ISSN 2319-3786
VOLUME 10, ISSUE 1, JANUARY 2022

M
J
M

Malaya Journal of Matematik

an international journal of mathematical sciences

MKD PUBLISHING HOUSE
Dawn To Researchers
5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),
Coimbatore- 641114, Tamil Nadu, India.
www.mkdpress.com | www.malayajournal.org

The Malaya Journal of Matematik is published quarterly in single volume annually and four issues constitute one volume appearing in the months of January, April, July and October.

Subscription

The subscription fee is as follows:

USD 350.00 For USA and Canada

Euro 190.00 For rest of the world

Rs. 4000.00 In India. (For Indian Institutions in India only)

Prices are inclusive of handling and postage; and issues will be delivered by Registered Air-Mail for subscribers outside India.

Subscription Order

Subscription orders should be sent along with payment by Cheque/ D.D. favoring "Malaya Journal of Matematik" payable at COIMBATORE at the following address:

MKD Publishing House

5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),

Coimbatore- 641114, Tamil Nadu, India.

Contact No. : +91-9585408402

E-mail : info@mkdpress.com; editorinchief@malayajournal.org; publishingeditor@malayajournal.org

Website : https://mkdpress.com/index.php/index/index

Vol. 10 No. 01 (2022): Malaya Journal of Matematik (MJM)

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 1–19.
http://doi.org/10.26637/mjm1001/001

A new analytical method to solve Klein-Gordon equations by using homotopy perturbation Mohand transform method

Ravi Shankar Dubey *[1], **Pranay Goswami**[2], **Tailor Gomati A.**[1], **Vinod Gill**[3]

[1] *Amity University, Jaipur, Rajasthan, India.*
[2] *School of Liberal Studies, Ambedkar University, Delhi-110006, India.*
[3] *Mathematics Department, Hisar College, Hariyana, India.*

***R*eceived 12 August 2021; *A*ccepted 17 December 2021**

Abstract. In this paper, we will study about Fractional-order partial differential equations in Mathematical Science and we will introduce and analyse fractional calculus with an integral operator that contains the Caputo- Fabrizio's fractional-order derivative. The advanced method is an appropriate union of the new integral transform named as 'Mohand transform' and the homotopy perturbation method. Some numerical examples are used to communicate the generality and clarity of the proposed method. We will also find the analytical solution of the linear and non-linear Klein-Gordan equation which originate in quantum field theory. The homotopy perturbation Mohand transform method (HPMTM) is a merged form of Mohand transform, homotopy perturbation method, and He's polynomials. Some numerical examples are used to indicate the generality and clarity of the proposed method.

Keywords: Mohand Transform, Homotopy Perturbation Method (HPM), Fractional Calculus

Contents

*Corresponding author. Email addresses: **ravimath13@gmail.com** (Ravi Shankar Dubey), **pranaygoswami83@gmail.com** (Pranay Goswami), **tailorgopi@gmail.com** (Tailor Gomati A.), **vinod.gill08@gmail.com** (Vinod Gill)

1. Introduction, Background and Preliminaries

Fractional calculus is an eminent phrase in each science and technology. Differential and crucial equations represent and outline various phenomena of technological knowledge and mold the difficulty in a new appearance. Fractional calculus is the generalization of regular differentiation and integration from linear to non-linear order. It extend with derivatives of actual or complex order.

In 1695 L'Hospital enquires Leibnitz that $D^n f$ could be what, if n is fractional. Leibnitz answers that it can be expand in the form of infinite series, such as $d^{1/2}xy$ and distant between infinite series also geometric series, we use only positive and negative integers in the finite series Leibniz also responded that $xd^{\left\{\frac{1}{2}\right\}} = x\sqrt{\mathrm{dx}:x}$ this is a clear paradox.

S.F. Larcroin developed a formula from a case of integer order which starts with$y = x^m$;m is a positive integer

$$\frac{d^n y}{\mathrm{dx}^n} = \frac{m! x^{m-n}}{(m-n)!}; m \succeq n$$

Like this many other mathematicians gave their definitions and formula.Fractional calculus attracted some mathematical minds like Fourier, Euler, Marquis de Laplace, and plenty of others due primarily to its incontestable applications in such different fields of science and engineering. The literature is full-fledged by generating, growing, working, modifications, and generalization of the facts, formulae, and definitions relating to fractional calculus. A whole historical development and progress of fragmental calculus operators seem in books of Kilbas, Srivastava, and Trujillo [1], Miller and Ross [2], Nishimoto [3], Oldham and Spanier [4], Podlubny [5] and, Ross [6], etc.

In mathematical analysis, there are several fields wherever fractional calculus operators are usefully employed in numerous branches like integral and differential equations, special functions, integral transforms, operational calculus (see [6],[7]), etc. because it start to be used fractional calculus in various areas as numerous varieties of operators came to light-weight and by the time they were got changed.

Mohand Transform is derived from the classical Fourier integral. Based on the mathematical simplicity of the Mohand transform and its fundamental properties. Mohand transform was introduced by Mohand Mahgoub to facilitate the process of solving ordinary and partial differential equations in the time domain. Typically, Fourier, Laplace, Elzaki, Aboodh, kamal and Sumudu transforms are the convenient mathematical tools for solving differential equations.

Mohand transform and some of its fundamental properties are also used to solve differential equations.

2. Mohand Transform

2.1. Definition

A new transform called the Mohand transform defined for function of exponential order we consider functions in the set A defined by: For a given function in the set A, the constant M must be finite number,k_1, k_2 may be finite or infinite.

$$A = \{f(t) : \exists M, k_1, k_2 > 0. |f(t)| < \mathrm{M}e^{\frac{|t|}{k_j}}, \mathrm{if} t \in (-1)^j \times [0, \infty)\} \tag{2.1}$$

The Mohand transform denoted by the operator M (.) defined by the integral equations

$$M[f(t)] = R(v) = v^2 \int_0^\infty f(t) e^{-\mathrm{vt}}\, dt, t \geq 0, k_1 \leq v \leq k_2 \tag{2.2}$$

The variable v in this transform is used to factor the variable t in the argument of the function f . this transform has deeper Connection with the Laplace ,Elzaki, and Aboodh transform.
The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving the linear differential equations.

2.2. Mohand Transform and Different Types of Results

Mohand Transform is derived from the classical Fourier integral based on the mathematical simplicity of the Mohand transform with its fundamental properties.Mohand transform was introduced by MohandMahgoub to facilitate the process of solving ordinary and partial differential equations in the time domain.Mohandtransform defined for the function of exponential order we consider functions in the set A defined by:

$$A = f(t) : M, K_1, K_2 > 0.|f(t)| < Me^{\frac{|t|}{K_t}} if\ t \in (-1)^j \times [0, \infty)$$

where M must be finite number and K_1, K_2 may be finite or infinite, for a given function in set A.
The integral equation defines the operator M(.) which represents Mohand transform i.e

$$M[f(X)] = R(v) = v^2 \int_0^\infty e^{vt} f(t)dt, t \geq 0, K_1 \leq v \leq K_2 \tag{2.3}$$

The variable v in this transform is used to factor the variable t in the argument of the function f.
If $R_1(t)$ and $R_2(t)$ represents Mohand transform for functions $F_1(t)$ and $F_2(t)$ respectively,then Mohand transform of their convolution $F_1(t) * F_2(t)$ is given by

$$M(F_1(t) * F_2(t)) = \tfrac{1}{v^2} MF_1(t)MF_2(t)$$

$$M(F_1(t) * F_2(t)) = \frac{1}{v^2} R_1(t)R_2(t) \tag{2.4}$$

where $F_1(t) * F_2(t)$ is defined by

$$F_1(t) * F_2(t) = \int_0^t F_1(t - x)F_2(t)dt = \int_0^t F_1(t)F_2(t - x)dt \tag{2.5}$$

Caputo fractional time derivative

$$D_t^\beta(h(t)) = \frac{M(\beta)}{1 - \beta} \int_\alpha^t h'(x)e^{\left[-\beta\left(\frac{t-x}{1-\beta}\right)\right]}dx \tag{2.6}$$

$M(\alpha)$ is function of normalization such as $M(0) = M(1) = 1$.

$$M\{D_t^\beta(h(t))\} = \frac{M(\beta)}{1 - \beta} v^2 \int_0^\infty e^{-vt} \int_\alpha^t h'(x)e^{\left[-\beta\left(\frac{t-x}{1-\beta}\right)\right]}dxdt \tag{2.7}$$

We use the convolution property of Mohand transform is defined as

$$M\{D_t^{\beta}(h(t))\} = \frac{M(\beta)}{1-\beta}[M(h'(x))] * M[e^{-\frac{\beta t}{1-\beta}}]$$

$$M\{D_t^{\beta}(h(t))\} = \frac{M(\beta)}{1-\beta}[vR(v) - h(0)v^2]\frac{v^2}{v-(\frac{-\beta}{1-\beta})}$$

$$M\{D_t^{\beta}(h(t))\} = M(\beta)[vR(v) - h(0)v^2]\frac{v^2}{v+(\beta(1-v))}$$

The solution of Caputo-Fabrizio fractional derivative is:

$$M\{D_t^{\beta}(h(t))\} = \frac{M(\beta)\left\{v^3 M[h(t)] - v^4 h(0)\right\}}{v+\beta(1-v)} \tag{2.8}$$

3. HPMTM for the model

In this section, we will study Klein-Gordon equation and its application by using homotopy perturbation Mohand transform method.

3.1. Solution of Klein-Gordon equation:

Klein-Gordon equation is

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + \text{a}u(x,t) = g(x,t) \tag{3.1}$$

with initial condition

$$u(x,0) = h(x), u_t(x,0) = f(x) \tag{3.2}$$

Taking the Mohand transform on both sides ofequ. (2.1), we get

$$M[u_{\text{tt}}(x,t)] = M[u_{\text{xx}}(x,t) - \text{a}u(x,t)] + M[g(x,t)] \tag{3.3}$$

Using the convolution property of Mohand transform, we get

$$v^2 R(x,v) - v^3 u(0) - v^2 u(0) = M[u_{\text{xx}}(x,t) - \text{a}u(x,t)] + M[g(x,t)] \tag{3.4}$$

On simplifying and initial conditions, we get

$$R(x,v) = f(x) + \text{v}h(x) + \frac{1}{v^2}M[u_{\text{xx}}(x,t) - \text{a}u(x,t)] + \frac{1}{v^2}M[g(x,t)] \tag{3.5}$$

Taking inverse Mohand transform on both sides ofequ. (2.5), we get

$$u(x,t) = G(x,t) + M^{-1}[\frac{1}{v^2}M[u_{\text{xx}}(x,t) - \text{a}u(x,t)]] \tag{3.6}$$

where G(x,t) represents the term arising from the function and the specified initial conditions.
Using the HPM method, we get

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{3.7}$$

Putting the equation (2.6) in equ. (2.7), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) + p\left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} p^n u_n(x,t) - a\sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right]\right) \tag{3.8}$$

On collecting the coefficients of exponents of p

$p^0 : u_0(x,t) = G(x,t)$

$$p^1 : u_1(x,t) = \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - \mathrm{au}_0(x,t)\right]\right]\right) \tag{3.9}$$

$p^2 : u_2(x,t) = \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_1(x,t) - \mathrm{au}_1(x,t)\right]\right]\right)$

$p^3 : u_3(x,t) = \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_2(x,t) - \mathrm{au}_2(x,t)\right]\right]\right)$

.
.
.

and similarly,

$$p^n : u_n(x,t) = \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_{(n-1)}(x,t) - \mathrm{au}_{(n-1)}(x,t)\right]\right]\right) \tag{3.10}$$

Hence, the solution is:

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,t) \tag{3.11}$$

3.2. Study of Mohand Transform Homotopy Perturbation Method (MTHPM)

Let a general non-linear non-homogeneous partial differential equation

$$\mathrm{Du}(x,t) + \mathrm{Ru}(x,t) + \mathrm{Nu}(x,t) = g(x,t) \tag{3.12}$$

With the initial conditions

$$u(x,0) = h(x), u_t(x,0) = f(x) \tag{3.13}$$

where D is the linear differential operator of order 2, R is a linear differential operator of less than D; N is the general nonlinear differential operator and is the source term.

Applying the Mohand transform on both sides ofequ. (2.12), we get

$$M[\mathrm{Du}(x,t)] = M[g(x,t)] - M[\mathrm{Ru}(x,t) + \mathrm{Nu}(x,t)] \tag{3.14}$$

Using the property of Mohand transform, we have

$$\left[v^2 R(x,v) - v^3 u(x,0) - v^2(x,0)\right] = M[g(x,t)] - M[\mathrm{Ru}(x,t) + \mathrm{Nu}(x,t)] \tag{3.15}$$

After the simplification and initial conditions, we get

$$R(x,v) = \mathrm{vh}(x) + f(x) + \frac{1}{v^2} M[g(x,t)] - \frac{1}{v^2} M[\mathrm{Ru}(x,t) + \mathrm{Nu}(x,t)] \tag{3.16}$$

Taking inverse Mohand transform on both sides of equ. (2.16), we get

$$u(x,t) = G(x,t) - M^{-1}\left[\frac{1}{v^2} M[\mathrm{Ru}(x,t) + \mathrm{Nu}(x,t)]\right] \tag{3.17}$$

where G(x,t) represents the term arising from the function and the specified initial conditions.

Now we use the HPM

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{3.18}$$

and the non-linear term can be written as

$$\mathrm{Nu}(x,t) = \sum_{n=0}^{\infty} p^n H_n(x,t) \tag{3.19}$$

where is $H_n(x,t)$ He's polynomials and given by

$$H_n\left(u_0, u_1, u_2, ...u_n\right) = \frac{1}{n!}\frac{\partial^2}{\partial p^2}\left[N \sum_{n=0}^{\infty} p^i u_i\right]_{p=0} \tag{3.20}$$

Substituting the equ. (2.19) and equ. (2.18) in equ. (2.17), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(M^{-1}\left[\frac{1}{v^2} M\left[R \sum_{n=0}^{\infty} p^n u_n(x,t) + N \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{3.21}$$

On collecting the coefficient of exponents of p

$p^0 : u_0(x,t) = G(x,t)$

$$p^1 : u_1(x,t) = -M^{-1}\left[\frac{1}{v^2} M\left[\mathrm{R}u_0(x,t) + H_0(u)\right]\right] \tag{3.22}$$

$p^2 : u_2(x,t) = -M^{-1}\left[\frac{1}{v^2} M\left[\mathrm{R}u_1(x,t) + H_1(u)\right]\right]$

$p^3 : u_3(x,t) = -M^{-1}\left[\frac{1}{v^2} M\left[\mathrm{R}u_2(x,t) + H_2(u)\right]\right]$

.
.
.

and similarly,

$$p^n : u_n(x,t) = -M^{-1}\left[\frac{1}{v^2}M\left[\mathrm{R}u_{n-1}(x,t) + H_{n-1}(u)\right]\right] \tag{3.23}$$

Hence, the solution is

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,t) \tag{3.24}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ...u_n(x,t) \tag{3.25}$$

4. Applications of the MTHPM

In this part, we applythe Mohand transform homotopy perturbation method (MTHPM) to solve the linear and nonlinear Klein-Gordon equation.

Example 1: Consider the linear Klein-Gordon equation

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + u(x,t) = 0 \tag{4.1}$$

With the initial conditions

$$u(x,0) = 0, u_t(x,0) = x \tag{4.2}$$

Taking Mohand transform on both sides of equ. (3.1), we get

$$M[u_{\text{tt}}(x,t)] = M[u_{\text{xx}}(x,t) - u(x,t)] \tag{4.3}$$

Using the convolution property of Mohand transform, we get

$$v^2R(x,v) - v^3u(0) - v^2u'(0) = M[u_{\text{xx}}(x,t) - u(x,t)] \tag{4.4}$$

On simplifying and above initial conditions, we get

$$R(x,v) = x + \frac{1}{v^2}M[u_{\text{xx}}(x,t) - u(x,t)] \tag{4.5}$$

Taking inverse Mohand transform on both sides of equ. (3.5), we get

$$u(x,t) = xt + M^{-1}[\frac{1}{v^2}M[u_{\text{xx}}(x,t) - u(x,t)]] \tag{4.6}$$

Now we use the HPM and the non-linear term then we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = xt + M^{-1}[\frac{1}{v^2} M[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(x,t)]] \tag{4.7}$$

Collecting the coefficients of exponents of p

$p^0 : u_0(x,t) = xt$

$$p^1 : u_1(x,t) = M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} u_0(x,t) - u_0(x,t)\right]\right] = -x\frac{t^3}{3!} \tag{4.8}$$

$$p^2 : u_2(x,t) = M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} u_1(x,t) - u_1(x,t)\right]\right] = -x\frac{t^5}{5!} \tag{4.9}$$

$$p^3 : u_3(x,t) = M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} u_2(x,t) - u_2(x,t)\right]\right] = -x\frac{t^7}{7!} \tag{4.10}$$

.
.
.

Similarly, we can obtain further values.

Hence the u(x,t) is

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\ &= x\left[\left(-\frac{t^7}{7!} + \frac{t^5}{5!} - \frac{t^3}{3!} + t...\right)....\right] \\ &= xsint \end{aligned} \tag{4.11}$$

Example 2: Consider the linear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 2\text{sinx} \tag{4.12}$$

With the initial conditions

$$u(x,0) = \text{sinx}, u_t(x,0) = 1 \tag{4.13}$$

Applyingthe Mohand transform on both sides of equ. (3.14), we get

$$M[u_{tt}(x,t)] = M[u_{xx}(x,t) - u(x,t)] + M[2\text{sinx}] \tag{4.14}$$

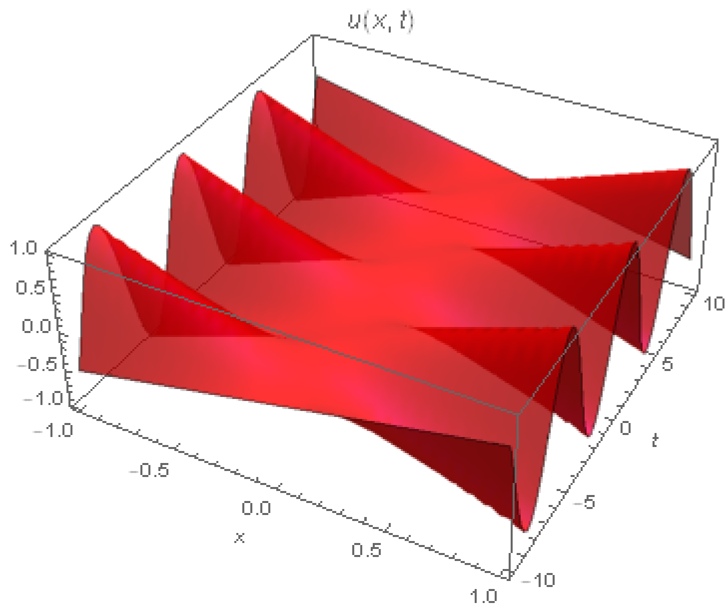

Figure 1: The Graph of $u(x,t) = xsint, t > 0. -\infty \leq x \leq \infty$

Using the convolution property of Mohand transform, we get

$$v^2 R(x,v) - v^3 u(x,0) - v^2 u'(x,0) = M[u_{xx}(x,t) - u(x,t)] + 2sinx(v) \tag{4.15}$$

After the simplification and above initial conditions, we get

$$R(x,v) = vsinx + 1 + 2\text{sinx}\frac{1}{v} + \frac{1}{v^2} M\left[u_{xx}(x,t) - u(x,t)\right] \tag{4.16}$$

Taking inverse Mohand transform on both sides of equ. (3.18), we get

$$u(x,t) = \text{sinx} + t^2\text{sinx} + t + M^{-1}\left[\frac{1}{v^2} M\left[u_{xx}(x,t) - u(x,t)\right]\right] \tag{4.17}$$

Now we use the HPM and the non-linear term then we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \text{sinx} + t + t^2\text{sinx} + p\left(M^{-1}\left[\frac{1}{v^2} M[\left[\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} p^n u_n(x,t)] - \sum_{n=0}^{\infty} p^n u_n(u)\right]\right]\right) \tag{4.18}$$

Collecting the coefficients of exponents of p

$$p^0 : u_0(x,t) = sinx + t + t^2 sinx \tag{4.19}$$

$$p^1 : u_1(x,t) = M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - u_0(x,t)\right]\right]$$
$$= -\left[\frac{t^3}{3!} + \frac{t^4}{3!}\text{sinx} + t^2\text{sinx}\right] \tag{4.20}$$

$$p^2 : u_2(x,t) = M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_1(x,t) - u_1(x,t)\right]\right]$$
$$= \left[\frac{t^5}{5!} + 8\frac{t^6}{6!}\text{sinx} + \frac{t^4}{3!}\text{sinx}\right] \tag{4.21}$$

$$p^3 : u_3(x,t) = M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_2(x,t) - u_2(x,t)\right]\right]$$
$$= -\left[\frac{t^7}{7!} + 8\frac{t^6}{6!}\text{sinx} + 16\frac{t^8}{8!}\text{sinx}\right] \tag{4.22}$$

.
.
.

Similarly, we can obtain further values.

Hence the u(x,t) is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ...$$
$$= \text{sinx} + \left[-\frac{t^7}{7!} + \frac{t^5}{5!} - \frac{t^3}{3!} + t\right]$$
$$= sinx + sint \tag{4.23}$$

Example 3: Consider the following nonlinear Klein-Gordon equation

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + u^2(x,t) = t^2x^2 \tag{4.24}$$

With the initial conditions

$$u(x,0) = 0, u_t(x,0) = x \tag{4.25}$$

Applyingthe Mohand transform on both sides of equ. (3.29), we get

$$M\left[u_{\text{tt}}(x,t)\right] = M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] + M\left[t^2x^2\right] \tag{4.26}$$

Using the convolution property of Mohand transform, we get

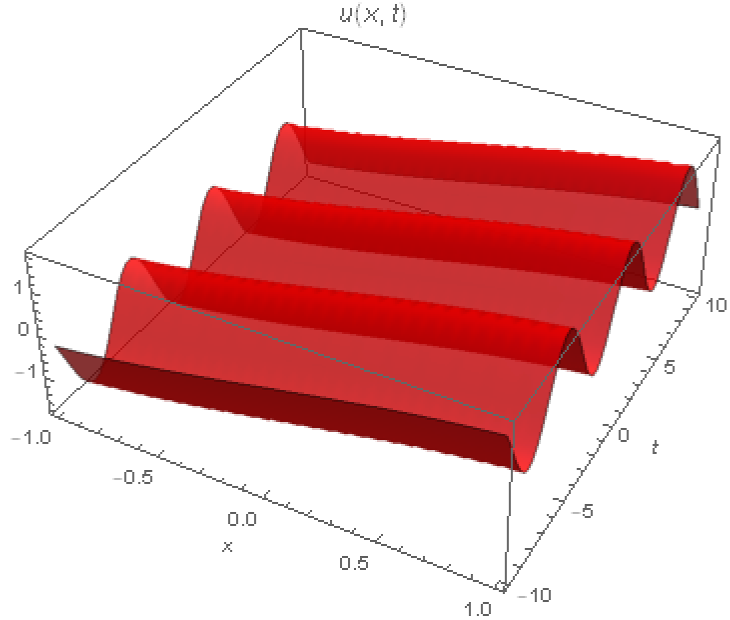

Figure 2: The Graph of $u(x,t) = sinx + sint, t > 0. -\infty \leq x \leq \infty$

$$v^2 R(x,v) - v^3 u(x,0) - v^2 u'(x,0) = 2x^2 \frac{1}{v} + M\left[u_{xx}(x,t) - u^2(x,t)\right] \tag{4.27}$$

On simplifying and above initial conditions, we get

$$R(x,v) = x + 2x^2 \frac{1}{v^3} + \frac{1}{v^2} M\left[u_{xx}(x,t) - u^2(x,t)\right] \tag{4.28}$$

Taking inverse Mohand transform on both sides of equ. (3.33), we get

$$u(x,t) = \text{xt} + \frac{x^2}{12} t^4 + \frac{1}{v^3} + M^{-1}\left[\frac{1}{v^2} M\left[u_{xx}(x,t) - u^2(x,t)\right]\right] \tag{4.29}$$

Now we use the HPM and the non-linear then we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \text{xt} + \frac{x^2}{12} t^4 + \frac{1}{v^3} + p\left(M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{4.30}$$

where $H_n(u)$ is represents the He's polynomial of nonlinear terms. The first few components of He's polynomials are given by

$$H_0(u) = (u_0)^2 \tag{4.31}$$

$$H_1(u) = 2u_0 u_1 \tag{4.32}$$

$$H_2(u) = 2u_0u_2 + (u_1)^2 \tag{4.33}$$

.
.
.

Equating the multipliers of exponents of p

$$p^0 : u_0(x,t) = \text{xt} + \frac{x^2}{12}t^4 \tag{4.34}$$

$$\begin{aligned} p^1 : u_1(x,t) &= \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - H_0(u)\right]\right]\right) \\ &= \left[\frac{t^{10}x^4}{12960} - \frac{t^7x^3}{252} + \frac{t^6}{180} - \frac{t^4x^2}{12}\right] \end{aligned} \tag{4.35}$$

$$\begin{aligned} p^2 : u_2(x,t) &= \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_1(x,t) - H_1(u)\right]\right]\right) \\ &= \left[\frac{t^{16}x^6}{18662400} + \frac{383t^{13}x^5}{15921360} - \frac{t^{12}x^2}{71280} + \frac{11t^{10}x^4}{45360} + \frac{t^7x^3}{252} - \frac{t^6}{180} - \frac{11\text{xt}^9}{22680}\right] \end{aligned} \tag{4.36}$$

.
.
.

similarly, we can obtain further values.

Hence, the u(x,t) is

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\ u(x,t) &= xt \end{aligned} \tag{4.37}$$

Example 4: Consider the following nonlinear Klein-Gordon equation

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + u^2(x,t) = 2x^2 - 2t^2 + t^4x^4 \tag{4.38}$$

With the initial conditions

$$u(x,0) = 0, u_t(x,0) = 0 \tag{4.39}$$

Applying the Mohand transform on both sides of equ. (3.45), we get

$$M\left[u_{\text{tt}}(x,t)\right] = M\left(2x^2 - 2t^2 + t^4x^4\right) + M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] \tag{4.40}$$

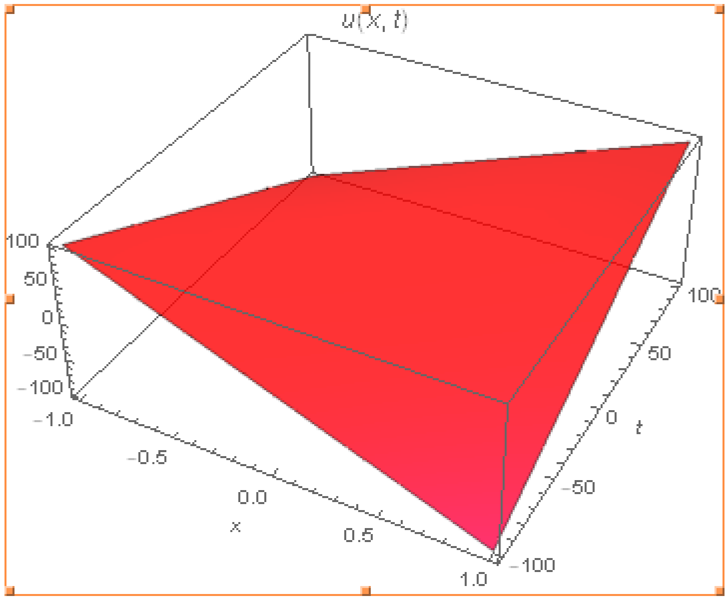

Figure 3: The Graph of $u(x,t) = xt, t > 0. -\infty \leq x \leq \infty$

Using the convolution property of Mohand transform, we get

$$v^2 R(x,v) - v^3 u(x,0) - v^2 u'(x,0) = 2x^2 v - \frac{4}{v} + \frac{24x^4}{v^3} + M\left[u_{\mathrm{xx}}(x,t) - u^2(x,t)\right] \tag{4.41}$$

On simplification and above initial conditions, we get

$$R(x,v) = \frac{2x^2}{v} - \frac{4}{v^3} + \frac{24x^4}{v^5} + \frac{1}{v^2} M\left[u_{\mathrm{xx}}(x,t) - u(x,t)\right] \tag{4.42}$$

Taking inverse Mohand transform on both sides ofequ. (3.49), we get

$$u(x,t) = t^2 x^2 - \frac{t^4}{6} + \frac{x^4}{30} t^6 + M^{-1}\left[\frac{1}{v^2} M\left[u_{\mathrm{xx}}(x,t) - u(x,t)\right]\right] \tag{4.43}$$

Now we use the HPM and the non-linear term we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^2 x^2 - \frac{t^4}{6} + \frac{x^4}{30} t^6 + p\left(M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{4.44}$$

where $H_n(u)$ is represents the He's polynomial of nonlinear terms. The first few components of He's polynomials are given by

$$H_0(u) = (u_0)^2 \tag{4.45}$$

$$H_1(u) = 2u_0 u_1 \tag{4.46}$$

$$H_2(u) = 2u_0 u_2 + (u_1)^2 \tag{4.47}$$

.
.
.

Equating the multipliers of exponents of p

$$p^0 : u_0(x,t) = t^2x^2 - \frac{t^4}{6} + \frac{x^4}{30}t^6 \tag{4.48}$$

$$\begin{aligned} p^1 : u_1(x,t) &= \left(\frac{1}{M}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - H_0 u\right]\right]\right) \\ &= \left[-\frac{532224t^{14}x^8}{14!} + \frac{4032t^{11}x^4}{39916800} - \frac{2688t^{10}x^6}{3628800} + \frac{288t^8x^2}{40320} - \frac{20t^8}{40320} + \frac{24t^7x^2}{5040} - \frac{t^6x^4}{30} + \frac{t^6}{6}\right] \end{aligned} \tag{4.49}$$

.
.
.

similarly, we can obtain further values.

Hence, the u(x,t) is

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\ u(x,t) &= x^2t^2 \end{aligned} \tag{4.50}$$

Example 5: Consider the following nonlinear Klein-Gordon equation

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + u^2(x,t) = 6\text{xt}\left(x^2 - t^2\right) + t^6x^6 \tag{4.51}$$

With the initial conditions

$$u(x,0) = 0, u_t(x,0) = x \tag{4.52}$$

Applyingthe Mohand transform on both sides of equ. (3.60), we get

$$M\left[u_{\text{tt}}(x,t)\right] = M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] + M\left[6\text{xt}\left(x^2 - t^2\right) + t^6x^6\right] \tag{4.53}$$

Using the convolution property of Mohand transform, we get

$$v^2R(x,v) - v^3u(x,0) - v^2u'(x,0) = 6x^3 - \frac{36x}{v^2} + \frac{720x^6}{v^5} + M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] \tag{4.54}$$

On simplification and above initial conditions, we get

$$R(x,v) = \frac{6x^3}{v^2} - \frac{36x}{v^4} + \frac{720x^6}{v^7} + \frac{1}{v^2}M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] \tag{4.55}$$

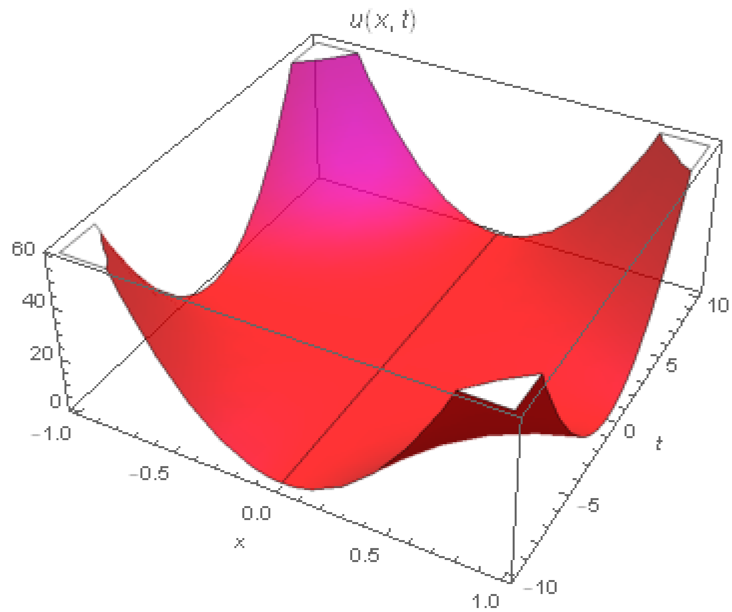

Figure 4: The Graph of $u(x,t) = x^2t^2, t > 0. -\infty \leq x \leq \infty$

Taking inverse Mohand transform on both sides ofequ. (3.64), we get

$$u(x,t) = \frac{6x^3}{v^2} - \frac{36x}{v^4} + \frac{720x^6}{v^7} + +M^{-1}\left[\frac{1}{v^2}M\left[u_{\text{xx}}(x,t) - u(x,t)\right]\right] \tag{4.56}$$

Now we use the HPM and the non-linear term we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^3x^3 - \frac{3\text{xt}^5}{10} + \frac{t^8x^6}{56} + p\left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{4.57}$$

where $H_n(u)$ is represents the He's polynomial of nonlinear terms. The first few components of He's polynomials are given by

$$H_0(u) = (u_0)^2 \tag{4.58}$$

$$H_1(u) = 2u_0u_1 \tag{4.59}$$

$$H_2(u) = 2u_0u_2 + (u_1)^2 \tag{4.60}$$

.
.
.

Equating the multipliers of exponents of p

$$p^0 : u_0(x,t) = t^3x^3 - \frac{3\mathrm{xt}^5}{10} + \frac{t^8x^6}{56} \tag{4.61}$$

$$\begin{aligned} p^1 : u_1(x,t) &= \left(\frac{1}{M}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - H_0u\right]\right]\right) \\ &= \left[\frac{t^{18}x^{12}}{653616} - \frac{3t^{15}x^7}{19600} + \frac{t^{13}x^9}{4368} + \frac{3t^{12}x^2}{4400} - \frac{53t^{10}x^4}{4200} + \frac{t^8x^6}{56} + \frac{3\mathrm{xt}^5}{10}\right] \end{aligned} \tag{4.62}$$

...

similarly, we can obtain further values.

Hence, the u(x,t) is

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\ u(x,t) &= x^3t^3 \end{aligned} \tag{4.63}$$

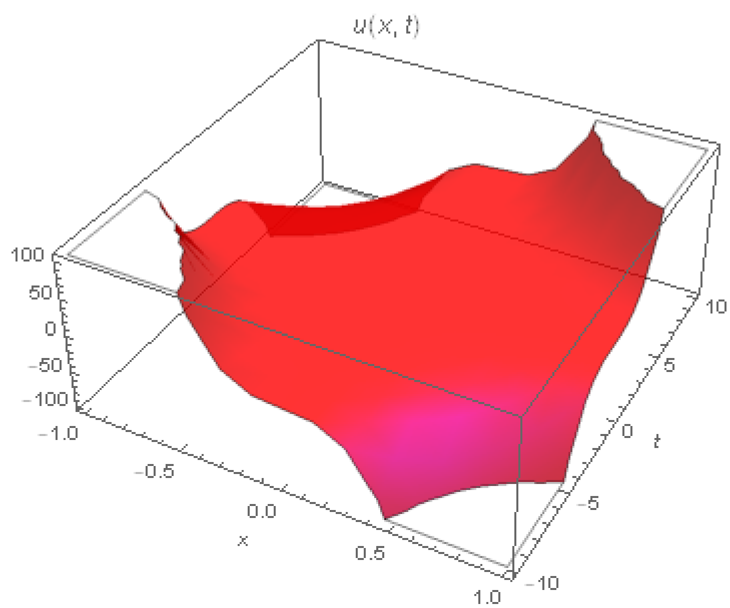

Figure 5: The Graph of $u(x,t) = x^3t^3, t > 0. -\infty \leq x \leq \infty$

5. Concluding Remarks and Observations

we have discussed the history, some definitions of fractional calculus, Riemann-Liouville differential and integral operator. We also knowing the Mittag-Leffler function and Caputo and Fabrizio fractional-order derivative. In this paper, we discussed some of the integral transforms (like Laplace Transform, Fourier Transform, and Mohand Transform). Homotopy perturbation Mohand transform method has been successfully operated to evaluating the linear and nonlinear Klein-Gordon equations with initial conditions. The method is good and simple to solve. In conclusion, the MTHPM may be considered as a nice simplification in numerical techniques and might find wide applications.

References

[1] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam 2006.*

[2] K.S. Miller and B. Ross, *An Introduction to The Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, and Singapore, 1993.

[3] K. Nishimoto, *Fractional Calculus*, Vol. 1, Vol. 2 and Vol. 3, Descartes Press, Koriyama, Japan, (1984), (1987), (1989).

[4] K.B. Oldham, and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, New York; and Dover Publications, New York, 1974.

[5] I. Podlubny, *Fractional Differential Equation*, Vol. 198, Academic Press, California, 1999.

[6] B. Ross, *Fractional Calculus and Its Applications*, (Proc. Internat. Conf., New Heaven, 1974), Lecture Notes in Math. Vol. 457, Springer Verlag, New York, 1978.

[7] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives:Theory and Applications*, Gordon and Breach, Amsterdam, 1993.

[8] L. Debnath and R. P. Feynman, Recent applications of fractional calculus to science and engineering, *Int. J. Math. Math. Sci.*, **54**(2003), 3413–3442.

[9] A. Prakash, Analytical method for space-fractional telegraph equation by homotopy perturbation transform method, *Nonlinear Eng.*, **5(2)**(2016), 123–128.

[10] M. Safari, D. D. Ganji, and M. Moslemi, Application of he's variational iteration method and Adomian's decomposition method to the fractional KdV-Burgers Kuramoto equation, *Comput. Math. Appl.*, **58(11-12)**(2009), 2091–2097.

[11] J. Singh, D. Kumar, D. Baleanu, and S. Rathore, An efficient numerical algorithm for the fractional Drinfeld-Sokolov-Wilson equation, *Appl. Math. Comput.*, **335**(2018), 12–24.

[12] Y. Chen, I. Petras, and D. Xue, Fractional-order control-A tutorial, *Proc. Amer. Control Conf.*, 2009, 1397–1411.

[13] J. D. Singh Kumar and D. Baleanu, On the analysis of fractional diabetes model with exponential law, *Adv. Difference Equ.*, **231**(2018).

[14] J. Singh, D. Kumar, and A. Kiliçman, Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological populations, *Abstr. Appl. Anal.*, 2014, Art. no. 535793.

[15] K. Hosseini, Y. J. Xu, P. Mayeli, A. Bekir, P. Yao, Q. Zhou, and O. Guner, A study on the conformable time-fractional Klein-Gordon equations with quadratic and cubic nonlinearities, *Optoelectron. Adv. Mater.-Rapid Commun.*, **11(7-8)**(2017), 423–429.

[16] B. Batiha, M. S. M. Noorani, I. Hashim, and K. Batiha, Numerical simulations of systems of PDEs by variational iteration method, *Phys. Lett. A,* **372(6)**(2008), 822–829.

[17] A. M. Wazwaz, The variational iteration method for solving linear and non-linear systems of PDEs, *Comput. Math. Appl.*, **54(7-8)**(2007), 895–902.

[18] H. Khan, R. Shah, D. Baleanu, and M. Arif, An efficient analytical technique, for the solution of fractional-order telegraph equations, *Mathematics*, **7(5)**(2019), 426–436.

[19] R. Shah, H. Khan, P. Kumam, M. Arif, and D. Baleanu, Natural transform de-composition method for solving fractional-order partial differential equations with proportional delay, *Mathematics*, **7(6)**(2019), 532–540.

[20] A. M. Wazwaz, *Partial Differential Equations: Methods and Applications*, Leiden, The Netherlands: Balkema Publishers, 2002.

[21] M. A. Abdou, Approximate solutions of a system of PDEEs arising in physics, *Int. J. Nonlinear Sci.*, **12(3)**(2011), 305–312.

[22] O. Özkan and A. Kurt, On conformable double Laplace transform, *Opt.Quantum Electron.*, **50(2)**(2018), 103–110.

[23] Y. Çenesiz, D. Baleanu, A. Kurt, and O. Tasbozan, New exact solutions of Burgers' type equations with conformable derivative, *Waves Random Complex Media*, **27(1)**(2017), 103–116.

[24] G. C. Wu and D. Baleanu, Variational iteration method for fractional calculus-a universal approach by Laplace transform, *Adv. Difference Equ.*, **18**(2013).

[25] O. Özkan, Approximate analytical solutions of systems of fractional partial differential equations, *Karaelmas Sci. Eng. J.*, **7(1)**(2017), 63–67.

[26] R. Shah, H. Khan, P. Kumam, and M. Arif, An analytical technique to solve the system of nonlinear fractional partial differential equations, *Mathematics*, **7(6)**(2019), 505–515.

[27] P. S. Kumar, P. Gomathi, S. Gowri, and A. Viswanathan, Applications of Mohand transform to mechanics and electrical circuit problems, *Int. J.Res. Advent Technol.*, **6(10)**(2018), 2838–2840.

[28] S. Aggarwal and R. Chauhan, A comparative study ofMohand and Aboodh transform, *Int. J. Res. Advent Technol.*, **7(1)**(2019), 520–529.

[29] S. Aggarwal and R. Chaudhary, A comparative study of Mohand and Laplace transform, *J. Emerg. Technol. Innov. Res.*, **6(2)**(2019), 230–240.

[30] S. Aggarwal and S. D. Sharma, A comparative study of Mohand and Sumudu transform, *J. Emerg. Technol. Innov. Res.*, **6(3)**(2019), 145–153.

[31] S. Aggarwal, R. Chauhan, and N. Sharma, Mohand transform of Bessel's functions, *Int. J. Res. Advent Technol.*, **6(11)**(2018), 3034–3038.

[32] S. Aggarwal, N. Sharma, and R. Chauhan, Solution of linear Volterra integral equations of a second kind using Mohand transform, *Int. J. Res.Advent Technol.*, **6(11)**(2018), 3098–3102.

[33] S. Aggarwal, S. D. Sharma, and A. R. Gupta, A new application of Mohand transform for handling Abel's integral equation, *J. Emerg. Technol. Innov. Res.*, **6(3)**(2019), 600–608.

[34] M. Mohand and A. Mahgoub, The new integral transform 'Mohand' transform, *Adv. Theor. Appl. Math.*, **12(2)**(2017), 113–120.

[35] S. Aggarwal, R. Mishra, and A. Chaudhary, A comparative study ofMohand and Elzaki transform, *Global J. Eng. Sci. Researches*, **6(2)**(2019), 203–213.

[36] K. Kothari, U. Mehta, and J. Vanualailai, A novel approach of fractional order time-delay system modeling based on Haar wavelet, *ISA Trans.*, **80**(2018), 371–380.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 20–35.
http://doi.org/10.26637/mjm1001/002

Exponential stability to a laminated beam in thermoelasticity of type III with delay

Madani Douib*[1,2], Salah Zitouni[3] and Abdelhak Djebabla[4]
[1,4] *Department of Mathematics, Faculty of Sciences, University of Annaba, P.O. Box 12, Annaba 23000, Algeria.*
[2] *Department of Mathematics, Teachers Higher College of Laghouat, Algeria.*
[3] *Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras 41000, Algeria.*

***R*eceived 04 September 2021; *A*ccepted 21 December 2021**

Abstract. In this paper, we study the well-posedness and asymptotic behaviour of solutions to a laminated beam in thermoelasticity of type III with delay term in the fourth equation. We first give the well-posedness of the system by using semigroup method and Lumer-Philips theorem. Then, by using the perturbed energy method and construct some Lyapunov functionals, we obtain the exponential decay result for the case of equal wave speeds.
AMS Subject Classifications: 35B40, 35L56, 93D20, 74F05.

Keywords: Laminated beam, thermoelasticity of type III, delay, well-posedness, exponential stability.

Contents

1. Introduction

In this work, we consider a coupled system of a laminated beam with thermoelasticity of type III and delay term in the fourth equation, which has the form

$$\begin{cases} \rho_1\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ \rho_2\left(3\omega - \psi\right)_{tt} - G\left(\psi - \varphi_x\right) - D\left(3\omega - \psi\right)_{xx} + \alpha\theta_x = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ \rho_2\omega_{tt} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma\omega + \frac{4}{3}\beta\omega_t - D\omega_{xx} = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \sigma\left(3\omega - \psi\right)_{ttx} - \mu_1\theta_{txx}\left(x,t\right) - \mu_2\theta_{txx}\left(x,t-\tau\right) = 0, & (x,t) \in (0,1)\times(0,+\infty), \end{cases} \tag{1.1}$$

with the following initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in [0,1], \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in [0,1], \\ \omega(x,0) = \omega_0(x), \omega_t(x,0) = \omega_1(x), & x \in [0,1], \\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), & x \in [0,1], \\ \theta_{tx}\left(x,t-\tau\right) = f_0\left(x,t-\tau\right), & (x,t) \in (0,1)\times(0,\tau), \\ \varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = 0, & t \in [0,+\infty), \\ \omega(0,t) = \omega\left(1,t\right) = \theta_x\left(0,t\right) = \theta_x\left(1,t\right) = 0, & t \in [0,+\infty), \end{cases} \tag{1.2}$$

***Corresponding author.** Email addresses: **madanidouib@gmail.com** (Madani Douib), **zitsala@yahoo.fr** (Salah Zitouni), **adjebabla@yahoo.com** (Abdelhak Djebabla)

where $\varphi(x,t)$ denotes the transverse displacement, $\psi(x,t)$ represents the rotation angle. $\omega(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x. $\theta(x,t)$ is the differential temperature, and $\rho_1,\rho_2,\rho_3,G,D,\alpha,\beta,\gamma,\delta,\sigma,\mu_1$ are positive constants, μ_2 is a real number and $\tau>0$ represents the time delay. Moreover, $\sqrt{\frac{G}{\rho_1}}$ and $\sqrt{\frac{D}{\rho_2}}$ are two wave speeds.

Laminated beam, which is a relevant research subject due to the high applicability of such materials in the industry, was firstly introduced by Hansen and Spies, see, for instance [15, 16]. Hansen [15] proposed a model of laminated beam based on the Timoshenko system which is one of particular interest. In [16], Hansen and Spies derived three mathematical models for two-layered beams with structural damping due to the interfacial slip. The system is given by the following equations

$$\begin{cases} \rho_1\varphi_{tt}+G\left(\psi-\varphi_x\right)_x=0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_2\left(3\omega-\psi\right)_{tt}-D\left(3\omega-\psi\right)_{xx}-G\left(\psi-\varphi_x\right)=0, & (x,t)\in(0,1)\times(0,+\infty),\\ 3\rho_2\omega_{tt}+3G\left(\psi-\varphi_x\right)+4\gamma\omega+4\beta\omega_t-3D\omega_{xx}=0, & (x,t)\in(0,1)\times(0,+\infty), \end{cases}$$

the coefficients ρ_1, G, ρ_2, D, γ and β are positive constants and represent density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. The third equation describes the dynamics of the slip. For asymptotic behavior results to laminated beams, we refer the reader to [1, 19, 21, 22, 31] and the references therein. In [26], Rivera and Racke established several exponential decay results for linear Timoshenko systems in classical thermoelasticity where the heat flux is given by Fourier's law. Since this theory predicts an infinite speed of heat propagation, many theories have emerged, to overcome this physical paradox. Green and Naghdi [11–13], suggest a replacing Fourier's law by the so- called thermoelasticity of type III. This is for heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed. For more details, see [2]. A large number of interesting decay results depending on the stability number have been established, (see [9, 24, 25, 27] and references therein). W. Liu et al. [23] considered a coupled system of a laminated beam with thermoelasticity of type III, which has the form

$$\begin{cases} \rho_1\varphi_{tt}+G\left(\psi-\varphi_x\right)_x=0, & (x,t)\in(0,1)\times(0,+\infty),\\ I_{\rho_1}\left(3\omega-\psi\right)_{tt}-D\left(3\omega-\psi\right)_{xx}-G\left(\psi-\varphi_x\right)+\alpha\theta_x=0, & (x,t)\in(0,1)\times(0,+\infty),\\ I_{\rho_1}\omega_{tt}-D\omega_{xx}+G\left(\psi-\varphi_x\right)+\frac{4}{3}\beta_1\omega+\frac{4}{3}\beta_2\omega_t=0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_2\theta_{tt}-\delta\theta_{xx}+\gamma\left(3\omega-\psi\right)_{ttx}-k\theta_{txx}=0, & (x,t)\in(0,1)\times(0,+\infty), \end{cases}$$

they used the energy method to prove an exponential decay result for the case of equal wave speeds.

Time delay appears in many physical, biological and economic problems, because, in most instances, the present state system does not depend only on the current state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research.The presence of delay may be a source of instability. It may turn a well-behaved system into a wild one. For example, it was shown in [4, 5, 14, 28, 32] that an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used. The stability issue of systems with delay is, therefore, of theoretical and practical great importance. In [29], Nicaise, Pignotti and Valein replaced the constant delay term in the boundary condition of [28] by a time-varying delay term and obtained an exponential decay result under an appropriate assumption on the weights of the damping and delay. Moreover, Kafini et al. [18] studied the following Timoshenko system of thermoelasticity of type III with delay of the form

$$\begin{cases} \rho_1\phi_{tt}-K\left(\phi_x+\psi\right)_x+\mu_1\phi_t\left(x,t\right)+\mu_2\phi_t\left(x,t-\tau\right)=0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_2\psi_{tt}-b\psi_{xx}+K\left(\phi_x+\psi\right)+\beta\theta_{tx}=0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_3\theta_{tt}-\delta\theta_{xx}+\gamma\psi_{tx}-k\theta_{txx}=0, & (x,t)\in(0,1)\times(0,+\infty), \end{cases}$$

under the initial and boundary conditions

$$\begin{cases} \theta(.,0)=\theta_0,\ \theta_t(.,0)=\theta_1,\psi(.,0)=\psi_0, & x\in[0,1],\\ \psi_t(.,0)=\psi_1,\phi(.,0)=\phi_0,\phi_t(.,0)=\phi_1, & x\in[0,1],\\ \phi_t\left(x,t-\tau\right)=f_0\left(x,t-\tau\right), & t\in(0,\tau),\\ \phi(0,t)=\phi(1,t)=\psi(0,t)=\psi(1,t)=\theta_x\left(0,t\right)=\theta_x\left(1,t\right)=0, t\in[0,+\infty), \end{cases}$$

the energy of system decays exponentially in the case of equal wave speeds. For other related results, we refer the reader to [3, 6–8, 17, 20]. Motivated by the above results, in the present work, we study the well-posedness and asymptotic behaviour of solutions to the laminated beam (1.1)-(1.2) in thermoelasticity of type III with delay term. The plan of the paper is as follows. In Section 2, we introduce some preliminaries. In Section 3, by using semigroup method and Lumer-Philips theorem, we state and prove the well posedness of the system. In Section 4, by using the perturbed energy method and construct some Lyapunov functionals, we then establish the exponential result if and only if $\frac{G}{\rho_1} = \frac{D}{\rho_2}$.

2. Preliminaries

In this section, we present some material that we shall use in order to present our results, to exhibit the dissipative nature of the system (1.1), we introduce some new variables

$$\Phi = \varphi_t \,, \Psi = \psi_t \,, W = \omega_t,$$

and we introduce as in [28] the new variable

$$z\left(x,\rho,t\right) = \theta_{tx}\left(x,t-\tau\rho\right), (x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty).$$

Then we have

$$\tau z_t\left(x,\rho,t\right) + z_\rho\left(x,\rho,t\right) = 0, (x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty).$$

Therefore, system (1.1) takes the form

$$\begin{cases} \rho_1\Phi_{tt} + G\left(\Psi-\Phi_x\right)_x = 0, \\ \rho_2\left(3W-\Psi\right)_{tt} - G\left(\Psi-\Phi_x\right) - D\left(3W-\Psi\right)_{xx} + \alpha\theta_{tx} = 0, \\ \rho_2 W_{tt} + G\left(\Psi-\Phi_x\right) + \frac{4}{3}\gamma W + \frac{4}{3}\beta W_t - DW_{xx} = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} - \mu_1\theta_{txx} - \mu_2 z_x\left(x,1,t\right) + \sigma\left(3W-\Psi\right)_{tx} = 0, \\ \tau z_t\left(x,\rho,t\right) + z_\rho\left(x,\rho,t\right) = 0, \end{cases} \tag{2.1}$$

where $(x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty)$, with the initial data and boundary conditions

$$\begin{cases} \Phi(x,0) = \Phi_0(x), \Phi_t(x,0) = \Phi_1(x), & x \in [0,1], \\ \Psi(x,0) = \Psi_0(x), \Psi_t(x,0) = \Psi_1(x), & x \in [0,1], \\ W(x,0) = W_0(x), W_t(x,0) = W_1(x), & x \in [0,1], \\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), & x \in [0,1], \\ z\left(x,\rho,0\right) = f_0\left(x,-\tau\rho\right), & (x,\rho) \in (0,1)\times(0,1), \\ z\left(x,0,t\right) = \theta_{tx}\left(x,t\right), & (x,t) \in (0,1)\times(0,\infty), \\ \Phi_x(0,t) = \Phi_x(1,t) = \Psi(0,t) = \Psi(1,t) = 0, & t \in [0,+\infty), \\ W(0,t) = W\left(1,t\right) = \theta_x\left(0,t\right) = \theta_x\left(1,t\right) = 0, & t \in [0,+\infty), \end{cases} \tag{2.2}$$

where

$$\begin{aligned} \Phi_0(x) &= \varphi_1,\ \ \Phi_1(x) = -\frac{G}{\rho_1}\left(\psi_0-\varphi_{0x}\right)_x,\ \Psi_0\left(x\right) = \psi_1, \\ \Psi_1(x) &= -\frac{4G}{\rho_2}\left(\psi_0-\varphi_{0x}\right) - \frac{D}{\rho_2}\left(3\omega_0-\psi_0\right)_{xx} + \frac{\alpha}{\rho_2}\theta_{1x} - \frac{4\gamma}{\rho_2}\omega_0 - \frac{4\beta}{\rho_2}\omega_1 + \frac{3D}{\rho_2}\omega_{0xx}, \\ W_0(x) &= \omega_1, W_1(x) = -\frac{G}{\rho_2}\left(\psi_0-\varphi_{0x}\right) - \frac{4\gamma}{3\rho_2}\omega_0 - \frac{4\beta}{3\rho_2}\omega_1 + \frac{D}{\rho_2}\omega_{0xx}, \end{aligned}$$

where $x \in [0,1]$. From equations $(2.1)_4$ and (2.2), we easily verify that

$$\frac{d^2}{dt^2}\int_0^1 \theta\left(x,t\right)dx = 0.$$

So, if we set

$$\overline{\theta}(x,t) := \theta(x,t) - \int_0^1 \theta_0(x)\,dx - t\int_0^1 \theta_1(x)\,dx,$$

then simple substitution shows that $\left(\Phi, \Psi, W, \overline{\theta}, z\right)$ satisfies (2.1), the boundary conditions in (2.2) and more importantly

$$\int_0^1 \overline{\theta}(x,t)\,dx = 0, \ \ \forall t > 0.$$

In this case, Poincaré's inequality is applicable for $\overline{\theta}$. In the sequel, we work with $\overline{\theta}$ but for convenience, we write θ instead. We will assume that

$$\mu_1 > |\mu_2|, \tag{2.3}$$

and show the well-posedness of the problem and that this condition is sufficient to prove the uniform decay of the solution energy.

3. Well-posedness of the problem

In this Section, we prove the existence and uniqueness of solutions for (2.1)-(2.2). Introducing the vector function

$$U = (\Phi, 3W - \Psi, W, \theta, \Phi_t, 3W_t - \Psi_t, W_t, \theta_t, z)^T,$$

system (2.1)-(2.2) can be written as

$$\begin{cases} \dfrac{dU(t)}{dt} = \mathcal{A}U(t), t > 0, \\ U(0) = U_0 = (\Phi_0, 3W_0 - \Psi_0, W_0, \theta_0, \Phi_1, 3W_1 - \Psi_1, W_1, \theta_1, f_0)^T, \end{cases} \tag{3.1}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\begin{pmatrix} \Phi \\ 3W - \Psi \\ W \\ \theta \\ \Phi_t \\ 3W_t - \Psi_t \\ W_t \\ \theta_t \\ z \end{pmatrix} = \begin{pmatrix} \Phi_t \\ 3W_t - \Psi_t \\ W_t \\ \theta_t \\ -\dfrac{G}{\rho_1}(\psi - \Phi_x)_x \\ \dfrac{G}{\rho_2}(\psi - \Phi_x) + \dfrac{D}{\rho_2}(3W - \Psi)_{xx} - \dfrac{\alpha}{\rho_2}\theta_{tx} \\ -\dfrac{G}{\rho_2}(\psi - \Phi_x) - \dfrac{4\gamma}{3\rho_2}W - \dfrac{4\beta}{3\rho_2}W_t + \dfrac{D}{\rho_2}W_{xx} \\ \dfrac{\delta}{\rho_3}\theta_{xx} - \dfrac{\sigma}{\rho_3}(3W - \Psi)_{tx} + \dfrac{\mu_1}{\rho_3}\theta_{txx} + \dfrac{\mu_2}{\rho_3}z_x(x,1,t) \\ -\tau^{-1}z_\rho \end{pmatrix}.$$

We consider the following spaces

$$L^2_*(0,1) = \left\{ w \in L^2(0,1) : \int_0^1 w(s)\,ds = 0 \right\}, \ H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1),$$
$$H^2_*(0,1) = \left\{ w \in H^2(0,1) : w_x(0) = w_x(1) = 0 \right\}.$$

Let

$$\mathcal{H} = H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H^1_*(0,1) \times L^2_*(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2_*(0,1) \times L^2\left((0,1), L^2(0,1)\right),$$

be the Hilbert space equipped with the inner product

$$\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} = \sigma\rho_1\int_0^1 \Phi_t\overline{\Phi}_t dx + \sigma G\int_0^1 (\Psi-\Phi_x)\left(\overline{\Psi}-\overline{\Phi}_x\right)dx + 4\sigma\gamma\int_0^1 W\overline{W}dx + 3\sigma\int_0^1 \rho_2 W_t\overline{W}_t dx$$
$$+\sigma\rho_2\int_0^1 (3W-\Psi)_t\left(3\overline{W}-\overline{\Psi}\right)_t dx + \sigma\int_0^1 D(3W-\Psi)_x\left(3\overline{W}-\overline{\Psi}\right)_x dx$$
$$+3\sigma D\int_0^1 W_x\overline{W}_x dx + \alpha\rho_3\int_0^1 \theta_t\overline{\theta}_t dx + \alpha\delta\int_0^1 \theta_x\overline{\theta}_x dx + \lambda\int_0^1\int_0^1 z\overline{z}d\rho dx,$$

where λ is the positive constant satisfying

$$\begin{cases} \tau\alpha\left|\mu_2\right| < \lambda < \tau\alpha\left(2\mu_1-\left|\mu_2\right|\right), & \text{if} \quad \left|\mu_2\right| < \mu_1, \\ \lambda = \tau\alpha\mu_1, & \text{if} \quad \left|\mu_2\right| = \mu_1. \end{cases} \tag{3.2}$$

Then, the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U\in\mathcal{H} \mid \Phi,\theta\in H_*^2(0,1)\cap H_*^1(0,1), \Psi, W\in H^2(0,1)\cap H_0^1(0,1), \\ \Psi_t, W_t\in H_0^1(0,1), \Phi_t,\theta_t\in H_*^1(0,1), \left(\delta+e^{-\tau}\mu_2\right)\theta+\mu_1\theta_t\in H_*^2(0,1), \\ z, z_\rho\in L^2\left((0,1), L^2(0,1)\right), z(x,0)=\theta_{tx}(x) \end{array} \right\}. \tag{3.3}$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

Theorem 3.1. *Assume that $U_0 \in \mathcal{H}$ and (2.3) holds. Then there exists a unique solution $U \in C(\mathbb{R}^+;\mathcal{H})$ of problem (3.1). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U\in C\left(\mathbb{R}^+; D(\mathcal{A})\cap C^1\left(\mathbb{R}^+;\mathcal{H}\right)\right).$$

Proof. The result follows from Lumer-Phillips theorem provided we prove that $\mathcal{A}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathcal{A}$ is dissipative and $Id-\mathcal{A}$ surjective.

Step 1. $\mathcal{A}$ is dissipative.

For any $U \in D(\mathcal{A})$, and using the inner product, we obtain

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}} = -4\sigma\beta\int_0^1 W_t^2 dx - \alpha\mu_1\int_0^1 \theta_{tx}^2 + \alpha\mu_2\int_0^1 z_x(x,1,t)\,\theta_t dx - \frac{\lambda}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)\,d\rho dx. \tag{3.4}$$

By using integration by parts and the fact that $z(x,0)=\theta_{tx}(x)$, the last term in the right-hand side of (3.4) gives

$$-\int_0^1\int_0^1 zz_\rho(x,\rho,t)\,d\rho dx = \frac{1}{2}\int_0^1 \theta_{tx}^2 dx - \frac{1}{2}\int_0^1 z^2(x,1,t)\,dx. \tag{3.5}$$

Substituting (3.5) in (3.4) yields

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}} = -4\sigma\beta\int_0^1 W_t^2 dx - \alpha\mu_1\int_0^1 \theta_{tx}^2 + \alpha\mu_2\int_0^1 z_x(x,1,t)\,\theta_t dx + \frac{\lambda}{2\tau}\int_0^1 \theta_{tx}^2 dx$$
$$-\frac{\lambda}{2\tau}\int_0^1 z^2(x,1,t)\,dx. \tag{3.6}$$

Also, using integration by parts and Young's inequality we obtain, from (3.6)

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}} \le -\left(\alpha\mu_1 - \frac{\alpha\left|\mu_2\right|}{2} - \frac{\lambda}{2\tau}\right)\int_0^1 \theta_{tx}^2 dx - \left(\frac{\lambda}{2\tau} - \frac{\alpha\left|\mu_2\right|}{2}\right)\int_0^1 z^2(x,1,t)\,dx - 4\sigma\beta\int_0^1 W_t^2 dx.$$

Keeping in mind condition (3.2), we observe that

$$\alpha\mu_1 - \frac{\alpha|\mu_2|}{2} - \frac{\lambda}{2\tau} \geq 0, \qquad \frac{\lambda}{2\tau} - \frac{\alpha|\mu_2|}{2} \geq 0.$$

Consequently, the operator $\mathcal{A}$ is dissipative.

Step 2. $Id - \mathcal{A}$ is surjective.

To prove that the operator $Id - \mathcal{A}$ is surjective, that is, for any $F = (f_1, ..., f_9) \in \mathcal{H}$, there exists $U = (\Phi, 3W - \Psi, W, \theta, \Phi_t, 3W_t - \Psi_t, W_t, \theta_t, z) \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})\, U = F, \tag{3.7}$$

which is equivalent to

$$\begin{cases} \Phi - \Phi_t = f_1, \\ (3W - \Psi) - (3W - \Psi)_t = f_2, \\ W - W_t = f_3, \\ \theta - \theta_t = f_4, \\ \rho_1\Phi_t - G\Phi_{xx} - G(3W - \Psi)_x + 3GW_x = \rho_1 f_5, \\ \rho_2 (3W - \Psi)_t + G\Phi_x + G(3W - \Psi) - 3GW - D(3W - \Psi)_{xx} + \alpha\theta_{tx} \\ = \rho_2 f_6, \\ \rho_2 W_t - G(3W - \Psi) + 3GW - G\Phi_x + \frac{4\gamma}{3}W + \frac{4\beta}{3}W_t - DW_{xx} = \rho_2 f_7, \\ \rho_3\theta_t - \delta\theta_{xx} + \sigma(3W - \Psi)_{tx} - \mu_1\theta_{txx} - \mu_2 z_x(x, 1, t) = \rho_3 f_8, \\ \tau z + z_\rho = \tau f_9. \end{cases} \tag{3.8}$$

From $(3.8)_1$–$(3.8)_4$, we have

$$\begin{cases} \Phi_t = \Phi - f_1, \\ (3W - \Psi)_t = (3W - \Psi) - f_2, \\ W_t = W - f_3, \\ \theta_t = \theta - f_4. \end{cases} \tag{3.9}$$

By combining (3.9) and (3.8), it can be $\Phi, 3W - \Psi, W, \theta$ shown that satisfy

$$\begin{cases} \rho_1\Phi - G\Phi_{xx} - G(3W - \Psi)_x + 3GW_x = \rho_1(f_1 + f_5), \\ \rho_2(3W - \Psi) + G\Phi_x + G(3W - \Psi) - 3GW - D(3W - \Psi)_{xx} + \alpha\theta_x \\ = \rho_2(f_2 + f_6) + \alpha\partial_x f_4, \\ \rho_2 W - G(3W - \Psi) + 3GW - G\Phi_x + \frac{4\gamma}{3}W + \frac{4\beta}{3}W - DW_{xx} \\ = \rho_2(f_3 + f_7) + \frac{4\beta}{3}f_3, \\ \rho_3\theta - \delta\theta_{xx} + \sigma(3W - \Psi)_x - \mu_1\theta_{xx} - \mu_2 z_x(x, 1, t) \\ = \rho_3(f_4 + f_8) + \sigma\partial_x f_2 + \mu_1\partial_{xx} f_4, \\ \tau z + z_\rho = \tau f_9. \end{cases} \tag{3.10}$$

Using the last equation in (3.10) we can find z with

$$z(x, 0) = \theta_{tx}(x),\ x \in (0, 1).$$

Following the same approach as in [28], we obtain, by using $(3.10)_5$,

$$z(x, \rho, \tau) = \theta_{tx}(x)\, e^{-\tau\rho} + \tau e^{-\tau\rho} \int_0^\rho e^{\tau s} f_9(x, s)\, ds.$$

From $(3.9)_4$, we obtain

$$z(x, \rho, \tau) = \theta_x e^{-\tau\rho} - \partial_x f_4(x)\, e^{-\tau\rho} + \tau e^{-\tau\rho} \int_0^\rho e^{\tau s} f_9(x, s)\, ds, \tag{3.11}$$

and in particular,

$$z\left(x,1,\tau\right)=\theta_{x}e^{-\tau}+z_{0}\left(x,\tau\right),$$

where

$$z_{0}\left(x,\tau\right)=-\partial_{x}f_{4}\left(x\right)e^{-\tau}+\tau e^{-\tau}\int_{0}^{1}e^{\tau s}f_{9}\left(x,s\right)ds.$$

In order to solve (3.8), we consider the following variational formulation

$$B\left(\left(\Phi,3W-\Psi,W,\theta\right)^{T},\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W},\tilde{\theta}\right)^{T}\right)=G\left(\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W},\tilde{\theta}\right)^{T}\right),\tag{3.12}$$

where $B:\left[H_{*}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{*}^{1}\left(0,1\right)\right]^{2}\longrightarrow\mathbb{R}$ is the bilinear form

$$\begin{aligned}
&B\left(\left(\Phi,3W-\Psi,W,\theta\right)^{T},\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W},\tilde{\theta}\right)^{T}\right)\\
&=\sigma\int_{0}^{1}G(\Psi-\Phi_{x})(\tilde{\Psi}-\tilde{\Phi}_{x})dx+\sigma\int_{0}^{1}\rho_{1}\Phi\tilde{\Phi}dx+\sigma\int_{0}^{1}\rho_{2}\left(3W-\Psi\right)\left(3\tilde{W}-\tilde{\Psi}\right)dx+\alpha\int_{0}^{1}\rho_{3}\theta\tilde{\theta}dx\\
&\quad+\left(3\sigma\rho_{2}+4\sigma\gamma+4\sigma\beta\right)\int_{0}^{1}W\tilde{W}dx+\sigma\int_{0}^{1}D\left(3W-\Psi\right)_{x}\left(3\tilde{W}-\tilde{\Psi}\right)_{x}dx+3\sigma\int_{0}^{1}DW_{x}\tilde{W}_{x}dx\\
&\quad+\alpha\left(\delta+\mu_{1}+e^{-\tau}\mu_{2}\right)\int_{0}^{1}\theta_{x}\tilde{\theta}_{x}dx+\sigma\alpha\int_{0}^{1}\left(3W-\Psi\right)_{x}\tilde{\theta}dx+\sigma\alpha\int_{0}^{1}\theta_{x}\left(3\tilde{W}-\tilde{\Psi}\right)dx,
\end{aligned}$$

and $G:H_{*}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{*}^{1}\left(0,1\right)\longrightarrow\mathbb{R}$ is the linear form

$$\begin{aligned}
&F\left(\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W},\tilde{\theta}\right)^{T}\right)\\
&=\sigma\int_{0}^{1}\rho_{1}\left(f_{1}+f_{5}\right)\tilde{\Phi}dx+\sigma\int_{0}^{1}\rho_{2}\left(f_{2}+f_{6}\right)\left(3\tilde{W}-\tilde{\Psi}\right)dx+3\sigma\int_{0}^{1}\rho_{2}\left(f_{3}+f_{7}\right)\tilde{W}dx\\
&\quad+4\sigma\int_{0}^{1}\beta f_{3}\tilde{W}dx+\alpha\int_{0}^{1}\rho_{3}\left(f_{4}+f_{8}\right)\tilde{\theta}dx+\alpha\sigma\int_{0}^{1}\partial_{x}f_{2}\tilde{\theta}dx+\alpha\mu_{1}\int_{0}^{1}\partial_{x}f_{4}\partial_{x}\tilde{\theta}dx\\
&\quad+\sigma\alpha\int_{0}^{1}\partial_{x}f_{4}\left(3\tilde{W}-\tilde{\Psi}\right)dx-\alpha\mu_{2}\int_{0}^{1}\partial_{x}z_{0}\tilde{\theta}dx.
\end{aligned}$$

Now, for

$$V=H_{*}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{*}^{1}\left(0,1\right),$$

equipped with the norm

$$\left\|\left(\Phi,3W-\Psi,W,\theta\right)\right\|_{V}^{2}=\left\|\Psi-\Phi_{x}\right\|_{2}^{2}+\left\|\Phi\right\|_{2}^{2}+\left\|\left(3W-\Psi\right)_{x}\right\|_{2}^{2}+\left\|W_{x}\right\|_{2}^{2}+\left\|\theta\right\|_{2}^{2}+\left\|\theta_{x}\right\|_{2}^{2},$$

one can easily see that $B(.,.)$ and $G(.)$ are bounded. Furthermore, using integration by parts, we obtain

$$B\left(\left(\Phi,3W-\Psi,W,\theta\right)^{T},\left(\Phi,3W-\Psi,W,\theta\right)^{T}\right)\geq c\left\|\left(\Phi,3W-\Psi,W,\theta\right)\right\|_{V}^{2},$$

for some $c>0$. Thus, $B(.,.)$ is coercive.

Consequently, by Lax-Milgram lemma, we obtain that (3.12) has a unique solution

$$\Phi\in H_{*}^{1}\left(0,1\right),\quad\left(3W-\Psi\right)\in H_{0}^{1}\left(0,1\right),\quad W\in H_{0}^{1}\left(0,1\right),\quad\theta\in H_{*}^{1}\left(0,1\right).$$

The substitution of $\Phi,3W-\Psi,W$ and θ into (3.9) yields

$$\Phi_{t}\in H_{*}^{1}\left(0,1\right),\quad\left(3W-\Psi\right)_{t}\in H_{0}^{1}\left(0,1\right),\quad W_{t}\in H_{0}^{1}\left(0,1\right),\quad\theta_{t}\in H_{*}^{1}\left(0,1\right).$$

Next, it remains to show that

$$\Phi \in \left(H_*^2(0,1) \cap H_*^1(0,1)\right), \quad (3W-\Psi) \in \left(H^2(0,1) \cap H_0^1(0,1)\right),$$
$$W \in \left(H^2(0,1) \cap H_0^1(0,1)\right), \quad \theta \in \left(H_*^2(0,1) \cap H_*^1(0,1)\right).$$

Taking $\left(3\tilde{W}-\tilde{\Psi}, \tilde{W}, \tilde{\theta}\right) = (0,0,0) \in H_0^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1)$ in (3.12), we get

$$\begin{aligned} & B\left((\Phi, 3W-\Psi, W, \theta)^T, \left(\tilde{\Phi}, 0,0,0\right)^T\right) \\ &= \sigma \int_0^1 \rho_1 \Phi \tilde{\Phi} dx + \sigma \int_0^1 G(-\Phi_{xx}\tilde{\Phi} - (3W-\Psi)_x \tilde{\Phi} + 3W_x \tilde{\Phi}) dx \\ &= \sigma \int_0^1 \rho_1 (f_1 + f_5) \tilde{\Phi} dx, \quad \forall \tilde{\Phi} \in H_*^1(0,1), \end{aligned} \tag{3.13}$$

which implies

$$G\Phi_{xx} = \rho_1 \Phi - G(3W-\Psi)_x + 3GW_x - \rho_1 (f_1+f_5) \in L_*^2(0,1). \tag{3.14}$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$\Phi \in H^2(0,1) \cap H_*^1(0,1).$$

Moreover, (3.13) is also true for any $\phi \in C^1[0,1] \subset H_*^1(0,1)$. Hence, we have

$$\int_0^1 G\Phi_x \phi_x dx + \int_0^1 \left(\rho_1 \Phi - G(3W-\Psi)_x + 3GW_x - \rho_1 (f_1+f_5)\right) \phi dx = 0$$

for all $\phi \in C^1[0,1]$. Thus, using integration by parts and bearing in mind (3.14), we obtain

$$\Phi_x(1)\phi(1) - \Phi_x(0)\phi(0) = 0, \forall \phi \in C^1[0,1].$$

Therefore, $\Phi_x(0) = \Phi_x(1) = 0$. Consequently, we obtain

$$\Phi \in H_*^2(0,1) \cap H_*^1(0,1).$$

In the same way, taking $\left(\tilde{\Phi}, \tilde{W}, \tilde{\theta}\right) = (0,0,0) \in H_*^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1)$ in (3.12), we get

$$\begin{aligned} & B\left((\Phi, 3W-\Psi, W, \theta)^T, \left(0, 3\tilde{W}-\tilde{\Psi}, 0, 0\right)^T\right) \\ &= \sigma \int_0^1 G\left(\Phi_x\left(3\tilde{W}-\tilde{\Psi}\right) + (3W-\Psi)\left(3\tilde{W}-\tilde{\Psi}\right) - 3W\left(3\tilde{W}-\tilde{\Psi}\right)\right) dx \\ &\quad + \sigma \int_0^1 \rho_2 (3W-\Psi)\left(3\tilde{W}-\tilde{\Psi}\right) dx + \sigma \int_0^1 D(3W-\Psi)_x \left(3\tilde{W}-\tilde{\Psi}\right)_x dx + \sigma\alpha \int_0^1 \theta_x \left(3\tilde{W}-\tilde{\Psi}\right) dx \\ &= \sigma \int_0^1 \rho_2 (f_2+f_6)\left(3\tilde{W}-\tilde{\Psi}\right) dx + \sigma\alpha \int_0^1 \partial_x f_4 \left(3\tilde{W}-\tilde{\Psi}\right) dx. \end{aligned}$$

Recalling $(3.8)_2$ and $(3.8)_4$, we arrive at

$$\begin{aligned} & \int_0^1 D(3W-\Psi)_x \left(3\tilde{W}-\tilde{\Psi}\right)_x dx \\ &= \int_0^1 \left[\rho_2 f_6 - G(\Phi_x + (3W-\Psi) - 3W) - \alpha\theta_{tx} - \rho_2 (3W-\Psi)_t\right] \left(3\tilde{W}-\tilde{\Psi}\right) dx \end{aligned} \tag{3.15}$$

for all $\left(3\tilde{W}-\tilde{\Psi}\right)\in H^{1}\left(0,1\right)$, which implies

$$\rho_{2}f_{6}-G\left(\Phi_{x}+\left(3W-\Psi\right)-3W\right)-\alpha\theta_{tx}-\rho_{2}\left(3W-\Psi\right)_{t}\in L^{2}\left(0,1\right).$$

Consequently, (3.15) takes the form

$$\int_{0}^{1}\left[-D\left(3W-\Psi\right)_{xx}+G\Phi_{x}+G\left(3W-\Psi\right)-3GW+\alpha\theta_{tx}+\rho_{2}\left(3W-\Psi\right)_{t}-\rho_{2}f_{6}\right]\left(3\tilde{W}-\tilde{\Psi}\right)dx=0.$$

We obtain

$$-D\left(3W-\Psi\right)_{xx}+G\left(\Phi_{x}+G\left(3W-\Psi\right)-3W\right)+\alpha\theta_{tx}+\rho_{2}\left(3W-\Psi\right)_{t}=\rho_{2}f_{6},$$

and

$$\left(3W-\Psi\right)\in H^{2}\left(0,1\right)\cap H_{0}^{1}\left(0,1\right),$$

which gives $(3.8)_6$. Similarly, we can show that

$$W\in H^{2}\left(0,1\right)\cap H_{0}^{1}\left(0,1\right),$$

and $(3.8)_7$ are satisfied. Also, if we take $\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W}\right)=(0,0,0)\in H_{*}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)$ in (3.12), then using $(3.8)_2$ and $(3.8)_4$, we get

$$\left(\delta+e^{-\tau}\mu_{2}\right)\theta_{xx}+\mu_{1}\theta_{txx}=\rho_{3}\theta_{t}-\rho_{3}f_{8}+\sigma\left(3W-\Psi\right)_{tx}+\mu_{2}\partial_{x}z_{0},$$

and we conclude that

$$\left(\delta+e^{-\tau}\mu_{2}\right)\theta+\mu_{1}\theta_{t}\in H^{2}\left(0,1\right).$$

Furthermore, it is obvious from

$$\left(\delta+e^{-\tau}\mu_{2}\right)\theta_{x}+\mu_{1}\theta_{tx}=\rho_{3}\int_{0}^{x}\theta_{t}dx-\rho_{3}\int_{0}^{x}f_{8}dx+\sigma\left(3W-\Psi\right)_{t}+\mu_{2}z_{0},$$

that

$$\left(\left(\delta+e^{-\tau}\mu_{2}\right)\theta_{x}+\mu_{1}\theta_{tx}\right)(0)=\left(\left(\delta+e^{-\tau}\mu_{2}\right)\theta_{x}+\mu_{1}\theta_{tx}\right)(1)=0,$$

then, we get

$$\left(\delta+e^{-\tau}\mu_{2}\right)\theta+\mu_{1}\theta_{t}\in H_{*}^{2}\left(0,1\right).$$

Finally, it follows, from (3.11), that

$$z\left(x,0\right)=\theta_{tx}\left(x\right)\quad\text{and } z,z_{\rho}\in L^{2}\left(\left(0,1\right),L^{2}\left(0,1\right)\right).$$

Hence, there exists a unique $U\in D\left(\mathcal{A}\right)$ such that (3.7) is satisfied, the operator $Id-\mathcal{A}$ is surjective. Moreover, it is easy to see that $D\left(\mathcal{A}\right)$ is dense in $\mathcal{H}$.

At last, by Lumer-Philips theorem (see [10, 30]) we have the well-posedness result stated in Theorem 3.1. ■

4. Exponential stability

In this section, we state and prove our stability result for the solution of problem (2.1)-(2.2), by using the multiplier technique. We first introduce the following energy functional

$$\begin{aligned}E\left(t\right):=\frac{1}{2}\int_{0}^{1}&\left[\sigma\rho_{1}\Phi_{t}^{2}+\sigma G\left(\Psi-\Phi_{x}\right)^{2}+\sigma\rho_{2}\left(3W-\Psi\right)_{t}^{2}+\sigma D\left(3W-\Psi\right)_{x}^{2}\right.\\&\left.+3\sigma\rho_{2}W_{t}^{2}+4\sigma\gamma W^{2}+3\sigma DW_{x}^{2}+\alpha\rho_{3}\theta_{t}^{2}+\alpha\delta\theta_{x}^{2}+\lambda\int_{0}^{1}z^{2}\left(x,\rho,t\right)d\rho\right]dx.\end{aligned}\tag{4.1}$$

To achieve our goal, we need the following lemmas.

Lemma 4.1. *Let (Φ,Ψ,W,θ,z) be the solution of problem (2.1)-(2.2). Then the energy functional $E(t)$ defined by (4.1) satisfies*

$$\frac{d}{dt}E\left(t\right)=-4\beta\sigma\int_0^1 W_t^2dx-C_1\int_0^1\theta_{tx}^2dx-C_2\int_0^1 z^2\left(x,1,t\right)dx\leq 0, \tag{4.2}$$

where

$$C_1=\mu_1\alpha-\frac{|\mu_2|\,\alpha}{2}-\frac{\lambda}{2\tau}\geq 0\quad,\quad C_2=\frac{\lambda}{2\tau}-\frac{|\mu_2|\,\alpha}{2}\geq 0.$$

Proof. Multiplying the first four equations in (2.1) by $\sigma\Phi_t$, $\sigma\left(3W-\Psi\right)_t$, $3\sigma W_t$, $\alpha\theta_t$ respectively, then, integrating over $(0,1)$, and multiplying $(2.1)_5$ by $\dfrac{\lambda}{\tau}z$ and integrating over $(0,1)\times(0,1)$ with respect to ρ and x, summing them up, we obtain

$$\begin{aligned}&\frac{d}{dt}\frac{\sigma}{2}\int_0^1\left[\rho_1\Phi_t^2+G\left(\Psi-\Phi_x\right)^2+\rho_2\left(3W_t-\Psi_t\right)^2+D\left(3W_x-\Psi_x\right)^2+3\rho_2W_t^2+4\gamma W^2+3DW_x^2\right]dx\\&+\frac{d}{dt}\frac{\alpha}{2}\int_0^1\left(\rho_3\theta_t^2+\delta\theta_x^2\right)dx+\frac{d}{dt}\frac{\lambda}{2}\int_0^1\int_0^1 z^2\left(x,\rho,t\right)d\rho dx\\=&-4\beta\sigma\int_0^1 W_t^2dx-\mu_1\alpha\int_0^1\theta_{tx}^2dx+\mu_2\alpha\int_0^1\theta_tz_x\left(x,1,t\right)dx-\frac{\lambda}{\tau}\int_0^1\int_0^1 zz_\rho\left(x,\rho,t\right)d\rho dx.\end{aligned} \tag{4.3}$$

The last two terms of the right side of (4.3) can be estimated as follows.

$$-\frac{\lambda}{\tau}\int_0^1\int_0^1 zz_\rho\left(x,\rho,t\right)d\rho dx=\frac{\lambda}{2\tau}\int_0^1\theta_{tx}^2dx-\frac{\lambda}{2\tau}\int_0^1 z^2\left(x,1,t\right)dx,$$

$$\mu_2\alpha\int_0^1\theta_tz_x\left(x,1,t\right)dx\leq\frac{|\mu_2|\,\alpha}{2}\int_0^1\theta_{tx}^2dx+\frac{|\mu_2|\,\alpha}{2}\int_0^1 z^2\left(x,1,t\right)dx.$$

Hence,

$$\frac{d}{dt}E\left(t\right)\leq-4\beta\sigma\int_0^1 W_t^2dx-\left(\mu_1\alpha-\frac{|\mu_2|\,\alpha}{2}-\frac{\lambda}{2\tau}\right)\int_0^1\theta_{tx}^2dx-\left(\frac{\lambda}{2\tau}-\frac{|\mu_2|\,\alpha}{2}\right)\int_0^1 z^2\left(x,1,t\right)dx.$$

Using (3.2), we obtain the result. ∎

Lemma 4.2. *Let (Φ,Ψ,W,θ,z) be the solution of problem (2.1)-(2.2). The functional*

$$F_1\left(t\right):=-\rho_1\int_0^1\Phi\Phi_tdx+\rho_2\int_0^1 WW_tdx \tag{4.4}$$

satisfies the estimate

$$\begin{aligned}F_1'\left(t\right)\leq&-\rho_1\int_0^1\Phi_t^2dx-\frac{2\gamma}{3}\int_0^1 W^2dx-\frac{D}{2}\int_0^1 W_x^2dx+C_3\int_0^1 W_t^2dx+C_4\int_0^1\left(\Psi-\Phi_x\right)^2dx\\&+\frac{D}{18}\int_0^1\left(3W_x-\Psi_x\right)^2dx,\end{aligned} \tag{4.5}$$

where

$$C_3=\rho_2+\frac{4\beta^2}{3\gamma}\quad,\quad C_4=G+\frac{9G^2}{2D}+\frac{3G^2}{4\gamma}.$$

Proof. By differentiating F_1 with respect to t, using $(2.1)_1$, $(2.1)_3$ and integrating by parts, we obtain

$$F_1'(t) = -\rho_1 \int_0^1 \Phi_t^2 dx - G\int_0^1 \Phi_x (\Psi - \Phi_x) dx + \rho_2 \int_0^1 W_t^2 dx - D\int_0^1 W_x^2 dx - G\int_0^1 W(\Psi - \Phi_x) dx$$
$$-\frac{4\gamma}{3}\int_0^1 W^2 dx - \frac{4\beta}{3}\int_0^1 WW_t dx.$$

Note that

$$-G\int_0^1 \Phi_x (\Psi - \Phi_x) dx = G\int_0^1 (\Psi - \Phi_x)^2 dx - G\int_0^1 \Psi(\Psi - \Phi_x) dx.$$

Then, we deduce that

$$F_1'(t) = -\rho_1 \int_0^1 \Phi_t^2 dx + G\int_0^1 (\Psi - \Phi_x)^2 dx - G\int_0^1 \Psi(\Psi - \Phi_x) dx + \rho_2 \int_0^1 W_t^2 dx - D\int_0^1 W_x^2 dx$$
$$-G\int_0^1 W(\Psi - \Phi_x) dx - \frac{4\gamma}{3}\int_0^1 W^2 dx - \frac{4\beta}{3}\int_0^1 WW_t dx.$$

Making use of Young's and Poincaré inequalities, we obtain

$$F_1'(t) \le -\rho_1 \int_0^1 \Phi_t^2 dx - \frac{2\gamma}{3}\int_0^1 W^2 dx - D\int_0^1 W_x^2 dx + \frac{D}{36}\int_0^1 \Psi_x^2 dx + \left(\rho_2 + \frac{4\beta^2}{3\gamma}\right)\int_0^1 W_t^2 dx$$
$$+\left(G + \frac{9G^2}{2D} + \frac{3G^2}{4\gamma}\right)\int_0^1 (\Psi - \Phi_x)^2 dx.$$

Note that

$$\int_0^1 \Psi_x^2 dx = \int_0^1 (\Psi_x - 3W_x + 3W_x)^2 dx \le 2\int_0^1 (3W_x - \Psi_x)^2 + 18\int_0^1 W_x^2 dx.$$

Then the estimate (4.5) is established. ■

Lemma 4.3. *Let $(\Phi, \Psi, W, \theta, z)$ be the solution of problem (2.1)-(2.2). The functional*

$$F_2(t) := \rho_2 \int_0^1 (3W - \Psi)(3W - \Psi)_t dx \tag{4.6}$$

satisfies the estimate

$$F_2'(t) \le -\frac{D}{2}\int_0^1 (3W_x - \Psi_x)^2 dx + \rho_2 \int_0^1 (3W_t - \Psi_t)^2 dx + \frac{G^2}{2D}\int_0^1 (\Psi - \Phi_x)^2 dx + \frac{\alpha^2}{D}\int_0^1 \theta_t^2 dx, \tag{4.7}$$

Proof. By differentiating F_2 with respect to t, using $(2.1)_2$ and integrating by parts, we get

$$F_2'(t) = G\int_0^1 (3W - \Psi)(\Psi - \Phi_x) dx - D\int_0^1 (3W_x - \Psi_x)^2 dx + \alpha\int_0^1 (3W_x - \Psi_x)\theta_t dx$$
$$+\rho_2 \int_0^1 (3W_t - \Psi_t)^2 dx.$$

Using Young's and Poincaré inequalities, we obtain the result. ■

Lemma 4.4. *Let $(\Phi, \Psi, W, \theta, z)$ be the solution of problem (2.1)-(2.2). The functional*

$$F_3(t) := \rho_2\rho_3 \int_0^1 (3W - \Psi)_t \int_0^x \theta_t(y,t) dy dx - \delta\rho_2 \int_0^1 \theta_x (3W - \Psi) dx \tag{4.8}$$

satisfies the estimate

$$F_3'(t) \le -\frac{\rho_2\sigma}{2}\int_0^1 (3W-\Psi)_t^2\,dx + \varepsilon_1\int_0^1 (\Psi-\Phi_x)^2\,dx + C_5(\varepsilon_1)\int_0^1 \theta_{xt}^2 dx + \varepsilon_1\int_0^1 (3W_x-\Psi_x)^2\,dx$$
$$+\frac{\rho_2\mu_2^2}{\sigma}\int_0^1 z^2(x,1,t)\,dx, \tag{4.9}$$

for any $\varepsilon_1 > 0$, *where*

$$C_5(\varepsilon_1) = \frac{\alpha\rho_3}{2} + \frac{\rho_2\mu_2^2}{\sigma} + \frac{D^2\rho_3^2}{8\varepsilon_1} + \frac{\delta^2\rho_2^2}{4\varepsilon_1} + \frac{G^2\rho_3^2}{16\varepsilon_1}.$$

Proof. By differentiating F_3 with respect to t, using $(2.1)_2$, $(2.1)_4$ and integrating by parts, we obtain

$$F_3'(t) = \rho_3\int_0^1 G(\Psi-\Phi_x)\int_0^x \theta_t(y,t)\,dydx - \delta\rho_2\int_0^1 \theta_{xt}(3W-\Psi)\,dx$$
$$+\left[\rho_3\left(-G\Phi + D(3W-\Psi)_x - \alpha\theta_t\right)\int_0^x \theta_t(y,t)\,dy\right]_{x=0}^{x=1} + \alpha\rho_3\int_0^1 \theta_t^2 dx - \rho_2\sigma\int_0^1 (3W-\Psi)_t^2\,dx$$
$$+\rho_2\mu_1\int_0^1 (3W-\Psi)_t\,\theta_{tx}dx - D\rho_3\int_0^1 \theta_t(3W-\Psi)_x\,dx + \rho_2\mu_2\int_0^1 (3W-\Psi)_t\,z(x,1,t)\,dx.$$

Note that

$$\int_0^1 \theta_t(y,t)\,dy = \frac{d}{dt}\int_0^1 \theta(y,t)\,dy = 0,$$

then, by Young's and Poincaré inequalities, with $\varepsilon_1 > 0$ to obtain (4.9). ■

Lemma 4.5. *Let* (Φ,Ψ,W,θ,z) *be the solution of problem (2.1)-(2.2). The functional*

$$F_4(t) := \int_0^1 \left[\rho_3\theta_t\theta + \frac{\mu_1}{2}\theta_x^2 + \sigma(3W-\Psi)_x\theta\right]dx \tag{4.10}$$

satisfies the estimate

$$F_4'(t) \le -\frac{\delta}{2}\int_0^1 \theta_x^2 dx + \left(\rho_3 + \frac{\sigma^2}{4\varepsilon_2}\right)\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 (3W-\Psi)_x^2\,dx + \frac{\mu_2^2}{2\delta}\int_0^1 z^2(x,1,t)\,dx, \tag{4.11}$$

for any $\varepsilon_2 > 0$.

Proof. By differentiating F_4 with respect to t, using $(2.1)_4$ and integrating by parts, we obtain

$$F_4'(t) = \int_0^1 \delta\theta_{xx}\theta dx + \int_0^1 \rho_3\theta_t^2 dx + \int_0^1 \mu_2 z_x(x,1,t)\,\theta dx + \int_0^1 \sigma(3W-\Psi)_x\,\theta_t dx.$$

Using Young's inequality with $\varepsilon_2 > 0$, we establish (4.11). ■

Lemma 4.6. *Let* (Φ,Ψ,W,θ,z)*be the solution of (2.1)-(2.2). Then the functional*

$$F_5(t) := \rho_2\int_0^1 (3W-\Psi)_t(\Phi_x-\Psi)\,dx + \frac{D\rho_1}{G}\int_0^1 (3W-\Psi)_x\,\Phi_t dx \tag{4.12}$$

satisfies the estimate

$$F_5'(t) \le -\frac{G}{2}\int_0^1 (\Psi-\Phi_x)^2\,dx + \frac{\alpha^2}{2G}\int_0^1 \theta_{tx}^2 dx + (\rho_2+\varepsilon_3)\int_0^1 (3W-\Psi)_t^2\,dx + \frac{9\rho_2^2}{4\varepsilon_3}\int_0^1 W_t^2 dx$$
$$+\left(\frac{D\rho_1}{G} - \rho_2\right)\int_0^1 (3W-\Psi)_{xt}\,\Phi_t dx, \tag{4.13}$$

for any $\varepsilon_3 > 0$.

Proof. By differentiating F_5 with respect to t, using $(2.1)_1$, $(2.1)_2$ and integrating by parts, we obtain

$$\begin{aligned} F_5'(t) = & -\int_0^1 G(\Psi - \Phi_x)^2 dx + \int_0^1 \alpha\theta_{tx}(\Psi - \Phi_x) dx - \rho_2 \int_0^1 (3W - \Psi)_t \Psi_t dx \\ & + \left(\frac{D\rho_1}{G} - \rho_2\right) \int_0^1 (3W - \Psi)_{xt} \Phi_t dx. \end{aligned}$$

Using Young's inequality with $\varepsilon_3 > 0$, we establish (4.13). ■

Lemma 4.7. *Let $(\Phi, \Psi, W, \theta, z)$be the solution of (2.1)-(2.2). Then the functional*

$$F_6(t) := \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t) \, d\rho dx \tag{4.14}$$

satisfies, for some $m, c > 0$, the following estimate

$$F_6'(t) \leq -m \int_0^1 \int_0^1 z^2(x, \rho, t) \, d\rho dx - \frac{c}{\tau} \int_0^1 z^2(x, 1, t) \, dx + \frac{1}{\tau} \int_0^1 \theta_{tx}^2 dx, \tag{4.15}$$

Proof. By differentiating F_6 with respect to t, using $(2.1)_5$ and integrating by parts, we obtain

$$\begin{aligned} F_6'(t) &= -\frac{2}{\tau} \int_0^1 \int_0^1 e^{-2\tau\rho} z(x, \rho, t) z_\rho(x, \rho, t) \, d\rho dx \\ &= -2 \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t) \, d\rho dx - \frac{1}{\tau} \int_0^1 \int_0^1 \frac{\partial}{\partial\rho} \left(e^{-2\tau\rho} z^2(x, \rho, t)\right) d\rho dx \\ &\leq -m \int_0^1 \int_0^1 z^2(x, \rho, t) \, d\rho dx - \frac{c}{\tau} \int_0^1 z^2(x, 1, t) \, dx + \frac{1}{\tau} \int_0^1 \theta_{tx}^2 dx. \end{aligned}$$

This gives (4.15). ■

The stability result reads as follows.

Theorem 4.8. *Assume that $\frac{G}{\rho_1} = \frac{D}{\rho_2}$ and (2.3) holds. Let $U_0 \in \mathcal{H}$, then there exist two positive constants c_0 and c_1, such that the energy $E(t)$ associated with problem (2.1)-(2.2) satisfies*

$$E(t) \leq c_0 E(0) e^{-c_1 t}, \ t \geq 0.$$

Proof. To establish the decay result, we assume $\frac{G}{\rho_1} = \frac{D}{\rho_2}$ and define a Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) := \delta_1 E(t) + F_1(t) + \delta_2 F_2(t) + \delta_3 F_3(t) + F_4(t) + \delta_4 F_5(t) + F_6(t),$$

where $\delta_1, \delta_2, \delta_3, \delta_4$ are positive constants to be chosen properly later.

Using Cauchy-Schwarz inequality and the Poincaré's inequality, one can easily see that all $F_i(t), i = 1, ..., 6$ are bounded by an expression with the existing terms in the energy $E(t)$. This leads to the equivalence of $\mathcal{L}(t)$ and $E(t)$.

Gathering the estimates in the previous lemmas and using

$$\int_0^1 \theta_t^2 dx \leq \int_0^1 \theta_{tx}^2 dx,$$

we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left[4\beta\sigma\delta_1 - C_3 - \frac{9\rho_2^2}{4\varepsilon_3}\delta_4\right]\int_0^1 W_t^2 dx - \frac{D}{2}\int_0^1 W_x^2 dx - \frac{\delta}{2}\int_0^1 \theta_x^2 dx \\ & -\left[\delta_1 C_1 - \frac{\alpha^2}{D}\delta_2 - C_5(\varepsilon_1)\delta_3 - \left(\rho_3 + \frac{\sigma^2}{4\varepsilon_2}\right) - \frac{\alpha^2}{2G}\delta_4 - \frac{1}{\tau}\right]\int_0^1 \theta_{tx}^2 dx \\ & -\left[\frac{G}{2}\delta_4 - C_4 - \frac{G^2}{2D}\delta_2 - \varepsilon_1\delta_3\right]\int_0^1 (\Psi - \Phi_x)^2 dx - \rho_1\int_0^1 \Phi_t^2 dx - \frac{2\gamma}{3}\int_0^1 W^2 dx \\ & -\left[\frac{D}{2}\delta_2 - \frac{D}{18} - \varepsilon_1\delta_3 - \varepsilon_2\right]\int_0^1 (3W_x - \Psi_x)^2 dx - \left[\frac{\rho_2\sigma}{2}\delta_3 - \rho_2\delta_2 - (\rho_2 + \varepsilon_3)\delta_4\right]\int_0^1 (3W_t - \Psi_t)^2 dx \\ & -\left[\delta_1 C_2 + \frac{c}{\tau} - \frac{\rho_2\mu_2^2}{\sigma}\delta_3 - \frac{\mu_2^2}{2\delta}\right]\int_0^1 z^2(x,1,t)\,dx - m\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx. \end{aligned} \tag{4.16}$$

At this point we will choose all the constants, carefully. First, we take δ_2 large enough and ε_2 small, such that

$$\frac{D}{2}\delta_2 - \frac{D}{18} - \varepsilon_2 > 0.$$

Then we can take δ_4 sufficiently large such that

$$\frac{G}{2}\delta_4 - C_4 - \frac{G^2}{2D}\delta_2 > 0.$$

Next, we pick ε_3 small and choose δ_3 large enough such that

$$\frac{\rho_2\sigma}{2}\delta_3 - \rho_2\delta_2 - (\rho_2 + \varepsilon_3)\delta_4 > 0.$$

After that, we then select ε_1 so small that

$$\frac{D}{2}\delta_2 - \frac{D}{18} - \varepsilon_2 - \varepsilon_1\delta_3 > 0\,, \quad \frac{G}{2}\delta_4 - C_4 - \frac{G^2}{2D}\delta_2 - \varepsilon_1\delta_3 > 0.$$

Finally, we choose δ_1 so large such that

$$\begin{gathered} 4\beta\sigma\delta_1 - C_3 - \frac{9\rho_2^2}{4\varepsilon_3}\delta_4 > 0\,, \quad \delta_1 C_2 + \frac{c}{\tau} - \frac{\rho_2\mu_2^2}{\sigma}\delta_3 - \frac{\mu_2^2}{2\delta} > 0, \\ \delta_1 C_1 - \frac{\alpha^2}{D}\delta_2 - C_5(\varepsilon_1)\delta_3 - \left(\rho_3 + \frac{\sigma^2}{4\varepsilon_2}\right) - \frac{\alpha^2}{2G}\delta_4 - \frac{1}{\tau} > 0. \end{gathered}$$

On the hand, from the above, we deduce that for some positive constants α_1, α_2 one has

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t)\,.$$

Therefore, (4.16) becomes

$$\mathcal{L}'(t) \leq -cE(t)\,.$$

For $c_1 = \dfrac{c}{\alpha_2}$, we get

$$\mathcal{L}'(t) \leq -c_1\mathcal{L}(t)\,, \forall t \geq 0. \tag{4.17}$$

Integrating (4.17) over $(0,t)$, yields

$$\mathcal{L}(t) \leq \mathcal{L}(0)\,e^{-c_1 t}, \forall t \geq 0. \tag{4.18}$$

At last, estimate (4.18) gives the desired result Theorem 4.8 when combined with the equivalence of $\mathcal{L}(t)$ and $E(t)$. ■

References

[1] T. A. Apalara, Uniform stability of a laminated beam with structural damping and second sound, *Z. Angew. Math. Phys.,* **68(2)**(2017), 1–16.

[2] D. S. Chandrasekharaiah, Hyperbolic thermoelasticity: a review of recent literature, *Appl. Mech. Rev.,* **51(12)**(1998), 705–729.

[3] M. M. Chen, W. J. Liu and W. C. Zhou, Existence and general stabilization of the Timoshenko system of thermo-viscoelasticity of type III with frictional damping and delay terms, *Adv. Nonlinear Anal.,* **7(4)**(2018), 547–569.

[4] R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, *SIAM J. Control Optim.,* **26(3)**(1988), 697–713.

[5] R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control Optim.,* **24**(1986), 152–156.

[6] A. Djebabla and N. E. Tatar, Exponential stabilization of the timoshenko system by a thermo-viscoelastic damping, *J. Dyn. Control Syst.,* **16**(2010), 189–210.

[7] M. Douib, S. Zitouni, and A. Djebabla, Well-posedness and exponential decay for a laminated beam in thermoelasticity of type III with delay term, *Mathematica,* **62**(2021), 58–76.

[8] S. Drabla, S. A. Messaoudi and F. Boulanouar, A general decay result for a multidimensional weakly damped thermoelastic system with second sound, *Discrete Contin. Dyn. Syst. Ser. B,* **22(4)**(2017), 1329–1339.

[9] A. Fareh and S. A. Messaoudi, Stabilization of a type III thermoelastic timoshenko system in the presence of a time-distributed delay, *Math. Nachr.,* **290(7)**(2017), 1017–1032.

[10] J. A. Goldstein, Semigroups of linear operators and applications, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1985.

[11] A. E. Green and P. M. Naghdi, A re-examination of the basic postulates of thermomechanics, *Proc. Roy. Soc. London Ser. A,* **432(1885)**(1991), 171–194.

[12] A. E. Green and P. M. Naghdi, On undamped heat waves in an elastic solid, *J. Thermal Stresses,* **15(2)**(1992), 253–264.

[13] A. E. Green and P. M. Naghdi, Thermoelasticity without energy dissipation, *J Elasticity,* **31**(1993), 189–208.

[14] A. Guesmia, Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay, *IMA J. Math. Control Inform.,* **30(4)**(2013), 507–526.

[15] S. W. Hansen, A model for a two-layered plate with interfacial slip, Control and estimation of distributed parameter systems: nonlinear phenomena (Vorau, 1993), Internat. *Ser. Numer. Math.,* Birkhauser, Basel, (1994), 143–170.

[16] S. W. Hansen and R. Spies, Structural damping in a laminated beam due to interfacial slip, *J. Sound Vibration,* **204(2)**(1997), 183–202.

[17] J. Hao and P. Wang, symptotical stability for memory-type porous thermoelastic system of type III with constant time delay , *Math. Methods Appl. Sci.,* **39**(2016), 3855–3865.

[18] M. Kafini, S. A. Messaoudi, M. I. Mustafa and T. Apalara, Well-posedness and stability results in a timoshenko-type system of thermoelasticity of type III with delay, *Z. Angew. Math. Phys.,* **66** **(4)**(2015), 1499–1517.

[19] G. Li, X. Y. Kong and W. J. Liu, General decay for a laminated beam with structural damping and memory: the case of non equal wave speeds, *J. Integral Equ. Appl.,* **30(1)**(2018), 95–116.

[20] W. J. Liu, K. W. Chen and J. Yu, Existence and general decay for the full von Kármán beam with a thermo-viscoelastic damping, frictional dampings and a delay term, *IMA J. Math. Control Inform.,* **34** **(2)**(2017), 521–542.

[21] W. J. Liu and W. F. Zhao, Stabilization of a thermoelastic laminated beam with past history, *Appl. Math. Optim.,* **80(1)**(2019), 103–133.

[22] A. Lo and N. E. Tatar, Stabilization of laminated beams with interfacial slip, *Electron. J. Differential Equations,* **2015(129)**(2015), 1–14.

[23] W. Liu, Y. Luan, Y. Liu and G. Li, Well-posedness and asymptotic stability to a laminated beam in thermoelasticity of type III, *Math. Methods Appl. Sci.,* **43(6)**(2020), 3148–3166.

[24] S. A. Messaoudi and T. A. Apalara, General stability result in a memory-type porous thermoelasticity system of type III, *Arab J. Math. Sci.,* **20(2)**(2014), 213–232.

[25] S. A. Messaoudi and A. Fareh, Energy decay in a timoshenko-type system of thermoelasticity of type III with different wave-propagation speeds, *Arab. J. Math. (Springer),* **2(2)**(2013), 199–207.

[26] J. E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear timoshenko systems- global existence and exponential stability, *J. Math. Anal. Appl.,* **276** **(1)**(2002), 248–278.

[27] M. I. Mustafa, On the decay rates for thermoviscoelastic systems of type III, *Appl. Math. Comput.,* **239**(2014), 29–37.

[28] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.,* **45(5)**(2006), 1561–1558.

[29] S. Nicaise, C. Pignotti and J. Valein, Exponential stability of the wave equation with boundary time-varying delay, *Discrete Contin. Dyn. Syst. Ser. S,* **4(3)**(2011), 693–722.

[30] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[31] N. E. Tatar, Stabilization of a laminated beam with interfacial slip by boundary controls, *Bound. Value Probl.,* **2015**(2015), 1–15.

[32] G. Q. Xu, S. P. Yung and L. K. Li, Stabilization of wave systems with input delay in the boundary control, *ESAIM Control Optim. Calc. Var.,* **12** **(4)**(2006), 770–785.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 36–46.
http://doi.org/10.26637/mjm1001/003

Householder's method for solving the p-adic polynomial equations

KECIES MOHAMED*[1]

[1] *Laboratoire LMPEA, Université de Jijel, Jijel, Algeria.*

Received 10 July 2021; Accepted 02 December 2021

Abstract. This work offers an analogue of Householder's Method for solving a root-finding problem $f(x) = 0$ in the p-adic setting. We apply this method to calculate the square roots of a p-adic number $a \in \mathbb{Q}_p$ where p is a prime number, and through the calculation of the approached solution of the p-adic polynomial equation $f(x) = x^2 - a = 0$. We establish the rate of convergence of this method. Finally, we also determine how many iterations are needed to obtain a specified number of correct digits in the approximate.

AMS Subject Classifications: 26E30, 11E95, 65H04.

Keywords: p-adic number, square roots, Householder iterative method, Hensel's lemma, rate of convergence.

Contents

1. Introduction and Background

Given a prime number p, the field of p-adic numbers $\mathbb{Q}_p$ were first introduced by Kurt Hensel at the end of the 19th century in a short paper written in German [8], which can be thought of as the completion of the field of rationals $\mathbb{Q}$ with respect to the p-adic norm, similar to how one constructs the field of real numbers $\mathbb{R}$ from $\mathbb{Q}$ (see [1], [3], [5], [6]). The p-adic numbers are useful because they provide another toolset for solving problems, one which is sometimes easier to work with than the real numbers. They have applications in number theory, analysis, algebra, and more. For about a century after the discovery of p-adic numbers, they were mainly considered as objects of pure mathematics. However, numerous applications of these numbers to theoretical physics have been proposed, to quantum mechanics, to p-adic - valued physical observables and many others. The field of p-adic numbers $\mathbb{Q}_p$ endowed with a metric d_p generated by p-adic valuation is also a fundamental example in the theory of ultrametric spaces. Nevertheless, many metric properties of the space $(\mathbb{Q}_p, d_p)$ remain unexplored now.

Finding the approximate solution of the nonlinear equation $f(x) = 0$ is one of the basic problems and frequently occurs in scientific work of various fields. Due to the higher order of the equation and the involvement of the transcendental functions, analytical methods for obtaining the exact root cannot be employed and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iteration procedure [4], [15]. If we come across a problem that the function f is not known explicitly or the derivatives of the function are difficult to compute, then a method that uses only computed values of the function is more appropriate.

*Corresponding author. Email address: **m.kecies@centre-univ-mila.dz** (Kecies Mohamed)

In fact, there are some results of the existence of square and cubic roots of p-adic numbers. For instance, in [13], the authors demonstrated how classical root-finding methods from numerical analysis can be employed to compute the multiplicative inverses of integers modulo $p^n, n \in \mathbb{N}$. A similar problem was addressed by Zerzaihi, Kecies, and Knapp [21] by using the fixed point iteration to compute the Hensel codes of square roots p-adic numbers. In [19] and [20] Zerzaihi and Kecies then extended the root-finding problem to the cube roots in $\mathbb{Q}_p$ of p-adic numbers by approximating the zeroes of $g(x) = x^3 - a, a \in \mathbb{Q}_p$, using the secant and Newton method. A related study was also carried out in [14] where Kecies considered the problem of finding the square roots of p-adic numbers in $\mathbb{Q}_p$ through the secant method. A similar problem also appeared in [11] wherein Ignacio et al. computed the square and cube roots of p-adic numbers via Newton-Raphson method.

Lately, a series of investigations explored the problem of finding square roots and the q-th roots of p-adic numbers. For instance, in [2], the authors proposed an analogue of Steffensen's method in finding roots of a general p-adic polynomial equation $f(x) = 0$ in $\mathbb{Z}_p$. Meanwhile, in [17], the author described an analogue of Halley's method for approximating roots of p-adic polynomial equations $f(x) = 0$ in $\mathbb{Z}_p$. A related study which examines a p-adic analogue of Olver's method was also considered in [16]. On the other hand, In [10], the authors gave the conditions for the existence of the q-th roots of p-adic numbers, and then applied the Newton-Raphson method to compute the q-th roots.

Our contribution in the present paper is to show how we can use classical root-finding method (Householder's method [9], [18]) to calculate the zero of a p-adic polynomial equation given by

$$f(x) = x^2 - a = 0, a \in \mathbb{Q}_p^*. \tag{1.1}$$

Our goal is to calculate the first numbers of the p-adic development of the solution of the previous equation, and this solution is approached by a sequence of the p-adic numbers $(x_n)_n \subset \mathbb{Q}_p$ constructed by the Householder method.

The rest of the paper is organized as follows. The next section recalls several concepts about $\mathbb{Q}_p$ which will be used through the paper. Our main contribution is formally stated and proved in Section 3, and a short concluding remark is given in the last section.

2. Preliminaries

Definition 2.1. *Fix a prime number $p \in \mathbb{Z}$. The p-adic valuation on $\mathbb{Z}$ is the function $v_p : \mathbb{Z} - \{0\} \longrightarrow \mathbb{R}$ defined as follows: for each integer $n \in \mathbb{Z}, n \neq 0$, let $v_p(n)$ be the unique positive integer satisfying*

$$n = p^{v_p(n)} n' \text{ with } p \nmid n'.$$

In other words, the p-adic valuation of n is the highest power of p that divides n.
We extend v_p to the field of rational numbers as follows: if $x = \frac{a}{b} \in \mathbb{Q}^*$, then

$$v_p(x) = v_p(a) - v_p(b).$$

Definition 2.2. *For any $x \in \mathbb{Q}$, we define the p-adic absolute value (or the p-adic norm) of x by*

$$|x|_p = p^{-v_p(x)},$$

if $x \neq 0$, and we set $|0|_p = 0$.
This norm satisfies the so called strong triangle inequality

$$|x + y|_p \leq \max\left\{|x|_p, |y|_p\right\} \text{ for all } x, y \in \mathbb{Q}, \tag{2.1}$$

and this is a non-Archimedean norm. The p-adic norm leads us to the p-adic metric on $\mathbb{Q}$ defined by

$$d_p(x, y) = |x - y|_p \text{ for all } x, y \in \mathbb{Q}. \tag{2.2}$$

We actually have something stronger than a metric. Thanks to the non-Archimedean property d_p is an ultrametric. Rather than the ordinary Triangle Inequality, d_p satisfies the Strong Triangle Inequality

$$d_p(x,y) \leq \max\{d_p(x,z), d_p(z,y)\} \text{ for all } x,y,z \in \mathbb{Q}. \tag{2.3}$$

We note that the range of the map $|\cdot|_p$ is the set $\{0\} \cup \{p^n : n \in \mathbb{Z}\}$ unlike the usual $|\cdot|_p$ on $\mathbb{R}$ whose values include all non-negative real numbers.

Definition 2.3. *For each prime p, the field of p-adic numbers denoted $\mathbb{Q}_p$ is the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ which contains the rational numbers $\mathbb{Q}$ as a dense subset.*
The elements of $\mathbb{Q}_p$ are equivalent classes of Cauchy sequences in $\mathbb{Q}$ with respect to the extension of the p-adic norm. For some $x \in \mathbb{Q}_p$ let $(x_n)_n$ be a Cauchy sequence of rational numbers representing x. Then by definition

$$|x|_p = \lim_{n \longrightarrow +\infty} |x_n|_p \,. \tag{2.4}$$

Each equivalence class of Cauchy sequences defining some element of $\mathbb{Q}_p$ contains a unique canonical representative Cauchy sequence. In order to describe its construction, we need the following theorem.

Theorem 2.4. *[7] Any p-adic number $\alpha \in \mathbb{Q}_p$ can be written in the form*

$$\alpha = \sum_{j=n}^{\infty} a_j p^j,$$

where each $a_j \in \mathbb{Z}$, and n is such that $|\alpha|_p = p^{-n}$. Moreover, if we choose each $a_j \in \{0,1,2,...,p-1\}$, then the expansion is unique. (In this case, the expansion is the canonical representation of α.)

Remark 2.5. *Notice that there is a one-to-one correspondence between the power series expansion*

$$\alpha = a_n p^n + a_{n+1} p^{n+1} + a_{n+2} p^{n+2} ... \tag{2.5}$$

and the abbreviated representation

$$\alpha = a_n a_{n+1} a_{n+2} ...$$

where only the coefficients of the powers of p are exhibited. Because of this correspondence we can use the power series expansion and the abbreviated representation interchangeably. In fact, we shall refer to each of them as the p-adic expansion for α. The abbreviated representation is completely analogous to the representation of the decimal expansion of a real number. In fact, we complete the analogy by introducing a p-adic point as a device for displaying the sign of n. Thus, we write

$$\alpha = \begin{cases} a_n a_{n+1} a_{n+2} ... a_{-2} a_{-1} \cdot a_0 a_1 a_2 ..., \text{ for } n < 0, \\ \cdot a_0 a_1 a_2 ..., \text{ for } n = 0, \\ \cdot 0 ... 0 a_n a_{n+1} ..., \text{ for } n > 0. \end{cases} \tag{2.6}$$

Definition 2.6.
(1) *A p-adic number is said to be a p-adic integer if its canonical expansion contains only nonnegative powers of p. The set of p-adic integers is denoted by $\mathbb{Z}_p$, so*

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : x = \sum_{j=0}^{\infty} a_j p^j \right\}. \tag{2.7}$$

It is easy to see that

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}. \tag{2.8}$$

In other words, $\mathbb{Z}_p$ appears as the closed unit ball in $\mathbb{Q}_p$.
(2) *Any p-adic integer whose first digit is non-zero is called a p-adic unit. The set of p-adic units is denoted by $\mathbb{Z}_p^\times$. Hence we have*

$$\mathbb{Z}_p^\times = \left\{ x = \sum_{j=0}^{\infty} a_j p^j : a_0 \neq 0 \right\} = \left\{ x \in \mathbb{Z}_p : |x|_p = 1 \right\}, \tag{2.9}$$

meaning that the group of units of $\mathbb{Z}_p$ is then the unit sphere in $\mathbb{Q}_p$.

The following proposition follows at once from the definition of the p-adic norm and the p-adic unit.

Proposition 2.7. *[12] Let x be a p-adic number of norm p^{-n}. Then x can be written as the product $x = p^n u$, where $u \in \mathbb{Z}_p^\times$.*

According to the above definition 2.3, $\mathbb{Q}_p$ is a complete metric space, and, consequently, every Cauchy sequence converges. Cauchy sequences are characterized as follows.

Theorem 2.8. *[1] A sequence (a_n) in $\mathbb{Q}_p$ is a Cauchy sequence, and therefore convergent, if and only if it satisfies*

$$\lim_{n \longrightarrow +\infty} |a_{n+1} - a_n|_p = 0. \tag{2.10}$$

Now let us consider a numerical series $\sum_{j=0}^{\infty} a_j, a_j \in \mathbb{Q}_p$. We say that this series converges if the sequence of its partial sums $s_n = \sum_{j=0}^{n} a_j$ converges in $\mathbb{Q}_p$, and it converges absolutely if the series $\sum_{j=0}^{\infty} |a_j|_p$ converges in $\mathbb{R}$. The following result is an important tool for determining whether a series of p-adic numbers converge in $\mathbb{Q}_p$ or not.

Proposition 2.9. *[1] A series $\sum_{n=0}^{\infty} a_n$ with $a_n \in \mathbb{Q}_p$ converges in $\mathbb{Q}_p$ if and only if $\lim_{n \longrightarrow +\infty} a_n = 0$, in which case*

$$\left| \sum_{n=0}^{\infty} a_n \right|_p \leq \max_n |a_n|_p. \tag{2.11}$$

Proposition 2.10. *[1] If*

$$\lim_{n \longrightarrow +\infty} x_n = x, x_n, x \in \mathbb{Q}_p, |x|_p \neq 0,$$

then the sequence of norms $\left\{ |x_n|_p : n \in \mathbb{N} \right\}$ must stabilize for sufficiently large n, i.e., there exists N such that

$$|x_n|_p = |x|_p, \forall n \geq N. \tag{2.12}$$

For fixed primes p the p-adic numbers have many applications to ordinary number theory especially to solving congruences modulo p. Important in this regard is Hensel's Lemma. The lemma says that if a polynomial equation has a simple root modulo a prime number p, then this root corresponds to a unique root of the same equation modulo any higher power of p. This root can be found by iteratively lifting the solution modulo successive powers of p and is an analog of Newton's method. First, we define congruence in $\mathbb{Q}_p$.

Definition 2.11. *We say that a and $b \in \mathbb{Q}_p$ are congruent $\mod p^n$ and write $a \equiv b \mod p^n$ if and only if $|a - b|_p \leq p^{-n}$.*

Theorem 2.12. *[5] (Hensel's Lemma)*
Let $f(x) = c_0 + c_1 x + ... + c_n x^n$ be a polynomial in $\mathbb{Z}_p[x]$ (coefficients are p-adic integers). Let $f'(x)$ be the formal derivative of $f(x)$. Suppose $\bar{a}_0 \in \mathbb{Z}_p$ with $f(\bar{a}_0) \equiv 0 \mod p$ and $f'(\bar{a}_0) \not\equiv 0 \mod p$. Then, there exists a unique p-adic integer a such that $f(a) = 0$ and $a \equiv \bar{a}_0 \mod p$.

As an application of the Hensel's lemma, we investigate the squares in $\mathbb{Q}_p$.

Corollary 2.13. *[6] Let $p \neq 2$ be a prime. An element $x \in \mathbb{Q}_p$ is a square if and only if it can be written $x = p^{2n} y^2$ with $n \in \mathbb{Z}$ and $y \in \mathbb{Z}_p^\times$ a p-adic unit.*

3. Main Results

Finding iterative methods for solving nonlinear equations is an important area of research in numerical analysis at it has interesting applications in several branches of pure and applied science can be studied in the general framework of the nonlinear equations $f(x) = 0$. Due to their importance, several numerical methods have been suggested and analyzed under certain condition. These numerical methods have been constructed using different techniques. It arises in a wide variety of practical applications in Physics, Chemistry, Biosciences, Engineering, etc.

Let us consider the nonlinear equation of the type

$$f(x) = 0. \tag{3.1}$$

For simplicity, we assume that r is a simple root of the equation (3.1) and x_0 is an initial guess sufficiently close to r. Using the Taylor's series expansion of the function f, we have

$$f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) = 0. \tag{3.2}$$

First two terms of the equation (3.2) gives the first approximation, as

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}. \tag{3.3}$$

This allows us to suggest the following one-step iterative method for solving the nonlinear equation (3.1).

For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, ... \tag{3.4}$$

Algorithm (3.4) is known as Newton method and has second-order convergence [4].

Again from (3.2) we have

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f''(x_0)(x - x_0)^2}{2f'(x_0)}. \tag{3.5}$$

Substitution again from (3.3) into the right hand side of (3.5) gives the second approximation

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f''(x_0) \left(f(x_0)\right)^2}{2 \left(f'(x_0)\right)^3}. \tag{3.6}$$

This formula allows us to suggest the following iterative methods for solving the nonlinear equation (3.1).

For a given x_0, compute approximates solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\left(f(x_n)\right)^2 f''(x_n)}{2 \left(f'(x_n)\right)^3}, n = 0, 1, 2, ... \tag{3.7}$$

Algorithm (3.7) is known as Householder method for solving the nonlinear equations [9]. This method is one of the famous methods in producing a sequence of approximation roots of (3.1) with initial point x_0.

To calculate the square root of a p-adic number $a \in \mathbb{Q}_p^*$, one studies the following problem

$$f(x) = x^2 - a = 0, a \in \mathbb{Q}_p^*. \tag{3.8}$$

The solution of the previous equation is approached by a sequence of the p-adic numbers $(x_n)_n \subset \mathbb{Q}_p$ constructed by the Householder method.

In this section we analyze the convergence of the method described previously. The important part of the convergence is about the convergence rate. In practice, a numerical method may take a large number of iterations to reach the optimum point. Therefore, it is important to employ methods having a faster rate of convergence.

The rate of convergence plays an important role in the theory of any iterative procedure that is producing a convergent sequence to the exact solution. The method converges faster to the solution for high order of convergence. Therefore, it requires a lesser number of iterations for a given accuracy. Rate of convergence of a numerical method is usually measured by the numbers of iterations and function evaluations needed to obtain an acceptable solution.

A practical method to calculate the rate of convergence is to calculate the sequence $(e_n)_n$ defined by

$$e_n = x_{n+n_0+1} - x_{n+n_0}. \tag{3.9}$$

with $n_0 \in \mathbb{N}$. Roughly speaking, if the rate of convergence of a method is s, then after each iteration the number of correct significant digits in the approximation increases by a factor of approximately s. Moreover, the number of iterations necessary to obtain the desired precision M which represents the number of p-adic digits in the development of $\sqrt{a}$ is very important for our objectives. it's all about finding n such that

$$|x_{n+n_0+1} - x_{n+n_0}|_p \leq p^{-M}, \tag{3.10}$$

this is equivalent to

$$v_p(e_n) \geq M. \tag{3.11}$$

Let $a \in \mathbb{Q}_p^*$ a p-adic number such that

$$|a|_p = p^{-v_p(a)} = p^{-2m}, m \in \mathbb{Z}. \tag{3.12}$$

If $(x_n)_n$ is a sequence of p-adic numbers that converges to a p-adic number $\alpha \neq 0$, then from a certain rank one has

$$|x_n|_p = |\alpha|_p\,. \tag{3.13}$$

We also know that if there exists a p-adic number α such that $\alpha^2 = a$, then $v_p(a)$ is even and

$$|x_n|_p = |\alpha|_p = p^{-m}. \tag{3.14}$$

We consider the following equation

$$f(x) = x^2 - a. \tag{3.15}$$

We know that the iterative formula of the Householder method is given by

$$\forall n \in \mathbb{N} : x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(f(x_n))^2 f''(x_n)}{2\,(f'(x_n))^3}.$$

Therefore the iteration of the Householder method associated with the function f given in (3.15) is written in the form

$$\forall n \in \mathbb{N} : x_{n+1} = x_n - \frac{1}{2x_n}\left(x_n^2 - a\right) - \frac{1}{8x_n^3}\left(x_n^2 - a\right)^2. \tag{3.16}$$

Theorem 3.1. *If x_{n_0} is the square root of a of order r, then*
1) *If $p \neq 2$, then x_{n+n_0} is the square root of a of order w_n, where the sequence $(w_n)_n$ is defined by*

$$\forall n \in \mathbb{N} : w_n = 3^n r + 2m(1 - 3^n). \tag{3.17}$$

2) *If $p = 2$, then x_{n+n_0} is the square root of a of order w'_n, where the sequence $(w'_n)_n$ is defined by*

$$\forall n \in \mathbb{N} : w'_n = 3^n r + (2m+3)\,(1 - 3^n). \tag{3.18}$$

Proof. Let $(x_n)_n$ be the sequence defined by (3.16). We have

$$\forall n \in \mathbb{N} : x_{n+1}^2 - a = \frac{1}{64}\frac{1}{x_n^6}\left(a - x_n^2\right)^3\left(a - 9x_n^2\right). \tag{3.19}$$

We assume that x_{n_0} is the square root of a of order r, i.e,

$$x_{n_0}^2 \equiv a \mod p^r, r \in \mathbb{N}. \tag{3.20}$$

Then

$$v_p\left(x_{n_0}^2 - a\right) \geq r,$$

hence we obtain

$$\left|x_{n_0}^2 - a\right|_p \leq p^{-r}.$$

On the other hand, we put

$$g(x) = \frac{1}{64}\frac{1}{x_n^6}\left(a - 9x_n^2\right). \tag{3.21}$$

Since

$$|64|_p = \begin{cases} 1, \text{ if } p \neq 2, \\ \frac{1}{64} = \frac{1}{2^6}, \text{ if } p = 2, \end{cases} \tag{3.22}$$

we have

$$|g(x_{n_0})|_p = \left|\frac{1}{64}\frac{1}{x_{n_0}^6}\left(a - 9x_{n_0}^2\right)\right|_p = \left|\frac{1}{64}\right|_p\left|\frac{1}{x_{n_0}^6}\right|_p\left|a - 9x_{n_0}^2\right|_p$$

This gives

$$|g(x_{n_0})|_p \leq \left|\frac{1}{64}\right|_p\left|\frac{1}{x_{n_0}^6}\right|_p \max\left\{|a|_p, \left|9x_{n_0}^2\right|_p\right\}$$

On the other hand, using the proposition 2.10, we get

$$|g(x_{n_0})|_p \leq \begin{cases} p^{6m}p^{-2m}, \text{ if } p \neq 2, \\ 2^6 2^{6m} 2^{-2m}, \text{ if } p = 2, \end{cases}$$

$$\leq \begin{cases} p^{4m}, \text{ if } p \neq 2, \\ 2^{4m+6}, \text{ if } p = 2. \end{cases}$$

We obtain

$$\left|x_{n_0+1}^2 - a\right|_p = |g(x_{n_0})|_p \left|a - x_n^2\right|_p^3,$$

and so we have

$$\begin{cases} \left|x_{n_0+1}^2 - a\right|_p \leq p^{4m}p^{-3r}, \text{ if } p \neq 2, \\ \left|x_{n_0+1}^2 - a\right|_2 \leq 2^{4m+6}2^{-3r}, \text{ if } p \neq 2. \end{cases}$$

Using the definition 2.11, we get

$$\begin{cases} x_{n_0+1}^2 - a \equiv 0 \mod p^{3r-4m} \text{ if } p \neq 2, \\ x_{n_0+1}^2 - a \equiv 0 \mod 2^{3r-4m-6} \text{ if } p = 2. \end{cases}$$

In this manner, we find that if $p \neq 2$, then

$$\forall n \in \mathbb{N} : x_{n+n_0}^2 - a \equiv 0 \mod p^{w_n}, \tag{3.23}$$

where the sequence $(w_n)_n$ is defined by

$$\forall n \in \mathbb{N} : \begin{cases} w_{n+1} = 3w_n - 4m, \\ \\ w_0 = r. \end{cases} \tag{3.24}$$

It is clear that $(w_n)_n$ is a linear recurrence sequence of order 1, whose general term is given by

$$\forall n \in \mathbb{N} : w_n = 3^n r + 2m(1 - 3^n). \tag{3.25}$$

Furthermore

$$v_p(x_{n+n_0}^2 - a) \geq w_n. \tag{3.26}$$

If $p = 2$, then

$$\forall n \in \mathbb{N} : x_{n+n_0}^2 - a \equiv 0 \mod 2^{w'_n}, \tag{3.27}$$

where the sequence $(w'_n)_n$ is defined by

$$\forall n \in \mathbb{N} : \begin{cases} w'_{n+1} = 3w'_n - (4m + 6), \\ \\ w'_0 = r. \end{cases} \tag{3.28}$$

Which give

$$\forall n \in \mathbb{N} : w'_n = 3^n r + (2m + 3)(1 - 3^n). \tag{3.29}$$

Furthermore

$$v_2(x_{n+n_0}^2 - a) \geq w'_n. \tag{3.30}$$

and so

$$\forall n \in \mathbb{N} : w'_n = w_n + 3(1 - 3^n). \tag{3.31}$$

This complete the proof. ■

Corollary 3.2. *If x_{n_0} is the square root of a of order r, then*
1) *If $p \neq 2$, then*

$$\forall n \in \mathbb{N} : x_{n+n_0+1} - x_{n+n_0} \equiv 0 \mod p^{s_n}, \tag{3.32}$$

where the sequence $(s_n)_n$ is defined by

$$\forall n \in \mathbb{N} : s_n = 3^n r + m(1 - 2 \cdot 3^n). \tag{3.33}$$

2) *If $p = 2$, then*

$$\forall n \in \mathbb{N} : x_{n+n_0+1} - x_{n+n_0} \equiv 0 \mod 2^{s'_n}, \tag{3.34}$$

such as

$$\forall n \in \mathbb{N} : s'_n = 3^n r + m(1 - 2 \cdot 3^n) - 3^{n+1}. \tag{3.35}$$

Proof. Let $(x_n)_n$ be the sequence defined by (3.16). We have

$$\forall n \in \mathbb{N} : x_{n+1} - x_n = -\frac{1}{8}\frac{1}{x_n^3}\left(a - x_n^2\right)\left(a - 5x_n^2\right). \tag{3.36}$$

This gives

$$\forall n \in \mathbb{N} : x_{n+n_0+1} - x_{n+n_0} = -\frac{1}{8}\frac{1}{x_{n+n_0}^3}\left(a - x_{n+n_0}^2\right)\left(a - 5x_{n+n_0}^2\right). \tag{3.37}$$

We put

$$h(x) = -\frac{1}{8}\frac{1}{x^3}\left(a - 5x^2\right).$$

Since

$$|8|_p = \begin{cases} 1, \text{ if } p \neq 2, \\ \frac{1}{8} = \frac{1}{2^3}, \text{ if } p = 2, \end{cases} \tag{3.38}$$

we have

$$\begin{aligned} |h(x_{n+n_0})|_p &= \left|-\frac{1}{8}\frac{1}{x_{n+n_0}^3}\left(a - 5x_{n+n_0}^2\right)\right|_p = \left|\frac{1}{8}\right|_p \left|\frac{1}{x_{n+n_0}^3}\right|_p \left|a - 5x_{n+n_0}^2\right|_p \\ &\leq \left|\frac{1}{8}\right|_p \left|\frac{1}{x_{n+n_0}^3}\right|_p \max\left\{|a|_p, \left|5x_{n+n_0}^2\right|_p\right\} \\ &\leq \begin{cases} p^{3m}p^{-2m}, \text{ if } p \neq 2 \\ 2^3 2^{3m} 2^{-2m}, \text{ if } p = 2 \end{cases} \\ &\leq \begin{cases} p^{m}, \text{ if } p \neq 2 \\ 2^{m+3}, \text{ if } p = 2. \end{cases} \end{aligned}$$

Hence we obtain

$$|x_{n+n_0+1} - x_{n+n_0}|_p = \left|h(x_{n+n_0})\left(a - x_{n+n_0}^2\right)\right|_p = |h(x_{n+n_0})|_p \cdot \left|a - x_{n+n_0}^2\right|_p$$

On the other hand, using (3.23) and (3.27), we get

$$|x_{n+n_0+1} - x_{n+n_0}|_p \leq \begin{cases} p^m p^{-w_n}, \text{ if } p \neq 2 \\ 2^{m+3} 2^{-w'_n}, \text{ if } p = 2, \end{cases}$$

and so

$$\begin{cases} x_{n+n_0+1} - x_{n+n_0} \equiv 0 \mod p^{w_n - m}, \text{ if } p \neq 2 \\ x_{n+n_0+1} - x_{n+n_0} \equiv 0 \mod 2^{w'_n - (m+3)}, \text{ if } p = 2. \end{cases}$$

Therefore, if $p \neq 2$, then

$$\forall n \in \mathbb{N} : x_{n+n_0+1} - x_{n+n_0} \equiv 0 \mod p^{s_n}, \tag{3.39}$$

where

$$\forall n \in \mathbb{N} : s_n = w_n - m = 3^n r + m(1 - 2 \cdot 3^n). \tag{3.40}$$

Furthermore

$$v_p(x_{n+n_0+1} - x_{n+n_0}) \geq s_n. \tag{3.41}$$

If $p = 2$, then

$$\forall n \in \mathbb{N} : x_{n+n_0+1} - x_{n+n_0} \equiv 0 \mod 2^{s'_n}, \tag{3.42}$$

where

$$\forall n \in \mathbb{N} : s'_n = w'_n - (m+3) = 3^n r + m\left(1 - 2 \cdot 3^n\right) - 3^{n+1} = s_n - 3^{n+1}. \tag{3.43}$$

Furthermore

$$v_2(x_{n+n_0+1} - x_{n+n_0}) \geq s'_n. \tag{3.44}$$

This complete the proof. ■

4. Conclusion

Our main results can be summarized as follows.

1. If $p \neq 2$, then the following are true.
   (a) The rate of convergence of the sequence $(x_n)_n$ is the order s_n.
   (b) If $r - 2m > 0$, then the number of iterations n to obtain M correct digits is

$$n = \left\lceil \frac{\ln\left(\frac{M-m}{r-2m}\right)}{\ln 3} \right\rceil. \tag{4.1}$$

2. If $p = 2$, then the following are true.
   (a) The rate of convergence of the sequence $(x_n)_n$ is the order s'_n.
   (b) If $r - (2m+3) > 0$, then the necessary number n of iterations to obtain M correct digits is

$$n = \left\lceil \frac{\ln\left(\frac{M-m}{r-2m-3}\right)}{\ln 3} \right\rceil. \tag{4.2}$$

3. In the p-adic setting, the Householder's method converges cubically insofar as the number of significant digits eventually triples with each iteration.

References

[1] S. Albeverio, A.Y. Khrennikov and V.M. Shelkovich, *Theory of p-adic distributions: linear and nonlinear models*, Cambridge University Press, 2010.

[2] J. B. Bacani, J. F. T. Rabago, Steffensen's analogue for approximating roots of p-adic polynomial equations. *Numerical Computations: Theory and Algorithms (NUMTA-2016).,* **1776(1)**(2016), 090038-1–090038-4.

[3] W.A. Coppel, *Number Theory: An introduction to mathematics*, Springer Science & Business Media, 2009.

[4] J.F. Epperson, *An introduction to numerical methods and analysis*, John Wiley & Sons, 2013.

[5] B. Fine and G. Rosenberger, *Number Theory: An Introduction via the Density of Primes*, Birkhäuser, 2016.

[6] F.Q. Gouvea, *p-adic Numbers: An Introduction*, Springer Science & Business Media, 2012.

[7] R.T. Gregory and E.V. Krishnamurthy, *Methods and applications of error-free computation*, Springer Science & Business Media, 2012.

[8] K. Hensel, Über eine neue Begründung der Theorie der algebraischen Zahlen. *JJahresber. Dtsch. Math.-Ver.,* **6**(1897), 83–88.

[9] A.S. Householder, *The Numerical Treatment of a Single Nonlinear Equation*, McGraw-Hill, New York, 1970.

[10] P.S.P. Ignacio, J.M. Addawe and J.A. Nable, P-adic Qth Roots Via Newton-Raphson Method, *Thai J. Math.,* **14(2)**(2016), 417–429.

[11] P. S. P. Ignacio, J. M. Addawe, W. V. Alangui and J. A. Nable, Computation of square and cube roots of p-adic numbers via Newton-Raphson method, *J.M.R.,* **5(2)**(2013), 31–38.

[12] S. Katok, *p-adic Analysis Compared with Real*, Vol. 37, American Mathematical Soc, 2007.

[13] M.P. Knapp and C. Xenophontos, Numerical Analysis meets Number Theory: Using rootfinding methods to calculate inverses $\mod p^n$, Appl. Anal. Discrete Math., **4**(2010), 23–31.

[14] M. Kecies, The Performance of the Secant Method in the Field of p-dic Numbers, *Malaya J. Mat.,* **9(2)**(2021), 28–38.

[15] A. Quarteroni, R. Sacco and F. Saleri, *Méthodes Numériques: Algorithmes, analyse et applications*, Springer Science & Business Media, 2008.

[16] J.F.T. Rabago, Olver's Method for Solving Roots of p-adic Polynomial Equations, *Italian J. Pure and Applied Math.,* **36**(2016), 739–748.

[17] J.F.T. Rabago, Halley's method for finding roots of p-adic polynomial equations, *Int. J. Math. Anal.,* **10(10)**(2016), 493–502.

[18] F. A. Shah and M. A. Noor, Variational iteration technique and some methods for the approximate solution of nonlinear equations,*Appl. Math. Inf. Sci. Lett.,* **2(3)**(2014), 85–93.

[19] T. Zerzaihi and M. Kecies, Computation of the cubic root of a p-adic number, *J.M.R.,* **3**(2011), 40–47.

[20] T. Zerzaihi and M. Kecies, General approach of the root of a p-adic number, *Filomat.,* **27**(2013), 431–436.

[21] T. Zerzaihi, M. Kecies and M.P. Knapp, Hensel codes of square roots of p-adic numbers, *Appl. Anal. Discrete Math.,* **4**(2010), 32–44.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 47–54.
http://doi.org/10.26637/mjm1001/004

Certain subclasses of Pseudo-type meromorphic bi-univalent functions

ADNAN GHAZY ALAMOUSH*[1]

[1] *Faculty of Science, Department of Mathematics, Taibah University, Madinah, Saudi Arabia.*

Received 26 September 2021; Accepted 19 December 2021

Abstract. In the present article, we define a new subclasses of pseudo-type meromorphic bi-univalent functions defined on $\triangle = \{z \mid : z \in \mathbb{C} \, and \;\; 1 < |z| < \infty\}$, and investigate the initial coefficient estimates $|b_0|$ and $|b_1|$. Further, several earlier results are also indicated.
AMS Subject Classifications: 30C50, 33C05.

Keywords: **Keywords:** Analytic functions, Univalent functions, Meromorphic univalent functions, Bi-Univalent functions, Meromorphic Bi-Univalent functions, Pseudo functions.

Contents

1. Introduction

Let A denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit open disk $U = \{z : z \in C, |z| < 1\}$. Also, we detote S be class of all functions in A which are univalent and normalized by the conditions

$$f(0) = 0 = f^{'}(0) - 1$$

in U. Let Σ' denote the class of meromorphic univalent functions g of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{1.2}$$

defined on the domain $\triangle = \{z : z \in U, 1 < |z| < \infty\}$. Since the function $g \in \Sigma'$ is univalent, then it has an inverse $g^{-1} = h$, defined by

$$g^{-1}(g(z)) = z \;\; (z \in \triangle),$$

*Corresponding author. Email address: **adnan_omoush@yahoo.com** (Adnan Ghazy ALAMOUSH)

and

$$g^{-1}(g(w)) = w \ \ (M < |w| < \infty,\ M > 0),$$

where

$$g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \frac{b_1}{w} - \frac{b_1 b_0 + b_2}{w^2} - \frac{b_1^2 + b_1 b_0^2 + 2b_0 b_2 + b_3}{w^3} + \cdots . \tag{1.3}$$

By a simple calculations, we have:

$$w = g(h(w)) = (b_0 + B_0) + w + \frac{b_1 + B_1}{w} + \frac{B_2 - b_1 B_0 + b_2}{w^2} + \frac{B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3}{w^3} + \cdots . \tag{1.4}$$

Comparing the initial coefficients in (1.4), we find that

$$\begin{aligned} b_0 + B_0 = 0 \qquad & \Rightarrow \ \ B_0 = -b_0 \\ b_1 + B_1 = 0 \qquad & \Rightarrow \ \ B_1 = -b_1 \\ B_2 - b_1 B_0 + b_2 = 0 \qquad & \Rightarrow B_2 = -(b_2 + b_1 b_0) \\ B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3 = 0 \qquad & \Rightarrow B_3 = -(b_3 + 2b_0 b_1 + b_1 b_0^2 + b_1^2). \end{aligned}$$

A function $g \in \Sigma'$ is said to be meromorphic bi-univalent if $g^{-1} \in \Sigma'$, and the family of all meromorphic bi-univalent functions is denoted by Σ'_*. The coefficient problem was widely investigated for various interesting subclasses of the meromorphic univalent functions; for example, Schiffer [19] obtained the estimate $|b_2| < \frac{3}{2}$ for meromorphic univalent functions $f \in S$ with $b_0 = 0$. In 1983, Duren [20] obtained the inequality $|b_2| < \frac{2}{n+1}$ for $f \in S$ with $b_k = 0, 1 \le k \le \frac{n}{2}$.

For the coefficients of inverses of meromorphic univalent functions, Springer [16] showed that

$$|B_3| < 1 \text{ and } |B_3 + \frac{1}{2} B_1^2| < \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \le \frac{(2n-2)!}{n!(n-1)!} \ (n = 1, 2, \cdots).$$

In 1977, Kubota [22] has proved that the Springer conjecture is true for n = 3; 4; 5. Furthermore, for $h \in \Sigma'$, Schober [15] obtained sharp bounds for $|B_{2n-1}|$ if $1 \le n \le 7$.

Recently, Some several researcher such as (for example [1], [2], [3], [4] [5], [6], [7], [8], [9], [10], [11], [12],[13], [14], [17], [21]) introduced new subclasses of bi-univalent functions and meromorphically bi-univalent functions and obtained estimates on the initial coefficients for functions in each of these subclasses.

In 2013, Babalola [18] introduced a new subclass λ-pseudo starlike function of order $0 \le \beta < 1$ satisfying the analytic condition

$$\Re\left\{\frac{z(f(z)')^{\lambda}}{f(z)}\right\} > \beta \ (\lambda \ge 1,\ z \in U). \tag{1.5}$$

In particular, Babalola [18] proved that all λ-pseudo-starlike functions are Bazilevic of type $1 - \frac{1}{\lambda}$ and order $\beta^{\frac{1}{\lambda}}$ and are univalent in open unit disk U.

In the present paper, we introduce two new subclasses of pseudo-type of meromorphically bi-univalent functions and obtained the estimates for the initial coefficients $|b_0|$ and $|b_1|$ of functions in these subclasses. Several some consequences of the new results are also pointed out.

2. Coefficient Bounds for the Function Class $\Sigma'_*(h,p,\lambda)$

We begin by introducing the function class $\Sigma'(h,p,\lambda)$ by means of the following definition.

Definition 2.1. *Let the functions $h;p:\triangle\to C$ be analytic functions and*

$$h(z)=1+\frac{h_1}{z}+\frac{h_2}{z^2}+\frac{h_3}{z^3}+\cdots,\;\; p(z)=1+\frac{p_1}{z}+\frac{p_2}{z^2}+\frac{p_3}{z^3}+\cdots,$$

such that

$$\min\{\Re(h(z)),\Re(p(z))\}>0, z\in\triangle.$$

A function $g(z)\in\Sigma'$ given by (1.2) is said to be in the class $\Sigma'_*(h,p,\lambda)$ if the following conditions are satisfied:

$$g\in\Sigma' \text{ and } \frac{z(g(z)')^{\lambda}}{g(z)}\in h(\triangle),(\ \lambda\geq 1,\ z\in\triangle), \tag{2.1}$$

and

$$\frac{w(h(w)')^{\lambda}}{h(w)}\in p(\triangle),(\ \lambda\geq 1,\ w\in\triangle), \tag{2.2}$$

where the function h is given by (1.3).

Theorem 2.2. *Let $g(z)$ be given by (1.2) be in the class $\Sigma'_*(h,p,\lambda)$. Then*

$$|b_0|\leq\min\left\{\sqrt{\frac{|h_1|^2+|p_1|^2}{2}},\sqrt{\frac{|h_2|+|p_2|}{2}}\right\} \tag{2.3}$$

and

$$|b_1|\leq\min\left\{\frac{|h_2|+|p_2|}{2|\lambda+1|},\frac{1}{\lambda+1}\left(\sqrt{\frac{|h_2|^2+|p_2|^2}{2}+\frac{(|h_1|^2+|p_1|^2)^2}{4}}\right)\right\}. \tag{2.4}$$

Proof. Let $g\in\Sigma'_*(h,p,\lambda)$. Then, by Definition 2.1 of meromorphically bi-univalent function class $\Sigma'_*(h,p,\lambda)$, the conditions (2.1) and (2.2) can be rewritten as follows:

$$\frac{z(g(z)')^{\lambda}}{g(z)}=h(z) \tag{2.5}$$

and

$$\frac{w(h(w)')^{\lambda}}{h(w)}=p(w), \tag{2.6}$$

respectively. Here, and in what follows,the functions $h(z)\in P$ and $p(w)\in P$ have the following forms:

$$p(z)=1+\frac{p_1}{z}+\frac{p_2}{z^2}+\frac{p_3}{z^3}+... \quad (z\in\triangle) \tag{2.7}$$

and

$$q(w)=1+\frac{q_1}{w}+\frac{q_2}{w^2}+\frac{q_3}{w^3}+... \quad (w\in\triangle). \tag{2.8}$$

Clearly, we have

$$\frac{z(g(z)')^{\lambda}}{g(z)}=1-\frac{b_0}{z}+\frac{b_0^2-(1+\lambda)b_1}{z^2}+\frac{b_0^3-(2\lambda)b_0b_1+(1+2\lambda)b_2}{z^3}+\cdots \tag{2.9}$$

and

$$\frac{w(h(w)')^\lambda}{h(w)} = 1 + \frac{b_0}{w} + \frac{b_0^2 + (1+\lambda))b_1}{w^2} + \frac{b_0^3 + (3(1+\lambda)b_0 b_1 + (1+2\lambda))b_2}{w^3} + \cdots . \tag{2.10}$$

Now, equating the Coefficients in (2.5) and (2.6), we get

$$-b_0 = h_1 \tag{2.11}$$

$$b_0^2 - (1+\lambda)b_1 = h_2 \tag{2.12}$$

$$b_0 = p_1 \tag{2.13}$$

$$b_0^2 + (1+\lambda)b_1 = p_2. \tag{2.14}$$

From (2.11) and (2.13), we find that

$$p_1 = -q_1 \tag{2.15}$$

and

$$2b_0^2 = h_1^2 + p_1^2 \tag{2.16}$$

that is,

$$|b_0|^2 \le \frac{|h_1|^2 + |p_1|^2}{2}. \tag{2.17}$$

Adding (2.12) and (2.14), we get

$$2b_0^2 = h_2 + p_2 \tag{2.18}$$

that is,

$$|b_0|^2 \le \frac{|h_2| + |p_2|}{2}. \tag{2.19}$$

From (2.18) and (2.19) we get the desired estimate on the coefficient $|b_0|$ as asserted in (2.3).

Next, in order to find the bound on $|b_0|$, by subtracting the equation (2.12) from the equation (2.14), we get

$$2(1+\lambda)b_1 = p_2 - h_2, \tag{2.20}$$

that is,

$$|b_1| \le \frac{|h_2| + |p_2|}{|2(1+\lambda)|}. \tag{2.21}$$

By squaring and adding (2.12) and (2.14), using (2.18) in the computation leads to

$$b_1^2 = \frac{1}{(1+\lambda)^2}\left(\frac{h_2^2 + p_2^2}{2} - \frac{[h_1^2 + p_1^2]^2}{4}\right). \tag{2.22}$$

that is,

$$|b_1| \le \frac{1}{1+\lambda}\left(\sqrt{\frac{|h_2|^2 + |p_2|^2}{2} + \frac{(|h_1|^2 + |p_1|^2)^2}{4}}\right). \tag{2.23}$$

From (2.21) and (2.23) we get the desired estimate on the coefficient $|b_1|$ as asserted in (2.4). ■

Remark 2.3. *If we take*

$$h(z) = p(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^\alpha = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \cdots , \ (0 < \alpha \le 1, \ z \in \triangle),$$

and

$$h(z) = p(z) = \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}} = 1 + \frac{2(1-\mu)}{z} + \frac{2(1-\mu)}{z^2}, \ (0 < \mu \le 1, \ z \in \triangle),$$

respectively, in the Theorem 2.2, we obtain the following results which is an improvement of estimates obtained by Srivastava el. at [17].

Corollary 2.4. *Let $g(z)$ be given by (1.2) be in the class $\Sigma'_*(\lambda, \alpha)$. Then*

$$|b_0| \le 2\alpha \tag{2.24}$$

and

$$|b_1| \le \frac{2\sqrt{5}\alpha^2}{\lambda+1}. \tag{2.25}$$

Corollary 2.5. *Let $g(z)$ be given by (1.2) be in the class $\Sigma'_*(\lambda, \mu, \alpha)$. Then*

$$|b_0| \le 2(1-\mu) \tag{2.26}$$

and

$$|b_1| \le \frac{2(1-\mu)\sqrt{4\mu^2 - 8\mu + 5\alpha^2}}{\lambda+1}. \tag{2.27}$$

3. Coefficient Bounds for the Function Class $\Sigma'_*(h, p, \lambda, \beta)$

We first introduce the function class $\Sigma'_*(h, p, \lambda, \beta)$ as follows.

Definition 3.1. *Let the functions $h; p : \triangle \to C$ be analytic functions and*

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots, \ \ p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots,$$

such that

$$\min\{\Re(h(z)), \Re(p(z))\} > 0, z \in \triangle.$$

A function $g(z) \in \Sigma'$ given by (1.2) is said to be in the class $\Sigma'_*(h, p, \lambda, \beta)$ if the following conditions are satisfied:

$$g \in \Sigma' \text{ and } (1-\beta)\left(\frac{g(z)}{z}\right)^\lambda + \beta\left(\frac{z(g(z)')^\lambda}{g(z)}\right) \in h(\triangle), \ (0 < \beta \le 1, \ \lambda \ge 1, z \in U^*), \tag{3.1}$$

and

$$(1-\beta)\left(\frac{h(w)}{w}\right)^\lambda + \beta\left(\frac{w(h(w)')^\lambda}{h(w)}\right) \in p(\triangle)(0 < \beta \le 1, \ \lambda \ge 1, \ w \in U^*), \tag{3.2}$$

where the function h is given by (1.3).

Next, we now derive the estimates on the Coefficients $|b_0|$ and $|b_1|$ for the meromorphically bi univalent function class $\Sigma'_{\lambda,\beta}(\mu)$.

Theorem 3.2. *Let $g(z)$ be given by (1.2) be in the class $\Sigma'_*(h, p, \lambda, \beta)$. Then*

$$|b_0| \le \min\left\{\sqrt{\frac{|h_1|^2 + |p_1|^2}{2(\lambda - \lambda\beta - \beta)^2}}, \sqrt{\frac{|h_1| + |p_1|}{|\lambda(\lambda-1)(1-\beta) + 2\beta|}}\right\} \tag{3.3}$$

and

$$|b_1| \le \min$$

$$\left\{\frac{|p_2| + |h_2|}{|2(\beta - \lambda + 2\lambda\beta)|}, \frac{1}{|\lambda - \beta - 2\lambda\beta|}\left(\sqrt{\frac{|h_2|^2 + |p_2|^2}{2} + \frac{[(\lambda(\lambda-1)(1-\beta) + 2\beta)]^2(|h_1|^2 + |p_1|^2)^2}{16(\lambda - \lambda\beta - \beta)^4}}\right)\right\}. \tag{3.4}$$

Proof. Let $g \in \Sigma'_*(h,p,\lambda,\beta)$. Then, by Definition 3.1 of meromorphically bi-univalent function class $\Sigma'_*(h,p,\lambda,\beta)$, the conditions (3.1) and (3.2) can be rewritten as follows:

$$(1-\beta)\left(\frac{g(z)}{z}\right)^{\lambda}+\beta\left(\frac{z(g(z)')^{\lambda}}{g(z)}\right)=h(z) \tag{3.5}$$

and

$$(1-\beta)\left(\frac{h(w)}{w}\right)^{\lambda}+\beta\left(\frac{w(h(w)')^{\lambda}}{h(w)}\right)=p(w), \tag{3.6}$$

respectively. Here, just as in our proof of Theorem 2.1, with the functions $p(z) \in P$ and $q(w) \in P$ have the forms in (2.7) and (2.7), and comparing the corresponding coefficients in (3.5) and (3.6), we have

$$(\lambda-\lambda\beta-\beta)b_0=h_1 \tag{3.7}$$

$$\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\lambda-\beta-2\lambda\beta)b_1=h_2 \tag{3.8}$$

$$-(\lambda-\lambda\beta-\beta)b_0=p_1 \tag{3.9}$$

$$\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\beta-\lambda+2\lambda\beta)b_1=p_2. \tag{3.10}$$

From (3.7) and (3.9), we obtain

$$h_1=-p_1 \tag{3.11}$$

and

$$2(\lambda-\lambda\beta-\beta)^2b_0^2=h_1^2+p_1^2 \tag{3.12}$$

that is,

$$|b_0|^2 \le \frac{|h_1|^2+|p_1|^2}{2(\lambda-\lambda\beta-\beta)^2}. \tag{3.13}$$

From (3.8) and (3.10), we get

$$(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2=h_1+p_1, \tag{3.14}$$

that is,

$$|b_0|^2 \le \frac{|h_1|+|p_1|}{|\lambda(\lambda-1)(1-\beta)+2\beta|}. \tag{3.15}$$

From (3.13) and (3.15) we get the desired estimate on the coefficient $|b_0|$ as asserted in (3.3).
Next, in order to find the bound on $|b_1|$, by subtracting the equation (3.8) from the equation (3.10), we get

$$2(\beta-\lambda+2\lambda\beta)b_1=p_2-h_2, \tag{3.16}$$

that is,

$$|b_1| \le \frac{|p_2|+|h_2|}{|2(\beta-\lambda+2\lambda\beta)|}. \tag{3.17}$$

By squaring and adding (3.8) and (3.10), using (3.14) in the computation leads to

$$b_1^2=\frac{1}{(\lambda-\beta-2\lambda\beta)^2}\left(\frac{h_2^2+p_2^2}{2}-\frac{[(\lambda(\lambda-1)(1-\beta)+2\beta)]^2[h_1^2+p_1^2]^2}{16(\lambda-\lambda\beta-\beta)^4}\right), \tag{3.18}$$

that is,

$$|b_1| \le \frac{1}{|\lambda-\beta-2\lambda\beta|}\left(\sqrt{\frac{|h_2|^2+|p_2|^2}{2}+\frac{([(\lambda(\lambda-1)(1-\beta)+2\beta)]^2(|h_1|^2+|p_1|^2)^2}{16(\lambda-\lambda\beta-\beta)^4}}\right). \tag{3.19}$$

From (3.17) and (3.19) we get the desired estimate on the coefficient $|b_1|$ as asserted in (3.4). ■

Future Work: For function $g \in \Sigma'_*(\lambda,\beta,\phi)$ given by (1.2) by taking $\phi=h(z)=p(z)$ as in Remark 2.3 or $(\phi=\frac{1+Az}{1+Bz}\ -1 \le B < A \le 1)$, we can obtain the initial coefficient estimates $|b_0|$ and $|b_1|$ by routine procedure (as in Theorem 2.2) and so we omit the details.

4. Acknowledgement

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

References

[1] A. A. Amourah, A. G. Al Amoush and M. Al-Kaseasbeh, Gegenbauer Polynomials and Bi-univalent Functions, *Palestine Journal of Mathematics*, **10(2)**(2021) , 625–632.

[2] A. A. Amourah, Faber polynomial coefficient estimates for a class of analytic bi-univalent functions, *AIP Conference Proceedings*, 2096(1)(2019), 020024.

[3] A. G. Alamoush, Coefficient Estimates for New Subclass of Pseudo-Type Meromorphic Bi-Univalent Functions, *Italian Journal of Pure and Applied Mathematics*, (46), accepted.

[4] A. G. Alamoush, A subclass of pseudo-type meromorphic bi-univalent functions, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **69(2)**(2020), 1025-1032.

[5] A. G. Alamoush, Certain subclasses of bi-univalent functions involving the Poisson distribution associated with Horadam polynomials, *Malaya Journal of Matematik*, 7(2018), 618–624.

[6] A. G. Alamoush, Coefficient Estimates for Certain Subclass of Bi-Bazilevic Functions Associated With Chebyshev Polynomials, *Acta Universitatis Apulensis*, **60**(2019), 53–59.

[7] A. G. Alamoush, Coefficient estimates for a new subclasses of lambda-pseudo bi-univalent functions with respect to symmetrical points associated with the Horadam Polynomials, *Turkish Journal of Mathematics*, **3**, 2865–2875.

[8] A. G. Alamoush, On subclass of analytic bi-close-to-convex functions, *Int. J. Open Problems Complex Analysis*, **13(1)**(2021), 10–18.

[9] A. G. Alamoush, On a subclass of bi-univalent functions associated to Horadam polynomials, *International Journal of Open Problems in Complex Analysis*, **12(1)**(2020), 58–56.

[10] A. G. Alamoush and M. Darus, Coefficient bounds for new subclasses of bi-univalent functions using Hadamard product, *Acta Universitatis Apulensis*, (2014), 153–161.

[11] A. G. Alamoush and M. Darus, Coefficients estimates for bi-univalent of fox-wright functions, *Far East Journal of Mathematical Sciences*, (2014), 249–262.

[12] A. G. Alamoush and M. Darus, Faber polynomial Coefficients estimates for a new subclass of meromorphic bi-univalent functions, *Advances in Inequalities and Applications*, (2016), 2016:3.

[13] A. G. Alamoush and M. Darus, On coefficient estimates for new generalized subclasses of bi-univalent functions, *AIP Conference Proceedings*, 1614(2014), 844.

[14] G. P. Kapoor and A.K. Mishra, Coefficients estimates for inverses of starlike functions of positive order, *J. Math. Anal. Appl.*, **329(2)**(2007), 922–934.

[15] G. Schobe, Coefficients of inverses of meromorphic univalent functions, *Proc. Amer. Math. Soc.*, **67(1)**(1977), 111–116.

[16] G. Springer, The Coefficients problem for schlicht mappings of the exterior of the unit circle, *Trans. Amer. Math. Soc.*, **70**(1951), 421–450.

[17] H. M. SRIVASTAVA, B. SANTOSH JOSHI, S. SAYALI JOSHI, H. PAWAR, Coefficient estimates for certain subclasses of meromorphically bi-univalent functions, *Pal. Jour. Math.*, **5**(2016), 250–258.

[18] K. O. BABALOLA, On λ-pseudo-starlike functions, *Jour. Class. Anal.*, **3**(2013), 137–147.

[19] M. SCHIFFER, On an extremum problem of conformal representation, *Bull. de la Soc. Math. de Fra.*, **66**(1938), 48–55.

[20] P. L. DUREN, Coefficients of meromorphic schlicht functions, *Proc. Amer. Math. Soc.*, **28**(1971), 169–172.

[21] S. G. HAMIDI , S.A. HALIM, J.M. JAHANGIRI, Coefficients estimates for a class of meromorphic bi-univalent functions, *C. R. Acad. Sci. Paris Sér. I*, **351**(2013), 349–352.

[22] Y. KUBOTA, Coefficients of meromorphic univalent functions, *Kod. Math. Sem. Rep.*, **28(2-3)**(1977), 253–261.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 55–62.
http://doi.org/10.26637/mjm1001/005

On isolate domination in hypergraphs

MEGHA M. JADHAV [1] AND KISHOR F. PAWAR*[2]

[1,2] *Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon-425001, India.*

Received 24 September 2021; Accepted 12 December 2021

Abstract. In this paper we introduced the notion of an isolate domination in hypergraphs. A set $D \subseteq V$ is called a dominating set of $\mathcal{H}$ if for every $v \in V \setminus D$ there exists $u \in D$ such that u and v are adjacent. A dominating set I of a hypergraph $\mathcal{H}$ is called an isolate dominating set of $\mathcal{H}$ if it contains at least one vertex $v \in I$ such that v is not adjacent to any vertex of I. The minimum cardinality of an isolate dominating set of $\mathcal{H}$ is called the isolate domination number γ_0 of $\mathcal{H}$. We determine the isolate domination number for some hypergraphs while the study on this parameter has been initiated. Furthermore, the effects of the removal of a vertex or an edge from the hypergraph upon the isolate domination number are examined.

AMS Subject Classifications: 05C65.

Keywords: Hypergraphs, domination number, isolate domination.

Contents

1. Introduction and Background

The concept of domination in graphs was initiated by de Jaenisch [14] during 1862 when he attempted to determine the minimum number of queens required to cover or dominate an $n \times n$ chess board. Similar problems posed by Ball [3] were studied by Yaglom brothers [18]. Berge [4] in 1958 and Ore [16] in 1962 introduced the idea of domination in graphs. Berge named domination as external stability and domination number as a coefficient of external stability while Ore used the words domination and domination number for the same idea. A survey of Cockayne and Hedetniemi [7] about domination motivates many researchers to work on it. Since then many researchers have been working on this topic and extending their contributions through research articles and books. An excellent treatment of fundamentals of domination in graph is given in Haynes et. al [11] while several advanced topics for domination can studied in [10]. Several variants of domination have been introduced and well-studied in the present literature such as edge domination, total domination, connected domination, global domination, equitable domination etc. and many others are being studied. For a detailed bibliography of papers on the concept of domination, the readers may to refer Hedetniemi and Laskar [12]. The notion of an isolate domination in graphs was introduced by Hamid and Balamurugan [8]. The theory of domination in graphs is well developed on the other hand, domination in hypergraph is a recent problem to study. However, as in case

***Corresponding author.** Email address: **meghachalisgaon@gmail.com** (Megha M. Jadhav), **kfpawar@nmu.ac.in** (Kishor F. Pawar)

of graphs, the domination in hypergraphs also has many interesting applications. The concept of domination in hypergraphs was initiated by Acharya [1], [2] and thereafter many researchers began to study domination in hypergraph. Reader may refer to the second part of the book [9] by Haynes et. al for the domination in hypergraph. Domination and related subset problems such as independence, irredundance, vertex covering and matching has became an extensively researched branch of graph theory, due to its wide applications and potential to solve many real life problems involving design and analysis of communication network as well as defense surveillance.

In this paper we introduced a new variant of domination in hypergraph and studied two new parameters of this domination. Later several important properties are studied and some results are found.

2. Preliminaries

We begin with recalling some basic definitions and results from [5], [6], [15], [13], [17] required for our purpose.

Definition 2.1. *A hypergraph $\mathcal{H}$ is a pair $\mathcal{H}(V, E)$ where V is a finite nonempty set and E is a collection of subsets of V. The elements of V are called vertices and the elements of E are called edges or hyperedges. And $\cup_{e_i \in E} e_i = V$ and $e_i \neq \phi$ are required for all $e_i \in E$. The number of vertices in $\mathcal{H}$ is called the order of the hypergraph and is denoted by $|V|$. The number of edges in $\mathcal{H}$ is called the size of $\mathcal{H}$ and is denoted by $|E|$. A hypergraph of order n and size m is called a (n, m) hypergraph. The number $|e_i|$ is called the degree (cardinality) of the edges e_i. The rank of a hypergraph $\mathcal{H}$ is $r(\mathcal{H}) = max_{e_i \in E} |e_i|$.*

Definition 2.2. *For any vertex v in a hypergraph $\mathcal{H}(V, E)$, the set*

$$N[v] = \{u \in V : u \text{ is adjacent to } v\} \cup \{v\}$$

is called the closed neighborhood of v in $\mathcal{H}$ and each vertex in the set $N[v] - \{v\}$ is called neighbor of v. The open neighborhood of the vertex v is the set $N[v] \setminus \{v\}$. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

Definition 2.3. *A simple hypergraph (or sperner family) is a hypergraph $\mathcal{H}(V, E)$ where $E = \{e_1, e_2, \cdots, e_m\}$ such that $e_i \subset e_j$ implies $i = j$.*

Definition 2.4. *For any hypergraph $\mathcal{H}(V, E)$ two vertices v and u are said to be adjacent if there exists an edge $e \in E$ that contains both v and u and non-adjacent otherwise.*

Definition 2.5. *For any hypergraph $\mathcal{H}(V, E)$ two edges are said to be adjacent if their intersection is nonempty. If a vertex $v_i \in V$ belongs to an edge $e_j \in E$ then we say that they are incident to each other.*

Definition 2.6. *The vertex degree of a vertex v is the number of vertices adjacent to the vertex v in $\mathcal{H}$. It is denoted by $d(v)$. The maximum (minimum) vertex degree of a hypergraph is denoted by $\Delta(\mathcal{H})$ $(\delta(\mathcal{H}))$.*

Definition 2.7. *The edge degree of a vertex v is the number of edges containing the vertex v. It is denoted by $d_E(v)$.*

The maximum (minimum) edge degree of a hypergraph is denoted by $\Delta_E(\mathcal{H})(\delta_E(\mathcal{H}))$. A vertex of a hypergraph which is incident to no edge is called an isolated vertex.

Definition 2.8. *A star hypergraph is an intersecting family of edges having a common element v. It is denoted by $\mathcal{H}(v)$ and the vertex v is called the center of $\mathcal{H}(v)$.*

Definition 2.9. *The hypergraph $\mathcal{H}(V, E)$ is called connected if for any pair of its vertices, there is a path connecting them. If $\mathcal{H}$ is not connected then it consists of two or more connected components, each of which is a connected hypergraph.*

Definition 2.10. *For $0 \leq r \leq n$, we define the complete r-uniform hypergraph to be the simple hypergraph $K_n^r = \mathcal{H}(V, E)$ such that $|V| = n$ and $E(K_n^r)$ coincides with all the r-subsets of V.*

Definition 2.11. *A complete r-partite hypergraph is an r-uniform hypergraph $\mathcal{H}(V, E)$ such that the set V can be partitioned into r non-empty parts, each edge contains precisely one vertex from each part, and all such subsets form E. It is denoted by $K^r_{n_1,n_2,\ldots,n_r}$, where n_i is the number of vertices in part V_i.*

Definition 2.12. *Let S be a set of vertices of a hypergraph $\mathcal{H}$ and let $u \in S$. Then the vertex v is said to be a private neighbor of u (with respect to S) if $N[v] \cap S = \{u\}$. The set of all private neighbors of u with respect to S is called private neighbor set of u with respect to S and is denoted by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.*

Definition 2.13. *For a hypergraph $\mathcal{H}(V, E)$, a set $D \subseteq V$ is called a dominating set of $\mathcal{H}$ if for every $v \in V \setminus D$ there exists $u \in D$ such that u and v are adjacent in $\mathcal{H}$, that is there exists $e \in E$ such that $u, v \in e$.*

Definition 2.14. *A dominating set D of a hypergraph $\mathcal{H}$ is called a minimal dominating set, if no proper subset of D is a dominating set of $\mathcal{H}$. The minimum(maximum) cardinality of a minimal dominating set in a hypergraph $\mathcal{H}$ is called the domination(upper domination) number of $\mathcal{H}$ and is denoted by $\gamma(\mathcal{H})(\Gamma(\mathcal{H}))$.*

3. Isolate Domination

In this section the notion of an isolate domination is given while the parameters like isolate domination number and upper isolate domination number are defined and verified with examples. Later we determine the values of these parameters for some hypergraphs and some bounds in terms of elements of $\mathcal{H}$ are obtained. Lastly, we investigate the properties of the hypergraphs for which $\gamma_0(\mathcal{H}) = n - \Delta(\mathcal{H})$.

Definition 3.1. *A dominating set I of a hypergraph $\mathcal{H}$ is called an isolate dominating set of $\mathcal{H}$ if it contains at least one vertex $v \in I$ such that v is not adjacent to any vertex of I i.e. $N(v) \cap I = \phi$, for at least one vertex $v \in I$.*

Definition 3.2. *An isolate dominating set I of a hypergraph $\mathcal{H}$ is called a minimal isolate dominating set if no proper subset of I is an isolate dominating set of $\mathcal{H}$.*

Definition 3.3. *The minimum (maximum) cardinality of a minimal isolate dominating set in a hypergraph $\mathcal{H}$ is called the isolate (upper isolate) domination number of $\mathcal{H}$ and is denoted by $\gamma_0(\mathcal{H})(\Gamma_0(\mathcal{H}))$.*
An isolate dominating set of cardinality $\gamma_0(\Gamma_0)$ is called a γ_0-set (Γ_0-set).

Example 3.4. *Consider the hypergraph $\mathcal{H}(V, E)$ where $V = \{v_1, v_2, \ldots, v_{14}\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$. In which the edges of $\mathcal{H}$ are defined as follows:*

$$\begin{aligned} e_1 &= \{v_1, v_2, v_3, v_4, v_5, v_6\}, \\ e_2 &= \{v_5, v_6, v_7, v_8\}, \\ e_3 &= \{v_6, v_9\}, \\ e_4 &= \{v_2, v_3, v_{10}, v_{11}\}, \\ e_5 &= \{v_1, v_2, v_{12}, v_{13}, v_{14}\}. \end{aligned}$$

Then the sets $I_1 = \{v_2, v_7, v_9\}$, $I_2 = \{v_4, v_6, v_{10}, v_{12}\}$ and $I_3 = \{v_4, v_7, v_9, v_{10}, v_{12}\}$ are the isolate dominating sets of $\mathcal{H}$. But among these only I_1 and I_3 are minimal isolate dominating sets but not I_2. In fact, I_1 is a minimal dominating set of $\mathcal{H}$ with minimum cardinality and I_3 is that of maximum cardinality. Hence $\gamma_0(\mathcal{H}) = 3$ and $\Gamma_0(\mathcal{H}) = 5$.

Theorem 3.5. *Let $\mathcal{H}$ be a disconnected hypergraph having $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots, \mathcal{H}_k$ as its components then*

1. $\gamma_0(\mathcal{H})$= *min*$_{1\leq i\leq k}\{s_i\}$, *where* $s_i = \gamma_0(\mathcal{H}_i) + \sum_{j=1,j\neq i}^{k} \gamma(\mathcal{H}_j)$.

2. $\Gamma_0(\mathcal{H})$= *max*$_{1\leq i\leq k}\{r_i\}$, *where* $r_i = \Gamma_0(\mathcal{H}_i) + \sum_{j=1,j\neq i}^{k} \Gamma(\mathcal{H}_j)$.

Proof. **1)** Suppose s_1 = min$\{s_1, s_2, ..., s_k\}$. Let I be a γ_0-set of $\mathcal{H}_1$ and D_i be a γ-sets of $\mathcal{H}_i$ for all $i \geq 2$. Then the set $I \cup (\cup_{i=2}^{k} D_i)$ is an isolate dominating set of $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) \leq \gamma_0(\mathcal{H}_1) + \sum_{j=2}^{k} \gamma(\mathcal{H}_j) = s_1 = \min_{1\leq i\leq k}\{s_i\}$.

Now let I be any minimal isolate dominating set of $\mathcal{H}$. Then the intersection of I and the vertex of $V(\mathcal{H}_i)$ of each component $\mathcal{H}_i$ is non-empty. In fact, the set $I \cap V(\mathcal{H}_i)$ a minimal dominating set of $\mathcal{H}_i$, for all $i = 1, 2, \ldots, k$. Further, for at least one i, say j we have $I \cap V(\mathcal{H}_j)$ is an isolate dominating set of $\mathcal{H}_j$. Therefore $|I| \geq \gamma_0(\mathcal{H}_j) + \sum_{i=1,i\neq j}^{k} \gamma(\mathcal{H}_i) = s_j \geq \min\{s_i\}$. Hence $\gamma_0(\mathcal{H})$= $\min_{1\leq i\leq k}\{s_i\}$.

2) Every Γ_0-set of $\mathcal{H}_i$ together with the set $\cup_{j=1,j\neq i}^{k} D_j$ forms a minimal isolate dominating set of $\mathcal{H}$, where D_j is a Γ-set of $\mathcal{H}_j$ and $1 \leq i \leq k$. Hence $\Gamma_0(\mathcal{H}) \geq \max_{1\leq i\leq k}\{r_i\}$.

Now let I be any minimal isolate dominating set of $\mathcal{H}$. Then $I \cap V(\mathcal{H}_i)$ is a minimal dominating set of $\mathcal{H}$ for every $i = 1, 2, \ldots, k$. Further for at least one i, say j we have $I \cap V(\mathcal{H}_j)$ is an isolate dominating set of $\mathcal{H}_j$.

Therefore $|I| \leq \Gamma_0(\mathcal{H}_j) + \sum_{i=1,i\neq j}^{k} \Gamma(\mathcal{H}_i) = r_j \leq \max_{1\leq i\leq k}\{r_i\}$.

Hence $\Gamma_0(\mathcal{H})$=$\max_{1\leq i\leq k}\{r_i\}$. ■

Observations 3.6. *If a hypergraph $\mathcal{H}$ contains an isolated vertex then $\gamma_0(\mathcal{H}) = \gamma(\mathcal{H})$ and $\Gamma_0(\mathcal{H}) = \Gamma(\mathcal{H})$.*

In light of the above observation, we restrict our attention to connected hypergraphs in the rest of this paper unless otherwise stated.

Theorem 3.7. *For complete r-uniform hypergraph $\mathcal{H} = K_n^r$, for $r \geq 2$, $\gamma_0(\mathcal{H}) = \Gamma_0(\mathcal{H}) = 1$ and for complete r-partite hypergraph $\mathcal{H} = K_{n_1,n_2,\ldots,n_r}^r$, $\gamma_0(\mathcal{H})$=min$\{n_1, n_2, \ldots, n_r\}$, $\Gamma_0(\mathcal{H})$=max$\{n_1, n_2, \ldots, n_r\}$.*

Proof. Any vertex in complete r-uniform hypergraph is adjacent to all vertices of $\mathcal{H}$. Hence $\gamma_0(K_n^r) = \Gamma_0(K_n^r) = 1$. Further from the definition of complete r-partite hypergraph $\mathcal{H}$, each r parts are the minimal isolate dominating sets of $\mathcal{H}$. Hence maximum and minimum values of the set $\{n_1, n_2, \ldots, n_r\}$ will be the $\gamma_0(\mathcal{H})$ and $\Gamma_0(\mathcal{H})$ respectively. ■

Observations 3.8. *If I is a minimal isolate dominating set of $\mathcal{H}$ then $V \setminus I$ is a dominating set of $\mathcal{H}$.*

In view of the above observation, complement of a minimal isolate dominating set is dominating but need not be an isolate dominating. But following theorem proves that like domination number of $\mathcal{H}$, the isolate domination number $\gamma_0(\mathcal{H})$ does not exceed half of the order of $\mathcal{H}$.

Theorem 3.9. *For a connected hypergraph $\mathcal{H}$, $\gamma_0(\mathcal{H}) \leq \frac{n}{2}$, where n is the number of vertices of $\mathcal{H}$. Moreover, if p and q are positive integers such that $q \geq 2p$ then there exists a hypergraph $\mathcal{H}$ of order q with $\gamma_0(\mathcal{H}) = p$.*

Proof. Let $\mathcal{H}$ be a connected hypergraph. Let D be a minimum dominating set of $\mathcal{H}$. If for any $v \in D$, we have $N(v) \cap D = \phi$ then D itself is a minimal isolate dominating set of $\mathcal{H}$ and the result follows. If $N(v) \cap D \neq \phi$, for every $v \in D$ then every vertex $v \in D$ has at least one private neighbor in $V \setminus D$ with respect to D. Let w be a vertex in D with minimum number of private neighbors, say m with respect to D. Then $\gamma(\mathcal{H}) + \gamma(\mathcal{H})m \leq n$. Further, the set $D - \{w\} \cup I$, where I is γ_0-set of $pn\,[w, D]$ is an isolate dominating set of $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) \leq \gamma(\mathcal{H}) - 1 + m$. Now we prove that $\gamma(\mathcal{H}) - 1 + m \leq \frac{\gamma(\mathcal{H})+\gamma(\mathcal{H})m}{2}$. The inequality is true when $\gamma(\mathcal{H}) = 2$. Now if $2(\gamma(\mathcal{H}) - 1 + m) > \gamma(\mathcal{H}) + \gamma(\mathcal{H})m$ and $\gamma(\mathcal{H}) \neq 2$, then we have $(\gamma(\mathcal{H}) - 2) > m(\gamma(\mathcal{H}) - 2)$, getting a contradiction as $m \geq 1$. Hence $\gamma_0(\mathcal{H}) \leq \gamma(\mathcal{H}) - 1 + m \leq \frac{\gamma(\mathcal{H})+\gamma(\mathcal{H})m}{2} \leq \frac{n}{2}$.

Now let p and q be any two positive integers such that $q > 2p$. Construct a hypergraph $\mathcal{H}$ of order q with $\gamma_0(\mathcal{H}) = p$. Firstly we consider an edge e' of cardinality p. Then the hypergraph $\mathcal{H}$ is obtained from that edge e' by attaching exactly one vertex at each $p - 1$ vertices and then adding one edge e containing the remaining one vertex from e' and $q - 2p + 1$ new vertices. It is clear to see that $\mathcal{H}$ is a hypergraph of order q with $\gamma_0(\mathcal{H}) = p$. ■

Observations 3.10. *For any vertex v in a hypergraph $\mathcal{H}$, the set $V \setminus N(v)$ is always an isolate dominating set of $\mathcal{H}$ and consequently $\gamma_0(\mathcal{H}) \leq n - \Delta(\mathcal{H})$.*

Theorem 3.11. *Let $\mathcal{H}$ be a hypergraph of order n with $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$ and let w be a vertex of degree $\Delta(\mathcal{H})$. Then $V \setminus N[w]$ is independent and $\Delta(\mathcal{H}) \geq \frac{n}{2}$.*

Proof. Let $\mathcal{H}$ be a given hypergraph. Suppose $V \setminus N[w]$ is not independent then there exists two vertices $p, q \in V \setminus N[w]$ such that p and q are adjacent. Consequently, the set $I = (V \setminus N[w] - \{p\}) \cup \{w\}$ is an isolate dominating set of $\mathcal{H}$ with cardinality $n - \Delta(\mathcal{H}) - 1$, a contradiction. Hence $V \setminus N[w]$ is independent. Now we prove that $\Delta(\mathcal{H}) \geq \frac{n}{2}$. Suppose $\Delta(\mathcal{H}) < \frac{n}{2}$. Before proving this, first we claim that each vertex of $N(w)$ is adjacent to at most one vertex in $V \setminus N[w]$. Suppose there exists a vertex $u \in N(w)$ having at least two neighbors say x and y in $V \setminus N[w]$. Since $\Delta(\mathcal{H}) < \frac{n}{2}$, it follows that $V \setminus N[w]$ contains at least $\Delta(\mathcal{H})$ vertices. Hence there exists a vertex z in $V \setminus N[w]$ which is not adjacent to u. Therefore the set $I = (V \setminus N[w] - \{x, y\}) \cup \{u, w\}$ is an isolate dominating set of $\mathcal{H}$ with cardinality less than or equal to $n - \Delta(\mathcal{H}) - 1$, which is a contradiction. Hence each vertex in $N(w)$ has at most one neighbor in $V \setminus N[w]$. Further $|V \setminus N[w]| \geq \Delta(\mathcal{H})$, together with the facts that $V \setminus N[w]$ is independent and each vertex of $N(w)$ has at most one neighbor in $V \setminus N[w]$, it follows that the sets $V \setminus N[w]$ and $N(w)$ have equal number of vertices . Hence a vertex in $N(w)$ together with its non-neighbors in $V \setminus N[w]$ form an isolate dominating set of $\mathcal{H}$ with cardinality $n - \Delta(\mathcal{H}) - 1$, a contradiction. Hence $\Delta(\mathcal{H}) \geq \frac{n}{2}$. ■

Theorem 3.12. *Let $\mathcal{H}$ be a connected hypergraph and let w be a vertex of degree $\Delta(\mathcal{H})$. If $V \setminus N[w]$ is an independent set and every vertex in $N(w)$ has at most one neighbor in $V \setminus N[w]$ then either $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$ or $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n - 1$. Further if $N(w)$ contains a vertex of degree 1 then $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$.*

Proof. Let $\mathcal{H}$ be a given hypergraph. Let I be an isolate dominating set of $\mathcal{H}$ with $|I| = \gamma_0(\mathcal{H})$. It is easy to see that the set $V \setminus N(w)$ is an isolate dominating set of $\mathcal{H}$ with cardinality $n - \Delta(\mathcal{H})$. Hence $\gamma_0(\mathcal{H}) \leq n - \Delta(\mathcal{H})$. Since $V \setminus N[w]$ is independent and every vertex in $N(w)$ has at most one neighbor in $V \setminus N[w]$, it follows that $|I| \geq |V \setminus N[w]| = n - \Delta(\mathcal{H}) - 1$. Therefore $n - \Delta(\mathcal{H}) - 1 \leq \gamma_0(\mathcal{H}) \leq n - \Delta(\mathcal{H})$. Hence $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$ or $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n - 1$. Further if $N(w)$ contains a vertex of degree 1. Let $u \in N(w)$ such that $d(u) = 1$. Then I must contain either u or w. Also I contains at least $|V \setminus N[w]|$ vertices for dominating all the vertices of $V \setminus N[w]$. Therefore $\gamma_0(\mathcal{H}) = |I| \geq n - \Delta(\mathcal{H})$. Hence $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$. This completes the proof. ■

4. Vertex Removal and Edge Removal

This section deals with the effects of vertex removal or edge removal on the isolate domination number and study the characteristics of vertices whose removal decreases or increases the isolate domination number of a hypergraph $\mathcal{H}$.

Definition 4.1. *[5] Let $\mathcal{H}$ be a hypergraph and $v \in V$. Then $\mathcal{H} \setminus \{v\}$ is a sub-hypergraph with vertex set $V \setminus \{v\}$ and edge set $\{e \setminus \{v\} : e \in E, e \setminus \{v\} \neq \phi\}$.*

Definition 4.2. *[5] Let $\mathcal{H}$ be a hypergraph and $e \in E$. Then $\mathcal{H} \setminus \{e\}$ is a sub-hypergraph with edge set $E \setminus \{e\}$, whose vertex set contains all vertices of $\mathcal{H}$ which are not pendant vertices in the deleted edge e.*

Theorem 4.3. *For a hypergraph $\mathcal{H}$ and $v \in V$, $\gamma_0(\mathcal{H} \setminus v) \geq \gamma_0(\mathcal{H}) - 1$.*

Proof. Let v be the vertex in $\mathcal{H}$ such that $\gamma_0(\mathcal{H} \setminus v) < \gamma_0(\mathcal{H})$. Let I be a γ_0-set of $\mathcal{H} \setminus v$. Then $N(v) \cap I = \phi$, otherwise I would be an isolate dominating set of $\mathcal{H}$ with cardinality less than $\gamma_0(\mathcal{H})$, which is a contradiction. Therefore the set $I \cup \{v\}$ forms an isolate dominating set of $\mathcal{H}$ with a vertex v such that $N(v) \cap I = \phi$. Thus $\gamma_0(\mathcal{H}) \leq |I \cup \{v\}| \leq \gamma_0(\mathcal{H} \setminus v) + 1$. Hence $\gamma_0(\mathcal{H} \setminus v) \geq \gamma_0(\mathcal{H}) - 1$. ■

Proposition 4.4. *Let $\mathcal{H}$ be a complete r-partite hypergraph with r-partitions $V_1, V_2, ...V_r$ then*

1. *If $|V_i| = 1$, for exactly one i, then $\gamma_0(\mathcal{H} \setminus v) \geq \gamma_0(\mathcal{H})$, for $v \in V$.*
2. *If $|V_i| = 1$, for more than one i, then isolate domination number remains unchanged on removal of any vertex v from H.*
3. *If each V_i contains at least two vertices then $\gamma_0(\mathcal{H} \setminus v) \leq \gamma_0(\mathcal{H})$, for $v \in V$.*

Proof. 1. Let $V_1 = \{w\}$. Then by definition, w dominates all the vertices of hypergraph $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) = 1$. Now if we remove a vertex w from hypergraph $\mathcal{H}$ then $\mathcal{H} \setminus w$ is a complete $(r-1)$ partite hypergraph with each part having at least two vertices. Thus $\gamma_0(\mathcal{H} \setminus w) \geq 2$, by theorem 3.7. Further, the removal of any vertex $v \neq w$ will not affect the value of $\gamma_0(\mathcal{H})$, as w is still there, to dominate all the vertices of hypergraph $\mathcal{H} \setminus v$. Hence $\gamma_0(\mathcal{H} \setminus v) \geq \gamma_0(\mathcal{H})$.

2. Let $V_1 = \{w_1\}$ and $V_2 = \{w_2\}$. Clearly V_1 and V_2 are the isolate dominating sets of $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) = 1$. Also the removal of any vertex v from $\mathcal{H}$ does not affect the value of $\gamma_0(\mathcal{H})$ as either w_1 or w_2 is present in $\mathcal{H} \setminus v$. Hence the result follows.

3. Let min$\{|V_i|\} = p$. Let the part V_k contains p vertices. Then by theorem 3.7, V_k is a γ_0-set of $\mathcal{H}$. Also each vertex of V_k is the only private neighbor of itself. Hence $\gamma_0(\mathcal{H} \setminus v) < \gamma_0(\mathcal{H})$, for $v \in V_k$. Further on removing any vertex $v \in V_i$ and $V_i \neq V_k$, we have $\gamma_0(\mathcal{H} \setminus v) = \gamma_0(\mathcal{H})$.

■

Theorem 4.5. *Let $\mathcal{H}$ be a hypergraph with $\gamma_0(\mathcal{H} \setminus v) = \gamma_0(\mathcal{H}) - 1$ iff there is a γ_0-set I with at least two vertices $u \in I$ such that $N(u) \cap I = \phi$ and $pn[v, I] = \{v\}$.*

Proof. Let $\gamma_0(\mathcal{H} \setminus v) = \gamma_0(\mathcal{H}) - 1$ and let I be a γ_0-set of $\mathcal{H} \setminus v$. Then $N(v) \cap I = \phi$. Thus the set $I \cup \{v\}$ is a γ_0-set of $\mathcal{H}$ with at least two vertices $u \in I$ such that $N(u) \cap I = \phi$ and also $pn[v, I] = \{v\}$. Conversely, suppose I be a γ_0-set of $\mathcal{H}$ with given conditions. Since $pn[v, I] = \{v\}$ and for at least two vertices of I, we have $N(v) \cap I = \phi$, it follows the set $I - v$ is an isolate dominating set of $\mathcal{H} \setminus v$. Therefore $\gamma_0(\mathcal{H} \setminus v) \leq |I| - 1 = \gamma(\mathcal{H}) - 1$. Hence by theorem 4.3, the result follows. ■

Theorem 4.6. *Let $\mathcal{H}$ be a hypergraph with at most one isolate vertex then $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$ if and only if*

1. *v is in every γ_0-set of $\mathcal{H}$.*
2. *No subset of $I \subseteq V \setminus N[v]$ with cardinality less than or equal to $\gamma_0(\mathcal{H})$ can be an isolate dominating set of $\mathcal{H} \setminus v$.*

Proof. Let $\mathcal{H}$ be a given hypergraph and $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$. Suppose v does not belong to γ_0-set I of $\mathcal{H}$. Then I will be an isolate dominating set of $\mathcal{H} \setminus v$. Consequently, $\gamma_0(\mathcal{H} \setminus v) \leq |I|$, which is a contradiction. Hence v is in every γ_0-set of $\mathcal{H}$ and 2 is obvious. Now conversely let 1) and 2) hold. Let I be a γ_0-set of $\mathcal{H} \setminus v$. If $I \subseteq V | N[v]$ then $|I| > \gamma_0(\mathcal{H})$, by condition 2. Hence $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$. If $I \cap N(v) \neq \phi$. Then I would be an isolate dominating set of $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) \leq |I|$. But by condition 1, $|I| > \gamma_0(\mathcal{H})$. Consequently, $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$. ■

Theorem 4.7. *Let $\mathcal{H}$ be a hypergraph with at most one isolated vertex. If u and v be the vertices in $\mathcal{H}$ such that $\gamma_0(\mathcal{H} \setminus u) < \gamma_0(\mathcal{H})$ and $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$ then u and v are not adjacent.*

Proof. Let $\mathcal{H}$ be a given hypergraph. Suppose u and v are adjacent. Let I be a γ_0-set of $\mathcal{H} \setminus u$. Then $I \cap N(u) = \phi$, otherwise I would form an isolate dominating set of $\mathcal{H}$ with cardinality less than $\gamma_0(\mathcal{H})$. Since u and v are adjacent, it follows $v \notin I$. Therefore the set $I \cup \{u\}$ would form a γ_0-set of $\mathcal{H}$, which is contradiction to the condition 1 of theorem 4.6,. Hence u and v are not adjacent. ■

Remark 4.8. *The following example illustrates that the converse is not true.*

Example 4.9. *Let $\mathcal{H}(V, E)$ be a hypergraph, where $V = \{v_1, v_2, \ldots, v_{11}\}$ and $E = \{e_1, e_2, \ldots, e_5\}$. In which the edges of $\mathcal{H}$ are defined as follows:*

$$\begin{aligned} e_1 &= \{v_1, v_2, v_3, v_4, v_5\}, \\ e_2 &= \{v_1, v_2, v_6, v_7\}, \\ e_3 &= \{v_1, v_8\}, \\ e_4 &= \{v_4, v_5, v_9, v_{10}\}, \\ e_5 &= \{v_4, v_{11}\}. \end{aligned}$$

The vertices v_8, v_{11} are not adjacent in $\mathcal{H}$ with $\gamma_0(\mathcal{H} \setminus v_8) < \gamma_0(\mathcal{H})$ and $\gamma_0(\mathcal{H} \setminus v_{11}) < \gamma_0(\mathcal{H})$. And the vertices v_6, v_9 are not adjacent in $\mathcal{H}$ with $\gamma_0(\mathcal{H} \setminus v_6) = \gamma_0(\mathcal{H})$, $\gamma_0(\mathcal{H} \setminus v_9) = \gamma_0(\mathcal{H})$.

Observations 4.10. *The isolate domination number $\gamma_0(\mathcal{H})$ of a hypergraph $\mathcal{H}$ may increase, decrease or remains unaltered when we remove an edge e from hypergraph $\mathcal{H}$. Moreover the differences $\gamma_0(\mathcal{H} \setminus e) - \gamma_0(\mathcal{H})$ and $\gamma_0(\mathcal{H}) - \gamma_0(\mathcal{H} \setminus e)$ can be made arbitrarily large.*

The following examples give the illustration of the above observation.

Example 4.11. *Consider two star hypergraphs $\mathcal{H}_1(u)$ and $\mathcal{H}_2(v)$ of size p whose centers are connected by an edge $e' = \{u, v\}$. Let $\mathcal{H}$ be that hypergraph. Then $\gamma_0(\mathcal{H}) = 1 + p$. Thus removing an edge e' from hypergraph $\mathcal{H}$ decrease the isolate domination number of $\mathcal{H}$ by $p - 1$.*

Example 4.12. *Consider the hypergraph $\mathcal{H}(V, E)$ where $V = \{v_1, v_2, \ldots, v_{p+2}, u_1, u_2, \ldots, u_{p+2}\}$ where p be any positive integer and $E = \{e_1, e_2, \ldots, e_{p+4}\}$. In which the edges of $\mathcal{H}$ are defined as follows:*

$$\begin{aligned} e_1 &= \{v_1, u_1\}, \\ e_2 &= \{v_2, u_2\}, \\ &\vdots \\ e_{p+2} &= \{v_{p+2}, u_{p+2}\}, \\ e_{p+3} &= \{v_1, v_2, \ldots v_{p+2}\}, \\ e_{p+4} &= \{u_1, u_2, \ldots u_{p+2}\}. \end{aligned}$$

Clearly, $\{v_1, u_2\}$ is an isolate dominating set of $\mathcal{H}$ and $\gamma_0(\mathcal{H}) = 2$. However, $\gamma_0(\mathcal{H} \setminus e_{p+3}) = p + 2$ and $\gamma_0(\mathcal{H} \setminus e_1) = 2$.

References

[1] B. D. ACHARYA, Domination in Hypergraphs, *AKCE Int. J. Graphs Combin.*, **4(2)**(2007), 117–126.

[2] B. D. ACHARYA, Domination in hypergraphs:II-New Directions, *Proc. Int. Conf. – ICDM*, (2008), 1–16.

[3] W. W. ROUSE BALL, *Mathematical Recreation and Problems of Past and Present Times*, 1892.

[4] C. BERGE, *Theory of Graphs and its Applications*, Methuen, London, (1962).

[5] C. BERGE, *Graphs and Hypergraphs*, North-Holland, Amsterdam, (1973).

[6] C. BERGE, *Hypergraphs, Combinatorics of Finite Sets*, North-Holland, Amsterdam, (1989).

[7] E. J. COCKAYNE AND S. T. HEDETNIEMI, Towards a theory of domination in graphs, *Networks*, **7**(1977), 247–261.

[8] I. SAHUL HAMID AND S. BALAMURUGAN, Isolate Domination in Graphs, *Arab Journal of Mathematical Sciences*, DOI:10.1016/j.ajmsc.2015.10.001.

[9] MICHAEL A. HENNING, STEPHEN T. HEDETNIEMI, TERESA W. HAYNES, *Structures of Domination in Graphs*, Springer International Publishing, 2021.

[10] T.W. HAYNES, S.T. HEDETNIEMI AND P.J. SLATER, *Domination in Graphs-Advanced Topics*, New York : Dekker, (1998).

[11] T.W. HAYNES, S.T. HEDETNIEMI AND P.J. SLATER, *Fundamentals of Domination in Graphs*, New York: Dekker (1998).

[12] S. T. HEDETNIEMI AND R. C. LASKAR, Bibliography on domination in graphs and some basic definitions of domination parameters, *Discrete Math.*, **86**(1990), 257–277.

[13] MEGHA M. JADHAV AND KISHOR F. PAWAR, On Edge Product Hypergraphs, *Journal of Hyperstructures*, **10(1)**(2021), 1–12.

[14] C. F. DE JAENISCH, *Traité des applications de l'analyse mathématique au jeu des échecs*, **3**(1862).

[15] BIBIN K. JOSE, ZS. TUZA, Hypergraph Domination and Strong Independence, *Appl. Anal. Discrete Math.*, **3** (2009), 347–358.

[16] OYSTEIN ORE, Theory of Graphs, *Amer. Math. Soc. Trans.*, Vol. 38 Amer. Math. Soc., providence, RI, (1962), 206-212.

[17] VITALY I. VOLOSHIN, *Introduction To Graph And Hypergraph Theory*, Nova Science Publishers, Inc. New York , 2009.

[18] A. M. YAGLOM AND I. M. YAGLOM, *Challenging mathematical problems with elementary solutions*, Volume 1 : Combinatorial Analysis and Probability Theory, 1964.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 63–78.
http://doi.org/10.26637/mjm1001/006

Existence results for $(p_1, ..., p_n)$-biharmonic systems under Navier boundary conditions

Têlé Jonas Doumate*[1,2], **Lawouè Robert Toyou**[2,3] **and Liamidi A. Leadi**[1,2]

[1] *Faculté des Sciences et Techniques, Département de Mathématiques, Université d'Abomey-Calavi, Bénin.*
[2] *Institut de Mathématiques et de Sciences Physiques, Université d'Abomey-Calavi, Bénin.*
[3] *Haute École de Commerce et de Management, Bénin.*

***R**eceived 19 September 2021; **A**ccepted 29 December 2021*

Abstract. We are concerned with the following $(p_1, \ldots, p_n)$-biharmonic system

$$\begin{cases} \Delta^2_{p_i} u_i - m(x)|u_i|^{\alpha_i - 1}u_i \prod\limits_{j=1, j\neq i}^{n} |u_j|^{\alpha_j+1} = \lambda m_i(x)|u_i|^{p_i-2}u_i, & \text{in } \Omega \\ u_i = \Delta u_i = 0, \text{ for } 1 \leq i \leq n, & \text{on } \partial\Omega. \end{cases}$$

The authors study the existence of weak solutions for the problem above via mountain pass theorem and establish semitrivial principal and strictly principal eigenvalues, positivity and simplicity results.

AMS Subject Classifications: 35D30, 35J35, 35J58, 35P30.

Keywords: Nonlinear eigenvalue problems, Variational methods, $(p_1, \ldots, p_n)$-biharmonic systems, Boundary value problems.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a non-empty bounded domain with smooth boundary $\partial\Omega$, $n \geq 1$ be an integer, α_i and p_i (with $i \in \{1, 2, \ldots, n\}$) be real constants such that $\alpha_i \geq 0$, $p_i > 1$ and $\sum\limits_{i=1}^{n} \frac{\alpha_i + 1}{p_i} = 1$.

The aim of this work is to study the following interesting problem

***Corresponding author.** Email address: **jonas.doumate@fast.uac.bj/doumatt@yahoo.fr** (Têlé Jonas Doumate), robert.toyou@imsp-uac.org (Lawouè Robert Toyou) and leadiare@imsp-uac.org (Liamidi A. Leadi)

$$
(Q):\begin{cases}
\Delta_{p_1}^2 u_1 - m(x)|u_1|^{\alpha_1-1}u_1 \prod_{i=2}^{n}|u_i|^{\alpha_i+1} = \lambda m_1(x)|u_1|^{p_1-2}u_1, & \text{in } \Omega\\
\dots\\
\Delta_{p_i}^2 u_i - m(x)|u_i|^{\alpha_i-1}u_i \prod_{j=1,j\neq i}^{n}|u_j|^{\alpha_j+1} = \lambda m_i(x)|u_i|^{p_i-2}u_i, & \text{in } \Omega\\
\dots\\
\Delta_{p_n}^2 u_n - m(x)|u_n|^{\alpha_n-1}u_n \prod_{i=1}^{n-1}|u_i|^{\alpha_i+1} = \lambda m_n(x)|u_n|^{p_n-2}u_n, & \text{in } \Omega\\
u_i = \Delta u_i = 0, \text{ for } 1 \le i \le n, & \text{on } \partial\Omega
\end{cases}
$$

where $\Delta_{p_i}^2 u_i = \Delta(|\Delta u_i|^{p_i-2}\Delta u_i)$ is the p_i-biharmonic operator and λ is a real parameter. Here, the coefficients m_i, (with $i = 1, 2, \dots, n$), $m \in L^\infty(\Omega)$ are assumed to be nonnegatives in Ω. Throughout this paper, we let W denote the Cartesian product of n Sobolev spaces $\left(W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)\right)$ for $i = 1, \dots, n$, i.e.,

$$
W = \left(W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega)\right) \times \left(W^{2,p_2}(\Omega) \cap W_0^{1,p_2}(\Omega)\right) \times \cdots \times \left(W^{2,p_n}(\Omega) \cap W_0^{1,p_n}(\Omega)\right)
$$

endowed with the norm

$$
\|(u_1, u_2, \dots, u_n)\| = \sum_{i=1}^{n} \|\Delta u_i\|_{p_i} \tag{1.1}
$$

where $\|.\|_p$ stands for the Lebesgue norm in L^p for all $p \in (1, \infty]$. We say that $((u_1, \dots, u_n), \lambda) \in W \times \mathbb{R}$ is a (weak) solution to the problem (Q) if

$$
\int_\Omega |\Delta u_i|^{p_i-2}\Delta u_i \Delta\varphi_i dx - \int_\Omega m \prod_{j=1,j\neq i}^{n} |u_j|^{\alpha_j+1}|u_i|^{\alpha_i-1}u_i\varphi_i dx = \lambda \int_\Omega m_i|u|^{p_i-2}u_i\varphi_i dx, \tag{1.2}
$$

for $1 \le i \le n$ and for all $(\varphi_1, \dots, \varphi_n) \in W$.

The study of nonlinear eigenvalue problems involving fourth-order differential equations has aroused a great interest in the scientific world and many applications have been made, including the study of deflections of elastic beams on nonlinear elastic foundations (see [2, 6, 19]), deformations of a rigid body and especially the study of traveling waves in suspension bridges (see, for instance, [14]) . A remarkable work of M. Talbi and N. Tsouli [19] has focused on the scalar version of (Q) with $m \equiv 0$, which reads

$$
(P_{a,p,\rho}):\begin{cases}
\Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda a(x)|u|^{p-2}u & \text{in } \Omega\\
u = \Delta u = 0 & \text{on } \partial\Omega
\end{cases} \tag{1.3}
$$

where $p \in (1, \infty)$, $\rho \in C(\bar{\Omega})$ such that $\rho > 0$ and $a \in L^\infty(\Omega)$. They proved that $(P_{a,p,\rho})$ possesses at least one non-decreasing sequence of eigenvalues and studied $(P_{a,p,\rho})$ in the particular one dimensional case. The authors, in the same reference gave the first eigenvalue $\lambda_{1,p,\rho}(a)$ and showed that if $a \ge 0$ almost everywhere in Ω and $a \in C(\bar{\Omega})$ then $\lambda_{1,p,\rho}(a)$ is simple, isolated and principal i.e. the associated eigenfunction, denoted by $\varphi_{p,\rho,a}$ is positive on Ω with

$$
\lambda_{1,p,\rho}(a) = \inf_{u\in\mathcal{A}} \int_\Omega \rho|\Delta u|^p dx \tag{1.4}
$$

where

$$
\mathcal{A} = \left\{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \int_\Omega a|u|^p dx = 1\right\}. \tag{1.5}
$$

By using a transformation of $(P_{a,p,\rho})$ to a known Poisson problem when $\rho \equiv 1$ and $a \equiv 1$, the authors in [9] proved the existence of a principal positive simple eigenvalue which is isolated and a description of all eigenvalues and associated eigenfunctions was given as well. The Dirichlet boundary conditions case was analyzed in [10] where it is shown that the spectrum contains at least one non-decreasing sequence of positive eigenvalues. On the other hand, J. Benedikt [4] gave the spectrum of the p-biharmonic operator with Dirichlet and Neumann boundary conditions in the special case $N = 1$, $\rho \equiv 1$ and $a \equiv 1$. They system (Q) in the absence of weights has drawn attention in [12] where the authors used the generalized three critical points of Ricceri, namely, three critical points theorem of Averna and Bonanno to prove the existence of at least three weak solutions for (Q) in case no weight is considered.

It is important to note that (u_1, λ) is solution of problem $(P_{m_1,p_1,1})$ if and only if $[(u_1, 0, \dots, 0), \lambda]$ is solution of (Q). This kind of solution is called "semitrivial" solution for (Q). Consequently, there are n "semitrivial" solutions of the problem (Q) that is $[(u_1, 0, \dots, 0), \lambda]$ with (u_1, λ) solution of problem $(P_{m_1,p_1,1})$, $[(0, \dots, u_i, \dots, 0), \lambda]$ with (u_i, λ) solution of problem $(P_{m_i,p_i,1})$ for $2 \le i \le n-1$ and $[(0, 0, \dots, u_n), \lambda]$ with (u_n, λ) solution of problem $(P_{m_n,p_n,1})$. In particular, when $\lambda = \lambda_{1,p_1,1}(m_1)$ (resp. $\lambda = \lambda_{1,p_i,1}(m_i)$) then $[(\varphi_{p_1,1,m_1}, 0), \lambda]$ (resp. $[(0, \varphi_{p_i,1,m_i}), \lambda]$ for $2 \le i \le n$) is called "semitrivial" solutions of the problem (Q) and $\lambda_{1,p_1,1}(m_1)$ (resp. $\lambda_{1,p_i,1}(m_i)$) is called "semitrivial" principal eigenvalue of (Q).

Recently, in a very interesting paper, L. A. Leadi and R. L. Toyou [17] studied the simplicity and the existence of the first strictly principal eigenvalue or semitrivial principal eigenvalue of problem (Q) in the particular case of $n = 2$. Motivated by their results we consider in this work the problem (Q), which generalizes the one in [17], and we intend to extend their findings in this general and challenging form of (Q). For this, we shall recall a bit of notations and basic results. The Sobolev space W endowed with the norm defined in (1.1) is a Banach and reflexive space [13, 18] and the weak convergence in W is denoted by $\rightharpoonup$. The positive and negative parts of a function w are denoted by $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$. Equalities (and inequalities) between two functions must be understood almost everywhere (a.e.). Notice that, for all $f \in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable in $X_p = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see for example [11]). We denote by Λ the inverse operator of $-\Delta : X_p \longmapsto L^p(\Omega)$. and the following lemma gives us some properties of the operator Λ:

Lemma 1.1. *[9, 19].*

1. *(Continuity) There exists a constant $c_p > 0$ such that*

$$\|\Lambda f\|_{W^{2,p}} \le c_p \|f\|_p$$

holds for all $p \in (1, \infty)$ and $f \in L^p(\Omega)$.

2. *(Continuity) Given $k \in \mathbb{N}^*$, there exists a constant $c_{p,k} > 0$ such that*

$$\|\Lambda f\|_{W^{k+2,p}} \le c_{p,k} \|f\|_{W^{k,p}}$$

holds for all $p \in (1, \infty)$ and $f \in W^{k,p}(\Omega)$.

3. *(Symmetry) The identity*

$$\int_\Omega \Lambda u . v dx = \int_\Omega u . \Lambda v dx$$

holds for $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $p' = \frac{p}{p-1}$ and $p \in (1, \infty)$.

4. *(Regularity) Given $f \in L^{\infty}(\Omega)$, we have $\Lambda f \in C^{1,\nu}(\bar{\Omega})$ for all $\nu \in (0,1)$. Moreover, there exists $c_\nu > 0$ such that*
$$\|\Lambda f\|_{C^{1,\nu}(\Omega)} \leq c_\nu \|f\|_\infty.$$
5. *(Regularity and Hopf-type maximum principle) Let $f \in C(\bar{\Omega})$ and $f \geq 0$ then $w = \Lambda f \in C^{1,\nu}(\bar{\Omega})$, for all $\nu \in (0,1)$ and w satisfies: $w > 0$ in Ω, $\frac{\partial w}{\partial n} < 0$ on $\partial\Omega$.*
6. *(Order preserving property) Given f, $g \in L^p(\Omega)$ if $f \leq g$ in Ω, then $\Lambda f < \Lambda g$ in Ω.*

The rest of the paper is organized as follows. The next section sets the functional framework, a review of tools and established results that help in our concern and constructs an eigencurve associated to the system (Q) as well as some well-known properties on obtained eigencurve. Section 3 is devoted to the study of semitrivial solutions and strictly principal eigenvalues of (Q). We thereby find the lowest eigenvalue of problem (Q) which is proved to be unique, positive, semitrivial principal or strictly principal and simple.

2. An eigenvalue curve associated to problem (Q)

We shall adopt the approach used in a number of papers (see for example [5], [7], [8], [16], [15]) by fixing λ and embed the system (Q) into a new system (Q_λ) in order to derive the existence of solutions for (Q) that is:

$$(Q_\lambda): \begin{cases} \Delta^2_{p_1} u_1 - m(x)|u_1|^{\alpha_1-1}u_1 \prod_{i=2}^{n} |u_i|^{\alpha_i+1} - \lambda m_1(x)|u_1|^{p_1-2}u_1 = \mu|u_1|^{p_1-2}u_1, & \text{in } \Omega \\ \dots \\ \Delta^2_{p_i} u_i - m(x)|u_i|^{\alpha_i-1}u_i \prod_{j=1,j\neq i}^{n} |u_j|^{\alpha_j+1} - \lambda m_i(x)|u_i|^{p_i-2}u_i = \mu|u_i|^{p_i-2}u_i, & \text{in } \Omega \\ \dots \\ \Delta^2_{p_n} u_n - m(x)|u_n|^{\alpha_n-1}u_n \prod_{i=1}^{n-1} |u_i|^{\alpha_i+1} - \lambda m_n(x)|u_n|^{p_n-2}u_n = \mu|u_n|^{p_n-2}u_n, & \text{in } \Omega \\ u_i = \Delta u_i = 0, \text{ for } 1 \leq i \leq n, & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where μ is a new real parameter. For convenience, we now give the following definitions:

Definition 2.1. .

1. *We say that $((u_1, \dots, u_n), \mu)$ is a (weak) solution to problem (Q_λ) if $((u_1, \dots, u_n), \mu) \in W \times \mathbb{R}$ and*
$$\int_\Omega |\Delta u_i|^{p_i-2}\Delta u_i \Delta\varphi_i dx - \int_\Omega m \prod_{j=1,j\neq i}^{n} |u_j|^{\alpha_j+1}|u_i|^{\alpha_i-1}u_i\varphi_i dx - \lambda \int_\Omega m_i|u_i|^{p_i-2}u_i\varphi_i dx = \mu \int_\Omega |u_i|^{p_i-2}u_i\varphi_i dx, \text{ for } 1 \leq i \leq n, \tag{2.2}$$
for all $(\varphi_1, \dots, \varphi_n) \in W$.
2. *We say that $((u_1, \dots, u_n), \lambda)$ is a (weak) solution to problem (Q) if $((u_1, \dots, u_n), \lambda) \in W \times \mathbb{R}$ and*
$$\int_\Omega |\Delta u_i|^{p_i-2}\Delta u_i \Delta\varphi_i dx - \int_\Omega m \prod_{j=1,j\neq i}^{n} |u_j|^{\alpha_j+1}|u_i|^{\alpha_i-1}u_i\varphi_i dx = \lambda \int_\Omega m_i|u|^{p_i-2}u_i\varphi_i dx, \tag{2.3}$$
for $1 \leq i \leq n$ and for all $(\varphi_1, \dots, \varphi_n) \in W$.
3. *If $((u_i, \dots, u_n), \lambda) \in W \times \mathbb{R}$ (resp. $((u_1, \dots, u_n), \mu) \in W \times \mathbb{R}$) is a (weak) solution to problem (Q) (resp. (Q_λ)), then $(u_1, \dots, u_n)$ shall be called an eigenfunction of the problem (Q) (resp. (Q_λ)) associated to the eigenvalue λ (resp. $\mu(\lambda)$).*

4. Let us agree to say that an eigenvalue of (Q) or (Q_λ) is strictly principal (resp. semitrivial principal) if it is associated to an eigenfunction $(u_1, \ldots, u_n)$ such that $u_i > 0$ or $u_i < 0$, $\forall i \in \{1, \ldots, n\}$ (resp. there exist $\emptyset \neq J_n \subset \{1, \ldots, n\}$ such that $u_k \equiv 0$, $\forall k \in J_n$ and $u_i > 0$ or $u_i < 0$, $\forall i \in \{1, \cdots, n\} \setminus J_n$).

Based on variational approach, for a fixed real λ, we define the energy functional

$$J_\lambda : W \longrightarrow \mathbb{R}$$

$$(u_1, \ldots, u_n) \longmapsto J_\lambda(u_1, \ldots, u_n) = \sum_{i=1}^{n} \frac{\alpha_i + 1}{p_i} \|\Delta u_i\|_{p_i}^{p_i} - V(u_1, \ldots, u_n) - \lambda M(u_1, \ldots, u_n),$$

where

$$V(u_1, \ldots, u_n) = \int_\Omega m \prod_{i=1}^{n} |u_i|^{\alpha_i + 1} dx, \quad \text{and} \quad M(u_1, \ldots, u_n) = \sum_{i=1}^{n} \frac{\alpha_i + 1}{p_i} M_i(u_i)$$

with

$$M_i(u_i) = \int_\Omega m_i |u_i|^{p_i} dx, \quad \forall (u_1, \ldots, u_n) \in W.$$

Equalities (2.2) are equivalent to

$$\nabla J_\lambda(u_1, \ldots, u_n) = \mu \nabla I(u_1, \ldots, u_n)$$

where

$$I(u_1, \ldots, u_n) = \sum_{i=1}^{n} \frac{\alpha_i + 1}{p_i} \|u_i\|_{p_i}^{p_i} \quad \forall (u_1, \ldots, u_n) \in W.$$

We now state the main result of this section which generalizes the result of Proposition 2.1 in [17] where (p, q)-biharmonic system case is treated.

Theorem 2.2. *The value*

$$\mu_1(\lambda) := \inf\{J_\lambda(u_1, \ldots, u_n) : (u_1, \ldots, u_n) \in \mathcal{M}\} \tag{2.4}$$

where

$$\mathcal{M} = \{(u_1, \ldots, u_n) \in W : I(u_1, \ldots, u_n) = 1\},$$

is the smallest eigenvalue of (Q_λ).

The proof of Theorem 2.2 relies on the following lemma:

Lemma 2.3. *Let $(\omega_1, \ldots, \omega_n) \in [L^\infty(\Omega)]^n$. If $\omega_1, \ldots, \omega_n > 0$ on Ω then there exist $n+1$ positive constants $c_1, \ldots, c_{n+1}$ such that*

$$\sum_{i=1}^{n} \|\Delta u_i\|_{p_i}^{p_i} \leq c_{n+1} J_\lambda(u_1, \ldots, u_n) + \sum_{i=1}^{n} c_i \int_\Omega \omega_i |u_i|^{p_i} dx \tag{2.5}$$

for every $(u_1, \ldots, u_n) \in W$.

Proof. We borrow ideas from [17]. Indeed, we know that $M_i(u_i) \leq \|m_i\|_\infty \|u_i\|_{p_i}^{p_i}$, for $1 \leq i \leq n$. Since $\sum\limits_{i=1}^{n} \frac{\alpha_i+1}{p_i} = 1$, it well known by Young inequality that:

$$V(u_1, \ldots, u_n) \leq \|m\|_\infty \int_\Omega \left(\sum_{i=1}^{n} \frac{\alpha_i + 1}{p_i} |u_i|^{p_i} \right) dx. \tag{2.6}$$

Setting $k_3 = \max\{k_1, k_2\}$ with

$$k_1 = \|m\|_\infty \max\left\{\frac{\alpha_i+1}{p_i},\ \text{for } 1\leq i\leq n\right\} \quad \text{and} \quad k_2 = |\lambda| \max\left\{\frac{\alpha_i+1}{p_i}\|m_i\|_\infty,\ \text{for } 1\leq i\leq n\right\},$$

one deduces that

$$V(u_1,\ldots,u_n) \leq k_1\left(\sum_{i=1}^n \|u_i\|_{p_i}^{p_i}\right) \quad \text{and} \quad |\lambda M(u,v)| \leq k_2\left(\sum_{i=1}^n \|u_i\|_{p_i}^{p_i}\right).$$

On the other hand for $\varepsilon > 0$ there exist $M_{i,\varepsilon} > 0$ for $1 \leq i \leq n$ such that

$$\|u_i\|_{p_i}^{p_i} \leq \varepsilon\|\Delta u_i\|_{p_i}^{p_i} + M_{i,\varepsilon}\int_\Omega \omega_i |u_i|^{p_i}dx.$$

Now, we have

$$\sum_{i=1}^n \left(\frac{\alpha_i+1}{p_i}\|\Delta u_i\|_{p_i}^{p_i}\right) = J_\lambda(u_1,\ldots,u_n) - V(u_1,\ldots,u_n) + \lambda M(u_1,\ldots,u_n).$$

Then,

$$\begin{aligned}\sum_{i=1}^n \left(\frac{\alpha_i+1}{p_i}\|\Delta u_i\|_{p_i}^{p_i}\right) &\leq J_\lambda(u_1,\ldots,u_n) + 2k_3\left(\sum_{i=1}^n \|u_i\|_{p_i}^{p_i}\right)\\ &\leq J_\lambda(u_1,\ldots,u_n) + 2\varepsilon k_3\left(\sum_{i=1}^n \|\Delta u_i\|_{p_i}^{p_i}\right) + 2k_3\left(\sum_{i=1}^n M_{i,\varepsilon}\int_\Omega \omega_i|u_i|^{p_i}dx\right).\end{aligned}$$

Let $\varepsilon > 0$ be such that $k_4 = \min\left\{\frac{\alpha_i+1}{p_i} - 2\varepsilon k_3,\ \text{for } 1\leq i\leq n\right\} > 0$.

Thus, it reads

$$k_4\sum_{i=1}^n \left(\|\Delta u_i\|_{p_i}^{p_i}\right) \leq J_\lambda(u_1,\ldots,u_n) + 2k_3\sum_{i=1}^n\left(M_{i,\varepsilon}\int_\Omega \omega_i|u_i|^{p_i}dx\right).$$

We deduce

$$\sum_{i=1}^n \left(\|\Delta u_i\|_{p_i}^{p_i}\right) \leq \frac{1}{k_4}J_\lambda(u_1,\ldots,u_n) + \sum_{i=1}^n\left(\frac{2k_3M_{i,\varepsilon}}{k_4}\int_\Omega \omega_i|u_i|^{p_i}dx\right)$$

and one can consequently take $c_{n+1} = \dfrac{1}{k_4}$, and $c_i = \dfrac{2k_3M_{i,\varepsilon}}{k_4}$ for $1\leq i\leq n$. This completes the proof. ■

Proof of Theorem 2.2 . By Lemma 2.3, for $\omega_i \equiv 1$ and $1 \leq i \leq n$, one can easily show that

$$\begin{aligned}0 \leq \sum_{i=1}^n \left(\|\Delta u_i\|_p^p\right) &\leq c_{n+1}J_\lambda(u_1,\ldots,u_n) + \sum_{i=1}^n\left(c_i\int_\Omega |u_i|^{p_i}dx\right)\\ &\leq c_{n+1}J_\lambda(u_1,\ldots,u_n) + c_0\sum_{i=1}^n\left(\frac{\alpha_i+1}{p_i}\int_\Omega |u_i|^{p_i}dx\right)\\ &= c_{n+1}J_\lambda(u_1,\ldots,u_n) + c_0, \forall(u_1,\ldots,u_n)\in\mathcal{M}\end{aligned}$$

where $c_0 = \max\{\frac{p_ic_i}{\alpha_i+1},\ \text{for } 1\leq i\leq n\}$, so that J_λ is bounded below on $\mathcal{M}$. Furthermore any sequence $(u_{1,k},\ldots,u_{n,k})$ that minimizes J_λ on $\mathcal{M}$ is bounded in W.

Thus there exists $(u_{1,0}, \dots, u_{n,0}) \in W$ such that, up to a subsequence, $(u_{1,k}, \dots, u_{n,k})$ converges weakly to $(u_{1,0}, \dots, u_{n,0})$ in W and strongly in $\prod_{i=1}^{n} L^{p_i}(\Omega)$. Hence

$$J_\lambda(u_{1,0}, \dots, v_{n,0}) \leq \lim_{k \longrightarrow \infty} J_\lambda(u_{1,k}, \dots, v_{n,k}) = \mu_1(\lambda), \quad (u_{1,0}, \dots, u_{n,0}) \in \mathcal{M}$$

and consequently $J_\lambda(u_{1,0}, \dots, u_{n,0}) = \mu_1(\lambda)$. By the Lagrange multipliers rule, $\mu_1(\lambda)$ is an eigenvalue for (Q_λ) and $(u_{1,0}, \dots, u_{n,0})$ is an associated eigenfunction.

Moreover for any eigenvalue $\mu(\lambda)$ associated to $(u_{\lambda,1}, \dots, u_{\lambda,n}) \in W \setminus \{(0, ..., 0)\}$, one has

$$J_\lambda(u_{\lambda,1}, \dots, u_{\lambda,n}) = \mu(\lambda) I(u_{\lambda,1}, \dots, u_{\lambda,n})$$

with $I(u_{\lambda,1}, \dots, u_{\lambda,n}) > 0$. Consequently

$$\mu_1(\lambda) \leq J_\lambda \left(\frac{u_{\lambda,1}}{I(u_{\lambda,1}, \dots, u_{\lambda,n})^{\frac{1}{p_1}}}, \dots, \frac{u_{\lambda,n}}{I(u_{\lambda,1}, \dots, u_{\lambda,n})^{\frac{1}{p_n}}} \right) = \frac{J_\lambda(u_{\lambda,1}, \dots, u_{\lambda,n})}{I(u_{\lambda,1}, \dots, u_{\lambda,n})} = \mu(\lambda).$$

All in all, we have proved that $\mu_1(\lambda)$ is the smallest eigenvalue of (Q_λ). ∎

Remark 2.4. *We can denote by*

$$\mu_0 := \inf \left\{ \sum_{i=1}^{n} \frac{\alpha_i + 1}{p_i} \|\Delta u_i\|_{p_i}^{p_i} : (u_1 ..., u_n) \in \mathcal{M} \right\} \tag{2.7}$$

for $m = m_i \equiv 0$, $\forall i \in \{1, \cdots, n\}$. Since the space $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)$, for $i \in \{1, \cdots, n\}$ does not contain any constant non trivial function, one has $\mu_0 > 0$.

It is straightforward proving the following:

Proposition 2.5. .

1. *μ_1 is concave and differentiable with $\mu_1^{'}(\lambda) = -M(u_{1,0}, ..., u_{n,0})$ where $(u_{1,0}, ..., u_{n,0})$ is some eigenfunction of (Q_λ) associated to $\mu_1(\lambda)$ for all $\lambda \in \mathbb{R}$.*
2. $\lim_{\lambda \to \infty} \mu_1(\lambda) = -\infty$.
3. *μ_1 is strictly decreasing.*

Proof. The proof is partly adapted from analogous technics in literature.

1. The concavity of μ_1 follows from the concavity of the mapping $\lambda \longmapsto J_\lambda(u_1, \dots, u_n)$, for a fixed $(u_1, \dots, u_n) \in W$. In particular μ_1 is continuous. Now let $\lambda_k \longrightarrow \lambda$ and $(u_{1,k}, \dots, u_{n,k})$, $(u_{\lambda,1}, \dots, u_{\lambda,n})$ be the I-normalized eigenfunctions related to $\mu_1(\lambda_k)$ and $\mu_1(\lambda)$ respectively. We apply Lemma 2.3 with $\omega_i \equiv 1$, for $1 \leq i \leq n$, to get

$$\begin{aligned} \sum_{i=1}^{n} (\|\Delta u_{i,k}\|_{p_i}^{p_i}) &\leq c_{n+1} J_\lambda(u_{1,k}, \dots, u_{n,k}) + \sum_{i=1}^{n} (c_i \int_\Omega |u_{i,k}|^{p_i} dx), \\ &\leq c_{n+1}\mu_1(\lambda_k) + \max \left\{ \frac{p_i c_i}{\alpha_i + 1}, \text{ for } 1 \leq i \leq n \right\}. \end{aligned}$$

In addition,

$$\lim_{k \to \infty} c_{n+1}\mu_1(\lambda_k) + \max \left\{ \frac{p_i c_i}{\alpha_i + 1}, \text{ for } 1 \leq i \leq n \right\} = c_{n+1}\mu_1(\lambda) + \max \left\{ \frac{p_i c_i}{\alpha_i + 1}, \text{ for } 1 \leq i \leq n \right\}.$$

So we conclude that $(u_{1,k},\dots,u_{n,k})_k$ is a bounded sequence in W. Hence there exists $(u_{1,0},\dots,u_{n,0})$ such that, up to a subsequence, $(u_{1,k},\dots,u_{n,k}) \rightharpoonup (u_{1,0},\dots,u_{n,0})$ in W, strongly in $\prod_{i=1}^{n} L^{p_i}(\Omega)$. Then $(u_{1,0},\dots,u_{n,0}) \in \mathcal{M}$ and from

$$J_\lambda(u_{1,0},\dots,u_{n,0}) \le \lim_{k\longrightarrow\infty} J_\lambda(u_{1,k},\dots,u_{n,k}) = \mu_1(\lambda)$$

we infer that $\mu_1(\lambda) = J_\lambda(u_{1,0},\dots,u_{n,0}) = J_\lambda(u_{\lambda,1},\dots,u_{\lambda,n})$ and $(u_{1,0},\dots,u_{n,0})$ is an eigenfunction of (Q_λ) associated to $\mu_1(\lambda)$. Furthermore

$$\begin{cases} \mu_1(\lambda_k) - \mu_1(\lambda) \ge -(\lambda_n - \lambda)M(u_{1,k},\dots,u_{n,k}) \\ \mu_1(\lambda_k) - \mu_1(\lambda) \le -(\lambda_n - \lambda)M(u_{1,0},\dots,u_{n,0}). \end{cases}$$

Hence

$$\begin{cases} -M(u_{1,k},\dots,u_{n,k}) \le \dfrac{\mu_1(\lambda_n)-\mu_1(\lambda)}{\lambda_k - \lambda} \le -M(u_{1,0},\dots,u_{n,0}), \text{ if } \lambda_k > \lambda \\ -M(u_{1,0},\dots,u_{n,0}) \le \dfrac{\mu_1(\lambda_k)-\mu_1(\lambda)}{\lambda_k - \lambda} \le -M(u_{1,k},\dots,u_{n,k}), \text{ if } \lambda_k < \lambda. \end{cases}$$

Passing to the limit we get $\mu_1^{'}(\lambda) = -M(u_{1,0},\dots,u_{n,0})$.

2. We know that m_1 is nonnegative, then there exists a function $u_1 \in X_{p_1}$ such that $M_1(u_1) > 0$ and $I(u_1, 0,\dots,0) = 1$.

   Then, for all $\lambda \in \mathbb{R}_+^*$, $\mu_1(\lambda) \le J_\lambda(u_1, 0,\dots,0)$. We deduce that

$$\lim_{\lambda\longrightarrow\infty} J_\lambda(u_1, 0,\dots,0) = \lim_{\lambda\longrightarrow\infty} E_m(u_1,0,\dots,0) - \lambda M(u_1,0,\dots,0) = -\infty$$

   where

$$E_m(u_1,\dots,u_n) = \sum_{i=1}^{n}\left(\frac{\alpha_i+1}{p_i}\|\Delta u_i\|_{p_i}^{p_i}\right) - \int_\Omega \left(m\prod_{i=1}^{n}|u_i|^{\alpha_i+1}\right)dx.$$

   Thus $\lim_{\lambda\longrightarrow\infty} \mu_1(\lambda) = -\infty$.

3. The result is clear from the fact that $M(u_{\lambda,1},\dots,u_{\lambda,n}) > 0$ for any $\lambda \in \mathbb{R}$. Indeed, if $\lambda_1 < \lambda_2$ then

$$\begin{aligned} \mu_1(\lambda_1) &= E_m(u_{\lambda_1,1},\dots,u_{\lambda_1,n}) - \lambda_1 M(u_{\lambda_1,1},\dots,u_{\lambda_1,n}) \\ &\ge E_m(u_{\lambda_1,1},\dots,u_{\lambda_1,n}) - \lambda_2 M(u_{\lambda_1,1},\dots,u_{\lambda_1,n}) \\ &\ge \mu_1(\lambda_2). \end{aligned}$$

■

3. Existence of solutions for the system (Q)

We address, in this section, the problem (Q) by looking for the zeros of the function $\mu_1(\lambda)$ which by construction solve the problem. Let us make this assumption

$$(H_m) : \|m\|_\infty < \mu_0.$$

We start by proving the following:

Lemma 3.1. *If (H_m) is satisfied, then $\mu_1(0) > 0$ and $\mu_1(\lambda) = 0$ if and only if $\lambda > 0$ is an eigenvalue of (Q).*

Proof. Assume that (H_m) is satisfied. By (2.6), we have $V(u_1, \dots, u_n) \leq \|m\|_\infty I(u_1, \dots, u_n)$, $\forall (u_1, \dots, u_n) \in W$. Then, one has

$$\sum_{i=1}^{n} \left(\frac{\alpha_i + 1}{p_i} \|\Delta u_i\|_{p_i}^{p_i} \right) - \|m\|_\infty I(u_1, \dots, u_n) \leq E_m(u_1, \dots, u_n), \forall (u_1, \dots, u_n) \in W.$$

This implies

$$\mu_0 \leq E_m(u_1, \dots, u_n) + \|m\|_\infty, \quad \forall (u_1, \dots, u_n) \in \mathcal{M},$$
$$\mu_0 - \|m\|_\infty \leq \inf\{E_m(u_1, \dots, u_n), (u_1, \dots, u_n) \in \mathcal{M}\} \leq \mu_1(0).$$

We then conclude that $\mu_1(0) > 0$ and $\mu_1(\lambda) = 0$ if and only if $\lambda > 0$ is an eigenvalue of (Q). ■

From now on, we denote

$$L(\Omega) := \left(\left[\prod_{i=1}^{n} L^{p_i}(\Omega) \right] \setminus \{(0, \dots, 0)\} \right) \times \mathbb{R}, \tag{3.1}$$

$$L_0(\Omega) := \left(\left[\prod_{i=1}^{n} L^{p_i}(\Omega) \right] \setminus \{(0, \dots, 0)\} \right) \times \{0\}. \tag{3.2}$$

We adapt and apply some results proved in [9] and some ideas used in [19] to establish the following.

Remark 3.2. .

1. *$\forall u \in X_p$, $\forall v \in L^p(\Omega)$ (with $p \in (1, \infty)$): $v = -\Delta u \Longleftrightarrow u = \Lambda v$.*
2. *Let N_p (with $p \in (1, \infty)$) be the Nemytskii operator defined by*

$$N_p(u)(x) = \begin{cases} |u(x)|^{p-2} u(x) & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0. \end{cases}$$

   We have

$$\forall v \in L^p(\Omega), \quad \forall w \in L^{p'}(\Omega): \quad N_p(v) = w \Longleftrightarrow v = N_{p'}(w) \tag{3.3}$$

   with $p' = \frac{p}{p-1}$.

3. *If $(u_1, \dots, u_n)$ is an eigenfunction of (Q_λ) associated with μ then $\varphi_i = -\Delta u_i$, for $1 \leq i \leq n$ satisfy:*

$$N_{p_j}(\varphi_j) = \Lambda \left([\mu(\lambda) + \lambda m_j] N_{p_j}(\Lambda \varphi_j) + m \prod_{i=1, i \neq j}^{n} |\Lambda \varphi_i|^{\alpha_i + 1} |\Lambda \varphi_j|^{\alpha_j - 1} \Lambda \varphi_j \right), \text{ for } 1 \leq j \leq n.$$

   Hence:

   (a) *$((u_{1,0}, \dots, u_{n,0}), \mu(\lambda))$ is a solution of (Q_λ) if and only if $((\varphi_{1,0}, \dots, \varphi_{n,0}), \mu(\lambda))$ is a solution of problem*

$$(Q_\lambda') : \begin{cases} \text{Find } ((\varphi_1, \dots, \varphi_n), \mu(\lambda)) \in L(\Omega) \text{ such that} \\ N_{p_j}(\varphi_j) = \Lambda \left([\mu(\lambda) + \lambda m_j] N_{p_j}(\Lambda \varphi_j) + m \prod_{i=1, i \neq j}^{n} |\Lambda(\varphi_i)|^{\alpha_i + 1} |\Lambda(\varphi_j)|^{\alpha_j - 1} \Lambda(\varphi_j) \right), \\ \text{for } 1 \leq j \leq n, \end{cases}$$

   with $\varphi_{j,0} = -\Delta(u_{j,0})$.

(b) $((\varphi_{1,0},\dots,u_{n,0}),\mu_1(\lambda))\in L_0(\Omega)$ *is a solution of* $(Q^{'}_\lambda)$ *if and only if* $((\varphi_{1,0},\dots,\varphi_{n,0}),\lambda)\in L(\Omega)$ *is a solution of problem*

$$(Q^{'}):\begin{cases}\text{Find } ((\varphi_1,\dots,\varphi_n),\lambda)\in L(\Omega) \text{ such that}\\ N_{p_j}(\varphi_j)=\Lambda\left(\lambda m_j N_{p_j}(\Lambda\varphi_j)+m\prod_{i=1,i\neq j}^{n}|\Lambda(\varphi_i)|^{\alpha_i+1}|\Lambda(\varphi_j)|^{\alpha_j-1}\Lambda(\varphi_j)\right),\\ \text{for } 1\le j\le n,\end{cases}$$

with $\varphi_{j,0}=-\Delta(u_{j,0})$.

(c)

$$\mu_1(\lambda):=\inf\left\{F_\lambda(\varphi_1,\dots,\varphi_n):(\varphi_1,\dots,\varphi_n)\in\prod_{i=1,i\neq j}^{n}L^{p_i}(\Omega),R(\varphi_1,\dots,\varphi_n)=1\right\}\tag{3.4}$$

where

$$F_\lambda(\varphi_1,\dots,\varphi_n)=\sum_{i=1}^{n}\left(\frac{\alpha_i+1}{p_i}\left[\int_\Omega|\varphi_i|^{p_i}dx-\lambda\int_\Omega m_i|\Lambda\varphi_i|^{p_i}dx\right]\right)-\int_\Omega m\prod_{i=1}^{n}|\Lambda\varphi_i|^{\alpha_i+1}dx,$$
$$R(\varphi_1,\dots,\varphi_n)=\sum_{i=1}^{n}\frac{\alpha_i+1}{p_i}\|\Lambda\varphi_i\|_{p_i}^{p_i}.$$

Lemma 3.3. *If* $((u_1,\dots,u_n),\mu(\lambda))$ *is a solution of* (Q_λ) *then* $-\Delta u_i\in C(\bar\Omega)$ *and* $u_i\in C^{1,\nu}(\bar\Omega)$, *for* $1\le i\le n$ *and for all* $\nu\in(0,1)$.

Proof. An easy adaptation of Lemma 3.2 in [17]. ■

Lemma 3.4. $((\varphi_{1,1},\dots,\varphi_{1,n}),\mu_1(\lambda))\in L(\Omega)$ *is a solution of problem* $(Q^{'}_\lambda)$, *if and only if*

$$G_\lambda(\varphi_{1,1},\dots,\varphi_{1,n})=0=\min_{(\varphi_1,\dots,\varphi_n)\in L^*(\Omega)}G_\lambda(\varphi_1,\dots,\varphi_n)\tag{3.5}$$

where

$$G_\lambda(\varphi_1,\dots,\varphi_n)=F_\lambda(\varphi_1,\dots,\varphi_n)-\mu_1(\lambda)R(\varphi_1,\dots,\varphi_n) \text{ and } L^*(\Omega)=\left[\prod_{i=1}^{n}L^{p_i}(\Omega)\right]\setminus\{(0,\dots,0)\}.$$

Proof. Assume that $((\varphi_{1,1},\dots,\varphi_{1,n}),\mu_1(\lambda))\in L(\Omega)$ is a solution of problem $(Q^{'}_\lambda)$.

Then $F_\lambda(\varphi_{1,1},\dots,\varphi_{1,n})=\mu_1(\lambda)R(\varphi_{1,1},\dots,\varphi_{1,n})$.

Hence $G_\lambda(\varphi_{1,1},\dots,\varphi_{1,n})=F_\lambda(\varphi_{1,1},\dots,\varphi_{1,n})-\mu_1(\lambda)R(\varphi_{1,1},\dots,\varphi_{1,n})=0$.

Put $\bar\varphi_i=\dfrac{\varphi_i}{[R(\varphi_1,\dots,\varphi_n)]^{\frac{1}{p_i}}}$ for every $(\varphi_1,\dots,\varphi_n)\in L^*(\Omega)$ and $1\le i\le n$. Then $R(\bar\varphi_1,\dots,\bar\varphi_n)=1$. We deduce that

$$\mu_1(\lambda)\le F_\lambda(\bar\varphi_1,\dots,\bar\varphi_n)=\frac{F_\lambda(\varphi_1,\dots,\varphi_n)}{R(\varphi_1,\dots,\varphi_n)}.\tag{3.6}$$

and

$$G_\lambda(\varphi_1,\dots,\varphi_n)=F_\lambda(\varphi_1,\dots,\varphi_n)-\mu_1(\lambda)R(\varphi_1,\dots,\varphi_n)\ge0\tag{3.7}$$

for all $(\varphi_1,\dots,\varphi_n)\in L^*(\Omega)$. We claim that (3.5) holds.

Now suppose that (3.5) holds. We deduce that $\nabla G_\lambda(\varphi_{1,1}, \dots, \varphi_{1,n}) = (0, \dots, 0)$. Then

$$\langle \frac{\partial G_\lambda}{\partial \varphi_i}(\varphi_{1,1}, \dots, \varphi_{1,n}), \Psi_i \rangle = 0, \text{ for } 1 \leq i \leq n, \tag{3.8}$$

for all $(\Psi_1, \dots, \Psi_n) \in \prod_{i=1}^{n} L^{p_i}(\Omega)$. Hence, $((\varphi_{1,1}, \dots, \varphi_{1,n}), \mu_1(\lambda)) \in L(\Omega)$ is a solution of $(Q_\lambda^{'})$. ■

Lemma 3.5. *If (H_m) holds and $((\varphi_{1,1}, \dots, \varphi_{1,n}), \mu_1(\lambda)) \in L_0(\Omega)$ is a solution of problem $(Q_\lambda^{'})$ then $((|\varphi_{1,1}|, \dots, |\varphi_{1,n}|), \mu_1(\lambda)) \in L_0(\Omega)$ is a solution of problem $(Q_\lambda^{'})$.*

Proof. Assume that (H_m) holds and $((\varphi_{1,1}, \dots, \varphi_{1,n}), \mu_1(\lambda)) \in L_0(\Omega)$ is a solution of problem $(Q_\lambda^{'})$. Then $G_\lambda(\varphi_{1,1}, \dots, \varphi_{1,n}) = 0$, $\mu_1(\lambda) = 0$ and $(|\varphi_{1,1}|, \dots, |\varphi_{1,n}|) \in \left[\prod_{i=1}^{n} L^{p_i}(\Omega)\right] \setminus \{(0, \dots, 0)\}$. Hence $G_\lambda(|\varphi_{1,1}|, \dots, |\varphi_{1,n}|) \geq 0$.

Additionally, one has $|\Lambda(|\varphi_i|)|^r \geq |\Lambda\varphi_i|^r$ for $1 \leq i \leq n$ and for all $r \in (1; \infty)$. We deduce that $F_\lambda(|\varphi_{1,1}|, \dots, |\varphi_{1,n}|) \leq F_\lambda(|\varphi_{1,1}|, \dots, |\varphi_{1,n}|)$ and $G_\lambda(|\varphi_{1,1}|, \dots, |\varphi_{1,n}|) \leq G_\lambda(|\varphi_{1,1}|, \dots, |\varphi_{1,n}|) = 0$.

Thus $G_\lambda(|\varphi_{1,1}|, \dots, |\varphi_{1,n}|) = 0$ and $((|\varphi_{1,1}|, \dots, |\varphi_{1,n}|), \mu_1(\lambda))$ is solution of $(Q_\lambda^{'})$. ■

Lemma 3.6. *[17].*

Let A, B, C and r be real numbers satisfying $A \geq 0$, $B \geq 0$, $C \geq \max\{B - A, 0\}$ and $r \in [1, +\infty)$. Then

$$|A + C|^r + |B - C|^r \geq A^r + B^r.$$

Lemma 3.7. *Let a_i and b_i be real numbers and $I_n = \{1, 2, \dots, n\}$, then*

$$\prod_{i=1}^{n}(a_i + b_i) = \sum_{J \subset I_n} \left(\prod_{i \in J} a_i\right)\left(\prod_{i \in I_n \setminus J} b_i\right).$$

Proof. Straightforward by recurrence on n. ■

Lemma 3.8. *Suppose that (H_m) holds.*

If $(\varphi_{1,1}, \dots, \varphi_{1,n})$ and $(\varphi_{2,1}, \dots, \varphi_{2,n})$ are positive eigenfunctions of problem $(Q_\lambda^{'})$ associated with $\mu_1(\lambda) = 0$, then $(w_{k,1}, \dots, w_{l,s}, \dots, w_{j,n})$ with:

$$\begin{cases} w_{1,i}(x) := \max\{\varphi_{1,i}(x), \varphi_{2,i}(x)\} = \varphi_{1,i}(x) + (\varphi_{2,i} - \varphi_{1,i})^+(x), \\ w_{2,i}(x) := \min\{\varphi_{1,i}(x), \varphi_{2,i}(x)\} = \varphi_{2,i}(x) - (\varphi_{2,i} - \varphi_{1,i})^+(x), \end{cases}$$

for all $x \in \Omega$, k, l, $j \in \{1, 2\}$, $s \in \{2, \dots, n-1\}$ and $i \in \{1, \dots, n\}$, are eigenfunctions of $(Q_\lambda^{'})$ associated with $\mu_1(\lambda) = 0$.

Proof. Assume that (H_m) holds and $(\varphi_{1,1}, \dots, \varphi_{1,n})$ and $(\varphi_{2,1}, \dots, \varphi_{2,n})$ are positive eigenfunctions of problem $(Q_\lambda^{'})$ associated with $\mu_1(\lambda) = 0$. By Lemma 3.6 we get

$$\begin{cases} |\Lambda w_{1,i}|^{p_i} + |\Lambda w_{2,i}|^{p_i} \geq |\Lambda\varphi_{1,i}|^{p_i} + |\Lambda\varphi_{2,i}|^{p_i} \\ |\Lambda w_{1,i}|^{\alpha_i+1} + |\Lambda w_{2,i}|^{\alpha_i+1} \geq |\Lambda\varphi_{1,i}|^{\alpha_i+1} + |\Lambda\varphi_{2,i}|^{\alpha_i+1}. \end{cases}.$$

Then, one has:

$$-\lambda \int_\Omega m_i |\Lambda w_{1,i}|^{p_i} dx - \lambda \int_\Omega m_i |\Lambda w_{2,i}|^{p_i} dx \leq -\lambda \int_\Omega m_i |\Lambda\varphi_{1,i}|^{p_i} dx - \lambda \int_\Omega m_i |\Lambda\varphi_{2,i}|^{p_i} dx. \tag{3.9}$$

Likewise, we have

$$Z_1(w_1,\dots,w_i,\dots,w_n) \le Z_1(\varphi_1,\dots,\varphi_i,\dots,\varphi_n) \le -\int_\Omega m\prod_{i=1}^{n}|\Lambda\varphi_{1,i}|^{\alpha_i+1}dx - \int_\Omega m\prod_{i=1}^{n}|\Lambda\varphi_{2,i}|^{\alpha_i+1}dx \tag{3.10}$$

with

$$Z_1(w_1,\dots,w_i,\dots,w_n) = -\sum_{J\subset I_n}\int_\Omega m\left(\prod_{i\in J}|\Lambda w_{1,i}|^{\alpha_i+1}\right)\left(\prod_{i\in I_n\setminus J}|\Lambda w_{2,i}|^{\alpha_i+1}\right)dx$$

and

$$Z_1(\varphi_1,\dots,\varphi_i,\dots,\varphi_n) = -\sum_{J\subset I_n}\int_\Omega m\left(\prod_{i\in J}|\Lambda\varphi_{1,i}|^{\alpha_i+1}\right)\left(\prod_{i\in I_n\setminus J}|\Lambda\varphi_{2,i}|^{\alpha_i+1}\right)dx.$$

Additionally, we have:

$$\int_\Omega |w_{1,i}|^{p_i}dx + \int_\Omega |w_{2,i}|^{p_i}dx = \int_\Omega |\varphi_{1,i}|^{p_i}dx + \int_\Omega |\varphi_{2,i}|^{p_i}dx. \tag{3.11}$$

By (3.9), (3.10) and (3.11) we deduce that:

$$\sum_{i\in J\subset I_n\setminus\{1,n\},\,\mathrm{k,\,l,\,j}\in\{1,2\}} F_\lambda(w_{k,1},\dots,w_{l,i},\dots,w_{j,n}) \le F_\lambda(\varphi_{1,1},\dots,\varphi_{1,n}) + F_\lambda(\varphi_{2,1},\dots,\varphi_{2,n})$$

and

$$\sum_{i\in J\subset I_n\setminus\{1,n\},\,\mathrm{k,\,l,\,j}\in\{1,2\}} G_\lambda(w_{k,1},\dots,w_{l,i},\dots,w_{j,n}) \le G_\lambda(\varphi_{1,1},\dots,\varphi_{1,n}) + G_\lambda(\varphi_{2,1},\dots,\varphi_{2,n}) = 0.$$

It follows that

$$G_\lambda(w_{k,1},\dots,w_{l,i},\dots,w_{j,n}) = 0, \text{ with } i\in J\subset I_n\setminus\{1,n\} \text{ and } k,l,j\in\{1,2\}.$$

Hence $(w_{k,1},\dots,w_{l,s},w_{j,n})$ with $s\in\{2,\dots,n-1\}$ and $k,l,j\in\{1,2\}$, are eigenfunctions of (Q'_λ) associated with $\mu_1(\lambda)=0$. ■

We are now in position to summarize the main existence result of this section in the following, which generalizes and extends result of Theorem 3.1 in [17].

Theorem 3.9. *Assume that (H_m) is satisfied. We have the following results:*

1. *If $\mu_1(\lambda)=0$ then λ is a semitrivial principal eigenvalue or strictly principal eigenvalue of problem (Q) and simple.*
2. *The lowest eigenvalue of problem (Q) is the value*

$$\lambda_1 := \min_{(u_1,\dots,u_n)\in\mathcal{S}} E_m(u_1,\dots,u_n). \tag{3.12}$$

where

$$\mathcal{S} = \{(u_1,\dots,u_n)\in W:\quad M(u_1,\dots,u_n)=1\}.$$

Moreover, λ_1 is unique, positive, strictly principal eigenvalue or strictly principal eigenvalue and simple.

Proof. Assume that (H_m) is satisfied.

1. If $\mu_1(\lambda) = 0$ then λ is an eigenvalue of the problem (Q) associated with $(u_1, \ldots, u_n) \in W \setminus \{(0, \ldots, 0)\}$.

   • If $u_i \not\equiv 0$, for $1 \le i \le n$, we deduce that

$$((\varphi_1, \ldots, \varphi_n), \mu_1(\lambda)) \in L_0(\Omega) \quad \text{and} \quad ((|\varphi_1|, \ldots, |\varphi_n|), \mu_1(\lambda)) \in L_0(\Omega)$$

are solution of problem (Q'_λ) with $\varphi_i = -\Delta u_i \not\equiv 0$, for $1 \le i \le n$. Since $|\varphi_i| \ge 0$, then $\Lambda(|\varphi_i|) > 0$, for $1 \le i \le n$. Therefore $N_{p_i}(\Lambda|\varphi_i|) > 0$ for $1 \le i \le n$,

$$\prod_{j=1, i\neq j}^{n} \left(\Lambda(|\varphi_j|)|^{\alpha_i+1}|\Lambda(|\varphi_i|)|^{\alpha_i}\right) > 0$$

and

$$\begin{cases} |\varphi_i| = N_{p'_i}\left(\Lambda\left[\lambda m_i N_{p_i}(\Lambda|\varphi_i|) + m \displaystyle\prod_{j=1, i\neq j}^{n} \left(\Lambda(|\varphi_j|)|^{\alpha_i+1}|\Lambda(|\varphi_i|)|^{\alpha_i}\right)\right]\right) > 0 \\ \text{for } 1 \le i \le n. \end{cases}$$

We then conclude that $((\varphi_1, \ldots, \varphi_n), \mu_1(\lambda))$ is solution of problem (Q'_λ) with $\varphi_i > 0$ or $\varphi_i < 0$, for $1 \le i \le n$. Since by Lemma 3.3, $\varphi_i \in C(\bar{\Omega})$, we have $u_i = \Lambda\varphi_i > 0$ or $u_i = \Lambda\varphi_i < 0$, for $1 \le i \le n$, from Lemma 1.1. Then λ is a strictly principal eigenvalue of (Q).

   • If $\exists i, j \in \{1, \ldots, n\}$ such that $[u_i \equiv 0$ and $u_j \not\equiv 0]$, then we also prove that $[u_i \equiv 0$ and $u_j > 0$ in Ω or $u_j < 0$ in $\Omega]$. Then λ is a semitrivial principal eigenvalue of (Q).

   It is now left with the simplicity and we argue by cases:

Case (1) λ is a strictly principal eigenvalue of (Q).

Let $(u_{1,1}, \ldots, u_{1,n})$ and $(u_{2,1}, \ldots, u_{2,n})$ be two eigenfunctions of (Q) associated with λ.

Then, $((\varphi_{1,1}, \ldots, \varphi_{1,n}), 0)$, $((\varphi_{2,1}, \ldots, \varphi_{2,n}), 0)$, $((|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|), 0)$, $((|\varphi_{2,1}|, \ldots, |\varphi_{2,n}|), 0) \in L_0(\Omega)$, are solutions of (Q'_λ) where $\varphi_{j,i} = -\Delta u_{j,i}$ with $\varphi_{j,i} > 0$ or $\varphi_{j,i} < 0$, for $j \in \{1, 2\}$ and $i \in \{1, \ldots, n\}$.

For $x_0 \in \Omega$, we set $k_i = \dfrac{\varphi_{2,i}(x_0)}{\varphi_{1,i}(x_0)}$, $w_{1,i}(x) = \max\{\varphi_{2,i}(x), k_i\varphi_{1,i}(x)\}$ for all $x \in \Omega$. From Lemma 3.8, $((w_{1,1}, \ldots, w_{1,n}), 0)$ is a solution of problem (Q'_λ) . We deduce that $N_{p_i}(\varphi_{1,i})$, $N_{p_i}(\varphi_{2,i})$, $N_{p_i}(w_{1,i}) \in C^{1,\nu}(\bar{\Omega})$ and $\dfrac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})} \in C^1(\Omega)$.

For any unit vector $e = (0, \ldots, e_j, \ldots, 0)$ with $j \in \{1, \ldots, N\}$ and $t \in \mathbb{R}$, we have

$$\begin{cases} N_{p_i}(\varphi_{2,i})(x_0 + te) - N_{p_i}(\varphi_{2,i})(x_0) \le N_{p_i}(w_{1,i})(x_0 + te) - N_{p_i}(w_{1,i})(x_0) \\ N_{p_i}(k\varphi_{1,i})(x_0 + te) - N_{p_i}(k\varphi_{1,i})(x_0) \le N_{p_i}(w_{1,i})(x_0 + te) - N_{p_i}(w_{1,i})(x_0) \end{cases}$$

Dividing these inequalities by $t > 0$ and $t < 0$ and letting t tend to $0^{\pm}$, we get

$$\begin{cases} \dfrac{\partial}{\partial x_j}[N_{p_i}(\varphi_{2,i})](x_0) \le \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \\ \dfrac{\partial}{\partial x_j}[N_{p_i}(k\varphi_{1,i})](x_0) \le \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \end{cases}$$

and

$$\begin{cases} \dfrac{\partial}{\partial x_j}[N_{p_i}(\varphi_{2,i})](x_0) \geq \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \\ \dfrac{\partial}{\partial x_j}[N_{p_i}(k\varphi_{1,i})](x_0) \geq \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \end{cases}$$

for all $j \in \{1, \ldots, N\}$. Thus,

$$\begin{cases} \dfrac{\partial}{\partial x_j}[N_{p_i}(\varphi_{2,i})](x_0) = \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \\ \dfrac{\partial}{\partial x_j}[N_{p_i}(k\varphi_{1,i})](x_0) = \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \end{cases}$$

for all $j \in \{1, \ldots, N\}$. Hence,

$$\nabla N_{p_i}(\varphi_{2,i})(x_0) = \nabla N_{p_i}(w_{1,i})(x_0) = \nabla N_{p_i}(k\varphi_{1,i})(x_0) = k^{p_i-1}\nabla N_{p_i}(\varphi_{1,i})(x_0).$$

Furthermore

$$\begin{aligned} \nabla\left(\frac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}\right)(x_0) &= \frac{\nabla(N_{p_i}(\varphi_{2,i}))(x_0)N_{p_i}(\varphi_{1,2})(x_0) - N_{p_i}(\varphi_{2,i})(x_0)\nabla(N_{p_i}(\varphi_{2,i}))(x_0)}{[N_{p_i}(\varphi_{1,i})(x_0)]^2} \\ &= \frac{[N_{p_i}(\varphi_{1,2})(x_0) - k_i^{1-p_i}N_{p_i}(\varphi_{2,i})(x_0)]\nabla(N_{p_i}(\varphi_{2,1}))(x_0)}{[N_{p_i}(\varphi_{1,i})(x_0)]^2} = 0. \end{aligned}$$

Then,

$$\frac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}(x) = const = \frac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}(x_0) = \left(\frac{\varphi_{2,i}(x_0)}{\varphi_{1,i}(x_0)}\right)^{p_i-1} = k_i^{p_i-1},$$

for all $x \in \Omega$. Consequently $\varphi_{2,i} = k_i\varphi_{1,i}$.

Accordingly, $(\varphi_{2,1}, \ldots, \varphi_{2,n}) = (k_1\varphi_{1,1}, \ldots, k_n\varphi_{1,n})$.

We deduce that $(u_{2,1}, \ldots, u_{2,n}) = (k_1u_{1,1}, \ldots, k_nu_{1,n})$ and the result follows.

Case (2) λ is a semitrivial principal eigenvalue of (Q).

Let $(\cdots, u_{1i}, \cdots)$ and $(\cdots, u_{2i}, \cdots)$ be two eigenfunctions of (Q) associated with λ (with $u_{1i} \not\equiv 0$, $u_{2i} \not\equiv 0$ and $i \in \{1, \cdots, n\}$). It is easy to see that there exist $k_i \neq 0$ real numbers such that $u_{1i} = k_iu_{2i}$.

2. By Lemma 3.1, $\mu_1(0) > 0$ and $\mu_1(\lambda) = 0$ if and only if $\lambda > 0$ is an eigenvalue of (Q).

   From Proposition 2.5, there exists a unique real $\lambda_1 \in (0, \infty)$ satisfying $\mu_1(\lambda_1) = 0$ and $\mu_1'(\lambda_1) = -M(u_{1,0}, \ldots, u_{n,0}) < 0$. On the other hand, $0 = \mu_1(\lambda_1) = E_m(u_{1,0}, \ldots, u_{n,0}) - \lambda_1M(u_{1,0}, \ldots, u_{n,0})$ with $(u_{1,0}, \ldots, u_{n,0}) \in \mathcal{M}$. Then,

$$E_m(u_{1,0}, \ldots, u_{n,0}) = \lambda_1M(u_{1,0}, \ldots, u_{n,0}) > 0$$

   and we can set

$$\overline{u}_{i,0} = \frac{u_{i,0}}{[M(u_{1,0}, \ldots, u_{n,0})]^{\frac{1}{p_i}}}.$$

   Thus, $(\overline{u}_{1,0}, \ldots, \overline{u}_{n,0}) \in \mathcal{S}$ and $E_m(\overline{u}_{1,0}, \ldots, \overline{u}_{n,0}) = \lambda_1$. Additionally, for every $(u_1, \ldots, u_n) \in \mathcal{S}$, one has

$$E_m\left(\frac{u_1}{[I(u_1, \ldots, u_n)]^{\frac{1}{p_1}}}, \ldots, \frac{u_n}{[I(u_1, \ldots, u_n)]^{\frac{1}{p_n}}}\right) \geq \lambda_1M\left(\frac{u_1}{[I(u_1, \ldots, u_n)]^{\frac{1}{p_1}}}, \ldots, \frac{u_n}{[I(u_1, \ldots, u_n)]^{\frac{1}{p_n}}}\right)$$

   i.e. $E_m(u_1, \ldots, u_n) \geq \lambda_1$. Consequently (3.12) holds. Moreover, from what has been previously proved, λ_1 is a strictly principal eigenvalue or strictly principal eigenvalue and simple.

■

References

[1] R. A. ADAMS, Sobolev Spaces, *Academic Press, New York*, (1975).

[2] G. BARLETTA AND R. LIVREA Infinitely many solutions for a class of differential inclusions involving the p-biharmonic, *Differ. Integral Equ.*, **26** (2013), 1157–1167.

[3] J. BENEDIKT, Uniqueness theorem for p-biharmonic equations, *Electron. J. Differential Equations*, **53** (2002), 1-17.

[4] J. BENEDIKT, On the discretness of the spectra of the Dirichlet and Neumann p-biharmonic problem, *Abstr. Appl. Anal.*, **293** (2004), 777-792.

[5] P. A. BINDING AND Y. X. HUANG, The eigencurve for the p-laplacian, *Differential Integral Equations*, **8(2)**(1995), 405-414.

[6] G. BONANNO AND B. DI BELLA, A boundary value problem for fourth-order elastic beam equations, *J. Math. Anal. Appl.*, **343** (2008), 1166–1176.

[7] M. CUESTA AND L. LEADI, Weighted eigenvalue problems for quasilinear elliptic operator with mixed Robin-Dirichlet boundary conditions, *J. Math. Anal. Appl.*, **422** (2015), 1-26.

[8] M. CUESTA AND HUMBERTO RAMOS QUOIRIN, A weighted eigenvalue problem for the p-laplacian plus a potential, *NoDEA Nonlinear Equations Appl.*, **16(4)** (2009), 469-491.

[9] P. DRÁBEK AND M. ÔTANI, Global bifurcation result for the p-biharmonic operator, *Electron. J. Differential Equations*, **48** (2001), 1-19.

[10] A. EL KHALIL, S. KELLATI AND A. TOUZANI, On the spectrum of the p-biharmonic operator In: 2002-Fez Conference on Partial Differential Equations, *Electron. J. Differential Equations*, **09** (2002), 161-170.

[11] D. GILBARG AND N. S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, *2nd ed., Springer, New York*, (1983).

[12] J. R. GRAEF AND S. HEIDARKHANI, Multiple solutions for a class of $(p_1, \ldots, p_n)$-biharmonic systems, *Communications on Pure and Applied Analysis*, **12(3)**(2013), doi:10.3934/cpaa.2013.12.1393.

[13] EL MILOUD HSSINI, MOHAMMED MASSAR AND NAJIB TSOULI, Infinitely many solutions for the Navier boundary (p,q)-biharmonic systems, *Electron. J. Differential Equations*, **2012(163)** (2012), 1-9.

[14] A. C. LAZER AND P. J. MCKENNA, Large amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis, *SIAM Rev.*, **32** (1990), 537–578.

[15] LIAMIDI LEADI AND HUMBERTO RAMOS QUOIRIN, Principal eigenvalue for quasilinear cooperative elliptic systems, *Differential and integral equations*, **24(11-12)**(2011), 1107–1124.

[16] L. LEADI AND A. MARCOS, A weighted eigencurve for Steklov problems with a potential, *NoDEA Nonlinear Differential Equations Appl.*, **16** (2013), 687-713.

[17] L. LEADI AND R. L. TOYOU, Principal Eigenvalue for Cooperative (p, q)-biharmonic Systems, *J. Partial Diff. Eq.*, **32** (2019), 33-51.

[18] LIN LI AND YU FU, Existence of Three Solutions for $(p_1, \ldots, p_n)$-biharmonic Systems, *International Journal of Nonlinear Sciences*, **10(4)**(2010), 495-506.

[19] M. TALBI AND N. TSOULI, On the spectrum of the weighted p-biharmonic operator with weight, *Mediterr. J. Math.*, **4** (2007), 73-86.

[20] WILLEM, Minimax Theorems, *Birkhäuser, Boston*, (1996).

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 79–89.
http://doi.org/10.26637/mjm1001/007

Fixed points of generalized (φ, ψ))-Jaggi contractions in orbitally complete partially ordered metric spaces

G. V. R. BABU*[1], K. K. M. SARMA[2] AND V. A. KUMARI[3]

[1,2] *Department of Mathematics, Andhra University, Visakhapatnam-530003, India.*
[3] *Department of Mathematics, D.R.N.S.C.V.S College, Chilakaluripet-522616, India.*

*R*eceived 22 September 2021; *A*ccepted 27 December 2021

Abstract. In this paper, we introduce generalized (φ, ψ))-Jaggi contraction mappings and prove the existence of fixed points for such mappings in orbitally complete partially ordered metric spaces. We provide examples in support of our results. Our results generalize the results of Harjani, Lopez and Sadarangani [3].

AMS Subject Classifications: 47H10, 54H25.

Keywords: Jaggi contraction, orbitally complete, orbitally continuous, generalized (φ, ψ))-Jaggi contraction.

Contents

1. Introduction and Background

A number of generalizations of the Banach contraction theorem were obtained in various directions by different authors. Generalization of contraction conditions and proving the existence of fixed points is an interesting aspect. In 1977, Jaggi [4] introduced a new concept namely 'rational type contraction mappings' and proved the existence of fixed points of such mappings.

Theorem 1.1. *[4] Let f be a continuous selfmap defined on a complete metric space (X, d). Suppose that f satisfies the following condition: there exist $\alpha,\ \gamma \in [0, 1)$ with $\alpha + \gamma < 1$ such that*

$$d(fx, fy) \leq \alpha \frac{d(x, fx)d(y, fy)}{d(x, y)} + \gamma d(x, y) \ for\ all\ x, y \in X,\ x \neq y. \tag{1.1}$$

Then f has a fixed point in X.

Here we note that a mapping $f : X \to X, X$ a metric space that satisfies (1.1) is called a Jaggi contraction map on X.

Harjani, Lopez and Sadarangani [3] extended Theorem 1.1 to the context of partially ordered complete metric spaces.

*[**Corresponding author.**] Email addresses: **gvr_babu@hotmail.com** (G. V. R. Babu), **sarmakmkandala@yahoo.in**(K. K. M. Sarma), **chinnoduv@rediffmail.com** (V. A. Kumari)

Theorem 1.2. *[3] Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a non-decreasing mapping such that*

$$d(fx, fy) \leq \alpha \frac{d(x, fx)d(y, fy)}{d(x, y)} + \gamma d(x, y) \tag{1.2}$$

for all $x, y \in X$ with $x \succeq y$, $x \neq y$ where $0 \leq \alpha, \gamma < 1$ with $\alpha + \gamma < 1$.
Also, assume either

(*i*) *f is continuous; (or)*

(*ii*) *if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \to x$ as $n \to \infty$ then $x = sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

In 2013, Samet, Vetro and Vetro[7] introduced a new type of contraction condition and proved fixed point theorems in complete metric spaces that generalize Banach contraction principle and Kannan fixed point results. For more works on the existence of fixed points in complete metric spaces, we refer [7].

Recently, Babu, Sailaja and Kidane[2] proved some new fixed point theorems in orbitally complete partially ordered metric spaces that generalize the fixed point theorems of Samet, Vetro and Vetro [7] and Ran and Reurings[6]. We denote
$\Psi_1 = \{\psi : [0, \infty) \to [0, \infty) / \psi$ is non-decreasing, continuous and $\psi(t) = 0 \Leftrightarrow t = 0\}$.
An element ψ in Ψ_1 is called an 'altering distance function', [5].

Theorem 1.3. (Babu, Sailaja and Kidane [2]) *Let $(X, \preceq)$ be a partially ordered set and d a metric on X. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$. Suppose that there exist a lower semi continuous function $\varphi : X \to [0, \infty)$ and $\psi \in \Psi$ such that the following condition holds.*
"For each $0 \leq a < b < \infty$, there exists $\gamma(a, b) \in [0, 1)$ such that
$a \leq \psi(d(x, y)) + \varphi(x) + \varphi(y) \leq b$ implies
$\psi(d(fx, fy)) + \varphi(fx) + \varphi(fy) \leq \gamma(a, b)M(x, y)$, where
$M(x, y) = \max\{\psi(d(x, y)) + \varphi(x) + \varphi(y), \psi(d(x, fx)) + \varphi(x) + \varphi(fx),$
$\psi(d(y, fy)) + \varphi(y) + \varphi(fy)\}$
for each $x, y \in \overline{O(x_0)}$ with $x \preceq y$."
Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n \to \infty} x_n = z,\ z \in X$.
Suppose that either

(*i*) *f is orbitally continuous at z; (or)*

(*ii*) *if $\{x_n\}$ is a non-decreasing sequence converging to $x \in X$, then $x_n \preceq x$, for all n.*

Then, z is a fixed point of f and $\varphi(z) = 0$.

Definition 1.4. *Let $(X, \preceq)$ be a partially ordered set. A map $f : X \to X$ is said to be non-decreasing if, for any $x, y \in X$ with $x \preceq y$ then $fx \preceq fy$.*

Definition 1.5. *Let X be a nonempty set and f be a selfmap of X. Let $x \in X$, we define the orbit of x w. r. t. f by $O(x) = \{f^n x / n = 0, 1, 2, ...\}$. Here $f^0 = I$, I is the identity map of X.*

Definition 1.6. *Let (X, d) be a metric space. Let $f : X \to X$ be a selfmap of X. A metric space X is said to be f-orbitally complete if every Cauchy sequence which is contained in $O(x)$ for all $x \in X$ converges to a point of X.*

Note: Every complete metric space is f-orbitally complete for any f; but every f-orbitally complete metric space need not be a complete metric space [9].

Definition 1.7. *A selfmap f of X is said to be $orbitally\ continuous$ at a point $z \in X$ with respect to $x\ inX$, if for any sequence $\{x_n\} \subset O(x)$ with $x_n \to z$ as $n \to \infty$ implies $fx_n \to fz$ as $n \to \infty$.*

Clearly, any continuous mapping of a metric space is orbitally continuous, but its converse need not be true [9].

We use the following lemma in our main result.

Lemma 1.8. *[1] Suppose that (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0\ as\ n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon,\ d(x_{m(k)-1}, x_{n(k)}) < \epsilon$ and*
(i) $\lim\limits_{k\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \epsilon$, (ii) $\lim\limits_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \epsilon$,
(iii) $\lim\limits_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \epsilon$, (iv) $\lim\limits_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \epsilon$,
(v) $\lim\limits_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon$, and (vi) $\lim\limits_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon$.

Motivated by Theorem 1.3, we define generalized (φ, ψ)-Jaggi contraction maps which contain rational expressions, in orbitally partially ordered metric spaces and prove the existence of fixed points.

In the following, Ψ_2 denotes the family of non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that ψ is continuous on $[0, \infty)$ and $\Sigma_{n=1}^{\infty}\psi^n(t) < +\infty$ for each t > 0, where ψ^n is the n^{th} iterate of ψ.

Remark 1.9. *Any function $\psi \in \Psi_2$ satisfies $\lim\limits_{n\to\infty} \psi^n(t) = 0$ and $\psi(t) < t$ for any $t > 0$.*

In the following, we observe that the classes of maps Ψ_1 and Ψ_2 are different.

Example 1.10. *We define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \lambda t$, where $\lambda \geq 1$.*
Then $\psi \in \Psi_1$ but $\psi \notin \Psi_2$.

Example 1.11. *We define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \begin{cases} 0 & if\ t \leq 1 \\ \frac{t-1}{2} & if\ t > 1. \end{cases}$*
Then $\psi \in \Psi_2$ but $\psi \notin \Psi_1$.

We now introduce generalized (φ, ψ)-Jaggi contraction in partially ordered metric spaces.

Definition 1.12. *Let $(X, \preceq)$ be a partially ordered metric space and suppose that $f : X \to X$ be a mapping. If there exist two functions $\varphi : X \to [0, \infty)$ lower semi continuous, $\psi \in \Psi_2$ and a point $x_0 \in X$ such that*

$$d(fx, fy) + \varphi(fx) + \varphi(fy) \leq \psi(M(x, y)), \tag{1.3}$$

where $M(x, y) = \max\{d(x, y) + \varphi(x) + \varphi(y), \frac{(d(x,fx)+\varphi(x)+\varphi(fx))(d(y,fy)+\varphi(y)+\varphi(fy))}{d(x,y)+\varphi(x)+\varphi(y)}\}$
for all $x, y \in \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$,
then we say that f is a $generalized\ (\varphi, \psi) - Jaggi\ contraction$.

Remark 1.13. *If $\varphi = 0$ in the inequality (1.3), then we say that f is a generalized ψ-Jaggi contraction.*

Note: In the context of partially ordered metric spaces, if f satisfies (1.2) with $\alpha + \gamma < 1$ then f is a generalized (φ, ψ)-Jaggi contraction with $\varphi = 0$ and $\psi(t) = (\alpha + \gamma)t$, $t \geq 0$ so that every Jaggi contraction is a generalized (φ, ψ)-Jaggi contraction. But, the following example suggests that its converse need not be true.

Example 1.14. *Let $X = [0,1)$ with the usual metric. We define partial order $\preceq$ on X as follows:*
$\preceq := \{(x,y) \in X \times X : x = y\} \cup \{(x,y) \in X \times X / x \preceq y \Leftrightarrow x \leq y$, *where $\leq$ is the usual order*$\}$.
We define $f : X \to X$ by $fx = \begin{cases} 0 & if\ x = 0 \\ \frac{x+1}{2} & if\ x \in (0, \frac{2}{5}) \\ \frac{3}{4} & if\ x \in [\frac{2}{5}, 1). \end{cases}$
We define $\varphi : X \to [0,\infty)$ by $\varphi(x) = \begin{cases} \frac{x}{2} & if\ x \in [0, \frac{3}{4}) \\ x - \frac{3}{4} & if\ x \in [\frac{3}{4}, 1) \end{cases}$ *and*
$\psi : [0,\infty) \to [0,\infty)$ *by* $\psi(t) = \frac{4}{5}t$ *for all* $t \geq 0$.
Let $x_0 = \frac{1}{8},\ fx_0 = \frac{9}{16}$ *then* $x_0 \preceq fx_0$. *Here* $O(x_0) = \{\frac{1}{8}, \frac{9}{16}, \frac{3}{4}, \frac{3}{4}, ...\}$ *and*
$\overline{O(x_0)} = \{\frac{1}{8}, \frac{9}{16}, \frac{3}{4}\} = O(x_0)$. *Let* $x, y \in O(x_0)$.
The following three cases arise to verify the inequality (1.3).
Case (i): $x = \frac{1}{8}$ *and* $y = \frac{9}{16}$.
In this case, $d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{15}{32}$ *and* $M(x,y) = \frac{25}{32}$.
$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{15}{32} \leq \psi(\frac{25}{32}) = \psi(M(x,y))$.
Case (ii): $x = \frac{9}{16}$ *and* $y = \frac{3}{4}$.
In this case, the inequality (1.3) *holds trivially.*
Case (iii): $x = \frac{1}{8}$ *and* $y = \frac{3}{4}$.
In this case, $d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{15}{32}$ *and* $M(x,y) = \frac{11}{16}$.
$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{15}{32} \leq \psi(\frac{11}{16}) = \psi(M(x,y))$.
Hence f is a generalized (φ,ψ)-Jaggi contraction.
Also we observe that the inequality (1.2) fails to hold.
For, by choosing $x = 0$ and $y = \frac{3}{4}$ we have
$d(f0, f(\frac{3}{4})) = \frac{3}{4} \nleq \alpha(0) + \gamma(\frac{3}{4}) < \frac{3}{4} = \alpha\frac{d(0,f0)d(\frac{3}{4},f\frac{3}{4})}{d(0,\frac{3}{4})} + \gamma d(0,\frac{3}{4})$.
$i.e.$, *f is not a $Jaggi\ contraction\ map$.*

Thus we conclude that the class of generalized (φ,ψ)-Jaggi contractions is more general than the class of Jaggi contraction maps.

In Section 2, we prove the existence of fixed points of generalized (φ,ψ)-Jaggi contraction mappings in orbitally complete partially ordered metric spaces. In Section 3, we deduce some corollaries to the main results and provide examples in support of our results.

2. Main Results

Theorem 2.1. *Let $(X,d,\preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$. Suppose that f is a generalized (φ,ψ)-Jaggi contraction and X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0,1,2,...$, is Cauchy in X. Let $\lim\limits_{n\to\infty} x_n = z,\ z \in X$. Assume that f is orbitally continuous at z. Then z is a fixed point of f and $\varphi(z) = 0$.*

Proof. Let $x_0 \in X$ be such that $x_0 \preceq fx_0$. We write $x_1 \in X$ so that $x_1 = fx_0$ then $x_0 \preceq x_1$. Since f is non-decreasing $x_1 = fx_0 \preceq fx_1$. Now, we write $x_2 \in X$ so that $x_2 = fx_1$ then $x_1 \preceq x_2$. On continuing this process, we get a sequence $\{x_n\} \subseteq \overline{O(x_0)}$ such that

$$x_{n+1} = fx_n\ for\ n = 0,1,2,... \tag{2.1}$$

satisfying $x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ...$.
If $x_n = x_{n+1}$ for some n, then the conclusion of the theorem trivially holds. Hence, without loss of generality, we assume that $x_n \neq x_{n+1}$ for all n. We denote

$$r_n = d(x_{n-1},x_n) + \varphi(x_{n-1}) + \varphi(x_n)\ for n = 1,2,... . \tag{2.2}$$

We consider $r_{n+1} = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) = d(fx_{n-1}, fx_n) + \varphi(fx_{n-1}) + \varphi(fx_n)$

$$\leq \psi(M(x,y)), \tag{2.3}$$

where
$M(x,y) = \max\{d(x_{n-1}, x_n)+\varphi(x_{n-1})+\varphi(x_n), \frac{(d(x_{n-1},fx_{n-1})+\varphi(x_{n-1})+\varphi(fx_{n-1}))(d(x_n,fx_n)+\varphi(x_n)+\varphi(fx_n))}{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)}\}$
$= \max\{r_n, \frac{r_n.r_{n+1}}{r_n}\} = \max\{r_n, r_{n+1}\}$.
If $\max\{r_n, r_{n+1}\} = r_{n+1}$ then from (2.3) we have
$r_{n+1} \leq \psi(r_{n+1}) < r_{n+1}$,
a contradiction.
Hence $\max\{r_n, r_{n+1}\} = r_n$ then from (2.3) we have

$$r_{n+1} \leq \psi(r_n) < r_n. \tag{2.4}$$

Thus it follows that $\{r_n\}$ is strictly decreasing sequence of non-negative real numbers and hence $\lim_{n\to\infty} r_n$ exists and it is r(say). $i.e.,$ $\lim_{n\to\infty} r_n = r \geq 0$.
We now show that $r = 0$.
Suppose that $r > 0$. Then from (2.4), we have
$r_{n+1} \leq \psi(r_n)$.
On letting $n \to \infty$, we have
$r \leq \lim_{n\to\infty} \psi(r_n) = \psi(\lim_{n\to\infty} r_n) = \psi(r) < r$,
a contradiction.
Hence $\lim_{n\to\infty} d(x_{n+1}, x_n) + \varphi(x_{n+1}) + \varphi(x_n) = 0$, which implies
$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ and $\lim_{n\to\infty} \varphi(x_n) = 0$.
We now show that $\{x_n\}$ is a Cauchy sequence in X.
Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exist $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \tag{2.5}$$

We choose $m(k)$, the least positive integer satisfying (2.5). Then, we have
$m(k) > n(k) > k$ with
$d(x_{m(k)}, x_{n(k)}) \geq \epsilon, \qquad d(x_{m(k)-1}, x_{n(k)}) < \epsilon.$
Now by Lemma 1.8, it follows that $\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon$.
Now from (1.3), we have
$d(x_{m(k)+1}, x_{n(k)+1}) + \varphi(x_{m(k)+1}) + \varphi(x_{n(k)+1}) = d(fx_{m(k)}, fx_{n(k)}) + \varphi(fx_{m(k)}) + \varphi(fx_{n(k)})$

$$\leq \psi(M(x,y)), \tag{2.6}$$

where $M(x,y) = \max\{d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}),$
$\frac{(d(x_{m(k)},x_{m(k)+1})+\varphi(x_{m(k)})+\varphi(x_{m(k)+1}))(d(x_{n(k)},x_{n(k)+1})+\varphi(x_{n(k)})+\varphi(x_{n(k)+1}))}{d(x_{m(k)},x_{n(k)})+\varphi(x_{m(k)})+\varphi(x_{n(k)})}\}$.
Now, on letting $k \to \infty$ in (2.6) we have
$\epsilon \leq \psi(\epsilon) < \epsilon$,
a contradiction.
Therefore $\{x_n\} \subset O(x_0)$ is a Cauchy sequence in (X, d). Since X is f-orbitally complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{2.7}$$

Since φ is lower semi continuous, we have
$\varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = 0.$
Hence $\varphi(z) = 0$.
Since f is orbitally continuous at z w.r.t. x_0, from (2.1), we have
$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fz$.
This completes the proof of the theorem. ■

Theorem 2.2. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0,\ \varphi : X \to \mathbb{R}^+$ lower semi continuous and $\psi \in \Psi_2$ such that*

$$d(fx, fy) + \varphi(fx) + \varphi(fy) \leq \psi(M(x, y)) \tag{2.8}$$

$M(x, y) = \max\{d(x, y) + \varphi(x) + \varphi(y), \frac{(d(x,fx)+\varphi(x)+\varphi(fx))(d(y,fy)+\varphi(y)+\varphi(fy))}{d(x,y)+\varphi(x)+\varphi(y)}\}$
for all $x, y \in \cup_{x_0 \preceq fx_0,\ x_0 \in X} \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$.
Assume the following:

(i) *if $\{x_n\}$ is a non-decreasing sequence converging to $z \in X$, then $x_n \preceq z$, for all n; and*

(ii) *if $\{x_n\}$ and $\{y_n\}$ are sequences in X with $x_n \preceq y_n$, for all n and $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ x,\ y \in X$ then $x \preceq y$.*

Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z,\ z \in X$. Then z is a fixed point of f and $\varphi(z) = 0$. Further, f is orbitally continuous at z.

Proof. Let $x_0 \in X$ be such that $x_0 \preceq fx_0$. On proceeding as in the proof of Theorem 2.1, we have $\{x_n\} \subset O(x_0)$ defined by (2.1)is a Cauchy sequence in (X, d). Since X is f-orbitally complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z \tag{2.9}$$

Since φ is lower semi continuous, we have
$\varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = 0.$
Hence $\varphi(z) = 0$.
Since $\{x_n\}$ is a non-decreasing sequence and $x_n \to z$, by (i) we have $x_n \preceq z$ for all n. Since f is non-decreasing, we have $fx_n \preceq fz$ for all n. $i.e.$, $x_{n+1} \preceq fz$ for all n. Moreover, as $x_n \preceq x_{n+1} \preceq fz$ for all n and by using (ii), we get $z \preceq fz$.

We now define a sequence $\{y_n\}$ as $y_0 = z,\ y_{n+1} = fy_n,\ n = 0, 1, 2, ...$. Then $y_0 \preceq fy_0$. Since f is non-decreasing, we obtain that $\{y_n\}$ is a non-decreasing sequence and $\{y_n\}$ is Cauchy (similar to the argument to show $\{x_n\}$ is Cauchy) $y_n \to y$ (say), $y \in X$. Again, by the condition (i), we have $y_n \preceq y$. Since $x_n \preceq z = y_0 \preceq fz = fy_0 \preceq y_n \preceq y$ for all n, we have $x_n \preceq y_n$ for all n, and hence $z \preceq y$.
If $x_n = y_n$ for some n, then $x_n \preceq z = y_0 \preceq fz = fy_0 \preceq y_n = x_n$ so that $fz = z$.
Hence we assume that $x_n \neq y_n$ for all n.
Suppose that $z \neq y$. Now from (2.8), we have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) + \varphi(x_{n+1}) + \varphi(y_{n+1}) &= d(fx_n, fy_n) + \varphi(fx_n) + \varphi(fy_n) \\ &\leq \psi(M(x, y)),\ where \end{aligned} \tag{2.10}$$

$M(x, y) = \max\{d(x_n, y_n) + \varphi(x_n) + \varphi(y_n), \frac{(d(x_n,fx_n)+\varphi(x_n)+\varphi(fx_n))(d(y_n,fy_n)+\varphi(y_n)+\varphi(fy_n))}{d(x_n,y_n)+\varphi(x_n)+\varphi(y_n)}\}$.
On letting $n \to \infty$ in (2.10), we have
$d(z, y) \leq \psi(d(z, y)) < d(z, y)$,
a contradiction.
Hence $z = y$, and we have $z \preceq fz = fy_0 \preceq y_n \preceq y = z$.
Therefore z is a fixed point of f. ■

Remark: Condition (ii) of Theorem 2.2 holds trivially in $\mathbb{R}$ with the usual order. But in partially ordered metric spaces it need not hold always. For more details, we refer [8].

Now we prove the uniqueness of fixed point of f by using 'condition (H)' and it is the following:

Condition (H): For all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.

Theorem 2.3. *In addition to the hypotheses of Theorem 2.1 $(Theorem\ 2.2)$ if condition (H) holds, then f has a unique fixed point.*

Proof. By Theorem 2.1, we have f has a fixed point. Suppose that $x, y \in X$ are two fixed points of f. By condition (H), there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.
Put $z = z_0,\ z_1 = fz_0$. and define a sequence $\{z_n\}$ in X by $z_{n+1} = fz_n$ for all $n \geq 0$. Then $x \preceq z_0$ and $y \preceq z_0$. By using the non-decreasing property of f, we have
$fx \preceq fz_0$ and $fy \preceq fz_0$. Hence $x \preceq z_1$ and $y \preceq z_1$.
On continuing this process, we have

$$x \preceq z_n \ \ and\, y \preceq z_n \ \ for \ \ n \geq 0. \tag{2.11}$$

In (2.11), if $x = z_n$ for some n, then $fx = fz_n$ so that $x = z_{n+1}$. In fact, we have $x = z_m$ for $m \geq n$ so that $\lim_{n\to\infty} z_n = x$.
If $x \neq z_n$ for all $n = 0, 1, 2, ...$ then by using (1.3), we have

$$\begin{aligned} d(x, z_{n+1}) + \varphi(x) + \varphi(z_{n+1}) &= d(fx, fz_n) + \varphi(fx) + \varphi(fz_n) \\ &\leq \psi(\max\{d(x, z_n) + \varphi(x) + \varphi(z_n), \tfrac{(d(x,fx)+\varphi(x)+\varphi(fx))(d(z_n,fz_n)+\varphi(z_n)+\varphi(fz_n))}{d(x,z_n)+\varphi(x)+\varphi(z_n)}\} \\ &= \psi(\max\{d(x, z_n) + \varphi(z_n), 0\}) = \psi(d(x, z_n) + \varphi(z_n)) \\ d(x, z_{n+1}) + \varphi(z_{n+1}) &\leq \psi(d(x, z_n) + \varphi(z_n)) = \psi(\psi(d(x, z_{n-1}) + \varphi(z_{n-1})) \\ &\leq \psi^2(d(x, z_{n-1}) + \varphi(z_{n-1})) \\ &\leq \psi^3(d(x, z_{n-2}) + \varphi(z_{n-2})) \leq ... \leq \psi^n(d(x, z_1) + \varphi(z_1)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

$$Therefore \ \lim_{n\to\infty} z_n = x. \tag{2.12}$$

Again, by applying the similar argument to $y \neq z_n$ for all $n = 0, 1, 2, ...$, it follows that

$$\lim_{n\to\infty} z_n = y. \tag{2.13}$$

From (2.12) and (2.13) we have $x = y$.
This completes the proof of the theorem. ■

3. Corollaries and examples

In the following, we deduce some corollaries to the main results of Section 2.

Corollary 3.1. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$. Suppose that there exists $\psi \in \Psi_2$ such that*

$$d(fx, fy) \leq \psi[\max\{d(x, y), \frac{d(x, fx)d(y, fy)}{d(x, y)}\}] \tag{3.1}$$

for all $x, y \in \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$.

Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by

$x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z,\ z \in X$. Suppose that f is orbitally continuous at z. Then z is a fixed point of f.

Proof. The inequality (3.1) implies the inequality (1.3) with $\varphi \equiv 0$ on X, and hence the conclusion of the corollary follows from Theorem 2.1. ■

Corollary 3.2. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$. Suppose that there exist a constant $k \in (0, 1)$ such that*

$$d(fx, fy) \leq k \max\{d(x, y), \frac{d(x, fx)d(y, fy)}{d(x, y)}\} \tag{3.2}$$

for all $x, y \in \overline{O(x_0)}$ *with* $x \preceq y$ *and* $x \neq y$.
Assume that X *is* f*-orbitally complete. Then, the sequence* $\{x_n\}$ *defined by* $x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ *is Cauchy in* X. *Let* $\lim\limits_{n\to\infty} x_n = z,\ z \in X$. *Suppose that* f *is orbitally continuous at* z. *Then* z *is a fixed point of* f.

Proof. By choosing $\psi(t) = kt,\ t \geq 0$ in the inequality (3.1), the conclusion of this corollary follows from Corollary 3.1. ■

Remark 3.3. *Theorem 1.2 follows as a corollary to Corollary 3.2, since the inequality (1.2) implies the inequality (3.2) with* $k = \alpha + \gamma < 1$.

Corollary 3.4. *Let* $(X, d, \preceq)$ *be a partially ordered metric space. Suppose that* $f : X \to X$ *is a non-decreasing map and* $x_0 \in X$ *such that* $x_0 \preceq fx_0$ *and* $\psi \in \Psi_2$ *such that*

$$d(fx, fy) \leq \psi(\max\{d(x, y), \frac{d(x, fx)d(y, fy)}{d(x, y)}\}) \tag{3.3}$$

for all $x, y \in \cup_{x_0 \preceq fx_0,\ x_0 \in X}\overline{O(x_0)}$ *with* $x \preceq y$ *and* $x \neq y$.
Assume the following:

(i) *if* $\{x_n\}$ *is a non-decreasing sequence converging to* $z \in X$, *then* $x_n \preceq z$, *for all* n*; and*

(ii) *if* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *with* $x_n \preceq y_n$, *for all* n *and* $\lim\limits_{n\to\infty} x_n = x,\ \lim\limits_{n\to\infty} y_n = y,\ x,\ y \in X$ *then* $x \preceq y$.

Assume that X *is* f*-orbitally complete. Then, the sequence* $\{x_n\}$ *defined by* $x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ *is Cauchy in* X. *Let* $\lim\limits_{n\to\infty} x_n = z,\ z \in X$.
Then z *is a fixed point of* f.

Proof. The inequality (3.3) implies the inequality (2.8) with $\varphi \equiv 0$ on X, and hence the conclusion of the corollary follows from Theorem 2.2. ■

Corollary 3.5. *Let* $(X, d, \preceq)$ *be a partially ordered metric space. Suppose that* $f : X \to X$ *is a non-decreasing map and* $x_0 \in X$ *such that* $x_0 \preceq fx_0$ *and there exist a constant* $k \in (0, 1)$ *such that*

$$d(fx, fy) \leq k \max\{d(x, y), \frac{d(x, fx)d(y, fy)}{d(x, y)}\} \tag{3.4}$$

for all $x, y \in \cup_{x_0 \preceq fx_0,\ x_0 \in X}\overline{O(x_0)}$ *with* $x \preceq y$ *and* $x \neq y$.
Assume the following:

(i) *if* $\{x_n\}$ *is a non-decreasing sequence converging to* $z \in X$, *then* $x_n \preceq z$, *for all* n*; and*

(ii) *if* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *with* $x_n \preceq y_n$*, for all* n *and*

$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ x,\ y \in X$ *then* $x \preceq y$.

Assume that X *is* f*-orbitally complete. Then, the sequence* $\{x_n\}$ *defined by* $x_{n+1} = fx_n,\ n = 0,1,2,...,$ *is Cauchy in* X*. Let* $\lim_{n\to\infty} x_n = z,\ z \in X$*. Then* z *is a fixed point of* f.

Proof. By choosing $\psi(t) = kt,\ t \geq 0$ in the inequality (3.3), the conclusion of this corollary follows from Corollary 3.4. ■

Remark 3.6. *Theorem 1.2 follows as a corollary to Corollary 3.5, since the inequality (1.2) implies the inequality (3.4) with* $k = \alpha + \gamma < 1$.

In the following, we provide examples in support of the results that are proved in Section 2.

Example 3.7. *Let* $X = [0,2)$ *with the usual metric. We define partial order* $\preceq$ *on* X *by*

$\preceq := \{(x,y) \in X \times X : x = y\} \cup \{(x,y)/x,y \in X,\ x \preceq y \Leftrightarrow x \geq y$*, where* $\geq$ *is the usual order*$\}$.

We define $f : X \to X$ *by* $fx = \begin{cases} 0 & if\ x = 0 \\ \frac{1}{8} & if\ x \in [\frac{1}{2},1) \\ \frac{x^2}{16} & if\ x \in [\frac{1}{4},\frac{1}{2}) \\ \frac{1}{2^{n+2}} & if\ x \in [\frac{1}{2^{n+1}},\frac{1}{2^n}), n \geq 2 \\ 2-x & if\ x \in [1,2). \end{cases}$

We define $\psi : [0,\infty) \to [0,\infty)$ *by* $\psi(t) = \frac{2}{3}t$ *for all* $t \geq 0$ *and*

$\varphi : X \to [0,\infty)$ *by* $\varphi(x) = \begin{cases} \frac{x}{3} & if\ x \in [0,\frac{5}{16}) \\ x - \frac{5}{16} & if\ x \in [\frac{5}{16},1). \end{cases}$

Let $x_0 = \frac{3}{8}$ *then* $x_0 \preceq fx_0$*. Here* $O(x_0) = \{\frac{3}{8},\frac{9}{2^{10}},\frac{1}{2^8},\frac{1}{2^9}...,\frac{1}{2^{2n+8}},....\} = \{\frac{3}{8},\frac{9}{2^{10}}\} \cup \{\frac{1}{2^n}/n \geq 8\}$ *and* $\overline{O(x_0)} = O(x_0) \cup \{0\}$.

We show that f *is a generalized* (φ,ψ)*-Jaggi contraction. The following are the possible four cases.*

<u>*Case* (i)*:*</u> $x = \frac{3}{8}$ *and* $y = \frac{9}{2^{10}}$.

In this case, $d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{7}{3.2^8}$ *and* $M(x,y) = \frac{442}{2^{10}}$.

$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{7}{3.2^8} \leq \psi(\frac{442}{2^{10}}) = \psi(M(x,y))$.

<u>*Case* (ii)*:*</u> $x = \frac{9}{2^{10}}$ *and* $y = \frac{1}{2^{i+3}},\ i \geq 2$.

In this case, $d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{2^i - 2^3}{3.2^{i+6}}$ *and* $M(x,y) = \frac{7}{2^3(9.2^i - 2^6)}$.

$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{2^i - 2^3}{3.2^{i+6}} \leq \psi(\frac{7}{2^3(9.2^i - 2^6)}) = \psi(M(x,y))$.

<u>*Case* (iii)*:*</u> $x = \frac{3}{8}$ *and* $y = \frac{1}{2^{i+1}},\ i \geq 2$*. In this case,*

$d(fx,fy) + \varphi(fx) + \varphi(fy) = \begin{cases} \frac{9.2^i - 2^7}{3.2^{i+8}} & if\ i \geq 2 \\ \frac{2^9 - 9.2^i}{3.2^{i+9}} & if\ i \leq 2 \end{cases}$ *and* $M(x,y) = \frac{663}{2^4(21.2^i - 16)}$.

<u>*Sub case* (a)*:*</u>

$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{9.2^i - 2^7}{3.2^{i+8}} \leq \psi(\frac{663}{2^4(21.2^i - 16)}) = \psi(M(x,y))$.

Sub case (b): $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2^9-9.2^i}{3.2^{i+9}}\leq\psi(\frac{663}{2^4(21.2^i-16)})=\psi(M(x,y))$.

Case (iv): $x=\frac{1}{2^i}$ *and* $y=\frac{1}{2^j},\ i\geq 2$ *and* $j\geq i$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}$ *and* $M(x,y)=\frac{4.2^j-2.2^i}{3.2^{i+j}}$.

$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}\leq\psi(\frac{4.2^j-2.2^i}{3.2^{i+j}})=\psi(M(x,y))$.

Hence, all the hypotheses of Theorem 2.1 hold and $0,\ 1$ *are two fixed points of* f *in* $\overline{O(x_0)}$. *Also* $\varphi(0)=0$.

Since, the inequality (1.3) *fails to hold at* $x=0,\ y=1$ *when* $\varphi\equiv 0$, f *is not a generalized* ψ*-Jaggi contraction. Further, we observe that at* $x=0$ *and* $y=1$, *we have*

$d(f0,f1)=1\nleq\alpha.0+\gamma.1=\alpha\frac{d(0,f0)d(1,f1)}{d(0,1)}+\gamma d(0,1)$

so that the inequality (1.2) does not hold for any α *and* γ *in* $[0,1)$ *with* $\alpha+\gamma<1$. $i.e.,$ f *is not a* $Jaggi\ contraction\ map$. *Therefore Theorem 1.2 is not applicable.*

Thus, it suggests that Theorem 2.1 is a generalization of Theorem 1.2.

Remark 3.8. *For* $x=0$ *and* $y=1$, *and for any* $z\in X$ *we have either* $0\npreceq z$ *or* $1\npreceq z$. *Hence condition (H) fails to hold and* f *has more than one fixed point namely* 0 *and* 1.

Example 3.9. *Let* $X=[0,1)$ *with the usual metric. We define partial order* $\preceq$ *on* X *by*

$\preceq:=\{(x,y)\in X\times X: x=y\}\cup\{(x,y)/x,y\in X,\ x\preceq y\Leftrightarrow x\geq y$, *where* $\geq$ *is the usual order*$\}$.

We define $f:X\to X$ *by* $fx=\begin{cases}\frac{x}{2} & if\ x\in[0,\frac{1}{2}]\\ \frac{x^2}{2} & if\ x\in(\frac{1}{2},1).\end{cases}$

We define $\psi:[0,\infty)\to[0,\infty)$ *by* $\psi(t)=\frac{5t}{6}$ *for all* $t\geq 0$ *and*

$\varphi:X\to[0,\infty)$ *by* $\varphi(x)=\begin{cases}\frac{x}{3} & if\ x\in[0,\frac{5}{16})\\ x-\frac{5}{16} & if\ x\in[\frac{5}{16},1).\end{cases}$

We choose $x_0=\frac{1}{2}$ *then* $x_0\preceq fx_0,\ O(x_0)=\{\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3}...,\frac{1}{2^n},....\}=\{\frac{1}{2^n}/n\geq 1\}$ *and* $\overline{O(x_0)}=O(x_0)\cup\{0\}$.

The following two cases arise to verify the inequality (2.8).

Case (i): $x=\frac{1}{2^i}$ *and* $y=\frac{1}{2^j},\ i\geq 2$ *and* $j\geq i$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}$ *and* $M(x,y)=\frac{4.2^j-2.2^i}{3.2^{i+j}}$.

$$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}$$
$$\leq\psi(\frac{4.2^j-2.2^i}{3.2^{i+j}})=\psi(M(x,y)).$$

Case (ii): $x=\frac{1}{2^i}$ *and* $y=0,\ i\geq 1$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2}{3.2^i}$ *and* $M(x,y)=\frac{4}{3.2^i}$.

$$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2}{3.2^i}$$
$$\leq\psi(\frac{4}{3.2^i})=\psi(M(x,y)).$$

Hence, all the hypotheses of Theorem 2.2 hold and 0 *is a fixed point of* f *in* $\overline{O(x_0)}$. *Also* $\varphi(0)=0$.

References

[1] G.V.R. BABU, AND P.D. SAILAJA, A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces, *Thai J. of Math.*, **9(1)**(2011), 1–10.

[2] G.V.R. BABU, P.D. SAILAJA, AND K.T. KIDANE, A fixed point theorem in orbitally complete partially ordered metric spaces, *J. of Oper.*, **2013**, 1–8, Article ID 404573.

[3] J. HARJANI, B. LOPEZ AND K. SADARANGANI, A fixed point theorem for mappings satisying a contractive condition of rational type on a partially ordered metric space, *Abstr. and Appl. Anal.,* 2010, (2010), 1–8.

[4] D.S. JAGGI, Some unique fixed point theorems, *Indian J. of Pure and Appl. Math.*, **8**(1977), 223 - 230.

[5] M.S. KHAN, M. SWALEH, AND S. SESSA, Fixed point theorems by altering distances between the points, *Bull. of Aust. Math. Soc.*, **30**(1984), 1–9.

[6] A. C. M. RAN AND M. C. B. REURINGS, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. of Amer. Math. Soc.*, **135(5)**(2004), 1435–1443.

[7] B. SAMET, C. VETRO AND F. VETRO, From metric spaces to partially metric spaces, *Fixed Point Theory and Applications*, **5**(2013), 1–10.

[8] K. P. R. SASTRY, G. V. R. BABU, K. K. M. SARMA AND P. H. KRISHNA, Fixed points of selfmaps on partially ordered metric spaces with control functions involving rational type expressions, *Jour. of Adv. Res. in Pure Math.*, **7(4)**(2015), 92–102.

[9] D. TURKOGLU, O. OZER AND B. FISHER, Fixed point theorems for T-orbitally complete space, *Mathematica*, **9**(1999), 211–218.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 90–97.
http://doi.org/10.26637/mjm1001/008

A study on Ss-Semilocal modules in view of singularity

ESRA ÖZTÜRK SÖZEN*[1]

[1] *Faculty of Sciences and Arts, Department of Mathematics, Sinop University, Sinop, Turkey.*

*R*eceived 20 September 2021; *A*ccepted 04 December 2021

Abstract. In this paper, we define weakly δ_{ss}- supplemented modules and give a characterization for them named with δ_{ss}-semilocal modules. In particular, we determine the suitable conditions for a δ_{ss}- semilocal module to be δ-semilocal and ss-semilocal, respectively. In addition to these we supply contrast examples pointing the relations are proper between these classes of modules.
AMS Subject Classifications: 16D10, 16D99, 16L99.

Keywords: semisimple module, (weakly) δ_{ss}- supplemented module, δ_{ss}- semilocal module, δ_{ss}-perfect ring.

Contents

1. Introduction and Background

Firstly, note that throughout this study the symbols R and M will denote an associative ring with identity and a unitary left R-module, respectively. The notations $A \leq M$ and $A \leq_{\oplus} M$ will indicate that A is a submodule of M and A is a direct summand of M. A submodule A of M is called *essential* (denoted by $A \trianglelefteq M$) if $A \cap K \neq \{0\}$ for any proper submodule K of M except for $\{0\}$. The intersection of all essential submodules of a module M is denoted by $Soc(M)$ which is the largest semisimple submodule of M. A submodule $B^{'} \leq M$ is called a *complement* of A in M if it is maximal in the set of submodules $B \leq M$ with $A \cap B = \{0\}$. A submodule A of M is called *small* (denoted by $A \ll M$) if $A + K \neq M$ for any proper submodule K of M. The sum of all small submodules of a module M is denoted by $Rad(M)$. A *(weak) supplement submodule* T of A in M is a submodule such that $A + T = M$ and $A \cap T \ll T$ $(A \cap T \ll M)$. A module M is called *(weakly) supplemented* if every submodule of M has a (weak) supplement in M [14].

In [15] and [6], the authors updated the small and supplemented modules via singularity as follows. A submodule $A \leq M$ is δ-small if and only if for all submodules $X \leq M$: if $A + X = M$, then $M = Y \oplus X$ for a projective semisimple submodule Y of A. Also the submodule A is called δ-*small* in M if $A + K \neq M$ for every proper submodule K of M with $\frac{M}{K}$ is singular (denoted by $A \ll_{\delta} M$) and the sum of all δ-small submodules of M denoted by $\delta(M)$. Clearly $Rad(M) \leq \delta(M)$.

A δ-supplement submodule T of A in M is a submodule such that $A + T = M$ and $A \cap T \ll_{\delta} T$. A *(generalized) weak δ-supplement submodule* T of A in M is a submodule such that $A + T = M$ and $(A \cap T \leq$

***Corresponding author.** Email address: **esozen@sinop.edu.tr** (Esra ÖZTÜRK SÖZEN)

$\delta(M)$) $A \cap T \ll_\delta M$ [11]. The module M is called *(weakly) δ-supplemented,* if every submodule of M has a (weak) δ-supplement in M. Clearly every (weakly) supplemented module is (weakly) δ-supplemented. In [5], the authors introduced ss-supplemented modules which are stronger than supplemented modules. A module M is called *ss-supplemented* if for every submodule A of M there exists a submodule T of M such that $A+T=M$ and $A \cap T \leq Soc_s(T)$ where $Soc_s(T) = Soc(T) \cap Rad(T)$. In [9], the authors generalized ss-supplemented modules to *weakly ss-supplemented modules* by taking $Soc_s(M)$ instead of $Soc_s(T)$ and gave a characterization for these modules named with ss-semilocal. In [13] the authors generalized (amply) ss-supplemented modules in view of singularity and introduced *(amply) δ_{ss}-supplemented modules and δ_{ss}-supplemented rings*.

In this article, in the light of the given studies we define weakly δ_{ss}-supplemented modules and obtain a new characterization for them named with *δ_{ss}-semilocal modules.* A module M is called δ_{ss}-semilocal whenever $\frac{M}{Soc_\delta(M)}$ is semisimple where $Soc_\delta(M) = Soc(M) \cap \delta(M)$. A module M is called δ-semilocal if $\frac{M}{\delta(M)}$ is semisimple. As $Soc_\delta(M) \leq \delta(M) \leq M$, every δ_{ss}-semilocal module is δ-semilocal and every ss-semilocal module is δ_{ss}-semilocal. We give examples on the converse implications might not be true. Also, we investigate suitable conditions when δ_{ss}semilocal modules are δ-semilocal and ss-semilocal. In particular, we obtain new characterizations for δ_{ss}-semilocal rings.

For undefined algebraic structures used here, such as δ-(semi)perfect and δ_{ss}-perfect rings, we refer to [15] and [13], respectively.

2. Weakly δ_{ss}-supplemented modules

A module M is called weakly δ-supplemented if for any submodule A of M there exists a submodule T of M such that $A+T=M$ and $A \cap T \ll_\delta M$ [11]. By means of this concept and the usuful lemma given in the following we will define weakly δ_{ss}-supplemented modules as a strongly version of weakly δ-supplemented modules.

Lemma 2.1. *Let $f: A \longrightarrow B$ be a module homomorphism. Then $f(Soc_\delta(A)) \leq Soc_\delta(B)$. In particular, we have $Soc_\delta(A) \leq Soc_\delta(B)$ whenever $A \leq B$.*

Proof. As f is a homomorphism we have $f(Soc(A)) \leq Soc(B)$ and $f(\delta(A)) \leq \delta(B)$. Therefore we get $f(Soc_\delta(A)) = f(Soc(A) \cap \delta(A)) \leq f(Soc(A)) \cap f(\delta(A)) \leq Soc(B) \cap \delta(B) = Soc_\delta(B)$. In particular, if the inclusion map from A to B is taken instead of f, then $Soc_\delta(A) \leq Soc_\delta(B)$ is obtained clearly. ■

Definition 2.2. *A module M is called weakly δ_{ss}-supplemented if for any submodule A of M there exists a submodule T of M such that $A+T=M$ and $A \cap T \leq Soc_\delta(M)$.*

It is a clear fact that every weakly δ_{ss}-supplemented module is weakly δ-supplemented but not vice versa. To verify this with an example we need the following lemma.

Lemma 2.3. *Let M be a weakly δ-supplemented module with $Soc(M)=0$. Then $M=0$.*

Proof. Let $A \leq M$. By hypothesis there exists a submodule T of M such that $A+T=M$ and $A \cap T \leq Soc_\delta(M)$. Since $Soc_\delta(M) = Soc(M) \cap \delta(M) = 0 \cap \delta(M) = 0$ then we have $A \cap T = \{0\}$. Therefore, M is a semisimple module as each submodule is a direct summand. Hence $M = Soc(M) = 0$. ■

Example 2.4. *It is a known fact that $\mathbb{Z}$-module $\mathbb{Q}$ is weakly δ-supplemented as it is weakly supplemented [3, 17.15 Example, 213p.]. On the orher hand, it is not weakly δ_{ss}-supplemented by Lemma 2.3.*

Now we give a characterization lemma for weak δ_{ss}-supplement submodules of a module.

Lemma 2.5. *Let M be a module and $A, T \leq M$. Then the following implications are equivalent:*

1. $M = A+T$ and $A \cap T \leq Soc_\delta(M)$.

2. T is a weak δ-supplement of A in M and $A \cap T$ is semisimple.

3. T is a generalized weak δ-supplement of A in M and $A \cap T$ is semisimple.

Proof. $(1) \Rightarrow (2)$: By hypothesis we have $M = A + T$, $A \cap T \leq \delta(M)$ and $A \cap T \leq Soc(M)$ as $Soc_\delta(M)$ is a submodule of both $\delta(M)$ and $Soc(M)$. Therefore, $A \cap T$ is semisimple and it is also δ-small in M by [13, Lemma 2.2].

$(2) \Rightarrow (3)$: It is clear.

$(3) \Rightarrow (1)$: By hypothesis we have $M = A + T$, $A \cap T \leq \delta(M)$ and $A \cap T$ is semisimple. Thus, $A \cap T \leq Soc(M)$. Hence, $A \cap T \leq Soc(M) \cap \delta(M) = Soc_\delta(M)$. ■

We say that a module M is called δ-semilocal if $\frac{M}{\delta(M)}$ is semisimple. And it is proven in [10, Theorem 3.7] that a module M with $\delta(M) \ll_\delta M$ and $\frac{M}{\delta(M)}$ is singular is δ-semilocal if and only if M is a generalized weakly δ-supplemented module. Motivated by this we give a similar characterization for our modules in the following theorem.

Theorem 2.6. *The following implications are equivalent for a module M:*

1. $\frac{M}{Soc_\delta(M)}$ is semisimple.

2. M is weakly δ_{ss}-supplemented.

3. M is a direct sum of two submodules M_1 and M_2 such that M_1 and $\frac{M_2}{Soc_\delta(M)}$ are semisimple, also $Soc_\delta(M) \trianglelefteq M_2$.

Proof. $(3) \Rightarrow (1)$: Let $M = M_1 \oplus M_2$. Then $\frac{M}{Soc_\delta(M)} = \frac{M_1 + Soc_\delta(M)}{Soc_\delta(M)} \oplus \frac{M_2}{Soc_\delta(M)}$ is semisimple as a direct sum of two semisimple modules.

$(1) \Rightarrow (2)$: For any $A \leq M$, $\frac{A + Soc_\delta(M)}{Soc_\delta(M)} \oplus \frac{T}{Soc_\delta(M)} = \frac{M}{Soc_\delta(M)}$ can be written by hypothesis. Then, $M = A + T$ and by modularity $(A + Soc_\delta(M)) \cap T = (A \cap T) + Soc_\delta(M) = Soc_\delta(M)$ are obtained. Thus, $A \cap T \leq Soc_\delta(M)$ is got.

$(1) \Rightarrow (3)$: Let M_1 be a complement of $Soc_\delta(M)$. Then, $M_1 \cong \frac{M_1 + Soc_\delta(M)}{Soc_\delta(M)} \leq_\oplus \frac{M}{Soc_\delta(M)}$ and so M_1 is semisimple as it is isomorphic to a submodule of a semisimple module. Additionally, there exists a semisimple direct summand $\frac{M_2}{Soc_\delta(M)}$ satisfying $\frac{M_1 + Soc_\delta(M)}{Soc_\delta(M)} \oplus \frac{M_2}{Soc_\delta(M)} = \frac{M}{Soc_\delta(M)}$. Clearly, $M = M_1 + M_2$. Furthermore, since $Soc_\delta(M) = (M_1 + Soc_\delta(M)) \cap M_2 = Soc_\delta(M) \oplus (M_1 \cap M_2)$ by modularity. Then we get $M_1 \cap M_2 \leq Soc_\delta(M)$ and $M_1 \cap M_2 \leq M_1$ which means $M_1 \cap M_2 \leq M_1 \cap Soc_\delta(M) = 0$ by the property of a complement. Thus $M = M_1 \oplus M_2$. For the remaining part of the proof let us show that $Soc_\delta(M) \trianglelefteq M_2$. As M_1 is the complement of $Soc_\delta(M)$ we have $M_1 \oplus Soc_\delta(M) \trianglelefteq M = M_1 \oplus M_2$ [3, 1.11(1)]. For the second injection map $i_2 : M_2 \longrightarrow M_1 \oplus M_2$, $i_2^{-1}(M_1 \oplus Soc_\delta(M)) \trianglelefteq M_2$ by [1, Theorem 9.1(3)].

$2 \Rightarrow 1$: For any $\frac{A}{Soc_\delta(M)} \leq \frac{M}{Soc_\delta(M)}$ we have $A + T = M$ and $A \cap T \leq Soc_\delta(M)$ for a submodule $T \leq M$ by hypothesis. Thus $\frac{A}{Soc_\delta(M)} \oplus \frac{T+M}{Soc_\delta(M)} = \frac{M}{Soc_\delta(M)}$, that is, $\frac{M}{Soc_\delta(M)}$ is semisimple. ■

From now on, we will call a module M is δ_{ss}-semilocal whenever M satisfies one of the equvalent conditions of the theorem given above.

3. δ_{ss}-Semilocal modules

In this part we will present the fundamental properties of our modules firstly. Before of all we need a useful lemma.

Lemma 3.1. *For a given family of R-modules $\{M_i\}_{i \in I}$, $Soc_\delta(\oplus_{i \in I} M_i) = \oplus_{i \in I} Soc_\delta(M_i)$.*

Proof. It is clear by Lemma 2.1 and [3, 6.2(3)]. ■

Theorem 3.2. *Let $\{M_i\}_{i\in I}$ be a family of δ_{ss}-semilocal modules. Then $M=\oplus_{i\in I}M_i$ is δ_{ss}-semilocal.*

Proof. As each $\frac{M_i}{Soc_\delta(M_i)}$ is semisimple, $\frac{M}{Soc_\delta(M)}=\frac{\oplus_{i\in I}M_i}{Soc_\delta(\oplus_{i\in I}M_i)}=\frac{\oplus_{i\in I}M_i}{\oplus_{i\in I}Soc_\delta(M_i)}\cong\oplus_{i\in I}\frac{M_i}{Soc_\delta(M_i)}$ is also semisimple by [4, Cor. 8.1.5] and Lemma 3.1. Hence, M is is δ_{ss}-semilocal. ■

Corollary 3.3. *The sum of δ_{ss}-semilocal modules is also δ_{ss}-semilocal.*

Theorem 3.4. *If M is a δ_{ss}-semilocal module, then so is any homomorphic image.*

Proof. Let us consider the module epimorphism $h:M\longrightarrow K$ where M is δ_{ss}-semilocal. Then the homomorphism $\overline{h}:\frac{M}{Soc_\delta(M)}\longrightarrow\frac{K}{Soc_\delta(K)}$ defined by $\overline{h}(x+Soc_\delta(M))=h(x)+Soc_\delta(K)$ for every $x+Soc_\delta(M)\in\frac{M}{Soc_\delta(M)}$ is epic. As $\frac{M}{Soc_\delta(M)}$is semisimple, then the homomorphic image $\frac{K}{Soc_\delta(K)}$ is also semisimple by [4, Cor. 8.1.5], that is $h(M)=K$ is δ_{ss}-semilocal. ■

Proposition 3.5. *Let M be a δ_{ss}-semilocal module and A be a submodule of M satisfying $\delta(A)=A\cap\delta(M)$. Then A is δ_{ss}-semilocal.*

Proof. Let $B\leq A$. Then there exists a submodule T of M such that $B+T=M$ and $B\cap T\leq Soc_\delta(M)$. Following this $A=(B+T)\cap A=B+(T\cap A)$ is obtained by using modularity. Now we will verify that $T\cap A$ is a weak δ_{ss}-supplement of B in A. As $B\cap(T\cap A)=B\cap T\leq Soc_\delta(M)\leq\delta(M)$ we have $B\cap(T\cap A)\leq\delta(M)\cap A=\delta(A)$. Thus, $B\cap T=B\cap(T\cap A)\leq Soc(A)\cap\delta(A)=Soc_\delta(A)$. Hence, A is δ_{ss}-semilocal. ■

Corollary 3.6. *Every δ_{ss}-supplement (and so δ-supplement) submodule of a δ_{ss}-semilocal module is δ_{ss}-semilocal.*

Recall that a module K is said to be M-generated, if there exists an epimorphism from $M^{(I)}$ to K where I is an index set.

Lemma 3.7. *Let M be a module. M is δ_{ss}-semilocal if and only if every M-generated module is δ_{ss}-semilocal.*

Proof. $(\Longrightarrow)$: It is clear by Corollary 3.3 and Theorem 3.4.
$(\Longleftarrow)$: It is clear. ■

In general, every amply δ_{ss}-supplemented module is δ_{ss}-supplemented [13]. now it is possible to think whether the analogous idea is valid for our modules. In the following proposition we show that δ_{ss}-semilocal modules already contain this property by themselves.

Proposition 3.8. *Let M be a δ_{ss}-semilocal module and $A,T\leq M$ with $A+T=M$. Then A has a weak δ_{ss}-supplement in M contained by T.*

Proof. As $A\cap T\leq M$, there is a submodule $B\leq M$ such that $(A\cap T)+B=M$ and $(A\cap T)\cap B\leq Soc_\delta(M)$ by hypothesis. By modularity, we have $T=T\cap M=T\cap[(A\cap T)+B]=(A\cap T)+(B\cap T)$. Thus, $M=A+T=A+(A\cap T)+(B\cap T)=A+(B\cap T)$ and $A\cap(B\cap T)=(A\cap B)\cap T\leq Soc_\delta(M)$. Hence, $B\cap T$ is a weak δ_{ss}-supplement of A in M contained by T. ■

As we pointed before every δ-semilocal module is δ_{ss}-semilocal. Under suitable conditions the converse might be provided as follows.

Proposition 3.9. *Let M be a δ-semilocal module with $\delta(M)\leq Soc(M)$. Then M is δ_{ss}-semilocal.*

Proof. Clearly, $Soc_\delta(M)=\delta(M)$ as $\delta(M)\leq Soc(M)$. Therefore, $\frac{M}{\delta(M)}=\frac{M}{Soc_\delta(M)}$ is semisimple. Hence, M is δ_{ss}-semilocal by Theorem 2.6. ■

Due to the consequences of the proposition given in the following we will obtain the ring characterization of δ_{ss}-semilocal modules in the next.

Proposition 3.10. *Let M be a δ_{ss}-semilocal module and $A \ll_\delta M$. Then $A \leq Soc_\delta(M)$.*

Proof. By hypothesis there exists a submodule T of M such that $A + T = M$ and $A \cap T \leq Soc_\delta(M)$. As $A \ll_\delta M$ we have $Y \oplus T = M$ for a projective semisimple submodule Y of A. From modularity we get $Y \oplus (T \cap A) = A$ and so A is semisimple as a direct sum of two semisimple modules. Hence $A \leq Soc(M) \cap \delta(M) = Soc_\delta(M)$. ■

Corollary 3.11. *Let M be a δ_{ss}-semilocal module and $\delta(M) \ll_\delta M$. Then $\delta(M) \leq Soc(M)$.*

As finitely generated modules have δ-small δ-radical we have the following corollary.

Corollary 3.12. *Let M be a finitely generated module. Then M is δ_{ss}-semilocal if and only if M is δ-semilocal and $\delta(M)$ is semisimple.*

Proof. $(\Longrightarrow)$: By hypothesis M is weakly δ_{ss}-supplemented and so it is weakly δ-supplemented. hence it can be shown that M is δ-semilocal by the similar way from [7, Prop. 2.1]. Also $\delta(M) \leq Soc(M)$ by Corollary 3.11 as $\delta(M) \ll_\delta M$.

$(\Longleftarrow)$: Let M be δ-semilocal with a semisimple δ-radical. Then $\delta(M) \leq Soc(M)$. Hence M is δ_{ss}-semilocal from Proposition 3.9. ■

Definition 3.13. *A module M is called weakly δ-radical δ-supplemented if every submodule of M containing $\delta(M)$ has a weak δ-supplement in M.*

Theorem 3.14. *Let M be a module with $\delta(M) \ll_\delta M$. Then the statements given in the following are equivalent:*

1. M is δ_{ss}-semilocal
2. M is δ-semilocal and $\delta(M)$ has a weak δ_{ss}-supplement in M.
3. M is δ-semilocal and $\delta(M) \leq Soc(M)$.
4. M is weakly δ-supplemented and $\delta(M) \leq Soc(M)$.
5. M is weakly δ-radical supplemented and $\delta(M) \leq Soc(M)$.

Proof. $(1) \Rightarrow (2)$: It is clear.

$(2) \Rightarrow (3)$: Let T be a weak δ-supplement of $\delta(M)$ in M. Then $\delta(M) + T = M$ and $\delta(M) \cap T \leq Soc_\delta(M) \leq Soc(M)$ and so $\delta(M) \cap T$ is semisimple. As $\delta(M) \ll_\delta (M)$ and $\delta(M) + T = M$ we have $M = Y \oplus T$ for a projective semisimple submodule Y of $\delta(M)$. By modularity we get $\delta(M) = Y \oplus (\delta(M) \cap T)$ and so $\delta(M)$ is semisimple by [4, Cor. 8.1.5]. Thus, $\delta(M) \leq Soc(M)$.

$(3) \Rightarrow (4)$: By hypothesis, for any $A \leq M$ there is a submodule $T \leq M$ such that $A + T = M$, $A \cap T \leq \delta(M)$ and $A \cap T$ is semisimple. Hence M is weakly δ-radical supplemented as $\delta(M) \ll_\delta M$.

$(4) \Rightarrow (5)$: It is clear.

$(5) \Rightarrow (1)$: For any $A \leq M$, $A \leq A + \delta(M)$ and so, there exists $T \leq M$ such that $[A + \delta(M)] + T = M$, $[A + \delta(M)] \cap T \ll_\delta M$. Following that $[A + \delta(M)] \cap T \leq \delta(M) \leq Soc(M)$. As $\delta(M) \ll_\delta M$, we have $P \oplus [A + T] = M$ for a projective semisimple submodule P of $\delta(M)$. Therefore, we get $A + (P \oplus T) = M$ and $A \cap (P \oplus T) \leq [P \cap (A + T)] + [T \cap (A + P)]$ where $P \cap (A + T)$ is δ-small and semisimple in M as P is projective semisimple and, $T \cap (A + P)$ is δ-small and semisimple in M as a submodule of $T \cap (A + \delta(P))$. ■

It is a clear fact that every δ_{ss}-semilocal module is weakly δ-supplemented but not vice versa. In the following example this is verified via Theorem 3.14.

Example 3.15. *Let us consider the $\mathbb{Z}$-module $\mathbb{Z}_8$. As it is local, it is supplemented and so δ-supplemented. Therefore, $\mathbb{Z}$-module $\mathbb{Z}_8$ is weakly δ-supplemented. On the other hand, since $\delta(\mathbb{Z}_8) = Rad(\mathbb{Z}_8) = 2\mathbb{Z}_8 \ll_\delta \mathbb{Z}_8$ and $Soc(\mathbb{Z}_8) = 4\mathbb{Z}_8$, $\mathbb{Z}_8$ is not a δ_{ss}-semilocal $\mathbb{Z}$-module by Theorem 3.14.*

Now we give a ring characterization theorem for δ_{ss}-perfect rings to be δ_{ss}-semilocal.

Corollary 3.16. *The following statements are equivalent for a ring R.*

1. ${}_RR$ is δ_{ss}-semilocal.
2. ${}_RR$ is δ-semilocal and $\delta(R) \leq Soc(R)$.
3. ${}_RR$ is δ_{ss}-perfect (δ_{ss}-supplemented).

Proof. $(1) \Longleftrightarrow (2)$: It is clear by Corollary 3.12

$(2) \Longrightarrow (3)$: As a ring R with unit is locally projective [8], $Soc(R) \ll_\delta R$ is got from [13, Prop. 5.2]. Thus, $\delta(R) = Soc(R)$ is obtained. Since $\frac{R}{\delta(R)} = \frac{R}{Soc(R)}$ and $Soc(R)$ is semisimple Artinian by hypothesis, then R is also Artinian and so it is δ-supplemented. Therefore R is δ-semiperfect by [6, Theorem 3.3]. Hence, R is δ_{ss}-perfect by [13, Theorem 5.3].

$(3) \Longrightarrow (1)$: Let R be a δ_{ss}-perfect ring. Then by [13, Theorem 5.3 (2)] R is δ-semiperfect and $\delta(R) = Soc(R)$. Therefore $Soc_\delta(R) = \delta(R)$ and so $\frac{R}{Soc_\delta(R)} = \frac{R}{Soc(R)}$ is semisimple by [15, Theorem 3.6]. Hence ${}_RR$ is δ_{ss}-semilocal. ∎

Owing to the following we will construct rings whose modules are δ_{ss}-semilocal. In addition to this a proper class of δ-perfect rings is obtained. It will be verified via Example 3.18.

Theorem 3.17. *The following statements are equivalent for a ring R:*

1. ${}_RR$ is δ_{ss}-semilocal.
2. Every R-module is δ_{ss}-semilocal.
3. R is δ-semilocal and $\delta(R) \leq Soc(R)$.

Proof. $(1) \Longrightarrow (2)$: Let M be an R-module. Since each R-module is R-generated, then there exists an epimorphism $h : R^{(I)} \longrightarrow M$. By hypothesis M is δ_{ss}-semilocal by Lemma 3.7.

$(2) \Longrightarrow (3)$: By hypothesis ${}_RR$ is δ_{ss}-semilocal. Then the proof is clear from Corollary 3.16.

$(3) \Longrightarrow (1)$:It is clear by Corollary 3.16. ∎

Example 3.18. *Let $\mathcal{F}$ be a field, $I = \begin{pmatrix} \mathcal{F} & \mathcal{F} \\ 0 & \mathcal{F} \end{pmatrix}$ and $R = \{(x_1, x_2, ..., x_n, x, x...) : n \in \mathbb{N}, x_i \in M_2(\mathcal{F}), x \in I\}$ be a ring with component-wise operations. Then, $Soc(R) = \{(x_1, x_2, ..., x_n, 0, 0...) : n \in \mathbb{N}, x_i \in M_2(\mathcal{F})\}$ and $\delta(R) = \{(x_1, x_2, ..., x_n, x, x...) : n \in \mathbb{N}, x_i \in M_2(\mathcal{F}), x \in J = \begin{pmatrix} 0 & \mathcal{F} \\ 0 & 0 \end{pmatrix}$. From [15, Example 4.3] it can be seen that R is a δ-perfect ring. But as $\delta(R) \neq Soc(R)$, R is not a δ_{ss}-semilocal ring by [13, Proposition 5.2].*

Every ss-semilocal module is δ_{ss}-semilocal. Now we investigate the suitable conditions satisfying the vice versa inspired by [2, Prop. 4.2].

Proposition 3.19. *Let M be a projective, semilocal and δ_{ss}-semilocal module with $Rad(M) \ll M$. Then M is ss-semilocal.*

Proof. As $Soc(M)$ is semisimple, the submodule $Soc_\delta(M)$ is a direct summand of $Soc(M)$. Then for a submodule X of M it can be written that $Soc(M) = Soc_s(M) \oplus X$. Besides there exists a submodule Y of M such that $M = X + Y$ and $X \cap Y \ll M$ since M is semilocal. Clearly, $X \cap Y \leq Rad(M)$. Following this we have $X \cap Y \leq X \cap Rad(M) = [X \cap Soc(M)] \cap Rad(M) = X \cap [Soc(M) \cap Rad(M)] = X \cap Soc_s(M) = 0$. Also we get $Rad(M) = Rad(X) \oplus Rad(Y) = Rad(Y)$ as X is semisimple. Here Y is projective as a direct summand of the projective module M. Now let us show that $\delta(Y) = Rad(Y)$. For thiswe have to verify that Y has no simple projective direct summand [12, Prop. 2.4]. Assume that S is a simple projective direct summand of Y. Then $Y = S \oplus K$ for $K \leq Y$. Therefore, $S \ll_\delta S \leq Y$ and so $S \leq Soc_\delta(Y) \leq Soc(Y)$ because S is semisimple projective. By modularity, $Soc(Y) = Soc(M) \cap Y = [Soc_s(M) \oplus X] \cap Y = [(Soc(M) \cap Rad(M)) \oplus X] \cap Y = [(Soc(M) \cap Rad(Y)) \oplus X] \cap Y =$

$[Soc(M) \cap Rad(Y)) \oplus (X \cap Y) = Soc(M) \cap Rad(Y) \leq Rad(Y)$ is got and using this $S \leq Soc(Y) \leq Rad(Y) = Rad(M) \ll M$ is obtained. As $Y \leq_\oplus M$, S is also small in Y and so this creates the contradiction $K = Y$. According to this it must be true that $\delta(Y) = Rad(Y)$. However, Y is also δ_{ss}-semilocal by Theorem 3.4 as M is δ_{ss}-semilocal. Then for any $U \leq Y$ there is a submodule V of Y such that $U + V = Y$ and $U \cap V \leq Soc_\delta(Y)$. From this fact $U \cap V \leq \delta(Y) = Rad(Y)$ and so $U \cap V \ll Y$. Thus, $U \cap V \leq Soc_s(Y)$. Hence, Y is an ss-semilocal module. By taking into account that X is an ss-semilocal by [9, Corollary 2.13]. ■

Corollary 3.20. *The following statements are equivalent for a ring R:*

1. ${}_RR$ is δ_{ss}-semilocal.
2. R is left δ_{ss}-perfect and semilocal.
3. R is left δ_{ss}-perfect and $\frac{Soc({}_RR)}{Soc_s({}_RR)}$ is finitely generated.

Proof. $(1) \Leftrightarrow (2)$: It is clear by Proposition 3.19 and Corollary 3.16.

$(2) \Leftrightarrow (3)$: It is clear by [13, Corollary 5.10] ■

Example 3.21. *Let $\mathcal{F}_i = \mathcal{Z}_2$ and $Q = \prod_{i=1}^{\infty} \mathcal{F}_i$. Then Q is a regular $(Rad(R) = 0)$ commutative ring with unity via component-wise operations. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} \mathcal{F}_i$ and 1_Q. Then it can be seen that $\delta(R) = Soc(R) = \bigoplus_{i=1}^{\infty} \mathcal{F}_i$. Since $\frac{R}{Soc_\delta(R)} \cong \mathcal{F}_i$ is simple then ${}_RR$ is δ_{ss}-semilocal. On the other hand, R is not a semilocal ring as $\frac{R}{Rad(R)} \cong R$ is not semisimple. Hence, R is not an ss-semilocal ring.*

4. Acknowledgement

The author is thankful to the referee for his/her valuable suggestions which improved the presentation of the paper.

References

[1] R. ALIZADE AND A. PANCAR, Homoloji cebire giriş, *Ondokuz Mayıs University, Samsun-Turkey* (1999), 177p.

[2] E. BÜYÜKAŞIK AND C. LOMP, When $\delta-$semiperfect rings are semiperfect, *Turkish Journal of Mathematics* , **34(3)**(2010), 317–324.

[3] J. CLARK, C. LOMP, N. VANAJA AND R. WISBAUER, Lifting modules, supplements and projectivity in module theory, *Frontiers in Mathematics, Birkhauser, Verlag-Basel*, (2006).

[4] F. KASCH, Modules and rings, *London Mathematical Society Monographs 17, Academic Press, Inc. [Harcourt Brace Jonavich, Publishers], London-New York.*, **()**(1982).

[5] E. KAYNAR, H. ÇALIŞICI AND E. TÜRKMEN, Ss-supplemented modules, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat..*, **69(1)**(2020), 473–485.

[6] M.T. KOŞAN, $\delta-$lifting and $\delta-$supplemented modules, *Algebra Colloquium*, **14(1)**(2007), 53–60.

[7] C. LOMP, On semilocal modules and rings, *Communications in Algebra*, **27(4)**(1999), 1921–1935.

[8] K. NISHIDA, Bicommutators of locally projective modules, *Communications in Algebra,* **9(1)**(1981), 81–94.

[9] A. OLGUN AND E. TÜRKMEN, On a class of perfect rings, *Honam Mathematical Journal*, **42(3)**(2020), 591–600.

[10] Y. TALEBI AND B.TALAEE, On generalized $\delta-$supplemented modules, *Vietnam Journal of Mathematics*, **37(4)**(2009), 515–525.

[11] Y. TALEBI AND A.R.M HAMZEKOLAEI, Closed weak $\delta-$supplemented modules, *JP Journal of Algebra, Number Theory and Applications*, **13(2)**(2009), 193–208.

[12] R. TRIBAK, When finitely generated $\delta-$supplemented modules are supplemented, *Algebra Colloquium*, **22(1)**)(2015), 119–130.

[13] B. NIŞANCI TÜRKMEN AND E. TÜRKMEN, $\delta_{ss}-$supplemented modules and rings, *Analele St. Univ. Ovidius Constanta*, **28(3)**(2020), 193–216.

[14] R. WISBAUER, Foundations of module and ring theory, *Gordon and Breach, Reading*, (1991).

[15] Y. ZHOU, Generalizations of perfect, semiperfect and semiregular rings, *Algebra Colloquium*, **7(3)**(2000), 305–318.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 98–109.
http://doi.org/10.26637/mjm1001/009

On the radio antipodal geometric mean number of ladder related graphs

M. Giridaran*[1], **T. Arputha Jose**[2] **and E. Anto Jeony**[3]

[1] *Department of Mathematics, DMI-St. Eugene University, Lusaka, Zambia.*
[2] *Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Kalavakkam-603110, India.*
[3] *Department of Mathematics, Holy Cross College, Nagercoil, India.*

***R*eceived 30 May 2021; *A*ccepted 04 December 2021**

Abstract. Let $G(V, E)$ be a graph with vertex set V and edge set E. A radio geometric mean labeling of a connected graph G is a one to one map from the vertex set $V(G)$ to the set of natural numbers N such that for two distinct vertices u and v of G, $d(u, v) + \lceil\sqrt{f(u)f(v)}\rceil \geq 1 + diam(G)$, where $d(u, v)$ represents the shortest distance between the vertices u and v and $diam(G)$ represents the diameter of G . Based on the concept of radio geometric mean labeling, a new graph labeling called *radio antipodal geometric mean labeling* is being introduced in this paper. A radio antipodal geometric mean labeling of a graph G is a mapping from the vertex set $V(G)$ to the set of natural numbers N such that for two distinct vertices u and v of G, $d(u, v) + \lceil\sqrt{f(u)f(v)}\rceil \geq diam(G)$. If $d(u, v) = diam(G)$, then the vertices u and v can be given the same label and if $d(u, v) \neq diam(G)$ then the vertices u and v should be assigned different labels. The radio antipodal geometric mean number of f, $r_{agmn}(f)$ is the maximum number assigned to any vertex of G. The radio antipodal geometric mean number of G, $r_{agmn}(G)$ is the minimum value taken over all radio antipodal geometric mean labeling f of G. In this paper, the radio antipodal geometric mean number of certain ladder related graphs have been investigated.

AMS Subject Classifications: 05C12, 05C15, 05C78.

Keywords: Radio labeling, Ladder graph, Triangular ladder graph, Circular ladder graph, Pagoda graph.

Contents

1. Introduction

In this paper, the graphs considered are simple, finite and undirected graphs. For definitions not given here, one can refer to [6]. In communication engineering, one of the major problem is *channel* or *frequency assignment problem* where we have to assign frequencies(channels) to different radio transmitters in such a way that the interference between any two radio transmitter is avoided. That is if the radio transmitters are close to each other, then the difference between the channel assigned should be large enough [1]. This problem was converted into a graph coloring problem by William Hale in 1980 [20]. Later graph labeling techniques were also developed to solve this problem. The process of assigning integers to the vertices, edges or to both based on certain condition is known as *graph labeling* [14]. The first paper on graph labeling was presented by A Rosa in 1966 [18] and up

*****Corresponding author.** Email address: **iamgiridaran@gmail.com** (M. Giridaran)

to till date, there are lot of researches going on graph labeling. The main reason is that it has many applications. To list a few among them, graph labeling techniques are useful in coding theory, astronomy, circuit design, communication network addressing, secret sharing [7, 12].

In order to solve the channel assignment problem, the first graph labeling technique was introduced by Jerrold R. Griggs and Roger K. Yeh [11] in the year 1992, known as $L(2,1)$ labeling or distance two labeling. The $L(2,1)$ labeling was defined as follows. Given a real number $d > 0$, an $L_d(2,1)$ - labeling of G is a non-negative real-valued function $f : V(G) \to [0,\infty)$ such that, whenever x and y are two adjacent vertices in V, then $\mid f(x) - f(y) \mid \geq 2d$, and whenever the distance between x and y is 2, then $\mid f(x) - f(y) \mid \geq d$. In the year 2001, Gary Chartrand et al. [4] modified the definition of $L(2,1)$ labeling and introduced a new graph labeling technique called Radio Labeling which was just an extension of the existing $L(2,1)$ labeling. A *radio labeling* of a graph G is a function $f : V(G) \to N$(set of natural numbers) such that, $d(u,v) + |f(u) - f(v)| \geq diam(G) + 1$. It has been proved that finding the radio number of an arbitrary graph is an *NP-complete problem* [13]. Gary Chartrand et al. [5] have also introduced the concept of radio antipodal labeling in the year 2002. A *radio antipodal labeling* of a graph G is a function $f : V(G) \to N$(set of natural numbers) such that, $d(u,v) + |f(u) - f(v)| \geq diam(G)$. The difference between radio labeling and radio antipodal labeling is that the former one is an one to one function whereas the latter one is not since the vertices which are at diametric distance can receive the same label in the latter. From this there are few new graph labeling techniques which were defined by modifying the definition of existing radio and radio antipodal labeling. One can refer to [2, 3, 8, 15, 17, 19] for different types of labeling techniques which were originated from radio labeling and radio antipodal labeling.

The concept of *radio geometric mean labeling* of graphs was first introduced by Hemalatha V et al. [8] in the year 2017. The radio geometric mean labeling of a graph G is a mapping from the vertex set $V(G)$ to the set of natural numbers N such that for two distinct vertices u and v of G, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq 1 + diam(G)$. The radio geometric mean number of f, $r_{gmn}(f)$ is the maximum number assigned to any vertex of G. The radio geometric mean number of G, $r_{gmn}(G)$ is the minimum value taken over all radio geometric mean labeling f of G. In that work, the authors have studied the radio geometric mean number of some star like graphs [8]. They have also investigated the radio geometric mean number of splitting of star and bistar [9]. The radio geometric mean number of some subdivision graphs have been obtained by Hemalatha V and Mohanaselvi V [10]. Based on the concept of radio geometric mean labeling, a new graph labeling called *radio antipodal geometric mean labeling* have been introduced in this paper by modifying the existing radio geometric mean labeling condition. A radio antipodal geometric mean labeling of a graph G is a mapping from the vertex set $V(G)$ to the set of natural numbers N such that for two distinct vertices u and v of G, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq diam(G)$. If $d(u,v) = diam(G)$, then the vertices u and v can be given the same label and if $d(u,v) \neq diam(G)$ then the vertices u and v should be assigned different labels. The radio antipodal geometric mean number of f, $r_{agmn}(f)$ is the maximum number assigned to any vertex of G. The radio antipodal geometric mean number of G, $r_{agmn}(G)$ is the minimum value taken over all radio geometric mean labeling f of G, which will be denoted as $ragmn(G)$.

We were motivated to study the radio antipodal geometric mean number of ladder related graphs, since ladder and ladder related graphs have wide range of applications in various fields. To name a few, ladder networks have been useful in electronics, electrical and wireless communication networks [16].

In this paper, the upper bounds of radio antipodal geometric mean number of ladder related graphs have been investigated.

2. Radio antipodal geometric mean number of ladder and triangular ladder graphs

In this section, the radio antipodal geometric mean number of ladder and triangular ladder graph have been obtained.

Definition 2.1. *[14] The Ladder graph denoted by $LG(n)$, is a graph obtained by the Cartesian product of two path graphs P_2 and P_n, $n \geq 2$. The n^{th} dimension of a ladder graph has $2n$ vertices and $3n-2$ edges. The* *diameter of $LG(n)$ is n. See Figure 1.*

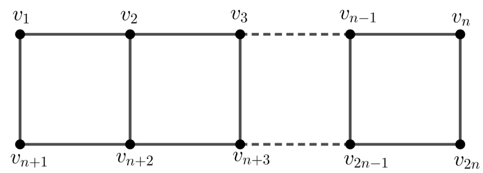

Figure 1: $LG(n)$

Definition 2.2. *[14] A Triangular ladder graph, denoted by $TLG(n)$, is a ladder graph obtained by adding the edges $(v_i, v_{n+i-1}), i = 2, 3, ..., n$. $TLG(n)$ has $2n$ vertices and its diameter is n. See Figure 2.*

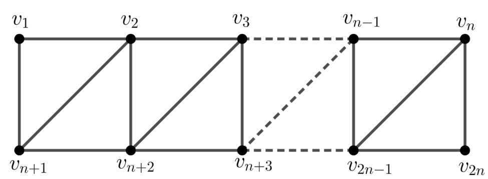

Figure 2: $TLG(n)$

Remark 2.3. *For our convenience, the vertex set of $LG(n)$ and $TLG(n)$ is partitioned into two disjoint sets V_1 and V_2, where $V_1 = \{v_i : 1 \leq i \leq n\}$ and $V_2 = \{v_i : n+1 \leq i \leq 2n\}$.*

Theorem 2.4. *The radio antipodal geometric mean number of ladder graph, $ragmn(LG(n)) \leq 3n - 6, n \geq 4$.*

Proof. Let $\{v_1, v_2, ..., v_n, v_{n+1}, ..., v_{2n}\}$ be the vertices of $LG(n)$.
In this vertex set, the vertices v_1 and v_{2n} are at diametric distance and hence they receive the same labeling. Therefore, $f(v_1) = f(v_{2n})$.
Similarly, the vertices v_n and v_{n+1} are at diametric distance and hence can be given same label, so that $f(v_n) = f(v_{n+1})$.
The remaining $2n - 2$ vertices of $LG(n)$ are labeled by the mapping,

$$f(v_i) = \begin{cases} n + i - 3, 1 \leq i \leq n - 2 \\ n - 3, i = n - 1 \\ 2n - 4, i = n \\ n + i - 5, n + 1 < i < 2n \end{cases} \tag{2.1}$$

Claim. The mapping (2.1) is a valid radio antipodal geometric mean labeling.
Let u, v be any two distinct vertices of $LG(n)$.
Case 1. Let $u, v \in V_1$.

Case 1.1. Let $u = v_i$ and $v = v_j, 1 \leq i, j \leq n-2$.
In this case, $d(u, v) \geq 1$.
By mapping (2.1), we have $f(v_i) = n+i-3$ and $f(v_j) = n+j-3$.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n+i-3)(n+j-3)} \rceil \geq n$.
Case 1.2. Let $u = v_i, 1 \leq i \leq n-2, v = v_{n-1}$.
By (2.1), we have $f(v_i) = n+i-3$ and $f(v_{n-1}) = n-3$.
Also, $d(u, v) \geq 1$.
This makes, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n+i-3)(n-3)} \rceil \geq n$.
Case 1.3. Suppose $u = v_i, 1 \leq i \leq n-2$ and $v = v_n$.
Here, $f(v_i) = n+i-3$ and $f(v_n) = 2n-4$. Also $d(u, v) \geq 2$.
Hence, $2 + \lceil \sqrt{(n+i-3)(2n-4)} \rceil \geq n$.
Case 1.4. If $u = v_{n-1}$ and $v = v_n$.
In this case, the distance between the vertices u and v will be 1.
Also, $f(v_{n-1}) = n-3$ and $f(v_n) = 2n-4$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 2. Let $u, v \in V_2$.
Case 2.1. Suppose $u = v_i$ and $v = v_j, n+2 \leq i, j \leq 2n-1$.
In this case, $d(u, v) \geq 1$.
By mapping (2.1), we have $f(v_i) = n+i-5$ and $f(v_j) = n+j-5$.
Consequently, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n+i-5)(n+j-5)} \rceil \geq n$.
Case 2.2. Let $u = v_{n+1}$ and $v = v_{2n}$.
Here, the distance between the vertices u and v will be $n-1$.
By mapping (2.1), we have $f(v_{n+1}) = 2n-4$ and $f(v_{2n}) = n-2$.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil > n$.
Case 3. Let $u \in V_1$ and $v \in V_2$.
Case 3.1. If $u = v_i, 1 \leq i \leq n-2$ and $v = v_j, n+1 < j < 2n$.
In this case, $d(u, v) \geq 1$.
Here by (2.1), we have $f(v_i) = n+i-3$ and $f(v_j) = n+j-5$.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n+i-3)(n+j-5)} \rceil \geq n$.
Case 3.2. If $u = v_n$ and $v = v_{n+1}$.
Here $d(u, v) = n$. Also, $f(u) = f(v) = 2n-4$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n + \lceil \sqrt{(2n-4)^2} \rceil > n$.
Case 3.3. Suppose $u = v_1$ and $v = v_{2n}$.
In this case, the distance between the vertices u and v will be n. As these two vertices are at diametric distance, $f(u) = f(v) = n-2$.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n + \lceil \sqrt{(n-2)^2} \rceil > n$.
Hence, in all the cases it can be seen that the mapping (2.1) satisfies the radio antipodal geometric mean labeling condition, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Therefore, (2.1) is a valid radio antipodal geometric mean labeling.
By the mapping (2.1) the vertex v_{2n-1} receives the maximum label which is given by,
$f(v_{2n-1}) = 3n-6$.
Hence, $ragmn(LG(n)) \leq 3n-6, n \geq 4$ ■

Theorem 2.5. *The radio antipodal geometric mean number of triangular ladder graph,* $ragmn(TLG(n)) \leq 3n-5, n \geq 5$.

Proof. Let $\{v_1, v_2, ..., v_n, v_{n+1}, ..., v_{2n}\}$ be the vertices of $TLG(n)$.
In these vertices v_1 and v_{2n} are at diametric distance and hence they receive the same labeling. Therefore, $f(v_1) = f(v_{2n})$.
The remaining $2n - 1$ vertices of $TLG(n)$ are labeled by the mapping,

$$f(v_i) = \begin{cases} n + i - 3, 1 \le i \le n - 2 \\ n - 3, i = n - 1 \\ 2n - 4, i = n \\ n + i - 4, n + 1 \le i < 2n \end{cases} \tag{2.2}$$

Claim. The mapping (2.2) is a valid radio antipodal geometric mean labeling.
Let u, v be any two vertices of $TLG(n)$.
Case 1. Let $u, v \in V_1$.
Case 1.1. Suppose $u = v_i$ and $v = v_j, 1 \le i, j \le n - 2$.
In this case, by (2.2), we have $f(v_i) = n + i - 3$ and $f(v_j) = n + j - 3$.
Also, $d(u, v) \ge 1$.
This assures, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \ge$
$1 + \lceil \sqrt{(n + i - 3)(n + j - 3)} \rceil \ge n$.
Case 1.2. Let $u = v_i, 1 \le i \le n - 2, v = v_{n-1}$.
In this case, $d(u, v) \ge 1$.
Also by (2.2), we have $f(v_i) = n + i - 3$ and $f(v_{n-1}) = n - 3$.
Consequently, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \ge$
$1 + \lceil \sqrt{(n + i - 3)(n - 3)} \rceil \ge n$.
Case 1.3. If $u = v_i, 1 \le i \le n - 2$ and $v = v_n$.
Here, $f(v_i) = n + i - 3$ and $f(v_n) = 2n - 4$. Also $d(u, v) \ge 2$.
Therefore, $2 + \lceil \sqrt{(n + i - 3)(2n - 4)} \rceil \ge n$.
Case 1.4. Let $u = v_{n-1}$ and $v = v_n$.
In this case, $d(u, v) = 1$.
Also, $f(v_{n-1}) = n - 3$ and $f(v_n) = 2n - 4$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \ge n$.
Case 2. Let $u, v \in V_2$.
Case 2.1. Suppose $u = v_i$ and $v = v_j, n + 1 \le i, j \le 2n - 1$.
Here, $d(u, v) \ge 1$.
By mapping (2.2), we have $f(v_i) = n + i - 4$ and $f(v_j) = n + j - 4$.
This gives, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \ge$
$1 + \lceil \sqrt{(n + i - 4)(n + j - 4)} \rceil \ge n$.
Case 2.2. Let $u = v_i, n + 1 \le i, j \le 2n - 1$ and $v = v_{2n}$.
By (2.2), we have $f(v_i) = n + i - 4$ and $f(v_{2n}) = n - 2$.
Also $d(u, v) \ge 1$.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \ge$
$1 + \lceil \sqrt{(n + i - 4)(n - 2)} \rceil \ge n$.
Case 3. Let $u \in V_1$ and $v \in V_2$.
Case 3.1. If $u = v_i, 1 \le i \le n - 2$ and $v = v_j, n + 1 \le j < 2n$.
In this case, the distance between the vertices u and v will be at least 1.
Here by (2.2), we have $f(v_i) = n + i - 3$ and $f(v_j) = n + j - 4$.
This assures, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \ge$
$1 + \lceil \sqrt{(n + i - 3)(n + j - 4)} \rceil \ge n$.
Case 3.2. If $u = v_n$ and $v = v_{n+1}$.
Here $d(u, v) = n - 1$. Also, $f(u) = 2n - 4$ and $f(v) = n + i - 4$.

Hence, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$(n-1) + \lceil \sqrt{(n+i-4)(2n-4)} \rceil > n.$
Case 3.3. Suppose $u = v_1$ and $v = v_{2n}$.
As these two vertices are at diametric distance,
$f(u) = f(v) = n-2.$
Here the distance between the vertices u and v will be n.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n + \lceil \sqrt{(n-2)^2} \rceil > n.$
Hence, in all the cases it can be seen that the mapping (2.2) satisfies the radio antipodal geometric mean labeling condition.
Therefore, (2.2) is a valid radio antipodal geometric mean labeling.
By the mapping (2.2), the vertex v_{2n-1} receive the maximum label and it is given by, $3n-5$.
Hence, $ragmn(TLG(n)) \leq 3n-5, n \geq 5$ ■

3. Radio antipodal geometric mean number of circular ladder and pagoda graphs

In this section, the radio antipodal geometric mean number of circular ladder and pagoda graphs have been investigated.

Definition 3.1. *[21] The circular ladder graph is a graph obtained from the Cartesian product $C_n \times K_2$, where K_2 is the complete graph on two vertices and C_n represents the cycle on n vertices. It is denoted by $CLG(n)$. The n^{th} dimension of $CLG(n)$ is shown in Figure 3.*

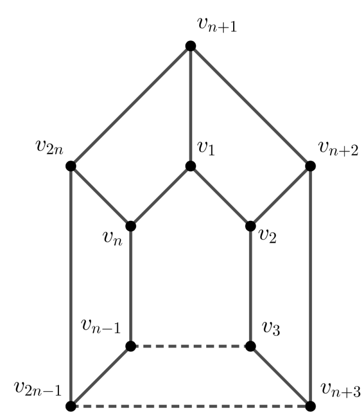

Figure 3: $CLG(n)$

Definition 3.2. *[14] A pagoda graph is a ladder graph formed by adding a vertex v_a in such a way that it is adjacent to the vertices v_1 and v_2. $PG(n)$ has $2n+1$ vertices and it's diameter is n. See Figure 4.*

Remark 3.3. *For our convenience, the vertices in the internal cycle $\{v_1, v_2, ..., v_n\}$ of $CLG(n)$ will be denoted as C_1 and the vertices in the outer cycle $\{v_{n+1}, v_{n+2}, ..., v_{2n}\}$ as C_2.*

Remark 3.4. *The vertex set of $PG(n)$ is partitioned into two disjoint sets V_1 and V_2, where $V_1 = \{v_{2i-1} : 1 \leq i \leq n\}$ and $V_2 = \{v_{2i} : 1 \leq i \leq n\}$.*

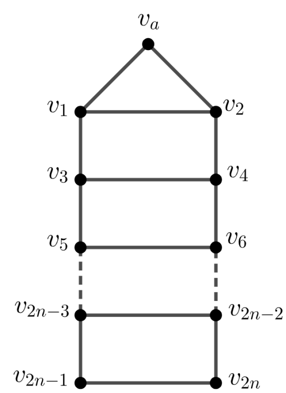

Figure 4: $PG(n)$

Theorem 3.5. *The radio antipodal geometric mean number of circular ladder graph,* $ragmn(CLG(n)) \leq 2n-3, n \equiv (1mod2), n \geq 5$.

Proof. The graph $CLG(n)$ has $2n$ vertices and $3n$ edges. In this $2n$ vertices there exists $\lceil \frac{n}{2} \rceil$ vertices which are at diametric distance and hence these vertices can receive the same label. These vertices are given by,
$f(v_i) = f(v_{n+\lceil \frac{n}{2} \rceil + i-1}), 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
The vertices of $CLG(n)$ are labeled by the mapping,

$$f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2, 1 \leq i \leq n + \lfloor \frac{n}{2} \rfloor \tag{3.1}$$

Claim. The mapping (3.1) is a valid radio antipodal geometric mean labeling.
Let u, v be any two distinct vertices of $CLG(n)$.
Case 1. Let $u, v \in C_1$.
In this case, $d(u, v) \geq 1$.
By mapping (3.1), we have $f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2$ and $f(v_j) = \lfloor \frac{n}{2} \rfloor + j - 2$.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(\lfloor \frac{n}{2} \rfloor + i - 2)(\lfloor \frac{n}{2} \rfloor + j - 2)} \rceil \geq d$.
Case 2. If the vertices $u, v \in C_2$.
Case 2.1. Let $u = v_i$ and $v = v_j, n+1 \leq i, j \leq n + \lfloor \frac{n}{2} \rfloor$.
Then, $f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2$ and $f(v_j) = \lfloor \frac{n}{2} \rfloor + j - 2$ by (3.1).
Also, $d(u, v) \geq 1$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 2.2. Suppose $u = v_i$ and $v = v_j, n + \lfloor \frac{n}{2} \rfloor \leq i, j \leq 2n$.
This case will be similar to Case 1.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 2.3. If $u = v_i, n+1 \leq n + \lfloor \frac{n}{2} \rfloor$ and $v = v_j, \leq n + \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq 2n$.
In this case, the distance between the vertices u and v will be at least 1.
By (3.1), we have $f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2$ and $f(v_j) = \lfloor \frac{n}{2} \rfloor + j - 2$.
This guarantees, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3. If $u \in C_1$ and $v \in C_2$.
Case 3.1. Suppose $u = v_i, 1 \leq i \leq n$ and $v = v_j, n+1 \leq j \leq n + \lfloor \frac{n}{2} \rfloor$.
In this case, $d(u, v) \geq 1$.

Also, $f(u) = \lfloor \frac{n}{2} \rfloor + i - 2$ and $f(v_i) = \lfloor \frac{n}{2} \rfloor + j - 2$.
This assures, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.2. Let $u = v_i, 1 \leq i \leq n$ and $v = v_i, n + \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq 2n$.
In this case, the vertices u and v will receive same labels as they are at diametric distance and hence $f(u_i) = f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2$.
This guarantees $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.

Accordingly, in all the cases it can be seen that the mapping (3.1) satisfies the radio antipodal geometric mean labeling condition, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Therefore, (3.1) is a valid radio antipodal geometric mean labeling.
By the mapping (3.1) the vertex $v_{n+\lfloor \frac{n}{2} \rfloor}$ receives the maximum label which is given by,
$f(v_{n+\lfloor \frac{n}{2} \rfloor}) = 2n - 3$.
Hence, $ragmn(CLG(n)) \leq 2n - 3, n \equiv (1 mod 2), n \geq 5$ ■

Remark 3.6. *It is easy to verify that* $ragmn(CLG(4)) = 4$ *and* $ragmn(CLG(6)) = 6$.

Theorem 3.7. *The radio antipodal geometric mean number of circular ladder graph,* $ragmn(CLG(n)) \leq 2n - 3, n \equiv (0 mod 2), n \geq 8$.

Proof. The graph $CLG(n)$ has $2n$ vertices out of which $\frac{n}{2}$ vertices are at diametric distance. Hence, these vertices can receive same label. These vertices are given as follows,
$f(v_i) = f(v_{n+\frac{n}{2}+i}), 1 \leq i \leq \frac{n}{2}$. The remaining vertices of $CLG(n)$ are labeled by the mapping:

$$f(v_i) = \begin{cases} \frac{n}{2} + i - 2, 1 \leq i \leq n - 2 \\ \frac{n}{2} - 2, i = n - 1 \\ n + \frac{n}{2} - 3, i = n \\ \frac{n}{2} + i - 3, n + 1 \leq i \leq n + \frac{n}{2} \end{cases} \quad (3.2)$$

We now claim that the mapping (3.2) is an valid radio antipodal geometric mean labeling.
Let u, v be any two distinct vertices of $CLG(n)$.
Case 1. If $u, v \in C_1$.
Case 1.1. Let $u = v_i, v = v_j, 1 \leq i, j \leq n - 2$.
In this context, by (3.2) we have $f(u) = \frac{n}{2} + i - 2$ and $f(v) = \frac{n}{2} + j - 2$. Also, $d(u, v) \geq 1$.
This makes, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 1.2. If $u = v_i, 1 \leq i \leq n - 2$ and $v = v_{n-1}$.
In this instance, by mapping (3.2) $f(u) = \frac{n}{2} + i - 2$ and $f(v_{n-1}) = \frac{n}{2} - 2$.
Further, $d(u, v) \geq 1$.
As a result, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 1.3. Let $u = v_{n-1}$ and $v = v_n$.
In this case, $d(u, v) = 1$.
Also by (3.2), we have $f(u) = \frac{n}{2} - 2$ and $f(v) = n + \frac{n}{2} - 3$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 1.4. If $u = v_i, 1 \leq i \leq n - 2$ and $v = v_n$.
In the considered case, $d(u, v) \geq 2$.
By (3.2), $f(u) = \frac{n}{2} + i - 2$ and $f(v) = n + \frac{n}{2} - 3$.
As a consequence of this, we have $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 2. Let $u, v \in C_2$.
Case 2.1. Suppose $u = v_i, v = v_j, n + 1 \leq i, j \leq n + \frac{n}{2}$.
By (3.2) we have $f(u) = \frac{n}{2} + i - 3$ and $f(v) = \frac{n}{2} + j - 3$.

Further more, $d(u,v) \geq 1$.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 2.2. If $u = v_i, v = v_j, n + \frac{n}{2} + 1 \leq i, j \leq 2n$.
This case will be similar to Case 1.1. and hence $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 2.3. If $u = v_i, n+1 \leq i \leq n + \frac{n}{2}$ and $v = v_j, n + \frac{n}{2} + 1 \leq j \leq 2n$.
In this case, the distance between the vertices u and v will be at least 1.
Also by (3.2), we have $f(u) = \frac{n}{2} + i - 3$ and $f(v) = \frac{n}{2} + j - 2$.
This assures $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3. Let $u \in C_1$ and $v \in C_2$.
Case 3.1. Suppose $u = v_i, 1 \leq i \leq n-2$ and $v = v_j, n+1 \leq j \leq n + \frac{n}{2}$.
In the situation under consideration, the distance between the vertices u and v will be at least 1.
By (3.2), $f(u) = \frac{n}{2} + i - 2$ and $f(v) = \frac{n}{2} + j - 3$.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.2. If $u = v_i, 1 \leq i \leq n-2$ and $v = v_j, n + \frac{n}{2} + 1 \leq j \leq 2n$.
This case will be similar to Case 1.1.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.3. Let $u = v_{n-1}$ and $v = v_j, n+1 \leq i \leq n + \frac{n}{2}$.
In this case, $f(v_{n-1}) = \frac{n}{2} - 2$ and $f(v) = \frac{n}{2} + i - 3$.
It can be seen that $d(u,v) \geq 3$.
Consequently, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.4. Let $u = v_{n-1}$ and $v = v_j, n + \frac{n}{2} + 1 \leq i \leq 2n$.
This case will be similar to Case 1.2. which guarantees $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.5. If $u = v_n$ and $v = v_j, n+1 \leq i \leq n + \frac{n}{2}$.
By (3.2), $f(v_n) = n + \frac{n}{2} - 3$ and $f(v) = \frac{n}{2} + i - 3$.
The distance between the vertices u and v will be at least 2.
Consequently, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.6. Suppose $u = v_n$ and $v = v_j, n + \frac{n}{2} + 1 \leq i \leq 2n$.
This case will be similar to Case 1.4. which assures that $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Hence the mapping (3.2) is an valid radio antipodal geometric mean labeling.
By (3.2), the vertex $v_{n+\frac{n}{2}}$ receives the maximum label which is given by $2n - 3$.
Therefore, $ragmn(CLG(n)) \leq 2n - 3, n \equiv (0 mod 2), n \geq 8$ ■

Theorem 3.8. *The radio antipodal geometric mean number of pagoda graph,* $ragmn(PG(n)) \leq 3n - 3, n \geq 3$.

Proof. Let $\{v_a, v_1, v_2, ..., v_n, v_{n+1}, ..., v_{2n}\}$ be the vertices of $PG(n)$.
Let $f(v_a) = n - 2$.
In these $2n+1$ vertices v_a and v_{2n} are at diametric distance and hence they receive the same labeling. Therefore, $f(v_a) = f(v_{2n})$.
The remaining $2n - 1$ vertices of $PG(n)$ are labeled by the mapping,

$$f(v_i) = n + i - 2, 1 \leq i < 2n. \tag{3.3}$$

Claim. The mapping (3.3) is a valid radio antipodal geometric mean labeling.
Let u, v be any two distinct vertices of $TLG(n)$.
Case 1. Let $u, v \in V_1$.
Case 1.1. If $u = v_i$ and $v = v_j, 1 \leq i, j \leq n$.
In this case, $d(u,v) \geq 1$.
By mapping (3.3), we have $f(v_i) = n + i - 2$ and $f(v_j) = n + j - 2$.
Thus, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 2. Let $u, v \in V_2$.

Case 2.1. Suppose $u = v_i$ and $v = v_j, 1 \leq i, j \leq n-1$.
In this case, $d(u, v) \geq 1$.
By mapping (3.3), we have $f(v_i) = n + i - 2$ and $f(v_j) = n + j - 2$.
Thus, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 2.2. Let $u = v_i, 1 \leq i \leq n-1$ and $v = v_{2n}$.
In this case, by (3.3), we have $f(v_i) = n + i - 2$ and $f(v_{2n}) = n - 2$.
Also, $d(u, v) \geq 1$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil > n$.
Case 3. Let $v_i \in V_1$ and $v_j \in V_2$.
Case 3.1. If $u = v_i, 1 \leq i \leq n$ and $v = v_j, 1 \leq j < n$.
In this case, $d(u, v) \geq 1$.
Here by (3.3), we have $f(v_i) = n + i - 2$ and $f(v_j) = n + j - 2$.
Thus, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n + i - 2)(n + j - 2)} \rceil \geq n$.
Case 3.2. If $u = v_a$ and $v = v_{2n}$.
Here $d(u, v) = n$.
Also, $f(u) = f(v) = n - 2$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 4. Suppose $u = v_a$ and $v \in V_1$ or $v \in V_2$.
We will have the following two sub cases:
Case 4.1 If $u = v_a$ and $v \in V_1$
By (3.3), $f(u) = n - 2$ and $f(v) = n + i - 2$.
Also, $d(u, v) \geq 1$.
This assures $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 4.2. Suppose $u = v_a$ and $v \in V_2$
Here, $d(u, v) \geq 1$.
By (3.3), $f(u) = n - 2$ and $f(v) = n + i - 2$.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n - 2)(n + i - 2)} \rceil \geq d$.
Hence, in all the cases it can be seen that the mapping (3.3) satisfies the radio antipodal geometric mean labeling condition.
Consequently, (3.3) is a valid radio antipodal geometric mean labeling.
By the mapping (3.3), the vertex v_{2n-1} receive the maximum label and the label is given by, $f(v_{2n-1}) = 3n - 3$.
Hence, $ragmn(PG(n)) \leq 3n - 3, n \geq 3$ ■

4. Conclusion

In this paper, a new graph labeling technique called radio antipodal geometric mean labeling have been introduced. By this technique the span of the given network can be minimized as the diametric opposite vertices can receive same labels. The upper bounds of ladder, triangular ladder, circular ladder and pagoda graphs have been investigated in this paper. This work can be extended further to other communication networks like honeycomb, butterfly, mesh.

References

[1] T. Arputha Jose, Daniel Raj, P. Venugopal and M. Giridaran, Radio Antipodal Mean Number of Quadrilateral Snake Families, *Indian Journal of Science and Technology*, **14(13)**(2021), 1071–1080.

[2] T. Arputha Jose and M. Giridaran, Radial Radio Mean Labeling of Mongolian Tent and Diamond Graphs, *International Journal for Research in Applied Science and Engineering Technology*, **8(7)**(2020), 2078–2083.

[3] T. Arputha Jose, P. Venugopal and M. Giridaran, Radio antipodal mean labeling of Triangular snake families, *Advances in Mathematics: A Scientific Journal*, **9(11)**(2020), 9739–9746.

[4] Gary Chartrand, David Erwin, P. Zhang and Frank Harary, Radio labeling's of graphs, *Bulletin of the Institute of Combinatorics and its Applications*, **33**(2001), 77–85.

[5] Gary Chartrand, David Erwin, and P. Zhang, Radio antipodal colorings of graphs, *Math. Bohem.*, **127**(2002), 57–69.

[6] Douglas Brent West, *Introduction to Graph Theory*, Upper Saddle River, Prentice Hall Ltd., 2001,

[7] M. Giridaran, Application of Super Magic Labeling in Cryptography, *International Journal of Innovative Research in Science, Engineering and Technology*, **9(6)**(2020), 4816–4822.

[8] V. Hemalatha, V. Mohanaselvi and K. Amuthavalli, Radio Geometric Mean Labeling of Some Star Like Graphs, *Journal of Informatics and Mathematical Sciences*, **9(3)**(2017), 969–977.

[9] V. Hemalatha, V. Mohanaselvi, K. Amuthavalli, Radio Geometric Mean Number of Splitting of Star and Bistar, *International Journal of Research in Advent Technology,* **6(7)**(2018), 1696–1700.

[10] V. Hemalatha, V. Mohanaselvi, Radio geometric mean number of some subdivision graphs, *International Journal of Pure and Applied Mathematics*, **113(6)**(2017), 362–374.

[11] Jerrold R. Griggs, Roger K. Yeh, Labelling graphs with a condition at distance 2, *SIAM Journal on Discrete Mathematics*, **5(4)**(1992), 586–595.

[12] Joseph A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of combinatorics*, **1**(2018).

[13] M. Kchikech, R. Khennoufa and O. Togni, Linear and cyclic radio k-labelings of trees, *Discussiones Mathematicae Graph Theory*, **27(1)**(2007), 105–123.

[14] Kins Yenoke, T. Arputha Jose and P. Venugopal, On the radio antipodal mean number of certain types of ladder graphs, *International Journal of Innovative Research in Science, Engineering and Technology*, **9(6)**(2020), 4607–4614.

[15] K. N. Meera, B. Sooryanarayana, Radio Secure number of a graph, *Electronic Notes in Discrete Mathematics*, **53**(2016), 271–286.

[16] M. I. Moussa, and E. M. Badr, Ladder and subdivision of ladder graphs with pendant edges are odd graceful, *Internat. J. Appl. Graph Theory Wireless Ad hoc Networks and Sensor Networks,(GRAPH-HOC),* **8(1)**(2016), 1–8.

[17] R. Ponraj, S. Sathish Narayanan and R. Kala, Radio mean labeling of a grah, *AKCE International Journal of Graphs and Combinatorics*, **12(2 - 3)**(2015), 224-228.

[18] A. Rosa, On certain valuations of the vertices of a graph, *In Theory of Graphs (Internat. Symposium, Rome)*, (1966), 349–355.

[19] Selvam Avadayappan, M. Bhuvaneshwari and S. Vimalajenifer, The Radial Radio Number and the Clique Number of a Graph, *International Journal of Engineering and Advanced Technology (IJEAT)*, **9**(2019), 1002–1006.

[20] WILLIAM K. HALE, Frequency Assignment: Theory and Applications, *Proceedings of the IEEE*, **68**(1980), 1497–1514.

[21] YICHAO CHEN, JONATHAN L. GROSS, AND TOUFIK MANSOUR, Total embedding distributions of circular ladders, *Journal of Graph Theory*, **74(1)**(2013), 32–57.