ISSN 2319-3786
VOLUME 11, ISSUE S, OCTOBER 2023

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Malaya Journal of Matematik

an international journal of mathematical sciences

MKD PUBLISHING HOUSE
Dawn To Researchers
5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),
Coimbatore- 641114, Tamil Nadu, India.
www.mkdpress.com | www.malayajournal.org

The Malaya Journal of Matematik is published quarterly in single volume annually and four issues constitute one volume appearing in the months of January, April, July and October.

Subscription

The subscription fee is as follows:

USD 350.00 For USA and Canada

Euro 190.00 For rest of the world

Rs. 4000.00 In India. (For Indian Institutions in India only)

Prices are inclusive of handling and postage; and issues will be delivered by Registered Air-Mail for subscribers outside India.

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Coimbatore- 641114, Tamil Nadu, India.

Contact No. : +91-9585408402

E-mail : info@mkdpress.com; editorinchief@malayajournal.org; publishingeditor@malayajournal.org

Website : https://mkdpress.com/index.php/index/index

Vol. 11 No. S (2023): Malaya Journal of Matematik (MJM): Special Issue Dedicated to Professor Gaston M. N'Guérékata's 70th Birthday

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 1–26.
http://doi.org/10.26637/mjm11S/001

Opial and Ostrowski type inequalities based on trigonometric and hyperbolic type Taylor formulae

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*R*eceived 22 March 2023; *A*ccepted 17 June 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. Based on [4], we produce a collection of interesting trigonometric and hyperbolic Taylor formulae with integral remainders. Using these we derive Opial and Ostrowski type corresponding inequalities of various kinds and norms.

AMS Subject Classifications: 26A24, 26D10, 26D15.

Keywords: Trigonometric and hyperbolic Taylor formula, Opial inequality, Ostrowski inequality.

Contents

1. Taylor formulae based on linear differential operators

This section is based entirely on [4]. Here K denotes $\mathbb{R}$ or $\mathbb{C}$.

Let I be an interval subset of $\mathbb{R}$, and $c = (c_0, ..., c_n) \in \mathbb{R}^{n+1}$, $c_n = 1$, $n \in \mathbb{N}$, and the n-th order linear differential operator D_c from $C_K^n(I)$ (n-times continuously differentiable K-valued functions defined on I) into $C_K(I)$ (continuous functions from I to K), where $D_c(f) := c_n f^{(n)} + ... + c_1 f' + c_0 f$, with $f \in C_K^n(I)$. Let $\omega_c \in C_{\mathbb{C}}^n(\mathbb{R})$ be the unique solution of initial value problem:

$$D_c(\omega_c) = 0, \;\; \omega_c^{(i)}(0) = \delta_{l,n-1} \;\; (l \in \{0, 1, ..., n-1\}).$$

By Theorem 3.2 of [4], p. 1131, we have that

$$f(x) = (T_{a,c} f)(x) + \int_a^x D_c(f)(t)\,\omega_c(x-t)\,dt, \tag{1.1}$$

where

$$(T_{a,c} f)(x) := \sum_{j=0}^{n-1} \left(f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(x-a) \right), \tag{1.2}$$

for all $x, a \in I$.

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Next, let $k \in \mathbb{N}_0$, $k < n$, $\gamma \in \mathbb{R}$, and we consider $\mathcal{J}_{n,k,\gamma}$ from $\mathbb{R}$ into $\mathbb{R}$, given by

$$\mathcal{J}_{n,k,\gamma}(t) := \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!}, \tag{1.3}$$

which converges for all t.

A further interpretation of (1.1), see Theorem 3.3, p. 1132 of [4], given us

$$f(x) = \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \sum_{j=k}^{n-1} f^{(j)}(a) \mathcal{J}_{n,k,\gamma}^{(n-j)}(x-a)$$
$$+ \int_a^x \left(f^{(n)}(t) - \gamma f^{(k)}(t) \right) \mathcal{J}'_{n,k,\gamma}(x-t)\, dt, \tag{1.4}$$

for all $f \in C_K^n(I)$, $n \in \mathbb{N}$ and $x, a \in I$.

Applications of (1.4) provide us the following interesting Taylor formulae of trigonometric and hyperbolic types:

Theorem 1.1. *([4]) For all $f \in C_K^2(I)$ and $a, x \in I$, we have*

$$f(x) = f(a)\cos(x-a) + f'(a)\sin(x-a) + \int_a^x (f''(t) + f(t))\sin(x-t)\, dt. \tag{1.5}$$

Theorem 1.2. *([4]) For all $f \in C_K^2(I)$ and $a, x \in I$, we have*

$$f(x) = f(a)\cosh(x-a) + f'(a)\sinh(x-a) + \int_a^x (f''(t) - f(t))\sinh(x-t)\, dt. \tag{1.6}$$

Theorem 1.3. *([4]) For all $f \in C_K^4(I)$ and $a, x \in I$, we have*

$$f(x) = f(a)\left(\frac{\cosh(x-a) + \cos(x-a)}{2}\right) + f'(a)\left(\frac{\sinh(x-a) + \sin(x-a)}{2}\right)$$
$$+ f''(a)\left(\frac{\cosh(x-a) - \cos(x-a)}{2}\right) + f'''(a)\left(\frac{\sinh(x-a) - \sin(x-a)}{2}\right)$$
$$+ \int_a^x \left(f''''(t) - f(t)\right)\left(\frac{\sinh(x-t) - \sin(x-t)}{2}\right) dt. \tag{1.7}$$

Theorem 1.4. *([4]) Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then, for all $f \in C_K^4(I)$ and $a, x \in I$, we have*

$$f(x) = f(a)\left(\frac{\beta^2\cos(\alpha(x-a)) - \alpha^2\cos(\beta(x-a))}{\beta^2 - \alpha^2}\right) +$$
$$f'(a)\left(\frac{\beta^3\sin(\alpha(x-a)) - \alpha^3\sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)}\right) +$$
$$f''(a)\left(\frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2 - \alpha^2}\right) +$$
$$f'''(a)\left(\frac{\beta\sin(\alpha(x-a)) - \alpha\sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)}\right) +$$
$$\int_a^x \left(f''''(t) + \left(\alpha^2 + \beta^2\right) f''(t) + \alpha^2\beta^2 f(t)\right)\left(\frac{\beta\sin(\alpha(x-t)) - \alpha\sin(\beta(x-t))}{\alpha\beta(\beta^2 - \alpha^2)}\right) dt. \tag{1.8}$$

Finally, we include the following result.

Theorem 1.5. *([4]) Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right) \neq 0$. Then, for all $f \in C_K^4(I)$ and $a, x \in I$, we have*

$$f(x)=f(a)\left(\frac{\beta^2 \cosh (\alpha(x-a))-\alpha^2 \cosh (\beta(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'(a)\left(\frac{\beta^3 \sinh (\alpha(x-a))-\alpha^3 \sinh (\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$
$$f''(a)\left(\frac{\cosh (\beta(x-a))-\cosh (\alpha(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'''(a)\left(\frac{\alpha \sinh (\beta(x-a))-\beta \sinh (\alpha(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$
$$\int_a^x\left(f''''(t)-\left(\alpha^2+\beta^2\right) f''(t)+\alpha^2\beta^2 f(t)\right)\left(\frac{\alpha \sinh (\beta(x-t))-\beta \sinh (\alpha(x-t))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right) dt. \tag{1.9}$$

In this article first we give the needed variants to Theorems 1.1-1.5, then based on these modified Taylor formulae we derive several Ostrowski type inequalities. In between we deal with Opial type inequalities.

2. Main Results

We are inspired by [1]-[3].

We give the following Taylor formula results of trigonometric and hyperbolic types.

Theorem 2.1. *For all $f \in C_K^2(I)$ and $a, x \in I$, we have*

$$f(x)-f(a)=f'(a)\sin(x-a)+2f''(a)\sin^2\left(\frac{x-a}{2}\right)+$$
$$\int_a^x\left[\left(f''(t)+f(t)\right)-\left(f''(a)+f(a)\right)\right]\sin(x-t)\,dt=$$
$$f'(a)\sin(x-a)+2f''(a)\sin^2\left(\frac{x-a}{2}\right)+ \tag{2.1}$$
$$\int_a^x\left[\left(f''(t)-f''(a)\right)+\left(f(t)-f(a)\right)\right]\sin(x-t)\,dt.$$

Proof. Here we use Theorem 1.1.

We have by (1.5) that

$$f(x)-f(a)=f(a)(\cos(x-a)-1)+f'(a)\sin(x-a)$$
$$+\int_a^x\left(f''(t)+f(t)\right)\sin(x-t)\,dt. \tag{2.2}$$

(By $\cos 2x=1-2\sin^2 x$, $\cos 2x-1=-2\sin^2 x$, and $\cos(x-a)-1=\cos 2\left(\frac{x-a}{2}\right)-1=-2\sin^2\left(\frac{x-a}{2}\right)$.)

Therefore it holds

$$f(x)-f(a)=-2f(a)\sin^2\left(\frac{x-a}{2}\right)+f'(a)\sin(x-a)$$
$$+\int_a^x\left(f''(t)+f(t)\right)\sin(x-t)\,dt= \tag{2.3}$$

$$-2f(a)\sin^2\left(\frac{x-a}{2}\right)+f'(a)\sin(x-a)+(f''(a)+f(a))(1-\cos(x-a))$$

$$+\int_a^x [(f''(t)+f(t))-(f''(a)+f(a))]\sin(x-t)\,dt$$

$$=-2f(a)\sin^2\left(\frac{x-a}{2}\right)+f'(a)\sin(x-a)+(f''(a)+f(a))\,2\sin^2\left(\frac{x-a}{2}\right)+ \quad (2.4)$$

$$\int_a^x [(f''(t)+f(t))-(f''(a)+f(a))]\sin(x-t)\,dt$$

$$=f'(a)\sin(x-a)+2f''(a)\sin^2\left(\frac{x-a}{2}\right)+$$

$$\int_a^x [(f''(t)+f(t))-(f''(a)+f(a))]\sin(x-t)\,dt.$$

Above notice that $(\cos(x-t))'=\sin(x-t)$.

The claim is proved. ■

Theorem 2.2. *For all $f\in C_K^2(I)$ and $a,x\in I$, we have*

$$f(x)-f(a)=f'(a)\sinh(x-a)+2f''(a)\sinh^2\left(\frac{x-a}{2}\right)+$$

$$\int_a^x [(f''(t)-f(t))-(f''(a)-f(a))]\sinh(x-t)\,dt=$$

$$f'(a)\sinh(x-a)+2f''(a)\sinh^2\left(\frac{x-a}{2}\right)+ \quad (2.5)$$

$$\int_a^x [(f''(t)-f''(a))-(f(t)-f(a))]\sinh(x-t)\,dt.$$

Proof. Here we use Theorem 1.2 (1.6).

Notice that $(-\cosh(x-t))'=\sinh(x-t)$.

By $\cosh 2x-1=2\sinh^2 x$, we get that $\cosh(x-a)-1=\cosh 2\left(\frac{x-a}{2}\right)-1=2\sinh^2\left(\frac{x-a}{2}\right)$.

By (1.6) we obtain

$$f(x)-f(a)=f(a)(\cosh(x-a)-1)+f'(a)\sinh(x-a)$$

$$+\int_a^x (f''(t)-f(t))\sinh(x-t)\,dt=$$

$$2f(a)\sinh^2\left(\frac{x-a}{2}\right)+f'(a)\sinh(x-a) \quad (2.6)$$

$$+\int_a^x (f''(t)-f(t))\sinh(x-t)\,dt=$$

$$2f(a)\sinh^2\left(\frac{x-a}{2}\right)+f'(a)\sinh(x-a)+(f''(a)-f(a))(\cosh(x-a)-1)$$

$$+\int_a^x [(f''(t)-f(t))-(f''(a)-f(a))]\sinh(x-t)\,dt$$

$$=2f(a)\sinh^2\left(\frac{x-a}{2}\right)+f'(a)\sinh(x-a)+(f''(a)-f(a))\left(2\sinh^2\left(\frac{x-a}{2}\right)\right)+ \quad (2.7)$$

$$\int_a^x [(f''(t) - f(t)) - (f''(a) - f(a))] \sinh(x-t)\, dt$$

$$= f'(a)\sinh(x-a) + 2f''(a)\sinh^2\left(\frac{x-a}{2}\right) +$$

$$\int_a^x [(f''(t) - f(t)) - (f''(a) - f(a))] \sinh(x-t)\, dt.$$

The claim is proved. ■

Theorem 2.3. *For all $f \in C_K^4(I)$ and $a, x \in I$, we have*

$$f(x) - f(a) = f'(a)\left(\frac{\sinh(x-a) + \sin(x-a)}{2}\right) +$$

$$f''(a)\left(\frac{\cosh(x-a) - \cos(x-a)}{2}\right) +$$

$$f'''(a)\left(\frac{\sinh(x-a) - \sin(x-a)}{2}\right) +$$

$$f''''(a)\left(\sinh^2\left(\frac{x-a}{2}\right) - \sin^2\left(\frac{x-a}{2}\right)\right) + \tag{2.8}$$

$$\int_a^x [(f''''(t) - f''''(a)) - (f(t) - f(a))]\left(\frac{\sinh(x-t) - \sin(x-t)}{2}\right) dt.$$

Proof. Here we use Theorem 1.3 (1.7).

We have that

$$f(x) - f(a) = f(a)\left(\frac{(\cosh(x-a) - 1) + (\cos(x-a) - 1)}{2}\right) +$$

$$f'(a)\left(\frac{\sinh(x-a) + \sin(x-a)}{2}\right) +$$

$$f''(a)\left(\frac{\cosh(x-a) - \cos(x-a)}{2}\right) + f'''(a)\left(\frac{\sinh(x-a) - \sin(x-a)}{2}\right) \tag{2.9}$$

$$+ \int_a^x (f''''(t) - f(t))\left(\frac{\sinh(x-t) - \sin(x-t)}{2}\right) dt =$$

$$f(a)\left(\sinh^2\left(\frac{x-a}{2}\right) - \sin^2\left(\frac{x-a}{2}\right)\right) +$$

$$f'(a)\left(\frac{\sinh(x-a) + \sin(x-a)}{2}\right) + \tag{2.10}$$

$$f''(a)\left(\frac{\cosh(x-a) - \cos(x-a)}{2}\right) + f'''(a)\left(\frac{\sinh(x-a) - \sin(x-a)}{2}\right)$$

$$+ (f''''(a) - f(a))\left(\sinh^2\left(\frac{x-a}{2}\right) - \sin^2\left(\frac{x-a}{2}\right)\right) +$$

$$\int_a^x [(f''''(t) - f(t)) - (f''''(a) - f(a))]\left(\frac{\sinh(x-t) - \sin(x-t)}{2}\right) dt =$$

$$f'(a)\left(\frac{\sinh(x-a) + \sin(x-a)}{2}\right) +$$

$$f''(a)\left(\frac{\cosh(x-a)-\cos(x-a)}{2}\right)+f'''(a)\left(\frac{\sinh(x-a)-\sin(x-a)}{2}\right)$$
$$+f''''(a)\left(\sinh^2\left(\frac{x-a}{2}\right)-\sin^2\left(\frac{x-a}{2}\right)\right)+ \tag{2.11}$$
$$\int_a^x\left[\left(f''''(t)-f''''(a)\right)-\left(f(t)-f(a)\right)\right]\left(\frac{\sinh(x-t)-\sin(x-t)}{2}\right)dt.$$

The claim is proved. ∎

We continue with

Theorem 2.4. *Let $\alpha,\beta\in\mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$. Then, for all $f\in C_K^4(I)$ and $a,x\in I$, we have*

$$f(x)-f(a)=f'(a)\left(\frac{\beta^3\sin(\alpha(x-a))-\alpha^3\sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$
$$f''(a)\left(\frac{\cos(\alpha(x-a))-\cos(\beta(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'''(a)\left(\frac{\beta\sin(\alpha(x-a))-\alpha\sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$
$$\frac{2\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\left(\beta^2\sin^2\left(\frac{\alpha(x-a)}{2}\right)-\alpha^2\sin^2\left(\frac{\beta(x-a)}{2}\right)\right)+ \tag{2.12}$$
$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\int_a^x\left[\left(f''''(t)+\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2f(t)\right)-\right.$$
$$\left.\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2f(a)\right)\right]\left[\beta\sin(\alpha(x-t))-\alpha\sin(\beta(x-t))\right]dt.$$

Proof. We see that

$$I^*:=f(a)\left(\frac{\beta^2\left(\cos(\alpha(x-a))-1\right)-\alpha^2\left(\cos(\beta(x-a))-1\right)}{\beta^2-\alpha^2}\right)=$$
$$f(a)\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\beta^2-\alpha^2\left(\cos(\beta(x-a))\right)+\alpha^2}{\beta^2-\alpha^2}\right)=$$
$$f(a)\left[\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\alpha^2\left(\cos(\beta(x-a))\right)}{\beta^2-\alpha^2}\right)-\left(\frac{\beta^2-\alpha^2}{\beta^2-\alpha^2}\right)\right]=$$
$$f(a)\left[\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\alpha^2\left(\cos(\beta(x-a))\right)}{\beta^2-\alpha^2}\right)-1\right]= \tag{2.13}$$
$$f(a)\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\alpha^2\left(\cos(\beta(x-a))\right)}{\beta^2-\alpha^2}\right)-f(a).$$

That is

$$I^*=f(a)\left(\frac{-2\beta^2\sin^2\left(\frac{\alpha(x-a)}{2}\right)+2\alpha^2\sin^2\left(\frac{\beta(x-a)}{2}\right)}{\beta^2-\alpha^2}\right)= \tag{2.14}$$
$$f(a)\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\alpha^2\left(\cos(\beta(x-a))\right)}{\beta^2-\alpha^2}\right)-f(a).$$

By Theorem 1.4, we obtain

$$f(x) - f(a) = f(a)\left(\frac{-2\beta^2 \sin^2\left(\frac{\alpha(x-a)}{2}\right) + 2\alpha^2 \sin^2\left(\frac{\beta(x-a)}{2}\right)}{\beta^2 - \alpha^2}\right) +$$

$$f'(a)\left(\frac{\beta^3 \sin(\alpha(x-a)) - \alpha^3 \sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right) +$$

$$f''(a)\left(\frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2-\alpha^2}\right) +$$

$$f'''(a)\left(\frac{\beta \sin(\alpha(x-a)) - \alpha \sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right) + \tag{2.15}$$

$$\frac{\left(f''''(a) + \left(\alpha^2+\beta^2\right) f''(a) + \alpha^2\beta^2 f(a)\right)}{(\beta^2-\alpha^2)}$$

$$\left(\frac{2\beta}{\alpha(\alpha\beta)} \sin^2\left(\frac{\alpha(x-a)}{2}\right) - \frac{2\alpha}{\beta(\alpha\beta)} \sin^2\left(\frac{\beta(x-a)}{2}\right)\right) +$$

$$\frac{1}{\alpha\beta(\beta^2-\alpha^2)} \int_a^x \left[\left(f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t)\right) -\right.$$

$$\left.\left(f''''(a) + \left(\alpha^2+\beta^2\right) f''(a) + \alpha^2\beta^2 f(a)\right)\right] \left[\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t))\right] dt.$$

Notice that

$$(\cos(\alpha(x-t)))' = -\sin(\alpha(x-t))(-\alpha) = \alpha \sin(\alpha(x-t)), \tag{2.16}$$

and

$$\left(\frac{1}{\alpha}\cos(\alpha(x-t))\right)' = \sin(\alpha(x-t)), \tag{2.17}$$

and

$$\int_a^x \beta(\sin(\alpha(x-t)))\, dt = \frac{\beta}{\alpha}(\cos(\alpha(x-t)))\Big|_a^x = \frac{\beta}{\alpha}(1 - \cos(\alpha(x-a)))$$

$$= \frac{2\beta}{\alpha}\left(\sin^2\left(\frac{\alpha(x-a)}{2}\right)\right). \tag{2.18}$$

Similarly, we get

$$-\int_a^x \alpha \sin(\beta(x-t))\, dt = -\frac{2\alpha}{\beta}\left(\sin^2\left(\frac{\beta(x-a)}{2}\right)\right). \tag{2.19}$$

Furthermore, we see that

$$\frac{\alpha^2\beta^2 f(a)}{(\beta^2-\alpha^2)}\left(\frac{2}{\alpha^2}\sin^2\left(\frac{\alpha(x-a)}{2}\right) - \frac{2}{\beta^2}\sin^2\left(\frac{\beta(x-a)}{2}\right)\right) =$$

$$\frac{f(a)}{(\beta^2-\alpha^2)}\left(2\beta^2\sin^2\left(\frac{\alpha(x-a)}{2}\right) - 2\alpha^2\sin^2\left(\frac{\beta(x-a)}{2}\right)\right). \tag{2.20}$$

The claim is proved. ■

We give

Theorem 2.5. *Let* $\alpha, \beta \in \mathbb{R}$ *and* $\alpha\beta\left(\alpha^2-\beta^2\right) \neq 0$. *Then, for all* $f \in C_K^4(I)$ *and* $a, x \in I$, *we have*

$$f(x)-f(a)=f'(a)\left(\frac{\beta^3\sinh(\alpha(x-a))-\alpha^3\sinh(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right)+$$
$$f''(a)\left(\frac{\cosh(\beta(x-a))-\cosh(\alpha(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'''(a)\left(\frac{\alpha\sinh(\beta(x-a))-\beta\sinh(\alpha(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right)+ \tag{2.21}$$
$$\frac{2\left(f''''(a)-\left(\alpha^2+\beta^2\right)f''(a)\right)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\left(\alpha^2\sinh^2\left(\frac{\beta(x-a)}{2}\right)-\beta^2\sin^2\left(\frac{\alpha(x-a)}{2}\right)\right)+$$
$$\frac{1}{\alpha\beta(\beta^2-\alpha^2)}\int_a^x\left[\left(f''''(t)-\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2f(t)\right)-\right.$$
$$\left.\left(f''''(a)-\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2f(a)\right)\right]\left[\alpha\sinh(\beta(x-t))-\beta\sinh(\alpha(x-t))\right]dt.$$

Proof. We see that

$$J:=f(a)\left(\frac{\beta^2(\cosh(\alpha(x-a))-1)-\alpha^2(\cosh(\beta(x-a))-1)}{\beta^2-\alpha^2}\right)= \tag{2.22}$$
$$f(a)\left(\frac{\beta^2(\cosh(\alpha(x-a)))-\alpha^2(\cosh(\beta(x-a)))}{\beta^2-\alpha^2}\right)-f(a).$$

Hence

$$J=f(a)\left(\frac{2\beta^2\sinh^2\left(\frac{\alpha(x-a)}{2}\right)-2\alpha^2\sinh^2\left(\frac{\beta(x-a)}{2}\right)}{\beta^2-\alpha^2}\right)= \tag{2.23}$$
$$f(a)\left(\frac{\beta^2\cosh(\alpha(x-a))-\alpha^2\cosh(\beta(x-a))}{\beta^2-\alpha^2}\right)-f(a).$$

By Theorem 1.5, we obtain

$$f(x)-f(a)=f(a)\left(\frac{2\beta^2\sinh^2\left(\frac{\alpha(x-a)}{2}\right)-2\alpha^2\sinh^2\left(\frac{\beta(x-a)}{2}\right)}{\beta^2-\alpha^2}\right)+$$
$$f'(a)\left(\frac{\beta^3\sinh(\alpha(x-a))-\alpha^3\sinh(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right)+$$
$$f''(a)\left(\frac{\cosh(\beta(x-a))-\cosh(\alpha(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'''(a)\left(\frac{\alpha\sinh(\beta(x-a))-\beta\sinh(\alpha(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right)+ \tag{2.24}$$
$$\frac{\left(f''''(a)-\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2f(a)\right)}{(\beta^2-\alpha^2)}$$
$$\left(\frac{2\alpha}{\beta(\alpha\beta)}\sinh^2\left(\frac{\beta(x-a)}{2}\right)-\frac{2\beta}{\alpha(\alpha\beta)}\sinh^2\left(\frac{\alpha(x-a)}{2}\right)\right)+$$
$$\frac{1}{\alpha\beta(\beta^2-\alpha^2)}\int_a^x\left[\left(f''''(t)-\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2f(t)\right)-\right.$$

$$\left(f''''(a)-\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2 f(a)\right)\big]\left(\alpha\sinh\left(\beta\left(x-t\right)\right)-\beta\sinh\left(\alpha\left(x-t\right)\right)\right)dt.$$

Notice that

$$\begin{gathered}\left(\cosh\left(\beta\left(x-t\right)\right)\right)'=\sinh\left(\beta\left(x-t\right)\right)\left(\beta x-\beta t\right)'=\\ \sinh\left(\beta\left(x-t\right)\right)\left(-\beta\right)=-\beta\sinh\left(\beta\left(x-t\right)\right),\end{gathered}\tag{2.25}$$

that is

$$\left(-\frac{1}{\beta}\cosh\left(\beta\left(x-t\right)\right)\right)'=\sinh\left(\beta\left(x-t\right)\right),\tag{2.26}$$

and

$$\left(-\frac{\alpha}{\beta}\cosh\left(\beta\left(x-t\right)\right)\right)'=\alpha\sinh\left(\beta\left(x-t\right)\right).\tag{2.27}$$

Thus, it holds

$$\begin{gathered}\int_a^x\alpha\left(\sinh\left(\beta\left(x-t\right)\right)\right)dt=-\frac{\alpha}{\beta}\left(\cosh\left(\beta\left(x-t\right)\right)\right)|_a^x=\\ -\frac{\alpha}{\beta}\left(1-\cosh\left(\beta\left(x-a\right)\right)\right)=-\frac{\alpha}{\beta}\left(-2\right)\sinh^2\left(\frac{\beta\left(x-a\right)}{2}\right)=\frac{2\alpha}{\beta}\sinh^2\left(\frac{\beta\left(x-a\right)}{2}\right).\end{gathered}\tag{2.28}$$

I.e.

$$\int_a^x\alpha\sinh\left(\beta\left(x-t\right)\right)dt=\frac{2\alpha}{\beta}\sinh^2\left(\frac{\beta\left(x-a\right)}{2}\right),\tag{2.29}$$

and

$$\int_a^x\left(-\beta\right)\sinh\left(\alpha\left(x-t\right)\right)dt=-\frac{2\beta}{\alpha}\sinh^2\left(\frac{\alpha\left(x-a\right)}{2}\right).\tag{2.30}$$

Furthermore

$$\begin{gathered}\frac{\alpha^2\beta^2 f(a)}{\left(\beta^2-\alpha^2\right)}\left(\frac{2}{\beta^2}\sinh^2\left(\frac{\beta\left(x-a\right)}{2}\right)-\frac{2}{\alpha^2}\sinh^2\left(\frac{\alpha\left(x-a\right)}{2}\right)\right)=\\ \frac{f(a)}{\left(\beta^2-\alpha^2\right)}\left(2\alpha^2\sinh^2\left(\frac{\beta\left(x-a\right)}{2}\right)-2\beta^2\sinh^2\left(\frac{\alpha\left(x-a\right)}{2}\right)\right).\end{gathered}\tag{2.31}$$

The claim is proved. ■

Next come Opial type inequalilies, for basics see [2].

Theorem 2.6. *Let $f\in C_K^2(I)$, with interval $I\subset\mathbb{R}$, $a,x\in I$, $a<x$, and $f(a)=f'(a)=0$, with $p,q>1$: $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\begin{gathered}\int_a^x\left|f(w)\right|\left|f''(w)+f(w)\right|dw\leq\\ 2^{-\frac{1}{q}}\left(\int_a^x\left(\int_a^w\left|\sin\left(w-t\right)\right|^p dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x\left|f''(w)+f(w)\right|^q dw\right)^{\frac{2}{q}}.\end{gathered}\tag{2.32}$$

Proof. By Theorem 1.1 we have

$$\begin{aligned}&f(x)=\textstyle\int_a^x\left(f''(t)+f(t)\right)\sin\left(x-t\right)dt,\\ &\text{and}\\ &f(w)=\textstyle\int_a^w\left(f''(t)+f(t)\right)\sin\left(w-t\right)dt,\end{aligned}\tag{2.33}$$

for $a\leq w\leq x$.

By Hölder's inequality we have

$$\left|f(w)\right|\leq\int_a^w\left|f''(t)+f(t)\right|\left|\sin\left(w-t\right)\right|dt\leq$$

$$\left(\int_a^w |\sin(w-t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^w |f''(t)+f(t)|^q dt\right)^{\frac{1}{q}}. \tag{2.34}$$

Call

$$z(w) := \int_a^w |f''(t)+f(t)|^q dt, \ z(a)=0, \ a \le w \le x. \tag{2.35}$$

Then

$$z'(w) = |f''(w)+f(w)|^q, \tag{2.36}$$

and

$$|f''(w)+f(w)| = (z'(w))^{\frac{1}{q}}, \ \text{ all } a \le w \le x.$$

Therefore we have (all $a \le w \le x$)

$$|f(w)||f''(w)+f(w)| \le$$

$$\left(\int_a^w |\sin(w-t)|^p dt\right)^{\frac{1}{p}} (z(w)z'(w))^{\frac{1}{q}}. \tag{2.37}$$

hence it holds

$$\int_a^x |f(w)||f''(w)+f(w)| dw \le$$

$$\int_a^x \left(\int_a^w |\sin(w-t)|^p dt\right)^{\frac{1}{p}} (z(w)z'(w))^{\frac{1}{q}} dw \le$$

$$\left(\int_a^x \left(\int_a^w |\sin(w-t)|^p dt\right) dw\right)^{\frac{1}{p}} \left(\int_a^x z(w)z'(w) dw\right)^{\frac{1}{q}} = \tag{2.38}$$

$$\left(\int_a^x \left(\int_a^w |\sin(w-t)|^p dt\right) dw\right)^{\frac{1}{p}} \left(\int_a^x z(w) dz(w)\right)^{\frac{1}{q}} =$$

$$\left(\int_a^x \left(\int_a^w |\sin(w-t)|^p dt\right) dw\right)^{\frac{1}{p}} \frac{z(x)^{\frac{2}{q}}}{2^{\frac{1}{q}}} =$$

$$\left(\int_a^x \left(\int_a^w |\sin(w-t)|^p dt\right) dw\right)^{\frac{1}{p}} \frac{\left(\int_a^x |f''(t)+f(t)|^q dt\right)^{\frac{2}{q}}}{2^{\frac{1}{q}}}. \tag{2.39}$$

The claim is proved. ■

Next come several similar results.

Theorem 2.7. *Let $f \in C_K^2(I)$, $a, x \in I$, $a < x$, and $f(a) = f'(a) = 0$, with $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^x |f(w)||f''(w)-f(w)| dw \le$$

$$2^{-\frac{1}{q}} \left(\int_a^x \left(\int_a^w |\sinh(w-t)|^p dt\right) dw\right)^{\frac{1}{p}} \left(\int_a^x |f''(w)-f(w)|^q dw\right)^{\frac{2}{q}}. \tag{2.40}$$

Proof. By Theorem 2.6, use of (1.6). ■

Theorem 2.8. *Let $f \in C_K^4(I)$, interval $I \subset \mathbb{R}$, let $a, x \in I$, $a < x$, $f(a) = f'(a) = f''(a) = f'''(a) = 0$, with $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^x |f(w)| \left| f^{(iv)}(w) - f(w) \right| dw \leq$$

$$2^{-\left(1+\frac{1}{q}\right)} \left(\int_a^x \left(\int_a^w |\sinh(w-t) - \sin(w-t)|^p \, dt \right) dw \right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{(iv)}(w) - f(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{2.41}$$

Proof. As in Theorem 2.6, use of (1.7). ■

Theorem 2.9. *All as in Theorem 2.8. Let $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) + \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}} \alpha\beta\left(\beta^2 - \alpha^2\right)} \left(\int_a^x \left(\int_a^w |\beta \sin(\alpha(w-t)) - \alpha \sin(\beta(w-t))|^p \, dt \right) dw \right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{(4)}(w) + \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{2.42}$$

Proof. As in Theorem 2.6, use of (1.8). ■

Theorem 2.10. *All as in Theorem 2.8. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) + 2\alpha^2 f''(w) + \alpha^4 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}+1}\alpha^3} \left(\int_a^x \left(\int_a^w |\sin(\alpha(w-t)) - \alpha(w-t)\cos(\alpha(w-t))|^p \, dt \right) dw \right)^{\frac{1}{p}} \tag{2.43}$$

$$\left(\int_a^x \left| f^{(4)}(w) + 2\alpha^2 f''(w) + \alpha^4 f(w) \right|^q dw \right)^{\frac{2}{q}}.$$

Proof. As in Theorem 2.6, use of Corollary 3.8 of [4], p. 1135. ■

Theorem 2.11. *All as in Theorem 2.9. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) - \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}} \alpha\beta\left(\beta^2 - \alpha^2\right)} \left(\int_a^x \left(\int_a^w |\alpha \sinh(\beta(w-t)) - \beta \sinh(\alpha(w-t))|^p \, dt \right) dw \right)^{\frac{1}{p}} \tag{2.44}$$

$$\left(\int_a^x \left| f^{(4)}(w) - \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right|^q dw \right)^{\frac{2}{q}}.$$

Proof. As in Theorem 2.6, use of (1.9). ■

Theorem 2.12. *All as in Theorem 2.10. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) - 2\alpha^2 f''(w) + \alpha^4 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}+1}\alpha^3} \left(\int_a^x \left(\int_a^w |\alpha(w-t)\cosh(\alpha(w-t)) - \sinh(\alpha(w-t))|^p \, dt \right) dw \right)^{\frac{1}{p}} \tag{2.45}$$

$$\left(\int_a^x \left| f^{(4)}(w) - 2\alpha^2 f''(w) + \alpha^4 f(w) \right|^q dw \right)^{\frac{2}{q}}.$$

Proof. As in Theorem 2.6, use of Corollary 3.10 of [4], p. 1135. ∎

We finish Opial type inequalities with the following general result.

Theorem 2.13. *Let $n \in \mathbb{N}$, $c = (c_0, ..., c_n) \in K^{n+1}$ with $c_n = 1$, $f \in C_K^n(I)$ and $x, a \in I$, $a < x$, with interval $I \subset \mathbb{R}$. Let, also $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume further that $f^{(j)}(a) = 0$, $j = 0, 1, ..., n-1$. Then*

$$\int_a^x |f(w)| |D_c(f)(w)| \, dw \leq \tag{2.46}$$

$$2^{-\frac{1}{q}} \left(\int_a^x \left(\int_a^w |\omega_c(w-t)|^p \, dt \right) dw \right)^{\frac{1}{p}} \left(\int_a^x |D_c(f)(w)|^q \, dw \right)^{\frac{2}{q}}.$$

Proof. By Theorem 3.2 of [4], p. 1131, for $x, a \in I$, we have

$$f(x) = (T_{a,c}f)(x) + \int_a^x D_c(f)(t)\, \omega_c(x-t)\, dt, \tag{2.47}$$

where

$$(T_{a,c}f)(x) := \sum_{j=0}^{n-1} \left(f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(x-a) \right). \tag{2.48}$$

Because $f^{(j)}(a) = 0$, $j = 0, 1, ..., n-1$, we get

$$f(x) = \int_a^x D_c(f)(t)\, \omega_c(x-t)\, dt. \tag{2.49}$$

The rest of the proof as similar to Theorem 2.6 is omitted. ∎

Next we present Ostrowski type inequalities involving $\|\cdot\|_\infty$. For basics see [2].

Theorem 2.14. *Let $f \in C_K^3([c,d])$, $a \in [c,d]$, such that $f'(a) = f''(a) = 0$. Then*

1)

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f(a) \right| \leq \|f''' + f'\|_\infty \frac{\left[(d-a)^3 + (a-c)^3 \right]}{6(d-c)}. \tag{2.50}$$

2) When $f'\left(\frac{c+d}{2}\right) = f''\left(\frac{c+d}{2}\right) = 0$, and $a = \frac{c+d}{2}$, we get

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f\left(\frac{c+d}{2}\right) \right| \leq \|f' + f'''\|_\infty \frac{(d-c)^2}{24}. \tag{2.51}$$

Proof. By Theorem 2.1 (2.1) we have

$$f(x) - f(a) = \int_a^x \left[(f''(t) + f(t)) - (f''(a) + f(a)) \right] \sin(x-t)\, dt. \tag{2.52}$$

Let $x \geq a$, then

$$|f(x) - f(a)| \leq \int_a^x |(f''(t) + f(t)) - (f''(a) + f(a))| \, |\sin(x-t)|\, dt$$

$$\leq \int_a^x |(f''(t) + f(t)) - (f''(a) + f(a))|\, dt$$

$$\leq \|f''' + f'\|_\infty \int_a^x (t-a)\, dt = \|f''' + f'\|_\infty \frac{(x-a)^2}{2}. \tag{2.53}$$

Let $x < a$, then

$$-(f(x) - f(a)) = \int_x^a \left[(f''(a) + f(a)) - (f''(t) + f(t)) \right] \sin(x-t)\, dt.$$

Hence

$$|f(x) - f(a)| \leq \int_x^a |(f''(a) + f(a)) - (f''(t) + f(t))| \, |\sin(x-t)|\, dt$$

$$\leq \|f''' + f'\|_\infty \int_x^a (a-t)\, dt = \|f''' + f'\|_\infty \frac{(a-t)^2}{2} |_a^x = \|f''' + f'\|_\infty \frac{(a-x)^2}{2}. \tag{2.54}$$

Therefore

$$|f(x) - f(a)| \leq \frac{\|f''' + f'\|_\infty}{2} (x-a)^2, \tag{2.55}$$

$\forall\, x \in [c, d]$.

We observe that

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f(a) \right| = \left| \frac{1}{d-c} \int_c^d (f(x) - f(a))\, dx \right| \leq$$

$$\frac{1}{d-c} \int_c^d |f(x) - f(a)|\, dx \leq \frac{1}{d-c} \left(\frac{\|f''' + f'\|_\infty}{2} \right) \int_c^d (x-a)^2\, dx =$$

$$\frac{\|f''' + f'\|_\infty}{2(d-c)} \left[\int_c^a (a-x)^2\, dx + \int_a^d (x-a)^2\, dx \right] =$$

$$\frac{\|f''' + f'\|_\infty}{6(d-c)} \left[(a-x)^3 |_a^c + (x-a)^3 |_a^d \right] = \tag{2.56}$$

$$\frac{\|f''' + f'\|_\infty}{6(d-c)} \left[(a-c)^3 + (d-a)^3 \right] =$$

$$\left(\frac{\|f''' + f'\|_\infty}{6(d-c)} \right) \left[(d-a)^3 + (a-c)^3 \right].$$

The claim is proved. ■

Using $|\sin x| \leq |x|$, we obtain an alternative Ostrowski type inequality.

Theorem 2.15. *All as in Theorem 2.14. Then*

1)

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \|f''' + f'\|_\infty \frac{\left[(d-a)^4 + (a-c)^4\right]}{24(d-c)}; \tag{2.57}$$

2) When $f'\left(\frac{c+d}{2}\right) = f''\left(\frac{c+d}{2}\right) = 0$, $a = \frac{c+d}{2}$, *we get that*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f\left(\frac{c+d}{2}\right)\right| \leq \|f' + f'''\|_\infty \frac{(d-c)^3}{192}. \tag{2.58}$$

Proof. Again by Theorem 2.1 (2.1) we have

$$f(x) - f(a) = \int_a^x \left[(f''(t) + f(t)) - (f''(a) + f(a))\right] \sin(x-t)\,dt. \tag{2.59}$$

Let $x \geq a$, then

$$|f(x) - f(a)| \leq \int_a^x \left|(f''(t) + f(t)) - (f''(a) + f(a))\right| |\sin(x-t)|\,dt$$

$$\leq \|f''' + f'\|_\infty \int_a^x (t-a)\,|\sin(x-t)|\,dt$$

$$\leq \|f''' + f'\|_\infty \int_a^x (x-t)^{2-1}(t-a)^{2-1}\,dt \tag{2.60}$$

$$\|f''' + f'\|_\infty \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}(x-a)^3 = \|f''' + f'\|_\infty \frac{(x-a)^3}{3!}.$$

That is

$$|f(x) - f(a)| \leq \|f''' + f'\|_\infty \frac{(x-a)^3}{3!}. \tag{2.61}$$

Let $x < a$, then

$$-(f(x) - f(a)) = \int_x^a \left[(f''(a) + f(a)) - (f''(t) + f(t))\right] \sin(x-t)\,dt. \tag{2.62}$$

Hence

$$|f(x) - f(a)| \leq \int_x^a \left|(f''(a) + f(a)) - (f''(t) + f(t))\right| |\sin(x-t)|\,dt$$

$$\leq \|f''' + f'\|_\infty \int_x^a (a-t)\,|x-t|\,dt = \|f''' + f'\|_\infty \int_x^a (a-t)^{2-1}(t-x)^{2-1}\,dt \tag{2.63}$$

$$= \|f''' + f'\|_\infty \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}(a-x)^3 = \|f''' + f'\|_\infty \frac{(a-x)^3}{3!}.$$

That is

$$|f(x) - f(a)| \leq \|f''' + f'\|_\infty \frac{(a-x)^3}{3!}. \tag{2.64}$$

Next we observe that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \frac{1}{d-c}\int_c^d |f(x) - f(a)|\,dx$$

$$\leq \frac{\|f''' + f'\|_\infty}{3!\,(d-c)} \left[\int_c^a (a-x)^3\, dx + \int_a^d (x-a)^3\, dx \right]$$

$$= \frac{\|f''' + f'\|_\infty}{4!\,(d-c)} \left[(a-c)^4 + (d-a)^4 \right]. \tag{2.65}$$

The claim is proved. ■

We continue with more involved Ostrowski type inequalities.

Theorem 2.16. *Let $f \in C_K^5([c,d])$, $a, x \in [c,d]$, and assume that $f^{(i)}(a) = 0$, $i = 1,2,3,4$. Here $\alpha, \beta \in \mathbb{R}$: $\alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then*

1)

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f(a) \right| \leq \tag{2.66}$$

$$\frac{(|\alpha| + |\beta|) \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty}{6\,|\alpha\beta\,(\beta^2 - \alpha^2)|\,(d-c)} \left[(d-a)^3 + (a-c)^3 \right],$$

2) When $f^{(i)}\left(\frac{c+d}{2}\right) = 0$, $i = 1,2,3,4$, we have:

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f\left(\frac{c+d}{2}\right) \right| \leq$$

$$\frac{(|\alpha| + |\beta|) \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty}{24\,|\alpha\beta\,(\beta^2 - \alpha^2)|} (d-c)^2. \tag{2.67}$$

Example 2.17. *(to (2.66)) Let $\alpha = 1$, $\beta = 10$, then*

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f(a) \right| \leq \tag{2.68}$$

$$\frac{\left\| f^{(5)} + 101 f^{(3)} + 100 f' \right\|_\infty}{540\,(d-c)} \left[(d-a)^3 + (a-c)^3 \right].$$

Example 2.18. *(to (2.67)) When $\alpha = 2$, $\beta = 1$, we get*

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f\left(\frac{c+d}{2}\right) \right| \leq \frac{\left\| f^{(5)} + 5 f^{(3)} + 4 f' \right\|_\infty}{48} (d-c)^2. \tag{2.69}$$

Proof. (of Theorem 2.16) By Theorem 2.4 (2.12) we have:

$$f(x) - f(a) = \frac{1}{\alpha\beta\,(\beta^2 - \alpha^2)}$$

$$\int_a^x \left[\left(f''''(t) + \left(\alpha^2 + \beta^2\right) f''(t) + \alpha^2\beta^2 f(t) \right) - \right.$$

$$\left. \left(f''''(a) + \left(\alpha^2 + \beta^2\right) f''(a) + \alpha^2\beta^2 f(a) \right) \right] \left[\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t)) \right] dt. \tag{2.70}$$

Let $x \geq a$, then

$$|f(x) - f(a)| \leq$$

$$\frac{(|\alpha| + |\beta|)}{|\alpha\beta\,(\beta^2 - \alpha^2)|} \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty \int_a^x (t-a)\, dt =$$

$$\frac{(|\alpha|+|\beta|)}{2\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2f'\right\|_{\infty}(x-a)^2. \tag{2.71}$$

Let $x<a$, then

$$-(f(x)-f(a))=\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}$$

$$\int_x^a\left[\left(f''''(t)+\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2f(t)\right)-\right.$$

$$\left.\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2f(a)\right)\right][\beta\sin(\alpha(x-t))-\alpha\sin(\beta(x-t))]\,dt. \tag{2.72}$$

Hence we have

$$|f(x)-f(a)|\leq$$

$$\frac{(|\alpha|+|\beta|)}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2f'\right\|_{\infty}\int_x^a(a-t)\,dt=$$

$$\frac{(|\alpha|+|\beta|)}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2f'\right\|_{\infty}\frac{(a-x)^2}{2}. \tag{2.73}$$

Therefore it holds

$$|f(x)-f(a)|\leq\frac{(|\alpha|+|\beta|)\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2f'\right\|_{\infty}}{2\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}(x-a)^2, \tag{2.74}$$

$\forall\, x\in[c,d]$.

We observe that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx-f(a)\right|\leq\frac{1}{d-c}\int_c^d|f(x)-f(a)|\,dx\leq$$

$$\frac{(|\alpha|+|\beta|)\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2f'\right\|_{\infty}}{2\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|(d-c)}\int_c^d(x-a)^2\,dx=$$

$$\frac{(|\alpha|+|\beta|)\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2f'\right\|_{\infty}}{6\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|(d-c)}\left[(a-c)^3+(d-a)^3\right]. \tag{2.75}$$

The claim is proved. ∎

A long alternative Ostrowski type inequality follows.

Theorem 2.19. *Let $f\in C_K^5([c,d])$, $a,x\in[c,d]$, and assume that $f^{(i)}(a)=0$, $i=1,2,3,4$. Here $\alpha,\beta\in\mathbb{R}$: $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$. Then*

1)

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx-f(a)\right|\leq \tag{2.76}$$

$$\frac{\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2f'\right\|_{\infty}}{12\left|\beta^2-\alpha^2\right|(d-c)}\left[(d-a)^4+(a-c)^4\right].$$

2) Above let $a=\frac{c+d}{2}$, then

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx-f\left(\frac{c+d}{2}\right)\right|\leq$$

$$\frac{\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2f'\right\|_{\infty}}{96\left|\beta^2-\alpha^2\right|}(d-c)^3. \tag{2.77}$$

Example 2.20. *(to (2.76)) Let* $\alpha = 1$, $\beta = 10$, *then*

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f(a) \right| \leq \tag{2.78}$$

$$\frac{\left\| f^{(5)} + 101 f^{(3)} + 100 f' \right\|_\infty}{1188\,(d-c)} \left[(d-a)^4 + (a-c)^4 \right].$$

Example 2.21. *(to (2.77)) When* $\alpha = 2$, $\beta = 1$, *we get:*

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f\left(\frac{c+d}{2} \right) \right| \leq \frac{\left\| f^{(5)} + 5 f^{(3)} + 4 f' \right\|_\infty}{288} (d-c)^3. \tag{2.79}$$

Proof. (of Theorem 2.19) By Theorem 2.4 (2.12) we have:

$$f(x) - f(a) = \frac{1}{\alpha\beta\left(\beta^2 - \alpha^2\right)}$$

$$\int_a^x \left[\left(f''''(t) + \left(\alpha^2 + \beta^2\right) f''(t) + \alpha^2\beta^2 f(t) \right) - \right. \tag{2.80}$$

$$\left. \left(f''''(a) + \left(\alpha^2 + \beta^2\right) f''(a) + \alpha^2\beta^2 f(a) \right) \right] \left[\beta \sin\left(\alpha(x-t)\right) - \alpha \sin\left(\beta(x-t)\right) \right] dt.$$

Let $x \geq a$ (by $|\sin x| \leq |x|$), then

$$|f(x) - f(a)| \leq \frac{1}{\left|\alpha\beta\left(\beta^2 - \alpha^2\right)\right|} \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty$$

$$\int_a^x (t-a) \left[|\beta| \left|\alpha(x-t)\right| + |\alpha| \left|\beta(x-t)\right| \right] dt \tag{2.81}$$

$$= \frac{2|\alpha\beta|}{\left|\alpha\beta\left(\beta^2 - \alpha^2\right)\right|} \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty \int_a^x (x-t)(t-a)\,dt$$

$$= \frac{2}{\left|\beta^2 - \alpha^2\right|} \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty \frac{(x-a)^3}{3!} \tag{2.82}$$

$$= \frac{1}{3\left|\beta^2 - \alpha^2\right|} \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty (x-a)^3.$$

So, when $x \geq a$, we get that

$$|f(x) - f(a)| \leq \frac{1}{3\left|\beta^2 - \alpha^2\right|} \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty (x-a)^3. \tag{2.83}$$

Let $x < a$, then

$$-\left(f(x) - f(a)\right) = \frac{1}{\alpha\beta\left(\beta^2 - \alpha^2\right)}$$

$$\int_x^a \left[\left(f''''(t) + \left(\alpha^2 + \beta^2\right) f''(t) + \alpha^2\beta^2 f(t) \right) - \right. \tag{2.84}$$

$$\left. \left(f''''(a) + \left(\alpha^2 + \beta^2\right) f''(a) + \alpha^2\beta^2 f(a) \right) \right] \left[\beta \sin\left(\alpha(x-t)\right) - \alpha \sin\left(\beta(x-t)\right) \right] dt.$$

Hence (by $|\sin x| \leq |x|$), we get that

$$|f(x) - f(a)| \leq \frac{1}{\left|\alpha\beta\left(\beta^2 - \alpha^2\right)\right|} \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty$$

$$\int_x^a (a-t)\left[|\beta|\,|\alpha|\,(t-x)+|\alpha|\,|\beta|\,(t-x)\right]dt$$

$$= \frac{2}{|\beta^2-\alpha^2|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty \int_x^a (a-t)(t-x)\,dt \tag{2.85}$$

$$= \frac{2}{|\beta^2-\alpha^2|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty \frac{(a-x)^3}{3!}$$

$$= \frac{1}{3|\beta^2-\alpha^2|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty (a-x)^3 .$$

So, when $x<a$, we got that

$$|f(x)-f(a)| \leq \frac{1}{3|\beta^2-\alpha^2|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty (a-x)^3 . \tag{2.86}$$

We observe that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \frac{1}{d-c}\int_c^d |f(x)-f(a)|\,dx \leq$$

$$\frac{1}{3|\beta^2-\alpha^2|(d-c)}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty$$

$$\left[\int_c^a (a-x)^3\,dx+\int_a^d (x-a)^3\,dx\right] = \tag{2.87}$$

$$\frac{\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty}{12|\beta^2-\alpha^2|(d-c)}\left[(d-a)^4+(a-c)^4\right].$$

The claim is proved. ■

More Ostrowski type inequalities follow regarding $\|\cdot\|_p$, $p\geq 1$.

Theorem 2.22. *Let $f\in C_K^2([c,d])$, $a\in[c,d]$, and assume that $f'(a)=f''(a)=0$. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{\left[(a-c)\|f''+f-f(a)\|_{L_1([c,a])}+(d-a)\|f''+f-f(a)\|_{L_1([a,d])}\right]}{d-c}. \tag{2.88}$$

Proof. As in the proof of Theorem 2.14 (2.53):

let $x\geq a$, then

$$|f(x)-f(a)| \leq \int_a^x |f''(t)+f(t)-f(a)|\,dt \leq$$

$$\int_a^d |f''(t)+f(t)-f(a)|\,dt = \|f''+f-f(a)\|_{L_1([a,d])} . \tag{2.89}$$

Next let $x<a$, by (2.54) we get that

$$|f(x)-f(a)| \leq \int_x^a |f(a)-f''(t)-f(t)|\,dt \leq$$

$$\int_c^a |f''(t)+f(t)-f(a)|\,dt = \|f''+f-f(a)\|_{L_1([c,a])} . \tag{2.90}$$

Hence it holds

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| = \frac{1}{d-c}\left|\int_c^d (f(x)-f(a))\,dx\right| \leq$$

$$\frac{1}{d-c}\int_c^d |f(x)-f(a)|\,dx =$$

$$\frac{1}{d-c}\left[\int_c^a |f(x)-f(a)|\,dx + \int_a^d |f(x)-f(a)|\,dx\right] \leq$$

$$\frac{1}{d-c}\left[\|f''+f-f(a)\|_{L_1([c,a])}(a-c) + \|f''+f-f(a)\|_{L_1([a,d])}(d-a)\right], \tag{2.91}$$

proving the claim. ■

Theorem 2.23. *All as in Theorem 2.22. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{\left[(a-c)^2\|f''+f-f(a)\|_{L_1([c,a])} + (d-a)^2\|f''+f-f(a)\|_{L_1([a,d])}\right]}{2(d-c)}. \tag{2.92}$$

Proof. As in (2.60) for $x \geq a$, we get

$$|f(x)-f(a)| \leq \int_a^x |(f''(t)+f(t))-f(a)|\,|\sin(x-t)|dt \leq$$

$$\int_a^d |(f''(t)+f(t))-f(a)|\,(x-t)\,dt \leq (x-a)\|f''+f-f(a)\|_{L_1([a,d])}. \tag{2.93}$$

Also, as in (2.63) for $x < a$, we have

$$|f(x)-f(a)| \leq \int_x^a |f(a)-f''(t)-f(t)|\,|\sin(x-t)|dt \leq$$

$$\int_x^a |f(a)-f''(t)-f(t)|\,(t-x)\,dt \leq (a-x)\|f''+f-f(a)\|_{L_1([c,a])}. \tag{2.94}$$

Hence it holds (see (2.91))

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x)-f(a)|\,dx + \int_a^d |f(x)-f(a)|\,dx\right] \leq$$

$$\frac{1}{d-c}\left[\|f''+f-f(a)\|_{L_1([c,a])}\int_c^a (a-x)\,dx + \right.$$

$$\left.\|f''+f-f(a)\|_{L_1([a,d])}\int_a^d (x-a)\,dx\right] = \tag{2.95}$$

$$\frac{1}{d-c}\left[\|f''+f-f(a)\|_{L_1([c,a])}\frac{(a-c)^2}{2} + \|f''+f-f(a)\|_{L_1([a,d])}\frac{(d-a)^2}{2}\right],$$

proving the claim. ■

Theorem 2.24. *All as in Theorem 2.22. Let also* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f(a) \right| \leq$$

$$\frac{\left[\|f'' + f - f(a)\|_{L_p([c,a])} (a-c)^{1+\frac{1}{q}} + \|f'' + f - f(a)\|_{L_p([a,d])} (d-a)^{1+\frac{1}{q}} \right]}{\left(1 + \frac{1}{q}\right)(d-c)}. \tag{2.96}$$

Proof. Let $x \geq a$, as in (2.89), we get

$$|f(x) - f(a)| \leq \int_a^x |(f''(t) + f(t)) - f(a)|\, dt \leq$$

$$\left(\int_a^x |f''(t) + f(t) - f(a)|^p\, dt \right)^{\frac{1}{p}} \left(\int_a^x 1 dt \right)^{\frac{1}{q}} \leq$$

$$\|f'' + f - f(a)\|_{L_p([a,d])} (x-a)^{\frac{1}{q}}. \tag{2.97}$$

Let $x < a$, as in (2.90), we get

$$|f(x) - f(a)| \leq \int_x^a |f''(t) + f(t) - f(a)|\, dt \leq$$

$$\left(\int_x^a |f''(t) + f(t) - f(a)|^p\, dt \right)^{\frac{1}{p}} \left(\int_x^a 1 dt \right)^{\frac{1}{q}} \leq$$

$$\|f'' + f - f(a)\|_{L_p([c,a])} (a-x)^{\frac{1}{q}}. \tag{2.98}$$

Acting as in (2.91) we obtain

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f(a) \right| \leq$$

$$\frac{1}{d-c} \left[\int_c^a |f(x) - f(a)|\, dx + \int_a^d |f(x) - f(a)|\, dx \right] \leq$$

$$\frac{1}{d-c} \left[\|f'' + f - f(a)\|_{L_p([c,a])} \int_c^a (a-x)^{\frac{1}{q}}\, dx + \right.$$

$$\left. \|f'' + f - f(a)\|_{L_p([a,d])} \int_a^d (x-a)^{\frac{1}{q}}\, dx \right] =$$

$$\frac{1}{d-c} \left[\|f'' + f - f(a)\|_{L_p([c,a])} \frac{(a-c)^{1+\frac{1}{q}}}{\left(1 + \frac{1}{q}\right)} + \right.$$

$$\left. \|f'' + f - f(a)\|_{L_p([a,d])} \frac{(d-a)^{1+\frac{1}{q}}}{\left(1 + \frac{1}{q}\right)} \right], \tag{2.99}$$

proving the claim. ■

Theorem 2.25. *All as in Theorem 2.24. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{\left[(a-c)^{2+\frac{1}{q}}\left\|f''+f-f(a)\right\|_{L_p([c,a])}+(d-a)^{2+\frac{1}{q}}\left\|f''+f-f(a)\right\|_{L_p([a,d])}\right]}{(q+1)^{\frac{1}{q}}\left(2+\frac{1}{q}\right)(d-c)}. \tag{2.100}$$

Proof. Let $x \geq a$, as in (2.93), we get

$$|f(x)-f(a)| \leq \int_a^x |(f''(t)+f(t))-f(a)|\,(x-t)\,dt \leq$$

$$\left(\int_a^x |f''(t)+f(t)-f(a)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^x (x-t)^q\,dt\right)^{\frac{1}{q}} \leq$$

$$\left\|f''+f-f(a)\right\|_{L_p([a,d])}\frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}. \tag{2.101}$$

Also, when $x < a$, by (2.94) we obtain

$$|f(x)-f(a)| \leq \int_x^a |f''(t)+f(t)-f(a)|\,(t-x)\,dt \leq$$

$$\left(\int_x^a |f''(t)+f(t)-f(a)|^p\,dt\right)^{\frac{1}{p}}\left(\int_x^a (t-x)^q\,dt\right)^{\frac{1}{q}} \leq$$

$$\left\|f''+f-f(a)\right\|_{L_p([c,a])}\frac{(a-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}. \tag{2.102}$$

As in (2.95) we derive

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x)-f(a)|\,dx + \int_a^d |f(x)-f(a)|\,dx\right] \leq$$

$$\frac{1}{d-c}\left[\left\|f''+f-f(a)\right\|_{L_p([c,a])}\frac{\int_c^a (a-x)^{\frac{q+1}{q}}\,dx}{(q+1)^{\frac{1}{q}}}+\right.$$

$$\left.\left\|f''+f-f(a)\right\|_{L_p([a,d])}\frac{\int_a^d (x-a)^{\frac{q+1}{q}}\,dx}{(q+1)^{\frac{1}{q}}}\right] = \tag{2.103}$$

$$\frac{1}{d-c}\left[\left\|f''+f-f(a)\right\|_{L_p([c,a])}\frac{(a-c)^{\frac{q+1}{q}+1}}{(q+1)^{\frac{1}{q}}\left(\frac{q+1}{q}+1\right)}\right.$$

$$\left.+\left\|f''+f-f(a)\right\|_{L_p([a,d])}\frac{(d-a)^{\frac{q+1}{q}+1}}{(q+1)^{\frac{1}{q}}\left(\frac{q+1}{q}+1\right)}\right] =$$

$$\frac{\left[\left\|f''+f-f(a)\right\|_{L_p([c,a])}(a-c)^{2+\frac{1}{q}}+\left\|f''+f-f(a)\right\|_{L_p([a,d])}(d-a)^{2+\frac{1}{q}}\right]}{(q+1)^{\frac{1}{q}}\left(2+\frac{1}{q}\right)(d-c)}, \tag{2.104}$$

proving the claim. ■

We continue with more involved L_p, $p \geq 1$, Ostrowski type inequalities.

Theorem 2.26. *Let $f \in C_K^4([c,d])$; $a \in [c,d] : f^{(i)}(a) = 0$, $i = 1,2,3,4$. Here $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|(d-c)}\right) \tag{2.105}$$

$$\Big[(a-c)\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])} +$$

$$(d-a)\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}\Big].$$

Proof. By (2.70) we obtain ($x \geq a$)

$$|f(x) - f(a)| \leq \left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)$$

$$\int_a^x \left|\left[\left(f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t)\right) - \right.\right.$$

$$\left.\left.\left(f''''(a) + \left(\alpha^2 + \beta^2\right)f''(a) + \alpha^2\beta^2 f(a)\right)\right]\right| dt \leq$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)\int_a^d \left|f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| dt = \tag{2.106}$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}.$$

From (2.72) we get ($x < a$)

$$|f(x) - f(a)| \leq$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)\int_x^a \left|f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| dt \leq \tag{2.107}$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)\int_c^a \left|f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| dt =$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}.$$

By (2.91) we get that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x) - f(a)|\,dx + \int_a^d |f(x) - f(a)|\,dx\right] \leq \tag{2.108}$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|(d-c)}\right)$$

$$\Big[\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}(a-c)$$

$$+\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}(d-a)\Big].$$

The claim is proved. ■

The counterpart of (2.105) follows.

Theorem 2.27. *All as in Theorem 2.26. Then*

$$\left|\frac{1}{d-c}\int_a^d f(x)\,dx - f(a)\right| \leq \frac{1}{(d-c)\left|\beta^2-\alpha^2\right|}$$

$$\Big[(a-c)^2 \left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])} +$$

$$(d-a)^2 \left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}\Big]. \tag{2.109}$$

Proof. By (2.80) we have ($x \geq a$)

$$|f(x) - f(a)| \leq \frac{1}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}$$

$$\int_a^x \left|f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| 2\left|\alpha\beta\right| (x-t)\,dt \leq \tag{2.110}$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\int_a^x \left|f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| (x-t)\,dt \leq$$

$$\frac{2(x-a)}{\left|\beta^2-\alpha^2\right|}\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}.$$

When $x < a$, by (2.84), we obtain

$$|f(x) - f(a)| \leq \frac{1}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}$$

$$\int_x^a \left|f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| 2\left|\alpha\beta\right| (t-x)\,dt \leq$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\int_x^a \left|f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| (t-x)\,dt \leq \tag{2.111}$$

$$\frac{2(a-x)}{\left|\beta^2-\alpha^2\right|}\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}.$$

By (2.91) we get that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x) - f(a)|\,dx + \int_a^d |f(x) - f(a)|\,dx\right] \leq$$

$$\frac{2}{(d-c)\left|\beta^2-\alpha^2\right|}\Big[\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}\int_c^a (a-x)\,dx$$

$$+\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}\int_a^d (x-a)\,dx\Big] = \tag{2.112}$$

$$\frac{1}{(d-c)\left|\beta^2-\alpha^2\right|}\Big[(a-c)^2\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}$$

$$+(d-a)^2\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}\Big].$$

The claim is proved. ■

The counterparts of the last two Theorems 2.26, 2.27, for $p > 1$, follow.

Theorem 2.28. *Let* $f \in C_K^4([c,d])$, $a \in [c,d] : f^{(i)}(a) = 0$, $i = 1,2,3,4$. *Here* $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. *Let also* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|(d-c)\left(1 + \frac{1}{q}\right)}\right)$$

$$\Big[(a-c)^{1+\frac{1}{q}}\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])} +$$

$$(d-a)^{1+\frac{1}{q}}\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])}\Big]. \tag{2.113}$$

Proof. Let $x \geq a$, by (2.106), we obtain

$$|f(x) - f(a)| \leq \left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)$$

$$\int_a^x \left|\left[\left(f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t)\right) - \alpha^2\beta^2 f(a)\right]\right| dt \leq$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)$$

$$\left(\int_a^x \left|f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|^p dt\right)^{\frac{1}{p}} (x-a)^{\frac{1}{q}} \leq \tag{2.114}$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])}(x-a)^{\frac{1}{q}}.$$

Let $x < a$, by (2.107), we get

$$|f(x) - f(a)| \leq \left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)$$

$$\int_x^a \left|\left(f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t)\right) - \alpha^2\beta^2 f(a)\right| dt \leq$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)$$

$$\left(\int_x^a \left|f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|^p dt\right)^{\frac{1}{p}} (a-x)^{\frac{1}{q}} \leq \tag{2.115}$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])}(a-x)^{\frac{1}{q}}.$$

By (2.91) we get that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x) - f(a)|\,dx + \int_a^d |f(x) - f(a)|\,dx\right] \leq$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|(d-c)}\right)$$

$$\Big[\left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])} \left(\int_c^a (a-x)^{\frac{1}{q}}\,dx\right)$$

$$+ \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])} \left(\int_a^d (x-a)^{\frac{1}{q}}\,dx\right)\Big] = \tag{2.116}$$

$$\left(\frac{(|\alpha|+|\beta|)}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|(d-c)\left(1+\frac{1}{q}\right)}\right)$$

$$\Big[(a-c)^{1+\frac{1}{q}} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])} +$$

$$(d-a)^{1+\frac{1}{q}} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])}\Big].$$

The claim is proved. ■

At last we give

Theorem 2.29. *All as in Theorem 2.28. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \frac{2}{\left|\beta^2-\alpha^2\right|(q+1)^{\frac{1}{q}}\left(2+\frac{1}{q}\right)(d-c)}$$

$$\Big[(a-c)^{2+\frac{1}{q}} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])} +$$

$$(d-a)^{2+\frac{1}{q}} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])}\Big]. \tag{2.117}$$

Proof. Let $x \geq a$, by (2.110), we get that

$$|f(x) - f(a)| \leq \frac{2}{\left|\beta^2-\alpha^2\right|}$$

$$\int_a^x \left| f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| (x-t)\,dt \leq$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\left(\int_a^x \left| f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|^p dt\right)^{\frac{1}{p}} \tag{2.118}$$

$$\left(\int_a^x (x-t)^q\,dt\right)^{\frac{1}{q}} \leq$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])} \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}.$$

Let $x < a$, by (2.111), we derive

$$|f(x) - f(a)| \leq \frac{2}{\left|\beta^2-\alpha^2\right|}$$

$$\int_x^a \left| f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| (t-x)\,dt \leq$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\left(\int_x^a \left| f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|^p dt\right)^{\frac{1}{p}}$$

$$\left(\int_x^a (t-x)^q \, dt\right)^{\frac{1}{q}} \leq$$

$$\frac{2}{|\beta^2-\alpha^2|} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_p([c,a])} \frac{(a-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}. \tag{2.119}$$

Finally, by (2.91) we get that

$$\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f(a) \right| \leq$$

$$\frac{1}{d-c} \left[\int_c^a |f(x)-f(a)|\, dx + \int_a^d |f(x)-f(a)|\, dx \right] \leq$$

$$\frac{2}{|\beta^2-\alpha^2| (q+1)^{\frac{1}{q}} (d-c)}$$

$$\left[\left(\int_c^a (a-x)^{\frac{q+1}{q}} \, dx \right) \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_9([c,a])} \right.$$

$$\left. + \left(\int_a^d (x-a)^{\frac{q+1}{q}} \, dx \right) \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_p([a,d])} \right] = \tag{2.120}$$

$$\frac{2}{|\beta^2-\alpha^2| (q+1)^{\frac{1}{q}} \left(2+\frac{1}{q}\right) (d-c)}$$

$$\left[(a-c)^{2+\frac{1}{q}} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_p([c,a])} + \right.$$

$$\left. (d-a)^{2+\frac{1}{q}} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_p([a,d])} \right].$$

The claim is proved. ■

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MALAYA JOURNAL OF MATEMATIK
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Exact solutions to interfacial flows with kinetic undercooling in a Hele-Shaw cell of time-dependent gap

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*R*eceived 22 April 2023 ; *A*ccepted 14 July 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. Hele-Shaw cells where the top plate is moving uniformly at a prescribed speed and the bottom plate is fixed have been used to study interface related problems. This paper focuses on interfacial flows with linear and nonlinear kinetic undercooling regularization in a radial Hele-Shaw cell with a time dependent gap. We obtain some exact solutions of the moving boundary problems when the initial shape is a circle, an ellipse or an annular domain. For the nonlinear case, a linear stability analysis is also presented for the circular solutions. The methodology is to use complex analysis and PDE theory.

AMS Subject Classifications: 35Q35, 76S05.

Keywords: Hele-Shaw flow, nonlinear kinetic undercooling, exact solution, Schwarz function, Laplace equation

Contents

1. Introduction and Background

There have been a huge number of studies in the problem of a less viscous fluid displacing a more viscous fluid in a Hele-Shaw cell since Saffman and Taylor's seminal papers [28, 35] in the 1950s. Neglecting the surface tension, Saffman and Taylor [28] found an one-parameter family of exact steady solutions, parameterized by width λ. This theoretical shape are usually referred to in the literature as the Saffman-Taylor finger. It was found [28] experimentally that an unstable planar interface evolves through finger competition to a steady translating finger, with relative finger width λ close to one half. However, in the zero-surface-tension steady-state theory, λ remained arbitary in the (0, 1) interval. The selection problem of λ was solved by incorporating the surface tension regularization, numerical and formal asymptotic calculations [22], [39],

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[19],[7, 8], [31], [17], [32, 33], [4] Rigorous results were later obtained in [34, 40].

The Hele-Shaw problem is similar to the Stefan problem in the context of melting or freezing. Besides surface tension, the physical effect most commonly incorporated in regularizing the ill-posed Stefan problem is kinetic undercooling [2, 14, 20], where the temperature on the moving interface is proportional to the normal velocity of the interface. For the Hele-Shaw problem, kinetic undercooling regularization first appeared in [26, 27]. Since then, results in different aspects were obtained ([1, 5, 9, 15, 24, 25, 41]).

Besides the classical Hele-Shaw setup, there are several variants related to the viscous fingering problem [3, 11, 12, 16]. One of the variants is interfacial flows in a Hele-Shaw cell where the top plate is lifted uniformly at a prescribed speed and the bottom plate is fixed (lifting plate problems) [6, 13, 23, 29, 30, 36, 37, 44]. In the lifting plate problem, the gap $b(t)$ between the two plates is increasing in time but uniform in space. As the plate is pulled, an inner viscous fluid shrinks in the center plane between the two plates and increases in the z-direction to preserve volume. An outer less viscous fluid invades the cell and generates fingering patterns. The patterns are visually similar to those in the classical radial Hele-Shaw problem, but the driving physics is different. In [29], the authors derived the governing equations for the lifting problem and they established the existence, uniqueness and regularity of solutions for analytic data when the surface tension is zero. Some exact solutions were also constructed, both with or without surface tension. Analytic results were also be generalized to higher dimensions in [38]. Numerical simulation and the pattern formation of the interface were presented in [29, 44].

Very recently, local existence of analytic and classical solutions was obtained in [42, 43] for the Hele-Shaw problem with time dependent gap where the kinetic undercooling regularization is used instead of the surface tension. In this paper we obtain some exact solutions of the moving boundary problem when the initial data is a circle or an ellipse. The methodology is to use complex analysis and Schwarz function.

2. Mathematical formulation

Our studies center on the free interface problem in a Hele-Shaw cell with a time dependent gap $b(t)$; see Figure 1. The upper plate is lifted or compressed perpendicular to the cell, while the lower plate stays fixed. We assume that

$$b(t) \in C^1([0,\infty)), \;\; b(t) \geq b_0 \text{ for some positive constants } b_0.$$

We consider the displacement of a viscous fluid by another fluid of negligible viscosity. Let $\Omega(t)$ in the (x, y) plane be the more viscous fluid domain with free boundary $\partial\Omega$. Following M. Shelley, F. Tian, and K. Wlodarski [29], we have the following governing equations: The fluid velocity is

$$u = -\frac{b^2(t)}{12\mu} \nabla p(x, y, t), \tag{2.1}$$

where p is the pressure and μ is the viscosity of the fluid. Conservation of mass equation is

$$\nabla \cdot u = -\frac{\dot{b}(t)}{b(t)} \text{ in } \Omega(t). \tag{2.2}$$

The kinematic boundary condition is

$$-\frac{b^2(t)}{12\mu}\frac{\partial p}{\partial n} = V_n \text{ on } \partial\Omega(t), \tag{2.3}$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the outward normal $\mathbf{n}$ to $\partial\Omega$, and V_n is the velocity of the $\partial\Omega(t)$ in the direction of outward normal vector $\mathbf{n}$; and the dynamic boundary condition is

$$p = \tau V_n \text{ on } \partial\Omega(t), \tag{2.4}$$

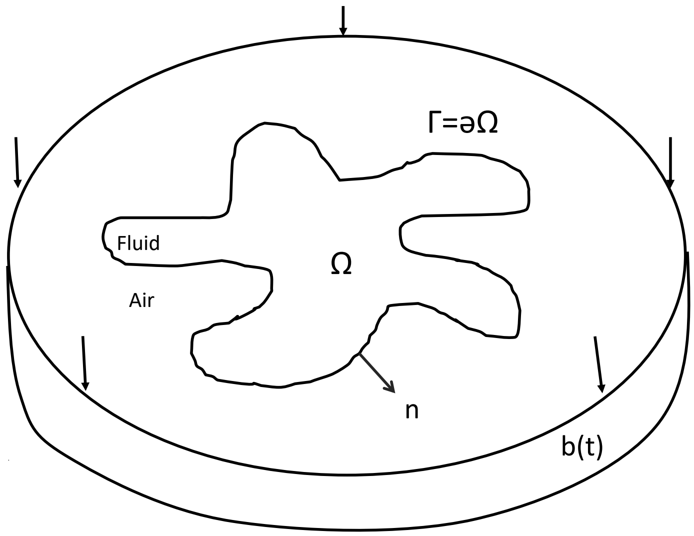

Figure 1: The Hele-Shaw flow with time dependent gap

or

$$p = \tau (V_n)^{\beta} \text{ on } \partial\Omega(t), \tag{2.5}$$

where τ is a kinetic undercooling coefficient and $\beta > 0$ is a real constant.

Non-dimensionalizing the length and time and the pressure, the nondimensional version of (2.1), (2.2), (2.3, (2.4) and (2.5) is

$$u = -b^2(t)\nabla p(x, y, t) \text{ in } \Omega(t), \tag{2.6}$$

$$\nabla \cdot u = -\frac{\dot{b}(t)}{b(t)} \text{ in } \Omega(t), \tag{2.7}$$

$$-b^2(t)\frac{\partial p}{\partial n} = V_n \text{ on } \partial\Omega(t), \tag{2.8}$$

$$p = cV_n \text{ on } \partial\Omega(t); \tag{2.9}$$

or

$$p = c(V_n)^{\beta} \text{ on } \partial\Omega(t); \tag{2.10}$$

where c is the nondimensional kinetic undercooling coefficient; β is a positive number and $\beta \neq 1$.
Plugging (2.6) into (2.7), we obtain

$$\nabla^2 p = \frac{\dot{b}(t)}{b^3(t)} \text{ in } \Omega(t). \tag{2.11}$$

We are going to consider the following two problems:
Problem one consists of the equations (2.8), (2.9) and (2.11) and
Problem Two consists of the equations (2.8), (2.10) and (2.11) with $\beta > 0$ and $\beta \neq 1$.

Let $\tilde{p} = p - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2)$, then $\tilde{p}$ satisfies

$$\nabla^2 \tilde{p} = 0, \text{ in } \Omega(t) \tag{2.12}$$

$$V_n = -b^2(t)\frac{\partial \tilde{p}}{\partial n} - \frac{\dot{b}(t)}{2b(t)}(x,y)\cdot \mathbf{n} \text{ on } \partial\Omega(t), \tag{2.13}$$

$$\tilde{p} = cV_n - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2), \text{ on } \partial\Omega(t), \tag{2.14}$$

or

$$\tilde{p} = c(V_n)^\beta - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2), \text{ on } \partial\Omega(t), \tag{2.15}$$

So equivalently, *Problem one* consists of the equations (2.12), (2.13) and (2.14) and *Problem Two* consists of the equations (2.12), (2.13) and (2.15) with $\beta \neq 1$.

3. Exact solutions of Problem One

It is noted in [42] that Problem One has a radially symmetric solution $\Omega(t) = \{(x,y) : r = \sqrt{x^2+y^2} < R(t)\}$, where

$$\frac{\dot{R(t)}}{R(t)} = -\frac{\dot{b(t)}}{2b(t)}, \quad R(t) = \frac{R(0)\sqrt{b(0)}}{\sqrt{b(t)}}; \tag{3.1}$$

$$p(t,r) = -\frac{cR(0)\sqrt{b(0)}\dot{b}(t)}{2b(t)\sqrt{b(t)}} - \frac{R^2(0)b(0)\dot{b}(t)}{4b^4(t)} + \frac{\dot{b}(t)}{4b^3(t)}r^2. \tag{3.2}$$

It was also obtained in [42] that the linear perturbation of solution (3.1) and (3.2) grows when $\dot{b}(t)$ is positive while the perturbation decays when $\dot{b}(t)$ is negative.
Next we are going to re-establish (3.1) and 3.2) using Schwarz function approach. We refer to [10] for properties of Schwarz function, we first give some preliminary lemmas.

Lemma 3.1. *Assume that $\partial\Omega(t)$ is an analytic curve, and $S(t,z)$ is the Schwarz Function of $\partial\Omega(t)$, where $z = x+iy$; then the outward normal velocity is*

$$V_n = -\frac{i\partial_t S(t,z)}{2\sqrt{S_z(t,z)}}. \tag{3.3}$$

The proof of the lemma can be found in [18, 21].
Let $W(z) = \tilde{p} + i\tilde{q}$, where $\tilde{q}$ is a harmonic conjugate of $\tilde{p}$. The analytic function $W(z)$ is called the complex velocity potential for Problem One. Let us obtain an equation for W_z in terms of the Schwarz function $S(t,z)$.We introduce s to be the arc length variable along $\partial\Omega(t)$. Using the properties of $S(z)$,(2.14) and (3.3), we have

$$\begin{aligned}
\partial_s \tilde{p} &= \frac{\partial \tilde{p}}{\partial z}\cdot\frac{\partial z}{\partial s} + \frac{\partial \tilde{p}}{\partial \bar{z}}\cdot\frac{\partial \bar{z}}{\partial s} \\
&= \frac{\partial \tilde{p}}{\partial z}\cdot\frac{1}{\sqrt{S_z(z)}} + \sqrt{S_z(z)}\frac{d\tilde{p}}{d\bar{z}} \\
&= \frac{1}{\sqrt{S_z(z)}}\left(-\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\bar{z} + \frac{\partial}{\partial z}c\,(V_n)\right) + \sqrt{S_z(z)}\left(-\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z + \frac{\partial}{\partial \bar{z}}cV_n\right) \\
&= \frac{1}{\sqrt{S_z(z)}}\left(-\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\bar{z} - c\frac{\partial}{\partial z}\left(\frac{i}{2}\frac{\partial_t S}{\sqrt{S_z}}\right)\right) + \sqrt{S_z(z)}\left(-\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\right) \\
&= -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\left(\frac{S(z)}{\sqrt{S_z(z)}} + z\sqrt{S_z(z)}\right) - \frac{ci}{2}\frac{1}{\sqrt{S_z(z)}}\frac{\partial}{\partial z}\left(\frac{\partial_t S}{\sqrt{S_z}}\right)
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\frac{\partial \tilde{p}}{\partial \vec{n}} &= -\frac{V_n}{b^2(t)} - \frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\frac{\partial}{\partial n}(z\bar{z}) \\ &= +\frac{i}{2}\frac{\partial_t S}{b^2\sqrt{S_z}} - \frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\left(\bar{z}\frac{\partial z}{\partial n} + z\frac{\partial \bar{z}}{\partial n}\right) \\ &= +\frac{i}{2}\frac{\partial_t S}{b^2\sqrt{Sz}} - \frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\left(S\cdot\frac{-i}{\sqrt{S_z(z)}} + iz\sqrt{S_z(z)}\right) \\ &= +\frac{i}{2}\frac{\partial_t S}{b^2\sqrt{S_z}} - \frac{i}{4}\frac{\dot{b}(t)}{b^3(t)}\left(\frac{-S}{\sqrt{S_z(z)}} + z\sqrt{S_z(z)}\right)\end{aligned} \tag{3.5}$$

Using (3.4) and (3.5), we have

$$\begin{aligned}W_z =&\frac{\partial_s W}{\partial_s z} = \frac{\partial_s(\tilde{p}+i\tilde{q})}{\partial_s z} \\ &= \frac{\partial_s\tilde{p}+i\partial_s\tilde{q}}{\partial_s z} = \frac{\partial_s\tilde{p}+i\partial_n\tilde{p}}{\partial_s z} \\ &= -\frac{1}{4}\frac{\dot{b}}{b^3}\left(S(z)+zS_z(z)\right) - \frac{ci}{2}\frac{\partial}{\partial z}\left(\frac{\partial_t S}{\sqrt{S_z}}\right) \\ &+ i\left(\frac{i}{2}\frac{\partial_t S}{b^2} - \frac{i}{4}\frac{b(t)}{b^3(t)}\left(-S+zS_z(z)\right)\right) \\ =& -\frac{ci}{2}\frac{\partial}{\partial z}\left(\frac{\partial_t S}{\sqrt{S_z}}\right) - \frac{\partial_t S}{2b^2} - \frac{1}{2}\frac{\dot{b}}{b^3}S\end{aligned} \tag{3.6}$$

From (2.13), (2.14) and (3.3), we have on $\partial\Omega(t)$:

$$-b^2(t)\frac{\partial \tilde{p}}{\partial n} = \frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}zS(z) - \frac{i\partial_t S(t,z)}{2\sqrt{S_z(t,z)}} \tag{3.7}$$

and

$$\tilde{p} = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\bar{z} + cV_n = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\bar{z} - \frac{ic\partial_t S(t,z)}{2\sqrt{S_z(t,z)}}. \tag{3.8}$$

Now we look at the following cases:

(1) Circular solutions: If the initial shape is circle, $\Omega(0) = \{(x,y) : x^2+y^2 \leq a_0^2\}$, then we are seeking circular solution for all $t > 0$, $\Omega(t) = \{(x,y) : x^2+y^2 \leq a(t)^2\}$. The Schwarz function is $S(z,t) = \frac{a^2(t)}{z}$, so $\partial_t S = 2a\frac{\dot{a}(t)}{z}, S_z = \frac{-a^2(t)}{z^2}$. From (3.6), we have

$$W_z = -\frac{a(t)}{b^2 z}\left(\dot{a} + \frac{\dot{b}a}{2b}\right); \tag{3.9}$$

Let $A(t)$ be the area of $\Omega(t)$, then

$$\begin{aligned}\frac{d}{dt}A(t) &= \frac{d}{dt}\iint_{\Omega(t)} 1dxdy = \oint_{\partial\Omega t} V_n ds \\ &= \oint_{\partial\Omega(t)} -b^2(t)\frac{\partial p}{\partial n}ds = -b^2(t)\oint_{\partial\Omega(t)}\frac{\partial p}{\partial n}ds \\ &= -b^2(t)\iint_{\Omega(t)}\Delta p dxdy = -b^2(t)\cdot\frac{\dot{b}(t)}{b^3(t)}A(t) = \frac{-\dot{b}(t)}{b(t)}A(t).\end{aligned} \tag{3.10}$$

Since $A(t) = \pi a^2(t)$, we have

$$2\dot{a} = -\frac{a(t)}{b(t)}\dot{b}(t) \tag{3.11}$$

(3.9) implies $W_z = 0$, so $W(t,z) = W(t)$ is independent of z, consequently $\tilde{p}(t,z) = \tilde{p}(t)$ is independent of z. Using (3.8) we have

$$\begin{aligned}\tilde{p} &= -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\cdot a^2(t) + cV_n = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}a^2(t) \\ &= -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}a^2(t) + c\dot{a}(t) = -\frac{1}{4}\frac{\dot{b}(t)a^2(t)}{b^3(t)} - \frac{ca(t))\dot{b}(t)}{2b(t)}.\end{aligned} \tag{3.12}$$

We note that (3.11) and (3.12) are equivalent to (3.1) and (3.2) respectively.

(2) Solutions of Ellipse shape: If $\Omega(0) = \{(x,y) : \frac{x^2}{a_0^2} + \frac{y^2}{h_0^2} \leqslant 1\}$, where $a_0 > h_0$.

Let us seek ellipse solution for $t > 0$, $\Omega(t) = \{(x,y) : \frac{x^2}{a^2(t)} + \frac{y^2}{h^2(t)} \leqslant 1\}$, then the Schwarz function is

$$S(z,t) = \frac{a^2+h^2}{d^2(t)}z - \frac{2a(t)h(t)}{d(t)^2}\sqrt{z^2-d^2}, \tag{3.13}$$

where $d(t) = \sqrt{a(t)^2 - h(t)^2}$.
Taking derivative in above, we have

$$\partial_t S(z,t) = z\frac{\partial}{\partial t}\left(\frac{a^2+h^2}{d^2}\right) - \sqrt{z^2-d^2}\frac{\partial}{\partial t}\left(\frac{2ah}{d^2}\right) + \frac{1}{\sqrt{z^2-d^2}}\frac{ah}{d^2}\frac{\partial}{\partial t}\left(d^2\right) \tag{3.14}$$

and

$$S_z(z,t) = \frac{\left(a^2+h^2\right)\sqrt{z^2-d^2} - 2ahz}{d^2\sqrt{z^2-d^2}} \tag{3.15}$$

Plugging (3.13), (3.14) and (3.15) into (3.6) and integrating we obtain,

$$\begin{aligned}W(t,z) = &\left(-\frac{ci}{2}\right)\frac{d\left[z\frac{\partial}{\partial t}\left(\frac{a^2+h^2}{d^2}\right)\sqrt{z^2-d^2} - \left(z^2-d^2\right)\frac{\partial}{\partial t}\left(\frac{2ah}{d^2}\right) + \frac{ah}{d^2}\frac{\partial}{\partial t}\left(d^2\right)\right]}{\left(z^2-d^2\right)^{\frac{1}{4}}\left(\left(a^2+h^2\right)\sqrt{z^2-d^2} - 2ahz\right)^{1/2}} \\ &- \frac{z^2}{4b^2}\left[\partial_t\left(\frac{a^2+h^2}{d^2}\right) - \frac{\left(a^2+h^2\right)\partial_t(ah)}{ahd^2}\right] \\ &- z\sqrt{z^2-d^2}\frac{ah\partial_t(d^2)}{2b^2d^4} + q(t)\end{aligned} \tag{3.16}$$

We examine the singularity of $W(t,z)$ in (3.16), W has to be analytic in $\Omega(t) = \frac{x^2}{a^2} + \frac{y^2}{h^2} \leqslant 1$.
We note that the zero of $\left(a^2+h^2\right)\sqrt{z^2-d^2} - 2ahz = 0$ are $z = \pm\frac{a^2+h^2}{d(t)} = \pm\frac{a^2+h^2}{\sqrt{a^2-h^2}}$, which are outside of $\Omega(t)$. So $\left(\left(a^2+h^2\right)\sqrt{z^2-d^2} - 2ahz\right)^{1/2}$ is analytic in $\Omega(t)$. That means that the singularities of the first term and the third term can not be canceled out to make $W(t,z)$ analytic. Hence we have the following result:

Proposition 3.2. *If the initial shape of Problem One is an ellipse, then for any $T > 0$, there is no solution of Problem One on $[0,T]$ which is of ellipse shape.*

(3) Solutions on annular domain:
If $\Omega(t) = \{(x, y) : r(t) < \sqrt{x^2 + y^2} < R(t)\}$ is an annular domain, then the problem will consist of the following equations:

$$\Delta p = \frac{\dot{b}}{b^3} \text{ in } \Omega(t); \tag{3.17}$$

$$p = c\dot{R} \; on \; x^2 + y^2 = R^2(t) \tag{3.18}$$

$$p = -c\dot{r} \; on \; x^2 + y^2 = r^2(t) \tag{3.19}$$

$$-b^2 \frac{\partial p}{\partial r} = \dot{R} \; on \; x^2 + y^2 = R^2(t) \tag{3.20}$$

$$-b^2 \frac{\partial p}{\partial r} = \dot{r} \; on \; x^2 + y^2 = r^2(t). \tag{3.21}$$

The general solution of (3.17) is given by

$$p(r, \theta) = \frac{1}{4} \frac{b}{b^3}(x^2 + y^2) + \alpha(t) \log \sqrt{x^2 + y^2} + \gamma(t) \tag{3.22}$$

where $\alpha(t)$ and $\gamma(t)$ with $R(t)$ and $r(t)$ are to be determined. Plugging above into (3.18)- (3.21) to obtain

$$c\dot{R} = \frac{1}{4} \frac{\dot{b}}{b^3} R^2 + \alpha(t) \log R + \gamma(t), \tag{3.23}$$

$$-c\dot{r} = \frac{1}{4} \frac{\dot{b}}{b^3} r^2 + \alpha(t) \log r + \gamma(t), \tag{3.24}$$

$$-\frac{1}{2} \frac{\dot{b}}{b} R - \frac{\alpha(t) b^2}{R} = \dot{R}, \tag{3.25}$$

$$-\frac{1}{2} \frac{\dot{b}}{b} r - \frac{\alpha(t) b^2}{r} = \dot{r}. \tag{3.26}$$

From (3.25) and (3.26) we have

$$-\frac{1}{2} \frac{\dot{b}}{b}(R^2 - r^2) = \frac{1}{2} \frac{d}{dt}(R^2 - r^2). \tag{3.27}$$

Solving (3.27) we derive

$$R^2 - r^2 = \frac{b(0)\big(R^2(0) - r^2(0)\big)}{b(t)} \tag{3.28}$$

which implies

$$R(t) = \sqrt{r(t)^2 + \frac{b(0)\big(R^2(0) - r^2(0)\big)}{b(t)}} \tag{3.29}$$

To solve $\alpha(t)$ we subtract (3.24) from (3.23), then use (3.25) and (3.26) to derive

$$\alpha(t) = \frac{-\frac{c}{2} \frac{\dot{b}}{b}(R + r) - \frac{1}{4} \frac{\dot{b}}{b^3} \frac{b(0)\left(R^2(0) - r^2(0)\right)}{b}}{\left[\log\left(\frac{R}{r}\right) + c\left(\frac{1}{R} + \frac{1}{r}\right)\right]} \tag{3.30}$$

Now plugging (3.30) into (3.26), we derive

$$\dot{r} = -\frac{1}{2} \frac{\dot{b}}{b} r + \frac{\frac{c}{2} \dot{b} b(R + r) + \frac{1}{4} \frac{\dot{b} b(0)\left(R^2(0) - r^2(0)\right)}{b^2}}{r\left[\log\left(\frac{R}{r}\right) + c\left(\frac{1}{R} + \frac{1}{r}\right)\right]} \tag{3.31}$$

Plugging (3.29) into (3.31), we obtain an ODE for $r(t)$

$$\dot{r} = -\frac{1}{2}\frac{\dot{b}}{b}r + \frac{\frac{cb\dot{b}}{2}\left(\sqrt{r^2+B(t)}+r\right)+\frac{1}{4}\frac{\dot{b}B(t)}{b}}{r\left[\frac{1}{2}\log\left(1+\frac{B(t)}{r^2}\right)+c\left(\frac{1}{\sqrt{r^2+B(t)}}+\frac{1}{r}\right)\right]}, \tag{3.32}$$

Where

$$B(t) = \frac{b(0)\big(R^2(0)-r^2(0)\big)}{b(t)}. \tag{3.33}$$

Once $r(t)$ is solved from (3.32), we solve $R(t)$ from (3.29), $\alpha(t)$ from (3.30) and $\gamma(t)$ from (3.24). In Figure 2, using MATLAB, we have numerically solved $r(t)$, $R(t)$, $\alpha(t)$ and $\gamma(t)$ from (3.32), (3.29), (3.30) and (3.24) when we take $b(t) = (1+t)/(2+t), c = 0.1, r(0) = 1, R(0) = 2$.

4. Exact solutions of Problem Two

Parallel to (3.6), (3.8) and Problem One, we have the following equations for Problem Two:

$$W_z = c\left[-\frac{i}{2}\frac{\partial}{\partial z}\left(\frac{\partial_t S}{\sqrt{S_z}}\right)\right]^\beta - \frac{\partial_t S}{2b^2} - \frac{1}{2}\frac{\dot{b}}{b^3}S \tag{4.1}$$

$$\tilde{p} = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\bar{z} + c(V_n)^\beta == -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\bar{z} + c\left[-\frac{i\partial_t S(t,z)}{2\sqrt{S_z(t,z)}}\right]^\beta. \tag{4.2}$$

(1) Circular solutions: If the initial shape is circle, $\Omega(0) = \{(x,y) : x^2+y^2 \le a_0^2\}$, then we are seeking circular solution for all $t > 0$, $\Omega(t) = \{(x,y) : x^2+y^2 \le a(t)^2\}$. Parallel to (3.9), (3.11) and (3.12) we have the following radially symmetric solution for Problem Two:

$$\frac{\dot{a(t)}}{a(t)} = -\frac{\dot{b(t)}}{2b(t)}, \quad a(t) = \frac{a_0\sqrt{b(0)}}{\sqrt{b(t)}}; \tag{4.3}$$

$$\begin{aligned}\tilde{p} &= -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\cdot a^2(t) + c[V_n]^\beta = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}a^2(t) + c[\dot{a}(t)]^\beta \\ &= -\frac{1}{4}\frac{\dot{b}(t)a^2(t)}{b^3(t)} - c\left[\frac{a(t))\dot{b}(t)}{2b(t)}\right]^\beta.\end{aligned} \tag{4.4}$$

Now we will discuss the linear stability of the circular solution (4.3) and (4.4). The perturbed interface is $\Big\{r(\theta,t) = a(t) + \varepsilon\delta(t)\cos(k\theta)\Big\} = \partial\Omega(t)$, then we have

$$\begin{aligned}\overrightarrow{n} &= \frac{< \partial_\theta r\ \sin\theta + r\cos\theta, -r_\theta\cos\theta + r\sin\theta >}{\sqrt{r_\theta^2+r^2}} \\ &= \frac{< a(t)\cos\theta, a(t)\sin\theta >}{a(t)\sqrt{1+\frac{2\varepsilon\delta(t)\cos(kt)}{a(t)}+O(\varepsilon^2)}} \\ &+ \frac{\varepsilon\delta(t)\langle -k\sin\theta\sin(k\theta)+\cos(k\theta)\cos\theta, k\cos\theta\sin(k\theta)+\cos(k\theta)\sin\theta\rangle}{a(t)\sqrt{1+\frac{2\varepsilon\delta(t)\cos(kt)}{a(t)}+O(\varepsilon^2)}} \\ &=< \cos\theta, \sin\theta > - \frac{\varepsilon\delta(t)k\sin(k\theta)}{a(t)} < \sin\theta, -\cos\theta > + O(\varepsilon^2)\end{aligned}$$

and

$$\begin{aligned} V_n &= < \dot{x}(t), \dot{y}(t) > \cdot \overrightarrow{n} \\ &= \big(a\dot{(t)} + \varepsilon\dot{\delta}(t)\cos(k\theta)\big) < \cos\theta, \sin\theta > \\ &\cdot \Big[< \cos\theta, \sin\theta > - \frac{\varepsilon\delta(t)k\sin(k\theta)}{a(t)} < \sin\theta, -\cos\theta > + O(\varepsilon^2)\Big] \\ &= \dot{a}(t) + \varepsilon\dot{\delta}(t)\cos(k\theta) + O(\varepsilon^2) \end{aligned} \tag{4.5}$$

Let

$$\widetilde{p}(t,r,\theta) = \widetilde{p}_0(t,r) + \varepsilon\widetilde{p}_1(t,r,\theta) + ... \tag{4.6}$$

then

$$\widetilde{p}(t,r,\theta)\Big|_{\partial\Omega(t)} = \widetilde{p}(t,r,\theta)\Big|_{r=a(t)} + \Big(\frac{\partial\widetilde{p}}{\partial r}\Big|_{r=a(t)}\Big)(\varepsilon\delta\cos(k\theta)) + O(\varepsilon^2) \tag{4.7}$$

from (2.15) and (4.5), we have on $\partial\Omega(t)$

$$\begin{aligned} \widetilde{p} &= c(\dot{a} + \varepsilon\dot{\delta}\cos(k\theta))^\beta - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2) \\ &= c(\dot{a})^\beta + c\beta(\dot{a})^{\beta-1}(\varepsilon\dot{\delta}\cos(k\theta)) - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2) + O(\varepsilon^2) \end{aligned} \tag{4.8}$$

From (4.6),(4.7) and (4.8) , we have

$$\widetilde{p}_1\Big|_{r=a(t)} = c\beta(\dot{a})^{\beta-1}\dot{\delta}\cos(k\theta) - \frac{\dot{b}}{2b^3(t)}a(t)\delta(t)\cos(k\theta); \tag{4.9}$$

From (2.13) and (4.5) we have

$$\dot{\delta}(t)\cos(k\theta) = -b^2(t)\frac{\partial\widetilde{p}_1}{\partial r}\Big|_{r=a(t)} = \frac{\delta(t)\dot{b}}{2b}\cos(k\theta) \tag{4.10}$$

$\Big(Note:\ \frac{\dot{b}}{2b} < x, y > \cdot \overrightarrow{n} = \frac{\dot{b}(t)}{2b(t)}a(t) + \frac{\varepsilon\delta(t)\dot{b}}{2b}\cos(k\theta) + O(\varepsilon^2)\ using\ (36).\Big)$ From (12), we have

$$\Delta\widetilde{p}_1 = 0 \text{ in } \{r < a(t)\} \tag{4.11}$$

Now using separation of variables. We solve the boundary value problem of (4.9) and (4.11), we have

$$\widetilde{p}_1(r,\theta,t) = \frac{c\beta(\dot{a})^{\beta-1}\dot{\delta} - \frac{\dot{b}}{2b^3}a(t)\delta(t)}{(a(t))^k}r^k\cos(k\theta) \tag{4.12}$$

From (4.12) we have

$$\frac{\partial\widetilde{p}_1}{\partial r}\Big|_{r=a(t)} = \frac{k\left(c\beta(\dot{a})^{\beta-1}\dot{\delta} - \frac{\dot{b}}{2b^3}a(t)\delta(t)\delta(t)\right)\cos(k\theta)}{a(t)} \tag{4.13}$$

Now using (4.10) and (4.13) we obtain

$$\dot{\delta}(t) = -\frac{kb^2\big[c\beta(\dot{a})^{\beta-1}\dot{\delta} - \frac{\dot{b}}{2b^3}a(t)\delta(t)\big]}{a(t)} - \frac{\delta\dot{b}}{2b} \tag{4.14}$$

We rewrite (4.14) as

$$\frac{\dot{\delta}(t)}{\delta(t)} = \frac{\dot{b}(t)}{2b(t)}\frac{(k-1)a(t)}{\big[a(t) + kc\beta b^2(\dot{a})^{\beta-1}\big]} \tag{4.15}$$

Using $\dot{a} = -\dfrac{\dot{b}}{2b}a$, we can rewrite (4.15) as

$$\frac{\dot{\delta}(t)}{\delta(t)} = \frac{\dot{b}(t)}{2b(t)} \frac{(k-1)a(t)}{\left[a(t) + kc\beta b^2\left(-\frac{\dot{b}}{2b}a\right)^{\beta-1}\right]} \tag{4.16}$$

So when $\dot{b} < 0$, $\dfrac{\dot{\delta}(t)}{\delta(t)} < 0$, the solution is stable

If $\dot{b} > 0$, when $\beta = 2m+1$ is positive odd numbers then $\dfrac{\dot{\delta}(t)}{\delta(t)} > 0$, the solution is unstable

If $\dot{b} > 0$, and $\beta = 2m$ is positive even numbers , then

$$\begin{aligned}
\frac{\dot{\delta}(t)}{\delta(t)} &= \frac{\dot{b}(t)}{2b(t)} \frac{(k-1)a(t)}{\left[a - 2kcmb^2\left(\frac{\dot{b}}{2b}\right)^{2m-1}a^{2m-1}\right]} \\
&= \frac{\dot{b}(t)}{2b(t)} \frac{(k-1)}{\left[1 - 2kcmb^2\left(\frac{\dot{b}}{2b}\right)^{2m-1}\left(\frac{a_0\sqrt{b(0)}}{\sqrt{b}}\right)^{2m-2}\right]} \\
&= \frac{\dot{b}(t)}{2b(t)} \frac{(k-1)}{\left[1 - 2^{2-2m}kcm(a_0\sqrt{b(0)})^{2m-2}(\dot{b})^{2m-1}b^{-(2m-4)}\right]}
\end{aligned}$$

the linear stability in this case will depend on the wave number k and the gap function $b(t)$.

(2) ellipse shape case: If there is an elliptic shape solution $\Omega(t) = (x, y) : \frac{x^2}{a^2(t)} + \frac{y^2}{h^2(t)} = 1$, then from (3.10) with $A(t) = \pi a(t)h(t)$

$$a(t)h(t) = \frac{\pi a(0)h(0)b(0)}{b(t)}. \tag{4.17}$$

Parallel to (3.13)-(3.16), we have

$$\begin{aligned}
W(t,z) = c &\left(\left(-\frac{i}{2}\right)\frac{d\left[z\frac{\partial}{\partial t}\left(\frac{a^2+h^2}{d^2}\right)\sqrt{z^2-d^2} - \left(z^2-d^2\right)\frac{\partial}{\partial t}\left(\frac{2ah}{d^2}\right) + \frac{ah}{d^2}\frac{\partial}{\partial t}\left(d^2\right)\right]}{(z^2-d^2)^{\frac{1}{4}}\left((a^2+h^2)\sqrt{z^2-d^2} - 2ahz\right)^{1/2}}\right)^{\beta} \\
&- \frac{z^2}{4b^2}\left[\partial_t\left(\frac{a^2+h^2}{d^2}\right) - \frac{\left(a^2+h^2\right)\partial_t(ah)}{ahd^2}\right] \\
&- z\sqrt{z^2-d^2}\frac{ah\partial_t(d^2)}{2b^2d^4} + q(t)
\end{aligned} \tag{4.18}$$

It is clear that singularities at $z = \pm d$ can not be removed in the case where β is an odd integer. It looks that from (4.18) that singularities can be removed for the case where β is positive even integer if $a(t)$ and $h(t)$ are properly chosen. However, by examining the case $\beta = 2$, we found that $a(t)$ and $h(t)$ must satisfies two independent ODEs which have no solution that also satisfies (4.18). Hence we have the following result.

Proposition 4.1. *If the initial shape of Problem Two is an ellipse, then for any $T > 0$, there is no solution of Problem Two which is of ellipse shape on $[0, T]$.*

(3) Solution on annular domain: For the nonlinear problem 2 on a annular domain, we have the same equations (32) (35) and (36) as for the linear problem 1, but instead of (33) and (34), the pressure conditions are

$$p = c\dot{R}^{\beta} \ on\ x^2 + y^2 = R^2, \tag{4.19}$$

and

$$p = c(-\dot{r})^{\beta} \text{ on } x^2 + y^2 = r^2. \tag{4.20}$$

We also note that (40)-(47) also hold for Problem 2. To solve for $\alpha(t)$, we use (37), (4.19) and (4.20) to obtain:

$$c(\dot{R})^{\beta} = \frac{1}{4}\frac{\dot{b}}{b^3}R^2 + \alpha(t)\log R + \gamma(t), \tag{4.21}$$

$$c(-\dot{r})^{\beta} = \frac{1}{4}\frac{\dot{b}}{b^3}r^2 + \alpha(t)\log r + \gamma(t), \tag{4.22}$$

Subtracting (4.21) and (4.22), and using (35)-(37), we have

$$c\Big(-\frac{1}{2}\frac{\dot{b}}{b}R - \frac{\alpha(t)b^2}{R}\Big)^{\beta} - c\Big(\frac{1}{2}\frac{\dot{b}}{b}r + \frac{\alpha(t)b^2}{r}\Big)^{\beta} = \frac{1}{4}\frac{\dot{b}}{b^3}B(t) + \alpha(t)\log\frac{R}{r}; \tag{4.23}$$

Where $B(t)$ is given by (3.33). We need to solve for $\alpha(t)$ from (4.23). We consider the case $\beta = 2$, then (4.23) becomes a quadratic equation

$$\alpha^2(t)cb^4\left(\frac{1}{r^2} - \frac{1}{R^2}\right) + \alpha(t)\left[\log\frac{R}{r}\right] + \frac{\dot{b}}{4b^2}\left(-c\dot{b} + \frac{1}{b}\right)B(t) = 0 \tag{4.24}$$

Solving (4.24) we have

$$\alpha(t) = \frac{-\log\frac{R}{r} + \sqrt{\log^2\frac{R}{r} - cb\dot{b}\left(\frac{1}{r^2} - \frac{1}{R^2}\right)B(t) + c^2b^2\dot{b}^2\left(\frac{1}{r^2} - \frac{1}{R^2}\right)B(t)}}{2cb^4\left(\frac{1}{r^2} - \frac{1}{R^2}\right)} \tag{4.25}$$

In (4.25), the expression under the square root is positive for sufficiently small c, and we take the positive square root because of

$$\lim_{c\to 0}\alpha(t) = -\frac{\dot{b}}{4b^3}\frac{B(t)}{\log\frac{R}{r}}. \tag{4.26}$$

Plugging the above into (3.26), we obtain

$$\dot{r} = -\frac{1}{2}\frac{\dot{b}}{b}r + \frac{b^2}{r}\frac{\log\frac{R}{r} - \sqrt{\log^2\frac{R}{r} - cb\dot{b}\left(\frac{1}{r^2} - \frac{1}{R^2}\right)B(t) + c^2b^2\dot{b}^2\left(\frac{1}{r^2} - \frac{1}{R^2}\right)B(t)}}{2cb^4\left(\frac{1}{r^2} - \frac{1}{R^2}\right)} \tag{4.27}$$

Now using (3.28), (3.29) and (3.33) we have

$$\dot{r} = -\frac{1}{2}\frac{\dot{b}}{b}r + \frac{b^2}{r}\frac{\log\sqrt{1 + \frac{B(t)}{r^2}}}{2cb^4\left(\frac{1}{r^2} - \frac{1}{r^2+B(t)}\right)} - \frac{b^2}{r}\frac{\sqrt{\log^2\sqrt{1 + \frac{B(t)}{r^2}} - cb\dot{b}\left(\frac{1}{r^2} - \frac{1}{r^2+B(t)}\right)B(t) + c^2b^2\dot{b}^2\left(\frac{1}{r^2} - \frac{1}{r^2+B(t)}\right)B(t)}}{2cb^4\left(\frac{1}{r^2} - \frac{1}{r^2+B(t)}\right)} \tag{4.28}$$

Once we have solved $r(t)$ from (4.28), we obtain $R(t)$ from (3.29), $\alpha(t)$ from (4.25) and $\gamma(t)$ from (4.22), so we have the problem for the case $\beta = 2$. In Figure 3, using MATLAB, we have numerically solved $r(t)$, $R(t)$, $\alpha(t)$ and $\gamma(t)$ from nonlinear problem (4.28), (3.29), (4.22) and (4.25) when we take $b(t) = (1+t)/(2+t), c = 0.1, r(0) = 1, R(0) = 2$.

Comparing Figure 2 to Figure 3, we note that the solution to the nonlinear problem 2 is qualitatively very similar to that to the linear problem 1, only the limit values of $R(t)$ and $r(t)$ as $t \to \infty$ are slightly different.

5. Conclusion and discussion

In this paper, we are concerned with exact solutions to some interfacial problems with kinetic undercooling regularization in a Hele-Shaw cell with time-dependent gap $b(t)$. For both linear and nonlinear regularization, using Schwarz function, we first recovered the circular solutions for the linear case obtained in [42], and we did linear stability analysis of circular solution for the nonlinear case. Then we found that the solution of the initial ellipse shape of the free boundary could not keep elliptic shape for any small time interval for both linear and nonlinear regularization. Using PDE theory, we obtained exact solutions for the linear case and some nonlinear cases.

In [42], the existence of analytic solution of the problem with linear regularization was obtained for any initial analytic shape of simply connected domain. We are going to study the existence and uniqueness of analytic solution to the problem with nonlinear regularization when the initial free boundary is analytic.

6. Acknowledgement

This paper is dedicated to the occasion of Professor Gaston M. N'Guerekata's 70th birthday. The authors would like to thank Professor N'Guerekata for his support over the years.

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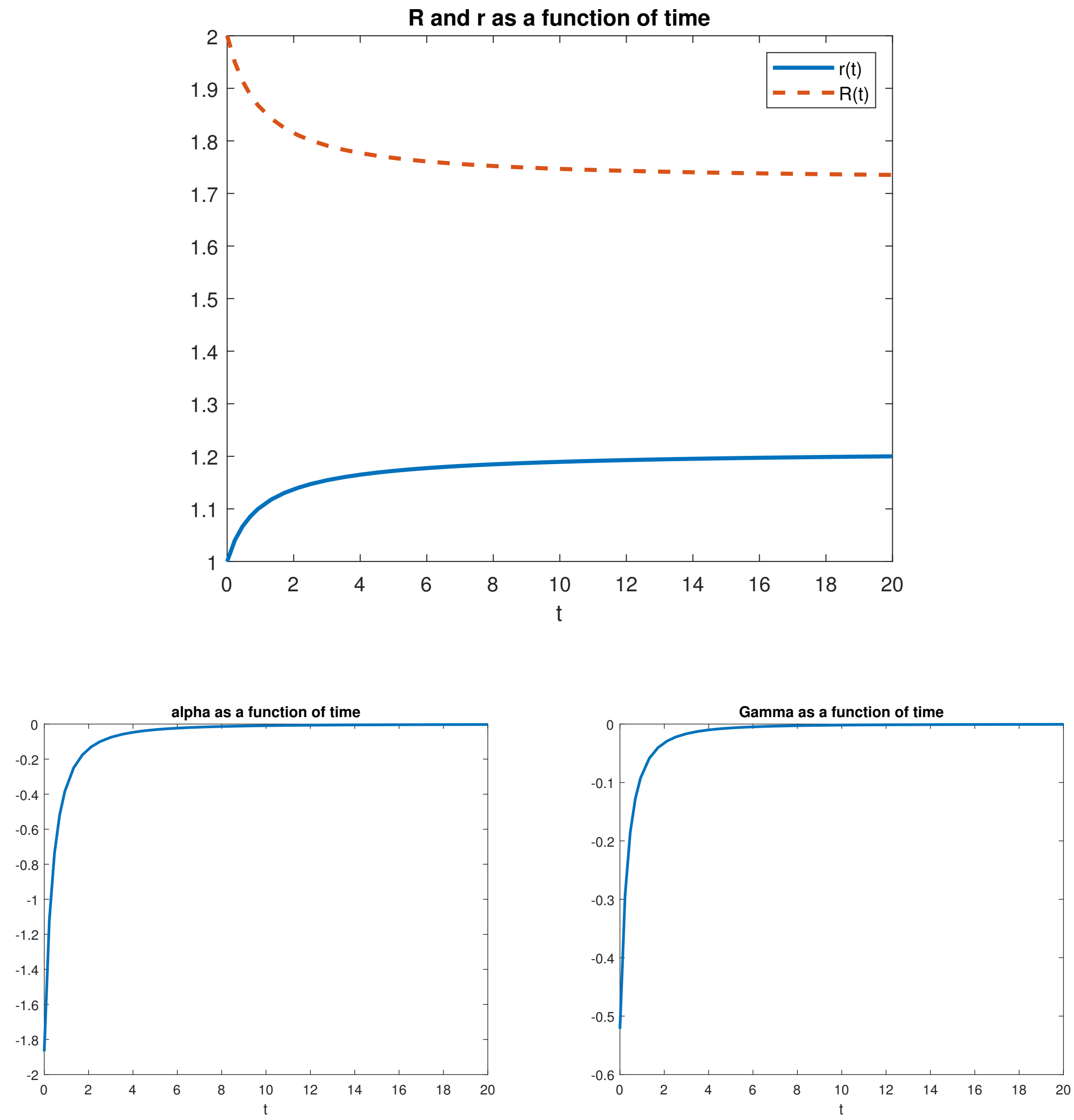

Figure 2: Numerical solution of $r(t)$, $R(t)$, $\alpha(t)$ and $\beta(t)$ from the linear problem 1 (3.23) - (3.26), when we take $b(t) = (1+t)/(2+t)$, $c = 0.1$, $r(0) = 1$, $R(0) = 2$.

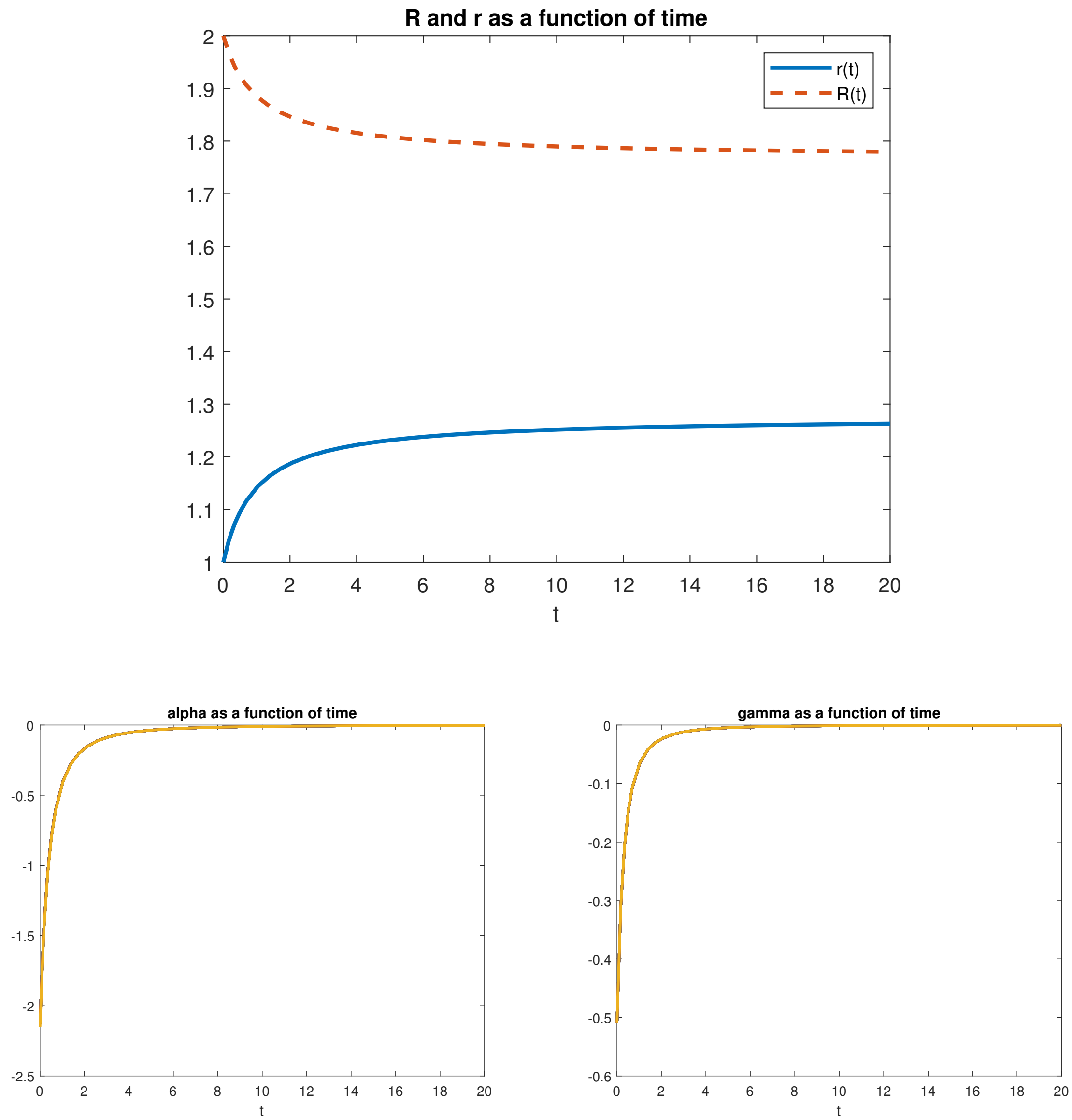

Figure 3: Numerical solution of $r(t)$, $R(t)$, $\alpha(t)$ and $\gamma(t)$ from the nonlinear problem 2 in a annular domain, when we take $\beta = 2, b(t) = (1+t)/(2+t), c = 0.1, r(0) = 1, R(0) = 2$.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 43–52.
http://doi.org/10.26637/mjm11S/003

Existence and stability analysis of solutions for fractional differential equations with delay

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*R*eceived 22 April 2023 ; *A*ccepted 19 July 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. In this manuscript, we establish the existence and stability of solutions for fractional differential equations with delay. We utilize the Bielecki Norm and the Ulam-Hyers stability for our results.
AMS Subject Classifications: 26A33.

Keywords: Existence, Stability, Fractional differential equations, Delay.

Contents

1. Introduction

The concept of the Deformable derivative was introduced by F. Zulfeqarr, A. Ujlayan, and P. Ahuja in 2017 [23]. It continuously deforms a function to a derivative, hence the name deformable derivative. This derivative is linearly related to the usual derivative. There are a few manuscripts pertaining to this fractional derivative. For more information, the reader could consult manuscripts such as [9, 10, 16–18, 23]. In [9], we established the existence and uniqueness of solutions to impulsive Cauchy problems involving the deformable derivative with local and nonlocal conditions.
In [10], we studied the existence of solutions for functional differential equations with infinite delay in the sense of the deformable derivative:

$$\begin{cases} D^\alpha y(t) = f(t, y_t), for\ t \in J = [0, b], \alpha \in (0, 1); \\ y(t) = \phi(t), t \in (-\infty, 0] \end{cases}$$

In this paper, we study the existence, uniqueness and the Ulam-Hyers type stability of solutions for the following fractional order differential equation:

$$\begin{cases} D^\alpha u(t) = f(t, u(t), u(\phi(t))) & t \in J = [0, T] \\ \quad u(t) = \mu(t) & t \in [-h, 0], \end{cases} \tag{1.1}$$

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where $0 < \alpha < 1, D^\alpha$ is the deformable derivative, $f \in C([0,T] \times \mathbb{R}^2, \mathbb{R}), \mu(t) \in C([-h,0], \mathbb{R}),\ \phi \in C([0,T],[-h,T])$; let $\phi(t) \leq t$.
The main motivation for this paper was the work of Develi and Duman (see [8]).

2. Preliminaries

In this section, $X := C([-h,T], \mathbb{R})$ stands for the Banach space of all continuous functions with the Bielecki norm:

$$\|u\|_{\mathcal{B}} := max\{|u(t)|e^{-\kappa t} : t \in [-h,T]\}.$$

Definition 2.1. *([23]) Let f be a real valued function on [a,b], $\alpha \in$ [0,1]. The Deformable derivative of f of order α at $t \in (a,b)$ is defined as:*

$$D^\alpha f(t) = \lim_{\epsilon \to 0} \frac{(1+\epsilon\beta) f(t+\epsilon\alpha) - f(t)}{\epsilon},$$

where $\alpha + \beta = 1$. If the limit exists, we say that f is α-differentiable at t.

Remark 2.2. *If $\alpha = 1$, then $\beta = 0$, we recover the usual derivative. This shows that the deformable derivative is more general than the usual derivative.*

Definition 2.3. *([23]) For f defined on $[a,b], \alpha \in (0,1]$, the α-integral of the function f is defined by*

$$I_a^\alpha f(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_a^t e^{\frac{\beta}{\alpha} x} f(x) dx, \quad t \in [a,b],$$

where $\alpha + \beta = 1$. When $a = 0$ we use the notation

$$I^\alpha f(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} x} f(x) dx.$$

Remark 2.4. *If $\alpha = 1$, then $\beta = 0$, we recover the usual Riemann integral. This also shows that the α-integral is more general than the usual Riemann integral.*

Theorem 2.5. ([23]) A differentiable function h at a point $t \in (a,b)$ is always α-differentiable at that point for any α. Moreover, we have

$$D^\alpha h(t) = \beta h(t) + \alpha D h(t).$$

Corollary 2.6. *([23]) An α - differentiable function f defined in (a,b) is differentiable as well.*

Theorem 2.7. ([17],[23]) *The operators D^α and I_a^α possess the following properties:*
Let $\alpha, \alpha_1, \alpha_2 \in (0,1]$ such that $\alpha + \beta = 1,\ \alpha_i + \beta_i = 1$ for $i = 1,2$.

1. *Let f be differentiable at a point t for some α. Then it is continuous there.*

2. *Suppose f and g are α-differentiable. Then*

$$\begin{aligned} D^\alpha (f \circ g)(t) &= \beta (f \circ g)(t) + \alpha D(f \circ g)(t) \\ &= \beta (f \circ g)(t) + \alpha f'(g(t)) g'(t). \end{aligned}$$

3. *Let f be continuous on [a,b]. Then $I_a^\alpha f$ is α- differentiable in (a,b), and we have*

$$D^\alpha (I_a^\alpha f(t)) = f(t), and$$

$$I_a^\alpha (D^\alpha f(t)) = f(t) - e^{\frac{\beta}{\alpha}(a-t)} f(a).$$

4. $D^{\alpha}\left(\frac{f}{g}\right) = \frac{gD^{\alpha}(f) - \alpha f Dg}{g^2}$.

5. *Linearity :* $D^{\alpha}(af + bg) = aD^{\alpha}f + bD^{\alpha}g$.

6. *Commutativity :* $D^{\alpha_1} \cdot D^{\alpha_2} = D^{\alpha_2} \cdot D^{\alpha_1}$.

7. *For a constant* $c, D^{\alpha}(c) = \beta c$.

8. $D^{\alpha}(fg) = (D^{\alpha}f)g + \alpha f Dg$.

9. *Linearity :* $I_a^{\alpha}(bf + cg) = bI_a^{\alpha}f + cI_a^{\alpha}g$.

10. *Commutativity :* $I_a^{\alpha_1}I_a^{\alpha_2} = I_a^{\alpha_2}I_a^{\alpha_1}$.

Definition 2.8. *Problem (1.1) is Ulam-Hyers stable if there exists a real number* $\zeta > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $\theta \in C([-h,T],\mathbb{R})$ *of the inequality*

$$|D^{\alpha}\theta(t) - f(t,\theta(t),\theta(\phi(t)))| \leq \epsilon, \ \ t \in [0,T], \tag{2.1}$$

there exists a solution u *in* $C([-h,T],\mathbb{R})$ *to problem (1.1) with*

$$|\theta(t) - u(t)| \leq \zeta\epsilon, \ \ t \in [-h,T].$$

Remark 2.9. *A function* $\theta \in C([0,T],\mathbb{R})$ *is a solution of the inequality (2.1) if and only if there exists a function* $\Omega \in C([0,T],\mathbb{R})$ *such that*
$(i)|\Omega(t)| \leq \epsilon$ *for all* $t \in [0,T]$,
$(ii)D^{\alpha}\theta(t) = f(t,\theta(t),\theta(\phi(t))) + \Omega(t)$ *for all* $t \in [0,T]$.

Remark 2.10. *It can readily be seen that using Definition 2.3 and Theorem 2.7, a solution* $\theta \in C([0,T],\mathbb{R})$ *of inequality (2) is also a solution to the following integral inequality:*

$$\left|\theta(t) - \theta(0)e^{\frac{-\beta}{\alpha}t} - \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s,\theta(s),\theta(\phi(s)))\right| ds \leq \frac{\epsilon}{\beta}$$

for all $t \in [0,T]$.

We derive the following inequality for our subsequent results:
For $\kappa > 0, 0 \leq s \leq t, t \in [0,T]$,

$$\begin{aligned}
\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}e^{\kappa s}ds &= \frac{1}{\alpha}\int_0^t e^{\frac{-\beta}{\alpha}(t-s)}e^{\kappa s}ds \\
&\leq \frac{1}{\alpha}\int_0^t e^{\kappa s}ds \\
&\leq \frac{e^{\kappa t}}{\kappa\alpha}.
\end{aligned}$$

Definition 2.11. *[21, 22]Let* (X,d) *be a metric space. An operator* $\mathcal{A} : X \to X$ *is said to be a Picard operator if there exists* $x^* \in X$ *such that*
(i) $F_{\mathcal{A}} = \{x^*\}$ *where* $F_{\mathcal{A}} = \{x \in X : \mathcal{A}(x) = x\}$ *is the fixed point set of* $\mathcal{A}$*;*
(ii) The sequence $(\mathcal{A}^n(x_0))_{n\in N}$ *converges to* x^* *for all* $x_0 \in X$.

Lemma 2.12. *[21, 22]Let* $(X,d,\leq)$ *be an ordered metric space and* $\mathcal{A} : X \to X$ *be an increasing Picard operator* $(F_{\mathcal{A}} = \{x^*\})$*. Then, for* $x \in X, x \leq \mathcal{A}(x) \implies x \leq x^*$ *while* $x \geq \mathcal{A}(x) \implies x \geq x^*$.

Lemma 2.13. *[6] If*

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds, \ \ t \in [t_0, T),$$

where all functions involved are continuous on $[t_0, T), T \leq +\infty$, *and* $k(t) \geq 0$, *then* $x(t)$ *satisfies*

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s) \ exp\left[\int_{s}^{t} k(\omega)d(\omega)\right] ds, \ t \in [t_0, T).$$

3. Existence and Uniqueness

In this section, we prove the existence and uniqueness of solutions for problem (1.1).

Definition 3.1. *A function* $u \in C([-h, T], \mathbb{R})$ *is said to be a mild solution of problem (1.1) if*

$$u(t) = \begin{cases} \mu(t), & t \in [-h, 0] \\ \mu(0)e^{\frac{-\beta}{\alpha}t} + \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s}f(s, u(s), u(\phi(s)))ds, & t \in [0, T]. \end{cases}$$

We investigate problem (1.1) with the following assumptions:
(H1) $f \in C([0, T] \times \mathbb{R}^2, \mathbb{R}), \phi \in C([0, T], [-h, T])$ and $\phi(t) \leq t$ on $[0, T]$,
(H2) There is a constant $L > 0$ such that
$|f(t, u_1, \theta_1) - f(t, u_2, \theta_2)| \leq L(|u_1 - u_2| + |\theta_1 - \theta_2|)$ for all $u_i, \theta_i \in C([-h, T], \mathbb{R})$ and $t \in [0, T]$.

Theorem 3.2. *Under the asumptions (H1)-(H2), if* $\kappa > \frac{2L}{\alpha}$, *then problem (1.1) has a unique mild solution.*

Proof. We first transform problem (1.1) into a fixed point problem.
Define $\mathcal{F} : C([-h, T], \mathbb{R}) \to C([-h, T], \mathbb{R})$ such that

$$\mathcal{F}u(t) = \begin{cases} \mu(t), & t \in [-h, 0] \\ \mu(0)e^{\frac{-\beta}{\alpha}t} + \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s}f(s, u(s), u(\phi(s)))ds, & t \in [0, T]. \end{cases} \tag{3.1}$$

Then we find a unique fixed point of $\mathcal{F}$, which is the unique solution. We consider the Banach space $X := C([-h, T], \mathbb{R})$ endowed with following norm

$$\|u\|_{\mathcal{B}} = \max_{t \in [-h, T]} |u(t)|e^{-\kappa t}. \tag{3.2}$$

Using Remark 2.10, we show that $\mathcal{F}$ is a contraction mapping on $(X, \| \cdot \|_{\mathcal{B}})$. For all $u(t), \theta(t) \in X, \mathcal{F}u(t) = \mathcal{F}\theta(t)$ if $t \in [-h, 0]$. For $t \in [0, T]$, we have

$$\begin{aligned} &|\mathcal{F}u(t) - \mathcal{F}\theta(t)| \\ &\leq \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}|f(s, u(s), u(\phi(s))) - f(s, \theta(s), \theta(\phi(s)))|ds \\ &\leq \frac{L}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}e^{\kappa s}\left(\max_{-h \leq s \leq T}|u(s) - \theta(s)|e^{-\kappa s} + \max_{-h \leq s \leq T}|u(\phi(s)) - \theta(\phi(s))|e^{-\kappa s}\right) \\ &\leq \frac{2L}{\alpha}e^{\frac{-\beta}{\alpha}t}\|u - \theta\|_{\mathcal{B}}\int_0^t e^{\frac{\beta}{\alpha}s}e^{\kappa s}ds \\ &\leq \frac{2L}{\kappa\alpha}\|u - \theta\|_{\mathcal{B}}e^{\kappa t}. \end{aligned}$$

Thus

$$\|\mathcal{F}u - \mathcal{F}\theta\|_{\mathcal{B}} \leq \eta\|u-\theta\|_{\mathcal{B}}, \text{where } \eta = \frac{2L}{\kappa\alpha}.$$

Since $\eta < 1$, we find a unique fixed point $\mathcal{F}$ by the Banach contraction principle. ■

Remark 3.3. *For a constant delay $\tau > 0,$ and $\phi(t) = t - \tau$, problem (1.1) becomes*

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), u(t-\tau)), & t \in [0, T] \\ u(t) = \mu(t), & t \in [-\tau, 0]. \end{cases} \tag{3.3}$$

The proof for the existence and uniqueness of solutions for the above fractional differential equation is obtained using the following three steps. To that end, we introduce the following Lipschitz condition.

Theorem 3.4. *Let $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume that there exists a positive constant L such that*

$$|f(t, u_1, \theta) - f(t, u_2, \theta)| \leq L|u_1 - u_2|$$

for all $u_i, \theta \in C([0, T], \mathbb{R}), (i = 1, 2, ...)$ and $t \in [0, T]$. And in addition, assume that $\kappa > \frac{L}{\alpha}$.Then (3.3) has a unique solution.

Proof. Problem (3.3) is equivalent to:

$$u(t) = \begin{cases} \mu(t), & -\tau \leq t \leq 0 \\ \mu(0)e^{\frac{-\beta}{\alpha}t} + \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s}f(s, u(s), u(s-\tau)ds, & 0 \leq t \leq T. \end{cases}$$

We partition the interval [0,T] into n sub-intervals of equal length S. And have the following for $0 < S < \tau$ and $nS = T$: $0 = S_0 < S_1 < \cdots < S_n = T, S_i - S_{i-1} = S$.
We see that $t \leq S_{i+1} \implies t - \tau \leq S_i$ using this argument:
$t \leq S_{i+1} \implies t - \tau \leq S_{i+1} - \tau \leq S_{i+1} - S = S_i$.
Step 1. let $(\mathcal{E}_1, \|\cdot\|_1)$ be a Banach space of continuous functions $u : [-\tau, S_1] \to \mathbb{R}$ with the following norm :

$$\|u\|_1 = \max_{t\in[-\tau,S_1]} |u(t)|e^{-\kappa t},$$

and $u(t) = \mu(t)$ for $-\tau \leq t \leq 0$. Define a mapping $\mathcal{F}_1 : \mathcal{E}_1 \to \mathcal{E}_1$ by:

$$\mathcal{F}_1u(t) = \begin{cases} \mu(t), & -\tau \leq t \leq 0 \\ \mu(0)e^{\frac{-\beta}{\alpha}t} + \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s}f(s, u(s), u(s-\tau)ds, & 0 \leq t \leq S_1. \end{cases}$$

For $u(t), \theta(t) \in \mathcal{E}_1, \mathcal{F}_1u(t) = \mathcal{F}_1\theta(t)$ if $t \in [-\tau, 0]$, For $t \in [0, S_1]$, we have

$$|\mathcal{F}_1u(t) - \mathcal{F}_1\theta(t)| \leq \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s, u(s), u(s-\tau) - f(s, \theta(s), \theta(s-\tau)))|ds.$$

Since $0 \leq s \leq S_1$ implies $(s-\tau) \in [-\tau, 0]$, and the definition of $\mathcal{E}_1$, we have

$$\begin{aligned}
&|\mathcal{F}_1 u(t) - \mathcal{F}_1 \theta(t)| \\
&\qquad \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, u(s), u(s-\tau)) - f(s, \theta(s), \theta(s-\tau))| ds \\
&\qquad \leq \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} e^{\kappa s} [\max_{-h \leq s \leq s_1} |u(s) - \theta(s)| e^{-\kappa s}] ds \\
&\qquad \leq \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t} \|u - \theta\|_1 \int_0^t e^{\frac{\beta}{\alpha} s} e^{\kappa s} ds \\
&\qquad \leq \frac{L}{\kappa \alpha} \|u - \theta\|_1 e^{\kappa s}.
\end{aligned}$$

Therefore,

$$\|\mathcal{F}_1 u - \mathcal{F}_1 \theta\|_1 \leq \eta \|u - \theta\|_1.$$

Since $\eta = \frac{L}{\kappa \alpha} < 1$, we get that $\mathcal{F}_1$ is a contraction mapping, and so there exists a unique fixed point $\mu_1 \in \mathcal{E}_1$ that satisfies (3.3) on $[-\tau, s_1]$.

Step 2: In this step, we extend the interval of step 1 into $[-\tau, S_2]$. Let $(\mathcal{E}_2, \|\cdot\|_2)$ be a complete normed space of continuous functions $u : [-\tau, S_2] \to \mathbb{R}$ with the following norm

$$\|u\|_2 = \max_{t \in [-r, S_2]} |u(t)| e^{-\kappa s}.$$

Let $u(t) = \mu_1(t)$ for $-\tau \leq t \leq S_1$. Continuing in like manner, define a mapping $\mathcal{F}_2 : \mathcal{E}_2 \to \mathcal{E}_2$ by

$$\mathcal{F}_2 u(t) = \begin{cases} \mu_1(t), & -\tau \leq t \leq S_1 \\ \mu(0) e^{\frac{-\beta}{\alpha} t} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, u(s), u(s-\tau)) ds, & S_1 \leq t \leq S_2. \end{cases}$$

For $u(t), \theta(t) \in \mathcal{E}_2, F_2 u(t) = F_2 \theta(t)$ if $t \in [-\tau, S_1]$; else we take $t \in [S_1, S_2]$. Thus

$$|F_2 u(t) - F_2 \theta(t)| \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, u(s), u(s-\tau)) - f(s, \theta(s), \theta(s-\tau))| ds.$$

Observe that $0 \leq s \leq S_2 \implies (s-r) \in [-\tau, S_1]$. Based on the the definition of $\mathcal{E}_2$, we may derive the following inequality:

$$\begin{aligned}
&|\mathcal{F}_2 u(t) - \mathcal{F}_2 \theta(t)| \\
&\qquad \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, u(s), \mu_1(s-\tau)) - f(s, \theta(s), \mu_1(s-\tau))| ds \\
&\qquad \leq \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} e^{\kappa s} (\max_{-h \leq s \leq S_2} |u(s) - \theta(s)| e^{-\kappa s}) ds \\
&\qquad \leq \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t} \|u - \theta\|_2 \int_0^t e^{\frac{\beta}{\alpha} s} e^{\kappa s} ds \\
&\qquad \leq \frac{L}{\kappa \alpha} \|u - \theta\|_2 e^{\kappa t}.
\end{aligned}$$

Thus, $\|\mathcal{F}_2 u - \mathcal{F}_2 \theta\|_2 \leq \eta \|u - \theta\|_2$, where $\eta < 1$ as aforementioned. Therefore, $\mathcal{F}_2$ has a unique fixed point μ_2 in $\mathcal{E}_2$ that satisfies (3.3) on $[-\tau, S_2]$.

Step 3: By following this method up to the the nth step, we can find that $\mathcal{F}_n$ has a unique fixed point μ_n in $\mathcal{E}_n$ satisfying (3.3) on $[-\tau, S_n] = [-\tau, T]$. ■

4. Ulam-Hyers stability.

Theorem 3.5. *Assume that conditions H1 and H2 are fulfilled. Then the first equation of problem (1.1) is Ulam-Hyers stable.*

Proof. Let θ be a solution to (2.1) and u be a unique solution to the following problem:

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), u(\phi(t)) & t \in [0, T] \\ u(t) = \theta(t) & t \in [-h, 0]. \end{cases}$$

Then

$$u(t) = \begin{cases} \theta(t) & t \in [-h, 0] \\ \theta(0)e^{\frac{-\beta}{\alpha}t} + \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s}f(s, u(s), u(\phi(s))ds & t \in [0, T]. \end{cases}$$

Observe that we have the following inequality from Remark 2.10:

$$|\theta(t) - \theta(0)e^{\frac{-\beta}{\alpha}t} - \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s, \theta(s), \theta(\phi(s)))ds| \leq \frac{\epsilon}{\beta}$$

for all $t \in [0, T]$, and $|\theta(t) - u(t)| = 0$ for all $t \in [-h, 0]$. For $t \in [0, T]$ we obtain from $H2$ that

$$\begin{aligned} |\theta(t) - u(t)| \leq & |\theta(t) - \theta(0)e^{\frac{-\beta}{\alpha}t} - \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s, \theta(s), \theta(\phi(s)))ds| \\ & + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}|f(s, \theta(s), \theta(\phi(s)) - f(s, u(s), u(\phi(s))|ds \\ \leq & \frac{\epsilon}{\beta} + \frac{L}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}(|\theta(s) - u(s)| + |\theta(\phi(s)) - u(\phi(s))|)ds. \end{aligned} \tag{3.4}$$

We define an operator for $v \in C([-h, T], \mathbb{R}^+)$:

$$\mathcal{A} := C([-h, T], \mathbb{R}^+) \to C([-h, T], \mathbb{R}^+),$$

given by

$$\mathcal{A}(v)(t) = \begin{cases} 0 & t \in [-h, 0] \\ \dfrac{\epsilon}{\beta} + \dfrac{L}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s}\left(v(s) + v(\phi(s))\right)ds & t \in [0, T]. \end{cases}$$

We show that $\mathcal{A}$ is a Picard operator via the contraction mapping principle. For $v, \tilde{v} \in C([-h, T], \mathbb{R}^+)$, one estimates

$$\begin{aligned} |\mathcal{A}v - \mathcal{A}\tilde{v}| &\leq \frac{L}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}(|v(s) - \tilde{v}(s)| + |v(\phi(s)) - \tilde{v}(\phi(s))|)ds \\ &\leq \frac{2L}{\alpha}\|v - \tilde{v}\|_{\mathcal{B}}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}e^{\kappa s}ds \\ &\leq \frac{2L}{\kappa\alpha}\|v - \tilde{v}\|_{\mathcal{B}}e^{\kappa t}ds, \end{aligned}$$

which means

$\|\mathcal{A}v - \mathcal{A}\tilde{v}\|_{\mathcal{B}} \leq \eta\|v - \tilde{v}\|_{\mathcal{B}}$ where $\eta = \frac{2L}{\kappa\alpha}$.

For $\kappa > \frac{2L}{\alpha} > 0$, we observe that $\eta < 1$, and consequently we get that $\mathcal{A}$ is a contraction mapping with respect to the Bielecki norm $\|\cdot\|_{\mathcal{B}}$ on $C([-h,T],\mathbb{R}^+)$. Thus, $\mathcal{A}$ is a Picard operator such that $F_{\mathcal{A}} = \{v^*\}$ and the Banach Contraction principle gives the equality:

$$v^*(t) = \frac{\epsilon}{\beta} + \frac{L}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}(v^*(s) + v^*(\phi(s)))ds$$

for $t \in [0,T]$. To show that v^* is increasing, let $m := \min_{t\in[0,T]}[v^*(t) + v^*(\phi(t))] \in \mathbb{R}^+$. For $0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned}
&v^*(t_2) - v^*(t_1)\\
&= \frac{L}{\alpha} e^{\frac{-\beta}{\alpha}t_2} \int_0^{t_2} e^{\frac{\beta}{\alpha}s}(v^*(s) + v^*(\phi(s)))ds - \frac{L}{\alpha} e^{\frac{-\beta}{\alpha}t_1} \int_0^{t_1} e^{\frac{\beta}{\alpha}s}(v^*(s) + v^*(\phi(s)))ds\\
&= \frac{L}{\alpha} \int_0^{t_2} e^{\frac{-\beta}{\alpha}t_2} e^{\frac{\beta}{\alpha}s}(v^*(s) + v^*(\phi(s)))ds - \frac{L}{\alpha} \int_0^{t_1} e^{\frac{-\beta}{\alpha}t_1} e^{\frac{\beta}{\alpha}s}(v^*(s) + v(\phi(s)))ds\\
&= \frac{L}{\alpha} \int_0^{t_1} (e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}) e^{\frac{\beta}{\alpha}s}(v^*(s) + v^*(\phi(s)))ds + \frac{L}{\alpha} \int_{t_1}^{t_2} e^{\frac{-\beta}{\alpha}t_2} e^{\frac{\beta}{\alpha}s}(v^*(s) + v^*(\phi(s)))ds\\
&\ge \frac{mL}{\alpha} \int_0^{t_1} (e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}) e^{\frac{\beta}{\alpha}s}ds + \frac{mL}{\alpha} \int_{t_1}^{t_2} e^{\frac{-\beta}{\alpha}t_2} e^{\frac{\beta}{\alpha}s}ds\\
&= \frac{mL}{\alpha} (e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}) \int_0^{t_1} e^{\frac{\beta}{\alpha}s}ds + \frac{mL}{\alpha} e^{\frac{-\beta}{\alpha}t_2} \int_{t_1}^{t_2} e^{\frac{\beta}{\alpha}s}ds\\
&= \frac{mL}{\beta}\left[e^{\frac{-\beta}{\alpha}(t_2-t_1)} - e^{\frac{-\beta}{\alpha}t_2} - 1 + e^{\frac{-\beta}{\alpha}t_1}\right] + \frac{mL}{\beta}\left[1 - e^{\frac{-\beta}{\alpha}(t_2-t_1)}\right]\\
&= \frac{mL}{\beta}\left[e^{\frac{-\beta}{\alpha}t_1} - e^{\frac{-\beta}{\alpha}t_2}\right] > 0.
\end{aligned}$$

Therefore, v^* is an increasing function, and so $v^*(\phi(t)) \le v^*(t)$ because $\phi(t) \le t$. It follows that

$$v^*(t) \le \frac{\epsilon}{\beta} + \frac{2L}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}(v^*(s))ds.$$

Using Lemma 2.13, one derives the following inequality

$$\begin{aligned}
v^*(t) &\le \frac{\epsilon}{\beta} + \frac{2L}{\alpha} \int_0^t e^{\frac{-\beta}{\alpha}t} e^{\frac{\beta}{\alpha}s} \frac{\epsilon}{\beta}\, exp\left[\int_s^t e^{\frac{-\beta}{\alpha}t} e^{\frac{\beta}{\alpha}\omega} d\omega\right] ds\\
&\le \frac{\epsilon}{\beta} + \frac{2L\epsilon}{\alpha\beta} \int_0^t e^{\frac{-\beta}{\alpha}t} e^{\frac{\beta}{\alpha}s}\, exp\left[\frac{\alpha}{\beta}\right] ds\\
&\le \frac{\epsilon}{\beta} + exp\left[\frac{\alpha}{\beta}\right] \frac{2L\epsilon}{\beta^2}\left[1 - e^{\frac{-\beta}{\alpha}t}\right]\\
&\le \frac{\epsilon}{\beta}\left[1 + \frac{2e^{\frac{\alpha}{\beta}}}{\beta}L\right]
\end{aligned}$$

for $t \in [-h,T]$. If $v = |\theta - u|$ in (3.4), then $v \le \mathcal{A}v$. So, we have $v < v^*$ because $\mathcal{A}$ is an increasing Picard operator. Consequently, we have

$$|\theta(t) - u(t)| \le \zeta\epsilon$$

where

$$\zeta = \frac{1}{\beta}\left[1 + \frac{2}{\beta} L e^{\frac{\alpha}{\beta}}\right].$$

Thus the first equation of problem (1.1) is Ulam-Hyers stable. The proof is complete. ■

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 53–69.
http://doi.org/10.26637/mjm11S/004

New subfamilies of univalent functions defined by Opoola differential operator and connected with modified Sigmoid function

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***R**eceived 25 April 2023; **A**ccepted 21 July 2023*

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. In this exploration, by making use of the Hadamard product of Opoola differential operator and modified sigmoid function, we define new subclasses of analytical and univalent functions $\mathcal{T}_n S^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ and $\mathcal{T}_n \mathcal{V}^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ and discussed some properties of the classes; such as the coefficient estimates, Growth and Closure theorems.
AMS Subject Classifications: 30C45, 30C50.

Keywords: Analytic Function, Hadamard product, modified sigmoid function, Opoola differential operator, Univalent Function.

Contents

1. Introduction and Background

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathcal{B}$ denote the class of functions $f(z)$ which are analytic in the open unit disk and of the form

$$f(z) = z + \sum_{t=2}^{\infty} a_t z^t. \tag{1.1}$$

Also, let

$$\gamma(s) = \frac{2}{(1 + e^{-s})}; \qquad s \geq 0 \tag{1.2}$$

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with $\gamma(s)=1$ for $s=0$ be the modified Sigmoid function. (See details in [6], [7], [10], [5], [11], [15]).

Additionally, let $T\in\mathcal{A}$ be the class of functions of the form

$$f(z)=z-\sum_{t=2}^{\infty}a_tz^t, \qquad a_t\geq 0 \tag{1.3}$$

For $f_\gamma(z)\in\mathcal{T}_\gamma$, [11] gave the following definition:

$$f_\gamma(z)=z-\sum_{t=2}^{\infty}\gamma(s)\,a_tz^t, \qquad a_t\geq 0 \tag{1.4}$$

as a consequence of (1.3).

Note that $\gamma(s)=1+\frac{1}{2}s-\frac{1}{24}s^3+\frac{1}{240}s^5-\frac{17}{40320}s^7+\cdots$ defined by (1.2). Furthermore, we define identity function for $\mathcal{T}_\gamma$ as

$$e_\gamma(z)=z. \tag{1.5}$$

For the purpose of defining the new differential operator of interest, the following definitions are required:

Definition 1.1. *[12] For $f(z)\in\mathcal{A}$, where $k\geq 0, 1\leq\mu\leq\rho, n\in\mathbb{N}_0$ and $z\in U$, the Opoola differential operator $D_k^n(\mu,\rho)f:\mathcal{A}\to\mathcal{A}$ is defined in [12] as*

$$\begin{aligned}
&D_0^k(\mu,\rho)f(z)=f(z)\\
&D_1^k(\mu,\rho)f(z)=tzf'(z)-z(\rho-\mu)k+(1+(\rho-\mu-1)k)f(z)\\
&D_n^k(\mu,\rho)f(z)=(D(D_k^{n-1}f(z))).
\end{aligned}$$

The $f(z)$ given in above (1.1) we get,

$$D_k^n(\mu,\rho,)f(z)=z+\sum_{t=2}^{\infty}[1+(t+\rho-\mu-1)k]^n\,a_tz^t. \tag{1.6}$$

It is evident that (1.6) reduces to Al- Oboudi differential operator [1, 3] and Salagean differential operator [22] by varying the involving parameters appropriately. We further note that, other works on (1.16) can be found in [4, 14, 16–18, 20–24, 26].

Definition 1.2. *[11] introduced the generalized differential operator $D^n_{\lambda,\omega}f_\gamma(z)$ involving sigmoid function which is a special case of (1.6):*

$$D^n_{\lambda,\omega}f_\gamma(z)=\gamma^n(s)z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)[(t-1)(\lambda-\omega)+t]^n\,a_tz^t \tag{1.7}$$

for $\lambda,\omega\geq 0$. For more information on this, interested reader may refer to [8].

Definition 1.3 (Hadamard product or convolution). *The Hadamard (or convolution) of two analytic functions $f(z)$ given by (1.1) and $g(z)=z+\sum_{t=2}^{\infty}b_tz^t$ is given by*

$$f(z)*g(z)=(f*g)(z)=z+\sum_{t=2}^{\infty}a_tb_t\,z^t, \qquad z\in U. \tag{1.8}$$

Following (1.8) for (1.6) and (1.7), a certain new differential operator associated with Sigmoid function involving convolution is defined as follows:

$$\begin{aligned}
&D^n_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)=(D^n_{\lambda,\omega}f_\gamma(z))*(D^n(\mu,\rho,k)f_\gamma(z))\\
&=\gamma^n(s)z+\sum_{t=2}^{\infty}\gamma^{n+1}(s)[1+(t+\rho-\mu-1)k)]^n[(t-1)(\lambda-\omega)+t]^na_tz^t
\end{aligned} \tag{1.9}$$

Remark 1.1. *When* $\gamma^n(s) = 1,\ \lambda = 1,\ \omega = 2$ *we have*

$$D^n_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) = \gamma^n(s)z + \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^n a_t z^t. \quad (1.10)$$

Remark 1.2. *When* $\gamma^n(s) = 1,\ \ for\ s = 0,\ \lambda = 1,\ \omega = 2,\ \mu = \rho,\ k = 1$ *we have the Salagean differential operator, see [15] and [8].*

Remark 1.3. *When* $\gamma^n(s) = 1,\ \lambda = 1,\ \omega = 2,\ \mu = \rho,$ *we have the Al-Oboudi differential operator, see [3] and [25].*

For the purpose of the main result, we rewrite equation (1.9) as follows for convenience

$$D^\beta_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) = \gamma^n(s)z + \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^\beta a_t z^t \quad (1.11)$$

where $\beta,\lambda,\omega \in \mathbb{R},\ \beta \geq\ 0,\ \lambda \geq 0,\ \omega \geq 0$ and it is denoted by $D^\beta_{\lambda,\omega}(\mu,\rho,k) : \mathcal{A} \to \mathcal{A}$.

Also, if $f \in C,\ f(z) = z - \sum_{t=2}^{\infty} a_t z^t, \quad a_t \geq 0,\ t = 2,\ z \in U.$

Then,

$$D^\beta_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) = z - \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^\beta a_t z^t \quad (1.12)$$

For convenience upon (1.11) we have the following definition

Definition 1.4. *We say that class* $\mathcal{T}_n\, S^k_p(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\gamma,\omega)$ *contain function* $f(z) \in \mathcal{T}$ *if and only if*

$$\left|\frac{\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1}{(M-L)\xi\Big(\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - \phi\Big) + L\lambda\Big(\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1\Big)}\right| < \delta \quad (1.13)$$

where $|z| < 1,\ 0 < \delta \leq 1,\ \frac{1}{2} \leq \xi \leq 1,\ k \geq 0,\ \mu \geq 0,\ \rho \geq 0, 0 \leq \phi \leq \frac{1}{2}\xi,\ \frac{1}{2} \leq \lambda \leq 1,\ \beta \geq 0,\ 0 < M \leq 1,\ -1 \leq L < M \leq 1,\ \omega \geq 0,\ \gamma^{n+1}(s) = 1,\ s = 0.$

Definition 1.5. *The class* $\mathcal{T}_n\, \mathcal{V}^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\gamma,\omega)$ *contain function* $f(z) \in \mathcal{T}$ *if and only if*

$$\left|\frac{\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1}{(M-L)\xi\Big(\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - \phi\Big) + L\lambda\Big(\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1\Big)}\right| < \delta \quad (1.14)$$

where $|z| < 1,\ 0 < \delta \leq 1,\ \frac{1}{2} \leq \xi \leq 1,\ k \geq 0,\ \mu \geq 0,\ \rho \geq 0, 0 \leq \phi \leq \frac{1}{2}\xi,\ \frac{1}{2} \leq \lambda \leq 1,\ \beta \geq 0,\ 0 < M \leq 1,\ -1 \leq L < M \leq 1,\ \omega \geq 0,\ \gamma^{n+1}(s) = 1,\ s = 0.$

Remark 1.4. *After putting* $\mu = \rho = 1,\ \gamma^n(s) = 1,\ \lambda = 1,\ \omega = 2,$ *we obtain the corresponding results of [25].*

Remark 1.5. *After putting* $\beta = 0,\ \mu = \rho = 1,\ \gamma^n(s) = 1,\ \lambda = 1,\ \omega = 2,$ *we get the corresponding sequal obtained by [9].*

Remark 1.6. *After putting* $\beta = 0,\ \mu = \rho = 1,\ k = 1,\ \lambda = 1,\ \omega = 2,$ *we get the corresponding sequal obtained by [2].*

Remark 1.7. *After putting* $\beta = 0,\ \mu = \rho = 1,\ k = 1,\ \xi = 1,\ \lambda = 1,\ \omega = 2,$ *we get the corresponding sequal obtained by [13].*

Let equation (1.11) be $\left|\frac{A}{B}\right| < \delta$

$$\left|\frac{A}{B}\right| < \delta = \frac{|A|}{|B|} < \delta \Rightarrow |A| < \delta|B| \tag{1.15}$$

when

$$A = \frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1 \tag{1.16}$$

$$A = \frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} \tag{1.17}$$

$$\begin{aligned} B = (M-L)\xi &\left(\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) - \phi D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}\right) + \\ L\lambda &\left(\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}\right) \end{aligned} \tag{1.18}$$

$$\left|\frac{A}{B}\right| = \frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{\begin{vmatrix}(M-L)\xi[D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) - \phi D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)] \\ +L\lambda[D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)]\end{vmatrix}} \tag{1.19}$$

2. Main Results

2.1. Coefficient Estimates

Theorem 2.1. *The class $\mathcal{T}_n\, S^k_p(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ contains a function $f(z)$ defined by (1.3) if and only if*

$$\begin{aligned} \sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} &\Big\{(t+\rho-\mu-1)k \\ [(t-1)(\lambda-\omega)+t]\,[1+L\lambda\delta+(M-L)\delta\xi] &+ (M-L)\delta\xi(1-\phi)\Big\}\, a_t \\ \leq (M-L)\delta\xi(1-\phi) & \end{aligned} \tag{2.1}$$

Proof. Suppose,

$$\begin{aligned} \sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} &\Big\{(t+\rho-\mu-1)k \\ [(t-1)(\lambda-\omega)+t]\,[1+L\lambda\delta+(M-L)\delta\xi] &+ (M-L)\delta\xi(1-\phi)\Big\}\, a_t \\ \leq (M-L)\delta\xi(1-\phi) & \end{aligned}$$

We have,

$$\begin{aligned} &\left|D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)\right| \\ &-\ \delta\Big|(M-L)\xi[D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) - \phi D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)] \\ &+ L\lambda[D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)]\Big| < 0 \end{aligned} \tag{2.2}$$

with the provision

$$\begin{aligned}
&\Bigg| z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t \\
&-\ z\ +\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t\Bigg| \\
&-\ \delta\Bigg|(M-L)\xi\Big[z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t \\
&-\ \phi z+\ \phi\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t\Big] \\
&+\ L\lambda\,\Big[z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t \\
&-\ z+\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t\Big]\ \Bigg|<0
\end{aligned} \tag{2.3}$$

Let $S_1=\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t$ and $S_0=\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t$.

The simplified expression becomes,

$$\Big|S_0-S_1\Big|-\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]+L\lambda[S_0-S_1]\Big|<0 \tag{2.4}$$

S_0-S_1 would have a negative sign. Thus, it would be more convenient to work with S_1-S_0 since $|S_0-S_1|=|S_1-S_0|$. Hence, we have

$$\Big|S_1-S_0\Big|<\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]-L\lambda[S_1-S_0]\Big| \tag{2.5}$$

We know that: $|A-B|\geq|A|-|B|$

$$\begin{aligned}
\Big|S_1-S_0\Big|<&\ \delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]-L\lambda[S_1-S_0]\Big| \\
&\geq\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]\Big|-\delta\Big|L\lambda[S_1-S_0]\Big|
\end{aligned} \tag{2.6}$$

$$\therefore\ \Big|S_1-S_0\Big|\leq\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]\Big|-\delta\Big|L\lambda[S_1-S_0]\Big| \tag{2.7}$$

Assuming $L\lambda$ is positive, we have

$$\Big|S_1-S_0\Big|(1+\delta L\lambda)\leq\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]\Big| \tag{2.8}$$

Rewriting the expression

$$z-S_1-\phi z+\phi S_0=z-S_1-\phi z+\phi S_0+\phi S_1-\phi S_1+S_0-S_0 \tag{2.9}$$

Collect the like terms

$$z-S_1-\phi z+\phi S_0=(1-\phi)(z-S_0)-(S_1-S_0) \tag{2.10}$$

The inequality (2.8) now becomes

$$\Big|S_1 - S_0\Big|(1+\delta L\lambda) \le \delta\Big|(M-L)\xi(1-\phi)(z-S_0) - (S_1-S_0)(M-L)\xi\Big| \tag{2.11}$$

Apply $|A-B| \ge |A| - |B|$, we have

$$\Big|S_1 - S_0\Big|(1+\delta L\lambda) \le \delta\Big|(M-L)\xi(1-\phi)(z-S_0)\Big| - \delta\Big|(M-L)\xi(S_1-S_0)\Big| \tag{2.12}$$

$$\therefore \Big|S_1 - S_0\Big|(1+\delta L\lambda) + \delta(M-L)\xi\Big|(S_1-S_0)\Big| \le \delta\Big|(M-L)\xi(1-\phi)(z-S_0)\Big| \tag{2.13}$$

Expand the right hand side and apply $|A-B| \ge |A| - |B|$

$$\Big|S_1 - S_0\Big|\Big((1+\delta L\lambda) + \delta(M-L)\xi\Big) \le \delta\Big|(M-L)\xi(1-\phi)z\Big| - \delta\Big|(M-L)\xi(1-\phi)S_0)\Big| \tag{2.14}$$

Add $\delta\Big|(M-L)\xi(1-\phi)S_0)\Big|$ to both sides

$$\Big|S_1 - S_0\Big|\Big((1+\delta L\lambda) + \delta(M-L)\xi\Big) + \delta\Big|(M-L)\xi(1-\phi)S_0)\Big| \le \delta\Big|(M-L)\xi(1-\phi)z\Big| \tag{2.15}$$

Therefore, substituting for $|S_1 - S_0|$ Also, remember $|z| = r < 1$

$$\begin{aligned}&\Big|S_1 - S_0\Big| = \sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ &\Bigg(\Big[(t+\rho-\mu-1)k\,(t-1)(\lambda-\omega)+t\Big]\Big((1+\delta L\lambda)+\delta(M-L)\xi\Big) \\ &+\ \delta(M-L)\xi(1-\phi)a_t r^t \le (M-L)\delta\xi(1-\phi)\Bigg)\end{aligned} \tag{2.16}$$

The proof is obtained. Hence, $f(z) \in \mathcal{T}_n\, S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho)$. ■

Theorem 2.2 (Second Class)**.** *The class $\mathcal{T}_n\, \mathcal{V}^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ contains a function $f(z) \in \mathcal{T}$ if and only if ,*

$$\left|\frac{\dfrac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1}{(M-L)\xi\Big(\dfrac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - \phi\Big) + L\lambda\Big(\dfrac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1\Big)}\right| < \delta$$

Proof. Let

$$\left|\frac{\dfrac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1}{(M-L)\xi\Big(\dfrac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - \phi\Big) + L\lambda\Big(\dfrac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1\Big)}\right| < \delta \tag{2.17}$$

$$= \left| \frac{\frac{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta+1} a_t\, z^t}{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} a_t\, z^t} - 1}{(M-L)\xi\Big(\frac{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta+1} a_t\, z^t}{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} a_t\, z^t} - \phi\Big) + L\lambda\Big(\frac{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta+1} a_t\, z^t}{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} a_t\, z^t} - 1\Big)} \right| < \delta \tag{2.18}$$

$$= \left| \frac{\sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} \times \left\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\right\} a_t\, z^t}{(M-L)\delta\xi(1-\phi) - \sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} \times \Big((M-L)\xi(1-\phi)+(M-L)\xi+L\lambda\Big)\left\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\right\}^{\beta} a_t\, z^t} \right| < \delta \tag{2.19}$$

As $|Ref(z)| \leq |z|$ for all z, we have

$$Re \left| \frac{\sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} \times \left\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\right\} a_t\, z^t}{(M-L)\delta\xi(1-\phi) - \sum_{t=2}^{\infty}\gamma^{n+1}(s)\left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} \times \Big((M-L)\xi(1-\phi)+(M-L)\xi+L\lambda\Big)\left\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\right\}^{\beta} a_t\, z^t} \right| < \delta \tag{2.20}$$

We choose z on real axis so that $\frac{D^{\beta+1}_{\lambda,\omega}}{D^{\beta}_{\lambda,\omega}}$ is real and clearing the denominator in the above relation and letting

$z \to 1$ over real values, we will get

$$\begin{aligned}
&\sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\
&\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\, a_t\, z^t \\
&<\ (M-L)\xi\delta z(1-\phi) - \sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \times \\
&(M-L)\delta\xi z(1-\phi) + \Big((M-L)\delta\xi + L\lambda\delta\Big)\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\, a_t\, z^t
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
&\Rightarrow \sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\
&\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\, a_t\, z^t + \\
&\sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\
&(M-L)\delta\xi z(1-\phi) + \Big((M-L)\delta\xi + L\lambda\delta\Big)\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\, a_t\, z^t \\
&<\ (M-L)\xi\delta z(1-\phi)
\end{aligned} \tag{2.22}$$

As $z \to 1$

$$\begin{aligned}
&= \sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\
&\Big[(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t] \\
&[1+L\lambda\delta+(M-L)\delta\xi] + (M-L)\delta\xi(1-\phi)\Big]\, a_t z^t - (M-L)\xi(1-\phi) < 0
\end{aligned} \tag{2.23}$$

Hence, the proof is obtained. ■

2.2. Growth and Distortion Theorem

Theorem 2.3. *If* $f(z) \in \mathcal{T}_n\, S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ *then*

$$r - r^2 \left\{ \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \times\ \Big[\Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ [1+L\lambda\delta+(M-L)\delta\xi] + (M-L)\delta\xi(1-\phi)\Big]\end{array}} \right\} \le |f(z)|$$

$$\leq r + r^2 \left\{ \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\left\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} \\ \times \left[\left\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\} \right.\\ \left.\times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\right]\end{array}} \right\}$$

Equality holds for

$$f(z) = z - \left\{ \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\left\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} \\ \times \left[\left\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\} \right.\\ \left.\times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\right]\end{array}} \right\}$$

Proof. From Theorem 2.1 we get $f(z) \in \mathcal{T}_n\, S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ if and only if

$$\begin{array}{l}\sum_{t=2}^{\infty} \gamma^{n+1}(s) \left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} \times \\ \left\{\left[(1+\rho-\mu)k[(t-1)(\lambda-\omega)+t]\right]\left\{1+L\lambda\delta+(M-L)\delta\xi\right\}+(M-L)\delta\xi(1-\phi)\right\}a_t \\ \leq (M-L)\delta\xi(1-\phi)\end{array} \tag{2.24}$$

Let $h = 1 - \dfrac{(M-L)\delta\xi(1-\phi)}{\left\{(1+\rho-\mu)k[(t-1)(\lambda-\omega)+t]\right\}[1+L\lambda\delta+(M-L)\delta\xi]}$

$\therefore f(z) \in \mathcal{T}_n\, S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ if and only if

$$\begin{array}{l}\sum_{t=2}^{\infty} \gamma^{n+1}(s) \left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} \times \\ \left\{(t+\rho-\mu-h)[(t-1)(\lambda-\omega)+t]\right\}a_t \leq (1-h)\end{array} \tag{2.25}$$

when $t = 2$,

$$\begin{array}{l}\left\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta}(2+\rho-\mu-h)\sum_{t=2}^{\infty} \gamma^{n+1}(s)\, a_t \leq \\ \sum_{t=2}^{\infty} \gamma^{n+1}(s) \left\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta} \times \\ \left\{(t+\rho-\mu-h)[(t-1)(\lambda-\omega)+t]\right\}a_t \leq (1-h)\end{array} \tag{2.26}$$

$$\begin{aligned}|f(z)| &\le r+\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t r^t \le r+r^2\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t \\ &\le r+r^2\Big[\frac{1-h}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big]\end{aligned} \tag{2.27}$$

Similarly,

$$\begin{aligned}|f(z)| &\ge r-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t r^t \ge r-r^2\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t \\ &\ge r-r^2\Big[\frac{1-h}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big]\end{aligned} \tag{2.28}$$

So

$$\begin{aligned}&r-r^2\Big[\frac{1-h}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big]\le|f(z)| \\ &\le r+r^2\Big[\frac{1-h}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big]\end{aligned} \tag{2.29}$$

Hence the result,

$$r-r^2\left\{\frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \times\ \Big[\Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}}\right\}\le|f(z)|$$

$$\le r+r^2\left\{\frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \times\ \Big[\Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}}\right\}$$

■

Corollary 2.4. *If* $f(z) \in \mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ *then* $\beta = 0, \xi = 1, \lambda = 1, \delta = 1, L = -1, M = 1$ *and* $\mu = \rho = 1$.

From the expression above, given $\rho = \mu,\ 1 + \rho - \mu = 1 + \rho - \rho = 1 + 0 = 1$.

$$= \frac{(1-\phi)}{k[(t-1)(1-\omega)+t](2-\phi)}$$

Hence the result

$$r - r^2 \left\{ \frac{(1-\phi)}{k[(t-1)(1-\omega)+t](2-\phi)} \right\} \leq |f(z)|$$

$$\leq r + r^2 \left\{ \frac{(1-\phi)}{k[(t-1)(1-\omega)+t](2-\phi)} \right\}$$

Theorem 2.5. *If* $f(z) \in \mathcal{T}_n\, \mathcal{V}^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ *then*

$$r - r^2 \left\{ \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l} \left\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta+1} \\ \times\ \left\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\} \\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi) \end{array}} \right\} \leq |f(z)|$$

$$\leq r + r^2 \left\{ \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l} \left\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta+1} \\ \times\ \left\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\} \\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi) \end{array}} \right\}$$

Proof. Similarly, we can prove this theorem as it is relevant to Theorem 2.3. So it is sufficient to substitute $\beta = \beta + 1$ in the above Theorem 2.3 and the subsequent corollary. ■

Theorem 2.6. *For* $f(z) \in \mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ *then*

$$1 - r\left\{\frac{(M-L)^2\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \times\ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}}\right\} \le |f(z)|$$

$$\le 1 + r\left\{\frac{(M-L)^2\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \times\ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}}\right\}$$

Proof. Since $f(z) \in \mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ we have by Theorem 2.3,

$$\begin{aligned}&\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}[(2+\rho-\mu-h)\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t\\ &\le \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\times\\ &\Big\{(t+\rho-\mu)k[(t-1)(\lambda-\omega)+t]\Big\}a_t \le (1-h)\end{aligned} \tag{2.30}$$

In look of Theorem 2.3 we have

$$\begin{aligned}\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t &= \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{(t+\rho-\mu-1)[(t-1)(\lambda-\omega)+t]\Big\}a_t + t\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t\\ &\le \frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\end{aligned} \tag{2.31}$$

$$\begin{aligned}&|f'(z)| \le 1+\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,ta_t|z|^{t-1} \le 1 + r\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,ta_t \le\\ &1+r\Big[\frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big]\end{aligned} \tag{2.32}$$

Similarly,

$$|f'(z)| \geq 1 - \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, ta_t |z|^{t-1} \geq 1 + r \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, ta_t \geq \\ 1 - r\Big[\frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big] \tag{2.33}$$

So,

$$1 - r\Big[\frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big] \leq |f'(z)| \leq \\ 1 - r\Big[\frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big] \tag{2.34}$$

Substituting the value of h in the above inequality, we have

$$r - r^2 \left\{ \begin{array}{l} \\ \\ \dfrac{(M-L)^2\delta\xi(1-\phi)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}} \\ \times\ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi) \end{array} \right\} \leq |f(z)|$$

$$\leq r + r^2 \left\{ \begin{array}{l} \\ \\ \dfrac{(M-L)^2\delta\xi(1-\phi)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}} \\ \times\ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi) \end{array} \right\}$$

Hence the proof is obtained. ■

Corollary 2.7. *For* $f(z) \in \mathcal{T}_n\, S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$.
In particular, if $\rho = \mu = 1$*, then we have* $1+\rho-\mu = 1+\rho-\rho = 1$.

$$= \frac{(M-L)^2\delta\xi(1-\phi)}{\Big\{k[(t-1)(\lambda-\omega)+t]\Big\}[1+L\lambda\delta+(M-L)\delta\xi]} \times \tag{2.35}$$

$$\frac{k[(t-1)(\lambda-\omega)+t](1+L\lambda\delta+(M-L)\delta\xi)}{\Big\{k[(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\Big\{k[(t-1)(\lambda-\omega)+t]\Big\}} \times$$
$$[1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)]$$

$$\frac{(M-L)^2\delta\xi(1-\phi)}{\Big\{[1+k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\Big\{k[(t-1)(\lambda-\omega)+t]\Big\}}$$
$$[1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)]$$

Hence, we obtain

$$1-r\left\{\begin{array}{l}\dfrac{(M-L)^2\delta\xi(1-\phi)}{\Big\{[1+k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\Big\{k[(t-1)(\lambda-\omega)+t]\Big\}}\\ [1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)]\end{array}\right\} \leq |f(z)|$$

$$\leq 1+r\left\{\begin{array}{l}\dfrac{(M-L)^2\delta\xi(1-\phi)}{\Big\{[1+k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\Big\{k[(t-1)(\lambda-\omega)+t]\Big\}}\\ [1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)]\end{array}\right\}$$

Theorem 2.8. *If* $f(z)\in\mathcal{T}_n\,\mathcal{V}^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ *then*

$$1-r\left\{\begin{array}{l}\dfrac{(M-L)^2\delta\xi(1-\phi)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}}\\ \times\ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}\right\} \leq |f(z)|$$

$$\leq 1+r\left\{\begin{array}{l}\dfrac{(M-L)^2\delta\xi(1-\phi)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}}\\ \times\ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}\right\}$$

Proof. Similarly, we can prove this theorem as it is analogous to Theorem 2.6. So it is sufficient to substitute $\beta=\beta+1$ in the above Theorem 2.6 and the subsequent corollary. ■

2.3. Closure Theorem

Theorem 2.9. *Let* $f_1(z) = z$ *and*

$$f_t(z) = \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \times\ \Big[\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}} z^t,$$

for $t = 2, 3, 4, \cdots$.

Then the class $\mathcal{T}_n S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ *contains a function* $f(z)$ *if and only if* $f(z) = \sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t f_t(z),\ \forall\, k_t \geq 0$ *and* $\sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t = 1.$

Proof. Let $f(z) = \sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t f_t(z),\ \forall\, k_t \geq 0$ and $\sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t = 1$. We have,

$$f(z) = \sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t f_t(z) = k_1 f_1(z) + \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, k_t f_t(z) \tag{2.36}$$

$$\therefore f(z) = z - \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, k_t \left\{ \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \times\ \Big[\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}} z^t, \right\} \tag{2.37}$$

Then

$$\begin{aligned}&\sum_{t=2}^{\infty} \gamma^{n+1}(s) \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \Big[\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}}\\ &\times \frac{\begin{array}{l}\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \Big[\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}}{(M-L)\delta\xi(1-\phi)}\\ &= \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, k_t = 1-k \leq 1\end{aligned} \tag{2.38}$$

Hence $f(z) \in \mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \gamma)$.
Conversely, suppose $f(z) \in \mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \gamma)$ then we have Theorem 2.1 ■

3. Acknowledgement

The authors are thankful to the editor and the referee(s) for their critical and valuable suggestions which improved this study.

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 70–81.
http://doi.org/10.26637/mjm11S/005

Application of homogenization and large deviations to a nonlocal parabolic semi-linear equation

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Received 16 June 2023; *Accepted* 21 August 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. We study the behavior of the solution for a class of nonlocal partial differential equation of parabolic-type with non-constant coefficients varying over length scale δ and nonlinear reaction term of scale $1/\varepsilon$, related to stochastic differential equations driven by multiplicative isotropic α-stable Lévy noise ($1 < \alpha < 2$). The behavior is required as ε tends to 0 with δ small compared to ε. Our homogenization method is probabilistic. Since δ decreases faster than ε, we may apply the large deviations principle with homogenized coefficients.
AMS Subject Classifications: 60H30, 60H10, 35B27, 35R09.

Keywords: Homogenization, Large deviation principle, nonlocal parabolic PDE, SDE with jumps, Feynman-Kac formula.

Contents

1. Introduction

Let $\varepsilon, \delta > 0$ small enough. Our aim in this article is to study the behavior of $u^{\varepsilon,\delta} : \mathbb{R}^d \longrightarrow \mathbb{R}$ of the following nonlocal partial differential equation (PDE) with parabolic-type :

$$\begin{cases} \dfrac{\partial u^{\varepsilon,\delta}}{\partial t}(t,x) = \mathcal{L}^{\alpha}_{\varepsilon,\delta} u^{\varepsilon,\delta}(t,x) + \frac{1}{\varepsilon} f\left(\frac{x}{\delta}, u^{\varepsilon,\delta}(t,x)\right), & x \in \mathbb{R}^d,\ 0 < t, \\ u^{\varepsilon,\delta}(0,x) = u_0(x), & x \in \mathbb{R}^d; \end{cases} \tag{1.1}$$

where the linear operator $\mathcal{L}^{\alpha}_{\varepsilon,\delta}$ is a nonlocal integro-differential operator of Lévy-type given by

$$\mathcal{L}^{\alpha}_{\varepsilon,\delta} f(x) := \int_{\mathbb{R}^d\setminus\{0\}} \left[f\left(x + \varepsilon\sigma\left(\frac{x}{\delta}, y\right)\right) - f(x) - \varepsilon\sigma^i\left(\frac{x}{\delta}, y\right)\partial_i f(x)\mathbf{1}_B(y)\right] \nu^{\alpha,\varepsilon^{-1}}(dy)$$
$$+ \left[\left(\frac{\varepsilon}{\delta}\right)^{\alpha-1} b^i_0\left(\frac{x}{\delta}\right) + b^i_1\left(\frac{x}{\delta}\right)\right]\partial_i f(x), \quad x \in \mathbb{R}^d.$$

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Here B is the unit open ball in $\mathbb{R}^d$ centering at the origin, and $\nu^{\alpha,\varepsilon^{-1}}(dy) := \frac{1}{\varepsilon}\nu^{\alpha}(dy) = \frac{\varepsilon^{-1}dy}{|y|^{d+\alpha}}$ is the isotropic α-stable Lévy measure. In this paper, we use Einstein's convention that the repeated indices in a product will be summed automatically.

The combinatorial effects of homogenization and large deviation principle (LDP) is a classical problem which goes back to P. Baldi [1] at the end of 20'th century . Such a problem has been most extensively investigated by Freidlin and Sowers [7] in stochastic differential equations (SDE) and linear parabolic PDE on the whole of $\mathbb{R}^d$. Huang et al. [9] recently studied a nonlocal problem from the mathematical point of view of homogenization theory. They considered the nonlocal parabolic linear equation without the viscosity (large deviations principle) parameter ε, with linear reaction term of scale $\frac{1}{\delta^{\alpha-1}}$. Inspired by [1, 7], the work in this paper is highly motivated by the consideration to combine the two principles in a compatible way, for a class of semilinear parabolic PDE. The present paper will only focus on the *subcritical* case $1 < \alpha < 2$. There are both probabilistic and analytical difficulties for the *supercritical* case $0 < \alpha \leqslant 1$. All things considered, the nonlocal part has lower order than the drift part, so that one cannot regard the drift as a perturbation of the nonlocal operator.
We first give the rate function $S_{0,t}$ of the large deviations, in fact since δ tends faster to zero than ε this function is expressed by the homogenized coefficients of the PDE (1.1), next we express the solution of PDE (1.1) by the use of Backward stochastic differential equations (BSDE) in [2] and the Feynman–Kac formula, then we consider an auxiliary equation solved by $\varepsilon \log u^{\varepsilon,\delta}$. The limit of this auxiliary equation helps us to find the limit of $u^{\varepsilon,\delta}$ when both ε, δ tend to zero. We show in the end that there exists a function V^* (which depends on $S_{0,t}$) such that $u^{\varepsilon,\delta}$ tends to zero if $(t,x) \in \{V^* < 0\}$ and tends to 1 in the interior of $\{V^* = 0\}$.

We organize the paper as follows. In Section (2), we present some general assumptions and definitions. Section (3) contains the results of large deviations principle. In Section (4), we study the behavior of the solution of the PDE (1.1).

2. Preliminaries

By B_r we means the open ball in $\mathbb{R}^d$ centering at the origin with radius $r > 0$, we shall omit the subscript when the radius is one. We denote by $\mathcal{C}^k$ $\left(\mathcal{C}_b^k\right)$ with integer $k \geqslant 0$ the space of (bounded) continuous functions possessing (bounded) derivatives of orders not greater than k. We shall explicitly write out the domain if necessary. Denote by $\mathcal{C}_b\left(\mathbb{R}^d\right) := \mathcal{C}_b^0\left(\mathbb{R}^d\right)$, it is a Banach space with the supremum norm $\|f\|_0 = \sup_{x\in\mathbb{R}^d}|f(x)|$. The space $\mathcal{C}_b^k\left(\mathbb{R}^d\right)$ is a Banach space endowed with the norm $\|f\|_k = \|f\|_0 + \sum_{j=1}^{k}\left\|\nabla^{\otimes j} f\right\|$. We also denote by $\mathcal{C}^{\text{Lip}}$ the class of all Lipschitz continuous functions. For a noninteger $\gamma > 0$, the Hölder spaces $\mathcal{C}^{\gamma}$ $(\mathcal{C}_b^{\gamma})$ are defined as the subspaces of $\mathcal{C}^{\lfloor\gamma\rfloor}$ $(\mathcal{C}_b^{\lfloor\gamma\rfloor})$ consisting of functions whose $\lfloor\gamma\rfloor$-th order partial derivatives are locally Hölder continuous (uniformly Hölder continuous) with exponent $\gamma - \lfloor\gamma\rfloor$. These two spaces $\mathcal{C}^{\lfloor\gamma\rfloor}$ and $\mathcal{C}_b^{\lfloor\gamma\rfloor}$ obviously coincide when the underlying domain is compact. The space $\mathcal{C}_b^{\lfloor\gamma\rfloor}\left(\mathbb{R}^d\right)$ is a Banach space endowed with the norm $\|f\|_{\gamma} = \|f\|_{\lfloor\gamma\rfloor} + [\nabla^{\lfloor\gamma\rfloor} f]_{\gamma-\lfloor\gamma\rfloor}$, where the seminorm $[\cdot]_{\gamma'}$ with $0 < \gamma' < 1$ is defined as $[f]_{\gamma'} := \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\gamma'}}$ (this seminorm can also be defined for the case $\gamma' = 1$, which is exactly the Lipschitz seminorm). In the sequel, the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ will be used frequently. Denote by $\mathcal{D} := \mathcal{D}\left(\mathbb{R}_+; \mathbb{T}^d\right)$ the space of all $\mathbb{T}^d$-valued càdlàg functions on $\mathbb{R}_+$, equipped with the Skorokhod topology. We shall always identify the periodic function on $\mathbb{R}^d$ of period 1 with its restriction on the torus $\mathbb{T}^d$.

For notational simplicity, we can organize all of this by considering $\delta_\varepsilon := \delta$, where $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$.

(**H.1**) We assume that $\lim_{\varepsilon\to 0}\dfrac{\delta_\varepsilon}{\varepsilon}=0.$

Let $\left(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geqslant 0}\right)$ be a filtered probability space endowed with a Poisson random measure $N^{\alpha,\varepsilon^{-1}}$ on $\mathbb{R}^d\setminus\{0\}\times\mathbb{R}_+$ with jump intensity measure $\nu^{\alpha,\varepsilon^{-1}}(dy)=\frac{1}{\varepsilon}\nu^\alpha(dy)=\frac{\varepsilon^{-1}dy}{|y|^{d+\alpha}}$ where $1<\alpha<2$, $\varepsilon>0$. Denote by $\tilde{N}$ the associated compensated Poisson random measure, that is, $\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds):=N^{\alpha,\varepsilon^{-1}}(dyds)-\nu^{\alpha,\varepsilon^{-1}}(dy)ds$. We assume that the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfies the usual conditions. Let $L^{\alpha,\varepsilon^{-1}}=\left\{L_t^{\alpha,\varepsilon^{-1}}\right\}_{t\geqslant 0}$ be a d-dimensional isotropic α-stable Lévy process given by

$$L_t^{\alpha,\varepsilon^{-1}}:=\int_0^t\int_{B\setminus\{0\}}y\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds)+\int_0^t\int_{B^c}yN^{\alpha,\varepsilon^{-1}}(dyds).$$

Given $\varepsilon>0, x\in\mathbb{R}^d$, consider the following:

$$dX_t^{\varepsilon,\delta_\varepsilon}=\left[\left(\frac{\varepsilon}{\delta_\varepsilon}\right)^{\alpha-1}b_0\left(\frac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)+b_1\left(\frac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)\right]dt+\varepsilon\sigma\left(\frac{X_{t-}^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},dL_t^{\alpha,\varepsilon^{-1}}\right),\quad X_0^{\varepsilon,\delta_\varepsilon}=x,\tag{2.1}$$

or more precisely,

$$\begin{aligned}X_t^{\varepsilon,\delta_\varepsilon}&=x+\int_0^t\left[\left(\frac{\varepsilon}{\delta_\varepsilon}\right)^{\alpha-1}b_0\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)+b_1\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)\right]ds\\&\quad+\int_0^t\int_{B\setminus\{0\}}\sigma\left(\frac{X_{s-}^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},y\right)\varepsilon\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds)+\int_0^t\int_{B^c}\sigma\left(\frac{X_{s-}^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},y\right)\varepsilon N^{\alpha,\varepsilon^{-1}}(dyds).\end{aligned}$$

Before continuing, we list some general assumptions for the PDE (1.1) and the nonlocal the SDE (2.1).
We consider $u_0\in\mathcal{C}_b\left(\mathbb{R}^d,\mathbb{R}_+\right)$ and we set

$$\sup_{x\in\mathbb{R}^d}u_0(x)=\overline{u}_0<\infty.$$

Let us set $U_0=\left\{x\in\mathbb{R}^d:u_0(x)>0\right\}$, since u_0 is continuous we have $\overline{\mathring{U}_0}=\overline{U}_0$.
We assume that $f:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ is periodic in each direction with respect to the first argument, and it verifies :

- $\forall x\in\mathbb{R}^d,\ f(x,1)=0;$
- There exists $c\in\mathcal{C}_b^\beta(\mathbb{R}^d\times\mathbb{R},\mathbb{R})$ such that

  $$f(x,y)=c(x,y)\cdot y,$$

  with

  $$c(x,y)>0,\ \forall x\in\mathbb{R}^d,\ y\in[0,1)\cup\mathbb{R}_-^*,\qquad\text{and}\qquad c(x,y)\leqslant 0,\ \forall x\in\mathbb{R}^d,\ y>1.$$

  And we assume that

  $$\max c(x,y)=c(x)=c(x,0)>0,\ \forall x\in\mathbb{R}^d.$$

$$(\textbf{H.2})\begin{cases}i)\text{ The functions }(b_0,b_1,u_0):\mathbb{R}^{3d}\longrightarrow\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+\text{ are all periodic of period 1 in each component.}\\ii)\text{ For every }y\in\mathbb{R}^d,\text{ the function }x\mapsto(\sigma(x,y),c(x,y))\text{ is periodic of period 1 in each component.}\\iii)\text{ The functions }b_0,b_1,c\text{ are of class }\mathcal{C}_b^\beta\text{ with exponent }\beta\text{ satisfying : }1-\dfrac{\alpha}{2}<\beta<1.\\iv)\text{ The initial functions }u_0\text{ is continuous.}\end{cases}$$

The function $\sigma : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies the following conditions (for some comments see, [9]).

(H.3)
$i)$ For every $x \in \mathbb{R}^d$, the function $y \mapsto \sigma(x,y)$ is of class $\mathcal{C}^2$. There exists $\alpha - 1 < \lambda \leqslant 1$ such that
$$\sup_{x\in\mathbb{R}^d} [\nabla_y \sigma(x,\cdot)]_\lambda < \infty.$$
There exists a constant $C > 0$, such that for any $x_1, x_2, y \in \mathbb{R}^d$,
$$|\sigma(x_1,y) - \sigma(x_1,y)| \leqslant C\,|x_1 - x_2|\,|y|\,.$$
$ii)$ The oddness condition : for all $x, y \in \mathbb{R}^d,\ \sigma(x,-y) = -\sigma(x,y)$.
$iii)$ The Jacobian matrix with respect to the second variable $\nabla_y \sigma(x,y)$ is non-degenerate $\forall x, y \in \mathbb{R}^d$, and there exists a constant $C > 0$ such that $\left\|(\nabla_y \sigma(x,y))^{-1}\right\|_{\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d)} \leqslant C$ for all $x, y \in \mathbb{R}^d$
$iv)$ There exists a positive bounded measurable function $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}_+$, such that for all $x, y \in \mathbb{R}^d$,
$$\phi(x)^{-1}|y| \leqslant \sigma(x,y) \leqslant \phi(x)|y|.$$

Let us introduce the linear operator $\mathcal{A}^{\sigma,\nu^\alpha}$ defined as

$$\mathcal{A}^{\sigma,\nu^\alpha} f(x) := \int_{\mathbb{R}^d\setminus\{0\}} \left[f\left(x + \sigma\left(x,y\right)\right) - f(x) - \sigma^i\left(x,y\right)\partial_i f(x)\mathbf{1}_B(y)\right]\nu^\alpha(dy), \quad x \in \mathbb{R}^d. \tag{2.2}$$

By virtue of the oddness condition and the symmetry of the jump intensity measure ν^α, we can rewrite the operator $\mathcal{A}^{\sigma,\nu^\alpha}$ as : (see, [9])

$$\mathcal{A}^{\sigma,\nu^\alpha} f(x) := \int_{\mathbb{R}^d\setminus\{0\}} \left[f\left(x + z\right) - f(x) - z^i\partial_i f(x)\mathbf{1}_B(z)\right]\nu^{\sigma,\alpha}(x,dz), \quad x \in \mathbb{R}^d. \tag{2.3}$$

where the kernel $\nu^{\sigma,\alpha}$ is given by

$$\nu^{\sigma,\alpha}(x,A) = \int_{\mathbb{R}^d\setminus\{0\}} \mathbf{1}_A\left(\sigma(x,y)\right)\nu^\alpha(dy), \quad A \in \mathcal{B}\left(\mathbb{R}^d\setminus\{0\}\right). \tag{2.4}$$

Next, to move the SDE (2.1) to the torus $\mathbb{T}^d$, we define $\tilde{X}_t^{\varepsilon,\delta_\varepsilon} := \frac{1}{\delta_\varepsilon} X^{\varepsilon,\delta_\varepsilon}_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})t}$, via the canonical quotient map $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^d/\mathbb{Z}^d$. It is easy to check that

$$d\tilde{X}_t^{\varepsilon,\delta_\varepsilon} = \left[b_0\left(\tilde{X}_t^{\varepsilon,\delta_\varepsilon}\right) + \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{\alpha-1} b_1\left(\tilde{X}_t^{\varepsilon,\delta_\varepsilon}\right)\right]dt + \frac{\varepsilon}{\delta_\varepsilon}\sigma\left(\tilde{X}_{t-}^{\varepsilon,\delta_\varepsilon}, \frac{\delta_\varepsilon}{\varepsilon}dL_t^\alpha\right), \quad \tilde{X}_0^{\varepsilon,\delta_\varepsilon} = \frac{x}{\delta_\varepsilon}, \tag{2.5}$$

where
$$L_t^\alpha := \int_0^t \int_{B\setminus\{0\}} y\tilde{N}^\alpha(dyds) + \int_0^t \int_{B^c} yN^\alpha(dyds),$$
and with $\left\{\frac{\varepsilon}{\delta} L^{\alpha,\varepsilon^{-1}}_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})t}\right\} \equiv \left\{\frac{\varepsilon}{\delta_\varepsilon} L^\alpha_{(\delta_\varepsilon/\varepsilon)^\alpha t}\right\} \overset{d}{:=} \{L_t^\alpha\}$ by virtue of the self-similarity.
We shall also consider the limit SDE (2.5), namely

$$d\tilde{X}_t = b_0\left(\tilde{X}_t\right)dt + \overline{\sigma}\left(\tilde{X}_{t-}, dL_t^\alpha\right), \quad \tilde{X}_0 = x, \tag{2.6}$$

where, heuristically by the L'Hôpital's rule, $\overline{\sigma}(x,y) = \nabla_y\sigma(x,0)y$ is the point-wise limit of $\frac{\varepsilon}{\delta_\varepsilon}\sigma\left(\cdot, \frac{\delta_\varepsilon}{\varepsilon}\cdot\right)$ as $\varepsilon \downarrow 0$. We need a stronger convergence as follows:

(H.4) For every $y \in \mathbb{R}^d$, $\frac{1}{\eta}\sigma(x,\eta y) \longrightarrow (\nabla_y\sigma(x,0))y$ uniformly in $x \in \mathbb{R}^d$, as $\eta \to 0$.

Let us set $\mathcal{L}^{\alpha}$ be the linear integro-partial differential operator given by

$$\mathcal{L}^{\alpha} := \mathcal{A}^{\overline{\sigma},\nu^{\alpha}} + b_0 \cdot \nabla. \tag{2.7}$$

By requirement there exists a $\mathcal{L}^{\alpha}$-Feller process on $\mathbb{R}^d$ and by periodicity assumption on the coefficients such a process induces a process $\tilde{X}$ which is a strong Markov process on the torus $\mathbb{T}^d$, moreover the $\mathcal{L}^{\alpha}$- process is ergodic (see, [9]). We denote by μ its unique invariant measure on $\left(\mathbb{T}^d, \mathcal{B}\left(\mathbb{T}^d\right)\right)$. In order to do the homogenization for the SDE $X^{\varepsilon,\delta_\varepsilon}$ (2.1), we need the following be in force ([3, 8, 10]):

(**H.5**) The centering condition : $\displaystyle\int_{\mathbb{T}^d} b_0(x)\mu(dx) = 0.$

Thanks to [9, Proposition 4.11], there is a unique periodic solution $\hat{b} \in \mathcal{C}^{\alpha+\beta}$ of the Poisson equation

$$\mathcal{L}^{\alpha}\hat{b} + b_0 = 0 \quad \text{such that} \quad \int_{\mathbb{T}^d} \hat{b}(x)\mu(dx) = 0, \tag{2.8}$$

which satisfies the estimate

$$\|\hat{b}\|_{\alpha+\beta} \leqslant C\left(\|\hat{b}\|_0 + \|b\|_\beta\right). \tag{2.9}$$

Now we set

$$\begin{aligned}
\overline{B} &:= \int_{\mathbb{T}^d} \left(I + \nabla\hat{b}\right) b_1(x)\mu(dx),\\
\overline{C} &:= \int_{\mathbb{T}^d} c(x)\mu(dx),\\
\overline{\nu}(A) &:= \int_{\mathbb{R}^d\setminus\{0\}} \int_{\mathbb{T}^d} \mathbf{1}_A\left(\sigma(x,y)\right)\mu(dx)\nu^{\alpha}(dy), \quad A \in \mathcal{B}\left(\mathbb{R}^d\setminus\{0\}\right).
\end{aligned}$$

3. Large deviation principle

The theory of large deviations is concerned with events A for which probability $\mathbb{P}\left(X^{\varepsilon,\delta_\varepsilon} \in A\right)$ converges to zero exponentially fast as $\varepsilon \to 0$ (see, [4]). The exponential decay rate of such probabilities is typically expressed in terms of a rate function $\mathcal{J}$ mapping $\mathbb{R}^d$ into $[0, +\infty]$. Our method allows us to characterize the LDP by analysing the logarithmic moment generating function [4, Chap. 2.3]. Initially the corresponding rate function is identified as the Legendre transform of the limit (when it exists) of the logarithmic moment generating function defined as:

$$\lim_{\varepsilon\to 0} g^{\varepsilon}_{t,x}(\theta) := \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\left\{\exp\left(\tfrac{1}{\varepsilon}\langle\theta, X^{\varepsilon,\delta_\varepsilon}_t\rangle\right)\right\}.$$

If we set

$$\hat{X}^{\varepsilon,\delta_\varepsilon}_t := X^{\varepsilon,\delta_\varepsilon}_t + \delta_\varepsilon\left[\hat{b}\left(\frac{X^{\varepsilon,\delta_\varepsilon}_t}{\delta_\varepsilon}\right) - \hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\right] \tag{3.1}$$

then we have by Itô's formula

$$\begin{aligned}
\hat{X}^{\varepsilon,\delta_\varepsilon}_t &= x + \int_0^t \left(I + \nabla\hat{b}_\varepsilon\right) b_{1_\varepsilon}\left(X^{\varepsilon,\delta_\varepsilon}_s\right) ds - \left(\frac{\varepsilon}{\delta_\varepsilon}\right)^{\alpha-1} \int_0^t \mathcal{A}^{\overline{\sigma},\nu^{\alpha}}\hat{b}\left(\frac{X^{\varepsilon,\delta_\varepsilon}_s}{\delta_\varepsilon}\right) ds\\
&+ \frac{\delta_\varepsilon}{\varepsilon}\int_0^t \mathcal{A}^{\varepsilon\sigma_\varepsilon,\nu^{\alpha}}\hat{b}_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_s\right) ds + \int_0^t \varepsilon\sigma_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_{s-}, dL^{\alpha,\varepsilon^{-1}}_s\right)\\
&+ \delta_\varepsilon\int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \left[\hat{b}_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_s + \varepsilon\sigma_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_{s-}, y\right)\right) - \hat{b}_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_s\right)\right]\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds),
\end{aligned} \tag{3.2}$$

where $\zeta_\varepsilon(x)=\zeta\left(\frac{x}{\delta_\varepsilon}\right)$ for $\zeta(x)$ in $\left\{b_1(x),\hat{b}(x),\nabla\hat{b},\sigma(x,\cdot)\right\}$. Note that $\nu^\alpha(\varepsilon A)=\varepsilon^{-\alpha}\nu^\alpha(A), A\in\mathcal{B}\left(\mathbb{R}\backslash\{0\}\right)$. Before proceeding, let us define for all $z\in\mathbb{T}^d$ and for all $\varphi\in\mathcal{C}^{\alpha+\beta}\left(\mathbb{T}^d\right)$

$$\begin{aligned}H^{\varepsilon,\varphi}\big(z,\cdot\big)&:=\varphi\Big(z+\frac{\varepsilon}{\delta_\varepsilon}\sigma\big(z,\frac{\delta_\varepsilon}{\varepsilon}\cdot\big)\Big)-\varphi(z),\\ \mathcal{Q}^{\varepsilon,\varphi}\big(z\big)&:=\mathcal{A}^{\frac{\varepsilon}{\delta_\varepsilon}\sigma(\cdot,(\delta_\varepsilon/\varepsilon)\cdot),\nu^\alpha}\varphi(z)-\mathcal{A}^{\overline{\sigma},\nu^\alpha}\varphi(z).\end{aligned}$$

Now, by Girsanov's formula, we have

$$\begin{aligned}g^\varepsilon_{t,x}(\theta)=\langle\theta,x\rangle+\varepsilon\log\tilde{\mathbb{E}}\Bigg\{&\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta,\Big(I+\nabla\hat{b}\Big)b_1\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right)\rangle ds\right)\\ \times\exp&\Bigg(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{Q}^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right)ds-\frac{\delta_\varepsilon}{\varepsilon}\left[\hat{b}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)t}\right)-\hat{b}\Big(\frac{x}{\delta_\varepsilon}\Big)\right]\Bigg)\\ \times\exp&\Bigg(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}}\Big\{e^{\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta,H^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\right)\right\rangle}-1-\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta,H^{\varepsilon,\hat{b}}\big(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy)ds\Bigg)\\ \times\exp&\Bigg(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}}\Big\{e^{\left\langle\theta,\sigma\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\right)\right\rangle}-1-\left\langle\theta,\sigma\big(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy)ds\Bigg)\Bigg\}\end{aligned}\tag{3.3}$$

where $\tilde{\mathbb{E}}$ is the expectation operator with respect to the probability $\tilde{\mathbb{P}}$ defined as

$$\begin{aligned}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}:=\exp&\Bigg(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta,H^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{s-},y\right)\rangle\tilde{N}^{\alpha,(\delta_\varepsilon/\varepsilon)^\alpha}(dyds)+\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta,\sigma\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{s-},dL^\alpha_s\right)\rangle\Bigg)\\ \times\exp&\Bigg(-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}}\Big\{e^{\left\langle\theta,H^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\right)\right\rangle}-1-\left\langle\theta,H^{\varepsilon,\hat{b}}\big(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy)ds\Bigg)\\ \times\exp&\Bigg(-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}}\Big\{e^{\left\langle\theta,\sigma\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\right)\right\rangle}-1-\left\langle\theta,\sigma\big(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy)ds\Bigg).\end{aligned}$$

Let us set, for all $z\in\mathbb{T}^d$, for all $\theta\in\mathbb{R}^d$:

$$\begin{aligned}\Phi^\varepsilon(z,\theta):=&\langle\theta,\Big(I+\nabla\hat{b}\Big)b_1\left(z\right)\rangle+\int_{\mathbb{R}^d\backslash\{0\}}\Big\{e^{\langle\theta,\sigma(z,y)\rangle}-1-\big\langle\theta,\sigma\big(z,y\big)\mathbf{1}_B(y)\big\rangle\Big\}\nu^\alpha(dy)\\ &+\int_{\mathbb{R}^d\backslash\{0\}}\Big\{e^{\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta,H^{\varepsilon,\hat{b}}(z,y)\right\rangle}-1-\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta,H^{\varepsilon,\hat{b}}\big(z,y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy),\end{aligned}\tag{3.4}$$

and let us set $\Psi^\varepsilon_\theta\in\mathcal{C}^{\alpha+\beta}\left(\mathbb{T}^d\right)$ be the unique solution of

$$\mathcal{L}^\alpha\Psi^\varepsilon_\theta(z)+\Phi^\varepsilon(z,\theta)=\int_{\mathbb{T}^d}\Phi^\varepsilon(z,\theta)\mu(dz)\quad\text{such that}\quad\int_{\mathbb{T}^d}\Psi^\varepsilon_\theta(z)\mu(dz)=0.$$

Such a solution Ψ^ε_θ must exist again by the assumptions on the coefficients and the Fredholm alternative. So applying Itô's formula to $\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\Psi^\varepsilon_\theta\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}\right)$, we have

$$\begin{aligned}\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\Phi^\varepsilon\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,\theta\right)=t\int_{\mathbb{T}^d}\Phi^\varepsilon(z,\theta)\mu(dz)+\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\left[\Psi^\varepsilon_\theta\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)}\right)-\Psi^\varepsilon_\theta\Big(\frac{x}{\delta_\varepsilon}\Big)\right]\\ \frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{Q}^{\varepsilon,\Psi^\varepsilon_\theta}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right)ds-\frac{\delta_\varepsilon^{2\alpha-1}}{\varepsilon^{2(\alpha-1)}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\nabla\Psi^\varepsilon_\theta b_1\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right)ds\\ -\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}}H^{\varepsilon,\Psi^\varepsilon_\theta}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\right)\tilde{N}^\alpha(dyds).\end{aligned}\tag{3.5}$$

Then putting (3.5) into the equation (3.3), we obtain

$$
\begin{aligned}
g^{\varepsilon}_{t,x}(\theta) = \langle \theta, x\rangle + t\int_{\mathbb{T}^d}\Phi^{\varepsilon}(z,\theta)\mu(dz) + \varepsilon \log \hat{\mathbb{E}}\Bigg\{ \exp\left(-\frac{\delta_{\varepsilon}^{2\alpha-1}}{\varepsilon^{2\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t} \nabla\Psi^{\varepsilon}_{\theta} b_1\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right) ds\right) \\
\times \exp\left(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t} \mathcal{Q}^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right) ds - \frac{\delta_\varepsilon}{\varepsilon}\left[\hat{b}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^{\alpha})t}\right) - \hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\right]\right) \\
\times \exp\left(-\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t} \mathcal{Q}^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right) ds + \frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\left[\Psi^{\varepsilon}_{\theta}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^{\alpha})t}\right) - \Psi^{\varepsilon}_{\theta}\left(\frac{x}{\delta_\varepsilon}\right)\right]\right) \\
\times \exp\left(-\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}\int_{\mathbb{R}^d\setminus\{0\}} \left\{ e^{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{\alpha} H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)} - 1 - \frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}} H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\mathbf{1}_B(y)\right\}\nu^{\alpha}(dy)ds\right)\Bigg\}.
\end{aligned}
\tag{3.6}
$$

where $\hat{\mathbb{E}}$ is the expectation operator with respect to the probability $\hat{\mathbb{P}}$ defined as

$$
\begin{aligned}
\frac{d\hat{\mathbb{P}}}{d\tilde{\mathbb{P}}} := \exp\left(-\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t} H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, \frac{\delta_\varepsilon}{\varepsilon}y\right)\tilde{N}^{\alpha}(dyds)\right) \\
\times \exp\left(\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}\int_{\mathbb{R}^d\setminus\{0\}} \left\{ e^{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{\alpha} H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)} - 1 - \frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}} H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\mathbf{1}_B(y)\right\}\nu^{\alpha}(dy)ds\right).
\end{aligned}
$$

Since the coefficients are bounded, we first notice that

$$
\begin{aligned}
\sup_{z\in\mathbb{T}^d}\Bigg\{ \exp\left(\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\left[\Psi^{\varepsilon}_{\theta}(z_t) - \Psi^{\varepsilon}_{\theta}\left(\frac{x}{\delta_\varepsilon}\right)\right] - \frac{\delta_\varepsilon}{\varepsilon}\left[\hat{b}(z_t) - \hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\right]\right) \\
\times \exp\left(-\frac{\delta_\varepsilon^{2\alpha-1}}{\varepsilon^{2\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t} \nabla\Psi^{\varepsilon}_{\theta} b_1(z_s)\, ds\right)\Bigg\} \leqslant \exp\left(\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}K_1 + \frac{\delta_\varepsilon}{\varepsilon}K_2 + \frac{1}{\varepsilon}\frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}}K_3\right).
\end{aligned}
\tag{3.7}
$$

Recall an elementary result.

Lemma 3.1 ([9]). *Let $0<\lambda\leqslant 1$ and $f\in\mathcal{C}^{1+\gamma}_b\left(\mathbb{R}^d\right)$. For any $x,u,v\in\mathbb{R}^d$, it holds that*

$$
\left|f(x+u) - f(x+v) - (u-v)\cdot\nabla f(x)\right| \leqslant \frac{1}{1+\lambda}[\nabla f]_{\lambda}\left|u-v\right|^{1+\lambda}.
$$

We let $r = (\delta_\varepsilon/\varepsilon)^{\gamma}$ for some $\gamma\in\mathbb{R}$ that will be chosen for B_r. It follows from Lemma 3.1 that for all $\varphi\in\mathcal{C}^{\alpha+\beta}\left(\mathbb{T}^d\right)$ (see [9, Appendix]) :

$$
\sup_{z\in\mathbb{T}^d}\left\{ \delta_\varepsilon\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t} \mathcal{Q}^{\varepsilon,\varphi}(z_s)\, ds\right\} \leqslant K_4\left(\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{\lambda(\alpha+\beta)-\alpha+1+\gamma[(1+\lambda)(\alpha+\beta)-\alpha]} + \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{1-\alpha(1+\gamma)}\right) \longrightarrow 0, \tag{3.8}
$$

if we select γ satisfying

$$
-\frac{\lambda(\alpha+\beta)-\alpha+1}{(1+\lambda)(\alpha+\beta)-\alpha} < \gamma < \frac{1}{\alpha} - 1.
$$

On the other hand, using a similar estimate once again, it follows

$$
\sup_{z\in\mathbb{T}^d}\left\{ \delta_\varepsilon\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t} H^{\varepsilon,\varphi}(z, \frac{\delta_\varepsilon}{\varepsilon}y)\right\} \longrightarrow 0. \tag{3.9}
$$

Therefore

$$\lim_{\varepsilon\to 0} g^{\varepsilon}_{t,x}(\theta) = \langle\theta, x\rangle + t\overbrace{\int_{\mathbb{R}^d\setminus\{0\}} \left(e^{\langle\theta,y\rangle} - 1 + \left\langle\theta, \overline{B} - y\mathbf{1}_B\right\rangle\right)\overline{\nu}(dy)}^{\mathcal{J}(\theta):=}. \tag{3.10}$$

Let $\overline{\mathcal{J}}$ denote the Fenchel-Legendre transform of $\mathcal{J}$. Then we have

$$\overline{\mathcal{J}}(\theta) := \int_{\mathbb{R}^d\setminus\{0\}} \varrho\left(\frac{\left|\theta - (\overline{B} - y\mathbf{1}_B)\right|}{|y|}\right)\overline{\nu}(dy), \tag{3.11}$$

where $\varrho(r) := r\log r - r + 1, \quad r \in \mathbb{R}^*_+$.
Now, we state our main result.

Theorem 3.2. *Fix $T > 0$ and assume **(H.1)** – **(H.5)** hold true. Then for every $x \in \mathbb{R}^d$, the family $\left\{X^{\varepsilon,\delta_\varepsilon} : \varepsilon > 0\right\}$ of $\mathbb{R}^d$-valued random variables has a large deviations principle with good rate function*

$$I_{T,x}(z) := T\overline{\mathcal{J}}\Big(\frac{z-x}{T}\Big).$$

Next, let us consider

$$S_{0,T}(\varphi) := \begin{cases} \int_0^T \overline{\mathcal{J}}\left(\dot{\varphi}(s)\right) ds & \text{if } \varphi \in \mathcal{D}\left([0,T], \mathbb{R}^d\right) \text{ and } \varphi(0) = x, \\ +\infty & \text{else.} \end{cases}$$

Since the function $\mathcal{J}$ is convex we can show that

$$\inf_{\substack{\varphi\in\mathcal{D}\left([0,T],\mathbb{R}^d\right)\\ \varphi(0)=x,\ \varphi(T)=z}} \int_0^T \overline{\mathcal{J}}\left(\dot{\varphi}(s)\right) ds := T\overline{\mathcal{J}}\left(\frac{z-x}{T}\right).$$

So we express the path space-LDP

Corollary 3.3. *Assume **(H.1)** – **(H.5)** hold true. Then the family $\left\{X^{\varepsilon,\delta_\varepsilon}\right\}_{\varepsilon>0}$ of $\mathcal{D}\left([0,T];\mathbb{R}^d\right)$-valued random variables has a large deviations principle with good rate function $S_{0,T}(\varphi)$ for all $\varphi \in \mathcal{D}\left([0,T];\mathbb{R}^d\right)$.*

From [4, Varadhan's Lemma], we have

Remark 3.4. *Let D be a Borel subset on $\mathcal{D}\left([0,t];\mathbb{R}^d\right)$ and c be an element of $\mathcal{C}^\beta\left(\mathbb{R}^d\right)$. Then we have*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon\log\mathbb{E}\left[\mathbf{1}_D\left(X^{\varepsilon,\delta_\varepsilon}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^t c\left(\frac{X^{\varepsilon,\delta_\varepsilon}_s}{\delta_\varepsilon}\right)ds\right\}\right] \geqslant \overline{C}t - \inf_{\phi\in\mathring{D}} S_{0,t}(\phi),$$
$$\limsup_{\varepsilon\downarrow 0} \varepsilon\log\mathbb{E}\left[\mathbf{1}_D\left(X^{\varepsilon,\delta_\varepsilon}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^t c\left(\frac{X^{\varepsilon,\delta_\varepsilon}_s}{\delta_\varepsilon}\right)ds\right\}\right] \leqslant \overline{C}t - \inf_{\phi\in\overline{D}} S_{0,t}(\phi).$$

4. Convergence of $u^{\varepsilon,\delta}$

Let us consider the progressive measurable process $(Y^{\varepsilon,\delta_\varepsilon}, U^{\varepsilon,\delta_\varepsilon,\cdot})$ solution of the BSDE:

$$\begin{cases} Y^{\varepsilon,\delta_\varepsilon}_t = u_0\left(X^{\varepsilon,\delta_\varepsilon}_t\right) + \frac{1}{\varepsilon}\int_s^t f\left(\frac{X^{\varepsilon,\delta_\varepsilon}_r}{\delta_\varepsilon}, Y^{x,\varepsilon,\delta_\varepsilon}_r\right)dr - \int_s^t U^{\varepsilon,\delta_\varepsilon}_r dL^\alpha_r,\ 0 \leqslant s \leqslant t, \\ \sqrt{\mathbb{E}\int_s^t \int_{\mathbb{R}^d\setminus\{0\}} U^{\varepsilon,\delta_\varepsilon}_r(y)^2 \nu^\alpha(dy)dr} < \infty. \end{cases}$$

By [2, 11], we have for all $(t,x) \in [0,+\infty[\times \mathbb{R}^d$, the solution $u^{\varepsilon,\delta_\varepsilon}(t,x)$ of the PDE (1.1) is of the form

$$Y_0^{x,\varepsilon,\delta_\varepsilon} = u^{\varepsilon,\delta_\varepsilon}(t,x),$$

and the Feynman-Kac formula implies that the solution of PDE (1.1) obeys

$$u^{\varepsilon,\delta_\varepsilon}(t,x) = \mathbb{E}\left\{ u_0\left(X_t^{\varepsilon,\delta_\varepsilon}\right) \exp\left(\frac{1}{\varepsilon}\int_0^t c\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, Y_s^{\varepsilon,\delta_\varepsilon}\right) ds\right)\right\}. \tag{4.1}$$

Remark 4.1.

- *If* $\overline{u}_0 \leqslant 1$, *then* $\forall \varepsilon > 0,\ 0 \leqslant Y_s^{\varepsilon,\delta_\varepsilon} \leqslant 1, \quad d\mathbb{P}\times ds\ a.s.$. .
- *On the other and, if* $c(x,y) \leqslant \kappa(y) < 0,\ (x,y) \in \mathbb{R}^d \times]1,+\infty[$, *where* κ *is Lipschitz continuous, then*

$$\limsup_{\varepsilon\to 0} Y_t^{\varepsilon,\delta_\varepsilon} \leqslant 1 \ \textit{ uniformly in any compact set of } \ ]0,+\infty[\times \mathbb{R}^d.$$

To prove this, we will use similar results proved in [12] .
Before continuing, let us introduce $v^{\varepsilon,\delta_\varepsilon(t,x)} = \varepsilon \log u^{\varepsilon,\delta_\varepsilon}(t,x)$, and let us set

$$\begin{aligned}\mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} v^{\varepsilon,\delta_\varepsilon}(t,x) := \int_{\mathbb{R}^d\setminus\{0\}} \Bigg[& e^{\left\{\frac{1}{\varepsilon} v^{\varepsilon,\delta_\varepsilon}\left(t,x+\varepsilon\sigma\left(\frac{x}{\delta},y\right)\right)\right\}} - 1 - \sigma\left(\frac{x}{\delta},y\right)\partial_i v^{\varepsilon,\delta_\varepsilon}(t,x)\,\mathbf{1}_B(y) \\ & - \left\{ v^{\varepsilon,\delta_\varepsilon}\left(t,x+\varepsilon\sigma\left(\frac{x}{\delta},y\right)\right) - v^{\varepsilon,\delta_\varepsilon}(t,x)\right\}\Bigg]\nu^\alpha(dy).\end{aligned}$$

Then, we observe that $v^{\varepsilon,\delta_\varepsilon}(t,x)$ is a viscosity solution of :

$$\begin{cases} \dfrac{\partial v^{\varepsilon,\delta_\varepsilon}}{\partial t}(t,x) = \mathcal{L}^\alpha_{\varepsilon,\delta_\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x) + \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} v^{\varepsilon,\delta_\varepsilon}(t,x) + c\Big(\dfrac{x}{\delta_\varepsilon}, \exp\left\{\frac{1}{\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x)\right\}\Big), & x \in \mathbb{R}^d, \\ v^{\varepsilon,\delta_\varepsilon}(0,x) = \varepsilon\log\left(u_0(x)\right), & x \in U_0, \\ \lim\limits_{t\to 0} v^{\varepsilon,\delta_\varepsilon}(t,x) = -\infty, & x \in \mathbb{R}^d\setminus U_0. \end{cases} \tag{4.2}$$

Let us define a distance in $\mathbb{R}_+ \times \mathbb{R}^d$, for $(t,x),(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d$:

$$d\left\{(t,x),(s,y)\right\} = \max\Big\{|t-s|, |x-y|\Big\},$$

and let us set

$$u^*(t,x) = \limsup_{\eta\to 0}\Big\{ v^{\varepsilon,\delta_\varepsilon}(s,y) :\ \varepsilon \leqslant \eta,\ (s,y) \in B\big((t,x),\eta\big)\Big\},$$
$$v^*(t,x) = \liminf_{\eta\to 0}\Big\{ v^{\varepsilon,\delta_\varepsilon}(s,y) :\ \varepsilon \leqslant \eta,\ (s,y) \in B\big((t,x),\eta\big)\Big\}.$$

Theorem 4.2. *Then* u^* *and* v^* *are sub and super viscosity solutions of :*

$$\begin{cases} \max\limits_{w}\left(\dfrac{\partial w}{\partial t}(t,x) - \mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}}\nabla w(t,x) - \overline{B}\cdot\nabla w(t,x) - \overline{C}\right) = 0 & x \in \mathbb{R}^d, t > 0, \\ w(0,x) = 0, & x \in U_0, \\ \lim\limits_{t\to 0} w(t,x) = -\infty, & x \in \mathbb{R}^d\setminus U_0, \end{cases}$$

where

$$\mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}} w := \int_{\mathbb{R}^d\setminus\{0\}} \Big\{ e^{\langle w,y\rangle} - 1 - \langle w,y\rangle\,\mathbf{1}_B(y)\Big\}\overline{\nu}(dy).$$

Proof. We use similar techniques as in Evans [5, 6]. Let us prove that u^* is a viscosity subsolution. The function $v^{\varepsilon,\delta_\varepsilon}(t,x)$ is viscosity solution of

$$\frac{\partial v^{\varepsilon,\delta_\varepsilon}}{\partial t}(t,x) - \mathcal{L}^{\alpha}_{\varepsilon,\delta_\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x) - \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} v^{\varepsilon,\delta_\varepsilon}(t,x) - c\Big(\frac{x}{\delta_\varepsilon}, \exp\Big\{\frac{1}{\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x)\Big\}\Big) = 0. \tag{4.3}$$

We notice that

$$\lim_{\varepsilon\to 0} \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} v = \mathcal{H}^{\boldsymbol{Id}_\sigma,\nu^\alpha} \nabla v.$$

Now, let Φ be a smooth function, (t_0,x_0) be a strict local maximum of $v^{\varepsilon,\delta_\varepsilon} - \Phi$, and $\psi \in \mathcal{C}^\beta\left(\mathbb{T}^d\right)$ be a periodic function solution of the following Poisson equation,

$$\begin{aligned}\mathcal{L}^\alpha \psi(z) + \Big(I + \nabla \hat{b}\Big) b_1(z) D\Phi(t_0,x_0) + \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha}\Phi(t_0,x_0) + c(z) \\ = \mathcal{H}^{\varepsilon,\sigma,\overline{\nu}}\Phi(t_0,x_0) + \overline{B}\cdot\nabla D\Phi(t_0,x_0) - \overline{C}.\end{aligned} \tag{4.4}$$

We consider now the perturbed test function

$$\Phi^\varepsilon(t,x) = \Phi(t,x) + \delta_\varepsilon \hat{b}\left(\frac{x}{\delta_\varepsilon}\right) D\Phi(t,x) + \frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\psi\left(\frac{x}{\delta_\varepsilon}\right). \tag{4.5}$$

Then we have

$$\frac{\partial \Phi^\varepsilon(t,x)}{\partial t} = \frac{\partial \Phi(t,x)}{\partial t} + \delta_\varepsilon \hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial}{\partial t} D\Phi(t,x), \tag{4.6}$$

$$D\Phi^\varepsilon(t,x) = \Big(I + \nabla\hat{b}\Big)\left(\frac{x}{\delta_\varepsilon}\right) D\Phi(t,x) + \delta_\varepsilon \hat{b}\left(\frac{x}{\delta_\varepsilon}\right) D^2\Phi(t,x) + \frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}} D\psi\left(\frac{x}{\delta_\varepsilon}\right). \tag{4.7}$$

There exists a sequence $(t_\varepsilon, x_\varepsilon)$ local maximum of $v^{\varepsilon,\delta_\varepsilon} - \Phi^\varepsilon$ converging towards (t_0,x_0). If we set $z_\varepsilon = \frac{x_\varepsilon}{\delta_\varepsilon}$, and getting ε small enough, and putting everything together in (4.3), we have

$$\begin{aligned}\frac{\partial\Phi}{\partial t}(t_0,x_0) - \mathcal{L}^\alpha\psi(z) - \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha}\Phi(t_0,x_0) - \Big(I+\nabla\hat{b}\Big) b_1(z) D\Phi(t_0,x_0) - c(z) \\ + \mathcal{A}^{\overline{\sigma},\nu^\alpha}\psi(z) + \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\mathcal{A}^{\overline{\sigma},\nu}\hat{b}(z) D\Phi(t_0,x_0) \\ - \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}} \underbrace{\Big[\Big(I+\nabla\hat{b}\Big) b_0(z) + \mathcal{A}^{\overline{\sigma},\nu}\hat{b}(z)\Big]}_{:=0} D\Phi(t_0,x_0) + o(1) \leqslant 0.\end{aligned}$$

So, from (4.5) we can observe that

$$\mathcal{A}^{\overline{\sigma},\nu^\alpha}\psi(z) = -\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\mathcal{A}^{\overline{\sigma},\nu}\hat{b}(z) D\Phi(t_0,x_0) + \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}\mathcal{A}^{\overline{\sigma},\nu}\left[\Phi^\varepsilon(t,x) - \Phi(t,x)\right].$$

By Lemma (3.1), we can observe that

$$\sup_{x\in\mathbb{R}^d}\left\{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}\mathcal{A}^{\overline{\sigma},\nu}\left[\Phi^\varepsilon(t,x) - \Phi(t,x)\right]\right\} \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

Hence, we deduce

$$\frac{\partial D\Phi}{\partial t}(t_0,x_0) - \mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}} D\Phi(t_0,x_0) - \overline{B}\cdot D\Phi(t_0,x_0) - \overline{C} \leqslant 0.$$

Let us now consider v^*. Let (t_0,x_0) such that $\overline{v}(t_0,x_0) < 0$. Let $\Phi \leqslant v^*$ be a smooth function such that $\Phi(t_0,x_0) = \overline{v}(t_0,x_0)$, and (t_0,x_0) is a strict local maximum of $\Phi - v^*$.

Considering the same perturbed function test Φ^ε as above. Hence, there exists a sequence $(t_\varepsilon, x_\varepsilon)$ locally maximizes $\Phi^\varepsilon - v^{\varepsilon,\delta_\varepsilon}$ and converges towards (t_0, x_0). By analogy,

$$\frac{\partial D\Phi}{\partial t}(t_0,x_0) - \mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}} D\Phi(t_0,x_0) - \overline{B}\cdot D\Phi(t_0,x_0) - \overline{C} \geqslant 0.$$

■

Let us now introduce some notations

$$\rho^2(t,x,y) := \inf\Big\{S_{0,t}(\varphi):\ \varphi(0)=x,\ \varphi(t)=y\Big\} \quad \text{and} \quad \rho^2(t,x,U_0) := \inf_{y\in U_0}\rho^2(t,x,y).$$

From this we easily show

Remark 4.3 ([12])**.** *Let u^* and v^* be respectively the sub- and supper-viscosity solutions of PDE (4.2). Assume that for all $(t,x)\in\,]0,\infty[\,\times\mathbb{R}^d$,*

$$-\rho^2(t,x,U_0) \leqslant v^*(t,x) \leqslant u^*(t,x) \leqslant \min\Big(\overline{C}t - \rho^2(t,x,U_0); 0\Big).$$

Then we have $v^* \geqslant u^*$.

Now, let $\mathcal{O}$ be a open subset in $\mathbb{R}\times\mathbb{R}^d$, define the function τ on $\mathbb{R}\times\mathcal{D}\left([0,\infty]\times\mathbb{R}^d\right)$ values into $[0,\infty]$,

$$\tau = \tau_{\mathcal{O}}(t,\phi) = \inf\{s : (t-s,\phi(s))\in\mathcal{O}\}$$

Take Θ the set of Markov functions τ. Let $V^*(t,x),\ t>0, x\in\mathbb{R}^d$ be the function :

$$V^*(t,x) = \inf_{\tau\in\Theta}\ \sup_{\{\phi\in\mathcal{D}([0,t],\mathbb{R}^d),\phi(0)=x,\phi(t)\in U_0\}} \Big\{\overline{C}\tau - S_{0,\tau}(\phi)\Big\}. \tag{4.8}$$

Hence, we have the uniform convergence

Remark 4.4 ([12])**.** *For $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d$,*

$$\lim_{\varepsilon\downarrow 0}\varepsilon\log u^{\varepsilon,\delta_\varepsilon}(t,x) = V^*(t,x) = \inf_{\tau\in\Theta}\ \sup_{\{\phi\in\mathcal{D}([0,t],\mathbb{R}^d),\phi(0)=x,\phi(t)\in U_0\}} \Big\{\overline{C}\tau - S_{0,\tau}(\phi)\Big\}.$$

Consider the partitions $\mathcal{M}$ and $\mathcal{E}$ of $\mathbb{R}_+\times\mathbb{R}^d$:

$$\begin{aligned}\mathcal{M} =& \Big\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d; V^*(t,x)=0\Big\},\\ \mathcal{E} =& \Big\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d; V^*(t,x)<0\Big\}.\end{aligned}$$

We have

Theorem 4.5. *By our assumptions.*

$$\lim_{\varepsilon\downarrow 0} u^{\varepsilon,\delta_\varepsilon}(t,x) = \begin{cases} 0 & \textit{uniformly from any compact } \mathcal{K} \textit{ of } \mathcal{E},\\ 1 & \textit{uniformly from any compact } \mathcal{K}' \textit{ of } \overset{\circ}{\mathcal{M}}. \end{cases}$$

5. Acknowledgement

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 82–91.
http://doi.org/10.26637/mjm11S/006

Combinatorics on words obtaining by k to k substitution and k to k exchange of a letter on modulo-recurrent words

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Received 16 June 2023; Accepted 22 August 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. We introduce two new concepts which are the k to k substitution and k to k exchange of a letter on infinite words. After studying the return words and the special factors of words obtaining by these applications on Sturmian words and modulo-recurrent words. Next, we establish the complexity functions of these words. Finally, we determine the palindromic complexity of these words.

AMS Subject Classifications: 68R15, 11B85, 03D15.

Keywords: Sturmian words, modulo-recursive, k to k substitution, k to k exchange, complexity function, palindrome.

Contents

1. Introduction

The complexity function, which counts the number of distinct factors of given length in some infinite word, is often used in characterization of some families of words [1]. For instance, Sturmian words are the infinite words non eventually periodic with minimal complexity [9, 10]. Over the past thirty years, Sturmian words are intensively studied. Thus, these investigations has led to numerous characterizations and various properties [5, 6, 8, 13, 14] on these words. In [2, 11, 12], the palindromic factors are used abundantly to study Sturmian words.

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The notion of k to k insertion of a letter on infinite words was introduced in [13], and widely studied in [4, 12]. Later in [3], authors has studied the notion of k to k erasure of letter on infinite words .

Now, we introduce the concepts of k to k substitution and k to k exchange of letter on infinite words. The first notion consists to substitute a letter steadily with step of length k in some infinite words namely Strumian words and modulo-recurrent words. Thus, the new word obtaining by this application is called *word by k to k substitution* of letter. The second consists to exchange a letter steadily with step of length k in the same infinite words. Therefore, the new word obtaining by this application is called *word by k to k exchange* of letter. This paper is focused in the studying of combinatorial properties of these two new types of words obtaining on strumian words and modulo-recurrent words.

The paper is organized as follow. In the Section 2, we give useful definitions and notations in combinatorics on words, and we recall some properties of Sturmian words and modulo-recurrent words. In Section 3, we study the special factors and we determine the complexity of words obtained by k to k substitution and by k to k exchange of letter on uniformly modulo-recurrent words. The study of some palindromic properties and the palindromic complexity of these words are established in Secttion 4.

2. Background

2.1. Combinatorial properties

An alphabet $\mathcal{A}$, is a non empty finite set whose the elements are called letters. A word is a finite or infinite sequence of elements over $\mathcal{A}$. The set of finite words over $\mathcal{A}$ is denoted $\mathcal{A}^*$ and ε, the empty word. For any $u \in \mathcal{A}^*$, the number of letters of u is called length of u and it is denoted $|u|$. Moreover, for any letter x of $\mathcal{A}$, $|u|_x$ is the number of occurrences of x in u. A word u of length n written with a unique letter x is simply denoted $u = x^n$.

Let $u = x_1x_2 \cdots x_n$ be a word such that $x_i \in \mathcal{A}$, for all $i \in \{1, 2, \ldots, n\}$. The image of u by the reversal map is the word denoted $\overline{u}$ and defined by $\overline{u} = x_n \cdots x_2x_1$. The word $\overline{u}$ is simply called reversal image of u. A finite word u is called palindrome if $\overline{u} = u$. If u and v are two finite words over $\mathcal{A}$, we have $\overline{uv}$=$\overline{v}\,\overline{u}$.

The set of infinite words over $\mathcal{A}$ is denoted $\mathcal{A}^\omega$ and we write $\mathcal{A}^\infty = \mathcal{A}^* \cup \mathcal{A}^\omega$, the set of finite and infinite words. An infinite word **u** is said to be aperiodic if there exist two words $v \in \mathcal{A}^*$ and $w \in \mathcal{A}^+$ such that $u = vw^\omega$. If $v = \varepsilon$, then u is periodic.

Let $\mathbf{u} \in \mathcal{A}^\infty$ and $w \in \mathcal{A}^*$. The word w is a factor of u if there exist $u_1 \in \mathcal{A}^*$ and $\mathbf{u}_2 \in \mathcal{A}^\infty$ such that $\mathbf{u} = u_1w\mathbf{u}_2$. The factor w is said to be a prefix (respectively, a suffix) if u_1 (respectively, $\mathbf{u}_2$) is the empty word.

A word **u** is said to be recurrent if each of its factors appears infinitly in **u**. A word **u** is said to be uniformly recurrent if for all integers n, there exists an integer N such that any factor of length N in **u** contains all the factors of length n.

A non-empty factor w of **u**, is said to be right (respectively, left) extendable by a letter x in **u** if wx (respectively, xw) appears in **u**. The number of right (respectively, left) extensions of w, is denoted $\partial^+ w$ (respectively, $\partial^- w$). The factor w is said to be right (respectively, left) special in **u** if $\partial^+ w > 1$ (respectively, $\partial^- w > 1$). If $\partial^+ w = 2$ (respectively, $\partial^+ w = 3$), we say that w have two-right (respectively, three-right) extensions in **u**. The factor w is said to be bispecial in **u** if w is both right and left special.

Let **u** be an infinite word over $\mathcal{A}$. The set of factors of length n in **u**, is written $\mathrm{L}_n(\mathbf{u})$ and the set of all factors in **u** is denoted by $\mathrm{L}(\mathbf{u})$. Let $\mathbf{u} = x_0x_1x_2 \cdots$, where $x_i \in \mathcal{A}$, $i \geq 0$ be an infinite word and w his factor. Then, w appears in **u** at the position l if $w = x_lx_{l+1} \cdots x_{l+n-1}$.

The complexity function of a given infinite word **u** is the map of $\mathbb{N}$ to $\mathbb{N}^*$ defined by $p_{\mathbf{u}}(n) = \#\mathrm{L}_n(\mathbf{u})$, where $\#\mathrm{L}_n(\mathbf{u})$ designates the cardinal of $\mathrm{L}_n(\mathbf{u})$.

This function is related to the special factors by the relation (see [7]):

$$p_{\mathbf{u}}(n+1) - p_{\mathbf{u}}(n) = \sum_{w \in \mathrm{L}_n(\mathbf{u})} (\partial^+(w) - 1).$$

We call first rigth (respectively, left) difference of the complexity function $p_{\mathbf{u}}$, the functions defined for any integer n by:

$$r_{\mathbf{u}}(n) = \sum_{w \in \mathrm{L}_n(\mathbf{u})} (\partial^+(w) - 1) \text{ and } l_{\mathbf{u}}(n) = \sum_{w \in \mathrm{L}_n(\mathbf{u})} (\partial^-(w) - 1).$$

The set of palindromic factors of length n in **u** is denoted $\mathrm{Pal}_n(\mathbf{u})$, and the set of all palindromic factors in **u**, is denoted $\mathrm{Pal}(\mathbf{u})$. The palindromic complexity function of **u** is the map of $\mathbb{N}$ to $\mathbb{N}$, defined by:

$$p^{al}_{\mathbf{u}}(n) = \#\mathrm{Pal}_n(\mathbf{u}).$$

A word $\mathbf{u} \in \mathcal{A}^\infty$ is said to be rich if for any factor w of **u**, the number of palindromic factor in w is $|w| + 1$.

Let $\mathbf{u} = x_0x_1x_2x_3\cdots$ be an infinite word. The window complexity function of **u** is the map, $p^f_{\mathbf{u}}$ of $\mathbb{N}$ into $\mathbb{N}^*$, defined by:

$$p^f_{\mathbf{u}}(n) = \#\left\{x_{kn}x_{kn+1}\cdots x_{n(k+1)-1} : k \geq 0\right\}.$$

The shift is the application S of $\mathcal{A}^\omega$ to $\mathcal{A}^\omega$ which erases the first letter of some given word. For instance, $S(\mathbf{u}) = x_1x_2\cdots$.

A morphism f is a map of $\mathcal{A}^*$ into itself such that $f(uv) = f(u)f(v)$, for any $u, v \in \mathcal{A}^*$.

2.2. Sturmian words and modulo-recurrent words

In this subsection, we recall some useful properties on Sturmian words and modulo-recurrent words.

Definition 2.1. *An infinite word* **u** *over* $\mathcal{A}_2 = \{a, b\}$ *is said to be Sturmian if for any integer* n, $p_{\mathbf{u}}(n) = n + 1$.

The well-known Sturmian word in the literature is the famous Fibonacci word. It is generated by the morphism φ defined over $\mathcal{A}_2 = \{a, b\}$ by:

$$\varphi(a) = ab \text{ and } \varphi(b) = a.$$

Definition 2.2. *[8] An infinite word* $\mathbf{u} = x_0x_1x_2\cdots$ *is said to be modulo-recurrent if, any factor* w *of* **u** *appears in* **u** *at all positions modulo* i*, for all* $i \geq 1$.

Proposition 2.1. *[8] Let* $\mathbf{u} \in \mathcal{A}^\infty$ *such that* $p_{\mathbf{u}}(n) = (\#\mathcal{A})^n$*, for all* $n \in \mathbb{N}$*. Then,* **u** *is a modulo-recurrent word.*

Definition 2.3. *Let* **u** *be a modulo-recurrent word. Then,* **u** *is called non-trivial if there exists an integer* n_0 *such that for all* $n \geq n_0$*:*

$$p_{\mathbf{u}}(n) < (\#\mathcal{A})^n.$$

Definition 2.4. *Let* **u** *be an infinite word. Then,* **u** *is said to be uniformly modulo-recurrent if it is uniformly recurrent and modulo-recurrent.*

Definition 2.5. *A factor* w *of some infinite word* **u** *is said to be a window factor when it appears in* **u** *at a mutiple position its length.*

Let us denote $\mathrm{L}^f_n(\boldsymbol{u})$, the set of n-window factors of u for a given factor of length n. Thus, his cardinal is $p^f_{\boldsymbol{u}}(n)$.

Theorem 2.6. *[13] Every Sturmian word is modulo-recurrent.*

It is clear that the Sturmian words are non-trivial and uniformly modulo-recurrent. The champernowne word is modulo-recurrent but does not uniformly recurrent.

The following theorem presents some characterization on Sturmian words.

Theorem 2.7. *[11] Let* **u** *be a Sturmian word. Then, for all $n \in \mathbb{N}$, we have:*

$$p_{\mathbf{u}}^{al}(n) = \begin{cases} 1 & if \quad n \quad is \quad even \\ 2 & otherwise. \end{cases}$$

The modulo-recurrent words can be characterized by their window complexity as follow:

Theorem 2.8. *[8] A recurrent word* **u** *is modulo-recurrent if and only if $p_{\mathbf{u}}^{f}(n) = p_{\mathbf{u}}(n)$, for all $n \geq 1$.*

3. k to k substitution and k to k exchange of letter in infinite words

We introduce two new concepts called k to k substitution and k to k exchange of a letter in infinite words with $k \geq 1$. We study the combinatorial properties of these words.

Definition 3.1. *Let* **u** *be an infinite word over $\mathcal{A}$ in the form:*

$$\mathbf{u} = x_0 m_0 x_1 m_1 x_2 m_2 x_3 m_3 \cdots x_i m_i \cdots$$

with $m_i \in L_k(\mathbf{u})$ et $x_i \in \mathcal{A}$, $i \in \mathbb{N}$.

Let us substitute the letters x_i with a letter $c \notin \mathcal{A}$ in **u***. Then, we obtain an infinite word:*

$$\mathbf{v} = c m_0 c m_1 c m_2 c m_3 c \cdots c m_i c \cdots .$$

This new word is called word by k to k substitution of letter in **u** *and is denoted $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$.*

Now, Let us denote $\tilde{.}$, the circular exchange map of letter defined over $\mathcal{A}$ and exchange the letters x_i in **u***. Thus, we obtain the word:*

$$\mathbf{w} = \tilde{x_0} m_0 \tilde{x_1} m_1 \tilde{x_2} m_2 \tilde{x_3} m_3 \cdots \tilde{x_i} m_i \cdots .$$

It is called word by k to k exchange of a letter in **u** *and denoted by $\mathbf{w} = \mathcal{E}_k(\mathbf{u})$.*

Example 3.1. Let us consider the Fibonacci word

$$\mathbf{f} = abaababaabaababaababaabababaabaababaa \cdots .$$

Then, we have:

$$\mathcal{S}_3^c(\mathbf{f}) = cbaacabacbaacabacbabcabacaabcabacaa \cdots ,$$

called word by 3 to 3 substitution of the letter c in **f**.

$$\mathcal{E}_2(\mathbf{f}) = bbaaaabbabbaaaabbabbabbabbaaaabbabbaaaaa \cdots ,$$

called word by 2 to 2 exchange of letter in **f**.

3.1. Return words and special factors in v

We study the return words of the extrenal letter c and special factors in **v**.

Proposition 3.1. *Let* **u** *be a recurrent word such that $\mathbf{v} = \mathcal{S}_3^c(\mathbf{u})$. Then, we have:*

$$Ret_{\mathbf{v}}(c) \subset \{cm_i; m_i \in \mathrm{L}_k(\mathbf{u})\}.$$

Proof. We have $Ret_{\mathbf{v}}(c) = \{cS(w_i);\ w_i \in \mathrm{L}_{k+1}^{f}(\mathbf{u})\}$. Moreover, we have $\{S(w_i); w_i \in \mathrm{L}_{k+1}^{f}(\mathbf{u})\} \subset \mathrm{L}_k(\mathbf{u})$. As a result, $\{cS(w_i); w_i \in \mathrm{L}_{k+1}^{f}(\mathbf{u})\} \subset \{cm_i; m_i \in \mathrm{L}_k(\mathbf{u})\}$. ■

Corollary 3.1. *Let* **u** *be a modulo-recurrent word such that* $\mathbf{v} = \mathcal{S}_3^c(\mathbf{u})$*. Then:*

$$\#Ret_{\mathbf{v}}(c) = p_{\mathbf{u}}(k).$$

Proof. According to Proposition 3.1, we have $Ret_{\mathbf{v}}(c) \subset \{cm_i; m_i \in \mathrm{L}_k(\mathbf{u})\}$. Let us consider $v_1 \in \{cm_i; m_i \in \mathrm{L}_k(\mathbf{u})\}$, then there exists $m_{i_0} \in \mathrm{L}_k(\mathbf{u})$ such that $v_1 = cm_{i_0}$. Since **u** is modulo-recurrent, then $cm_{i_0} \in \mathrm{L}(\mathbf{v})$, i.e. $v_1 \in \mathrm{L}(\mathbf{v})$. Hence, $\{cm_i;\ m_i \in \mathrm{L}_k(\mathbf{u})\} \subset Ret_{\mathbf{v}}(c)$. So, $Ret_{\mathbf{v}}(c) = \{cm_i;\ m_i \in \mathrm{L}_k(\mathbf{u})\}$, i.e. $\#Ret_{\mathbf{v}}(c) = p_{\mathbf{u}}(k)$. ∎

Lemma 3.1. *Let* **u** *be a Sturmian word and* w *a factor of* $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$*. Then,* w *admits three-right (respectively, three-left) extensions in* **v** *if and only if* $|w| < k$ *and* w *is right (respectively, left) special in* **u**.

Proof. Let us consider $w \in \mathrm{L}(\mathbf{v})$.

CS: Let us assume that $|w| < k$ such that w is a rigth special factor of **u**. Then, wa and wb are factors of $textbfu$. In addition, $|wa| = |wb| \leq k$. Thus $wa, wb \in \mathrm{L}(\mathbf{v})$ since **u** is modulo-recurrent. Therefore, **u** being modulo-recurrent then by Corollary 3.1, we have $wc \in \mathrm{L}(\mathbf{v})$. So, w have three-right extensions in **v**.

CN: Let us suppose that w have three-right extensions in **v**. We discuss over the length of w.

Case 1: $|w| < k$. Since $wc \in \mathrm{L}(\mathbf{v})$, then $w \in \mathrm{L}(\mathbf{u})$ and $wa, wb \in \mathrm{L}(\mathbf{v})$. Thus, we have $wa, wb \in \mathrm{L}(\mathbf{u})$.

Case 2: $|w| = k$. Then, w have three-right extensions in **v**, i.e. $w \in \mathrm{L}(\mathbf{u})$. In addition, any factor of length $(k+1)$ in **v** contains exactly one occurrence of the letter c. That implies $wa, wb \notin \mathrm{L}(\mathbf{v})$. This contradicts the fact that w have three-right extensions in **v**.

Case 3: $|w| > k$. Then, w is in the form $w = w_0cw_1$ with $|w_1| = k$. Since w is right special in **v** then $w_1 \in \mathrm{L}(\mathbf{u})$. By a similar reasoning to the case 2, we have a contradiction.

Hence, in all cases, w have three-right extensions in **v** if $|w| < k$.

With the same arguments we can show that w have also three-left extensions in **v**. ∎

Theorem 3.2. *Let* **u** *be a Sturmian word over* $\mathcal{A}_2$ *and* $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$*. Then, the number of three-right (respectively, three-left) extensions of length* n *in* **v** *is:*

$$\boldsymbol{T}rip_{\mathbf{v}}(n) = \begin{cases} 1 & if \quad n < k \\ 0 & otherwise. \end{cases}$$

Proof. Let us consider $w \in \mathrm{L}(\mathbf{v})$. Then, by Lemma 3.1, w have three-right (respectively, three-left) extensions in **v** if and only if w is right (respectively, left) special in **u** and $|w| < k$. Since **u** is Sturmian, then for each length n, **u** have only one right (respectively, left) special factor. Consequently, **v** produces three-right (respectively, three-left) extensions factor of length n if $n < k$ and neither otherwise. ∎

Theorem 3.3. *Let* **u** *be a Sturmian word over* $\mathcal{A}_2$ *and* $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$*. Then, the number of two-right (respectively, two-left) extensions of length* n *in* **v** *is:*

$$\boldsymbol{B}ip_{\mathbf{v}}(n) = \begin{cases} 2n+1 & if \quad n < k \\ k+1 & otherwise. \end{cases}$$

Proof. Let us designate by $\mathcal{B}r_{\mathbf{v}}(n)$, the set of factors of length n having two-right extensions in **v** and denote w_n the right special factor of length n in **u**. It follows that:

- For $n \leq k$, we have:

$$\mathcal{B}r_{\mathbf{v}}(n) = \mathrm{L}_n(\mathbf{u})\backslash\{w_n\} \bigcup \{t_0ct_1 : t_0xt_1 = w_n; |t_1| = 0, 1\cdots, n-1\} \bigcup \{tc : t = w_{n-1}\}.$$

Consequently, $\mathbf{B}ip_{\mathbf{v}}(n) = 2n+1$.

• For $n > k$, we have:

$$\mathcal{B}r_{\mathbf{v}}(n) = \{t_0ct_1ct_2c\cdots ct_q : t_0x_1t_1x_2t_2x_3\cdots x_qt_q = w_n\}\bigcup\{tc : t = w_{n-1}\}$$

with $|t_0| \leq k, |t_q| \leq k-1$ and $|t_i| = k, \forall i = 1, 2, \cdots, q-1$.

Hence, $\mathbf{B}ip_{\mathbf{v}}(n) = k+1$. ■

Remark 3.1. Let **u** be a Sturmian word over $\mathcal{A}_2$ and $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$. Then, any right (respectively, left) special factor of length n in **v** with ($n \geq k$), give two-right (respectively, two-left) extensions in **v**.

3.2. Complexity function of the infinite word v

In this subsection, we determine the complexity function for the word **v** obtaining by k to k substitutiton of letter in some infinite word **u**.

Let $u_1 \in \mathrm{L}(\mathbf{u})$ for a given infinite word **u**. Then, we denote by $\mathcal{F}_k^c(u_1)$, the set of words obtaining by substitution of the letter c in u_1, i.e.

$$\mathcal{F}_k^c(u_1) = \{t_0ct_1ct_2c\cdots ct_q : t_0x_1t_1x_2t_2x_3\cdots x_qt_q = u_1; |t_0|, |t_q| \leq k, |t_i| = k, \forall i = 1, 2, \ldots, q-1\}.$$

Proposition 3.2. *Let* **u** *be a modulo-recurrent word such that* $u_1 \in \mathrm{L}(\mathbf{u})$ *and* $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$. *Then:*

$$\mathcal{F}_k^c(u_1) \subset \mathrm{L}(\mathbf{v}).$$

Remark 3.2. Any factor of length n in **v** comes from a factor of length n in **u**.

$$\#\mathcal{F}_k^c(u_1) = \begin{cases} |u_1| & if \quad |u_1| \geq k \\ k+1 & otherwise. \end{cases}$$

Lemma 3.2. *Let us consider* $v_1, v_2 \in \mathrm{L}_n(\mathbf{v})$. *Then:*

$$||v_1|_c - |v_2|_c| \leq 1.$$

Proof. By Remark 3.2, the k to k substitution conserve the lengths of factors. ■

Proposition 3.3. *Let* **u** *be an infinite word such that* $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$. *Then, for any integer* $n > 1$, *we have:*

$$p_{\mathbf{v}}(n) \leq \begin{cases} (n+1)p_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1) & if \quad n \leq k \\ (k+1)p_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1) & if \quad n > k. \end{cases}$$

Proof. Firstly, the substitutions starting with the first letter of extensions to the left of the same special factor to the left give the same factor. The same applies to substitutions ending in the last letter extensions to the right of the same special factor to the right give the same factor. Thus, according to the values of k and n we have two cases.

Case 1 : $n \leq k$. Then, some factors of **u** of length n are also factors of **v**. Moreover, the other factors of **v** of length n are produced from factors of **u** of length n by substitution of one letter. Thus, we have:

$$\mathrm{L}_n(\mathbf{v}) \subseteq \mathrm{L}_n(\mathbf{u})\bigcup\{rcs; \quad rxs \in \mathrm{L}_n(\mathbf{u}), x \in \mathcal{A}\}.$$

Consequently, we have the following inequality: $p_{\mathbf{v}}(n) \leq p_{\mathbf{u}}(n) + np_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1)$, i.e. $p_{\mathbf{v}}(n) \leq (n+1)p_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1)$, for all $n \leq k$.

Case 2 : $n > k$. Then, any factor of length n in **v** contains at least one occurrence of the letter c. By Remark 3.2, any factor of length n in **u** prodruces at most $(k+1)$ factors of length n in **v**. Hence, $p_{\mathbf{v}}(n) \leq (k+1)p_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1)$, for all $n > k$. ■

Corollary 3.2. *Let* **u** *be a uniformly modulo-recurrent word and* $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$*. Then, the complexity function of* **v** *is given by:*

$$p_{\mathbf{v}}(n) = \begin{cases} \#(\mathcal{A}) + 1 & if \quad n = 1 \\ (n+1)p_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1) & if \quad 1 < n \leq k. \end{cases}$$

Proof. Since **u** being modulo-recurrent, then by Proposition 3.2, all substitutions of factors of **u** give the factors of **v**. By using the Proposition 3.3, we deduce
$p_{\mathbf{v}}(n) = (n+1)p_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1) \quad$ if $\quad 1 < n \leq k.$ ■

For the following, we need the definition below.

Definition 3.4. *Let* **u** *be a uniformly modulo-recurrent word and* u_1 *be a factor of* **u**. *Then* u_1 *is said to be sufficiently long if, it contains all the* $(k+1)$*-window factors of* **u**.

Now, we denote n_0, the minimal length of the sufficiently long factors of ***u***. The next Lemma allowed to determine the complexity function of **v**.

Lemma 3.3. *Let* **u** *be a uniformly modulo-recurrent word over* $\mathcal{A}$*. Let* $u_1,\ u_2$ *be two distinct sufficiently long factors of* **u** *such that* $S(u_1) \neq S(u_2)$ *with* $u_1x^{-1} \neq u_2y^{-1}$ *where* $x, y \in \mathcal{A}$*. Then, the words obtained by* k *to* k *substitution of letter in* u_1 *and* u_2 *are distinct in* $\mathcal{S}_k^c(\mathbf{u})$.

Proof. Let us consider $|S(u_1)| \geq n_0$ and $|S(u_2)| \geq n_0$. We have two cases.

Case 1 : $|u_1| \neq |u_2|$. Then u_1 and u_2 give distinct factors, since the k to k substitution preserve the lengths.

Case 2 : $|u_1| = |u_2|$. Then, let us assume that $S(u_1) \neq S(u_2)$ and $u_1x^{-1} \neq u_2y^{-1}$. Since ***u*** is a modulo-recurrent and $|u_1|, |u_2| \geq n_0$, then we have $\mathcal{F}_k^c(u_1) \cap \mathcal{F}_k^c(u_2) = \emptyset$. ■

Theorem 3.5. *Let* **u** *be a uniformly modulo-recurrent word and* $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$*. Then, the complexity function of* **v** *is given by:*

$$p_{\mathbf{v}}(n) = (k+1)p_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1), \textit{ for all } n \geq n_0.$$

Proof. Let us suppose that **u** is uniformly modulo-recurrent. Then, by Proposition 3.2, all substitutions of factors of **u** give the factors of **v**. According to Lemma 3.3, we obtain the existence of n_0. Thus, the Proposition 3.5 allows us to deduce:

$$p_{\mathbf{v}}(n) = (k+1)p_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1), \text{ for all } n \geq n_0.$$

■

Theorem 3.6. *Let* **u** *be a uniformly modulo-recurrent word and* $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$*. Then, the window complexity function of* **v** *satisfies:*

$$p_{\mathbf{v}}^f(n) = \begin{cases} p_{\mathbf{u}}(n-1) & if \quad n \equiv 0 \ \ mod\ (k+1) \\ p_{\mathbf{v}}(n) & otherwise. \end{cases}$$

Proof. Let us consider $n \in \mathbb{N}$ and **u** modulo-recurrent. Then, we have:

• If $n \equiv 0 \ \ mod\ (k+1)$, then let us put $n = q(k+1)$, $q \geq 1$. It follows that:
$\mathrm{L}_n^f(\mathbf{v}) = \{cm_1cm_2c\cdots cm_q :\ x_1m_1x_2\cdots x_qm_q \in \mathrm{L}_n^f(\mathbf{u});\ |m_i| = k;\ i = 1, \ldots, q\}$
$\mathrm{L}_n^f(\mathbf{v}) = \{cm_1cm_2c\cdots cm_q :\ m_1x_2\cdots x_qm_q \in \mathrm{L}_{n-1}(\mathbf{u});\ |m_i| = k;\ i = 1, \ldots, q\}$.
Thus, $\#\mathrm{L}_n^f(\mathbf{v}) = \#\mathrm{L}_{n-1}(\mathbf{u})$, i.e. $p_{\mathbf{v}}^f(n) = p_{\mathbf{u}}(n-1)$.

• Otherwise we have allways that $\#\mathrm{L}_n^f(\mathbf{v}) = \#\mathrm{L}_n^f(\mathbf{u}) = \#\mathrm{L}_n(\mathbf{u})$. ■

Corollary 3.3. *Let* **u** *be a Sturmian word and* $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$. *Then, we have:*

• *for* $n \leq k$,

$$p_{\mathbf{v}}(n) = \begin{cases} 3 & if \quad n = 1 \\ n^2 + 2n - 1 & if \quad 1 < n \leq k; \end{cases}$$

• *for* $n \geq n_0$,

$$p_{\mathbf{v}}(n) = kn + n + k - 1, \quad for\ all \quad n \geq n_0.$$

Proof. We have $\mathrm{L}_1(\mathbf{v}) = \{a, b, c\}$, so $p_{\mathbf{v}}(1) = 3$. According to Definition 2.1, we have $r_{\mathbf{u}}(n) = l_{\mathbf{u}}(n) = 1$, for all $n \in \mathbb{N}$. Hence, by Corollary 3.2 we have $p_{\mathbf{v}}(n) = n^2 + 2n - 1$ if $1 < n \leq k$ and by using the Theorem 3.5, we deduce that $p_{\mathbf{v}}(n) = kn + n + k - 1$, for all $n \geq n_0$. ■

Corollary 3.4. *Let* **u** *be a non-trivial and uniformly modulo-recurrent word over* $\mathcal{A}$ *and* $\mathbf{v} = \mathcal{S}_k^x(\mathbf{u})$ *with* $x \in \mathcal{A}$. *Then, the complexity function of* **v** *satisfies:*

$$p_{\mathbf{v}}(n) = (k+1)p_{\mathbf{u}}(n) - r_{\mathbf{u}}(n-1) - l_{\mathbf{u}}(n-1), \text{ for all } n \geq n_0.$$

Proof. The k to k substitution of an internal and external letter in the modulo-recurrent words are the same for longer lengths. In addition, ***u*** being uniformly recurrent, thus by Lemma 3.3, we have the existence of n_0. Hence, by Theorem 3.5, we deduce the equalty. ■

3.3. Complexity function of the infinite word w

Here we are focus our study on the complexity function of the word **w**.

Proposition 3.4. *Let* **u** *be an infinite word and* $\mathbf{w} = \mathcal{E}_k(\mathbf{u})$. *Then, we have:*

$$p_{\mathbf{w}}(n) \leq \begin{cases} (n+1)p_{\mathbf{u}}(n) & if \quad n \leq k \\ (k+1)p_{\mathbf{u}}(n) & if \quad n > k. \end{cases}$$

Proof. The demonstration is similar to Proposition 3.3. ■

Theorem 3.7. *Let* **u** *be a non-trivial and uniformly modulo-recurrent word and* $\mathbf{w} = \mathcal{E}_k(\mathbf{u})$. *Then, the complexity function of* **w** *is given by:*

$$p_{\mathbf{w}}(n) = (k+1)p_{\mathbf{u}}(n), \ for\ all \ \ n \geq n_0.$$

Proof. Since **u** is uniformly modulo-recurrent. Then, by Definition 3.4, any factor of length n ($n \geq n_0$) in **u** produces $(k+1)$ distinct factors of length n in **w**. Hence, $p_{\mathbf{w}}(n) = (k+1)p_{\mathbf{u}}(n)$. ■

Remark 3.3. Let $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$ and $\mathbf{w} = \mathcal{E}_k(\mathbf{u})$ be two infinite words for a given word **u**. Then, we have:

(i) The word **v** (respectively, **w**) is aperiodic if and only if **u** is aperiodic.

(ii) The word **v** (respectively, **w**) is recurrent if and only if **u** is recurrent.

4. Palindromic study of v and w

In this section, we prove that **v** and **w** have the same palindromic complexity function if **u** is non-trivial and uniformly modulo-recurrent and then we give this complexity.

Proposition 4.1. *Let* **u** *be a modulo-recurrent word such that* $\mathbf{v} = \mathcal{S}_k^x(\mathbf{u})$ *and* $\mathbf{w} = \mathcal{E}_k(\mathbf{u})$ *with* $x \in \mathcal{A}$. *Then, the language* $\mathrm{L}(\mathbf{v})$ *(respectively,* $\mathrm{L}(\mathbf{w})$*) is stable by reversal map if and only if* $\mathrm{L}(\mathbf{u})$ *is stable by reversal map.*

Remark 4.1. Any palindromic factor of **v** (respectively, **w**) comes from a palindromic factor of ***u***.

Proposition 4.2. *Let* **u** *be an infinite word. Then, the palindromic complexity function of* **v** *is given by:*

- *for $n \leq k$,*

$$p_{\mathbf{v}}^{al}(n) \leq \begin{cases} p_{\mathbf{u}}^{al}(n) & if \quad n \ even \\ 2p_{\mathbf{u}}^{al}(n) & otherwise. \end{cases}$$

- *for $n > k$, there exists two integers q and r such that $n = (k+1)q + r$ with $0 \leq r \leq k$ and $q \geq 1$;*

$$p_{\mathbf{v}}^{al}(n) \leq \begin{cases} 2p_{\mathbf{u}}^{al}(n) & if \quad k, r \ odd \\ 0 & if \quad k \ odd, r \ even \\ p_{\mathbf{u}}^{al}(n) & otherwise. \end{cases}$$

Proof. • For $n \leq k$, the factors of length n in **v** are factors of **u** and those of **v** in the form rcs with $rxs \in \mathrm{L}_n(\mathbf{u})$. We obtain:

$$\mathrm{Pal}_n(\mathbf{v}) \subseteq \mathrm{Pal}_n(\mathbf{u}) \bigcup \left\{tc\bar{t};\quad tx\bar{t} \in \mathrm{Pal}_n(\mathbf{u}), x \in \mathcal{A}_2\right\}.$$

- For $n > k$, we have $n = (k+1)q + r$ with $0 \leq r \leq k$, $q \geq 1$. According to Lemma 3.2, for $v_1 \in L_n(\mathbf{v})$ we have $|v_1|_c \in \{q, q+1\}$ with $q = \dfrac{n-r}{k+1} = \left\lfloor \dfrac{n}{k+1} \right\rfloor$.

$$\mathrm{Pal}_n(\mathbf{v}) \subseteq \left\{m_0cm_1cm_2c\cdots cm_q : m_0x_1m_1x_2\cdots x_qm_q \in \mathrm{Pal}_{(k+1)q+r}(\mathbf{u}); |m_0| = |m_q| = \frac{k+r}{2}\right\}$$

$$\bigcup \left\{m_0cm_1c\cdots cm_qcm_{q+1} : m_0x_1m_1\cdots x_{q+1}m_{q+1} \in \mathrm{Pal}_{(k+1)q+r}(\mathbf{u}); |m_0| = |m_{q+1}| = \frac{r-1}{2}\right\}.$$

  - If k and r are odd, then $k+r$ and $r-1$ are even. Thus, we have $p_{\mathbf{v}}^{al}(n) \leq 2p_{\mathbf{u}}^{al}(n)$.
  - If k and r are even, then $k+r$ is even and $r-1$ is odd. So, we have $p_{\mathbf{v}}^{al}(n) \leq p_{\mathbf{u}}^{al}(n)$.
  - If k is even and r is odd, then $k+r$ is odd and $r-1$ is even. Thus, we deduce $p_{\mathbf{v}}^{al}(n) \leq p_{\mathbf{u}}^{al}(n)$.
  - If k is odd and r is even, then $k+r$ and $r-1$ are odd. Consequently, **v** does not admit palindromic factor of length n.

■

Corollary 4.1. *Let **u** be a uniformly modulo-recurrent word over $\mathcal{A}$ such that $awa, bwb \in \mathrm{Pal}(\mathbf{u})$ implies $a = b$, for all $a, b \in \mathcal{A}$. Then, the inequality of the Proposition 4.2 becomes an eqality, for all $n > n_0$.*

Remark 4.2. Let **u** be a non-trivial and uniformly modulo-recurrent word over $\mathcal{A}$ such that $awa, bwb \in \mathrm{Pal}(\mathbf{u})$ implies $a = b$, for all $a, b \in \mathcal{A}$. Then, $\mathbf{v} = \mathcal{S}_k^x(\mathbf{u})$ and $\mathbf{w} = \mathcal{E}_k(\mathbf{u})$ verifie:

$$p_{\mathbf{w}}^{al}(n) = p_{\mathbf{v}}^{al}(n), \text{ for all } n > n_0.$$

Remark 4.3. The k to k substitution and k to k exchange do not preserve:

(i) the palindromic richness with exception where $k = 1$;

(ii) the modulo-recursive.

Theorem 4.1. *Let **u** be a Sturmian word over $\mathcal{A}_2$ and $\mathbf{v} = \mathcal{S}_k^c(\mathbf{u})$. Then, we have:*

- *for $n \leq k$,*

$$p_{\mathbf{v}}^{al}(n) = \begin{cases} 3 & if \quad n = 1 \\ 1 & if \quad n \ even \\ 4 & otherwise \end{cases}.$$

- *for* $n > n_0$,

$$p^{al}_{\mathbf{v}}(n) = \begin{cases} 4 & if \quad k, n\ odd \\ 2 & if \quad k\ even, n\ odd \\ 1 & if \quad k, n\ even \\ 0 & if \quad k\ odd, n\ even. \end{cases}$$

Proof. • For $n \leq k$. Then for the initial rank $n = 1$, we have $\mathrm{Pal}_1(\mathbf{v}) = \{a, b, c\}$.
Since **u** is Sturmian then, it is modulo-recurrent. So, by Theorem 3.5, for $1 < n \leq k$, we have:

$$\mathrm{Pal}_n(\mathbf{v}) = \mathrm{Pal}_n(\mathbf{u}) \bigcup \left\{tc\bar{t};\quad tx\bar{t} \in \mathrm{Pal}_n(\mathbf{u}), x \in \mathcal{A}_2\right\}.$$

By Theorem 2.7, **v** admits 1 (respectively, 4) palindromic factors if n is even (respectively, odd).

- $n > k$, we write $n = (k+1)q + r$ with $0 \leq r \leq k$. By using the parity of k and n in Proposition 4.2 and Corollary 4.1, we have the result by Theorem 2.7 .

■

Remark 4.4. If **u** is Sturmian then $\mathbf{v} = \mathcal{S}^c_k(\mathbf{u})$ is not erasing Sturmian.

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 92–114.
http://doi.org/10.26637/mjm11S/007

On the asymptotic behavior of a size-structured model arising in population dynamics

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*R*eceived 16 June 2023; *A*ccepted 24 August 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. We study Perron's theorem of a size-structured population model with delay when the nonlinearity is small in some sense. The novelty in this work is that the operator governing the linear part of the equation does not generate a compact semigroup unlike in the results present in literature. In such a case the spectrum does not consist wholly of eigenvalues but also has a non-trivial component called Browder's essential spectrum. To overcome the lack of compactness, we give a localization of Browder's essential spectrum of the operator governing the linear part and we use the Perron-Frobenius spectral analysis adapted to semigroups of positive operators in Banach lattices to investigate the long time behavior of the system.

AMS Subject Classifications: 35B40, 35R10, 47D06.

Keywords: Perron-Frobenius, positive operators, structured population models, Browder's essential spectrum, asymptotic behavior, semigroups of operators.

Contents

1. Introduction

Many areas of applied mathematics involve delay partial differential equations. Dynamical systems found in biology, physics, or economics depend not only on the present state of the dynamic but also on the past states. One of the simplest delay models describing a population of species struggling for a common food is the logistic model [13, 19]

$$\dot{N}(t) = \gamma \left(1 - \frac{N(t-r)}{K}\right) N(t). \tag{1.1}$$

The delay r here is the production time of food resources. The food resources at time t are determined by the population number at time $t-r$. The constant γ is related to the reproduction of species, and represents the difference between birth and death rates. Usually, γ is called the *Maltus coefficient* of linear growth. The constant

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K is the average population number, and is related to the ability of the environment to sustain the population. At the same time, Equation (1.1) can be used to study hatching periods, pregnancy duration, egg-laying, etc.

However, individuals in every biological population differ in their physiological characteristics. This gives an importance to structured partial differential equations to understand the dynamics of such populations. We refer the interested reader to the monographs [20] for basic concepts and results in the theory of structured populations, and [24, 30] for the theory of structured populations models using the semigroup approach.

In this work, we study the asymptotic behavior of the following size structured population model:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,s) = -\gamma \dfrac{\partial}{\partial s}u(t,s) - \mu(s)u(t,s) + \displaystyle\int_{-r}^{0} \nu(s,\sigma)u(t+\sigma,s)d\sigma \\ \qquad\qquad + \displaystyle\int_{0}^{\infty}\int_{-r}^{0} \beta(\sigma,s,b)u(t+\sigma,b)d\sigma db + f(t,u(t,s)) \qquad \text{for } t\geq 0,\ s\in\mathbb{R}^+ \\ u(t,0)=0 \qquad \text{for } t\geq 0 \\ u(\sigma,s)=\varphi(\sigma,s) \qquad \text{for } (\sigma,s)\in[-r,0]\times\mathbb{R}^+ \end{cases} \tag{1.2}$$

when the nonlinear perturbation f is small in some sense. To achieve this task, we will use a functional analytic approach involving semigroups of operators.

The theory of strongly continuous semigroups of operators have been applied with great success to partial differential equations with delay. This idea goes back to N. Krasovskii [21], who showed that solutions of delay differential equations generate a semigroup of operators on an appropriate function space, known as history or phase space. J. Hale [15] and S. N. Shimanov [28] were the first to formulate a general theory. Subsequently, using semigroup theory, J. Hale and S. Verduyn Lunel [16] described the asymptotic properties of the solution in the finite-dimensional case. Other works in this direction include [1, 5, 10, 18, 29]. The idea is to rewrite delay partial differential equations in the following form:

$$\frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t,x(t)), \tag{1.3}$$

where A is a linear (unbounded) operator acting on a Banach space X, x_t is the history function and L is a linear operator acting on the delay space with values in X. If X is finite dimensional and $L=0$, then A is a matrix and Equation (1.3) is an ordinary differential equation. If X is infinite dimensional, then the operator A is usually considered to be unbounded and generates a strongly continuous semigroup of operators $(T(t))_{t\geq 0}$ [12]. The so called Perron's Theorem for the asymptotic behavior of solutions of differential equations have been the subject of many studies, see [3, 4, 7, 23, 25–27]. For ordinary differential equations, we refer the reader to the books [8, 9, 11, 17]. Let us recall the original Perron's Theorem for ordinary differential equations.

Theorem. [9] *Consider the following ordinary differential equation*

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + f(t,x(t)) \quad \textit{for } t\geq 0 \\ x(0)=x_0\in\mathbb{C}^n, \end{cases} \tag{1.4}$$

where A is an $n\times n$ constant complex matrix and $f:[0,\infty)\times\mathbb{C}^n\to\mathbb{C}^n$ is a continuous function such that

$$|f(t,z)| \leq \gamma(t)|z| \quad \textit{for } t\geq 0 \textit{ and } z\in\mathbb{C}^n,$$

where $\gamma:[0,\infty)\to[0,\infty)$ is a continuous function satisfying:

$$\int_t^{t+1}\gamma(s)ds\to 0 \quad \textit{as } t\to\infty.$$

If $x(.)$ is a solution of Equation (1.4)*, then either*

$$x(t)=0 \quad \textit{for all large } t,$$

or

$$\lim_{t\to\infty} \frac{\log|x(t)|}{t} = Re\,\lambda_0,$$

where λ_0 is one of the eigenvalues of A.

In [26], the author proved a Perron's Theorem for Equation (1.3), when $A = 0$, with a finite delay and the space X is finite dimensional. In [22], the authors studied the case when X is infinite dimensional and the delay is infinite. They assumed that the operator A is the infinitesimal generator of a **compact** strongly continuous semigroup on X. A typical example of such an operator A is the differential operator in reaction diffusion equations on bounded regular domains Ω.

The aim of this work is to investigate the asymptotic behavior of the semilinear partial differential equation (1.2). Unlike in most models of semilinear reaction diffusion equations, the linear part of our equation is governed by a semigroup which is not compact. In such a case the spectrum does not consist wholly of eigenvalues but also has a non-trivial component called the essential spectrum. In the literature there are many different ways of looking at the essential spectrum, but a notable result in this area is that due to Nussbaum and (independently) Lebow and Schechter: the radius of the essential spectrum is the same for all the commonly used definitions of essential spectrum. To overcome the lack of compactness in our system, we will first give a localization of Browder's essential spectrum of the operator governing the linear part. This allows us to investigate the asymptotic behavior of the semilinear equation via a spectral decomposition by splitting the spectrum of the linear part with vertical lines $i\mathbb{R} + \rho$, $\rho \in \mathbb{R}$ "far" from the essential spectrum. Finally, we give a sufficient condition for extinction of the population in terms of the coefficients of the system. To achieve this task, we use the semigroup version of the Perron-Frobenius theory of positive operators in Banach lattices [2, 14].

This work is organized as follows: In Section 2, we give a localization of Browder's essential spectrum of the linear model. In Section 3, we will study the effect of small nonlinear perturbations on the original linear model. Moreover, we give a sufficient condition for extinction of the population using a Perron-Frobenius type theory of positive operators.

2. The linear model: localization of the essential spectrum

We consider a population of individuals that are distinguished by their individual size. Therefore, the density of population of size s at time t can be described by the number $u(t,s)$. More precisely $\int_{s_1}^{s_2} u(t,s)ds$ is the number of individuals that at time t have size s between s_1 and s_2. As time passes, the following processes are supposed to take place in this population:

- Individuals grow linearly in time at constant rate $\gamma > 0$.
- Individuals are subject to a size-dependent mortality denoted by μ.
- It is assumed that individuals may have different sizes at birth, and therefore $\beta(\sigma, s, b)$ gives the rate at which an individual of size b produces offspring of the size s. This process is assumed to occur with a continuous time delay smaller than r (e.g. pregnancy duration).
- The population is subject to a density-dependent migration process with continuous time lags smaller then r represented by the term $\int_{-r}^{0} \nu(s,\sigma)u(t+\sigma,s)d\sigma$.

From those assumptions the following evolution equation can be derived:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,s) = -\gamma\dfrac{\partial}{\partial s}\left(u(t,s)\right) - \mu(s)u(t,s) + \displaystyle\int_{-r}^{0} \nu(s,\sigma)u(t+\sigma,s)d\sigma \\ \qquad\qquad + \displaystyle\int_{0}^{\infty}\int_{-r}^{0} \beta(\sigma,s,b)u(t+\sigma,b)d\sigma db \qquad \text{for } t \geq 0 \text{ and } s \in \mathbb{R}^+ \\ u(t,0) = 0 \qquad \text{for } t \geq 0 \\ u(\sigma,s) = \varphi(\sigma,s) \qquad \text{for } (\sigma,s) \in [-r,0]\times\mathbb{R}^+. \end{cases} \tag{2.1}$$

In the sequel, we assume that: $\mu \in L^{\infty}(\mathbb{R}^{+},\mathbb{R}^{+})$ and $\nu \in L^{\infty}(\mathbb{R}^{+}\times[-r,0],\mathbb{R}^{+})$. The birth function $\beta:[-r,0]\times\mathbb{R}^{+}\times\mathbb{R}^{+}\to\mathbb{R}^{+}$ satisfies:

$$\sup_{\substack{-r\leq\sigma\leq 0\\ b\geq 0}}\int_{0}^{\infty}\beta(\sigma,s,b)ds<\infty. \tag{2.2}$$

An example of such function is given by

$$\beta(\sigma,s,b)=\beta_{1}(\sigma)\beta_{2}(b)e^{-s}, \tag{2.3}$$

where β_1 and β_2 are bounded functions respectively on $[-r,0]$ and $\mathbb{R}^{+}$.
To write this equation in an abstract form, we introduce the Banach lattice $X=L^{1}(\mathbb{R}^{+})$ and the operator A defined on X by

$$\begin{cases} D(A)=\left\{z\in W^{1,1}(\mathbb{R}^{+}):z(0)=0\right\}\\ \left(Az\right)(s)=-\gamma z'(s)-\mu(s)z(s)\ \text{ for } s\in\mathbb{R}^{+}.\end{cases}$$

The operator A generates a c_0-semigroup on X the given by

$$\left(T(t)z\right)(s)=\begin{cases}0 & \text{for } s<\gamma t\\ e^{-\frac{1}{\gamma}\int_{s-\gamma t}^{s}\mu(b)db}z(s-\gamma t) & \text{for } s>\gamma t.\end{cases} \tag{2.4}$$

We introduce the delay operator $\Phi: L^{1}\left([-r,0],X\right)\to X$ defined for each $\varphi\in L^{1}\left([-r,0],X\right)$ and $s\geq 0$ by:

$$\left(\Phi\varphi\right)(s):=\int_{-r}^{0}\nu(s,\sigma)\varphi(\sigma)(s)d\sigma+\int_{0}^{\infty}\int_{-r}^{0}\beta(\sigma,s,b)\varphi(\sigma)(b)d\sigma db. \tag{2.5}$$

If we write $u(t,.)=u(t)$, then system (2.1) is written on the Banach lattice $X=L^{1}(\mathbb{R}^{+})$ as follows:

$$\begin{cases}\dot{u}(t)=Au(t)+\Phi(u_t) \qquad \text{for } t\geq 0,\\ u(0)=y\in X,\\ u_0=\varphi\in L^{1}([-r,0],X),\end{cases} \tag{2.6}$$

To rewrite this equation as an abstract equation, we introduce the product space $\mathcal{X}=X\times L^{1}([-r,0],X)$ and the function

$$\mathcal{U}(t):=\begin{pmatrix}u(t)\\ u_t\end{pmatrix}\in\mathcal{X}.$$

In this case we have

$$|\mathcal{U}(t)|=|u(t,.)|_{L^1}+\int_{-r}^{0}|u(t+\theta,.)|_{L^1}\,d\theta.$$

Further, on this product space we define the following operator

$$\begin{cases} D(\mathcal{A}):=\left\{\begin{pmatrix}z\\ \varphi\end{pmatrix}\in D(A)\times W^{1,1}([-r,0],X):\varphi(0)=z\right\}\\ \mathcal{A}:=\begin{pmatrix}A & \Phi\\ 0 & \frac{d}{d\sigma}\end{pmatrix},\end{cases}$$

where $\frac{d}{d\sigma}$ denotes the derivative with respect to σ.

The following result is a consequence of [5, Corollary 3.5]:

Proposition 2.1. *Equation* (2.6) *is equivalent to the following abstract Cauchy problem*

$$\begin{cases} \dot{\mathcal{U}}(t) = \mathcal{A}\mathcal{U}(t), \quad t \geq 0, \\ \mathcal{U}(0) = \begin{pmatrix} y \\ \varphi \end{pmatrix} \end{cases}$$

on $\mathcal{X}$.

To show that $\mathcal{A}$ generates a c_0-semigroup on $\mathcal{X}$, we split it as

$$\mathcal{A} := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} + \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} =: \mathcal{A}_0 + \mathcal{A}_\Phi, \tag{2.7}$$

where

$$\begin{cases} D(\mathcal{A}_0) := D(\mathcal{A}) \\ \mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \end{cases} \quad \text{and} \quad \begin{cases} D(\mathcal{A}_\Phi) := D(\mathcal{A}) \\ \mathcal{A}_\Phi := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \end{cases}$$

The following result is a consequence of [5, Theorem 3.25]:

Proposition 2.2. *The operator* $\mathcal{A}_0$ *generates a* c_0*-semigroup given explicitly by the following formula:*

$$\mathcal{T}_0(t) := \begin{pmatrix} T(t) & 0 \\ T_t & T_l(t) \end{pmatrix}, \tag{2.8}$$

where $(T_l(t))_{t\geq 0}$ *is the nilpotent left shift semigroup on* $L^1([-r,0],X)$ *and* $T_t : X \to L^1([-r,0],X)$ *is defined for each* $z \in X$ *by*

$$(T_t\, z)(\tau) := \begin{cases} T(t+\tau)z, & \textit{if } -t < \tau \leq 0, \\ 0, & \textit{if } -r \leq \tau \leq -t. \end{cases}$$

One can see that the perturbation operator $\mathcal{A}_\Phi$ is bounded. Moreover, we can see that the semigroup $(T(t))_{t\geq 0}$ (see (2.4)) and the delay operator Φ (see (2.5)) are positive. Thus from [12, Theorem 1.10], we have the following result.

Proposition 2.3. *The operator* $\mathcal{A}$ *generates a positive* c_0*-semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *on* $\mathcal{X}$.

For a bounded subset B of a Banach space Z, the Kuratowski measure of noncompactness $\alpha(B)$ is defined by

$$\alpha(B) := \inf\{d > 0 : \text{ there exist finitely many sets of diameter at most } d \text{ which cover } B\}.$$

Moreover, for a bounded linear operator K on Z, we define $\alpha(K)$ by

$$\alpha(K) := \inf\{k > 0 : \ \alpha(K(B)) \leq k\alpha(B) \ \text{ for any bounded set } B \text{ of } Z\}.$$

Definition 2.4. [6] Let $\mathcal{C}$ be a closed linear operator with dense domain in a Banach space Z. Let $\sigma(\mathcal{C})$ denote the spectrum of the operator $\mathcal{C}$. The Browder's essential spectrum of $\mathcal{C}$ denoted by $\sigma_{ess}(\mathcal{C})$ is the set of $\lambda \in \sigma(\mathcal{C})$ such that one of the following conditions holds:
(i) $Im(\lambda I - \mathcal{C})$ is not closed,
(ii) the generalized eigenspace $M_\lambda(\mathcal{C}) := \bigcup_{k\geq 1} Ker(\lambda I - \mathcal{C})^k$ is of infinite dimension,
(iii) λ is a limit point of $\sigma(\mathcal{C})$.

The essential radius of $\mathcal{C}$ is defined by

$$r_{ess}(\mathcal{C}) = \sup\{|\lambda| : \ \lambda \in \sigma_{ess}(\mathcal{C})\}.$$

We recall some important facts about c_0-semigroups. Let $(R(t))_{t\geq 0}$ be a c_0-semigroup on a Banach space Z and $\mathcal{A}_R$ its infinitesimal generator.

Definition 2.5. [12, 30] The growth bound $\omega_0(R)$ of the c_0-semigroup $(R(t))_{t\geq 0}$ is defined by

$$\omega_0(R) := \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0} e^{-\omega t}|R(t)| < \infty\right\}.$$

Definition 2.6. [30] The essential growth bound (or α-growth bound) $\omega_{ess}(R)$ of the c_0-semigroup $(R(t))_{t\geq 0}$ is defined by:

$$\omega_{ess}(R) := \lim_{t\to\infty} \frac{\log \alpha(R(t))}{t} = \inf_{t>0} \frac{\log \alpha(R(t))}{t}. \tag{2.9}$$

The relation between $r_{ess}(R(t))$ and $\omega_{ess}(R)$ is given by the following formula ([30, Proposition 4.13])

$$r_{ess}(R(t)) = e^{t\omega_{ess}(R)} \quad \text{and} \quad e^{t\sigma_{ess}(A_R)} \subset \sigma_{ess}(R(t)). \tag{2.10}$$

Let A_R be the generator of $(R(t))_{t\geq 0}$. Then

$$\sigma_{ess}(A_R) \subset \{\lambda \in \sigma(A_R) : \; Re\,\lambda \leq \omega_{ess}(R)\}. \tag{2.11}$$

This means that if $\lambda \in \sigma(A_R)$ and $Re\lambda > \omega_{ess}(R)$, then λ does not belong to $\sigma_{ess}(A_R)$. Therefore λ is an isolated eigenvalue of A_R ([30, Proposition 4.11]).
The spectral bound $s(A_R)$ of the infinitesimal generator A_R is defined by:

$$s(A_R) := \sup\{Re\,\lambda : \; \lambda \in \sigma(A_R)\}.$$

Recall the following formula [30]

$$\omega_0(R) = \max\{\omega_{ess}(R), s(A_R)\}.$$

Consider the operator Φ_λ defined on X for each $\lambda \in \mathbb{C}$ and $z \in X$ by

$$\Phi_\lambda(z)(s) := \Phi(e^{\lambda(.)}z)(s) = \left(\int_{-r}^{0} \nu(s,\sigma)e^{\lambda\sigma}d\sigma\right) z(s) + \int_{0}^{\infty}\left(\int_{-r}^{0} \beta(\sigma,s,b)e^{\lambda\sigma}d\sigma\right) z(b)db.$$

Since the perturbation operator $\mathcal{A}_\Phi$ is bounded.

Lemma 2.7. *[5, Theorem 6.15] For each $\lambda \in \mathbb{R}$, if $s(A+\Phi_\lambda) \leq \lambda$, then $s(\mathcal{A}) \leq \lambda$.*

Lemma 2.8. *[12, Chapter VI, Theorem 1.15] Let B be the generator of a positive c_0-semigroup $(S(t))_{t\geq 0}$ on the Banach lattice $L^p(\Omega,\mu)$, $1 \leq p < \infty$. Then $\omega_0(S) = s(B)$ holds.*

The following result gives a localization of Browder's essential spectrum of the operator $\mathcal{A}$.

Theorem 2.9. *Let λ_0 be the unique real solution of the following equation:*

$$\overline{\nu}\frac{(1-e^{-r\lambda})}{\lambda} = \lambda + \underline{\mu}$$

where $\overline{\nu} = \sup_{s\geq 0,\, \sigma\in[-r,0]} \nu(s,\sigma)$ and $\underline{\mu} = \inf_{s\geq 0} \mu(s)$. If

$$\lim_{\alpha\to\infty} \sup_{\substack{-r\leq\sigma\leq 0\\ b\geq 0}} \int_{\alpha}^{\infty} \beta(\sigma,s,b)ds = 0 \tag{2.12}$$

and

$$\lim_{h\to 0} \sup_{\substack{-r\leq\sigma\leq 0\\ b\geq 0}} \int_{0}^{\infty} |\beta(\sigma,s+h,b) - \beta(\sigma,s,b)|\,ds = 0. \tag{2.13}$$

Then, $\omega_{ess}(\mathcal{T}) \leq \lambda_0$, thus $\sigma_{ess}(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : Re\lambda \leq \lambda_0\}$.

Proof. Consider the following decomposition

$$\Phi = \Phi^1 + \Phi^2,$$

where Φ_1 is defined for each $\varphi \in W^{1,1}([-r,0], X)$ and $s \geq 0$ by

$$\left(\Phi^1\varphi\right)(s) := \int_{-r}^{0} \nu(s,\sigma)\varphi(\sigma)(s)d\sigma$$

and Φ_2 is defined for each $\varphi \in L^1([-r,0], X)$ and $s \geq 0$ by

$$\left(\Phi^2\varphi\right)(s) := \int_{0}^{\infty}\int_{-r}^{0} \beta(\sigma,s,b)\varphi(\sigma)(b)d\sigma db.$$

Note that condition (2.2) implies that Φ^2 is bounded.
Consider the following decomposition of the operator $\mathcal{A}$

$$\mathcal{A} = \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} = \begin{pmatrix} A & \Phi^1+\Phi^2 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} = \begin{pmatrix} A & \Phi^1 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} + \begin{pmatrix} 0 & \Phi^2 \\ 0 & 0 \end{pmatrix} = \mathcal{A}_1 + \mathcal{K},$$

where $\mathcal{A}_1$ is the operator defined by

$$\begin{cases} D(\mathcal{A}_1) := D(\mathcal{A}) \\ \mathcal{A}_1 := \begin{pmatrix} A & \Phi^1 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \end{cases}$$

and $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is the bounded operator given by

$$\mathcal{K} = \begin{pmatrix} 0 & \Phi^2 \\ 0 & 0 \end{pmatrix}.$$

Using again [5, Theorem 1.37] and [5, Theorem 6.10], $\mathcal{A}_1$ generates a positive c_0-semigroup $(\mathcal{T}_1(t))_{t\geq 0}$ on the Banach lattice $\mathcal{X}$. Using the Frï¿œchet-Kolmogorov Theorem [32, page 275], one can see that conditions (2.12) and (2.13) imply that the operator $\mathcal{K}$ is compact. Hence by [12, Proposition IV.2.12]

$$\omega_{ess}(\mathcal{T}) = \omega_{ess}(\mathcal{T}_1) \leq \omega_0(\mathcal{T}_1). \tag{2.14}$$

The space $L^1([-r,0], X)$ is canonically isomorphic to $L^1([-r,0] \times \mathbb{R}^+)$ and the space $X \times L^1([-r,0] \times \mathbb{R}^+)$ with norm $|(z,\varphi)| = |z|_{L^1(\mathbb{R}^+)} + |\varphi|_{L^1([-r,0]\times\mathbb{R}^+)}$ is again an L^1-space.
By Lemma 2.8, we deduce that

$$\omega_0(\mathcal{T}_1) = s(\mathcal{A}_1). \tag{2.15}$$

Let

$$\xi(\lambda) = \overline{\nu}\frac{(1-e^{-r\lambda})}{\lambda} - \lambda - \underline{\mu}.$$

Since ξ is strictly decreasing on $\mathbb{R}$, $\lim_{\lambda\to-\infty}\xi(\lambda) = \infty$ and $\lim_{\lambda\to\infty}\xi(\lambda) = -\infty$, then the following equation

$$\overline{\nu}\frac{(1-e^{-r\lambda})}{\lambda} = \lambda + \underline{\mu}$$

has a unique real solution λ_0. The operator $A + \Phi^1_{\lambda_0}$ is given by

$$\left(\left(A + \Phi^1_{\lambda_0}\right)z\right)(s) = -\gamma z'(s) - \left(\mu(s) - \int_{-r}^{0}\nu(s,\sigma)e^{\lambda\sigma}d\sigma\right)z(s) \quad \text{for } z \in D(A). \tag{2.16}$$

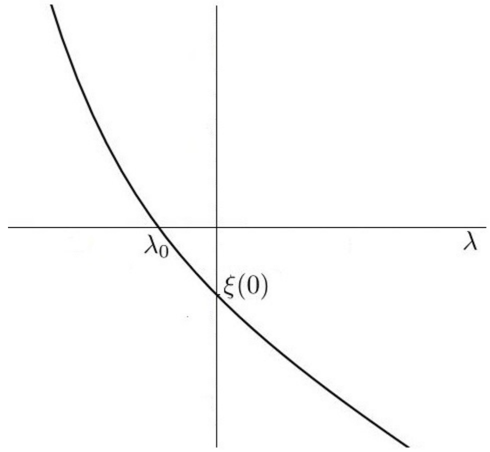

Figure 1: Graph of the function $\xi(\lambda)$

Let $\left(T^1_{\lambda_0}(t)\right)_{t\geq 0}$ be the c_0-semigroup generated by $\left(A+\Phi^1_{\lambda_0}, D(A)\right)$. The c_0-semigroup $\left(T^1_{\lambda_0}(t)\right)_{t\geq 0}$ is given explicitly for each $z \in X$ by

$$\left(T^1_{\lambda_0}(t)z\right)(s) = \begin{cases} 0 & \text{for } s < \gamma t \\ \exp\left(\frac{1}{\gamma}\int_{s-\gamma t}^{s}\left(\int_{-r}^{0}\nu(b,\sigma)e^{\lambda_0\sigma}d\sigma - \mu(b)\right)db\right) z(s-\gamma t) & \text{for } s > \gamma t. \end{cases} \tag{2.17}$$

Moreover,

$$\left|T^1_{\lambda_0}(t)z\right| \leq e^{\left(\overline{\nu}\frac{(1-e^{-r\lambda_0})}{\lambda_0} - \underline{\mu}\right)t}\, |z|\,.$$

Thus

$$s\left(A+\Phi^1_{\lambda_0}\right) = \omega_0\left(T^1_{\lambda_0}\right) \leq \overline{\nu}\frac{(1-e^{-r\lambda_0})}{\lambda_0} - \underline{\mu} = \lambda_0.$$

It follows by Lemma 2.7 that $s(\mathcal{A}_1) \leq \lambda_0$ and thus by (2.14) and (2.15) we have $\omega_{ess}(\mathcal{T}) \leq \lambda_0$. Therefore, by (2.11) we conclude that $\sigma_{ess}(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : Re\lambda \leq \lambda_0\}$. ■

Remark. If $\overline{\nu}r < \underline{\mu}$ then $\xi(0) = \overline{\nu}r - \underline{\mu} < 0$ and thus $\lambda_0 < 0$ (see Figure 1). It follows that the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is quasicompact, namely, $\omega_{ess}(\mathcal{T}) < 0$.

Remark. The growth rate γ does not have an effect on the asymptotic behavior of the c_0-semigroup $(\mathcal{T}_1(t))_{t\geq 0}$.

3. Nonlinear small perturbations

Consider the following model:

$$\begin{cases} \frac{\partial}{\partial t}u(t,s) = -\gamma\frac{\partial}{\partial s}\left(u(t,s)\right) - \mu(s)u(t,s) + \int_{-r}^{0}\nu(s,\sigma)u(t+\sigma,s)d\sigma & \\ \qquad + \int_{0}^{\infty}\int_{-r}^{0}\beta(\sigma,s,b)u(t+\sigma,b)d\sigma db + f(t,u(t,s)) & \text{for } t\geq 0,\ s\in\mathbb{R}^+ \\ u(t,0) = 0 & \text{for } t \geq 0 \\ u(\sigma,s) = \varphi(\sigma,s) & \text{for } (\sigma,s)\in[-r,0]\times\mathbb{R}^+. \end{cases} \tag{3.1}$$

Assume that $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses:

- For all $t \geq 0$ and $z \in L^1(\mathbb{R}^+)$: $s \mapsto f(t, z(s)) \in L^1(\mathbb{R}^+)$.
- For all $(t,z),(t_n,z_n) \in \mathbb{R}^+ \times L^1(\mathbb{R}^+)$ with $t_n \to t$ and $z_n \to z$ in $L^1(\mathbb{R}^+)$: $\int_0^\infty |f(t_n, z_n(s)) - f(t, z(s))|\, ds \to 0$ as $n \to \infty$.
- f is globally Lipschitz with respect to the second variable.
- $|f(t,x)| \leq p(t)\,|x|$ for all $t \geq 0$ and $x \in \mathbb{R}$, where $p : [0,\infty) \to [0,\infty)$ is a continuous function satisfying $\lim_{t\to\infty} \int_t^{t+1} p(s)ds = 0$.

An example of such a function is $f(t,x) = \dfrac{e^{-t}x}{1+x^2}$.

We write (3.1) in the space $\mathcal{X} = X \times L^1([-r,0],X)$ in the following form

$$\begin{cases} \dot{\mathcal{U}}(t) = \mathcal{A}\mathcal{U}(t) + \mathcal{F}(t,\mathcal{U}(t)) \;\text{ for }\; t \geq 0, \\ \mathcal{U}(0) = \begin{pmatrix} y \\ \varphi \end{pmatrix}, \end{cases} \tag{3.2}$$

where $\mathcal{U}(t) := \begin{pmatrix} u(t) \\ u_t \end{pmatrix}$, $\mathcal{F}(t,\mathcal{U}(t)) = \begin{pmatrix} F(t,u(t)) \\ 0 \end{pmatrix}$ and $F(t,u(t))(s) := f(t,u(t,s))$ for all $s \geq 0$. It follows that the nonlinear function $\mathcal{F} : \mathbb{R}^+ \times \mathcal{X} \to \mathcal{X}$ is continuous and globally Lipschitz with respect to the second variable. Thus we have the following result [31]

Theorem 3.1. *Equation* (3.1) *has a unique solution $\mathcal{U}$ defined on $\mathbb{R}^+$.*

In the sequel, we will assume that the birth rate has the following form

$$\beta(\sigma,s,b) = \beta_1(s)\beta_2(\sigma,b),$$

where $\beta_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and $\beta_2 : [-r,0] \times \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta_1 \neq 0$.

In reality individuals with large sizes cannot give birth, then without loss of generality we can assume that the birth function component $\beta_2(\sigma,s)$ vanishes for $s \geq m$ where m is the maximal size of fertility. Thus Condition (2.2) becomes

$$\sup_{\substack{-1\leq\sigma\leq 0 \\ 0\leq b\leq m}} \beta_2(\sigma,b) < \infty \quad \text{and} \quad \int_0^\infty \beta_1(s)ds < \infty. \tag{3.3}$$

We state the first main result of this section:

Theorem 3.2. *Let λ_0 be the unique real solution of the following equation*

$$\overline{\nu}\frac{(1-e^{-r\lambda})}{\lambda} = \lambda + \underline{\mu}.$$

Assume that the solution $\mathcal{U}$ does not vanish for sufficiently large t. Then, we have either

$$\limsup_{t\to\infty} \frac{\log\left(|u(t,.)|_{L^1} + \int_{-r}^0 |u(t+\theta,.)|_{L^1}\, d\theta\right)}{t} \leq \lambda_0 \tag{3.4}$$

or

$$\lim_{t\to\infty} \frac{\log\left(|u(t,.)|_{L^1} + \int_{-r}^0 |u(t+\theta,.)|_{L^1}\, d\theta\right)}{t} = Re\,\lambda, \tag{3.5}$$

where λ is a solution of the equation

$$\gamma = \int_0^m \left(\int_{-r}^0 e^{\lambda\theta}\beta_2(\theta,s)d\theta\right)\left(\int_0^s \exp\left(\frac{1}{\gamma}\int_b^s \left(-\lambda - \mu(c) + \int_{-r}^0 \nu(c,\sigma)e^{\lambda\sigma}d\sigma\right)dc\right)\beta_1(b)db\right)ds.$$

Since by Theorem 2.9 we have $\sigma_{ess}(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : Re\lambda \leq \lambda_0\}$, then each $\lambda \in \sigma(\mathcal{A})$ with $Re\lambda > \lambda_0$ is an isolated eigenvalue of the operator $\mathcal{A}$. Let $\rho > \lambda_0$ be such that

$$\sigma(\mathcal{A}) \cap (i\mathbb{R} + \rho) = \emptyset.$$

Consider the set

$$\Sigma_\rho := \{\lambda \in \sigma(\mathcal{A}) : \quad Re\,\lambda \geq \rho\}. \tag{3.6}$$

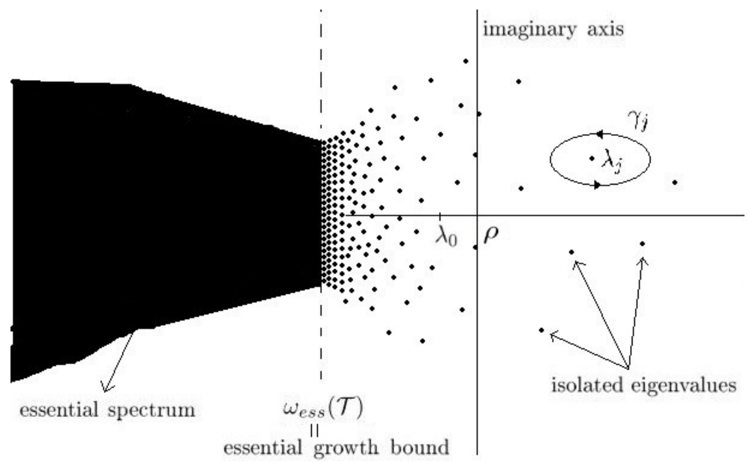

Figure 2: Spectrum of the operator $\mathcal{A}$

From [12, Corollary IV.2.11 and Theorem V.3.1] and (2.11), the set Σ_ρ is finite and $\Sigma_\rho \cap \sigma_{ess}(\mathcal{A}) = \emptyset$. Thus Σ_ρ contains only isolated eigenvalues of $\mathcal{A}$. Let $\Sigma_\rho = \{\lambda_1, \ldots, \lambda_n\}$ and define the following operators

$$\Pi_j := \frac{1}{2\pi i} \int_{\gamma_j} R(\lambda, \mathcal{A}) d\lambda$$

for each $1 \leq j \leq n$, where γ_j is a positively oriented closed curve in $\mathbb{C}$ enclosing the isolated singularity λ_j, but no other points of $\sigma(\mathcal{A})$ (see Figure 2). Then Π_j is a projection in $\mathcal{X}$ and $\Pi_j \Pi_h = 0$ for $j \neq h$. Let $U_j := R(\Pi_j)$ be the range of Π_j, then $\mathcal{A}$ restricted to U_j is a bounded operator with spectrum consisting of the single point λ_j. Let $P_1 := \sum_{j=1}^{n} \Pi_j$, $P_2 = I - P_1$, $S_\rho = R(P_2)$ and $U_\rho = U_1 \oplus \cdots \oplus U_n$. Then P_1 and P_2 are projections on U_ρ and S_ρ respectively and

$$\mathcal{X} = U_\rho \oplus S_\rho, \tag{3.7}$$

and U_ρ and S_ρ are closed subspaces of $\mathcal{X}$ which are invariant under the semigroup $(\mathcal{T}(t))_{t\geq 0}$. Let $\Pi^{U_\rho} := P_1$ and $\Pi^{S_\rho} := P_2$. The subspace U_ρ is finite-dimensional. Moreover, for every sufficiently small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{cases} |\mathcal{T}(t)\,\mathcal{Z}| \leq C_\varepsilon e^{(\rho-\varepsilon)t}\,|\mathcal{Z}| & \text{for } t \geq 0 \text{ and } \mathcal{Z} \in S_\rho \\ |\mathcal{T}(t)\,\mathcal{Z}| \leq C_\varepsilon e^{(\rho+\varepsilon)t}\,|\mathcal{Z}| & \text{for } t \leq 0 \text{ and } \mathcal{Z} \in U_\rho. \end{cases} \tag{3.8}$$

For more details, we refer the reader to [30, Proposition 4.15].

In what follows, $\mathcal{T}^{U_\rho}(t)$ and $\mathcal{T}^{S_\rho}(t)$ denote the restrictions of $\mathcal{T}(t)$ on U_ρ and S_ρ respectively. Then $\left(\mathcal{T}^{U_\rho}(t)\right)_{t\in\mathbb{R}}$ is a group of operators and

$$\mathcal{T}^{U_\rho}(t) = e^{t\mathcal{A}_{U_\rho}} \quad \text{with } \mathcal{A}_{U_\rho} \in \mathcal{L}(U_\rho).$$

Let $\varepsilon_\rho > 0$ be such that $\sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \ \rho - \varepsilon_\rho \leq Re\lambda \leq \rho + \varepsilon_\rho\} = \emptyset$. Put

$$\rho_1 := \rho - \varepsilon_\rho \quad \text{and} \quad \rho_2 := \rho + \varepsilon_\rho. \tag{3.9}$$

We deduce from (3.8) that there exists a constant $C_\rho > 0$ such that for each $t \geq 0$

$$\left\|\mathcal{T}^{S_\rho}(t)\right\| \leq C_\rho e^{\rho_1 t} \quad \text{and} \quad \left\|\mathcal{T}^{U_\rho}(-t)\right\| \leq C_\rho e^{-\rho_2 t}.$$

We introduce the new norm defined on $\mathcal{X}$ by

$$|\mathcal{Z}|_{\mathcal{T}} := \sup_{t\geq 0} e^{-\rho_1 t}\left|\mathcal{T}^{S_\rho}(t)\,\Pi^{S_\rho}\mathcal{Z}\right| + \sup_{t\geq 0} e^{\rho_2 t}\left|\mathcal{T}^{U_\rho}(-t)\,\Pi^{U_\rho}\mathcal{Z}\right|.$$

Lemma 3.3. [10, 22, 26] *The two norms $|.|$ and $|.|_{\mathcal{T}}$ are equivalent, namely, for all $\mathcal{Z} \in \mathcal{X}$, we have*

$$|\mathcal{Z}| \leq |\mathcal{Z}|_{\mathcal{T}} \leq C_2\,|\mathcal{Z}|, \tag{3.10}$$

where $C_2 := C_\rho\left(\left\|\Pi^{S_\rho}\right\| + \left\|\Pi^{U_\rho}\right\|\right)$. In addition, for all $\mathcal{Z} \in \mathcal{X}$

$$|\mathcal{Z}|_{\mathcal{T}} = \left|\Pi^{S_\rho}\mathcal{Z}\right|_{\mathcal{T}} + \left|\Pi^{U_\rho}\mathcal{Z}\right|_{\mathcal{T}}. \tag{3.11}$$

The corresponding operator norms $\left\|\mathcal{T}^{S_\rho}(t)\right\|_{\mathcal{T}}$ and $\left\|\mathcal{T}^{U_\rho}(-t)\right\|_{\mathcal{T}}$ satisfy

$$\left\|\mathcal{T}^{S_\rho}(t)\right\|_{\mathcal{T}} \leq e^{\rho_1 t} \quad \text{and} \quad \left\|\mathcal{T}^{U_\rho}(-t)\right\|_{\mathcal{T}} \leq e^{-\rho_2 t} \quad \text{for } t \geq 0. \tag{3.12}$$

Lemma 3.4. *Let $\mathcal{U}$ be the solution of Equation (3.1). Then for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) \geq 1$ such that*

$$|\mathcal{U}(t)| \leq C(\varepsilon)\, e^{(\omega_0(\mathcal{T})+\varepsilon)(t-\sigma)} \exp\left(C(\varepsilon)\int_\sigma^t p(s)\,ds\right)|\mathcal{U}(\sigma)| \quad \text{for } 0 \leq \sigma \leq t. \tag{3.13}$$

In particular, there exists a constant $C_1 \geq 0$ such that for $m \in \mathbb{N}$ and $m \leq t \leq m+1$, we have

$$\frac{1}{C_1}\,|\mathcal{U}(m+1)| \leq |\mathcal{U}(t)| \leq C_1\,|\mathcal{U}(m)|. \tag{3.14}$$

Proof. Using the variation of constants formula, we have for $0 \leq \sigma \leq t$

$$\mathcal{U}(t) = \mathcal{T}(t-\sigma)\,\mathcal{U}(\sigma) + \int_\sigma^t \mathcal{T}(t-s)\,\mathcal{F}(s,\mathcal{U}(s))\,ds. \tag{3.15}$$

Let $\varepsilon > 0$. Then, there exists $C(\varepsilon) \geq 1$ such that

$$\|\mathcal{T}(t)\| \leq C(\varepsilon)\, e^{(\omega_0(\mathcal{T})+\varepsilon)t} \quad \text{for } t \geq 0. \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$|\mathcal{U}(t)| \leq C(\varepsilon)\, e^{(\omega_0(\mathcal{T})+\varepsilon)(t-\sigma)}\,|\mathcal{U}(\sigma)| + C(\varepsilon)\int_\sigma^t e^{(\omega_0(\mathcal{T})+\varepsilon)(t-s)} p(s)\,|\mathcal{U}(s)|\,ds.$$

It follows that

$$e^{-(\omega_0(\mathcal{T})+\varepsilon)t}\,|\mathcal{U}(t)| \leq C(\varepsilon)\, e^{-(\omega_0(\mathcal{T})+\varepsilon)\sigma}\,|\mathcal{U}(\sigma)| + C(\varepsilon)\int_\sigma^t e^{-(\omega_0(\mathcal{T})+\varepsilon)s}\,|\mathcal{U}(s)|\,p(s)\,ds.$$

The Gronwall's Lemma implies that for $0 \leq \sigma \leq t$

$$e^{-(\omega_0(\mathcal{T})+\varepsilon)t}\,|\mathcal{U}(t)| \leq C(\varepsilon)\, e^{-(\omega_0(\mathcal{T})+\varepsilon)\sigma}\,|\mathcal{U}(\sigma)| \exp\left(C(\varepsilon)\int_\sigma^t p(s)\,ds\right).$$

Therefore we get the inequality (3.13). Now let $m \in \mathbb{N}$ and $m \leq t \leq m+1$. By taking $\varepsilon = 1$ and $\sigma = m$ in (3.13), we get

$$\begin{aligned}|\mathcal{U}(t)| &\leq C\left(1\right) e^{(\omega_0(\mathcal{T})+1)(t-m)} \left|\mathcal{U}(m)\right| \exp\left(C\left(1\right) \int_m^t p\left(s\right) ds\right) \\ &\leq C_1 \left|\mathcal{U}(m)\right|,\end{aligned}$$

where $C_1 := C\left(1\right) \max\left\{1, e^{(\omega_0(\mathcal{T})+1)}\right\} e^{C(1)Q}$ and $Q := \sup\limits_{m\geq 0} \int_m^{m+1} p\left(s\right) ds$. Similarly, we get

$$|\mathcal{U}(m+1)| \leq C_1 \left|\mathcal{U}(t)\right|.$$

■

Remark. By (3.14) and (3.10), we can see that for $m \in \mathbb{N}$ and $m \leq t \leq m+1$

$$\frac{1}{C_3} \left|\mathcal{U}(m+1)\right|_{\mathcal{T}} \leq \left|\mathcal{U}(t)\right|_{\mathcal{T}} \leq C_3 \left|\mathcal{U}(m)\right|_{\mathcal{T}}, \tag{3.17}$$

where $C_3 := C_1 C_2$.

Proposition 3.5. *Let $\mathcal{U}$ be the solution of Equation* (3.1). If $\mathcal{U}(t)$ does not vanish for sufficiently large t*, then we have*

$$\limsup_{t\to\infty} \frac{\log\left|\mathcal{U}(t)\right|}{t} \leq \omega_0\left(\mathcal{T}\right).$$

Remark. It is clear from Lemma 3.4 that if $\mathcal{U}(t_0) = 0$ for some $t_0 \geq 0$, then $\mathcal{U}(t) = 0$ for all $t \geq t_0$.

Proof of Proposition 3.5**.** Let $\varepsilon > 0$, from Lemma 3.4, we deduce that for $t \geq 0$

$$\frac{\log\left|\mathcal{U}(t)\right|}{t} \leq \frac{\log\left(C_0\left(\varepsilon\right)\left|\mathcal{U}(0)\right|\right)}{t} + \omega_0\left(\mathcal{T}\right) + \varepsilon + C_0\left(\varepsilon\right) \frac{\int_0^t p\left(s\right) ds}{t}. \tag{3.18}$$

Since $\dfrac{\int_0^t p\left(s\right) ds}{t} \to 0$ as $t \to \infty$, then by taking $t \to \infty$ in (3.18), we obtain that

$$\limsup_{t\to\infty} \frac{\log\left|\mathcal{U}(t)\right|}{t} \leq \omega_0\left(\mathcal{T}\right) + \varepsilon. \tag{3.19}$$

Now by letting $\varepsilon \to 0$ in (3.19) we obtain the desired estimation. ■

We fix a real number ρ such that $\rho > \lambda_0$ and $\sigma\left(\mathcal{A}\right) \cap \left(i\mathbb{R} + \rho\right) = \emptyset$. Let $\mathcal{U}$ be the solution of Equation (3.2). Define for $m \in \mathbb{N}$

$$\mathcal{U}^U\left(m\right) := \left|\Pi^{U_\rho}\mathcal{U}(m)\right|_{\mathcal{T}}, \quad \mathcal{U}^S\left(m\right) := \left|\Pi^{S_\rho}\mathcal{U}(m)\right|_{\mathcal{T}} \tag{3.20}$$

and

$$\widetilde{p}\left(m\right) := C_1 C_2^2 \max\left\{1, e^{\rho_1}, e^{\rho_2}\right\} \int_m^{m+1} p\left(s\right) ds, \tag{3.21}$$

where ρ_1 and ρ_2 are the real numbers defined by (3.9).

Lemma 3.6. *The following estimations hold:*

$$\mathcal{U}^S\left(m+1\right) \leq e^{\rho_1}\mathcal{U}^S\left(m\right) + \widetilde{p}\left(m\right)\left(\mathcal{U}^S\left(m\right) + \mathcal{U}^U\left(m\right)\right), \tag{3.22}$$

and

$$\mathcal{U}^U\left(m+1\right) \geq e^{\rho_2}\mathcal{U}^U\left(m\right) - \widetilde{p}\left(m\right)\left(\mathcal{U}^S\left(m\right) + \mathcal{U}^U\left(m\right)\right). \tag{3.23}$$

Proof. Using the variation of constants formula, we obtain for each $m \in \mathbb{N}$

$$\mathcal{U}(m+1) = \mathcal{T}(1)\mathcal{U}(m) + \int_m^{m+1} \mathcal{T}(m+1-s) f(s,\mathcal{U}(s))\, ds. \tag{3.24}$$

By projecting the formula (3.24) onto the subspace S_ρ and using (3.12), (3.10) and (3.14), we have

$$\begin{aligned}\left|\Pi^{S_\rho}\mathcal{U}(m+1)\right|_{\mathcal{T}} &\leq \left|\mathcal{T}^{S_\rho}(1)\Pi^{S_\rho}\mathcal{U}(m)\right|_{\mathcal{T}} + \int_m^{m+1} \left|\mathcal{T}^{S_\rho}(m+1-s)\Pi^{S_\rho} f(s,\mathcal{U}(s))\right|_{\mathcal{T}} ds \\ &\leq e^{\rho_1}\left|\Pi^{S_\rho}\mathcal{U}(m)\right|_{\mathcal{T}} + C_2^2 \max\{1, e^{\rho_1}\} \int_m^{m+1} p(s)\,|\mathcal{U}(s)|\, ds \\ &\leq e^{\rho_1}\left|\Pi^{S_\rho}\mathcal{U}(m)\right|_{\mathcal{T}} + C_1 C_2^2 \max\{1, e^{\rho_1}\} \int_m^{m+1} p(s)\, ds\, |\mathcal{U}(m)|_{\mathcal{T}}.\end{aligned}$$

Using (3.11) and the above inequality, we conclude that (3.22) holds.

Now from (3.12), we have for $\phi \in U_\rho$

$$\left|\mathcal{T}^{U_\rho}(1)\phi\right|_{\mathcal{T}} \geq e^{\rho_2}\,|\phi|_{\mathcal{T}}.$$

By projecting the formula (3.24) onto the subspace U_ρ using (3.12), (3.10), (3.14) and (3.11), we deduce that

$$\begin{aligned}\left|\Pi^{U_\rho}\mathcal{U}(m+1)\right|_{\mathcal{T}} &= \left|\mathcal{T}^{U_\rho}(1)\left(\Pi^{U_\rho}\mathcal{U}(m) + \int_m^{m+1} \mathcal{T}^{U_\rho}(m-s)\Pi^{U_\rho} f(s,\mathcal{U}(s))\, ds\right)\right|_{\mathcal{T}} \\ &\geq e^{\rho_2}\mathcal{U}^U(m) - e^{\rho_2}\int_m^{m+1} e^{\rho_2(m-s)}\left|\Pi^{U_\rho} f(s,\mathcal{U}(s))\right|_{\mathcal{T}} ds \\ &\geq e^{\rho_2}\mathcal{U}^U(m) - e^{\rho_2} C_2^2 \max\left\{1, e^{-\rho_2}\right\}\int_m^{m+1} p(s)\, C_1\,|\mathcal{U}(m)|\, ds \\ &\geq e^{\rho_2}\mathcal{U}^U(m) - C_1 C_2^2 \max\{e^{\rho_2}, 1\}\int_m^{m+1} p(s)\, ds\left(\mathcal{U}^U(m) + \mathcal{U}^S(m)\right).\end{aligned}$$

Therefore, we get the estimation (3.23). ■

In what follows, we assume that the solution $\mathcal{U}$ *does not vanish for sufficiently large* t. We have the following Lemma.

Lemma 3.7. *Either*

$$\lim_{m\to\infty}\frac{\mathcal{U}^U(m)}{\mathcal{U}^S(m)} = 0 \tag{3.25}$$

or

$$\lim_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)} = 0. \tag{3.26}$$

Proof. The proof follows the same approach as in [22, 26]. From (3.10), one can see that $|\mathcal{U}(t)|_{\mathcal{T}} > 0$ for $t \geq 0$. Suppose that (3.25) fails, then there exists $\varepsilon > 0$ such that

$$\frac{\mathcal{U}^U(m)}{\mathcal{U}^S(m)} \geq \varepsilon,$$

for infinitely many m. Next we will show that (3.26) must hold. From (3.21) we can see that

$$\lim_{m\to\infty}\widetilde{p}(m) = 0. \tag{3.27}$$

By (3.27), there exists $m_1 \geq 0$ such that for $m \geq m_1$

$$e^{\rho_2} - \frac{1+\varepsilon}{\varepsilon}\widetilde{p}\,(m) > 0$$

and

$$\frac{e^{\rho_1} + (1+\varepsilon)\,\widetilde{p}\,(m)}{\varepsilon e^{\rho_2} - (1+\varepsilon)\,\widetilde{p}\,(m)} < \frac{1}{\varepsilon}. \tag{3.28}$$

Since (3.25) fails then there exists $m_2 \geq m_1$ such that

$$\mathcal{U}^U(m_2) \geq \varepsilon \mathcal{U}^S(m_2).$$

Next we show that for all $m \geq m_2$

$$\mathcal{U}^U(m) \geq \varepsilon \mathcal{U}^S(m). \tag{3.29}$$

Suppose by induction that this inequality holds for some $m \geq m_2$. Then it follows from (3.22) that

$$\begin{aligned}\mathcal{U}^S(m+1) &\leq e^{\rho_1}\frac{\mathcal{U}^U(m)}{\varepsilon} + \widetilde{p}\,(m)\frac{\mathcal{U}^U(m)}{\varepsilon} + \widetilde{p}\,(m)\mathcal{U}^U(m)\\ &= \left(\frac{e^{\rho_1}}{\varepsilon} + \frac{\widetilde{p}\,(m)}{\varepsilon} + \widetilde{p}\,(m)\right)\mathcal{U}^U(m).\end{aligned}$$

Now from (3.23) we have

$$\begin{aligned}\mathcal{U}^U(m+1) &\geq e^{\rho_2}\mathcal{U}^U(m) - \widetilde{p}(m)\frac{\mathcal{U}^U(m)}{\varepsilon} - \widetilde{p}\,(m)\mathcal{U}^U(m)\\ &= \left(e^{\rho_2} - \frac{\widetilde{p}(m)}{\varepsilon} - \widetilde{p}\,(m)\right)\mathcal{U}^U(m).\end{aligned} \tag{3.30}$$

It follows that

$$\begin{aligned}\mathcal{U}^S(m+1) &\leq \left(\frac{e^{\rho_1}}{\varepsilon} + \frac{\widetilde{p}\,(m)}{\varepsilon} + \widetilde{p}\,(m)\right)\mathcal{U}^U(m)\\ &\leq \left(\frac{e^{\rho_1}}{\varepsilon} + \frac{\widetilde{p}\,(m)}{\varepsilon} + \widetilde{p}\,(m)\right)\frac{1}{e^{\rho_2} - \widetilde{p}\,(m) - \frac{\widetilde{p}(m)}{\varepsilon}}\mathcal{U}^U(m+1)\\ &= \frac{e^{\rho_1} + \widetilde{p}(m) + \varepsilon\widetilde{p}\,(m)}{\varepsilon e^{\rho_2} - \varepsilon\widetilde{p}\,(m) - \widetilde{p}(m)}\mathcal{U}^U(m+1).\end{aligned}$$

Now from (3.28), we deduce that

$$\mathcal{U}^U(m+1) \geq \varepsilon\mathcal{U}^S(m+1).$$

Thus by induction, the inequality (3.29) holds for all $m \geq m_2$. From (3.22) and (3.30), we deduce that for $m \geq m_2$

$$\begin{aligned}\frac{\mathcal{U}^S(m+1)}{\mathcal{U}^U(m+1)} &\leq \frac{e^{\rho_1}\mathcal{U}^S(m) + \widetilde{p}\,(m)\left(\mathcal{U}^S(m) + \mathcal{U}^U(m)\right)}{\left(e^{\rho_2} - \widetilde{p}\,(m) - \frac{\widetilde{p}(m)}{\varepsilon}\right)\mathcal{U}^U(m)}\\ &= \frac{e^{\rho_1} + \widetilde{p}\,(m)}{\left(e^{\rho_2} - \widetilde{p}\,(m) - \frac{\widetilde{p}(m)}{\varepsilon}\right)}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)} + \frac{\widetilde{p}\,(m)}{\left(e^{\rho_2} - \widetilde{p}\,(m) - \frac{\widetilde{p}(m)}{\varepsilon}\right)}.\end{aligned}$$

It follows by (3.27) that

$$\limsup_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)} \leq \frac{e^{\rho_1}}{e^{\rho_2}}\limsup_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}.$$

That is

$$\left(1-e^{\rho_1-\rho_2}\right)\limsup_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}\leq 0.$$

But since $\rho_1<\rho_2$ and $\limsup_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}\geq 0$, we deduce that $\limsup_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}=0$. Therefore

$$\lim_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}=0.$$

This ends the proof of Lemma 3.7. ■

The proof of Theorem 3.2 is based on the following principal Lemma.

Lemma 3.8. *Either*

$$\limsup_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t}<\rho \tag{3.31}$$

or

$$\liminf_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t}>\rho. \tag{3.32}$$

Proof. By Lemma 3.7, we have to discuss two cases:

Case 1. Assume that (3.25) holds. Then we have $\mathcal{U}^U(m)<\mathcal{U}^S(m)$ for all large integers m, where $\mathcal{U}^U(m)$ and $\mathcal{U}^S(m)$ are given by (3.20). Let ε be a positive real number. Then by (3.27), there exists a large positive integer m_ε such that for $m\geq m_\varepsilon$,

$$\widetilde{p}(m)<\varepsilon \ \text{ and } \ \mathcal{U}^U(m)<\mathcal{U}^S(m). \tag{3.33}$$

Using (3.22) and (3.33) we have $\mathcal{U}^S(m+1)\leq(e^{\rho_1}+2\varepsilon)\mathcal{U}^S(m)$ for $m\geq m_\varepsilon$. It follows that

$$\mathcal{U}^S(m)\leq(e^{\rho_1}+2\varepsilon)^{m-m_\varepsilon}\mathcal{U}^S(m_\varepsilon)=K_\varepsilon\left(e^{\rho_1}+2\varepsilon\right)^m,$$

where $K_\varepsilon:=(e^{\rho_1}+2\varepsilon)^{-m_\varepsilon}\mathcal{U}^S(m_\varepsilon)>0$. For $t\geq m_\varepsilon$, we have $[t]\geq m_\varepsilon$, where $[.]$ is the floor function. Since $[t]\leq t\leq[t]+1$, it follows from (3.10), (3.11), (3.17) and (3.33) that

$$|\mathcal{U}(t)|\leq|\mathcal{U}(t)|_{\mathcal{T}}\leq C_3\left|\mathcal{U}_{[t]}\right|_{\mathcal{T}}\leq 2C_3\mathcal{U}^S([t])\leq 2C_3K_\varepsilon\left(e^{\rho_1}+2\varepsilon\right)^{[t]}.$$

Hence,

$$\frac{\log|\mathcal{U}(t)|}{t}\leq\frac{\log(2C_3K_\varepsilon)}{t}+\frac{[t]}{t}\log\left(e^{\rho_1}+2\varepsilon\right).$$

Let $t\to\infty$, then

$$\limsup_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t}\leq\log\left(e^{\rho_1}+2\varepsilon\right).$$

Now by taking $\varepsilon\to 0$, we obtain that

$$\limsup_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t}\leq\log\left(e^{\rho_1}\right)=\rho_1<\rho,$$

that is, (3.31) holds.

Case 2. Suppose that (3.26) holds. Note that $\mathcal{U}^S(m)<\mathcal{U}^U(m)$ for all large integers m. Let ε such that $0<\varepsilon<\frac{e^{\rho_2}}{2}$. By (3.27), there exists a large positive integer m_ε such that for $m\geq m_\varepsilon$,

$$\widetilde{p}(m)<\varepsilon \ \text{ and } \ \mathcal{U}^S(m)<\mathcal{U}^U(m). \tag{3.34}$$

Using (3.23) and (3.34) we have $\mathcal{U}^U(m+1) \geq (e^{\rho_2} - 2\varepsilon)\mathcal{U}^U(m)$ for $m \geq m_\varepsilon$, which implies that

$$\mathcal{U}^U(m) \geq (e^{\rho_2} - 2\varepsilon)^{m-m_\varepsilon} \mathcal{U}^U(m_\varepsilon) = K_\varepsilon (e^{\rho_2} - 2\varepsilon)^m,$$

where $K_\varepsilon := (e^{\rho_2} - 2\varepsilon)^{-m_\varepsilon} \mathcal{U}^U(m_\varepsilon) > 0$. For $t \geq m_\varepsilon$, we have $[t] + 1 \geq m_\varepsilon$. Since $[t] \leq t \leq [t] + 1$, it follows from (3.10), (3.17) that

$$|\mathcal{U}(t)| \geq \frac{|\mathcal{U}(t)|_{\mathcal{T}}}{C_2} \geq \frac{\left|\mathcal{U}_{[t]+1}\right|_{\mathcal{T}}}{C_2 C_3} \geq \frac{\mathcal{U}^U([t]+1)}{C_2 C_3} \geq \frac{K_\varepsilon (e^{\rho_2} - 2\varepsilon)^{[t]+1}}{C_2 C_3}.$$

Hence,

$$\frac{\log |\mathcal{U}(t)|}{t} \geq \frac{\log\left(\frac{K_\varepsilon}{C_2 C_3}\right)}{t} + \frac{[t]+1}{t} \log (e^{\rho_2} - 2\varepsilon).$$

By taking $t \to \infty$ we get that

$$\liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} \geq \log (e^{\rho_2} - 2\varepsilon).$$

Now by taking $\varepsilon \to 0$, we obtain that

$$\liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} \geq \log (e^{\rho_2}) = \rho_2 > \rho,$$

that is, (3.32) holds. This completes the proof. ■

3.1. Proof of Theorem 3.2

Proof of Theorem 3.2. Let $\mathcal{U}$ be the solution of Equation (3.2) such that $|\mathcal{U}(t)| > 0$ for all $t \geq 0$. Suppose that

$$\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} > \lambda_0.$$

Since $\omega_{ess}(\mathcal{T}) \leq \lambda_0$, it follows from Proposition 3.5 that

$$\omega_0(\mathcal{T}) > \omega_{ess}(\mathcal{T}).$$

Therefore

$$\omega_0(\mathcal{T}) = \max\{s(\mathcal{A}), \omega_{ess}(\mathcal{T})\} = s(\mathcal{A})$$

and

$$\Lambda := \{\lambda \in \sigma(\mathcal{A}) : \ Re\,\lambda > \omega_{ess}(\mathcal{T})\} \neq \emptyset.$$

We claim that there exists $\lambda \in \Lambda$ such that

$$\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} = Re\,\lambda.$$

In fact, if $\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} = \rho \notin \{Re\,\lambda : \ \lambda \in \Lambda\}$, with $\rho > \omega_{ess}(\mathcal{T})$, then condition (3.31) in Lemma 3.8 fails. Hence, we must have

$$\liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} > \rho.$$

However, this implies that

$$\rho = \limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} \geq \liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} > \rho,$$

which is a contradiction. Therefore, there exists $\lambda \in \Lambda$ such that

$$\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} = Re\,\lambda.$$

Since $Re\,\lambda > \omega_{ess}(\mathcal{T})$, then there exists $\rho_0 \notin \{Re\,\lambda : \ \lambda \in \Lambda\}$ such that $Re\,\lambda > \rho_0 > \omega_{ess}(\mathcal{T})$. That is

$$\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} = Re\,\lambda > \rho_0. \tag{3.35}$$

By applying Lemma 3.8 to ρ_0 using (3.35), we obtain that

$$\liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} > \rho_0 > \omega_{ess}(\mathcal{T}).$$

We claim that

$$\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} = \liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t}.$$

In fact if $\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} > \liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t}$, then there exists $\rho_1 \notin \{Re\,\lambda : \ \lambda \in \Lambda\}$ with $\rho_1 > \omega_{ess}(\mathcal{T})$ such that

$$\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} \ > \ \rho_1 \tag{3.36}$$

and

$$\liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} \ < \ \rho_1. \tag{3.37}$$

By applying Lemma 3.8 to ρ_1 using (3.36) , we obtain

$$\liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} > \rho_1,$$

which contradicts (3.37). Therefore, we have

$$\lim_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} = \limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} = \liminf_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} = Re\,\lambda.$$

But since $Re\,\lambda > \omega_{ess}(\mathcal{T})$, then $\lambda \in \sigma_p(\mathcal{A})$, which is true if and only if $\lambda \in \sigma_p(A + \Phi_\lambda)$ (see [5, Lemma 3.20 page 58]). The operator $A + \Phi_\lambda$ is given by

$$(A + \Phi_\lambda)(z)(s) := -\gamma z'(s) - \mu(s)z(s) + \int_{-r}^{0} \nu(s,\sigma)e^{\lambda\sigma} d\sigma z(s) + \int_0^m \int_{-r}^0 \beta(\sigma, s, b)e^{\lambda\sigma} z(b) d\sigma db$$

Thus $\lambda \in \sigma_p(A + \Phi_\lambda)$ if and only if there exists $z \in D(A) = \{z \in W^{1,1}(\mathbb{R}^+) : z(0) = 0\}$, $z \neq 0$ such that

$$(A + \Phi_\lambda)z = \lambda z.$$

It follows that z satisfies the following differential equation

$$z'(s) = \left(-\lambda - \mu(s) + \int_{-r}^0 \nu(s,\sigma)e^{\lambda\sigma} d\sigma\right) z(s) + \frac{1}{\gamma} \int_0^m \int_{-r}^0 \beta(\sigma, s, u)e^{\lambda\sigma} z(u) d\sigma du.$$

By solving this equation using $\beta(\sigma, b, u) = \beta_1(b)\beta_2(\sigma, u)$, we get

$$z(s) = \frac{1}{\gamma} C_z \int_0^s \exp\left(\frac{1}{\gamma} \int_b^s \left(-\lambda - \mu(c) + \int_{-r}^0 \nu(c,\sigma)e^{\lambda\sigma} d\sigma\right) dc\right) \beta_1(b) db. \tag{3.38}$$

where $C_z := \left(\int_0^m \int_{-r}^0 \beta_2(\sigma,u)e^{\lambda\sigma}z(u)d\sigma du\right)$. Multiply the above equation by $e^{\lambda\theta}\beta_2(\theta,s)$ and integrating, we get

$$C_z = C_z\frac{1}{\gamma}\int_0^m \left(\int_{-r}^0 e^{\lambda\theta}\beta_2(\theta,s)d\theta\right)\left(\int_0^s \exp\left(\frac{1}{\gamma}\int_b^s\left(-\lambda-\mu(c)+\int_{-r}^0 \nu(c,\sigma)e^{\lambda\sigma}d\sigma\right)dc\right)\beta_1(b)db\right)ds.$$

Since $z \neq 0$ then by (3.38), we have $C_z \neq 0$. Therefore

$$1 = \frac{1}{\gamma}\int_0^m \left(\int_{-r}^0 e^{\lambda\theta}\beta_2(\theta,s)d\theta\right)\left(\int_0^s \exp\left(\frac{1}{\gamma}\int_b^s\left(-\lambda-\mu(c)+\int_{-r}^0 \nu(c,\sigma)e^{\lambda\sigma}d\sigma\right)dc\right)\beta_1(b)db\right)ds.$$

This proves the theorem. ■

3.2. Extinction of population

In the following, we give a sufficient condition for the extinction of the population.

Theorem 3.9. *Assume that*

$$s \mapsto \int_0^s \beta_1(b)db \in L^1(\mathbb{R}^+), \tag{3.39}$$

$$\int_0^m \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma\right)\left(\int_0^s \beta_1(b)e^{-\frac{1}{\gamma}\int_b^s\left(\overline{\nu}r-\underline{\mu}+\mu(c)-\int_{-r}^0\nu(c,\sigma)d\sigma\right)dc}db\right)ds > \gamma \tag{3.40}$$

and

$$\int_0^m \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma\right)\left(\int_0^s \beta_1(b)e^{-\frac{1}{\gamma}\int_b^s\left(\mu(c)-\int_{-r}^0\nu(c,\sigma)d\sigma\right)dc}db\right)ds < \gamma. \tag{3.41}$$

Then there exists $c > 0$ such that for t large enough

$$|u(t,.)|_{L^1} \leq e^{-ct}.$$

Remark. One can interpret Theorem 3.9 in this way: (3.41) shows that if the birth rate and the density-dependent migration are small enough with respect to the mortality and growth rate, then the population goes extinct.

The following lemma is needed in the proof of Theorem 3.9.

Lemma 3.10. [14, Corollary 1.7] *Let $S(t)_{t\geq 0}$ be a positive c_0-semigroup on a Banach lattice and let B be its infinitesimal generator. If there exist t_0 and a compact operator K such that $r\left(S(t_0)-K\right) < r\left(S(t_0)\right)$, then $s(B)$ is an eigenvalue of B.*

Proof of Theorem 3.9 From Proposition 3.5 we have

$$\limsup_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t} \leq \omega_0\left(\mathcal{T}\right). \tag{3.42}$$

We will prove that $\omega_0(\mathcal{T}) < 0$. Since by Lemma 2.8 $\omega_0(\mathcal{T}) = s(\mathcal{A})$, it is sufficient to prove that $s(\mathcal{A}) < 0$. To do this we will prove that $s(A+\Phi_0) < 0$ and use Lemma 2.7 to conclude. We first claim that $s(A+\Phi_0)$ is an eigenvalue of $A+\Phi_0$. In fact, the operator $A+\Phi_0$ is given by

$$\left(\left(A+\Phi_0\right)z\right)(s) = -\gamma z'(s) - \left(\mu(s) - \int_{-r}^0 \nu(s,\sigma)d\sigma\right)z(s) + \beta_1(s)\int_0^m\int_{-r}^0 \beta_2(\sigma,b)z(b)d\sigma db.$$

Consider the following decomposition

$$A+\Phi_0=\left(A+\Phi_0^1\right)+\Phi_0^2, \tag{3.43}$$

where $A+\Phi_0^1$ is defined by

$$\left(\left(A+\Phi_0^1\right)z\right)(s)=-\gamma z'(s)-\left(\mu(s)-\int_{-r}^{0}\nu(s,\sigma)d\sigma\right)z(s)$$

and Φ_0^2 is given by

$$\left(\Phi_0^2 z\right)(s)=\beta_1(s)\int_0^m\int_{-1}^0\beta_2(\sigma,b)z(b)d\sigma db \quad \text{for } z\in X.$$

The operator Φ_0^2 is of finite rank and thus compact. The operator $\left(A+\Phi_0^1, D(A)\right)$ generates the semigroup $\left(T_0^1(t)\right)_{t\geq 0}$ given explicitly for each $z\in X$ by

$$\left(T_0^1(t)z\right)(s)=\begin{cases}0 & \text{for } s<\gamma t\\ \exp\left(\frac{1}{\gamma}\int_{s-\gamma t}^{s}\left(\int_{-r}^{0}\nu(b,\sigma)d\sigma-\mu(b)\right)db\right)z(s-\gamma t) & \text{for } s>\gamma t.\end{cases} \tag{3.44}$$

Moreover,

$$\left|T_0^1(t)z\right|\leq e^{\left(\overline{\nu}r-\underline{\mu}\right)t}\left|z\right|.$$

Thus

$$s\left(A+\Phi_0^1\right)=\omega_0\left(T_0^1\right)\leq\overline{\nu}r-\underline{\mu}. \tag{3.45}$$

Being a bounded perturbation of the operator $A+\Phi_0^1$, the operator $A+\Phi_0$ generates a positive semigroup $(T_0(t))_{t\geq 0}$. Using [12, Proposition IV.2.12], we deduce from the decomposition (3.43) and the compactness of the operator Φ_0^2 that the operator $T_0(t)-T_0^1(t)$ is compact for $t>0$. Let $K:=T_0(t_0)-T_0^1(t_0)$ for some $t_0>0$. Thus from (3.45) we have

$$r(T_0(t_0)-K)=r(T_0^1(t_0))=e^{\omega_0\left(T_0^1\right)t_0}\leq e^{\left(\overline{\nu}r-\underline{\mu}\right)t_0}.$$

Since $r(T_0(t))=e^{\omega_0(T_0)t}$ for all $t\geq 0$, to show that $r(T_0(t_0)-K)<r(T_0(t_0))$ it suffices to show that

$$\overline{\nu}r-\underline{\mu}<\omega_0(T_0). \tag{3.46}$$

Note that $\omega_0(T_0)=s(A+\Phi_0)$ again by Lemma 2.8. To prove (3.46), we will find a real eigenvalue of $A+\Phi_0$ such that $\overline{\nu}r-\underline{\mu}<\lambda_0$. Consider the function ξ defined by

$$\xi(\lambda)=\int_0^m\left(\int_{-r}^0\beta_2(\sigma,s)d\sigma\right)\left(\int_0^s\beta_1(b)e^{-\frac{1}{\gamma}\int_b^s\left(\lambda+\mu(c)-\int_{-r}^0\nu(c,\sigma)d\sigma\right)dc}db\right)ds-\gamma.$$

We have $\lim_{\lambda\to\infty}\xi(\lambda)=-\gamma$ and $\lim_{\lambda\to-\infty}\xi(\lambda)=\infty$ and ξ is decreasing. This implies that there exists a unique $\lambda_0\in\mathbb{R}$ such that

$$\xi(\lambda_0)=0. \tag{3.47}$$

We claim that λ_0 is an eigenvalue of $A+\Phi_0$ with an eigenvector given by

$$z_0(s)=\int_0^s\beta_1(b)e^{-\frac{1}{\gamma}\int_b^s\left(\lambda_0+\mu(c)-\int_{-r}^0\nu(c,\sigma)d\sigma\right)dc}db.$$

Notice that

$$\xi(\lambda_0)=\int_0^m\left(\int_{-r}^0\beta_2(\theta,s)d\theta\right)z_0(s)ds-\gamma. \tag{3.48}$$

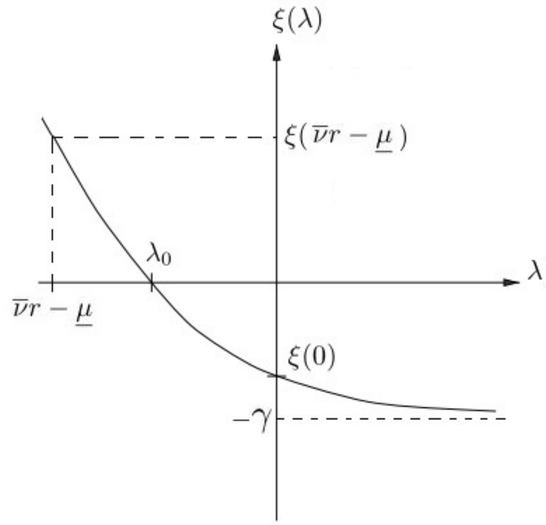

Figure 3: Graph of ξ

We have

$$z_0'(s) = -\frac{1}{\gamma}\Big(\lambda_0 + \mu(s) - \int_{-r}^{0} \nu(s,\sigma)d\sigma\Big)z_0(s) + \beta_1(s). \tag{3.49}$$

Thus using (3.47), (3.48) and (3.49), we obtain that

$$\begin{aligned}((A+\Phi_0)\,z_0)\,(s) &= -\gamma z_0'(s) - (\mu(s) - \int_{-r}^{0}\nu(s,\sigma)d\sigma)z_0(s) + \beta_1(s)\int_0^m\left(\int_{-r}^{0}\beta_2(\sigma,b)d\sigma\right)z_0(b)db\\ &= -\gamma z_0'(s) - (\mu(s) - \int_{-r}^{0}\nu(s,\sigma)d\sigma)z_0(s) + \beta_1(s)\,(\xi(\lambda_0)+\gamma)\\ &= -\gamma z_0'(s) - (\mu(s) - \int_{-r}^{0}\nu(s,\sigma)d\sigma)z_0(s) + \gamma\beta_1(s)\\ &= \lambda_0 z_0(s).\end{aligned}$$

Note that (3.40) is equivalent to $\xi(\overline{\nu}r - \underline{\mu}) > 0$ which implies by the monotony of ξ (see Figure 3) that

$$\overline{\nu}r - \underline{\mu} < \lambda_0. \tag{3.50}$$

One can see that (3.50) together with (3.39) insures that $z_0 \in L^1(\mathbb{R}^+)$. Now by (3.3) and (3.49) we have $z_0' \in L^1(\mathbb{R}^+)$. We conclude that $z_0 \in W^{1,1}(\mathbb{R}^+)$ and thus $z_0 \in D(A+\Phi_0)$ because $z_0(0) = 0$. Since $z_0 \neq 0$, we deduce that λ_0 is an eigenvalue of $A+\Phi_0$. As a consequence, (3.50) implies that $\overline{\nu}r - \underline{\mu} < s(A+\Phi_0)$ and thus $r(T_0(t_0) - K) < r(T_0(t_0))$. By applying Lemma 3.10, we deduce that $\lambda_1 := s(A+\Phi_0)$ is an eigenvalue of the operator $A+\Phi_0$. Thus there exists $z \in D(A)$ with $z \neq 0$ such that

$$((A+\Phi_0)\,z)\,(s) = \lambda_1 z(s),$$

that is

$$z'(s) = -\frac{1}{\gamma}\left(\lambda_1 + \mu(s) - \int_{-r}^{0}\nu(s,\sigma)d\sigma\right)z(s) + \frac{1}{\gamma}\beta_1(s)\int_0^m\left(\int_{-r}^{0}\beta_2(\sigma,b)d\sigma\right)z(b)db. \tag{3.51}$$

By solving (3.51) taking into account the fact that $z(0) = 0$ we get

$$z(s) = C_z \left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\lambda_1 + \mu(c) - \int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db \right), \tag{3.52}$$

where C_z is the constant given by $C_z := \dfrac{1}{\gamma}\displaystyle\int_0^m \left(\int_{-r}^0 \beta_2(\sigma, b)d\sigma \right) z(b)db$. Note that $C_z \neq 0$ because $z \neq 0$. Multiplying (3.52) by $\int_{-1}^0 \beta_2(\sigma, s)d\sigma$, we get that

$$\left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma \right) z(s) = C_z \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma \right) \left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\lambda_1 + \mu(c) - \int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db \right)$$
$$\gamma = \int_0^m \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma \right) \left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\lambda_1 + \mu(c) - \int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db \right) ds$$

Now, by integrating the above equation and using the fact that $C_z \neq 0$ we get that

$$\gamma = \int_0^m \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma \right) \left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\lambda_1 + \mu(c) - \int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db \right) ds,$$

that is $\xi(\lambda_1) = 0$. Thus $s(A + \Phi_0) = \lambda_1 = \lambda_0$ because λ_0 is the only real zero of ξ. Note that (3.41) is equivalent to

$$\xi(0) = \int_0^m \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma \right) \left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\mu(c) - \int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db \right) ds - \gamma < 0$$

which implies by monotony of ξ that $s(A + \Phi_0) = \lambda_0 < 0$ (see Figure 3). Therefore using Lemma 2.7 we conclude that $\omega_0(\mathcal{T}) = s(\mathcal{A}) < 0$. The proof is now complete by using (3.42) and the fact that $|u(t,.)|_{L^1} \leq |\mathcal{U}(t)|$. ■

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 115–124.
http://doi.org/10.26637/mjm11S/008

Monotone traveling waves in a general discrete model for populations with long term memory

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*R*eceived 21 June 2023; *A*ccepted 11 September 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. In this paper we consider the existence of monotone traveling waves for a class of general integral difference models for populations that are dependent on the previous state term and also on long term memory. This allows us to consider multiple past states. For this model we will have to deal with the non-compactness of the evolution operator when we prove the existence of a fixed point. This difficulty will be overcome by using the Monotone Iteration Method and Dini's Theorem to show uniform convergence of an iterative evolution operator to a continuous wave function.

AMS Subject Classifications: 92D25, 37N25, 39A22.

Keywords: Traveling waves; spreading speed; partially sedentary population; delay effect

Contents

1. Introduction

Many papers use the following integrodifference model to study the dynamics of certain populations

$$u_{n+1}(x) = \int_{-\infty}^{\infty} K(x-y) f(u_n(y)) dy. \tag{1.1}$$

Here $u_n(x)$ is the density of the population at the location x at time n, $x \in \mathbb{R}$ is a location in the "habitat" of the population and $n \in \mathbb{Z}$ is the observable time.

In the above mentioned model, given the "fecundity function" f and the diffusion kernel K, one assumes that the dynamics of the system depends only on its status at the last time. A more realistic model should reflect the effect of the history of the system in the past rather than only at the last time. For simplicity, we assume that it depends on the status of the system at time n and $n-1$ as other more general settings could be treated in a similar manner. This allows us to consider long term memory of the following evolution operator.

$$u_{n+1}(x) = \int_{-\infty}^{\infty} K_1(x-y) f_1(u_n(y)) dy + \int_{-\infty}^{\infty} K_2(x-y) f_2(u_{n-1}(y)) dy, \tag{1.2}$$

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where K_i, f_i are functions of the same nature as above, though they may be different.

Pioneering works by Weinberger, and contempories studied spreading speed and asymptotic behavior of wave front solutions in population models, see [1, 2, 8, 10, 11, 32–34]. Furthermore, delay models are often considered in population dynamics, this is due to the fact populations often are reliant on previous states in time, see the manuscripts [3, 35]. The spreading speed of wavefront solutions have been studied in several types of models. For example, the dynamics in certain competition and cooperation models were studied in [8, 11, 20, 21, 34].

The dynamics, spreading speed and asymptotic behavior of nonmonotone waves have been studied for discrete and integrodifference equations in [4, 10]. Results on traveling waves for certain plant populations, including seed banks and dormant seed banks can found in [7, 9, 13–15, 19, 22, 23, 25, 28, 29, 31].

In Li, [9] and Thuc, and Nguyen, [7], a version of the following model was studied

$$u_{n+1}(x) = (1-\gamma)\int_{-\infty}^{+\infty} k(x-y)g(u_n(y))dy + \gamma\rho u_n(x), \tag{1.3}$$

where $u_n(x)$ represents the density of the mature plant population at time n, and k is seed dispersal kernel. In this model it is understood that $k(x-y)$ is the density function of the probability $P(x,y)$ of an individual to migrate from location y to location x in the habitat. This is a model for seed banks introduced by MacDonald and Watkinson, [19]. The spreading speed and asymptotic behavior for nonlinear integral equations was also studied by Thieme, [26, 27].

This paper is interested in dynamics in certain plant populations. In particular, we are interested in monotone wave fronts for Equation (1.2). In section 2, we show the existence of traveling waves for an integrodifferential model of the form Equation (1.2) loses compactness, so standard fixed point theorems do not suffice. To get around this issue we will use Dini's Theorem to show uniform convergence of an evolution operator to a continuous wave front. This is similar to the results found in Thuc and Nguyen, [7].

However, instead of considering the Ricker function, we consider a general Lipchitz function that is bounded on $\mathbb{R}$ and is increasing on a certain interval. Furthermore, we strengthen the results by eliminating the continuity condition found the Standing Assumption of the kernel function found in Thuc and Nguyen, [7]. Lastly, in section 3 we provide a brief discussion of our results and some interesting questions about extending our idea to more general models, including the addition of discrete terms.

Notations and Assumptions

We denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ the set of natural numbers, set of integers, and set of the reals, respectively. We also denote by $BM(\mathbb{R},\mathbb{R})$ ($BC(\mathbb{R},\mathbb{R})$, respectively) the space of all measurable and bounded real valued functions on $\mathbb{R}$ (the space of all bounded continuous real valued functions on $\mathbb{R}$, respectively) with sup-norm. For a constant α we will denote the constant function $\mathbb{R} \ni x \mapsto \alpha$ by this number α for convenience if this does not cause any confusion. C_M stands for the set $\{f \in BC(\mathbb{R},\mathbb{R}) | f(x) \in [0,M]\}$, and B_M stands for $B_M := \{f \in BM(\mathbb{R},\mathbb{R}) | f(x) \in [0,M]\}$. The metric on C_M is defined by the sup norm. In $BM(\mathbb{R},\mathbb{R})$ we use the natural order defined as $u \leq v$ if and only if $u(x) \leq v(x)$ for all $x \in \mathbb{R}$.

2. Main Results

For model (1.2) the compactness will disappear after a standard conversion of the delayed equation into a non-delayed equation. In fact, by assuming $K := K_1 = K_2$ and $f := f_1 = f_2$, and setting

$$w_{n+1}(x) = \begin{bmatrix} u_n(x) \\ u_{n-1}(x) \end{bmatrix},$$

we obtain an equation

$$w_{n+1}(x) = \begin{bmatrix} u_n(x) \\ u_{n-1}(x) \end{bmatrix} \tag{2.1}$$

$$= \begin{bmatrix} \int_{-\infty}^{\infty} K_1(x-y)f_1(u_n(y))dy + \int_{-\infty}^{\infty} K_2(x-y)f_2(u_{n-1}(y))dy \\ u_{n-1}(x) \end{bmatrix} \tag{2.2}$$

$$= \begin{bmatrix} \int_{-\infty}^{\infty} K(x-y)\left[f(u_n(y)) + f(u_{n-1}(y))\right] dy \\ u_{n-1}(x) \end{bmatrix}. \tag{2.3}$$

Next, we denote the projection $\mathbb{R}^2 \ni (x,y)^T \to x \in \mathbb{R}$ by P_1 and $\mathbb{R}^2 \ni (x,y)^T \to y \in \mathbb{R}$ by P_2. Then, we obtain the equation we obtain an equation

$$w_{n+1}(x) = \begin{bmatrix} \int_{-\infty}^{\infty} K(x-y)\left[f(P_1 w_n(y)) + f(P_2 w_n(y))\right] dy \\ P_2 w_n(x) \end{bmatrix}$$

$$= \begin{bmatrix} \int_{-\infty}^{\infty} K(x-y)F(w_n(y))dy \\ P_2 w_n(x) \end{bmatrix}. \tag{2.4}$$

Denote the evolution operator as

$$A[w_n](x) = \int_{-\infty}^{\infty} K(x-y)F(w_n(y))dy,$$

where $F(w_n(y)) = f(P_1 w_n(y)) + f(P_2 w_n(y))$ is bounded. Moreover, notice that the projection operator $B[w_n](x) = P_2 w_n(x)$ is a constant, so we lose compactness. Thus, we have the following operator

$$Q[w_n](x) = \begin{bmatrix} A[w_n](x) \\ B[w_n](x) \end{bmatrix}$$

We also impose the following conditions on the convolution kernel and fecundity function.

Standing Assumption

(P1) $K(x) \geq 0$ and measurable for all $x \in \mathbb{R}$.

(P2) For all $\mu \geq 0$

$$\int_{-\infty}^{\infty} K(x)dx = 1, \quad \int_{-\infty}^{\infty} K(x)e^{\mu|x|}dx < \infty.$$

(P3) $0 \leq F(\alpha) \leq r, F(0) = 0, F(r) = r,$ where $0 < r \leq 1, \alpha \in \mathbb{R}, F(\alpha) > \alpha, \alpha < r, F(\alpha) < \alpha, \alpha > r.$

(P4) $F(\cdot)$ is Lipchitz continuous.

Lemma 2.1. *Assume the standing assumptions hold, then the following are true.*

1. $A[\cdot]$ *maps monotone functions to monotone functions of the same orientation.*
2. $A[0] = 0, A[r] = r, A[\alpha] > \alpha$ *when* $0 < \alpha < r$.
3. *For all* $v \leq u$, *where* $u, v \in B_r$, *then* $A[v] \leq A[u]$.
4. *Let* $w_n \in BC(\mathbb{R}, \mathbb{R})$ *such that* w_n *converges uniformly to some non-negative real constant,* w *on each bounded subset of* $\mathbb{R}$, *then* $A[w_n](x)$ *converges uniformly to* $A[w](x)$ *for every* $x \in \mathbb{R}$.
5. *If* $\alpha > r$, *then* $A[\alpha] < \alpha$.

6. *There is a constant* $\overline{\gamma}$ *such that* $\gamma < A[\gamma] < \overline{\gamma}$ *for all* $0 \leq \gamma < \overline{\gamma}$, *and* $A[\overline{\gamma}] = \overline{\gamma}$.

Proof. We only show the increasing portion for $i.)$ since the non-increasing portion can be shown similarly. We note if $w(\cdot)$ is increasing, so is $P_i(w(\cdot)), i = 1, 2$. Furthermore, assume that $F(\cdot)$ is increasing when $r \in (0, 1)$, fix $x \geq x_0$.

$$\begin{aligned}
&\int_{-\infty}^{\infty} K(x-y)F(w(y))dy - \int_{-\infty}^{\infty} K(x_0-y)F(w(y))dy \\
&= \int_{-\infty}^{\infty} K(\xi)F(w(x-\xi))d\xi - \int_{-\infty}^{\infty} K(\xi)F(w(x_0-\xi))d\xi \\
&= \int_{-\infty}^{\infty} K(\xi)\left[F(w(x-\xi)) - F(w(x_0-\xi))\right]d\xi \geq 0.
\end{aligned}$$

For $ii.)$ $A[0] = 0$, due to $(P3)$ of the standing assumption. $A[r] = r$ follows similarly. Take α as some constant, then by $(P4)$ we have $F(\alpha) > \alpha$, so the result follows by $(P2)$.

For $iii.)$ note $F(x)$ is increasing when $x \in [0, 1]$. The result follows.

For $iv.)$ assume w_n converges uniformly to a non-negative constant w. By Lebesgue's Dominated Convergence theorem we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} K(x-y)F(w_n(y))dy = \int_{-\infty}^{\infty} K(x-y)F(w(y))dy.$$

Take $\alpha > r$ as some constant, then by $(P4)$ we have $F(\alpha) < \alpha$, so $v.)$ follows by $(P2)$.

Take $\overline{\gamma} = r$, the $A[\gamma] < A[\overline{\gamma}]$, when $0 \leq \gamma < \overline{\gamma}$ due to $(P4)$, so $vi)$ holds. This proves the lemma. ■

Using Lemma 2.1 allows the theory on spreading speed developed by Weinberger, [32] can be used. Moreover, we extend spreading speed results from [7, 9, 10, 15]. To this end, define a function $\varphi : \mathbb{R} \to \mathbb{R}$ that enjoys the following properties:

1. φ is continuous and non-increasing,
2. $\varphi(-\infty) = \lim_{t\to-\infty} \varphi(t) \in (0, r)$,
3. $\varphi(s) = 0$ for all $s \geq 0$.

It is now possible to define the following operator $R_c[\cdot]$ on the space C_r for a speed, c as

$$R_c[u](s) = \max\{\varphi(s), A[u(c + \cdot)](s)\}, s \in \mathbb{R}.$$

Moreover, define an iterative sequence

$$a_{n+1} = R_c[a_n], \quad a_0 = \varphi.$$

From [7, 32] the sequence $\{a_n(c; \cdot)\}$ is bounded and non-increasing for all $s \in \mathbb{R}$. Therefore, we obtain the pointwise limit

$$\lim_{n\to\infty} a_n(c; s) = a(c; s).$$

The spreading speed is defined as

$$c^* = \sup\{c : a(c; \infty) = r\},$$

where $0 \leq a(c;s) \leq r$. We can also define a number c^*_- as

$$R_c[u](s) = \max\{\varphi(s), A[u(c-\cdot)](s)s \in \mathbb{R}.$$

Then, we can define a sequence of functions

$$b_{n+1} = R_c[u_n], \quad b_0 = \varphi,$$

with a pointwise limit

$$\lim_{n\to\infty} b_n(c;s) = b(c;s).$$

Clearly, $0 \leq b(c;s) \leq r$.

$$c^*_- = \sup\{c : b(c;\infty) = r\}.$$

Defining a non-negative, bounded, real-valued measure $m(x, dx)$ where

$$A[u](x) \leq \int_{-\infty}^{\infty} u(x-y)m(y,dy),$$

then

$$c^* \leq \inf_{\mu>0} \frac{1}{\mu} \ln\left(\int_{-\infty}^{\infty} e^{\mu x} m(x,dx)\right) \tag{2.5}$$

$$c^*_- \leq \inf_{\mu>0} \frac{1}{\mu} \ln\left(\int_{-\infty}^{\infty} e^{-\mu x} m(x,dx)\right). \tag{2.6}$$

Now, using the theory in [7, 9, 32] we can define

$$m(x,dx) = K(x)F'(0)dx.$$

This gives,

$$\begin{aligned} c^* &\leq \inf_{\mu>0} \frac{1}{\mu} \ln\left(\int_{-\infty}^{\infty} e^{\mu x} m(x,dx)\right) \\ &= \inf_{\mu>0} \frac{1}{\mu} \ln\left(F'(0)\int_{-\infty}^{\infty} e^{\mu x} K(x)dx\right), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} c^*_- &\leq \inf_{\mu>0} \frac{1}{\mu} \ln\left(\int_{-\infty}^{\infty} e^{-\mu x} m(x,dx)\right) \\ &= \inf_{\mu>0} \frac{1}{\mu} \ln\left(F'(0)\int_{-\infty}^{\infty} e^{-\mu x} K(x)dx\right). \end{aligned} \tag{2.8}$$

Proposition 2.2. *Assume the standing assumption holds then the spreading speeds c^*, c^*_- are finite.*

Proof. The result follows from the definition of Eq(2.5, 2.7) and the standing assumption. ■

Existence of Traveling Waves

We are now ready to prove our main result.

Definition 2.3. *A monotone traveling wave solution with a speed, c connecting 0 and r of Eq. (2.4) can be defined as a non-increasing continuous function with the following behavior.*

$$\lim_{x\to\infty} w(x) = 0, \lim_{x\to-\infty} w(x) = r, w(x) = u(x-nc), n \in \mathbb{N}.$$

Substituting the wave transformation into Eq. (2.4) gives

$$w((x-(n+1)c) = \begin{bmatrix} \int_{-\infty}^{\infty} K(x-y)F(w(y-nc))dy \\ P_2 w(x-nc) \end{bmatrix}. \tag{2.9}$$

Making the change of variables $\xi = x-(n+1)c, \mu = y-nc$ gives

$$u(\xi) = \begin{bmatrix} \int_{-\infty}^{\infty} K(\xi+c-\mu)F(w(\mu))dy \\ P_2 w(\xi+c) \end{bmatrix}. \tag{2.10}$$

This yields the following operator

$$Q_c[w](\xi) = \begin{bmatrix} A_c[w](\xi) \\ B_c[w](\xi) \end{bmatrix}. \tag{2.11}$$

Lemma 2.4. *Assume the standing assumption holds, then $A_c : B_r \to BC(\mathbb{R}, \mathbb{R})$ and A_c is Lipchitz continuous.*

Proof. We first show that A_c maps B_r into $BC(\mathbb{R}, \mathbb{R})$. To this end, fix $u \in BC(\mathbb{R}, [0, r]), x, x_0 \in \mathbb{R}$. Then we can use $(P2)$. This means the operator A_c is uniformly convergent on $\mathbb{R}$. Thus, for any $\varepsilon > 0$ it is possible to find two constants, $T, \delta > 0$, dependent upon ε such that when $|x - x_0| < \delta$

$$\left| \int_{-\infty}^{-T} \left(K(x+c-\mu) - K(x_0+c-\mu)\right) F(u(\mu))d\mu + \int_{T}^{\infty} \left(K(x+c-\mu) - K(x_0+c-\mu)\right) F(u(\mu))d\mu \right| < \frac{\varepsilon}{3}.$$

This means

$$\begin{aligned} |A_c[u](x_0) - A_c[u](x)| &= \left| \int_{\infty}^{-\infty} K(x+c-\mu)F(u(\mu))d\mu - \int_{\infty}^{-\infty} K(x_0+c-\mu)F(u(\mu))d\mu \right| \\ &< \left| \int_{-T}^{T} K(x+c-\mu)F(u(\mu))d\mu - \int_{-T}^{T} K(x_0+c-\mu)F(u(\mu))d\mu \right| + \frac{\varepsilon}{3}. \end{aligned}$$

Using the following change of variable $\xi = x - \mu, H(\xi; \cdot) = K(\xi)F(\cdot)$. Then,

$$\begin{aligned} &\left| \int_{T}^{-T} H(\xi; x-\xi)d\xi - \int_{-T+\delta}^{T-\delta} H(\xi; x_0-\xi)d\xi \right| \\ &\leq \left| \int_{-T}^{-T+\delta} H(\xi; x-\xi)d\xi - \int_{T-\delta}^{T} H(\xi; x-\xi)d\xi \right| + \left| \int_{-T+\delta}^{T-\delta} \left(H(\xi; x-\xi) - H(\xi; x_0-\xi)\right) d\xi \right| \\ &\leq 2\delta T + \left| \int_{-T+\delta}^{T-\delta} \left(H(\xi; x-\xi) - H(\xi; x_0-\xi)\right) d\xi \right|. \end{aligned}$$

Using the fact that $F(u(x))$ is continuous, then for there exists positive constant N such that

$$|F(u(x)) - F(u(x_0))| < \frac{\varepsilon}{6NT}, \text{ when } |x - x_0| < \delta.$$

Since K is measurable with finite measure, then Lusin's Theorem [24, Chapter 3] allows us to see that K is continuous almost everywhere on the real line. In particular, there are finite discontinuities of K in $[-T, T]$, say $T_k, k = 1, 2, \cdots, N$. Moreover, take $T_0 = -T, T_{N+1} = T$. Then, for any $\mu \in (-T_k + \delta, T_k - \delta)$ we have

$$\begin{aligned} |A_c[u](x_0) - A_c[u](x)| &\leq 2\delta T + \left| \sum_{k=0}^{N} \int_{-T_k+\delta}^{T_k-\delta} \left(H(\xi; x-\xi) - H(\xi; x_0-\xi)\right) d\xi \right| \\ &\leq 2\delta T + 2NT\frac{\varepsilon}{6NT}. \end{aligned}$$

Lastly, take $\delta < \frac{\varepsilon}{6T}$, then

$$|A_c[v](x_0) - A_c[v](x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Now, using the fact that $K(x)$ is essentially bounded and $F(u(x))$ is bounded we have shown that A_c maps into $BC(\mathbb{R}, \mathbb{R})$.

Now, we show the operator is Lipchitz, we fix $u, v \in BC(\mathbb{R}, [0, r]), x \in \mathbb{R}$.

$$|A_c[v](x) - A_c[u](x)| = \left| \int_{-\infty}^{\infty} K(x + c - \mu) \left[F(v(\mu)) - F(u(\mu))\right] d\mu \right|$$
$$\leq \int_{-\infty}^{\infty} |K(x + c - \mu) \left[F(v(\mu)) - F(u(\mu))\right] d\mu|$$
$$\leq F'(0)||v - u||,$$

by $(P4)$ in the standing assumption. Thus, the operator is Lipchitz. ■

Theorem 2.5. *Assume the standing assumption holds and*

$$c \geq \inf_{\mu > 0} \frac{1}{\mu} \ln \left(F'(0) \int_{-\infty}^{\infty} K(x) e^{\mu x} dx \right),$$

then there is a monotone wave front to Eq. (2.4) with a wave speed, c that connects 0 *and* r.

Proof. Again, the operator A_c only needs to be considered, because the projection will follow. To this end, define $\phi_{n+1} = A_c[\phi_n], n \in \mathbb{N}$, then

$$\Phi_{n+1} = \begin{bmatrix} A_c[\phi_n] \\ B_c[\phi_n] \end{bmatrix}. \text{ and } \phi_1(s) = a(c; s).$$

The function $a(c; s)$ is non-increasing and bounded, so it is measurable on $\mathbb{R}$. Thus, $a(c; s) \in B_r$. Using the fact that the translation operator is invariant, the for the sequence $\{a_n(c; \cdot)\}$

$$a_{n+1}(c; s) = \max \varphi(s), A_c[a_n(c; \cdot + s + c)](0)$$
$$= \max \varphi(s), A_c[a_n(c; \cdot + c)](s).$$

Thus, $a(c; s) = \max \varphi(s), A_c[a(c; \cdot)](s)$. This leaves the following estimate $a(c; s) \geq A_c[a(c; s)]$, which means $\phi_1 \geq \phi_2$. Moreover, the operators A_c, B_c preserve order, so we have by an inductive argument

$$\phi_n \geq \phi_{n+1}, n \in N.$$

Therefore, since the sequence $\{\phi_n(x)\}$ is non-negative and non-increasing for every fixed $x \in \mathbb{R}$ it is convergent to some non-negative function, non-increasing measurable function, say $W(x)$. This allows us to use Lebesgue's Dominated Convergence Theorem to see

$$\lim_{n \to \infty} A_c[\phi_n] = \lim_{n \to \infty} \int_{-\infty}^{\infty} K(x + c - y) F(\phi_n(y)) dy$$
$$= \int_{-\infty}^{\infty} K(x + c - y) F(W(y)) dy$$
$$= A_c[W](x) = W(x).$$

Thus, W is continuous and is a fixed point of A_c in B_r. Now, we note that B_c is a constant, so it is non-increasing. Moreover, projections are continuous in any Banach Space and W is a continuous limit of a non-increasing monotone sequence of continuous functions. For every compact subset on $\mathbb{R} \times \mathbb{R}$ equipped with the standard ordering and sup norm allows us to invoke Dini's Theorem. Note that a version of Dini's Theorem for functions taking values in $\mathbb{R}^2$ can be easily proved. Thus, the convergence of Q_c is uniform to a continuous function. Lastly, we need to show W is a traveling wave solution to Eq. (2.4). Using the theory found in [7, 9, 32], we see

$$\lim_{t\to-\infty} a(c;t) = r, \quad \lim_{t\to\infty} a(c;t) = 0.$$

Since the sequence $\{\phi_n\}$ converges uniformly on every compact interval of $\mathbb{R}$ we can use [32] to see

$$\lim_{t\to-\infty} W(t) = r, \quad 0 \leq \lim_{t\to\infty} W(t) \leq \lim_{t\to\infty} \phi_1(t).$$

However,

$$\lim_{t\to\infty} \phi_1(t) = \lim_{t\to\infty} a(c,t) = 0, \text{ thus } \lim_{t\to\infty} W(t) = 0.$$

The theorem is proved. ■

3. Discussion

In this paper we assumed that the fecundity function $F(x)$ to be Lipchitz, bounded and increasing on the interval $0 < x \leq r \leq 1$. We proved the existence of monotone traveling waves on the whole real line. Moreover, we relaxed the standing assumption on the kernel function. Some interesting questions would be to add discrete terms into the model and study if the results hold. In fact, a generalization for long term memory would be an interesting addition to study. One example may be

$$N_{n+1}(x) = \sum_{i=1}^{L} \eta_i(f_1(N_n(x-c_i)) + f_2(N_{n-1}(x-c_i)) \\ + \int_{-\infty}^{\infty} K_1(x-y) f_1(N_n(y))dy + \int_{-\infty}^{\infty} K_2(x-y) f_2(N_{n-1}(y))dy, \tag{3.1}$$

where $N_n(x)$ is the population densities at year n, and $\eta_i, c_i > 0$ such that

$$\sum_{i=1}^{L} \eta_i \leq 1.$$

Furthermore, the points $n = 1, \cdots, L$, are fixed locations in the habitat taken to be $\mathbb{R}$.

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 125–148.
http://doi.org/10.26637/mjm11S/009

The spectrum theory of the discrete Schrödinger operator and its application

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Received 26 June 2023; *Accepted* 24 September 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. This paper introduces the spectrum theory of discrete Schrödinger operators with different kinds of potentials, including bounded, unbounded, periodic, or complex potentials. The paper also provides exponential estimates of the Green's function and eigenfunctions of the discrete Schrödinger operators. As an application, I review some of our results on standing wave solutions of discrete Schrödinger equations.

AMS Subject Classifications: 35Q55, 35Q51, 37L60, 39A12, 39A70, 47J30, 78A40.

Keywords: Discrete Schrödinger operator, spectrum, bounded/unbounded potential, complex potential, periodic potential, Green's function, standing waves

Contents

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1. Introduction and Preliminaries

The spectrum theory of discrete Schrödinger operators constitutes a fundamental framework in the field of mathematical physics and quantum mechanics. These operators play a crucial role in understanding the behavior of quantum systems on discrete spaces, such as lattices or graphs.

In the spectrum theory of discrete Schrödinger operators, the focus lies on the investigation of the eigenvalues and eigenfunctions associated with these operators. One key aspect of this theory is the analysis of different types of potentials that can influence the behavior of the discrete Schrödinger operators. These potentials can vary in nature, including bounded potentials, unbounded potentials, periodic potentials, and even complex potentials. Understanding the impact of these diverse potential profiles on the spectrum is very important for unraveling the intricacies of quantum systems in discrete settings.

The spectrum theory encompasses the study of various spectral properties, such as the existence of band gaps, spectral gaps, and the presence of absolutely continuous, singular continuous, or discrete spectra. These properties shed light on the system's spectral structure, revealing essential information about its stability, resonances, and localization properties.

In addition to the spectral analysis, the estimation of Green's functions associated with discrete Schrödinger operators is a crucial topic within this theory. Green's functions provide insights into the propagator behavior, which describes the evolution of quantum states in time.

Numerous research works have contributed to the development and advancement of the spectrum theory of discrete Schrödinger operators. Seminal works by Kirsch and Simon [2], and Remling [3] have provided significant insights into the spectral analysis of discrete Schrödinger operators. Additionally, the monographs by Teschl [4] and Simon [5] offer comprehensive treatments of the subject, covering various aspects of the spectrum theory and its applications.

The spectrum theory of the discrete Schrödinger operators has been extensively applied in research on nonlinear discrete Schrödinger equations, including the investigation into the existence of standing waves(see [26–30]). This paper is organized as follows:

- In section 1 we introduce some basic results on the spaces of sequences;
- In section 2 we study the basic spectrum theorem of the discrete Schrödinger operators with bounded, unbounded or complex potentials;
- In section 3 we provide exponential estimates of Green's function and eigenfunctions of the discrete Schrödinger operators;
- In section 4 we investigate the spectrum structure of discrete Schrödinger operators with periodic potentials;
- In section 5 we review some results on standing wave solutions of discrete Schrödinger equations as an application of the spectrum theory.

1.1. Spaces of Sequences

In this paper, we focus solely on real or complex-valued sequences that are involved in our research on discrete Schrödinger equations. For a more comprehensive understanding of Banach sequences, we recommend referring to classical functional analysis books such as [6, 7, 11, 13], as well as [16].

Let $\mathbb{K}$ be the real ($\mathbb{R}$) or complex field ($\mathbb{C}$), $\mathbb{Z}$ be the set of integers and d be a positive integer. Let $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$ and

$$|n| = \sum_{1\le i\le d} |n_i|\,.$$

Any function from $\mathbb{Z}^d$ to $\mathbb{K}$ is called a sequence. We denote the set of all sequence by $l(\mathbb{Z}^d)$ and the set of finitely supported sequence bon discrete Schrödinger equation. For more details on Banach sequences we refer readers to classical functional analysis books such as [6, 7, 11, 13] as well as to [16].

Let $\mathbb{K}$ be the real ($\mathbb{R}$) or complex field ($\mathbb{C}$), $\mathbb{Z}$ be the set of integers and d be a positive integer. Let $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$ and

$$|n| = \sum_{1 \le i \le d} |n_i| \, .$$

Any function from $\mathbb{Z}^d$ to $\mathbb{K}$ is called a sequence. We denote the set of all sequence by $l(\mathbb{Z}^d)$ and the set of finitely supported sequence by

$$l_0(\mathbb{Z}^d) = \{u | u : \mathbb{Z}^d \to \mathbb{K}, u = \{u(n)\}, u(n) = 0, \text{ for all but finitely many } n\},$$

Obviously, these are vector spaces with respect to standard operations.

We define some Banach sequence spaces as follows:

- $c_0(\mathbb{Z}^d) = \{u | u : \mathbb{Z}^d \to \mathbb{K}, u = \{u(n)\}, \lim_{|n| \to \infty} |u(n)| = 0\}$,
- $l^p(\mathbb{Z}^d) = \{u | u : \mathbb{Z}^d \to \mathbb{K}, u = \{u(n)\}, \sum_{n \in \mathbb{Z}^d} |u(n)|^p < \infty\}, \ 1 \le p < \infty$,
- $l^\infty(\mathbb{Z}^d) = \{u | u : \mathbb{Z}^d \to \mathbb{K}, u = \{u(n)\}, \sup_{n \in \mathbb{Z}^d} |u(n)| < \infty\}$.

It is well known that these sequence spaces are Banach spaces when equipped with the following norms:

- $\|u\|_\infty = \sup_{n \in \mathbb{Z}^d} |u(n)|$, for $u \in c_0(\mathbb{Z}^d)$,
- $\|u\|_p = (\sum_{n \in \mathbb{Z}^d} |u(n)|^p)^{1/p}$, for $u \in l^p(\mathbb{Z}^d)$ and $1 \le p < \infty$,
- $\|u\|_\infty = \sup_{n \in \mathbb{Z}^d} |u(n)|$, for $u \in l^\infty(\mathbb{Z}^d)$.

Furthermore, $l^2(\mathbb{Z})$ is a Hilbert space with the inner product

$$(u, v) = \sum_{n \in \mathbb{Z}^d} u(n) \overline{v(n)} \, ,$$

where as usual $\bar{a}$ stands for the complex conjugate of $a \in \mathbb{C}$.

The following embeddings hold:

If $1 \le p_1 < p_2 \le \infty$, then $\|u\|_{p_2} \le \|u\|_{p_1}$, for all $u \in l^{p_1}(\mathbb{Z}^d)$; therefore, we have

$$l^{p_1}(\mathbb{Z}^d) \subset l^{p_2}(\mathbb{Z}^d).$$

These embeddings are dense if $p_2 < \infty$.

It is easy to see for all $1 \le p < \infty$ we have

$$l^p(\mathbb{Z}^d) \subset c_0(\mathbb{Z}^d) \subset l^\infty(\mathbb{Z}^d)$$

The representation of the dual spaces is entirely analogous to the classical result that is listed as follows:

- $c_0(\mathbb{Z}^d)^* = l^1(\mathbb{Z}^d)$,
- $l^p(\mathbb{Z}^d)^* = l^q(\mathbb{Z}^d)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,
- $l^1(\mathbb{Z}^d)^* = l^\infty(\mathbb{Z}^d)$.

A Banach space is said to be reflexive if the dual space of its dual space is isomorphic to itself under the canonical embedding. From the representation of dual space of $l^p(\mathbb{Z}^d)$ we know that $l^p(\mathbb{Z}^d)$ is reflexive if $1 < p < \infty$ and $c_0(\mathbb{Z}^d), l^1(\mathbb{Z}^d)$ and $l^\infty(\mathbb{Z}^d)$ are nonreflexive.

Assume that v_k is a sequence of elements of $l^p(\mathbb{Z}^d)$ and $v \in l^p(\mathbb{Z}^d)$, $1 \le p \le \infty$. Let $l^q(\mathbb{Z}^d)$ be the dual space of $l^p(\mathbb{Z}^d)$, then we have $\frac{1}{p} + \frac{1}{q} = 1$ and $q = \infty$ if $p = 1$. Therefore we can define the sequence convergence as follows:

(i) v_k is norm (or strongly) convergent to v in $l^p(\mathbb{Z}^d)$, denoted by $v_k \to v$, if $\lim_{k\to\infty} \|v_k - v\|_p = 0$, for $1 \le p \le \infty$;

(ii) for $1 \le p < \infty$, v_k is weakly convergent to v in $l^p(\mathbb{Z}^d)$, denoted by $v_k \rightharpoonup v$, if for all $u \in l^q(\mathbb{Z}^d)$

$$\lim_{k\to\infty} \sum_{n\in\mathbb{Z}^d} (v_k(n) - v(n))u(n) = 0.$$

(iii) for $1 < p \le \infty$, v_k is weakly* convergent to v in $l^p(\mathbb{Z}^d) = l^q(\mathbb{Z}^d)^*$, denoted by $v_k \to^{w^*} v$, if for all $u \in l^q(\mathbb{Z}^d)$

$$\lim_{k\to\infty} \sum_{n\in\mathbb{Z}^d} (v_k(n) - v(n))u(n) = 0.$$

Some important theorems are introduced here without proof([16]).

Theorem 1.1. *If $1 \le p < \infty$ and v_k is a sequence of elements of $l^p(\mathbb{Z}^d)$ and $v \in l^p(\mathbb{Z}^d)$, then $v_k \to v$ if and only if:*

(i) $\lim_{k\to\infty} |v_k(n) - v(n)| = 0$ *for all $n \in \mathbb{Z}^d$;*

(ii) $\lim_{k\to\infty} \|v_k\|_p = \|v\|_p$.

Theorem 1.2. *If $1 \le p < \infty$ and v_k is a sequence of elements of $l^p(\mathbb{Z}^d)$ and $v \in l^p(\mathbb{Z}^d)$, then $v_k \rightharpoonup v$ if and only if:*

(i) $\lim_{k\to\infty} |v_k(n) - v(n)| = 0$ *for all $n \in \mathbb{Z}^d$;*

(ii) *there exists an $M > 0$ such that $(\sum_{n\in\mathbb{Z}^d} |u_k(n)|^p)^{1/p} \le M$ for all $k \ge 1$.*

Theorem 1.3. *If $1 < p \le \infty$ and v_k is a sequence of elements of $l^p(\mathbb{Z}^d)$ and $v \in l^p(\mathbb{Z}^d)$, then $v_k \to^{w^*} v$ if and only if:*

(i) $\lim_{k\to\infty} |v_k(n) - v(n)| = 0$ *for all $n \in \mathbb{Z}^d$;*

(ii) *there exists an $M > 0$ such that $(\sum_{n\in\mathbb{Z}^d} |u_k(n)|^p)^{1/p} \le M$ for all $k \ge 1$.*

Recall that a Banach space E has the Radon-Riesz property if and only if the following statement is true: if v_k is a sequence in E and $v \in E$ such that $v_k \rightharpoonup v$ and $\|v_k\| \to \|v\|$, then $\|v_k - v\| \to 0$.

Theorem 1.4. *If $1 \le p < \infty$, then $l^p(\mathbb{Z}^d)$ has the Radon-Riesz property, that is, $v_k \to v$ if and only if:*

(i) $v_k \rightharpoonup v$,

(ii) $\|v_k\|_p \to \|v\|_p$.

The next result gives a criterion for compactness of a subset $K \subset l^p(\mathbb{Z}^d), 1 \le p < \infty$, and is completely similar to the classical theorem ([17]).

Theorem 1.5. *If $1 \le p < \infty$, then $K \subset l^p(\mathbb{Z}^d)$ is compact if and only if*

(i) *K is closed and bounded,*

(ii)*given any $\varepsilon > 0$, there exist a positive integer $N = N(\varepsilon)$ (depending only on ε) such that $(\sum_{|n|>N} |u(n)|^p)^{1/p} < \varepsilon$ for all $u \in K$.*

A subset K of a Banach space E is said to be weakly sequentially compact if and only if every sequence in K contains a subsequence that converges weakly to a point in E.

Theorem 1.6. (i) *If $1 < p < \infty$, then $K \subset l^p(\mathbb{Z}^d)$ is weakly sequentially compact if and only if K is bounded;*

(ii) *$K \subset l^1(\mathbb{Z}^d)$ is weakly sequentially compact if and only if K is strongly conditionally compact.*

1.2. Some Operators in Spaces of Sequences

We introduce the canonical basis $\{\mathbf{e}_i : i = 1, \cdots, d\}$ of the free Abelian group $\mathbb{Z}^d$ as follows:

$$\mathbf{e}_1 = (1, 0, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e}_d = (0, 0, 0, \cdots, 1).$$

For $m \in \mathbb{Z}^d$ we define the translation operator on $l(\mathbb{Z}^d)$, denoted by T_m, as follows $(T_m u)(n) = u(n - m)$. In particular, we obtain the frequently used right shift operator $S_i = T_{\mathbf{e}_i}$ and left shift operator $T_i = T_{-\mathbf{e}_i}$ on $l(\mathbb{Z}^d)$ as follows

$$(S_i u)(n) = u(n - \mathbf{e}_i), \quad (T_i u)(n) = u(n + \mathbf{e}_i), \quad \forall i = 1, \cdots, d.$$

Obviously, translations are linear operators.

We define the forward partial difference ($\nabla_i^+ = T_i - I$) and backward partial difference ($\nabla_i^- = I - S_i$) as follows

$$(\nabla_i^+ u)(n) = u(n + \mathbf{e}_i) - u(n), \quad (\nabla_i^- u)(n) = u(n) - u(n - \mathbf{e}_i).$$

$T_i S_i = S_i T_i = I$ implies

$$\nabla_i^+ \nabla_i^- = \nabla_i^- \nabla_i^+ = S_i + T_i - 2I.$$

Operators T_i and S_i act as isometric operators in $l^p(\mathbb{Z}^d)$, $1 \le p \le \infty$. As consequence, difference operators ∇_i^+ and ∇_i^- are bounded linear operators in all $l^p(\mathbb{Z}^d)$, $1 \le p \le \infty$.

The following proposition is the analogue (or discrete version) of the product rule of derivative and its proof is straightforward.

Proposition 1.7. *For any $u, v \in l(\mathbb{Z}^d)$*

$$\nabla_i^+(uv) = u\nabla_i^+ v + T_i v \nabla_i^+ u$$

and

$$\nabla_i^-(uv) = u\nabla_i^- v + S_i v \nabla_i^- u\,.$$

Making use of elementary identities

$$S_i \nabla_i^+ = \nabla_i^+ S_i = \nabla_i^-$$

and

$$T_i \nabla_i^- = \nabla_i^- T_i = \nabla_i^+\,,$$

we obtain the following statement.

Corollary 1.8. *If $u \in l(\mathbb{Z}^d)$ and $v \in l(\mathbb{Z}^d)$, then*

$$\begin{aligned}\nabla_i^- \nabla_i^+(uv) &= u\nabla_i^- \nabla_i^+ v + (\nabla_i^- u)(\nabla_i^- v + \nabla_i^+ v) \\ &+ (\nabla_i^- \nabla_i^+ u)(T_i v)\end{aligned}$$

and

$$\begin{aligned}\nabla_i^+ \nabla_i^-(uv) &= u\nabla_i^+ \nabla_i^- v + (\nabla_i^+ u)(\nabla_i^- v + \nabla_i^+ v) \\ &+ (\nabla_i^+ \nabla_i^- u)(S_i v)\,.\end{aligned}$$

If $d = 1$, then the classical Abel's summation by parts formula reads

$$\sum_{n=k}^{m} u(n)(\nabla^+ v)(n) = u(m)v(m+1) - u(k-1)v(k) - \sum_{n=k}^{m} (\nabla^- u)(n)v(n)$$

(here we skip the index in the notation of difference operators). The formula can be extended to the case $d > 1$ but we do not use such an extension in the following. We only need the following particular case.

Proposition 1.9. *Assume that either $u \in l(\mathbb{Z}^d)$ and $v \in l_0(\mathbb{Z}^d)$, or $u \in l_0(\mathbb{Z}^d)$ and $v \in l(\mathbb{Z}^d)$. Then*

$$\sum_{n\in\mathbb{Z}^d} u(n)(\nabla_i^+ v)(n) = -\sum_{n\in\mathbb{Z}^d} (\nabla_i^- u)(n)v(n)$$

for all $i = 1, \ldots, d$.

Corollary 1.10. *In the space $l^2(\mathbb{Z}^d)$ operators ∇_i^+ and ∇_i^- are mutually skew-adjoint,* i.e.*,*

$$(\nabla_i^+)^* = -\nabla_i^-$$

for all $i = 1, \ldots, d$.

2. Discrete Schrödinger operators

From now on all sequence spaces are supposed to be complex valued, and we drop $\mathbb{C}$ in the notation of spaces. The norm and inner product in $l^2(\mathbb{Z}^d)$ are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. We use the standard notation $\sigma(A)$ and $\rho(A) = \mathbb{C}\setminus\sigma(A)$ for the spectrum and resolvent set of a linear operator A, respectively.

2.1. Discrete Laplacian

The discrete Laplacian $-\Delta$ on $\mathbb{Z}^d$ is defined by

$$\begin{aligned} -\Delta &= -\nabla^-\cdot\nabla^+ \\ &= -\nabla^+\cdot\nabla^- \\ &= -\sum_{i=1}^d \nabla_i^-\nabla_i^+ . \end{aligned}$$

Here the second equality follows from the fact that operators ∇_j^+ and ∇_j^-, $j = 1, \ldots, d$, commutes. In more details, for any $u \in l(\mathbb{Z}^d)$

$$(-\Delta u)(n) = \sum_{|m-n|=1} u(m) - du(n) .$$

This is a linear operator in the space $l(\mathbb{Z}^d)$. It is easily seen that $-\Delta$ leaves the space $l_0(\mathbb{Z}^d)$ invariant, and acts a bounded linear operator in all spaces $l^p(\mathbb{Z}^d)$, $p \in [1,\infty]$.

The following proposition follows immediately from the summation by parts formula.

Proposition 2.1. *The operator $-\Delta$ is a bounded, self-adjoint operator in $l^2(\mathbb{Z}^d)$. Furthermore, $-\Delta$ is a nonnegative operator,* i.e.*,*

$$(-\Delta u, u) \geq 0\,, \quad u \in l^2(\mathbb{Z}^d)\,.$$

Proposition 2.2. *The spectrum $\sigma(-\Delta)$ is purely continuous and coincides with* $[0, 4d]$.

Proof For any $u \in l^2(\mathbb{Z}^d)$ we consider its Fourier transform

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{d/2}} \sum_{n\in\mathbb{Z}^d} u(n)\exp(2\pi\xi\cdot n)\,.$$

Then $\hat{u} \in L^2([-\pi,\pi])$. By Parseval's theorem, $\|\hat{u}\|_{L^2} = \|u\|$, and the map $u \mapsto \hat{u}$ is an isometric isomorphism between $l^2(\mathbb{Z}^d)$ and $L^2([-\pi,\pi])$. A straightforward calculation shows that

$$-\hat{\Delta} u(\xi) = a(\xi)\hat{u}(\xi)\,,$$

where

$$a(\xi) = 2\sum_{i=1}^{d}(\cos\xi_i - 1)\,,$$

i.e., $-\Delta$ is unitary equivalent to the multiplication operator by $a(\xi)$ in $L^2([-\pi,\pi])$ and, therefore, the spectra of these two operators coincide. It is easily seen that the spectrum of multiplication operator by $a(\xi)$ is precisely the range of $a(\xi)$ which is equal to $[0, 4d]$. The proof is complete.

2.2. Self-adjoint Discrete Schrödinger Operator

Let $V \in l(\mathbb{Z}^d)$ be a real sequence. We associate with V the multiplication operator by V. In what follows we do not distinguish notationally between the sequence V and the associated multiplication operator. Such operators can be considered as, generally, unbounded operators in various sequence spaces. The most important case for us is the l^2 case. More precisely, the multiplication operator by V in $l^2(\mathbb{Z}^d)$ is defined on the domain

$$D(V) = \{u \in l^2(\mathbb{Z}^d) : Vu \in l^2(\mathbb{Z}^d)\}\,.$$

Obviously, $D(V)$ is dense in $l^2(\mathbb{Z}^d)$. As a diagonal operator, the operator V is self-adjoint. It is easily seen that V is a bounded operator and $D(V) = l^2(\mathbb{Z}^d)$ if and only if $V \in l^\infty(\mathbb{Z}^d)$.

The discrete Schrödinger operator with potential $V \in l(\mathbb{Z}^d)$ is defined by

$$L = -\Delta + V\,,$$

where V is regarded as the operator of multiplication by V. Mainly we consider L as an operator in the basic Hilbert space $l^2(\mathbb{Z}^d)$ though time by time we shall need to study its action in other spaces. Note that both $-\Delta$ and V are self-adjoint operators in $l^2(\mathbb{Z}^d)$, and the first one is bounded. Therefore, the classical result on the sum of self-adjoint operators in its simplest form immediately yields the following statement.

Proposition 2.3. *The Schrödinger operator L is a self-adjoint operator in $l^2(\mathbb{Z}^d)$ with the domain $D(L) = D(V)$. In particular, L is bounded if and only if the sequence V is bounded.*

If L (equivalently, V) is unbounded, we equip $D(L) = D(V)$ with the graph norm. It is convenient to use the graph norm associated with V

$$\|u\|_L = (\|u\|^2 + \|Vu\|^2)^{1/2}\,, \quad u \in D(L)\,, \tag{2.1}$$

Then the domain becomes a Hilbert space with inner product

$$(u,v)_L = (u,v) + (Vu, Vv)\,, \quad u \in D(L)\,, v \in D(L)\,. \tag{2.2}$$

Notice that the embedding $D(L) \in l^2(\mathbb{Z}^d)$ is continuous and dense. Furthermore, for every $\lambda \in \rho(L)$ the operator $(L-\lambda I)^{-1}$ maps $l^2(\mathbb{Z}^d)$ onto $D(L)$ isomorphically. We say that the operator L has *compact resolvent* if for some (hence, for all) $\lambda \in \rho(L)$ the operator $(L-\lambda I)^{-1}$ is compact. Equivalently, this means that the embedding $D(L) \subset l^2(\mathbb{Z}^d)$ is compact. Also the compactness of resolvent is equivalent to the property that the spectrum $\sigma(L)$ is purely discrete, *i.e.*, consists of countably many eigenvalues of finite multiplicity with the only accumulation point at infinity. These results have been applied in our research on standing waves of nonlinear discrete Schrödinger equations with unbounded potential (see [26–30]).

Theorem 2.4. *The spectrum of L is purely discrete if and only if $|V(n)| \to \infty$ as $|n| \to \infty$.*

Proof Due to the second resolvent identity,

$$(L-iI)^{-1} - (V-iI)^{-1} = (L-iI)^{-1}\Delta(V-iI)^{-1}\,.$$

This implies immediately that L has compact resolvent if and only if so does V.

Assume that $|V(n)| \to \infty$ as $|n| \to \infty$ and prove that the embedding $D(V) \subset l^2(\mathbb{Z}^d)$ is compact. With this aim it is enough to show that the set

$$\begin{aligned} B &= \{u \in l^2(\mathbb{Z}^d) : \|u\|^2 + \|Vu\|^2 \leq 1\} \\ &= \{u \in l^2(\mathbb{Z}^d) : \sum_{n \in \mathbb{Z}^d} (1 + |V(n)|^2)|u(n)|^2 \leq 1\} \end{aligned}$$

is precompact in $l^2(\mathbb{Z}^d)$. For any $\varepsilon > 0$ there exists $N > 0$ such that

$$1 + |V(n)|^2 \geq \varepsilon^{-1}$$

whenever $|n| \geq N$. Then

$$\sum_{|n| \geq N} |u(n)|^2 \leq \varepsilon \sum_{|n| \geq N} (1 + |V(n)|^2)|u(n)|^2 \leq \varepsilon \,.$$

Since B is obviously bounded in $l^2(\mathbb{Z}^d)$, Theorem 1.5 implies that B is precompact in $l^2(\mathbb{Z}^d)$.

Now we prove that the compactness of embedding $D(L) \subset l^2(\mathbb{Z}^d)$ implies that $|V(n)| \to \infty$ as $|n| \to \infty$. Assuming the contrary, we see that there exists an infinite set $S \subset \mathbb{Z}^d$ such that V is bounded on S. Then on the subspace

$$\{u \in l^2(\mathbb{Z}^d) : u(n) = 0 \quad \forall n \notin S\} \subset D(L)$$

the l^2-norm and the graph norm are equivalent, and, therefore, the embedding $D(L) \subset l^2(\mathbb{Z}^d)$ is not compact.

The proof is complete.

Remark 2.5. *If* $d = 1$*, then all isolated eigenvalues of* L *are simple.*

Assume now that the potential V is bounded below, say,

$$V(n) \geq \alpha \,, \quad n \in \mathbb{Z}^d \,,$$

for some $\alpha \in \mathbb{R}$. Then the operator L is semi-bounded below, *i.e.*,

$$(Lu, u) \geq \alpha \|u\|^2 \,, \quad u \in D(L) \,.$$

In this case the associated sesquilinear and quadratic forms have explicit representations

$$\begin{aligned} q_L(u, v) &= (\nabla^+ u, \nabla^+ v) + (Vu, v) \\ &= (\nabla^- u, \nabla^- v) + (Vu, v) \,. \end{aligned}$$

and

$$\begin{aligned} q_L(u) &= \|\nabla^+ u\|^2 + \|Vu\|^2 \\ &= \|\nabla^- u\|^2 + \|Vu\|^2 \,. \end{aligned}$$

We remind that $q_L(u) = q_L(u, u)$.

The domain $D(q_L)$, *i.e.* the form domain, or energy space $E = E_L$ of L, is a Hilbert space with the inner product

$$(u, v)_E = q_L(u, v) + C(u, v) \,,$$

where C is large enough. Notice that all these inner products are equivalent. If the operator L is positive definite, *i.e.* $\alpha > 0$, the most natural inner product is $(\cdot, \cdot)_E = q_L(\cdot, \cdot)$. Also we note that E consists of all $u \in l^2(\mathbb{Z}^d)$ such that $|V|^{1/2} u \in l^2(\mathbb{Z}^d)$.

Making use of the arguments similar to those in the proof of Theorem 2.4, we obtain the following proposition.

Proposition 2.6. *Assume that the potential* V *is bounded below. Then the following statements are equivalent.*

(i) *The embedding* $D(q_L) \subset l^2(\mathbb{Z}^d)$ *is compact.*

(ii) $V(n) \to +\infty$ *as* $|n| \to \infty$.

2.3. Dissipative Discrete Schrödinger Operator

First we remind some general results (see, *e.g.*, [20, 23] and, in the case of operators in real Hilbert spaces, [8]).

Let A be a linear operator in a Hilbert space H, with domain $D(A)$. The operator A is said to be *dissipative* if

$$\mathrm{Re}(Au,u) \le 0$$

for all $u \in D(A)$. It is called *m-dissipative* if, in addition, A is closed and the range $R(A-\lambda_0 I)$ is dense in H for some $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}\lambda_0 > 0$.

Proposition 2.7. *Let A be a closed, dissipative operator. Then the following statements are equivalent:*

(a) *A is m-dissipative;*

(b) *there exists $\lambda_0 \in \mathbb{C}$, with $\mathrm{Re}\lambda_0 > 0$, such that $\lambda_0 \in \rho(A)$;*

(c) *$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\} \subset \rho(A)$;*

(d) *the domain $D(A)$ is dense in H and A^* is m-dissipative.*

If A is m-dissipative, then

$$\|(A-\lambda I)^{-1}\| \le (\mathrm{Re}\lambda)^{-1}$$

whenever $\mathrm{Re}\lambda > 0$.

Proposition 2.8. *A linear operator A in H is m-dissipative if and only if its domain $D(A)$ is dense in H, A is closed, $(0,\infty) \subset \rho(A)$, and*

$$\|(A-\lambda I)^{-1}\| \le \lambda^{-1}$$

for all $\lambda > 0$.

Proposition 2.9. *Let A be a densely defined closed linear operator in H. If both A and A^* are dissipative, then A is m-dissipative.*

Remark 2.10. *A linear, dissipative operator A in H is called* maximal dissipative *if for any disipative operator $\tilde{A}$ such that $D(A) \subset D(\tilde{A})$ and $\tilde{A}_{|D(A)} = A$ we have $D(\tilde{A}) = D(A)$ and, hence, $\tilde{A} = A$. In other words, A is maximal dissipative if it has no proper dissipative extensions. In fact, the classes of m-dissipative and maximal dissipative operators coincide (see,* e.g.*, [8]). Thus the term 'm-dissipative' is an abbreviation for the term 'maximal dissipative'.*

Now we consider the discrete Schrödinger operator with complex potential $V \in l(\mathbb{Z}^d)$. We keep the notation V for the operator of multiplication by the sequence V acting in $l^2(\mathbb{Z}^d)$, with the domain

$$D(V) = \{u \in l^2(\mathbb{Z}^d) : Vu \in l^2(\mathbb{Z}^d)\}\,.$$

As in Subsection 2.2, this is a closed linear operator, and it is bounded if and only if $V \in l^\infty(\mathbb{Z}^d)$. Since the operator V is diagonal, for its adjoint operator we have that $V^* = \bar{V}$, where $\bar{V}$ is the complex conjugate of V, and $D(V^*) = D(\bar{V}) = D(V)$.

As usual, the Schrödinger operator with complex potential V is defined by

$$Lu = -\Delta u + Vu\,, \quad u \in D(L)\,,$$

with the domain

$$D(L) = D(V)\,.$$

Since Δ is a bounded operator, the operator L is closed, $D(L^*) = D(L)$, and

$$L^*u = -\Delta u + \bar{V}u\,, \quad D(L)\,.$$

Proposition 2.11. *Assume that* $\mathrm{Im} V(n) \geq 0$ *for all* $n \in \mathbb{Z}^d$*. Then the operator* $\mathrm{i}L$ *is m-dissipative.*

Proof Let $V = V_0 + \mathrm{i}V_1$. Due to Proposition 2.9, it is enough to show that both $\mathrm{i}L$ and $-\mathrm{i}L^*$ are dissipative. Since $|V_0(n)| \leq |V(n)|$ and $|V_1(n)| \leq |V(n)|$ for all $n \in \mathbb{Z}^d$, we have $D(V) \subset D(V_0)$ and $D(V) \subset D(V_1)$. Then, for all $u \in D(L)$,

$$(\mathrm{i}Lu, u) = -\mathrm{i}(-\Delta u, u) + \mathrm{i}(V_0 u, u) - (V_1 u, u)\,,$$

and, by the assumption of proposition, $\mathrm{Re}(\mathrm{i}Lu, u) \leq 0$ for all $u \in D(L)$. Thus, $\mathrm{i}L$ is dissipative. The dissipativity of $-\mathrm{i}L^*$ follows similarly.

3. Exponential Estimates

In this section we consider Green's function of discrete Schrödinger operator and eigenfunctions with isolated eigenvalues of finite multiplicity.

Let $\{\delta_k\}_{k\in\mathbb{Z}^d}$ be the standard orthonormal basis in $l^2(\mathbb{Z}^d)$, *i.e.*,

$$\delta_k(n) = \begin{cases} 1\,, & n = k\,, \\ 0\,, & n \neq k\,. \end{cases}$$

For any $\lambda \in \rho(L)$ we define Green's function $G(n, k; \lambda)$ by

$$G(n, k; \lambda) = ((L - \lambda I)^{-1}\delta_k, \delta_n)\,, \quad k, n \in \mathbb{Z}^d\,.$$

The following symmetry identities are straightforward:

$$G(k, n; \lambda) = G(n, k; \lambda)$$

and

$$G(n, k; \bar{\lambda}) = \overline{G(n, k; \lambda)}$$

for all $n \in \mathbb{Z}^d$, $k \in \mathbb{Z}^d$ and $\lambda \in \rho(L)$.

The main result on Green's function is the following theorem.

Theorem 3.1. *Let* K *be a compact subset of* $\rho(L)$*. There exist constants* $C = C_K > 0$ *and* $\alpha = \alpha_K > 0$ *such that*

$$|G(k, n; \lambda)| \leq C \exp(-\alpha|n - k|) \tag{3.1}$$

for all $n \in \mathbb{Z}^d$*,* $k \in \mathbb{Z}^d$ *and* $\lambda \in K$*.*

As consequence, we obtain the following representation of resolvent.

Proposition 3.2. *If* $\lambda \in \rho(L)$*, then for all* $f \in l^2(\mathbb{Z}^d)$

$$((L - \lambda I)^{-1} f)(n) = \sum_{k\in\mathbb{Z}^d} G(n, k; \lambda) f(k)\,. \tag{3.2}$$

Furthermore, the right-hand side of (3.2) converges for $f \in l^p(\mathbb{Z}^d)$*, and defines a bounded linear operator in* $l^p(\mathbb{Z}^d)$ *for all* $p \in [1, \infty]$*.*

Remark 3.3. *By Proposition 3.2, the resolvent* $(L - \lambda I)^{-1}$*,* $\lambda \in \rho(L)$*, extends to a bounded linear operator in* $l^p(\mathbb{Z}^d)$ *for all* $p \in [1, \infty]$*. Actually, the operator* L *can be considered as a closed, in general unbounded, linear operator in* $l^p(\mathbb{Z}^d)$*,* $p \in [1, \infty]$*. The resolvent set of such extension contains* $\rho(L)$*, and the resolvent of extension is given by the right-hand side of (3.2) for* $\lambda \in \rho(L)$*. In fact, one can show that the spectrum of* L *considered as an operator in* $l^p(\mathbb{Z}^d)$*,* $p \in [1, \infty]$*, is independent of* p *but we do not use this result.*

As the first step toward the proof of Theorem 3.1 we introduce the action of discrete Schrödinger operators in certain weighted l^2 spaces. Let

$$\varphi_{\varepsilon,k}(n) = e^{-\varepsilon|n-k|}, \quad n \in \mathbb{Z}^d,$$

and let

$$l^2_{\varepsilon,k}(\mathbb{Z}^d) = \{u \in l(\mathbb{Z}^d) : \varphi_{\varepsilon,k}u \in l^2(\mathbb{Z}^d)\},$$

where $\varepsilon \in \mathbb{R}$. Endowed with the norm $\|u\|_{\varepsilon,k} = \|\varphi_{\varepsilon,k}u\|$, this is a Banach (actually, Hilbert) space. We denote by $\Phi_{\varepsilon,k}$ the multiplication operator by $\varphi_{\varepsilon,k}$

$$\Phi_{\varepsilon,k}u = \varphi_{\varepsilon,k}u\ .$$

Then $\Phi_{\varepsilon,k}$ maps $l^2_{\varepsilon,k}(\mathbb{Z}^d)$ onto $l^2(\mathbb{Z}^d)$ isometrically, and the inverse operator

$$\Phi^{-1}_{\varepsilon,k} : l^2(\mathbb{Z}^d) \to l^2_{\varepsilon,k}(\mathbb{Z}^d)$$

is represented by $\Phi_{-\varepsilon,k}$.

Now we introduce the operator $L_{\varepsilon,k}$ in $l^2_{\varepsilon,k}(\mathbb{Z}^d)$ as follows. Its domain $D(L_{\varepsilon,k})$ is given by

$$D(L_{\varepsilon,k}) = \Phi^{-1}_{\varepsilon,k}D(L) = \Phi^{-1}_{\varepsilon,k}D(V),$$

and the action of $L_{\varepsilon,k}$ is given by

$$L_{\varepsilon,k}u = -\Delta u + Vu$$

for all $u \in D(L_{\varepsilon,k})$. It is easily seen that $L_{\varepsilon,k}$ is a closed linear operator in the space $l^2_{\varepsilon,k}(\mathbb{Z}^d)$. Notice that it is bounded if and only if the potential V is bounded. The operator $L_{\varepsilon,k}$ is isometrically equivalent to the following operator

$$L^{\varepsilon,k} = \Phi_{\varepsilon,k}L_{\varepsilon,k}\Phi_{-\varepsilon,k}$$

in the space $l^2(\mathbb{Z}^d)$. Its domain coincides with $D(L)$.

In the notation just introduced we suppress k whenever $k = 0$.

Lemma 3.4. *Let K be a compact subset of $\rho(L)$. Then there exists a constant $\varepsilon_0 > 0$ such that for every $\lambda \in K$ the operator $L_{\varepsilon,k} - \lambda I$ has a bounded inverse operator for all $k \in \mathbb{Z}^d$ and all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Furthermore, the norm of $(L_{\varepsilon,k} - \lambda I)^{-1}$ is bounded above by a constant independent of $\lambda \in K$, $k \in \mathbb{Z}^d$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

Proof Since operators $L_{\varepsilon,k}$ and $L^{\varepsilon,k}$ are isometrically equivalent, it is enough to prove the statement with $L_{\varepsilon,k}$ replaced by $L^{\varepsilon,k}$.

Making use of Corollary 1.8, we have

$$L^{\varepsilon,k} = L + B_{\varepsilon,k},$$

where

$$B_{\varepsilon,k}u = -\sum_{i=1}^{d}[\varphi_{\varepsilon,k}(\nabla^-_i\varphi_{-\varepsilon,k})(\nabla^+_i + \nabla^-_i)u + \varphi_{\varepsilon,k}(\nabla^-_i\nabla^+_i\varphi_{-\varepsilon,k})T_iu].$$

We claim that $B_{\varepsilon,k}$ is a bounded linear operator in $l^2(\mathbb{Z}^d)$ and

$$\|B_{\varepsilon,k}\| = o(|\varepsilon|)$$

uniformly with respect to $k \in \mathbb{Z}^d$. Indeed, an elementary calculation shows that

$$\varphi_{\varepsilon,k}(n)(\nabla^-_i\varphi_{-\varepsilon,k})(n) = 1 - e^{\pm\varepsilon},$$

depending on whether $n_i - k_i > 0$ or not, and therefore is $o(|\varepsilon|)$. Similarly,

$$\varphi_{\varepsilon,k}(n)(\nabla^-_i\nabla^+_i\varphi_{-\varepsilon,k})(n) = o(\varepsilon^2)$$

uniformly with respect to $k \in \mathbb{Z}^d$.

Since the resolvent $(L - \lambda I)^{-1}$ is uniformly bounded as $\lambda \in K$, there exists $\varepsilon_0 > 0$ such that

$$\|B_{\varepsilon,k}\|\|(L - \lambda I)^{-1}\| \leq \alpha$$

for some $\alpha \in (0, 1)$. Then the operator

$$I + B_{\varepsilon,k}(L - \lambda I)^{-1}, \quad \lambda \in K,$$

is invertible in $l^2(\mathbb{Z}^d)$, and its inverse is uniformly bounded. Hence, the operator

$$L^{\varepsilon,k} - \lambda I = (I + B_{\varepsilon,k}(L - \lambda I)^{-1})(L - \lambda I)$$

has the inverse operator which is uniformly bounded if $\lambda \in K$.

The proof is complete.

Remark 3.5. *From the proof of Lemma 3.4 it is clear that* $(L^{\varepsilon,k} - \lambda I)^{-1}$ *depends continuously on* $(\lambda, \varepsilon) \in K \times [-\varepsilon_0, \varepsilon_0]$.

Proof of Theorem 3.1: Since

$$G(\cdot, k; \lambda) = (L_{(-\varepsilon_0,k)} - \lambda I)^{-1}\delta_k$$

and $\|\delta_k\|_{-\varepsilon_0,k} = 1$ we have, by Lemma 3.4,

$$\|G(\cdot, k; \lambda)\|^2_{-\varepsilon_0,k} = \sum_{n \in \mathbb{Z}^d} e^{2\varepsilon_0|n-k|}|G(n, k; \lambda)|^2 \leq \|(L_{(-\varepsilon_0,k)} - \lambda I)^{-1}\|^2\|\delta_k\|^2_{-\varepsilon_0,k} \leq C.$$

The result follows with $\alpha = \varepsilon_0$.

Proposition 3.6. *Assume that* $\sigma(L) = \Sigma_0 \cup \Sigma_1$, *where* Σ_0 *and* Σ_1 *are disjoint closed sets, and* Σ_0 *is bounded. Then the spectral projectors* P_0 *and* P_1 *that correspond to the spectral components* Σ_0 *and* Σ_1, *respectively, are continuous with respect to* l^p *norm for all* $p \in [1, \infty]$.

Proof Since $P_1 = I - P_0$, it suffice to prove l^p-continuity only for P_0. Let $\Gamma \subset \mathbb{C}$ be a smooth, closed, connected, counterclockwise oriented curve surrounding the set Σ_0 and such that $\Gamma \cap \Sigma_1 = \emptyset$. Then P_0 possesses the representation

$$P_0 = -\frac{1}{2\pi i}\int_\Gamma (L - \lambda I) d\lambda,$$

and the result follows from Proposition 3.2.

Now we turn to discrete eigenvalues.

Theorem 3.7. *Let* λ_0 *be an isolated eigenvalue of* L *with finite multiplicity, and* $u \in l^2(\mathbb{Z}^d)$ *be an associated eigenfunction. Then there exist constants* $\alpha > 0$ *and* $C > 0$ *such that*

$$|u(n)| \leq C \exp(-\alpha|n|), \quad n \in \mathbb{Z}^d.$$

Proof Let Γ be a circle centered at λ_0, counterclockwise oriented, and such that it does not intersect $\sigma(L)$. Then the eigenspace E of L that corresponds to the eigenvalue λ_0 is the image of the Riesz projector

$$P = -\frac{1}{2\pi i}\int_\Gamma (L - \lambda I)^{-1} d\lambda,$$

and $k = \dim E$ is the multiplicity of λ_0. By Remark 3.5, in a small neighborhood of $\varepsilon = 0$ the operator $(L^\varepsilon - \lambda I)^{-1}$ is a continuous function of ε and $\lambda \in \Gamma$. Hence, the Riesz projector

$$P^\varepsilon = -\frac{1}{2\pi i}\int_\Gamma (L^\varepsilon - \lambda I)^{-1} d\lambda$$

as a bounded operator in $l^2(\mathbb{Z}^d)$ depends continuously on ε in that neighborhood, and $\dim E^\varepsilon = k < \infty$ is independent of ε. Notice that $P^0 = P$ and $E^0 = E$.

As isometrically equivalent to L^ε, the operator L_ε has the same spectrum. Its Riesz projector P_ε that corresponds to the part of spectrum inside Γ is isometrically equivalent to P^ε. Indeed,

$$\begin{aligned}\Phi_\varepsilon P_\varepsilon \Phi_{-\varepsilon} &= -\frac{1}{2\pi i}\int_\Gamma \Phi_\varepsilon (L_\varepsilon - \lambda I)^{-1}\Phi_{-\varepsilon} d\lambda \\ &= -\frac{1}{2\pi i}\int_\Gamma (\Phi_\varepsilon (L - \lambda I)\Phi_{-\varepsilon})^{-1} d\lambda = P^\varepsilon .\end{aligned}$$

As consequence, the image E_ε of P_ε is isomorphic to E^ε. Since both spaces are finite dimensional, $\dim E_\varepsilon = k$.

If $\varepsilon = -\alpha < 0$, then

$$l^2_\varepsilon(\mathbb{Z}^d) \subset l^2(\mathbb{Z}^d),$$

$$D(L_\varepsilon) \subset D(L)$$

and the operator L_ε is the restriction of L to $D(L_\varepsilon)$. Therefore, the resolvent $(L_\varepsilon - \lambda I)^{-1}$ is the restriction of $(L - \lambda I)^{-1}$ to the space $l^2(\mathbb{Z}^d)$. Hence, the projector P_ε is the restriction of P, and $E_\varepsilon \subset E$. Since both these spaces have the same dimension k, we see that $E = E_\varepsilon \subset l^2(\mathbb{Z}^d)$. Thus, for any eigenfunction $u \in E$, we have

$$\exp(\alpha|\cdot|)u \in l^2(\mathbb{Z}^d) \subset l^\infty(\mathbb{Z}^d).$$

This yields immediately the required, and the proof is complete.

Corollary 3.8. *If $u \in l^2(\mathbb{Z}^d)$ is an eigenfunction of L associated to an isolated eigenvalue of finite multiplicity, then $u \in l^1(\mathbb{Z}^d)$.*

4. Periodic Discrete Schrödinger Operators

In this section we consider the Schrödinger operator with periodic potential. We fix $N = (N_1, \dots, N_d) \in \mathbb{Z}^d$ such that $N_i > 1$ for all $i = 1, \dots, d$. Assume that the potential V is N-periodic, *i.e.*,

$$V(n + N) = V(n), \quad n \in \mathbb{Z}^d.$$

Notice that in this case the operator

$$L = -\Delta + V$$

is a bounded self-adjoint operator in $l^2(\mathbb{Z}^d)$.

The *periodicity cell* $\square_N$ is defined by

$$\square_N = \{n \in \mathbb{Z}^d : 0 \le n_i \le N_i - 1, i = 1, \dots, d\}.$$

The cardinality of $\square_N$ is equal to

$$|\square_N| = N_1 N_2 \cdots N_d .$$

The *lattice of periods* G_N is the subgroup of $\mathbb{Z}^d$ generated by the vectors $N_i \mathbf{e}_i$, $i = 1, \dots, d$. We denote by G^*_N the dual lattice to G^*_N which consists of all vectors $\kappa \in \mathbb{R}^d$ such that $\kappa \cdot \gamma \in 2\pi\mathbb{Z}$ for all $\gamma \in G_N$. Here $\cdot$ stands for the usual dot product in $\mathbb{R}^d$. More explicitly, G_N^* is the subgroup of $\mathbb{R}^d$ generated by the vectors $2\pi N_i^{-1}\mathbf{e}_i$, $i = 1, \ldots, d$.

Recall that (unitary) *characters* of the group G_N, *i.e.*, group homomorphisms

$$G_N \to \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\},$$

are of the form

$$\chi_\xi(\gamma) = e^{i\xi\cdot\gamma}, \quad \gamma \in G_N,$$

where $\xi \in \mathbb{R}^d$. According to physics terminology, vectors ξ are called *quasi-momenta*. It is easily seen that

$$\chi_{\xi+\kappa} = \chi_\xi$$

for all $\kappa \in G_N^*$. Therefore, we can restrict the values of quasi-momenta to the set

$$B_N = \{\xi \in \mathbb{R}^d : -\frac{\pi}{N_i} < \xi_i \leq \frac{\pi}{N_i}, i = 1, \ldots, d\}.$$

In physics the set B_N is called *Brillouin zone*.

For a sequence $u \in l^2(\mathbb{Z}^d)$ we define its *Floquet transform* by

$$\hat{u}(n,\xi) = \frac{|\square_N|^{1/2}}{(2\pi)^{d/2}} \sum_{\gamma\in G_N} u(n+\gamma)e^{-i\xi\cdot\gamma}. \tag{4.1}$$

For any $n \in \mathbb{Z}^d$ the series in the right-hand side of (4.1) converges in the sense of $L^2(B_N)$, and $\hat{u}(n,\xi)$ is a G_N^*-periodic function with respect to ξ:

$$\hat{u}(n,\xi+\kappa) = \hat{u}(n,\xi), \quad n \in \mathbb{Z}^d,$$

for all $\kappa \in G_N^*$. Also it is easily seen that

$$\hat{u}(n+\gamma,\xi) = e^{i\xi\cdot\gamma}\hat{u}(n,\xi), \quad \gamma \in G_N. \tag{4.2}$$

As consequence, $\hat{u}(\cdot,\xi)$ is completely determined by its restriction to $\square_N$, and we can consider the Floquet transform of u as a function $\hat{u}(\xi)$ with values in the space F_N of complex functions on $\square_N$. We equip F_M with the standard inner product of l^2 type. The function u also can be considered as a function on G_N with values in F_N. In this context the Floquet transform becomes the Fourier transform for F_N-valued functions on the group G_N. Hence, the mapping $u \longmapsto \hat{u}$ is a unitary equivalence between $l^2(\mathbb{Z}^d)$ and $L^2(B_N; F_N)$, and we have the following inversion formula

$$u(\gamma+n) = \frac{|\square_N|^{1/2}}{(2\pi)^{d/2}} \int_{B_N} \hat{u}(n,\xi)e^{i\xi\cdot\gamma}d\xi, \quad \gamma \in G_N, n \in \square_N. \tag{4.3}$$

Now we look for a representation of operator L in terms of the Floquet transform. More precisely, let us define the operator $\hat{L}$ by

$$(\hat{L}\hat{u})(\xi) = \widehat{Lu}(\xi), \quad \xi \in B_N.$$

Proposition 4.1. *There exists a real analytic function $M(\xi)$, with values in the set of self-adjoint operators acting in the spaces F_N, such that $(\hat{L}\hat{u})(\xi) = M(\xi)\hat{u}(\xi)$.*

Proof We represent the operator $\hat{L}$ in the form

$$\hat{L} = -\sum_{j=1}^{d} \hat{\nabla}_j^- \hat{\nabla}_j^+ + \hat{V},$$

where

$$(\hat{\nabla}_j^{\pm}\hat{u})(\xi) = (\widehat{\nabla_j^{\pm}u})(\xi)$$

and

$$(\hat{V}\hat{u}) = (\widehat{Vu})(\xi)\,.$$

Making use of periodicity of V, we have

$$\frac{(2\pi)^{d/2}}{|\square|^{1/2}}(\widehat{Vu})(n,\xi) = \sum_{\gamma\in G_N} V(n+\gamma)u(n+\gamma) = V(n)\sum_{\gamma\in G_N} u(n+\gamma)\,,$$

i.e., $\hat{V}$ is the operator of multiplication by V, and does not depend on ξ. Straightforward calculations show that, for $j = 1, \ldots, d$,

$$(\hat{\nabla}_j^{+})u(n) = \begin{cases} u(n+\mathbf{e}_j) - u(n)\,, & n_j < N_j\,, \\ e^{iN_j\xi_j}u(n - N_j\mathbf{e}_j) - u(n)\,, & n_j = N_j\,, \end{cases}$$

and

$$(\hat{\nabla}_j^{-})u(n) = \begin{cases} u(n) - u(n-\mathbf{e}_j)\,, & n_j \geq 0\,, \\ u(n) - e^{-iN_j\xi_j}u(n + (N_j-1)\mathbf{e}_j)\,, & n_j = 0\,. \end{cases}$$

Since

$$\hat{L} = -\sum_{j=1}^{d}\hat{\nabla}_j^{-}\hat{\nabla}_j^{+} + \hat{V}\,,$$

the result follows.

Remark 4.2. *Notice that the matrix $M(\xi)$ is G_N^*-periodic in ξ.*

The following theorem provides an information about the spectrum of periodic discrete Schrödinger operator.

Theorem 4.3. *The spectrum of discrete Schrödinger operator with N-periodic potential is equal to the union of $|\square_N|$ bounded closed intervals B_k, $k = 1, \ldots, |\square_N|$.*

Proof For the sake of simplicity, we set $r = |\square_N|$. Let

$$\mu_1(\xi) \leq \mu_2(\xi) \leq \cdots \leq \mu_r(\xi)\,,$$

be the eigenvalues of the matrix $M(\xi)$. Due to Remark 4.2, the eigenvalues are G_N^*-periodic functions of ξ. By Proposition 4.1, $\lambda \in \sigma(L)$ if and only if $\lambda = \mu_k(\xi)$ for some $k = 1, \ldots, r$ and some $\xi \in \bar{B}_N$. Furthermore, the matrix $M(\xi)$ depends analytically on ξ. Perturbation theory of finite dimensional self-adjoint operators implies that the functions $\mu_k(\xi)$ are continuous and piece-wise analytic. Hence, the range of $\mu_k(\xi)$ is a bounded closed interval B_k, $k = 1, \ldots, r$, and the proof is complete.

The intervals B_k are called *spectral bands*. It may happen that some, or even all, bands are separated by open intervals free of spectrum. Such open intervals are called *spectral gaps*. Certainly, there are two infinite intervals free of spectrum, above and below $\sigma(L)$. Sometimes these intervals are also called (infinite) gaps. In physics literature the multi-valued function $\sigma(M(\xi))$ is called the *dispersion relation*.

A detailed discussion of the discrete Floquet theory in dimension $d = 1$ can be found in [24] (see also [18]). Notice, that the case $d > 1$ does not appear in the literature. The presentation in this section follows [10], where operators on periodic discrete and quantum graphs are considered (see also [14]). For the Floquet theory of ordinary and partial differential equations we refer to [9, 12, 15, 22].

5. Standing Wave Solutions

In this section, as an application of the spectrum theory, we review some results (in [30]) on the existence of nontrivial standing wave solution of the discrete nonlinear Schrödinger equation with the growing potential at infinity. We combine the variational method with Proposition 2.6 to demonstrate the existence of nontrivial standing wave solutions.
We consider the one-dimensional discrete nonlinear Schrödinger (DNLS) equation,

$$i\dot{\psi}_n + \Delta\psi_n - v_n\psi_n + \sigma\gamma_n f(\psi_n) = 0, \quad n \in \mathbb{Z}, \tag{5.1}$$

where $\sigma = \pm 1$ and

$$\Delta\psi_n = \psi_{n+1} - 2\psi_n + \psi_{n-1} \tag{5.2}$$

is the discrete Laplacian operator.

5.1. Assumptions and Main results

(A1) Assume that the nonlinearity $f(u)$ is gauge invariant, that is, $f(e^{i\omega}u) = e^{i\omega}f(u)$ for any $\omega \in \mathbb{R}$.

Thus we can consider the special solutions of the equation (5.1) of the form $\psi_n = e^{-it\omega}u_n$. These solutions are called *standing waves* or breather solutions. Inserting the ansatz of a standing wave solution into the equation (5.1) we see that any standing wave solution satisfies the infinite nonlinear system of algebraic equations

$$-(\Delta u)_n + v_n u_n - \omega u_n - \sigma\gamma_n f(u_n) = 0 \tag{5.3}$$

(A2) Assume that there exist two constants $0 < \underline{\gamma} \leq \overline{\gamma}$ such that for any $n \in \mathbb{Z}$,

$$\underline{\gamma} \leq \gamma_n \leq \overline{\gamma}. \tag{5.4}$$

(A3) Assume that the discrete potential $V = \{v_n\}_{n\in\mathbb{Z}}$ is bounded from below and satisfies

$$\lim_{|n|\to\infty} v_n = \infty. \tag{5.5}$$

Without losing the generality we assume that $V \geq 1$ and denote $H = -\Delta + V$ which is well-defined on $l^2(\mathbb{Z})$. Let

$$E = \{u \in l^2(\mathbb{Z}) : \ (-\Delta + V)^{1/2}u \in l^2(\mathbb{Z})\}, \quad \|u\|_E = \|(-\Delta + V)^{1/2}u\|_{l^2(\mathbb{Z})}. \tag{5.6}$$

We denote by λ_1 the smallest eigenvalue of H. With the help of Proposition 2.6, under slightly strengthened assumption (A2) with $\underline{\gamma} = 0$, using Nehari manifold approach we proved (see [29]) the existence of standing wave solutions for the case $\omega < \lambda_1$ and the power nonlinearity

$$f(u) = |u|^{p-2}u, \quad 2 < p < \infty. \tag{5.7}$$

Theorem 5.1. *Assume that the equation (5.3) satisfies (5.4), (5.5) and (5.7). Then we have*
(1) if $\sigma = -1, \omega \leq \lambda_1$, there is no nontrivial solution for the equation (5.3);
(2) if $\sigma = 1, \omega < \lambda_1$, there is at least a pair of nontrivial solution $\pm u$ in $l^2(\mathbb{Z})$ for the equation (5.3);
(3) The solutions obtained in case (2) exponentially decay at infinity, that means, there exist two positive constants C and α such that

$$|u_n| \leq Ce^{-\alpha|n|}, \qquad n \in \mathbb{Z}.$$

We rewrite the equation (5.3) as

$$Hu_n - \omega u_n - \sigma\gamma_n f(u_n) = 0. \tag{5.8}$$

Now we list some basic assumptions on the nonlinearity $f(u)$ here.
(f1) Assume that $f(u) \in \mathcal{C}^1(\mathbb{R})$. The assumption (A1) implies $f(u)$ is an odd function.
(f2) There exist a positive constants C_1 and $2 < p < \infty$ such that

$$|f(u)| \leq C_1(1 + |u|^{p-1}). \tag{5.9}$$

(f3) Assume that f is superlinear near 0, that is,

$$\lim_{u\to 0} \frac{f(u)}{|u|} = 0. \tag{5.10}$$

(f4) There is a $2 < q < \infty$ such that

$$0 < qF(u) \leq uf(u), \quad ,\forall u \neq 0, \tag{5.11}$$

where

$$F(u) = \int_0^u f(s)ds. \tag{5.12}$$

Combining (5.9) and (5.11) we can conclude that $q \leq p$ and there is $C_2 > 0$ such that

$$F(u) \geq C_2|u|^q, \quad \forall u \in \mathbb{R}. \tag{5.13}$$

From (5.9) and (5.10) it is easy to show that for any given $\varepsilon > 0$, there exists $A \equiv A(\varepsilon) > 0$ such that for any $u \in \mathbb{R}$

$$f(u)u \leq \varepsilon|u|^2 + A|u|^p, \tag{5.14}$$

$$F(u) \leq \frac{\varepsilon}{2}|u|^2 + \frac{A}{p}|u|^p. \tag{5.15}$$

A typical example for f is the following power nonlinearity, for some $2 < p < \infty$, $q = p$

$$f(u) = |u|^{p-1}u, \quad f'(u) = (p-1)|u|^{p-2}, \quad F(u) = \frac{1}{p}|u|^p.$$

Now we can define the action functional

$$J(u) = \frac{1}{2}((H-\omega)u, u) - \sigma\Sigma_{n\in\mathbb{Z}}\gamma_n F(u_n), \tag{5.16}$$

The assumption (5.9) and Proposition 2.6 imply that $J(u) \in C^1(E, \mathbb{R})$ and

$$(J'(u), v) = ((H-\omega)u, v) - \sigma\Sigma_{n\in\mathbb{Z}}\gamma_n f(u_n)v_n. \tag{5.17}$$

Now we summarize our main results as follows.

Theorem 5.2. *Assume that the equation (5.3) satisfies the assumptions (A1)-(A3) and the nonlinearity f satisfies the assumptions (f1)-(f4). Then*
(1) if $\sigma = 1, \omega \in \mathbb{R}$, there is at least a pair of nontrivial solution $\pm u$ in $l^2(\mathbb{Z})$ for the equation (5.3);
(2) the solutions obtained in (1) exponentially decay at infinity, that means, there exist two positive constants C and α such that

$$|u_n| \leq Ce^{-\alpha|n|}, \qquad n \in \mathbb{Z};$$

(3) if $\sigma = 1, \omega \in \mathbb{R}$, there exists an unbounded sequence of critical values of the functional $J(u)$. Consequently, there exist infinitely many pair of exponentially decaying standing wave solutions in $l^2(\mathbb{Z})$ for the equation (5.3).

5.2. The Palais-Smale condition and Linking Geometry

The following lemma, proven in [30], establishes that the functional J satisfies the so-called Palais-Smale (PS) condition.

Lemma 5.3. *For $\sigma = \pm 1$ and $\omega \in \mathbb{R}$, $J(u)$ satisfies the (PS) condition, that is, any sequence $u^{(k)} \in E$ such that $J(u^{(k)})$ is bounded and $J'(u^{(k)}) \to 0$ contains a convergent subsequence.*

By Theorem 2.1 and Remark 2.5 the spectrum of the Hamiltonian operator H is discrete and without losing the generality we can assume that

$$1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty.$$

Let ϕ_k be the associated normalized eigenfunction with λ_k for each k, that is,

$$H\phi_k = \lambda_k \phi_k, \quad \|\phi_k\|_{l^2} = 1.$$

Moreover, $\{\phi_k \ : \ k = 1, 2, \cdots\}$ is an orthonormal basis of $l^2(\mathbb{Z})$.
For any $\omega \geq \lambda_1$, there exists a unique k such that $\omega \in [\lambda_k, \lambda_{k+1})$. Let

$$Y = \text{Span}\{\phi_1, \cdots, \phi_k\}, \quad \text{dim}Y = k < \infty,$$

for $\omega < \lambda_1$, we take $Y = \{0\}$, then the Hilbert space E can be decomposed into the direct sum

$$E = Y \oplus Z, \quad Z = Y^{\perp} = \overline{\text{Span}\{\phi_j | j \geq k+1\}}^{\|\cdot\|_E}.$$

Notice that the linking geometry will be reduced to the mountain geometry as $Y = \{0\}$. Therefore the Mountain Pass theorem can be viewed as a special case of the Linking theorem.
Let $z \in Z, \ \|z\|_E = 1$ and define

$$N = \{u \in Z | \|u\|_E = r\}, \quad M = \{u = y + \lambda z | y \in Y, \|u\|_E \leq \rho, \lambda \geq 0\}$$

and the boundary of M

$$\begin{aligned}\partial M &= \{u = y + \lambda z | y \in Y, \|u\|_E = \rho, \lambda \geq 0 \text{ or } \|u\|_E \leq \rho, \lambda = 0\} \\ &= \{u = y + \lambda z | y \in Y, \|u\|_E = \rho, \lambda > 0\} \cup \{y \in Y | \|y\|_E \leq \rho\}.\end{aligned}$$

According to the linking theorem in the Appendix we need the following lemma (linking geometry) to prove our main result Theorem 5.2.

Lemma 5.4. *There exist two positive constants $\rho > r > 0$ such that*

$$\inf_{v \in N} J(v) > \sup_{v \in \partial M} J(v).$$

Proof. Let $y = \sum_{i=1}^{k} a_i \phi_i \in Y$ and $z = \sum_{i=k+1}^{\infty} b_i \phi_i \in Z$ with $\|z\|_E = 1$, that is,

$$\|H^{1/2} z\|_{l^2} = 1 \Leftrightarrow \sum_{i=k+1}^{\infty} \lambda_i b_i^2 = 1.$$

By a simple calculation we obtain

$$\|y\|_E^2 = \sum_{i=1}^{k} \lambda_i a_i^2, \quad \|y + \lambda z\|_E^2 = \sum_{i=1}^{k} \lambda_i a_i^2 + \lambda^2.$$

Let $u = \sum_{i=k+1}^{\infty} \beta_i \phi_i \in Z$,

$$J(u) = \frac{1}{2}((H-\omega)u, u) - \sum_{n\in\mathbb{Z}} \gamma_n F(u_n).$$

For any $\varepsilon > 0$, there exists $A = A(\varepsilon) > 0$, such that

$$0 \le F(u) \le \varepsilon |u|^2 + A|u|^p.$$

Since

$$\frac{1}{2}((H-\omega)u, u) = \frac{1}{2}\sum_{i=k+1}^{\infty} (\lambda_i - \omega)\beta_i^2,$$

by virtue of (5.4) we have

$$\begin{aligned} J(u) &\ge \frac{1}{2}\sum_{i=k+1}^{\infty} (\lambda_i - \omega)\beta_i^2 - \overline{\gamma}[\varepsilon \sum_{i=k+1}^{\infty} \beta_i^2 + A\|u\|_{l^p}^p] \\ &\ge \frac{1}{2}\sum_{i=k+1}^{\infty} \lambda_i\beta_i^2 - (\omega/2 + \overline{\gamma}\varepsilon)\sum_{i=k+1}^{\infty} \beta_i^2 - \overline{\gamma}A(\sum_{i=k+1}^{\infty} \beta_i^2)^{p/2}. \end{aligned}$$

Let $\delta = \lambda_{k+1} - \omega > 0$ and $0 < \varepsilon < \frac{\delta}{4\overline{\gamma}}$. If $u \in N$, then

$$\sum_{i=k+1}^{\infty} \lambda_i\beta_i^2 = \|u\|_E^2 = r^2 \ge \lambda_{k+1}\sum_{i=k+1}^{\infty} \beta_i^2,$$

which implies

$$\sum_{i=k+1}^{\infty} \beta_i^2 \le r^2/\lambda_{k+1},$$

thus

$$J(u) \ge \frac{\delta}{4\lambda_{k+1}} r^2 - \frac{\overline{\gamma}A}{\lambda_{k+1}^{p/2}} r^p \equiv f(r).$$

Notice that $f(r)$ reaches its maximum value at

$$r = (\frac{\delta}{2p\overline{\gamma}A})^{\frac{1}{p-2}} \lambda_{k+1}^{1/2}, \tag{5.18}$$

and

$$J(u) \ge \frac{(p-2)\delta}{4p}(\frac{\delta}{2p\overline{\gamma}A})^{\frac{2}{p-2}} > 0. \tag{5.19}$$

Consider a special $z = \phi_{k+1}/\lambda_{k+1}^{1/2}$, then $z \in Z$ and $\|z\|_E = 1$. Let $y = \sum_{i=1}^{k} a_i\phi_i$ and

$$u = y + \lambda z \in \partial M \subset \text{Span}\{\phi_1, \phi_2, \cdots, \phi_{k+1}\} = Y \oplus \{s\phi_{k+1} : \ s \in \mathbb{R}\}.$$

We distinguish two cases.
(1) $\lambda = 0, \|y\|_E \le \rho$, then

$$\sum_{i=1}^{k} \lambda_i a_i^2 \le \rho^2,$$

and

$$J(u) = J(y) \le \frac{1}{2}((H-\omega)y, y) = \frac{1}{2}\sum_{i=1}^{k}(\lambda_i - \omega)a_i^2 \le 0.$$

(2)$\lambda \geq 0$ and $\|y+\lambda z\|_E = \rho$, that is

$$\sum_{i=1}^{k} \lambda_i a_i^2 = \rho^2 - \lambda^2,$$

then

$$\begin{aligned} J(u) = J(y+\lambda z) &= \frac{1}{2}((H-\omega)u,u) - \sum_{n\in\mathbb{Z}} \lambda_n F(u_n) \\ &= \frac{1}{2}\|y+\lambda z\|_E^2 - \frac{\omega}{2}\|y+\lambda z\|_{l^2}^2 - \sum_{n\in\mathbb{Z}} \lambda_n F(u_n) \\ &= \frac{\rho^2}{2} - \frac{\omega\lambda^2}{2\lambda_{k+1}} - \frac{\omega}{2}\sum_{i=1}^{k} a_i^2 - \sum_{n\in\mathbb{Z}} \lambda_n F(u_n) \\ &\leq \frac{\rho^2}{2} - \frac{\omega\lambda^2}{2\lambda_{k+1}} - \frac{\omega(\rho^2-r^2)}{2\lambda_k} - \underline{\gamma} C_2 \|y+\lambda z\|_{l^q}^q \\ &= \frac{\rho^2}{2}(1-\frac{\omega}{\lambda_k}) + \frac{\omega}{2}(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2 - \underline{\gamma} C_2 \|y+\lambda z\|_{l^q}^q. \end{aligned}$$

Notice that all norms in a finite dimensional space are equivalent and $y+\lambda z$ belongs to a finite dimensional space, then there exists a positive constant K depending on k and q such that

$$\|y+\lambda z\|_{l^q} \geq K\|y+\lambda z\|_{l^2}.$$

Thus for $0 \leq \lambda \leq \rho$

$$\begin{aligned} J(u) = J(y+\lambda z) &\leq \frac{\rho^2}{2}(1-\frac{\omega}{\lambda_k}) + \frac{\omega}{2}(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2 - \underline{\gamma} C_2 K^q (\sum_{i=1}^{k} a_i^2 + \lambda^2/\lambda_{k+1})^{q/2} \\ &\leq \frac{\rho^2}{2}(1-\frac{\omega}{\lambda_k}) + \frac{\omega}{2}(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2 - \underline{\gamma} C_2 K^q (\frac{\rho^2}{\lambda_k} - (\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2)^{q/2} \equiv \widetilde{g}(\lambda). \end{aligned}$$

Notice that for $0 \leq \lambda \leq \rho$

$$\widetilde{g}'(\lambda) = \omega\lambda(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}) + \underline{\gamma} C_2 K^q q\lambda(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})(\frac{\rho^2}{\lambda_k} - (\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2)^{\frac{q-2}{2}} \geq 0,$$

thus

$$J(u) \leq \max_{0\leq\lambda\leq\rho} \widetilde{g}(\lambda) = \frac{\delta}{2\lambda_{k+1}}\rho^2 - \frac{\underline{\gamma} C_2 K^q}{\lambda_{k+1}^{q/2}}\rho^q \equiv g(\rho).$$

Therefore there exists $\rho > r > 0$ such that $g(\rho) < 0$. By the choice of r 5.18 we know that

$$\inf_{u\in N} J(u) \geq f(r) > 0 > g(\rho) \geq \sup_{u\in\partial M} J(u).$$

■

5.3. Exponential Decay

The following theorem about exponential decay of standing waves was proved in [30].

Theorem 5.5. *Let $u \in l^2(\mathbb{Z})$ be a solution to the equation (5.3). If u satisfies furthermore*

$$\lim_{|n|\to\infty} \gamma_n f(u_n) = 0, \tag{5.20}$$

then there exists two positive constants C and α such that

$$|u_n| \leq Ce^{-\alpha|n|}, \qquad n \in \mathbb{Z}. \tag{5.21}$$

5.4. Proof of Theorem 5.2

Now we prove our main result Theorem 5.2 as follows (also see [30]). Actually, by Lemma 5.3 and Lemma 5.4 we know that the functional $J(u)$ satisfies the Palais-Smale condition and the linking geometry. Thus (1) becomes a natural consequence of the linking theorem 5.7. (2) is just a corollary of Theorem 5.5. Therefore we only need to prove (3). To this end we need one more lemma.
By Remark 2.5 we can define the nested sequence of finite dimensional space $\{E_m\}$ in Theorem 5.8 as follows. For $\omega < \lambda_1$, $E_m \equiv \mathrm{Span}\{\phi_1, \cdots, \phi_m\}$, and for $\lambda_k \le \omega < \lambda_{k+1}$, $k \ge 1$, $E_m \equiv \mathrm{Span}\{\phi_1, \cdots, \phi_{k+m}\}$, for $m = 1, 2, \cdots$.

Lemma 5.6. *There exist two positive constants c_1 and c_2 depending on k and m such that for any $u \in E_m$,*

$$J(u) \le c_1\|u\|_E^2 - c_2\|u\|_E^q. \tag{5.22}$$

We can see that the assumption (B2) in Theorem 5.8 is an immediate consequence of Lemma 5.6 since $q > 2$. Since the assumption (B1) has been verified in the proof of Lemma 5.4, (3) of Theorem 5.2 becomes a consequence of Theorem 5.8. Therefore we can complete the proof of Theorem 5.2 now by showing Lemma 5.6.

Proof of Lemma 5.6 For the case $m = 1$, it has been done essentially in the proof of Lemma 5.4 if we notice that for any $u \in E_1$, there exist unique $y \in Y$ and $\lambda \in \mathbb{R}$ such that $u = y + \lambda z$, where Y and z are defined in the proof of Lemma 5.4. Therefor by a similar calculation in the proof of Lemma 5.4 we obtain for $u \in E_1$

$$J(u) \le \frac{\lambda_{k+1} - \omega}{2\lambda_{k+1}}\|u\|_E^2 - \frac{\gamma C_2 K^q}{\lambda_{k+1}^{q/2}}\|u\|_E^q \tag{5.23}$$

For the case $m > 1$, let $\omega_m \equiv (\lambda_{k+m-1} + \lambda_{k+m})/2$. We define a functional

$$J_m(u) = \frac{1}{2}((H - \omega_m)u, u) - \sum_{n\in\mathbb{Z}} \gamma_n F(u_n),$$

which is just the function $J(u)$ with a different frequency $\omega = \omega_m$. Notice that $\lambda_{k+m-1} \le \omega_m < \lambda_{k+m}$, by (5.23) with $k+1$ replaced by $k+m$ we obtain for any $u \in E_m$,

$$J(u) \le \frac{\lambda_{k+m} - \omega_m}{2\lambda_{k+m}}\|u\|_E^2 - \frac{\gamma C_2 K^q}{\lambda_{k+m}^{q/2}}\|u\|_E^q. \tag{5.24}$$

Thus let $u = \sum_{i=1}^{k+m} a_i\phi_i$, from

$$\|u\|_E^2 = \sum_{i=1}^{k+m} \lambda_i a_i^2 \ge \lambda_1 \sum_{i=1}^{k+m} a_i^2,$$

we obtain for any $u \in E_m$,

$$\begin{aligned} J(u) &= J_m(u) + \frac{1}{2}(\omega_m - \omega)\sum_{i=1}^{k+m} a_i^2 \\ &\le \frac{\lambda_{k+m} - \omega_m}{2\lambda_{k+m}}\|u\|_E^2 - \frac{\gamma C_2 K^q}{\lambda_{k+m}^{q/2}}\|u\|_E^q + \frac{\omega_m - \omega}{2\lambda_1}\|u\|_E^2 \\ &\le \frac{\lambda_{k+m} - \omega}{2\lambda_1}\|u\|_E^2 - \frac{\gamma C_2 K^q}{\lambda_{k+m}^{q/2}}\|u\|_E^q \end{aligned}$$

which implies (5.22). Therefore Lemma 5.6 holds.

5.5. Appendix: Linking theorem and Multiple Critical Points

Here we recall the so-called linking theorem (see [19, 21, 25]). Let $E = Y \oplus Z$ be a Banach space decomposed into the direct sum of two closed subspaces Y and Z, with $\dim Y < \infty$. Let $\rho > r > 0$ and let $z \in Z$ be a fixed vector, $\|z\| = 1$. Define

$$M = \{u = y + \lambda z : \ y \in Y, \|u\| \leq \rho, \lambda \geq 0\} \quad N = \{u \in Z : \ \|u\| = r\}.$$

The boundary of M is denoted by ∂M

$$\partial M = \{u = y + \lambda z : \ y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0, \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\}.$$

Theorem 5.7. *Let $J(u) \in \mathcal{C}^1(E, \mathbb{R})$ and assume that J satisfies the Palais-Smale (PS) condition, i.e. any sequence $u^{(k)} \in E$ such that $J(u^{(k)})$ is bounded and $J'(u^{(k)}) \to 0$ contains a convergent subsequence. Assume also that J possesses the following so-called linking geometry*

$$\beta \equiv \inf_{u \in N} J(u) > \sup_{u \in \partial M} J(u) \equiv \alpha. \tag{5.25}$$

Let $\Gamma = \{\gamma \in C(M, E) : \ \gamma = id \text{ on } \partial M\}$. Then

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in M} J(\gamma(u))$$

is a critical value of J and

$$\beta \leq c \leq \sup_{u \in M} J(u). \tag{5.26}$$

Multiple Critical Points Here we recall a $\mathbb{Z}_2$ version of the Mountain Pass Theorem (see Theorem 9.12 in [1] or [21]).

Theorem 5.8. *Let E be an infinite dimensional Banach space and let $J \in \mathcal{C}^1(E, \mathbb{R})$ be even, satisfy the Palais-Smale condition, and $J(0) = 0$. If $E = Y \oplus Z$, where Y is finite dimensional and J satisfies*
(B1) there are constants $r, \alpha > 0$ such that $J|_{\partial B_r \cap Z} \geq \alpha$, and
(B2) for a nested sequence $E_1 \subset E_2 \subset \cdots$ of increasing finite dimension, there exist $\rho_i \equiv \rho(E_i) > 0$ such that $J \leq 0$ on $B^c_{\rho_i} \equiv \{x \in E_i \mid \|x\| > \rho_i\}$, for $i = 1, 2, \cdots$,
then J possesses an unbounded sequence of critical values.

6. Acknowledgement

The author is thankful to the referees for their quick reviews.

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 149–165.
http://doi.org/10.26637/mjm11S/010

Generalized almost periodic solutions of Volterra difference equations

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***R**eceived 11 July 2023; **A**ccepted 24 September 2023*

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. In this paper, we investigate several new classes of generalized ρ-almost periodic sequences in the multi-dimensional setting. We specifically analyze the class of Levitan ρ-almost periodic sequences and the class of remotely ρ-almost periodic sequences. We provide many important applications of the established theoretical results to the abstract Volterra difference equations.

AMS Subject Classifications: 39A24, 43A60, 34C27.

Keywords: Levitan ρ-almost periodic sequences, remotely ρ-almost periodic sequences, abstract Volterra difference equations.

Contents

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1. Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a complex Banach space. An X-valued sequence $(x_k)_{k\in\mathbb{Z}}$ is called (Bohr) almost periodic if and only if, for every $\epsilon > 0$, there exists a natural number $K_0(\epsilon)$ such that among any $K_0(\epsilon)$ consecutive integers in $\mathbb{Z}$, there exists at least one integer $\tau \in \mathbb{Z}$ satisfying that

$$\left\|x_{k+\tau} - x_k\right\| \leq \epsilon, \quad k \in \mathbb{Z};$$

as in the case of functions, this number is said to be an ϵ-period of sequence (x_k). Further on, an X-valued sequence $(x_k)_{k\in\mathbb{Z}}$ is said to be almost automorphic if and only if, for every sequence $(h'_k)_{k\in\mathbb{Z}}$ of integer numbers, there exists a subsequence $(h_k)_{k\in\mathbb{Z}}$ of $(h'_k)_{k\in\mathbb{Z}}$ and an X-valued sequence $(y_m)_{m\in\mathbb{Z}}$ satisfying that

$$\lim_{k\to\infty} x_{m+h_k} = y_m, \ n \in \mathbb{Z} \ \text{ and } \ \lim_{k\to\infty} y_{m-h_k} = x_m, \ m \in \mathbb{Z}.$$

Any almost periodic sequence $(x_k)_{k\in\mathbb{Z}}$ is almost automorphic while the converse statement is not true in general. It is well known that a sequence $(x_k)_{k\in\mathbb{Z}}$ in X is almost periodic (almost automorphic) if and only if there exists an almost periodic (compactly almost automorphic) function $f : \mathbb{R} \to X$ such that $x_k = f(k)$ for all $k \in \mathbb{Z}$; see e.g., the proof of [5, Theorem 2] for the almost periodic setting and [7, Theorem 1, p. 92] for the almost automorphic setting (the notion of an almost periodic function $f : \mathbb{R} \to X$ and the notion of a compactly almost automorphic function $f : \mathbb{R} \to X$ can be found in [8], e.g.).

Several new classes of generalized ρ-almost periodic type sequences, like (equi)-Weyl-(p, ρ)-almost periodic sequences, Doss (p, ρ)-almost periodic sequences and Besicovitch-p-almost periodic sequences, have recently been considered in [10]. The main aim of this paper is to continue the above-mentioned research study by investigating some classes of Levitan ρ-almost periodic type sequences and remotely ρ-almost periodic type sequences. We also aim to provide certain applications of our results to the abstract Volterra difference equations.

The paper is quite simply organized; after collecting the basic results about principal fundamental matrix solutions, Green functions and exponential dichotomies in Subsection 1.1, we analyze the Levitan ρ-almost periodic type sequences and the remotely ρ-almost periodic type sequences in Section 2 and Section 3, respectively. The main aim of Section 4, which is broken down into two separate subsections, is to provide certain applications of the established results to the abstract Volterra difference equations; the final section of paper is reserved for some conclusions and final remarks about the introduced notion.

Notation and terminology. Suppose that X, Y, Z and T are given non-empty sets. Let us recall that a binary relation between X into Y is any subset $\rho \subseteq X \times Y$. If $\rho \subseteq X \times Y$ and $\sigma \subseteq Z \times T$ with $Y \cap Z \neq \emptyset$, then we define $\rho^{-1} \subseteq Y \times X$ and $\sigma \cdot \rho = \sigma \circ \rho \subseteq X \times T$ by $\rho^{-1} := \{(y, x) \in Y \times X : (x, y) \in \rho\}$ and $\sigma \circ \rho := \{(x, t) \in X \times T : \exists y \in Y \cap Z$ such that $(x, y) \in \rho$ and $(y, t) \in \sigma\}$, respectively. As is well known, the domain and range of ρ are defined by $D(\rho) := \{x \in X : \exists y \in Y$ such that $(x, y) \in X \times Y\}$ and $R(\rho) := \{y \in Y : \exists x \in X$ such that $(x, y) \in X \times Y\}$, respectively; $\rho(x) := \{y \in Y : (x, y) \in \rho\}$ $(x \in X)$, $x \, \rho \, y \Leftrightarrow (x, y) \in \rho$. If ρ is a binary relation on X and $n \in \mathbb{N}$, then we define ρ^n inductively; $\rho^{-n} := (\rho^n)^{-1}$ and $\rho^0 := \Delta_X := \{(x, x) : x \in X\}$. Set $\rho(X') := \{y : y \in \rho(x)$ for some $x \in X'\}$ $(X' \subseteq X)$ and $\mathbb{N}_n := \{1, \cdots, n\}$ $(n \in \mathbb{N})$. An unbounded subset $A \subseteq \mathbb{Z}$ is called syndetic if and only if there exists a strictly increasing sequence (a_n) of integers such that $A = \{a_n : n \in \mathbb{Z}\}$ and $\sup_{n\in\mathbb{Z}}(a_{n+1} - a_n) < +\infty$. Set, for every $\mathbf{t}_0 \in \mathbb{R}^n$ and $l > 0$, $B(\mathbf{t}_0, l) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$, where $|\cdot - \cdot|$ denotes the Euclidean distance in $\mathbb{R}^n$. We will always assume henceforth that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces as well as that $\rho \subseteq Y \times Y$ is a given binary relation. By I we denote the identity operator on Y; $\mathcal{B}$ stands for any non-empty collection of non-empty subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. The space of all linear continuous operators from X into Y is denoted by $L(X, Y)$; $L(Y) \equiv L(Y, Y)$.

Before proceeding further, we need to recall the following notion (cf. [4] for more details on the subject):

Definition 1.1. *Suppose that $F : \mathbb{R}^n \to Y$ is a continuous function and $T \in L(Y)$. Then we say that the function $F(\cdot)$ is:*

(i) *Levitan T-pre-almost periodic if and only if $F(\cdot)$ is for each $N > 0$ and $\epsilon > 0$ there exists a finite real number $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l)$ such that*

$$\left\| F(\mathbf{t}+\tau) - TF(\mathbf{t}) \right\| \leq \epsilon \text{ for all } \mathbf{t} \in \mathbb{R}^n \text{ with } |\mathbf{t}| \leq N;$$

by $E(\epsilon, T, N)$ we denote the set of all such points τ which we also call (ϵ, N, T)-almost periods of $F(\cdot)$.

(ii) *strongly Levitan T-almost periodic if and only if $F(\cdot)$ is Levitan T-pre-almost periodic and, for every real numbers $N > 0$ and $\epsilon > 0$, there exist a finite real number $\eta > 0$ and the relatively dense sets $E^j_{\eta;N}$ in $\mathbb{R}$ ($1 \leq j \leq n$) such that the set $E_{\eta;N} \equiv \prod_{j=1}^n E^j_{\eta;N}$ consists solely of (η, N, T)-almost periods of $F(\cdot)$ and $E_{\eta;N} \pm E_{\eta;N} \subseteq E(\epsilon, T, N)$.*

1.1. Principal fundamental matrix solutions, Green functions and exponential dichotomies

In order to analyze the existence and uniqueness of solutions for a class of discrete dynamical systems, we shall first remind the readers of the notion of discrete exponential dichotomy, which plays an important role in the setup of the main results.

Definition 1.2 ([12, Definition 5]). *Let $X(t)$ be the principal fundamental matrix solution of the linear homogeneous system*

$$x(t+1) = A(t)x(t), \ t \in \mathbb{Z}; \quad x(t_0) = x_0 \in \mathbb{C}^n, \tag{1.1}$$

where $A(t)$ is a matrix function which is invertible for all $t \in \mathbb{Z}$. Then we say that (1.1) *admits an exponential dichotomy if and only if there exist a projection P and positive constants $\alpha_1, \ \alpha_2, \ \beta_1$ and β_2 such that*

$$\left\| X(t)PX^{-1}(s) \right\| \leq \beta_1 \left(1+\alpha_1\right)^{s-t}, \ t \geq s,$$
$$\left\| X(t)\left(I-P\right)X^{-1}(s) \right\| \leq \beta_2 \left(1+\alpha_2\right)^{t-s}, \ s \geq t.$$

We define the Green function by

$$G\left(t,s\right) := \begin{cases} X\left(t\right)PX^{-1}\left(s\right) \text{ for } t \geq s \\ X\left(t\right)\left(I-P\right)X^{-1}\left(s\right) \text{ for } s \geq t \end{cases}.$$

We will use the following result later on (cf. [12, Theorem 2]):

Theorem 1.3. *If the system* (1.1) *admits an exponential dichotomy and the function $f(\cdot)$ is bounded, then the nonhomogeneous system*

$$x(t+1) = A(t)x(t) + f\left(t\right), \ t \in \mathbb{Z}; \quad x(t_0) = x_0 \tag{1.2}$$

has a bounded solution of the form

$$x\left(t\right) = \sum_{j=-\infty}^{\infty} G\left(t, j+1\right) f\left(j\right). \tag{1.3}$$

2. Levitan ρ-almost periodic type sequences

In a joint research article with B. Chaouchi and D. Velinov [4], the first named author has recently analyzed Levitan ρ-almost periodic type functions and uniformly Poisson stable functions. We will use the following notions (cf. also [4, Definition 2.1, Definition 2.13]):

Definition 2.1. *Suppose that $\emptyset \neq I \subseteq \mathbb{Z}^n$, $\emptyset \neq I' \subseteq \mathbb{Z}^n$, $i + i' \in I$ for all $i \in I$, $i' \in I'$ and $F : I \times X \to Y$. Then we say that the sequence $F(\cdot; \cdot)$ is:*

(i) *Levitan-pre-$(\mathcal{B}, I', \rho)$-almost periodic if and only if for every $\epsilon > 0$, $B \in \mathcal{B}$ and $N > 0$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in I'$, there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that, for every $x \in B$ and $i \in I$ with $|i| \leq N$, there exists $y_{i;x} \in \rho(F(i;x))$ such that*

$$\left\|F(i+\tau;x) - y_{i;x}\right\| \leq \epsilon, \quad x \in B;$$

by $E_{\epsilon;N;B}$ we denote the set consisting of all such numbers $\tau \in I'$.

(ii) *Levitan $(\mathcal{B}, \rho)$-almost periodic if and only if $F(\cdot;\cdot)$ is Levitan-pre-$(\mathcal{B}, I', \rho)$-almost periodic with $I' = I$, $\rho = \mathrm{I}$ and, for every $\epsilon > 0$, $B \in \mathcal{B}$ and $N > 0$, there exist a number $\eta > 0$ and a relatively dense set $E_{\eta;N;B}$ in I (i.e., for every $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t} \in I$ there exists $\tau \in B(\mathbf{t}, l) \cap E_{\eta;N;B}$) such that $E_{\eta;N;B} \subseteq I'$ and $E_{\eta;N;B} \pm E_{\eta;N;B} \subseteq E_{\epsilon;N;B}$.*

(iii) *strongly Levitan $(\mathcal{B}, \rho)$-almost periodic if and only if $F(\cdot;\cdot)$ is Levitan $\mathcal{B}$-almost periodic and the set $E_{\eta;N;B}$ from the part* (ii) *can be written as $E_{\eta;N;B} = \prod_{j=1}^{n} E^{j}_{\eta;N;B}$, where the set $E^{j}_{\eta;N;B}$ is relatively dense in the j-th projection of the set I.*

We omit the term "$\mathcal{B}$" from the notation for the sequences $F : I \to Y$; furthermore, we omit the term "ρ" from the notation if $\rho = \mathrm{I}$.

Using the same argumentation as in the proofs of [5, Theorem 2], [10, Theorem 2.3, Proposition 2.4, Theorem 2.6] and the fact that strongly Levitan N-almost periodic functions form the vector space with the usual operations, we may deduce the following important results (we will provide the main details of the proofs, only; by a strongly Levitan almost periodic sequence (function), we mean a strongly Levitan I-almost periodic sequence (function)):

Theorem 2.2. *Suppose that $\rho = T \in L(Y)$ and $F : \mathbb{Z}^n \to Y$. Then the following holds:*

(i) *If $F : \mathbb{Z}^n \to Y$ is a Levitan T-pre-almost periodic sequence, then there exists a continuous Levitan T-pre-almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$ and $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}^n$. Furthermore, if $F(\cdot)$ is bounded, then $\tilde{F}(\cdot)$ is uniformly continuous.*

(ii) *If $F : \mathbb{Z}^n \to Y$ is a (strongly) Levitan T-almost periodic sequence, then there exists a continuous (strongly) Levitan T-almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$ and $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}^n$. Furthermore, if $F(\cdot)$ is bounded, then $\tilde{F}(\cdot)$ is uniformly continuous.*

Proof. We will present all relevant details of the proof of (ii) in the two-dimensional setting; cf. also the proof of [5, Theorem 2] with $c = 1$ and $\delta = 1/2$. Consider first the statement (i). If $t = (t_1, t_2) \in \mathbb{R}^2$ is given, then there exist the uniquely detemined numbers $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$ such that $t_1 \in [k, k+1)$ and $t_2 \in [m, m+1)$. Define first $\tilde{F}(t_1, m) := \tilde{F}_{\mathbb{Z}}(k, m)$ if $t_1 \in [k, k+(1/2))$ and $\tilde{F}(t_1, m) := 2(\tilde{F}_{\mathbb{Z}}(k+1, m) - \tilde{F}_{\mathbb{Z}}(k, m))(t_1 - k - (1/2)) + \tilde{F}_{\mathbb{Z}}(k, m)$ if $t_1 \in [k+(1/2), k+1)$; we similarly define $\tilde{F}(t_1, m+1) := \tilde{F}_{\mathbb{Z}}(k, m+1)$ if $t_1 \in [k, k+(1/2))$ and $\tilde{F}(t_1, m+1) := 2(\tilde{F}_{\mathbb{Z}}(k+1, m+1) - \tilde{F}_{\mathbb{Z}}(k, m+1))(t_1 - k - (1/2)) + \tilde{F}_{\mathbb{Z}}(k, m+1)$ if $t_1 \in [k+(1/2), k+1)$. After that, we define $\tilde{F}(t_1, t_2) := \tilde{F}(t_1, m)$ if $t_2 \in [m, m+(1/2))$ and $\tilde{F}(t_1, t_2) := 2(\tilde{F}(t_1, m+1) - \tilde{F}(t_1, m))(t_2 - m - (1/2)) + \tilde{F}(t_1, m)$ if $t_2 \in [m+(1/2), m+1)$. Then the function $\tilde{F}(\cdot)$ is continuous, $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$, $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}^n$ and the function $\tilde{F}(\cdot)$ is uniformly continuous provided that $F(\cdot)$ is bounded. As in the proof of [10, Theorem 2.3], we may show that $\tilde{F}(\cdot)$ is a Levitan T-pre-almost periodic function provided that $F(\cdot)$ is a Levitan T-pre-almost periodic sequence. ■

Theorem 2.3. *Suppose that $F : \mathbb{Z}^n \to Y$. If $F : \mathbb{R}^n \to Y$ is a strongly Levitan almost periodic function and $F(\cdot)$ is uniformly continuous, then $F_{\mathbb{Z}^n} : \mathbb{Z}^n \to Y$ is a strongly Levitan almost periodic sequence.*

Proof. Let $\epsilon > 0$ and $N > 0$ be given; we will consider the non-trivial case $Y \neq 0$, only. Since $F(\cdot)$ is uniformly continuous, we can find a number $\delta \in (0, \epsilon)$ such that the assumptions $x,\ y \in \mathbb{R}^n$ and $|x - y| \leq \delta$ implies $\|F(x) - F(y)\|_Y \leq \epsilon$. Since the strongly Levitan almost periodic functions form a vector space with the usual operations, we know that there exists a number $\eta \in (0, \delta)$ and relatively dense sets $E^j_{\eta;N}$ in $\mathbb{R}$ such that the set $E_{\eta;N} \equiv \prod_{j=1}^n E^j_{\eta;N}$ consists solely of common (η, N)-almost periods of the function $F(\cdot)$ and the functions $G_j(\cdot)$ defined below $(1 \leq j \leq n)$ as well as that $E_{\eta;N} \pm E_{\eta;N} \subseteq E(\epsilon, N)(F, G_1, ..., G_n)$; here, we use the same notion and notation as in [4]. Therefore, if $\tau = (\tau_1, ..., \tau_n)$ in $E_{\eta;N}$, then we have $\|F(\mathbf{t} + \tau) - F(\mathbf{t})\|_Y \leq \eta$ for all $\mathbf{t} \in \mathbb{R}^n$ with $|\mathbf{t}| \leq N$, and $\|G_j(\mathbf{t} + \tau) - G_j(\mathbf{t})\|_Y \leq \eta$ for all $\mathbf{t} \in \mathbb{R}^n$ with $|\mathbf{t}| \leq N$ and $j \in \mathbb{N}_n$, where the Bohr $\mathcal{B}$-almost periodic function $G_j : \mathbb{R}^n \to Y$ is defined as the usual periodic extension of the function $G_{j;0}(\mathbf{t}) := (1 - |1 - t_j|)y$, $\mathbf{t} = (t_1, ..., t_j, ..., t_n) \in [0, 2]^n$ to the space $\mathbb{R}^n$ (the non-zero element $y \in Y$ is fixed in advance). As in the one-dimensional setting, this simply implies that there exist two vectors $p \in \mathbb{Z}^n$ and $w = (w_1, ..., w_n) \in B(0, \eta)$ such that $\tau = 2p + \omega$. Therefore, we have:

$$\begin{aligned} &\left\|F(\mathbf{t} + 2p) - F(\mathbf{t})\right\|_Y \\ &\leq \left\|F(\mathbf{t} + 2p) - F(\mathbf{t} + 2p + w)\right\|_Y + \left\|F(\mathbf{t} + 2p + w) - F(\mathbf{t})\right\|_Y \\ &\leq \epsilon + \eta < 2\epsilon, \quad \mathbf{t} \in \mathbb{R}^n, \ |\mathbf{t}| \leq N. \end{aligned}$$

This simply implies that $F_{\mathbb{Z}^n}(\cdot)$ is a Levitan almost periodic sequence and the second condition from the formulation of Definition 2.1(iii) holds, so that $F_{\mathbb{Z}^n}(\cdot)$ is a strongly Levitan almost periodic sequence. ■

Remark 2.4. *It is very difficult to state a satisfactory analogue of* Theorem 2.3 *if the function $F(\cdot)$ is not uniformly continuous. In connection with this issue, we would like to mention that many intriguing examples of unbounded Levitan almost periodic functions $F : \mathbb{R} \to \mathbb{R}$ which are not uniformly continuous have recently been constructed by A. Nawrocki in [16]; the discretizations of such functions cannot be simply analyzed by means of* Theorem 2.3.

We continue by providing the following illustrative example:

Example 2.5. *Suppose that*

$$F(t) := \frac{1}{2 + \cos t + \cos(\sqrt{2}t)}, \quad t \in \mathbb{R}.$$

*Then we know that the function $F(\cdot)$ is Levitan almost periodic, unbounded and not uniformly continuous ([13, 14]). Furthermore, the sequence $(F(k))_{k\in\mathbb{Z}}$ is unbounded, as easily approved, and Levitan almost periodic, which can be proved as follows (*Theorem 2.3 *is inapplicable here). The argumentation contained on [14, p. 59] shows that for each $\epsilon > 0$ and $N > 0$ there exists a sufficiently small number $\delta > 0$ such that any integer which is δ-almost period of the function $2 + \cos\cdot + \cos(\sqrt{2}\cdot)$ is also a Levitan (ϵ, N)-almost period of the function $F(\cdot)$; it is well known that the set of all such integers which are δ-almost periods is relatively dense in $\mathbb{R}$. If $\epsilon > 0$ and $N > 0$ are given, then we can simply choose the number $\eta = \delta/2$ in* Definition 2.1(ii) *and the set $E_{\eta;N}$ consisting of all integer $(\delta/2)$-almost periods of the function $2 + \cos\cdot + \cos(\sqrt{2}\cdot)$. Observe finally that, due to [17, Corollary 1], for each $\epsilon > 0$ there exists $M_\epsilon > 0$ such that $F(k) \leq M_\epsilon |k|^{2+\epsilon}$ for all $k \in \mathbb{Z}$.*

The notion of a strongly Levitan almost periodic sequence and the notion of a Levitan almost periodic sequence coincide in the one-dimensional setting. Without going into any further details concerning the multi-dimensional setting, where the famous Bogolyubov theorem does not admit a satisfactory reformulation (cf. [4] for more details), we will only formulate here the following important consequence of Theorem 2.2 and Theorem 2.3:

Theorem 2.6. *Suppose that $F : \mathbb{Z} \to Y$ is bounded. Then $(F(k))_{k\in\mathbb{Z}}$ is a Levitan almost periodic sequence if and only if $(F(k))_{k\in\mathbb{Z}}$ is an almost automorphic sequence.*

Proof. If $(F(k))_{k\in\mathbb{Z}}$ is a Levitan almost periodic sequence, then Theorem 2.2(ii) implies that there exists a uniformly continuous, Levitan almost periodic function $\tilde{F} : \mathbb{R} \to Y$ such that $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}$. Due to [18, Theorem 3.1], we have that $\tilde{F} : \mathbb{R} \to Y$ is compactly almost automorphic so that $(F(k))_{k\in\mathbb{Z}}$ is an almost automorphic sequence. On the other hand, if $(F(k))_{k\in\mathbb{Z}}$ is an almost automorphic sequence, then there exists a compactly almost automorphic function $\tilde{F}:\mathbb{R}\rightarrow Y$ such that $\tilde{F}(k)=F(k)$ for all $k\in\mathbb{Z}$. Clearly, $\tilde{F}(\cdot)$ is uniformly continuous; applying again [18, Theorem 3.1], we get that $\tilde{F}(\cdot)$ is Levitan almost periodic. Therefore, the final conclusion simply follows from an application of Theorem 2.3. ■

3. Remotely ρ-almost periodic type sequences

The following notion is a special case of the notion introduced in [9, Definition 4.1] (see also [11, Definition 3.1, Definition 3.2]):

Definition 3.1. *Suppose that $\mathbb{D}\subseteq I\subseteq\mathbb{Z}^n$, $\emptyset\neq I'\subseteq\mathbb{Z}^n$, $\emptyset\neq I\subseteq\mathbb{Z}^n$, the sets $\mathbb{D}$ and I' are unbounded, $I+I'\subseteq I$ and $F:I\times X\rightarrow Y$ is a given function. Then we say that:*

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B},I',\rho)$-almost periodic if and only if for every $B\in\mathcal{B}$ and $\epsilon>0$ there exists a finite real number $l>0$ such that for each $\mathbf{t}_0\in I'$ there exists $\tau\in B(\mathbf{t}_0,l)\cap I'$ such that, for every $x\in B$, there exists a function $G_x\in Y^{\mathbb{D}}$, the set of all functions from $\mathbb{D}$ into Y, such that $G_x(\mathbf{t})\in\rho(F(\mathbf{t};x))$ for all $\mathbf{t}\in\mathbb{D}$, $x\in B$ and

$$\limsup_{|\mathbf{t}|\rightarrow+\infty;\mathbf{t}\in\mathbb{D}}\sup_{x\in B}\left\|F(\mathbf{t}+\tau;x)-G_x(\mathbf{t})\right\|_Y\leq\epsilon.$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-remotely $(\mathcal{B},I',\rho)$-almost periodic if and only if $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B},I',\rho)$-almost periodic and, for every $B\in\mathcal{B}$, the function $F(\cdot;\cdot)$ is bounded and uniformly continuous on $I\times B$.

Remark 3.2. *If $X=\{0\}$ in* (ii)*, then the boundedness and the uniform continuity on $I\times B$ is equivalent with the boundedness on I.*

If $X=\{0\}$, then we omit the term "$\mathcal{B}$" from the notation; further on, we omit the term "I'" from the notation if $I'=I$ and we omit the term "ρ" from the notation if $\rho=\mathrm{I}$. The usual notion is obtained by plugging $X=\{0\}$, $\mathbb{D}=I'=I$ and $\rho=\mathrm{I}$, when we also say that the function $F(\cdot)$ is quasi-asymptotically almost periodic (remotely almost periodic). If $\mathbb{D},\ I',\ I\subseteq\mathbb{R}^n$, then we accept the same terminology for the functions.

The following result, which establishes a bridge between remotely almost periodic functions on continuous and discrete time domains, can be deduced with the help of the argumentation contained in the proof of [19, Theorem 2.1]:

Theorem 3.3. *A necessary and sufficient condition for a function $F:\mathbb{Z}^n\rightarrow Y$ to be remotely almost periodic is that there exists a remotely almost periodic function $H:\mathbb{R}^n\rightarrow Y$ so that $F(t)=H(t)$ for all $t\in\mathbb{Z}^n$.*

We perform the proof of the following composition principle by exactly pursuing the same direction of the proof of [19, Lemma 3.4]; the same proof works for the functions and can be adapted for the almost automorphic sequences (functions):

Theorem 3.4. *Suppose that $(Z,\|\cdot\|_Z)$ is a complex Banach space, $F:\mathbb{Z}^n\times X\rightarrow Y$ is $\mathcal{B}$-remotely almost periodic and $G:\mathbb{Z}^n\times Y\rightarrow Z$ is $\mathcal{B}'$-remotely almost periodic, where $\mathcal{B}$ denotes the family of all bounded subsets of X and $\mathcal{B}'$ denotes the family of all bounded subsets of Y. Suppose, further, that for each bounded subset B' of Y there exists a finite real constant $L_{B'}>0$ such that*

$$\left\|G\left(\mathbf{t};y_1\right)-G\left(\mathbf{t};y_2\right)\right\|_Z\leq L_{B'}\left\|y_1-y_2\right\|_Y,\quad\mathbf{t}\in\mathbb{Z}^n,\ y_1,\ y_2\in B.\tag{3.1}$$

Then the sequence $H:\mathbb{Z}^n\times X\rightarrow Z$, defined by $H(\mathbf{t};x):=G(\mathbf{t};F(\mathbf{t};x))$, $\mathbf{t}\in\mathbb{Z}^n$, $x\in X$, is $\mathcal{B}$-remotely almost periodic.

Proof. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be given. Then the set $B' := \{F(\mathbf{t}; x) : \mathbf{t} \in \mathbb{Z}^n,\ x \in B\}$ is bounded and there exists $L_{B'} > 0$ such that (3.1) holds. This yields

$$\begin{aligned}\left\|H(\mathbf{t}'; x') - H(\mathbf{t}; x)\right\|_Z &\leq \left\|G(\mathbf{t}'; F(\mathbf{t}'; x')) - G(\mathbf{t}'; F(\mathbf{t}; x))\right\|_Z \\ &+ \left\|G(\mathbf{t}'; F(\mathbf{t}; x)) - G(\mathbf{t}; F(\mathbf{t}; x))\right\|_Z \\ &\leq L_{B'}\left\|F(\mathbf{t}'; x') - F(\mathbf{t}; x)\right\|_Y + \sup_{y \in B'}\left\|G(\mathbf{t}'; y) - G(\mathbf{t}'; y)\right\|_Z,\end{aligned}$$

where $\mathbf{t},\ \mathbf{t}' \in \mathbb{Z}^n$and $x,\ x' \in B$ which simply implies that the function $H(\cdot; \cdot)$ is bounded and uniformly continuous on $I \times B$. Further on, let us denote by $l_\infty(B : Y)$ the Banach space of all bounded functions from B into Y, equipped with the sup-norm. Then the function $F_B : \mathbb{Z}^n \to l_\infty(B : Y)$, given by $[F_B(\mathbf{t})](x) := F(\mathbf{t}; x)$, $\mathbf{t} \in \mathbb{Z}^n$, $x \in B$, is remotely almost periodic and the function $G_{B'} : \mathbb{Z}^n \to l_\infty(B' : Y)$, given by $[G_{B'}(\mathbf{t})](y) := G(\mathbf{t}; y)$, $\mathbf{t} \in \mathbb{Z}^n$, $y \in B'$, is remotely almost periodic. Consequently, the set

$$\tau(H, \varepsilon) := \left\{p \in \mathbb{Z}^n : \limsup_{|\mathbf{t}| \to \infty} \sup_{y \in B'} \|G(\mathbf{t} + p; y) - G(\mathbf{t}; y)\|_Z \right.$$
$$\left. + \limsup_{|\mathbf{t}| \to \infty} \sup_{x \in B} \|F(\mathbf{t} + p; x) - F(\mathbf{t}; x)\|_Y \right\} < \varepsilon$$

is relatively dense in $\mathbb{Z}^n$. The final conclusion follows from the next computation ($\mathbf{t} \in \mathbb{Z}^n$, $x \in B$):

$$\begin{aligned}&\left\|G(\mathbf{t} + p; F(\mathbf{t} + p; x)) - G(\mathbf{t}; F(\mathbf{t}; x))\right\|_Z \leq \left\|G(\mathbf{t} + p; F(\mathbf{t} + p; x)) - G(\mathbf{t}; F(\mathbf{t} + p; x))\right\|_Z \\ &+ \|G(\mathbf{t}; F(\mathbf{t} + p; x)) - G(\mathbf{t}; F(\mathbf{t}; x))\|_Z \\ &\leq \|G(\mathbf{t} + p; F(\mathbf{t} + p; x)) - G(\mathbf{t}; F(\mathbf{t} + p; x))\|_Z + L_{B'}\|F(\mathbf{t} + p; x) - F(\mathbf{t}; x)\|_Y,\end{aligned}$$

and the sub-additivity of operation $\limsup_{|\mathbf{t}| \to \infty} \cdot$. ■

We end this section by noting that the space of remotely almost periodic sequences $RDAP(\mathbb{Z}^n : Y)$ is, in fact, a closed subspace of the Banach space of bounded sequences on $\mathbb{Z}^n$ so that $RDAP(\mathbb{Z}^n : Y)$ is a Banach space when endowed by the sup-norm.

4. Applications to the abstract Volterra difference equations

In this section, we will provide some applications of our results and introduced notion to the abstract Volterra difference equations. We divide the material into two individual subsections.

4.1. On the abstract difference equation $u(k + 1) = Au(k) + f(k)$, its fractional and multi-dimensional analogues

In [3, Section 3], D. Araya, R. Castro and C. Lizama have investigated the almost automorphic solutions of the first-order linear difference equation

$$u(k + 1) = Au(k) + f(k), \quad k \in \mathbb{Z}, \tag{4.1}$$

where $A \in L(X)$ and $(f_k \equiv f(k))_{k \in \mathbb{Z}}$ is an almost automorphic sequence. In this subsection, we will reconsider the obtained results by assuming that $(f_k)_{k \in \mathbb{Z}}$ is a Levitan almost periodic type sequence (cf. also [10]).

Suppose first that $A = \lambda \mathrm{I}$, where $\lambda \in \mathbb{C}$ and $|\lambda| \neq 1$. We already know that the almost automorphy of sequence $(f_k)_{k\in\mathbb{Z}}$ implies the existence of a unique almost automorphic solution $u(\cdot)$ of (4.1), given by

$$u(k) = \sum_{m=-\infty}^{k} \lambda^{k-m} f(k-1), \quad k \in \mathbb{Z}, \tag{4.2}$$

if $|\lambda| < 1$, and

$$u(k) = -\sum_{m=k}^{\infty} \lambda^{k-m-1} f(k), \quad k \in \mathbb{Z}, \tag{4.3}$$

if $|\lambda| > 1$. We also have the following:

Proposition 4.1. *Suppose that $\rho = T \in L(X)$ and $f(\cdot)$ is a bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence). Then a unique bounded Levitan pre-(I', T)-almost periodic solution (Levitan T-almost periodic solution) of* (4.1) *is given by* (4.2) *if* $|\lambda| < 1$, *and* (4.3) *if* $|\lambda| > 1$.

Proof. We will only prove that the sequence $(u(k))_{k\in\mathbb{Z}}$ is bounded and Levitan pre-(I', T)-almost periodic provided that $|\lambda| < 1$ and $f(\cdot)$ is a bounded Levitan pre-(I', T)-almost periodic sequence. This is clear for the boundedness; suppose now that the numbers $\epsilon > 0$ and $N > 0$ are fixed. Then there exists a natural number $v' \in \mathbb{N} \setminus \{1\}$ such that

$$\sum_{v=v'}^{\infty} |\lambda|^v \|f(j+\tau-v-1) - Tf(j-v-1)\| \leq (1+\|T\|)\|f\|_\infty \sum_{v=v'}^{\infty} |\lambda|^v < \epsilon/2, \tag{4.4}$$

where $\tau \in \mathbb{Z}$. Set $N' := N + 1 + v'$. Let $\tau \in I'$ be any $(\epsilon(1-|\lambda|)/2, N')$-almost period of the sequence $(f(k))_{k\in\mathbb{Z}}$, with the meaning clear. Then we have

$$\begin{aligned}
\|u(j+\tau) - Tu(j)\| &\leq \sum_{v=0}^{\infty} |\lambda|^v \|f(j+\tau-v-1) - Tf(j-v-1)\| \\
&\leq \sum_{v=0}^{v'-1} |\lambda|^v \|f(j+\tau-v-1) - Tf(j-v-1)\| \\
&+ \sum_{v=v'}^{\infty} |\lambda|^v \|f(j+\tau-v-1) - Tf(j-v-1)\| \\
&\leq \sum_{v=0}^{v'-1} |\lambda|^v (\epsilon(1-|\lambda|)/2) + (\epsilon/2) \leq \epsilon, \quad j \in \mathbb{Z},\ |j| \leq N.
\end{aligned}$$

This implies the required conclusion. ■

Similarly, we can prove the following (without going into further details, we will only note that the statement of [3, Theorem 3.2] can be simply reformulated for the bounded Levitan T-almost periodic type sequences, as well):

Proposition 4.2. *Suppose that $\rho = T \in L(X)$, $A \in L(X)$ and $f(\cdot)$ is a bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence) and $\|A\| < 1$. Then there exists a unique bounded Levitan pre-(I', T)-almost periodic solution (Levitan T-almost periodic solution) of* (4.1).

Before investigating some fractional difference equations below, we would like to make the following important observations:

Remark 4.3. *Suppose that there exist two finite real constants $M \geq 1$ and $k \in \mathbb{N}$ such that $\|f(j)\| \leq M(1+|j|)^k$ for all $j \in \mathbb{Z}$. Then the solution $u(\cdot)$ from* Proposition 4.1 *is still well-defined and we have $u(j) = \sum_{v=0}^{\infty} \lambda^v f(j-v-1)$ for all $j \in \mathbb{Z}$, so that*

$$\|u(j)\| \leq M \sum_{v=0}^{\infty} |\lambda|^v (1+|j|+|\, v|)^k \leq M3^{k-1} \sum_{v=0}^{\infty} |\lambda|^v (1+|j|)^k$$
$$+M3^{k-1} \sum_{v=0}^{\infty} |\lambda|^v v^k \leq M'(1+|j|)^k, \quad j \in \mathbb{Z},$$

for some finite real constant $M' \geq 1$. But, it is not clear how we can prove that $u(\cdot)$ is Levitan pre-(I', T)-almost periodic (Levitan T-almost periodic); in the newly arisen situation, the main problem is the existence of a sufficiently large natural number $v' \in \mathbb{N}$, depending only on $\epsilon > 0$ and $N > 0$, such that (4.4) *holds true. We have not been able to find a solution of this problem even for the Levitan almost periodic sequence $(F(k) \equiv 1/(2+\cos k + \cos(\sqrt{2}k)))_{k \in \mathbb{Z}}$ from* Example 2.5*, with $I' = \mathbb{Z}$ and $T = \mathrm{I}$.*

1. Fractional analogues of $u(k+1) = Au(k) + f(k)$. In [2], E. Alvarez, S. Díaz and C. Lizama have recently analyzed the existence and uniqueness of (N, λ)-periodic solutions for the abstract fractional difference equation

$$\Delta^{\alpha} u(k) = Au(k+1) + f(k), \quad k \in \mathbb{Z}, \tag{4.5}$$

where A is a closed linear operator on X, $0 < \alpha < 1$ and $\Delta^{\alpha} u(k)$ denotes the Caputo fractional difference operator of order α; see [2, Definition 2.3] for the notion. We will use the same notion and notation as in the above-mentioned paper.

Let A be a closed linear operator on X such that $1 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A, and let $\|(\mathrm{I} - A)^{-1}\| < 1$. Due to [2, Theorem 3.4], we know that A generates a discrete (α, α)-resolvent sequence $\{S_{\alpha,\alpha}(v)\}_{v \in \mathbb{N}_0}$ such that $\sum_{v=0}^{+\infty} \|S_{\alpha,\alpha}(v)\| < +\infty$. Furthermore, if $(f_k)_{k \in \mathbb{Z}}$ is a bounded sequence, then we know that the function

$$u(k) = \sum_{l=-\infty}^{k-1} S_{\alpha,\alpha}(k-1-l) f(l), \quad k \in \mathbb{Z} \tag{4.6}$$

is a mild solution of (4.5). Since $\sum_{v=0}^{+\infty} \|S_{\alpha,\alpha}(v)\| < +\infty$, the argumentation contained in the proof of Proposition 4.1 enables one to deduce the following analogue of this result:

Proposition 4.4. *Suppose that $\rho = T \in L(X)$ and $f(\cdot)$ is a bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence). Then a mild solution of* (4.5)*, given by* (4.6)*, is bounded Levitan pre-(I', T)-almost periodic (Levitan T-almost periodic).*

Before proceeding further, we will only note that we can similarly analyze the existence and uniqueness of Levitan T-almost periodic type solutions for the following class of Volterra difference equations with infinite delay:

$$u(k+1) = \alpha \sum_{l=-\infty}^{k} a(k-l) u(l) + f(k), \quad k \in \mathbb{Z},\ \alpha \in \mathbb{C};$$

cf. [1, Theorem 3.1, Theorem 3.3] for more details in this direction.

2. Multi-dimensional analogues of $u(k+1) = Au(k) + f(k)$. In [10, Subsection 4.3], we have briefly explained how the results established so far can be employed in the analysis of some multi-dimensional analogues of the abstract difference equation $u(k+1) = Au(k) + f(k)$.

In the first concept, we assume that $f : \mathbb{Z}^n \to X$, $\lambda_1, \ \lambda_2, ..., \ \lambda_n$ are given complex numbers and

$$\max(|\lambda_1|, |\lambda_2|, ..., |\lambda_n|) < 1.$$

Consider the function

$$u(k_1, k_2, ..., k_n) := \sum_{l_1 \leq k_1, l_2 \leq k_2, ..., l_n \leq k_n} \lambda_1^{k_1 - l_1} \lambda_2^{k_2 - l_2} \cdot ....\lambda_n^{k_n - l_n} f(l_1 - 1, l_2 - 1, ..., l_n - 1)$$
$$= \sum_{v_1 \geq 0, v_2 \geq 0, ..., v_n \geq 0} \lambda_1^{v_1} \lambda_2^{v_2} \cdot .... \cdot \lambda_n^{v_n} f(k_1 - v_1 - 1, k_2 - v_2 - 1, ..., k_n - v_n - 1) \tag{4.7}$$

defined for any $(k_1, k_2, ..., k_n) \in \mathbb{Z}^n$. Then it is not difficult to find the form of function $F : \mathbb{Z}^n \to X$ such that

$$u(k_1 + 1, k_2 + 1, ..., k_n + 1) = \lambda_1 \lambda_2 ... \lambda_n \cdot u(k_1, k_2, ..., k_n) + F(k_1, k_2, ..., k_n), \tag{4.8}$$

for all $(k_1, k_2, ..., k_n) \in \mathbb{Z}^n$. Arguing as in the proof of Proposition 4.1, we may conclude the following: If $\rho = T \in L(X)$ and $f(\cdot)$ is a bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence), then a mild solution of (4.8), given by (4.7), is bounded Levitan pre-(I', T)-almost periodic (Levitan T-almost periodic).

In the second concept, we consider the solution $u_j : \mathbb{Z} \to X$ of the equation $u_j(k+1) = \lambda u_j(k) + f_j(k)$, $k \in \mathbb{Z}$, where $f_j(\cdot)$ is a bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence) for $1 \leq j \leq n$ and $\lambda \in \mathbb{C}$ satisfies $|\lambda| < 1$. Define $u(k_1, ..., k_n) := u_1(k_1) + u_2(k_2) + ... + u_n(k_n)$ and $f(k_1, ..., k_n) := f_1(k_1) + f_2(k_2) + ... + f_n(k_n)$ for all $k_j \in \mathbb{Z}$ ($1 \leq j \leq n$). Then we have

$$u(k_1 + 1, ..., k_n + 1) = \lambda u(k_1, ..., k_n) + f(k_1, ..., k_n), \quad (k_1, ..., k_n) \in \mathbb{Z}^n;$$

moreover, the sequence $u(\cdot)$ is likewise bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence); here, $\rho = T \in L(X)$.

Before proceeding to the next subsection, we will only observe that all results established in this subsection can be formulated if the term "bounded Levitan pre-(I', T)-almost periodic" is replaced with the term "remotely (I', T)-almost periodic". Then the solution $u(\cdot)$ will be also remotely (I', T)-almost periodic; for example, in the case of consideration of Proposition 4.1, we can apply the following computation:

$$\limsup_{|j| \to +\infty} \|u(j + \tau) - Tu(j)\| \leq \sum_{v=0}^{\infty} |\lambda|^v \limsup_{|j| \to +\infty} \|f(j + \tau - v - 1) - Tf(j - v - 1)\|$$
$$= \sum_{v=0}^{\infty} |\lambda|^v \limsup_{|j| \to +\infty} \|f(j + \tau) - Tf(j)\| \leq \epsilon \sum_{v=0}^{\infty} |\lambda|^v,$$

where τ is a remote ϵ-almost period of the forcing term $f(\cdot)$.

4.2. The existence and uniqueness of remotely ρ-almost periodic type solutions for the equation (1.2)

We start this subsection by stating the following result concerning the inhomogeneous discrete dynamical system (1.2):

Theorem 4.5. *Let $I' \subseteq \mathbb{Z}$, $\inf I' = -\infty$ and $\sup I' = +\infty$. Assume that $f : \mathbb{Z} \to \mathbb{R}^n$ is bounded and quasi-asymptotically (I', T)-almost periodic, where $T \in L(\mathbb{C}^n)$, and the homogeneous part of* (1.2) *admits an exponential dichotomy. If for each $p \in I'$ we have*

$$\limsup_{|t| \to +\infty} \sum_{j \in \mathbb{Z}} \left\| G(t + p, j + p) - G(t, j) \right\| = 0, \tag{4.9}$$

then the bounded solution $x(t)$ of (1.2)*, given by* (1.3)*, is quasi-asymptotically (I', T)-almost periodic.*

Proof. By Theorem 1.3, the bounded solution of (1.2) is given by

$$x(t) = \sum_{j=-\infty}^{\infty} G(t, j+1) f(j).$$

Further on, we have:

$$\begin{aligned}
\|x(t+p) - Tx(t)\| &= \left\| \sum_{j=-\infty}^{\infty} G(t+p, j+1) f(j) - T \sum_{j=-\infty}^{\infty} G(t, j+1) f(j) \right\| \\
&= \left\| \sum_{j=-\infty}^{\infty} G(t+p, j+p+1) f(j+p) - T \sum_{j=-\infty}^{\infty} G(t, j+1) f(j) \right\| \\
&\leq \left\| \sum_{j=-\infty}^{\infty} \left(G(t+p, j+p+1) - G(t, j+1)\right) f(j+p) \right\| \\
&+ \left\| \sum_{j=-\infty}^{\infty} G(t, j+1) \left(f(j+p) - Tf(j)\right) \right\| \\
&\leq \|f\|_{\infty} \sum_{j=-\infty}^{\infty} \|G(t+p, j+p+1) - G(t, j+1)\| \\
&+ \sum_{j=-\infty}^{\infty} \beta (1+\alpha)^{-|t-j-1|} \|f(j+p) - Tf(j)\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\limsup_{t\to\pm\infty} \|x(t+p) - x(t)\| &\leq \|f\|_{\infty} \limsup_{t\to\pm\infty} \sum_{j=-\infty}^{\infty} \|G(t+p, j+p+1) - G(t, j+1)\| \\
&+ \limsup_{t\to\pm\infty} \sum_{j=-\infty}^{\infty} \beta (1+\alpha)^{-|t-j-1|} \|f(j+p) - f(j)\|.
\end{aligned}$$

Let $\epsilon > 0$ be given. Then there exists $l > 0$ such that every interval I of length l contains a point p such that there exists an integer $M(\epsilon, p) > 0$ such that

$$\|f(j+p) - Tf(j)\| \leq \epsilon, \quad |j| \geq M(\epsilon, p). \tag{4.10}$$

This implies

$$\begin{aligned}
\limsup_{t\to\pm\infty} \|x(t+p) - x(t)\| &\leq \|f\|_{\infty} \limsup_{t\to\pm\infty} \sum_{j=-\infty}^{\infty} \|G(t+p, j+p+1) - G(t, j+1)\| \\
&+ 2\|f\|_{\infty} \limsup_{t\to\pm\infty} \sum_{|j|<M(\epsilon,p)} \beta (1+\alpha)^{-|t-j-1|} \\
&+ \epsilon \limsup_{t\to\pm\infty} \sum_{|j|\geq M(\epsilon,p)} \beta (1+\alpha)^{-|t-j-1|}
\end{aligned}$$

$$= \|f\|_\infty \limsup_{t\to\pm\infty} \sum_{j=-\infty}^{\infty} \|G(t+p,j+p+1)-G(t,j+1)\|$$
$$+ \epsilon \limsup_{t\to\pm\infty} \sum_{|j|\geq M(\epsilon,p)} \beta(1+\alpha)^{-|t-j-1|}$$
$$\leq \|f\|_\infty \limsup_{t\to\pm\infty} \sum_{j=-\infty}^{\infty} \|G(t+p,j+p+1)-G(t,j+1)\|$$
$$+ 2\beta\epsilon \sum_{j\in\mathbb{Z}} (1+\alpha)^{-|j|}.$$

An application of (4.9) completes the proof. ■

Remark 4.6. *The assumption that for each $p \in I'$ we have* (4.9) *is a little bit redundant. This assumption holds if the Green function $G(t,s)$ is bi-periodic in the usual sense, with appropriately chosen set I'; in particular, this situation occurs if the functions $A_\pm(\cdot)$ from the formulation of [15, Theorem 2] are p-periodic for some $p \in \mathbb{N}$ (see the equation [15, (21), Lemma 2]), when we can choose $I' := p\mathbb{N}$.*

Consider now the situation in which the functions $A_\pm(\cdot)$ from the formulation of [15, Theorem 2] are remotely almost periodic and the sequence $f(\cdot)$ is remotely almost periodic ($I' = \mathbb{Z}$, $\rho = \mathrm{I}$). Then the remotely almost periodic extension $\tilde{f}(\cdot)$ of the sequence $f(\cdot)$ to the real line can share the same set of remote ϵ-periods with the functions $A_\pm(\cdot)$. We can apply again the equation [15, (21), Lemma 2] and a simple calculation in order to see that the solution $x(\cdot)$ will be remotely almost periodic.

Without going into further details, we would like to emphasize here that the proofs of [15, Theorem 3, Theorem 4] are not completely correct because the authors have not proved that, in general case, there exists a common set of remote ϵ-bi-almost periods of $G(t,s)$ and remote ϵ-almost periods of forcing term $f(\cdot)$.

We continue by stating the following result:

Theorem 4.7. *Consider the nonlinear discrete dynamical system*

$$x(t+1) = A(t)x(t) + g(t, x(t)), \; x(t_0) = x_0, \tag{4.11}$$

where $g : \mathbb{Z}\times\mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{B}$-remotely almost periodic with $\mathcal{B}$ being the collection of all bounded subsets of $\mathbb{R}^n$, and the homogeneous part of (4.11) *admits an exponential dichotomy which satisfies that for each $p \in \mathbb{Z}$ we have* (4.9)*. If the function $g(\cdot;\cdot)$ satisfies the Lipschitz condition*

$$\|g(t,x) - g(t,y)\| \leq L\|x-y\| \text{ for all } x \text{ and } y \in \mathbb{R}^n,$$

and if

$$L\left(\frac{2\beta}{\alpha} + \beta\right) = \lambda < 1,$$

then the functional system (4.11) *has a unique remotely almost periodic solution.*

Proof. Suppose that $x(\cdot)$ is remotely almost periodic. Then Theorem 3.4 implies that the function $g(\cdot; x(\cdot))$ is remotely almost periodic. Let us introduce the mapping $H : RDAP(\mathbb{Z} : \mathbb{R}^n) \to RDAP(\mathbb{Z} : \mathbb{R}^n)$ by

$$[H(x(\cdot))](t) := \sum_{j=-\infty}^{\infty} G(t, j+1)\, g(j, x(j)), \quad t \in \mathbb{Z}.$$

Then Theorem 4.5 indicates that H maps $RDAP(\mathbb{Z}:\mathbb{R}^n)$ into itself. If $x,\ y\in RDAP(\mathbb{Z}:\mathbb{R}^n)$, then we have

$$\begin{aligned}\|H(x)-H(y)\| &\leq \sum_{j=-\infty}^{\infty}\|G(t,j+1)\|\cdot\|g(j,x(j))-g(j,y(j))\| \\ &= \sum_{j=-\infty}^{t-1}\|G(t,j+1)\|\cdot\|g(j,x(j))-g(j,y(j))\| \\ &+ \sum_{j=t}^{\infty}\|G(t,j+1)\|\cdot\|g(j,x(j))-g(j,y(j))\| \\ &\leq L\|x-y\|\left(\sum_{j=-\infty}^{t-1}\beta(1+\alpha)^{j+1-t}+\sum_{j=t}^{\infty}\beta(1+\alpha)^{t-j-1}\right) \\ &= L\|x-y\|\left(\frac{2\beta}{\alpha}+\beta\right) \\ &< \lambda\|x-y\|,\end{aligned}$$

which shows that H is a contraction. The Banach fixed point theorem implies that the nonlinear discrete dynamical system (4.11) has a unique remotely almost periodic solution. ■

Now we turn our attention into a more specific discrete dynamical system, which is a non-convolution type Volterra difference system with infinite delay given in the form

$$x(t+1)=A(t)x(t)+\sum_{j=-\infty}^{t}B(t,j)x(j)+f(t),\ t\in\mathbb{Z}, \tag{4.12}$$

where A and B are $n\times n$ matrix functions and $f(\cdot)$ is a vector function. Indeed, almost periodic solutions of Volterra difference equations have taken prominent attention in the existing literature, and there is a vast literature based on the existence of discrete almost periodic solutions for numerous kind of Volterra difference equations. In pioneering paper of S. Elaydi (see [6]) the investigation of sufficient conditions for the existence of discrete almost periodic solutions was stated as an open problem, and [12] (2018) provided a solution to this open problem by using the discrete variant of exponential dichotomy and the fixed point theory. It is clear that the space $RDAP(\mathbb{Z}:\mathbb{R}^n)$ is a much more larger space than the space of discrete almost periodic functions. In this paper, we consider the remotely almost periodic solutions of (4.12).

By a remotely almost periodic solution of the Volterra system (4.12), we mean a vector-valued remotely almost periodic function $x^{\xi}(\cdot)$ on $\mathbb{Z}$, which satisfies (4.12) for all $t\in\mathbb{Z}_+$ and $x^{\xi}(t)=\xi(t)$ for all $t\in\mathbb{Z}_-$, where $\mathbb{Z}_-$ is the set of negative integers ($\mathbb{Z}_+$ is the set of nonnegative integers), and $\xi:\mathbb{Z}_-\to\mathbb{R}^n$ is the bounded initial vector function with $\sup_{t\in\mathbb{Z}_-}|\xi(t)|<U_{\xi}<\infty$.

Initially, we make the following assumption:

A1 The homogeneous part of the Volterra system (4.12) admits an exponential dichotomy.

As in [12], we define the following mapping

$$\left(Tx^{\xi}\right)(t):=\begin{cases}\xi(t)\ ,\ t\in\mathbb{Z}_- \\ \displaystyle\sum_{j=-\infty}^{\infty}G(t,j+1)W(j,x(j))\ ,\ t\in\mathbb{Z}_+\end{cases},$$

where

$$W(j,x(j)):=\sum_{k=-\infty}^{j}B(j,k)x(k)+f(j).$$

In the remainder of the manuscript, we assume the following conditions:

A2 The sequence $f(\cdot)$ is remotely almost periodic.

A3 For each $p \in \mathbb{Z}$, (4.9) holds with the function $G(\cdot;\cdot)$ replaced by the function $B(\cdot;\cdot)$. Also, we ask that there exists a positive constant $U_B > 0$ such that

$$0 < \sup_{t \in \mathbb{Z}_+} \sum_{k=-\infty}^{t} \|B(t,k)\| \le U_B < \infty. \tag{4.13}$$

A4 For each $p \in \mathbb{Z}$, we have (4.9).

The following result follows from an application of Theorem 4.5:

Lemma 4.8. *If the function $x(\cdot)$ is remotely almost periodic, then $W(\cdot, x(\cdot))$ is remotely almost periodic, too.*

Theorem 4.9 (Schauder)**.** *Let $\mathbb{B}$ be a Banach space. Assume that K is a closed, bounded and convex subset of $\mathbb{B}$. If $T : K \to K$ is a compact operator, then T has a fixed point in K.*

In order to establish the final outcome of our paper, we introduce the following set

$$\Theta_U := \left\{ x^\xi \in RDAP(\mathbb{Z} : \mathbb{R}^n) : \left\| x^\xi \right\| \le U \right\}$$

for a fixed positive constant $U > 0$. Clearly, Θ_U is a bounded, closed and convex subset of $RDAP(\mathbb{Z} : \mathbb{R}^n)$.

Theorem 4.10. *Assume that the conditions* (A1-A4) *are satisfied. Then the Volterra difference system* (4.12) *has a remotely almost periodic solution.*

Proof. As the initial task, we have to show that $T : \Theta_U \to \Theta_U$. Pick $x^\xi \in \Theta_U$. Then, $W(\cdot, x(\cdot))$ is remotely almost periodic, and consequently, $T\left(x^\xi\right)(\cdot)$ is remotely almost periodic. We skip the proof of this assertion since one may easily show this claim by exactly repeating the same steps of the proof of Theorem 4.5. Further on, we have

$$\begin{aligned} \|(Tx)(t)\| &\le \sum_{j=-\infty}^{\infty} \|G(t, j+1)\| \cdot \|W(j, x(j))\| \\ &\le U_W \sum_{j=-\infty}^{\infty} \|G(t, j+1)\| \\ &\le U_W \left(\frac{2\beta}{\alpha} + \beta \right), \end{aligned}$$

where U_W stands for the upper bound of the remotely almost periodic function W. Set

$$U := \max \left\{ U_\xi, U_W \left(\frac{2\beta}{\alpha} + \beta \right) \right\},$$

and observe that T maps Θ_U into itself. Suppose now that $\varphi_1,\ \varphi_2 \in \Theta_U$ and define $\delta = \delta(\varepsilon) > 0$ by

$$\delta := \frac{\varepsilon}{U_B \left(\frac{2\beta}{\alpha} + \beta \right)}.$$

Next, we pursue the proof by showing that T is continuous. If $\|\varphi_1 - \varphi_2\| < \delta$, then we have

$$\begin{aligned}\|(T\varphi_1)(t) - (T\varphi_2)(t)\| &\leq \sum_{j=-\infty}^{\infty} \|G(t, j+1)\| \cdot \|W(j, \varphi_1(j)) - W(j, \varphi_2(j))\| \\ &\leq U_B \|\varphi_1 - \varphi_2\| \sum_{j=-\infty}^{\infty} \|G(t, j+1)\| \\ &\leq U_B \|\varphi_1 - \varphi_2\| \left(\frac{2\beta}{\alpha} + \beta\right) \\ &< \varepsilon, \quad t \in \mathbb{Z},\end{aligned}$$

which implies the continuity of T.

As the final step of our proof, we aim to show that $T(\Theta_U)$ is precompact by using diagonalization. Suppose that the sequence $\{x_k\} \in \Theta_U$, and consequently, $\{x_k(t)\}$ is a bounded sequnce for $t \in \mathbb{Z}$. Thus, it has a convergent subseqeunce $\{x_k(t_k)\}$. By repeating the diagonalization for each $k \in \mathbb{Z}_+$, we get a convergent subseqeunce $\{x_{k_l}\}$ of $\{x_k\}$ in Θ_U. Since T is continuous, $\{T(x_k)\}$ has a convergent subsequence in $T(\Theta_U)$; therefore, $T(\Theta_U)$ is precompact. The conclusion follows from Schauder's theorem, which shows that there exists a function $x \in \Theta_U$ so that $(Tx^{\xi})(t) = x(t)$ for all $t \in \mathbb{Z}_+$. Equivalently, the non-convolution type Volterra difference system has a remotely almost periodic solution. ■

We can similarly analyze the existence of discrete almost automorphic solutions to (4.12).

5. Conclusions and final remarks

In this paper, we have investigated the class of Levitan ρ-almost periodic type sequences and the class of remotely ρ-almost periodic type sequences. We have provided many structural results, remarks and useful examples about the introduced notion. Several applications of established theoretical results to the abstract Volterra difference equations are given.

Let us finally mention a few topics not considered in our previous work and some perspectives for further investigations of the abstract Volterra difference equations.

1. Many recent papers analyze the class of almost periodic functions in view of the Lebesgue measure μ; cf. [16] and references cited therein. In this paper, we will not consider the discretizations of the almost periodic functions in view of the Lebesgue measure μ; cf. also [16, Lemma 2.8].

2. Suppose that $\emptyset \neq I \subseteq \mathbb{Z}^n$, $\emptyset \neq I' \subseteq \mathbb{Z}^n$, $i + i' \in I$ for all $i \in I$, $i' \in I'$ and $F : I \times X \to Y$. The following notion is also meaningful: a sequence $F(\cdot;\cdot)$ is said to be Bebutov-$(\mathcal{B}, I', \rho)$-almost periodic if and only if, for every $\epsilon > 0$, $B \in \mathcal{B}$ and $N > 0$, there exist a sequence $(\tau_k)_{k\in\mathbb{N}}$ in I' such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and a positive integer $k_0 \in \mathbb{N}$ such that, for every $x \in B$ and $i \in I$ with $|i| \leq N$, there exists $y_{i;x} \in \rho(F(i;x))$ such that

   $$\left\|F\bigl(i + \tau_k; x\bigr) - y_{i;x}\right\| \leq \epsilon, \quad x \in B,\ k \geq k_0.$$

   We will skip all details concerning the class of Bebutov-$(\mathcal{B}, I', \rho)$-almost periodic sequences.

3. It is worth noting that the notion of quasi-asymptotically almost periodicity and the notion of remote almost periodicity have not been considered in the sense of Bochner's approach. We can also consider the following notion: Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$, $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq \mathbb{R}^n$, the sets $\mathbb{D}$ and I' are unbounded, $I + I' \subseteq I$ and $F : I \times X \to Y$ is a given function. Then we say that:

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bochner $(\mathcal{B}, I', \rho)$-almost periodic if and only if, for every $B \in \mathcal{B}$ and for every unbounded sequence $(\tau_k')_{k\in\mathbb{N}}$ in I', there exists a subsequence $(\tau_k)_{k\in\mathbb{N}}$ of $(\tau_k')_{k\in\mathbb{N}}$ such that, for every $x \in B$, there exists a function $G_x \in Y^{\mathbb{D}}$ such that $G_x(\mathbf{t}) \in \rho(F(\mathbf{t}; x))$ for all $\mathbf{t} \in \mathbb{D}$, $x \in B$ and

$$\lim_{k\to+\infty} \limsup_{|\mathbf{t}|\to+\infty;\mathbf{t}\in\mathbb{D}} \sup_{x\in B} \big\| F(\mathbf{t}+\tau_k; x) - G_x(\mathbf{t}) \big\|_Y = 0.$$

(ii) $F(\cdot;\cdot)$ is Bochner $\mathbb{D}$-remotely $(\mathcal{B}, I', \rho)$-almost periodic if and only if $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bochner $(\mathcal{B}, I', \rho)$-almost periodic and, for every $B \in \mathcal{B}$, the function $F(\cdot;\cdot)$ is uniformly continuous on $I \times B$.

We will consider this notion somewhere else.

4. Without going into further details, we will only note that our results can be also applied in the qualitative analysis of solutions to the semilinear abstract difference equation $u(k+1) = Au(k) + f(k, u(k))$ ans its fractional analogue

$$\Delta^{\alpha} u(k) = Au(k+1) + f(k; u(k)), \quad k \in \mathbb{Z};$$

cf. [2] and [10] for more details.

5. As a special case of the notion which has recently been introduced in [9, Definition 2.1, Definition 2.2; Definition 3.1, Definition 3.2], we can also consider some classes of $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, ρ)-periodic type sequences and $(\mathbb{D}, \mathcal{B}, \rho)$-slowly oscillating type sequences. Further analysis of these classes will be carried in a forthcoming research study.

6. Acknowledgement

The authors are thankful to the referee for his valuable suggestions which improved the presentation of the paper.

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 166–196.
http://doi.org/10.26637/mjm11S/011

Square-mean pseudo almost automorphic solutions of infinite class under the light of measure theory

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***R**eceived 14 July 2023; **A**ccepted 25 September 2023*

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. The aim of this work is to present new concept of square-mean pseudo almost automorphic of infinite class using the measure theory. We use the (μ,ν)-ergodic process to define the spaces of (μ,ν)-pseudo almost automorphic processes of infinite class in the square-mean sense. We present many interesting results on those spaces like completeness and composition theorems and we study the existence and the uniqueness of the square-mean (μ,ν)-pseudo almost automorphic solutions of infinite class for of the stochastic evolution equation. We provide an example to illustrate ours results.

AMS Subject Classifications: 47D09, 34G10, 60J65.

Keywords: measure theory; ergodicity; (μ,ν)-pseudo almost automorphic function; evolution equations; partial functional differential equations; Stochastic processes; stochastic evolution equations.

Contents

1. Introduction

In this work, we study the basic properties of the square-mean (μ,ν)-pseudo almost automorphic process using the measure theory and used those results to study the following stochastic evolution equations in a Hilbert space H,

$$dx(t) = [Ax(t) + L(x_t) + f(t)]dt + g(t)dW(t), \tag{1.1}$$

where $A : D(A) \subset H$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on H, $f, g : \mathbb{R} \to L^2(P,H)$ are two stochastic processes, $W(t)$ is a two-sided and standard one-dimensional Brownian notion defined on the filtered probability space $(\Omega,\mathcal{F},P,\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(u) - W(v) \mid u,v \leqslant t\}$ and L is a bounded linear operator from $\mathcal{B}$ into $Ł^2(P,H)$. The phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty,0]$ into X for

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every $t \geqslant 0$, x_t denotes the history function of $\mathcal{B}$ defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in]-\infty, 0]$
We assume $(H, ||\cdot||)$ is real separable Hilbert space and $L^2(P,H)$ is the space of all H-valued random variables x such that

$$\mathbb{E}||x||^2 = \int_\Omega ||x||^2 dP < +\infty.$$

This work is an extension of [10] whose authors have studied equation (4.1) in the deterministic case. Some recent contributions concerning square-mean pseudo almost automorphic solutions for abstract differrential equations similar to equation (4.1) have been made. For example in [7] the authors studied equation(4.1) without the operator L. They showed that the equation has a unique square-mean μ-pseudo almost automorphic mild solution on $\mathbb{R}$ when f and g are square mean pseudo almost automorphic functions.
In [4] the authors studied the square-mean almost automorphic solutions to a class of nonautonomous stochastic differential equations without our operator L and without delay on a separable real Hilbert space. They established the existence and uniqueness of a square-mean almost automorphic mild solution to those nonautonomous stochastic differential equations with the 'Acquistapace-Terreni' conditions.
In [8] The authors established the existence, uniqueness and stability of square-mean μ-pseudo almost periodic(resp. automorphic) mild solution to a linear and semilinear case of the stochastic evolution equations in case when the functions forcing are both continuous and $S^2-\mu-$pseudo almost periodic (resp. automorphic) and verify some suitable assumptions.
This work is organized as follow, in section 2, we study spectral decomposition of phase space then in section 3 we study square-mean (μ,ν)-Pseudo almost automorphic process, in section 4 we study square-mean pseudo almost automorphic solutions of infinite class and we finish with application of our theory.

2. Variation of constants formula and spectral decomposition

In this work, the state space $(\mathcal{B}, |.|_\mathcal{B})$ is a normed linear space of functions mapping $]-\infty,0]$ into $L^2(P,H)$ and satisfying the following fundamental axioms.

$(\mathbf{A}_1)$ There exist a positive constant H and functions $K(.), M(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $u :]-\infty, a] \to L^2(P,H)$, $u_\sigma \in \mathcal{B}$, and $u(.)$ is continuous on $[\sigma, \sigma+a]$, then for every $t \in [\sigma, \sigma+a]$ the following conditions hold

(i) $u_t \in \mathcal{B}$,

(ii) $|u(t)| \leq H|u_t|_\mathcal{B}$, which is equivalent to $|\varphi(0)| \leq H|\varphi|_\mathcal{B}$ for every $\varphi \in \mathcal{B}$

(iii) $|u_t|_\mathcal{B} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |u(s)| + M(t-\sigma)|u_\sigma|_\mathcal{B}$.

$(\mathbf{A}_2)$ For the function $u(.)$ in $(\mathbf{A}_1)$, $t \mapsto u_t$ is a $\mathcal{B}$-valued continuous function for $t \in [\sigma, \sigma+a]$.

(B) The space $\mathcal{B}$ is a Banach space.

Assume that:

$(\mathbf{C_1})$ If $(\varphi_n)_{n\geq 0}$ is a sequence in $\mathcal{B}$ such that $\varphi_n \to 0$ in $\mathcal{B}$ as $n \to +\infty$, then $(\varphi_n(\theta))_{n\geq 0}$ converges to 0 in $L^2(P,H)$.

Let $C(]-\infty,0], L^2(P,H))$ be the space of continuous functions from $]-\infty,0]$ to $L^2(P,H)$. Suppose the following assumptions:

($\mathbf{C_2}$) $\mathcal{B} \subset C(]-\infty, 0], L^2(P, H))$.

($\mathbf{C_3}$) there exists $\lambda_0 \in \mathbb{R}$ such that, for all $\lambda \in \mathbb{C}$ with Re$\lambda > \lambda_0$ and $x \in L^2(P, H)$, $e^{\lambda \cdot}x \in \mathcal{B}$ and

$$K_0 = \sup_{\substack{\text{Re}\lambda > \lambda_0, x \in L^2(P,H) \\ x \neq 0}} \frac{|e^{\lambda \cdot}x|_{\mathcal{B}}}{|x|} < \infty,$$

where $(e^{\lambda \cdot}x)(\theta) = e^{\lambda\theta}x \quad \text{for} \quad \theta \in]-\infty, 0]$ and $x \in L^2(P, H)$.

To equation (4.1), associate the following initial value problem

$$\begin{cases} du_t = [Au(t) + L(u_t) + f(t)]dt + g(t)dW(t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{2.1}$$

where $f : \mathbb{R}^+ \to L^2(P, H)$ is a continuous function.

Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ which defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : \ Ax \in \overline{D(A)}\} \\ A_0 x = Ax \text{ for } \ x \in D(A_0) \end{cases}$$

The following assumption is supposed:

($\mathbf{H_0}$) A satisfies the Hille-Yosida condition.

Lemma 2.1. *[2] A_0 generates a strongly continuous semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A)}$.*

The phase space $\mathcal{B}_A$ of equation (2.1) is defined by

$$\mathcal{B}_A = \{\varphi \in \mathcal{B} : \ \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, the linear operator $\mathcal{U}(t)$ on $\mathcal{B}_A$ is defined by

$$\mathcal{U}(t) = v_t(., \varphi)$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt} v_t = Av(t) + L(v_t) \text{ for } t \geq 0 \\ \\ v_0 = \varphi \in \mathcal{B}. \end{cases}$$

Proposition 2.2. *[3] $(\mathcal{U}(t))_{t \geq 0}$ is a strongly continuous semigroup of linear operators on $\mathcal{B}_A$. Moreover, $(\mathcal{U}(t))_{t \geq 0}$ satisfies, for $t \geq 0$ and $\theta \in]-\infty, 0]$, the following translation property*

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) \textit{ for } t + \theta \geq 0 \\ \\ \varphi(t+\theta) \textit{ for } t + \theta \leq 0. \end{cases}$$

Theorem 2.3. *[3] Assume that $\mathcal{B}$ satisfies (A_1), (A_2), (B), (C_1) and (C_2). Then $\mathcal{A_U}$ defined on $\mathcal{B}_A$ by*

$$\begin{cases} D(\mathcal{A_U}) = \Big\{\varphi \in C^1(]-\infty,0];X)\cap\mathcal{B}_A;\ \varphi'\in\mathcal{B}_A,\ \varphi(0)\in\overline{D(A)}\ \ and\ \ \varphi'(0)=A\varphi(0)+L(\varphi)\Big\} \\ \\ \mathcal{A_U}\varphi=\varphi'\ \ for\ \ \varphi\in D(\mathcal{A_U}). \end{cases}$$

is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ on $\mathcal{B}_A$.

Let $\langle X_0\rangle$ be the space defined by

$$\langle X_0\rangle = \{X_0x:\ x\in X\}$$

where the function X_0x is defined by

$$(X_0x)(\theta) = \begin{cases} 0 & \text{if} \quad \theta\in]-\infty,0[, \\ \\ x & \text{if} \quad \theta = 0. \end{cases}$$

The space $\mathcal{B}_A\oplus\langle X_0\rangle$ equipped with the norm $|\phi+X_0c|_{\mathcal{B}}=|\phi|_{\mathcal{B}}+|c|$ for $(\phi,c)\in\mathcal{B}_A\times X$ is a Banach space and consider the extension $\mathcal{A_U}$ defined on $\mathcal{B}_A\oplus\langle X_0\rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A_U}}) = \Big\{\varphi\in C^1(]-\infty,0];X):\ \varphi\in D(A)\ \text{ and } \varphi'\in\overline{D(A)}\Big\} \\ \widetilde{\mathcal{A_U}}\varphi=\varphi'+X_0(A\varphi+L(\varphi)-\varphi'). \end{cases}$$

Lemma 2.4. *[3] Assume that $\mathcal{B}$ satisfies (A_1), (A_2), (B), (C_1), (C_2) and (C_3). Then, $\widetilde{\mathcal{A_U}}$ satisfies the Hille-Yosida condition on $\mathcal{B}_A\oplus\langle X_0\rangle$.*

Now, start the variation of constants formula associated to equation (2.1).

Let C_{00} be the space of X-valued continuous function on $]-\infty,0]$ with compact support. Assume that:

(**D**) If $(\varphi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{B}$ and converges compactly to φ on $]-\infty,0]$, then $\varphi\in\mathcal{B}$ and $|\varphi_n-\varphi|\to 0$.

Definition 2.5. *The semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if*

$$\sigma(\mathcal{A_U})\cap i\mathbb{R}=\emptyset$$

Let $(S_0(t))_{t\geq 0}$ be the strongly continuous semigroup defined on the subspace

$$\mathcal{B}_0=\{\varphi\in\mathcal{B}:\ \varphi(0)=0\}$$

by

$$(S_0(t)\phi)(\theta) = \begin{cases} \phi(t+\theta) & \text{if} \ \ t+\theta\leq 0 \\ \\ 0 & \text{if} \ \ t+\theta\geq 0 \end{cases}$$

Definition 2.6. *Assume that the space $\mathcal{B}$ satisfies Axioms (B) and (D), $\mathcal{B}$ is said to be a fading memory space, if for all $\varphi\in\mathcal{B}_0$,*

$$|S_0(t)|\to 0\ \ as\ \ t\to+\infty\ in\ \mathcal{B}_0.$$

Moreover, $\mathcal{B}$ is said to be a uniform fading memory space, if

$$|S_0(t)|\to 0\ \ as\ \ t\to+\infty.$$

Lemma 2.7. *If $\mathcal{B}$ is a uniform fading memory space, then the function K can be chosen to be constant and the function M such that $M(t) \to 0$ as $t \to +\infty$.*

Proposition 2.8. *If the phase space $\mathcal{B}$ is a fading memory space, then the space $BC(]-\infty, 0], X)$ of bounded continuous X-valued functions on $]-\infty, 0]$ endowed with the uniform norm topology, is continuous embedding in $\mathcal{B}$. In particular $\mathcal{B}$ satisfies ($\boldsymbol{C_3}$), for $\lambda_0 > 0$.*

For the sequel, make the following assumption:

($\mathbf{H_1}$) $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

($\mathbf{H_2}$) $\mathcal{B}$ is a uniform fading memory space.

Theorem 2.9. *[3] Assume that $\mathcal{B}$ satisfies ($\boldsymbol{A_1}$), ($\boldsymbol{A_2}$), ($\boldsymbol{B}$), ($\boldsymbol{C_1}$) and ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$) hold. Then the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is decomposed on $\mathcal{B}_A$ as follows*

$$\mathcal{U}(t) = \mathcal{U}_1(t) + \mathcal{U}_2(t) \ \ for \ \ t \geq 0$$

where $(\mathcal{U}_1(t))_{t\geq 0}$ is an exponentially stable semigroup on $\mathcal{B}_A$, which means that there are positive constants α_0 and N_0 such that

$$|\mathcal{U}_1(t)| \leq N_0 e^{-\alpha_0 t} |\varphi| \ \ for \ \ t \geq 0 \ \ and \ \ \varphi \in \mathcal{B}_A$$

and $(\mathcal{U}_2(t))_{t\geq 0}$ is compact for for every $t > 0$.

The following result on the spectral decomposition of the phase space $\mathcal{B}_A$ is obtained.

Theorem 2.10. *[3] Assume that $\mathcal{B}$ satisfies ($\boldsymbol{A_1}$), ($\boldsymbol{A_2}$), ($\boldsymbol{B}$), ($\boldsymbol{C_1}$), and ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$) hold. Then the space $\mathcal{B}_A$ is decomposed as a direct sum*

$$\mathcal{B}_A = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restricted semigroup on $\mathcal{U}$ is a group and there exist positive constants $\overline{M}$ and ω such that

$$|\mathcal{U}(t)\varphi| \leq \overline{M} e^{-\omega t} |\varphi| \ \ for \ \ t \geq 0 \ \ and \ \ \varphi \in S$$

$$|\mathcal{U}(t)\varphi| \leq \overline{M} e^{\omega t} |\varphi| \quad for \ \ t \leq 0 \ \ and \ \ \varphi \in U,$$

where S and U are called respectively the stable and unstable space.

Let $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$.

Definition 2.11. *Let $x : \mathbb{R} \to L^2(P, H)$ be a stochastic process.*

1. *x said to be stochastically bounded if there exists $C > 0$ such that*

$$\mathbb{E}||x(t)||^2 \leqslant C \ \forall \ t \in \mathbb{R}.$$

2. *x is said to be stochastically continuous if*

$$\lim_{t\to s} \mathbb{E}||x(t) - x(s)||^2 = 0 \ \forall \ s \in \mathbb{R}.$$

Denote by $SBC(\mathbb{R}, L^2(P,H))$, the space of all stochastically bounded and continuous process. Otherwise, this space endowed the following norm

$$||x||_\infty = \sup_{t\in\mathbb{R}} \left(\mathbb{E}||x(t)||^2\right)^{\frac{1}{2}}$$

is a Banach space.

Definition 2.12. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be square-mean $(\mu,\nu)-$ ergodic if $f \in SBC(\mathbb{R}, L^2(P,H))$ and satisfied*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau} \mathbb{E}||f(\theta)||^2 d\mu(t) = 0.$$

We denote by $\mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu)$, the space of all such process.

Definition 2.13. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be square-mean $(\mu,\nu)-$ ergodic of infinite class if $f \in SBC(\mathbb{R}, L^2(P,H))$ and satisfied*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) = 0.$$

We denote by $\mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$, the space of all such process.
For $\mu, \nu \in \mathcal{M}$ and $a \in \mathbb{R}$, we denote by μ_a and ν_a positives measures on $(\mathbb{R}, \mathcal{N})$ respectively defined by

$$\mu_a(A) = \mu(a+b : b \in A) \text{ and } \nu_a(A) = \nu(a+b : b\in A) \text{ for } A \in \mathcal{N}. \tag{2.2}$$

From $\mu, \nu \in \mathcal{M}$, we formulate the following hypothesis.
(**H$_2$**): For all $a \in \mathbb{R}$, there exists $\beta > 0$ and a bounded intervall I such that $\mu_a(A) \leqslant \beta\mu(A)$ when $A \in \mathcal{N}$ satisfies $A \cap I = \emptyset$.
(**H$_3$**) For all a, b and $c \in \mathbb{R}$, such that $0 \leqslant a < b \leq c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \geqslant \delta_0 \implies \mu(a+\delta, b+\delta) \geqslant \alpha_0 \mu(\delta, c+\delta).$$

(**H$_4$**) Let $\mu, \nu \in \mathcal{M}$ be such that $\limsup_{\tau\to+\infty} \dfrac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \alpha < \infty$.

Proposition 2.14. *Assume that (**H$_4$**) holds. Then $\mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ is a Banach space with the norm $||\cdot||_\infty$.*

Proof. It is easy to see that $\mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$ is a vector subspace of $SBC(\mathbb{R}, L^2(P,H)))$. To complete the proof, it is enough to prove that $\mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$ is closed in $SBC(\mathbb{R}; L^2(P,H))$. Let $(f_n)_n$ be a sequence in $\mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$ such that $\lim_{n\to+\infty} f_n = f$ uniformly in $SBC(\mathbb{R}, L^2(P,H))$. From $\nu(\mathbb{R}) = +\infty$, it follows $\nu([-\tau,\tau]) > 0$ for τ sufficiently large. Let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|f_n - f\|_\infty < \varepsilon$. Let $n \geq n_0$, then

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]} \mathbb{E}||f(\theta)||^2\Big)d\mu(t) &\leq \frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]} \mathbb{E}||f_n(\theta)-f(\theta)||^2\Big)d\mu(t)\\
&\quad+\frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]} \mathbb{E}||f_n(\theta)||^2\Big)d\mu(t)\\
&\leq \frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}} \mathbb{E}||f_n(t)-f(t)||^2\Big)d\mu(t)\\
&\quad+\frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]} \mathbb{E}||f_n(\theta)||^2\Big)d\mu(t)\\
&\leq 2\|f_n - f\|_\infty^2 \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} + \frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]} \mathbb{E}|f_n(\theta)|\Big)d\mu(t).
\end{aligned}$$

Consequently

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\leq 2\alpha\varepsilon\ \text{ for any }\varepsilon>0.\blacksquare$$

The following theorem is a characterization of square-mean (μ,ν)-ergodic processes(eventually $I=\emptyset$).

Theorem 2.15. *Assume that $f\in SBC(\mathbb{R},L^2(P,H))$. Then the following assertions are equivalent:*

i) $\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$

ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2d\mu(t)=0$

iii) For any $\varepsilon>0$, $\displaystyle\lim_{\tau\to+\infty}\frac{\mu\Big\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}}{\nu([-\tau,\tau]\setminus I)}=0$

Proof. The proof is made like the proof of Theorem(2.13) in [6].
First, we will show that i) is equivalent to ii).
Denote by $A=\nu(I)$, $B=\displaystyle\int_I\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)$. A and B belong to $\mathbb{R}$, since the interval I is bounded and the process f is stochastically bounded and continuous. For $\tau>0$ such that $I\subset[-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I)>0$, it follows

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)=\frac{1}{\nu([-\tau,\tau])-A}\Big[\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)-B\Big]$$
$$=\frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau])-A}\Big[\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)-\frac{B}{\nu([-\tau,\tau])}\Big].$$

From above equalities and the fact that $\nu(\mathbb{R})=+\infty$, ii) is equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)=0,$$

that is i).

Then, we will show that iii) implies ii).
Denote by A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}\ \text{ and }\ B_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I):\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\leq\varepsilon\Big\}.$$

Assume that iii) holds, that is

$$\lim_{\tau\to+\infty}\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}=0. \tag{2.3}$$

From the equality

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)=\int_{A_\tau^\varepsilon}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)+\int_{B_\tau^\varepsilon}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t),$$

Then for τ sufficiently large

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\leq\|f\|_\infty^2\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}.$$

By using ($\mathbf{H_4}$), it follows that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\leq\alpha\varepsilon,\quad\text{for any}\quad\varepsilon>0,$$

consequently ii) holds.

Thus, we shall show that ii) implies iii).
Assume that ii) holds. From the following inequality

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\geq\int_{A^\varepsilon_\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)$$
$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\geq\varepsilon\frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau]\setminus I)}$$
$$\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\geq\frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau]\setminus I)},$$

for τ sufficiently large, equation (2.3) is obtained, that is iii).■

Definition 2.16. *Let $f\in SBC(\mathbb{R},L^2(P,H))$ and $\tau\in\mathbb{R}$. We denote by f_τ the function defined by $f_\tau(t)=f(t+\tau)$ for $t\in\mathbb{R}$. A subset $\mathfrak{F}$ of $SBC(\mathbb{R},L^2(P,H))$ is said to translation invariant if for all $f\in\mathfrak{F}$ we have $f_\tau\in\mathfrak{F}$ for all $\tau\in\mathbb{R}$.*

Definition 2.17. *Let μ_1 and $\mu_2\in\mathcal{M}$. μ_1 is said to be equivalent to μ_2 ($\mu_1\sim\mu_2$) if there exist constants α and $\beta>0$ and a bounded interval I(eventually $I=\emptyset$) such that $\alpha\mu_1(A)\leqslant\mu_2(A)\leqslant\beta\mu_1(A)$ for $A\in\mathcal{N}$ satisfying $A\cap I=\emptyset$.*

Remark 2.18. *The relation $\sim$ is an equivalence relation on $\mathcal{M}$.*

Theorem 2.19. *Let $\mu_1,\mu_2,\nu_1,\nu_2\in\mathcal{M}$. If $\mu_1\sim\mu_2$ and $\nu_1\sim\nu_2$, then*
$\mathcal{E}(\mathbb{R},L^2(P,H),\mu_1,\nu_1,\infty)=\mathcal{E}(\mathbb{R},L^2(P,H),\mu_2,\nu_2,\infty)$.

Proof. Since $\mu_1\sim\mu_2$ and $\nu_1\sim\nu_2$ there exist some constants $\alpha_1,\alpha_2,\beta_1,\beta_2>0$ and a bounded interval I (eventually $I=\emptyset$) such that $\alpha_1\mu_1(A)\leq\mu_2(A)\leq\beta_1\mu_1(A)$ and $\alpha_2\nu_1(A)\leq\nu_2(A)\leq\beta_2\nu_1(A)$ for each $A\in\mathcal{N}$ satisfies $A\cap I=\emptyset$ i.e

$$\frac{1}{\beta_2\nu_1(A)}\leq\frac{1}{\nu_2(A)}\leq\frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1\sim\mu_2$ and $\mathcal{N}$ is the Lebesgue σ-field, then for τ sufficiently large, it follows that

$$\frac{\alpha_1\mu_1\Big(\Big\{t\in[-\tau,\tau]\setminus I:\sup\limits_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}\Big)}{\beta_2\nu_1([-\tau,\tau]\setminus I)}\leq\frac{\mu_2\Big(\Big\{t\in[-\tau,\tau]\setminus I:\sup\limits_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}\Big)}{\nu_2([-\tau,\tau]\setminus I)}$$
$$\leq\frac{\beta_1\mu_1\Big(\Big\{t\in[-\tau,\tau]\setminus I:\sup\limits_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}\Big)}{\alpha_2\nu_1([-\tau,\tau]\setminus I)}$$

Consequently by Theorem 3.2, $\mathcal{E}(\mathbb{R},X,\mu_1,\nu_1,\infty)=\mathcal{E}(\mathbb{R},X,\mu_2,\nu_2,\infty)$.■
Let μ, $\nu\in\mathcal{M}$ denote by

$$cl(\mu,\nu)=\{\omega_1,\omega_2:\mu\sim\omega_1\text{ and }\nu\sim\omega_2\}.$$

Lemma 2.20. *[5] Let $\mu \in \mathcal{M}$. Then μ satisfies ($\boldsymbol{H_2}$) if and only if the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 2.21. *[6] ($\boldsymbol{H_3}$) implies for all σ, $\limsup_{\tau\to\infty} \dfrac{\mu([-\tau-\sigma,\tau+\sigma])}{\mu([-\tau,\tau])} < +\infty$.*

Theorem 2.22. *Let $\mu,\nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_2}$). Then $\mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ is translation invariant.*

Proof. The proof of this theorem is inspired of Theorem (3.5) in [5]. Let $f \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ and $a \in \mathbb{R}$. Since $\nu(\mathbb{R}) = +\infty$. there exists $a_0 > 0$ such that $\nu([-\tau-|a|, \tau+|a|]) > 0$ for $|a| \geqslant a_0$. Let us denote by

$$M_a(\tau) = \frac{1}{\nu_a([-\tau,\tau])}\int_{[-\tau,\tau]}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\right)d\mu_a(t)\ \ \forall\tau>0 \text{ and } a\in\mathbb{R},$$

where ν_a is the positive measure defined by equation(4.3). By using Lemma (2.20), it follows that ν and ν_a are equivalent, μ and μ_a are equivalent by using Theorem (2.19) we have $\mathcal{E}(\mathbb{R}, L^2(P,H), \mu_a, \nu_a, \infty) = \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ therefore $f \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu_a, \nu_a, \infty)$ that is $\lim_{\tau\to+\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.
For all $A \in \mathcal{N}$, we denote by $\mathcal{X}_A$ the characteristic function of A, by using definition of the measure μ_a, we obtain that

$$\int_{[-\tau,\tau]}\mathcal{X}_A(t)d\mu_a(t) = \int_{[-\tau,\tau]}\mathcal{X}_A(t)d\mu(t+a) = \int_{[-\tau+a,\tau+a]}d\mu(t) \text{ for all } A\in\mathcal{N}$$

and since $t \mapsto \sup_{\theta\in]\infty,t]}\mathbb{E}||f(\theta)||^2$ is the pointwise limit of an increasing sequence of linear combinations of functions [[12]; Theorem 1.17 p.15], we deduce that

$$\int_{[-\tau,\tau]}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2 d\mu_a(t) = \int_{[-\tau+a,\tau+a]}\sup_{\theta\in]-\infty,t-a]}\mathbb{E}||f(\theta)||^2 d\mu(t).$$

If we denote by $a^+ := \max(a,0)$ and $a^- := \max(-a,0)$ we have $|a|+a = 2a^+$ and $|a|-a = 2a^-$, and then $[-\tau+a-|a|, \tau+a|a|] = [-\tau-2a^-, \tau+2a^+]$. Therefore we obtain

$$M_a(\tau+|a|) = \frac{1}{\nu([-\tau-2a^-,\tau+2a^+])}\int_{[-\tau-2a^-,\tau+2a^+]}\sup_{\theta\in]-\infty,\,t-a]}\mathbb{E}||f(\theta)||^2 d\mu(t). \tag{2.4}$$

From equation (2.4) and the following inequality

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in]-\infty,\,t-a]}\mathbb{E}||f(\theta)||^2 d\mu(t) \leqslant \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau-2a^-,\tau+2a^+]}\sup_{\theta\in]-\infty,\,t-a]}\mathbb{E}||f(\theta)||^2 d\mu(t)$$

we obtain

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in]-\infty,\,t-a]}\mathbb{E}||f(\theta)||^2 d\mu(t) \leqslant \frac{\nu([-\tau-2a^-,\tau+2a^+]}{\nu([-\tau,\tau])}\times M_a(\tau+|a|).$$

That implies ,
$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in]-\infty,\,t-a]}\mathbb{E}||f(\theta)||^2 d\mu(t) \leqslant \frac{\nu([-\tau-2a^-,\tau+2a^+]}{\nu([-\tau,\tau])}\times M_a(\tau+|a|)$$
That implies

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in]-\infty,\,t-a]}\mathbb{E}||f(\theta)||^2 d\mu(t) \leqslant \frac{\nu([-\tau-2|a|,\tau+2|a|])}{\nu([-\tau,\tau])}\times M_a(\tau+|a|). \tag{2.5}$$

From equation(2.4) and equation(2.5) and using Lemma (2.21) we deduce that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in]-\infty,\,t-a]}\mathbb{E}||f(\theta)||^2d\mu(t)=0$$

which equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in]-\infty,\,t]}\mathbb{E}||f(\theta-a)||^2d\mu(t)=0.$$

That is $f_{-a}\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$. We have proved that $f\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ then $f_{-a}\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ for $a\in\mathbb{R}$. That is $\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ is translation invariant.

Proposition 2.23. *Let ν, $\mu\in\mathcal{M}$ satisfy . Then $SPAA(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ is translation invariant, that is for all $\alpha\in\mathbb{R}$ and $f\in SPAA(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$, $f_\alpha\in SPAA(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$.*

Lemma 2.24. *(Ito's isometry). [13] Let $W:[0,T]\times\Omega\to\mathbb{R}$ denote the canonical real-valued Wiener process defined up to time $T>0$, and let $X:[0,T]\times\Omega\to\mathbb{R}$ be a stochastic process that is adapted to the natural filtration $\mathcal{F}_*^W$ of the Wiener process. Then*

$$\mathbb{E}\left[\left(\int_0^T X_t\,\mathrm{d}W_t\right)^2\right]=\mathbb{E}\left[\int_0^T X_t^2\,\mathrm{d}t\right],$$

where $\mathbb{E}$ denotes expectation with respect to classical Wiener measure.

3. Square-Mean (μ,ν)-Pseudo Almost automorphic Process

In this section, we define square-mean (μ,ν)-pseudo almost automorphic and we study their basic properties.

Definition 3.1. *Let $f:\mathbb{R}\to L^2(P,H)$ be a continuous stochastic process. f is said be square-mean almost automorphic process if for every sequence of real numbers $(t'_n)_n$, we can extract a subsequence $(t_n)_n$ such that, for some stochastic process $g:\mathbb{R}\to L^2(P,H)$, we have*

$$\lim_{n\to+\infty}\mathbb{E}||f(t+t_n)-g(t)||^2=0 \text{ for all } t\in\mathbb{R}$$

and

$$\lim_{n\to+\infty}\mathbb{E}||g(t-t_n)-f(t)||^2=0 \text{ for all } t\in\mathbb{R}$$

We denote the space of all such stochastic process by $SAA(\mathbb{R},L^2(P,H))$.

Theorem 3.2. *[11] $SAA(\mathbb{R},L^2(P,H))$ equiped with the norm $||\cdot||_\infty$ is a Banach space.*

Definition 3.3. *Let $f:\mathbb{R}\to L^2(P,H)$ be a bounded continuous stochastic process. f is said be square-mean compact almost automorphic process if for every sequence of real numbers $(t'_n)_n$, we can extract a subsequence $(t_n)_n$ such that, for some stochastic process $h:\mathbb{R}\to L^2(P,H)$, we have*

$$\lim_{n\to+\infty}\mathbb{E}||f(t+t_n)-h(t)||^2=0 \text{ for all } t\in\mathbb{R}$$

and

$$\lim_{n\to+\infty}\mathbb{E}||h(t-t_n)-f(t)||^2=0 \text{ for all } t\in\mathbb{R}$$

uniformly on compact subsets of $\mathbb{R}$. We denote the space of all such stochastic process by $SAA_c(\mathbb{R},L^2(P,H))$.

Theorem 3.4. *$SAA_c(\mathbb{R},L^2(P,H))$ equiped with the norm $||\cdot||_\infty$ is a Banach space.*

Definition 3.5. *A function $f : \mathbb{R} \times L^2(P,H) \to L^2(P,H)$, $(t,x) \mapsto f(t,x)$, which is jointly continuous, is said to be square mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(P,H)$ if for every sequence of real numbers $(t'_n)_n$, there exist a subsequence $(t_n)_n$ such that for some function g*
$\lim_{n\to+\infty} \mathbb{E}||f(t+t_n,x) - g(t,x)||^2 = 0$ and $\lim_{n\to+\infty} \mathbb{E}||g(t-t_n,x) - f(t,x)||^2 = 0$
for each $t \in \mathbb{R}$ and each $x \in L^2(P,H)$.

We denote the space off all such stochastic processes by $SAA(\mathbb{R} \times L^2(P,H), L^2(P,H))$.

Definition 3.6. *Let $\mu, \nu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(P,H)$ be a continuous stochastic process.*
f is said be (μ,ν)-square mean pseudo almost automorphic process if it can be decomposed as follows

$$f = g + \phi,$$

where $g \in SAA(\mathbb{R}, L^2(P,H))$ and $\phi \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu)$.

We denote the space of all such stochastic processes by $SPAA(\mathbb{R}, L^2(P,H), \mu, \nu)$.

Definition 3.7. *Let $\mu, \nu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(P,H)$ be a continuous stochastic process.*
f is said be (μ,ν)-square mean compact pseudo almost automorphic process if it can be decomposed as follows

$$f = g + \phi,$$

where $g \in SAA_c(\mathbb{R}, L^2(P,H))$ and $\phi \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu)$.

We denote the space of all such stochastic processes by $SPAA_c(\mathbb{R}, L^2(P,H), \mu, \nu)$.
Hence, together with Theorem 2.22 and Definition 3.7, we arrive at the following conclusion.

Theorem 3.8. *Let μ, $\nu \in \mathcal{M}$ and $f \in SPAA(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ be such that*

$$f = g + \phi,$$

where $g \in SAA(\mathbb{R}, L^2(P,H))$ and $\phi \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$. If $SPAA(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ is translation invariant, then

$$\overline{\{f(t), t \in \mathbb{R}\}} \supset \{g(t), t \in \mathbb{R}\}. \tag{3.1}$$

The proof of Theorem 3.8 is similar to the proof of Theorem 4.1 in [5]

Theorem 3.9. *Let $\mu, \nu \in \mathcal{M}$. Assume that $SPAA(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ is a Banach space with the norm $||\cdot||_\infty$.*

The proof of the theorem above is similar to the proof of Theorem 4.9 in [5].
Next, we study the composition of square-mean (μ,ν) pseudo almost automorphic processes.

Definition 3.10. *Let $\mu, \nu \in \mathcal{M}$. A continuous function $f(t,x) : \mathbb{R} \times L^2(P,H) \to L^2(P,H)$ is said to be square mean (μ,ν)-pseudo almost automorphic in t for any $x \in L^2(P,H)$ if it can be decomposed as $f = g + \phi$, where $g \in SAA(\mathbb{R} \times L^2(P,H), L^2(P,H))$, $\phi \in \mathcal{E}(\mathbb{R} \times L^2(P,H), \mu, \nu, \infty)$. Denote the set of all such stochastically continuous processes by $SPAA(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu, \nu, \infty)$*

Theorem 3.11. *[11] Let $f : \mathbb{R} \times L^2(P,H) \to L^2(P,H)$, $(t,x) \mapsto f(t,x)$ be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(P,H)$, and assume that f satisfies the Lipschitz condition in the following sense:*

$$\mathbb{E}||f(t,x) - f(t,y)||^2 \leqslant L.\mathbb{E}||x-y||^2$$

for all x, $y \in L^2(P,H)$ and for each $t \in \mathbb{R}$, where $L > 0$ is independent of t. Then for any square-mean almost automorphic process $x : \mathbb{R} \to L^2(P,H)$, the stochastic process $F : \mathbb{R} \to L^2(P,H)$ given by $F(t) := f(t,x(t))$ is square-mean almost automorphic.

Theorem 3.12. *Let* $\mu, \nu \in \mathcal{M}$, $\phi = \phi_1 + \phi_2 \in SPAA(\mathbb{R} \times L^2(P,H); L^2(P,H), \mu, \nu, \infty)$ *with* $\phi_1 \in SAA(\mathbb{R} \times L^2(P,H); L^2(P,H))$, $\phi_2 \in \mathcal{E}(\mathbb{R} \times L^2(P,H); L^2(P,H), \mu, \nu, \infty)$ *and* $h \in SPAA(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$. *Assume:*

$i)$ $\phi_1(t,x)$ *is uniformly continuous on any bounded subset uniformly for* $t \in \mathbb{R}$.
$ii)$ *there exist a nonnegative function* $L_\phi \in L^p(\mathbb{R})$, $(1 \leq p \leq \infty)$ *such that*

$$\mathbb{E}||\phi(t,x_1) - \phi(t,x_2)||^2 \leq L_\phi(t)\mathbb{E}||x_1 - x_2||^2, \quad \textit{for all } \ t \in \mathbb{R} \quad \textit{and for all } \ x_1, x_2 \in L^2(P,H). \tag{3.2}$$

If

$$\beta = \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta \in]-\infty,t]} L_\phi(\theta) \Big) d\mu(t) < \infty \tag{3.3}$$

then the function $t \to \phi(t, h(t))$ *belongs to* $SPAA(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$.

To prove the theorem, we need the following lemma.

Lemma 3.13. *Assume* (***H₃***) *holds and let* $f \in SBC(\mathbb{R}; L^2(P,H))$. *Then* $f \in \mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$ *if and only if for any* $\varepsilon > 0$,

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau]} = 0$$

where

$$M_{\tau,\varepsilon}(f) = \{t \in [-\tau,\tau] : \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 \geq \varepsilon\}.$$

Proof. Suppose that $f \in \mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$. Then

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) &= \frac{1}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(f)} \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) \\
&\quad + \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau] \setminus M_{\tau,\varepsilon}(f)} \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) \\
&\geq \frac{1}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(f)} \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) \\
&\geq \frac{\varepsilon \mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])}.
\end{aligned}$$

Consequently

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau]} = 0.$$

Suppose that $f \in SBC(\mathbb{R}; L^2(P,H))$ such that for any $\varepsilon > 0$,

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau]} = 0.$$

Assume $\mathbb{E}||f(t)||^2 \leq N$ for all $t \in \mathbb{R}$, then using (**H₃**), it follows that

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2 d\mu(t) &= \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2 d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2 d\mu(t)\\
&\leq \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)} d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2 d\mu(t)\\
&\leq \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)} d\mu(t) + \frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} d\mu(t)\\
&\leq \frac{N\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])} + \frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])}.
\end{aligned}$$

Which implies that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2 d\mu(t) \leq \alpha\varepsilon \text{ for any } \varepsilon > 0.$$

Therefore $f \in \mathcal{E}(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$.■

The following proof is for the Theorem(3.12).

Proof. Assume that $\phi = \phi_1 + \phi_2,\ h = h_1 + h_2$ where $\phi_1 \in AA(\mathbb{R}\times L^2(P,H); L^2(P,H)),\ \phi_2 \in \mathcal{E}(\mathbb{R}\times L^2(P,H); L^2(P,H),\mu,\nu,\infty)$ and $h_1 \in AA(\mathbb{R}; L^2(P,H)),\ h_2 \in \mathcal{E}(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$. Consider the following decomposition

$$\phi(t,h(t)) = \phi_1(t,h_1(t)) + [\phi(t,h(t)) - \phi(t,h_1(t))] + \phi_2(t,h_1(t)).$$

From [11], $\phi_1(.,h_1(.)) \in SAA(\mathbb{R}; L^2(P,H))$. It remains to prove that both $\phi(.,h(.)) - \phi(.,h_1(.))$ and $\phi_2(.,h_1(.))$ belong to $\mathcal{E}(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$. Clearly, $\phi(t,h(t)) - \phi(t,h_1(t))$ is bounded and continuous. Assume $\mathbb{E}||\phi(t,h(t)) - \phi(t,h_1(t))||^2 \leq N,\ \forall t \in \mathbb{R}$. Since $h(t),\ h_1(t)$ are bounded, choose a bounded subset $B \subset \mathbb{R}$ such that $h(\mathbb{R}), h_1(\mathbb{R}) \subset B$. Under assumption (ii), for a given $\varepsilon > 0$, $\mathbb{E}||x_1 - x_2||^2 \leq \varepsilon$, implies that $\mathbb{E}||\phi(t,x_1) - \phi(t,x_2)||^2 \leq \varepsilon L_\phi(t)$, for all $t \in \mathbb{R}$. Since for $\delta \in \mathcal{E}(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$, Lemma 3.13 yields that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]}\mu(M_{\tau,\varepsilon}(\delta)) = 0.$$

Consequently

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))||^2\Big)d\mu(t)\\
&=\frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\delta)}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))||^2\Big)d\mu(t)\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(\delta)}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))||^2\Big)d\mu(t)\\
&\le\frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\delta)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(\delta)}\Big(\sup_{\theta\in]-\infty,t]}|L_\phi(\theta)|\Big)d\mu(t)\\
&\le\frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\delta)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}|L_\phi(\theta)|\Big)d\mu(t)\\
&\le\frac{N\mu(M_{\tau,\varepsilon}(\delta))}{\nu([-\tau,\tau])}+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}|L_\phi(\theta)|\Big)d\mu(t).
\end{aligned}$$

Which implies that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))||^2\Big)d\mu(t)\le\varepsilon\beta\ \text{ for any }\varepsilon>0,$$

which shows that $t\mapsto\phi(t,h(t))-\phi(t,h_1(t))$ is (μ,ν)-ergodic of infinite class.
Now to complete the proof, it is enough to prove that $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of infinite class. Since ϕ_2 is uniformly continuous on the compact set $\Omega=\overline{\{h_1(t):\ t\in\mathbb{R}\}}$ with respect to the second variable x, then for given $\varepsilon>0$, there exists $\delta>0$ such that, for all $t\in\mathbb{R}$, ξ_1 and $\xi_2\in\Omega$, one has

$$\mathbb{E}||\xi_1-\xi_2||^2\le\delta\Rightarrow\mathbb{E}||\phi_2(t,\xi_1(t))-\phi_2(t,\xi_2(t))||^2\le\varepsilon.$$

Therefore, there exist $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)}\subset\Omega$, such that

$$\Omega\subset\bigcup_{i=1}^{n(\varepsilon)}B_\delta(z_i,\delta)$$

and then

$$\mathbb{E}||\phi_2(t,h_1(t))||^2\le\varepsilon+\sum_{n=1}^{n(\varepsilon)}\mathbb{E}||\phi_2(t,z_i)||^2$$

Since

$$\forall i\in\{1,...,n(\varepsilon)\},\quad\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi_2(\theta,z_i)||^2\Big)d\mu(t)=0,$$

then

$$\forall\varepsilon>0,\quad\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi_2(\theta,h_1(t))||^2\Big)d\mu(t)\le\varepsilon,$$

that implies

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi_2(\theta,h_1(t))||^2\Big)d\mu(t)=0.$$

Consequently $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of infinite class.■

4. Square-mean pseudo almost automorphic solutions of infinite class

($\mathbf{H_5}$): g is a stochastically bounded process.

Theorem 4.1. *Assume that* ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_4}$) *and* ($\boldsymbol{H_5}$) *hold and the semigroup* $(\mathcal{U}(t))_{t\geqslant 0}$ *is hyperbolic. If* f *is bounded and continuous on* $\mathbb{R}$*, then there exists a unique bounded solution* u *of equation (1.1) on* $\mathbb{R}$ *given by*

$$\begin{aligned} u_t = &\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\ &+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda(X_0 g(s))dW(s)\end{aligned}$$

$\forall \quad t \geq 0$*, where* $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}}_{\mathcal{U}})^{-1}$, Π^s *and* Π^u *are the projections of* $\mathcal{B}_A$ *onto the stable and unstable subspaces.*

Proof. Let $u_t = v(t) + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)$
$+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\forall \ t\geq 0$, where
$v(t) = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds$

Let us first prove that u_t exists. The existence of $v(t)$ have proved by [1]. Now, we show that the limit $\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)$ exist.
For $t\in\mathbb{R}$ and using the Ito's isometry property of the stochastic integral we have,

$$\begin{aligned}\mathbb{E}\left|\left|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\ (\widetilde{B}_\lambda X_0 g(s))dW(s)\right|\right|^2 &\leqslant \mathbb{E}\int_{-\infty}^{t}\overline{M}^2 e^{-2w(t-s)}|\Pi^s|^2||\ (\widetilde{B}_\lambda X_0 g(s))||^2 ds \\ &\leqslant \overline{M}^2\mathbb{E}\int_{-\infty}^{t}e^{-2w(t-s)}|\Pi^s|^2||\ (\widetilde{B}_\lambda X_0 g(s))||^2 ds \\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\int_{-\infty}^{t}e^{-2w(t-s)}||g(s)||^2 ds \\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{\infty}\mathbb{E}\left(\int_{t-n}^{t-n+1}e^{-2w(t-s)}||g(s)||^2 ds\right).\end{aligned}$$

then, using the Holders inequality, we obtain

$$\begin{aligned}&\mathbb{E}\left|\left|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\ (\widetilde{B}_\lambda X_0 g(s))dW(s)\right|\right|^2 \\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{+\infty}\left(\int_{t-n}^{t-n+1}e^{-4w(t-s)}ds\right)^{\frac{1}{2}}\mathbb{E}\left(\int_{t-n}^{t-n+1}||g(s)||^2 ds\right)^{\frac{1}{2}} \\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}\sum_{n=1}^{\infty}\left(e^{-4w(n-1)}-e^{-4wn}\right)^{\frac{1}{2}}\mathbb{E}\left(\int_{t-n}^{t-n+1}||g(s)||^2 ds\right)^{\frac{1}{2}} \\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}(e^{4wn}-1)^{\frac{1}{2}}\sum_{n=1}^{\infty}e^{-2wn}\times\mathbb{E}\left(\int_{t-n}^{t-n+1}||g(s)||^2 ds\right)^{\frac{1}{2}}\end{aligned}$$

Since the serie $\sum_{n=1}^{\infty} e^{-2wn}$ is convergent, then it exists a constant $c > 0$ such that $\sum_{n=1}^{\infty} e^{-2wn} \leqslant c$, moreover it follows that

$$\begin{aligned}\mathbb{E}\Big|\Big|\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\,(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big|\Big|^2 &\leqslant \overline{M}\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}(e^{4w}-1)^{\frac{1}{2}}\mathbb{E}||g(s)||\sum_{n=1}^{\infty} e^{-2wn}\\ &\leqslant \gamma\sum_{n=1}^{\infty} e^{-2wn}\\ &\leqslant \gamma c,\end{aligned}$$

where, $\gamma = \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}(e^{4w}-1)^{\frac{1}{2}}\mathbb{E}||g(s)||$.

Let $F(n,s,t) = \mathcal{U}^s(t-s)\Pi^s(\,\widetilde{B}_\lambda X_0 g(s))$ for $n \in \mathbb{N}$ for $s \leqslant t$.

For n is sufficiently large and $\sigma \leqslant t$ and using the Ito's isometry property of the stochastic integral we get the following result

$$\begin{aligned}&\mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma} F(n,s,t)dW(s)\Big|\Big|^2\\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{+\infty}\Big(\int_{\sigma-n}^{\sigma-n+1} e^{-4w(t-s)}ds\Big)^{\frac{1}{2}} \times \mathbb{E}\left(\int_{\sigma-n}^{\sigma-n+1}||g(s)||^2ds\right)^{\frac{1}{2}}\\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{\omega}}\Big(\sum_{n=1}^{\infty}\left(e^{-4\omega(t-\sigma+n-1)} - e^{-4\omega(t-\sigma+n)}\right)^{\frac{1}{2}}\\ &\times \mathbb{E}\Big(\int_{\sigma-n}^{\sigma-n+1}||g(s)||^2ds\Big)^{\frac{1}{2}}\Big)\\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{\omega}}e^{-2\omega(t-\sigma)}(e^{4\omega}-1)^{\frac{1}{2}}\sum_{n=1}^{\infty} e^{-2\omega n} \times \mathbb{E}\left(\int_{\sigma-n}^{\sigma-n+1}||g(s)||^2ds\right)^{\frac{1}{2}}\\ &\leqslant \gamma c e^{-2w(t-\sigma)}\end{aligned}$$

It follow that for n and m sufficiently large and $\sigma \leqslant t$, we have

$$\begin{aligned}\mathbb{E}\Big|\Big|\int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{\infty}^{t} F(m,s,t)dW(s)\Big|\Big|^2 &\leqslant \mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma} F(n,s,t)dW(s) + \int_{\sigma}^{t} F(n,s,t)dW(s)\\ &- \int_{-\infty}^{\sigma} F(m,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big|\Big|\\ &\leqslant 3\mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma} F(n,s,t)dW(s)\Big|\Big|^2 + 3\mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma} F(m,s,t)dW(s)\Big|\Big|^2\\ &+ 3\mathbb{E}\Big|\Big|\int_{\sigma}^{t} F(n,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big|\Big|^2\\ &\leqslant 6\gamma c e^{-2\omega(t-\sigma)} + 3\mathbb{E}\Big|\Big|\int_{\sigma}^{t} F(n,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big|\Big|^2\end{aligned}$$

Since $\lim_{n\to+\infty}\mathbb{E}\Big|\Big|\int_{\sigma}^{t} F(n,s,t)dW(s)\Big|\Big|^2$ exists, then

$$\limsup_{n,m\to+\infty}\mathbb{E}\Big|\Big|\int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big|\Big|^2 \leqslant 6\gamma c e^{-2\omega(t-\sigma)}$$

If $\sigma \to -\infty$, then

$$\limsup_{n,m\to+\infty} \mathbb{E}\Big|\Big|\int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big|\Big|^2 = 0.$$

We deduce that the limit

$$\lim_{n\to+\infty} \mathbb{E}\Big|\Big|\int_{-\infty}^{t} F(n,s,t)dW(s)\Big|\Big|^2 = \lim_{n\to+\infty} \mathbb{E}\Big|\Big|\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)\Big|\Big|^2$$

exists. Therefore, $\lim_{n\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)$ exists. In addition, one can show that the function

$$t \to \lim_{n\to+\infty} \mathbb{E}\Big|\Big|\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))ds\Big|\Big|^2$$

is bounded on $\mathbb{R}$. Similary, we can show that the function

$$t \to \lim_{n\to+\infty}\int_{t}^{+\infty} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_n X_0 g(s))dW(s)$$

is well defined and bounded on $\mathbb{R}$.■

Theorem 4.2. *Assume that (**H3**) holds. Let $\mu,\nu \in \mathcal{M}$ and $\phi \in SPAA_c(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$ then the function $t \to \phi_t$, belongs to $SPAA_c(C(]-\infty,0],L^2(P,H)),\mu,\nu,\infty)$.*

Proof. Assume that $\phi = v + h$, where $v \in SAA_c(\mathbb{R}, L^2(P,H))$ and $h \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$. We have $\phi_t = v_t + h_t$. Firstly, we show that $v_t \in SAA_c(\mathbb{R}, L^2(P,H))$.
Let $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $w \in SBC(\mathbb{R}, L^2(P,H)))$ such that $v(s+s_n) \longrightarrow w(s)$ uniformly on compact subsets of $\mathbb{R}$. Let $K \subset [-L;L]$. For $\varepsilon > 0$ fix $N_{\varepsilon,L} \in \mathbb{N}$ such that $\mathbb{E}||v(s+s_n) - w(s)||^2 \leqslant \varepsilon$ for $s \in [-L;L]$. Whenerver $n \geqslant N_{\varepsilon,L}$. For $t \in K$ and $n \geqslant N_{\varepsilon,L}$ we have

$$\begin{aligned}\mathbb{E}||v_{t+s_n} - w_t||^2 &\leqslant \sup_{\theta\in[-L;L]} \mathbb{E}||v(\theta+s_n) - w(\theta)||^2\\ &\leqslant \varepsilon\end{aligned}$$

then, v_{t-s_n} converges to w_t uniformly in K. Simlarly, we can show prove that w_{t-s_n} converges to v_t uniformly in K.
Finaly, we show that $h_t \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$

$$M_\alpha = \frac{1}{\nu_\alpha([-\tau,\tau])}\int_{-\tau}^{\tau}(\sup_{\theta\in]-\infty,t]} \mathbb{E}||h(\theta)||^2)d\mu_\alpha(t).$$

Where μ_α and ν_α are the positive measures defined by equation (4.3). By using Lemma (2.20), it follows that μ_α and μ are equivalent and ν_α and ν are also equivalent. Then by using Theorem (3.8) we have $\mathcal{E}(\mathbb{R}, L^2(P,H),\mu_\alpha,\nu_\alpha,\infty) = \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$ therefore $h \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu_\alpha,\nu_\alpha,\infty)$ that is $\lim_{\tau\to+\infty} M_\alpha(\tau) = 0$ for all $\alpha \in \mathbb{R}$. On the other hand, for $r > 0$ we have

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\left(\sup_{\eta\in]-\infty,0]}\left(\mathbb{E}||h(\theta+\eta)||^2\right)\right)d\mu(t) &\leqslant \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\left(\mathbb{E}||h(\theta)||^2\right)d\mu(t)\\
&\leqslant \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big[\sup_{\theta\in]-\infty,t-r]}\left(\mathbb{E}||h(\theta)||^2\right) + \sup_{\theta\in]-\infty,t]}\left(\mathbb{E}||h(\theta)||^2\right)\Big]d\mu(t)\\
&\leqslant \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,\tau+r]}\left(\mathbb{E}||h(\theta)||^2\right)d\mu(t+r) + \int_{-\tau}^{\tau}\sup_{\theta\in]\infty,t]}\left(\mathbb{E}||h(\theta)||^2\right)d\mu(t)\\
&\leqslant \frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\times\frac{1}{\nu([-\tau-r,\tau+r])}\int_{-\tau-r}^{\tau+r}\sup_{\theta\in]\infty,t]}\mathbb{E}||h(\theta)||^2 d\mu(t+r)\\
&+ \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||h(\theta)||^2 d\mu(t)\end{aligned}$$

Consequently,

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\left(\sup_{\eta\in]-\infty,0]}\left(\mathbb{E}||h(\theta+\eta)||^2\right)\right)d\mu(t) \leqslant \frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\times M_r(\tau+r) + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||h(\theta)||^2 d\mu(t)$$

which shows using Lemma(2.21) and Lemma (2.20) that ϕ_t belongs to $SPAA_c(C(]-\infty,0],L^2(P,H)),\mu,\nu,\infty)$. Thus, we obtain the desired result ■

Theorem 4.3. *Let $f,\, g\in SAA_c(\mathbb{R},X)$ and Γ be the mapping defined for $t\in\mathbb{R}$ by*

$$\Gamma(f,g)(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big](0)$$

Then $\Gamma(f,g)\in SAA_c(\mathbb{R},L^2(P,H))$.

Proof. Let $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $v,h\in SBC(\mathbb{R},L^2(P,H))$ such that $f(t+s_n)$ converges to $v(t)$ and $g(t+s_n)$ converges to $h(t)$ uniformly on compact subsets of $\mathbb{R}$. using Lemma 2.4 and Theorem 2.10, we get the following estimates

$$\lim_{\lambda\to+\infty}||\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))||^2 \leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2e^{-2\omega(t-s)}||f(s)||^2 \tag{4.1}$$

$$\lim_{\lambda\to+\infty}||\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))||^2 \leqslant \overline{M}^2\widetilde{M}^2|\Pi^u|^2e^{2\omega(t-s)}||f(s)||^2 \tag{4.2}$$

$$\lim_{\lambda\to+\infty}||\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))||^2 \leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2e^{-2\omega(t-s)}||g(s)||^2 \tag{4.3}$$

and

$$\lim_{\lambda\to+\infty}||\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))||^2 \leqslant \overline{M}^2\widetilde{M}^2|\Pi^u|^2e^{2\omega(t-s)}||g(s)||^2 \tag{4.4}$$

Therefore, if

$$w(t+s_n) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_n))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_n))ds + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s)\Big]$$

then by Equations.(4.1), (4.2), (4.3) and (4.4) and the Lebesgue Dominated convergence Theorem, we have $w(t+s_n)$ that converges to $v(t)$.

$$v(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big]$$

Now, It remains to prove that the convergence is uniform on all compact subset of $\mathbb{R}$. Let $K\subset\mathbb{R}$ be an arbitrary compact and let $\varepsilon>0$. We fix $L>0$ and $N_\varepsilon\in\mathbb{N}$ such that $K\subset\left[\frac{-L}{2};\frac{L}{2}\right]$ with,

$$\int_{\frac{L}{2}}^{+\infty}e^{-2\omega s}ds<\varepsilon.$$

$$\mathbb{E}||f(s+s_n)-v(s)||^2 \leqslant \varepsilon \text{ for } n \geqslant N_\varepsilon \text{ and } s \in [-L,L]. \tag{4.5}$$

and

$$\mathbb{E}||g(s+s_n)-h(s)||^2 \leqslant \varepsilon \text{ for } n \geqslant N_\varepsilon \text{ and } s \in [-L,L]. \tag{4.6}$$

Then, for each $t \in K$, ones has

$$\begin{aligned}
&\mathbb{E}||w(t+s_n)-z(t)||^2\\
&= \mathbb{E}\Big|\Big| \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_n))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_n))ds\\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s)\\
&- \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 v(s))ds - \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 v(s))ds\\
&- \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))dW(s) - \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))dW(s)\Big|\Big|^2\\
&\leqslant 4\Big(\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\\
&+ \mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\\
&+ \mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda(X_0 g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\\
&+ \mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0(g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\Big)
\end{aligned}$$

progressively, we increase each terms of previous inegalitie.

$$\begin{aligned}
&\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\Big|\Big|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\Big)\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\Big|\Big|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\Big)\\
&\leqslant \mathbb{E}\Big(\int_{-\infty}^{t}\overline{M}^2\widetilde{M}^2e^{-2\omega(t-s)}|\Pi^s|^2\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\Big)\\
&\leqslant \int_{-\infty}^{t}\overline{M}^2\widetilde{M}^2e^{-2\omega(t-s)}|\Pi^s|^2\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&\leqslant \widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\Big|\Big|\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\Big)\\
&\leqslant \mathbb{E}\Big(\int_{t}^{+\infty}\overline{M}^2\widetilde{M}^2e^{-2\omega(t-s)}|\Pi^u|^2\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\Big)\\
&\leqslant \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds
\end{aligned}$$

Using Ito's isometry property of stochastic integral, we obtain that

$$\begin{aligned}
&\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda(X_0 g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\Big|\Big|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda(X_0 g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\Big)\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\Big|\Big|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda(X_0 g(s+s_n)-h(s)))ds\Big|\Big|^2\Big)\\
&\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds
\end{aligned}$$

and,

$$\begin{aligned}
&\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda(X_0 g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\Big|\Big|\int_{-\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda(X_0 g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\Big)\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\Big|\Big|\mathcal{U}^s(t-s)\Pi^u(\widetilde{B}_\lambda(X_0 g(s+s_n)-h(s)))ds\Big|\Big|^2\Big)\\
&\leqslant \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}||w(t+s_n)-z(t)||^2 \leqslant\ & 4\Big(\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\Big)\\
\mathbb{E}||w(t+s_n)-z(t)||^2 \leqslant\ & 4\Big(2\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}||w(t+s_n)-z(t)||^2 &\leqslant 4\Big(2\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{t+L}^{+\infty}e^{-2\omega s}ds \\
&+ \overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)\int_{-L}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2 ds \\
&+ \overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)\int_{-L}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2 ds\Big) \\
&\leqslant 4\Big(2\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{\frac{L}{2}}^{+\infty}e^{-2\omega s}ds+2\varepsilon\overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)\int_0^{+\infty}e^{-2\omega s}ds\Big) \\
&\leqslant \Big(8\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2+8\overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)\int_0^{+\infty}e^{-2\omega s}ds\Big)\varepsilon \\
&\leqslant \Big(8\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2+\frac{4\overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)}{\omega}\Big)\varepsilon
\end{aligned}$$

which proves that the convergence is uniform on K, by the fact that the last estimate is independent of $t\in K$. Proceeding as previously, one can similarly prove that $z(t-s_n)$ converges to w uniformly on compact subsets in $\mathbb{R}$. This completes the proof.■

Theorem 4.4. *Assume that* $(\boldsymbol{H_3})$ *and* $(\boldsymbol{H_5})$ *holds. Let* $f,g\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ *then* $\Gamma(f,g)\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$.

Proof.

$$\begin{aligned}
\Gamma(f,g)(t) &= \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\Big|\Big|\Gamma(f,g)(\theta)\Big|\Big|^2 &= \mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big|\Big|^2.
\end{aligned}$$

$$\begin{aligned}
\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||\Gamma(f,g)(\theta)||^2 d\mu(t) &\leqslant \int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big[4\mathbb{E}\Big(\widetilde{M}^2\overline{M}^2\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2||f(s)||^2ds \\
&+\widetilde{M}^2\overline{M}^2\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2||f(s)||^2ds \\
&+\widetilde{M}^2\overline{M}^2\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2||g(s)||^2ds \\
&+\widetilde{M}^2\overline{M}^2\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2||g(s)||^2ds\Big)\Big]d\mu(t) \\
&\leqslant 4\widetilde{M}^2\overline{M}^2\Big[\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2\mathbb{E}||f(s)||^2ds\Big)d\mu(t) \\
&+\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2\mathbb{E}||f(s)||^2ds\Big)d\mu(t)
\end{aligned}$$

$$+\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2\mathbb{E}||g(s)||^2ds+\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2\mathbb{E}||g(s)||^2ds\Big)d\mu(t)\Big]$$
$$\leqslant 4\widetilde{M}^2\overline{M}^2\Big[|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$+|\Pi^u|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)\Big]$$

one the one hand using Fubini's theorem, we have

$$|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds$$
$$\leqslant e^{2\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant e^{2\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant e^{2\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant e^{2\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\Big(\int_{0}^{+\infty}e^{-2\omega s}(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2)ds\Big)d\mu(t)$$
$$\leqslant e^{2\omega r}|\Pi^s|^2\int_{0}^{+\infty}e^{-2\omega s}\int_{-\tau}^{\tau}\Big(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2\Big)d\mu(t)ds$$

By using Theorem(2.22) we deduce that
$\lim_{\tau\to+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2\Big)d\mu(t)\to 0$ for all $s\in\mathbb{R}^+$ and
$\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2\Big)d\mu(t)\leqslant\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\Big(||f||_\infty^2+||g||_\infty^2\Big)$
Since f and g are bounded functions, then the function $s\mapsto\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\Big(||f||_\infty^2+||g||_\infty^2\Big)$ belongs to $L^1([0,+\infty[)$ in view of the Lebesgue dominated convergence Theorem, it follows that $e^{\omega r}\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2\Big)d\mu(t)ds\to 0.$
On the other hand by Fubinis theorem, we also have

$$|\Pi^u|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant |\Pi^u|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{t-r}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant |\Pi^u|^2\int_{-\tau}^{\tau}\Big(\int_{t-r}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$

$$\leqslant |\Pi^u|^2 \int_{-\tau}^{\tau} \Big(\int_{-\infty}^{r} e^{2\omega s} (\mathbb{E}||f(s)||^2 + \mathbb{E}||g(s)||^2) ds \Big) d\mu(t)$$
$$\leqslant |\Pi^u|^2 \int_{-\infty}^{r} \Big(\int_{-\tau}^{\tau} e^{2\omega s} (\mathbb{E}||f(s)||^2 + \mathbb{E}||g(s)||^2) d\mu(t) \Big) ds$$

Since the function $s \mapsto \dfrac{e^{2\omega s}}{\nu([-\tau,\tau])}\Big(||f||_\infty^2 + ||g||_\infty^2 \Big)$ belongs to $L^1(]-\infty, r])$ resoning like above, it follows that

$$\lim_{\tau\to+\infty} \int_{-\infty}^{r} e^{\omega s} \times \frac{1}{\nu([-\tau,\tau])} \Big(\int_{-\tau}^{\tau} e^{2\omega s} (\mathbb{E}||f(s)||^2 + \mathbb{E}||g(s)||^2) d\mu(t) \Big) ds = 0$$

Consequently

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}||\Gamma(f,g)(\theta)||^2 d\mu(t) = 0$$

Thus, we obtain the desired result.■

For proof of existence of square-mean compact pseudo almost automorphic solution of infinite class , we need the following assertion.

($\mathbf{H_6}$) $f, g : \mathbb{R} \longrightarrow L^2(P,H)$ are square-mean compact pseudo almost automorpic of infinite class

Theorem 4.5. *Assume (**H_0**), (**H_1**) and (**H_6**) hold. Then Eq (4.1) has a unique pseudo almost automorpic solution of infinite class*

Proof. Since f and g are pseudo almost periodic functions, f has a decomposition $f = f_1 + f_2$ and $g = g_1 + g_2$ where $f_1, g_1 \in SAA_c(\mathbb{R}; L^2(P,H))$ and $f_2, g_2 \in \mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$. Using Theorem 4.1, Theorem 4.3 and Theorem 4.4, we get the desired result. ■

Our next objective is to show the existence of square mean (μ,ν)-pseudo almost automorphic solutions of infinite class for the following problem

$$du(t) = [Au(t) + L(u_t) + f(t,u_t)]dt + g(t,u_t)dW(t) \text{ for } t \in \mathbb{R} \tag{4.7}$$

where $f : \mathbb{R} \times \mathcal{B} \to L^2(P,H)$ and $g : \mathbb{R} \times \mathcal{B} \to L^2(P,H)$ are two stochastic continuous processes. To prove our result, we formulate the following assumptions

($\mathbf{H_7}$) Let μ, $\nu \in \mathcal{M}$ and $f : \mathbb{R} \times C(]-\infty,0], L^2(P,H)) \to L^2(P,H)$ square mean $cl(\mu,\nu)$-pseudo automorphic periodic of infinite class such that there exists a function L_f such that $\mathbb{E}\Big|\Big| f(t,\phi_1) - f(t,\phi_2) \Big|\Big|^2 \leqslant L_f(t)\mathbb{E}||\phi_1 - \phi_2||^2$ for all $t \in \mathbb{R}$ and $\phi_1, \phi_2 \in C(]-\infty,0], L^2(P,H))$.

($\mathbf{H_8}$) Let μ, $\nu \in \mathcal{M}$ and $g : \mathbb{R} \times C(]-\infty,0], L^2(P,H)) \to L^2(P,H)$ square mean $cl(\mu,\nu)$-pseudo almost periodic of infinite class such that there exists a function L_g such that $\mathbb{E}\Big|\Big| g(t,\phi_1) - g(t,\phi_2) \Big|\Big|^2 \leqslant L_g(t)\mathbb{E}||\phi_1 - \phi_2||^2$ for all $t \in \mathbb{R}$ and ϕ_1, $\phi_2 \in C(]-\infty,0], L^2(P,H))$. Where L_f and $L_g \in L^p(\mathbb{R})$, $(1 \leqslant p < \infty)$

($\mathbf{H_9}$) Let $k = \max(L_f, L_g)$.

($\mathbf{H_{10}}$) The instable space $U \equiv \{0\}$

Theorem 4.6. *Assume that $\mathcal{B}$ satisfies (**A_1**), (**A_2**), **(B)**, (**C_1**), (**C_2**) and (**H_0**), (**H_1**), (**H_3**) , (**H_4**), (**H_4**), (**H_6**), (**H_7**), (**H_8**), (**H_9**) and (**H_{10}**) hold. Then Eq.(4.7) has a unique $cl(\mu,\nu)$- square mean pseudo compact almost automorphic mild solution of infinite class.*

Proof. Let x be a function in $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$ from Theorem 4.2 the function $t \to x_t$ belongs to $SPAA_c(C(]-\infty,0]; L^2(P,H)),\mu,\nu,\infty)$. Hence Theorem implies that the function $g(.) := f(., x_.)$ is in $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$. Since the instable space $U \equiv \{0\}$, then $\Pi^u \equiv 0$. Consider now the following mapping

$$\mathcal{H} : SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty) \to SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$$

defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s,x_s))ds + \lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s,x_s))dW(s)\Big](0)$$

From Theorem 4.3, Theorem 4.4, Theorem 4.4 and Theorem 4.1 we obtain that $\mathcal{H}$ maps $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$ into $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$.
It remains now to show that the operator $\mathcal{H}$ has a unique fixed point in $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$.
Since $\mathcal{B}$ is a uniform fading memory space, by the Lemma (2.7), choose the function K constant and the function M such that $M(t) \to 0$ as $t \to +\infty$. Let $\eta = \max_{t\in\mathbb{R}}\Big\{\sup_{t\in\mathbb{R}}|K(t)|^2, \sup_{t\in\mathbb{R}}|M(t)|^2\Big\}$ **Case 1:** $L_f, L_g \in L^1(\mathbb{R},\mathbb{R}^+)$
Let $x_1, x_2 \in SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$. Then we have

$$\begin{aligned}\mathbb{E}\Big|\Big|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)\Big|\Big|^2 &\leqslant 2\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s,x_{1s}) - f(s,x_{2s}))ds\Big|\Big|^2 \\ &+ 2\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0\Big[g(s,x_{1s}) - g(s,x_{2s})\Big])dW(s)\Big|\Big|^2\end{aligned}$$

Using Ito's isometry we have

$$\begin{aligned}&\mathbb{E}\Big|\Big|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)\Big|\Big|^2 \leqslant 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t e^{-2\omega(t-s)}L_f(s)\mathbb{E}||x_{1s} - x_{2s}||^2_{\mathcal{B}}ds \\ &+ 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t e^{-2\omega(t-s)}L_g(s)\mathbb{E}||x_{1s} - x_{2s}||^2_{\mathcal{B}}ds \\ &\leqslant 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t e^{-2\omega(t-s)}L_f(s)\mathbb{E}\Big(K(s)\sup_{0\leqslant\xi\leqslant s}||x_1(\xi) - x_2(\xi)|| + M(s)||x_{1_0} - x_{2_0}||\Big)^2 ds \\ &+ 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t e^{-2\omega(t-s)}L_g(s)\mathbb{E}\Big(K(s)\sup_{0\leqslant\xi\leqslant s}||x_1(\xi) - x_2(\xi)|| + M(s)||x_{1_0} - x_{2_0}||\Big)^2 ds \\ &\leqslant 4\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t e^{-2\omega(t-s)}k(s)\mathbb{E}\Big(K(s)\sup_{0\leqslant\xi\leqslant s}||x_1(\xi) - x_2(\xi)|| + M(s)||x_{1_0} - x_{2_0}||\Big)^2 ds \\ &\leqslant 8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t e^{-2\omega(t-s)}k(s)\mathbb{E}\Big(K^2(s)\sup_{0\leqslant\xi\leqslant s}||x_1(\xi) - x_2(\xi)||^2 + M^2(s)||x_{1_0} - x_{2_0}||^2\Big)ds \\ &\leqslant 8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t e^{-2\omega(t-s)}k(s)\Big(K^2(s)\sup_{0\leqslant\xi\leqslant s}\mathbb{E}||x_1(\xi) - x_2(\xi)||^2 + M^2(s)\mathbb{E}||x_{1_0} - x_{2_0}||^2\Big)ds \\ &\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^t k(s)ds\Big)||x_1 - x_2||^2_\infty\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\Big|\Big|\mathcal{H}^2x_1(t)-\mathcal{H}^2x_2(t)\Big|\Big|^2 &\leqslant 2\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s\Big(\widetilde{B}_\lambda X_0\Big[f(s,\mathcal{H}x_{1s})-f(s,\mathcal{H}x_{2s})\Big]\Big)ds\Big|\Big|^2\\ &+2\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s\Big(\widetilde{B}_\lambda X_0\Big[g(s,\mathcal{H}x_{1s})-g(s,\mathcal{H}x_{2s})\Big]\Big)dW(s)\Big|\Big|^2\\ &\leqslant \Big(16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big)^2\Big(\int_{-\infty}^t k(s)ds\Big)^2||x_1-x_2||_\infty^2\end{aligned}$$

By induction on n we obtain the following inequalitie

$$\mathbb{E}\Big|\Big|\mathcal{H}^nx_1(t)-\mathcal{H}^nx_2(t)\Big|\Big|^2\leqslant(16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta)^n\Big(\int_{-\infty}^t k(s)ds\Big)^n||x_1-x_2||_\infty^2$$

Therefore

$$\Big|\Big|\mathcal{H}^nx_1(t)-\mathcal{H}^nx_2(t)\Big|\Big|_\infty\leqslant(4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta})^n|k|^n_{L^1(\mathbb{R})}||x_1-x_2||_\infty$$

Let n_0 be such that $(4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta})^{n_0}|k|^{n_0}_{L^1(\mathbb{R})}<1$. By Banach fix point Theorem, $\mathcal{H}$ has a unique point fixed and this fixed point satisfies the integral equation

$$u_t=\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s\widetilde{B}_\lambda(X_0f(s))ds+\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s\widetilde{B}_\lambda(X_0g(s))dW(s)$$

Case 2: $L_g, L_f\in L^p(\mathbb{R})$; $(1<p<\infty)$
First, put

$$\mu(t)=\int_{-\infty}^t(k(s))^pds.$$

Then we define an equivalent norm over $SPAA(\mathbb{R},L^2(P,H),\mu,\nu,r)$ as follows

$$||f||_c=\sup_{t\in\mathbb{R}}\Big(e^{-c\mu(t)}\mathbb{E}||f(t)||^2\Big)^{\frac{1}{2}}$$

where c is a fixed positive number to be precised later. Using the Holder inequality and Ito's isometry we have

$$\begin{aligned}\mathbb{E}\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|^2 &\leqslant 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t e^{-2\omega(t-s)}L_f(s)\mathbb{E}||x_{1s}-x_{2s}||^2_{\mathcal{B}}ds\\ &+2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t e^{-2\omega(t-s)}L_g(s)\mathbb{E}||x_{1s}-x_{2s}||^2_{\mathcal{B}}ds\end{aligned}$$

$$\leqslant 8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}e^{c\mu(s)}e^{-c\mu(s)}k(s)\Big(K^2(s)\sup_{0\leqslant\xi\leqslant s}\mathbb{E}||x_1(\xi)-x_2(\xi)||^2$$
$$+M^2(s)\mathbb{E}||x_{1_0}-x_{2_0}||^2\Big)ds$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\int_{-\infty}^{t}e^{-2\omega(t-s)}e^{c\mu(s)}k(s)\Big(\sup_{s\in\mathbb{R}}e^{-c\mu(s)}\mathbb{E}||x_1(\xi)-x_2(\xi)||^2\Big)ds$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\int_{-\infty}^{t}e^{-2\omega(t-s)}e^{c\mu(s)}k(s)\Big(\sup_{s\in\mathbb{R}}\Big(e^{-c\mu(s)}\mathbb{E}||x_1(\xi)-x_2(\xi)||^2\Big)^{\frac{1}{2}}\Big)^2ds$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}e^{c\mu(s)}k(s)ds\Big)||x_1-x_2||_c^2$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^{t}e^{-2q\omega(t-s)}ds\Big)^{\frac{1}{q}}\Big(\int_{-\infty}^{t}e^{pc\mu(s)}k^p(s)ds\Big)^{\frac{1}{p}}||x_1-x_2||_c^2$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^{t}e^{-2q\omega(t-s)}ds\Big)^{\frac{1}{q}}\Big(\int_{-\infty}^{t}e^{pc\mu(s)}\mu'(s)ds\Big)^{\frac{1}{p}}||x_1-x_2||_c^2$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\frac{1}{(2\omega q)^{\frac{1}{q}}}\times\frac{1}{(pc)^{\frac{1}{p}}}\Big)e^{c\mu(t)}||x_1-x_2||_c^2$$
$$e^{-c\mu(t)}\mathbb{E}\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|^2\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\frac{1}{(2\omega q)^{\frac{1}{q}}}\times\frac{1}{(pc)^{\frac{1}{p}}}\Big)||x_1-x_2||_c^2$$
$$\Big(e^{-c\mu(t)}\mathbb{E}\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|^2\Big)^{\frac{1}{2}}\leqslant 4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta}\Big(\frac{1}{(2\omega q)^{\frac{1}{2q}}}\times\frac{1}{(pc)^{\frac{1}{2p}}}\Big)||x_1-x_2||_c^2$$

Consequently,

$$||\mathcal{H}x_1(t)-\mathcal{H}x_2(t)||_c\leqslant\frac{4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta}}{(2\omega q)^{\frac{1}{2q}}\times(pc)^{\frac{1}{2p}}}||x_1-x_2||_c$$

Fix $c>0$ so large, then the function $c\mapsto\dfrac{1}{(pc)^{\frac{1}{2p}}}$ converges to 0 when c converges to $+\infty$. It follows that for $c>0$ so large we have $\dfrac{4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta}}{(2\omega q)^{\frac{1}{2q}}\times(pc)^{\frac{1}{2p}}}<1$. Thus $\mathcal{H}$ is a contractive mapping. we conclude that there is a unique pseudo almost automorphic integral solution to Eq.(4.7).

Proposition 4.7. *Assume that $\mathcal{B}$ is a uniform fading space and (A_1), (A_2), (C_1), (C_2), (**H₀**), (**H₁**), (**H₂**), (**H₄**) and (**H₅**) hold f and g are lipschitz continuous with respect the second argument if* $\max(Lip(f),Lip(g))<\dfrac{\omega}{2\sqrt{2M}\widetilde{M}|\Pi^s|\eta}$*.Then Eq(4.7) has a unique square-mean $cl(\mu,\nu)$-pseudo almost automorphic solution of infinite class, where $Lip(f)$ and $Lip(g)$ are respectively Lipschitz constant of f and g.*

Proof. Let us pose $k=\max(Lip(f),Lip(g))$, we have

$$\mathbb{E}\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|^2\leqslant 8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}k(s)\Big(K^2(s)\sup_{0\leqslant\xi\leqslant s}\mathbb{E}||x_1(\xi)-x_2(\xi)||^2+M^2(s)\mathbb{E}||x_{1_0}-x_{2_0}||^2\Big)ds$$
$$\leqslant 16k\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}ds\Big)||x_1-x_2||_\infty^2$$
$$\leqslant\frac{8\eta\overline{M}^2\widetilde{M}^2|\Pi^s|^2k}{\omega}||x_1-x_2||_\infty^2$$
$$\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|\leqslant\frac{2\sqrt{2M}\widetilde{M}|\Pi^s|k\eta}{\omega}||x_1-x_2||_\infty$$

Consequently $\mathcal{H}$ is a strict contraction if $k < \dfrac{\omega}{2\sqrt{2M}\widetilde{M}|\Pi^s|\eta}$. ■

5. Application

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} dz(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x)dt + \Big[\displaystyle\int_{-\infty}^{0} G(\theta)z(t+\theta,x)d\theta + \sin\Big(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\Big) + \arctan(t) \\ + \displaystyle\int_{-\infty}^{0} e^{\omega\theta}h(\theta, z(t+\theta,x)d\theta\Big]dt + \Big[\cos\Big(\frac{1}{\sin(t)+\sin(\sqrt{2}t)}\Big) + \sin(t) + \int_{-\infty}^{0} e^{\omega\theta}h(\theta, z(t+\theta,x))d\theta\Big]dW(t) \\ z(t,0) = z(t,\pi) = 0 \text{ for } t \in \mathbb{R} \end{cases} \tag{5.1}$$

Where $G :]-\infty, 0] \to \mathbb{R}$ define by $G(\theta) = e^{(\gamma+1)\theta}$ is a continuous function and $h :]-\infty, 0] \times \mathbb{R} \to \mathbb{R}$ is continuous, Lipschitzian with respect to the second argument and ω is a positive positive real number.
For example, take $h(\theta, x) = \theta^3 + \cos\left(\dfrac{x}{3}\right)$ for $(\theta, x) \in]-\infty, 0] \times \mathbb{R}$, it follows that

$$\Big|h(\theta, x_1) - h(\theta, x_2)\Big| \leqslant \frac{1}{2}|x_1 - x_2|$$

which implies $h :]-\infty, 0] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and lipschitzian with respect to the second argument. $W(t)$ is a two-sided and standard one-dimensional Brownian notion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(u) - W(v) \mid u, v \leqslant t\}$.
The phase $\mathcal{B} = C_\gamma, \gamma > 0$ where

$$C_\gamma = \Big\{\phi \in C(]-\infty, 0]; L^2(P, H)) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } L^2(P, H)\Big\}$$

With the following norm

$$||\phi||_\gamma = \sup_{\theta \leqslant 0}\Big(\mathbb{E}||e^{\gamma\theta}\phi(\theta)||^2\Big)^{\frac{1}{2}}$$

To rewrite equation (5.1) in the abstract form , we introduce the space $H = L^2((0,\pi))$. Let $A : D(A) \to L^2((0,\pi))$ defined by

$$\begin{cases} D(A) = \mathbf{H}^1((0,\pi)) \cap \mathbf{H}_0^1((0,1)) \\ Ay(t) = y''(t) \ \text{ for } t \in (0,\pi) \text{ and } y \in D(A) \end{cases}$$

Then A generates a C_0-semigroup $(\mathcal{U}(t))_{t \geqslant 0}$ on $L^2((0,\pi))$ given by

$$(\mathcal{U}(t)x)(r) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < x, e_n >_{L^2} e_n(r)$$

Where $e_n(r) = \sqrt{2}\sin(n\pi r)$ for $n = 1, 2, ...,$ and $||\mathcal{U}(t)|| \leqslant e^{-\pi^2 t}$ for all $t \geqslant 0$. Thus $\overline{M} = 1$ and $\omega = \pi^2$. Then A satisfied the Hille-Yosida condition in $L^2((0,\pi))$. Moreover the part A_0 of A in $\overline{D(A)}$. It follows that $(\mathbf{H_0})$ and $(\mathbf{H_1})$ are satisfied.
We define $f : \mathbb{R} \times \mathcal{B} \to L^2((0,\pi))$, $g : \mathbb{R} \times \mathcal{B} \to L^2((0,\pi))$ and $L : \mathcal{B} \to L^2((0,\pi))$ as follows

$$f(t,\phi)(x) = \sin\Big(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\Big) + \arctan(t) + \int_{-\infty}^{0} e^{\omega\theta}h(\theta, \phi(\theta)(x))d\theta$$

$$g(t,\phi)(x) = \cos\Big(\frac{1}{\sin(t)+\sin(\sqrt{2}t)}\Big) + \sin(t) + \int_{-\infty}^{\theta} e^{\omega\theta} h(\theta,\phi(\theta)(x))d\theta$$

$$L(\phi)(x) = \int_{-\infty}^{\theta} G(\theta,\phi(\theta)(x))d\theta \text{ for } -\infty < \theta \leqslant 0 \text{ and } x \in (0,\pi)$$

let us pose $v(t) = z(t,x)$. Then equation(5.1) takes the following abstract form

$$dv(t) = [Av(t) + L(v_t) + f(t,v_t)]dt + g(t,v_t)dW(t) \text{ for } t \in \mathbb{R}$$

Consider the measures μ and ν where its Radon-Nikodyn derivative are respectively ρ_1, $\rho_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_1(t) = \begin{cases} 1 \text{ for } t > 0 \\ e^t \text{ for } t \leqslant 0 \end{cases}$$

and

$$\rho_2(t) = |t| \ for \ t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \int_A \rho_2(t)dt \text{ for } A \in \mathcal{B}.$$

From [6] $\mu, \nu \in \mathcal{M}$, μ, ν satisfy $(\mathbf{H_4})$, $\sin\Big(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\Big)$ and $\cos\Big(\frac{1}{\sin(t)+\sin(\sqrt{2}t)}\Big)$ are almost automorphic.
We have

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\int_{-\tau}^{0} e^t dt + \int_0^{\tau} dt}{2\int_{-\tau}^{0} t dt} = \limsup_{\tau\to+\infty} \frac{1-e^{-\tau}+\tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H_2})$ is satisfied.
For all $\theta \in \mathbb{R}$, $-1 \leqslant \sin(\theta) \leqslant 1$ then,

$$\begin{aligned} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\Big|\sin(\theta)\Big|^2 dt &\leqslant \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} d\mu(t) \\ &\leqslant \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \to 0 \text{ as } \tau \to +\infty \end{aligned}$$

Consequently,

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\Big|\sin(\theta)\Big|^2 d\mu(t) = 0$$

It follows that $t \mapsto \sin(t)$ is square mean (μ,ν)-ergodic of infinite class , consequently, g is uniformly square mean (μ,ν)-pseudo almost automorphic of infinite class.
For all $\theta \in \mathbb{R}$, $\frac{-\pi}{2} < \arctan\theta < \frac{\pi}{2}$ then,

$$\begin{aligned} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\Big|\arctan(\theta)\Big|^2 dt &\leqslant \frac{\pi}{2} \times \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} d\mu(t) \\ &\leqslant \frac{\pi}{2} \times \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \to 0 \text{ as } \tau \to +\infty \end{aligned}$$

Consequently,

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}\Big|\arctan\theta\Big|^2 d\mu(t)=0$$

It follows that $t\mapsto \arctan t$ is square mean (μ,ν)-ergodic of infinite class , consequently, f is uniformly square mean (μ,ν)-pseudo almost automorphic of infinite class.
For $\phi\in C_\gamma$, $\gamma\in C(]-\infty,0];L^2(P,H))$ and $\lim_{\theta\to-\infty}e^{\gamma\theta}\phi(\theta)=x_0$ exist in $L^2(P,H)$, then there exists $M\geqslant 0$ such that $\mathbb{E}||e^{\gamma\theta}\phi(\theta)||^2\leqslant M$ for $\theta\in]-\infty,0]$.

$$\begin{aligned}
\mathbb{E}||L(\phi)(x)||^2 &= \mathbb{E}\Big|\Big|\int_{-\infty}^0 G(\theta)\phi(\theta)(x)d\theta\Big|\Big|^2\\
&\leqslant \int_{-\infty}^0 \mathbb{E}||G(\theta)\phi(\theta)(x)||^2 d\theta\\
&\leqslant \int_{-\infty}^0 e^{2(\gamma+1)\theta}\mathbb{E}||e^{-\gamma\theta}e^{\gamma\theta}G(\theta)\phi(\theta)(x)||^2 d\theta\\
&\leqslant \int_{-\infty}^0 e^{2(\gamma+1)\theta}\times e^{-\gamma\theta}\mathbb{E}||e^{\gamma\theta}G(\theta)\phi(\theta)(x)||^2 d\theta\\
&\leqslant \int_{-\infty}^0 e^{2\theta}\mathbb{E}||e^{\gamma\theta}G(\theta)\phi(\theta)(x)||^2 d\theta\\
&\leqslant M\int_{-\infty}^0 e^{2\theta}d\theta<\infty
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathbb{E}||L(\phi)(x)||^2 &\leqslant \Big(\int_{-\infty}^0 e^{2\theta}d\theta\Big)\sup_{\theta\leqslant 0}\mathbb{E}||e^{\gamma\theta}\phi(\theta)(x)||^2\\
&\leqslant \Big(\int_{-\infty}^0 e^{2\theta}d\theta\Big)||\phi||^2_{\mathcal{B}}
\end{aligned}$$

Then L is well defined and L is bounded linear operator from $\mathcal{B}$ to $L^2(P,L^2((0,\pi))$.

$$\begin{aligned}
\mathbb{E}||f(t,\phi_1)(x)-f(t,\phi_2)(x)||^2 &= \mathbb{E}\Big|\Big|\int_{-\infty}^0 e^{\omega\theta}\Big[h(\theta,\phi_1(\theta)(x))-h(\theta,\phi_2(\theta)(x))\Big]\Big|\Big|^2 d\theta\\
&\leqslant \int_{-\infty}^0 e^{2\omega\theta}\mathbb{E}\Big|\Big|h(\theta,\phi_1(\theta)(x))-h(\theta,\phi_2(\theta)(x))\Big|\Big|^2 d\theta\\
&\leqslant \frac{1}{9}\int_{-\infty}^0 e^{2\omega\theta}e^{-\frac{1}{2}\gamma\theta}e^{\frac{1}{2}\gamma\theta}\mathbb{E}\Big|\Big|\phi_1(\theta)(x)-\phi_2(\theta)(x)\Big|\Big|^2 d\theta\\
&\leqslant \frac{1}{9}\int_{-\infty}^0 e^{(2\omega-\frac{1}{2}\gamma)\theta}\mathbb{E}\Big|\Big|e^{2\gamma\theta}(\phi_1(\theta)(x)-\phi_2(\theta)(x))\Big|\Big|^2 d\theta\\
&\leqslant \frac{1}{9}\Big(\int_{-\infty}^0 e^{(2\omega-\frac{1}{2}\gamma)\theta}d\theta\Big)\Big|\Big|\phi_1-\phi_2\Big|\Big|^2_{\mathcal{B}}
\end{aligned}$$

Consequently, we conclude that f and g are Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost automorphic of infinite class .
Moreover, since h is Lipschitzian by consequently bounded i.e there exists a constant M_1 positive real number

such that $|h(\theta, x)| \leqslant M_1$, then we have

$$\begin{aligned}\mathbb{E}||g(t,\phi)(x)||^2 &\leqslant 2+\int_{-\infty}^{0} e^{\omega\theta}\mathbb{E}\Big|h(\theta,\phi(\theta)(x))\Big|^2 d\theta \\ &\leqslant 2+M_1^2\int_{-\infty}^{0} e^{\omega\theta}d\theta<\infty\end{aligned}$$

Which implies that g verifies $(\mathbf{H_5})$

Lemma 5.1. *[9] If* $\int_{-\infty}^{0}|G(\theta)|d\theta < 1$, *then the semigroup* $(\mathcal{U}(t))_{t\geqslant 0}$ *is hyperbolic and the instable space* $U \equiv \{0\}$.

Observe that
$\int_{-\infty}^{0}|G(\theta)|d\theta = \lim_{t\to+\infty}\int_{-r}^{0} e^{(\gamma+1)\theta}d\theta = \lim_{r\to+\infty}\left[\frac{1}{\gamma+1}e^{(\gamma+1)\theta}\right]_{-r}^{0} = \frac{1}{\gamma+1} < 1$, then $(\mathbf{H_8})$ holds Then by Proposition(4.7) we deduce the following result.

Theorem 5.2. *Under the above assumptions, then equation (5.1) has a unique square mean* $cl(\mu,\nu)$*-pseudo almost automorphic solution of infinite class .*

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 197–207.
http://doi.org/10.26637/mjm11S/012

Approximation results for PBVPs of nonlinear first order ordinary functional differential equations in a closed subset of the Banach space

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*R*eceived 09 August 2023; *A*ccepted 26 September 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. In this paper we prove the approximation results for existence and uniqueness of the solution of PBVPs of nonlinear first order ordinary functional differential equations in a closed subset of the Banach space. We employ the Dhage monotone iteration method based on a recent hybrid fixed point theorem of Dhage (2022) and Dhage *et al.* (2022) for the main results of this paper. Finally an example is indicated to illustrate the abstract ideas involed in the approximation results.

AMS Subject Classifications: 34A12, 34A34.

Keywords: Functional periodic boundary value problem; Dhage monotone iteration method; Existence and approximation theorem.

Contents

1. Introduction

The study of periodic boundary value problems (in short PBVPs) and functional PBVPs of first order ordinary differential equations for existence and approximations using hybrid fixed point theory is initiated by Dhage and Dhage [9] and Dhage [5] respectively.Then after several results appeared in the literature for different types of hybrid PBVPs in the partially ordered Banach space. But to the knowledge of the present authors such results are not proved in the closed subsets of the Banach space. For details of functional differential equations and their importance, the readers are referred to Hale [15]. In this paper we prove the existence and approximation results for a PBVP more general than that studied in Dhage and Dhage [9] using the monotone iteration method of Dhage. This method relies on a recent hybrid fixed point theorem of Dhage *et al.* [12] in a partially ordered Banach space. Before stating the proposed PBVP, we give some preliminaries.

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Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$ and let $J = [-r, T]$. By $\mathcal{C} = C(I_0, \mathbb{R})$ we denote the space of continuous real-valued functions defined on I_0 with he norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|x\|_{\mathcal{C}} = \sup_{-r\leq\theta\leq 0} |x(\theta)|. \tag{1.1}$$

The Banach space $\mathcal{C}$ with this supremum norm is called the history space of the functional differential equation in question. For any continuous function $x : J \to \mathbb{R}$ and for any $t \in I$, we denote by x_t the element of the space $\mathcal{C}$ defined by

$$x_t(\theta) = x(t + \theta),\ -r \leq \theta \leq 0. \tag{1.2}$$

Now, given a history function $\phi \in \mathcal{C}$, we consider the PBVP of nonlinear first order ordinary functional differential equations (in short functional PBVP),

$$\left.\begin{aligned} x'(t) + h(t)x(t) = f(t, x(t), x_t),\ \text{a. e. } t \in I, \\ x(0) = \phi(0) = x(T), \\ x_0 = \phi, \end{aligned}\right\} \tag{1.3}$$

where $h : I \to \mathbb{R}$ and $f : I \times \mathbb{R} \times \mathcal{C}$ are continuous functions.

Definition 1.1. *A function $x \in AC(J, \mathbb{R})$ is said to be a solution of the functional PBVP* (1.3) *if*

(i) $x_0 = \phi$*,*

(ii) $x_t \in \mathcal{C}$ *for each* $t \in I$*, and*

(iii) x *satisfies the equations in* (1.3) *on* J*,*

where $AC(J, \mathbb{R})$ is the space of absolutely continuous real-valued functions defined on J.

In this paper we obtain the existence and approximation theorem for the functional PBVP (1.3) in a closed subset of the relevant function space. The rest of the paper is organized as follows. Below in Section 2, we give the auxiliary results needed later in the subsequent part of the paper. The main existence and uniqueness theorems are proved in Section 3 and a couple of illustrative examples are presented in Section 4.

2. Auxiliary Results

First we convert the functional PBVP (1.3) into an equivalent integral equation, because the integrals are easier to handle than differentials. We need the following result similar to Nieto [16, 17] and Dhage [2] which can proved by using the theory of calculus.

Lemma 2.1. *For any $h \in L^1(J, \mathbb{R}^+)$ and $\sigma \in L^1(J, \mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x' + h(t)x(t) = \sigma(t)\ \text{a. e. } t \in I, \\ x(0) = \phi(0) = x(T), \\ x_0 = \phi, \end{aligned}\right\} \tag{2.1}$$

if and only if it is a solution of the integral equation

$$\left.\begin{aligned} x(t) &= \int_0^T G_h(t, s)\sigma(s)\, ds,\ \ t \in I, \\ x_0 &= \phi, \end{aligned}\right\} \tag{2.2}$$

where,

$$G_h(t,s) = \begin{cases} \dfrac{e^{H(s)-H(t)}}{1-e^{-H(T)}}, & 0 \le s \le t \le T, \\[2ex] \dfrac{e^{H(s)-H(t)-H(T)}}{1-e^{-H(T)}}, & 0 \le t < s \le T, \end{cases} \tag{2.3}$$

and $H(t) = \int_0^t h(s)\,ds$.

Notice that the Green's function G_h is nonnegative on $J \times J$ and the number

$$M_h := \max\{\,|G_h(t,s)| \,:\, t,s \in [0,T]\,\},$$

exists for all $L^1(J,\mathbb{R}^+)$. Note also that $H(t) > 0$ for all $t > 0$.

We need the following definition in the sequel.

Definition 2.2. *A mapping $\beta : I \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ is said to be* **Carathéodory** *if*

(i) $t \mapsto \beta(t,x,y)$ is measurable for each $x \in \mathbb{R}$, $y \in \mathcal{C}$, and

(ii) $(x,y) \mapsto \beta(t,x,y)$ is jointly continuous almost everywhere for $t \in I$.

Again a Carathéodory function $\beta(t,x,y)$ is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $m_r \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x,y)| \le m_r(t) \ \text{ a.e. } \ t \in J,$$

for all $x \in \mathbb{R}$ and $y \in \mathcal{C}$ with $|x| \le r$ and $\|y\|_{\mathcal{C}} \le r$.

The following lemma is proved using the arguments similar to that given in Dhage and Dhage [9]. See also Dhage [2, 5] and references therein.

Lemma 2.3. *Suppose that there exists a function $u \in AC(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t) + h(t)u(t) \le f(t,u(t),u_t) \ \text{ a.e. } \ t \in I, \\ u(0) = \phi(0) \le u(T), \\ u_0 \le \phi. \end{aligned}\right\} \tag{2.4}$$

Then,

$$\left.\begin{aligned} & u(t) \le \int_0^T G_h(t,s)\sigma(s)\,ds, \quad t \in I, \\ & u_0 \le \phi. \end{aligned}\right\} \tag{2.5}$$

Similarly, if there exists a function $v \in AC(J,\mathbb{R})$ such that the inequalities in (2.4) are satisfied with reverse sign, then the inequalities in (2.5) *hold with reverse sign.*

It is well-known that the fixed point theoretic technique is very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations. See Granas and Dugundji [14], Zeidler [18] and the references therein. Here, we employ the Dhage monotone iteration method based on the following two hybrid fixed point theorems of Dhage [8] and Dhage *et al.* [12].

Theorem 2.4 (Dhage [8])**.** *Let S be a non-empty partially compact subset of a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ with every chain C in S is Janhavi set and let $\mathcal{T} : S \to S$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.5 (Dhage [8])**.** *Let S be a non-empty partially closed subset of a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq\big)$ and let $\mathcal{T} : S \to S$ be a monotone nondecreasing nonlinear partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a unique comparable solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. Moreover, ξ^* is unique provided every pair of elements in E has a lower bound or an upper bound.*

Remark 2.6. *We note that every every pair of elements in a partially ordered set* $(\mathbf{poset})$ *$(E, \preceq)$ has a lower or upper bound if $(E, \preceq)$ is a lattice, that is, $\preceq$ is a lattice order in E. In this case the poset $(E, \|\cdot\|, \preceq)$ is called a* **partially lattice ordered Banach space***. There do exist several lattice partially ordered Banach spaces which are useful for applications in nonlinear analysis. For example, every Banach lattice is a partially lattice ordered Banach space. The details of the lattice structure of the Banach spaces appear in Birkhoff [1].*

As a consequence of Remark 2.6, we obtain

Theorem 2.7 (Dhage [8])**.** *Let S be a non-empty partially closed subset of a regular partially lattice ordered Banach space $\big(E, \|\cdot\|, \preceq\big)$ and let $\mathcal{T} : S \to S$ be a monotone nondecreasing nonlinear partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a unique solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

If a Banach X is partially ordered by an order cone K in X, then in this case we simply say X is an **ordered Banach space** which we denote it by (X, K). Similarly, an ordered Banach space (X, K), where partial order $\preceq$ defined by the con K is a lattice order, then (X, K) is called the **lattice ordered Banach space**. Clearly, an ordered Banach space $\big(C(J, \mathbb{R}), K\big)$ of continuous real-valued functions defined on the closed and bounded interval J is lattice ordered Banach space, where the cone K is given by $K = \{x \in CJ, \mathbb{R}) \mid x \succeq 0\}$. The details of the cones and their properties appear in Guo and Lakshmikantham [13]. Then, we have the following useful results concerning the ordered Banach spaces proved in Dhage [7, 8].

Lemma 2.8 (Dhage [7, 8])**.** *Every ordered Banach space (X, K) is regular.*

Lemma 2.9 (Dhage [7, 8])**.** *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 2.8 and 2.9 we obtain the following hybrid fixed point theorem which we need in what follows.

Theorem 2.10 (Dhage [8] and Dhage et al. [12])**.** *Let S be a non-empty partially compact subset of an ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid operator equation $\mathcal{T}x = x$ has a solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.11 (Dhage [8] and Dhage et al. [12])**.** *Let S be a non-empty partially closed subset of a lattice ordered complete normed linear space $\big(X, K\big)$ and let $\mathcal{T} : S \to S$ be a monotone nondecreasing partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid operator equation $\mathcal{T}x = x$ has a unique solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

The details of the notions of partial order, Janhavi set, regularity, monotonicity, partial continuity, partial closure, partial compactness and nonlinear partial contraction along with their applications may be found in Guo and Lakshmikatham [13], Dhage [3, 6, 7], Dhage and Dhage [9, 10] and references therein.

3. Existence and Approximation Results

We place the nonlinear integral equation corresponding to the PBVP (1.3) in the Banach space $C(J, \mathbb{R})$ equipped with the norm $\|\cdot\|$ and the order relation $\preceq$ defined by

$$\|x\| = \sup_{t \in J} |x(t)|, \tag{3.1}$$

and

$$x \preceq y \iff y - x \in K, \tag{3.2}$$

where K is a cone in $C(J, \mathbb{R})$ given by

$$K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \; \forall \, t \in J\}. \tag{3.3}$$

It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice with respect to the *meet* and *join* lattice the operations $x \wedge y = \min\{x\,,\,y\}$ and $x \vee y = \max\{x\,,\,y\}$. Therefore, every pair of elements of $C(J, \mathbb{R})$ has a lower and an upper bound. See Dhage [6, 7] and the references therein. The following useful lemma concerning the partial compactness of the subsets of of $C(J, \mathbb{R})$ follows easily and is often times used in the theory of nonlinear differential and integral equations.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), K\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\preceq$ defined by (3.1) and (3.2) respectively. Then every compact subset S of $C(J, \mathbb{R})$ is partially compact, but the converse may not be true.*

We introduce an order relation $\preceq_{\mathcal{C}}$ in $\mathcal{C}$ induced by the order relation $\preceq$ defined in $C(J, \mathbb{R})$. Thus, for any $x, y \in \mathcal{C}$, $x \preceq_{\mathcal{C}} y$ implies $x(\theta) \leq y(\theta)$ for all $\theta \in I_0$. Note that if $x, y \in C(J, \mathbb{R})$ and $x \preceq y$, then $x_t \preceq_{\mathcal{C}} y_t$ for all $t \in I$ (Cf. Dhage [4, 5]).

Let $C_{eq}(J, \mathbb{R})$ denote the subset of all equicontinuous functions in $C(J, \mathbb{R})$. Then for a constant $M > 0$, by $C_{eq}^M(J, \mathbb{R})$ we denote the class of equicontinuous functions in $C(J, \mathbb{R})$ defined by

$$C_{eq}^M(J, \mathbb{R}) = \{x \in C_{eq}(J, \mathbb{R}) \mid \|x\| \leq M\}.$$

Clearly, $C_{eq}^M(J, \mathbb{R})$ is a closed and uniformly bounded subset of the set of equicontinuous functions of the Banach space $C(J, \mathbb{R})$ which is compact in view of Arzelá-Ascoli theorem.

We need the following definition in what follows.

Definition 3.2. *A function $u \in C_{eq}^M(J, \mathbb{R})$ is said to be a* lower solution *of the PBVP (1.3) if the conditions (i) and (ii) of Definition 1.1 hold and u satisfies the inequalities*

$$\left.\begin{aligned} u'(t) + h(t)u(t) \leq f(t, u(t), u_t) \;\; \textit{a.e.} \;\; t \in I, \\ u(0) = \phi(0) \leq u(T), \\ u_0 \leq \phi. \end{aligned}\right\} \tag{3.4}$$

Similarly, a function $v \in C_{eq}^M(J, \mathbb{R})$ is called an upper solution of the functional PBVP (1.3) if the above inequality is satisfied with reverse sign. By a solution *of the PBVP (1.3) in a subset $C_{eq}^M(J, \mathbb{R})$ of the Banach space $C(J, \mathbb{R})$ we mean a function $x \in C_{eq}^M(J, \mathbb{R})$ which is both lower and upper solution of the functional PBVP (1.3) defined on J.*

We consider the following set of hypotheses in what follows:

(H$_1$) There exist constants $\ell_1 > 0$, $\ell_2 > 0$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \ell_1(x_1 - y_2) + \ell_2\|x_2 - y_2\|_{\mathcal{C}},$$

for all $t \in J$, where $x_1, y_1 \in \mathbb{R}$ and $x_2, y_2 \in \mathcal{C}$ with $x_1 \geq y_1$, $x_2 \succeq_{\mathcal{C}} y_2$.

(H$_2$) The function f is L^1-Carathéodory on $I \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$.

(H$_3$) $f(t, x, y)$ is monotone nondecreasing in x and y for each $t \in I$.

(H$_4$) The functional PBVP (1.3) has a lower solution $u \in C_{eq}^M(J, \mathbb{R})$.

(H$_5$) The functional PBVP (1.3) has an upper solution $v \in C_{eq}^M(J, \mathbb{R})$.

Theorem 3.3. *Suppose that hypotheses (H$_2$) through (H$_4$) hold. Furthermore, if the inequality*

$$\|\phi\|_{\mathcal{C}} + M_h \|m_M\|_{L^1} \leq M, \tag{3.5}$$

holds, then the PBVP (1.3) has a solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\left.\begin{aligned} x_0(t) &= u(t),\ \ t \in J, \\ x_{n+1}(t) &= \begin{cases} \displaystyle\int_0^T G_h(t,s) f(s, x_n(s), x_s^n)\, ds, & t \in I, \\ \phi(t), & t \in I_0, \end{cases} \end{aligned}\right\} \tag{3.6}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, is monotone nondecreasing and converges to x^*.

Proof. Set $S = C_{eq}^M(J, \mathbb{R})$. Then, S is a uniformly bounded and equicontinuous subset of the ordered Banach space (X, K). Hence S is compact in view of Arzellá-Ascoli theorem. Consequently, S is partially compact subset of (X, K). Define an operator $\mathcal{T} : S \to C(J, \mathbb{R})$ by

$$\mathcal{T}x(t) = \begin{cases} \displaystyle\int_0^T G_h(t,s) f(s, x(s), x_s)\, ds, & t \in I, \\ \phi(t), & t \in I_0. \end{cases} \tag{3.7}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.7 in a series of following steps.

Step I: *$\mathcal{T}$ is well defined and $\mathcal{T} : S \to S$.*

Clearly, $\mathcal{T}$ is well defined in view of continuity of the functions k and f on $J \times J$ and $J \times \mathbb{R} \times \mathbb{R}$ receptively. We show that $\mathcal{T}(S) \subset S$. Let $x \in S$ be arbitrary. Now by hypothesis (H$_2$),

$$\begin{aligned} |\mathcal{T}x(t)| &\leq \begin{cases} \displaystyle\int_0^T G_h(t,s) |f(s, x(s), x_s)|\, ds, & t \in I, \\ |\phi(t)|, & t \in I_0, \end{cases} \\ &\leq \begin{cases} \displaystyle\int_0^T M_h\, m_M(s)\, ds, & t \in I, \\ |\phi(t)|, & t \in I_0, \end{cases} \\ &\leq \|\phi\|_{\mathcal{C}} + M_h \|m_M\|_{L^1} \\ &= M, \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{T}x\| \leq M$ for all $x \in C_{eq}^M(J, \mathbb{R})$.

Next, we prove that $\mathcal{T}(S) \subset S$. Let $y \in \mathcal{T}(S)$ be arbitrary. Then there is an $x \in S$ such that $y = \mathcal{T}x$. Now we consider the following three cases:

Case I : Suppose that $t_1, t_2 \in I$. Then, we have

$$\begin{aligned}
|y(t_1) - y(t_2)| &= \big|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\big| \\
&\leq \int_0^T \big|G_h(t_1, s) - G_h(t_2, s)\big| \, |f(s, x(s), x_s)| \, ds \\
&\leq \big[(\ell_1 + \ell_2)L + F_0\big] \int_0^T |G_h(t_1, s) - G_h(t_2, s)| \, ds. \qquad (*)
\end{aligned}$$

Since k is continuous on compact $J \times J$, it is uniformly continuous there. Therefore, for each fixed $s \in J$, we have

$$|k(t_1, s) - k(t_2, s)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly. This further in view of inequality (*) implies that

$$\big|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\big| \to 0 \quad \text{as} \quad t_1 \to t_2, \qquad (i)$$

uniformly for all $x \in S$.

Case II : Suppose that $t_1, t_2 \in I_0$. Then, we have

$$|y(t_1) - y(t_2)| = \big|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\big| = |\phi(t_1) - \phi(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2,$$

uniformly for $x \in S$.

Case III : Let $t_1 \in I_)$ and $t_2 \in I$. Then we obtain

$$|y(t_1) - y(t_2)| = \big|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\big| \leq \big|\mathcal{T}x(t_1) - \mathcal{T}x(0)\big| + \big|\mathcal{T}x(0) - \mathcal{T}x(t_2)\big|.$$

If $t_1 \to t_2$, that is, $|t_1 - t_2| \to 0$, then $t_1 \to 0$ and $t_2 \to 0$ which in view of inequalities (i) and (ii) implies that

$$|y(t_1) - y(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2 \qquad (iii)$$

uniformly for all $y \in \mathcal{T}(S)$. From above three cases (i)-(iii) it follows that $\mathcal{T}x \in S$ for all $x \in S$. As a result $\mathcal{T}(S) \subseteq S$.

Step II: *$\mathcal{T}$ is a monotone nondereasing operator on S.*

Let $x, y \in S$ be such that $x \succeq y$. Then, $x_t \succeq y_t$ for each $t \in I$. Therefore, by hypothesis (H$_2$), we get

$$\begin{aligned}
\mathcal{T}x(t) &= \begin{cases} \displaystyle\int_0^T G_h(t, s) f(s, x(s), x_s) \, ds, \ , \ t \in I, \\ \phi(t), \ \ t \in I_0, \end{cases} \\
&\geq \begin{cases} \displaystyle\int_0^T G_h(t, s) f(s, y(s), y_s) \, ds, \ , \ t \in I, \\ \phi(t), \ \ t \in I_0, \end{cases} \\
&= \mathcal{T}y(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}x \succeq \mathcal{T}y$ and consequently the operator $\mathcal{T}$ is monotone nondecreasing on S.

Step III: *$\mathcal{T}$ is partially continuous on S.*

Let C be a chain in the closed and bounded subset $C_{eq}^M(J,\mathbb{R})$ of the ordered Banach space $\big(C(J,\mathbb{R}),K\big)$ and let $\{x_n\}$ be a sequence of points in C such that $x_n \to x$ as $n\to\infty$. Then, by definition of the operator $\mathcal{T}$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\mathcal{T}x_n &= \lim_{n\to\infty}\begin{cases}\displaystyle\int_0^T G_h(t,s)f(s,x_n(s),x_s^n)\,ds, \ , \ t\in I,\\ \phi(t), \ \ t\in I_0,\end{cases}\\ &=\begin{cases}\displaystyle\int_0^T G_h(t,s)\Big[\lim_{n\to\infty}f(s,x_n(s),x_s^n)\Big]\,ds, \ , \ t\in I,\\ \phi(t), \ \ t\in I_0,\end{cases}\\ &=\begin{cases}\displaystyle\int_0^T G_h(t,s)f(s,x(s),x_s)\,ds, \ , \ t\in I,\\ \phi(t), \ \ t\in I_0,\end{cases}\\ &=\mathcal{T}x(t),\end{aligned}$$

for all $t\in J$. This shows that $\mathcal{T}x_n \to \mathcal{T}x$ pointwise on J. Next, by following the arguments asin Step II, it is proved that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of points in S. This shows that $\mathcal{T}x_n\to\mathcal{T}x$ uniformly on J. Consequently $\mathcal{T}$ is a partially continuous operator on S into itself.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.7 on a partially compact subset S of the Banach space $C(J,\mathbb{R})$. Hence $\mathcal{T}$ has a fixed point $x^*\in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^\infty$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the PBVP (1.3) has a solution x^* on J and the sequence $\{x_n\}_{n=0}^\infty$ successive approximations defined by (3.6) converges monotone nondecreasingly to x^*. This completes the proof. □

Theorem 3.4. *Suppose that the hypotheses (H$_1$) and (H$_4$) hold. Furthermore, if $M_hT(\ell_1+\ell_2)<1$, then the PBVP (1.3) has a unique solution x^* defined on J and the sequence $\{x_n\}_{n=0}^\infty$ of successive approximations defined by (3.6) is monotone nondecreasing and converges to x^*.*

Proof. Set $S=C_{eq}^M(J,\mathbb{R})$. Then $S=C_{eq}^M(J,\mathbb{R})$ is a closed subset of an ordered Banach space (X,K) and soo it is partially closed set in (X,K). Define an operator $\mathcal{T}$ on S by (3.7). Then $\mathcal{T}$ is well defined. We shall show that $\mathcal{T}$ is a partial contraction on S.

Let $x,y\in S$ be such that $x\succeq y$. Then, by hypothesis (H$_1$), we have

$$\begin{aligned}\big|\mathcal{T}x(t)-\mathcal{T}y(t)\big| &= \Big|\int_0^T G_h(t,s)\big[f(s,x(s),x_s)-f(s,y(s),y_s)\big]\,ds\Big|\\ &\le \int_0^T G_h(t,s)\big|f(s,x(s),x_s)-f(s,y(s),y_s)\big|\,ds\\ &\le \int_0^T G_h(t,s)\big[f(s,x(s),x_s)-f(s,y(s),y_s)\big]\,ds\\ &\le \int_0^T G_h(t,s)\big[\ell_1|x(s)-y(s))|+\ell_2\|x_s-y_s\|_{\mathcal{C}}\big]\,ds\\ &\le \int_0^T G_h(t,s)(\ell_1+\ell_2)\|x-y\|\,ds\\ &\le M_hT(\ell_1+\ell_2)\|x-y\|,\end{aligned}$$

for all $t\in J$. Taking the supremum over t, we obtain

$$\|\mathcal{T}x-\mathcal{T}y\|\le M_hT(\ell_1+\ell_2)\|x-y\|$$

for all comparable elements $x, y \in S$. This shows that $\mathcal{T}$ is a partial contraction on S. We know that every partially Lipschitz operator is partially continuous, so $\mathcal{T}$ is a partially continuous operator on S. Now, we apply Theorem **??** to the operator $\mathcal{T}$ and conclude that $\mathcal{T}$ has a unique fixed point $x^* \in S$ and the sequence $\{\mathcal{T}^n u\}_{n=0}^{\infty}$ of successive iterations converges to x^*. This further implies that functional PBVP (1.3) has a unique solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.6) is monotone nondecreasing and converges to x^*. □

Remark 3.5. *The conclusion of existence and uniqueness theorems, Theorems 3.3 and 3.4 for the problem* (1.3) *also remains true if we replace the hypothesis (H$_4$) by (H$_5$). In this case the sequence $\{y_n\}_{n=0}^{\infty}$ defined similar to* (3.6) *converges monotone nonincreasingly to the solution x^* of the functional PBVP (1.3) defined on J.*

4. An Example

Example 4.1. *Let $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I == [0,1]$ be two closed and bounded intervals in $\mathbb{R}$, the set of real number and let $J = \left[-\frac{\pi}{2}, 0\right] \bigcup [0,1] = \left[-\frac{\pi}{2}, 1\right]$. Given a history function $\phi(t) = \sin t$, $t \in \left[-\frac{\pi}{2}, 0\right]$, consider the nonlinear two point functional BVP*

$$\left.\begin{aligned} x'(t) + x(t) = f_1(t, x(t), x_t), \;\; t \in [0,1], \\ x(0) = \phi(0) = 0 = x(1), \\ x_0 = \phi, \end{aligned}\right\} \tag{4.1}$$

for all $t \in [0,1]$, where $x_t(\theta) = x(t+\theta)$, $\theta \in \left[-\frac{\pi}{2}, 0\right]$ and the function f_2 is given by

$$f_1(t,x,y) = \begin{cases} 0, & \textit{if} \quad x \le 0, \; y \preceq_{\mathcal{C}} 0, \\ \dfrac{1}{4}\,\dfrac{x}{1+x}, & \textit{if} \quad x > 0, \; y \preceq_{\mathcal{C}} 0, \\ \dfrac{1}{4}\,\dfrac{\|y\|_{\mathcal{C}}}{1+\|y\|_{\mathcal{C}}}, & \textit{if} \quad x \le 0, \; y \succeq_{\mathcal{C}} 0, \; y \ne 0, \\ \dfrac{1}{4}\left[\dfrac{x}{1+x} + \dfrac{\|y\|_{\mathcal{C}}}{1+\|y\|_{\mathcal{C}}}\right], & \textit{if} \quad x > 0, \; y \succeq_{\mathcal{C}} 0, \; y \ne 0, \end{cases}$$

for all $t \in [0,1]$.

Here, $h = 1 = T$ and f_1 defines a continuous function $f : [0,1] \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$. We shall show that f_1 satisfies all the conditions of Theorem 3.4. Now, let $x_1, y_1 \in \mathbb{R}$ and $x_2, y_2 \in \mathcal{C}$ be such that $x_1 \ge y_1 \ge 0$ and $x_2 \succeq_{\mathcal{C}} y_2 \succeq 0$. Therefore, we have

$$\begin{aligned} 0 &\le f(t,x_1,x_2) - f(t,y_1,y_2) \\ &\le \frac{1}{4}\cdot\frac{x_1}{1+x_1} - \frac{y_1}{1+y_1} + \frac{\|x_2\|_{\mathcal{C}}}{1+\|x_2\|_{\mathcal{C}}} - \frac{\|y_2\|_{\mathcal{C}}}{1+\|y_2\|_{\mathcal{C}}} \\ &\le \frac{1}{4}\cdot\frac{x_1-y_1}{1+x_1-y_1} + \frac{1}{4}\cdot\frac{\|x_2\|_{\mathcal{C}} - \|y_2\|_{\mathcal{C}}}{1+\|x_2\|_{\mathcal{C}} - \|y_2\|_{\mathcal{C}}} \qquad (\because |x_1 - y_1| \le |x_1| + |y_1|) \\ &\le \frac{1}{4}\cdot\frac{|x_1-y_1|}{1+|x_1-y_1|} + \frac{1}{4}\cdot\frac{\big|\,\|x_2\|_{\mathcal{C}} - \|y_2\|_{\mathcal{C}}\big|}{1+\big|\,\|x_2\|_{\mathcal{C}} - \|y_2\|_{\mathcal{C}}} \\ &\le \frac{1}{4}\cdot\frac{|x_1-y_1|}{1+|x_1-y_1|} + \frac{1}{4}\cdot\frac{\|x_2-y_2\|_{\mathcal{C}}}{1+\|x_2-y_2\|_{\mathcal{C}}} \\ &\le \frac{1}{4}\cdot|x_1-y_1| + \frac{1}{4}\cdot\|x_2-y_2\|_{\mathcal{C}}, \end{aligned}$$

for all $t \in [0,1]$. Similarly, we get the same estimate for other values of the function f_1. So the hypothesis (H_1) holds with $\ell_1 = \frac{1}{4}$ and $\ell_2 = \frac{1}{4}$. Again, the Green's function G is continuous and nonnegative on $[0,1] \times [0,1]$ with bound $M_1 \approx 1.6$, so that the hypothesis (H_3) holds. Moreover, here we have

$$M_h\, T(\ell_1 + \ell_2) \approxeq 1.6 \left(\frac{1}{4} + \frac{1}{4} \right) = 0.8 < 1,$$

and so all the conditions of Theorem 3.4 are satisfied for $M = \frac{9}{5}$. Finally, the functions u and v defined by

$$u(t) = \begin{cases} -\displaystyle\int_0^1 G_1(t,s)\, ds, & t \in [0,1], \\ \sin t, & t \in \left[-\dfrac{\pi}{2}, 0 \right], \end{cases}$$

and

$$v(t) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^1 G_1(t,s)\, ds, & t \in [0,1], \\ \sin t, & t \in \left[-\dfrac{\pi}{2}, 0 \right], \end{cases}$$

satisfy respectively the inequalities of the lower solution and upper solution of the functional PBVP (4.1) *with $u \preceq v$ on J. Hence the functional PBVP (4.1) has a unique solution $x^* \in C_{eq}^{9/5}(J, \mathbb{R})$ defined on $J = \left[-\frac{\pi}{2}, 1 \right]$. Moreover, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_0(t) = u(t), \quad t \in \left[-\frac{\pi}{2}, 1 \right],$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^1 G_1(t,s) f(s, x_n(s), x_s^n)\, ds, & t \in [0,1], \\ \sin t, & t \in \left[-\dfrac{\pi}{2}, 0 \right], \end{cases}$$

is monotone nondecreasing and converges to x^*. Similarly, the sequence $\{y_n\}_{n=0}^{\infty}$ defined by

$$y_0(t) = v(t), \quad t \in \left[-\frac{\pi}{2}, 1 \right],$$

$$y_{n+1}(t) = \begin{cases} \displaystyle\int_0^1 G_1(t,s) f(s, y_n(s), y_s^n)\, ds, & t \in [0,1], \\ \sin t, & t \in \left[-\dfrac{\pi}{2}, 0 \right], \end{cases}$$

is monotone nonincreasing and converges to the unique dolution $x^* \in C_{eq}^{9/5}(J, \mathbb{R})$. Thus, we have

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x^* \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0.$$

5. Remarks and Conclusion

We observe that the existence of solutions of the PBVP (1.3) can also be obtained by an application of topological Schauder fixed point principle under the hypothesis (H_2), but in that case we do not get any sequence of successive approximations that converges to the solution. Again, we can not apply analytical or geometric Banach contraction mapping principle to the problem (3.1) under the considered hypotheses (H_1) and (H_3) in order to get the desired conclusion, because here the nonlinear function f does not satisfy the usual Lipschitz condition on the domain $I \times \mathbb{R} \times \mathcal{C}$. Similarly, we can not apply algebraic Knaster-Tarski fixed point theorem to PBVP (1.3) for proving the existence of solution, because $C(J, \mathbb{R})$ is not a complete lattice. Therefore, all these arguments show that our hybrid fixed point principle, Theorem 2.4 has more advantages than classical fixed point theorems to get more information about the solution of nonlinear equations in the subject of nonlinear analysis.

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 208–226.
http://doi.org/10.26637/mjm11S/013

On Tribonacci functions and Gaussian Tribonacci functions

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*R*eceived 31 August 2023; *A*ccepted 28 September 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. In this work, Gaussian Tribonacci functions are defined and investigated on the set of real numbers $\mathbb{R}$, *i.e.*, functions $f_G : \mathbb{R} \to \mathbb{C}$ such that for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $f_G(x+n) = f(x+n) + if(x+n-1)$ where $f : \mathbb{R} \to \mathbb{R}$ is a Tribonacci function which is given as $f(x+3) = f(x+2) + f(x+1) + f(x)$ for all $x \in \mathbb{R}$. Then the concept of Gaussian Tribonacci functions by using the concept of f-even and f-odd functions is developed. Also, we present linear sum formulas of Gaussian Tribonacci functions. Moreover, it is showed that if f_G is a Gaussian Tribonacci function with Tribonacci function f, then $\lim\limits_{x\to\infty} \frac{f_G(x+1)}{f_G(x)} = \alpha$ and $\lim\limits_{x\to\infty} \frac{f_G(x)}{f(x)} = \alpha + i$, where α is the positive real root of equation $x^3 - x^2 - x - 1 = 0$ for which $\alpha > 1$. Finally, matrix formulations of Tribonacci functions and Gaussian Tribonacci functions are given. In the literature, there are several studies on the functions of linear recurrent sequences such as Fibonacci functions and Tribonacci functions. However, there are no study on Gaussian functions of linear recurrent sequences such as Gaussian Tribonacci and Gaussian Tetranacci functions and they are waiting for the investigating.
We also present linear sum formulas and matrix formulations of Tribonacci functions which have not been studied in the literature.

AMS Subject Classifications: 28A10.

Keywords: Tribonacci numbers, Tribonacci functions, Gaussian Tribonacci functions, f-even functions.

Contents

1. Introduction

A function f defined on the real numbers $\mathbb{R}$ is said to be a Fibonacci function if it satisfies the following relation

$$f(x+2) = f(x+1) + f(x)$$

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for all $x \in \mathbb{R}$. First and foremost, Elmore [2], Parker [8] and Spickerman [12] discovered useful properties of the Fibonacci functions. Later, many renowned researchers such as Fergy and Rabago [3], Han, et al. [5], Sroysang [14], and Gandhi [4], have devoted their study to the analysis of many properties of the Fibonacci function.

A function f defined on the real numbers $\mathbb{R}$ is said to be a Tribonacci function if it satisfies the following relation

$$f(x+3) = f(x+2) + f(x+1) + f(x)$$

for all $x \in \mathbb{R}$ (a short review on Tribonacci functions will be given in this section below). Some references on Tribonacci functions are Arolkar [1], Magnani [6], Parizi [7] and Sharma [10].

A function f defined on the real numbers $\mathbb{R}$ is said to be a Tetranacci function if it satisfies the following relation

$$f(x+4) = f(x+3) + f(x+2) + f(x+1) + f(x)$$

for all $x \in \mathbb{R}$. See Sharma [11] for more information on Tetranacci functions.

More generally, a function f defined on the real numbers $\mathbb{R}$ is said to be a k-step Fibonacci function if it satisfies the following relation

$$f(x+k) = f(x+k-1) + f(x+k-2) + f(x+k-3) + ... + f(x)$$

for all $x \in \mathbb{R}$. See Sriponpaew and Sassanapitax [13], and Wolfram [18] for more information on k-step Fibonacci functions.

Before giving a short review on Tribonacci functions, we recall the definition of a Tribonacci sequence. A Tribonacci sequence $\{T_n\}_{n\geq 0} = \{V_n(T_0, T_1, T_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

with the initial values $T_0 = 0, T_1 = 1, T_2 = 1$.

Next, we present the first few values of the Tribonacci numbers with positive and negative subscripts:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504	927
T_{-n}	....	0	1	-1	0	2	-3	1	4	-8	5	7	-20	18

If we let $u_0 = 0, u_1 = 1, u_2 = 1$, then we consider the full (bilateral) Tribonacci sequence $\{u_n\}_{n=-\infty}^{\infty}$: . . . , $-3, 2, 0, -1, 1, 0, 0, 1, 1, 2, 4, 7, 13, ...$, i.e. $T_{-n} = T_n^2 + T_{2n} + T_{n+2}T_n - 4T_{n+1}T_n$ [see 5, Corollary 7] for $n > 0$ and $u_n = T_n$, the nth Tribonacci number.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Tribonacci function if it satisfies the formula

$$f(x+3) = f(x+2) + f(x+1) + f(x)$$

for all $x \in \mathbb{R}$ or equivalently

$$f(x) = f(x-1) + f(x-2) + f(x-3)$$

for all $x \in \mathbb{R}$.

Note that

$$f(x+n) = T_{n-1}f(x+2) + (T_{n-2} + T_{n-3})f(x+1) + T_{n-2}f(x) \tag{1.1}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

We next present the Binet's formula of f.

Lemma 1.1. *[5, p.141] The Binet's formula of f is*

$$f(x) = K_1\alpha^x + K_2\alpha^{-x/2}\cos(\theta x) + \alpha^{-x/2}\left(\frac{K_2\left(\frac{1-\alpha}{2}\right) + K_3}{\sqrt{\frac{1}{\alpha} - \left(\frac{\alpha-1}{2}\right)^2}}\right)\sin(\theta x),$$

where

$$
\begin{aligned}
K_1 &= \frac{f(0)}{\alpha(\alpha-1)(3\alpha+1)} + \frac{f(1)}{3\alpha+1} + \frac{f(2)}{(\alpha-1)(3\alpha+1)}, \\
K_2 &= f(0) - K_1, \\
K_3 &= \frac{K_1 - \alpha(f(2)-f(1)-f(0))}{\alpha^2}, \\
\theta &= \arccos\left(\frac{1-\alpha}{2}\sqrt{\alpha}\right).
\end{aligned}
$$

Note that this formula does not use complex roots.

Next, we list some examples of Tribonacci functions.

Example 1.2.

(a) *[5, Example 2.1]*

$$f:\mathbb{R}\to\mathbb{R},\ f(x)=\alpha^x$$

is a Tribonacci function, where α is a positive root of equation $x^3-x^2-x-1=0$ and α is greater than one and given as

$$\alpha = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}.$$

The other two roots are

$$\beta = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

and

$$\gamma = \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3}.$$

(b) *[5, Example 2.2] Let $\{u_n\}_{n=-\infty}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}$ and $\{w_n\}_{n=-\infty}^{\infty}$ be full Tribonacci sequences and define a function $f(x)$ by $f(x)=u_{\lfloor x\rfloor}+v_{\lfloor x\rfloor}t+w_{\lfloor x\rfloor}t^2$, where $t=x-\lfloor x\rfloor\in(0,1)$ and $x\in\mathbb{R}$, $\lfloor x\rfloor$ is the greatest integer function (floor function). Then*

$$f(x+3)=f(x+2)+f(x+1)+f(x)$$

so that f is a Tribonacci function.

(c) *[5, Proposition 2.3 and Example 2.4] Let f be a Tribonacci function and define $g(x)=f(x+t+t^2)$ for any $x\in\mathbb{R}$, where $t\in\mathbb{R}$. Then g is also a Tribonacci function. If $f(x)=\alpha^x$ which is a Tribonacci function, then $g(x)=\alpha^{x+t+t^2}=\alpha^{t+t^2}f(x)$ is a Tribonacci function.*

We now present the concepts of f-even and f-odd functions which were defined by Han, et al. [5] in 2012.

Definition 1.3. *Suppose that $a(x)$ is a real-valued function of real variable such that if $a(x)h(x)=0$ and $h(x)$ is continuous. Then $h(x)\equiv 0$. The map $a(x)$ is called an f-even function if $a(x+1)=a(x)$ and f-odd function if $a(x+1)=-a(x)$) for all $x\in\mathbb{R}$.*

We present an f-even and an f-odd function.

Example 1.4.

(a) *If $a(x)=x-\lfloor x\rfloor$ then $a(x)$ is an f-even function.*

(b) *If $a(x) = \sin(\pi x)$ then $a(x)$ is an f-odd function.*

Solution.

(a) If $a(x) = x - \lfloor x \rfloor$ then $a(x)h(x) = 0$ implies $h(x) \equiv 0$ if $x \notin \mathbb{Z}$. By continuity of $h(x)$, it follows that $h(n) = \lim_{x \to n} h(x) = 0$ for any integer $n \in \mathbb{Z}$ and therefore $h(x) \equiv 0$. Since

$$a(x+1) = (x+1) - \lfloor x+1 \rfloor = (x+1) - (\lfloor x \rfloor + 1) = x - \lfloor x \rfloor = a(x),$$

we see that $a(x)$ is an f-even function.

(b) If $a(x) = \sin(\pi x)$, then $a(x)h(x) \equiv 0$ implies that $h(x) = 0$ if $x \neq n\pi$ for any integer $n \in \mathbb{Z}$. Since $h(x)$ is continuous, it follows that $h(n\pi) = \lim_{x \to n\pi} h(x) = 0$ for $n \in \mathbb{Z}$, and therefore, $h(x) \equiv 0$. Since

$$a(x+1) = \sin(\pi x + \pi) = \sin(\pi x)\cos(\pi) = -\sin(\pi x) = -a(x),$$

we see that $a(x)$ is an f-odd function. □

The following theorem is given in ([7], Theorem 3.3).

Theorem 1.5. *Assume that $f(x) = a(x)g(x)$ is a function, where $a(x)$ is an f-even function and $g(x)$ is a continuous function. Then $f(x)$ is a Tribonacci function if and only if $g(x)$ is a Tribonacci function.*

The following theorem shows that the limit of quotient of a Tribonacci function exists.

Theorem 1.6. *If $f(x)$ is a Tribonacci function, then the limit of quotient $\frac{f(x+1)}{f(x)}$ exists (5, Theorem 4.1]) and $\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = \alpha$ (take $p = 1$ in Magnani [5, Theorem 14].*

We also have a result on the limit of quotient $\frac{f(x+2)}{f(x)}$ which we need for calculation of the limit of the quotient of a Gaussian Tribonacci function.

Theorem 1.7. *If $f(x)$ is a Tribonacci function, then, for $0 \leq k, m \in \mathbb{N}$ we have*

$$\lim_{x \to \infty} \frac{f(x+k)}{f(x+m)} = \alpha^{k-m}. \tag{1.2}$$

Proof. If $0 \leq k, m \leq 1$ then (1.2) is true (Theorem 1.6). We give the proof in three stages:
Stage I:

$$\lim_{x \to \infty} \frac{f(x+2)}{f(x)} = \alpha^2.$$

Stage II: for $3 \leq k \in \mathbb{N}$,

$$\lim_{x \to \infty} \frac{f(x+k)}{f(x)} = \alpha^k.$$

Stage III: for $3 \leq k, m \in \mathbb{N}$,

$$\lim_{x \to \infty} \frac{f(x+k)}{f(x+m)} = \alpha^{k-m}.$$

Proof of Stage I:
Given $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $x = y + n$. Then, using the formula

$$f(x+n) = T_{n-1}f(x+2) + (T_{n-2} + T_{n-3})f(x+1) + T_{n-2}f(x),$$

we get

$$\begin{aligned}\frac{f(x+2)}{f(x)} &= \frac{f(y+n+2)}{f(y+n)} \\ &= \frac{T_{n+1}f(y+2)+(T_n+T_{n-1})f(y+1)+T_nf(y)}{T_{n-1}f(y+2)+(T_{n-2}+T_{n-3})f(y+1)+T_{n-2}f(y)} \\ &= \frac{T_n}{T_{n-2}}\frac{\frac{T_{n+1}}{T_n}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n-1}}{T_n})\frac{f(y+1)}{f(y)}+1}{\frac{T_{n-1}}{T_{n-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n-3}}{T_{n-2}})\frac{f(y+1)}{f(y)}+1}\end{aligned}$$

Since

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)}=\lim_{y\to\infty}\frac{f(y+1)}{f(y)}=\alpha$$

and

$$\lim_{n\to\infty}\frac{T_{n+p}}{T_{n+q}}=\alpha^{p-q},\ \ p,q\in\mathbb{Z}$$

and

$$\lim_{x\to\infty}\frac{f(x+2)}{f(x)}=\lim_{y\to\infty}\frac{f(y+2)}{f(y)}=u\quad\text{(say)}$$

we obtain

$$u=\alpha^2\frac{\alpha u+(1+\alpha^{-1})\alpha+1}{\alpha u+(1+\alpha^{-1})\alpha+1}$$

and so

$$u=\lim_{x\to\infty}\frac{f(x+2)}{f(x)}=\alpha^2.$$

This completes the proof of Stage I.

Proof of Stage II:

Given $x\in\mathbb{R}$, there exists $y\in\mathbb{R}$ and $n\in\mathbb{N}$ such that $x=y+n$. Then, by using Stage I, we get

$$\begin{aligned}\frac{f(x+k)}{f(x)} &= \frac{f(y+n+k)}{f(y+n)} \\ &= \frac{T_{n+k-1}f(y+2)+(T_{n+k-2}+T_{n+k-3})f(y+1)+T_{n+k-2}f(y)}{T_{n-1}f(y+2)+(T_{n-2}+T_{n-3})f(y+1)+T_{n-2}f(y)} \\ &= \frac{T_{n+k-2}}{T_{n-2}}\frac{\frac{T_{n+k-1}}{T_{n+k-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n+k-3}}{T_{n+k-2}})\frac{f(y+1)}{f(y)}+1}{\frac{T_{n-1}}{T_{n-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n-3}}{T_{n-2}})\frac{f(y+1)}{f(y)}+1}\end{aligned}$$

and so

$$\begin{aligned}\lim_{x\to\infty}\frac{f(x+k)}{f(x)} &= \lim_{y\to\infty}\lim_{n\to\infty}\frac{T_{n+k-2}}{T_{n-2}}\frac{\frac{T_{n+k-1}}{T_{n+k-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n+k-3}}{T_{n+k-2}})\frac{f(y+1)}{f(y)}+1}{\frac{T_{n-1}}{T_{n-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n-3}}{T_{n-2}})\frac{f(y+1)}{f(y)}+1} \\ &= \alpha^k\frac{\alpha\alpha^2+(1+\alpha^{-1})\alpha+1}{\alpha\alpha^2+(1+\alpha^{-1})\alpha+1} \\ &= \alpha^k\end{aligned}$$

which completes the proof of Stage II.

Proof of Stage III:

By using Stage II, we obtain

$$\lim_{x\to\infty}\frac{f(x)}{f(x+m)}=\frac{1}{\lim_{x\to\infty}\frac{f(x+m)}{f(x)}}=\frac{1}{\alpha^m}=\alpha^{-m}.$$

Now, it follows that

$$\lim_{x\to\infty}\frac{f(x+k)}{f(x+m)}=\lim_{x\to\infty}\frac{f(x+k)}{f(x)}\lim_{x\to\infty}\frac{f(x)}{f(x+m)}=\alpha^k\alpha^{-m}=\alpha^{k-m}$$

which completes the proof of Stage III. □

2. Gaussian Tribonacci Function

Gaussian Tribonacci numbers $\{GT_n\}_{n\geq 0}=\{GT_n(GT_0,GT_1,GT_2)\}_{n\geq 0}$ are defined by

$$GT_n=GT_{n-1}+GT_{n-2}+GT_{n-3},$$

with the initial conditions $GT_0=0$, $GT_1=1$ and $GT_2=1+i$. Note that

$$GT_n=T_n+iT_{n-1}.$$

The first few values of Gaussian Tribonacci numbers with positive and negative subscript are given in the following table.

n	0	1	2	3	4	5	6	7	8	9
GT_n	0	1	$1+i$	$2+i$	$4+2i$	$7+4i$	$13+7i$	$24+13i$	$44+24i$	$81+44i$
GT_{-n}	0	i	$1-i$	-1	$2i$	$2-3i$	$-3+i$	$1+4i$	$4-8i$	$-8+5i$

The full Gaussian Tribonacci sequence, where $Gu_n=GT_n$ the n^{th} Gaussian Tribonacci numbers, are: . . . , $-3+i, 2-3i, 2i, -1, 1-i, i, 0, 1, 1+i, 2+i, 4+2i, 7+4i, 13+7i, \ldots$.

Definition 2.1. *A Gaussian function f_G on the real numbers $\mathbb{R}$ is said to be a Gaussian Tribonacci function if it satisfies the formula*

$$f_G(x+n)=GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x) \tag{2.1}$$

for all $x\in\mathbb{R}$ and $n\in\mathbb{Z}$ where f is a Tribonacci function.

To emphasize which Tribonacci function used we can say that f_G is a Gaussian Tribonacci function with Tribonacci function f.

The following theorem gives an equivalent characterization of a Gaussian Tribonacci function.

Theorem 2.2. *A Gaussian function f_G on the real numbers $\mathbb{R}$ is a Gaussian Tribonacci function if and only if*

$$f_G(x+n)=f(x+n)+if(x+n-1) \tag{2.2}$$

for $x\in\mathbb{R}$, $n\in\mathbb{Z}$ where f is a Tribonacci function.

Proof.

(:⇒) Assume that f_G is a Gaussian Tribonacci function, i.e., f_G satisfies (2.1). Then

$$\begin{aligned} f_G(x+n) &= (T_{n-1}+iT_{n-2})f(x+2)+((T_{n-2}+iT_{n-3}) \\ &\quad +(T_{n-3}+iT_{n-4}))f(x+1)+(T_{n-2}+iT_{n-3})f(x) \\ &= T_{n-1}f(x+2)+(T_{n-2}+T_{n-3})f(x+1)+T_{n-2}f(x)+ \\ &\quad i(T_{n-2}f(x+2)+(T_{n-3}+T_{n-4})f(x+1)+T_{n-3}f(x)) \\ &= f(x+n)+if(x+n-1) \end{aligned}$$

since

$$GT_n=T_n+iT_{n-1}$$

and

$$f(x+n) = T_{n-1}f(x+2) + (T_{n-2}+T_{n-3})f(x+1) + T_{n-2}f(x)$$

($\Leftarrow$:) If we suppose that (2.2) holds then by (1.1), we obtain

$$\begin{aligned}
& f_G(x+n) \\
=\ & f(x+n) + if(x+n-1) \\
=\ & T_{n-1}f(x+2) + (T_{n-2}+T_{n-3})f(x+1) + T_{n-2}f(x) + \\
& +(iT_{n-2}f(x+2) + (T_{n-3}+T_{n-4})f(x+1) + T_{n-3}f(x)) \\
=\ & (T_{n-1}+iT_{n-2})f(x+2) + ((T_{n-2}+iT_{n-3}) + (T_{n-3}+iT_{n-4}))f(x+1) + (T_{n-2}+iT_{n-3})f(x) \\
=\ & GT_{n-1}f(x+2) + (GT_{n-2}+GT_{n-3})f(x+1) + GT_{n-2}f(x)
\end{aligned}$$

□

Remark 2.3. *Using the Binet's formula of Tribonacci function f (see Lemma 1.1) and (2.1) or equivalenly (2.2), the Binet's formula of Gaussian Tribonacci function can be found.*

Now, we present an example of a Tribonacci function.

Example 2.4. *The function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\alpha^x$, considered in Example 1.2, is a Tribonacci function. Then*

$$f_G(x+n) = f(x+n) + if(x+n-1) = \alpha^{x+n} + i\alpha^{x+n-1} = (1+i\alpha^{-1})\alpha^{x+n}$$

is a Gaussian Tribonacci function.

The following example shows that using floor function, a Tribonacci function and a Gaussian Tribonacci function can be obtained.

Example 2.5. *Let $\{Gu_n\}_{n=-\infty}^{\infty}$, $\{Gv_n\}_{n=-\infty}^{\infty}$, and $\{Gw_n\}_{n=-\infty}^{\infty}$, be full (bilateral) Gaussian Tribonacci sequences. We define a function f_G by $f_G(x+n) = Gu_{\lfloor x\rfloor+n} + Gv_{\lfloor x\rfloor+n}t + Gw_{\lfloor x\rfloor+n}t^2 = Gu_{\lfloor x+n\rfloor} + Gv_{\lfloor x+n\rfloor}t + Gw_{\lfloor x+n\rfloor}t^2$ and $f(x) = u_{\lfloor x\rfloor} + v_{\lfloor x\rfloor}t + w_{\lfloor x\rfloor}t^2$, where $t = x-\lfloor x\rfloor \in (0,1)$ and $x\in\mathbb{R}$. Then, f is a Tribonacci function and f_G is a Gaussian Tribonacci function.*

Solution. Since

$$\begin{aligned}
f(x) &= u_{\lfloor x\rfloor} + v_{\lfloor x\rfloor}t + w_{\lfloor x\rfloor}t^2 \\
f(x+1) &= u_{\lfloor x+1\rfloor} + v_{\lfloor x+1\rfloor}t + w_{\lfloor x+1\rfloor}t^2 = u_{\lfloor x\rfloor+1} + v_{\lfloor x\rfloor+1}t + w_{\lfloor x\rfloor+1}t^2 \\
f(x+2) &= u_{\lfloor x+2\rfloor} + v_{\lfloor x+2\rfloor}t + w_{\lfloor x+2\rfloor}t^2 = u_{\lfloor x\rfloor+2} + v_{\lfloor x\rfloor+2}t + w_{\lfloor x\rfloor+2}t^2
\end{aligned}$$

and

$$\begin{aligned}
f(x+2)+f(x+1)+f(x) &= (u_{\lfloor x\rfloor+2} + v_{\lfloor x\rfloor+2}t + w_{\lfloor x\rfloor+2}t^2) + (u_{\lfloor x\rfloor+1} + v_{\lfloor x\rfloor+1}t + w_{\lfloor x\rfloor+1}t^2) \\
&\quad +u_{\lfloor x\rfloor} + v_{\lfloor x\rfloor}t + w_{\lfloor x\rfloor}t^2 \\
&= (u_{\lfloor x\rfloor+2} + u_{\lfloor x\rfloor+1} + u_{\lfloor x\rfloor}) + (v_{\lfloor x\rfloor+2} + v_{\lfloor x\rfloor+1} + v_{\lfloor x\rfloor})t \\
&\quad +(w_{\lfloor x\rfloor+2} + w_{\lfloor x\rfloor+1} + w_{\lfloor x\rfloor})t^2 \\
&= u_{\lfloor x\rfloor+3} + v_{\lfloor x\rfloor+3}t + w_{\lfloor x\rfloor+3}t^2 = u_{\lfloor x+3\rfloor} + v_{\lfloor x+3\rfloor}t + w_{\lfloor x+3\rfloor}t^2 \\
&= f(x+3)
\end{aligned}$$

f is a Tribonacci function and since

$$\begin{aligned} Gu_{\lfloor x\rfloor+n} &= u_{\lfloor x\rfloor+n} + iu_{\lfloor x\rfloor+n-1}, \\ Gv_{\lfloor x\rfloor+n} &= v_{\lfloor x\rfloor+n} + iv_{\lfloor x\rfloor+n-1}, \\ Gw_{\lfloor x\rfloor+n} &= w_{\lfloor x\rfloor+n} + iw_{\lfloor x\rfloor+n-1}, \end{aligned}$$

we get

$$\begin{aligned} f_G(x+n) &= Gu_{\lfloor x\rfloor+n} + Gv_{\lfloor x\rfloor+n}t + Gwu_{\lfloor x\rfloor+n}t^2 \\ &= (u_{\lfloor x\rfloor+n} + iu_{\lfloor x\rfloor+n-1}) + (v_{\lfloor x\rfloor+n} + iv_{\lfloor x\rfloor+n-1})t + (w_{\lfloor x\rfloor+n} + iw_{\lfloor x\rfloor+n-1})t^2 \\ &= (u_{\lfloor x\rfloor+n} + v_{\lfloor x\rfloor+n}t + w_{\lfloor x\rfloor+n}t^2) + (u_{\lfloor x\rfloor+n-1} + v_{\lfloor x\rfloor+n-1}t + w_{\lfloor x\rfloor+n-1}t^2)i \\ &= f(x+n) + if(x+n-1). \end{aligned}$$

Therefore, f_G is a Gaussian Tribonacci function. □

Lemma 2.6. *Let f_G be a Gaussian Tribonacci function, i.e., $f_G(x+n) = f(x+n) + if(x+n-1)$ for $x \in \mathbb{R}$, $n \in \mathbb{Z}$ where f is a Tribonacci function. We define $g_G(x+n) = f_G(x+t+n)$ and $g(x) = f(x+t)$ for any $x \in \mathbb{R}$ where $t \in \mathbb{R}$. Then g is a Tribonacci function and g_G is a Gaussian Tribonacci function.*

Proof. Let $x \in \mathbb{R}$. Since f_G is a Gaussian Tribonacci function and f is a Tribonacci function, it follows that

$$\begin{aligned} g(x+3) &= f(x+3+t) = f(x+t+3) \\ &= f(x+t+2) + f(x+t+1) + f(x+t) \\ &= g(x+2) + g(x+1) + g(x) \end{aligned}$$

which shows that g is a Tribonacci function and

$$\begin{aligned} g_G(x+n) &= f_G(x+t+n) \\ &= f(x+t+n) + if(x+t+n-1) \\ &= g(x+n) + ig(x+n-1) \end{aligned}$$

which shows that g_G is a Gaussian Tribonacci function. □

Lemma 2.7. *Let $\{u_n\}$ and $\{Gu_n\}$ be the full Tribonacci and Gaussian Tribonacci sequences, respectively. Then*

$$\begin{aligned} Gu_{\lfloor x\rfloor+n} &= GT_{n-1}u_{\lfloor x\rfloor+2} + (GT_{n-2} + GT_{n-3})u_{\lfloor x\rfloor+1} + GT_{n-2}u_{\lfloor x\rfloor}, \\ Gu_{\lfloor x\rfloor+n-1} &= GT_{n-1}u_{\lfloor x\rfloor+1} + (GT_{n-2} + GT_{n-3})u_{\lfloor x\rfloor} + GT_{n-2}u_{\lfloor x\rfloor-1}, \\ Gu_{\lfloor x\rfloor+n-2} &= GT_{n-1}u_{\lfloor x\rfloor} + (GT_{n-2} + GT_{n-3})u_{\lfloor x\rfloor-1} + GT_{n-2}u_{\lfloor x\rfloor-2}. \end{aligned}$$

Proof. The functions $f_G(x+n) = Gu_{\lfloor x\rfloor+n} + Gv_{\lfloor x\rfloor+n}t + Gw_{\lfloor x\rfloor+n}t^2$ and $f(x) = u_{\lfloor x\rfloor} + v_{\lfloor x\rfloor}t + w_{\lfloor x\rfloor}t^2$ where $t = x - \lfloor x\rfloor \in (0,1)$ and $x \in \mathbb{R}$, considered in Example 2.5, are Gaussian Tribonacci and Tribonacci functions, respectively. So, if we let $v_{\lfloor x\rfloor} = u_{\lfloor x\rfloor-1}$, $Gv_{\lfloor x\rfloor+n} = Gu_{\lfloor x\rfloor+n-1}$ and $w_{\lfloor x\rfloor} = u_{\lfloor x\rfloor-2}$, $Gw_{\lfloor x\rfloor+n} = Gu_{\lfloor x\rfloor+n-2}$ then $f(x)$ and $f_G(x)$ are Tribonacci function and Gaussian Tribonacci function, respectively. Note that

$$\begin{aligned} f(x) &= u_{\lfloor x\rfloor} + v_{\lfloor x\rfloor}t + w_{\lfloor x\rfloor}t^2 = u_{\lfloor x\rfloor} + u_{\lfloor x\rfloor-1}t + u_{\lfloor x\rfloor-2}t^2 \\ f(x+1) &= u_{\lfloor x+1\rfloor} + v_{\lfloor x+1\rfloor}t + w_{\lfloor x+1\rfloor}t^2 = u_{\lfloor x\rfloor+1} + v_{\lfloor x\rfloor+1}t + w_{\lfloor x\rfloor+1}t^2 = u_{\lfloor x\rfloor+1} + u_{\lfloor x\rfloor}t + u_{\lfloor x\rfloor-1}t^2 \\ f(x+2) &= u_{\lfloor x+2\rfloor} + v_{\lfloor x+2\rfloor}t + w_{\lfloor x+2\rfloor}t^2 = u_{\lfloor x\rfloor+2} + v_{\lfloor x\rfloor+2}t + w_{\lfloor x\rfloor+2}t^2 = u_{\lfloor x\rfloor+2} + u_{\lfloor x\rfloor+1}t + u_{\lfloor x\rfloor}t^2 \end{aligned}$$

Then we obtain

$$\begin{aligned}
& Gu_{\lfloor x\rfloor+n}+Gu_{\lfloor x\rfloor+n-1}t+Gu_{\lfloor x\rfloor+n-2}t^2 \\
&= Gu_{\lfloor x\rfloor+n}+Gv_{\lfloor x\rfloor+n}t+Gw_{\lfloor x\rfloor+n}t^2=f_G(x+n) \\
&= GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x) \\
&= GT_{n-1}(u_{\lfloor x\rfloor+2}+u_{\lfloor x\rfloor+1}t+u_{\lfloor x\rfloor}t^2)+(GT_{n-2}+GT_{n-3})(u_{\lfloor x\rfloor+1}+u_{\lfloor x\rfloor}t+u_{\lfloor x\rfloor-1}t^2) \\
&\quad GT_{n-2}(u_{\lfloor x\rfloor}+u_{\lfloor x\rfloor-1}t+u_{\lfloor x\rfloor-2}t^2) \\
&= (GT_{n-1}u_{\lfloor x\rfloor+2}+GT_{n-1}u_{\lfloor x\rfloor+1}t+GT_{n-1}u_{\lfloor x\rfloor}t^2) \\
&\quad +((GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor+1}+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor}t+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor-1}t^2) \\
&\quad (GT_{n-2}u_{\lfloor x\rfloor}+GT_{n-2}u_{\lfloor x\rfloor-1}t+GT_{n-2}u_{\lfloor x\rfloor-2}t^2) \\
&= (GT_{n-1}u_{\lfloor x\rfloor+2}+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor+1}+GT_{n-2}u_{\lfloor x\rfloor}) \\
&\quad +(GT_{n-1}u_{\lfloor x\rfloor+1}+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor}+GT_{n-2}u_{\lfloor x\rfloor-1})t \\
&\quad +(GT_{n-1}u_{\lfloor x\rfloor}+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor-1}+GT_{n-2}u_{\lfloor x\rfloor-2})t^2
\end{aligned}$$

This completes the proof. □

By taking $\{u_n\}=\{T_n\}$ in the last theorem, we have the following corollary.

Corollary 2.8. *For $x\in\mathbb{R}$, we have the following formulas:*

$$\begin{aligned}
GT_{\lfloor x\rfloor+n} &= GT_{n-1}T_{\lfloor x\rfloor+2}+(GT_{n-2}+GT_{n-3})T_{\lfloor x\rfloor+1}+GT_{n-2}T_{\lfloor x\rfloor}, \\
GT_{\lfloor x\rfloor+n-1} &= GT_{n-1}T_{\lfloor x\rfloor+1}+(GT_{n-2}+GT_{n-3})T_{\lfloor x\rfloor}+GT_{n-2}T_{\lfloor x\rfloor-1}, \\
GT_{\lfloor x\rfloor+n-2} &= GT_{n-1}T_{\lfloor x\rfloor}+(GT_{n-2}+GT_{n-3})T_{\lfloor x\rfloor-1}+GT_{n-2}T_{\lfloor x\rfloor-2}.
\end{aligned}$$

By taking $\lfloor x\rfloor=m\in\mathbb{Z}$ in the last corollary, we see that for all integers m,n we have

$$\begin{aligned}
GT_{m+n} &= GT_{n-1}T_{m+2}+(GT_{n-2}+GT_{n-3})T_{m+1}+GT_{n-2}T_m, \\
GT_{m+n-1} &= GT_{n-1}T_{m+1}+(GT_{n-2}+GT_{n-3})T_m+GT_{n-2}T_{m-1}, \\
GT_{m+n-2} &= GT_{n-1}T_m+(GT_{n-2}+GT_{n-3})T_{m-1}+GT_{n-2}T_{m-2}.
\end{aligned}$$

Theorem 2.9. *Let $f_G(x)=a(x)g_G(x)$ be a function, $g(x)$ and $f(x)=a(x)g(x)$ be Tribonacci functions, where $a(x)$ is an f-even function, and suppose that $g_G(x)$ and $g(x)$ are continuous functions. Then $f_G(x)$ is a Gaussian Tribonacci function with Tribonacci function $f(x)$ if and only if $g_G(x)$ is a Gaussian Tribonacci function with Tribonacci function $g(x)$.*

Proof. By definition of the function f_G and since $a(x)$ is an f-even function, we have

$$f_G(x+n)=a(x+n)g_G(x+n)=a(x)g_G(x+n). \tag{2.3}$$

Suppose that f_G is a Gaussian Tribonacci function. Then, since $a(x)$ is an f-even function, we obtain

$$\begin{aligned}
f_G(x+n) &= f(x+n)+if(x+n-1) \\
&= a(x+n)g(x+n)+ia(x+n-1)g(x+n-1) \\
&= a(x)g(x+n)+ia(x)g(x+n-1) \\
&= a(x)(g(x+n)+ig(x+n-1)).
\end{aligned} \tag{2.4}$$

From the equations (2.3) and (2.4), we get

$$a(x)(g_G(x+n)-g(x+n)-ig(x+n-1))\equiv 0$$

and so

$$g_G(x+n) - g(x+n) - ig(x+n-1) \equiv 0$$

i.e.,

$$g_G(x+n) = g(x+n) + ig(x+n-1).$$

Therefore, g_G is a Gaussian Tribonacci function.
On the other hand, if g_G is a Gaussian Tribonacci function, then

$$g_G(x+n) = g(x+n) + ig(x+n-1) \tag{2.5}$$

Since $f(x) = a(x)g(x)$ and $a(x)$ is an f-even function, we obtain

$$\begin{aligned} f(x+n) &= a(x+n)g(x+n) = a(x)g(x+n), \\ f(x+n-1) &= a(x+n-1)g(x+n-1) = a(x)g(x+n-1). \end{aligned}$$

Then, since $f_G(x) = a(x)g_G(x)$ and $a(x)$ is an f-even function, the equation (2.5) implies that

$$\begin{aligned} f_G(x+n) &= a(x+n)g_G(x+n) = a(x)g_G(x+n) \\ &= a(x)(g(x+n) + ig(x+n-1)) \\ &= a(x)g(x+n) + ia(x)g(x+n-1) \\ &= f(x+n) + if(x+n-1). \end{aligned}$$

Hence, f_G is a Gaussian Tribonacci function. □

3. Sums of Tribonacci and Gaussian Tribonacci Functions

In this section, we discuss the sums of the terms of a Tribonacci function and a Gaussian Tribonacci function. The following corollary gives linear sum formulas of Tribonacci numbers.

Corollary 3.1. *For $n \geq 0$, Tribonacci numbers have the following property.*

$$\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+3} - T_{n+1} - 1).$$

Proof. For a proof see [17]. □

The following theorem gives linear sum formulas of Tribonacci functions.

Theorem 3.2. *Suppose that f is a Tribonacci function. Then for all $x \in \mathbb{R}$ and $n \geq 0$, the following sum formula holds:*

$$\sum_{k=0}^{n} f(x+k) = \frac{1}{2}(f(x+n+4) - f(x+n+2) - 2f(x+n+1) - f(x+2) + f(x)).$$

Proof. We use corollary 3.1. Since

$$\begin{aligned} \sum_{k=0}^{n} T_{k-1} &= \frac{1}{2}(T_{n+3} - T_{n+1} - 2T_n - 1), \\ \sum_{k=0}^{n} T_{k-2} &= \frac{1}{2}(T_{n+3} - T_{n+1} - 2T_{n-1} - 2T_n + 1), \\ \sum_{k=0}^{n} T_{k-3} &= \frac{1}{2}(T_{n+3} - T_{n+1} - 2T_n - 2T_{n-1} - 2T_{n-2} - 1), \end{aligned}$$

and

$$f(x+n)=T_{n-1}f(x+2)+(T_{n-2}+T_{n-3})f(x+1)+T_{n-2}f(x),$$

we obtain

$$\begin{aligned}\sum_{k=0}^{n}f(x+k) &= f(x+2)\sum_{k=0}^{n}T_{k-1}+f(x+1)\sum_{k=0}^{n}(T_{k-2}+T_{k-3})+f(x)\sum_{k=0}^{n}T_{k-2}\\ &= f(x+2)\times\frac{1}{2}(T_{n+3}-T_{n+1}-2T_{n}-1)\\ &\quad +f(x+1)\times\frac{1}{2}(T_{n+2}+T_{n+1}-(T_{n}+T_{n-1})-2(T_{n-1}+T_{n-2}))\\ &\quad +f(x)\times\frac{1}{2}(T_{n+2}-T_{n}-2T_{n-1}+1)\\ &= \frac{1}{2}(f(x+n+4)-f(x+n+2)-2f(x+n+1)-f(x+2)+f(x))\end{aligned}$$

□

Note that if we consider the Tribonacci function

$$f(x)=\alpha^{x}$$

then, using Theorem 3.2, we have the sum formula

$$\sum_{k=0}^{n}\alpha^{x+k}=\frac{1}{2}(\alpha^{x+n+4}-\alpha^{x+n+2}-2\alpha^{x+n+1}-\alpha^{x+2}+\alpha^{x})$$

for all $x\in\mathbb{R}$ and $n\geq 0$.

The following corollary gives linear sum formulas of Gaussian Tribonacci numbers.

Corollary 3.3. *For $n\geq 1$ we have the following formulas:*

$$\sum_{k=1}^{n}GT_{k}=\frac{1}{2}(GT_{n+3}-GT_{n+1}-(1+i)).$$

Proof. It is given in [16].

The following theorem gives linear sum formulas of Gaussian Tribonacci functions.

Theorem 3.4. *Suppose that f_G is a Gaussian Tribonacci function with Tribonacci function f. Then for all $x\in\mathbb{R}$ and $n\geq 1$ the following sum formula holds:*

$$\sum_{k=1}^{n}f_{G}(x+k)=\frac{1}{2}(f_{G}(x+n+3)-f_{G}(x+n+1)-(1+i)f\left(x+2\right)+(-1+i)f(x))$$

Proof. We use corollary 3.3. Since

$$\begin{aligned}\sum_{k=1}^{n}GT_{k} &= \frac{1}{2}(GT_{n+3}-GT_{n+1}-(1+i)),\\ \sum_{k=1}^{n}GT_{k-1} &= \frac{1}{2}(GT_{n+2}-GT_{n}-(1+i)),\\ \sum_{k=1}^{n}GT_{k-2} &= \frac{1}{2}(GT_{n+1}-GT_{n-1}-1+i),\\ \sum_{k=1}^{n}GT_{k-3} &= \frac{1}{2}(GT_{n}-GT_{n-2}+1-i),\end{aligned}$$

and

$$f_G(x+n) = GT_{n-1}f(x+2) + (GT_{n-2} + GT_{n-3})f(x+1) + GT_{n-2}f(x),$$

we get

$$\begin{aligned}\sum_{k=1}^{n} f_G(x+k) &= f(x+2)\sum_{k=1}^{n} GT_{k-1} + f(x+1)(\sum_{k=1}^{n} GT_{k-2} + \sum_{k=1}^{n} GT_{k-3}) + f(x)\sum_{k=1}^{n} GT_{k-2} \\ &= \frac{1}{2}(f_G(x+n+3) - f_G(x+n+1) - (1+i)f(x+2) + (-1+i)f(x)).\end{aligned}$$

□

Note that if we consider the Tribonacci function

$$f(x) = \alpha^x$$

and the Gaussian Tribonacci function

$$f_G(x+n) = (1+i\alpha^{-1})\alpha^{x+n}$$

then, using Theorem 3.4, we have the sum formula

$$\sum_{k=1}^{n}(1+i\alpha^{-1})\alpha^{x+k} = \frac{1}{2}((1+i\alpha^{-1})\alpha^{x+n+3} - (1+i\alpha^{-1})\alpha^{x+n+1} - (1+i)\alpha^{x+2} + (-1+i)\alpha^x)$$

for all $x \in \mathbb{R}$ and $n \geq 1$.

4. Ratio of Gaussian Tribonacci Functions

In this section, we discuss the limit of the quotient of a Gaussian Tribonacci function. Note that since

$$\lim_{n\to\infty} \frac{T_{n+p}}{T_{n+q}} = \alpha^{p-q}, \quad p, q \in \mathbb{Z}$$

we have

$$\begin{aligned}\lim_{n\to\infty} \frac{GT_{n+p}}{GT_{n+q}} &= \lim_{n\to\infty} \frac{T_{n+p} + iT_{n+p-1}}{T_{n+q} + iT_{n+q-1}} \\ &= \lim_{n\to\infty} \frac{\frac{T_{n+p}}{T_{n+q}} + i\frac{T_{n+p-1}}{T_{n+q}}}{\frac{T_{n+q}}{T_{n+q}} + i\frac{T_{n+q-1}}{T_{n+q}}} \\ &= \frac{\alpha^{p-q} + i\alpha^{p-1-q}}{1+i\alpha^{-1}}.\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{GT_{n+p}}{T_{n+q}} &= \lim_{n\to\infty} \frac{T_{n+p} + iT_{n+p-1}}{T_{n+q}} \\ &= \lim_{n\to\infty} \frac{T_{n+p}}{T_{n+q}} + i\lim_{n\to\infty} \frac{T_{n+p-1}}{T_{n+q}} \\ &= \alpha^{p-q} + i\alpha^{p-1-q}.\end{aligned}$$

Theorem 4.1. *If f_G is a Gaussian Tribonacci function, then the limit of quotient*

$$\frac{f_G(x+k)}{f_G(x+m)}$$

exists and given by

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)} = \alpha^{k-m}$$

for all $k, m \in \mathbb{Z}$.

Proof. Suppose that f_G is a Gaussian Tribonacci function with Tribonacci function f. Note that from Theorem 1.6 and Theorem 1.7, the limit of quotients $\frac{f(x+1)}{f(x)}$ and $\frac{f(x+2)}{f(x)}$ exists and $\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \alpha$ and $\lim_{x\to\infty}\frac{f(x+2)}{f(x)} = \alpha^2$. We use the formula, by definition,

$$f_G(x+n) = GT_{n-1}f(x+2) + (GT_{n-2} + GT_{n-3})f(x+1) + GT_{n-2}f(x).$$

Given $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $x = y + n$.Then

$$\begin{aligned}\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)} &= \lim_{y\to\infty}\lim_{n\to\infty}\frac{f_G(y+n+k)}{f_G(y+n+m)} = \lim_{x\to\infty}\lim_{n\to\infty}\frac{f_G(x+n+k)}{f_G(x+n+m)} \\ &= \lim_{x\to\infty}\lim_{n\to\infty}\frac{\frac{GT_{n+k-1}}{GT_{n+m-1}}\frac{f(x+2)}{f(x)} + (\frac{GT_{n+k-2}}{GT_{n+m-1}} + \frac{GT_{n+k-3}}{GT_{n+m-1}})\frac{f(x+1)}{f(x)} + \frac{GT_{n+k-2}}{GT_{n+m-1}}}{\frac{f(x+2)}{f(x)} + (\frac{GT_{n+m-2}}{GT_{n+m-1}} + \frac{GT_{n+m-3}}{GT_{n+m-1}})\frac{f(x+1)}{f(x)} + \frac{GT_{n+m-2}}{GT_{n+m-1}}}.\end{aligned}$$

Since

$$\begin{aligned}\lim_{n\to\infty}\frac{T_{n+p}}{T_{n+q}} &= \alpha^{p-q}, \quad p, q \in \mathbb{Z}, \\ \lim_{n\to\infty}\frac{GT_{n+p}}{GT_{n+q}} &= \frac{\alpha^{p-q} + i\alpha^{p-1-q}}{1 + i\alpha^{-1}}, \quad p, q \in \mathbb{Z}, \\ \lim_{x\to\infty}\frac{f(x+1)}{f(x)} &= \alpha, \\ \lim_{x\to\infty}\frac{f(x+2)}{f(x)} &= \alpha^2,\end{aligned}$$

it follows that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)} = \alpha^{k-m}$$

□

Note that if we consider the Tribonacci function

$$f(x) = \alpha^x$$

and the Gaussian Tribonacci function

$$f_G(x+n) = (1 + i\alpha^{-1})\alpha^{x+n}$$

then, we see that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)} = \lim_{x\to\infty}\frac{(1+i\alpha^{-1})\alpha^{x+k}}{(1+i\alpha^{-1})\alpha^{x+m}} = \lim_{x\to\infty}\frac{1}{\alpha^{m+x}}\alpha^{k+x} = \alpha^{k-m}.$$

Also, it follows from Theorem 4.1, that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)} = \alpha^{k-m}.$$

Corollary 4.2. *If f_G is a Gaussian Tribonacci function, then*

$$\lim_{x\to\infty}\frac{f_G(x+1)}{f_G(x)}=\alpha.$$

Proof. Take $k=1, m=0$ in Theorem 4.1. □

Theorem 4.3. *If f_G is a Gaussian Tribonacci function with Tribonacci function f, then the limit of quotient*

$$\frac{f_G(x+k)}{f(x)}$$

exists and given by

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=GT_{k-1}\alpha^2+(GT_{k-2}+GT_{k-3})\alpha+GT_{k-2}$$

for all $k\in\mathbb{Z}$.

Proof. Suppose that f_G is a Gaussian Tribonacci function with Tribonacci function f. Note that from Theorem 1.6, the limit of quotients $\frac{f(x+1)}{f(x)}$ and $\frac{f(x+2)}{f(x)}$ exists. Using the formula, by definition,

$$f_G(x+n)=GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x).$$

we get

$$\begin{aligned}\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}&=\lim_{x\to\infty}\frac{GT_{k-1}f(x+2)+(GT_{k-2}+GT_{k-3})f(x+1)+GT_{k-2}f(x)}{f(x)}\\&=GT_{k-1}\lim_{x\to\infty}\frac{f(x+2)}{f(x)}+(GT_{k-2}+GT_{k-3})\lim_{x\to\infty}\frac{f(x+1)}{f(x)}+GT_{k-2}.\end{aligned}$$

Hence, since the limit of quotients $\frac{f(x+1)}{f(x)}$ and $\frac{f(x+2)}{f(x)}$ exists, $\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}$ exists and

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=GT_{k-1}\alpha^2+(GT_{k-2}+GT_{k-3})\alpha+GT_{k-2}.$$

□

Note that if we consider the Tribonacci function

$$f(x)=\alpha^x$$

and the Gaussian Tribonacci function

$$f_G(x+n)=(1+i\alpha^{-1})\alpha^{x+n}$$

then, we see that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=\lim_{x\to\infty}\frac{(1+i\alpha^{-1})\alpha^{x+k}}{\alpha^x}=\lim_{x\to\infty}(1+i\alpha^{-1})\alpha^k=(1+i\alpha^{-1})\alpha^k. \tag{4.1}$$

Also,from Theorem 4.3, we know that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=GT_{k-1}\alpha^2+(GT_{k-2}+GT_{k-3})\alpha+GT_{k-2}. \tag{4.2}$$

Therefore, comparing (4.1) and (4.2), we obtain

$$GT_{k-1}\alpha^2+(GT_{k-2}+GT_{k-3})\alpha+GT_{k-2}=(1+i\alpha^{-1})\alpha^k$$

for $k\in\mathbb{Z}$.

Corollary 4.4. *If f_G is a Gaussian Tribonacci function with Tribonacci function f, then*

$$\lim_{x\to\infty}\frac{f_G(x)}{f(x)} = 1+i(\alpha^2-\alpha-1),$$
$$\lim_{x\to\infty}\frac{f_G(x+1)}{f(x)} = \alpha+i,$$
$$\lim_{x\to\infty}\frac{f_G(x+2)}{f(x)} = \alpha^2+i\alpha.$$

Proof. Take $k=0,1,2$ in Theorem 4.3, respectively. □

We can generalize Theorem 4.3 as follows.

Theorem 4.5. *If f_G is a Gaussian Tribonacci function with Tribonacci function f, then the limit of quotient*

$$\frac{f_G(x+k)}{f(x+m)}$$

exists and given by

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x+m)} = (\alpha+i)\alpha^{k-m-1}$$

for all $k,m\in\mathbb{Z}$.

Proof. Suppose that f_G is a Gaussian Tribonacci function with Tribonacci function f. Note that from Theorem 1.7, the limit of quotients $\frac{f(x+1)}{f(x)}$ and $\frac{f(x+2)}{f(x)}$ exists and $\lim\limits_{x\to\infty}\frac{f(x+1)}{f(x)}=\alpha$ and $\lim\limits_{x\to\infty}\frac{f(x+2)}{f(x)}=\alpha^2$. Given $x\in\mathbb{R}$, there exists $y\in\mathbb{R}$ and $n\in\mathbb{N}$ such that $x=y+n$. By using the formulas

$$f_G(x+n) = GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x)$$

and

$$f(x+n) = T_{n-1}f(x+2)+(T_{n-2}+T_{n-3})f(x+1)+T_{n-2}f(x)$$

we get

$$\begin{aligned}\lim_{x\to\infty}\frac{f_G(x+k)}{f(x+m)} &= \lim_{y\to\infty}\lim_{n\to\infty}\frac{f_G(y+n+k)}{f(y+n+m)}\\ &= \lim_{x\to\infty}\lim_{n\to\infty}\frac{f_G(x+n+k)}{f(x+n+m)}\\ &= \lim_{x\to\infty}\lim_{n\to\infty}\frac{\frac{GT_{n+k-1}}{T_{n+m-1}}\frac{f(x+2)}{f(x)}+(\frac{GT_{n+k-2}}{T_{n+m-1}}+\frac{GT_{n+k-3}}{T_{n+m-1}})\frac{f(x+1)}{f(x)}+\frac{GT_{n+k-2}}{T_{n+m-1}}}{\frac{f(x+2)}{f(x)}+(\frac{T_{n+m-2}}{T_{n+m-1}}+\frac{T_{n+m-3}}{T_{n+m-1}})\frac{f(x+1)}{f(x)}+\frac{T_{n+m-2}}{T_{n+m-1}}}\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{GT_{n+p}}{T_{n+q}} = \alpha^{p-q}+i\alpha^{p-1-q},\quad p,q\in\mathbb{Z},$$
$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \alpha,$$
$$\lim_{x\to\infty}\frac{f(x+2)}{f(x)} = \alpha^2,$$

it follows that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x+m)} = (\alpha+i)\alpha^{k-m-1}. \quad \square$$

5. Matrix Formulation of $f(x)$ and $f_G(x+n)$

The matrix method is very useful method in order to obtain some identities for special sequences. We define the square matrix M of order 3 as:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det M = 1$. Note that for all $n \in \mathbb{Z}$, we have

$$M^n = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}. \tag{5.1}$$

Matrix formulation of T_n can be given as

$$\begin{pmatrix} T_{n+2} \\ T_{n+1} \\ T_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} T_2 \\ T_1 \\ T_0 \end{pmatrix}. \tag{5.2}$$

Consider the matrices N_T, E_T defined by as follows:

$$N_T = \begin{pmatrix} 1+i & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 1-i \end{pmatrix},$$

$$E_T = \begin{pmatrix} GT_{n+2} & GT_{n+1} & GT_n \\ GT_{n+1} & GT_n & GT_{n-1} \\ GT_n & GT_{n-1} & GT_{n-2} \end{pmatrix}.$$

The following theorem presents the relations between M^n, N_T and E_T.

Theorem 5.1. *For all $n \in \mathbb{Z}$, we have*

$$M^n N_T = E_T.$$

Proof. For a proof, see [16]. □

Define

$$A_f = \begin{pmatrix} f(x+2) & f(x+1) & f(x) \\ f(x+1) & f(x) & f(x-1) \\ f(x) & f(x-1) & f(x-2) \end{pmatrix},$$

$$B_f = \begin{pmatrix} f(x+n+2) & f(x+n+1) & f(x+n) \\ f(x+n+1) & f(x+n) & f(x+n-1) \\ f(x+n) & f(x+n-1) & f(x+n-2) \end{pmatrix}.$$

Theorem 5.2. *For all integers $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have*

$$M^n A_f = B_f \tag{5.3}$$

Proof. By using

$$f(x+n) = T_{n-1} f(x+2) + (T_{n-2} + T_{n-3}) f(x+1) + T_{n-2} f(x)$$

and

$$f(x+3) = f(x+2) + f(x+1) + f(x),$$

the case $n \geq 0$ can be proved by mathematical induction. Then for the case $n \leq 0$, we take $m = -n$ in 5.3 and then the case $m \geq 0$ can be proved by mathematical induction, as well. □

Note that if we consider the Tribonacci function

$$f(x) = \alpha^x$$

then, we see that

$$A_f = \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} & \alpha^x \\ \alpha^{x+1} & \alpha^x & \alpha^{x-1} \\ \alpha^x & \alpha^{x-1} & \alpha^{x-2} \end{pmatrix}, B_f = \begin{pmatrix} \alpha^{x+n+2} & \alpha^{x+n+1} & \alpha^{x+n} \\ \alpha^{x+n+1} & \alpha^{x+n} & \alpha^{x+n-1} \\ \alpha^{x+n} & \alpha^{x+n-1} & \alpha^{x+n-2} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} & \alpha^x \\ \alpha^{x+1} & \alpha^x & \alpha^{x-1} \\ \alpha^x & \alpha^{x-1} & \alpha^{x-2} \end{pmatrix} = \begin{pmatrix} \alpha^{x+n+2} & \alpha^{x+n+1} & \alpha^{x+n} \\ \alpha^{x+n+1} & \alpha^{x+n} & \alpha^{x+n-1} \\ \alpha^{x+n} & \alpha^{x+n-1} & \alpha^{x+n-2} \end{pmatrix}$$

for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

Define

$$D_{GT} = \begin{pmatrix} GT_{n+1} & GT_n + GT_{n-1} & GT_n \\ GT_n & GT_{n-1} + GT_{n-2} & GT_{n-1} \\ GT_{n-1} & GT_{n-2} + GT_{n-3} & GT_{n-2} \end{pmatrix}$$

and

$$C_{f_G} = \begin{pmatrix} f_G(x+n+2) & f_G(x+n+1) & f_G(x+n) \\ f_G(x+n+1) & f_G(x+n) & f_G(x+n-1) \\ f_G(x+n) & f_G(x+n-1) & f_G(x+n-2) \end{pmatrix}.$$

Theorem 5.3. *For all integers $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have*

$$D_{GT} A_f = C_{f_G} \tag{5.4}$$

Proof. By using

$$f_G(x+n) = GT_{n-1} f(x+2) + (GT_{n-2} + GT_{n-3}) f(x+1) + GT_{n-2} f(x)$$

and

$$f(x+3) = f(x+2) + f(x+1) + f(x),$$

the case $n \geq 0$ can be proved by mathematical induction. Then for the case $n \leq 0$, we take $m = -n$ in 5.4 and then the case $m \geq 0$ can be proved by mathematical induction, as well. □

Note that if we consider the Tribonacci function

$$f(x) = \alpha^x$$

and the Gaussian Tribonacci function

$$f_G(x+n) = (1 + i\alpha^{-1})\alpha^{x+n}$$

then, we see that

$$A_f = \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} & \alpha^x \\ \alpha^{x+1} & \alpha^x & \alpha^{x-1} \\ \alpha^x & \alpha^{x-1} & \alpha^{x-2} \end{pmatrix}, D_{GT} = \begin{pmatrix} GT_{n+1} & GT_n + GT_{n-1} & GT_n \\ GT_n & GT_{n-1} + GT_{n-2} & GT_{n-1} \\ GT_{n-1} & GT_{n-2} + GT_{n-3} & GT_{n-2} \end{pmatrix}$$

and

$$C_{f_G} = \begin{pmatrix} (1+i\alpha^{-1})\alpha^{x+n+2} & (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n} \\ (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n} & (1+i\alpha^{-1})\alpha^{x+n-1} \\ (1+i\alpha^{-1})\alpha^{x+n} & (1+i\alpha^{-1})\alpha^{x+n-1} & (1+i\alpha^{-1})\alpha^{x+n-2} \end{pmatrix},$$

and so

$$\begin{pmatrix} GT_{n+1} & GT_n + GT_{n-1} & GT_n \\ GT_n & GT_{n-1} + GT_{n-2} & GT_{n-1} \\ GT_{n-1} & GT_{n-2} + GT_{n-3} & GT_{n-2} \end{pmatrix} \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} & \alpha^{x} \\ \alpha^{x+1} & \alpha^{x} & \alpha^{x-1} \\ \alpha^{x} & \alpha^{x-1} & \alpha^{x-2} \end{pmatrix}$$
$$= \begin{pmatrix} (1+i\alpha^{-1})\alpha^{x+n+2} & (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n} \\ (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n} & (1+i\alpha^{-1})\alpha^{x+n-1} \\ (1+i\alpha^{-1})\alpha^{x+n} & (1+i\alpha^{-1})\alpha^{x+n-1} & (1+i\alpha^{-1})\alpha^{x+n-2} \end{pmatrix}.$$

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 227–240.
http://doi.org/10.26637/mjm11S/014

Mild solutions for some nonautonomous evolution equations with state-dependent delay governed by equicontinuous evolution families

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Received 31 August 2023; Accepted 28 September 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. In this work, we study the existence solutions and the dependence continuous with the initial data for some nondensely nonautonomous partial functional differential equations with state-dependent delay in Banach spaces. We assume that the linear part is not necessarily densely defined, satisfies the well-known hyperbolic conditions and generate a noncompact evolution family. Our existence results are based on Sadovskii fixed point Theorem. An application is provided to a reaction-diffusion equation with state-dependent delay.

AMS Subject Classifications: 35R10, 47J35.

Keywords: Nondensely nonautonomous evolution equations, Mild solution, Kuratowski measure of non-compactness, State-dependent delay.

Contents

1. Introduction

Partial differential equations with delay are important for investigating some problems raised from natural phenomena. They have been successfully used to study a number of areas of biological, physical, engineering applications, and such equations have received much attention in recent years. It is generally known that taking into account the past states of the model, in addition to the present one, makes the model more realistic. This leads to the so called functional differential equations. In recent years, nonlinear evolution equations with

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state-dependent delay have been studied by several authors and some interesting results have been obtained, see [9–11, 14, 15, 17, 19].

In 1970, Kato in [12] initiated a study of the evolution family solution of hyperbolic linear evolution equations of the form

$$\begin{cases} x'(t) = A(t)x(t), & t \geq s, \\ x(s) = x_s \in X. \end{cases} \tag{1.1}$$

in a Banach space X. Some fundamental and basic results about the well posedness and dynamical behavior of equation (1.1) were established under the so called stability condition, ((B_2) in Section 2). The autors focus on the nonautonomous linear case.

In 2011, Belmekki et al investigated in [5] several results on the existence of solutions of the initial value problem for a new class of abstract evolution equations with state-dependent delay in Banach space X,

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t - \rho(x(t)))), & t \in [0; a], \\ x(t) = \varphi(t), & t \in [-r; 0]. \end{cases} \tag{1.2}$$

where $f : [0; +\infty) \times X \longrightarrow X$ is a suitable nonlinear function, the initial data $\varphi : [-r; 0] \longrightarrow X$ is a continuous function, ρ is a positive bounded continuous function defined on X and r is the maximal delay given by $r = \sup\limits_{x \in X} \rho(x)$. The autors focus on the case where the differential operator in the main part is nondensely define and independent of time t in $[0, a]$. Here the equation is autonomous partial functional differential equations with state-dependent delay. Their approach is based on a nonlinear alternative of Leray-Schauder and integrated semigroup $(S(t))_{t\geq}$ which is considered to be compact for $t > 0$.

In 2019, Kpoumie et al. investigated in [15] several results on the existence of solutions of the following nonautomous equations:

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t - \rho(x(t)))), & t \in [0; a], \\ x(t) = \varphi(t), & t \in [-r; 0]. \end{cases} \tag{1.3}$$

in a Banach space $(X, \|.\|)$, where the family of closed linear operator $(A(t))_{t\geq 0}$ on X is not necessarily densely, satisfying the hyperbolic conditions (B_1) through (B_3) and $\varphi : [-r; 0] \longrightarrow X$ the continuous function. Their approach is based on a nonlinear alternative of Leray-Schauder under the assumption of the compactness of evolution family generated by $(A(t))_{t\geq 0}$. They get the existence of mild solution under the Carathéodory condition on f.

In 2019, Chen and al. investigated in [7] several results on the existence of solutions of the nonautonomous parabolic evolution equations with non-instantaneous impulses in Banach space E:

$$\begin{cases} x'(t) - A(t)x(t) = f(t, x(t)), & t \in \bigcup_{k=0}^{m} (s_k, t_{k+1}], \\ x(t) = \gamma_k(t, x(t)), & t \in \bigcup_{k=1}^{m} (t_k, s_k], \\ x(0) = x_0). \end{cases} \tag{1.4}$$

by introducing the concepts of mild and classical solutions, where $A : D(A) \subset E \longrightarrow E$ is the generator of a $C_0 - semigroup$ of bounded linear operator $T(t)_{t\geq 0}$ defined on E, $u_0 \in E,\ 0 < t_1 < t_2 < \cdots < t_m < t_{m+1} := a,\ a > 0$ is a constant, $s_0 := 0$ and $s_k \in (t_k, t_{k+1})$ for each $k = 1, 2, \cdots, m,\ f : [0, a] \times E \longrightarrow E$ is a suitable nonlinear function, $\gamma_k : (t_k, s_k] \times E \longrightarrow E$ is continuous non-instantaneous impulsive function for all $k = 1, 2, \cdots, m$. Their results are based on Sadovskii fixed point

Theorem and they consider that evolution family is noncompact.

Therefore, it is for great significance and interesting to study the nonautonomous evolution equation where the family of closed linear operator $(A(t))_{t\geq 0}$ on X is not necessarily densely define and generates the noncompact evolution famillies. Driven by the above aspects, we will investigate the existence of mild solutions and the dependent continuous on the initial data of the following nonautonomous partial functional differential equations with state-dependent delay governed by noncompact evolution families of the form

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t-\rho(x(t)))), & t \geq 0, \\ x(t) = \varphi(t), & t \in [-r; 0]. \end{cases} \tag{1.5}$$

in a Banach space $(X, \|.\|)$, where the family of closed linear operator $(A(t))_{t\geq 0}$ on X is not necessarily densely and satisfying the hyperbolic conditions (B_1) through (B_3) introduced by Kato in [12] that will be specified later. $f : [0; +\infty) \times X \longrightarrow X$ is a suitable nonlinear function satisfying some conditions which will be specified later. The initial data $\varphi : [-r; 0] \longrightarrow X$ is a continuous function and ρ is a positive bounded continuous function on X. The constant r is the maximal delay defined by $r = \sup_{x\in X} \rho(x)$.

We point out that the work of this paper is the following of [5, 7, 12, 15]. But under appropriate circonstances, evolutionary families are not compact. Our work is organized as follows: First, we recall some preliminary results about the evolution family generated by $(A(t))_{t\geq 0}$ and recall also some preliminary results concern Kuratowski measure. Second, we use the alternative of Sadovskii fixed point Theorem to prove the existence of at least one mild solution and the dependent continuous on initial data. Third, we propose an application to illustrate the main result.

2. Preliminary results

Our notations in this section are the usual in the theory of evolution equations. In particular, we denote by $\mathcal{C}(E, F)$ the space of continuous functions from E into F and $\mathcal{C}^2(E, F)$ denotes the space of twice continuously differentiable functions from E into F.
We mention here some results on nonautonomous differential equations with nondense domaine. We cite [12, 13, 16, 18, 19]. We recall some properties and Theorems.

In the whole of this work, we assume the following hyperbolic assumptions:
(B_1) $D(A(t)) := D$ independent of t and not necessarily densely defined ($\overline{D} \subsetneq X$) .
(B_2) The family $(A(t)_{t\geq 0}$ is stable in the sense that there are constants $M \geq 1$ and $w \in \mathbb{R}$ such that $(\omega, +\infty) \subset \varrho(A(t))$ (resolvent set of A(t)) for $t \in [0, +\infty)$ and

$$\|\prod_{j=1}^{k} R(\lambda, A(t_j))\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and every finite sequence $\{t_j\}_{j=1}^{k}$ with $0 \leq t_1 \leq t_2 \leq \ \leq t_k$ and $k = 1, 2, ...$
(B_3) The mapping $t \longmapsto A(t)x$ is continuously differentiable in X for all $x \in D$.
We follow by recall the classical result which gives us the existence and explicit formula of the evolution family generated by $(A(t))_{t\geq 0}$ due to Kato [12]. Let $\lambda > 0$, $0 \leq s \leq t$ and $x \in \overline{D}$,

$$U_\lambda(t, s)x = \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(i\lambda))^{-1}x.$$

Theorem 2.1. *[1, 12] Assuming the three conditions* $(B_1)-(B_3)$*. Then the limit*

$$U(t,s)x = \lim_{\lambda\longrightarrow 0^+} U_\lambda(t,s)x \tag{2.1}$$

exists for $x \in \overline{D}$ *and* $0 \le s \le t$*, where the convergence is uniform on* $\Gamma := \{(t,s) : 0 \le s \le t\}$*. Moreover, the family* $\{U(t,s) : (t,s) \in \Gamma\}$ *satisfies the following properties:*

(i) $U(t,s)D(s) \subset D(t)$ *for all* $0 \le s \le t$*, where* $D(t)$ *is defined by*

$$D(t) := \{x \in D : A(t)x \in D\}$$

,

(ii) $U(t,s) : \overline{D} \longrightarrow \overline{D}$ *for* $(t,s) \in \Gamma$

(iii) $U_\lambda(t,t)x = x$ *and* $U_\lambda(t,s)x = U_\lambda(t,r)U_\lambda(r,s)x$ *for* $x \in \overline{D}$*,* $\lambda > 0$ *and* $0 \le s \le r \le t$

(iv) $U(t,t)x = x$ *and* $U(t,s)x = U(t,r)U(r,s)x$ *for* $x \in \overline{D}$ *and* $0 \le s \le r \le t$*,*

(v) *the mapping* $(t,s) \longmapsto U(t,s)x$ *is continuous on* Γ *for any* $x \in \overline{D}$*,*

(vi) *for all* $x \in D(s)$ *and* $t \ge s$*, the function* $t \longmapsto U(t,s)x$ *is continuously differentiable with* $\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x$*, and* $\frac{\partial^+}{\partial s}U(t,s)x = -U(t,s)A(s)x$*.*

Corollary 2.2. *[1] Assume the condition* (B_2)*. Then there exists* $M \ge 1$ *and* $\omega \in \mathbb{R}$ *such that*

$$\Big\|U(t,s)x\Big\| \le Me^{\omega(t-s)}\|x\|, \qquad \text{for } x \in \overline{D} \text{ and } 0 \le s \le t.$$

Remark 2.3. *Since* (B_2)*,* $\lambda > \omega$ *and hence for (2.1), we get that* ω *is non positive. And by using Corollary 2.2, we have* $\|U(t,s)x\| \le M\|x\|$ *for each* $x \in \overline{D}$ *and* $0 \le s \le t$*.*

Definition 2.4. *[4, 7] An evolution family* $\{U(t,s) : 0 \le s \le t \le a\}$ *is said to be equicontinuous if for any* $s \ge 0$*, the function* $t \longmapsto U(t,s)$ *is continuous by operator norm for* $t \in (s;+\infty)$*.*

In the following, we give some results on the existence of solutions for the following nondensely nonautonomous partial functional differential equation

$$\begin{cases} x'(t) = A(t)x(t) + f(t), & t \in [0;a], \\ x(0) = x_0. \end{cases} \tag{2.2}$$

where $f : [0,a] \longrightarrow X$ is a function. The following Theorem gives us the generalized variation of constants formula of equation (2.2).

Theorem 2.5. *[9] Let* $x_0 \in \overline{D}$ *and* $f \in L^1([0,a];X)$*. Then the limit*

$$x(t) := U(t,0)x_0 + \lim_{\lambda\longrightarrow 0^+}\int_0^t U_\lambda(t,r)f(r)dr \tag{2.3}$$

exists uniformly for $t \in [0;a]$*,* x *is a continuous function on* $[0,a]$ *and*

$$\|x(t)\| \le Me^{\omega t}\|x_0\| + \int_0^t Me^{\omega(t-s)}\|f(s)\|ds \le M\|x_0\| + \int_0^t M\|f(s)\|ds \tag{2.4}$$

.

Definition 2.6. *[12] For $x_0 \in \overline{D}$, a continuous function $x : [0, a] \longrightarrow X$ is called a mild solution of equation (2.2) if it satisfies the equation (2.3).*

We introduce some basic definitions and properties of the Kuratowski noncompactness measure, this will be used to demonstrate our main result.

Definition 2.7. *[4, 7] The Kuratowski measure of noncoampactness $\mu(.)$ defined on bounded set V of Banach space E is*

$$\mu(V) := \inf\{\delta > 0 : V = \bigcup_{i=1}^{m} V_i \; and \; diam(V_i) \leq \delta \; for \; i = 1, 2, ..., m\}.$$

Definition 2.8. *[4, 7] Consider a Banach space X, and a nonempty subset E of X. A continuous operator $G : E \longrightarrow X$ is called to be $\lambda-$set-contractive if there exists a constant $\lambda \in [0; 1)$ such that, for every bounded set $B \subset E$,*

$$\mu\Big(G(B)\Big) \leq \lambda\mu(B).$$

Theorem 2.9. *[4, 7] Let E be a Banach space and $U, V \subseteq E$ be bounded. The following properties are satisfied:*

(a) $\mu(V) = 0$, if and only if $\overline{V}$ is compact, where $\overline{V}$ means the closure hull of V;

(b) $\mu(U) = \mu(\overline{U}) = \mu(convU)$, where $convU$ means the convex hull of U;

(c) $\mu(\lambda U) = |\lambda|\mu(U)$ for any $\lambda \in \mathbb{R}$;

(d) $U \subset V$ implies $\mu(U) \leq \mu(V)$;

(e) $\mu(U \bigcup V) = \max\{\mu(U), \mu(V)\}$;

(f) $\mu(U + V) \leq \mu(U) + \mu(V)$, where $U + V = \{x/x = u + v, u \in U, v \in V\}$;

(g) If $G : \mathcal{D}(G) \subset E \longrightarrow X$ is Lipschitz continuous with constant λ, then $\mu(G(V)) \leq \lambda\mu(V)$ for any bounded subset $V \subset D(G)$, where X is another Banach space.

For more details about properties of the Kuratowski measure of noncompactness, we refer to the monographs of Bana's and Goebel [4] and Deimling [7].

Theorem 2.10. *[4, 7] Consider a Banach space E, and $B \subset E$ bounded. Then, there exists a countable set $B_0 \subset B$, such that $\mu(B) < 2\mu(B_0)$.*

Theorem 2.11. *[4, 7] Let E be a Banach space and $B = \{u_n : n \in \mathbb{N}\} \subset \mathcal{C}([\alpha; \beta], E)$ be a bounded and countable set for constants $-\infty < \alpha < \beta < +\infty$. Then, $t \longmapsto \mu(B(t))$ is Lebesgue integral on $[\alpha; \beta]$, and*

$$\mu\Big(\Big\{ \int_\alpha^\beta u_n(t)dt \mid n \in \mathbb{N}\Big\}\Big) \leq 2 \int_\alpha^\beta \mu(B(t))dt.$$

Theorem 2.12. *[4, 7] Consider E a Banach space, and $B \subset \mathcal{C}([\alpha; \beta], E)$ a bounded and equicontinuous. Then, the mapping $t \longmapsto \mu(B(t))$ is continuous on $[\alpha; \beta]$, and $\mu(B) = \max_{t \in [\alpha;\beta]} \mu(B(t))$.*

The following Sadovskii fixed point theorem plays a key role in the proof of our main results.

Theorem 2.13. *[4, 7] Consider a Banach space E and suppose that, $\Omega \subset E$ is bounded, closed and convex. If the operator $G : \Omega \longrightarrow \Omega$ is condensing, which means that $\mu(G(\Omega)) < \mu(\Omega)$, then G has at least one fixed point in Ω.*

3. Existence of mild solution

In this section, we try our self to prove the existence of global mild solutions for equation (1.5) using the equicontinuity of $\{U(t,s) : 0 \leq s \leq t < +\infty\}$. We begin by define the mild solution that correspond to the definition in (1.5) and denote $\mathcal{C}_r := \mathcal{C}[-r, 0]$ with $r > 0$.

Definition 3.1. *Let $\varphi \in \mathcal{C}_r$ such that $\varphi(0) \in \overline{D}$. We say that a continuous function $x : (-r; +\infty) \longrightarrow X$ is a mild solution of the equation (1.5), if it satisfies the following equation*

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \lim\limits_{\lambda \to 0^+} \displaystyle\int_0^t U_\lambda(t,s) f(s, x(t - \rho(s, x(s))))ds, & for \quad t \geq 0, \\ \varphi(t) \qquad -r < t \leq 0. \end{cases} \tag{3.1}$$

Firstly we study the local mild solution of equation (1.5).
To obtain our result, we consider the following assumptions :

($\mathbf{H}_1$) The nonlinear function $f : [0; \infty) \times X \longrightarrow X$ is continuous; and for some $r > 0$ there exist a constant $\delta_1 > 0$ and $\phi_r \in L^1([0,a], \mathbb{R}^+)$ such that for all $t \in [0,a]$ and $u \in \mathcal{C}([-r,a], X)$ satisfying $\|u\| \leq r$, $\|f(t,u)\| \leq \phi_r(t)$ and $\limsup\limits_{r \to +\infty} \dfrac{\|\phi_r\|_{L^1([0,a],\mathbb{R}^+)}}{r} = \delta_1 < +\infty$.

($\mathbf{H}_2$) There exists positive constant L_1 such that for any countable set $D \subset X$,

$$\mu(f(t,D)) \leq L_1 \mu(D), \quad t \in [0;a].$$

($\mathbf{H}_3$) We assume that the evolution family $\Big(U(t,s)\Big)_{t \geq s \geq 0}$ is equicontinuous i.e for any $s \geq 0$, the function $t \longmapsto U(t,s)$ is continuous by operator norm for $t \in (s; +\infty)$.

Theorem 3.2. *Let $a > 0$ and assume that the family of linear operators $\Big(A(t)\Big)_{t \geq 0}$ satisfies the hyperbolic conditions ($\boldsymbol{B}_1$)-($\boldsymbol{B}_3$), the assumptions ($\boldsymbol{H}_1$) - ($\boldsymbol{H}_3$) and $\varphi(0) \in \overline{D}$. Then the problem (1.5) has at least one local mild solution defined on $[-r, a]$. Moreover, the mild solution depends continuously on the initial data.*

Proof. Our proof is based on Sadoskii's fixed Point Theorem.
Let $(G_1 u)(t) = U(t,0)\varphi(0)$ and $(G_2 u)(t) = \lim\limits_{\lambda \longrightarrow 0^+} \displaystyle\int_0^t U_\lambda(t,s) f(s, u(s - \rho(u(s))))ds$ for each $u \in \mathcal{C}([-r;a];X)$ and $0 \leq s \leq t \leq a$.
We claim that $G = G_1 + G_2$ is well defined on $\mathcal{C}([0;a];X)$ to itself. Let $u \in \mathcal{C}([0;a];X)$, we show that $Gu \in \mathcal{C}([0;a];X)$. By the strongly continuity of the evolution family $\{U(t,s) : 0 \leq s \leq t \leq a\}$, we get for $0 \leq s \leq t \leq a$ that:

$$\begin{aligned} \|(Gu)(t) - (Gu)(s)\| &\leq \|U(t,0)\varphi(0) - U(s,0)\varphi(0)\| \\ &\quad + \|\lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,r) f(r, u(r - \rho(u(r))))dr - \lim_{\lambda \longrightarrow 0^+} \int_0^s U_\lambda(s,r) f(r, u(r - \rho(u(r))))dr\| \\ &\leq \|U(t,0)\varphi(0) - U(s,0)\varphi(0)\| + \|\lim_{\lambda \longrightarrow 0^+} \int_0^s \Big(U_\lambda(t,r) - U_\lambda(s,r)\Big)(f(r, u(r - \rho(u(r))))dr\| \\ &\quad + \|\lim_{\lambda \longrightarrow 0^+} \int_s^t U_\lambda(t,r) f(r, u(r - \rho(u(r))))dr\|. \end{aligned}$$

By ($\mathbf{H}_3$), we get that: $\lim_{s\to t}\|U(t,0)\varphi(0) - U(s,0)\varphi(0)\| = 0$ and using ($\mathbf{H}_1$) we deduce from the Lebesgue dominated convergence theorem that:

$$\lim_{s\to t}\| \lim_{\lambda\longrightarrow 0^+} \int_s^t U_\lambda(t,r) f(r, u(r-\rho(u(r))))dr\| = 0$$

and

$$\lim_{s\to t}\| \lim_{\lambda\longrightarrow 0^+} \int_0^s \Big(U_\lambda(t,r) - U_\lambda(s,r)\Big)(f(r, u(r-\rho(u(r))))dr\| = 0.$$

Thus

$$\lim_{s\to t}\|(Gu)(t) - (Gu)(s)\| = 0.$$

Therefore, our operator $Gu \in \mathcal{C}([0;a];X)$ for any $u \in \mathcal{C}([-r;a];X)$.

Case 1: Assume that

$$Ma\max\{\delta_1, 4L_1\} < 1 \tag{3.2}$$

We claim that there exists a constant $R > 0$ such that $G(B_R) \subset B_R$ where

$$B_R = \{u \in \mathcal{C}([0,a],X), \|u\| \leq R\}$$

By virtue of (3.2), we choose R such that $R \geq \frac{M\|\varphi(0)\|}{1-M\delta_1 a}$. Let $u \in B_R$ and ($\mathbf{H}_1$) hypothesis , we get that

$$\begin{aligned}
\|(Gu)(t)\| &\leq \|U(t,0)\varphi(0)\| + \| \lim_{\lambda\longrightarrow 0^+} \int_0^t U_\lambda(t,s) f(s, u(s-\rho(u(s))))ds\| \\
&\leq M\|\varphi(0)\| + \int_0^t M\|f(s, u(s-\rho(u(s))))\|ds \\
&\leq M\|\varphi(0)\| + \int_0^t M\phi_r(s)ds \\
&\leq M\|\varphi(0)\| + \int_0^t MR\delta_1 ds \\
&\leq M\|\varphi(0)\| + MR\delta_1 t
\end{aligned}$$

Then, $\max_{s\in[0;t]}\|(Gu)(s)\| \leq M\|\varphi(0)\| + MR\delta_1 a \leq R.$

Therefore $G(B_R) \subset B_R$.

We claim that $G : B_R \longrightarrow B_R$ is continuous.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in B_R such that $u_n \to u \in B_R$ as $n \to +\infty$. From ($\mathbf{H}_1$), we consider the definition of the operator G, the continuity of ρ and ($\mathbf{H}_2$) hypothesis. We get for any $t \in [0;a]$ that:

$$\begin{aligned}
\|(Gu_n)(t) - (Gu)(t)\| &\leq \int_0^t M\|f(s, u_n(s-\rho(u_n(s)))) - f(s, u(s-\rho(u(s))))\|ds \\
&\leq \int_0^t M\|f(s, u_n(s-\rho(u_n(s)))) - f(s, u_n(s-\rho(u(s))))\|ds \\
&+ \int_0^t M\|f(s, u_n(s-\rho(u(s)))) - f(s, u(s-\rho(u(s))))\|ds
\end{aligned}$$

Since $(u_n)_{n\in\mathbb{N}} \subset B_R$, then for each $n \in \mathbb{N}$, u_n is continuous on $[-r,a]$. And by using ($\mathbf{H}_1$) we have:

$$\|f\Big(s,u_n(s-\rho(u_n(s)))\Big)-f\Big(s,u_n(s-\rho(u(s)))\Big)\|\to 0\ as\ n\to+\infty$$

and

$$\|f\Big(s,u_n(s-\rho(u(s)))\Big)-f\Big(s,u(s-\rho(u(s)))\Big)\|\to 0\ as\ n\to+\infty$$

By using the Lebesgue dominated convergence theorem, we get that:

$$\lim_{n\to+\infty}\|(Gu_n)(t)-(Gu)(t)\|=0\ \ for\ \ each\ \ t\in[0,a].$$

Consequently, $Gu_n\to Gu$ as $n\to+\infty$. So the operator G is continuous in B_R.

We claim that the operator $G:B_R\longrightarrow B_R$ is equicontinuous. For all $u\in B_R$, $0<t_1<t_2\leq a$ and $\varepsilon>0$ small enough, by using $(\mathbf{H}_3)$, we get that:

$$\begin{aligned}
\|(Gu)(t_2)\ -\ (Gu)(t_1)\| &\leq \|U(t_2,0)\varphi(0)-U(t_1,0)\varphi(0)\|\\
&+\ \|\lim_{\lambda\longrightarrow 0^+}\Big[\int_0^{t_2}U_\lambda(t_2,s)f(s,u(s-\rho(u(s))))ds-\int_0^{t_1}U_\lambda(t_1,s)f(s,u(s-\rho(u(s))))ds\Big]\|\\
&\leq\ \|[U(t_2,0)-U(t_1,0)]\varphi(0)\|+\|\lim_{\lambda\longrightarrow 0^+}\int_{t_1}^{t_2}U_\lambda(t_2,s)f(s,u(s-\rho(u(s))))ds\|\\
&+\ \|\lim_{\lambda\longrightarrow 0^+}\int_0^{t_1-\varepsilon}[U_\lambda(t_2,s)-U_\lambda(t_1,s)]f(s,u(s-\rho(u(s))))ds\|\\
&+\ \|\lim_{\lambda\longrightarrow 0^+}\int_{t_1-\varepsilon}^{t_1}[U_\lambda(t_2,s)-U_\lambda(t_1,s)]f(s,u(s-\rho(u(s))))ds\|\\
&\leq\ \|U(t_2,0)-U(t_1,0)\|_{\mathcal{L}(X)}\|\varphi(0)\|+M\int_{t_1}^{t_2}\|f(s,u(s-\rho(u(s))))\|ds\\
&+\ \sup_{s\in[0,t_1-\varepsilon]}\|\lim_{\lambda\longrightarrow 0^+}[U_\lambda(t_2,s)-U_\lambda(t_1,s)]\|_{\mathcal{L}(X)}\int_0^{t_1-\varepsilon}\|f(s,u(s-\rho(u(s))))\|ds\\
&+\ \sup_{s\in[t_1-\varepsilon,t_1]}\|\lim_{\lambda\longrightarrow 0^+}[U_\lambda(t_2,s)-U_\lambda(t_1,s)]\|_{\mathcal{L}(X)}\int_{t_1-\varepsilon}^{t_1}\|f(s,u(s-\rho(u(s))))\|ds\\
&\to 0\ as\ t_2\to t_1\ \ and\ \ \varepsilon\to 0.
\end{aligned}$$

We claim that the operator $G:B_R\longrightarrow B_R$ is condensing.
For any $B\subset B_R$, B is bounded. By using Theorem 2.10, there exists a countable set $A=\{v_n:n\in\mathbb{N}\}\subset B$ such that

$$\mu(G(B))\leq 2\mu(G(A)). \tag{3.3}$$

Because $A\subset B\subset B_R$, we get that $G(A)\subset G(B_R)$ then $G(A)$ is bounded. And since the operator $G:B_R\longrightarrow B_R$ is equicontinuous, by the Theorem 2.12 we get that

$$\mu(G(A))=\max_{t\in[0;a]}\mu(G(A)(t)). \tag{3.4}$$

By using the definition of the operator G_1, we get that $(G_1u)(t)=U(t,0)\varphi(0)$ for all $u\in B$ and $0\leq t\leq a$. Therefore $G_1(B)(t)=\{U(t,0)\varphi(0)\}$ for $t\in[0;a]$. From the definition of μ, we have

$\mu(G_1(B)(t)) = 0$ for all $t \in [0;a]$ and according to the Theorem 2.12, we get $\mu(G_1(B)) = 0$.
By using Theorem 2.9, Theorem 2.11, the assumptions ($\mathbf{H}_1$) and the definition of G_2, we have

$$\begin{aligned}\mu(G_2(A)(t)) &= \mu(\{\lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,s) f(s, u_n(s-\rho(u_n(s))))ds \mid n \in \mathbb{N}\}) \\ &\leq M\mu(\{\int_0^t f(s, u_n(s-\rho(u_n(s))))ds \mid n \in \mathbb{N}\}) \\ &\leq M\mu(\{\int_0^t f(s, u_n(s))ds \mid n \in \mathbb{N}\}) \\ &\leq 2M \int_0^t \mu(f(s, A(s))ds \\ &\leq 2ML_1 \int_0^t \mu(A(s))ds \\ &\leq 2ML_1 \int_0^t \mu(A)ds \\ &\leq 2ML_1 t\mu(A) \\ &\leq 2ML_1 a\mu(A). \end{aligned} \tag{3.5}$$

We know that $A \subset B$, and using Theorem 2.9,

$$\mu(A) \leq \mu(B). \tag{3.6}$$

$$\mu(G(A)) = \mu(G_1(A) + G_2(A)) \leq \mu(G_1(A)) + \mu(G_2(A)) = \mu(G_2(A)). \tag{3.7}$$

By using (3.3)-(3.7), we have

$$\mu(G(B)) \leq 4ML_1 a\mu(B). \tag{3.8}$$

Since (3.2) and (3.8), we have

$$\mu(G(B)) < \mu(B). \tag{3.9}$$

The inequality (3.9) proves that the operator $G : B_R \longrightarrow B_R$ is condensing. From the Theorem 2.13, the problem (1.5) has at least one local mild solution defined on $[-r, a]$.

Case 2: We assume that $Ma \max\{\delta_1, 4L_1\} \geq 1$.

We know that $\frac{4ML_1 a}{k} \to 0 \;\; as \;\; k \to +\infty$ and $\frac{M\delta_1 a}{k} \to 0 \;\; as \;\; k \to +\infty$. Then there exists a constance $n \in \mathbb{N}\backslash\{0, 1\}$ such that $\frac{4ML_1 a}{n} < 1$ and $\frac{M\delta_1 a}{n} < 1$. Let $b = \frac{a}{n}$, hence $nb = a$ and $4ML_1 b < 1$ and $M\delta_1 b < 1$. We deduce from **Case 1** that there exists at least one local mild solution $x_1 : [-r; b] \longrightarrow X$ of the problem (1.5).

We denote $\varphi_1 \in \mathcal{C}([-r; 0], X)$ such that $\varphi_1(t) = x_1(t + b)$ for any $t \in [-r - b; 0]$ and $\mathcal{C}_{\varphi_1}([b, 2b], X) := \{y \in \mathcal{C}([b, 2b], X) : y(b) = \varphi_1(0)\}$. We consider the following problem

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t - \rho(x(t)))), & t \in [b; 2b], \\ x(t) = x_1(t), & t \in [-r; b]. \end{cases} \tag{3.10}$$

The problem (3.10) is equivalent to the following problem

$$\begin{cases} y'(s) = B(s)y(s) + f_1(s, y(s - \rho(y(s)))), & s \in [0; b], \\ y(s) = \varphi_1(s), & s \in [-r; 0]. \end{cases} \tag{3.11}$$

where $s = t - b, \quad y(s) = x(s+b), \;\; B(s) = A(b+s) \;\; and \;\; f_1(s,.) = f(b+s,.)$.
In this case, f_1 satisfies (H_1) and (H_2). And the family of linear operator $\{B(t) : 0 \le t \le b\}$ satisfies $(B_1)-(B_3)$ and its evolution family satisfies all condition that the evolution family generated by $\{A(t) : 0 \le t \le a\}$ does. It follows from **Case 1** that there exists at least one local mild solution $y : [-r; b] \longrightarrow X$ of the problem (3.11). Then the problem (1.5) has at least one local solution in $[b, 2b]$ defined by $x_2(t) = y(t) \quad for \;\; t \in [b; 2b]$.
By use the inductive reasoning, we get that the problem (1.5) has at least one local solution x_k in $[(k-1)b, kb], \;\; k = 1, 2, \cdots, n$. Hence, the problem (1.5) has at least one local solution defined by:

$$x(t) = x_k(t) \quad for \;\; t \in [(k-1)b; kb], \;\; k = 1, 2, \cdots, n.$$

Therefore, the problem (1.5) has at least one local mild solution on $[-r, a]$.

Let $y = y(., \varphi)$ and $z = z(., \psi)$ be two solutions of equation (1.5) corresponding respectively to initial data $\varphi, \psi \in \mathcal{B}$ with $\varphi(0), \psi(0) \in \overline{D}$. Then

$$\begin{aligned}
\|y(t) - z(t)\| \le & \|U(t,0)\Big[\varphi(0) - \psi(0)\Big]\| \\
& + \|\lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,s)\Big[f(s, y(s - \rho(y(s)))) - f(s, z(s - \rho(z(s))))\Big] ds\| \\
& \le M\|\varphi(0) - \psi(0)\| + M\int_0^t \|f(s, y(s - \rho(y(s)))) - f(s, z(s - \rho(z(s))))\| ds \\
& \le M\|\varphi - \psi\|_\infty + M\int_0^t \|f(s, y(s - \rho(y(s)))) - f(s, y(s - \rho(z(s))))\| ds \\
& + M\int_0^t \|f(s, y(s - \rho(z(s)))) - f(s, z(s - \rho(z(s))))\| ds.
\end{aligned}$$

For all $\epsilon > 0$, we find $\delta > 0$ such that $\|\varphi - \psi\|_\infty < \delta \Rightarrow \max_{0 \le s \le a} \|y(s) - z(s)\| < \epsilon$.
ρ is continuous function, then there exists $\delta_1 > 0$ such that:

$$\|y(s) - z(s)\| < \epsilon \Rightarrow \|\rho(y(s)) - \rho(z(s))\| < \delta_1, \;\; s \in [0, a].$$

y is continuous function, then there exists $\delta_2 > 0$ such that:

$$\|\rho(y(s)) - \rho(z(s))\| < \delta_1 \Rightarrow \|y(s - \rho(y(s))) - y(s - \rho(z(s)))\| < \delta_2, \;\; s \in [0, a].$$

f is continuous function, then there exists $\delta_3 > 0$ such that

$$\|y(s - \rho(y(s))) - y(s - \rho(z(s)))\| < \delta_2 \Rightarrow$$

$$\|f(s, y(s - \rho(y(s)))) - f(s, y(s - \rho(z(s))))\| < \frac{\delta_3}{2aM}$$

and

$$\|f(s, y(s - \rho(z(s)))) - f(s, z(s - \rho(z(s))))\| < \frac{\delta_3}{2aM}, \;\; s \in [0, a].$$

Consequently,

$$\|\varphi-\psi\|_\infty < \delta \Rightarrow \max_{0\le s\le a} \|f(s,y(s-\rho(y(s)))) - f(s,y(s-\rho(z(s))))\| < \frac{\delta_3}{2aM}$$

and

$$\max_{0\le s\le a} \|f(s,y(s-\rho(z(s)))) - f(s,z(s-\rho(z(s))))\| < \frac{\delta_3}{2aM}.$$

Therefore

$$\begin{aligned} M\|\varphi-\psi\|_\infty &+ M\int_0^t \|f(s,y(s-\rho(y(s)))) - f(s,y(s-\rho(z(s))))\|ds \\ &+ M\int_0^t \|f(s,y(s-\rho(z(s)))) - f(s,z(s-\rho(z(s))))\|ds < \epsilon. \end{aligned} \tag{3.12}$$

Relation (3.12) implies that:

$$M\delta+\delta_3<\epsilon. \quad \text{Then,} \quad \delta < \frac{\epsilon-\delta_3}{M}.$$

Choose $\epsilon \ and \ \delta_3$ such that $\epsilon > \delta_3$ and take $\delta = \frac{\epsilon-\delta_3}{2M}$.
Therefore, the mild solution of (1.5) depends continuously on the initial data.

■

Our subsequent objective is to establish the global mild solution of problem (1.5).

($\mathbf{H}_4$) The nonlinear function $f : [0;\infty)\times X \longrightarrow X$ is continuous; and for some $R > 0$ there exist a constant $\delta_0 > 0$ and $\phi_r \in L^1([0,+\infty),\mathbb{R}^+)$ such that for all $t \ge 0$ and $u \in \mathcal{C}([-r,+\infty),X)$ satisfying $\|u\| \le R$, $\|f(t,u)\| \le \phi_R(t)$ and $\limsup_{R\to+\infty} \frac{\|\phi_R\|_{L^1([0,+\infty),\mathbb{R}^+)}}{R} = \delta_0 < +\infty$.

($\mathbf{H}_5$) There exists positive constant L_0 such that for any countable set $D \subset X$,

$$\mu(f(t,D)) \le L_0\mu(D), \quad t \ge 0.$$

Theorem 3.3. *Assume that the family of linear operators $\Big(A(t)\Big)_{t\ge0}$ satisfies the hyperbolic conditions ($\boldsymbol{B}_1$)-($\boldsymbol{B}_3$), the evolution family $\Big(U(t,s)\Big)_{t\ge s\ge 0}$ is equicontinuous, ($\boldsymbol{H}_3$) − ($\boldsymbol{H}_5$) hold. Then problem (1.5) has at least one global mild solution on $[-r;+\infty)$.*

Proof. Using Theorem 3.2, We deduce that there exists an unique local mild solution x^n of problem (1.5) defined on $[-r;n]$ for each $n \in \mathbb{N}$. It is clear that $x^{n+1}|_{[-r;n]} = x^n$ for each $n \in \mathbb{N}$. Hence the problem (1.5) has at least one global mild solution $x(.)$ on $[-r;+\infty)$ and it is defined by $x(t) = x^n(t) \ \ for \ \ each \ \ -r \le t \le n$ and for all $n \in \mathbb{N}$. ■

4. Application

In this section, we apply our results to the following non-autonomous partial differential equation of evolution.

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \theta(t)\dfrac{\partial^2}{\partial x^2}u(t,x) + \dfrac{t}{6(t+1)^3 + \mid u(t-\psi(u(t,x),x)\mid}, \ for \ (t,x) \in [0;+\infty)\times\Omega, \\ u(t,x) = 0, \ for \ (t,x) \in [0;+\infty)\times\partial\Omega, \\ u(t,x) = \phi(t,x), \ for \ (t,x) \in [-T;0]\times\Omega. \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}$ is a bounded and closed domain with smooth boundary $\partial\Omega$ and the function $\theta \in \mathcal{C}^1([0;+\infty),\mathbb{R})$. The delay function ψ is a bounded positive continuous function in $\mathbb{R}$ and let T be its upper bound element in $\mathbb{R}$ and the function $\phi \in \mathcal{C}^2([-T;0]\times\Omega;\mathbb{R})$.

Theorem 4.1. *The problem (4.1) has at least one mild solution.*

Proof. We consider X, the Banach space defined by $X = \mathcal{C}(\Omega;\mathbb{R})$ and the operator $A : D \subset X \longrightarrow X$ defined by

$$\begin{cases} D = D(A) = \{z \in \mathcal{C}^2(\Omega;\mathbb{R}) : z(x) = 0, x \in \partial\Omega\}, \\ Az(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x), \;\; (t,x) \in [0;\infty)\times\Omega. \end{cases}$$

We have $\overline{D} = \{z \in \mathcal{C}(\Omega;\mathbb{R}) : z(x) = 0, x \in \partial\Omega\} \neq X$. We know from [**?**] that

$$(0,+\infty) \subset \varrho(A) \;\; and \;\; \|R(\lambda,A)\| \leq \frac{1}{\lambda} \;\; for \;\; \lambda > 0. \tag{4.2}$$

Let $(A(t))_{t\geq 0}$ be a family of operators defined by $A(t) = \theta(t)\dfrac{\partial^2}{\partial x^2}$.
For any $t \geq 0$, we have $D(A(t)) = D$ independent of t.
Then it is well know that for every $t \geq 0$

$$R(\lambda, A(t)) = \frac{1}{\theta(t)} R(\frac{\lambda}{\theta(t)}, A) \tag{4.3}$$

Since (4.2), we have $\|R(\lambda,A)\| \leq \dfrac{1}{\lambda}$ for each $\lambda \in (0,+\infty)\cap\varrho(A)$.
Then, by adding the (4.3) and for $0 \leq t_1 \leq t_2 \leq ... \leq t_k < +\infty$ we get

$$\|\prod_{j=1}^{k} R(\lambda, A(t_j))\| \leq \frac{1}{\lambda^k}$$

Using the definition of the function θ and Banach's space X, the mapping $t \longmapsto A(t)x$ is continuously differentiable in X for all $x \in D$.
Hence, the family of linear operator $(A(t))_{t\geq 0}$ on X satisfies the assumptions $(B_1) - (B_3)$.
Since [2], the operator $A_0(.)$ of $A(.)$ in $\overline{D(A)}$ generates an evolution family $\Big(U(s,t)\Big)_{t\geq s\geq 0}$ given by

$$U(s,t) = T_0\Big(\int_s^t \theta(r)dr\Big)$$

which is equicontinuous for each $t \geq s \geq 0$. where the operator $\triangle_0$ of $\triangle$ in $\overline{D(\triangle)}$ give by

$$\begin{cases} D(\triangle_0) = \{x \in D(\triangle) : \triangle x \in \overline{D(\triangle)}\}, \\ \triangle_0 x = \triangle x, \end{cases}$$

generates the semigroup $\Big(T_0(t)\Big)_{t\geq 0}$ such that

$$\|T_0(t)\| \leq e^{-t} \;\; for \;\; each \;\; t \geq 0.$$

Hence,

$$\|U(t,s)\| \leq 1 \;\; for \;\; each \;\; t \geq s \geq 0.$$

We get that $M = 1$.
Let $f : [0;+\infty) \times X \longrightarrow X$ defined by $f(t,z)x = \dfrac{t}{6(t+1)^3 + \mid z(x) \mid}$ for $x \in \Omega$ and $t \geq 0$. The initial data φ is defined by $\varphi(t)x = \phi(t,x)$ for $x \in \Omega$ and $t \geq 0$.
and $z(t)x = u(t,x)$.
Therefore (4.1) becomes

$$\begin{cases} z'(t) = A(t)z(t) + f(t, z(t - \psi(z(t)))),\ for\ t \geq 0, \\ z(t) = \varphi(t),\ for\ t \in [-T;0]. \end{cases} \tag{4.4}$$

For every $t \geq 0\ \ and\ \ z, y \in X$,

$$\begin{aligned} |f(t,z) - f(t,y)| &= \frac{t}{[6(t+1)^3 + |z|][6(t+1)^3 + |y|]} ||y| - |z|| \\ &\leq \frac{t}{[6(t+1)^3 + |z|][6(t+1)^3 + |y|]} |z - y| \\ &\leq \frac{t}{36(t+1)^6} |z - y| \\ &\leq \frac{1}{36} |z - y| \end{aligned}$$

we get that $L_0 = \dfrac{1}{36}$. Thus (H_5) is verified.
Let $r > 0$, for every $t \geq 0$, $u \in B_r$, we get that:

$$\|f(t,u)\| \leq \frac{t}{6(t+1)^3} \leq \frac{(t+1)}{6(t+1)^3} \leq \frac{r+1}{(t+1)^2} = \phi_r(t), \quad \phi_r \in L^1([0,+\infty), \mathbb{R}^+)$$

$$\|\phi_r\|_{L^1([0,+\infty),\mathbb{R}^+)} = r + 1$$

and

$$\lim_{r \to +\infty} \frac{\|\phi_r\|_{L^1([0,+\infty),\mathbb{R}^+)}}{r} = 1.$$

Then (H_4) is verified and we take $\delta_0 = 1$. Therefore, by using Theorem 3.3, we get that the problem (4.4) has at least one global mild solution $u : [-T;+\infty) \longrightarrow X$. ■

5. Acknowledgement

The authors would like to thank the referee for his valuable comments and suggestions which allowed to improve the original manuscript.

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MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 241–253.
http://doi.org/10.26637/mjm11S/015

Fixed point of almost generalized $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$-contractive type mappings in weak partial metric spaces

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Received 31 August 2023; Accepted 30 September 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. This study introduces almost generalized $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$-contractive type mappings and investigates some fixed point theorems for such mappings in weak partial metric spaces.Our results extend the implications of Altun and Durmaz [15] and other prior results in this area. Additionally, We present some examples that support our findings.

AMS Subject Classifications: 47H10, 54H25.

Keywords: Triangular $\hat{\alpha}$-admissible mapping, almost generalized $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$-contractive type mapping, weak partial metric spaces.

Contents

1. Introduction and Background

Banach fixed point theorem has been expanded in numerous ways and it has undergone numerous generalisations in various metric spaces. Partial metric space (PMS), which Matthews [1] introduced in 1992, is a very intriguing generalisation of the metric space in which the self distance not required to be zero.By establishing a new class of contractive type mappings known as $\hat{\alpha} - \hat{\psi}$ contractive type mappings, Samet et al. [3] further expanded and generalised the Banach contraction principle. The $\hat{\alpha} - \hat{\psi}$ contractive type mappings were generalised by Karapinar and Samet[4].On the other hand, Berinde [7, 8] introduced the concept of almost contractions in metric spaces. The concept of weak partial metric spaces, a generalisation of partial metric spaces, was first introduced by Heckmann [14] in 1999. Some results for mappings in weak partial metric spaces have recently been obtained in [17], [18],[19] and [20].

Definition 1.1. *[12] Let Ψ be the set of functions $\hat{\psi} : [0, \infty) \to [0, \infty)$ such that*

(a) $\hat{\psi}$ *is non decreasing and continuous;*

(b) $\hat{\psi}(u) = 0 \Leftrightarrow u = 0$.

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Definition 1.2. *[3] Let $\Gamma : W_p \to W_p$ and $\hat{\alpha} : W_p \times W_p \to [0,\infty)$. Γ is said to $\hat{\alpha}$-admissible if*

$$\hat{\alpha}(\eta_p, \zeta_p) \geq 1 \Rightarrow \hat{\alpha}(\Gamma\eta_p, \Gamma\zeta_p) \geq 1$$

for all $\eta_p, \zeta_p \in W_p$.

Definition 1.3. *[5] Let $\Gamma : W_p \to W_p$ and $\hat{\alpha} : W_p \times W_p \to [0,\infty)$ be two functions. Then Γ is said to triangular $\hat{\alpha}$-admissible if Γ is $\hat{\alpha}$-admissible and for $\eta_p, \zeta_p, \delta_p \in W_p$, $\hat{\alpha}(\eta_p, \delta_p) \geq 1$ and $\hat{\alpha}(\delta_p, \zeta_p) \geq 1 \Rightarrow \hat{\alpha}(\eta_p, \zeta_p) \geq 1$.*

Lemma 1.4. *[5] Let $\Gamma : W_p \to W_p$ be a triangular $\hat{\alpha}$-admissible mapping. Suppose that there exists $\eta_{p_0} \in W_p$ such that $\hat{\alpha}(\eta_{p_0}, \Gamma\eta_{p_0}) \geq 1$.If we define a sequence $\{\eta_{p_i}\}$ by $\eta_{p_{i+1}} = \Gamma\eta_{p_i}$ for every $i \in \mathbb{N}_0$. Then we have $\hat{\alpha}(\eta_{p_j}, \eta_{p_i}) \geq 1$ for all $j, i \in \mathbb{N}$ with $j > i$.*

In 1992, Matthews [1] presented generalization of metric space as follows:

Definition 1.5. *([1]) Let W_p be a set which is non-empty. A mapping $\mathfrak{d}_\varrho : W_p \times W_p \to [0,\infty)$ is known as partial metric on W_p if the following conditions are satisfied:*

(PMS1) $\eta_p = \zeta_p \Leftrightarrow \mathfrak{d}_\varrho(\eta_p, \eta_p) = \mathfrak{d}_\varrho(\zeta_p, \zeta_p) = \mathfrak{d}_\varrho(\eta_p, \zeta_p)$ *;*

(PMS2) $\mathfrak{d}_\varrho(\eta_p, \eta_p) \leq \mathfrak{d}_\varrho(\eta_p, \zeta_p)$;

(PMS3) $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \mathfrak{d}_\varrho(\zeta_p, \eta_p)$:

(PMS4) $\mathfrak{d}_\varrho(\eta_p, \zeta_p) \leq \mathfrak{d}_\varrho(\eta_p, \delta_p) + \mathfrak{d}_\varrho(\delta_p, \zeta_p) - \mathfrak{d}_\varrho(\delta_p, \delta_p)$. *for all $\eta_p, \zeta_p, \delta_p \in W_p$.*

Lemma 1.6. *([1]) Let $(W_p, \mathfrak{d}_\varrho)$ be a partial metric space.*

(a) A sequence $\{\eta_{p_i}\}$ in the space $(W_p, \mathfrak{d}_\varrho)$ converges to a point $\eta_p \in W_p \Leftrightarrow$

$$\mathfrak{d}_\varrho(\eta_p, \eta_p) = \lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_p),$$

(b) If $\lim_{j,i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j})$ exists and finite then the sequence $\{\eta_{p_i}\}$ is a Cauchy sequence in space $(W_p, \mathfrak{d}_\varrho)$,

(c) If every Cauchy sequence $\{\eta_{p_i}\}$ in W_p converges to a point $\eta_p \in W_p$, such that

$$\mathfrak{d}_\varrho(\eta_p, \eta_p) = \lim_{j,i\to\infty} \mathfrak{d}_\varrho(\eta_{p_j}, \eta_{p_i}) = \lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_p) = \mathfrak{d}_\varrho(\eta_p, \eta_p)$$

Then $(W_p, \mathfrak{d}_\varrho)$ is complete.

Lemma 1.7. *([11],[1],[2]) Let $\mathfrak{d}_\varrho$ be a partial metric on W_p, then the mapping $\mathfrak{d}^m_\varrho : W_p \times W_p \to \mathbb{R}^+$ such that*

$$\begin{aligned}\mathfrak{d}^m_\varrho(\eta_p, \zeta_p) =& \max\{\mathfrak{d}_\varrho(\eta_p, \zeta_p) - \mathfrak{d}_\varrho(\eta_p, \eta_p), \mathfrak{d}_\varrho(\eta_p, \zeta_p) - \mathfrak{d}_\varrho(\zeta_p, \zeta_p)\}\\ =&\mathfrak{d}_\varrho(\eta_p, \zeta_p) - \min\{\mathfrak{d}_\varrho(\eta_p, \eta_p), \mathfrak{d}_\varrho(\zeta_p, \zeta_p)\}\end{aligned} \tag{1.1}$$

is metric on W_p. Furthermore, $(W_p, \mathfrak{d}^m_\varrho)$ is metric space.
Let $(W_p, \mathfrak{d}^m_\varrho)$ be a partial metric space. Then

1. *A sequence $\{\eta_{p_i}\}$ in $(W_p, \mathfrak{d}^m_\varrho)$ is a Cauchy sequence $\Leftrightarrow$ $\{\eta_{p_i}\}$ is a Cauchy sequence in the metric space $(W_p, \mathfrak{d}^m_\varrho)$,*

2. *$(W_p, \mathfrak{d}^m_\varrho)$ is complete $\Leftrightarrow$ $(W_p, \mathfrak{d}_\varrho)$ is complete. Moreover*
$\lim_{i\to\infty} \mathfrak{d}^m_\varrho(\eta_{p_i}, \eta_p) = 0 \Leftrightarrow \mathfrak{d}_\varrho(\eta_p, \eta_p) = \lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_p) = \lim_{i,j\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j})$.

Lemma 1.8. *([18]) Suppose that $\{\eta_{p_i}\}$ be a sequence $\eta_{p_i} \to \delta_p$ as $i \to \infty$ in a partial metric space $(W_p, \mathfrak{d}_\varrho)$ such that $\mathfrak{d}_\varrho(\delta_p, \delta_p) = 0$. Then $\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \zeta_p) = \mathfrak{d}_\varrho(\delta_p, \zeta_p)$ for every $\zeta_p \in W_p$.*

Lemma 1.9. *[18] If $\{\eta_{p_i}\}$be a sequence with $\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) = 0$ such that $\{\eta_{p_i}\}$ is not a Cauchy sequence in $(W_p, \mathfrak{d}_\varrho)$, and there exist two sequences $\{i(u)\}$ and $\{j(u)\}$ of positive integers such that $i(u) > j(u) > u$, then following sequences*

$$\mathfrak{d}_\varrho(\eta_{p_{j(u)}}, \eta_{p_{i(u)+1}}), \mathfrak{d}_\varrho(\eta_{p_{j(u)}}, \eta_{p_{i(u)}}),$$
$$\mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)+1}}), \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)}})$$

tend to $\mu_p > 0$ when $u \to \infty$

Lemma 1.10. *([12], [16])Let W_p be a set which is non-empty. Suppose that $(W_p, \mathfrak{d}_\varrho)$ be a partial metric space.*

1. *If $\eta_p \neq \zeta_p$ then $\mathfrak{d}_\varrho(\eta_p, \zeta_p) > 0$,*
2. *if $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = 0$ then $\eta_p = \zeta_p$.*

By omitting the small self-distance axiom in partial metric spaces, Heckmann [14] introduced the concept of weak partial metric space as follows:

Definition 1.11. *[14] Let W_p be a set which is non-empty . A mapping $\mathfrak{d}_\varrho : W_p \times W_p \to [0, \infty)$ is known as weak partial metric on W_p if the following conditions are satisfied:*

(WPMS1) $\eta_p = \zeta_p \Leftrightarrow \mathfrak{d}_\varrho(\eta_p, \eta_p) = \mathfrak{d}_\varrho(\zeta_p, \zeta_p) = \mathfrak{d}_\varrho(x, \zeta_p)$;

(WPMS2) $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \mathfrak{d}_\varrho(\zeta_p, \eta_p)$;

(WPMS3) $\mathfrak{d}_\varrho(\eta_p, \zeta_p) \leq \mathfrak{d}_\varrho(\eta_p, \delta_p) + \mathfrak{d}_\varrho(\delta_p, \zeta_p) - \mathfrak{d}_\varrho(\delta_p, \delta_p)$. *for all $\eta_p, \zeta_p, \delta_p \in W_p$.*

and the pair $(W_p, \mathfrak{d}_\varrho)$ is called weak partial metric space (in short WPMS).
Additionally, Heckmann [14] demonstrates that the weak small self-distance feature follows if $\mathfrak{d}_\varrho$ is a weak partial metric on W_p i.e.

$$\mathfrak{d}_\varrho(\eta_p, \zeta_p) \geq \frac{\mathfrak{d}_\varrho(\eta_p, \eta_p) + \mathfrak{d}_\varrho(\zeta_p, \zeta_p)}{2}$$

for all $\eta_p, \zeta_p \in W_p$.
Every partial metric space is obviously a weak partial metric space, but the converse may not be true. For example, for $\eta_p, \zeta_p \in \mathbb{R}$ the function $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \frac{e^{\eta_p} + e^{\zeta_p}}{2}$ is a weak partial metric space but not a partial metric on $\mathbb{R}$.

Lemma 1.12. *[15] Let $(W_p, \mathfrak{d}_\varrho)$ be a weak partial metric space(WPMS).*

(i) $\{\eta_{p_i}\}$ *is a Cauchy sequence in $(W_p, \mathfrak{d}_\varrho) \Leftrightarrow$ it is a Cauchy sequence in $(W_p, \mathfrak{d}_\varrho^m)$;*

(ii) $(W_p, \mathfrak{d}_\varrho)$ *is complete* $\Leftrightarrow (W_p, \mathfrak{d}_\varrho^m)$ *is complete.*

Lemma 1.13. *[17] Let $(W_p, \mathfrak{d}_\varrho)$ be a weak partial metric space and $\{\eta_{p_i}\}$ is a sequence in $(W_p, \mathfrak{d}_\varrho)$. If $\lim_{i\to\infty} \eta_{p_i} = \eta_p$ and $\mathfrak{d}_\varrho(\eta_p, \eta_p) = 0$, then $\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \zeta_p) = \mathfrak{d}_\varrho(\eta_p, \zeta_p)$, for all $\zeta_p \in W_p$.*

Definition 1.14. *[13] Let Φ be the set of all functions $\hat{\varphi} : [0, \infty) \to [0, \infty)$ satisfying the following conditions:*

(i) $\hat{\varphi}(u) < \hat{\psi}(u)$ *for all* $u > 0$

(ii) $\hat{\varphi}(0) = 0$

Definition 1.15. *[21] Let Θ be the set of functions $\hat{\vartheta} : [0, \infty) \to [0, \infty)$ such that*

(i) $\hat{\vartheta}$ *is continuous;*

(ii) $\hat{\vartheta}(u) = 0 \Leftrightarrow u = 0$.

Remark 1.16. *The convergence of sequences, Cauchy sequences, and completeness in a weak partial metric space are defined as being in a partial metric space.*

2. Main Results

Definition 2.1. *Let* $(W_p, \mathfrak{d}_\varrho)$ *be a weak partial metric space and* $\Gamma : W_p \to W_p$ *be a given self map. We say that* Γ *is almost generalized* $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$*-contractive mapping if there exists* $\hat{\alpha} : W_p \times W_p \to [0, \infty)$ *and* $\hat{\psi} \in \Psi$*,* $\hat{\varphi} \in \Phi$*,* $\hat{\vartheta} \in \Theta$ *and* $L \geq 0$ *such that for all* $\eta_p, \zeta_p \in W_p$ *we have*

$$\hat{\alpha}(\eta_p, \zeta_p)\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p)) \leq \hat{\varphi}(\tilde{\mathcal{M}}(\eta_p, \zeta_p)) + L\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_p, \zeta_p)) \tag{2.1}$$

Where

$$\tilde{\mathcal{M}}(\eta_p, \zeta_p) = \max\left\{\mathfrak{d}_\varrho(\eta_p, \zeta_p), \mathfrak{d}_\varrho(\eta_p, \Gamma\eta_p), \mathfrak{d}_\varrho(\zeta_p, \Gamma\zeta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p, \Gamma\zeta_p) + \mathfrak{d}_\varrho(\zeta_p, \Gamma\eta_p)]\right\} \tag{2.2}$$

and

$$\tilde{\mathcal{N}}(\eta_p, \zeta_p) = \min\{\mathfrak{d}^m_\varrho(\eta_p, \Gamma\eta_p), \mathfrak{d}^m_\varrho(\zeta_p, \Gamma\eta_p)\} \tag{2.3}$$

Theorem 2.2. *Let* $(W_p, \mathfrak{d}_\varrho)$ *be a complete weak partial metric space and* $\Gamma : W_p \to W_p$ *be self mapping. Suppose* $\hat{\alpha} : W_p \times W_p \to [0, \infty)$ *be the mapping satisfying the conditions:*

(i) Γ *is triangular* $\hat{\alpha}$*-admissible;*

(ii) Γ *is almost generalized* $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$*-contractive mapping;*

(iii) There exists $\eta_{p_0} \in W_p$ *such that* $\hat{\alpha}(\eta_{p_0}, \Gamma\eta_{p_0}) \geq 1$*;*

(iv) Γ *is continuous.*

Then Γ *has a fixed point in* W_p*.*

Proof. Let there be an arbitrary point η_{p_0} such that $\hat{\alpha}(\eta_{p_0}, \Gamma\eta_{p_0}) \geq 1$. Suppose there is a sequence $\{\eta_{p_i}\}$ in W_p such that $\eta_{p_{i+1}} = \Gamma\eta_{p_i}$ for all $i \in \mathbb{N}_0$.
If $\eta_{p_i} = \eta_{p_{i+1}}$ for some $i \in \mathbb{N}_0$, then η_{p_i} is a fixed point of Γ and then proof of existence part of fixed point is finished. Suppose $\eta_{p_i} \neq \eta_{p_{i+1}}$ for every $i \in \mathbb{N}_0$, Then $\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) > 0$ by Lemma 1.10. Now, since Γ is $\hat{\alpha}$-admissible, so

$$\begin{aligned}\hat{\alpha}(\Gamma\eta_{p_0}, \Gamma\eta_{p_1}) = \hat{\alpha}(\eta_{p_1}, \eta_{p_2}) \geq 1\\ \hat{\alpha}(\Gamma\eta_{p_1}, \Gamma\eta_{p_2}) = \hat{\alpha}(\eta_{p_2}, \eta_{p_3}) \geq 1\end{aligned}$$

and using induction we have $\hat{\alpha}(\eta_{p_i}, \eta_{p_{i+1}}) \geq 1$ for all $i \in \mathbb{N}$.

Now, from (2.1) we have

$$\begin{aligned}\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) = \hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i-1}}, \Gamma\eta_{p_i})) &\leq \hat{\alpha}(\eta_{p_{i-1}}, \eta_{p_i})\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i-1}}, \Gamma\eta_{p_i}))\\ &\leq \hat{\varphi}(\tilde{\mathcal{M}}(\eta_{p_{i-1}}, \eta_{p_i})) + L\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_{p_{i-1}}, \eta_{p_i}))\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\tilde{\mathcal{N}}(\eta_{p_{i-1}}, \eta_{p_i}) &= \min\{\mathfrak{d}^m_\varrho(\eta_{p_{i-1}}, \Gamma\eta_{p_{i-1}}), \mathfrak{d}^m_\varrho(\eta_{p_i}, \Gamma\eta_{p_{i-1}})\}\\ &= \min\{\mathfrak{d}^m_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}^m_\varrho(\eta_{p_i}, \eta_{p_i})\}\\ &= 0\end{aligned} \tag{2.5}$$

and

$$\tilde{\mathcal{M}}(\eta_{p_{i-1}}, \eta_{p_i}) = \max\left\{\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \Gamma\eta_{p_{i-1}}), \mathfrak{d}_\varrho(\eta_{p_i}, \Gamma\eta_{p_i}), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \Gamma\eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \Gamma\eta_{p_{i-1}})]\right\}$$
$$= \max\left\{\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_{i+1}}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i})]\right\} \tag{2.6}$$

Now, using the condition(*WPMS3*) we have

$$\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_{i+1}}) \le \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) - \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i})$$

Therefore

$$\frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_{i+1}}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i})] \le \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) - \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i})]$$
$$= \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})]$$
$$\le \max\{\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})\} \tag{2.7}$$

By (2.6) and (2.7) we get that

$$\tilde{\mathcal{M}}(\eta_{p_{i-1}}, \eta_{p_i}) \le \max\{\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})\} \tag{2.8}$$

Now, using (2.5) and (2.8) in (2.4) and the fact that and $\hat{\vartheta}(u) = 0 \Leftrightarrow u = 0$, we get that

$$\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) \le \hat{\varphi}(\max\{\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}), \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}\}) \tag{2.9}$$

Now, if $\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) > \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i})$ using definition that $\hat{\varphi}(u) < \hat{\psi}(u)$ for $u > 0$ we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) \le \hat{\varphi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) < \hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}))$$

which is a contradiction. Hence

$$\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) \le \hat{\varphi}(\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i})) < \hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i})) \tag{2.10}$$

We get a sequence of non-negative real numbers $\{\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) : i \in \mathbb{N}\}$ that decreases. Therefore there exists $\lambda_0 \ge 0$ such that

$$\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) = \lambda_0$$

Let $\lambda_0 > 0$. Then taking limit $i \to \infty$ in (2.10) we get

$$\hat{\psi}(\lambda_0) \le \hat{\varphi}(\lambda_0) < \hat{\psi}(\lambda_0)$$

This is contradiction. Hence

$$\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) = 0 \tag{2.11}$$

We now show that $\{\eta_{p_i}\}$ is a Cauchy sequence in W_p. i.e. $\lim_{i,j\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) = 0$.
By contradiction, we prove it.
Let

$$\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) \ne 0$$

Then, with reference to lemma 1.9 all sequences tends to $\mu_p > 0$, when $u \to \infty$.
So we can see that

$$\lim_{u\to\infty} \mathfrak{d}_\varrho(\eta_{p_{j(u)}}, \eta_{p_{i(u)}}) = \mu_p \tag{2.12}$$

Further corresponding to $j(u)$, we can choose $i(u)$ in such a way that it is smallest integer with $i(u) > j(u) > u$. Then

$$\lim_{u\to\infty} \mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)}}) = \mu_p \tag{2.13}$$

Again,

$$\mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)-1}}) \le \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)}}) + \mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \eta_{p_{i(u)-1}}) - \mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \eta_{p_{i(u)}})$$

Letting $u \to \infty$ and using lemma 1.9 we get

$$\lim_{u\to\infty} \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)-1}}) = \mu_p \tag{2.14}$$

Again note that

Now, since Γ is triangular $\hat{\alpha}$-admissible, from Lemma 1.4 we derive that $\hat{\alpha}(\eta_{p_i}, \eta_{p_j}) \ge 1$ for all $i > j \in \mathbb{N}_0$.
Replacing η_p by $\eta_{p_{i(u)}}$ and ζ_p by $\eta_{p_{j(u)}}$ in (2.1) respectively, we get

$$\begin{aligned}\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \eta_{p_{j(u)}})) = \hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{j(u)-1}})) &\le \hat{\alpha}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}})\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{j(u)-1}}))\\ &\le \hat{\varphi}(\tilde{\mathcal{M}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}})) + L(\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}})))\end{aligned} \tag{2.15}$$

Where

$$\begin{aligned}\tilde{\mathcal{M}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}) &= \max\Big\{\mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}), \mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{i(u)-1}}), \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \Gamma\eta_{p_{j(u)-1}}),\\ &\qquad \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{j(u)-1}}) + \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \Gamma\eta_{p_{i(u)-1}})]\Big\}\\ &= \max\Big\{\mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}), \mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{i(u)}}), \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{j(u)}}),\\ &\qquad \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)}}) + \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)}})]\Big\}\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}) &= \min\{\mathfrak{d}_\varrho^m(\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{i(u)-1}}), \mathfrak{d}_\varrho^m(\eta_{p_{j(u)-1}}, \Gamma\eta_{p_{i(u)-1}})\}\\ &= \min\{\mathfrak{d}_\varrho^m(\eta_{p_{i(u)-1}}, \eta_{p_{i(u)}}), \mathfrak{d}_\varrho^m(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)}})\}\end{aligned} \tag{2.16}$$

Letting $u \to \infty$ in (**??**) and (2.16) and using (2.11), (2.12), (2.13), (2.14) and lemma 1.9 we get

$$\lim_{u\to\infty} \tilde{\mathcal{M}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}) = \max\{\mu_p, 0, 0, \mu_p\} = \mu_p \tag{2.17}$$

and

$$\lim_{u\to\infty} \tilde{\mathcal{N}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}) = 0. \tag{2.18}$$

Now Letting $u \to \infty$ in (2.15) and using (2.17) and (2.18) we get

$$\hat{\psi}(\mu_p) \leq \hat{\varphi}(\mu_p) < \hat{\psi}(\mu_p)$$

This is a contradiction, Therefore

$$\lim_{i,j\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) = 0 \tag{2.19}$$

This implies that $\{\eta_{p_i}\}$ is a Cauchy sequence in $(W_p, \mathfrak{d}_\varrho)$. On the other hand, since

$$\begin{aligned}\mathfrak{d}_\varrho^m(\eta_{p_i}, \eta_{p_j}) &= \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) - \min\{\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_j}, \eta_{p_j})\}\\ &\leq \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j})\end{aligned}$$

Now, taking the limit as $j, i \to \infty$ and using (2.19) we get that

$$\lim_{i,j\to\infty} \mathfrak{d}_\varrho^m(\eta_{p_i}, \eta_{p_j}) = 0 \tag{2.20}$$

This shows that $\{\eta_{p_i}\}$ is also a Cauchy sequence in the metric space $(W_p, \mathfrak{d}_\varrho^m)$. Since $(W_p, \mathfrak{d}_\varrho)$ is complete, then from Lemma 1.12, the sequence $\{\eta_{p_i}\}$ converges in the metric space $(W_p, \mathfrak{d}_\varrho^m)$, say $\lim_{i\to\infty} \mathfrak{d}_\varrho^m(\eta_{p_i}, \delta_p) = 0$. Again from Lemma 1.12 we have

$$\mathfrak{d}_\varrho(\delta_p, \delta_p) = \lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \delta_p) = \lim_{j,i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) \tag{2.21}$$

Therefore, from (2.21) and (2.19) we get that

$$\mathfrak{d}_\varrho(\delta_p, \delta_p) = \lim_{n\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \delta_p) = \lim_{j,i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i}) = 0 \tag{2.22}$$

Moreover, As Γ is continuous, we have

$$\delta_p = \lim_{i\to\infty} \eta_{p_{i+1}} = \lim_{i\to\infty} \Gamma\eta_{p_i} = \Gamma\delta_p$$

■

In the following, we omit the continuity assumption of Γ in Theorem 2.2.

Theorem 2.3. *Let $(W_p, \mathfrak{d}_\varrho)$ be a complete weak partial metric space and $\Gamma : W_p \to W_p$ be self mapping. Suppose $\hat{\alpha} : W_p \times W_p \to [0, \infty)$ be the mappings satisfying the conditions:*

(i) Γ is triangular $\hat{\alpha}$-admissible;

(ii) Γ is almost generalized $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$-contractive mapping;

(iii) There exists $\eta_{p_0} \in W_p$ such that $\hat{\alpha}(\eta_{p_0}, \Gamma\eta_{p_0}) \geq 1$;

(iv) If $\{\eta_{p_i}\}$ is a sequence in W_p such that $\eta_{p_i} \to \eta_p \in W_p$, $\hat{\alpha}(\eta_{p_i}, \eta_{p_{i+1}}) \geq 1$ for all i, there exists a subsequence $\{\eta_{p_{i(u)}}\}$ of $\{\eta_{p_i}\}$ such that $\hat{\alpha}(\eta_{p_{i(u)}}, \eta_p) \geq 1$ for all u.

Then Γ has a fixed point in W_p. Further if δ_p, δ_q are fixed points of Γ such that $\hat{\alpha}(\delta_p, \delta_q) \geq 1$ then $\delta_p = \delta_q$.

Proof. From the proof of the Theorem 2.2, the sequence η_{p_i} defined by $\eta_{p_{i+1}} = \Gamma\eta_{p_i}$ is Cauchy in W_p and converges to $\delta_p \in W_p$. According to the assumptions, there is a subsequence of $\{\eta_{p_{i(u)}}\}$ of $\{\eta_{p_i}\}$ such that $\hat{\alpha}(\eta_{p_{i(u)}}, \delta_p) \geq 1$ for all u. We will now demonstrate that δ_p is a fixed point of Γ. Consider the alternative, then $\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p) > 0$.
Now in (2.1) replacing η_p by $\eta_{p_{i(u)}}$ and ζ_p by δ_p we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_{i(u)+1}}, \Gamma\delta_p)) = \hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i(u)}}, \Gamma\delta_p)) \leq \hat{\alpha}(\eta_{p_{i(u)}}, \delta_p)\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i(u)}}, \Gamma\delta_p))$$
$$\leq \hat{\varphi}(\tilde{\mathcal{M}}(\eta_{p_{i(u)}}, \delta_p)) + L(\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_{p_{i(u)}}, \delta_p))) \quad (2.23)$$

Where

$$\tilde{\mathcal{M}}(\eta_{p_{i(u)}}, \delta_p) = \max\left\{\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \delta_p), \mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \Gamma\eta_{p_{i(u)}}), \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \Gamma\delta_p) + \mathfrak{d}_\varrho(\delta_p, \Gamma\eta_{p_{i(u)}})]\right\}$$
$$= \max\left\{\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \delta_p), \mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \eta_{p_{i(u)+1}}), \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \Gamma\delta_p) + \mathfrak{d}_\varrho(\delta_p, \eta_{p_{i(u)+1}})]\right\} \quad (2.24)$$

and

$$\tilde{\mathcal{N}}(\eta_{p_{i(u)}}, \delta_p) = \min\{\mathfrak{d}^m_\varrho(\eta_{p_{i(u)}}, \Gamma\eta_{p_{i(u)}}), \mathfrak{d}^m_\varrho(\delta_p, \Gamma\eta_{p_{i(u)}})\}$$
$$= \min\{\mathfrak{d}^m_\varrho(\eta_{p_{i(u)}}, \eta_{p_{i(u)+1}}), \mathfrak{d}^m_\varrho(\delta_p, \eta_{p_{i(u)+1}})\} \quad (2.25)$$

Now, taking $u \to \infty$ in (2.24) and ((2.25) and using the fact that due to (2.22) we have $\mathfrak{d}_\varrho(\delta_p, \delta_p) = 0$, we get

$$\lim_{u\to\infty} \tilde{\mathcal{M}}(\eta_{p_{i(u)}}, \delta_p) = \max\{0, 0, \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p) + 0]\} = \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p) \quad (2.26)$$

and

$$\lim_{u\to\infty} \tilde{\mathcal{N}}(\eta_{p_{i(u)}}, \delta_p) = 0 \quad (2.27)$$

Now, taking $u \to \infty$ in (2.23) and using (2.26), (2.27) and definitions of $\hat{\psi}$, $\hat{\varphi}$ and $\hat{\vartheta}$ we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p)) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p)) < \hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p))$$

which is a contradiction. Therefore $\Gamma\delta_p = \delta_p$ i.e. δ_p is a fixed point.
Further, suppose δ_p and δ_q be two fixed point of Γ such that $\mathfrak{d}_\varrho(\delta_p, \delta_q) > 0$ and $\hat{\alpha}(\delta_p, \delta_q) \geq 1$ then replacing η_p by δ_p and ζ_p by δ_q in (2.1) we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \delta_q)) = \hat{\psi}(\mathfrak{d}_\varrho(\Gamma\delta_p, \Gamma\delta_q)) \leq \hat{\alpha}(\delta_p, \delta_q)\mathfrak{d}_\varrho(\Gamma\delta_p, \Gamma\delta_q)$$
$$\leq \hat{\varphi}(\tilde{\mathcal{M}}(\delta_p, \delta_q)) + L(\hat{\vartheta}(\tilde{\mathcal{N}}(\delta_p, \delta_q))) \quad (2.28)$$

Where

$$\tilde{\mathcal{M}}(\delta_p, \delta_q) = \max\left\{\mathfrak{d}_\varrho(\delta_p, \delta_q), \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p), \mathfrak{d}_\varrho(\delta_q, \Gamma\delta_q), \frac{1}{2}[\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_q) + \mathfrak{d}_\varrho(\delta_q, \Gamma\delta_p)]\right\}$$
$$= \max\left\{\mathfrak{d}_\varrho(\delta_p, \delta_q), \mathfrak{d}_\varrho(\delta_p, \delta_p), \mathfrak{d}_\varrho(\delta_q, \delta_q), \frac{1}{2}[\mathfrak{d}_\varrho(\delta_p, \delta_q) + \mathfrak{d}_\varrho(\delta_q, \delta_p)]\right\}$$
$$= \max\left\{\mathfrak{d}_\varrho(\delta_p, \delta_q), 0, 0, \frac{1}{2}[\mathfrak{d}_\varrho(\delta_p, \delta_q) + \mathfrak{d}_\varrho(\delta_p, \delta_q)]\right\} \textit{by(WPMS2)}$$
$$= \mathfrak{d}_\varrho(\delta_p, \delta_q) \quad (2.29)$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(\delta_p, \delta_q) &= \min\{\mathfrak{d}^m_\varrho(\delta_p, \Gamma\delta_p), \mathfrak{d}^m_\varrho(\delta_q, \Gamma\delta_p)\}\\ &= \min\{\mathfrak{d}^m_\varrho(\delta_p, \delta_p), \mathfrak{d}^m_\varrho(\delta_q, \delta_p)\}\\ &= 0 \end{aligned} \tag{2.30}$$

By putting (2.29), (2.30) in (2.28) and using the definitions of $\hat{\psi}, \hat{\varphi}$ and $\hat{\vartheta}$ we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \delta_q)) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\delta_p, \delta_q)) < \hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \delta_q))$$

This is contradictory. As a result, Γ has a unique fixed point. The evidence is now complete.

■

The theorems' consequences are given below.

Corollary 2.4. *Let $(W_p, \mathfrak{d}_\varrho)$ be a complete weak partial metric space. $\Gamma : W_p \to W_p$ satisfy the criterion by self-mapping with*

$$\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p)) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\eta_p, \zeta_p)) + L(\tilde{\mathcal{N}}(\eta_p, \zeta_p)) \tag{2.31}$$

For all $\eta_p, \zeta_p \in W_p$, $\hat{\psi} \in \Psi$, $\hat{\varphi} \in \Phi$ and $L \geq 0$. Then Γ has a unique fixed point in W_p.

Corollary 2.5. *Let $(W_p, \mathfrak{d}_\varrho)$ be a complete weak partial metric space. A self-mapping $\Gamma : W_p \to W_p$ be such that*

$$\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p) \leq k(\tilde{\mathcal{M}}(\eta_p, \zeta_p))$$

For all $\eta_p, \zeta_p \in W_p$, $k \in (0, 1)$, where

$$\tilde{\mathcal{M}}(\eta_p, \zeta_p) = \max\left\{\mathfrak{d}_\varrho(\eta_p, \zeta_p), \mathfrak{d}_\varrho(\eta_p, \Gamma\eta_p), \mathfrak{d}_\varrho(\zeta_p, \Gamma\zeta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p, \Gamma\zeta_p) + \mathfrak{d}_\varrho(\zeta_p, \Gamma\eta_p)]\right\} \tag{2.32}$$

Then Γ has a unique fixed point in W_p.

Example 2.6. *Let $W_p = [0, 1]$ and $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \frac{1}{2}(\eta_p + \zeta_p)$, Then $\mathfrak{d}^m_\varrho(\eta_p, \zeta_p) = \frac{1}{2}|\eta_p - \zeta_p|$. Therefore, since $(W_p, \mathfrak{d}^m_\varrho)$ is complete, the by Lemma 1.12 $(W_p, \mathfrak{d}_\varrho)$ is a complete weak partial metric space (WPMS).*

Consider the mapping $\Gamma : W_p \to W_p$ defined by $\Gamma(\eta_p) = \frac{\eta_p}{3}$ and let $\hat{\psi}, \hat{\varphi}, \hat{\vartheta} : [0, \infty) \to [0, \infty)$ be such that $\hat{\psi}(u) = 2u$, $\hat{\varphi}(u) = \frac{2u}{3}$ and $\hat{\vartheta}(u) = u$ for all $u \geq 0$. If we define the functions $\hat{\alpha} : W_p \times W_p \to [0, \infty)$ as

$$\hat{\alpha}(\eta_p, \zeta_p) = \begin{cases} 1 & \eta_p, \zeta_p \in [0, \frac{1}{2}] \\ o & \eta_p, \zeta_p \in (\frac{1}{2}, 1] \end{cases} \tag{2.33}$$

We show that contractive condition of Theorem 2.2 is satisfied.
Let $\eta_p, \zeta_p \in [0, \frac{1}{2}]$ we get

$$\begin{aligned}\hat{\alpha}(\eta_p, \zeta_p)\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p)) &= \hat{\alpha}(\eta_p, \zeta_p)\hat{\psi}(\mathfrak{d}_\varrho(\frac{\eta_p}{3}, \frac{\zeta_p}{3}))\\ &= \hat{\psi}(\frac{1}{2}(\frac{\eta_p + \zeta_p}{3}))\\ &= \frac{2}{3}\mathfrak{d}_\varrho(\eta_p, \zeta_p) \end{aligned} \tag{2.34}$$

On the other side

$$\begin{aligned}\tilde{\mathcal{M}}(\eta_p,\zeta_p) &= \max\big\{\mathfrak{d}_\varrho(\eta_p,\zeta_p),\mathfrak{d}_\varrho(\eta_p,\Gamma\eta_p),\mathfrak{d}_\varrho(\zeta_p,\Gamma\zeta_p),\frac{1}{2}[\mathfrak{d}_\varrho(\eta_p,\Gamma\zeta_p)+\mathfrak{d}_\varrho(\zeta_p,\Gamma\eta_p)]\big\}\\ &= \max\big\{\mathfrak{d}_\varrho(\eta_p,\zeta_p),\mathfrak{d}_\varrho(\eta_p,\frac{\eta_p}{3}),\mathfrak{d}_\varrho(\zeta_p,\frac{\zeta_p}{3}),\frac{1}{2}[\mathfrak{d}_\varrho(\eta_p,\frac{\zeta_p}{3})+\mathfrak{d}_\varrho(\zeta_p,\frac{\eta_p}{3})]\big\}\\ &= \max\big\{\frac{\eta_p+\zeta_p}{2},\frac{2\eta_p}{3},\frac{2\zeta_p}{3},\frac{\eta_p+\zeta_p}{3}\big\}\\ &= \frac{\eta_p+\zeta_p}{2} = \mathfrak{d}_\varrho(\eta_p,\zeta_p) \end{aligned} \tag{2.35}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(\eta_p,\zeta_p) &= \min\{\mathfrak{d}^m_\varrho(\eta_p,\Gamma\zeta_p),\mathfrak{d}^m_\varrho(\zeta_p,\Gamma\eta_p)\}\\ &= \min\{\mathfrak{d}^m_\varrho(\eta_p,\frac{\eta_p}{3}),\mathfrak{d}^m_\varrho(\zeta_p,\frac{\eta_p}{3})\}\end{aligned} \tag{2.36}$$

Therefore from (2.35) we get

$$\begin{aligned}\hat{\varphi}(\tilde{\mathcal{M}}(\eta_p,\zeta_p))+L(\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_p,\zeta_p))) &= \hat{\varphi}(\frac{\eta_p+\zeta_p}{2})+L(\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_p,\zeta_p)))\\ &= \frac{2}{3}(\frac{\eta_p+\zeta_p}{2})+L(\tilde{\mathcal{N}}(\eta_p,\zeta_p))\\ &= \frac{2}{3}\mathfrak{d}_\varrho(\eta_p,\zeta_p)+L(\tilde{\mathcal{N}}(\eta_p,\zeta_p))\end{aligned} \tag{2.37}$$

Now since $L(\tilde{\mathcal{N}}(\eta_p,\zeta_p)) = L(\min\{\mathfrak{d}^m_\varrho(\eta_p,\frac{\eta_p}{3}),\mathfrak{d}^m_\varrho(\zeta_p,\frac{\eta_p}{3})\}) \geq 0$ for all $\eta_p,\zeta_p \in W_p$, and from (2.34) and (2.37) we get

$$\frac{2}{3}\mathfrak{d}_\varrho(\eta_p,\zeta_p) \leq \frac{2}{3}\mathfrak{d}_\varrho(\eta_p,\zeta_p)+L(\tilde{\mathcal{N}}(\eta_p,\zeta_p)) \tag{2.38}$$

for all $\eta_p,\zeta_p \in W_p$.

Now, let $\eta_p,\zeta_p \in (\frac{1}{2},1]$, in this case the contractive conditions of theorem 2.2 is already satisfied since $\hat{\alpha}(\eta_p,\zeta_p)=0$. It is clear that all the conditions of Theorem 2.2 hold. Hence Γ has a fixed point, which in this case is 0.

Example 2.7. *Let $W_p=[0,1]$ and $\mathfrak{d}_\varrho(\eta_p,\zeta_p)=\frac{1}{2}(\eta_p+\zeta_p)$, Then $\mathfrak{d}^m_\varrho(\eta_p,\zeta_p)=\frac{1}{2}|\eta_p-\zeta_p|$. Therefore, since $(W_p,\mathfrak{d}^m_\varrho)$ is complete, the by lemma 1.12 $(W_p,\mathfrak{d}_\varrho)$ is a complete weak partial metric space (WPMS).*

Consider the mapping $\Gamma: W_p \to W_p$ defined by $\Gamma(\eta_p)=\begin{cases}\eta_p^2 & \eta_p \in [0,\frac{1}{2}]\\ o & \eta_p \in (\frac{1}{2},1]\end{cases}$ and let $\hat{\psi},\hat{\varphi}:[0,\infty)\to[0,\infty)$ be such that $\hat{\psi}(u)=u$, $\hat{\varphi}(u)=\frac{u}{2}$ for all $u \geq 0$.
Now, we show that contractive condition of corollary 2.4 is satisfied for $L=1$, i.e.,

$$\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p,\Gamma\zeta_p)) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\eta_p,\zeta_p))+L(\tilde{\mathcal{N}}(\eta_p,\zeta_p)) \tag{2.39}$$

for all $\eta_p,\zeta_p \in W_p$. Let $\eta_p,\zeta_p \in [0,\frac{1}{2}]$, then

$$\begin{aligned}\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p,\Gamma\zeta_p)) &= \hat{\psi}(\frac{\eta_p^2+\zeta_p^2}{2}) = \frac{\eta_p^2+\zeta_p^2}{2} \leq \frac{1}{2}(\frac{\eta_p+\zeta_p}{2}) = \frac{1}{2}\mathfrak{d}_\varrho(\eta_p,\zeta_p)\\ &\leq \frac{1}{2}\mathfrak{d}_\varrho(\eta_p,\zeta_p)+\min\{\mathfrak{d}^m_\varrho(\eta_p,\Gamma\eta_p),\mathfrak{d}^m_\varrho(\zeta_p,\Gamma\eta_p)\}\\ &= \hat{\varphi}(\mathfrak{d}_\varrho(\eta_p,\zeta_p))+\min\{\mathfrak{d}^m_\varrho(\eta_p,\Gamma\eta_p),\mathfrak{d}^m_\varrho(\zeta_p,\Gamma\eta_p)\}\end{aligned}$$

Now, let $\eta_p, \zeta_p \in (\frac{1}{2}, 1]$, then result is clear since in this case $\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p) = 0$. As a result, all requirements of corollary 2.4 are completely satisfied. As a result, it has a fixed point, which in this instance is 0.

Now, we demonstrate that the contractive requirement of Corollary 2.5 is met.

Example 2.8. *Let $W_p = [0,1]$ and $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \frac{1}{2}(\eta_p + \zeta_p)$, Then $\mathfrak{d}^m_\varrho(\eta_p, \zeta_p) = \frac{1}{2}|\eta_p - \zeta_p|$. Therefore, since $(W_p, \mathfrak{d}^m_\varrho)$ is complete, the by lemma 1.12 $(W_p, \mathfrak{d}_\varrho)$ is a complete weak partial metric space (WPMS).*

Consider the mapping $\Gamma : W_p \to W_p$ defined by $\Gamma(\eta_p) = \frac{\eta_p}{3}$ Then

$$\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p) = \mathfrak{d}_\varrho(\frac{\eta_p}{3}, \frac{\zeta_p}{3}) = \frac{1}{3}\mathfrak{d}_\varrho(\eta_p, \zeta_p) \tag{2.40}$$

On the other hand side

$$\begin{aligned}
\tilde{\mathcal{M}}(\eta_p, \zeta_p) &= \max\{\mathfrak{d}_\varrho(\eta_p, \zeta_p), \mathfrak{d}_\varrho(\eta_p, \Gamma\eta_p), \mathfrak{d}_\varrho(\zeta_p, \Gamma\zeta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p, \Gamma\zeta_p) + \mathfrak{d}_\varrho(\zeta_p, \Gamma\eta_p)]\} \\
&= \max\{\mathfrak{d}_\varrho(\eta_p, \zeta_p), \mathfrak{d}_\varrho(\eta_p, \frac{\eta_p}{3}), \mathfrak{d}_\varrho(\zeta_p, \frac{\zeta_p}{3}), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p, \frac{\zeta_p}{3}) + \mathfrak{d}_\varrho(\zeta_p, \frac{\eta_p}{3})]\} \\
&= \max\{\frac{\eta_p + \zeta_p}{2}, \frac{2\eta_p}{3}, \frac{2\zeta_p}{3}, \frac{\eta_p + \zeta_p}{3}\} \\
&= \frac{\eta_p + \zeta_p}{2} = \mathfrak{d}_\varrho(\eta_p, \zeta_p)
\end{aligned} \tag{2.41}$$

From (2.40) and (2.41) we get

$$\frac{1}{3}\mathfrak{d}_\varrho(\eta_p, \zeta_p) \leq k\mathfrak{d}_\varrho(\eta_p, \zeta_p) \tag{2.42}$$

for $k \in [\frac{1}{3}, 1)$. i.e.

$$\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p) \leq k(\tilde{\mathcal{M}}(\eta_p, \zeta_p))$$

for $k \in [\frac{1}{3}, 1)$.It is evident from (2.42) that it satisfies the requirement of Corollary 2.5. As a result, it has a fixed point, which in this instance is 0.

3. Conclusion

In this study, we proved certain fixed point theorems in the context of complete weak partial metric spaces using triangular $\hat{\alpha}$-admissible mappings and provided some implications of the main findings. We included some examples to support our results. The results in this article expand upon and generalise several results from the existing literature.

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