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On the DNA codes

ABDULLAH DERTLI *[1], YASEMIN CENGELLENMIS [2]

[1] *Department of Mathematics, Ondokuz Mayıs University Samsun, Turkey.*
[2] *Department of Mathematics, Trakya University, Edirne, Turkey.*

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Abstract. In this paper, by using three different methods, the DNA codes are obtained from some codes over a family of the rings $D_i = D_1 [w_2, ..., w_i] / \langle w_i^2 - w_i, w_i w_j - w_j w_i \rangle$, where $i = 2, ..., r, j = 1, 2, ...r$ and $D_1 = F_2 + uF_2 + w_1(F_2 + uF_2), u^2 = 0, w_1^2 = w_1, uw_1 = w_1 u, F_2 = \{0, 1\}$.

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Contents

1. Introduction

The transmission and storage of information take place in digital platform and the coding theory is necessary in order to correct and detect errors in the platform. There is another platform. In the platform, the correcting and detecting errors are necessary but it does not take place in digital. It is DNA.

It is well known that DNA contains genetic program for the biological development of life and has two strands which are linked by Watson-Crick pairing so that every A is linked with a T and every C with a G, and vice versa, where A, T, C, G are the four bases of a DNA sequence.

The idea of computing with DNA was given by T. Head in [7]. L. Adleman performed the computation using DNA strands in [1].

To perform computation using DNA strands, a specific set of DNA sequences are required with particular properties. The aim of this paper is to obtain the set of DNA strands satisfying various constraints, by using the some error correcting codes over a family of finite rings which enjoy DNA properties. One of the constraints is

*Corresponding author. Email address: **abdullah.dertli@gmail.com** (Abdullah DERTLI), **ycengellenmis@gmail.com** (Yasemin CENGELLENMIS)

reverse constaint. This leads to reversible codes.The other one is reverse complement constraint. This leads to reversible complement codes.

In order to obtain reversible DNA codes, some authors considered skew cyclic codes. The reversibility problem for DNA 8-bases and DNA $2^{s+1}k$-bases is solved in [4] and [5] respectively by using skew cyclic codes over the finite rings $F_{16} + uF_{16} + vF_{16} + uvF_{16}$ where $u^2 = u, v^2 = v, uv = vu$ and $F_{4^{2k}}[u_1, ..., u_s]/\langle u_1{}^2 - u_1, ..., u_s{}^2 - u_s\rangle$ where $k, s \geqslant 1, u_iu_j = u_ju_i$. Reversibility problem arises from the fact that the pairing of nucleotides in two different strands of a DNA sequence is done in opposite direction and reverse order. For example, let us consider the codeword (DNA string) $GTTAGGCA$ which corresponds to a codeword (a_1, a_2). The reverse of (a_1, a_2) is (a_2, a_1). However, the vector (a_2, a_1) corresponds to $GGCAGTTA$ which is not the reverse of $GTTAGGCA$. The reverse of $GTTAGGCA$ is $ACGGATTG$.

In order to obtain the DNA codes, some authors used cyclic DNA codes of length n that enjoy some of the properties of DNA. In [9], by introducing a map, a family of cyclic codes over the ring $F_2[u]/ < u^4 - 1 >$ is mapped to DNA codes.

In [10], the design of linear codes over $D_1 = F_2 + uF_2 + vF_2 + uvF_2, u^2 = 0, v^2 = v, uv - vu, F_2 = \{0, 1\}$ is presented by using σ-set, where σ is a nontrivial automorphism on the finite ring D_1. By using these linear codes, the authors obtained DNA codes with the other method.

In this paper, firstly, a non-trivial automorphism θ_i over $D_i = D_{i-1} + w_iD_{i-1}$, where $i = 2, 3, ..., r, w_i^2 = w_i, D_1 = F_2 + uF_2 + w_1(F_2 + uF_2), u^2 = 0, w_1^2 = w_1, uw_1 = w_1u, F_2 = \{0, 1\}$ is defined. By introducing skew cyclic codes over a family of the finite rings $D_i = D_1 [w_2, ..., w_i] / \langle w_i^2 - w_i, w_iw_j - w_jw_i\rangle$, where $i = 2, ..., r, j = 1, 2, ..., r$, the reversible DNA codes are obtained from them. With the other method, the necessary and sufficient conditions of cyclic codes over D_i, where $i = 1, ..., r$ to be reversible and reversible complement are given. By introducing a map, the DNA codes are obtained from these type codes. As a last, the linear codes over D_i are designed, by using θ_i-set for $i = 2, 3, ..., r$. By using these type codes, the reversible or reversible complement DNA codes are obtained.

2. Preliminaries

A family of the finite rings $D_i = D_{i-1} + w_iD_{i-1}$, where $i = 2, 3, ..., r, w_i^2 = w_i, D_1 = F_2 + uF_2 + w_1(F_2 + uF_2), u^2 = 0, w_1^2 = w_1, uw_1 = w_1u$ contains the commutative finite rings with characteristic 2 and cardinality 4^{2^i} for $i = 1, 2, ..., r$.

The finite rings of the family are written as recursively

$$D_i = D_{i-1} + w_iD_{i-1}$$

where $i = 2, 3, ..., r, w_i^2 = w_i, D_1 = F_2 + uF_2 + w_1(F_2 + uF_2), u^2 = 0, w_1^2 = w_1, uw_1 = w_1u, F_2 = \{0, 1\}$.

In [10], the map φ_1 was defined as follows

$$\begin{aligned} \varphi_1 \quad &: \quad D_1 \longrightarrow (F_2 + uF_2)^2 \\ a + bw_1 &\longmapsto (a, a + b) \end{aligned}$$

where $a, b \in F_2 + uF_2, u^2 = 0, w_1^2 = w_1$.

We define the map on D_i where $i = 2, ..., r$ as follows

$$\begin{aligned} \varphi_i \quad &: \quad D_i \longrightarrow D_{i-1}^2 \\ x_{i-1} + y_{i-1}w_i &\longmapsto (x_{i-1}, x_{i-1} + y_{i-1}) \end{aligned}$$

where $x_{i-1}, y_{i-1} \in D_{i-1}, w_i^2 = w_i$ for $i = 2, 3, ..., r$.

In [10], they defined a ξ_1 correspondence between the elements of the finite ring $D_1 = F_2 + uF_2 + w_1F_2 + uw_1F_2$, where $u^2 = 0, w_1^2 = w_1, uw_1 = w_1u$ and DNA double pairs as follows

Elements α	Gray images	DNA double pairs $\xi_1(\alpha)$
0	$(0,0)$	AA
1	$(1,1)$	GG
u	(u,u)	TT
w_1	$(0,1)$	AG
uw_1	$(0,u)$	AT
$1+u$	$(1+u,1+u)$	CC
$1+w_1$	$(1,0)$	GA
$u+w_1$	$(u,1+u)$	TC
$u+uw_1$	$(u,0)$	TA
w_1+uw_1	$(0,1+u)$	AC
$1+uw_1$	$(1,1+u)$	GC
$1+u+uw_1$	$(1+u,1)$	CG
$1+u+w_1$	$(1+u,u)$	CT
$1+w_1+uw_1$	$(1,u)$	GT
$u+w_1+uw_1$	$(u,1)$	TG
$1+u+w_1+uw_1$	$(1+u,0)$	CA

By using the map φ_2 and ξ_1, we established ξ_2 correspondence between the element of D_2 and DNA 4-bases $x_1 + y_1w_2 \longmapsto (\xi_1(x_1), \xi_1(x_1+y_1))$ as follows

Elements β	DNA 4-bases $\xi_2(\beta)$
0	$AAAA$
1	$GGGG$
u	$TTTT$
w_1	$AGAG$
w_2	$AAGG$
...	

By using the matching and the elements of D_1 and $S_{D_{16}} = \{AA, TT, ..., GG\}$ and by using the Gray map from D_i to D_{i-1}^2, we can define ξ_i correspondence between the elements of the finite ring D_i and DNA 2^i-bases for $i = 2, ..., r$ as follows

$$\begin{aligned} \xi_i \quad &: \quad D_i \longrightarrow D_{i-1}^2 \longrightarrow \{A,T,G,C\}^{2^i} \\ x_{i-1} + y_{i-1}w_i &\longmapsto (x_{i-1}, x_{i-1}+y_{i-1}) \longmapsto q \end{aligned}$$

where $q = (\xi_{i-1}(x_{i-1}), \xi_{i-1}(x_{i-1}+y_{i-1}))$.

It can be written that $\xi_i = \gamma_i\varphi_i$, where a map γ_i is defined from D_{i-1}^2 to 2^i-bases as follows,

$$\gamma_i(s_{i-1}, t_{i-1}) = (\xi_{i-1}(s_{i-1}), \xi_{i-1}(t_{i-1}))$$

where $s_{i-1}, t_{i-1} \in D_{i-1}$ for $i = 2, ..., r$.

In [10], a nontrivial automorphism was defined on D_1 as follows

$$\begin{aligned} \theta_1 \quad &: \quad D_1 \longrightarrow D_1 \\ x_0 + y_0w_1 &\longmapsto x_0 + (1+w_1)y_0 \end{aligned}$$

where $x_0, y_0 \in F_2 + uF_2, u^2 = 0$.

By defining a nontrivial automorphism on D_i as follows, for $i = 2, ..., r$, we can define the skew cyclic codes over D_i, for $i = 2, ..., r$.

$$\begin{aligned} \theta_i \quad &: \quad D_i \longrightarrow D_i \\ x_{i-1} + y_{i-1}w_i &\longmapsto \theta_i(x_{i-1} + y_{i-1}w_i) = l \end{aligned}$$

where $l = \theta_{i-1}(x_{i-1}) + (1 + w_i)\theta_{i-1}(y_{i-1})$ and $x_{i-1}, y_{i-1} \in D_{i-1}$, for $i = 2, ..., r$.

The order of θ_i, for $i = 1, 2, ..., r$ is 2.

The rings

$$D_i[x, \theta_i] = \{b_0^i + b_1^i x + ... + b_{n-1}^i x^{n-1} : b_j^i \in D_i, n \in N, i = 2, ..., k, j = 0, 1, ..., n-1\}$$

are skew polynomial rings with the usual polynomial addition and the multiplication as follows

$$(a_i x^s)(b_i x^j) = a_i \theta_i^s(b_i) x^{s+j}$$

where $a_i, b_i \in D_i$, for $i = 1, ..., r$. They are non-commutative rings.

Definition 2.1. *A subset C_i of D_i^n, where $i = 1, ..., r$ is called a skew cyclic code of length n if C_i satisfies the following conditions,*

1. *C_i is a submodule of D_i^n*
2. *If $\mathbf{c}_i = (c_0^i, c_1^i, ..., c_{n-1}^i) \in C_i$, then $\theta_i(\mathbf{c}_i) = \left(\theta_i(c_{n-1}^i), \theta_i(c_0^i), ..., \theta_i(c_{n-2}^i)\right) \in C_i$, where θ_i is the skew cyclic shift operator.*

In polynomial representation, a skew cyclic code of length n over D_i is defined as a left ideal of the quotient ring $D_{i_{\theta_i,n}} = D_i[x, \theta_i]/\langle x^n - 1\rangle$, if the order of θ_i divides n, that is, if n is even. If the order of θ_i does not divide n, a skew cyclic code of length n over D_i is defined as a left $D_i[x, \theta_i]$-submodule of $D_{i_{\theta_i,n}}$, since the set $D_{i_{\theta_i,n}} = D_i[x, \theta_i]/\langle x^n - 1\rangle = \{f_i(x) + \langle x^n - 1\rangle : f_i(x) \in D_i[x, \theta_i]\}$ is a left $D_i[x, \theta_i]$-module with the multiplication from left defined by

$$r_i(x)(f_i(x) + \langle x^n - 1\rangle) = r_i(x)f_i(x) + \langle x^n - 1\rangle$$

for any $r_i(x) \in D_i[x, \theta_i]$.

In either case, the following holds.

Theorem 2.2. *Let C_i be a skew cyclic code over D_i and let $f_i(x)$ be a polynomial in C_i of minimal degree. If the leading coefficient of $f_i(x)$ is a unit in D_i, then $C_i = \langle f_i(x)\rangle$, where $f_i(x)$ is a right divisor of $x^n - 1$.*

3. Reversible DNA codes

In this section, the reversible DNA codes are obtained by using the skew cyclic codes over D_i for $i = 1, 2, ..., r$.

Definition 3.1. *For $\mathbf{x}_i = \left(x_0^i, x_1^i, ..., x_{n-1}^i\right) \in D_i^n$, the vector $\left(x_{n-1}^i, x_{n-2}^i, ..., x_1^i, x_0^i\right)$ is called the reverse of $\mathbf{x}_i$ and is denoted by $\mathbf{x}_i^r$. A linear code C_i of length n over D_i is said to be reversible if $\mathbf{x}_i^r \in C_i$ for every $\mathbf{x}_i \in C_i$, where $i = 1, 2, ..., r$.*

We can express the matching the elements of D_1 and $S_{D_{16}} = \{AA, TT, ..., GG\}$ by means of the automorphism θ_1 as follows.

Each element $\alpha_1 = x + yw_1 \in D_1$, where $x, y \in F_2 + uF_2, u^2 = 0$ and $\theta_1(\alpha_1)$ are mapped to DNA double pairs which are reverse of each other. Since a correspondence the elements of the finite ring D_1 and DNA double pairs is ξ_1, so we have $\xi_1(w_1) = AG$, while $\xi_1(\theta_1(w_1)) = GA$.

By using a map $\xi_i = \gamma_i \circ \varphi_i$, where the map γ_i is from D_{i-1}^2 to 2^i-bases as follows,

$$\gamma_i(s_{i-1}, t_{i-1}) = (\xi_{i-1}(s_{i-1}), \xi_{i-1}(t_{i-1}))$$

where $s_{i-1}, t_{i-1} \in D_{i-1}$ for $i = 2, ..., r$, we can explain a relationship between skew cyclic codes and DNA codes. $\xi_i(s_i)$ and $\xi_i(\theta_i(s_i))$ are DNA reverse of each other $s_i = a_{i-1} + w_i b_{i-1}, a_{i-1}, b_{i-1} \in D_{i-1}$, where $a_{i-1}, b_{i-1} \in D_{i-1}, i = 2, ..., r$.

For $s_i = a_{i-1} + w_i b_{i-1} \in D_i, i = 2, ..., r$, we have

$$\begin{aligned} \xi_i(s_i) &= \gamma_i(\varphi_i(a_{i-1} + w_i b_{i-1})) \\ &= \gamma_i(a_{i-1}, a_{i-1} + b_{i-1}) \\ &= (\xi_{i-1}(a_{i-1}), \xi_{i-1}(a_{i-1} + b_{i-1})). \end{aligned}$$

On the other hand,

$$\begin{aligned} \xi_i(\theta_i(s_i)) &= \xi_i(\theta_{i-1}(a_{i-1}) + (1 + w_i)\theta_{i-1}(b_{i-1})) \\ &= \xi_i(\theta_{i-1}(a_{i-1} + b_{i-1}) + w_i\theta_{i-1}(b_{i-1})) \\ &= \gamma_i(\varphi_i(\theta_{i-1}(a_{i-1} + b_{i-1}) + w_i\theta_{i-1}(b_{i-1}))) \\ &= \gamma_i(\theta_{i-1}(a_{i-1} + b_{i-1}), \theta_{i-1}(a_{i-1})) \\ &= (\xi_{i-1}(\theta_{i-1}(a_{i-1} + b_{i-1})), \xi_{i-1}(\theta_{i-1}(a_{i-1}))) \end{aligned}$$

where $i = 2, ..., r$.

This map can be extended as follows. For any $\mathbf{d}_i = (d_0^i, ..., d_{n-1}^i) \in D_i^n$, where $i = 2, ..., r$

$$\left(\xi_i(d_0^i), \xi_i(d_1^i), \ldots, \xi_i(d_{n-1}^i)\right)^r = \left(\xi_i(\theta_i(d_{n-1}^i)), \ldots, \xi_i(\theta_i(d_1^i)), \xi_i(\theta_i(d_0^i))\right).$$

Example 3.2. *Let $i = 2$. If $d_2 = (1 + uw_1) + w_2(1 + u + w_1) \in D_2$, then we get*

$$\begin{aligned} \xi_2(d_2) &= \gamma_2(\varphi_2(d_2)) = \gamma_2(1 + uw_1, u + w_1 + uw_1) \\ &= (\xi_1(1 + uw_1), \xi_1(u + w_1 + uw_1)) = GCTG. \end{aligned}$$

On the other hand,

$$\begin{aligned} \xi_2(\theta_2(d_2)) &= \xi_2(\theta_1(1 + uw_1) + (1 + w_2)\theta_1(1 + u + w_1)) \\ &= \xi_2(\theta_1(u + w_1 + uw_1) + w_2\theta_1(1 + u + w_1)) \\ &= \gamma_2(\varphi_2(\theta_1(u + w_1 + uw_1) + w_2\theta_1(1 + u + w_1))) \\ &= \gamma_2(\theta_1(u + w_1 + uw_1), \theta_1(1 + uw_1)) \\ &= (\xi_1(\theta_1(u + w_1 + uw_1)), \xi_1(\theta_1(1 + uw_1))) \\ &= GTCG. \end{aligned}$$

Example 3.3. *Let $i = 3$. If $d_3 = [(1 + uw_1) + w_2(1 + u + w_1)] + w_3(1 + w_2) \in D_3$, then we get*

$$\begin{aligned} \xi_3(d_3) &= \gamma_3(\varphi_3(d_3)) = \gamma_3((1 + uw_1) + w_2(1 + u + w_1), uw_1 + w_2(u + w_1)) \\ &= (\xi_2((1 + uw_1) + w_2(1 + u + w_1)), \xi_2(uw_1 + w_2(u + w_1))) \\ &= GCTGATTG. \end{aligned}$$

On the other hand,

$$\begin{aligned}
\xi_3\left(\theta_3(d_3)\right) &= \xi_3\left(\theta_2((1+uw_1)+w_2(1+u+w_1))+(1+w_3)\theta_2(1+w_2)\right) \\
&= \xi_3\left(\theta_2(uw_1+w_2(u+w_1))+w_3\theta_2(1+w_2)\right) \\
&= \gamma_3(\varphi_3\left(\theta_2(uw_1+w_2(u+w_1))+w_3\theta_2(1+w_2)\right)) \\
&= \gamma_3\left(\theta_2(uw_1+w_2(u+w_1)),\theta_2(1+uw_1+w_2(1+u+w_1))\right) \\
&= \left(\xi_2\left(\theta_2(uw_1+w_2(u+w_1))\right),\xi_2\left(\theta_2(1+uw_1+w_2(1+u+w_1))\right)\right) \\
&= GTTAGTCG.
\end{aligned}$$

Definition 3.4. *Let C_i be a code of length n over D_i, for $i=1,...,r$. If $\xi_i(\mathbf{d}_i)^r \in \xi_i(C_i)$ for all $\mathbf{d}_i \in C_i$, then C_i or equivalently $\xi_i(C_i)$ is called a reversible DNA code.*

Definition 3.5. *Let $g_i(x)=a_0^i+a_1^ix+a_2^ix^2+...+a_s^ix^s$ be a polynomial of degree s over D_i. $g_i(x)$ is called a palindromic polynomial if $a_t^i=a_{s-t}^i$ for all $t\in\{0,1,...,s\}$. $g_i(x)$ is called a θ_i-palindromic polynomial if $a_t^i=\theta_i(a_{s-t}^i)$ for all $t\in\{0,1,...,s\}$, for $i=1,...,r$.*

As the order of θ_i is 2, a skew cyclic code of odd length n over D_i with respect to θ_i is an ordinary cyclic code. So we will take the length n to be even.

The next two theorems show that palindromic and θ_i-palindromic polynomials generate reversible DNA codes.

Theorem 3.6. *Let $C_i=\langle f_i(x)\rangle$ be a skew cyclic code of length n over D_i, where $f_i(x)$ is a right divisor of x^n-1 and $\deg(f_i(x))$ is odd. If $f_i(x)$ is a θ_i-palindromic polynomial, then $\xi_i(C_i)$ is a reversible DNA code, for $i=1,...,r$.*

Proof. Let $f_i(x)$ be a θ_i-palindromic polynomial and $f_i(x)=a_0^i+a_1^ix+...+a_{2s-1}^ix^{2s-1}$. So $a_t^i=\theta_i(a_{2s-1-t}^i)$, for all $t=0,1,...,s-1$. Let $h_i(x)=h_0^i+h_1^ix+\cdots+h_{2k-1}^ix^{2k-1}$. Let b_l^i be the coefficient of x^l in $h_i(x)f_i(x)$, where $l=0,1,\ldots,n-1$. For any $p<n/2$, the coefficient of x^p in $h_i(x)f_i(x)$ is

$$b_p^i=\sum_{j=0}^{p}h_j^i\theta_i^j(a_{p-j}^i)$$

and the coefficient of x^{n-p} is

$$b_{n-p}^i=\sum_{j=0}^{p}h_{2k-1-j}^i\theta_i^{2k-1-j}(a_{2s-1-(p-j)}^i).$$

The polynomial $h_i(x)f_i(x)=\sum_{d=0}^{2k-1}h_d^ix^df_i(x)$ corresponds to a vector $\mathbf{b}_i=(b_0^i,b_1^i,...,b_{n-1}^i)\in C_i$.

The vector $\xi_i(\mathbf{b}_i)^r=((\xi_i(b_0^i),...,\xi_i(b_{n-1}^i)))^r$ is equal to the vector $\xi_i(\mathbf{z}_i)$, where the vector $\mathbf{z}_i$ corresponds to the polynomial $\sum_{d=0}^{2k-1}\theta_i(h_d^i)x^{2k-1-d}f_i(x)$.

So $\xi_i(C_i)$ is a reversible DNA code. ■

Theorem 3.7. *Let $C_i=\langle f_i(x)\rangle$ be a skew cyclic code of length n over D_i, where $f_i(x)$ is a right divisor of x^n-1 and $\deg(f_i(x))$ is even. If $f_i(x)$ is a palindromic polynomial, then $\xi_i(C_i)$ is a reversible DNA code, for $i=1,...,r$.*

Proof. Let $f_i(x)$ be a palindromic polynomial with even degree so that $f_i(x)=a_0^i+a_1^ix+...+a_{2s}^ix^{2s}$ and $a_t^i=a_{2s-t}^i$, for all $t=0,1,...,s$. Let $h_i(x)=h_0^i+h_1^ix+\cdots+h_{2k}^ix^{2k}$. Let b_l^i be the coefficient of x^l in $h_i(x)f_i(x)$, where $l=0,1,\ldots,n-1$. For any $p<n/2$, the coefficient of x^p in $h_i(x)f_i(x)$ is

$$b^i_p = \sum_{j=0}^{p} h^i_j \theta^j_i (a^i_{p-j})$$

and the coefficient of x^{n-p} is

$$b^i_{n-p} = \sum_{j=0}^{t} h^i_{(2k)-j} \theta_i^{(2k)-j} (a^i_{2s-(p-j)}).$$

The polynomial $h_i(x)f_i(x) = \sum_{d=0}^{2k} h^i_d x^d f_i(x)$ corresponds to a vector $\mathbf{b}_i = (b^i_0, b^i_1, ..., b^i_{n-1}) \in C_i$.

The vector $\xi_i(\mathbf{b}_i)^r = ((\xi_i(b^i_0), ..., \xi_i(b^i_{n-1})))^r$ is equal to the vector $\xi_i(\mathbf{z}_i)$, where the vector $\mathbf{z}_i$ corresponds to the polynomial $\sum_{d=0}^{2k} \theta_i(h^i_d) x^{2k-d} f_i(x)$. So $\xi_i(C_i)$ is a reversible DNA code. ■

Theorem 3.8. *Let $x^n - 1 = h_i(x)f_i(x) \in D_i[x, \theta_i]$, where the degree of $f_i(x)$ is odd. If $f_i(x)$ is a θ_i-palindromic polynomial, then $h_i(x)$ is a palindromic polynomial.*

Proof. Let $f_i(x) = a^i_0 + a^i_1 x + ... + a^i_{2s-1} x^{2s-1}$. As the length n is even, then $h_i(x) = h^i_0 + h^i_1 x + \cdots + h^i_{2k-1} x^{2k-1}$. Since $f_i(x)$ is a θ_i-palindromic polynomial, then $a^i_t = \theta_i(a^i_{2s-1-t})$ for all $t = 0, 1, ..., s-1$. Let b^i_l be the coefficient of x^l in $h_i(x)f_i(x)$, where $l = 0, 1, \ldots, n-1$. For any $p < n/2$, the coefficient of x^p in $h_i(x)f_i(x)$ is

$$b^i_p = \sum_{j=0}^{p} h^i_j \theta^j_i (a^i_{p-j})$$

and the coefficient of x^{n-p} is $b^i_{n-p} = \sum_{j=0}^{p} h^i_{2k-1-j} \theta_i^{2k-1-j} (a^i_{2s-1-(p-j)})$. By using the fact that $b^i_0 = b^i_n = 0$ and $b^i_t = 0$ for all $t = 1, 2, ..., n-1$, it can be shown that $h^i_t = h^i_{2k-1-t}$ for all $t = 0, 1, .., k-1$ by induction, as in [6]. ■

4. Reversible and reversible complement codes over D_r

In this section, the necessary and sufficient conditions of cyclic codes over D_i to be reversible and reversible complement are given. By using the map, the DNA codes are obtained from these codes.

In [10], they characterized the reversible codes over D_1 as follows.

Theorem 4.1. *[10] Let $C_1 = w_1 C_0^1 \oplus (1 + w_1) C_0^2$ be a cyclic code of arbitrary length n over D_1. Then C_1 is reversible if and only if C_0^1 and C_0^2 are reversible codes over $F_2 + uF_2, u^2 = 0$ and both of them are cyclic codes over $F_2 + uF_2, u^2 = 0$.*

In [3] and [8], the necessary and sufficient conditions of cyclic codes over the ring $F_2 + uF_2, u^2 = 0$ to be reversible were given in case of the length n is odd or even, respectively.

In [2], the reversible codes over D_2 were characterized as follows;

Theorem 4.2. *[2] Let $C_2 = w_2 C_1^1 \oplus (1 + w_2) C_1^2$ be a cyclic code of arbitrary length n over D_2. Then C_2 is reversible if and only if C_1^1 and C_1^2 are reversible codes over D_1 and both of them are cyclic codes over D_1.*

Firstly, we characterize the reversible codes over D_i, where $i = 3, ..., r$.

Theorem 4.3. *Let $C_i = w_i C_{i-1}^1 \oplus (1 + w_i) C_{i-1}^2$ be a cyclic code of arbitrary length n over D_i, where $i = 3, ..., r$. Then C_i is reversible if and only if C_{i-1}^1 and C_{i-1}^2 are reversible codes over D_{i-1}, where $i = 3, ..., r$ and both of them are cyclic codes over D_{i-1}, where $i = 3, ..., r$.*

Proof. Let C^{1}_{i-1},C^{2}_{i-1} be reversible codes. For any $\mathbf{b}_i \in C_i$, $\mathbf{b}_i = w_i\mathbf{b}^1_{i-1} + (1+w_i)\mathbf{b}^2_{i-1}$, where $\mathbf{b}^1_{i-1} \in C^1_{i-1}$, $\mathbf{b}^2_{i-1} \in C^{2}_{i-1}$. As C^{1}_{i-1}, C^{2}_{i-1} are reversible codes, $\left(\mathbf{b}^1_{i-1}\right)^r \in C^1_{i-1}$, $\left(\mathbf{b}^2_{i-1}\right)^r \in C^{2}_{i-1}$, so $\mathbf{b}^r_i = w_i\left(\mathbf{b}^1_{i-1}\right)^r + (1+w_i)\left(\mathbf{b}^2_{i-1}\right)^r \in C_i$. Hence C_i is reversible codes.

On the other hand, let C_i be a reversible code over D_i. So for any $\mathbf{b}_i = w_i\mathbf{b}^1_{i-1} + (1+w_i)\mathbf{b}^2_{i-1} \in C_i$, where $\mathbf{b}^1_i \in C^1_{i-1}$,$\mathbf{b}^2_{i-1} \in C^{2}_{i-1}$, we get $\mathbf{b}^r_i = w_i\left(\mathbf{b}^1_{i-1}\right)^r + (1+w_i)\left(\mathbf{b}^2_{i-1}\right)^r \in C_i$. Let $\mathbf{b}^r_i = w_i\left(\mathbf{b}^1_{i-1}\right)^r + (1+w_i)\left(\mathbf{b}^2_{i-1}\right)^r = w_i\mathbf{s}^1_{i-1} + (1+w_i)\mathbf{s}^2_{i-1}$,where $\mathbf{s}^1_{i-1} \in C^1_{i-1}$,$\mathbf{s}^2_{i-1} \in C^{2}_{i-1}$. Therefore C^{1}_{i-1} and C^{2}_{i-1} are reversible codes over D_{i-1}. ■

In [10] and [2], they characterized the reversible complement codes over D_1 and D_2, respectively. Secondly, we characterize the reversible complement codes over D_i, where $i = 3, ..., r$.

Definition 4.4. *For $\mathbf{x}_i = \left(x^i_0, x^i_1, ..., x^i_{n-1}\right) \in D^n_i$, the vector $\left(\overline{x^i_{n-1}}, \overline{x^i_{n-2}}, ..., \overline{x^i_1}, \overline{x^i_0}\right)$ is called the reversible complement of $\mathbf{x}_i$ and is denoted by $\mathbf{x}^{rc}_i$, where $\overline{x^i_j}$ represents the complement of the elements x^i_j, $0 \le j \le n-1$. A linear code C_i of length n over D_i is said to be reversible complement if $\mathbf{x}^{rc}_i \in C_i$, for every $\mathbf{x}_i \in C_i$.*

Lemma 4.5. *For any $c_i \in D_i$,where $i = 1, ..., r$ we have $c_i + \overline{c_i} = u$.*

Lemma 4.6. *Let $a_i, b_i \in D_i$, where $i = 1, ..., r$, then $\overline{a_i + b_i} = \overline{a_i} + \overline{b_i} + u$.*

Theorem 4.7. *[10] Let $C_1 = w_1C^{1}_0 \oplus (1+w_1)C^{2}_0$ be a cyclic code of arbitrary length n over D_1. Then C_1 is reversible complement if and only if C_1 is reversible and $(\overline{0}, \overline{0}, ..., \overline{0}) \in C_1$, where C^{1}_0, C^{2}_0 are both cyclic codes over $F_2 + uF_2, u^2 = 0$.*

Theorem 4.8. *[2] Let $C_2 = w_2C^{1}_1 \oplus (1+w_2)C^{2}_1$ be a cyclic code of arbitrary length n over D_2. Then C_2 is reversible complement if and only if C_2 is reversible and $(\overline{0}, \overline{0}, ..., \overline{0}) \in C_2$, where C^{1}_1, C^{2}_1 are both cyclic codes over D_1.*

Theorem 4.9. *Let $C_i = w_iC^{1}_{i-1} \oplus (1+w_i)C^{2}_{i-1}$ be a cyclic code of arbitrary length n over D_i, where $i = 3, ..., r$. Then C_i is reversible complement if and only if C_i is reversible and $(\overline{0}, \overline{0}, ..., \overline{0}) \in C_i$, where C^{1}_{i-1}, C^{2}_{i-1} are both cyclic codes over $D_{i-1}, i = 3, ..., r$.*

Proof. Since C_i is reversible complement, for any $\mathbf{d}_i = (d^i_0, ...d^i_{n-1}) \in C_i, \mathbf{d}^{rc}_i = (\overline{d^i}_{n-1}, ..., \overline{d^i}_0) \in C_i$. Since C_i is a linear code, so $(0, 0, ..., 0) \in C_i$. By using Lemma 4.5, we get

$$\mathbf{d}^r_i = (d^i_{n-1}, ..., d^i_0) = (\overline{d^i}_{n-1}, ..., \overline{d^i}_0) + (u, u, u, ..., u) \in C_i.$$

Hence for any $\mathbf{d}_i \in C_i$, we have $\mathbf{d}^r_i \in C_i$.

On the other hand, let C_i be reversible code over D_i. So, for any $\mathbf{d}_i = (d^i_0, ...d^i_{n-1}) \in C_i$, then $\mathbf{d}^r_i = (d^i_{n-1}, ..., d^i_0) \in C_i$. For any $\mathbf{d}_i \in C_i$,

$$\mathbf{d}^{rc}_i = (\overline{d^i}_{n-1}, ..., \overline{d^i}_0) = (d^i_{n-1}, ...d^i_0) + (u, ..., u) \in C_i.$$

So, C_i is reversible complement code over D_i. ■

By a cyclic DNA code over D_i of length n, we mean a cyclic code that has the reverse complement property, where $i = 1, 2, .., r$.

Corollary 4.10. *Let C_i be a cyclic DNA code of length n over D_i and minimum Hamming distance d, where $i = 1, 2, .., r$. Then $\xi_i(C_i)$ is a DNA code of length $2^i n$ over the alphabet $\{A, T, C, G\}$ with minimum Hamming distance at least d.*

5. Reversible and Reversible Complement DNA Codes

In [10], the design of linear codes over D_1 was presented. It was obtained DNA codes from them.

In this section, we will design linear codes over D_i, where $i = 2, ..., r$, by using θ_i-set, where θ_i is a non trivial automorphism for $i = 2, .., r$ in order to obtain DNA codes.

Definition 5.1. *Let* $f_{0,1}, ..., f_{0,2^i}$ *be polynomials dividing* $x^n - 1$ *over* $F_2 + uF_2, u^2 = 0$ *and* $f_{i-1,1}, f_{i-1,2}$ *be polynomials with* $\deg f_{i-1,1} = t_{i-1,1}, \deg f_{i-1,2} = t_{i-1,2}$ *and both are over* D_{i-1}*, for* $i = 2, ..., r$*. Let* $f_i = w_i f_{i-1,1} + (1 + w_i) f_{i-1,2} \in D_i[x]$ *and*

$$\begin{aligned} f_{i-1,1} &= w_{i-1} f_{i-2,1} + (1 + w_{i-1}) f_{i-2,2}, \\ f_{i-1,2} &= w_{i-1} f_{i-2,3} + (1 + w_{i-1}) f_{i-2,4}, \\ f_{i-2,1} &= w_{i-2} f_{i-3,1} + (1 + w_{i-2}) f_{i-3,2}, \\ f_{i-2,2} &= w_{i-2} f_{i-3,3} + (1 + w_{i-2}) f_{i-3,4}, \\ f_{i-2,3} &= w_{i-2} f_{i-3,5} + (1 + w_{i-2}) f_{i-3,6}, \\ f_{i-2,4} &= w_{i-2} f_{i-3,7} + (1 + w_{i-2}) f_{i-3,8}, \\ &\vdots \\ f_{1,1} &= w_1 f_{0,1} + (1 + w_1) f_{0,2}, \\ f_{1,2} &= w_1 f_{0,3} + (1 + w_1) f_{0,4}, \\ &\vdots \\ f_{1,2^{i-1}} &= w_1 f_{0,2^i-1} + (1 + w_1) f_{0,2^i}. \end{aligned}$$

Let $m_i = \min\{n - t_{i-1,1}, n - t_{i-1,2}\}$*. The set* $L(f_i)$ *is called a* θ_i*-set and is defined as*

$$L(f_i) = \{E_0, E_1, ..., E_{m_i-1}, F_0, F_1, ..., F_{m_i-1}\},$$

where $E_j = x^j f_i, F_j = x^j \theta_i(h_i), 0 \le j \le m_i - 1, i = 2, ..., r.$

If deg $f_{0,2s} \ge deg f_{0,2s-1}$,

$$h_{i,1,s} = w_1 x^{deg f_{0,2s} - deg f_{0,2s-1}} f_{0,2s-1} + (1 + w_1) f_{0,2s}$$

otherwise,

$$h_{i,1,s} = w_1 f_{0,2s-1} + (1 + w_1) x^{deg f_{0,2s-1} - deg f_{0,2s}} f_{0,2s}$$

where $s = 1, 2, ..., 2^{i-1}$ *and*

if $deg h_{i,1,2t} \ge deg h_{i,1,2t-1}$

$$h_{i,2,t} = w_2 x^{deg h_{i,1,2t} - deg h_{i,1,2t-1}} h_{i,1,2t-1} + (1 + w_2) h_{i,1,2t}$$

otherwise,

$$h_{i,2,t} = w_2 h_{i,1,2t-1} + (1 + w_2) x^{deg h_{i,1,2t-1} - deg h_{i,1,2t}} h_{i,1,2t}$$

where $t = 1, 2, ..., 2^{i-2}$ *and*

$$\vdots$$

if $deg h_{i,i-2,2v} \ge deg h_{i,i-2,2v-1}$

$$h_{i,i-1,v} = w_{i-1} x^{deg h_{i,i-2,2v} - deg h_{i,i-2,2v-1}} h_{i,i-2,2v-1} + (1 + w_{i-1}) h_{i,i-2,2v}$$

otherwise,

$$h_{i,i-1,v} = w_{i-1}h_{i,i-2,2v-1} + (1+w_{i-1})x^{degh_{i,i-2,2v-1}-degh_{i,i-1,2v}}h_{i,i-2,2v}$$

where $v = 1, 2$ *and*

If $degh_{i,i-1,2} \geq degh_{i,i-1,1}$,

$$h_i = w_i x^{degh_{i,i-1,2}-degh_{i,i-1,1}}h_{i,i-1,1} + (1+w_i)h_{i,i-1,2}$$

otherwise,

$$h_i = w_i h_{i,i-1,1} + (1+w_i)x^{degh_{i,i-1,1}-degh_{i,i-1,2}}h_{i,i-1,2}.$$

$L(f_i)$ *generates a linear code* C_i *over* D_i*, where* $i = 2, ..., r$*. It will be denoted by* $C_i = \langle f_i \rangle_{\theta_i}$ *or* $C_i = \langle L(f_i) \rangle$*. It means that it is* D_i*-submodule generated by the set* $L(f_i)$*, where* $i = 2, ..., r$*.*

Let $f_i = a_0^i + a_1^i x + ... + a_t^i x^t \in D_i[x], \theta_i(h_i) = b_0^i + b_1^i x + ... + b_s^i x^s$*, where* $i = 2, .., r$*. The* D_i*-submodule can be considered to be generated by the rows of the following matrix*

$$L(f_i) = \begin{bmatrix} E_0 \\ F_0 \\ E_1 \\ F_1 \\ E_2 \\ F_2 \\ \cdots \end{bmatrix}$$

$$= \begin{bmatrix} a_0^i & a_1^i & a_2 & \cdots & a_t^i & 0 & \cdots & \cdots & & \cdots & 0 \\ b_0^i & b_1^i & \cdots & \cdots & b_t^i & b_{t+1}^i & \cdots & b_s^i & 0 & \cdots & 0 \\ 0 & a_0^i & a_1^i & a_2^i & \cdots & a_t^i & 0 & 0 & & \cdots & 0 \\ 0 & b_0^i & b_1^i & b_2^i & \cdots & \cdots & \cdots & \cdots & b_s^i & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & & \cdots & \vdots \end{bmatrix}.$$

Theorem 5.2. *Let* $f_{0,1}, ..., f_{0,2^i}$ *be self reciprocal polynomials dividing* $x^n - 1$ *over* $F_2 + uF_2, u^2 = 0$*. So* $C_i = \langle L(f_i) \rangle$ *is a linear code over* D_i *and* $\xi_i(C_i)$ *is a reversible DNA code, where* ξ_i *is from* C_i *to* $S_{D_4}^{2^i n}$*, for* $i = 2, ..., r$*.*

Proof. It is proved as in the proof of the Theorem 4.3 in [10]. ■

Corollary 5.3. *Let* $f_{0,1}, ..., f_{0,2^i}$ *be self reciprocal polynomials dividing* $x^n - 1$ *over* $F_2 + uF_2, u^2 = 0$ *and* $C_i = \langle L(f_i) \rangle$ *be a cyclic code over* D_i*. If* $u\frac{x^n-1}{x-1} \in C_i$*, then* $\xi_i(C_i)$ *is a reversible complement DNA code.*

Example 5.4.

$$f_{0,1}(x) = x + 1,$$
$$f_{0,2}(x) = x^2 + x + 1,$$
$$f_{0,3}(x) = x^6 + x^3 + 1,$$
$$f_{0,4}(x) = x + 1,$$

where all of them divide $x^9 - 1$ *over* F_2*. Hence*

$$f_2 = w_2 (w_1 f_{0,1} + (1+w_1) f_{0,2}) + (1+w_2)(w_1 f_{0,3} + (1+w_1) f_{0,4})$$

over D_2*. That is*

$$f_2 = w_1(1+w_2)x^6 + w_1(1+w_2)x^3 + w_2(1+w_1)x^2 + (w_1(1+w_2)+1)x + 1.$$

Since $h_{2,1,1} = w_1 x f_{0,1} + (1+w_1) f_{0,2}$ *and* $h_{2,1,2} = w_1 f_{0,3} + x^5(1+w_1) f_{0,4}$*, we get* $h_2 = w_2 h_{2,1,1} + (1+w_2) h_{2,1,2}$ *Then we have* $h_2 = x^6 + (1+w_1+w_1w_2)\,x^5 + x^4\,(1+w_1)\,w_2 + (1+w_2)\,w_1x^3 + w_1\,(1+w_2)$*. So* $\theta_2\,(h_2) = x^6 + (1+w_2\,(1+w_1))\,x^5 + w_1\,(1+w_2)\,x^4 + (1+w_1)\,w_2x^3 + w_2\,(1+w_1)$.
Since $m_2 = 3$*, we consider the generator matrix of* C*,*

$$\begin{bmatrix} E_0 \\ F_0 \\ E_1 \\ F_1 \\ E_2 \\ F_2 \end{bmatrix}$$

where $E_0 = F_2, E_1 = xf_2, E_2 = x^2 f_2, F_0 = \theta_2(h_2), F_1 = x\theta_2(h_2), F_2 = x^2\theta_2(h_2)$*. If we take* $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 1, \beta_0 = 0, \beta_1 = 0, \beta_2 = 1$*, then* $\alpha_0 E_0 + \alpha_1 E_1 + \alpha_2 E_2 + \beta_0 F_0 + \beta_1 F_1 + \beta_2 F_2 = x + x^2\,(w_1+w_2) + w_2\,(1+w_1)\,x^3 + (w_1+w_2)\,x^4 + (w_1+w_2)\,x^5 + w_1\,(1+w_2)\,x^6 + (1+w_1+w_2)\,x^7 + (1+w_1+w_1w_2)\,x^8$*. It is correspondence to the codeword*

$$\mathbf{d}_1 = \begin{pmatrix} 0, 1, w_1+w_2, w_2\,(1+w_1)\,, w_1+w_2, w_1+w_2, \\ w_1\,(1+w_2)\,, 1+w_1+w_2, 1+w_1+w_1w_2 \end{pmatrix}.$$

Hence $\xi_2(\mathbf{d}_1) = AAAAGGGGAGGAAAGAAGGAAGGAAGAAGAAGGAGG$.

Moreover $\theta_2\,(\alpha_0)\,F_2 + \theta_2\,(\alpha_1)\,F_1 + \theta_2\,(\alpha_2)\,F_0 + \theta_2\,(\beta_0)\,E_2 + \theta_2\,(\beta_1)\,E_1 + \theta_2\,(\beta_2)\,E_0 = 1 + w_2\,(1+w_1) + x\,(1+w_1+w_2) + x^2\,(w_2\,(1+w_1)) + x^3\,(w_1+w_2) + x^4\,(w_1+w_2) + x^5\,(1+w_1+w_2) + x^6\,(1+w_1+w_2) + x^7$ *correspondences to the codeword*

$$\mathbf{d}_2 = \begin{pmatrix} 1+w_2\,(1+w_1)\,, 1+w_1+w_2, w_2\,(1+w_1)\,, w_1+w_2, \\ w_1+w_2, 1+w_1+w_2, 1+w_1+w_2, 1, 0 \end{pmatrix}.$$

Hence $\xi_2(\mathbf{d}_2) = GGAGGAAGAAGAAGGAAGGAAGAAAGGAGGGGAAAA$.

So $(\xi_2(\mathbf{d}_2))^r = \xi_2(\mathbf{d}_1)$.

6. Conclusion

By using three different methods, the DNA codes are obtained from the some error correcting codes over the family of finite rings.

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Approximating local solutions of IVPs of nonlinear first order ordinary hybrid integrodifferential equations

Janhavi B. Dhage [1] **and Bapurao C. Dhage** [*2]

[1,2] *Kasubai, Gurukul Colony, Thodga Road, Ahmedpur - 413515, Dist. Latur, Maharashtra, India.*

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Abstract. In this paper, we prove a couple of approximation results for local existence and uniqueness of the solution of a IVP of nonlinear first order ordinary hybrid integrodifferential equations by using the Dhage monotone iteration method based on a hybrid fixed point theorem of Dhage (2022) and Dhage et al. (2022). An approximation result for the Ulam-Hyers stability of the local solution of the considered hybrid integrodifferential equation is also established. Finally, our main abstract results are illustrated with a couple of numerical examples.

AMS Subject Classifications: 34A12, 34A34.

Keywords: Integrodifferential equation; Hybrid fixed point principle; Dhage Monotone iteration method; Approximation theorem; Ulam-Hyers stability.

Contents

1. Introduction

The nonlinear differential and integral equations arise in several natural and physical phenomena of the universe, see for example, Li *et al.* [18], Ramosa [20], Shah *et al.* [21] and the references therein. The iterative method is a powerful technique useful for finding the approximate solution of nonlinear problems which is used since long time in nonlinear analysis and became popular among the mathematicians all over the world. The different iteration methods used in nonlinear analysis have different characterizations and different advantages and limitations. The iteration methods used in Al-Jawary *et al.* [1] are due to Temimi and Ansari [22] and put no condition on the nonlinearity of the differential equations, however these methods yield the convergent power series expansion of the solution. The Picard's iteration used in Lyons *et al.* [19] employs the Lipschitz condition on the nonlinear function involved in the equations and the solution is obtained in the form of a convergent

[*]**Corresponding author.** Email address: **jbdhage@gmail.com** (Janhavi B. Dhage) **bcdhage@gmail.com** (Bapurao C. Dhage)

sequence of successive approximations (see also Coddington [2]). Similarly, the variational iteration method used in Wang and He [24] uses the Lagrange's multiplier in the successive iterations. Here, we discuss the considered nonlinear equation via Dhage iteration method under certain monotonicity condition but without using the usual Lipschitz condition on the nonlinearity and Lagrange's multiplier in the successive approximations which goes monotonically to the solution.

Given a closed and bounded interval $J = [t_0, t_0 + a]$ of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$, we consider the IVP of nonlinear first order ordinary hybrid integrodifferential equation (HIGDE),

$$\left.\begin{aligned} \frac{dx}{dt} &= \int_{t_0}^{t} f(s, x(s)), \quad t \in J, \\ x(t_0) &= \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where the function $f : J \times \mathbb{R} \to \mathbb{R}$ satisfies some hybrid, that is, mixed hypotheses from algebra, analysis and topology to be specified later.

Definition 1.1. *A function $x \in C(J, \mathbb{R})$ is said to be a* solution *of the HIGDE (1.1 if it satisfies the equations in (1.1) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J. If the solution x lies in a closed ball $\overline{B_r(x_0)}$ centered at a point $x_0 \in C(J, \mathbb{R})$ of radius $r > 0$, then we say it is a* local solution *or* neighborhood solution *(in short* nbhd solution*) of the HIGDE (1.1 on J.*

The HIGDE (1.1 is familiar in the subject of nonlinear analysis and can be studied for a variety of different aspects of the solution by using different methods form nonlinear functional analysis. The existence of local solution can be proved by using the Schauder fixed point principle, see for example, Coddington [2], Lakshmikantham and Leela [17], Granas and Dugundji [15] and references therein. The approximation result for uniqueness of solution can be proved by using the Banach fixed point theorem under a Lipschitz condition which is considered to be very strong in the area of nonlinear analysis. But to the knowledge the present authors, the approximation result for local existence and uniqueness theorems without using the Lipschitz condition is not discussed so far in the theory of nonlinear differential and integral equations. In this paper, we discuss the approximation results for local existence and uniqueness of the solution under weaker Lipschitz condition but via construction of an algorithm based on Dhage iteration method and a hybrid fixed point theorem of Dhage [6] and Dhage *et al.* [10].

The rest of the paper is organized as follows. Section 2 deals with the auxiliary results and main hybrid fixed point theorems involved in the Dhage iteration method. The hypotheses and main approximation results for the local existence and uniqueness of solution are given in Section 3. The approximation of the Ulam-Hyer stability is discussed in Section 4 and a couple of illustrative examples are presented in Section 5. Finally, some concluding remarks are mentioned in Section 6.

2. Auxiliary Results

We place the problem of HDE (1.1) in the function space $C(J, \mathbb{R})$ of continuous, real-valued functions defined on J. We introduce a supremum norm $\|\cdot\|$ in $C(J, \mathbb{R})$ defined by

$$\|x\| = \sup_{t \in J} |x(t)|, \tag{2.1}$$

and an order relation $\preceq$ in $C(J, \mathbb{R})$ by the cone K given by

$$K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \ \forall \ t \in J\}. \tag{2.2}$$

Thus,

$$x \preceq y \iff y - x \in K, \tag{2.3}$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t) \ \forall \ t \in J.$$

It is known that the Banach space $C(J, \mathbb{R})$ together with the order relations $\preceq$ becomes a partially ordered Banach space which we denote for convenience, by $\big(C(J, \mathbb{R}), K\big)$. We denote the open and closed spheres centered at $x_0 \in C(J, \mathbb{R})$ of radius r, for some $r > 0$, by

$$B_r(x_0) = \{x \in C(J, \mathbb{R}) \mid \|x - x_0\| < r\} = B(x, r),$$

and

$$B_r[x_0] = \{x \in C(J, \mathbb{R}) \mid \|x - x_0\| \leq r\} = \overline{B(x, r)},$$

receptively. It is clear that $B_r[x_0] = \overline{B_r(x_0)}$. Let $M > 0$ be a real number. Denote

$$B_r^M[x_0] = \big\{x \in B_r[x_0] \,\big|\, |x(t_1) - x(t_2)| \leq M\,|t_1 - t_2| \text{ for } t_1, t_2 \in J\big\}. \tag{2.4}$$

Then, we have the following result.

Lemma 2.1. *The set $B_r^M[x_0]$ is compact in $C(J, \mathbb{R})$.*

Proof. By definition $B_r[x_0]$ is a closed and bounded subset of the Banach space $C(J, \mathbb{R})$. Moreover, $B_r^M[x_0]$ is an equicontinuous subset of $C(J, \mathbb{R})$ in view of the condition (2.1). Now by an application of Arzelá-Ascoli theorem, $B_r^M[x_0]$ is compact set in $C(J, \mathbb{R})$ and the proof of the lemma is complete. □

It is well-known that the fixed point theoretic technique is very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations. See Granas and Dugundji [15] and the references therein. Here, we employ the Dhage monotone iteration method or simply *Dhage iteration method* based on the following two hybrid fixed point theorems of Dhage [6] and Dhage *et al.* [10].

Theorem 2.2 (Dhage [6])**.** *Let S be a non-empty partially compact subset of a regular partially ordered Banach space $\big(E, ||\cdot\|, \preceq,\big)$ with every chain C in S is Janhavi set and let $\mathcal{T} : S \to S$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.3 (Dhage [6])**.** *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a regular partially ordered Banach space $\big(E, ||\cdot\|, \preceq,\big)$ and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying*

$$\|x_0 - \mathcal{T}x_0\| \leq (1 - q)r, \tag{2.5}$$

for some real number $r > 0$, then $\mathcal{T}$ has a unique comparable fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. Furthermore, if every pair of elements in X has a lower or upper bound, then ξ^* is unique.

Remark 2.4. *We note that every every pair of elements in a partially ordered set $(\mathbf{poset})$ $(E, \preceq)$ has a lower or upper bound if $(E, \preceq)$ is a lattice, that is, $\preceq$ is a lattice order in E. In this case the poset $(E, \|\cdot\|, \preceq)$ is called a* **partially lattice ordered Banach space***. There do exist several lattice partially ordered Banach spaces which are useful for applications in nonlinear analysis.*

If a Banach X is partially ordered by an order cone K in X, then in this case we simply say X is ordered Banach space which we denote it by (X, K). Similarly, an ordered Banach space (X, K), where partial order $\preceq$ defined by the con K is a lattice order, then (X, K) is called the **lattice ordered Banach space**. Then, we have the following useful results concerrning the ordered Banach space proved in Dhage [4, 5].

Lemma 2.5 (Dhage [4, 5]). *Every ordered Banach space (X, K) is regular.*

Lemma 2.6 (Dhage [4, 5]). *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 2.5 and 2.6 we obtain the following hybrid fixed point theorem which we need in what follows.

Theorem 2.7 (Dhage [6] and Dhage *et al.* [10]). *Let S be a non-empty partially compact subset of an ordered Banach space (X, K) and let $\mathcal{T} : S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.8 (Dhage [6]). *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in an ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (2.5), then $\mathcal{T}$ has a unique comparable fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. Furthermore, if every pair of elements in X has a lower or upper bound, then ξ^* is unique.*

Theorem 2.9 (Dhage [6]). *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a lattice ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (2.5), then $\mathcal{T}$ has a unique fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

The details of the notions of partial order, Janhavi set, regularity, monotonicity, partial continuity, partial closure, partial compactness and partial contraction etc. and related applications appear in Dhage [3–5], Dhage and Dhage [7, 8], Dhage *et al.* [9], Dhage *et al.* [10], Dhage and Dhage [11], Dhage *et al.* [12–14] and references therein.

3. Local Approximation Results

We consider the following set of hypotheses in what follows.

(H$_1$) The function f is continuous and bounded on $J \times \mathbb{R}$ with bound M_f.

(H$_2$) There exists a constant $k > 0$ such that

$$0 \leq f(t, x) - f(t, y) \leq k(x - y),$$

for all $x, y \in \mathbb{R}$ with $x \geq y$, where $k\,a^2 < 1$.

(H$_3$) $f(t, x)$ is nondecreasing in x for each $t \in J$.

(H$_4$) $f(t, \alpha_0) \geq 0$ for all $t \in J$.

Then we have the following useful lemma.

Lemma 3.1. *If $h \in L^1(J, \mathbb{R})$, then the IVP of ordinary first order linear differential equation*

$$\frac{dx}{dt} = \int_{t_0}^{t} h(s), ds, \;\; t \in J, \quad x(t_0) = \alpha_0, \tag{3.1}$$

is equivalent to the integral equation

$$x(t) = \alpha_0 + \int_{t_0}^{t} (t - s)\, h(s)\, ds, \;\; , t \in J. \tag{3.2}$$

Theorem 3.2. *Sppose that the hypotheses (H$_1$), (H$_3$) and (H$_4$) hold. Furthermore, if $M_f\,a^2 \leq r$ and $2\,M_f\,a \leq M$, then the HIGDE (1.1) has a solution x^* in $B_r^M[x_0]$, where $x_0 \equiv \alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= \alpha_0, \;\; t \in J, \\ x_{n+1}(t) &= \alpha_0 + \int_{t_0}^{t} (t-s)\,f(s, x_n(s))\,ds, \;\; t \in J, \end{aligned}\right\} \tag{3.3}$$

where $n = 0, 1, \ldots$; is monotone nondecreasing and converges to x^*.

Proof. Set $X = C(J, \mathbb{R})$. Clearly, (X, K) is a partially ordered Banach space. Let x_0 be a constant function on J such that $x_0(t) = \alpha_0$ for all $t \in J$ and define a closed ball $B_r^M[x_0]$ in X defined by (2.3). By Lemma 2.1, $B_r^M[x_0]$ is a compact subset of X. By Lemma 3.1, the HIGDE (1.1) is equivalent to the nonlinear hybrid integral equation (HIE)

$$x(t) = \alpha_0 + \int_{t_0}^{t} (t-s)\,f(s, x(s))\,ds, \;\; t \in J. \tag{3.4}$$

Now, define an operator $\mathcal{T}$ on $B_r^M[x_0]$ into X by

$$\mathcal{T}x(t) = \alpha_0 + \int_{t_0}^{t} (t-s)\,f(s, x(s))\,ds, \;\; t \in J. \tag{3.5}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.2 on $B_r^M[x_0]$ in the following series of steps.

Step I: *The operator $\mathcal{T}$ maps $B_r^M[x_0]$ into itself.*

Firstly, we show that $\mathcal{T}$ maps $B_r^M[x_0]$ into itself. Let $x \in B_r^M[x_0]$ be arbitrary element. Then,

$$\begin{aligned} |\mathcal{T}x(t) - x_0(t)| &= \left| \int_{t_0}^{t} (t-s)\,f(s, x(s))\,ds \right| \\ &\leq \int_{t_0}^{t} |t-s|\,\big|f(s, x(s))\big|\,ds \\ &= M_f\,a \int_{t_0}^{t_0+a} ds \\ &= M_f\,a^2 \leq r. \end{aligned}$$

Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - x_0\| \leq M_f\,a^2 \leq r,$$

which implies that $\mathcal{T}x \in B_r[x_0]$ for all $x \in B_r^M[x_0]$. Next, let $t_1, t_2 \in J$ be arbitrary. Then, we have

$$\begin{aligned}
\left|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\right| &\le \left|\int_{t_0}^{t}(t_1 - s)\, f(s, x(s))\, ds - \int_{t_0}^{t_2}(t_2 - s)\, f(s, x(s))\, ds\right| \\
&\le \left|\int_{t_0}^{t_1}(t_1 - s)\, f(s, x(s))\, ds - \int_{t_0}^{t_1}(t_2 - s)\, f(s, x(s))\, ds\right| \\
&\quad + \left|\int_{t_0}^{t_1}(t_2 - s)\, f(s, x(s))\, ds - \int_{t_0}^{t_2}(t_2 - s)\, f(s, x(s))\, ds\right| \\
&\le \int_{t_0}^{t_1} |t_1 - t_2|\, |f(s, x(s))|\, ds + \left|\int_{t_1}^{t_2} |t_2 - s|\, |f(s, x(s))|\, ds\right| \\
&\le \int_{t_0}^{t_0+a} |t_1 - t_2|\, M_f\, ds + \left|\int_{t_1}^{t_2} a\, M_f\, ds\right| \\
&\le 2M_f\, a\, |t_1 - t_2| \\
&\le M\, |t_1 - t_2|
\end{aligned}$$

where, $2M_f\, a \le M$. Therefore, $\mathcal{T}x \in B_r^M[x_0]$ for all $x \in B_r^M[x_0]$ As a result, we have $\mathcal{T}(B_r^M[x_0]) \subset B_r^M[x_0]$.

Step II: *$\mathcal{T}$ is a monotone nondecreasing operator on $B_r^M[x_0]$.*

Let $x, y \in B_r^M[x_0]$ be any two elements such that $x \succeq y$. Then, by hypothesis (H_3),

$$\begin{aligned}
\mathcal{T}x(t) &= \alpha_0 + \int_{t_0}^{t}(t - s)\, f(s, x(s))\, ds \\
&\ge \alpha_0 + \int_{t_0}^{t}(t - s)\, f(s, y(s))\, ds \\
&= \mathcal{T}y(t),
\end{aligned}$$

for all $t \in J$. So, $\mathcal{T}x \succeq \mathcal{T}y$, that is, $\mathcal{T}$ is monotone nondecreasing on $B_r^M[x_0]$.

Step III: *$\mathcal{T}$ is a partially continuous operator on $B_r^M[x_0]$.*

Let C be a chain in $B_r^M[x_0]$ and let $\{x_n\}$ be a sequence in C converging to a point $x \in C$. Then, by dominated cnonvergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n &= \lim_{n\to\infty}\left[\alpha_0 + \int_{t_0}^{t}(t - s)\, f(s, x_n(s))\, ds\right] \\
&= \alpha_0 + \lim_{n\to\infty}\int_{t_0}^{t}(t - s)\, f(s, x_n(s))\, ds \\
&= \alpha_0 + \int_{t_0}^{t}(t - s)\, \left[\lim_{n\to\infty} f(s, x_n(s))\right]\, ds \\
&= \alpha_0 + \int_{t_0}^{t}(t - s)\, f(s, x(s))\, ds \\
&= \mathcal{T}x(t),
\end{aligned}$$

for all $t \in J$. Therefore, $\mathcal{T}x_n \to \mathcal{T}x$ pointwise on J. As $\{\mathcal{T}x_n\} \subset B_r^M[x_0]$, $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of points in X. As a reult, we have that $\mathcal{T}x_n \to \mathcal{T}x$ uniformly on J. Hence $\mathcal{T}$ is partially continuouus operator on $B_r^M[x_0]$.

Step IV: *The element $x_0 \in B_r^M[x_0]$ satisfies the relation $x_0 \preceq \mathcal{T}x_0$.*

Since (H_4) holds, one has

$$\begin{aligned} x_0(t) &= \alpha_0 + \int_{t_0}^{t} (t-s) f(s, x_0(s))\, ds \\ &\leq x_0(t) + \int_{t_0}^{t} (t-s) f(s, \alpha_0)\, ds \\ &= \alpha_0 + \int_{t_0}^{t} (t-s) f(s, x_0(s))\, ds \\ &= \mathcal{T} x_0(t), \end{aligned}$$

for all $t \in J$. This shows that the constant function x_0 in $B_r^M[x_0]$ serves as to satisfy the operator inequality $x_0 \preceq \mathcal{T} x_0$.

Thus, the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.2, and so $\mathcal{T}$ has a fixed point x^* in $B_r^M[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the HIE (3.4) and consequently the HIGDE (1.1) has a local solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to x^*. This completes the proof. □

Next, we prove an approximation result for existence and uniqueness of the solution simultaneously under weaker form of Lipschitz condition.

Theorem 3.3. *Suppose that the hypotheses (H_1), (H_2) and (H_4) hold. Furthermore, if*

$$M_f a \leq (1 - ka^2) r, \quad ka^2 < 1, \tag{3.6}$$

for some real number $r > 0$, then the HIGDE (1.1) has a unique solution x^* in $B_r[x_0]$ defined on J, where $x_0 \equiv \alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to x^*.

Proof. Set $(X, K) = \big(C(J, \mathbb{R}), \preceq\ \big)$ which is a lattice w.r.t. the lattice *join* and *meet* operations defined by $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$, and so every pair of elements of X has a lower and an upper bound. Let x_0 be a constant function on J such that $x_0(t) = \alpha_0$ for all $t \in J$ and consider the closed sphere $B_r[x_0]$ centered at $x_0 \in C(J, \mathbb{R})$ of radius r,for some fixed $r > 0$, in the partially ordered Banach space (X, K).

Define an operator $\mathcal{T}$ on X into X by (3.5). Clearly, $\mathcal{T}$ is monotone nondecreasing on X. To see this, let $x, y \in X$ be two elements such that $x \succeq y$. Then, by hypothesis (H_2),

$$\mathcal{T}x(t) - \mathcal{T}y(t) = \int_{t_0}^{t} (t-s) \big[f(s, x(s)) - f(s, y(s)) \big]\, ds \geq 0,$$

for all $t \in J$. Therefore, $\mathcal{T}x \succeq \mathcal{T}y$ and consequently $\mathcal{T}$ is monotone nondecresing on X.

Next, we show that $\mathcal{T}$ is a partial contraction on X. Let $x, y \in X$ be such that $x \succeq y$. Then, by hypothesis (H_2), we obtain

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}y(t)| &= \left| \int_{t_0}^{t} (t-s) \big[f(s, x(s)) - f(s, y(s)) \big]\, ds \right| \\ &\leq \left| \int_{t_0}^{t} k(t-s) \big(x(s) - y(s) \big)\, ds \right| \\ &= \int_{t_0}^{t} k\, a\, |x(s) - y(s)|\, ds \\ &\leq k\, a \int_{t_0}^{t_0 + a} \|x - y\|\, ds \\ &= k\, a^2\, \|x - y\|, \end{aligned}$$

for all $t \in J$, where $k\,a^2 < 1$. Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - \mathcal{T}y\| \leq k\,a^2\,\|x - y\|,$$

for all comparable elements $x, y \in X$. This shows that $\mathcal{T}$ is a partial contraction on X with contraction constant $k\,a$. Furthermore, it can be shown as in the proof of Theorem 3.2 that the element $x_0 \in B_r^M[x_0]$ satisfies the relation $x_0 \preceq \mathcal{T}x_0$ in view of hypothesis (H$_4$). Finally, by hypothesis (H$_1$) and condition (3.6), one has

$$\begin{aligned}\|x_0 - \mathcal{T}x_0\| &= \sup_{t\in J}\left|\int_{t_0}^{t}(t-s)\,f(s, x_0(s))\,ds\right| \\ &\leq \sup_{t\in J}\int_{t_0}^{t}|t-s|\,|f(s,\alpha_0)|\,ds \\ &\leq M_f\,a^2 \\ &\leq (1-k\,a^2)r,\end{aligned}$$

which shows that the condition (2.5) of Theorem 2.9 is satisfied. Hence $\mathcal{T}$ has a unique fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the HIE (3.4) and consequently the HIGDE (1.1) has a unique local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations converges monotone nondecreasingly to x^*. This completes the proof. □

Remark 3.4. *The conclusion of Theorems 3.2 and 3.3 also remains true if we replace the hypothesis (H$_4$) with the following one.*

(H$_4$) The function f satisfies $f(t, \alpha_0) \leq 0$ for all $t \in J$.

In this case, the HDE (1.1) has a local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nonincreasing and converges to the solution x^*.

Remark 3.5. *If the initial condition in the equation (1.1) is such that $\alpha_0 > 0$, then under the conditions of Theorem 3.2, the HIGDE (1.1) has a local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) monotone nondecreasing and converges to the positive solution x^*. Similarly, under the conditions of Theorem 3.3, the HIGDE (1.1) has a unique local positive solution x^* defined on J and the sequence of successive approximations defined by (3.3) $\{x_n\}_{n=0}^{\infty}$ monotone nondecreasing and converges to the unique positive solution x^*.*

4. Approximation of Local Ulam-Hyers Stability

The Ulam-Hyers stability for various dynamic systems has already been discussed by several authors under the conditions of classical Schauder fixed point theorem (see Tripathy [23], Huang *et al.* [16] and references therein). Here, in the present paper, we discuss the approximation of the Ulam-Hyers stability of local solution of the HIGDE (1.1) under the conditions of hybrid fixed point principle stated in Theorem 2.3. We need the following definition in what follows.

Definition 4.1. *The HIGDE (1.1) is said to be locally Ulam-Hyers stable if for $\epsilon > 0$ and for each local solution $y \in B_r[x_0]$ of the inequality*

$$\left.\begin{aligned}\left|\frac{dy}{dt} - \int_{t_0}^{t} f(s, y(s))\,ds\right| &\leq \epsilon, \quad t \in J,\\ y(t_0) = \alpha_0 &\in \mathbb{R},\end{aligned}\right\} \tag{$*$}$$

there exists a constant $K_f > 0$ such that

$$\big|y(t) - \xi(t)\big| \leq K_f\epsilon, \tag{$**$}$$

for all $t \in J$, where $\xi \in B_r[x_0]$ is a local solution of the HIGDE (1.1) defined on J. The solution ξ of the HIGDE (1.1) is called Ulam-Hyers stable local solution on J.

Theorem 4.2. *Assume that all the hypotheses of Theorem 3.3 hold. Then the HIGDE (1.1) has a unique Ulam-Hyers stable local solution $x^* \in B_r[x_0]$, where $x_0 \equiv \alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations given by (3.3) is monotone nondecreasing and converges to x^*.*

Proof. Let $\epsilon > 0$ be given and let $y \in B_r[x_0]$ be a solution of the functional inequality (*) on J, that is, we have

$$\left.\begin{aligned} &\left|\frac{dy}{dt} - \int_{t_0}^{t} f(s, y(s))\, ds\right| \leq \epsilon, \ \ t \in J, \\ &y(t_0) = \alpha_0 \in \mathbb{R}. \end{aligned}\right\} \tag{4.1}$$

By Theorem 3.3, the HIGDE (1.1) has a unique local solution $\xi \in B_r[x_0]$. Then by Lemma 2.1, one has

$$\xi(t) = \alpha_0 + \int_{t_0}^{t} (t-s)\, f(s, \xi(s))\, ds, \quad t \in J. \tag{4.2}$$

Now, by integration of (4.1) yields the estimate:

$$\left|y(t) - \alpha_0 - \int_{t_0}^{t} (t-s)\, f(s, y(s))\, ds\right| \leq a\, \epsilon, \tag{4.3}$$

for all $t \in J$.

Next, from (4.2) and (4.3), we obtain

$$\begin{aligned} \big|y(t) - \xi(t)\big| &= \left|y(t) - \alpha_0 - \int_{t_0}^{t} (t-s)\, f(s, \xi(s))\, ds\right| \\ &\leq \left|y(t) - \alpha_0 - \int_{t_0}^{t} (t-s)\, f(s, y(s)\, ds\right| \\ &\quad + \left|\int_{t_0}^{t} (t-s)\, f(s, y(s))\, ds - \int_{t_0}^{t} (t-s)\, f(s, \xi(s))\, ds\right| \\ &\leq a\, \epsilon + \int_{t_0}^{t} |t-s|\, \big|f\big(s, y(s)\big) - f(s, \xi(s))\big|\, ds \\ &\leq a\, \epsilon + k\, a^2 (\|y - \xi\|). \end{aligned}$$

Taking the supremum over t, we obtain

$$\|y - \xi\| \leq a\, \epsilon + ka^2 \|y - \xi\|,$$

or

$$\|y - \xi\| \leq \left[\frac{a\, \epsilon}{1 - ka^2}\right],$$

where, $ka^2 < 1$. Letting $K_f = \left[\dfrac{a}{1 - ka^2}\right] > 0$, we obtain

$$\big|y(t) - \xi(t)\big| \leq K_f \epsilon,$$

for all $t \in J$. As a result, ξ is a Ulam-Hyers stable local solution of the HIGDE (1.1) on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to ξ. Consequently the HIGDE (1.1) is a locally Ulam-Hyers stable on J. This completes the proof. □

Remark 4.3. *If the given initial condition in the equation (1.1) is such that $\alpha_0 > 0$, then under the conditions of Theorem 4.2, the HIGDE (1.1) has a unique Ulam-Hyers stable local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges is monotone nondecreasing and converges to x^*.*

5. The Examples

In this section we give a couple of numerical examples to illustrate the hypotheses and abstract ideas involved in the main approximation results of the previous Sections 3 and 4.

Example 5.1. *Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the IVP of nonlinear first order HIGDE,*

$$\frac{dx}{dt} = \int_0^t \tanh x(s)\, ds, \;\; t \in [0,1]; \quad x(0) = \frac{1}{4}. \tag{5.1}$$

Here $\alpha_0 = \frac{1}{4}$ and $f(t,x) = \tanh x$ for $(t,x) \in [0,1] \times \mathbb{R}$. We show that f satisfies all the conditions of Theorem 3.2. Clearly, f is bounded on $[0,1] \times \mathbb{R}$ with bound $M_f = 1$ and so the hypothesis (H_1) is satisfied. Also the function $f(t,x)$ is nondecreasing in x for each $t \in [0,1]$. Therefore, hypothesis (H_3) is satisfied. Moreover, $f(t,\alpha_0) = f(t,\frac{1}{4}) = \tanh(\frac{1}{4}) \geq 0$ for each $t \in [0,1]$, and so the hypothesis (H_4) holds. If we take $r = 1$ and $M = 2$, all the conditions of Theorem 3.2 are satisfied. Hence, the HIGDE (5.1) has a local solution x^* in the closed ball $B_1^2[\frac{1}{4}]$ of $C(J,\mathbb{R})$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = \frac{1}{4}, \;\; t \in [0,1],$$

$$x_{n+1}(t) = \frac{1}{4} + \int_0^t (t-s)\, \tanh x_n(s)\, ds, \;\; t \in [0,1],$$

converges monotone nondecreasingly to x^*.

Example 5.2. *Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the IVP of nonlinear first order HIGDE,*

$$\frac{dx}{dt} = \frac{1}{2} \int_0^t \tan^{-1} x(s)\,, \;\; t \in [0,1]; \quad x(0) = \frac{1}{4}. \tag{5.2}$$

Here $\alpha_0 = \dfrac{1}{4}$ and $f(t,x) = \dfrac{1}{2}\, \tan^{-1} x$ for $(t,x) \in [0,1] \times \mathbb{R}$. We show that f satisfies all the conditions of Theorem 3.3. Clearly, f is bounded on $[0,1] \times \mathbb{R}$ with bound $M_f = \frac{22}{28}$ and so, the hypothesis (H_1) is satisfied. Next, let $x, y \in \mathbb{R}$ be such that $x \geq y$. Then there exists a constant ξ with $x_1 < \xi < y$ satisfying

$$0 \leq f(t,x) - f(t,y) \leq \frac{1}{2} \cdot \frac{1}{1+\xi^2}(x-y) \leq \frac{1}{2} \cdot (x-y),$$

for all $t \in [0,1]$. So the hypothesis (H_2) holds with $k = \frac{1}{2}$. Moreover, $f(t,\alpha_0) = f\left(t, \frac{1}{4}\right) = \frac{1}{2}\, \tan^{-1}\left(\frac{1}{4}\right) \geq 0$ for each $t \in [0,1]$, and so the hypothesis (H_4) holds. If we take $r = 2$, then we have

$$M_f a = \frac{11}{14} \leq \left(1 - \frac{1}{2}\right) \cdot 2 = (1 - ka^2) r,$$

and so, the condition (3.6) is satisfied. Thus, all the conditions of Theorem 3.3 are satisfied. Hence, the HIGDE (5.2) has a unique local solution x^* in the closed ball $B_2[\frac{1}{4}]$ of $C(J,\mathbb{R})$. This further in view of Remark 3.5 implies that the HDE (5.2) has a unique local positive solution x^* and and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = \frac{1}{4}, \;\; t \in [0,1],$$

$$x_{n+1}(t) = \frac{1}{4} + \int_0^t (t-s)\, \tan^{-1} x_n(s)\, ds, \;\; t \in [0,1],$$

converges monotone nondecreasingly to x^*. Moreover, the unique local solution x^* is Ulam-Hyers stable on $[0,1]$ in view of Definition 4.1. Consequently the HIGDE (5.2) is a locally Ulam-Hyers stable on the interval $[0,1]$.

6. Concluding Remark

Finally, while concluding this paper, we remark that unlike the Schauder fixed point theorem we do not require any convexity argument in the proof of main existence theorem, Theorem 3.2. Similarly, we do not require the usual Lipschitz condition in the proof of uniqueness theorem, Theorem 3.3, but a weaker form of one sided or partial Lipschitz condition is enough to serve the purpose. However, in both the cases we are able to achieve the approximation of local solution by monotone convergence of the successive approximations. Moreover, for simplicity and in order to illustrate the underlined ideas and the procedure of finding the approximate solution, a simple form of a integrodifferential equation (1.1) is considered in this paper, however other complex nonlinear IVPs of HDEs with integer or fractional orders may also be considered and the present study can also be extended to such sophisticated nonlinear inegrodifferential equations with appropriate modifications. These and other such problems form the further research scope in the subject of nonlinear differential and integral equations with applications. Some of the results in this direction will be reported elsewhere.

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On determination of discontinuous Sturm-Liouville operator from Weyl function

OZGE AKCAY KARAKUS *[1]

[1] *Faculty of Engineering, Department of Computer Engineering, Munzur University, 62000 Tunceli, Turkey.*

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Abstract. In this paper, the Weyl function for the Sturm-Liouville operator which contains the discontinuous coefficient and discontinuity conditions at an interior point of the finite interval is defined and examined. The uniqueness theorem of solution of the inverse spectral problem for the discontinuous Sturm-Liouville operator according to Weyl function is proved.

AMS Subject Classifications: 34B24, 34A55, 47E05.

Keywords: Inverse problem, Sturm-Liouville equation, discontinuity conditions and discontinuous coefficient, Weyl function.

Contents

1. Introduction and Background

This paper deals with the discontinuous Sturm-Liouville operator which contains both the discontinuous coefficient and the discontinuity conditions at an interior point $x = \xi \in (0, \pi)$ of the finite interval:

$$-\omega'' + q(x)\omega = \tau^2 r(x)\omega, \quad 0 < x < \pi \tag{1.1}$$

$$\omega\left(\xi + 0\right) = c\omega\left(\xi - 0\right), \quad \omega'\left(\xi + 0\right) = c^{-1}\omega'\left(\xi - 0\right) \tag{1.2}$$

$$\omega'(0) - b_1\omega(0) = 0, \quad \omega'(\pi) + b_2\omega(\pi) = 0, \tag{1.3}$$

where real valued function $q(x)$ belongs to $L_2(0, \pi)$, $c > 0$, b_1 and b_2 are real constants, τ is a spectral parameter, the discontinuous coefficient $r(x)$ is in the following form:

$$r(x) = \begin{cases} 1, & 0 < x < \xi, \\ a^2, & \xi < x < \pi, \end{cases}$$

$0 < a \neq 1$ and assume that $\xi > \frac{a\pi}{a+1}$.

In recent years, many works on the discontinuous boundary value problems have been done and there has been a significant increase in interest on this subject. We indicate that such problems are connected with discontinuous material properties, so the investigations on this problems are attractive in the mathematics, physics and engineering (for details see [8]).

*Corresponding author. Email address: **ozgeakcay@munzur.edu.tr** (Ozge AKCAY KARAKUS)

The purpose of this study is to examine the inverse spectral problem for the discontinuous Sturm-Liouville problem (1.1)-(1.3) and this problem is stated in the following way: given the Weyl function, construct the boundary value problem (1.1)-(1.3). Therefore, firstly we define and examine the Weyl function of the problem (1.1)-(1.3) and then the uniqueness theorem for the solution of this inverse spectral problem is proved.

Differently from other studies, considered problem contains both the discontinuous coefficient $r(x)$ and the discontinuity conditions at $x = \xi \in (0, \pi)$. In the special cases, i.e., as $c = 1$, the inverse problems for Sturm-Liouville operator with discontinuous coefficient by Weyl function are examined in [1, 4, 11] and as $r(x) \equiv 1$, the inverse problems for Sturm-Liouville operator with discontinuity conditions by Weyl function are investigated in [6, 7]. Moreover, the various works on the inverse problems for the discontinuous Sturm-Liouville operators can be given as follows: [3, 5, 9, 10, 12–18] and the references therein.

The spectral properties of the boundary value problem (1.1)-(1.3) are studied in [2]; namely, the integral representation of the solution of (1.1) with discontinuity conditions (1.2) is obtained and using this solution, the asymptotic formulas of the eigenvalues and eigenfunctions of this problem are investigated. Note that the constructed integral representation is not transformation operator, moreover; the kernel of this solution has a discontinuity along the line $t = -a(x - \xi) + a$, for $\xi < a < \pi$. Unlike other studies, using this constructed integral representation we prove the uniqueness theorem of the inverse spectral problem (1.1)-(1.3) by the Weyl function.

Theorem 1.1. *[2] The integral representation of the solution $f(x, \tau)$ of equation (1.1) with discontinuity conditions (1.2) satisfying the conditions $f(0, \tau) = 1$, $f'(0, \tau) = i\tau$ has the form:*

$$f(x, \tau) = f_0(x, \tau) + \int_{-\alpha(x)}^{\alpha(x)} k(x, t)e^{i\tau t}dt, \tag{1.4}$$

where

$$f_0(x, \tau) = \begin{cases} e^{i\tau x}, & 0 < x < \xi, \\ \kappa_1 e^{i\tau(a(x-\xi)+\xi)} + \kappa_2 e^{i\tau(-a(x-\xi)+\xi)}, & \xi < x < \pi, \end{cases}$$

with $\kappa_1 = \frac{1}{2}\left(c + \frac{1}{ac}\right)$ and $\kappa_2 = \frac{1}{2}\left(c - \frac{1}{ac}\right)$,

$$\alpha(x) = \begin{cases} x, & 0 < x < \xi, \\ a(x - \xi) + \xi, & \xi < x < \pi, \end{cases}$$

the kernel $k(x, .) \in L_1(-\alpha(x), \alpha(x))$ for each fixed $x \in (0, \pi)$ and satisfies the inequality

$$\int_{-\alpha(x)}^{\alpha(x)} |k(x, t)|dt \leq e^{p\sigma(x)} - 1$$

with

$$\sigma(x) = \int_0^x (x - u)|q(u)|du, \quad p = (a + 4)|\kappa_1| + (a + 2)|\kappa_2|.$$

Remark 1.2. *The function $k(x, t)$ has following properties:*

$$k(x, \alpha(x)) = \begin{cases} \frac{1}{2}\int_0^x q(u)du, & 0 < x < \xi, \\ \frac{\kappa_1}{2}\int_0^x \frac{1}{\sqrt{r(u)}}q(u)du, & \xi < x < \pi, \end{cases}$$

$$k(x, -a(x - \xi) + \xi + 0) - k(x, -a(x - \xi) + \xi - 0) = \frac{-\kappa_2}{2}\left(\int_0^\xi q(u)du - \frac{1}{a}\int_\xi^x q(u)du\right), \quad \xi < x < \pi,$$

$$k(x, -\alpha(x)) = 0.$$

Moreover, it is seen that the real-valued function $k(x, t)$ has a discontinuity along the line $t = -a(x - \xi) + \xi$ for $\xi < x < \pi$.

Now, take into account the case of $b_1 = \infty$ in the boundary condition (1.3). Then, the boundary condition is as follows:

$$\omega(0) = \omega'(\pi) + b_2\omega(\pi) = 0 \tag{1.5}$$

and consider the boundary value problems (1.1)-(1.3) and (1.1),(1.2),(1.5).

Denote $u(x,\tau)$ and $v(x,\tau)$ by the solutions of the equation (1.1) with the condition (1.2) under the initial conditions

$$u(0,\tau) = 1, \quad u'(0,\tau) = b_1,$$

$$v(0,\tau) = 0, \quad v'(0,\tau) = 1.$$

Using the integral representation (1.4), we express the solutions $u(x,\tau)$ and $v(x,\tau)$ in the following forms:

$$u(x,\tau) = u_0(x,\tau) + \int_0^{\alpha(x)} \left(h(x,t)\cos\tau t + \tilde{h}(x,t)\frac{b_1 \sin\tau t}{\tau} dt \right),$$

and

$$v(x,\tau) = v_0(x,\tau) + \int_0^{\alpha(x)} \tilde{h}(x,t)\frac{\sin\tau t}{\tau} dt,$$

where for $0 < x < \xi$:

$$u_0(x,\tau) = \cos\tau x + \frac{b_1 \sin\tau x}{\tau}, \quad v_0(x,\tau) = \frac{\sin\tau x}{\tau}$$

and for $\xi < x < \pi$:

$$u_0(x,\tau) = \kappa_1\left(\cos\tau\upsilon^+(x) + \frac{b_1\sin\tau\upsilon^+(x)}{\tau}\right) + \kappa_2\left(\cos\tau\upsilon^-(x) + \frac{b_1\sin\tau\upsilon^-(x)}{\tau}\right),$$

$$v_0(x,\tau) = \kappa_1\frac{\sin\tau\upsilon^+(x)}{\tau} + \kappa_2\frac{\sin\tau\upsilon^-(x)}{\tau}$$

with $\upsilon^{\pm}(x) = \pm a(x-\xi) + \xi$, $h(x,t) = k(x,t) + k(x,-t)$ and $\tilde{h}(x,t) = k(x,t) - k(x,-t)$, respectively.

Let $\phi(x,\tau)$ be the solution of the equation (1.1) with the condition (1.2) under the initial conditions

$$\phi(\pi,\tau) = -1, \quad \phi'(\pi,\tau) = b_2.$$

The characteristic functions $\chi(\tau)$ and $\varphi(\tau)$ of the problems (1.1)-(1.3) and (1.1), (1.2) and (1.5) can be given as follows:

$$\chi(\tau) = u'(\pi,\tau) + b_2 u(\pi,\tau) = \phi'(0,\tau) - b_1\phi(0,\tau) \tag{1.6}$$

$$\varphi(\tau) = v'(\pi,\tau) + b_2 v(\pi,\tau) = -\phi(0,\tau), \tag{1.7}$$

respectively. It is known from [2] that

$$|\chi(\tau)| \geq C_\delta|\tau|e^{|Im\tau|\upsilon^+(\pi)}, \quad \tau \in G_\delta, \tag{1.8}$$

where $G_\delta = \{\tau : |\tau - \tilde{\tau}_n| \geq \delta\}$, here $\tilde{\tau}_n = \frac{n\pi}{\upsilon^+(\pi)} + d_n$, $\sup_n |d_n| = d < \infty$ and $\delta << \frac{s}{2}$ is a sufficiently small positive number with $s = \inf_{n\neq k}|\tilde{\tau}_n - \tilde{\tau}_k| > 0$. Moreover, from the expression of the solution $v(x,\tau)$, we have

$$|\varphi(\tau)| \leq Ce^{|Im\tau|\upsilon^+(\pi)}. \tag{1.9}$$

Theorem 1.3. *[2] The boundary value problem (1.1)-(1.3) has a countable set of eigenvalues $\{\tau_n^2\}_{n\geq 1}$:*

$$\tau_n = \tilde{\tau}_n + \frac{s_n}{\tilde{\tau}_n} + \frac{t_n}{n},$$

where s_n is a bounded sequence and $\{t_n\} \in l_2$.

The norming constants γ_n of the problem (1.1)-(1.3) are defined by

$$\gamma_n = \int_0^\pi u^2(x,\tau)r(x)dx.$$

Moreover, the following asymptotic formulas of the solutions $u(x,\tau)$, $v(x,\tau)$ and $\phi(x,\tau)$ are valid for $|\tau| \to \infty$:

$$u(x,\tau) = O(e^{|Im\tau|\alpha(x)}), \quad v(x,\tau) = O\left(\frac{e^{|Im\tau|\alpha(x)}}{|\tau|}\right)$$
$$\phi(x,\tau) = O(e^{|Im\tau|(\alpha(\pi)-\alpha(x))}). \tag{1.10}$$

Note that when $q(x) \equiv 0$ in the equation (1.1), the solution $\phi_0(x,\tau)$ has the representation:

$$\begin{aligned}\phi_0(x,\tau) &= -a(\kappa_1 \cos\tau(\upsilon^+(\pi)-x) - \kappa_2 \cos\tau(\upsilon^-(\pi)-x)) \\ &\quad -b_2\left(\kappa_1 \frac{\sin\tau(\upsilon^+(\pi)-x)}{\tau} + \kappa_2 \frac{\sin\tau(\upsilon^-(\pi)-x)}{\tau}\right), \quad 0<x<\xi,\end{aligned}$$

$$\phi_0(x,\tau) = -\cos\tau(\upsilon^+(\pi)-\upsilon^+(x)) - \frac{b_2 \sin\tau(\upsilon^+(\pi)-\upsilon^+(x))}{a\tau}, \quad \xi<x<\pi.$$

2. Main Results

Now, let us examine the Weyl solution and Weyl function for the boundary value problem (1.1)-(1.3).

Denote $\psi(x,\tau)$ by a solution of the equation (1.1) with the condition (1.2) satisfying the conditions

$$\psi'(0,\tau) - b_1\psi(0,\tau) = 1, \quad \psi'(\pi,\tau) + b_2\psi(\pi,\tau) = 0.$$

Then, it is obtained that

$$\psi(x,\tau) = \frac{\phi(x,\tau)}{\chi(\tau)} = v(x,\tau) + m(\tau)u(x,\tau), \tag{2.1}$$

where $m(\tau) = \psi(0,\tau)$. The functions $\psi(x,\tau)$ and $m(\tau)$ are called the *Weyl solution* and *Weyl function*, respectively. Moreover, taking into account (1.7), we can write

$$m(\tau) = \frac{\phi(0,\tau)}{\chi(\tau)} = -\frac{\varphi(\tau)}{\chi(\tau)}. \tag{2.2}$$

Hence, it can be seen that Weyl function $m(\tau)$ is meromorphic function with simple poles in the points $\tau = \tau_n$, $n \geq 1$. The squares of the poles and zeros of $m(\tau)$ coincide with the eigenvalues of the problems (1.1)-(1.3) and (1.1), (1.2), (1.5), respectively.

Theorem 2.1. *The representation is valid:*

$$m(\tau) = \sum_{n=1}^{\infty} \frac{1}{\gamma_n(\tau^2 - \tau_n^2)}. \tag{2.3}$$

Proof. Taking into account (1.8), (1.9) and (2.2) we have for sufficiently large $\tau^* > 0$

$$|m(\tau)| \leq \frac{C_\delta}{|\tau|}, \quad \tau \in G_\delta, \ |\tau| \geq \tau^*. \tag{2.4}$$

Using the relations $\dot{\chi}(\tau_n) = 2\tau_n\gamma_n\mu_n$ and $\phi(x,\tau_n) = \mu_n u(x,\tau_n)$ with $\mu_n \neq 0$ (see [2]), we find $\varphi(\tau_n) = -\phi(0,\tau_n) = -\mu_n$. Then, it follows from this relation that

$$\underset{\tau=\tau_n}{Res}\, m(\tau) = -\frac{\varphi(\tau_n)}{\dot{\chi}(\tau_n)} = \frac{1}{2\tau_n\gamma_n}. \tag{2.5}$$

Now, consider the contour integral

$$J_N(\tau) = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{m(\zeta)}{\zeta-\tau}d\zeta, \quad \tau \in int\Gamma_N,$$

where $\Gamma_N = \{\tau : |\tau| = |\tilde{\tau}_n| + \frac{s}{2}\}$. It follows from (2.4) that $\lim\limits_{N\to\infty} J_N(\tau) = 0$. Moreover, applying the residue theorem and from (2.5), we find

$$\begin{aligned} J_N(\tau) &= m(\tau) + \sum_{n=1}^{N} \frac{1}{2\tau_n\gamma_n}\left(\frac{1}{(\tau_n-\tau)} - \frac{1}{(\tau_n+\tau)}\right) \\ &= m(\tau) - \sum_{n=1}^{N} \frac{1}{\gamma_n(\tau^2-\tau_n^2)}. \end{aligned}$$

Thus, as $N \to \infty$, since $\lim\limits_{N\to\infty} J_N(\tau) = 0$, we obtain the relation (2.3). ■

Now, we examine the inverse problem indicated in the following way: given the Weyl function $m(\tau)$, determine the boundary value problem (1.1)-(1.3).

Let us demonstrate the uniqueness theorem of the solution for this inverse problem. Then, we specify the boundary value problem (1.1)-(1.3) as $L = L(q(x), b_1, b_2)$ and we take the problem $\hat{L} = L(\hat{q}(x), \hat{b}_1, \hat{b}_2)$ which has a similar form to L but with different potential and coefficients in the boundary conditions.

Theorem 2.2. *If $m(\tau) = \hat{m}(\tau)$, then $L = \hat{L}$. Namely, the Weyl function uniquely determines the problem (1.1)-(1.3).*

Proof. Denote the matrix $U(x,\tau) = [U_{k\ell}(x,\tau)]_{k,\ell=1,2}$ by the relation

$$U(x,\tau)\begin{pmatrix} \hat{u}(x,\tau) & \hat{\psi}(x,\tau) \\ \hat{u}'(x,\tau) & \hat{\psi}'(x,\tau)\end{pmatrix} = \begin{pmatrix} u(x,\tau) & \psi(x,\tau) \\ u'(x,\tau) & \psi'(x,\tau)\end{pmatrix}. \tag{2.6}$$

It follows from the equality

$$\langle u(x,\tau), \psi(x,\tau)\rangle = 1 \tag{2.7}$$

and the formula (2.6) that

$$\begin{aligned} U_{k1}(x,\tau) &= u^{(k-1)}(x,\tau)\hat{\psi}'(x,\tau) - \psi^{(k-1)}(x,\tau)\hat{u}'(x,\tau), \\ U_{k2}(x,\tau) &= \psi^{(k-1)}(x,\tau)\hat{u}(x,\tau) - u^{(k-1)}(x,\tau)\hat{\psi}(x,\tau), \quad k=1,2 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} u(x,\tau) &= U_{11}(x,\tau)\hat{u}(x,\tau) + U_{12}(x,\tau)\hat{u}'(x,\tau), \\ \psi(x,\tau) &= U_{11}(x,\tau)\hat{\psi}(x,\tau) + U_{12}(x,\tau)\hat{\psi}'(x,\tau). \end{aligned} \tag{2.9}$$

Using (2.1), (2.7) and (2.8), we obtain

$$U_{11}(x,\tau) = 1 + u(x,\tau)\left(\frac{\hat{\phi}'(x,\tau)}{\hat{\chi}(\tau)} - \frac{\phi'(x,\tau)}{\chi(\tau)}\right) + \frac{\phi(x,\tau)}{\chi(\tau)}(u'(x,\tau) - \hat{u}'(x,\tau))$$

and

$$U_{12}(x,\tau) = \hat{u}(x,\tau)\frac{\phi(x,\tau)}{\chi(\tau)} - u(x,\tau)\frac{\hat{\phi}(x,\tau)}{\hat{\chi}(\tau)}.$$

With the help of the asymptotic formulas (1.10) and the inequality (2.4), we find

$$\lim_{\substack{|\tau|\to\infty \\ \tau\in G_\delta}} \max_{0\le x\le\pi} |U_{11}(x,\tau)-1| = \lim_{\substack{|\tau|\to\infty \\ \tau\in G_\delta}} \max_{0\le x\le\pi} |U_{12}(x,\tau)| = 0. \tag{2.10}$$

According to (2.1) and (2.8), we can write

$$U_{11}(x,\tau) = u(x,\tau)\hat{v}'(x,\tau) - v(x,\tau)\hat{u}'(x,\tau) + u(x,\tau)\hat{u}'(x,\tau)(\hat{m}(\tau) - m(\tau)),$$

$$U_{12}(x,\tau) = v(x,\tau)\hat{u}(x,\tau) - u(x,\tau)\hat{v}(x,\tau) + u(x,\tau)\hat{u}(x,\tau)(m(\tau) - \hat{m}(\tau)).$$

If $m(\tau) = \hat{m}(\tau)$, then the functions $U_{11}(x,\tau)$ and $U_{12}(x,\tau)$ are entire in τ and according to (2.10), we have $U_{11}(x,\tau) \equiv 1$ and $U_{12}(x,\tau) \equiv 0$. Putting these relations into (2.9), we find $u(x,\tau) \equiv \hat{u}(x,\tau)$ and $\psi(x,\tau) \equiv \hat{\psi}(x,\tau)$, thus we obtain $L = \hat{L}$. As a result, it is shown that the problem (1.1)-(1.3) is uniquely determined by the Weyl function $m(\tau)$. ■

Remark 2.3. *Taking into account the Weyl function expansion (2.3), it can be seen that the Weyl function $m(\tau)$ is represented by the spectral data $\{\tau_n^2, \gamma_n\}_{n\ge1}$. Then, we can state that the spectral data $\{\tau_n^2, \gamma_n\}_{n\ge1}$ uniquely determines the boundary value problem (1.1)-(1.3).*

Considering the relation (2.2) it is appeared that the poles and zeros of the Weyl function $m(\tau)$ coincide with the zeros τ_n and λ_n of the characteristic functions $\chi(\tau)$ and $\varphi(\tau)$, respectively. Thus, the Weyl function $m(\tau)$ is determined by two spectra $\{\tau_n^2\}$ and $\{\lambda_n^2\}$ and the problem (1.1)-(1.3) is uniquely specified by two spectra.

Consequently, the inverse problems of the boundary value problem (1.1)-(1.3) by spectral data and two spectra are special cases of the inverse problem of the problem (1.1)-(1.3) by Weyl function.

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Screen invariant lightlike hypersurfaces of almost product-like statistical manifolds and locally product-like statistical manifolds

Ömer Aksu[1], **Esra Erkan**[*2] **and Mehmet Gülbahar**[3]

[1,2,3] *Faculty of Arts and Sciences, Department of Mathematics, Harran University, Şanlıurfa, Turkey.*

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Abstract. The main formulas and relations are presented for screen invariant lightlike hypersurfaces. Concurrent and recurrent vector fields are investigated and several formulas are obtained for screen invariant lightlike hypersurfaces.

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Keywords: Hypersurface, statistical structure, product manifold.

Contents

1. Introduction

One of the techniques to characterize a Riemannian manifold is to review the geometry of some appropriate vector fields. The appropriate vector fields that have been widely studied in the literature recently are torse-forming, concircular concurrent, geodesic and recurrent vector fields, etc. The impression of concurrent vector fields is firstly announced by K. Yano [22] in such a way:

Let (L, h) be a Riemannian manifold equipped with a metric h and D be the Riemannian connection on (L, h). A vector field ζ is entitled concurrent if

$$D_Z \zeta = Z$$

holds for each tangent vector field Z.

There exist remarkable applications dealing with concurrent vector fields into submanifolds of Riemannian manifolds admitting differential structures [10, 13, 14, 18, 23, 24], etc. Besides these facts, statistical structures on Riemannian manifolds have been widely studied lately with interesting geometrical properties. The impression of statistical manifolds was initially announced by S. Amari [2] and the basic properties of hypersurfaces were

*Corresponding author. Email address: **esraerkan@harran.edu.tr** (Esra ERKAN)

revealed by H. Furuhata in [11, 12]. Later, this concept admitting complex, contact and product structures was examined by various authors in [4, 5, 15–17, 21].

An interesting perspective on statistical manifolds came from K. Takano's definition of Hermite-like manifolds, which is a generalization of Hermitian manifolds. A Riemannian manifold (L, h) included two almost complex structures J and $J^{\star}$ is entitled a Hermite-like manifold [19, 20] if

$$h(JZ_1, Z_2) = -h(Z_1, J^{\star}Z_2)$$

holds for each tangent vector fields Z_1 and Z_2. One of the interesting aspects of Hermite-like manifolds is that although there are no examples in classical Euclidean spaces, there are examples of Hermite-like manifolds in non-Euclidean geometry. With a similar idea, product-like manifolds were introduced and the geometry of some special type hypersurfaces of these manifolds was investigated in [1, 7].

The primary objective of this paper is to review screen invariant lightlike hypersurfaces of an almost product-like statistical manifold. With the aid of statistical structures, some main formulas and relations are obtained and concurrent vector fields are examined on these hypersurfaces.

2. Almost product-like manifolds and their lightlike hypersurfaces

A differentiable manifold $\widetilde{L}$ is entitled an almost product manifold if it includes a tensor field providing $F^2 = I$, where I expresses the identity transformation. We note that the eigenvalues of F are $+1$ and -1. If we put

$$T = \frac{1}{2}(I + F), \;\; Q = \frac{1}{2}(I - F)$$

then we find

$$T + Q = I, \;\; T^2 = T, \;\; Q^2 = Q, \;\; TQ = QT = 0$$

and

$$F = T - Q.$$

If a Riemannian metric $\widetilde{h}$ on $\widetilde{L}$ provides

$$\widetilde{h}(FZ_1, Z_2) = \widetilde{h}(Z_1, FZ_2) \tag{2.1}$$

for each $Z_1, Z_2 \in \Gamma(T\widetilde{L})$, then $(\widetilde{L}, \widetilde{h}, F)$ is called an almost product Riemannian manifold.

Now, we remind the following definition [7]:

Definition 2.1. *Let F and $F^{\star}$ be two almost product structures on $\widetilde{L}$. If the equation*

$$\widetilde{h}(FZ_1, Z_2) = \widetilde{h}(Z_1, F^{\star}Z_2) \tag{2.2}$$

is provided then $(\widetilde{L}, \widetilde{h}, F)$ is entitled an almost product-like semi-Riemannian manifold.

If we indite FZ_1 in place of of Z_1 in (2.2), we obtain that

$$\widetilde{h}(FZ_1, F^{\star}Z_2) = \widetilde{h}(Z_1, Z_2) \tag{2.3}$$

is provided.

Example 2.2. *Let F be a tensor field on $\mathbb{R}^4_1$ such that*

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then we find $(\mathbb{R}_1^4, F)$ *is an almost product manifold. If we write*

$$\widetilde{h} = \begin{bmatrix} -e^{x_1} & 0 & 0 & 0 \\ 0 & e^{x_1} & 0 & 0 \\ 0 & 0 & e^{x_1} & 0 \\ 0 & 0 & 0 & e^{x_1} \end{bmatrix} \quad \textit{and} \quad F^{\star} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

then we obtain $(\mathbb{R}_1^4, \widetilde{h}, F)$ *is provided (2.2).*

Presume that $\widetilde{D}$ is a torsion-free connection on $(\widetilde{L}, \widetilde{h}, F)$. If $\widetilde{D}g$ is symmetric, then $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is entitled an almost product-like statistical manifold. For each $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$, we indite another torsion-free connection satisfying

$$Z_3\widetilde{h}(Z_1, Z_2) = \widetilde{h}(\widetilde{D}_{Z_3}Z_1, Z_2) + \widetilde{h}(Z_1, \widetilde{D}^{\star}_{Z_3}Z_2) \tag{2.4}$$

for each $Z_1, Z-2, Z_3 \in \Gamma(T\widetilde{L})$. $\widetilde{D}^{\star}$ is called the dual connection of $\widetilde{D}$. In addition, we indite

$$\widetilde{D}^0_{Z_1}Z_2 = \frac{1}{2}(\widetilde{D}_{Z_1}Z_2 + \widetilde{D}^{\star}_{Z_1}Z_2), \tag{2.5}$$

where $\widetilde{D}^0$ is the Levi-Civita connection of $(\widetilde{L}, \widetilde{h}, F)$.

Definition 2.3. *Let* $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ *be an almost product-like statistical manifold. If* F *is parallel with regard to* $\widetilde{D}$*, then* $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ *is entitled a locally product-like statistical manifold.*

In view of (2.4), we find the following equation is satisfied:

$$\widetilde{h}((\widetilde{D}_{Z_1}F)Z_2, Z_3) = \widetilde{h}(Z_2, (\widetilde{D}^{\star}_{Z_1}F^{\star})Z_3). \tag{2.6}$$

From (2.6), it is clear that

$$\widetilde{D}F = 0 \Leftrightarrow \widetilde{D}^{\star}F^{\star} = 0.$$

Therefore, $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is a locally product-like statistical manifold if and only if so is $(\widetilde{L}, \widetilde{h}, \widetilde{D}^{\star}, F^{\star})$.

Example 2.4. *Let* $(\mathbb{R}_1^4, \widetilde{h}, F)$ *be an almost product-like Lorentzian manifold of Example 2.2. By a straightforward computation, we put*

$$\begin{aligned}
&\widetilde{D}_{\partial_1}\partial_1 = \widetilde{D}_{\partial_4}\partial_4 = \frac{1}{2}\partial_1, \\
&\widetilde{D}_{\partial_1}\partial_2 = \widetilde{D}_{\partial_2}\partial_1 = \widetilde{D}_{\partial_3}\partial_4 = \widetilde{D}_{\partial_4}\partial_3 = \frac{1}{2}\partial_2, \\
&\widetilde{D}_{\partial_1}\partial_3 = \widetilde{D}_{\partial_3}\partial_1 = \widetilde{D}_{\partial_2}\partial_4 = \widetilde{D}_{\partial_4}\partial_2 = \frac{1}{2}\partial_3, \\
&\widetilde{D}_{\partial_1}\partial_4 = \widetilde{D}_{\partial_4}\partial_1 = \frac{1}{2}\partial_4, \\
&\widetilde{D}_{\partial_2}\partial_2 = \widetilde{D}_{\partial_3}\partial_3 = \frac{1}{2}\partial_1 + \Gamma^2_{22}\partial_2 + \Gamma^3_{22}\partial_3, \\
&\widetilde{D}_{\partial_2}\partial_3 = \widetilde{D}_{\partial_3}\partial_2 = \Gamma^3_{22}\partial_2 + \Gamma^2_{22}\partial_3 + \frac{1}{2}\partial_4
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{D}^{\star}_{\partial_1}\partial_1 = \widetilde{D}^{\star}_{\partial_4}\partial_4 = \frac{1}{2}\partial_1,\\
&\widetilde{D}^{\star}_{\partial_1}\partial_2 = \widetilde{D}^{\star}_{\partial_2}\partial_1 = -\widetilde{D}^{\star}_{\partial_3}\partial_4 = -\widetilde{D}^{\star}_{\partial_4}\partial_3 = \frac{1}{2}\partial_2,\\
&\widetilde{D}^{\star}_{\partial_1}\partial_3 = \widetilde{D}^{\star}_{\partial_3}\partial_1 = -\widetilde{D}^{\star}_{\partial_2}\partial_4 = -\widetilde{D}^{\star}_{\partial_4}\partial_2 = \frac{1}{2}\partial_3,\\
&\widetilde{D}^{\star}_{\partial_1}\partial_4 = \widetilde{D}^{\star}_{\partial_4}\partial_1 = \frac{1}{2}\partial_4,\\
&\widetilde{D}^{\star}_{\partial_2}\partial_2 = \widetilde{D}^{\star}_{\partial_3}\partial_3 = \frac{1}{2}\partial_1 - \Gamma^2_{22}\partial_2 - \Gamma^3_{22}\partial_3,\\
&\widetilde{D}^{\star}_{\partial_2}\partial_3 = \widetilde{D}^{\star}_{\partial_3}\partial_2 = -\Gamma^3_{22}\partial_2 - \Gamma^2_{22}\partial_3 - \frac{1}{2}\partial_4,
\end{aligned}$$

where Γ^2_{22} and Γ^3_{22} are any functions on $\mathbb{R}^4_1$ and $\{\partial_1,\partial_2,\partial_3,\partial_4\}$ is the natural basis of $\mathbb{R}^4_1$. Then we obtain that $(\mathbb{R}^4_1,\widetilde{h},\widetilde{D},F)$ is a locally product-like statistical manifold.

Let (L,h) be a hypersurface of $(\widetilde{L},\widetilde{h},F)$ with the induced metric h from $\widetilde{h}$. If h is degenerate on L, then (L,h) is entitled a lightlike hypersurface. For any lightlike hypersurface, the radical distribution $Rad(TL)$ is given as follows:

$$Rad(TL) = \text{span}\{\xi : h(\xi,Z) = 0, \forall Z \in \Gamma(TL)\}.$$

Denote a complementary vector bundle of $Rad(TL)$ in TL by $S(TL)$. The distribution $S(TL)$ is called a screen distribution of (L,h) and thus we write

$$TL = Rad(TL) \oplus_{orth} S(TL),$$

where $\oplus_{orth}$ stands for the orthogonal direct sum. It is known that the screen distribution is not unique since h is degenerate. There is a unique null section N providing

$$\widetilde{h}(\xi,N) = 1,\ \ \widetilde{h}(N,N) = \widetilde{h}(N,Z) = 0$$

for any $Z \in \Gamma(S(TL))$. We note that the vector bundle $ltr(TL) = \text{span}\{N\}$ is called the transversal bundle of $(L,h,S(TL))$ [8, 9].

The Gauss and Weingarten formulas with regard to the Levi-Civita connection $\widetilde{\nabla}^0$ are formulated by

$$\widetilde{D}^0_{Z_1}Y = D^0_{Z_1}Y + B^0(Z_1,Y)N \tag{2.7}$$

and

$$\widetilde{D}^0_{Z_1}N = -A^0_N Z_1 + \tau^0(Z_1)N, \tag{2.8}$$

where D^0 is the induced connection, A^0_N is the shape operator and τ^0 is a $1-$form.

The hypersurface $(L,h,S(TL))$ is called

i) totally geodesic if $B^0 = 0$,

ii) totally umbilical if there is a differentiable function μ such that $B^0(Z_1,Z_2) = \mu h(Z_1,Z_2)$,

iii) minimal if $trace_{S(TL)}B^0 = 0$, where $trace_{S(TM)}$ is the trace with regard to $S(TL)$.

Similar formulas and definitions could be given with regard to $\widetilde{D}$.

The Gauss and Weingarten type formulas with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$ is written by

$$\widetilde{D}_{Z_1}Y = D_{Z_1}Y + B(Z_1,Z_2)N, \tag{2.9}$$

$$\widetilde{D}_{Z_1}N = -A^{\star}_N Z_1 + \tau^{\star}(Z_1)N \tag{2.10}$$

and

$$\widetilde{D}^{\star}_{Z_1} Z_2 = D^{\star}_{Z_1} Z_2 + B^{\star}(Z_1, Z_2)N, \tag{2.11}$$
$$\widetilde{D}^{\star}_{Z_1} N = -A_N Z_1 + \tau(Z_1)N, \tag{2.12}$$

where $D_X Y, DZ_1{}^{\star} Z_2, A_N Z_1, A^{\star}_N Z_1 \in \Gamma(TL)$, τ and $\tau^{\star}$ are $1-$forms. Also, the Gauss and Weingarten type formulas on $S(TL)$ could be given as follows:

$$D_{Z_1} P Z_2 = \overline{D}_{Z_1} P Z_2 + C(Z_1, PZ_2)\xi, \tag{2.13}$$
$$D_{Z_1}\xi = -\overline{A}_{\xi} Z_1 - \tau(Z_1)\xi \tag{2.14}$$

and

$$D^{\star}_{Z_1} P Z_2 = \overline{D}^{\star}_{Z_1} P Z_2 + C^{\star}(Z_1, PZ_2)\xi, \tag{2.15}$$
$$D^{\star}_{Z_1}\xi = -\overline{A}^{\star}_{\xi} Z_1 - \tau^{\star}(Z_1)\xi, \tag{2.16}$$

where P is the projection morphism from $\Gamma(TL)$ onto $\Gamma(S(TL))$, $\overline{D}_{Z_1} P Z_2, \overline{D}^{\star}_{Z_1} P Z_2 \in \Gamma(S(TL))$ and $\overline{A}_{\xi}, \overline{A}^{\star}_{\xi} \in \Gamma(S(TL))$.

A lightlike hypersurface $(L, h, S(TL))$ is called screen conformal with regard to $\widetilde{D}$ if there exists a smooth function α satisfying

$$A_N = \alpha A_{\xi} \tag{2.17}$$

and it is called screen conformal with regard to $\widetilde{D}^{\star}$ if there exists a smooth function $\alpha^{\star}$ satisfying

$$A^{\star}_N = \alpha^{\star}\overline{A}^{\star}_{\xi}. \tag{2.18}$$

Furthermore, the following concepts could be given:

A lightlike hypersurface $(L, h, S(TL))$ of $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is called

i) totally geodesic with regard to $\widetilde{D}$ if $B = 0$,

ii) totally geodesic with regard to $\widetilde{D}^{\star}$ if $B^{\star} = 0$,

iii) $S(TL)-$geodesic with regard to $\widetilde{D}$ if $C = 0$,

iv) $S(TL)-$geodesic with regard to $\widetilde{D}^{\star}$ if $C^{\star} = 0$,

v) totally tangential umbilical with regard to D if $B(Z_1, Z_2) = kh(Z_1, Z_2)$,

vi) totally tangential umbilical with regard to $D^{\star}$ if $B^{\star}(Z_1, Z_2) = k^{\star}h(Z_1, Z_2)$,

vii) totally normally umbilical with regard to D if $A^{\star}_N Z_1 = kZ_1$,

viii) totally normally umbilical with regard to $D^{\star}$ if $A_N Z_1 = k^{\star} Z_1$,

where k and $k^{\star}$ are smooth functions on L.

For any lightlike hypersurface $(M, g, S(TM))$, the following equalities are satisfied [6]:

$$B(Z_1, \xi) + B^{\star}(Z_1, \xi) = 0, \quad h(A_N Z_1 + A^{\star}_N Z_1, Z_2) = 0, \tag{2.19}$$
$$C(Z_1, PZ_2) = h(A_N Z_1, PZ_2), \quad C^{\star}(Z_1, PZ_2) = h(A^{\star}_N Z_1, PZ_2), \tag{2.20}$$
$$B(Z_1, Z_2) = h(\overline{A}^{\star}_{\xi} Z_1, Z_2) + B^{\star}(Z_1, \xi)\widetilde{h}(Z_2, N), \tag{2.21}$$
$$B^{\star}(Z_1, Z_2) = h(\overline{A}_{\xi} Z_1, Z_2) + B(Z_1, \xi)\widetilde{h}(Z_2, N). \tag{2.22}$$

3. Screen invariant lightlike hypersurfaces

Definition 3.1. *Let $(L,h,S(TL))$ be a lightlike hypersurface of $(\widetilde{L},\widetilde{h},F)$. If $F(S(TL))$ belongs to $S(TL)$, then $(L,h,S(TL))$ is called a screen invariant lightlike hypersurface.*

In view of (2.2), we obtain that if $(L,h,S(TL))$ is a screen invariant lightlike hypersurface, then $F^{\star}(S(TL))$ belongs to $S(TL)$. Thus, we can write

$$F\xi=\lambda_1\xi+\mu_1 N,\quad F^{\star}\xi=\mu_2\xi+\mu_1 N, \tag{3.1}$$

$$FN=\lambda_2\xi+\mu_2 N,\quad F^{\star}N=\lambda_2\xi+\lambda_1 N, \tag{3.2}$$

where $\lambda_1,\lambda_2,\mu_1,\mu_2$ are smooth functions. Using the fact that $F^2\xi=\xi$, we find

$$\begin{aligned}\xi&=F(\lambda_1\xi+\mu_1 N)\\&=\lambda_1^2\xi+\lambda_1\mu_1 N+\mu_1\lambda_2\xi+\mu_1\mu_2 N\end{aligned}$$

which yields

$$\lambda_1^2+\mu_1\lambda_2=1\ \text{ and }\ \lambda_1\mu_1+\mu_1\mu_2=0. \tag{3.3}$$

Moreover, using the fact that $(F^{\star})^2\xi=\xi$, we find

$$\mu_2^2+\mu_1\lambda_2=1\ \text{ and }\ \mu_2\mu_1+\mu_1\lambda_1=0. \tag{3.4}$$

Now, we write a tangent vector field Z in $\Gamma(TL)$ by

$$Z=PZ+\eta(Z)\xi, \tag{3.5}$$

where $\eta(Z)=\widetilde{g}(Z,N)$ and P is the projection morphism from $\Gamma(TL)$ onto $\Gamma(S(TL))$.

In view of (3.1), (3.2) and (3.5), we put

$$\begin{aligned}FZ&=FPZ+\eta(Z)F\xi\\&=\varphi Z+\eta(Z)\lambda_1\xi+\eta(Z)\mu_1 N\end{aligned} \tag{3.6}$$

and

$$F^{\star}Z=\varphi^{\star}Z+\eta(Z)\mu_1\xi+\eta(Z)\mu_1 N, \tag{3.7}$$

where φZ and $\varphi^{\star}Z$ belong to $\Gamma(S(TM))$. Using (2.2), (3.6) and (3.7), we find

$$h(\varphi Z_1,Z_2)=h(Z_1,\varphi^{\star}Z_2) \tag{3.8}$$

for any $Z_1,Z_2\in\Gamma(TM)$.

Example 3.2. *Let $(\mathbb{R}^4_1,\widetilde{h},F)$ be an almost product-like Lorentzian manifold of Example 2.2. Consider a hypersurface M given by*

$$L=\{(x_1,x_2,x_3,x_4):x_1=x_4\}.$$

Then the induced metric of M becomes

$$h=\begin{bmatrix}0&0&0\\0&e^{x_1}&0\\0&0&e^{x_1}\end{bmatrix}.$$

By a straightforward computation, we obtain

$$\begin{aligned}Rad(TL)&=span\{\xi=\partial_1+\partial_4\},\\S(TL)&=span\{e_1=\partial_2,\ \ e_2=\partial_3\}\end{aligned}$$

and

$$ltr(TL) = span\left\{N = \frac{1}{2e^{x_1}}(-\partial_1 + \partial_4)\right\}.$$

Then, we find $F(S(TL)) \subset S(TL)$, which yields to $(L, h, S(TL))$ is a screen invariant lightlike hypersurface of $(\mathbb{R}^4_1, \widetilde{h}, \widetilde{D}, F)$.

Proposition 3.3. *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$. Then we have the following equalities:*

$$(\widetilde{D}_Z\lambda_2)\xi + \lambda_2 D_Z\xi - \mu_2 A^\star_N Z = -\varphi A^\star_N Z - \eta(A^\star_N Z)\lambda_1\xi + \tau^\star(Z)\lambda_2\xi \tag{3.9}$$

and

$$\lambda_2 B(Z, \xi) + \widetilde{D}_Z\mu_2 = -\eta(A^\star_N Z)\mu_1. \tag{3.10}$$

Proof. From (3.2), we have

$$\begin{aligned}\widetilde{D}_Z FN &= \widetilde{D}_Z(\lambda_2\xi + \mu_2 N)\\ &= (\widetilde{D}_Z\lambda_2)\xi + \lambda_2\widetilde{D}_Z\xi + (\widetilde{D}_Z\mu_2)N + \mu_2\widetilde{D}_Z N.\end{aligned} \tag{3.11}$$

Putting (2.9) in (3.11), we obtain

$$\begin{aligned}\widetilde{D}_Z FN = (\widetilde{D}_Z\lambda_2)\xi + \lambda_2\nabla_Z\xi + \lambda_2 B(Z, \xi)N + (\widetilde{D}_Z\mu_2)N - \mu_2 A^\star_N Z\\ + \mu_2\tau^\star(Z)N.\end{aligned} \tag{3.12}$$

Besides this fact, using (2.10) we have

$$\begin{aligned}F\widetilde{D}_Z N &= F(-A^\star_N Z + \tau^\star(Z)N)\\ &= -FA^\star_N Z + \tau^\star(Z)FN.\end{aligned} \tag{3.13}$$

Putting (3.2) and (3.6) in (3.13), we find

$$\begin{aligned}F\widetilde{D}_Z N = -\varphi A^\star_N Z - \eta(A^\star_N Z)\lambda_1\xi - \eta(A^\star_N Z)\mu_1 N + \tau^\star(Z)\lambda_2\xi\\ + \tau^\star(Z)\mu_2 N.\end{aligned} \tag{3.14}$$

Using the fact that $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is a locally product-like statistical manifold in (3.14), we get (3.9) and (3.10). ■

As a result of Proposition 3.3, we find

Theorem 3.4. *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. If $A^\star_N = 0$, then ξ is a recurrent vector field with regard to D and*

$$B(Z, \xi) = -\frac{1}{\lambda_2}\widetilde{D}_Z\mu_2 \tag{3.15}$$

is satisfied.

Proof. Under the assumption, if we write $A^\star_N Z = 0$ in (3.9), we obtain

$$D_Z\xi = \frac{1}{\lambda_2}(\tau^\star(Z)\lambda_2 - \widetilde{D}_Z\lambda_2)\xi, \tag{3.16}$$

which shows that ξ is a recurrent vector field. Putting $A^\star_N Z = 0$ in (3.10), we obtain (3.15). ■

Corollary 3.5. *Let $(L, h, S(TH))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. If ξ is a recurrent vector field, then one of the following situations occurs:*

i) $A^{\star}_N Z$ *is in the direction of* ξ.

ii) $A^{\star}_N Z = 0$.

Corollary 3.6. *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then,* $A^{\star}_N = 0$ *and* λ_2 *is constant if and only if* B *vanishes on* $Rad(TL)$.

With similar arguments in the proof of Proposition 3.3, we find

Proposition 3.7. *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then the following equalities hold for any* $Z \in \Gamma(TL)$*:*

$$(\widetilde{D}^{\star}_Z \lambda_2)\xi + \lambda_2 D^{\star}_Z \xi - \lambda_1 A_N Z = -\varphi^{\star} A_N Z - \eta(A_N Z)\mu_1 \xi + \tau(Z)\lambda_2 \xi \tag{3.17}$$

and

$$\lambda_2 B^{\star}(Z, \xi) + \widetilde{D}^{\star}_Z \lambda_1 = -\eta(A_N Z)\mu_1. \tag{3.18}$$

Theorem 3.8. *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. If* $A_N X = 0$*, then* ξ *is a recurrent vector field with regard to* $D^{\star}$ *and the following equality is satisfied:*

$$B^{\star}(Z, \xi) = -\frac{1}{\lambda_2}(\widetilde{D}^{\star}_Z \lambda_1). \tag{3.19}$$

Corollary 3.9. *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. If* ξ *is a recurrent vector field, then one of the following situations occurs:*

i) A_N *is in the direction of* ξ.

ii) $A_N = 0$.

Corollary 3.10. *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold.* $A_N = 0$ *and* λ_2 *is constant if and only if* $B^{\star}$ *vanishes on* $Rad(TL)$.

Proposition 3.11. *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface. Then the following relations are satisfied:*

$$(\widetilde{D}_Z \lambda_1)\xi + \lambda_1 D_Z \xi - \mu_1 A^{\star}_N Z = \varphi D_Z \xi + \eta(D_Z \xi)\lambda_1 \xi + B(Z, \xi)\lambda_2 \xi \tag{3.20}$$

and

$$\lambda_1 B(Z, \xi) + \widetilde{D}_Z \mu_1 + \mu_1 \tau^{\star}(Z) = \eta(D_Z \xi)\mu_1 + B(Z, \xi)\mu_2. \tag{3.21}$$

Proof. From (3.1), we have

$$\widetilde{D}_Z F\xi = \widetilde{D}_Z(\lambda_1 \xi + \mu_1 N). \tag{3.22}$$

Using (2.9) and (2.10) in (3.22), we obtain

$$\begin{aligned}\widetilde{D}_Z F\xi = (\widetilde{D}_Z \lambda_1)\xi + \lambda_1 D_Z \xi + \lambda_1 B(Z, \xi)N + (\widetilde{D}_Z \mu_1)N - \mu_1 A^{\star}_N Z \\ + \mu_1 \tau^{\star}(Z)N.\end{aligned} \tag{3.23}$$

On the other hand, using (2.9), (2.10) and (3.1), we have

$$\begin{aligned}F\widetilde{D}_Z \xi = \varphi \nabla_Z \xi + \eta(D_Z \xi)\lambda_1 \xi + \eta(D_Z \xi)\mu_1 N + B(Z, \xi)\lambda_2 \xi \\ + B(Z, \xi)\mu_2 N.\end{aligned} \tag{3.24}$$

Using the fact that $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is a locally product-like statistical manifold, we find (3.20) and (3.21) immediately. ■

As a result of (3.20), we find

Theorem 3.12. *If ξ is a parallel vector field with regard to D, then one of the following relations holds:*

i) $A^{\star}_N$ *is in the direction of* ξ.

ii) $A^{\star}_N = 0$.

Proof. Under the assumption, if ξ is a parallel vector field with regard to D, we obtain from (3.20) that

$$A^{\star}_N Z = \frac{1}{\mu_1}(\widetilde{D}_Z \lambda_1 + B(Z,\xi)\lambda_2)\xi,$$

which shows that $A^{\star}_N$ is in the direction of ξ or $A^{\star}_N = 0$. ■

With similar arguments as in the proof of Proposition 3.11, we get the followings:

Proposition 3.13. *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then the following relations hold:*

$$(\widetilde{D}^{\star}_Z \mu_2)\xi + \mu_2 D^{\star}_Z \xi - \mu_1 A_N Z = \varphi^{\star} D^{\star}_Z \xi + \eta(D^{\star}_Z \xi)\mu_1 \xi + B^{\star}(Z,\xi)\lambda_2 \xi \tag{3.25}$$

and

$$\mu_2 B^{\star}(Z,\xi) + \widetilde{D}^{\star}_Z \mu_1 + \mu_1 \tau(Z) = \eta(D^{\star}_Z \xi)\mu_1 + B^{\star}(Z,\xi)\lambda_1. \tag{3.26}$$

Theorem 3.14. *If ξ is a parallel vector field with regard to $D^{\star}$, then one of the following situations holds:*

i) A_N *is in the direction of* ξ.

ii) $A_N = 0$.

Proposition 3.15. *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then we have the following formulas:*

$$\begin{aligned}(\nabla_{Z_1}\varphi)Z_2 = {} & \eta(D_{Z_1} Z_2)\lambda_1 \xi + B(Z_1, Z_2)\lambda_2 \xi - C(Z_1, PZ_2)\lambda_1 \xi + g(A^{\star}_N Z_1, Z_2)\lambda_1 \xi \\ & - \eta(Z_2)\lambda_1 D_{Z_1}\xi + \eta(Z_2)\mu_1 A^{\star}_N Z_1\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}& B(Z_1, \varphi Z_2) + \eta(Z_2)\lambda_1 B(Z_1, \xi) + C(Z_1, PZ_2)\mu_1 - g(A^{\star}_N Z_1, Z_2)\mu_1 + \widetilde{D}_{Z_1}\mu_1 \eta(Z_2) \\ & + \eta(Z_2)\mu_1 \tau^{\star}(Z_1) = \eta(D_{Z_1} Z_2)\mu_1 + B(Z_1, Z_2)\mu_2.\end{aligned} \tag{3.28}$$

Proof. Using (3.6), we have

$$\begin{aligned}\widetilde{D}_{Z_1} F Z_2 = {} & \widetilde{D}_{Z_1}\varphi Z_2 + Z_1 g(Z_2, N)\lambda_1 \xi + \widetilde{D}_{Z_1}\lambda_1 \eta(Z_2)\xi + \eta(Z_2)\lambda_1 \widetilde{D}_{Z_1}\xi \\ & + Z_1 g(Z_2, N)\mu_1 N + \widetilde{D}_{Z_1}\mu_1 \eta(Z_2) N + \eta(Z_2)\mu_1 \widetilde{D}_{Z_1} N.\end{aligned} \tag{3.29}$$

Considering (2.4), (2.9) and (2.10) in (3.29), it follows that

$$\begin{aligned}\widetilde{D}_{Z_1} F Z_2 = {} & D_{Z_1}\varphi Z_2 + B(Z_1, \varphi Z_2) N + C(Z_1, PZ_2)\lambda_1 \xi - g(A^{\star}_N Z_1, Z_2)\lambda_1 \xi \\ & + \widetilde{D}_{Z_1}\lambda_1 \eta(Z_2)\xi + \eta(Y)\lambda_1 D_{Z_1}\xi + \eta(Z_2)\lambda_1 B(Z_1, \xi) N \\ & + C(Z_1, PZ_2)\mu_1 N - g(A^{\star}_N Z_1, Z_2)\mu_1 N + \widetilde{D}_{Z_1}\mu_1 \eta(Z_2) N \\ & - \eta(Z_2)\mu_1 A^{\star}_N Z_1 + \eta(Z_2)\mu_1 \tau^{\star}(Z_1) N.\end{aligned} \tag{3.30}$$

Besides the above fact, we have from (3.6) that

$$\begin{aligned} F\widetilde{D}_{Z_1}Z_2 &= FD_{Z_1}Z_2 + B(Z_1,Z_2)FN \\ &= \varphi D_{Z_1}Z_2 + \eta(D_{Z_1}Z_2)\lambda_1\xi + \eta(D_{Z_1}Z_2)\mu_1 N + B(Z_1,Z_2)\lambda_2\xi \\ &\quad + B(Z_1,Z_2)\mu_2 N. \end{aligned} \tag{3.31}$$

Considering the tangential and transversal parts of (3.30), (3.31) and using the fact that $\widetilde{D}_{Z_1}FZ_2 = F\widetilde{D}_{Z_1}Z_2$, we get (3.27) and (3.28) immediately. ■

As a result of (3.27), we get the following theorem:

Theorem 3.16. *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then φ is parallel with regard to D.*

Proof. For a special case, if we choose $Z_2 \in \Gamma(TL)$ in (3.27), then we get

$$(D_{Z_1}\varphi)Z_2 = [\eta(D_{Z_1}Z_2)\lambda_1 + B(Z_1,Z_2)\lambda_2 - C(Z_1,Z_2)\lambda_1 + h(A^\star_N Z_1, Z_2)\lambda_1]\xi,$$

which is a contradiction to Z_2 belonging $\Gamma(S(TL))$. Thus, φ is parallel with regard to D. ■

Proposition 3.17. *Let $(L, h, S(TL))$ be a totally geodesic screen invariant lightlike hypersurface with regard to $\widetilde{D}$. Then the following relation is satisfied:*

$$C(Z_1, PZ_2) = \eta(D_{Z_1}Z_2) + h(A^\star_N Z_1, Z_2). \tag{3.32}$$

Proof. Putting $B(Z_1, Z_2) = 0$ for any $Z_1, Z_2 \in \Gamma(TL)$, the proof is easy to follow from (3.27) or (3.28). ■

4. Concurrent vector fields

Let $(\widetilde{L}, \widetilde{h}, \widetilde{D})$ be a statistical manifold. A vector field ζ is called a concurrent vector field with regard to $\widetilde{D}$ (resp. $\widetilde{D}^\star$) if $\widetilde{D}_Z\zeta = Z$ (resp. $\widetilde{D}^\star_Z\zeta = Z$) for each $Z \in \Gamma(T\widetilde{L})$.

If ζ is a concurrent vector field with respect to $\widetilde{D}$ and $\widetilde{D}^\star$, we obtain from (2.4) that

$$\widetilde{h}(\widetilde{D}_{Z_2}Z_1, \zeta) = \widetilde{h}(\widetilde{D}^\star_{Z_2}Z_1, \zeta)$$

is satisfied for each $Z_1, Z_2 \in \Gamma(T\widetilde{L})$. Also, we get from (2.5) that if ζ is a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^\star$, then it is also concurrent with regard to the Levi-Civita connection $\widetilde{D}^0$.

Now, we recall the definition of rigged metric for lightlike hypersurfaces [3]:

Definition 4.1. *Let $(L, h, S(TL))$ be a lightlike hypersurface and ψ be a vector field such that $\psi_p \notin T_pL$ for any $p \in L$. If we define a $1-$form η satisfying*

$$\eta(X) = \widetilde{h}(Z, \psi)$$

then ψ is called a rigging vector field.

If we choose $\psi = N$, then a rigged metric $\overline{h}$ with regard to N is defined by

$$\overline{h}(Z_1, Z_2) = h(Z_1, Z_2) + \eta(Z_1)\eta(Z_2) \tag{4.1}$$

for each $Z_1, Z_2 \in \Gamma(TL)$. It is easy to see that $\overline{h}$ is non-degenerate and the following relations are satisfied:

$$\overline{h}(N, Z) = \eta(Z), \;\; \overline{h}(\xi, \xi) = 1 \tag{4.2}$$

and

$$\overline{h}(Z_1, Z_2) = h(Z_1, Z_2), \ \forall Z_1, Z_2 \in \Gamma(TL) \tag{4.3}$$

It is known that the gradient of a smooth function could not be defined on a degenerate metric h since the inverse of h does not exist. But the gradient of a function f could be defined by using a rigged metric as follows:

$$grad f = \sum_{i=1}^{n} h^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j},$$

where $[h^{ij}]$ is the inverse of $\overline{h}$. We note that $[h^{ij}]$ is also known as the pseudo-inverse of h [3].

Let $(L, h, S(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{D})$ and ζ be a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$. Then we can write

$$\zeta = \zeta^T + \zeta^N, \tag{4.4}$$

where ζ^T is the tangential part, while ζ^N is the transversal part of ζ. In view of (4.1) and (4.4), we obtain

$$\begin{aligned} \overline{h}(\zeta^T, \xi) &= h(\zeta^T, \xi) + \eta(\zeta^T)\eta(\xi). \\ &= \widetilde{h}(\zeta, N). \end{aligned}$$

In view of (2.4), we find

$$\begin{aligned} Z\overline{h}(\zeta^T, \xi) &= X\widetilde{h}(\zeta, N) \\ &= \widetilde{h}(\widetilde{D}_Z \zeta, N) + \widetilde{h}(\widetilde{D}^{\star}_Z N, \zeta) \\ &= \eta(Z) - \widetilde{h}(A_N Z, \zeta) + \tau(Z)\eta(\zeta). \end{aligned} \tag{4.5}$$

Moreover, (4.5) could be written as

$$\begin{aligned} Z\overline{h}(\zeta^T, \xi) &= \widetilde{h}(\widetilde{D}^{\star}_Z \zeta, N) + \widetilde{h}(\widetilde{D}_Z N, \zeta) \\ &= \eta(Z) - \widetilde{g}(A^{\star}_N Z, \zeta) + \tau^{\star}(Z)\eta(\zeta). \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we find

Proposition 4.2. *Let $(L, h, S(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{D})$. If ζ is a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$, then*

$$\widetilde{h}(A_N Z, \zeta) - \tau(Z)\eta(\zeta) = \widetilde{h}(A^{\star}_N Z, \zeta) - \tau^{\star}(Z)\eta(\zeta) \tag{4.7}$$

is satisfied for any $Z \in \Gamma(TL)$. In particular, if $\eta(\zeta) = 0$ then

$$\widetilde{h}(A_N Z, \zeta) = \widetilde{h}(A^{\star}_N Z, \zeta) \tag{4.8}$$

is satisfied.

As a result of (2.20) and Proposition 4.2, we have

Corollary 4.3. *Let $(L, h, S(TL))$ be an $S(TL)-$geodesic lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{D})$. If ζ is a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$, then*

$$\tau(Z) = \tau^{\star}(Z) \tag{4.9}$$

is satisfied for any $Z \in \Gamma(TL)$.

Proposition 4.4. *Let $(L, h, S(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{D})$. Then we have the following situations:*

i) If ζ is a concurrent vector field with regard to $\widetilde{D}$, then $B(Z,\zeta)=0$ is satisfied for any $Z\in\Gamma(TL)$.

ii) If ζ is a concurrent vector field with regard to $\widetilde{D}^{\star}$, then $B^{\star}(Z,\zeta)=0$ is satisfied for any $Z\in\Gamma(TL)$.

As a result of Proposition 4.4, (2.9) and (2.11), we see that if v is a concurrent vector field with regard to $\widetilde{D}$ (resp. $\widetilde{D}^{\star}$), then it is also concurrent with regard to D (resp $D^{\star}$). We note that the converse part of this claim is not correct in general.

Example 4.5. *Let $(L,h,S(TL))$ be a screen invariant lightlike hypersurface of Example 3.2. In view of Example 2.4, it is clear that $\zeta=\partial_1$ is a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$.*

Now, suppose that ζ belongs to $\Gamma(TL)$. Then we write

$$\zeta=P\zeta+a\xi, \tag{4.10}$$

where $P\zeta\in\Gamma(S(TL))$ and $\eta(\zeta)=a$. Thus, we find

Proposition 4.6. *Let ζ be a concurrent vector field with regard to $\widetilde{D}$. Then we have*

$$h(A_N\zeta,\zeta)=\frac{1}{2}a\tau(\zeta). \tag{4.11}$$

Proof. From (4.5), it follows that

$$\begin{aligned}\zeta\overline{h}(\zeta,\xi)&=\eta(\zeta)-g(A_N\zeta,\zeta)+\tau(\zeta)\eta(\zeta)\\&=a-g(A_N\zeta,\zeta)+a\tau(\zeta).\end{aligned} \tag{4.12}$$

Now, we compute the left-hand side of (4.12). From (4.1), we find

$$\zeta\overline{h}(\zeta,\xi)=\eta(\zeta)+h(A_N\zeta,\zeta). \tag{4.13}$$

The proof is easy to follow from (4.12) and (4.13). ■

In a similar way to Proposition 4.6, we find

Proposition 4.7. *Let ζ be a concurrent vector field with regard to $\widetilde{D}$. Then we have*

$$h(A^{\star}_N\zeta,\zeta)=\frac{1}{2}a\tau^{\star}(\zeta). \tag{4.14}$$

Theorem 4.8. *Let $(L,h,S(TL))$ be an $S(TL)-$geodesic lightlike hypersurface with regard to $\widetilde{D}$ and ζ be a concurrent vector field with regard to $\widetilde{D}$ such that $\zeta\in\Gamma(TL)$. Then ζ could not be concurrent with regard to $\widetilde{D}^{\star}$.*

Proof. Under the assumption and from (4.13), we get $A_N\zeta=\tau(\zeta)=0$. If we put this equation in (2.12), we find $\widetilde{D}_\zeta N=0$, which shows that ζ could not be concurrent with regard to $\widetilde{D}^{\star}$. ■

In a similar way to Theorem 4.8, we obtain

Theorem 4.9. *Let $(L,h,S(TL))$ be an $S(TL)-$geodesic lightlike hypersurface with regard to $\widetilde{D}^{\star}$ and ζ be a concurrent vector field with regard to $\widetilde{D}^{\star}$ such that $\zeta\in\Gamma(TL)$. Then ζ could not be concurrent with regard to $\widetilde{D}$.*

Now, we shall investigate concurrent vector fields in screen invariant lightlike hypersurfaces.

Proposition 4.10. *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold and ζ be a concurrent vector field with regard to $\widetilde{D}$. Then we have the following relations:*

$$A_N^\star(\zeta) = \frac{1}{\mu_1}(\widetilde{D}_v \lambda_1)\xi \tag{4.15}$$

and

$$\tau^\star(v) = 1 - \frac{1}{\mu_1}(\widetilde{D}_\zeta \mu_1). \tag{4.16}$$

Proof. Since ζ is concurrent with regard to $\widetilde{D}$, we write

$$\widetilde{D}_\zeta F\xi = F\xi. \tag{4.17}$$

Using the fact that if ζ is a concurrent vector field with regard to $\widetilde{D}$, then it is also concurrent with regard to D and using (3.23), we obtain

$$\widetilde{D}_\zeta F\xi = (\widetilde{D}_\zeta \lambda_1)\xi + \lambda_1\xi + (\widetilde{D}_\zeta \mu_1)N - \mu_1 A_N^\star \zeta + \mu_1 \tau^\star(\zeta)N. \tag{4.18}$$

From (3.1), (4.17) and (4.18), the proof is easy to follow. ■

As a result of Proposition 4.10, we obtain

Corollary 4.11. *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold and ζ be a concurrent with regard to $\widetilde{D}$, then the following situations occur:*

i) If $\widetilde{D}_\zeta \lambda_1 = 0$, then $A_N^\star(\zeta) = 0$.

ii) If $\widetilde{D}_\zeta \mu_1 = 0$, then $\tau^\star(\zeta) = 1$.

iii) If $\mu_1 = 0$, then $\widetilde{D}_\zeta \lambda_1 = 0$ and $\tau^\star(\zeta) = 1$.

By a similar arguments to Proposition 4.10, we get

Proposition 4.12. *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface and ζ be a concurrent vector field with regard to $\widetilde{D}^\star$. Then we have the following relations:*

$$A_N \zeta = \frac{1}{\mu_1}(\widetilde{D}_\zeta \mu_2)\xi \tag{4.19}$$

and

$$\tau^\star(\zeta) = 1 - \frac{1}{\mu_1}(\widetilde{D}_\zeta \mu_1). \tag{4.20}$$

Corollary 4.13. *Let ζ be concurrent with regard to $\widetilde{D}^\star$. Then the following situations occur:*

i) If $\widetilde{D}_\zeta \mu_2 = 0$, then $A_N \zeta = 0$.

ii) If $\widetilde{D}_\zeta \mu_1 = 0$, then $\tau(\zeta) = 1$.

iii) If $\mu_1 = 0$, then $\widetilde{D}_\zeta \mu_2 = 0$ and $\tau(\zeta) = 1$.

Corollary 4.14. *Let $(L, h, S(TL))$ be an $S(TL)-$umbilical screen invariant lightlike hypersurface with regard to $\widetilde{D}^\star$ (resp. $\widetilde{D}$) and $\zeta \notin \Gamma(Rad(TL))$. Then ζ is not concurrent with regard to $\widetilde{D}$ (resp $\widetilde{D}^\star$).*

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Weighted pseudo S-asymptotically Bloch type periodic solutions for a class of mean field stochastic fractional evolution equations

MAMADOU MOUSTAPHA MBAYE[1*], AMADOU DIOP[2] AND MOUSTAPHA DIEYE[3]

[1] *Département de Mathématiques et Informatique, Faculté des Sciences et Technique, Université Cheikh Anta Diop, Dakar, BP5005, Sénégal.*
[2] *Laboratoire d'Analyse Numérique et Informatique, Université Gaston Berger, Saint-Louis, BP234, Sénégal.*
[3] *Ecole Polytechnique de Thiès, Thiès, BPA10, Sénégal.*

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Abstract. This paper concerns a class of mean-field stochastic fractional evolution equations. Initially, we establish some auxiliary results for weighted pseudo S-asymptotically Bloch type periodic stochastic processes. Without a compactness assumption on the resolvent operator and some additional conditions on forced terms, the existence and uniqueness of weighted pseudo S-asymptotically Bloch type periodic mild solutions on the real line of the referred equation are obtained. In addition, we show the existence of weighted pseudo S-asymptotically Bloch type periodic mild solutions with sublinear growth assumptions on the drift term and compactness conditions. Finally, an example is provided to verify the main outcomes.

AMS Subject Classifications: 30D45, 34C25, 60H15, 60G22.

Keywords: Stochastic processes, stochastic fractional evolution equations, asymptotically Bloch type periodicity, Brownian motion, distribution, Mean field.

Contents

*[*] **Corresponding author.** Email address: **mamadoumoustapha3.mbaye@ucad.edu.sn** (M.M Mbaye), **amadou.diop@ugb.edu.sn** (A. Diop), **mdieye@ept.sn** (M. Dieye)

1. Introduction

Periodicity is a key topic in the qualitative property of differential equations, because of its importance in both pure mathematics and applications. However, not all the phenomena in the real world can satisfy the periodicity criteria. Some phenomena have a behaviour that is not periodic, but rather almost periodic, asymptotically almost periodic, ω-periodicity, asymptotically ω-periodicity, Bloch periodic, and so on. Bloch periodic phenomena are related to the conductivity of crystalline solids, as F. Bloch showed in [10]. N'Guérékata and Hasler introduced the concept of Bloch-type periodic functions in [19], which generalizes the classical notions of ω-periodicity and ω-anti-periodicity. Some publications have also explored the effects of perturbations on Bloch periodic functions, by defining some quasi-Bloch periodicity concepts. For instance, [18, 20, 32] studied asymptotically Bloch periodic functions and their applications, while [31–34] investigated (pseudo) S-asymptotically Bloch periodic functions and their applications. These quasi-Bloch periodic functions are extensions of the corresponding asymptotically ω-periodic and (pseudo) S-asymptotically ω-periodic functions in the deterministic case. In [27, 28], the authors investigated the notion of S-asymptotically ω-periodicity for stochastic processes and established some results on their existence, uniqueness, and asymptotic stability. Recently, in [5], the concepts of square-mean (weighted) pseudo S-asymptotically Bloch-type periodicity for stochastic process was introduced, which is a type of periodicity that can capture more stochastic phenomena. Moreover, the authors of [5] investigated the existence and uniqueness of the mild solution of some stochastic evolution equations.

In contrast, mean-field stochastic differential equations (SDEs), also known as McKean-Vlasov equations, represent weak interactions between particles within a large system. Kac [17] first investigated it in relation to the Boltzmann equation for particle density in diluted monoatomic gases. He also studied it in the stochastic toy model for the Vlasov kinetic equation for plasma. McKean [13] examined how chaos spreads in physical systems containing N particles interacting with one another. In his work, he emphasized the importance of the Boltzmann equation, which describes the statistical behavior of gases with low densities. In [3, 4], Sznitman examines chaos and the limit equation from a different perspective. As with the previously mentioned SDEs, he described the limit equation using an evolution equation. A study of the dynamics of the polymers was carried out by E and Shen in [30]. To approximate the description of the polymers, they used stochastic partial differential equations of the mean field type. In [2, 11] they addressed similar issues related to stochastic differential equations in infinite dimensional spaces.

A number of current research investigations have focused primarily on the existence and uniqueness of solutions for stochastic fractional order evolution equations of the McKean-Vlasov type, with little or no results from periodic or quasi-periodic solutions for the referred class of equations. Therefore, the above literature motivates us to explore the existence and uniqueness of weighted pseudo S-asymptotically Bloch type periodicity mild solutions of the following abstract mean field stochastic fractional evolution equations

$$\begin{cases} \partial_t^\alpha v(t) = Av(t) + \displaystyle\int_{-\infty}^{t} b(t-s)Av(s)ds + g(t, v(t), \mathbb{P}_{v(t)}) \\ \qquad\qquad + f(t, v(t), \mathbb{P}_{v(t)})\dfrac{d\mathbb{W}(t)}{dt}, \quad t \in \mathbb{R}, \\ \mathbb{P}_{v(t)} = \text{ Probability distribution of } v(t). \end{cases} \tag{1.1}$$

Here ∂_t^α denotes the Weyl fractional derivative of order $\alpha > 0$, $A : D(A) \subseteq \mathbb{L}^2(\Omega, \mathbb{H}) \to \mathbb{L}^2(\Omega, \mathbb{H})$ is a closed linear operator on a complex Hilbert space $\mathbb{L}^2(\Omega, \mathbb{H})$ (where $\mathbb{L}^2(\Omega, \mathbb{H})$ is an appropriate function space specified in Section 2) and generate an α-resolvent family $\{\mathcal{R}_\alpha(t)\}_{t\geq 0}$ on $\mathbb{H}$, g, f are $\mathbb{H}$-valued stochastic processes. Here $(\mathbb{W}(t))_{t\in\mathbb{R}}$ represents a two-sided and standard one-dimensional Brownian motion on $\mathbb{H}$ and $\mathbb{P}_{v(t)} = \mathbb{P} \circ [v(t)]^{-1}$ is the probability distribution of $v(t)$ under $\mathbb{P}$ (i.e $\mathbb{P}_{v(t)}(K) = \mathbb{P}(\{x \in \Omega : v(t, x) \in K\})$ for each $K \in \mathcal{B}(\mathbb{H})$, where $\mathcal{B}(\mathbb{H})$ represents the Borel class on $\mathbb{H}$.

Returning to the literature, when $\alpha = 1$ and $b(t) = 0$, the problem (1.1) degrades to a classical stochastic differential equations of Mckean–Vlasov type for which have been investigated by many researchers through different methods [7, 14–16, 21, 22]. For example, let consider an q-particle system $v^{q,1}, \cdots, v^{q,q}$ given by the following weakly interacting stochastic partial differential equations

$$dv^{q,j}(t) = Av^{q,j}(t)dt + g(t, v^{q,j}(t), \mu^q(t))dt + f(t, v^{q,j}(t), \mu^q(t))dW^j(t), \tag{1.2}$$

$j = 1, 2, \cdots, q$, where $\mu^q(t) = \dfrac{1}{q}\sum_{j=1}^{q} \delta_{v^{q,j}(t)}$, $t \in \mathbb{R}_+$ represents the empirical distributions, $(W^j(t))_{j=1,\cdots,q}$ are independent standard cylindrical Brownian motions and A generates a C_0-semigroup. It is proved that under suitable conditions on the Equ.(1.2), it's possible to describe the limit by the following McKean-Vlasov equation

$$dv(t) = Av(t)dt + g(t, v(t), \mathbb{P}_{v(t)}) + f(t, v(t), \mathbb{P}_{v(t)})dW(t) \tag{1.3}$$

Moreover, if $f \equiv 0$ and $g(t, v(t), \mathbb{P}_{v(t)}) \equiv g(t, v(t))$, the existence and uniqueness of almost automorphic, asymptotically periodic, almost periodic, asymptotically ω-periodic solutions, S-asymptotically ω-periodic solutions, asymptotically almost periodic and asymptotically almost automorphic, (ω, c)-periodic and pseudo S-asymptotically (ω, k)-Bloch periodic mild solutions of problem (1.1) have been investigated in deterministic cases by various authors [6, 8, 9, 24, 35, 36, 38]. In this work, the problem (1.1) captures fading memory behaviors, and randomness of the dynamical processes. We examine a more general class of above mentioned problems under the situation that diffusion and drift terms f, g are weighted speudo-S-asymptotically Bloch periodic, and depend on the probability distribution of the process at times. The obtained outcomes are mainly relied upon on the Wassertein topology, resolvent operator theory, Banach and Krasnoselki's fixed point theorem and stochastic analysis. We firstly provide some convolutions and composition results under some suitable conditions and continuity assumptions. Next, we establish existence and uniqueness result (see Theorem 3.10) which needs no compactness condition on the resolvent operator under global Lipschitz conditions on f, g and additional suitable conditions. Finally, we relax the Lipschitz condition of g to some sublinear growth conditions (see Theorem 3.14). Consequently, our research study can be viewed as an extension and continuation of investigation in [5, 6, 8, 9, 24, 35, 36, 38]. Additionally, this work generalize various papers on S-asymptotically ω-antiperiodic (or ω-periodic) mild solutions of to square-mean weighted pseudo S-asymptotically (ω, k)-periodic mild solutions for some stochastic fractional evolution equations.

The remainder of the paper is arranged as follows: Section 2 discusses some basic results regarding weighted pseudo square-mean S-asymptotically Bloch type periodicity processes. Section 3.2 is devoted to the existence and uniqueness of weighted pseudo S-asymptotically Bloch type periodicity mild solutions of Eq.(1.1). To summarize this work, we provide an example that illustrates our results, in Section 4.

2. Background

We suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ represents a probability space, $\mathbb{H}$ is complex separable Hilbert space, and $\mathbb{K}$ indicates a real separable Hilbert space. For convenience, the same notations $\|\cdot\|$ and $(\cdot,\cdot)$ are applied to denote the norms and the inner products in $\mathbb{H}$ and $\mathbb{K}$. We denote by $\mathcal{L}(\mathbb{K}, \mathbb{H})$ the Banach space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$ endowed with the topology defined by the operator norm, and $\mathbb{L}^2(\Omega, \mathbb{H})$ stands for the collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, which is a complex Hilbert space endowed with the norm

$$\|v\|_{\mathbb{L}^2} = (\mathsf{E}\|v\|^2)^{1/2},\ v \in \mathbb{L}^2(\Omega, \mathbb{H})$$

where $\mathsf{E}(\cdot)$ is the expectation defined by $\mathsf{E}\|v\|^2 = \displaystyle\int_\Omega \|v\|^2 d\mathbb{P}$.

Definition 2.1. *A stochastic process $v : \mathbb{R} \to \mathbb{L}^2(\Omega, \mathbb{H})$ is said to be*

(i) *stochastically bounded if there exists a constant $M > 0$ such that*

$$\mathsf{E}\|v(t)\|^2 = \int_{\Omega} \|v(t)\|^2 d\mathbb{P} < M \quad \textit{for all} \quad t \in \mathbb{R};$$

(ii) *stochastically continuous if* $\lim_{t\to s} \mathsf{E}\|v(t) - v(s)\|^2 = 0 \quad \textit{for all} \quad s \in \mathbb{R}.$

We denote by $\mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})\right)$ the complex Banach space of all bounded and continuous stochastic processes v from $\mathbb{R}$ into $\mathbb{L}^2(\Omega, \mathbb{H})$ with the norm $\|v\|_\infty = \left(\sup_{t\in\mathbb{R}} \mathsf{E}\|v(t)\|^2\right)^{1/2}$. We denote by $\mathbb{P}_{v(t)} = \mathbb{P} \circ [v(t)]^{-1} = \mu(v(t))$ the distribution of all random variable $v(t) : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{H}$.

2.1. Wasserstein distances

Let $(\mathbb{H}, d)$ be a separable complete metric space and $\mathcal{P}(\mathbb{H})$ be the space of Borel probability measures on $\mathbb{H}$. For $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{H})$, we define

$$d_{BL}(\mu_1, \mu_2) = \sup_{\|\psi\|_{BL}\le 1} \left| \int_{\mathbb{H}} \psi \, d(\mu_1 - \mu_2) \right|, \tag{2.1}$$

where ψ are Lipschitz continuous functions on $\mathbb{H}$ with the norm

$$\|\psi\|_L = \sup\left\{ \frac{|\psi(z_1) - \psi(z_2)|}{\|z_1 - z_2\|} \; ; \; z_1, \; z_2 \in \mathbb{H}, \; z_1 \neq z_2 \right\}$$

$$\|\psi\|_{BL} = \max\{\|\psi\|_\infty, \|\psi\|_L\} \; , \; \|\psi\|_\infty := \sup_{k\in\mathbb{Y}} |\psi(k)| < \infty.$$

It is known that d_{BL} is a complete metric on $\mathcal{P}(\mathbb{Y})$ which generates the weak topology [29]. For any $p \ge 1$,we denote by $\mathcal{P}_p(\mathbb{H})$ the subspace of $\mathcal{P}(\mathbb{H})$ consisting of the probability measures of order p. For any $p \ge 1$ and $u, \tilde{u} \in \mathcal{P}_p(\mathbb{H})$, the p-Wasserstein distance $\mathsf{W}_p(u, \tilde{u})$ is defined by :

$$\mathsf{W}_p(u, \tilde{u}) = \inf\left\{ \left[\int_{\mathbb{H}\times\mathbb{H}} |x - y|^p \mu(dx, dy) \right]^{1/p} : \mu \in \mathcal{P}_p(\mathbb{H} \times \mathbb{H}) \text{ with marginals } u \text{ and } \tilde{u} \right\}$$

The following lemma is of great importance in our analysis.

Lemma 2.2 (Carmona and Delarue [25], Corollary 5.4)**.** *If $(\mathcal{P}_1(\mathbb{H}), d)$ is a complete separable metric space, and $\mu, \tilde{\mu} \in \mathcal{P}_1(\mathbb{H})$, then*

$$\mathsf{W}_1(\mu, \tilde{\mu}) = \sup_{\psi:|\psi(x)-\psi(y)|\le d(x,y)} \int_{\mathbb{H}} \psi(z)(\mu - \tilde{\mu})(dz)$$

where the supremum is taken over all the $1-$lipschitz functions..

Notice that if v and $\widetilde{v}$ are random variables of order p, then

$$\mathsf{W}_p(\mathbb{P}_v, \mathbb{P}_{\widetilde{v}}) \le (\mathbb{E}\|v - \widetilde{v}\|^p)^{1/p}$$

and the Hölder inequality implies that

$$\mathsf{W}_p(\mu, \widetilde{\mu}) \le \mathsf{W}_q(\mu, \widetilde{\mu}), \; \mu, \widetilde{\mu} \in \mathcal{P}_p(\mathbb{H}), \; 1 \le p \le q.$$

2.2. Weighted square-mean S-asymptotically Bloch type periodic process

In this segment, we recall some definitions and properties of weighted square-mean S-asymptotically Bloch type periodic processes. We refer to [5] for a more detailed analysis. Let Λ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and each $\rho \in \Lambda$, we set

$$m(r, \rho) = \int_{-r}^{r} \rho(s) ds.$$

Throughout the work, we suppose that following condition hold:

$$(\mathbf{H}^{\rho}) \;\; : \;\; \text{For all } \;\zeta \in \mathbb{R}, \;\; \limsup_{|t| \to \infty} \frac{\rho(t+\zeta)}{\rho(t)} < +\infty.$$

Define the $\Lambda_{\infty} = \{\rho \in \Lambda : \lim_{r \to +\infty} m(r, \rho) = \infty\}$.

Definition 2.3 ([5]). *A stochastic process $v \in \mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})\right)$ is said to be square-mean weighted pseudo-S-asymptotically (ω, k)-Bloch periodic if for given $\omega \in \mathbb{R}, k \in \mathbb{R}$,*

$$\lim_{r \to +\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \mathbb{E}\|v(t+\omega) - e^{ik\omega} v(t)\|^2 \rho(t) dt = 0,$$

for each $t \in \mathbb{R}$. We denote the space of all such processes by $\mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ and and

$$\mathcal{WSABP}_{\omega,k}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H})) = \Big\{ h(\cdot, v, \mu) \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho) \;\; \textit{for any} \quad v \in \mathbb{L}^2(\Omega, \mathbb{H}), \; \mu \in \mathcal{P}_2(\mathbb{H}) \Big\}.$$

From definition 2.3, we can formulate, the following concepts. By taking

1. $k\omega = \pi$, we obtain the notion of square-mean weighted pseudo-S-asymptotically ω-antiperiodic stochastic processes, i.e
$$\lim_{r \to +\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \mathbb{E}\|v(t+\omega) + v(t)\|^2 \rho(t) dt = 0, \text{ for each } t \in \mathbb{R};$$
2. $k\omega = 2\pi$, we get the concept of square-mean weighted pseudo-S-asymptotically ω-periodic stochastic processes, i.e
$$\lim_{r \to +\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \mathbb{E}\|v(t+\omega) - v(t)\|^2 \rho(t) dt = 0, \text{ for each } t \in \mathbb{R}.$$

Lemma 2.4 ([5]). *Let $\rho \in \Lambda_{\infty}$, and $X_1, X_2, X \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$, then the following results hold:*

(a) *$X_1 + X_2 \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$, and $aX \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ for each $a \in \mathbb{C}$.*

(b) *$X_a \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ for each $a \in \mathbb{R}$, where $X_a(t) := X(t+a)$ for each $t \in \mathbb{R}$.*

(c) *$\mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ is a Banach space endowed with the norm $\|\cdot\|_{\infty}$.*

Throughout the paper, we define the set

$$\mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H})) = \Big\{ h(\cdot, v, \mu) \in \mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})\right) \;\; \text{for any} \quad v \in \mathbb{L}^2(\Omega, \mathbb{H}), \; \mu \in \mathcal{P}_2(\mathbb{H}) \Big\}.$$

3. Discussion on solutions existence

This section of the paper is mainly concerned with demonstrating that for each weighted pseudo S-asymptotically (ω, k)-periodic input, the output is still a bounded and continuous mild solutions to the fractional stochastic evolution equation (1.1), which is also weighted pseudo S-asymptotically (ω, k)-periodic. To achieve that, we provide some auxiliary outcomes where we establish some useful superposition results.

3.1. Some auxiliary results

Let $h \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ and consider the following assumptions:

(H0) For all $u \in \mathbb{L}^2(\Omega, \mathbb{H})$,

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, u, \mathbb{P}_u) - e^{ik\omega} h(t, e^{-ik\omega}u, \mathbb{P}_{e^{-ik\omega}u})\|^2 \rho(t)dt = 0$$

uniformly on any bounded set of $\mathbb{L}^2(\Omega, \mathbb{H})$.

(H1) There exists a number $L > 0$ such that for any $u, v \in \mathbb{L}^2(\Omega, \mathbb{H})$ and $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{H})$,

$$\mathsf{E}\|h(t, u, \mu_1) - h(t, v, \mu_2)\|^2 \leq L.\left(\mathsf{E}\|u - v\|^2 + \mathsf{W}_2^2(\nu_1, \nu_2)\right),$$

uniformly for all $t \in \mathbb{R}$.

(H*1) For any $\epsilon > 0$ and any bounded subset $D \subset \mathbb{L}^2(\Omega, \mathbb{H})$, there exist constants $T_{\epsilon,D} > 0$ and $\delta_{\epsilon,D} > 0$ such that

$$\mathsf{E}\|h(t, v_1, \mathbb{P}_{v_1}) - h(t, v_2, \mathbb{P}_{v_2})\|^2 \leq \epsilon$$

for all $v_1, v_2 \in D$ with $\mathsf{E}\|v_1 - v_2\|^2 \leq \delta_{\epsilon,D}$ and $t \geq T_{\epsilon,D}$.

Remark 3.1. *The condition* **(H*1)** *mean that* $h : \mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H})$ *is asymptotically uniformly continuous on bounded sets of* $\mathbb{L}^2(\Omega, \mathbb{H})$.

Remark 3.2. *Particularly, by choosing*

1. $k\omega = \pi$, *condition* **(H0)** *degrades to assumption* **(H*0)** *given by: for all* $v \in \mathbb{L}^2(\Omega, \mathbb{H})$,

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v, \mathbb{P}_v) + h(t, -v, \mathbb{P}_{-v})\|^2 \rho(t)dt = 0$$

uniformly on any bounded set of $\mathbb{L}^2(\Omega, \mathbb{H})$.

2. $k\omega = 2\pi$, *condition* **(H0)** *degrades to assumption* **(H**0)** *given by: for all* $v \in \mathbb{L}^2(\Omega, \mathbb{H})$,

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v, \mathbb{P}_v) - h(t, v, \mathbb{P}_v)\|^2 \rho(t)dt = 0$$

uniformly on any bounded set of $\mathbb{L}^2(\Omega, \mathbb{H})$.

We have the following composition theorem.

Theorem 3.3. *Let* $\rho \in \Lambda_\infty$. *If* $h \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ *satisfies* $(\mathbf{H0}) - (\mathbf{H1})$, *then we have* $h(\cdot, v(\cdot), \mathbb{P}_{v(\cdot)}) \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ *for every* $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$.

Proof. Since $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ then $\sup\limits_{t\in\mathbb{R}} \mathsf{E}\|v(t)\|^2 < \infty$. Therefore

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, v(t), \mathbb{P}_{v(t)})\|^2 &\leq \sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t, 0, 0\|^2 + \sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, 0, 0)\|^2 \\
&\leq L \sup_{t\in\mathbb{R}} \Big(\mathsf{E}\|v(t)\|^2 + \mathsf{W}_2^2(\mathbb{P}_{v(t)}, 0) \Big) + \sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, 0, 0)\|^2 \\
&\leq 2L \sup_{t\in\mathbb{R}} \mathsf{E}\|v(t)\|^2 + \sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, 0, 0)\|^2 < \infty.
\end{aligned}$$

Let $t, t_0 \in \mathbb{R}$ and $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$. Then

$$\begin{aligned}
\mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t_0), \mathbb{P}_{v(t_0)})\right\|^2 &\leq 3\mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t), \mathbb{P}_{v(t)})\right\|^2 \\
&\quad + 3\mathsf{E}\left\|h(t_0, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t_0), \mathbb{P}_{v(t)})\right\|^2 \\
&\quad + 3\mathsf{E}\left\|h(t_0, v(t_0), \mathbb{P}_{v(t)}) - h(t_0, v(t_0), \mathbb{P}_{v(t_0)})\right\|^2 \\
&\leq 3\mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t), \mathbb{P}_{v(t)})\right\|^2 \\
&\quad + 3L\mathsf{E}\left\|v(t) - v(t_0)\right\|^2 + 3L\mathsf{W}_2^2\big(v(t), v(t_0)\big) \\
&\leq 3\mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t), \mathbb{P}_{v(t)})\right\|^2 \\
&\quad + 6L\mathsf{E}\left\|v(t) - v(t_0)\right\|^2.
\end{aligned}$$

Since $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ and $h \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ then

$$\lim_{t\to t_0} \mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t), \mathbb{P}_{v(t)})\right\|^2 = \lim_{t\to t_0} \mathsf{E}\left\|v(t) - v(t_0)\right\|^2 = 0.$$

It follows that $h(\cdot, v(\cdot), \mathbb{P}_{v(\cdot)}) \in \mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})\right)$. Next, we prove that

$$\lim_{r\to+\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)})) - e^{ik\omega} h(t, v(t), \mathbb{P}_{v(t)}))\|^2 \rho(t)dt = 0.$$

We have

$$\begin{aligned}
&\frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)})) - e^{ik\omega} h(t, v(t), \mathbb{P}_{v(t)}))\|^2 \rho(t)dt \\
\leq &\frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)}) - e^{ik\omega} h(t, e^{-ik\omega} v(t+\omega), \mathbb{P}_{e^{-ik\omega} v(t+\omega)})\|^2 \rho(t)dt \\
&+ \frac{2}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|e^{ik\omega} h(t, e^{-ik\omega} v(t+\omega), \mathbb{P}_{e^{-ik\omega} v(t+\omega)}) - e^{ik\omega} h(t, v(t), \mathbb{P}_{v(t)}))\|^2 \rho(t)dt \\
\leq &\frac{2}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)}) - e^{ik\omega} h(t, e^{-ik\omega} v(t+\omega), \mathbb{P}_{e^{-ik\omega} v(t+\omega)})\|^2 \rho(t)dt \\
&+ \frac{2L}{m(r,\rho)} \int_{-r}^{r} \Big(\mathsf{E}\|e^{-ik\omega} v(t+\omega)) - v(t)\|^2 + \mathsf{W}_2^2\big(\mathbb{P}_{e^{-ik\omega} v(t+\omega)}, \mathbb{P}_{v(t)}\big) \Big) \rho(t)dt \\
\leq &\frac{2}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)}) - e^{ik\omega} h(t, e^{-ik\omega} v(t+\omega), \mathbb{P}_{e^{ik\omega} v(t+\omega)})\|^2 \rho(t)dt \\
&+ \frac{4L}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|e^{-ik\omega} v(t+\omega) - v(t)\|^2 \rho(t)dt \\
&\to 0 \text{ as } r \to +\infty \text{ by } (\mathbf{H0}) \text{ and the fact that } v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho).
\end{aligned}$$

Then it follows that

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|h(t+\omega,v(t+\omega),\mathbb{P}_{v(t+\omega)}))-e^{ik\omega}h(t,v(t),\mathbb{P}_{v(t)}))\|^2\rho(t)dt=0.$$

Which means that $h(\cdot,v(\cdot),\mathbb{P}_{v(\cdot)})\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. ∎

From Theorem 3.3, we derive the following corollaries.

Corollary 3.4. *Let $\rho\in\Lambda_\infty$ and v be a square-mean weighted pseudo-S-asymptotically ω-antiperiodic stochastic process. If $h\in\mathcal{BC}(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H})\times\mathcal{P}_2(\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H}))$ satisfies* **(H*0)-(H1)**, *then $t\longmapsto h(t,v(t),\mathbb{P}_{v(t)})$ is square-mean weighted pseudo-S-asymptotically ω-antiperiodic.*

Corollary 3.5. *Let $\rho\in\Lambda_\infty$ and v be a square-mean weighted pseudo-S-asymptotically ω-periodic stochastic process. If $h\in\mathcal{BC}(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H})\times\mathcal{P}_2(\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H}))$ satisfies* **(H**0)-(H1)**, *then $t\longmapsto h(t,v(t),\mathbb{P}_{v(t)})$ is square-mean weighted pseudo-S-asymptotically ω-periodic.*

Theorem 3.6. *Let $\rho\in\Lambda_\infty$. If $h\in\mathcal{BC}(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H})\times\mathcal{P}_2(\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H}))$ satisfies* **(H0)-(H*1)**, *then we have $t\longrightarrow h(t,v(t),\mathbb{P}_{v(t)})\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$ for every $v\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$.*

Proof. The demonstration can be done similarly to Theorem 3.3 and Theorem 2.6 in [5] with minor modifications. ∎

Now, we present some convolutions results.

Lemma 3.7. *If $\{\mathcal{K}(t)\}_{t\geq0}\subset\mathcal{B}(\mathbb{H})$ is uniformly $1-$integrable and strongly continuous family of operators, and $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$, then*

$$t\longmapsto\Phi(t)=\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)ds\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho).$$

Proof. Let $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. Since the operator family $\{\mathcal{K}(t)\}_{t\geq0}$ is uniformly $1-$integrable then there exist $M>0$ such that

$$\int_0^\infty\|\mathcal{K}(t)\|dt\leq M.$$

Then it is easy to show that $\Phi\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ using the Lebesgue dominated convergence theorem and the fact that $X\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. For any $r>0$ we have

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\Phi(t+\omega)-e^{ik\omega}\Phi(t)\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t+\omega}\mathcal{K}(t+\omega-s)X(s)ds-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)ds\right\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)X(s+\omega)ds-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)ds\right\|^2d\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)[X(s+\omega)-e^{ik\omega}X(s)]ds\right\|^2\rho(t)dt\\
&\leq\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{-\infty}^{t}\|\mathcal{K}(t-s)\|\,ds\int_{-\infty}^{t}\|\mathcal{K}(t-s)\|\;\mathsf{E}\|X(s+\omega)-e^{ik\omega}X(s)\|^2ds\right]\rho(t)dt\\
&\leq M\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{-\infty}^{t}\|\mathcal{K}(t-s)\|\;\mathsf{E}\|X(s+\omega)-e^{ik\omega}X(s)\|^2ds\right]\rho(t)dt\\
&\leq M\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{0}^{\infty}\|\mathcal{K}(s)\|\;\mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2ds\right]\rho(t)dt.
\end{aligned}$$

From the Fubini theorem, it follows that

$$\begin{aligned}
&M\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_0^{\infty}\|\mathcal{K}(s)\|\ \mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2ds\right]\rho(t)dt\\
&=M\int_0^{\infty}\|\mathcal{K}(s)\|\left[\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2\rho(t)dt\right]ds\\
&=M\int_0^{\infty}\|\mathcal{K}(s)\|\left[\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X_{-s}(t+\omega)-e^{ik\omega}X_{-s}(t)\|^2\rho(t)dt\right]ds.
\end{aligned}$$

Since $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$, thanks to Lemma 2.4, we know that for any $s\in\mathbb{R}$, we have

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X_{-s}(t+\omega)-e^{ik\omega}X_{-s}(t)\|^2\rho(t)dt=0.$$

Then Lebesgue dominated convergence theorem yield that

$$\int_0^{\infty}\|\mathcal{K}(s)\|\left[\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X_{-s}(t+\omega)-e^{ik\omega}X_{-s}(t)\|^2\rho(t)dt\right]ds\to 0 \text{ as } r\to+\infty.$$

Therefore

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\Phi(t+\omega)-e^{ik\omega}\Phi(t)\|^2\rho(t)dt=0.$$

Which proves that $\Phi\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. ■

Lemma 3.8. *If $\{\mathcal{K}(t)\}_{t\geq 0}\subset\mathcal{B}(\mathbb{H})$ is uniformly $2-$integrable and strongly continuous family of operators, and $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$, then*

$$t\longmapsto\widetilde{\Phi}(t)=\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)d\mathbb{W}(s)\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho).$$

Proof. Since the operator family $\{\mathcal{K}(t)\}_{t\geq 0}$ is uniformly $2-$integrable then by the Ito's isometry property of stochastic integral, the Lebesgue dominated convergence theorem and the fact that $X\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, it is easy to show that $\widetilde{\Phi}\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ using the Lebesgue dominated convergence theorem and the fact that $X\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. We have that

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\widetilde{\Phi}(t+\omega)-e^{ik\omega}\widetilde{\Phi}(t)\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t+\omega}\mathcal{K}(t+\omega-s)X(s)d\mathbb{W}(s)-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)d\mathbb{W}(s)\right\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)X(s+\omega)d\mathbb{W}(s+\omega)-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)d\mathbb{W}(s)\right\|^2\rho(t)dt.
\end{aligned}$$

Let $\widetilde{W}(s)=\mathbb{W}(s+\omega)-\mathbb{W}(\omega)$. We know that $\widetilde{W}$ is a Brownian motion and has the same distribution as $\mathbb{W}$.

Using the Ito's isometry property of stochastic integral and the Fubini's theorem, we obtain that

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\widetilde{\Phi}(t+\omega)-e^{ik\omega}\widetilde{\Phi}(t)\|^2 d\mu(t)\\
&\quad=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)X(s+\omega)d\widetilde{W}(s)-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)d\widetilde{W}(s)\right\|^2\rho(t)dt\\
&\quad=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)\left[X(s+\omega)-e^{ik\omega}X(s)\right]d\widetilde{W}(s)\right\|^2\rho(t)dt\\
&\quad=\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-\infty}^{t}\mathsf{E}\left\|\mathcal{K}(t-s)\left[X(s+\omega)-e^{ik\omega}X(s)\right]\right\|^2 ds\rho(t)dt\\
&\quad\le\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{-\infty}^{t}\|\mathcal{K}(t-s)\|^2\,\mathsf{E}\|X(s+\omega)-e^{ik\omega}X(s)\|^2ds\right]\rho(t)dt\\
&\quad\le\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{0}^{\infty}\|\mathcal{K}(s)\|^2\,\mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2ds\right]\rho(t)dt\\
&\quad\le\int_{0}^{\infty}\|\mathcal{K}(s)\|^2\left[\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2\rho(t)dt\right]ds.
\end{aligned}$$

Since $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$ and $\{\mathcal{K}(t)\}_{t\ge 0}$ is uniformly $2-$integrable, therefore invoking Lemma 2.4 and Lebesgue dominated converge theorem, we get that

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\widetilde{\Phi}(t+\omega)-e^{ik\omega}\widetilde{\Phi}(t)\|^2\rho(t)dt=0.$$

which proves that $\widetilde{\Phi}\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. ■

It is worthwhile to point out that if $k\omega=\pi$ (resp. $k\omega=2\pi$), from Lemmas 3.7 and 3.8, we can get some convolutions results for weighted pseudo S-asymptotically ω-anti-periodic (resp. ω-periodic) stochastic processes.

3.2. Existence of mild solution in $\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$

We will begin by recollecting some facts about the Weyl fractional integrals and derivatives of order $\alpha>0$, as well as the α-resolvent operators that will be employed to develop the main results. For more details on properties α-resolvent operators, one can make reference to [24]. Suppose that $\mathbb{X}$ is a Banach space. For given function $h:\mathbb{R}\to\mathbb{X}$, the Weyl fractional integral of order $\alpha>0$ is defined by

$$\partial_t^{-\alpha}h(t):=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}h(s)ds,\qquad t\in\mathbb{R},$$

when this integral is convergent. The Weyl fractional derivative ∂_t^{α} of order α is defined by

$$\partial_t^{\alpha}h(t)=\frac{d^n}{dt^n}\partial_t^{-(n-\alpha)}h(t),\quad t\in\mathbb{R},$$

where $n=[\alpha]+1$, and the notation $[\alpha]$ represents the integer part of α. Now, Let A be a closed and linear operator with domain $\mathcal{D}(A)$ defined on a Banach space X, and $\alpha>0$. For a given kernel $b(\cdot)\in L^1_{loc}(\mathbb{R}_+)$, it is said that A is the generator of an α-resolvent family if there exists $\xi>0$ and a strongly continuous family $\mathcal{R}_\alpha:\mathbb{R}_+\to\mathcal{B}(\mathbb{X})$ such that

$$\left\{\frac{\lambda^\alpha}{1+\hat{b}(\lambda)}:\text{Re}(\lambda)>\xi\right\}\subseteq\rho(A)$$

and for all $y \in \mathbb{X}$,

$$(\lambda^\alpha - (1+\hat{b}(\lambda))A)^{-1}y = \frac{1}{1+\hat{b}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{b}(\lambda)} - A\right)^{-1} y = \int_0^\infty e^{-\lambda t}\mathcal{R}_\alpha(t)y dt, \quad \text{Re}\lambda > \xi.$$

$\{\mathcal{R}_\alpha(t)\}_{t\geq 0}$ is called the α-resolvent family generated by the operator A. Motivated by Ponce [24], we present the concept of mild solutions for Eq.(1.1). For each $t \in \mathbb{R}$, $\mathbb{W}(t)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{\mathbb{W}(r) - \mathbb{W}(s) \,|\, r, s \leq t\}$.

Definition 3.9. *An $\mathcal{F}_t$-progressively measurable process $\{v(t)\}_{t\in\mathbb{R}}$ is called a mild solution of problem* (1.1) *if it satisfies the following stochastic integral equation*

$$v(t) = \int_{-\infty}^t \mathcal{R}_\alpha(t-s)g(s, v(s), \mathbb{P}_{v(s)})ds + \int_{-\infty}^t \mathcal{R}_\alpha(t-s)f(s, v(s), \mathbb{P}_{v(s)})d\mathbb{W}(s)$$

for all $t \in \mathbb{R}$, where $\{\mathcal{R}_\alpha(t)\}_{t\geq 0}$ the resolvent family generated by the operator A.

We establish the existence and uniqueness of weighted square-mean S-asymptotically Bloch type periodic mild solution for Eq.(1.1) under global Lipschitz-type conditions on the second variable of functions.

Theorem 3.10. *Suppose that the operator A generates an α-resolvent operator $\{\mathcal{R}_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(\mathbb{H})$ such that for $t \geq 0$, $\|\mathcal{R}_\alpha(t)\| \leq \phi_\alpha(t)$ where $\phi_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Further, assume that $g,\ f \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ satisfy* **(H0)** *and there exists constants $L, L' > 0$ such that for any $v_1, v_2 \in \mathbb{L}^2(\Omega, \mathbb{H})$ and $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{H})$,*

$$\mathsf{E}\|g(t, v_1, \mu_1) - g(t, v_2, \mu_2)\|^2 \leq L\left(\mathsf{E}\|v_1 - v_2\|^2 + \mathsf{W}_2^2(\nu_1, \nu_2)\right),$$
$$\mathsf{E}\|f(t, v_1, \mu_1) - f(t, v_2, \mu_2)\|^2 \leq L'\left(\mathsf{E}\|v_1 - v_2\|^2 + \mathsf{W}_2^2(\nu_1, \nu_2)\right),$$

uniformly for all $t \in \mathbb{R}$.
Then Eq.(1.1) *has a unique mild solution $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$, provided*

$$\|\phi_\alpha\|_{L^1}^2 L + L'\|\phi_\alpha\|_{L^2}^2 < \frac{1}{4}. \tag{3.1}$$

Proof. From Theorem 3.3, for each $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$, the stochastic processes $s \mapsto f(s, v(s), \mathbb{P}_{v(s)})$, $s \mapsto g(s, v(s), \mathbb{P}_{v(s)})$ belongs to $\mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$. From Lemmas 3.7, 3.8 and 2.4-(a), we can define the operator

$$\mathcal{S} : \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho) \to \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$$

$$\text{by}\ \ (\mathcal{S}v)(t) = \int_{-\infty}^t \mathcal{R}_\alpha(t-s)g(s, v(s), \mathbb{P}_{v(s)})ds + \int_{-\infty}^t \mathcal{R}_\alpha(t-s)f(s, v(s), \mathbb{P}_{v(s)})d\mathbb{W}(s)\,.$$

Let $v_1, v_2 \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ and $t \in \mathbb{R}$. Then using Cauchy-Schwartz's inequality and Ito's isometry property of stochastic integral, we have

$$\begin{aligned}
&\mathsf{E}\|(\mathcal{S}v_1)(t)-(\mathcal{S}v_2)(t)\|^2\\
&\quad\le 2\mathsf{E}\|\int_{-\infty}^{t}\mathcal{R}_\alpha(t-s)[g(s,v_1(s),\mathbb{P}_{v_1(s)})-g(s,v_2(s),\mathbb{P}_{v_2(s)})]ds\|^2\\
&\qquad+2\mathsf{E}\|\int_{-\infty}^{t}\mathcal{R}_\alpha(t-s)[f(s,v_1(s),\mathbb{P}_{v_1(s)})-f(s,v_2(s),\mathbb{P}_{v_2(s)})]d\mathbb{W}(s)\|^2\\
&\quad\le 2\int_{-\infty}^{t}\phi_\alpha(t-s)ds\left(\int_{-\infty}^{t}\phi_\alpha(t-s)\mathsf{E}\|[g(s,v_1(s),\mathbb{P}_{v_1(s)})-g(s,v_2(s),\mathbb{P}_{v_2(s)})]\|^2\,ds\right)\\
&\qquad+2\left(\int_{-\infty}^{t}\phi_\alpha^2(t-s)\mathsf{E}\|[f(s,v_1(s),\mathbb{P}_{v_1(s)})-f(s,v_2(s),\mathbb{P}_{v_2(s)})]\|^2\,ds\right)\\
&\quad\le 2L\|\phi_\alpha\|_{L^1}\left(\int_{-\infty}^{t}\phi_\alpha(t-s)\Big[\mathsf{E}\|v_1(s)-v_2(s)\|^2+\mathsf{W}_2^2\big(v_1(s),v_2(s)\big)\Big]ds\right)\\
&\qquad+2L'\left(\int_{-\infty}^{t}\phi_\alpha^2(t-s)\Big[\mathsf{E}\|v_1(s)-v_2(s)\|^2+\mathsf{W}_2^2\big(v_1(s),v_2(s)\big)\Big]ds\right)\\
&\quad\le 4L\,\|\phi_\alpha\|_{L^1}^2\sup_{s\in\mathbb{R}}\mathsf{E}\|v_1(s)-v_2(s)\|^2+4L'\|\phi_\alpha\|_{L^2}^2\sup_{s\in\mathbb{R}}\mathsf{E}\|v_1(s)-v_2(s)\|^2\\
&\quad\le 4\left(\|\phi_\alpha\|_{L^1}^2L+L'\|\phi_\alpha\|_{L^2}^2\right)\|v_1-v_2\|_\infty^2.
\end{aligned}$$

Therefore we have

$$\|\mathcal{S}v_1-\mathcal{S}v_2\|_\infty\le 2\sqrt{\|\phi_\alpha\|_{L^1}^2L+L'\|\phi_\alpha\|_{L^2}^2}\,\|v_1-v_2\|_\infty\,.$$

The conclusion follows from the Banach fixed point theorem.

Remark 3.11. *By taking $k\omega=\pi$, we can derive some existence results for square-mean weighted pseudo S-asymptotically ω-antiperiodic mild solutions to problem* (1.1) *from Theorems 3.10. Moreover, choosing $k\omega=2\pi$, we can derive some existence results for square-mean weighted pseudo S-asymptotically ω-periodic mild solutions to problem* (1.1) *from Theorems 3.10.*

■

In the rest of this section, we prove the existence of the weighted pseudo S-asymptotic Bloch type periodic mild solution for Eq.(1.1) under sublinear growth conditions on g and global Lipschitz assumption on f . First, suppose that $\Psi:\mathbb{R}\to\mathbb{R}$ is a continuous functions which satisfies $\Psi(t)\ge 1$ for all $t\in\mathbb{R}$ and $\lim_{|t|\to\infty}\Psi(t)=\infty$. We define the space

$$\mathcal{C}_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})):=\left\{v\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)\ :\ \lim_{|t|\to\infty}\frac{\mathsf{E}\|v(t)\|^2}{\Psi(t)}=0\right\}.$$

$\mathcal{C}_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$ is a Banach space equipped with the norm $\|v\|_\Psi=\left(\sup_{t\in\mathbb{R}}\dfrac{\mathsf{E}\|v(t)\|^2}{\Psi(t)}\right)^{1/2}$.

Lemma 3.12 ([12])**.** *A set $\mathcal{U}\subset\mathcal{C}_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$ is relatively compact if the following conditions hold*

(a) $\lim_{|t|\to\infty}\dfrac{\mathsf{E}\|u(t)\|^2}{\Psi(t)}=0$ *uniformly for any $u\in\mathcal{U}$;*

(b) *$\mathcal{U}$ is equicontinuous.*

(c) *The set $\mathcal{U}(t)=\{u(t)\ :\ u\in\mathcal{U}\}$ is relatively compact in $\mathbb{L}^2(\Omega,\mathbb{H})$ for each $t\in\mathbb{R}$.*

In order to accomplish that, we will need the following conditions:

(H2) The functions $g,\ f \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ satisfy

1. $f(t, z, \mu)$ and $g(t, z, \mu)$ are uniformly continuous in any bounded subset $D \subseteq \mathbb{L}^2(\Omega, \mathbb{H})$ uniformly for $t \in \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{H})$
2. For all $\mu \in \mathcal{P}_2(\mathbb{H})$, there is a continuous nondecreasing function $\mathcal{X}_g : [0, +\infty) \to [0, +\infty)$ and positive constant $M_g := M_g(\mu)$ such that

$$\mathsf{E}\|g(t, z, \mu)\|^2 \leq M_g\, \mathcal{X}_g(\mathsf{E}\|z\|^2) \text{ for all } t \in \mathbb{R},\ z \in \mathbb{L}^2(\Omega, \mathbb{H}).$$

3. For each $\epsilon > 0$ there exist $\delta > 0$ such that for every $y, z \in C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$, $\|y - z\|_\Psi \leq \delta$ implies that

$$\int_{-\infty}^{t} \phi_\alpha(t-s)\mathbb{E}\|g(t, y, \mathbb{P}_y) - g(t, z, \mathbb{P}_z)\|^2 ds \leq \frac{\epsilon}{4(\|\phi_\alpha\|_{L^1} + 1)} \text{ for all } t \in \mathbb{R},$$
$$\int_{-\infty}^{t} \phi_\alpha^2(t-s)\mathbb{E}\|f(t, y, \mathbb{P}_y) - f(t, z, \mathbb{P}_z)\|^2 ds \leq \frac{\epsilon}{4} \text{ for all } t \in \mathbb{R} \text{ and}$$
$$J := \sup_{t\in\mathbb{R}} \Big(\int_{-\infty}^{t} \phi_\alpha^2(t-s)\Psi(s) ds \Big) < \infty.$$

Our strategy is based on the following Krasnoselskii's fixed point theorem.

Lemma 3.13 ([1])**.** *Suppose* $\mathbf{B}$ *is a closed convex and nonempty subset of a Banach space* $\mathbb{Y}$ *and* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are two operators verifying*

1. *If* $y, z \in \mathbf{B}$*; then* $\mathcal{S}_1 y + \mathcal{S}_2 z \in \mathbf{B}$*;*
2. $\mathcal{S}_1$ *is compact and continuous;*
3. $\mathcal{S}_2$ *is a mapping contraction.*

Then, there exists $y \in \mathbf{B}$ *such that* $y = \mathcal{S}_1 y + \mathcal{S}_2 y$*.*

We have the following existence result.

Theorem 3.14. *Suppose that the operator* A *generates a compact* α*-resolvent operator* $\{\mathcal{R}_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(\mathbb{H})$ *for* $t > 0$ *such that* $\|\mathcal{R}_\alpha(t)\| \leq \phi_\alpha(t)$ *where* $\phi_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ *for* $t \geq 0$*. Assume that* $g,\ f \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ *verify assumptions* **(H0)** *and* **(H2)***. Moreover, suppose that* g *satisfies condition* **(H*1)** *and there exists constants* $L' > 0$ *such that for any* $v_1, v_2 \in \mathbb{L}^2(\Omega, \mathbb{H})$ *and* $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{H})$*,*

$$\mathsf{E}\|f(t, v_1, \mu_1) - f(t, v_2, \mu_2)\|^2 \leq L' \bigg(\mathsf{E}\|v_1 - v_2\|^2 + \mathsf{W}_2^2(\nu_1, \nu_2) \bigg).$$

Then the problem (1.1) *has at least one mild solution in* $\mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ *provided that*

$$2\, L' J < 1.$$

Proof. Define the operator $\mathcal{S} : C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})) \to C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$ by

$$\begin{aligned}(\mathcal{S}v)(t) &= \int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) g(s, v(s), \mathbb{P}_{v(s)}) ds + \int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) f(s, v(s), \mathbb{P}_{v(s)}) d\mathbb{W}(s)\\ &= (\mathcal{S}_1 v)(t) + (\mathcal{S}_2 v)(t).\end{aligned}$$

where

$$(\mathcal{S}_1 v)(t) = \int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) g(s, v(s), \mathbb{P}_{v(s)}) ds \text{ and}$$
$$(\mathcal{S}_2 v)(t) = \int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) f(s, v(s), \mathbb{P}_{v(s)}) d\mathbb{W}(s).$$

In order to show that $\mathcal{S}$ has at least one fixed point in $\mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ through the Krasnoselskii's fixed point theorem, we will divide the proof in several steps.
Step 1. We claim that $\mathcal{S} : C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})) \to C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$.
Let $v \in C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$. First, from Lemmas 3.7 and 3.8, $\mathcal{S}(\Phi) \in \mathcal{C}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$.

Next, we have that

$$\begin{aligned}
&\mathsf{E}\|(\mathcal{S}v)(t)\|^2 \\
\leq & 2\mathsf{E}\| \int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) g(s, v(s), \mathbb{P}_{v(s)}) ds\|^2 + 2\mathsf{E}\| \int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) f(s, v(s), \mathbb{P}_{v(s)}) d\mathbb{W}(s)\|^2 \\
\leq & 2 \int_{-\infty}^{t} \phi_\alpha(t-s) ds \left(\int_{-\infty}^{t} \phi_\alpha(t-s) \mathsf{E}\|g(s, v(s), \mathbb{P}_{v(s)})\|^2 \, ds \right) \\
& + 2 \left(\int_{-\infty}^{t} \phi_\alpha^2(t-s) \mathsf{E}\|f(s, v(s), \mathbb{P}_{v(s)}) - f(s, 0, \mathbb{P}_{v(s)}) + f(s, 0, \mathbb{P}_{v(s)})\|^2 \, ds \right) \\
\leq & 2\|\phi_\alpha\|_{L^1} \left(\int_{-\infty}^{t} \phi_\alpha(t-s) \left[M_g \mathcal{X}_g(\|v\|_\infty^2) \right] \, ds \right) \\
& + 4 \left(\int_{-\infty}^{t} \phi_\alpha^2(t-s)) \left[L' \mathsf{E}\|v(s)\|^2 + \mathsf{E}\|f(s, 0, \mathbb{P}_{v(s)})\|^2 \right] \, ds \right) \\
\leq & 2 M_g \mathcal{X}_g(\|v\|_\infty^2) \|\phi_\alpha\|_{L^1}^2 + 4\|\phi_\alpha\|_{L^2}^2 \Big(L' \|v\|_\infty^2 + M_f \Big),
\end{aligned}$$

where $M_f \equiv M_f(\mu) = \sup_{t \in \mathbb{R}} \mathsf{E}\|f(t, 0, \mu)\|^2$ for all $\mu \in \mathcal{P}_2(\mathbb{H})$.
Since $v \in C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$ and by invoking condition **(H2)-(3)**, we derive that

$$\lim_{|t| \to \infty} \frac{\mathsf{E}\|(\mathcal{S}v)(t)\|^2}{\Psi(t)} \leq \lim_{|t| \to \infty} \frac{2 M_g \mathcal{X}_g(\|v\|_\infty^2) \|\phi_\alpha\|_{L^1}^2 + 4\|\phi_\alpha\|_{L^2}^2 \Big(L' \|v\|_\infty^2 + M_f \Big)}{\Psi(t)} = 0.$$

This prove that $\mathcal{S} : C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})) \to C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$.

Step 2. We claim that $\mathcal{S}$ is continuous on $C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$
Let $\epsilon > 0$. From condition **(H1)-(3)**, there exist a real positive constant $\delta > 0$ such that for each $y, z \in C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$ with $\|y - z\|_\Psi \leq \delta$, we have

$$\int_{-\infty}^{t} \phi_\alpha(t-s) \mathbb{E}\|g(t, y, \mathbb{P}_y) - g(t, z, \mathbb{P}_z)\|^2 ds \leq \frac{\epsilon}{4(\|\phi_\alpha\|_{L^1} + 1)} \text{ for all } t \in \mathbb{R},$$
$$\int_{-\infty}^{t} \phi_\alpha^2(t-s) \mathbb{E}\|f(t, y, \mathbb{P}_y) - f(t, z, \mathbb{P}_z)\|^2 ds \leq \frac{\epsilon}{4} \text{ for all } t \in \mathbb{R}.$$

Now, we obtain that

$$\begin{aligned}
\mathsf{E}\|(\mathcal{S}y)(t)-&(\mathcal{S}z)(t)\|^2 \\
\leq& 2\mathsf{E}\|\int_{-\infty}^{t}\mathcal{R}_\alpha(t-s)[g(s,y(s),\mathbb{P}_{y(s)})-g(s,z(s),\mathbb{P}_{z(s)})]ds\|^2 \\
&+2\mathsf{E}\|\int_{-\infty}^{t}\mathcal{R}_\alpha(t-s)[f(s,y(s),\mathbb{P}_{y(s)})-f(s,z(s),\mathbb{P}_{z(s)})]d\mathbb{W}(s)\|^2 \\
\leq& 2\int_{-\infty}^{t}\phi_\alpha(t-s)ds\left(\int_{-\infty}^{t}\phi_\alpha(t-s)\mathsf{E}\|[g(s,y(s),\mathbb{P}_{y(s)})-g(s,z(s),\mathbb{P}_{z(s)})]\|^2\,ds\right) \\
&+2\left(\int_{-\infty}^{t}\phi_\alpha^2(t-s)\mathsf{E}\|[f(s,y(s),\mathbb{P}_{y(s)})-f(s,z(s),\mathbb{P}_{z(s)})]\|^2\,ds\right) \\
\leq& 2\|\phi_\alpha\|_{L^1}\left(\int_{-\infty}^{t}\phi_\alpha(t-s)\mathsf{E}\|[g(s,y(s),\mathbb{P}_{y(s)})-g(s,z(s),\mathbb{P}_{z(s)})]\|^2\,ds\right) \\
&+2\left(\int_{-\infty}^{t}\phi_\alpha^2(t-s)\mathsf{E}\|[f(s,y(s),\mathbb{P}_{y(s)})-f(s,z(s),\mathbb{P}_{z(s)})]\|^2\,ds\right) \\
\leq& (2\|\phi_\alpha\|_{L^1})\times\left(\frac{\epsilon}{4(\|\phi_\alpha\|_{L^1}+1)}\right)+2\times\left(\frac{\epsilon}{4}\right) \\
\leq& \epsilon.
\end{aligned}$$

Which implies that

$$\|(\mathcal{S}y)-(\mathcal{S}z)\|_\Psi=\left(\sup_{t\in\mathbb{R}}\frac{1}{\Psi(t)}\mathsf{E}\|(\mathcal{S}y)(t)-(\mathcal{S}z)(t)\|^2\right)^{1/2}\longrightarrow 0 \text{ as } y\to z.$$

This show that $\mathcal{S}$ is continuous on $C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$.

Step 3. We show that there is $\mathbf{k}>0$ such that $\mathcal{S}(\mathbf{B_k})\subseteq\mathbf{B_k}$, where

$$\mathbf{B_k}\equiv\mathbf{B_k}\left(C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))\right):=\{z\in C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))\text{ such that }\|z\|_\Psi\leq\mathbf{k}\}$$

represents the closed ball with center at 0 and radius $\mathbf{k}$ in the space $C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$. Arguing by contradiction, suppose that for each $\mathbf{k}>0$ there exist $z_\mathbf{k}\in\mathbf{B_k}$ such that $\|\mathcal{S}z_\mathbf{k}\|_\Psi>\mathbf{k}$.
We have

$$\mathsf{E}\|(\mathcal{S}z_\mathbf{k})(t)\|^2\leq 2M_g\mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2+4\|\phi_\alpha\|_{L^2}^2\Big(L'\|v\|_\infty^2+M_f\Big).$$

For all $t\in\mathbb{R}$, we get

$$\frac{\mathsf{E}\|(\mathcal{S}z_\mathbf{k})(t)\|^2}{\Psi(t)}\leq\frac{1}{\Psi(t)}\left(2M_g\mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2+4\|\phi_\alpha\|_{L^2}^2\Big(L'\|v\|_\infty^2+M_f\Big)\right)$$

We get that

$$\begin{aligned}
\mathbf{k}<\|(\mathcal{S}z_\mathbf{k})\|_\Psi&=\sup_{t\in\mathbb{R}}\frac{\mathsf{E}\|(\mathcal{S}z_\mathbf{k})(t)\|^2}{\Psi(t)} \\
&\leq\sup_{t\in\mathbb{R}}\frac{1}{\Psi(t)}\left(2M_g\mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2+4\|\phi_\alpha\|_{L^2}^2\Big(L'\|v\|_\infty^2+M_f\Big)\right)=0.
\end{aligned}$$

Which is a contradiction.

Step 4. We show that $\mathcal{S}_1$ is completely continuous and $\mathcal{S}_2$ is a contraction.
By similar computations as in the proof of theorem 3.10, it's easy to see that $\mathcal{S}_2$ is a contraction provided $4\,L'\|v_\alpha\|_{L^2}^2 < 1$. On the other hand, it easy to see that $\mathcal{S}_1 : \mathbf{B_k} \to \mathbf{B_k}$ is continuous.
Let $\mathcal{U} = \mathcal{S}_1(\mathbf{B_k})$ and $u(t) = (\mathcal{S}_1 v)(t)$ for $v \in \mathbf{B_k}$ and $t \in \mathbb{R}$. We aim to prove that $\mathcal{U}$ is relatively compact with the aid of lemma 3.12. For more clarity, we split this step in three claims.

Claim 1. $\mathcal{U}(t)$ is a relatively compact subset of $\mathbb{L}^2(\Omega, \mathbb{H})$ for each $t \in \mathbb{R}$.
We know that $s \longmapsto \phi_\alpha(s)$ is integrable on $\mathbb{R}_+$. Hence, for $\epsilon > 0$, we can choose $b > 0$ such that

$$\int_b^\infty \phi_\alpha(s)ds \leq \frac{\epsilon}{\|\phi_\alpha\|_{L^1} M_g \mathcal{X}_g(\|v\|_\infty^2) + 1}.$$

Since for any $0 < a < b < \infty$, let

$$u_a(t) = \int_a^b \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds + \int_b^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds,$$

and

$$\mathcal{U}_a(t) := \{u_a(t) \ : \ 0 < a < b < \infty\}.$$

We derive that

$$\mathsf{E}\left\|\int_b^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds\right\|^2 \leq \|\phi_\alpha\|_{L^1} M_g \mathcal{X}_g(\|v\|_\infty^2) \int_b^\infty \phi_\alpha(s)ds < \epsilon.$$

and by invoking the mean value theorem for Bochner integral, we get

$$u_a(t) \in (b-a)\overline{co(\mathcal{O})} + \mathbf{B}_\epsilon(\mathbb{L}^2(\Omega, \mathbb{H})),$$

where $co(\mathcal{O})$ represents the convex hull of $\mathcal{O}$ and

$$\mathcal{O} := \{\mathcal{R}_\alpha(s) g(\xi, v, \mu) \ : \ a \leq s \leq b, \ t-b \leq \xi \leq t-a, \ \|v\|_\Psi \leq \mathbf{k}, \ \mu \in \mathcal{P}_2(\mathbb{H})\}$$

Furthermore, by the compactness of $\mathcal{R}_\alpha(t)$ for $t > 0$, it follows that $\mathcal{O}$ is relatively compact. Hence, we deduce that $\mathcal{U}_a(t) \subseteq (b-a)\overline{co(\mathcal{O})} + \mathbf{B}_\epsilon(\mathbb{L}^2(\Omega, \mathbb{H}))$ is also relatively compact for any $a > 0$.
By Lebesgue dominated convergence theorem, we have that

$$\begin{aligned}\mathbb{E}\|u(t) - u_a(t)\|^2 =& \mathsf{E}\Bigg\|\int_0^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds - \int_a^b \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds \\ & - \int_b^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds, \Bigg\|^2 \\ =& \mathsf{E}\left\|\int_0^a \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds\right\|^2 \\ \leq& \|\phi_\alpha\|_{L^1} M_g \mathcal{X}_g(\|v\|_\infty^2) \int_0^a \phi_\alpha(s)ds \to 0 \text{ as } a \to 0.\end{aligned}$$

Thus there exists relatively compact sets arbitrarily close to the set $\mathcal{U}(t)$. This proves that $\mathcal{U}(t)$ is relatively compact.

Claim 2. $\mathcal{U}$ **is equicontinuous.**
Simple computations yield that

$$
\begin{aligned}
& u(t+r)-u(t) \\
=& \int_0^\infty \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds - \int_0^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
=& \Big[\int_0^r \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds \\
& \quad + \int_0^a \mathcal{R}_\alpha(s+r) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
& \quad + \int_a^\infty \mathcal{R}_\alpha(s+r) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \Big] \\
& - \Big[\int_0^a \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds + \int_a^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \Big] \\
=& \int_0^r \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds \\
& \quad + \int_0^a [\mathcal{R}_\alpha(s+r) - \mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
& \quad + \int_a^\infty [\mathcal{R}_\alpha(s+r) - \mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
=& F_1(a,v,t,r) + F_2(a,v,t,r),
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(a,v,t,r) &= \int_0^r \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds \\
&\quad + \int_0^a [\mathcal{R}_\alpha(s+r) - \mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
F_2(a,v,t,r) &= \int_a^\infty [\mathcal{R}_\alpha(s+r) - \mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds.
\end{aligned}
$$

It follows that

$$
\mathsf{E}\|u(t+r)-u(t)\|^2 \le 2\left(\mathsf{E}\|F_1(a,v,t,r)\|^2 + \mathsf{E}\|F_2(a,v,t,r)\|^2 \right)
$$

By using Cauchy-Schwartz's inequality and condition **(H2)-(2)**, we obtain the following estimations

$$
\begin{aligned}
& \mathsf{E}\|F_1(a,v,t,r)\|^2 \\
\le & 2\mathsf{E}\left\| \int_0^r \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds \right\|^2 \\
& + 2\mathsf{E}\left\| \int_0^a [\mathcal{R}_\alpha(s+r) - \mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \right\|^2 \\
\le & 2\|\phi_\alpha\|_{L^1} \int_0^r \phi_\alpha(s) \mathsf{E}\|g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)})\|^2 ds \\
& + \left(\int_0^a \|\mathcal{R}_\alpha(s+r) - \mathcal{R}_\alpha(s)\| ds \right) \left(\int_0^a \|\mathcal{R}_\alpha(s+r) - \mathcal{R}_\alpha(s)\| \, \mathsf{E}\|g(t-s, v(t-s), \mathbb{P}_{v(t-s)})\|^2 ds \right) \\
\le & 2\|\phi_\alpha\|_{L^1} M_g \mathcal{X}_g(\|v\|_\infty^2) \int_0^r \phi_\alpha(s) ds + M_g \mathcal{X}_g(\|v\|_\infty^2) \left(\int_0^a \|\mathcal{R}_\alpha(s+r) - \mathcal{R}_\alpha(s)\| ds \right)^2,
\end{aligned}
$$

and

$$\begin{aligned}
&\mathsf{E}\|F_2(a,t,r)\|^2\\
&=\mathsf{E}\|\int_a^{\infty}[\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)]g(t-s,v(t-s),\mathbb{P}_{v(t-s)})ds\|^2\\
&\leq\int_a^{\infty}\|\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)\|ds\int_a^{\infty}\|\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)\|\ \mathsf{E}\|g(t-s,v(t-s),\mathbb{P}_{v(t-s)})\|^2\,ds\\
&\leq M_g\mathcal{X}_g(\|v\|_\infty^2)\left(\int_a^{\infty}\|\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)\|ds\right)^2.
\end{aligned}$$

By the continuity of $(\mathcal{R}_\alpha(t))_{t\geq 0}$ for $t>0$ in the operator norm topology, the right side of the above two inequalities tends to zero as $r\to 0$. Consequently,

$$\lim_{r\to 0}\mathsf{E}\|F_2(a,v,t,r)\|^2=\lim_{r\to 0}\mathsf{E}\|F_2(a,v,t,r)\|^2=0.$$

We deduce that

$$\begin{aligned}
\mathsf{E}\|(\mathcal{S}_1\Phi)(t+r)-(\mathcal{S}_1\Phi)(t)\|^2=&\mathsf{E}\|u(t+r)-u(t)\|^2\\
\leq&2\mathsf{E}\|F_1(a,v,t,r)\|^2+2\mathsf{E}\|F_2(a,v,t,r)\|^2\\
&\longrightarrow 0 \text{ as } r\to 0.
\end{aligned}$$

This prove that $\mathcal{U}$ is equicontinuous.

Claim 3. We show that $\lim\limits_{|t|\to\infty}\dfrac{\mathsf{E}\|u(t)\|^2}{\Psi(t)}=0$**.**

We have that

$$\frac{\mathsf{E}\|u(t)\|^2}{\Psi(t)}\leq\frac{M_g\mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2}{\Psi(t)}\longrightarrow 0 \text{ as } |t|\to\infty.$$

and this convergence in independent of $v\in\mathbf{B_k}(C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})))$.

Hence, by claims 1, 2, 3 and lemma 3.12, we deduce that $\mathcal{U}$ is relatively compact in $C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$. Hence $\mathcal{S}_1$ is completely continuous

Step 5. $\mathcal{S}$ **has a fixed point in** $\overline{\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}}^{\Psi}$

From Theorem 3.3 and 3.6, for each $v\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$, the stochastic processes $s\mapsto f(s,v(s),\mathbb{P}_{v(s)})$, $s\mapsto g(s,v(s),\mathbb{P}_{v(s)})$ belongs to $\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. From Lemmas 3.7, 3.8 and 2.4-(a), we obtain

$$\mathcal{S}(\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho))\subseteq\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho).$$

From step 1, it follows that

$$\mathcal{S}\left(\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}\right)\subseteq\left(\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}\right).$$

On the other hand, by the continuity of $\mathcal{S}$, we have that

$$\begin{aligned}
\mathcal{S}\left(\overline{\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}}^{\Psi}\right)&\subseteq\overline{\mathcal{S}\left(\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}\right)}^{\Psi}\\
&\subseteq\overline{\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}}^{\Psi}.
\end{aligned}$$

Thanks to Krasnoselski Theorem 3.13, we deduce that $\mathcal{S}$ admits a fixed point

$$v\in\overline{\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}}^{\Psi}.$$

Step 7. We prove that $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H}), \rho)$.
Let $\{v_n\} \subset \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H}), \rho) \cap \mathbf{B_k}$ such that $\|v_n - v\|_\Psi \to 0$. We obtain

$$\begin{aligned}\mathsf{E}\|(\mathcal{S}v_n)(t) - v(t)\|^2 &= \mathsf{E}\|(\mathcal{S}v_n)(t) - (\mathcal{S}v)(t)\|^2 \\ &\leq 2\|\phi_\alpha\|_{L^1}\left(\int_{-\infty}^t \phi_\alpha(t-s)\mathsf{E}\|[g(s,v_n(s),\mathbb{P}_{v_n(s)}) - g(s,v(s),\mathbb{P}_{v(s)})]\|^2\,ds\right) \\ &\quad + 2\left(\int_{-\infty}^t \phi_\alpha^2(t-s)\mathsf{E}\|[f(s,v_n(s),\mathbb{P}_{v_n(s)}) - f(s,v(s),\mathbb{P}_{v(s)})]\|^2\,ds\right).\end{aligned}$$

By using **(H2)-(3)**, we derive that $\{\mathcal{S}v_n\}$ converges to $\{\mathcal{S}v\}$ uniformly in $\mathbb{R}$. Thus, $v = \mathcal{S}v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H}), \rho)$ is mild solution of the problem (1.1). ■

Remark 3.15. *By taking $k\omega = \pi$, we can derive some existence results for square-mean weighted pseudo S-asymptotically ω-antiperiodic mild solutions to problem* (1.1) *from Theorems 3.10 and 3.14. Moreover, choosing $k\omega = 2\pi$, we can derive some existence results for square-mean weighted pseudo S-asymptotically ω-periodic mild solutions to problem* (1.1) *from Theorems 3.10 and 3.14.*

For example, we have the following results.

Corollary 3.16. *Suppose that the operator A generates a compact α-resolvent operator $\{\mathcal{R}_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(\mathbb{H})$ for $t > 0$ such that $\|\mathcal{R}_\alpha(t)\| \leq \phi_\alpha(t)$ where $\phi_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for $t \geq 0$ and the functions $g,\ f \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega,\mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega,\mathbb{H}))$ verify assumptions* **(H*0)-(H2)**. *Moreover, suppose that g satisfies condition* **(H*1)**, *and there exists constants $L' > 0$ such that for any $v_1, v_2 \in \mathbb{L}^2(\Omega,\mathbb{H})$ and $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{H})$,*

$$\mathsf{E}\|f(t,v_1,\mu_1) - f(t,v_2,\mu_2)\|^2 \leq L'\left(\mathsf{E}\|v_1 - v_2\|^2 + \mathsf{W}_2^2(\nu_1,\nu_2)\right).$$

Then the problem (1.1) *has at least one square-mean weighted pseudo S-asymptotically ω-antiperiodic mild solution provided that $2\,L'J < 1$..*

Corollary 3.17. *Suppose that the operator A generates a compact α-resolvent operator $\{\mathcal{R}_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(\mathbb{H})$ for $t > 0$ such that $\|\mathcal{R}_\alpha(t)\| \leq \phi_\alpha(t)$ where $\phi_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for $t \geq 0$ and the functions $g,\ f \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega,\mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega,\mathbb{H}))$ verify assumptions* **(H**0)-(H2)**. *Moreover, suppose that g satisfies condition* **(H*1)**, *and there exists constants $L' > 0$ such that for any $v_1, v_2 \in \mathbb{L}^2(\Omega,\mathbb{H})$ and $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{H})$,*

$$\mathsf{E}\|f(t,v_1,\mu_1) - f(t,v_2,\mu_2)\|^2 \leq L'\left(\mathsf{E}\|v_1 - v_2\|^2 + \mathsf{W}_2^2(\nu_1,\nu_2)\right).$$

Then the problem (1.1) *has at least one square-mean weighted pseudo S-asymptotically ω-periodic mild solution provided that $2\,L'J < 1$..*

4. Example

To illustrate our theoretical results, we consider

$$\rho(t) = 1 + t^2 \quad \text{for} \quad t \in \mathbb{R}.$$

Then, $\rho \in \Lambda_\infty$ and satisfies $(\mathbf{H}^\rho)$.
Let $\mathbb{H} = L^2[0,\pi]$, $1 < \alpha < 2$, $\nu > 0$ and consider the following problem

$$\begin{cases} \partial_t^\alpha v(t,\xi) = & -\nu\, v(t,\xi) - \dfrac{\nu^2}{4}\displaystyle\int_{-\infty}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(t,\xi)ds \\ & +g\big(t, v(t,\xi,\mathbb{P}_{v(t,\xi)}\big) + f\big(t, v(t,\xi), \mathbb{P}_{v(t,\xi)}\big)\dfrac{\partial\mathbb{W}(t)}{\partial t}, \quad (t,\xi) \in \mathbb{R}\times(0,\pi) \\ & v(t,0) = v(t,\pi) = 0, \end{cases} \tag{4.1}$$

where $\mathbb{W}(t)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$. The problem (4.1) can be written into the form (1.1) with $v(t)(\xi) = v(t,\xi)$, $b(t) = \dfrac{\nu}{4}\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $A = -\nu I$, I is the identity operator on the Hilbert space $\mathbb{H}$. It follows from [24, Example 4.17], that A generates a $\alpha-$resolvent family $\{\mathcal{R}_\alpha(t)\}_{t\geq 0}$ with its Laplace transform satisfying

$$\hat{\mathcal{R}}_\alpha(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + \nu/2)^2} = \frac{\lambda^{\alpha-\nu/2}}{(\lambda^\alpha+\nu/2)} \cdot \frac{\lambda^{\alpha-\nu/2}}{(\lambda^\alpha+\nu/2)}$$

and

$$\mathcal{R}_\alpha(t) = (r * r)(t) \text{ where } r(t) = t^{\frac{\alpha}{2}} \mathcal{E}_{\alpha,\alpha/2}\Big(-\frac{\nu}{2}t^\alpha\Big)$$

and $\mathcal{E}_{\alpha,\alpha/2}(\cdot)$ is the Mittag-Leffter function (see [26]). From [23, Theorem 4.12], there exists a constant $C > 0$, depending only on α, such that, for $t \geq 0$

$$\|\mathcal{R}_\alpha(t)\| \leq \frac{C}{1+\nu t^\alpha} := \phi_\alpha(t).$$

Simple calculations yield that :

$$\|\phi_\alpha\|_{L^1} = \frac{C}{\alpha \nu^{1/\alpha}} \mathsf{B}\left(\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right) < \infty$$

and

$$\|\phi_\alpha\|^2_{L^2} = \frac{C^2}{\alpha \nu^{(1/\alpha)-1}} \mathsf{B}\left(\frac{1}{\alpha}, 2-\frac{1}{\alpha}\right) < \infty,$$

where $\mathsf{B}(\cdot,\cdot)$ denotes the Beta function.

First, to illustrate the Theorem 3.10, let take the forcing terms are follows:

$$f(t,z,\mathbb{P}_z)(\xi) = M_1(t,z)(\xi) + \widetilde{M}_1(t,z,\mathbb{P}_z)(\xi) \quad \text{and}$$
$$g(t,z,\mathbb{P}_z)(\xi) = M_2(t,z)(\xi) + \widetilde{M}_2(t,z,\mathbb{P}_z)(\xi),$$

where $M_1(t,z)(\xi) = \gamma(t)\sigma_1(z(t)(\xi))$, $\quad M_2(t,z)(\xi) = \gamma(t)\sigma_2(z(t)(\xi))$

$$\widetilde{M}_1(t,z,\mathbb{P}_z)(\xi) = \frac{\gamma(t)}{1+t^2}\left[\cos(z(t)(\xi)) + \int_{\mathbb{L}^2(0,\pi)} \ell(x)\mathbb{P}_{z(t,\xi)}(dx)\right] \text{ and}$$
$$\widetilde{M}_2(t,z,\mathbb{P}_z)(\xi) = \frac{\gamma(t)}{1+t^2}\left[\sin(z(t)(\xi)) + \int_{\mathbb{L}^2(0,\pi)} \ell(x)\mathbb{P}_{z(t,\xi)}(dx)\right].$$

We suppose that $\gamma(t)$ is bounded continuous function such that $\gamma(t+\omega) = \gamma(t)$ with $\omega \in \mathbb{R}$ and $\ell : \mathbb{L}^2(0,\pi) \to \mathbb{R}$ is a 1-Lipschitz continuous function. Furthermore, the functions σ_i $(i = 1,2)$ are such that

$$\sigma_i(e^{ik\omega}x) = e^{ik\omega}\sigma_i(x), \text{ and } \quad \mathsf{E}\|\sigma_i(u) - \sigma_i(v)\|^2_{\mathbb{H}} \leq L_i\, \mathsf{E}\|u - v\|^2_{\mathbb{H}},\ L_i \geq 0 \quad \text{for} \quad i = 1,2.$$

Now, for $(i = 1,2)$, we have that

$$M_i(t+\omega,z)(\xi) = \gamma(t+\omega)\sigma_i(z)(\xi) = \gamma(t)e^{ik\omega}\sigma_i(e^{-ik\omega}z)(\xi) = e^{ik\omega}M_i(t,e^{-ik\omega}z)(\xi),$$

then we get following estimation for $z \in \mathbb{L}^2(\Omega, \mathbb{H})$ and $r > 0$:

$$\begin{aligned}
\frac{1}{m(r,\rho)} \int_{-r}^{r} & \mathsf{E}\|f(t+\omega, z, \mathbb{P}_z) - e^{ik\omega} f(t, e^{-ik\omega} z, \mathbb{P}_{e^{ik\omega} z})\|_{\mathbb{H}}^2 \rho(t) dt \\
&= \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|\gamma(t+\omega) M_1(z) + \widetilde{M}_1(t+\omega, z, \mathbb{P}_z) \\
&\qquad - e^{ik\omega} \left(\gamma(t) M_1(e^{-ik\omega} z) + \widetilde{M}_1(t, e^{-ik\omega} z, \mathbb{P}_{e^{ik\omega} z}) \right) \|_{\mathbb{H}}^2 \rho(t) dt \\
&= \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|\widetilde{M}_1(t+\omega, z, \mathbb{P}_z) - e^{ik\omega} \widetilde{M}_1(t, e^{-ik\omega} z, \mathbb{P}_{e^{ik\omega} z})\|_{\mathbb{H}}^2 \rho(t) dt \\
&\leq \frac{\|\gamma\|_\infty^2}{m(r,\rho)} \int_{-r}^{r} \mathsf{E} \Bigg\| \frac{1}{\rho(t+\omega)} \left[\cos(z) + \int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_z(dx) \right] \\
&\qquad - \frac{1}{\rho(t)} e^{ik\omega} \left[\cos(e^{-ik\omega} z) + \int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_{e^{-ik\omega} z}(dx) \right] \Bigg\|_{\mathbb{H}}^2 \rho(t) dt
\end{aligned}$$

$$\begin{aligned}
\leq \frac{2\|\gamma\|_\infty^2}{m(r,\rho)} \int_{-r}^{r} & \left(\mathsf{E} \left\| \frac{1}{\rho(t+\omega)} \left[\cos(z) + \int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_z(dx) \right] \right\|_{\mathbb{H}}^2 \right. \\
&\left. + \mathsf{E} \left\| \frac{1}{\rho(t)} e^{ik\omega} \left[\cos(e^{-ik\omega} z) + \int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_{e^{-ik\omega} z}(dx)\,(\xi) \right] \right\|_{\mathbb{H}}^2 \right) \rho(t) dt.
\end{aligned}$$

By lemma 2.2, Hölder's inequality and the representation of Wasserstein distance in terms of random variables, we have

$$\int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_z(dx) \leq \mathsf{W}_1(\mathbb{P}_z, 0) \leq \mathsf{W}_2(\mathbb{P}_z, 0) \leq (\mathbb{E}\|z\|^2)^{1/2} < \infty$$

$$\begin{aligned}
&\frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|f(t+\omega, z, \mathbb{P}_z) - e^{ik\omega} f(t, e^{-ik\omega} z, \mathbb{P}_{e^{ik\omega} z})\|_{\mathbb{H}}^2 \rho(t) dt \\
&\leq \frac{2\|\gamma\|_\infty^2}{m(r,\rho)} \int_{-r}^{r} \left(\left| \frac{1}{1+(t+\omega)^2} \right|^2 2(1 + \mathsf{E}\|z\|_{\mathbb{H}}^2) + \left| \frac{1}{1+t^2} \right|^2 2(1 + \mathsf{E}\|e^{ik\omega} z\|_{\mathbb{H}}^2) \right) \rho(t) dt \\
&\leq \frac{4\,(1 + \mathsf{E}\|z\|_{\mathbb{H}}^2) \|\gamma\|_\infty^2}{m(r,\rho)} \int_{-r}^{r} \left(\left| \frac{1}{1+(t+\omega)^2} \right|^2 + \left| \frac{1}{1+t^2} \right|^2 \right) \rho(t) dt.
\end{aligned}$$

We have

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \left| \frac{1}{1+t^2} \right|^2 \rho(t) dt = \frac{1}{m(r,\rho)} \int_{-r}^{r} \frac{dt}{1+t^2} = \frac{\arctan(r)}{r + \frac{r^3}{3}} \to 0 \text{ as } r \to \infty. \tag{4.2}$$

Note that by the assumption $(\mathbf{H}^\rho)$, there exits a constant $b > 0$ such that for a.e $t \in \mathbb{R}$, we have

$$\frac{\rho(t-\omega)}{\rho(t)} \leq b, \qquad \frac{m(t-|\omega|,\rho)}{m(t,\rho)} \leq b.$$

Then

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\left|\frac{1}{1+(t+\omega)^2}\right|^2 \rho(t)dt\\
\leq&\frac{m(r+|\omega|,\rho)}{m(r,\rho)}\left(\frac{1}{m(r+|\omega|,\rho)}\int_{-r-|\omega|}^{r+|\omega|}\left|\frac{1}{1+t^2}\right|^2 \frac{\rho(t-\omega)}{\rho(t)}\rho(t)dt\right)\\
\leq&\frac{b^2}{m(r+|\omega|,\rho)}\int_{-r-|\omega|}^{r+|\omega|}\left|\frac{1}{1+t^2}\right|^2 \rho(t)dt \to 0 \text{ as } r\to\infty \text{ (similarly to (4.2))}.
\end{aligned}$$

Hence

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|f(t+\omega,z,\mathbb{P}_z))-e^{ik\omega}f(t,e^{-ik\omega}z,\mathbb{P}_{e^{ik\omega}z}))\|_{\mathbb{H}}^2\rho(t)dt=0.$$

Similarly, we have

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|g(t+\omega,z,\mathbb{P}_z))-e^{ik\omega}g(t,e^{-ik\omega}z,\mathbb{P}_{e^{ik\omega}z}))\|_{\mathbb{H}}^2\rho(t)dt=0.$$

Therefore f,g satisfy **(H0)**. Let $u,v\in\mathbb{L}^2(\Omega,\mathbb{H})$ and $t\in\mathbb{R}$ and $\mu_1,\mu_2\in\mathcal{P}_2(\mathbb{H})$, then we have following estimation:

$$\begin{aligned}
&\mathsf{E}\|f(t,u,\mu_1)-f(t,v,\mu_2)\|_{\mathbb{H}}^2\\
&\quad\leq 3\|\gamma\|_\infty^2\Bigg(\mathsf{E}\|\sigma_1(u)-\sigma_1(v)\|_{\mathbb{H}}^2\\
&\qquad\qquad+\mathsf{E}\|\cos(u)-\cos(v)\|+\left\|\int_{\mathbb{L}^2(0,\pi)}\ell(x)\mu_1(dx)-\int_{\mathbb{L}^2(0,\pi)}\ell(x)\mu_2(dx)\right\|_{\mathbb{H}}^2\Bigg)\\
&\quad\leq 3\|\gamma\|_\infty^2\left(L_1\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\|\int_{\mathbb{L}^2(0,\pi)}\ell(x)\mu_1(dx)-\int_{\mathbb{L}^2(0,\pi)}\ell(x)\mu_2(dx)\|_{\mathbb{H}}^2\right)\\
&\quad\leq 3\|\gamma\|_\infty^2\left(L_1\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\|\int_{\mathbb{L}^2(0,\pi)}\ell(x)(\mu_1-\mu_2)(dx)\|_{\mathbb{H}}^2\right).
\end{aligned}$$

By Lemma 2.2 and Hölder's inequality, we have that

$$\begin{aligned}
\mathsf{E}\|f(t,u,\mu_1)-f(t,v,\mu_2)\|_{\mathbb{H}}^2&\leq 3\|\gamma\|_\infty^2\Big[(L_1+1)\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{W}_1^2(\mu_1,\mu_2)\Big]\\
&\leq 3\|\gamma\|_\infty^2(L_1+1)\Big[\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{W}_2^2(\mu_1,\mu_2)\Big].
\end{aligned}$$

Similarly, we obtain

$$\mathsf{E}\|g(t,u,\mu_1)-g(t,v,\mu_2)\|_{\mathbb{H}}^2\leq 3\|\gamma\|_\infty^2(L_2+1)\Big(\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{W}_2^2(\mu_1,\mu_2)\Big).$$

Putting $L=3\|\gamma\|_\infty^2(L_2+1)$ and $L'=3\|\gamma\|_\infty^2(L_2+1)$, we obtain that

$$\begin{aligned}
&\|\phi_\alpha\|_{L^1}^2L+L'\|\phi_\alpha\|_{L^2}^2\\
:=&\frac{C}{\alpha\nu^{1/\alpha}}\mathsf{B}\left(\frac{1}{\alpha},1-\frac{1}{\alpha}\right)3\|\gamma\|_\infty^2(L_1+1)+3\|\gamma\|_\infty^2(L_2+1)\frac{C^2}{\alpha\nu^{(1/\alpha)-1}}\mathsf{B}\left(\frac{1}{\alpha},2-\frac{1}{\alpha}\right).
\end{aligned}$$

Hence, condition (3.1) of Theorem 3.10 is fulfilled by choosing $\|\gamma\|_\infty$ is small enough. Therefore, by Theorem 3.10, the problem (4.1) has a unique square-mean weighted pseudo S-asymptotically Bloch type periodic mild solution on $\mathbb{R}$.

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A quasistatic elastic-viscoplastic contact problem with wear and frictionless

AHMED HAMIDAT*[1] AND ADEL AISSAOUI[2]

[1] *Laboratory of Operator Theory and PDE: Foundations and Applications, Faculty of Exact Sciences, University of El Oued 39000, El Oued, Algeria.*
[2] *Department of Mathematics, University of El Oued 39000 El Oued, Algeria.*

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Abstract. We consider here a frictionless contact problem for elastic-viscoplastic materials, in a quasi-static process. The contact with a rigid base is modeled without friction with condition of wear and damage. The damage the elastic deformations of the material is modeled by an internal variable of the body called the damage field. The problem formula is given as a system that includes a variational equation with respect to the displacement field, and a variational inequality of the parabolic type with respect to the damage field. We prove a weak solution existence and uniqueness theorem relating to the problem. The methods utilised are grounded in the concept of monotonic operators, followed by fixed-point arguments.

AMS Subject Classifications: 74C10, 49J40, 74M15, 74R20

Keywords: frictionless, quasistatic, damage, wear, fixed point.

Contents

1. Introduction

Contact-related problems, whether involving friction or not, between deformable bodies or between a rigid body and a deformable one, are frequently encountered in both industrial settings and everyday experiences. Considering the importance and the multitude of these phenomena, vast studies have been undertaken, also the literature concerning contact mechanics is vast and addresses as many different subjects as are modeling, mathematical analysis or approximation numerical contact problems, see the works [1, 2, 10, 11].

This paper explores an investigation concerning boundary conditions that mirror real-world phenomena like contact, material wear and damage. In our study, we adopt an elastic-viscoplastic constitutive law to describe the behavior of the material.

To illustrate the procedure of deformation of an elastic-viscoplastic body with wear when it contacts with a rigid body foundation, been touched on many quasi-static elastic-viscoplastic frictional Contact problems involving wear have been introduced and investigated under various conditions. For further details, we direct the reader to [5, 6] and the cited references therein.

*Corresponding author. Email address: **hamidat-ahmed@univ-eloued.dz** (Ahmed Hamidat), **aissaouiadel@gmail.com** (Adel Aissaoui)

Chen et al.[4] were among the first to provide error estimates for fully discrete schemes designed to solve quasi-static viscoplastic frictional contact problems with wear. Gasinski et al. [7] introduced a mathematical model to describe quasi-static frictional contact with wear between a thermo-viscoelastic body and a moving foundation. In a recent development, Jureczka and Ochal [9] conducted numerical analysis and simulations for the quasi-static elastic frictional contact problem that accounts for wear.

There are other real phenomena which are very important. Such as material damage and body adhesion. The consideration of damage holds fundamental significance in the field of design engineering since it has a direct impact on the useful lifespan of the designed structure or component. There exists a substantial body of engineering literature devoted to this subject. Mathematical models that incorporate the influence of internal material damage on the contact process have been thoroughly examined. In [8], novel comprehensive damage models have been derived based on the principle of virtual power. Further mathematical analyses of one-dimensional problems related to this topic can be found in [3]. the material damage is described by capacity damage. The damage function α varies between 0 and 1. When $\alpha = 1$ there is no damage in the material, when $\alpha = 0$ the material is completely damaged, when $0 \prec \alpha \prec 1$ the damage is partial. This work is a continuation in this line of research to the mathematical study of a frictionlessly contact problem for Viscoplastic materials, in a quasi-static process. The contact with a rigid base is modeled without friction with condition of wear and damage. Our focus is to establish the existence of a unique weak solution for the abstract problem with regularized boundary conditions. The structure of the remainder of this paper is as follows: In Section 2, we provide an inventory of notations and outline the assumptions concerning the problem data. Additionally, we state our primary result regarding the existence and uniqueness of solutions. In Section 3, we delve into the proof of the theorem, where we consider the existence and uniqueness of the solution, utilizing arguments derived from the theory of monotonic operators and the Banach fixed-point theorem. In Section 4, we present an illustrative example that demonstrates the practical application of the abstract result.

Problem $\mathcal{P}$

Find the displacement field $\mathbf{u} : [0,T] \to V$, the stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$, the damage field $\alpha : [0,T] \to \mathbb{R}$.

$$\begin{aligned}(A\dot{\mathbf{u}}(t), \mathbf{v})_V + (B\mathbf{u}(t), \mathbf{v})_V + \left(\int_0^t F(\boldsymbol{\sigma}(s) - A\dot{\mathbf{u}}(t), \mathbf{u}(s), \alpha(s))\, ds, \mathbf{v}\right)_{\mathcal{H}} \\ = (\boldsymbol{f}(t), \mathbf{v})_V \quad a.e.\ t \in (0,T),\end{aligned} \tag{1.1}$$

$$\begin{aligned}&(\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ &\quad \geq (S(\boldsymbol{\sigma}(s) - A\dot{\mathbf{u}}(t), \mathbf{u}(t), \alpha(t)), \xi - \alpha(t))_{L^2(\Omega)}, \xi \in K, \text{a.e } t \in (0,T),\end{aligned} \tag{1.2}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \alpha(0) = \alpha_0. \tag{1.3}$$

We have three spaces denoted as V, $\mathcal{H}$, and K. These spaces correspond to admissible displacements, stress, and damage, and they are all Hilbert spaces. Notably, K is a nonempty, closed, and convex set within the space V. It is defined as follows:

$$K = \{\zeta \in V \mid \quad 0 \leq \zeta(x) \leq 1 \text{ a.e. } x \in \Omega\}.$$

The operators A, B, and F are associated with the constitutive law governing an elastic-viscoplastic material with damage. The functional S is determined by the source function of the damage and the friction occurring on part Γ_3. The data $\boldsymbol{f}$ relates to both traction forces and body forces. The functions $\mathbf{u}_0$ and α_0 represent the initial data for displacement and damage, respectively. We denote the displacement field as $\mathbf{u}$ and the stress tensor field as $\boldsymbol{\sigma}$. The constitutive law applied here pertains to an elastic-viscoplastic material with damage. The interval $[0,T]$ signifies the time span of observation. A dot above $\mathbf{u}$ and α indicates the derivative of displacement $\mathbf{u}$ and the derivative of damage α with respect to the variable t.

2. Preliminaries and notion

In this section, we introduce important tools for our main results. Specifically, we denote:

$\mathbb{S}^d$ as the space comprising second-order symmetric tensors defined on $\Omega \subset \mathbb{R}^d$ (where $d = 2, 3$), and with a smooth boundary $\partial\Omega = \Gamma$. We designate Γ_3 as the boundary contact .

We define $\boldsymbol{\nu} = (\nu_i)$ as the unit outward normal vector, and $x \in \overline{\Omega} = \Omega \cup \partial\Omega$ represents the position vector. It's worth noting that unless specified otherwise, the indices i, j range from 1 to d, and we apply the summation convention to repeated indices. For the sake of simplicity, we do not explicitly indicate the variables' dependence on x.

The inner products and norms for $\mathbb{R}^d$ and $\mathbb{S}^d$ are denoted as follows:

$$\mathbf{u} \cdot \mathbf{w} = u_i w_i \quad \|\mathbf{w}\|_{\mathbb{R}^d} = (\mathbf{w}, \mathbf{w})^{1/2} \text{ for all } \mathbf{u} = (u_i), \mathbf{w} = (w_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}.\boldsymbol{\vartheta} = \sigma_{ij}\vartheta_{ij} \quad \|\boldsymbol{\vartheta}\|_{\mathbb{S}^d} = (\boldsymbol{\vartheta}, \boldsymbol{\vartheta})^{1/2} \text{ for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\vartheta} = (\vartheta_{ij}) \in \mathbb{S}^d,$$

We denote the following quantities:

$\mathbf{u} = (u_i)$ represents the displacement vector.

$\boldsymbol{\sigma} = (\sigma_{ij})$ denotes the stress tensor.

$\varepsilon(\mathbf{u}) = (\varepsilon(\mathbf{u}_{ij}))$ represents the linear strain tensor.

Furthermore, we use the following notation for components of displacement $\mathbf{u}$ on Γ:

Normal component: $u_\nu = u.\nu$

Tangential component: $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$

Similar notation is applied to $\dot{u_\nu}$ and $\dot{\mathbf{u}}_\tau$, which represent the normal and tangential velocities on the boundary, respectively.

Regarding the stress field $\boldsymbol{\sigma}$ on the boundary, we define its components as:

Normal component: $\sigma_\nu = (\boldsymbol{\sigma\nu}).\boldsymbol{\nu}$

Tangential component: $\boldsymbol{\sigma_\tau} = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$

We use the following notations

$$H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\right\}, \quad H_1 = \{\mathbf{u} = (u_i) \mid \varepsilon(\mathbf{u}) \in \mathcal{H}\},$$
$$\mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\sigma} \in H\}.$$

The deformation operator ε and the divergence operator Div are defined as follows:

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces H, H_1, $\mathcal{H}$, and $\mathcal{H}_1$ are real Hilbert spaces equipped with the canonical inner products defined as follows:

$$(\mathbf{u}, \mathbf{w})_H = \int u_i w_i dx, \quad \forall \mathbf{u}, \mathbf{w} \in H,$$
$$(\mathbf{u}, \mathbf{w})_{H_1} = (\mathbf{u}, \mathbf{w})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{w}))_{\mathcal{H}}, \forall \mathbf{u}, \mathbf{w} \in H_1,$$
$$(\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{\mathcal{H}} = \int \sigma_{ij}\vartheta_{ij} dx, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\vartheta} \in \mathcal{H},$$
$$(\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\vartheta})_H, \ \forall \boldsymbol{\sigma}, \boldsymbol{\vartheta} \in \mathcal{H}_1.$$

The associated norm in the space H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$, is denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}_1}$, respectively.

When $\boldsymbol{\sigma}$ is a regular function. The following Green-type formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{w}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \mathbf{w})_H = \int_\Gamma \boldsymbol{\sigma\nu}.\mathbf{w} da \quad \forall \mathbf{w} \in H_1. \tag{2.1}$$

For the displacement field, we necessitate the closed subspace of H_1 defined as

$$V = \{\mathbf{w} \in H_1 \mid \mathbf{w} = \mathbf{0},\ on\ \Gamma_1\}.$$

Given that $meas(\Gamma_1) > 0$, Korn's inequality is satisfied, and there exists a positive constant C_k, which solely depends on Ω and Γ_1, such that

$$\|\varepsilon(\mathbf{w})\|_{\mathcal{H}} \geq C_k \|\mathbf{w}\|_{H^1(\Omega)^d}, \quad \forall \mathbf{w} \in V.$$

We define inner product on V by

$$(\mathbf{u}, \mathbf{w})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{w}))_{\mathcal{H}}, \quad \|\mathbf{w}\|_V = \|\varepsilon(\mathbf{w})\|_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{w} \in V, \tag{2.2}$$

and let $\|.\|_V$ be the associated norm. Consequently, the norms $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_V$ are equivalent on V, and as a result, $(V, (,)_V)$ forms a real Hilbert space. Furthermore, in accordance with the Sobolev trace theorem, there exists a constant $\tilde{C}_0$, which relies solely on Ω, Γ_1, and Γ_3, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq \tilde{C}_0 \|\mathbf{v}\|_V, \quad \forall \mathbf{v} \in V. \tag{2.3}$$

We recall some spaces $W^{k,p}(0,T;V), H^k(0,T;V)$, $C(0;T;V)$ and $C^1(0;T;V)$ for a Banach space V equipped with the norm $\|.\|_V$ for $1 < p < +\infty$ and $k \geq 1$. Let $W^{k,p}(0,T;V)$ be the space of all functions from $[0,T]$ to V with the norm

$$\|\omega\|_{W^{k,p}(0,T;V)} = \begin{cases} \left(\displaystyle\int_0^T \sum_{1 \leq l \leq k} \left\| \partial_t^l \omega \right\|_V^p dt \right)^{1/p}, & \text{if } 1 \leq p < +\infty \\ \max_{0 \leq l \leq k_0 \leq t \leq T} \sup_t \left\| \partial_t^l \omega \right\|_V, & \text{if } p = +\infty. \end{cases}$$

When $p = 2$ or $k = 0$, $W^{k,2}([0,T];V)$ is written as $H^k([0,T];V)$ or $L^p([0,T];V)$, respectively. We denote by $C([0,T];V)$ the space of continuous functions from $[0,T]$ to V, and by $C^1(0,T;V)$ the space of continuously differentiable functions from $(0,T)$ to V. These spaces are equipped with the following norms:

$$\|\omega\|_{C([0,T];V)} = \max_{t \in [0,T]} \|\omega(t)\|_V.$$

$$\|\omega\|_{C^1([0,T];V)} = \max_{t \in [0,T]} \|\omega(t)\|_V + \max_{t \in [0,T]} \|\dot{\omega}(t)\|_V.$$

Clearly, $C([0,T];V)$, $W^{k,p}([0,T];V)$ and $H^k([0,T];V)$ are all Banach spaces when V is a Banach space.

In order to solve Problem $\mathcal{P}$, we impose the following assumptions.

We consider operators $A, B : V \to V$, $F : \mathcal{H} \times \mathcal{H} \times H^1(\Omega) \to V$, the damage source function $S : \mathcal{H} \times \mathcal{H} \times H^1(\Omega) \to \mathbb{R}$, and two initial values $u_0 \in V$ and $\alpha_0 \in K$. These operators and values satisfy the following properties

There exists a constant $M_A \succ 0$ such that

$$(A\mathbf{v}_1 - A\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \geq M_A \|\mathbf{v}_1 - \mathbf{v}_2\|^2, \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{2.4}$$

There exists a constant $L_A \succ 0$ such that

$$\|A\mathbf{v}_1 - A\mathbf{v}_2\|_{V'} \leq L_A \|\mathbf{v}_1 - \mathbf{v}_2\|_V, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{2.5}$$

There exists a constant $L_B \succ 0$ such that

$$\|B\mathbf{v}_1 - B\mathbf{v}_2\|_V \leq L_B \|\mathbf{v}_1 - \mathbf{v}_2\|, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{2.6}$$

The $\boldsymbol{f}$ function satisfies:

$$\boldsymbol{f} \in L^2(0,T;V). \tag{2.7}$$

There exists a constant $L_F \succ 0$ such that

$$\|F(\boldsymbol{\sigma}_1, \mathbf{u}_1, \zeta_1) - F(\boldsymbol{\sigma}_2, \mathbf{u}_2, \zeta_2)\| \leq L_F (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\| + \|\zeta_1 - \zeta_2\|), \tag{2.8}$$

for all $\boldsymbol{\sigma}_i \in \mathcal{H},\ \mathbf{u}_i \in V,\ \zeta_i \in H^1(\Omega),\ i = 1, 2.$

There exists $M_S \succ 0$ such that

$$\|S(\boldsymbol{\sigma}_1, \mathbf{u}_1, \zeta_1) - S(\boldsymbol{\sigma}_2, \mathbf{u}_2, \zeta_2)\| \leq M_S (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\| + \|\zeta_1 - \zeta_2\|), \tag{2.9}$$

for all $\boldsymbol{\sigma}_i \in \mathcal{H},\ \mathbf{u}_i \in V,\ \forall \zeta_i \in H^1(\Omega),\ i = 1, 2.$

Now let problem $\mathcal{P}_1$ as it follows

Problem $\mathcal{P}_1$

Find $\mathbf{u} \in C^1(0, T; V)$ such that

$$\begin{cases} A\mathbf{u}(t) = \mathbf{f}, \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{2.10}$$

Theorem 2.1. *If conditions (2.4),(2.5) and (2.7) are satisfied Then there exists $\mathbf{u} \in C^1(0, T; V)$ solution to the problem $\mathcal{P}_1$ satisfying*

$$\mathbf{u} \in H^1(0, T; V) \cap C^1(0, T; H). \tag{2.11}$$

The previous result is a special case of the Minty-Browder Theorem.

Problem $\mathcal{P}_2$

Find $\alpha(t) \in K$ such that

$$(\dot{\alpha}(t), \rho - \alpha(t))_{V' \times V} + a(\dot{\alpha}(t), \rho - \alpha(t)) \geq (S(t), \rho - \alpha(t))_{L^2(\Omega)},\ \forall \rho \in K, \tag{2.12}$$

$$\alpha(0) = \alpha_0. \tag{2.13}$$

We consider two real Hilbert spaces, denoted as V and H. It is important to note that V is densely embedded in H, and this injection map is continuous. Furthermore, we identify the space H with both its own dual and as a subspace of the dual space V' of V. In other words, we express this relationship as $V \subset H \subset V'$, and this set of inclusions is what defines a Gelfand triple.

The following is a well-established result for parabolic variational inequalities, and you can find it in standard references such as [12].

Theorem 2.2. *Consider a Gelfand triple $V \subset H \subset V'$, where K is a nonempty, closed, and convex set in V. Assume the existence of a continuous and symmetric bilinear form $a(.,.) : V \times V \to \mathbb{R}$ satisfying the following inequality for constants λ and γ:*

$$a(\alpha, \alpha) + \gamma \|\alpha\|_H^2 \geq \lambda \|\alpha\|_V^2, \quad \forall \alpha \in V.$$

Under these conditions, for any initial value $\alpha_0 \in K$ and source function $S \in L^2(0, T; H)$, there exists a unique function $\alpha \in H^1(0, T; H) \cap L^2(0, T; V)$ such that $\alpha(0) = \alpha_0$ and $\alpha(t) \in K$ for all $t \in [0, T]$. This α is the unique solution to Problem $\mathcal{P}_2$.

The next section is dedicated to investigating the existence of a unique solution to Problem $\mathcal{P}$.

3. Proof of the main result

Theorem 3.1. *Under the assumptions (2.4)-(2.9), there exists a unique solution of the problem $\mathcal{P}$, Moreover the solution satisfies:*

$$\mathbf{u} \in H^1(0,T;V) \cap C^1(0,T;H), \tag{3.1}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad \operatorname{Div} \boldsymbol{\sigma} \in L^2(0,T;H), \tag{3.2}$$

$$\alpha \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \tag{3.3}$$

The proof of Theorem 3.1 is conducted through several sequential steps and relies on the subsequent abstract result concerning evolutionary variational inequalities.

Suppose we have $\eta \in L^2(0,T;V)$, and let's consider the following problem

Problem $\mathcal{P}_\eta$

Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$, such that

$$\begin{cases} (A\dot{\mathbf{u}}_\eta(t), \mathbf{v})_V + (\boldsymbol{\eta}(t), \mathbf{v})_V = (\boldsymbol{f}, \mathbf{v})_V, \\ \qquad \text{a.e. } t \in (0,T), \quad \forall \mathbf{v} \in V, \\ \qquad \mathbf{u}_\eta(0) = \mathbf{u}_0. \end{cases} \tag{3.4}$$

Here is the given result concerning $\mathcal{P}_\eta$.

Lemma 3.2. *A unique solution $\mathbf{u}_\eta \in C^1(0,T;V)$ to the problem $\mathcal{P}_\eta$ exists, and it satisfies the condition (3.1).*

Proof. We apply Theorem 2.1, The Riesz representation theorem allows us to define $\boldsymbol{f}_\eta : [0,T] \to V$, by $\left(\boldsymbol{f}_\eta(t), \mathbf{v}\right)_V = (f(t) - \boldsymbol{\eta}(t), \mathbf{v})_V$. Using hypotheses (2.4)-(2.7), and $\mathbf{u}_\eta(t) = \mathbf{u}_0 + \int_0^t \dot{\mathbf{u}}_\eta(s)ds, \quad \forall t \in (0,T)$, we directly find the result. ■

Subsequently, introduce $\theta \in L^2(0,T;L^2(\Omega))$, and let's examine the following problem

Problem $\mathcal{P}_\theta$

Find the damage field $\alpha_\theta : [0,T] \to \mathbb{R}$,

$$\begin{aligned} \alpha_\theta(t) \in K, (\dot{\alpha}_\theta(t), \rho - \alpha_\theta(t))_{L^2(\Omega)} + a\left(\alpha_\theta(t), \rho - \alpha_\theta(t)\right) \\ \geq (\theta(t), \rho - \alpha_\theta(t))_{L^2(\Omega)}, \forall \rho \in K, \text{ a.e.t } \in (0,T), \end{aligned} \tag{3.5}$$

$$\alpha_\theta(0) = \alpha_0. \tag{3.6}$$

Lemma 3.3. *problem $\mathcal{P}_\theta$ has a unique solution α_θ such that*

$$\alpha_\theta \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \tag{3.7}$$

For the proof, we apply Theorem 2.2.

Finally, in the concluding step, formulate the subsequent Cauchy problem for the stress field

Problem $\mathcal{P}_{\eta,\theta}$

Find the stress field $\boldsymbol{\sigma}_{\eta,\theta} : (0,T) \to \mathcal{H}$, solution of the problem

$$\boldsymbol{\sigma}_{\eta,\theta}(t) = B\mathbf{u}_\eta(t) + \int_0^t F\left(\boldsymbol{\sigma}_{\eta,\theta}(s) - A\dot{\mathbf{u}}_\eta(s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds, \text{ a.e.t } \in (0,T). \tag{3.8}$$

Lemma 3.4. *The problem $\mathcal{P}_{\eta,\theta}$ has a unique solution. Additionally, if $\mathbf{u}_{\eta_i}$, α_{θ_i}, and $\boldsymbol{\sigma}_{\eta_i,\theta_i}$ represent the solutions to problems $\mathcal{P}_\eta$, $\mathcal{P}_\theta$, and $\mathcal{P}_{\eta,\theta}$ for $i = 1,2$, then there exists a positive constant C such that*

$$\begin{aligned}\|\boldsymbol{\sigma}_{\eta_1,\theta_1}(t) - \boldsymbol{\sigma}_{\eta_2,\theta_2}(t)\|_{\mathcal{H}}^2 \leq & C\left(\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V^2 + \int_0^t \|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|_V^2 \, ds\right. \\ & \left. + \int_0^t \|\alpha_{\theta_1}(s) - \alpha_{\theta_2}(s)\|_{L^2(\Omega)}^2 \, ds\right).\end{aligned} \tag{3.9}$$

Proof. Consider the mapping $\sum\limits_{\eta,\theta} : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ defined as

$$\sum_{\eta,\theta} \boldsymbol{\sigma}_{\eta,\theta}(t) = B\mathbf{u}_\eta(t) + \int_0^t F\left(\boldsymbol{\sigma}_{\eta,\theta}(s) - A\dot{\mathbf{u}}_\eta(s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds. \tag{3.10}$$

let $\boldsymbol{\sigma}_i \in L^2(0,T;\mathcal{H})$, $\quad i = 1,2$ and $t_1 \in (0,T)$, we use the assumption (2.8) and the HÖlder inequality we find

$$\left\|\sum_{\eta,\theta} \boldsymbol{\sigma}_1(t_1) - \sum_{\eta,\theta} \boldsymbol{\sigma}_2(t_1)\right\|_{\mathcal{H}}^2 \leq L_F^2 T \int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 \, ds. \tag{3.11}$$

We have more

$$\begin{aligned}&\left\|\sum_{\eta,\theta}\left(\sum_{\eta,\theta} \boldsymbol{\sigma}_1(t_1)\right) - \sum_{\eta,\theta}\left(\sum_{\eta,\theta} \boldsymbol{\sigma}_2(t_1)\right)\right\|_{\mathcal{H}}^2 \\ &\qquad \leq L_F^2 T \int_0^{t_1} \left\|\sum_{\eta,\theta} \boldsymbol{\sigma}_1(t_1) - \sum_{\eta,\theta} \boldsymbol{\sigma}_2(t_1)\right\|_{\mathcal{H}}^2 dt_2 \\ &\qquad \leq L_F^4 T^2 \int_0^{t_1}\int_0^{t_2} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 \, ds dt_2.\end{aligned}$$

By extending the inequality through recurrence, we deduce that for all $t_1, t_2, ..., t_n \in (0,T)$,

$$\left\|\sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_1(t_n) - \sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_2(t_n)\right\|_{\mathcal{H}}^2 \leq L_F^{2n} T^n \int_0^{t_1}\int_0^{t_2} \dots \int_0^{t_n} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 \, ds dt_n \dots dt_2.$$

Thus, we can deduce by integrating with respect to $(0,T)$ the following inequality

$$\left\|\sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_1 - \sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_2\right\|_{\mathcal{H}}^2 \leq \frac{L_F^{2n} T^{2n}}{n!} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathcal{H}}^2. \tag{3.12}$$

Then from (3.12), for n sufficiently large, the operator $\sum_{\eta,\theta}^{(n)}$, is a contraction on space $L^2(0,T;\mathcal{H})$ and according to the Banach fixed point theorem, there is a single element $\boldsymbol{\sigma}_{\eta,\theta} \in L^2(0,T;\mathcal{H})$ such that

$\sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_{\eta,\theta} = \boldsymbol{\sigma}_{\eta,\theta}$, which represents the unique solution of problem $\mathcal{P}_{\eta,\theta}$. Moreover, if $\mathbf{u}_{\eta_i}$, α_{θ_i} and $\boldsymbol{\sigma}_{\eta_i,\theta_i}$, represents the solutions of problem $\mathcal{P}_{\eta_i}$,$\mathcal{P}_{\theta_i}$ and $\mathcal{P}_{\eta_i,\theta_i}$ respectively. For $i = 1, 2$. designate $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \boldsymbol{\sigma}_{\eta_i,\theta_i} = \boldsymbol{\sigma}_i, \alpha_{\theta_i} = \alpha_i$.

We have

$$\boldsymbol{\sigma}_i(t) = B\mathbf{u}_i(t) + \int_0^t F\left(\boldsymbol{\sigma}_i(s) - A\dot{\mathbf{u}}_i(s),\ \mathbf{u}_i(s), \alpha_i(s)\right) ds,\ \text{a.e. } t \in (0,T),$$

we use the assumption (2.6),(2.8)), we find

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}}^2 \leq C \Bigg(& \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\, ds \\ & + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\, ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2\, ds \Bigg).\end{aligned}$$

We employ the Gronwall argument within the resulting inequality to derive (3.9). ■

Now, let's contemplate the mapping

$$\Lambda : L^2(0,T;\mathcal{H} \times L^2(\Omega)) \to L^2(0,T;\mathcal{H} \times L^2(\Omega)),$$

$$\Lambda\left(\boldsymbol{\eta},\theta\right)(t) = \left(\Lambda^1\left(\boldsymbol{\eta},\theta\right)(t), \Lambda^2\left(\boldsymbol{\eta},\theta\right)(t)\right), \tag{3.13}$$

defined by equalities

$$\Lambda^1(\boldsymbol{\eta},\theta)(t), = B\mathbf{u}_\eta(t) + \int_0^t F\left(\boldsymbol{\sigma}_{\eta,\theta}(s) - A\dot{\mathbf{u}}(s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds, \tag{3.14}$$

$$\Lambda^2(\boldsymbol{\eta},\theta)(t) = S\left(\left(\boldsymbol{\sigma}_{\eta,\theta}(t), \mathbf{u}_\eta(t)\right), \alpha_\theta(t)\right). \tag{3.15}$$

We have the following result.

Lemma 3.5. *For $(\boldsymbol{\eta},\theta) \in L^2\left(0,T;\mathcal{H} \times L^2(\Omega)\right)$, the operator $\Lambda(\boldsymbol{\eta},\theta) : [0,T] \to \mathcal{H} \times L^2(\Omega)$ have a unique fixed point denoted as $(\boldsymbol{\eta}^*,\theta^*) \in L^2\left(0,T;\mathcal{H} \times L^2(\Omega)\right)$, satisfying*

$$\Lambda\left(\boldsymbol{\eta}^*,\theta^*\right) = (\boldsymbol{\eta}^*,\theta^*).$$

Proof. Let $t \in (0,T)$ and $(\boldsymbol{\eta}_1,\theta_1), (\boldsymbol{\eta}_2,\theta_2) \in L^2\left(0,T;\mathcal{H} \times L^2(\Omega)\right)$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i,\ \dot{\mathbf{u}}_{\eta_i} = \dot{\mathbf{u}}_i,\ \alpha_{\eta_i} = \alpha_i,\ \boldsymbol{\sigma}_{\boldsymbol{\eta}_i,\theta_i} = \boldsymbol{\sigma}_i$,For $i = 1, 2$ and using the assumptions (2.5),(2.6) and (2.8)

$$\begin{aligned}&\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}}^2 \\ &\quad= \|B\mathbf{u}_1(t) + \int_0^t F\left(\boldsymbol{\sigma}_1(s) - A\dot{\mathbf{u}}_1(s), \mathbf{u}_1(s), \alpha_1(s)\right) ds \\ &\quad- B\mathbf{u}_2(t) - \int_0^t F\left(\boldsymbol{\sigma}_2(s) - A\dot{\mathbf{u}}_2(s), \mathbf{u}_2(s), \alpha_2(s)\right) ds\|_{\mathcal{H}}^2 \\ &\quad\leq L_B\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + L_F \int_0^t (\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 + \\ &\quad L_A\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds.\end{aligned}$$

We utilise the estimate (3.9) to derive

$$\begin{aligned}&\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}}^2 \\ &\quad\leq C(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ &\quad+ \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2))ds.\end{aligned}$$

On the other hand, we know that $\mathbf{u}_i(t) = \mathbf{u}_0 + \int_0^t \dot{\mathbf{u}}_i(s)ds$, for all $t \in (0,T)$

$$\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 \leq \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 ds. \tag{3.16}$$

By Apply the inequality (3.16) becomes

$$\begin{aligned} \|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}}^2 \leq C\int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds. \end{aligned} \tag{3.17}$$

By a similar argument, from (3.9),(3.15) and (2.9) it follows that

$$\begin{aligned} \|\Lambda^2(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^2(\boldsymbol{\eta}_2,\theta_2)(t)\|_{L^2(\Omega)}^2 \leq C(\int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds \\ + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2). \end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned} \|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}^2 \leq C(\int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds \\ + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2). \end{aligned} \tag{3.19}$$

Combine the inequality (3.16) with (3.19) to obtain

$$\begin{aligned} \|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}^2 \leq C\int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds. \end{aligned} \tag{3.20}$$

Using the inequality (3.4), by adding the results obtained we have

$$(A\dot{\mathbf{u}}_1(t) - A\dot{\mathbf{u}}_2(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t))_V = (\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t))_V,\ t \in (0,T), \tag{3.21}$$

using inequality (2.4), we find

$$M_A\|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_V^2 \leq \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_V \|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_V.$$

Therefore

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leq C\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V, \quad \forall t \in [0,T].$$

Let's use (3.16)

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \leq C\int_0^t \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V ds, \quad \forall t \in [0,T]. \tag{3.22}$$

Using (3.5) we find

$$(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \leq (\theta_1 - \theta_2, \alpha_1 - \alpha_2)_{L^2(\Omega)},$$

$$a \cdot e \cdot t \in (0,T),$$

By integrating the inequality with respect to time and incorporating the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$, along with the inequality $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \geq 0$, we combine this inequality with Gronwall's lemma, resulting in the following result

$$\|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Omega)} \leq C \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)} ds, \forall t \in [0, T]. \tag{3.23}$$

From the previous inequality and estimates (3.20), (3.22) and (3.23) it follows that now

$$\begin{aligned} &\|\Lambda(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda(\boldsymbol{\eta}_2, \theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)} \\ &\quad \leq C\left(\int_0^t \|(\boldsymbol{\eta}_1, \theta_1)(s) - (\boldsymbol{\eta}_2, \theta_2)(s)\|^2_{\mathcal{H}\times L^2(\Omega)} ds\right). \end{aligned}$$

Let is introduce the following notations

$$\begin{cases} I_1 = \displaystyle\int_0^t \|(\boldsymbol{\eta}_1, \theta_1)(s) - (\boldsymbol{\eta}_2, \theta_2)(s)\|_{\mathcal{H}\times L^2(\Omega)} ds, \\ \vdots \\ I_k = \displaystyle\int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_1} \|(\boldsymbol{\eta}_1, \theta_1)(r) - (\boldsymbol{\eta}_2, \theta_2)(r)\|_{\mathcal{H}\times L^2(\Omega)}, \end{cases}$$

Through an inductive process, denoting the m^{th} power of the operator Λ as Λ^m, we arrive at the following conclusion

$$\begin{aligned} &\|\Lambda^m(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda^m(\boldsymbol{\eta}_2, \theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)} \\ &\quad \leq C^m\left(\sum_{k=1}^m C_m^k I^{m-k} \|(\boldsymbol{\eta}_1, \theta_1)(t) - (\boldsymbol{\eta}_2, \theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}\right), \end{aligned} \tag{3.24}$$

for all $t \in [0, T]$ and $m \in \mathbb{N}$,

$$\begin{aligned} I^{m-k}((\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)) &= \int_{(m-k)\text{ fois}} . \int \|(\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)\| \\ &\leq \int_0^s \int \cdots \int_{(m-k)\text{ fois}} \|(\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)\|_{L^2(0,T;\mathcal{H}\times L^2(\Omega))} \\ &\leq \frac{t^{m-k}}{k!} \|(\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)\|_{L^2(0,T;\mathcal{H}\times L^2(\Omega))} \\ &\leq \frac{T^{m-k}}{k!} \|(\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)\|_{L^2(0,T;\mathcal{H}\times L^2(\Omega))}, \end{aligned}$$

Consequently,

$$\begin{aligned} &\|\Lambda^m(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda^m(\boldsymbol{\eta}_2, \theta_2)(t)\|^2_{L^2(0,T;\mathcal{H}\times L^2(\Omega))} \\ &\quad \leq C^m\left(\sum_{k=1}^m C_m^k \frac{T^{m-k}}{k!} \|(\boldsymbol{\eta}_1, \theta_1)(t) - (\boldsymbol{\eta}_2, \theta_2)(t)\|^2_{L^2(0,T;\mathcal{H}\times L^2(\Omega))}\right) \\ &\quad \leq \frac{(CT)^m}{m!} \|(\boldsymbol{\eta}_1, \theta_1)(t) - (\boldsymbol{\eta}_2, \theta_2)(t)\|^2_{L^2(0,T;\mathcal{H}\times L^2(\Omega))}, \end{aligned}$$

this implies that for m large enough, the operator Λ^m of Λ is a contraction on Banach space $L^2\left(0, T; \mathcal{H} \times L^2(\Omega)\right)$. So Λ^m has a unique fixed point $(\boldsymbol{\eta}^*, \theta^*) \in L^2\left(0, T; \mathcal{H} \times L^2(\Omega)\right)$, and therefore $(\boldsymbol{\eta}^*, \theta^*)$ is the only fixed point of Λ. ■

Existence

Let $(\boldsymbol{\eta}^*, \theta^*) \in L^2\left(0,T;\mathcal{H}\times L^2(\Omega)\right)$, be the fixed point of Λ defined by (3.14)-(3.15) and let $\mathbf{u}_\eta,\ \alpha_\theta$, be the solutions of problems $\mathcal{P}_\eta$, $\mathcal{P}_\theta$, for $\eta=\eta^*,\ \theta=\theta^*,\ \mathbf{u}=\mathbf{u}_{\eta^*},\ \alpha=\alpha_{\theta^*}$, we find $(\mathbf{u},\boldsymbol{\sigma},\alpha)$ is a solution of problem $\mathcal{P}$. properties (3.1)-(3.3) follow from lemma 3.2, 3.3, 3.4.

Uniqueness

The uniqueness of the solution is a result of the uniqueness of the fixed point of operator Λ.

4. Application

In this section, we will utilise the main result from Section 3 to analyse a problem of contact without friction with condition of wear and damage. between an elastic-viscoplastic body and a rigid base in a quasistatic process. We provide the physical context for the contact problem and introduce certain notations that will be employed in the subsequent discussion. We consider a elastic-viscoplastic body which occupies a domain $\Omega\subset\mathbb{R}^d$, where $d=2,3$, such that the boundary $\Gamma=\partial\Omega$ is Lipschitz continuous. The boundary $\partial\Omega$ is divided into three disjoint measurable parts Γ_1,Γ_2 and Γ_3 with $meas(\Gamma_1)>0$. We are interested in an evolution of the body in a finite time interval $(0,T)$.

We consider the following classical formulation of the problem

Problem P

Find a displacement field $\mathbf{u}:\Omega\times[0,T]\to\mathbb{R}^d$, the stress field $\boldsymbol{\sigma}:\Omega\times[0,T]\to\mathbb{S}^d$, the damage field $\alpha:\Omega\times[0,T]\to\mathbb{R}$.

$$0=\mathrm{Div}\boldsymbol{\sigma}+f_0, \qquad \text{in } \Omega\times(0,T), \tag{4.1}$$

$$\boldsymbol{\sigma}(t)=\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t))+\mathcal{B}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{F}\left(\boldsymbol{\sigma}(s)-\mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)),\varepsilon\left(\mathbf{u}(s)\right),\alpha(s)\right)ds \qquad \text{in } \Omega\times(0,T), \tag{4.2}$$

$$\dot{\alpha}-k_0\Delta\alpha+\partial\varphi_K(\alpha)\ni\phi(\boldsymbol{\sigma},\varepsilon(\mathbf{u}),\alpha), \qquad \text{in } \Omega\times(0,T), \tag{4.3}$$

$$\mathbf{u}=0, \qquad \text{on } \Gamma_1\times(0,T), \tag{4.4}$$

$$\sigma\nu=f_2, \qquad \text{on } \Gamma_2\times(0,T), \tag{4.5}$$

$$\begin{cases}-\sigma_\nu=k\|\dot{u}_\nu\|\\ \boldsymbol{\sigma}_\tau=0\end{cases} \qquad \text{on } \Gamma_3\times(0,T), \tag{4.6}$$

$$\frac{\partial\alpha}{\partial\nu}=0, \qquad \text{on } \Gamma\times(0,T), \tag{4.7}$$

$$\mathbf{u}(0)=\mathbf{u}_0,\quad \alpha(0)=\alpha_0, \qquad \text{in } \Omega. \tag{4.8}$$

Equation (4.1) describes the equation of motion, where f_0 stands for the density of the voluminal forces exerted upon the deformable body Ω. Equation (4.2) describes the constitutive law applicable to an elastic-viscoplastic material with damage, (4.3) represents a differential inclusion describing the evolution of the damage field where S is a damage source function. φ_K is the sub-differential of the indicator function of the set of admissible damage functions K. The conditions (4.4) and (4.5) are displacement-traction conditions, (4.6) represent the boundary contact conditions with wear and without friction. (4.7) represents the boundary condition of Neumann, Finally,(4.8) represents the initial conditions.

Next, we outline the assumptions concerning the data of the problem, starting with the viscosity operator $\mathcal{A}:\Omega\times\mathbb{S}^d\longrightarrow\mathbb{S}^d$ satisfied

$$\begin{cases}(a)\text{ There exists } L_{\mathcal{A}}>0 \quad \text{such that}\\ \|\mathcal{A}(\mathbf{x},\boldsymbol{\upsilon}_1)-\mathcal{A}(\mathbf{x},\boldsymbol{\upsilon}_2)\|\leq L_{\mathcal{A}}\|\boldsymbol{\upsilon}_1-\boldsymbol{\upsilon}_2\|,\ \forall\boldsymbol{\upsilon}_1,\boldsymbol{\upsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \mathbf{x}\in\Omega.\\ (b)\text{ There exists } m_{\mathcal{A}}>0 \quad \text{such that}\\ (\mathcal{A}(\mathbf{x},\boldsymbol{\upsilon}_1)-\mathcal{A}(\mathbf{x},\boldsymbol{\upsilon}_2)).(\boldsymbol{\upsilon}_1-\boldsymbol{\upsilon}_2)\geq m_{\mathcal{A}}\|\boldsymbol{\upsilon}_1-\boldsymbol{\upsilon}_2\|^2,\ \forall\boldsymbol{\upsilon}_1,\boldsymbol{\upsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \mathbf{x}\in\Omega.\\ (c)\text{ The mapping } \mathbf{x}\mapsto\mathcal{A}(\mathbf{x},\boldsymbol{\upsilon}) \text{ is lebesgue measurable on } \Omega, \forall\boldsymbol{\upsilon}\in\mathbb{S}^d.\\ (d)\text{ The mapping } \mathbf{x}\mapsto\mathcal{A}(\mathbf{x},\mathbf{0})\in\mathcal{H}.\end{cases} \tag{4.9}$$

The elasticity operator $\mathcal{B}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d$ satisfied

$$\begin{cases}(a)\text{ There exists } L_{\mathcal{B}}>0 \quad \text{such that}\\ \|\mathcal{B}(\mathbf{x},\boldsymbol{\upsilon}_1)-\mathcal{B}(\mathbf{x},\boldsymbol{\upsilon}_2)\|\leq L_{\mathcal{B}}\|\boldsymbol{\upsilon}_1-\boldsymbol{\upsilon}_2\|,\quad \forall\boldsymbol{\upsilon}_1,\boldsymbol{\upsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \mathbf{x}\in\Omega.\\ (b)\text{ There exists } m_{\mathcal{B}}>0 \quad \text{such that}\\ (\mathcal{B}(\mathbf{x},\boldsymbol{\upsilon}_1)-\mathcal{B}(\mathbf{x},\boldsymbol{\upsilon}_2)).(\boldsymbol{\upsilon}_1-\boldsymbol{\upsilon}_2)\geq m_{\mathcal{B}}\|\boldsymbol{\upsilon}_1-\boldsymbol{\upsilon}_2\|^2,\ \forall\boldsymbol{\upsilon}_1,\boldsymbol{\upsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \mathbf{x}\in\Omega.\\ (c)\text{ The mapping } \mathbf{x}\mapsto\mathcal{B}(\mathbf{x},\boldsymbol{\upsilon}) \text{ is lebesgue measurable on } \Omega,\\ \qquad \forall\boldsymbol{\upsilon}\in\mathbb{S}^d.\\ (d)\text{ The mapping } \mathbf{x}\mapsto\mathcal{B}(\mathbf{x},\mathbf{0})\in\mathcal{H}.\end{cases} \tag{4.10}$$

The relaxation function $\mathcal{F}:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{S}^d$, satisfied

$$\begin{cases}(a)\text{ There exists } L_{\mathcal{F}}>0 \quad \text{such that}\\ \|\mathcal{F}(\mathbf{x},\boldsymbol{\sigma}_1,\boldsymbol{\upsilon}_1,\alpha_1)-\mathcal{F}(\mathbf{x},\boldsymbol{\sigma}_2,\boldsymbol{\upsilon}_2,\alpha_2)\|\leq\\ \qquad\qquad L_{\mathcal{F}}(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+|\boldsymbol{\upsilon}_1-\boldsymbol{\upsilon}_2\|+\|\alpha_1-\alpha_2\|)\\ \forall\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\upsilon}_1,\boldsymbol{\upsilon}_2\in\mathbb{S}^d,\forall\alpha_1,\alpha_2\in\mathbb{R},\forall t\in[0,T],\ \text{a.e. } \mathbf{x}\in\Omega.\\ (b)\text{ The mapping } \mathbf{x}\mapsto\mathcal{F}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\upsilon},\alpha) \text{ is lebesgue measurable on } \Omega,\\ \forall\boldsymbol{\sigma},\boldsymbol{\upsilon}\in\mathbb{S}^d,\ \forall t\in[0,T],\ \forall\alpha\in\mathbb{R}.\\ (c)\text{ The mapping } x\mapsto\mathcal{F}(\mathbf{x},\mathbf{0},\mathbf{0},0)\in\mathcal{H},\forall t\in[0,T].\end{cases} \tag{4.11}$$

The function describing the source of damages, denoted as $\phi:\Omega\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{R}$, is satisfied

$$\begin{cases}(a)\text{ There exists } M_{\phi}>0 \quad \text{such that}\\ \|\phi(\mathbf{x},\boldsymbol{\upsilon}_1,\alpha_1)-\phi(\mathbf{x},\boldsymbol{\upsilon}_2,\alpha_2)\|\leq M_{\phi}(\|\boldsymbol{\upsilon}_1-\boldsymbol{\upsilon}_2\|+\|\alpha_1-\alpha_2\|),\\ \forall\boldsymbol{\upsilon}_1,\boldsymbol{\upsilon}_2\in\mathbb{S}^d,\forall\alpha_1,\alpha_2\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Omega.\\ (b)\text{ The mapping } \mathbf{x}\mapsto\phi(\mathbf{x},\boldsymbol{\upsilon},\alpha) \text{ is lebesgue measurable on } \Omega,\\ \forall\boldsymbol{\upsilon}\in\mathbb{S}^d,\forall\alpha\in\mathbb{R}.\\ (c)\text{ The mapping } \mathbf{x}\mapsto\phi(\mathbf{x},\mathbf{0},0)\in L^2(\Omega).\end{cases} \tag{4.12}$$

The body force $\mathbf{f}_0$, surface traction $\mathbf{f}_2$, coefficient of friction k, initial conditions u_0, have the following properties

$$\begin{cases}\mathbf{f}_0\in L^2(0,T;H),\\ \mathbf{f}_2\in L^2\left(0,T;L^2(\Gamma_2)^d\right),\\ k\in L^\infty(\Gamma_3),\quad k(x)\geq 0 \text{ for a.e. } x\in\Gamma_3,\\ \mathbf{u}_0\in V.\end{cases} \tag{4.13}$$

We establish the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ as follows

$$a(\xi, \zeta) = k_0 \int_\Omega \nabla\xi \nabla\zeta dx \tag{4.14}$$

and the micro crack diffusion coefficient verifies $k_0 > 0$.

The initial damage α_0 field satisfies

$$\alpha_0 \in K. \tag{4.15}$$

To consider the field of displacements, we require the closed subspace V within the space H_1, defined by:

$$V = \{\mathbf{u} \in H_1 \mid \mathbf{u} = \mathbf{0} \;\; on\, \Gamma_1 \;\}. \tag{4.16}$$

Using Riesz's representation theorem, we find

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Gamma \mathbf{f}_0 \cdot \mathbf{v} dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in V, t \in [0, T]. \tag{4.17}$$

It's important to observe that condition (4.13) results in the implication that

$$\mathbf{f} \in L^2\left(0, T; V\right). \tag{4.18}$$

Now, consider the application $j : V \times V \to \mathbb{R}$, defined as follows

$$j\left(\mathbf{u}, \mathbf{v}\right) = \int\limits_{\Gamma_3} k \left\| u_\nu \right\| v_\nu da. \tag{4.19}$$

The variational formulation for problem P is presented as follows

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) \\ &\quad + \int_0^t \mathcal{F}\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon\left(\mathbf{u}(s)\right), \alpha(s)\right) ds \quad \text{a.e } .t \in (0, T), \end{aligned} \tag{4.20}$$

$$(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) = (\mathbf{f}, \mathbf{v})_V, \quad \forall \mathbf{v} \in V, \tag{4.21}$$

$$\begin{aligned} \alpha(t) \in K, \quad & (\dot{\alpha}(t), \zeta - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \zeta - \alpha(t)) \\ & \geq (\phi(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{u}(t)), \alpha(t)), \zeta - \alpha(t))_{L^2(\Omega)}, \quad \forall \zeta \in K, t \in [0, T], \end{aligned} \tag{4.22}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \alpha(0) = \alpha_0. \tag{4.23}$$

Utilising Riesz's representation theorem, we define the operator $A : V \to V$ as follows:

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}(\varepsilon(\mathbf{u})), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.24}$$

We will verify the hypotheses (2.4),(2.5). Let $\mathbf{u}_1, \mathbf{u}_2 \in V$. Using (4.9),(4.24) and the definition of j given by (4.19), we let's find

$$\begin{aligned} \|A\mathbf{u}_1 - A\mathbf{u}_2\|_V &= \|\mathcal{A}\varepsilon(\mathbf{u}_1) - \mathcal{A}\varepsilon(\mathbf{u}_2)\|_{\mathcal{H}} + C_0^2 \|k\|_{L^\infty(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_V \\ &\leq L_{\mathcal{A}} \|\varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2)\|_{\mathcal{H}} + C_0^2 \|k\|_{L^\infty(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_V \\ &= \left(L_{\mathcal{A}} + C_0^2 \|k\|_{L^\infty(\Gamma_3)}\right) \|\mathbf{u}_1 - \mathbf{u}_2\|_V, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in V. \end{aligned} \tag{4.25}$$

Similarly for all $\mathbf{u}_1, \mathbf{u}_2 \in V$ we have

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \geq \left(m_{\mathcal{A}} - C_0^2 \|k\|_{L^\infty(\Gamma_3)}\right) \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in V. \tag{4.26}$$

Let $\gamma_0 = \dfrac{m_{\mathcal{A}}}{C_0^2}$, it is clear that γ_0 is positive which depends on Ω_1,Γ_3, and $\mathcal{A}$. Then A is strongly monotonic on V if

$$\|k\|_{L^\infty(\Gamma_3)} < \gamma_0.$$

After confirming that all the assumptions of Theorem 3.1 are met, we can conclude that a unique weak solution to problem P exists, satisfying (4.20)-(4.23), along with the regularity conditions (3.1)-(3.3).

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Generalized Hyers-Ulam stability of a 3D additive-quadratic functional equation in Banach spaces: A study with counterexamples

G. Yagachitradevi[*1], **S. Lakshminarayanan**[2] **and P. Ravindiran**[3]

[1] *Department of Mathematics, Siga College of Management and Computer Science, Villupuram - 605601, Tamil Nadu, India.*
[2,3] *Department of Mathematics, Arignar Anna Government Arts College, Villupuram - 605 602, Tamil Nadu, India.*

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Abstract. In this research, we focus on solving a mixed type additive-quadratic functional equation expressed as:

$$h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3) =12\tilde{h}(s_1)+8\tilde{h}(s_2)+2\tilde{h}(s_3)+12h(s_1)$$

where $\tilde{h}(s_1)=h(s_1)+h(-s_1)$ is derived. We proceed to investigate the generalized Hyers-Ulam stability of this equation within the framework of Banach spaces, employing the Hyers direct method. Additionally, examples of non-stable cases are also provided.

AMS Subject Classifications: 39B52, 39B72, 39B82.

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Contents

1. Introduction

Ulam's seminal work on the stability of group homomorphisms [30] sparked a new line of inquiry into the stability of functional equations. Hyers gave a favorable answer to this topic in the context of Banach spaces, making significant progress [12]. Credit for extending this research to the broader topic of Generalized Hyers-Ulam stability inside functional equations belongs to Aoki [2] and Rassias [23]. Aoki expanded Rassias's original approach to incorporate additive mappings, which included employing an infinite Cauchy difference for linear mappings. In 1994, Gavruta [11] proposed the generalized control function $\phi(s_1,s_2)$ as a substantial alternative to the boundless Cauchy difference. In 2008, following this work, Ravi et al. [27] used the product and sum

[*]**Corresponding author.** Email addresses: **yagachitradevi@gmail.com** (G. Yagachitradevi), **slnvpm29@gmail.com** (S. Lakshminarayanan), **p.ravindiran@gmail.com** (P. Ravindiran)

of two $p-$norms to prove a particular version of Gavruta's theorem. Many researchers have thoroughly studied the stability issues of several functional equations and there are numerous noteworthy outcomes related to this problem, as seen in [1, 3, 8–10, 13, 16, 18–20, 22, 24–26, 29] and other cited references.

The Cauchy equation, which has the form:

$$h(s_1 + s_2) = h(s_1) + h(s_2), \tag{1.1}$$

is one of the most well-known functional equations in mathematics. Functions that have this relationship are called *additive functions*.

The quadratic functional equation

$$h(s_1 + s_2) + h(s_1 - s_2) = 2h(s_1) + 2h(s_2) \tag{1.2}$$

is connected to a symmetric bi-additive function, as shown by the work of [1, 17]. Each of the solutions to this equation is a *quadratic function*. Skof [28] addressed a stability issue connected to the quadratic functional equation (1.2) by studying functions $h : K \to L$, where K is a normed space and L is a Banach space. An Abelian group may stand in for the domain K without affecting the validity of the argument, as noted by Cholewa [6], who elaborated on Skof's work. Czerwik [7] adds to the expanding body of evidence supporting the stability of the quadratic functional equation by demonstrating that it is Hyers-Ulam-Rassias stable.

The quadratic and additive functional equation

$$h(s_1 + ds_2) + dh(s_1 - s_2) = h(s_1 - ds_2) + dh(s_1 + s_2) \tag{1.3}$$

was studied by Jun and Kim [14], who examined the general solution and the generalized Hyers-Ulam stability for any positive integer d with $d \neq -1, 0, 1$. Additionally, Najati and Moghimi [21] investigated the quadratic and additive functional equation

$$h(2s_1 + s_2) + h(2s_1 - s_2) = 2h(s_1 + s_2) + 2h(s_1 - s_2) + 2h(2s_1) - 4h(s_1). \tag{1.4}$$

K. Balamurugan et al. [5] obtained the general solution to the cubic functional equation

$$\begin{aligned} &g(3s_1 + 2s_2 + s_3) + g(3s_1 + 2s_2 - s_3) + g(3s_1 - 2s_2 + s_3) + g(3s_1 - 2s_2 - c) \\ &= 24[g(s_1 + s_2) + g(s_1 - s_2)] + 6[g(s_1 + s_3) + g(s_1 - s_3)] + g(s_1) \end{aligned} \tag{1.5}$$

and investigated its generalized Hyers-Ulam stability.

M. Arunkumar et al. [4] have recently developed a general solution and generalized Hyers-Ulam stability for the three-dimensional additive-quadratic functional equation

$$\begin{aligned} &g(s_1 + 2s_2 + 3s_3) + g(s_1 + 2s_2 - 3s_3) + g(s_1 - 2s_2 + 3s_3) + g(-s_1 + 2s_2 + 3s_3) \\ =&g(s_1 + s_2 + s_3) + g(s_1 + s_2 - s_3) + g(s_1 - s_2 + s_3) + g(-s_1 + s_2 + s_3) \\ &+ 2g(s_2) + 4g(s_3) + 5[g(s_2) + g(-s_2)] + 14[g(s_3) + g(-s_3)] \end{aligned} \tag{1.6}$$

using a direct and fixed point approach in Banach space and non-Archimedean fuzzy Banach space.

In this study, we provide a general solution to the additive-quadratic functional equation

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &= 12\tilde{h}(s_1) + 8\tilde{h}(s_2) + 2\tilde{h}(s_3) + 12h(s_1), \end{aligned} \tag{1.7}$$

where $\tilde{h}(s_1) = h(s_1) + h(-s_1)$ and investigate the generalized Hyers - Ulam stability of this equation with the Hyers direct technique. In addition, unstable counterexamples are provided.

In Section 2, we provide a generic solution to (1.7). Using the direct method and the concept of generalized Hyers-Ulam, we demonstrate the stability of equation (1.7) for odd, even, and mixed mappings, with counterexamples provided in Sections 3, 4, and 5.

2. General Solution of (1.7)

This section examines the general solution of the functional equation (1.7) when $\mathcal{K}$ and $\mathcal{L}$ are treated as real vector spaces.

Theorem 2.1. *If an odd function $h : \mathcal{K} \to \mathcal{L}$ meets the requirements of the functional equation* (1.7) *for all $s_1, s_2, s_3 \in \mathcal{K}$, then it must also meet the functional equation* (1.1) *for all $s_1, s_2 \in \mathcal{K}$ and vice versa.*

Proof. Consider the odd function $h : \mathcal{K} \to \mathcal{L}$ to satisfy the functional equation (1.7). By inputting (s_1, s_2, s_3) as $(0, 0, 0)$ in (1.7), we determine that $h(0) = 0$. By setting s_3 to 0 in (1.7) and using the property that h is odd, we can deduce that

$$h(3s_1 + 2s_2) + h(3s_1 - 2s_2) = 6h(s_1), \tag{2.1}$$

for any $s_1, s_2 \in \mathcal{K}$. Additionally, by setting s_2 to 0 in this equation, we find that

$$h(3s_1) = 3h(s_1) \tag{2.2}$$

for any $s_1 \in \mathcal{K}$. By substituting $\frac{s_1}{3}$ for s_1 in this equation, we arrive

$$h\left(\frac{s_1}{3}\right) = \frac{1}{3}h(s_1) \tag{2.3}$$

for any $s_1 \in \mathcal{K}$. Finally, by replacing s_1 with $\frac{s_1}{3}$ and s_2 with $\frac{s_1}{2}$ in (2.1) and using (2.3), we can conclude that

$$h(2s_1) = 2h(s_1) \tag{2.4}$$

for any $s_1 \in \mathcal{K}$. Hence, for any positive whole number b,

$$h(bs_1) = bh(s_1) \tag{2.5}$$

for any $s_1 \in \mathcal{K}$. By entering (s_1, s_2) as $\left(\frac{s_1}{3}, \frac{s_2}{2}\right)$ into (2.1) and using (2.3), we infer that

$$h(s_1 + s_2) + h(s_1 - s_2) = 2h(s_1), \tag{2.6}$$

for any $s_1, s_2 \in \mathcal{K}$. By switching the positions of s_1 and s_2 and applying the characteristic of h being an odd function, we arrive

$$h(s_1 + s_2) - h(s_1 - s_2) = 2h(s_2), \tag{2.7}$$

for any $s_1, s_2 \in \mathcal{K}$. By merging equations (2.6) and (2.7), we reach the desired outcome of (1.1).

Let us suppose, on the other hand, that an atypical odd mapping $h : \mathcal{K} \to \mathcal{L}$ satisfies the conditions stated in functional equation (1.1). By plugging in $s_1 = 0$ and $s_2 = 0$ into equation (1.1), we find that $h(0) = 0$. By also plugging in s_1 for s_2 and $2s_1$ for s_2 into (1.1), we get two new equations:

$$h(2s_1) = 2h(s_1) \quad \text{and} \quad h(3s_1) = 3h(s_1) \tag{2.8}$$

for any $s_1 \in \mathcal{K}$. By induction, for any natural number c, we have

$$h(cs_1) = ch(s_1) \tag{2.9}$$

for any $s_1 \in \mathcal{K}$. We start with the equation (1.1) and replace the variable s_2 with $s_2 + s_3$ and use (1.1). This gives us

$$h(s_1 + s_2 + s_3) = h(s_1) + h(s_2) + h(s_3) \tag{2.10}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. Next, substitute (s_1, s_2, s_3) with $(3s_1, 2s_2, s_3)$ in (2.10) and we use (2.8) to obtain

$$h(3s_1 + 2s_2 + s_3) = 3h(s_1) + 2h(s_2) + h(s_3) \tag{2.11}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. We then change the sign of s_2 in (2.11) to get

$$h(3s_1 - 2s_2 + s_3) = 3h(s_1) + 2h(-s_2) + h(s_3) \tag{2.12}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$ and repeat the process with s_3 to get

$$h(3s_1 + 2s_2 - s_3) = 3h(s_1) + 2h(s_2) + h(-s_3) \tag{2.13}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. We then substitute both s_2 and s_3 with their negative versions in (2.11) to obtain

$$h(3s_1 - 2s_2 - s_3) = 3h(s_1) + 2h(-s_2) + h(-s_3) \tag{2.14}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. By adding together equations (2.11), (2.12), (2.13) and (2.14), we arrive at

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &= 12h(s_1) + 4h(s_2) + 4h(-s_2) + 2h(s_3) + 2h(-s_3) \end{aligned} \tag{2.15}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. We then add $12h(s_1) + 4h(s_2)$ to both sides of equation (2.15) to get

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) + 12h(s_1) + 4h(s_2) \\ &= 12h(s_1) + 4h(s_2) + 4h(-s_2) + 2h(s_3) + 2h(-s_3) + 12h(s_1) + 4h(s_2) \end{aligned} \tag{2.16}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. From (2.15), we can conclude

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &= 12h(s_1) + 4h(s_2) + 4h(-s_2) + 2h(s_3) + 2h(-s_3) + 12h(s_1) + 4h(s_2) - 12h(s_1) - 4h(s_2) \end{aligned} \tag{2.17}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. The fact that h is an odd function has allowed us to convincingly demonstrate our conclusion. ■

Theorem 2.2. *If an even function $h : \mathcal{K} \to \mathcal{L}$ meets the requirements of the functional equation* (1.7) *for all $s_1, s_2, s_3 \in \mathcal{K}$, then it must also meet the functional equation* (1.2) *for all $s_1, s_2 \in \mathcal{K}$ and vice versa.*

Proof. Consider the even function $h : \mathcal{K} \to \mathcal{L}$ to satisfy the functional equation (1.7). By inputting (s_1, s_2, s_3) as $(0, 0, 0)$ in (1.7), we determine that $h(0) = 0$. By setting (s_1, s_2, s_3) as $(0, s_1, s_2)$ in (1.7) and using the property that h is even, we can deduce that

$$h(2s_1 + s_2) + h(2s_1 - s_2) = 8h(s_1) + 2h(s_2), \tag{2.18}$$

for any $s_1, s_2 \in \mathcal{K}$. Additionally, by setting s_2 to 0 in this equation, we find that

$$h(2s_1) = 4h(s_1) \tag{2.19}$$

for any $s_1 \in \mathcal{K}$. By substituting $\dfrac{s_1}{2}$ for s_1 in this equation, we arrive

$$h\left(\frac{s_1}{2}\right) = \frac{1}{4}h(s_1) \tag{2.20}$$

for any $s_1 \in \mathcal{K}$. Finally, by replacing s_2 with s_1 in (2.18), we can conclude that

$$h(3s_1) = 9h(s_1) \tag{2.21}$$

for any $s_1 \in \mathcal{K}$. Hence, for any positive whole number a,

$$h(as_1) = a^2 h(s_1) \tag{2.22}$$

for any $s_1 \in \mathcal{K}$. By substituting $\frac{s_1}{2}$ for s_1 in equation (2.18) and using (2.20), we arrive at equation(1.2), as intended.

Let us suppose, on the other hand, that an atypical even mapping $h : \mathcal{K} \to \mathcal{L}$ satisfies the conditions stated in functional equation (1.2). By substituting $s_1 = 0$ and $s_2 = 0$ into (1.2), we can determine that $h(0) = 0$. Additionally, by inputting s_1 for s_2 and $2s_1$ for s_2 into the same equation and taking into account that h is an even function, we obtain two additional equations:

$$h(2s_1) = 4h(s_1) \quad \text{and} \quad h(3s_1) = 9h(s_1) \tag{2.23}$$

for any $s_1 \in \mathcal{K}$. We can prove that for any natural number c through the method of induction, we have

$$h(cs_1) = c^2 h(s_1) \tag{2.24}$$

for any $s_1 \in \mathcal{K}$. By replacing s_1 with $3s_1$ and s_2 with $2s_2$ in (1.2) and using (2.24), we obtain

$$h(3s_1 + 2s_2) + h(3s_1 - 2s_2) = 18h(s_1) + 8h(s_2) \tag{2.25}$$

for any $s_1, s_2 \in \mathcal{K}$. Again replacing s_1 with $3s_1 + 2s_2$ and s_2 with s_3 in (1.2), we have

$$h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) = 2h(3s_1 + 2s_2) + 2h(s_3) \tag{2.26}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. We then change the sign of s_2 in (2.26) to get

$$h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) = 2h(3s_1 - 2s_2) + 2h(s_3) \tag{2.27}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. Adding both (2.26) and (2.27), we obtain

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &\quad = 2[h(3s_1 + 2s_2) + h(3s_1 - 2s_2)] + 4h(s_3) \end{aligned} \tag{2.28}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. Using (2.25) in (2.28) and the property of h being even, we achieve

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &\quad = 36h(s_1) + 16h(s_2) + 4h(s_3) = 12\tilde{h}(s_1) + 8\tilde{h}(s_2) + 2\tilde{h}(s_3) + 12h(s_1), \end{aligned} \tag{2.29}$$

where $\tilde{h}(s_1) = h(s_1) + h(-s_1)$, for any $s_1, s_2, s_3 \in \mathcal{K}$. ■

Hearafter, throughout this analysis, we will presume that $\mathcal{K}$ is a normed space and $\mathcal{L}$ is a Banach space, and we will introduce the mapping $Dh : \mathcal{K}^3 \to \mathcal{L}$ in the following manner:

$$\begin{aligned} Dh(s_1, s_2, s_3) =& h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) \\ &+ h(3s_1 - 2s_2 - s_3) - 12\tilde{h}(s_1) - 8\tilde{h}(s_2) - 2\tilde{h}(s_3) - 12h(s_1), \end{aligned}$$

where $\tilde{h}(s_1) = h(s_1) + h(-s_1)$, for all $s_1, s_2, s_3 \in \mathcal{K}$.

3. Stability of (1.7) for odd mappings

In this paper, we examine the generalized Hyers-Ulam stability of the functional equation (1.7), in particular for the case of an odd mapping.

Theorem 3.1. *Let $s = \pm 1$ and $\xi : \mathcal{K}^3 \to [0, \infty)$ be a mapping such that*

$$\sum_{i=0}^{\infty} \frac{\xi\left(6^{si} s_1, 6^{si} s_2, 6^{si} s_3\right)}{6^{si}} < \infty \tag{3.1}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Let $h : \mathcal{K} \to \mathcal{L}$ be an odd mapping that satisfies

$$\|Dh(s_1, s_2, s_3)\| \leq \xi\left(s_1, s_2, s_3\right) \tag{3.2}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ satisfying (1.7) *and*

$$\|h(s_1) - \mathcal{A}(s_1)\| \leq \frac{1}{6} \sum_{i=\frac{1-s}{2}}^{\infty} \frac{\psi\left(6^{si} s_1\right)}{6^{si}} \tag{3.3}$$

where $\psi : \mathcal{K} \to \mathcal{L}$ and $\mathcal{A}(s_1)$ are given by

$$\psi\left(s_1\right) = \xi\left(s_1, s_1, s_1\right) + \frac{1}{2} \xi\left(s_1, 0, s_1\right) \tag{3.4}$$

and

$$\mathcal{A}(s_1) = \lim_{i \to \infty} \frac{h(6^{si} s_1)}{6^{si}} \tag{3.5}$$

for all $s_1 \in \mathcal{K}$, respectively.

Proof. Assuming that s is equal to 1. By substituting (s_1, s_2, s_3) with (s_1, s_1, s_1) in (3.2) and make use of the oddness of h, we arrive at the inequality

$$\|h(6s_1) + h(4s_1) + h(2s_1) - 12h(s_1)\| \leq \xi\left(s_1, s_1, s_1\right) \tag{3.6}$$

for all $s_1 \in \mathcal{K}$. Similarly, substituting (s_1, s_2, s_3) with $(s_1, 0, s_1)$ in (3.2) and using the oddness of h, we get

$$\|h(4s_1) + h(2s_1) - 6h(s_1)\| \leq \frac{1}{2} \xi\left(s_1, 0, s_1\right) \tag{3.7}$$

for all $s_1 \in \mathcal{K}$. Combining these two inequalities, we find that

$$\begin{aligned} \|h(6s_1) - 6h(s_1)\| &\leq \|h(6s_1) + h(4s_1) + h(2s_1) - 12h(s_1)\| + \|h(4s_1) + h(2s_1) - 6h(s_1)\| \\ &\leq \xi\left(s_1, s_1, s_1\right) + \frac{1}{2} \xi\left(s_1, 0, s_1\right) \end{aligned} \tag{3.8}$$

for all $s_1 \in \mathcal{K}$. Dividing the preceding inequality by 6 yields

$$\left\| \frac{h(6s_1)}{6} - h(s_1) \right\| \leq \frac{\psi(s_1)}{6} \tag{3.9}$$

where

$$\psi(s_1) = \xi\left(s_1, s_1, s_1\right) + \frac{1}{2} \xi\left(s_1, 0, s_1\right)$$

for all $s_1 \in \mathcal{K}$. By plugging in $6s_1$ in place of s_1 and dividing by 6 in (3.9), we acquire

$$\left\| \frac{h(6^2 s_1)}{6^2} - \frac{h(6s_1)}{6} \right\| \leq \frac{\xi(6s_1)}{6^2} \tag{3.10}$$

for all $s_1 \in \mathcal{K}$. From (3.9) and (3.10), we obtain

$$\begin{aligned}\left\|\frac{h(6^2 s_1)}{6^2}-h(s_1)\right\| &\leq \left\|\frac{h(6s_1)}{6}-h(s_1)\right\|+\left\|\frac{h(6^2 s_1)}{6^2}-\frac{h(6s_1)}{6}\right\| \\ &\leq \frac{1}{6}\left[\xi(s_1)+\frac{\xi(6s_1)}{6}\right] \end{aligned} \tag{3.11}$$

for all $s_1 \in \mathcal{K}$. Then, by using induction to a positive integer n, we have

$$\begin{aligned}\left\|\frac{h(6^n s_1)}{6^n}-h(s_1)\right\| &\leq \frac{1}{6}\sum_{i=0}^{n-1}\frac{\xi(6^i s_1)}{6^i} \\ &\leq \frac{1}{6}\sum_{i=0}^{\infty}\frac{\xi(6^i s_1)}{6^i}\end{aligned} \tag{3.12}$$

for all $s_1 \in \mathcal{K}$. Substituting $6^m s_1$ for s_1 and dividing by 6^m in (3.12), we see that the sequence $\left\{\frac{h(6^n s_1)}{6^n}\right\}$ converges. It follows that for any m and n in the positive integer range, we can conclude that

$$\begin{aligned}\left\|\frac{h(6^{n+m} s_1)}{6^{(n+m)}}-\frac{h(6^m s_1)}{6^m}\right\| &= \frac{1}{6^m}\left\|\frac{h(6^n\cdot 6^m s_1)}{6^n}-h(6^m s_1)\right\| \\ &\leq \frac{1}{6}\sum_{i=0}^{n-1}\frac{\xi(6^{i+m} s_1)}{6^{(i+m)}} \\ &\leq \frac{1}{6}\sum_{i=0}^{\infty}\frac{\xi(6^{i+m} s_1)}{6^{(i+m)}} \\ &\to 0 \quad \text{as} \quad m\to\infty\end{aligned}$$

for all $s_1 \in \mathcal{K}$. Thus $\left\{\frac{h(6^n s_1)}{6^n}\right\}$ is Cauchy. For complete set $\mathcal{L}$, a mapping $\mathcal{A}:\mathcal{K}\to\mathcal{L}$ exists with

$$\mathcal{A}(s_1)=\lim_{n\to\infty}\frac{h(6^n s_1)}{6^n}, \quad \forall \ s_1\in\mathcal{K}.$$

When we plug in (3.12), where n may go to infinity, we get that (3.3) is true for every $s_1 \in \mathcal{K}$. To show that $\mathcal{A}$ satisfies (1.7), we substitute $(6^n s_1, 6^n s_2, 6^n s_3)$ for (s_1, s_2, s_3) in (3.2) and divide by 6^n to get

$$\frac{1}{6^n}\left\|Dh(6^n s_1,6^n s_2,6^n s_3)\right\| \leq \frac{1}{6^n}\xi(6^n s_1,6^n s_2,6^n s_3)$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Using the definition of $\mathcal{A}(s_1)$ and the aforementioned inequality, we can show that $\mathcal{A}(s_1)$ as n goes to infinity. Hence For all $s_1, s_2, s_3 \in \mathcal{K}$, $\mathcal{A}$ fulfils (1.7). If $\mathcal{A}$ is not unique, we may show that $\mathcal{C}(s_1)$ is also an additive mapping fulfilling (1.7) and (3.3), as

$$\begin{aligned}\|\mathcal{A}(s_1)-\mathcal{C}(s_1)\| &= \frac{1}{6^n}\|\mathcal{A}(6^n s_1)-\mathcal{C}(6^n s_1)\| \\ &\leq \frac{1}{6^n}\left\{\|\mathcal{A}(6^n s_1)-h(6^n s_1)\|+\|h(6^n s_1)-\mathcal{C}(6^n s_1)\|\right\} \\ &\leq \frac{1}{3}\sum_{i=0}^{\infty}\frac{\xi(6^{(i+n)} s_1)}{6^{(i+n)}} \\ &\to 0 \quad \text{as} \quad n\to\infty\end{aligned}$$

for every $s_1 \in \mathcal{K}$. For this reason, $\mathcal{A}$ cannot be found anywhere else. This proves that the theory is correct when s equals 1. Substituting $\frac{s_1}{6}$ for s_1 in inequality (3.8) leads to the conclusion that

$$\left\|h(s_1) - 6h\left(\frac{s_1}{6}\right)\right\| \leq \xi\left(\frac{s_1}{6}, \frac{s_1}{6}, \frac{s_1}{6}\right) + \frac{1}{2}\xi\left(\frac{s_1}{6}, 0, \frac{s_1}{6}\right)$$

for every $s_1 \in \mathcal{K}$. The remainder of the proof for $s = -1$ is the same as it is for $s = 1$. Therefore, the theorem is valid for both $s = 1$ and $s = -1$. The theorem has been proven at this point. ■

The next Corollary is directly derived from Theorem 3.1 concerning the stability of Equation (1.7).

Corollary 3.2. *Let t be a positive real value, and assume $\nu \geq 0$. For any $s_1, s_2, s_3 \in \mathcal{K}$, let $h : \mathcal{K} \to \mathcal{L}$ be a function that fulfils the inequality*

$$\|Dh(s_1, s_2, s_3)\| \leq \begin{cases} \nu, & \\ \nu\left\{||s_1||^t + ||s_2||^t + ||s_3||^t\right\}, & t \neq 1; \\ \nu||s_1||^t||s_2||^t||s_3||^t, & 3t \neq 1; \\ \nu\left\{||s_1||^t||s_2||^t||s_3||^t + ||s_1||^{3t} + ||s_2||^{3t} + ||s_3||^{3t}\right\}, & 3t \neq 1. \end{cases} \tag{3.13}$$

If so, then for every $s_1 \in \mathcal{K}$, there is a unique additive function $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ such that

$$\|h(s_1) - \mathcal{A}(s_1)\| \leq \begin{cases} \frac{3\nu}{10}, & \\ \frac{4\nu||s_1||^t}{|6^t - 6|}, & t \neq 1; \\ \frac{\nu||s_1||^t}{|6^{3t} - 6|}, & 3t \neq 1; \\ \frac{5\nu||s_1||^t}{|6^{3t} - 6|}, & 3t \neq 1. \end{cases} \tag{3.14}$$

To demonstrate that (1.7) is not stable at $t = 1$, as stated in Corollary 3.2, we will now present an illustration.

Example 3.3. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \nu s_1, & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant; the function $h : \mathbb{R} \to \mathbb{R}$, defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^n}$$

for all $s_1 \in \mathbb{R}$, fulfills the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq 432\nu(|s_1| + |s_2| + |s_3|) \tag{3.15}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. If this is the case, then there cannot be an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ with a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{A}(s_1)| \leq \kappa|s_1| \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{3.16}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|\xi(6^n s_1)|}{|6^n|} \leq \sum_{n=0}^{\infty} \frac{\nu}{6^n} = \frac{6\nu}{5}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (3.15). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1| + |s_2| + |s_3| \geq \frac{1}{6}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{(|s_1| + |s_2| + |s_3|)} \leq \frac{6\nu}{5} \times 60 \times 6 = 432\nu$$

and hence (3.15) is obvious. Take into account the scenario where

$$0 < |s_1| + |s_2| + |s_3| < \frac{1}{6}.$$

In the above scenario, there is a positive whole number m such that

$$\frac{1}{6^{(m+1)}} \leq |s_1| + |s_2| + |s_3| < \frac{1}{6^m}. \tag{3.17}$$

This implies that $6^{m-1}x < \frac{1}{6}, 6^{m-1}y < \frac{1}{6}$, and $6^{m-1}z < \frac{1}{6}$. As a result,

$$6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3),$$
$$6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m - 1$, the values of

$$6^n(3s_1 + 2s_2 + s_3), 6^n(3s_1 - 2s_2 + s_3), 6^n(3s_1 + 2s_2 - s_3), 6^n(3s_1 - 2s_2 - s_3),$$
$$6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)$$

are also within the range of $(-1, 1)$. Due to the fact that ξ is linear over this range, we may conclude that

$$\xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) + \xi(6^n(3s_1 - 2s_2 + s_3)) + \xi(6^n(3s_1 - 2s_2 - s_3))$$
$$- 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] - 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] - 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) = 0,$$

for $n = 0, 1, \ldots, m - 1$. Using (3.17) and the definition of h, we may calculate

$$\begin{aligned}\frac{|Dh(s_1, s_2, s_3)|}{(|s_1| + |s_2| + |s_3|)} \leq \sum_{n=m}^{\infty} \frac{1}{6^n(|s_1| + |s_2| + |s_3|)}\Big|&\xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) \\ &+ \xi(6^n(3s_1 - 2s_2 + s_3)) + \xi(6^n(3s_1 - 2s_2 - s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] \\ &- 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] - 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1))\Big| \\ \leq \sum_{k=0}^{\infty} \frac{60\nu}{6^k 6^m(|s_1| + |s_2| + |s_3|)} \leq \sum_{k=0}^{\infty} \frac{360\nu}{6^k} = 432\nu.\end{aligned}$$

Consequently, for all $s_1, s_2, s_3 \in \mathbb{R}$ with $0 < |s_1| + |s_2| + |s_3| < \frac{1}{6}$, h fulfills (3.15). According to Corollary 3.2, the additive functional equation (1.7) is unstable at $t = 1$. Let us assume, however, that there is an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ obeying (3.16), where R is the set of all real numbers and $\kappa > 0$. Since h is bounded and continuous for every $s_1 \in \mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{A}$ is also bounded and continuous within the interval. $\mathcal{A}$ must have the form $\mathcal{A}(s_1) = cs_1$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)| \leq (\kappa + |c|)\,|s_1|. \tag{3.18}$$

being true. However, by choosing a positive integer m with $m\nu > \kappa + |c|$, we can find $s_1 \in \left(0, \frac{1}{6^{m-1}}\right)$ such that $6^n s_1 \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this s_1, we get

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^n} \geq \sum_{n=0}^{m-1} \frac{\nu \times 6^n s_1}{6^n} = m\nu s_1 > (\kappa + |c|)\, s_1$$

which defies (3.18). Based on the inequality (3.13), it may be concluded that the equation (1.7) is not stable in the Hyers-Ulam-Rassias sense while $t = 1$. ■

Here we present an example to show that, as mentioned in Corollary 3.2, the functional equation (1.7) is unstable for $t = \frac{1}{3}$.

Example 3.4. *Suppose t is such that $0 < t < \frac{1}{3}$. Then, there exists a function $h : \mathbb{R} \to \mathbb{R}$ and a constant $\nu > 0$ such that for all real numbers $s_1, s_2, s_3 \in \mathbb{R}$,*

$$|Dh(s_1, s_2, s_3)| \leq \nu |s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{1-2t}{3}} \tag{3.19}$$

and for all additive mappings $\mathcal{A} : \mathbb{R} \to \mathbb{R}$

$$\sup_{s_1 \neq 0} \frac{|h\,(s_1) - \mathcal{A}\,(s_1)|}{|s_1|} = +\infty. \tag{3.20}$$

Proof. If we set $h(s_1) = s_1 \ln |s_1|$, if $s_1, \neq 0$, and $h(0) = 0$, then we may deduce that

$$\begin{aligned}
\sup_{s_1 \neq 0} \frac{|h\,(s_1) - \mathcal{A}\,(s_1)|}{|x|} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|h\,(n) - \mathcal{A}\,(n)|}{|n|} \\
&= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|n \ln |n| - n\, \mathcal{A}\,(1)|}{|n|} \\
&= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \ |\ln |n| - \mathcal{A}\,(1)|\ = \infty.
\end{aligned}$$

We need to show that (3.19).

Case (i): If $s_1, s_2, s_3 > 0$ in (3.19) then,

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= |(3s_1 + 2s_2 + s_3) \ln |3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3) \ln |3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3) \ln |3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3) \ln |3s_1 - 2s_2 - s_3| \\
&\quad - 12[s_1 \ln |s_1| - s_1 \ln |-s_1|] - 8[s_2 \ln |s_2| - s_2 \ln |-y|] \\
&\quad -2[s_3 \ln |s_3| - s_3 \ln |-s_3|] - 12 s_1 \ln |s_1||
\end{aligned}$$

If we set $s_1 = j, s_2 = k$, and $s_3 = l$, then we get

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= |(3s_1 + 2s_2 + s_3) \ln |3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3) \ln |3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3) \ln |3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3) \ln |3s_1 - 2s_2 - s_3|
\end{aligned}$$

$$
\begin{aligned}
&- 12[s_1 \ln|s_1| - s_1 \ln|-s_1|] - 8[s_2 \ln|s_2| - s_2 \ln|-y|] \\
&-2[s_3 \ln|s_3| - s_3 \ln|-s_3|] - 12 s_1 \ln|s_1|| \\
&= |(3j+2k+l)\ln|3j+2k+l| + (3j+2k-l)\ln|3j+2k-l| \\
&+ (3j-2k+l)\ln|3j-2k+l| + (3j-2k-l)\ln|3j-2k-l| \\
&- 12[j\ln|j| - j\ln|-j|] - 8[k\ln|k| - k\ln|-k|] \\
&-2[l\ln|l| - l\ln|-l|] - 12j\ln|j|| \\
&|h(3j+2k+l) + h(3j+2k-l) + h(3j-2k+l) + h(3j-2k-l) \\
&-12[h(j)+h(-j)] - 8[h(k)+h(-k)] - 2[h(l)+h(-l)] - 12h(j)| \\
&\le \nu|j|^{\frac{t}{3}}\ |k|^{\frac{t}{3}}\ |c|^{\frac{1-2t}{3}} = \nu|s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{1-2t}{3}}
\end{aligned}
$$

Case (ii): If $s_1, s_2, s_3 < 0$ in (3.19) then,

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= |(3s_1+2s_2+s_3)\ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)\ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)\ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)\ln|3s_1-2s_2-s_3| \\
&- 12[s_1 \ln|s_1| - s_1 \ln|-s_1|] - 8[s_2 \ln|s_2| - s_2 \ln|-y|] \\
&-2[s_3 \ln|s_3| - s_3 \ln|-s_3|] - 12 s_1 \ln|s_1||
\end{aligned}
$$

If we set $s_1 = -j, s_2 = -k$, and $s_3 = -l$, then we get

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= |(3s_1+2s_2+s_3)\ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)\ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)\ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)\ln|3s_1-2s_2-s_3| \\
&- 12[s_1 \ln|s_1| - s_1 \ln|-s_1|] - 8[s_2 \ln|s_2| - s_2 \ln|-y|] \\
&-2[s_3 \ln|s_3| - s_3 \ln|-s_3|] - 12 s_1 \ln|s_1|| \\
&= |(-j-2k-l)\ln|-3j-2k-l| + (-3j-2k+l)\ln|-3j-2k+l| \\
&+ (-3j+2k-l)\ln|-3j+2k-l| + (-3j+2k+l)\ln|-3j+2k+l| \\
&- 12[-j\ln|-j| + j\ln|j|] - 8[-k\ln|-k| + k\ln|k|] \\
&-2[-l\ln|-l| + l\ln|l|] + 12j\ln|-j|| \\
&|h(-j-2k-l) + h(-3j-2k+l) + h(-3j+2k-l) + h(-3j+2k+l) \\
&-12[h(-j)+h(j)] - 8[h(-k)+h(k)] - 2[h(-l)+h(l)] - 12h(-j)| \\
&\le \nu|-j|^{\frac{t}{3}}\ |-k|^{\frac{t}{3}}\ |-c|^{\frac{1-2t}{3}} = \nu|s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{1-2t}{3}}
\end{aligned}
$$

Case (iii): If $s_1 > 0, s_2, s_3 < 0$ in (3.19) then,

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= |(3s_1+2s_2+s_3)\ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)\ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)\ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)\ln|3s_1-2s_2-s_3| \\
&- 12[s_1 \ln|s_1| - s_1 \ln|-s_1|] - 8[s_2 \ln|s_2| - s_2 \ln|-y|] \\
&-2[s_3 \ln|s_3| - s_3 \ln|-s_3|] - 12 s_1 \ln|s_1||
\end{aligned}
$$

If we set $s_1 = j, s_2 = -k$, and $s_3 = -l$, then we get

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= |(3s_1 + 2s_2 + s_3)\ln|3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3)\ln|3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3)\ln|3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3)\ln|3s_1 - 2s_2 - s_3| \\
&\quad - 12[s_1\ln|s_1| - s_1\ln|-s_1|] - 8[s_2\ln|s_2| - s_2\ln|-y|] \\
&\quad -2[s_3\ln|s_3| - s_3\ln|-s_3|] - 12s_1\ln|s_1|| \\
&= |(3j - 2k - l)\ln|3j - 2k - l| + (3j - 2k + l)\ln|3j - 2k + l| \\
&\quad + (3j + 2k - l)\ln|3j + 2k - l| + (3j + 2k + l)\ln|3j + 2k + l| \\
&\quad - 12[j\ln|j| - j\ln|-j|] - 8[-k\ln|-k| + k\ln|k|] \\
&\quad -2[-l\ln|-l| + l\ln|l|] - 12j\ln|j|| \\
&|h(3j - 2k - l) + h(3j - 2k + l) + h(3j + 2k - l) + h(3j + 2k + l) \\
&\quad -12[h(j) + h(-j)] - 8[h(-k) + h(k)] - 2[h(-l) + h(l)] - 12h(j)| \\
&\le \nu|j|^{\frac{t}{3}}\ |-k|^{\frac{t}{3}}\ |-c|^{\frac{1-2t}{3}} \\
&= \nu|s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{1-2t}{3}}
\end{aligned}$$

Case (iv): If $s_1 < 0, s_2, s_3 > 0$ in (3.19) then,

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= |(3s_1 + 2s_2 + s_3)\ln|3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3)\ln|3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3)\ln|3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3)\ln|3s_1 - 2s_2 - s_3| \\
&\quad - 12[s_1\ln|s_1| - s_1\ln|-s_1|] - 8[s_2\ln|s_2| - s_2\ln|-y|] \\
&\quad -2[s_3\ln|s_3| - s_3\ln|-s_3|] - 12s_1\ln|s_1||
\end{aligned}$$

If we set $s_1 = -j, s_2 = k$, and $s_3 = l$, then we get

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= |(3s_1 + 2s_2 + s_3)\ln|3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3)\ln|3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3)\ln|3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3)\ln|3s_1 - 2s_2 - s_3| \\
&\quad - 12[s_1\ln|s_1| - s_1\ln|-s_1|] - 8[s_2\ln|s_2| - s_2\ln|-y|] \\
&\quad -2[s_3\ln|s_3| - s_3\ln|-s_3|] - 12s_1\ln|s_1|| \\
&= |(-3j + 2k + l)\ln|-3j + 2k + l| + (-3j + 2k - l)\ln|-3j + 2k - l| \\
&\quad + (-3j - 2k + l)\ln|-3j - 2k + l| + (-3j - 2k - l)\ln|-3j - 2k - l| \\
&\quad - 12[-j\ln|-j| + j\ln|-j|] - 8[k\ln|k| - k\ln|-k|] \\
&\quad -2[l\ln|l| - l\ln|-l|] + 12j\ln|-j|| \\
&|h(-3j + 2k + l) + h(-3j + 2k - l) + h(-3j - 2k + l) + h(-3j - 2k - l) \\
&\quad -12[h(-j) + h(-j)] - 8[h(k) + h(-k)] - 2[h(l) + h(-l)] - 12h(-j)| \\
&\le \nu|-j|^{\frac{t}{3}}\ |k|^{\frac{t}{3}}\ |l|^{\frac{1-2t}{3}} \\
&= \nu|s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{1-2t}{3}}
\end{aligned}$$

Case (v): If $s_1 = s_2 = s_3 = 0$ in (3.19), then the statement is obvious. ■

Here we present an example to show that, as mentioned in Corollary 3.2, the functional equation (1.7) is unstable for $t = \frac{1}{3}$.

Example 3.5. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \nu s_1, & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant; the function $h : \mathbb{R} \to \mathbb{R}$, defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^n}$$

for all $s_1 \in \mathbb{R}$, fulfills the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq 432\nu \left(|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3|\right) \tag{3.21}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. If this is the case, then there cannot be an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ with a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{A}(s_1)| \leq \kappa |s_1| \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{3.22}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|\xi(6^n s_1)|}{|6^n|} \leq \sum_{n=0}^{\infty} \frac{\nu}{6^n} = \frac{6\nu}{5}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (3.15). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3| \geq \frac{1}{6}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{\left(|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3|\right)} \leq \frac{6\nu}{5} \times 60 \times 6 = 432\nu$$

and hence (3.15) is obvious. Take into account the scenario where

$$0 < |s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3| < \frac{1}{6}.$$

In the above scenario, there is a positive whole number m such that

$$\frac{1}{6^{(m+1)}} \leq |s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3| < \frac{1}{6^m}. \tag{3.23}$$

This implies that $6^{m-1} s_1^{\frac{1}{3}} y^{\frac{1}{3}} z^{\frac{1}{3}} < \frac{1}{6}, 6^{m-1} x < \frac{1}{6}, 6^{m-1} y < \frac{1}{6}$ and $6^{m-1} z < \frac{1}{6}$. As a result,

$$6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3),$$
$$6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$6^n(3s_1 + 2s_2 + s_3), 6^n(3s_1 - 2s_2 + s_3), 6^n(3s_1 + 2s_2 - s_3), 6^n(3s_1 - 2s_2 - s_3),$$

$$6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)$$

are also within the range of $(-1,1)$. Due to the fact that ξ is linear over this range, we may conclude that

$$\begin{aligned}&\xi(6^n(3s_1+2s_2+s_3))+\xi(6^n(3s_1+2s_2-s_3))+\xi(6^n(3s_1-2s_2+s_3))\\&+\xi(6^n(3s_1-2s_2-s_3))-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]\\&-2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))=0,\end{aligned}$$

for $n=0,1,\ldots,m-1$. Utilising (3.17) and the definition of h, we may calculate

$$\begin{aligned}&\frac{|Dh(s_1,s_2,s_3)|}{(|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}}+|s_1|+|s_2|+|s_3|)}\\&\leq\sum_{n=m}^{\infty}\frac{1}{6^n(|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}}+|s_1|+|s_2|+|s_3|)}\Big|\xi(6^n(3s_1+2s_2+s_3))\\&+\xi(6^n(3s_1+2s_2-s_3))+\xi(6^n(3s_1-2s_2+s_3))+\xi(6^n(3s_1-2s_2-s_3))\\&-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]\\&-2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))\Big|\\&\leq\sum_{k=0}^{\infty}\frac{60\nu}{6^k6^m(|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}}+|s_1|+|s_2|+|s_3|)}\leq\sum_{k=0}^{\infty}\frac{360\nu}{6^k}=432\nu.\end{aligned}$$

Consequently, for all $s_1,s_2,s_3\in\mathbb{R}$ with $0<|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}}+|s_1|+|s_2|+|s_3|<\frac{1}{6}$, h fulfills (3.15). According to Corollary 3.2, the additive functional equation (1.7) is unstable at $t=\frac{1}{3}$. Let us assume, however, that there is an additive mapping $\mathcal{A}:\mathbb{R}\to\mathbb{R}$ obeying (3.16), where R is the set of all real numbers and $\kappa>0$. Since h is bounded and continuous for every $s_1\in\mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{A}$ is also bounded and continuous within the interval. $\mathcal{A}$ must have the form $\mathcal{A}(s_1)=cs_1$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)|\leq(\kappa+|c|)\,|s_1|. \tag{3.24}$$

being true. However, by choosing a positive integer m with $m\nu>\kappa+|c|$, we can find $s_1\in\left(0,\frac{1}{6^{m-1}}\right)$ such that $6^ns_1\in(0,1)$ for all $n=0,1,\ldots,m-1$. For this s_1, we get

$$h(s_1)=\sum_{n=0}^{\infty}\frac{\xi(6^ns_1)}{6^n}\geq\sum_{n=0}^{m-1}\frac{\nu\times6^ns_1}{6^n}=m\nu s_1>(\kappa+|c|)\,s_1$$

which defies (3.18). Based on the inequality (3.13), it may be concluded that the equation (1.7) is unstable in the Hyers-Ulam-Rassias sense while $t=\frac{1}{3}$. ■

4. Stability of (1.7) for even mappings

In this paper, we examine the generalized Hyers-Ulam stability of the functional equation (1.7), in particular for the case of an even mapping.

Theorem 4.1. *Let $s=\pm1$ and $\xi:\mathcal{K}^3\to[0,\infty)$ be a mapping such that*

$$\sum_{i=0}^{\infty}\frac{\xi\left(6^{si}s_1,6^{si}s_2,6^{si}s_3\right)}{6^{2si}}<\infty \tag{4.1}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$*. Let* $h : \mathcal{K} \to \mathcal{L}$ *be an even mapping fulfills*

$$\|Dh(s_1, s_2, s_3)\| \leq \xi\left(s_1, s_2, s_3\right) \tag{4.2}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$*. Then there is only one quadratic mapping* $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ *that fulfills* (1.7) *and*

$$\|h(s_1) - \mathcal{B}(s_1)\| \leq \frac{1}{6^2} \sum_{i=\frac{1-s}{2}}^{\infty} \frac{\psi\left(6^{si}s_1\right)}{6^{2si}} \tag{4.3}$$

where $\psi : \mathcal{K} \to \mathcal{L}$ *and* $\mathcal{B}(s_1)$ *are defined by*

$$\psi\left(s_1\right) = \xi\left(s_1, s_1, s_1\right) + \frac{1}{2}\xi\left(s_1, 0, s_1\right) \tag{4.4}$$

and

$$\mathcal{B}(s_1) = \lim_{i\to\infty} \frac{h(6^{si}s_1)}{6^{2si}} \tag{4.5}$$

for all $s_1 \in \mathcal{K}$*, respectively.*

Proof. Assuming that s is equal to 1. By substituting (s_1, s_2, s_3) with (s_1, s_1, s_1) in (4.2) and make use of the evenness of h, we arrive at the inequality

$$\|h(6s_1) + h(4s_1) + h(2s_1) - 56h(s_1)\| \leq \xi\left(s_1, s_1, s_1\right) \tag{4.6}$$

for all $s_1 \in \mathcal{K}$. Similarly, substituting (s_1, s_2, s_3) with $(s_1, 0, s_1)$ in (4.2) and using the even property of h, we get

$$\|h(4s_1) + h(2s_1) - 20h(s_1)\| \leq \frac{1}{2}\xi\left(s_1, 0, s_1\right) \tag{4.7}$$

for all $s_1 \in \mathcal{K}$. Combining these two inequalities, we find that

$$\begin{aligned}\|h(6s_1) - 36h(s_1)\| &= \|h(6s_1) + h(4s_1) + h(2s_1) - 56h(s_1)\| + \|h(4s_1) + h(2s_1) - 20h(s_1)\| \\ &\leq \xi\left(s_1, s_1, s_1\right) + \frac{1}{2}\xi\left(s_1, 0, s_1\right)\end{aligned} \tag{4.8}$$

for all $s_1 \in \mathcal{K}$. Dividing the preceding inequality by 6^2 yields

$$\left\|\frac{h(6s_1)}{6^2} - h(s_1)\right\| \leq \frac{\psi(s_1)}{6^2} \tag{4.9}$$

where

$$\psi(s_1) = \xi\left(s_1, s_1, s_1\right) + \frac{1}{2}\xi\left(s_1, 0, s_1\right)$$

for all $s_1 \in \mathcal{K}$. By plugging in $6s_1$ in place of s_1 and dividing by 6^2 in (4.9), we acquire

$$\left\|\frac{h(6^2s_1)}{6^4} - \frac{h(6s_1)}{6^2}\right\| \leq \frac{\xi(6s_1)}{6^4} \tag{4.10}$$

for all $s_1 \in \mathcal{K}$. From (4.9) and (4.10), we obtain

$$\begin{aligned}\left\|\frac{h(6^2s_1)}{6^4} - h(s_1)\right\| &\leq \left\|\frac{h(6s_1)}{6^2} - h(s_1)\right\| + \left\|\frac{h(6^2s_1)}{6^4} - \frac{h(6s_1)}{6^2}\right\| \\ &\leq \frac{1}{6^2}\left[\xi(s_1) + \frac{\xi(6s_1)}{6^2}\right]\end{aligned} \tag{4.11}$$

for all $s_1 \in \mathcal{K}$. Then, by induction to a postive integer n, we have

$$\left\|\frac{h(6^n s_1)}{6^{2n}} - h(s_1)\right\| \leq \frac{1}{6^2}\sum_{i=0}^{n-1}\frac{\xi(6^i s_1)}{6^{2i}} \leq \frac{1}{6^2}\sum_{i=0}^{\infty}\frac{\xi(6^i s_1)}{6^{2i}} \tag{4.12}$$

for all $s_1 \in \mathcal{K}$. Substituting $6^m s_1$ for s_1 and dividing by 6^{2m} in (4.12), we see that the sequence $\left\{\frac{h(6^n s_1)}{6^{2n}}\right\}$ converges. It follows that for any m and n in the positive integer range, we can conclude that

$$\begin{aligned}\left\|\frac{h(6^{n+m}s_1)}{6^{2(n+m)}} - \frac{h(6^m s_1)}{6^{2m}}\right\| &= \frac{1}{6^{2m}}\left\|\frac{h(6^n \cdot 6^m s_1)}{6^{2n}} - h(6^m s_1)\right\| \\ &\leq \frac{1}{6^2}\sum_{i=0}^{n-1}\frac{\xi(6^{i+m}s_1)}{6^{2(i+m)}} \leq \frac{1}{6^2}\sum_{i=0}^{\infty}\frac{\xi(6^{i+m}s_1)}{6^{2(i+m)}} \\ &\to 0 \quad \text{as} \quad m \to \infty\end{aligned}$$

for all $s_1 \in \mathcal{K}$. Thus $\left\{\frac{h(6^n s_1)}{6^{2n}}\right\}$ is Cauchy. For complete set $\mathcal{L}$, a mapping $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ exists with

$$\mathcal{B}(s_1) = \lim_{n\to\infty}\frac{h(6^n s_1)}{6^{2n}}, \quad \forall\ s_1 \in \mathcal{K}.$$

When we plug in (4.12), where n may go to infinity, we get that (4.3) is true for every $s_1 \in \mathcal{K}$. To show that $\mathcal{A}$ satisfies (1.7), we substitute $(6^n s_1, 6^n s_2, 6^n s_3)$ for (s_1, s_2, s_3) in (4.2) and divide by 6^{2n} to get

$$\frac{1}{6^{2n}}\left\|Dh(6^n s_1, 6^n s_2, 6^n s_3)\right\| \leq \frac{1}{6^{2n}}\xi(6^n s_1, 6^n s_2, 6^n s_3)$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Using the definition of $\mathcal{B}(s_1)$ and the aforementioned inequality, we can show that $\mathcal{B}(s_1)$ as n goes to infinity. Hence For all $s_1, s_2, s_3 \in \mathcal{K}$, $\mathcal{B}$ fulfils (1.7). If $\mathcal{B}$ is not unique, we may show that $\mathcal{D}(s_1)$ is also an additive mapping fulfilling (1.7) and (4.3), as

$$\begin{aligned}\|\mathcal{B}(s_1) - \mathcal{D}(s_1)\| &= \frac{1}{6^{2n}}\|\mathcal{B}(6^n s_1) - \mathcal{D}(6^n s_1)\| \\ &\leq \frac{1}{6^{2n}}\left\{\|\mathcal{B}(6^n s_1) - h(6^n s_1)\| + \|h(6^n s_1) - \mathcal{D}(6^n s_1)\|\right\} \\ &\leq \frac{2}{6^2}\sum_{i=0}^{\infty}\frac{\xi(6^{(i+n)}s_1)}{6^{2(i+n)}} \to 0 \quad \text{as} \quad n \to \infty\end{aligned}$$

for all $s_1 \in \mathcal{K}$. For this reason, $\mathcal{B}$ cannot be found anywhere else. This proves that the theory is correct when s equals 1. Substituting $\frac{s_1}{6}$ for s_1 in inequality (4.8) leads to the conclusion that

$$\left\|h(s_1) - 36h\left(\frac{s_1}{6}\right)\right\| \leq \xi\left(\frac{s_1}{6}, \frac{s_1}{6}, \frac{s_1}{6}\right) + \frac{1}{2}\xi\left(\frac{s_1}{6}, 0, \frac{s_1}{6}\right)$$

for all $s_1 \in \mathcal{K}$. The remainder of the proof for $s = -1$ is the same as it is for $s = 1$. Therefore, the theorem is valid for both $s = 1$ and $s = -1$. The theorem has been proven at this point. ■

The next Corollary is directly derived from Theorem 4.1 concerning the stability of Equation (1.7).

Corollary 4.2. *Let t be a positive real value, and assume $\nu \geq 0$. For any $s_1, s_2, s_3 \in \mathcal{K}$, let $h : \mathcal{K} \to \mathcal{L}$ be a function that fulfills the inequality*

$$\|Dh(s_1, s_2, s_3)\| \leq \begin{cases} \nu, & \\ \nu \left\{||s_1||^t + ||s_2||^t + ||s_3||^t\right\}, & t \neq 2; \\ \nu||s_1||^t||s_2||^t||s_3||^t, & 3t \neq 2; \\ \nu \left\{||s_1||^t||s_2||^t||s_3||^t + \left\{||s_1||^{3t} + ||s_2||^{3t} + ||s_3||^{3t}\right\}\right\}, & 3t \neq 2. \end{cases} \tag{4.13}$$

If so, then for every $s_1 \in \mathcal{K}$, there is a unique quadratic function $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ such that

$$\|h(s_1) - \mathcal{B}(s_1)\| \leq \begin{cases} \frac{3\nu}{70}, & \\ \frac{4\nu||s_1||^t}{|6^t - 6^2|}, & t \neq 2; \\ \frac{\nu||s_1||^t}{|6^{3t} - 6^2|}, & 3t \neq 2; \\ \frac{5\nu||s_1||^t}{|6^{3t} - 6^2|}, & 3t \neq 2. \end{cases} \tag{4.14}$$

To demonstrate that (1.7) is not stable at $t = 1$, as stated in Corollary 4.2, we will now present an illustration.

Example 4.3. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \nu s_1^2, & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant, and the function $h : \mathbb{R} \to \mathbb{R}$, which is defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^{2n}}$$

for all $s_1 \in \mathbb{R}$ fulfills the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq \frac{15552\nu}{7}(|s_1|^2 + |s_2|^2 + |s_3|^2) \tag{4.15}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. If this is the case, then there cannot be a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ with a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{B}(s_1)| \leq \kappa|s_1|^2 \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{4.16}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|\xi(6^n s_1)|}{|6^{2n}|} \leq \sum_{n=0}^{\infty} \frac{\nu}{6^{2n}} = \frac{36\nu}{35}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (4.15). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1|^2 + |s_2|^2 + |s_3|^2 \geq \frac{1}{6^2}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{(|s_1|^2 + |s_2|^2 + |s_3|^2)} \leq \frac{36\nu}{35} \times 60 \times 6^2 = \frac{15552\nu}{7}$$

and hence (4.15) is obvious. Take into account the scenario where

$$0 < |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^2}.$$

In the above scenario , there is a positive whole number m such that

$$\frac{1}{6^{2(m+1)}} \leq |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^{2m}}. \tag{4.17}$$

This implies that $6^{m-1}x < \frac{1}{6}, 6^{m-1}y < \frac{1}{6}$, and $6^{m-1}z < \frac{1}{6}$. As a result,

$$6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3),$$
$$6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$6^n(3s_1 + 2s_2 + s_3), 6^n(3s_1 - 2s_2 + s_3), 6^n(3s_1 + 2s_2 - s_3), 6^n(3s_1 - 2s_2 - s_3),$$
$$6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)$$

are also within the range of $(-1, 1)$. Due to the fact that ξ is quadratic over this range, we may conclude that

$$\begin{aligned}
&\xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) + \xi(6^n(3s_1 - 2s_2 + s_3)) \\
&+ \xi(6^n(3s_1 - 2s_2 - s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] - 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] \\
&- 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) = 0,
\end{aligned}$$

for integers $n = 0 \to m-1$. Utilising (4.17) and the definition of h, we may calculate

$$\begin{aligned}
\frac{|Dh(s_1, s_2, s_3)|}{(|s_1|^2 + |s_2|^2 + |s_3|^2)} \leq& \sum_{n=m}^{\infty} \frac{1}{36^n(|s_1|^2 + |s_2|^2 + |s_3|^2)} \Big| \xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) \\
&+ \xi(6^n(3s_1 - 2s_2 + s_3)) + \xi(6^n(3s_1 - 2s_2 - s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] \\
&- 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] - 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) \Big| \\
\leq& \sum_{k=0}^{\infty} \frac{60\nu}{36^k 36^m(|s_1|^2 + |s_2|^2 + |s_3|^2)} \leq \sum_{k=0}^{\infty} \frac{2160\nu}{36^k} = \frac{15552\nu}{7}.
\end{aligned}$$

Consequently, for all $s_1, s_2, s_3 \in \mathbb{R}$ with $0 < |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^2}$, h fulfills (4.15). According to Corollary 4.2, the quadratic functional equation (1.7) is unstable at $t = 1$. Let us assume, however, that there is a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ obeying (4.16), where R is the set of all real numbers and $\kappa > 0$. Since h is continuous and bounded for every $s_1 \in \mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{B}$ is also continuous and bounded within the interval. $\mathcal{B}$ must have the form $\mathcal{B}(s_1) = cs_1^2$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)| \leq (\kappa + |c|)\, s_1^2. \tag{4.18}$$

being true. However, by choosing a positive integer m with $m\nu > \kappa + |c|$, we can find $s_1 \in \left(0, \frac{1}{6^{m-1}}\right)$ such that $6^n s_1 \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. For this s_1, we get

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^{2n}} \geq \sum_{n=0}^{m-1} \frac{\nu \times 6^{2n} x^2}{6^{2n}} = m\nu s_1^2 > (\kappa + |c|)\, s_1^2$$

which defies (4.18). Based on the inequality (4.13), it may be concluded that the equation (1.7) is unstable in the Hyers-Ulam-Rassias sense while $t = 1$. ■

Here we present an example to show that, as mentioned in Corollary 4.2, the functional equation (1.7) is unstable for $t = \frac{2}{3}$.

Example 4.4. *Suppose t is such that $0 < t < \frac{2}{3}$. Then, there exists a function $h : \mathbb{R} \to \mathbb{R}$ and a constant $\nu > 0$ such that for all real numbers $s_1, s_2, s_3 \in \mathbb{R}$,*

$$|Dh(s_1, s_2, s_3)| \leq \nu |s_1|^{\frac{t}{3}} \; |s_2|^{\frac{t}{3}} \; |s_3|^{\frac{2-2t}{3}} \tag{4.19}$$

and for all quadratic mappings $\mathcal{B} : \mathbb{R} \to \mathbb{R}$

$$\sup_{s_1 \neq 0} \frac{|h(s_1) - \mathcal{B}(s_1)|}{|s_1|} = +\infty. \tag{4.20}$$

Proof. If we set $h(s_1) = s_1^2 \ln |s_1|$, if $s_1 \neq 0$, and $h(0) = 0$, then we may deduce that

$$\begin{aligned}
\sup_{x \neq 0} \frac{|h(s_1) - \mathcal{B}(s_1)|}{|s_1|} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|h(n) - \mathcal{B}(n)|}{|n|} \\
&= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left| n^2 \ln |n| - n^2 \, \mathcal{B}(1) \right|}{|n|} \\
&= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \; |\ln |n| - \mathcal{B}(1)| \; = \infty.
\end{aligned}$$

We need to show that (4.19).

Case (i): If $s_1, s_2, s_3 > 0$ in (4.19) then,

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= \big|(3s_1 + 2s_2 + s_3)^2 \ln |3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3)^2 \ln |3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3)^2 \ln |3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3)^2 \ln |3s_1 - 2s_2 - s_3| \\
&\quad - 12[s_1^2 \ln |s_1| + s_1^2 \ln |-s_1|] - 8[s_2^2 \ln |s_2| + s_2^2 \ln |-s_2|] \\
&\quad -2[s_3^2 \ln |s_3| + s_3^2 \ln |-s_3|] - 12 s_1^2 \ln |s_1|\big|
\end{aligned}$$

If we set $s_1 = j, s_2 = k$, and $s_3 = l$, then we get

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= \big|(3s_1 + 2s_2 + s_3)^2 \ln |3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3)^2 \ln |3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3)^2 \ln |3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3)^2 \ln |3s_1 - 2s_2 - s_3| \\
&\quad - 12[s_1^2 \ln |s_1| + s_1^2 \ln |-s_1|] - 8[s_2^2 \ln |s_2| + s_2^2 \ln |-s_2|] \\
&\quad -2[s_3^2 \ln |s_3| + s_3^2 \ln |-s_3|] - 12 s_1^2 \ln |s_1|\big| \\
&= \big|(3j + 2k + l)^2 \ln |3j + 2k + l| + (3j + 2k - l)^2 \ln |3j + 2k - l| \\
&\quad + (3j - 2k + l)^2 \ln |3j - 2k + l| + (3j - 2k - l)^2 \ln |3j - 2k - l| \\
&\quad - 12[j^2 \ln |j| + j^2 \ln |-j|] - 8[k^2 \ln |k| + k^2 \ln |-k|] \\
&\quad -2[l^2 \ln |l| + l^2 \ln |-l|] - 12 j^2 \ln |j|\big| \\
&|h(3j + 2k + l) + h(3j + 2k - l) + h(3j - 2k + l) + h(3j - 2k - l) \\
&\quad -12[h(j) + h(-j)] - 8[h(k) + h(-k)] - 2[h(l) + h(-l)] - 12h(j)|
\end{aligned}$$

$$\leq \nu|j|^{\frac{t}{3}}\ |k|^{\frac{t}{3}}\ |l|^{\frac{2-2t}{3}}$$
$$= \nu|s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{2-2t}{3}}$$

Case (ii): If $s_1, s_2, s_3 < 0$ in (4.19) then,

$$\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3)\\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)|\\
&= \big|(3s_1+2s_2+s_3)^2\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)^2\ln|3s_1+2s_2-s_3|\\
&\quad +(3s_1-2s_2+s_3)^2\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)^2\ln|3s_1-2s_2-s_3|\\
&\quad -12[s_1^2\ln|s_1|+s_1^2\ln|-s_1|]-8[s_2^2\ln|s_2|+s_2^2\ln|-s_2|]\\
&\quad -2[s_3^2\ln|s_3|+s_3^2\ln|-s_3|]-12s_1^2\ln|s_1|\big|
\end{aligned}$$

If we set $s_1=-j, s_2=-k$, and $s_3=-l$, then we get

$$\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3)\\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)|\\
&= \big|(3s_1+2s_2+s_3)^2\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)^2\ln|3s_1+2s_2-s_3|\\
&\quad +(3s_1-2s_2+s_3)^2\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)^2\ln|3s_1-2s_2-s_3|\\
&\quad -12[s_1^2\ln|s_1|+s_1^2\ln|-s_1|]-8[s_2^2\ln|s_2|+s_2^2\ln|-s_2|]\\
&\quad -2[s_3^2\ln|s_3|+s_3^2\ln|-s_3|]-12s_1^2\ln|s_1|\big|\\
&= \big|(-3j-2k-l)^2\ln|-3j-2k-l|+(-3j-2k+l)^2\ln|-3j-2k+l|\\
&\quad +(-3j+2k-l)^2\ln|-3j+2k-l|+(-3j+2k+l)^2\ln|-3j+2k+l|\\
&\quad -12[j^2\ln|j|+j^2\ln|-j|]-8[k^2\ln|k|+k^2\ln|-k|]\\
&\quad -2[l^2\ln|l|+l^2\ln|-l|]-12j^2\ln|j|\big|\\
&|h(-3j-2k-l)+h(-3j-2k+l)+h(-3j+2k-l)+h(-3j+2k+l)\\
&\quad -12[h(-j)+h(j)]-8[h(-k)+h(k)]-2[h(-l)+h(l)]-12h(-j)|\\
&\leq \nu|j|^{\frac{t}{3}}\ |-k|^{\frac{t}{3}}\ |-l|^{\frac{2-2t}{3}}\\
&= \nu|s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{2-2t}{3}}
\end{aligned}$$

Case (iii): If $s_1 > 0, s_2, s_3 < 0$ in (4.19) then

$$\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3)\\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)|\\
&= \big|(3s_1+2s_2+s_3)^2\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)^2\ln|3s_1+2s_2-s_3|\\
&\quad +(3s_1-2s_2+s_3)^2\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)^2\ln|3s_1-2s_2-s_3|\\
&\quad -12[s_1^2\ln|s_1|+s_1^2\ln|-s_1|]-8[s_2^2\ln|s_2|+s_2^2\ln|-s_2|]\\
&\quad -2[s_3^2\ln|s_3|+s_3^2\ln|-s_3|]-12s_1^2\ln|s_1|\big|
\end{aligned}$$

If we set $s_1=j, s_2=-k$, and $s_3=-l$, then we get

$$\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3)\\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)|\\
&= \big|(3s_1+2s_2+s_3)^2\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)^2\ln|3s_1+2s_2-s_3|\\
&\quad +(3s_1-2s_2+s_3)^2\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)^2\ln|3s_1-2s_2-s_3|
\end{aligned}$$

$$
\begin{aligned}
&- 12[s_1^2 \ln|s_1| + s_1^2 \ln|-s_1|] - 8[s_2^2 \ln|s_2| + s_2^2 \ln|-s_2|] \\
&-2[s_3^2 \ln|s_3| + s_3^2 \ln|-s_3|] - 12 s_1^2 \ln|s_1|\big| \\
&= \big|(3j-2k-l)^2 \ln|3j-2k-l)| + (3j-2k+l)^2 \ln|j-2k+l| \\
&+ (3j+2k-l)^2 \ln|3j+2k-l| + (3j+2k+l)^2 \ln|3j+2k+l| \\
&- 12[j^2 \ln|j| + j^2 \ln|-j|] - 8[k^2 \ln|k| + k^2 \ln|-k|] \\
&-2[l^2 \ln|l| + l^2 \ln|-l|] - 12 j^2 \ln|j|\big| \\
&|h(3j-2k-l) + h(3j-2k+l) + h(3j+2k-l) + h(3j+2k+l) \\
&-12[h(j)+h(-j)] - 8[h(-k)+h(k)] - 2[h(-l)+h(l)] - 12h(j)| \\
&\le \nu |j|^{\frac{t}{3}} \, |-k|^{\frac{t}{3}} \, |-l|^{\frac{2-2t}{3}} \\
&= \nu |s_1|^{\frac{t}{3}} \, |s_2|^{\frac{t}{3}} \, |s_3|^{\frac{2-2t}{3}}
\end{aligned}
$$

Case (iv): If $s_1 < 0, s_2, s_3 > 0$ in (4.19) then,

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= \big|(3s_1+2s_2+s_3)^2 \ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)^2 \ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)^2 \ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)^2 \ln|3s_1-2s_2-s_3| \\
&- 12[s_1^2 \ln|s_1| + s_1^2 \ln|-s_1|] - 8[s_2^2 \ln|s_2| + s_2^2 \ln|-s_2|] \\
&-2[s_3^2 \ln|s_3| + s_3^2 \ln|-s_3|] - 12 s_1^2 \ln|s_1|\big|
\end{aligned}
$$

If we set $s_1 = -j, s_2 = k$, and $s_3 = l$, then we get

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= \big|(3s_1+2s_2+s_3)^2 \ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)^2 \ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)^2 \ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)^2 \ln|3s_1-2s_2-s_3| \\
&- 12[s_1^2 \ln|s_1| + s_1^2 \ln|-x|] - 8[s_2^2 \ln|s_2| + s_2^2 \ln|-s_2|] \\
&-2[s_3^2 \ln|s_3| + s_3^2 \ln|-s_3|] - 12 s_1^2 \ln|s_1|\big| \\
&= \big|(-3j+2k+l)^2 \ln|-3j+2k+l| + (-3j+2k-l)^2 \ln|-3j+2k-l| \\
&+ (-3j-2k+l)^2 \ln|-3j-2k+l| + (-3j-2k-l)^2 \ln|-3j-2k-l| \\
&- 12[j^2 \ln|-j| + j^2 \ln|j|] - 8[k^2 \ln|k| + k^2 \ln|-k|] \\
&-2[l^2 \ln|l| + l^2 \ln|-l|] - 12 j^2 \ln|-j|\big| \\
&|h(-3j+2k+l) + h(-3j+2k-l) + h(-3j-2k+l) + h(-3j-2k-l) \\
&-12[h(-j)+h(j)] - 8[h(k)+h(-k)] - 2[h(l)+h(-l)] - 12h(-j)| \\
&\le \nu |-j|^{\frac{t}{3}} \, |k|^{\frac{t}{3}} \, |l|^{\frac{2-2t}{3}} \\
&= \nu |s_1|^{\frac{t}{3}} \, |s_2|^{\frac{t}{3}} \, |s_3|^{\frac{2-2t}{3}}
\end{aligned}
$$

Case (v): If $s_1 = s_2 = s_3 = 0$ in (4.19), then the statement is obvious. ■

Here we present an example to show that, as mentioned in Corollary 4.2, the functional equation (1.7) is unstable for $t = \frac{2}{3}$.

Example 4.5. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \nu s_1^2, & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant; the function $h : \mathbb{R} \to \mathbb{R}$, defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^{2n}}$$

for all $s_1 \in \mathbb{R}$, fulfills the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq \frac{15552}{7} \left(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2\right) \tag{4.21}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. If this is the case, then there cannot be a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ with a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{B}(s_1)| \leq \kappa s_1^2 \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{4.22}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|\xi(6^n s_1)|}{|6^{2n}|} \leq \sum_{n=0}^{\infty} \frac{\nu}{6^{2n}} = \frac{36\nu}{35}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (4.15). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2 \geq \frac{1}{6^2}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{\left(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2\right)} \leq \frac{36\nu}{35} \times 60 \times 6^2 = \frac{15552}{7}\nu$$

and hence (4.15) is obvious. Take into account the scenario where

$$0 < |s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^2}.$$

In the above scenario, there is a positive whole number m such that

$$\frac{1}{6^{2(m+1)}} \leq |s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^{2m}}. \tag{4.23}$$

This implies that $6^{m-1}x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} < \frac{1}{6}, 6^{m-1}x < \frac{1}{6}, 6^{m-1}y < \frac{1}{6}$ and $6^{m-1}z < \frac{1}{6}$. As a result,

$$6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3),$$
$$6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$6^n(3s_1 + 2s_2 + s_3), 6^n(3s_1 - 2s_2 + s_3), 6^n(3s_1 + 2s_2 - s_3), 6^n(3s_1 - 2s_2 - s_3),$$
$$6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)$$

are also within the range of $(-1, 1)$. Due to the fact that ξ is quadratic over this range, we may conclude that

$$\xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) + \xi(6^n(3s_1 - 2s_2 + s_3))$$

$$+\xi(6^n(3s_1-2s_2-s_3))-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]$$
$$-2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))=0,$$

for $n=0,1,\ldots,m-1$. Utilising (4.17) and the definition of h, we may calculate

$$\begin{aligned}\frac{|Dh(s_1,s_2,s_3)|}{\left(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}}+|s_1|^2+|s_2|^2+|s_3|^2\right)} &\leq \sum_{n=m}^{\infty}\frac{1}{6^{2n}\left(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}}+|s_1|^2+|s_2|^2+|s_3|^2\right)}\\
&\Big|\xi(6^n(3s_1+2s_2+s_3))+\xi(6^n(3s_1+2s_2-s_3))\\
&+\xi(6^n(3s_1-2s_2+s_3))+\xi(6^n(3s_1-2s_2-s_3))\\
&-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]\\
&-2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))\Big|\\
&\leq \sum_{k=0}^{\infty}\frac{60\nu}{36^k36^m(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}}+|s_1|^2+|s_2|^2+|s_3|^2)}\\
&\leq \sum_{k=0}^{\infty}\frac{2160\nu}{36^k}=\frac{15552}{7}\nu.\end{aligned}$$

Consequently, for all $s_1,s_2,s_3\in\mathbb{R}$ with $0<|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}}+|s_1|^2+|s_2|^2+|s_3|^2<\frac{1}{6^2}$, h fulfills (4.15). According to Corollary 4.2, the quadratic functional equation (1.7) is unstable at $t=\frac{2}{3}$. Let us assume, however, that there is a quadratic mapping $\mathcal{B}:\mathbb{R}\to\mathbb{R}$ obeying (3.16), where R is the set of all real numbers and $\kappa>0$. Since h is bounded and continuous for every $s_1\in\mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{B}$ is also bounded and continuous within the interval. $\mathcal{B}$ must have the form $\mathcal{B}(s_1)=cs_1^2$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)|\leq(\kappa+|c|)\,s_1^2. \tag{4.24}$$

being true. However, by choosing a positive integer m with $m\nu>\kappa+|c|$, we can find $s_1\in\left(0,\frac{1}{6^{m-1}}\right)$ such that $6^ns_1\in(0,1)$ for all $n=0,1,\ldots,m-1$. For this s_1, we get

$$h(s_1)=\sum_{n=0}^{\infty}\frac{\xi(6^ns_1)}{6^{2n}}\geq\sum_{n=0}^{m-1}\frac{\nu\times 6^{2n}s_1^2}{6^{2n}}=m\nu s_1^2>(\kappa+|c|)\,s_1^2$$

which defies (4.18). Based on the inequality (4.13), it may be concluded that the equation (1.7) is unstable in the Hyers-Ulam-Rassias sense while $t=\frac{2}{3}$. ■

5. Stability of (1.7) for mixed mappings

In this section, we will examine the generalised Hyers-Ulam stability of the functional equation (1.7) in the case where the mapping is a mixture of odd and even mappings.

Theorem 5.1. *Let $s=\pm1$ and $\xi:\mathcal{K}^3\to[0,\infty)$ be a mapping such that*

$$\sum_{i=0}^{\infty}\frac{\xi\left(6^{si}s_1,6^{si}s_2,6^{si}s_3\right)}{6^{si}}<\infty \text{ and } \sum_{i=0}^{\infty}\frac{\xi\left(6^{si}s_1,6^{si}s_2,6^{si}s_3\right)}{6^{2si}}<\infty \tag{5.1}$$

for all $s_1,s_2,s_3\in\mathcal{K}$. Let $h:\mathcal{K}\to\mathcal{L}$ be a mapping satisfying the inequality

$$\|Dh(s_1,s_2,s_3)\|\leq\xi\left(s_1,s_2,s_3\right) \tag{5.2}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Then, a unique additive mapping $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ and a unique quadratic mapping $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ exist with

$$\|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)\| \leq \frac{1}{2} \sum_{i=\frac{1-s}{2}}^{\infty} \left\{ \frac{1}{6^{(si+1)}} + \frac{1}{6^{2(si+1)}} \right\} \left[\psi\left(6^{si} s_1\right) + \psi\left(-6^{si} s_1\right) \right]$$

where $\psi(s_1), \mathcal{A}(s_1)$ and $\mathcal{B}(s_1)$ are defined as in (3.4), (3.5) *and* (4.5) *for all $s_1 \in \mathcal{K}$, respectively.*

Proof. Define a mapping $h_a : \mathcal{K} \to H$ by

$$h_a(s_1) = \frac{1}{2}[h(s_1) - h(-s_1)] \tag{5.3}$$

for all $s_1 \in \mathcal{K}$. Then $h_a(0) = 0$ and $h_a(-s_1) = -h_a(s_1)$ for all $s_1 \in \mathcal{K}$. Hence

$$\|Dh_a(s_1, s_2, s_3)\| \leq \frac{1}{2} \left[\xi\left(s_1, s_2, s_3\right) + \xi\left(-x, -s_2, -s_3\right)\right] \tag{5.4}$$

for all $s_1 \in \mathcal{K}$. By Theorem 3.1, we have

$$\|h_a(s_1) - \mathcal{A}(s_1)\| \leq \frac{1}{2} \sum_{i=\frac{1-s}{2}}^{\infty} \frac{\psi\left(6^{si} s_1\right) + \psi\left(-6^{si} s_1\right)}{6^{(si+1)}} \tag{5.5}$$

where $\psi(s_1)$ and $\mathcal{A}(s_1)$ are defined as in (3.4) and (3.5) for all $s_1 \in \mathcal{K}$, respectively. Also, define a mapping $h_q : \mathcal{K} \to H$ by

$$h_q(s_1) = \frac{1}{2}[h(s_1) + h(-s_1)] \tag{5.6}$$

for all $s_1 \in \mathcal{K}$. Then $h_q(0) = 0$ and $h_q(-s_1) = h_q(s_1)$ for all $s_1 \in \mathcal{K}$. Hence

$$\|Dh_q(s_1, s_2, s_3)\| \leq \frac{1}{2} \left[\xi\left(s_1, s_2, s_3\right) + \xi\left(-s_1, -s_2, -s_3\right)\right] \tag{5.7}$$

for all $s_1 \in \mathcal{K}$. By Theorem 4.1, we have

$$\|h_q(s_1) - \mathcal{B}(s_1)\| \leq \frac{1}{2} \sum_{i=\frac{1-s}{2}}^{\infty} \frac{\psi\left(6^{si} s_1\right) + \psi\left(-6^{si} s_1\right)}{6^{2(si+1)}} \tag{5.8}$$

where $\psi : \mathcal{K} \to \mathcal{L}$ and $\mathcal{B}(s_1)$ are defined as in (3.4) and (4.5) for all $s_1 \in \mathcal{K}$, respectively. From (5.3) and (5.5), we have

$$h(s_1) = h_a(s_1) + h_q(s_1) \tag{5.9}$$

for all $s_1 \in \mathcal{K}$. Using (5.5), (5.8) and (5.9), we get

$$\begin{aligned} \|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)\| &= \|h_a(s_1) + h_q(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)\| \\ &\leq \|h_a(s_1) - \mathcal{A}(s_1)\| + \|h_q(s_1) - \mathcal{B}(s_1)\| \\ &\leq \frac{1}{2} \sum_{i=\frac{1-s}{2}}^{\infty} \left\{ \frac{1}{6^{(si+1)}} + \frac{1}{6^{2(si+1)}} \right\} \left[\psi\left(6^{si} s_1\right) + \psi\left(-6^{si} s_1\right) \right] \end{aligned}$$

where $\psi(s_1), \mathcal{A}(s_1)$ and $\mathcal{B}(s_1)$ are defined as in (3.4), (3.5) and (4.5) for all $s_1 \in \mathcal{K}$, respectively. ■

The Corollary that follows is a direct result of Theorem 4.1, which pertains to the stability of equation (1.7).

Corollary 5.2. *Assume that t be a positive real value and $\nu \geq 0$. Let $h : \mathcal{K} \to \mathcal{L}$ be a function fulfills the inequality*

$$\|Dh(s_1, s_2, s_3)\| \leq \begin{cases} \nu, & \\ \nu\left\{||s_1||^t + ||s_2||^t + ||s_3||^t\right\}, & t \neq 1,2; \\ \nu||s_1||^t||s_2||^t||s_3||^t, & 3t \neq 1,2; \\ \nu\left\{||s_1||^t||s_2||^t||s_3||^t + \left\{||s_1||^{3t} + ||s_2||^{3t} + ||s_3||^{3t}\right\}\right\}, & 3t \neq 1,2; \end{cases} \tag{5.10}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Then, a unique additive mapping $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ and a unique quadratic mapping $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ exist with

$$\|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)\| \leq \begin{cases} 3\nu\left(\dfrac{1}{10} + \dfrac{1}{70}\right), & \\ 4\nu||s_1||^t\left(\dfrac{1}{|6^t - 6|} + \dfrac{1}{|6^t - 6^2|}\right), & t \neq 1,2; \\ \nu||s_1||^{3t}\left(\dfrac{1}{|6^{3t} - 6|} + \dfrac{1}{|6^{3t} - 6^2|}\right), & 3t \neq 1,2; \\ 5\nu||s_1||^{3t}\left(\dfrac{1}{|6^{3t} - 6|} + \dfrac{1}{|6^{3t} - 6^2|}\right), & 3t \neq 1,2; \end{cases} \tag{5.11}$$

for all $s_1 \in \mathcal{K}$.

To demonstrate that (1.7) is not stable at $t = 1$, as stated in Corollary 4.2, we will now present an illustration.

Example 5.3. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \dfrac{\nu}{2}(s_1 + s_1^2), & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant, and the function $h : \mathbb{R} \to \mathbb{R}$, which is defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{(6^n + 1)}{6^{2n}} \xi(6^n s_1)$$

for all $s_1 \in \mathbb{R}$, satisfies the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq \frac{936 \times 6\nu}{7}(|s_1| + |s_2| + |s_3|) \tag{5.12}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. Then there is no existence of an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$, a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)| \leq \kappa(|s_1| + |s_1|^2) \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{5.13}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|6^n + 1|}{|6^{2n}|} |\xi(6^n s_1)| \leq \sum_{n=0}^{\infty} \frac{\nu(6^n + 1)}{6^{2n}} = \frac{78\nu}{35}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (5.12). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1| + |s_2| + |s_3| \geq \frac{1}{6}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{(|s_1| + |s_2| + |s_3|)} \leq \frac{78\nu}{35} \times 60 \times 6 = \frac{936 \times 6\nu}{7}$$

and hence (5.12) is obvious. Take into account the scenario where

$$0 < |s_1| + |s_2| + |s_3| < \frac{1}{6}.$$

In the above scenario , there is a positive whole number m such that

$$\frac{1}{6^{(m+1)}} \leq |s_1| + |s_2| + |s_3| < \frac{1}{6^m}. \tag{5.14}$$

This implies that $6^{m-1}s_1 < \frac{1}{6}, 6^{m-1}s_2 < \frac{1}{6}$, and $6^{m-1}s_3 < \frac{1}{6}$. As a result,

$$6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3),$$
$$6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$6^n(3s_1 + 2s_2 + s_3), 6^n(3s_1 - 2s_2 + s_3), 6^n(3s_1 + 2s_2 - s_3), 6^n(3s_1 - 2s_2 - s_3),$$
$$6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)$$

Due to the fact that ξ is additive-quadratic over this range, we may conclude that

$$\begin{aligned}&\xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) + \xi(6^n(3s_1 - 2s_2 + s_3))\\&+ \xi(6^n(3s_1 - 2s_2 - s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] - 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))]\\&- 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) = 0,\end{aligned}$$

for $n = 0, 1, \ldots, m-1$. Utilising (5.14) and from the definition of h, we may calculate

$$\begin{aligned}\frac{|Dh(s_1, s_2, s_3)|}{(|s_1| + |s_2| + |s_3|)} &\leq \sum_{n=m}^{\infty} \frac{(6^n + 1)}{6^{2n}(|s_1| + |s_2| + |s_3|)} \Big| \xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3))\\&+ \xi(6^n(3s_1 - 2s_2 + s_3)) + \xi(6^n(3s_1 - 2s_2 - s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))]\\&- 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] - 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) \Big|\\&\leq \sum_{n=m}^{\infty} \frac{(6^n + 1)60\nu}{6^{2n}(|s_1| + |s_2| + |s_3|)} \leq \frac{78\nu}{35} \times \frac{60}{6^m(|s_1| + |s_2| + |s_3|)} \leq \frac{936 \times 6\nu}{7}.\end{aligned}$$

Consequently, for all $s_1, s_2, s_3 \in \mathbb{R}$ with $0 < |s_1| + |s_2| + |s_3| < \frac{1}{6}$, h fulfills (5.12). According to Corollary 5.2, the the additive-quadratic functional equation (1.7) is unstable at $t = 1$. Let us assume, however, that there exist an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$, a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ that satisfies (5.13). Since h is continuous and bounded for every $s_1 \in \mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{A}$ and $\mathcal{B}$ are also continuous and bounded within the interval. $\mathcal{A}$ must have the form $\mathcal{A}(s_1) = cs_1$ and $\mathcal{B}$ must have the form $\mathcal{B}(s_1) = cs_1^2$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)| \leq (\kappa + |c|)\,(|s_1| + |s_1|^2). \tag{5.15}$$

being true. However, by choosing a positive integer m with $m\nu > \kappa + |c|$, we can find $s_1 \in \left(0, \frac{1}{6^{m-1}}\right)$ such that $6^n s_1 \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. For this s_1, we get

$$h(s_1) = \sum_{n=0}^{\infty} \frac{(6^n + 1)}{6^{2n}} \xi(6^n s_1) \geq \sum_{n=0}^{m-1} \frac{\nu(6^n + 1)}{6^{2n}} (6^n s_1 + 6^{2n} s_1^2)$$

$$= \sum_{n=0}^{m-1} \frac{\nu(6^n+1)}{6^n}(s_1 + 6^n s_1^2) \geq \sum_{n=0}^{m-1} \nu(s_1 + s_1^2)$$
$$= m\nu(s_1 + s_1^2) > (\kappa + |c|)(s_1 + s_1^2)$$

which defies (5.15). Based on the inequality (5.10), it may be concluded that the equation (1.7) is unstable in the Hyers-Ulam-Rassias sense while $t = 1$. ■

Here we present an example to show that, as mentioned in Corollary 5.2, the functional equation (1.7) is unstable for $t = 2$

Example 5.4. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \frac{\nu}{2}(s_1 + s_1^2), & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant, and the function $h : \mathbb{R} \to \mathbb{R}$, which is defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{(6^{2n}+1)}{6^{4n}} \xi(6^{2n} s_1)$$

for all $s_1 \in \mathbb{R}$, satisfies the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq \frac{31536 \times 6^2 \nu}{259} \left(|s_1|^2 + |s_2|^2 + |s_3|^2\right) \tag{5.16}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. Then there is no existence of an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$, a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)| \leq \kappa(|s_1| + |s_1|^2) \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{5.17}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq 2\nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|6^{2n}+1|}{|6^{4n}|} |\xi(6^{2n} s_1)| \leq \sum_{n=0}^{\infty} \frac{\nu(6^{2n}+1)}{6^{4n}} = \frac{2628\nu}{1295}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (5.16). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1|^2 + |s_2|^2 + |s_3|^2 \geq \frac{1}{6^2}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{(|s_1|^2 + |s_2|^2 + |s_3|^2)} \leq \frac{2628\nu}{1295} \times 60 \times 6^2 = \frac{31536 \times 6^2 \nu}{259}$$

and hence (5.16) is obvious. Take into account the scenario where

$$0 < |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^2}.$$

In the above scenario , there is a positive whole number m such that

$$\frac{1}{6^{2(m+1)}} \leq |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^{2m}}. \tag{5.18}$$

This implies that $6^{m-1}s_1 < \frac{1}{6}$, $6^{m-1}s_2 < \frac{1}{6}$, and $6^{m-1}s_3 < \frac{1}{6}$. As a result,

$$6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3),$$

$$6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)$$

are all within the range of $(-1,1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$\begin{aligned}&6^n(3s_1+2s_2+s_3), 6^n(3s_1-2s_2+s_3), 6^n(3s_1+2s_2-s_3), 6^n(3s_1-2s_2-s_3),\\&6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)\end{aligned}$$

are also within the range of $(-1,1)$. Due to the fact that ξ is additive-quadratic over this range, we may conclude that

$$\begin{aligned}&\xi(6^n(3s_1+2s_2+s_3)) + \xi(6^n(3s_1+2s_2-s_3)) + \xi(6^n(3s_1-2s_2+s_3))\\&+\xi(6^n(3s_1-2s_2-s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] - 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))]\\&-2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) = 0,\end{aligned}$$

for $n = 0, 1, \dots, m-1$. Utilising (5.18) and from the definition of h, we may calculate

$$\begin{aligned}\frac{|Dh(s_1,s_2,s_3)|}{(|s_1|^2+|s_2|^2+|s_3|^2)} &\le \sum_{n=m}^{\infty} \frac{(6^{2n}+1)}{6^{4n}(|s_1|^2+|s_2|^2+|s_3|^2)} \Big| \xi(6^n(3s_1+2s_2+s_3)) + \xi(6^n(3s_1+2s_2-s_3))\\&+\xi(6^n(3s_1-2s_2+s_3)) + \xi(6^n(3s_1-2s_2-s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))]\\&-8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] - 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1))\Big|\\&\le \sum_{n=m}^{\infty} \frac{(6^{2n}+1)}{6^{4n}} \times \frac{60\nu}{(|s_1|^2+|s_2|^2+|s_3|^2)}\\&\le \frac{2628\nu}{1295} \times \frac{60}{6^{2m}(|s_1|^2+|s_2|^2+|s_3|^2)} = \frac{31536\times 6^2\nu}{259}.\end{aligned}$$

Thus h satisfies (5.16) with $0 < |s_1|^2+|s_2|^2+|s_3|^2 < \frac{1}{6^2}$ for all $s_1, s_2, s_3 \in \mathbb{R}$. We assert that the additive-quadratic functional equation (1.7) is not stable when $t = 2$ as stated in Corollary 5.2. To contradict this, let's assume that there exists an additive mapping $\mathcal{A}: \mathbb{R} \to \mathbb{R}$, a quadratic mapping $\mathcal{B}: \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ that satisfies (5.17). Since h is bounded and continuous for all $s_1 \in \mathbb{R}$, $\mathcal{A}$ and $\mathcal{B}$are bounded within a range and continuous at the origin when s_1 is in an open interval containing the origin. In light of Theorem 5.1, $\mathcal{A}$ must have the form $\mathcal{A}(s_1) = cs_1$ and $\mathcal{B}(s_1) = cs_1^2$ for any s_1 in $\mathbb{R}$. This leads to

$$|h(s_1)| \le (\kappa + |c|)\,(|s_1| + |s_1|^2). \tag{5.19}$$

being true. However, by choosing a positive integer m with $m\nu > \kappa + |c|$, we can find $s_1 \in \left(0, \frac{1}{6^{m-1}}\right)$ such that $6^n s_1 \in (0,1)$ for all $n = 0, 1, \dots, m-1$. For this s_1, we get

$$\begin{aligned}h(s_1) &= \sum_{n=0}^{\infty} \frac{(6^{2n}+1)}{6^{4n}} \xi(6^{2n}s_1) \ge \sum_{n=0}^{m-1} \frac{\nu(6^{2n}+1)}{6^{4n}} (6^{2n}s_1 + 6^{4n}s_1^2)\\&= \sum_{n=0}^{m-1} \frac{\nu(6^{2n}+1)}{6^{2n}} (s_1 + 6^{2n}s_1^2) \ge \sum_{n=0}^{m-1} \nu(s_1+s_1^2)\\&= m\nu(s_1+s_1^2) > (\kappa+|c|)\,(s_1+s_1^2)\end{aligned}$$

which contradicts (5.19). Therefore, the additive-quadratic functional equation (1.7) is not stable in the sense of Hyers-Ulam-Rassias when $t = 1$, as stated in the inequality (5.10). ■

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On Kenmotsu metric spaces satisfying some conditions on the W_7 -curvature tensor

Pakize Uygun*[1] and Mehmet Atçeken[2]

[1,2] *Department of Mathematics, Faculty of Arts and Sciences, Aksaray University, 68100, Aksaray, Turkey.*

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Abstract. This research article is about the geometry of the Kenmotsu manifold. Some important properties such as the $W_7 \cdot W_5 = 0$, $W_7 \cdot W_6 = 0$, $W_7 \cdot W_7 = 0$, $W_7 \cdot W_8 = 0$, $W_7 \cdot W_9 = 0$ and $W_7 \cdot W_0^{\star} = 0$ curvature conditions of the Kenmotsu manifold have been investigated.

AMS Subject Classifications: 53C25, 53C35

Keywords: Kenmotsu manifold, η-Einstein manifold, Einstein manifold, Riemannian curvature tensor.

Contents

1. Introduction and Background

In 1963, K. Kobayashi and K. Nomizu demonstrated that Any two complete Riemannian manifolds that are simply connected and have a constant curvature k, are isometric to one another. [7]. Following that, several scholars, including [8–10], explored manifolds curvature in various methods to varying degrees.

According to D. B. Abdussattar's research, tensor $\widetilde{C}$ must disappear identically in order for a space time to be conharmonic to a flat space time. If a space time is conharmonically flat, it is either empty, in which case it is flat, or filled with a distribution defined by an energy momentum tensor T that has an electromagnetic field's algebraic structure while also conforming to a flat space time [1].

Let M be an $n-$dimensional differentiable manifold of differentiability class C^{r+1} with a (1,1) tensor field ϕ, the connected vector field ξ, a contact form η and the related Riemannian metric g. Kenmotsu described the differential geometric features of class manifolds in 1972. The structure developed is known as the Kenmotsu structure. A Sasakian structures are distinct from Kenmotsu structures. [6].

This study aims to examine a Kenmotsu metric manifold's curvature tensor's characteristics. In addition, we take research $W_7 \cdot W_5 = 0$, $W_7 \cdot W_6 = 0$, $W_7 \cdot W_7 = 0$, $W_7 \cdot W_8 = 0$, $W_7 \cdot W_9 = 0$ and $W_7 \cdot W_0^{\star} = 0$ where W_5, W_6, W_7, W_8, W_9, and $W_0^{\star}$ denote the curvature tensors of Kenmotsu manifold, respectively.

*Corresponding author. Email addresses: **pakizeuygun@hotmail.com** (Pakize Uygun) and **mehmetatceken@aksaray.edu.tr** (Mehmet Atçeken)

2. Preliminaries

We have collected some fundamental information regarding contact metric manifold in this part. With a $(2n+1)$-dimensional linked structure, let M be an almost contact metric manifold. (φ, ξ, η, g), that is, φ is an $(1,1)$-tensor field, ξ is a vector field, η is a 1-form and the Riemanniann metric g satisfying

$$\varphi^2(\theta_1) = -\theta_1 + \eta(\theta_1)\xi, \quad \eta(\varphi\theta_1) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\varphi = 0 \tag{2.2}$$

for all $\theta_1, \theta_2 \in \Gamma(TM)$[11]. Let g be Riemannian metric compatible with (φ, ξ, η),
that is

$$g(\varphi\theta_1, \varphi\theta_2) = g(\theta_1, \theta_2) - \eta(\theta_1)\eta(\theta_2), \tag{2.3}$$

or equivalently,

$$g(\theta_1, \varphi\theta_2) = -g(\varphi\theta_1, \theta_2) \quad \text{and} \quad g(\theta_1, \xi) = \eta(\theta_1) \tag{2.4}$$

for all $\theta_1, \theta_2 \in \Gamma(TM)$[4]. If in addition to above relations

$$(\nabla_{\theta_1}\varphi)\theta_2 = -\eta(\theta_2)\varphi\theta_1 - g(\theta_1, \varphi\theta_2)\xi, \tag{2.5}$$

and

$$\nabla_{\theta_1}\xi = \theta_1 - \eta(\theta_1)\xi, \tag{2.6}$$

where g holds Riemannian connection is indicated by the symbol, the manifold $(M, \varphi, \xi, \eta, g)$ is referred to as an almost Kenmotsu manifold. In a Kenmotsu manifold M, the following relation holds[5, 6]:

$$(\nabla_{\theta_1}\eta)\theta_2 = g(\theta_1, \theta_2) - \eta(\theta_1)\eta(\theta_2), \tag{2.7}$$

$$R(\theta_1, \theta_2)\xi = \eta(\theta_1)\theta_2 - \eta(\theta_2)\theta_1, \tag{2.8}$$

$$R(\xi, \theta_1)\theta_2 = \eta(\theta_2)\theta_1 - g(\theta_1, \theta_2)\xi, \tag{2.9}$$

$$S(\theta_1, \xi) = -2n\eta(\theta_1), \tag{2.10}$$

$$Q\xi = -2n\xi, \tag{2.11}$$

where r is scalar curvature of the connection ∇, As defined by $S(\theta_1, \theta_2) = g(Q\theta_1, \theta_2)$, where Q is the Ricci operator, S is the Ricci tensor, and R is the Riemannian curvature tensor. It submits to

$$S(\varphi\theta_1, \varphi\theta_2) = S(\theta_1, \theta_2) + 2n\eta(\theta_1)\eta(\theta_2). \tag{2.12}$$

Unknown Kenmotsu manifold if M's Ricci tensor S has the following structure, M is allegedly η-Einstein manifold.

$$S(\theta_1, \theta_2) = ag(\theta_1, \theta_2) + b\eta(\theta_1)\eta(\theta_2) \tag{2.13}$$

in which a and b are functions on (M^{2n+1}, g) for any arbitrary vector fields θ_1, θ_2. $\eta-$ Einstein manifold becomes Einstein manifold if $b = 0$ [6, 14]. Let M be a Kenmotsu manifold of dimension $(2n+1)$. According to the relationship, the curvature tensor R of M is determined by

$$\widetilde{R}(\theta_1, \theta_2)\theta_3 = \widetilde{\nabla}_{\theta_1}\widetilde{\nabla}_{\theta_2}\theta_3 - \widetilde{\nabla}_{\theta_2}\widetilde{\nabla}_{\theta_1}\theta_3 - \widetilde{\nabla}_{[\theta_1, \theta_2]}\theta_3. \tag{2.14}$$

Following that, in a Kenmotsu manifold, we arrive

$$\widetilde{R}(\theta_1, \theta_2)\theta_3 = R(\theta_1, \theta_2)\theta_3 + g(\theta_2, \theta_3)\theta_1 - g(\theta_1, \theta_3)\theta_2, \tag{2.15}$$

where $R(\theta_1,\theta_2)\theta_3 = \nabla_{\theta_1}\nabla_{\theta_2}\theta_3 - \nabla_{\theta_2}\nabla_{\theta_1}\theta_3 - \nabla_{[\theta_1,\theta_2]}\theta_3$, is the curvature tensor of M with respect to the connection ∇ [15, 16, 19]. The idea that W_5-curvature tensor was explained by [13]. W_5-curvature tensor,

W_6-curvature tensor, W_7-curvature tensor, W_8-curvature tensor, W_9-curvature tensor and $W_0^\star$-curvature tensor of a $(2n+1)$-dimensional Riemannian manifold are, respectively, specified as

$$W_5(\theta_1,\theta_2)\theta_3 = R(\theta_1,\theta_2)\theta_3 - \frac{1}{2n}[S(\theta_1,\theta_3)\theta_2 - g(\theta_1,\theta_3)Q\theta_2], \tag{2.16}$$

$$W_6(\theta_1,\theta_2)\theta_3 = R(\theta_1,\theta_2)\theta_3 - \frac{1}{2n}[S(\theta_2,\theta_3)\theta_1 - g(\theta_1,\theta_2)Q\theta_3], \tag{2.17}$$

$$W_7(\theta_1,\theta_2)\theta_3 = R(\theta_1,\theta_2)\theta_3 - \frac{1}{2n}[S(\theta_2,\theta_3)\theta_1 - g(\theta_2,\theta_3)Q\theta_1], \tag{2.18}$$

$$W_8(\theta_1,\theta_2)\theta_3 = R(\theta_1,\theta_2)\theta_3 - \frac{1}{2n}[S(\theta_2,\theta_3)\theta_1 - S(\theta_1,\theta_2)\theta_3], \tag{2.19}$$

$$W_9(\theta_1,\theta_2)\theta_3 = R(\theta_1,\theta_2)\theta_3 + \frac{1}{2n}[S(\theta_1,\theta_2)\theta_3 - g(\theta_2,\theta_3)Q\theta_1], \tag{2.20}$$

$$W_0^\star(\theta_1,\theta_2)\theta_3 = R(\theta_1,\theta_2)\theta_3 + \frac{1}{2n}[S(\theta_2,\theta_3)\theta_1 - g(\theta_1,\theta_3)Q\theta_2], \tag{2.21}$$

for all $\theta_1,\theta_2,\theta_3 \in \Gamma(TM)$[12, 13].

3. Some curvature characterizations on Kenmotsu metric spaces

The key findings for this article are presented in this section.

When we designate the W_5 curvature tensor from (2.16) and assume that M is a $(2n+1)$-dimensional Kenmotsu metric manifold, we obtain for subsequent consideration.

$$W_5(\theta_1,\theta_2)\xi = 2\eta(\theta_1)\theta_2 - \eta(\theta_2)\theta_1 + \frac{1}{2n}\eta(\theta_1)Q\theta_2. \tag{3.1}$$

Adding $\theta_1 = \xi$ to (3.1)

$$W_5(\xi,\theta_2)\xi = 2\theta_2 - \eta(\theta_2)\xi + \frac{1}{2n}Q\theta_2. \tag{3.2}$$

In (2.17) choosing $\theta_3 = \xi$ and using (2.8), we obtain

$$W_6(\theta_1,\theta_2)\xi = \eta(\theta_1)\theta_2 - g(\theta_1,\theta_2)\xi. \tag{3.3}$$

In (3.3), it follows

$$W_6(\xi,\theta_2)\xi = \theta_2 - \eta(\theta_2)\xi. \tag{3.4}$$

From (2.18) and (2.8), we arrive

$$W_7(\theta_1,\theta_2)\xi = \eta(\theta_1)\theta_2 + \frac{1}{2n}\eta(\theta_2)Q\theta_1. \tag{3.5}$$

Setting $\theta_1 = \xi$, in (2.18)

$$W_7(\xi,\theta_2)\theta_3 = \eta(\theta_3)\theta_2 - 2g(\theta_2,\theta_3)\xi - \frac{1}{2n}S(\theta_2,\theta_3)\xi, \tag{3.6}$$

and

$$W_7(\xi,\theta_2)\xi = \theta_2 - \eta(\theta_2)\xi. \tag{3.7}$$

The same applies, putting $\theta_3=\xi$ in (2.19) and using (2.8), we have

$$W_8(\theta_1,\theta_2)\xi=\eta(\theta_1)\theta_2+\frac{1}{2n}S(\theta_1,\theta_2)\xi. \tag{3.8}$$

In (3.8), using $\theta_1=\xi$, we get

$$W_8(\xi,\theta_2)\xi=\theta_2-\eta(\theta_2)\xi. \tag{3.9}$$

Choosing $\theta_3=\xi$, in (2.20), we obtain

$$W_9(\theta_1,\theta_2)\xi=\eta(\theta_1)\theta_2-\eta(\theta_2)\theta_1+\frac{1}{2n}(S(\theta_1,\theta_2)\xi-\eta(\theta_2)Q\theta_1). \tag{3.10}$$

In (3.10) it follows

$$W_9(\xi,\theta_2)\xi=\theta_2-\eta(\theta_2)\xi. \tag{3.11}$$

In (2.21), choosing $\theta_3=\xi$ and using (2.8), we obtain

$$W_0^{\star}(\theta_1,\theta_2)\xi=\eta(\theta_1)\theta_2-2\eta(\theta_2)\theta_1-\frac{1}{2n}\eta(\theta_1)Q\theta_2. \tag{3.12}$$

Setting $\theta_1=\xi$, in (3.12)

$$W_0^{\star}(\xi,\theta_2)\xi=\theta_2-2\eta(\theta_2)\xi-\frac{1}{2n}Q\theta_2. \tag{3.13}$$

Theorem 3.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then M is a $W_7\cdot W_5=0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that M is a $W_7\cdot W_5=0$. This implies that

$$\begin{aligned}(W_7(\theta_1,\theta_2)W_5)(\theta_4,\theta_5)\theta_3 =\ & W_7(\theta_1,\theta_2)W_5(\theta_4,\theta_5)\theta_3-W_5(W_7(\theta_1,\theta_2)\theta_4,\theta_5)\theta_3\\ & -W_5(\theta_4,W_7(\theta_1,\theta_2)\theta_5)\theta_3\\ & -W_5(\theta_4,\theta_5)W_7(\theta_1,\theta_2)\theta_3=0,\end{aligned} \tag{3.14}$$

for any $\theta_1,\theta_2,\theta_3,\theta_4,\theta_5\in\Gamma(TM)$. Taking $\theta_1=\theta_3=\xi$ in (3.14), with the usage of (3.6) and (3.7), for $p_1=\frac{1}{2n}$, we have

$$\begin{aligned}(W_7(\xi,\theta_2)W_5)(\theta_4,\theta_5)\xi =\ & W_7(\xi,\theta_2)(2\eta(\theta_4)\theta_5-\eta(\theta_5)\theta_4+p_1\eta(\theta_4)Q\theta_5)\\ & -W_5(\eta(\theta_4)\theta_2)-2g(\theta_2,\theta_4)\xi-p_1S(\theta_2,\theta_4)\xi,\theta_5)\xi\\ & -W_5(\theta_4,\eta(\theta_5)\theta_2-2g(\theta_2,\theta_5)\xi-p_1S(\theta_2,\theta_5)\xi)\xi\\ & -W_5(\theta_4,\theta_5)(\theta_2-\eta(\theta_2)\xi)=0.\end{aligned} \tag{3.15}$$

While considering (3.1), (3.2), (3.6) in (3.15), we obtain

$$\begin{aligned}& -W_5(\theta_4,\theta_5)\theta_2-4\eta(\theta_4)g(\theta_5,\theta_2)\xi-2\eta(\theta_4)S(\theta_5,\theta_2)\xi\\ & +\eta(\theta_5)S(\theta_2,\theta_4)\xi-2np_1\eta(\theta_4)\eta(\theta_5)\theta_2-p_1\eta(\theta_4)S(\theta_2,Q\theta_5)\xi\\ & +2p_1g(\theta_2,\theta_4)Q\theta_5+2p_1S(\theta_2,\theta_4)\theta_5-p_1\eta(\theta_5)S(\theta_2,\theta_4)\xi\\ & -p_1\eta(\theta_4)\eta(\theta_5)Q\theta_2-4g(\theta_2,\theta_5)\theta_4+2\eta(\theta_4)g(\theta_2,\theta_5)\xi\\ & -2p_1g(\theta_2,\theta_5)Q\theta_4-2p_1S(\theta_2,\theta_5)\theta_4-p_1^2S(\theta_2,\theta_5)Q\theta_4=0.\\ & +4g(\theta_2,\theta_4)\theta_5+p_1^2S(\theta_2,\theta_4)Q\theta_5=0.\end{aligned} \tag{3.16}$$

Using the formulas (2.16), (2.4), (2.11), choosing the value $\theta_5=\xi$ for the product that is contained on both sides of (3.16) by $\xi\in\chi(M)$, we arrive

$$\begin{aligned}& [1+p_1-2np_1^2]S(\theta_2,\theta_4)=[1-4+4np_1]g(\theta_2,\theta_4)\\ & +[(2np_1)^2+4n^2p_1-4np_1-8n+5]\eta(\theta_4)\eta(\theta_2)=0.\end{aligned} \tag{3.17}$$

and from (3.17) and using (2.10), we conclude

$$S(\theta_2, \theta_4) = -g(\theta_2, \theta_4) + (8 - 6n)\eta(\theta_2)\eta(\theta_4).$$

M is a -Einstein manifold as a result. On the other hand, consider $M^{2n+1}(\varphi, \xi, \eta, g)$ as $\eta-$Einstein manifold, i.e. $S(\theta_2, \theta_4) = -g(\theta_2, \theta_4) + (8 - 6n)\eta(\theta_2)\eta(\theta_4)$, then from equations (3.17), (3.16), (3.15) and (3.14), we obtain $W_7 \cdot W_5 = 0$. Which verifies our assertion. ■

Theorem 3.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_6 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Let us say M is a $W_7 \cdot W_6 = 0$. This gives way to

$$\begin{aligned}(W_7(\theta_1, \theta_2)W_6)(\theta_4, \theta_5)\theta_3 = {} & W_7(\theta_1, \theta_2)W_6(\theta_4, \theta_5)\theta_3 - W_6(W_7(\theta_1, \theta_2)\theta_4, \theta_5)\theta_3 \\ & -W_6(\theta_4, W_7(\theta_1, \theta_2)\theta_5)\theta_3 \\ & -W_6(\theta_4, \theta_5)W_7(\theta_1, \theta_2)\theta_3 = 0, \end{aligned} \tag{3.18}$$

for any $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Gamma(TM)$. Taking $\theta_1 = \theta_3 = \xi$ in (3.18) and using (3.3), (3.6), (3.7), for $p_1 = -\frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi, \theta_2)W_6)(\theta_4, \theta_5)\xi = {} & W_7(\xi, \theta_2)(\eta(\theta_4)\theta_5 - g(\theta_4, \theta_5)\xi) \\ & -W_6(\eta(\theta_4)\theta_2 - 2g(\theta_4, \theta_2)\xi + p_1 g(\theta_2, \theta_4)\xi, \theta_5)\xi \\ & -W_6(\theta_4, \eta(\theta_5)\theta_2 - 2S(\theta_5, \theta_2)\xi + p_1 g(\theta_2, \theta_5)\xi)\xi \\ & -W_6(\theta_4, \theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0. \end{aligned} \tag{3.19}$$

and we arrive

$$\begin{aligned}& \eta(\theta_4)W_7(\xi, \theta_2)\theta_5 - g(\theta_4, \theta_5)W_7(\xi, \theta_2)\xi - \eta(\theta_4)W_6(\theta_2, \theta_5)\xi \\ & +2g(\theta_2, \theta_4)W_6(\xi, \theta_5)\xi - p_1 S(\theta_4, \theta_2)W_6(\xi, \theta_5)\xi \\ & -\eta(\theta_5)W_6(\theta_4, \theta_2)\xi + 2g(\theta_2, \theta_5)W_6(\theta_4, \xi)\xi \\ & -p_1 S(\theta_5, \theta_2)W_6(\theta_4, \xi)\xi - W_6(\theta_4, \theta_5)\theta_2 + \eta(\theta_2)W_6(\theta_4, \theta_5)\xi = 0. \end{aligned} \tag{3.20}$$

Taking into account that (3.6), (3.4) and (3.3) in (3.20), we get

$$\begin{aligned}& -W_6(\theta_4, \theta_5)\theta_2 - S(\theta_5, \theta_4)\theta_2 + \eta(\theta_4)g(\theta_5, \theta_2)\xi \\ & +2p_4 g(\theta_4, \theta_2)\theta_5 - 2\eta(\theta_5)g(\theta_2, \theta_4)\xi - p_1 S(\theta_2, \theta_4)\theta_5 \\ & +p_1\eta(\theta_5)S(\theta_2, \theta_4)\xi + \eta(\theta_5)g(\theta_2, \theta_4)\xi \\ & -g(\theta_2, \theta_5)\theta_4 + p_1 S(\theta_5, \theta_2)\theta_4 = 0. \end{aligned} \tag{3.21}$$

Putting $\theta_5 = \xi$, using (2.17) and using the inner product on both sides of (3.21) by $\theta_3 \in \chi(M)$, and lastly $\theta_4 = \xi$, we draw a conclusion

$$S(\theta_2, \theta_5) = 2ng(\theta_2, \theta_5) - 4n\eta(\theta_2)\eta(\theta_5).$$

M is therefore $\eta-$Einstein manifold. Let $M^{2n+1}(\varphi, \xi, \eta, g)$ instead be $\eta-$Einstein manifold, i.e. $S(\theta_2, \theta_5) = 2ng(\theta_2, \theta_5) - 4n\eta(\theta_2)\eta(\theta_5)$, then from equations (3.21), (3.20), (3.19) and (3.18), we obtain $W_7 \cdot W_6 = 0$. This completes of the proof. ■

Theorem 3.3. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_7 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Assume that M is a $W_7 \cdot W_7 = 0$. This conforms to

$$\begin{aligned}(W_7(\theta_1,\theta_2)W_7)(\theta_4,\theta_5)\theta_3 = & W_7(\theta_1,\theta_2)W_7(\theta_4,\theta_5)\theta_3 - W_7(W_7(\theta_1,\theta_2)\theta_4,\theta_5)\theta_3 \\ & -W_7(\theta_4,W_7(\theta_1,\theta_2)\theta_5)\theta_3 \\ & -W_7(\theta_4,\theta_5)W_7(\theta_1,\theta_2)\theta_3 = 0, \end{aligned} \tag{3.22}$$

for any $\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_3 \in \Gamma(TM)$. Taking $\theta_1 = \theta_3 = \xi$ in (3.22) and using (3.5), (3.7), (3.6), for $p_1 = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi,\theta_2)W_7)(\theta_4,\theta_5)\xi = & W_7(\xi,\theta_2)(\eta(\theta_4)\theta_5 + p_1\eta(\theta_5)Q\theta_4) \\ & -W_7(\eta(\theta_4)\theta_2 - 2g(\theta_2,\theta_4)\xi - p_1S(\theta_2,\theta_4)\xi,\theta_5)\xi \\ & -W_7(\theta_4,\eta(\theta_5)\theta_2 - 2g(\theta_2,\theta_5)\xi - p_1S(\theta_2,\theta_5)\xi)\xi \\ & -W_7(\theta_4,\theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0. \end{aligned} \tag{3.23}$$

and we have

$$\begin{aligned} & \eta(\theta_4)W_7(\xi,\theta_2)\theta_5 + p_1\eta(\theta_5)W_7(\xi,\theta_2)Q\theta_4 - \eta(\theta_4)W_7(\theta_2,\theta_5)\xi \\ & +2g(\theta_2,\theta_4)W_7(\xi,\theta_5)\xi + p_1S(\theta_2,\theta_4)W_7(\xi,\theta_5)\xi - W_7(\theta_4,\theta_5)\theta_2 \\ & -\eta(\theta_5)W_7(\theta_4,\theta_2)\xi + g(\theta_2,\theta_5)W_7(\theta_4,\xi)\xi + p_1S(\theta_2,\theta_5)W_7(\theta_4,\xi)\xi \\ & +\eta(\theta_2)W_7(\theta_4,\theta_5)\xi = 0. \end{aligned} \tag{3.24}$$

Taking into account that (3.5) and (3.6) in (3.24), we get

$$\begin{aligned} & -W_7(\theta_4,\theta_5)\theta_2 - 2np_1\eta(\theta_5)\eta(\theta_4)\theta_2 - p_1^2\eta(\theta_5)S(Q\theta_4,\theta_2)\xi \\ & -2p_1\eta(\theta_5)S(\theta_4,\theta_2)\xi - p_1\eta(\theta_5)\eta(\theta_4)Q\theta_2 + 2g(\theta_4,\theta_2)\theta_5 \\ & +2g(\theta_2,\theta_4)\eta(\theta_5)\xi + p_1S(\theta_4,\theta_2)\theta_5 - p_1\eta(\theta_5)S(\theta_2,\theta_4)\xi \\ & -2g(\theta_2,\theta_5)\theta_4 - p_1S(\theta_2,\theta_5)\theta_4 = 0. \end{aligned} \tag{3.25}$$

Choosing $\theta_4 = \xi$, making use of (3.5) and inner product both sides of (3.25) by $\theta_3 \in \chi(M)$ and using $\theta_5 = \xi$,we get

$$p_1S(\theta_2,\theta_3) = -2np_1g(\theta_2,\theta_3) + [2np_1 - 4n^2p_1^2 - 1]\eta(\theta_2)\eta(\theta_3) = 0. \tag{3.26}$$

From (3.26) and by using (2.10), we set

$$S(\theta_2,\theta_3) = -2n\left(g(\theta_2,\theta_3) + \eta(\theta_2)\eta(\theta_3)\right).$$

Thus, M is an $\eta-$Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an $\eta-$Einstein manifold, i.e. $S(\theta_2,\theta_3) = -2n\left(g(\theta_2,\theta_3) + \eta(\theta_2)\eta(\theta_3)\right)$, then from equations (3.26), (3.25), (3.24), (3.23) and (3.22) we obtain $W_7 \cdot W_7 = 0$. Which verifies our assertion. ■

Theorem 3.4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_8 = 0$ if and only if M is an $\eta-$Einstein manifold..*

Proof. If M is a $W_7 \cdot W_8 = 0$, that is. As a result,

$$\begin{aligned}(W_7(\theta_1,\theta_2)W_8)(\theta_4,\theta_5)\theta_3 = & W_7(\theta_1,\theta_2)W_8(\theta_4,\theta_5)\theta_3 - W_8(W_7(\theta_1,\theta_2)\theta_4,\theta_5)\theta_3 \\ & -W_8(\theta_4,W_7(\theta_1,\theta_2)\theta_5)\theta_3 \\ & -W_8(\theta_4,\theta_5)W_7(\theta_1,\theta_2)\theta_3 = 0, \end{aligned} \tag{3.27}$$

for any $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Gamma(TM)$. Setting $\theta_1 = \theta_3 = \xi$ in (3.27) and making use of (3.8), (2.8), (2.9), for $p_1 = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi,\theta_2)W_8)(\theta_4,\theta_5)\xi = & W_7(\xi,\theta_2)(\eta(\theta_4)\theta_5 + p_1 S(\theta_4,\theta_5)\xi) \\ & -W_8(\eta(\theta_4)\theta_2 - 2g(\theta_4,\theta_2)\xi - p_1 S(\theta_2,\theta_4)\xi,\theta_5) \\ & -W_8(\theta_4,\eta(\theta_5)\theta_2 - 2g(\theta_5,\theta_2)\xi - p_1 S(\theta_2,\theta_5)\xi)\xi \\ & -W_8(\theta_4,\theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0. \end{aligned} \tag{3.28}$$

Using of (3.8), (3.9), (3.6)and (3.28), we get

$$\begin{aligned} & -W_8(\theta_4,\theta_5)\theta_2 + p_1 S(\theta_5,\theta_4)\theta_2 - p_1\eta(\theta_4)S(\theta_5,\theta_2)\xi \\ & +2g(\theta_2,\theta_4)\theta_5 - 2\eta(\theta_5)S(\theta_2,\theta_4)\xi + p_1 S(\theta_2,\theta_4)\theta_5 \\ & -2p_1\eta(\theta_5)S(\theta_2,\theta_4)\xi - 2g(\theta_5,\theta_2)\theta_4 - p_1 S(\theta_2,\theta_5)\theta_4 = 0. \end{aligned} \tag{3.29}$$

Inner product both sides of (3.29) by $\xi \in \chi(M)$, using $\theta_4 = \xi$ and putting (2.11), we have

$$3p_1 S(\theta_5,\theta_2) = -g(\theta_5,\theta_2) + [-1 - p_1]\eta(\theta_2)\eta(\theta_5) = 0. \tag{3.30}$$

From (3.30) and by using (2.10), we set

$$S(\theta_5,\theta_2) = -\frac{2n}{3}g(\theta_5,\theta_2) - \left(\frac{2n+1}{3}\right)\eta(\theta_5)\eta(\theta_2).$$

M is an $\eta-$Einstein manifold, hence. Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an η-Einstein manifold in contrast, i.e. $S(\theta_5,\theta_2) = -\frac{2n}{3}g(\theta_5,\theta_2) - \left(\frac{2n+1}{3}\right)\eta(\theta_5)\eta(\theta_2)$, then from equations (3.30), (3.29), (3.28) and (3.27), we obtain $W_7 \cdot W_8 = 0$. This completes of the proof. ■

Theorem 3.5. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_9 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Let us say M is a $W_7 \cdot W_9 = 0$. It follows that

$$\begin{aligned}(W_7(\theta_1,\theta_2)W_9)(\theta_4,\theta_5,\theta_3) = & W_7(\theta_1,\theta_2)W_9(\theta_4,\theta_5)\theta_3 - W_9(W_7(\theta_1,\theta_2)\theta_4,\theta_5)\theta_3 \\ & -W_9(\theta_4,W_7(\theta_1,\theta_2)\theta_5)\theta_3 \\ & -W_9(\theta_4,\theta_5)W_7(\theta_1,\theta_2)\theta_3 = 0, \end{aligned} \tag{3.31}$$

for any $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Gamma(TM)$. Setting $\theta_1 = \theta_3 = \xi$ in (3.31) and making use of (3.10), (3.6), for $p_1 = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi,\theta_2)W_9)(\theta_4,\theta_5)\xi = & W_7(\xi,\theta_2)(\eta(\theta_4)\theta_5 - \eta(\theta_5)\theta_4 + p_1 S(\theta_4,\theta_5)\xi \\ & -p_1\eta(\theta_5)Q\theta_4) - W_9(\eta(\theta_4)\theta_2 - 2g(\theta_4,\theta_2)\xi \\ & -p_1 S(\theta_2,\theta_4)\xi,\theta_5)\xi - W_9(\theta_4,\eta(\theta_5)\theta_2 - 2g(\theta_5,\theta_2)\xi \\ & -p_1 g(\theta_2,\theta_5)\xi)\xi - W_9(\theta_4,\theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0. \end{aligned} \tag{3.32}$$

Using (3.6) and (3.11) in (3.32), we get

$$\begin{aligned} & -W_9(\theta_4,\theta_5)\theta_2 + 2\eta(\theta_5)g(\theta_4,\theta_2)\xi + p_1 S(\theta_4,\theta_5)\theta_2 + 2np_1\eta(\theta_5)\eta(\theta_4)Q\theta_5 \\ & +p_1^2\eta(\theta_5)S(\theta_2,Q\theta_4)\xi + 2g(\theta_2,\theta_4)\theta_5 - 2g(\theta_4,\theta_2)\eta(\theta_5)\xi \\ & +p_1 S(\theta_4,\theta_2)\theta_5 + p_1\eta(\theta_5)S(\theta_2,\theta_4)\xi - 2g(\theta_2,\theta_5)\theta_4 \\ & -p_1 S(\theta_2,\theta_5)\theta_4 - p_1\eta(\theta_4)S(\theta_2,\theta_5)\xi + p_1\eta(\theta_5)\eta(\theta_4)Q\theta_2 = 0. \end{aligned} \tag{3.33}$$

Utilizing (2.20), picking $\theta_4 = \xi$ and the inner product on both sides of (3.33) by $\xi \in \chi(M)$, we have

$$2p_1 S(\theta_2, \theta_5) = -2g(\theta_2, \theta_5) + [4n^2 p_1^2 - 4n^2 p_1 - 8np_1 + 2]\eta(\theta_5)\eta(\theta_2) \tag{3.34}$$

from which, we conclude

$$S(\theta_2, \theta_5) = -2ng(\theta_2, \theta_5) - (1 + 2n)\eta(\theta_2)\eta(\theta_5).$$

As a result, M is an η-Einstein manifold. On the other hand, consider $M^{2n+1}(\varphi, \xi, \eta, g)$ as an η-Einstein manifold, i.e. $S(\theta_2, \theta_5) = -2ng(\theta_2, \theta_5) - (1 + 2n)\eta(\theta_2)\eta(\theta_5)$, then from equations (3.34), (3.33), (3.32) and (3.31), we obtain $W_7 \cdot W_9 = 0$. Which verifies our assertion. ■

Theorem 3.6. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_0^\star = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Consider M to be a $W_7 \cdot W_0^\star = 0$. This means that

$$\begin{aligned}(W_7(\theta_1, \theta_2)W_0^\star)(\theta_4, \theta_5, \theta_3) = {} & W_7(\theta_1, \theta_2)W_0^\star(\theta_4, \theta_5)\theta_3 - W_0^\star(W_7(\theta_1, \theta_2)\theta_4, \theta_5)\theta_3 \\ & - W_0^\star(\theta_4, W_7(\theta_1, \theta_2)\theta_5)\theta_3 \\ & - W_0^\star(\theta_4, \theta_5)W_7(\theta_1, \theta_2)\theta_3 = 0,\end{aligned} \tag{3.35}$$

for any $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Gamma(TM)$. Setting $\theta_1 = \theta_3 = \xi$ in (3.35) and making use of (3.12), (3.6), (3.7), for $p_1 = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi, \theta_2)W_0^\star)(\theta_4, \theta_5)\xi = {} & W_7(\xi, \theta_2)(\eta(\theta_4)\theta_5 - 2\eta(\theta_5)\theta_4 - p_1\eta(\theta_4)Q\theta_5) \\ & - W_0^\star(\eta(\theta_4)\theta_2 - 2g(\theta_2, \theta_4)\xi - p_1 S(\theta_2, \theta_4)\xi, \theta_5)\xi \\ & - W_0^\star(\theta_4, \eta(\theta_5)\theta_2 - 2g(\theta_2, \theta_5)\xi - p_1 S(\theta_2, \theta_5)\xi)\xi \\ & - W_0^\star(\theta_4, \theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0.\end{aligned} \tag{3.36}$$

Using (3.12) and (3.13) in (3.36), we get

$$\begin{aligned}& -W_0^\star(\theta_4, \theta_5)\theta_2 - 2\eta(\theta_4)g(\theta_2, \theta_5)\xi + 2\eta(\theta_5)g(\theta_2, \theta_4)\xi + 2np_1\eta(\theta_4)\eta(\theta_5)\theta_2 \\ & - p_1^2\eta(\theta_4)S(\theta_2, Q\theta_5)\xi - p_1\eta(\theta_4)\eta(\theta_2)Q\theta_5 + 2g(\theta_2, \theta_4)\theta_5 - 4g(\theta_2, \theta_4)\eta(\theta_5) \\ & + p_1 S(\theta_2, \theta_4)\theta_5 - p_1^2 S(\theta_2, \theta_4)Q\theta_5 + p_1(\theta_4)\eta(\theta_5)Q\theta_2 - 2g(\theta_2, \theta_5)\theta_4 \\ & + 4\eta(\theta_4)g(\theta_2, \theta_5)\xi + 2p_1 g(\theta_2, \theta_5)Q\theta_4 - p_1 S(\theta_2, \theta_5)\theta_4 + 2p_1\eta(\theta_4)S(\theta_2, \theta_5)\xi \\ & + p_1^2 S(\theta_2, \theta_5)Q\theta_4 - p_1(\theta_4)\eta(\theta_2)Q\theta_5 - 2p_1 g(\theta_2, \theta_4)Q\theta_5 = 0.\end{aligned} \tag{3.37}$$

Making use of (2.21), using $\theta_2 = \theta_4 = \xi$ and inner product both sides of (3.37) by $\theta_3 \in \chi(M)$, we have

$$2np_1^2 S(\theta_3, \theta_5) = -g(\theta_3, \theta_5) + [5 - 4n^2 p_1^2 + 2n - 4n^2 p_1]\eta(\theta_3)\eta(\theta_5). \tag{3.38}$$

Finally, from (2.10) and (3.38), we arrive

$$S(\theta_3, \theta_5) = -2ng(\theta_3, \theta_5) + 8n\eta(\theta_3)\eta(\theta_5).$$

This indicates that M is an $\eta-$Einstein manifold. Let M instead be an η-Einstein manifold, i.e. $S(\theta_3, \theta_5) = -2ng(\theta_3, \theta_5) + 8n\eta(\theta_3)\eta(\theta_5)$, then from (3.38), (3.37), (3.36) and (3.35), we have $W_7 \cdot W_0^\star = 0$. This completes of the proof. ■

Conclusion 3.7. *Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4, Theorem 3.5 and Theorem 3.6 than we have. Assume that $M^{2n+1}(\varphi, \xi, \eta, g)$ is a Kenmotsu manifold. M is thus $W_7 \cdot W_5 = 0$, $W_7 \cdot W_6 = 0$, $W_7 \cdot W_7 = 0$, $W_7 \cdot W_8 = 0$, $W_7 \cdot W_9 = 0$ and $W_7 \cdot W_0^{\star} = 0$ if and only if M is an $\eta-$Einstein manifold.*

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On the dissipative conformable fractional singular Sturm-Liouville operator

Yüksel YALÇINKAYA *[1], **Bilender P. ALLAHVERDİEV**[2] **and Hüseyin TUNA**[3]

[1] *Faculty of Science, Department of Mathematics, Süleyman Demirel University, Isparta, Turkey.*
[2] *Department of Mathematics, Khazar University, AZ1096 Baku, Azerbaijan and Research Center of Econophysics,UNEC-Azerbaijan State University of Economics,Baku, Azerbaijan.*
[3] *Faculty of Science, Department of Mathematics, Akif Ersoy University, Burdur, Turkey.*

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Abstract. In this study, a dissipative conformable fractional singular Sturm–Liouville operator is studied. For this operator, a completeness theorem is proved by Krein's theorem.

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Contents

1. Introduction and Background

In recent years, Khalil and his friends ([9]) defined conformable fractional derivative as

$$T_\alpha u(\zeta) = \lim_{\varepsilon \to \infty} \frac{u\left(\zeta + \varepsilon \zeta^{1-\alpha}\right) - u\left(\zeta\right)}{\varepsilon}, \tag{1.1}$$

where $0 < \alpha < 1$ and $u : (0, \infty) \longrightarrow \mathbb{R} : =(-\infty, \infty)$ is a function. Conformable fractional derivative aims to expand the derivative definition as known by providing the natural characteristics of classical derivative and to pain new perspectives for differential equation theory with the help of conformable differential equations obtained as using this derivative definition ([10]). Later in ([1]), Abdeljawad defined the right and left conformable fractional derivatives, the fractional chain rule and fractional integrals of higher orders.

In [2], the authors studied a conformable fractional Sturm–Liouville (CFSL) problem. In [4], Belanau et al. conctructed Weyl's theory for the conformable sequential equation with distributional potentials.

In this paper, using Krein's theorems, we prove that the system of all eigenvectors and associated vectors of dissipative CFSL operator is complete.

Now, some preliminary concepts related to conformable fractional calculus and the essentials of Krein's theorem are given.

*[**Corresponding author.**] Email addresses: **matyuksel@hotmail.com** (Yüksel YALÇINKAYA), **bilenderpasaoglu@sdu.edu.tr** (Bilender P. ALLAHVERDİEV), **hustuna@gmail.com** (Hüseyin TUNA)

Definition 1.1 ([1]). *The conformable fractional entire is given by*

$$(I_\alpha u)(\zeta) = \int_0^\zeta s^{\alpha-1} u(s) ds = \int_0^\zeta u(s) d_\alpha s.$$

Let

$$L_\alpha^2(I) = \left\{ z : \left(\int_0^a |z(\zeta)|^2 d_\alpha \zeta \right)^{1/2} < \infty \right\},$$

where $I = [0, a)$ and $0 < a < \infty$. $L_\alpha^2(I)$ is a Hilbert space with the inner product

$$\langle u, z \rangle := \int_0^a u(\zeta) \overline{z(\zeta)} d_\alpha \zeta, \text{ where } u, z \in L_\alpha^2(I).$$

Theorem 1.2 (Krein [7]). *The system of root vectors of a compact dissipative operator B with nuclear imaginary component is complete in the Hilbert space H so long as at least one of the following two conditions is fulfilled:*

$$\lim_{\sigma\to\infty} \frac{n_+(\sigma, B_R)}{\sigma} = 0, \text{ or } \lim_{\sigma\to\infty} \frac{n_-(\sigma, B_R)}{\sigma} = 0,$$

where $n_+(\sigma, B_R)$ and $n_-(\sigma, B_R)$ denote the numbers of the characteristic values of the real component B_R of the operator B in the intervals $[0, \sigma]$ and $[-\sigma, 0]$, respectively.

Definition 1.3. *Let Ξ be an entire function. If for each $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$, such that*

$$|\Xi(\mu)| \le C_\varepsilon e^{\varepsilon|\mu|}, \quad \mu \in \mathbb{C} \tag{1.2}$$

then Ξ is called an entire function of the order ≤ 1 of growth and minimal type ([7]).

Theorem 1.4 ([11]). *If the entire function Ξ satisfies (1.2), then*

$$\lim_{\sigma\to\infty} \frac{n_+(\sigma, \Xi)}{\sigma} = \lim_{\sigma\to\infty} \frac{n_-(\sigma, \Xi)}{\sigma} = 0,$$

where $n_+(\sigma, \Xi)$ and $n_-(\sigma, \Xi)$ denote the numbers of the zeros of Ξ in the intervals $[0, \sigma]$ and $[-\sigma, 0]$, respectively.

2. Main Results

Consider the following singular problem

$$l[z] = -T_\alpha^2 z(\zeta) + q(\zeta) z(\zeta) = \mu y, \ \zeta \in I = [0, a), \tag{2.1}$$

where q is a real-valued function on I, $q \in L^1_{\alpha,loc}(I)$, and a is a singular point.

The maximal operator is given by

$$L_{\max} z := l[z],$$

where

$$D_{\max} := \left\{ z \in L_\alpha^2(I) : z, T_\alpha z \in AC_{\alpha,loc}(I), \ l[z] \in L_\alpha^2(I) \right\}.$$

Green's formula [2] is defined as

$$\int_0^a l[z_1](\zeta) \overline{z_2(\zeta)} d_\alpha \zeta - \int_0^a z_1(\zeta) \overline{l[z_2](\zeta)} d_\alpha \zeta = [z_1, z_2](a) - [z_1, z_2](0), \tag{2.2}$$

where $z_1, z_2 \in D_{\max}$ and

$$[z_1, z_2](\zeta) = z_1(\zeta)\overline{T_\alpha z_2(\zeta)} - T_\alpha z_1(\zeta)\overline{z_2(\zeta)} = W\left(z_1, \overline{z_2}\right).$$

Set

$$D_{\min} := \left\{z \in D_{\max} : z\left(0\right) = T_\alpha z\left(0\right) = 0,\ [z, \chi](a) = 0\right\},$$

for arbitrary $\chi \in D_{\max}$.The minimal *operator* $L_{\min}$ is the restriction of $L_{\max}$ to $D_{\min}$ and $L_{\max} = L^*_{\min}$ ([4, 6, 12, 15]).

In this paper, we will assume that $L_{\min}$ has the deficiency indices $(2, 2)$.

We will denote by $\phi\left(\zeta, \mu\right), \psi\left(\zeta, \mu\right)$ two linearly independent solutions of Eq. (2.1) satisfying the following conditions

$$\begin{aligned} &\phi\left(0, \mu\right) = \cos\beta,\ T_\alpha\phi\left(0, \mu\right) = \sin\beta, \\ &\psi\left(0, \mu\right) = -\sin\beta,\ T_\alpha\psi\left(0, \mu\right) = \cos\beta, \end{aligned} \tag{2.3}$$

where $\beta \in \mathbb{R}$. $\phi\left(\zeta, \mu\right)$ and $\psi\left(\zeta, \mu\right)$ are entire functions of μ ([2]). Due to $L_{\min}$ has the deficiency indices $(2, 2)$, $\phi\left(\zeta, \mu\right), \psi\left(\zeta, \mu\right) \in L^2_\alpha(I)$.

Let $r(\zeta) = \phi\left(\zeta, 0\right)$ and $v(\zeta) = \psi(\zeta, 0)$. Then we have

$$\begin{aligned} &r\left(0\right) = \cos\beta,\ T_\alpha r\left(0\right) = \sin\beta, \\ &v\left(0\right) = -\sin\beta,\ T_\alpha v\left(0\right) = \cos\beta. \end{aligned} \tag{2.4}$$

Clearly, $r, v \in L^2_\alpha(I)$ and $r, v \in D_{\max}$.

Let

$$D(L) = \left\{z \in D_{\max} : \begin{array}{c} z\left(0\right)\cos\beta + T_\alpha z(0)\sin\beta = 0, \\ \left[z, r\right]\left(a\right) - h\left[z, v\right]\left(a\right) = 0 \end{array}\right\}, \tag{2.5}$$

where $h \in \mathbb{C}$ and $\operatorname{Im} h > 0$. Then, for all $z \in D(L)$, the operator L is defined by $Ly = l[z]$.

Theorem 2.1. *L is a dissipative operator.*

Proof. Let $z \in D(L)$. From (2.2), we find

$$\langle Lz, z\rangle - \langle z, Lz\rangle = \left[z, z\right]\left(a\right) - \left[z, z\right]\left(0\right). \tag{2.6}$$

By (2.5), we obtain

$$\left[r, z\right]\left(a\right) = -\overline{h\left[z, v\right]}\left(a\right)$$

and

$$\left[z, z\right]\left(0\right) = 0.$$

Since

$$\left[z_1, z_2\right]\left(\zeta\right) = \left[z_1, v\right]\left(\zeta\right)\left[r, z_2\right]\left(\zeta\right) - \left[z_1, r\right]\left(\zeta\right)\left[v, z_2\right]\left(\zeta\right),\ \zeta \in I, \tag{2.7}$$

where $z_1, z_2 \in D_{\max}$, we see that

$$\begin{aligned} \left[z, z\right]\left(a\right) &= \left[z, v\right]\left(a\right)\left[r, z\right]\left(a\right) - \left[z, r\right]\left(a\right)\left[v, z\right]\left(a\right) \\ &= -\overline{h}|\left[z, v\right]\left(a\right)|^2 + h|\left[z, v\right]\left(a\right)|^2 \\ &= 2i(\operatorname{Im} h)|\left[z, v\right]\left(a\right)|^2. \end{aligned}$$

Hence

$$\operatorname{Im}\langle Lz, z\rangle = (\operatorname{Im} h)|\left[z, v\right]\left(a\right)|^2 \geq 0. \tag{2.8}$$

■

Corollary 2.2. *Since the operator L is dissipative, all eigenvalues of L lie in the closed upper half-plane* $\operatorname{Im}\mu \geq 0$.

Theorem 2.3. *L has no real eigenvalue.*

Proof. Assume that μ_0 is a real eigenvalue of L and $\psi_0 = \psi_0(\zeta, \mu_0)$ is a corresponding eigenfunction. Due to

$$\operatorname{Im}(L\psi_0, \psi_0) = \operatorname{Im}(\mu_0, ||\psi_0||^2) = 0,$$

and we see that $[\psi_0, v]\,(a) = 0$. From (2.5), we obtain $[\psi_0, r]\,(a) = 0$. By (2.7), we conclude that

$$1 = W_0(\phi_0, \psi_0) = W_a(\phi_0, \psi_0) = [\phi_0, \overline{\psi_0}]\,(a)$$

$$= [\phi_0, v]\,(a)\,[r, \overline{\psi_0}]\,(a) - [\phi_0, r]\,(a)\,[\overline{v}, \psi_0]\,(a) = 0,$$

a contradiction. ■

Theorem 2.4 ([3]). *Every nontrivial solution z of Eq.(2.1) and $T_\alpha z$ are entire functions of μ of the order at most $\frac{1}{2}$ in the interval $[0, c]$, $c < a$.*

Let

$$\Theta_1(\mu) = [\psi(\zeta, \mu), r(\zeta)]\,(a)\,,$$

$$\Theta_2(\mu) = [\psi(\zeta, \mu), v(\zeta)]\,(a)\,,$$

where $\psi(\zeta, \mu)$ is the solution of Eq.(2.1). Clearly,

$$\sigma_p(L) = \{\mu \in \mathbb{C} : \ \Theta(\mu) = 0\}\,,$$

where

$$\Theta(\mu) = \Theta_1(\mu) - hc_2(\mu), \tag{2.9}$$

and $\sigma_p(L)$ is the point spectrum of L.

Theorem 2.5. *The functions $\Theta_1(\mu)$ and $\Theta_2(\mu)$ are entire functions of order ≤ 1 of growth and minimal type.*

Proof. Let

$$\Theta_{a_k,1}(\mu) = [\psi(\zeta, \mu), r(\zeta)]\,(a_k)\,,$$

$$\Theta_{a_k,2}(\mu) = [\psi(\zeta, \mu), v(\zeta)]\,(a_k)\,,$$

where $a_k \in I$.

By Theorem 8, $\psi(a_k, \mu)$ and $T_\alpha\psi(a_k, \mu)$ are entire functions of order $\frac{1}{2}$ for arbitrary fixed a_k. Consequently, $\Theta_{a_k,1}(\mu)$ and $\Theta_{a_k,2}(\mu)$ are entire functions of order $\frac{1}{2}$.

If we define

$$\Xi_1(\zeta, \mu) = [z, r]\,(\zeta)\,,$$

$$\Xi_2(\zeta, \mu) = [z, v]\,(\zeta)\,,$$

then we see that $\Xi_1(\zeta, \mu)$ and $\Xi_2(\zeta, \mu)$ satisfy the following system

$$T_{\alpha,\zeta}\Xi_1(\zeta, \mu) = \lambda z(\zeta, \mu)r(\zeta), \ T_{\alpha,\zeta}\Xi_2(\zeta, \mu) = \lambda z(\zeta, \mu)v(\zeta), \ \zeta \in I. \tag{2.10}$$

Using (2.10), we deduce that

$$T_{\alpha,\zeta}\Xi(\zeta, \mu) = \lambda\Omega(\zeta)\Xi(\zeta, \mu), \ \ \zeta \in I. \tag{2.11}$$

$$T_{\alpha,\zeta}\Xi(\zeta, \mu) = T_{\alpha,\zeta}\begin{bmatrix}\Xi_1(\zeta, \mu)\\ \Xi_2(\zeta, \mu)\end{bmatrix} = \begin{bmatrix}\lambda z(\zeta, \mu)r(\zeta)\\ \lambda z(\zeta, \mu)v(\zeta)\end{bmatrix}$$

$$= \mu \begin{bmatrix} [z,v]r^2 - [z,r]vr \\ [z,v]vr - [z,v]v^2 \end{bmatrix} = \mu \begin{bmatrix} \Xi_2 r^2 - \Xi_1 vr \\ \Xi_2 vr - \Xi_1 v^2 \end{bmatrix}$$

$$= \mu \begin{bmatrix} -vr & r^2 \\ -v^2 & rv \end{bmatrix} \begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix},$$

where

$$\Xi(\zeta,\mu) = \begin{bmatrix} \Xi_1(\zeta,\mu) \\ \Xi_2(\zeta,\mu) \end{bmatrix},\ \Omega(\zeta) = \begin{bmatrix} -r(\zeta)v(\zeta) & r^2(\zeta) \\ -v^2(\zeta) & r(\zeta)v(\zeta) \end{bmatrix},$$

and the elements $\Omega(\zeta)$ are in $L^1_\alpha(I)$. For

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

we put $\|w\| = |w_1| + |w_2|$. The inclusion $\|\Omega(\zeta)\| \in L^1_\alpha(I)$ holds.

If $z(\zeta,\mu) = \psi(\zeta,\mu)$, then (2.11) is equivalent to the following equation

$$\Xi(\zeta,\mu) = \Xi(a_k,\mu) + \mu \int_{a_k}^{\zeta} \Omega(s)\Xi(s,\mu) d_\alpha s,\ \ \zeta \in I, \tag{2.12}$$

where

$$\Xi(a_k,\mu) = \begin{bmatrix} \Xi_1(a_k,\mu) \\ \Xi_2(a_k,\mu) \end{bmatrix} = \begin{bmatrix} [z,r]\,(a_k) \\ [z,v]\,(a_k) \end{bmatrix} = \begin{bmatrix} \Theta_{a_k,1}(\mu) \\ \Theta_{a_k,2}(\mu) \end{bmatrix},$$

$$\Xi(0,\mu) = \begin{bmatrix} \Xi_1(0,\mu) \\ \Xi_2(0,\mu) \end{bmatrix} = \begin{bmatrix} [z,r]\,(0) \\ [z,v]\,(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

$$\Xi(a,\mu) = \begin{pmatrix} \Theta_1(\mu) \\ \Theta_2(\mu) \end{pmatrix}.$$

Using Gronwall's inequality in (2.12), we find

$$\|\Xi(\zeta,\mu)\| \le \|\Xi(a_k,\mu)\| \exp\left(|\mu| \int_{a_k}^{\zeta} \|\Omega(s)\|\, d_\alpha s \right);$$

hence

$$\|\Xi(a,\mu) - \Xi(a_k,\mu)\| \le |\mu| \exp\left\{ |\mu| \int_0^a \|\Omega(s)\|\, ds \right\} \int_{a_k}^{a} \|\Omega(s)\|\, d_\alpha s, \tag{2.13}$$

$$\|\Xi(a,\mu)\| \le \exp\left(|\mu| \int_{a_k}^{a} \|\Omega(s)\|\, d_\alpha s \right) \|\Xi(a_k,\mu)\|\,. \tag{2.14}$$

(2.13) shows that $\Theta_{a_k,j}(\mu) \to \Theta_j(\mu)$ as $a_k \to a$, uniformly in μ in a compact set. Thus, $\Theta_{,j}(\mu),\ j = 1,2$, are entire functions.

By (2.14), we obtain

$$\|\Xi(a,\mu)\| \le \exp\left(|\mu| \int_0^a \|\Omega(x)\|\, d_\alpha x \right).$$

Therefore $\Theta_{,j}(\mu)$ are of not higher than the first order. By (2.14), we see that $\Theta_{,j}(\mu),\ j = 1,2$, are of minimal type. ■

From (2.2), we conclude that

$$\Theta_1(\mu) = [\psi(\zeta,\mu), r(\zeta)]\,(a) = -1 + \mu \int_0^a \psi(\zeta,\mu) r(\zeta) d_\alpha \zeta \tag{2.15}$$

$$\Theta_2(\mu) = [\psi(\zeta,\mu), v(\zeta)]\,(a) = \mu \int_0^a \psi(\zeta,\mu) v(\zeta) d_\alpha \zeta. \tag{2.16}$$

From (2.9), (2.15) and (2.16) we find that $\Theta(0) = -1$.

By Theorem 7, the inverse operator L^{-1} exists. Now we obtain L^{-1}.

Define $u(\zeta) = r(\zeta) - hv(\zeta)$. Clearly, we see that $v, u \in L^2_\alpha(I)$ and $W(v,u) = -1$.

Let

$$Y\Xi = \int_0^a G(\zeta,t)\Xi(x) d_\alpha t, \tag{2.17}$$

where $\Xi \in L^2_\alpha(I)$ and

$$G(\zeta,t) = \begin{cases} v(\zeta)u(t), & 0 \le \zeta \le t \\ v(t)u(\zeta) & t \le \zeta < a. \end{cases} \tag{2.18}$$

Since

$$\int_0^a \int_0^a |G(\zeta,t)|^2 \, d_\alpha t d_\alpha t < \infty,$$

we see that $Y = L^{-1}$ and the operator Y is Hilbert–Schmidt([2]). Therefore, the root lineals of L and Y coincide and the completeness of the system of all eigen- and associated functionss of L is equivalent to the completeness of those for Y.

Each eigenvector of L may have only a finite number of linear independent associated vectors due to the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite.

Theorem 2.6. *The system of all root vectors of L (also of Y) is complete in $L^2_\alpha(I)$.*

Proof. Due to $u(\zeta) = r(\zeta) - hv(\zeta)$, setting $h = h_1 + ih_2$ we get from (2.17) in view of (2.18) that $Y = Y_1 + iY_2$, where

$$Y_1\Xi = \langle G_1(t,\zeta), \overline{\Xi(\zeta)} \rangle,$$

$$Y_2\Xi = \langle G_2(t,\zeta), \overline{\Xi(\zeta)} \rangle$$

and

$$G_1(t,\zeta) = \begin{cases} v(t)\,[u(\zeta) - h_1 v(\zeta)], \; 0 \le \zeta \le t \le a, \\ v(\zeta)\,[u(t) - h_1 v(t)], \; 0 \le \zeta \le t \le a, \end{cases}$$

$$G_2(t,\zeta) = -h_2 v(t)\, v(\zeta), \; h_2 = \operatorname{Im} h > 0.$$

Y_1 is the self-adjoint Hilbert–Schmidt operator in $L^2_\alpha(I)$, and Y_2 is the self-adjoint one-dimensional operator in $L^2_\alpha(I)$.

Let us denote by L_1 the operator in $L^2_\alpha(I)$ generated by $l[z]$ and the following conditions

$$z(0)\cos\alpha + T_\alpha z(0)\sin\alpha = 0,$$

$$[z,r]\,(a) - h_1[z,v]\,(a) = 0,$$

where $h_1 = \operatorname{Re} h$. It is obviously that Y_1 is the inverse of the operator L_1. Let

$$\rho_p(L_1) = \{\mu : \mu \in \mathbb{C}, \; \Psi(\mu) = 0\}, \tag{2.19}$$

where

$$\Psi(\mu) := \Theta_1(\mu) - h_1\Theta_2(\mu). \tag{2.20}$$

Thus we obtain

$$|\Psi(\mu)| \leq C_\varepsilon e^{\varepsilon|\mu|},\ \forall \mu \in \mathbb{C}. \tag{2.21}$$

Let $Z = -Y$ and $Z = Z_1 + iZ_2$, where $Z_1 = -Y_1,\ Z_2 = -Y_2$. It follows from (2.19), (2.21) and Theorem 5 that

$$\lim_{\sigma\to\infty} \frac{m_+(\sigma, Z_1)}{\sigma} = 0 \text{ or } \lim_{\sigma\to\infty} \frac{m_-(\sigma, Z_1)}{\sigma} = 0,$$

where $m_+(\sigma, Z_1)$ and $m_-(\sigma, Z_1)$ denote the numbers of the characteristic values of the real component $Z_R = Z_1$ in the intervals $[0, \sigma]$ and $[-\sigma, 0]$, respectively. Thus the dissipative operator Z (also of Y) carries out all the conditions of Krein's theorem on completeness. ■

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Graceful and felicitous labeling of Stem-Lotus graph

A. ELUMALAI[*1]

[1] *Department of Mathematics, R.M.K Engineering College (An Autonomous Institution), Gummidipoondi-601206, Tamil Nadu, India.*

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Abstract. A graph obtained from a shell graph $C(2n+3, 2n)$ where $n \geq 1$ by adding a vertex in between each pair of adjacent vertices on the cycle, adding an edge in apex and two more chords is called a Stem-Lotus graph. In this paper, I have proved that a Stem-Lotus graph is graceful and also felicitous.

AMS Subject Classifications: 05C78, 05C38.

Keywords: Graceful labeling, Felicitous labeling, Stem-Lotus graph.

Contents

1. Introduction

We begin with simple, finite, undirected and connected graph $G = (p, q)$. An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, 2, \ldots, q\}$ is called graceful, if the edge labels induced by $|f(x) - f(y)|$ for each edge xy are distinct. A graph which admits graceful labeling is called graceful graph. An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, 2, \ldots, q\}$ is called felicitous, if the edge labels induced by $(f(x) + f(y))(\text{mod } q)$ for each edge xy are distinct. A graph which admits felicitous labeling is called felicitous graph.

From the excellent survey of Gallian Dec 2, 2022, one can find many families of cycle related graphs, on which Shell graph is an important family. Deb & Limaye (2002) have defined a shell graph as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. Shell graphs are denoted as $C(n, n-3)$. In this paper, a new family of graph called a Stem-Lotus graph is introduced by adding a vertex in between each pair of adjacent vertices on the cycle of a shell graph. Here I have proved that Stem-Lotus graph is graceful and felicitous.

A graph obtained from a shell graph $C(2n+3, 2n)$ where $n \geq 1$ by adding a vertex in between each pair of adjacent vertices on the cycle, adding an edge in apex and two more chords is called a Stem-Lotus graph. So clearly a Stem-Lotus graph must have a pollen grain (C_4), a stem (P_1), two leafs (each P_3) and some pairs of petals (each petal is P_3) as displayed in Figure 1.

[*]**Corresponding author.** Email address: **aeseven2000@gmail.com** (A. Elumalai)

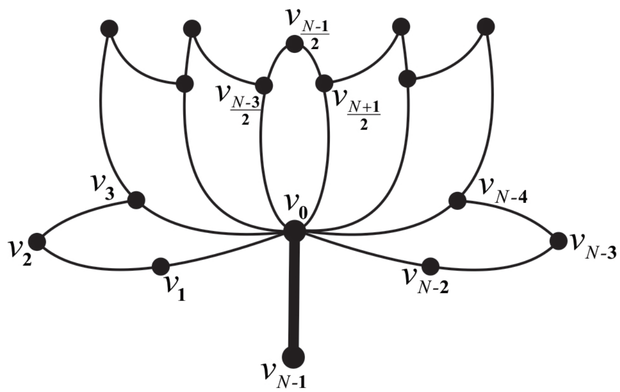

Figure 1: Stem-Lotus graph

In this direction, I have proved that Stem-Lotus graph with n-pairs of petals is graceful and felicitous for all $n \geq 1$.

2. Stem-Lotus graph with n-pairs of petals is graceful and felicitous

In this section, I have proved that Stem-Lotus graph with n-pairs of petals is graceful and felicitous for all $n \geq 1$.

Theorem 2.1. *A Stem-Lotus graph with n-pairs of petals is graceful for all $n \geq 1$.*

Proof. Let G be a Stem-Lotus graph as displayed in Figure 1. It is observed that, there are totally $M = 6n + 11$ edges and $N = 4n + 9$ vertices for the graph G. Let $v_0, v_1, v_2, \ldots, v_{N-1}$ be the N vertices of G.

Define the vertex labeling of G as follows,

$$\begin{aligned} f(v_0) &= 0 \\ f(v_{N-1}) &= \frac{3N-5}{2} \\ f(v_{2i}) &= \frac{N-3+2i}{2}, 1 \leq i \leq \frac{N-3}{2} \\ f(v_{2i-1}) &= \frac{3N-5-2i}{2}, 1 \leq i \leq \frac{N-1}{2} \end{aligned}$$

From the above vertex labeling, the set $\left\{f(v_{2i})/1 \leq i \leq \frac{N-3}{2}\right\}$ form a monotonically increasing sequence and the sets $\{f(v_{N-1})\}$ and $\left\{f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2}\right\}$ form a monotonically decreasing sequence. It is observed that,

$$\max\left\{f(v_{2i})/1 \leq i \leq \frac{N-3}{2}\right\} < \min\left\{\{f(v_{N-1})\} \cup \left\{f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2}\right\}\right\}.$$

Therefore the labels of all vertices of G are distinct.

Now the edge values of G in the following manner,

- Let A_1 denote the edge $\{v_0 v_{N-1}\}$ of G.
- Let A_2 denote the set of $\frac{M+1}{3}$ edges $\{v_0v_1, v_0v_3, v_0v_5, \ldots, v_0v_{N-4}, v_0v_{N-2}\}$ of G.
- Let A_3 denote the set of $\frac{2M-4}{3}$ edges $\{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{N-4}v_{N-3}, v_{N-3}v_{N-2}\}$ of G.

Let A_1', A_2' & A_3' be the sets of observed edge values of the sets A_1, A_2 & A_3.

$$A_1' = \{M\}$$
$$A_2' = \left\{M-1, M-2, \ldots, \frac{2M-1}{3}\right\}$$
$$A_3' = \left\{\frac{2M-4}{3}, \frac{2M-7}{3}, \ldots, 3, 2, 1\right\}$$

It is observed that, $A_1' \cup A_2' \cup A_3' = \{M, M-1, \ldots, 3, 2, 1\}$ and the values in the sets A_1', A_2' & A_3' are all distinct, hence G is graceful. ■

Theorem 2.2. *A Stem-Lotus graph with n-pairs of petals is felicitous for all $n \geq 1$.*

Proof. Let G be a Stem-Lotus graph as displayed in Figure 1. It is observed that, there are totally $M = 6n + 11$ edges and $N = 4n + 9$ vertices for the graph G. Let $v_0, v_1, v_2, \ldots, v_{N-1}$ be the N vertices of G. Define the vertex labeling of G as follows,

$$f(v_0) = 0$$
$$f(v_{N-1}) = N-2$$
$$f(v_{2i}) = \frac{N-3+2i}{2}, 1 \leq i \leq \frac{N-3}{2}$$
$$f(v_{2i-1}) = N-2+i, 1 \leq i \leq \frac{N-1}{2}$$

From the above vertex labeling, the set $\left\{f(v_{2i})/1 \leq i \leq \frac{N-3}{2}\right\}$ form a monotonically increasing sequence and the sets $\{f(v_{N-1})\}$ and $\left\{f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2}\right\}$ form a monotonically decreasing sequence.

It is observed that,

$$\max\left\{f(v_{2i})/1 \leq i \leq \frac{N-3}{2}\right\} < \min\left\{\{f(v_{N-1})\} \cup \left\{f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2}\right\}\right\}.$$

Therefore the labels of all vertices of G are distinct.

Now the edge values of G in the following manner,

- Let A_1 denote the edge $\{v_0 v_{N-1}\}$ of G.
- Let A_2 denote the set of $\frac{M+1}{3}$ edges $\{v_0v_1, v_0v_3, v_0v_5, \ldots, v_0v_{N-4}, v_0v_{N-2}\}$ of G.
- Let A_3 denote the set of $\frac{2M-4}{3}$ edges $\{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{N-4}v_{N-3}, v_{N-3}v_{N-2}\}$ of G.

Let A_1', A_2' & A_3' be the sets of observed edge values of the sets A_1, A_2 & A_3.

$$A_1' = \left\{\frac{2M-1}{3}\right\}$$
$$A_2' = \left\{\frac{2M+2}{3}, \frac{2M+5}{3}, \ldots, M-1, 0\right\}$$
$$A_3' = \left\{1, 2, 3, \ldots, \frac{2M-7}{3}, \frac{2M-4}{3}\right\}$$

It is observed that, $A_1' \cup A_2' \cup A_3' = \{M-1, M-2, \ldots, 2, 1, 0\}$ and the values in the sets A_1', A_2' & A_3' are all distinct, hence G is felicitous. ■

Illustrative example of the labeling given in the proof of Theorem 2.1 is displayed in Figure 2.

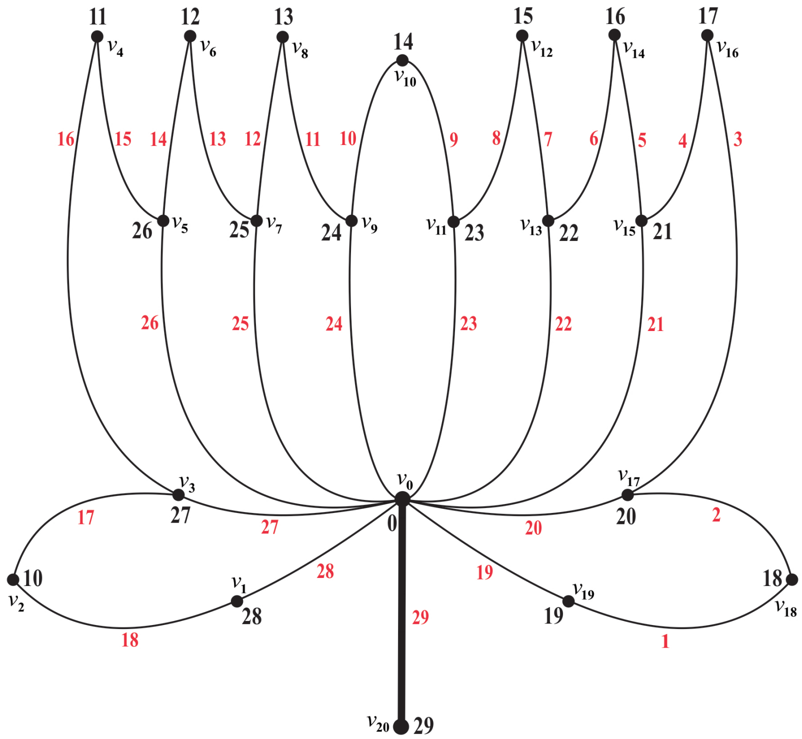

Figure 2: Graceful labeling of Stem-Lotus graph with 3-pairs of petals

Illustrative example of the labeling given in the proof of Theorem 2.2 is displayed in Figure 3.

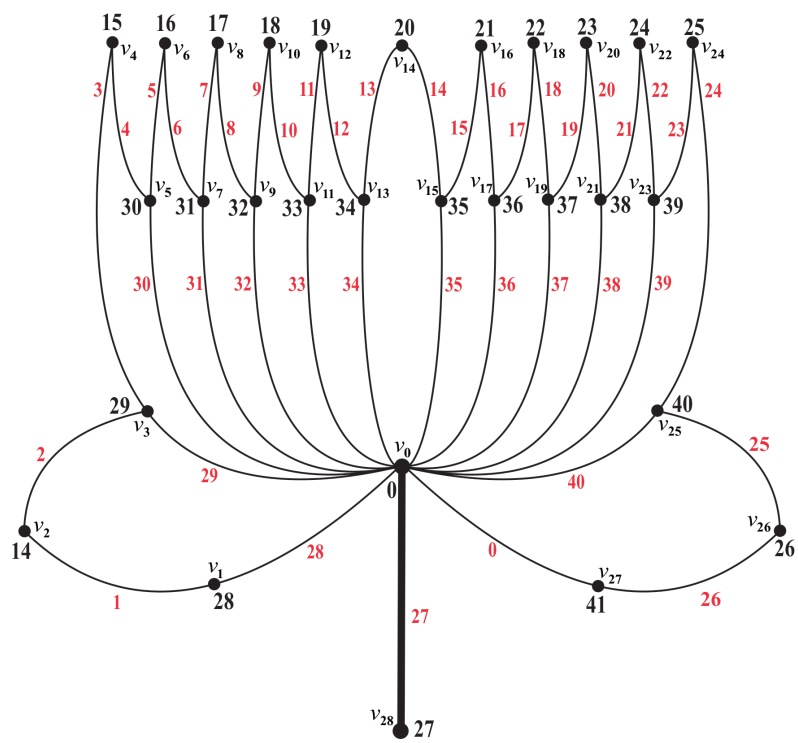

Figure 3: Felicitous labeling of Stem-Lotus graph with 5-pairs of petals

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