k,n+2}&\mathbb{H}_{k,n+1}&\mathbb{H}_{k,n}\\\mathbb{H}_{k,n+1}&\mathbb{H}_{k,n}&\mathbb{H}_{k,n-1}\\\mathbb{H}_{k,n}&\mathbb{H}_{k,n-1}&\mathbb{H}_{k,n-2}\end{bmatrix}.$$

Proof. Through the finite induction principle, for $n=2$, we have:

$$\begin{aligned}\begin{bmatrix}0&1&k\\1&0&0\\0&1&0\end{bmatrix}^2\begin{bmatrix}\mathbb{H}_{k,2}&\mathbb{H}_{k,1}&\mathbb{H}_{k,0}\\\mathbb{H}_{k,1}&\mathbb{H}_{k,0}&\mathbb{H}_{k,-1}\\\mathbb{H}_{k,0}&\mathbb{H}_{k,-1}&\mathbb{H}_{k,-2}\end{bmatrix}&=\begin{bmatrix}1&k&0\\0&1&k\\1&0&0\end{bmatrix}\begin{bmatrix}\mathbb{H}_{k,2}&\mathbb{H}_{k,1}&\mathbb{H}_{k,0}\\\mathbb{H}_{k,1}&\mathbb{H}_{k,0}&\mathbb{H}_{k,-1}\\\mathbb{H}_{k,0}&\mathbb{H}_{k,-1}&\mathbb{H}_{k,-2}\end{bmatrix}\\&=\begin{bmatrix}\mathbb{H}_{k,2}+k\mathbb{H}_{k,1}&\mathbb{H}_{k,1}+k\mathbb{H}_{k,0}&\mathbb{H}_{k,0}+k\mathbb{H}_{k,-1}\\\mathbb{H}_{k,1}+k\mathbb{H}_{k,0}&\mathbb{H}_{k,0}+k\mathbb{H}_{k,-1}&\mathbb{H}_{k,-1}+k\mathbb{H}_{k,-2}\\\mathbb{H}_{k,2}&\mathbb{H}_{k,1}&\mathbb{H}_{k,0}\end{bmatrix}\\&=\begin{bmatrix}\mathbb{H}_{k,4}&\mathbb{H}_{k,3}&\mathbb{H}_{k,2}\\\mathbb{H}_{k,3}&\mathbb{H}_{k,2}&\mathbb{H}_{k,1}\\\mathbb{H}_{k,2}&\mathbb{H}_{k,1}&\mathbb{H}_{k,0}\end{bmatrix}.\end{aligned}$$

Checking the validity for any $n = z, z \in \mathbb{N}$, one has:

$$\begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^z \begin{bmatrix} \mathbb{H}_{k,2} & \mathbb{H}_{k,1} & \mathbb{H}_{k,0} \\ \mathbb{H}_{k,1} & \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} \\ \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} & \mathbb{H}_{k,-2} \end{bmatrix} = \begin{bmatrix} \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} \\ \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} \\ \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} & \mathbb{H}_{k,z-2} \end{bmatrix}.$$

Therefore, it turns out to be valid for $n = z + 1 = 1 + z$:

$$\begin{aligned} \begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{1+z} \begin{bmatrix} \mathbb{H}_{k,2} & \mathbb{H}_{k,1} & \mathbb{H}_{k,0} \\ \mathbb{H}_{k,1} & \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} \\ \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} & \mathbb{H}_{k,-2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^z \begin{bmatrix} \mathbb{H}_{k,2} & \mathbb{H}_{k,1} & \mathbb{H}_{k,0} \\ \mathbb{H}_{k,1} & \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} \\ \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} & \mathbb{H}_{k,-2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} \\ \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} \\ \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} & \mathbb{H}_{k,z-2} \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{H}_{k,z+1} + k\mathbb{H}_{k,z} & \mathbb{H}_{k,z} + k\mathbb{H}_{k,z-1} & \mathbb{H}_{k,z-1} + k\mathbb{H}_{k,z-2} \\ \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} \\ \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{H}_{k,z+3} & \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z+1} \\ \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z} \\ \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} \end{bmatrix}. \end{aligned}$$

■

4. Conclusion

The study allowed for a mathematical analysis of the k-Padovan sequence and its complex form. Thus, the hyperbolic k-Padovan quaternion sequence was introduced, addressing some mathematical properties and theorems. It is noteworthy that for the particular case of $k = 1$, it is possible to notice that we have the hyperbolic quaternionic Padovan sequence.

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