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Positivity and dynamics preserving discretization schemes for nonlinear evolution equations

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Abstract. Discretization of a continuous-time system of differential equations becomes inevitable due to the lack of analytical solutions. Standard discretization techniques, however, have many things that could be improved, e.g., the positivity of the solution and dynamic consistency may be lost, and stability and convergence may depend on the step length. A nonstandard finite difference (NSFD) scheme is sometimes used to avoid inconsistencies. There are two fundamental issues regarding the construction of NSFD models. First, how to construct the denominator function of the discrete first-order derivative? Second, how to discretize the nonlinear terms of a given differential equation with nonlocal terms? We define here a uniform technique for nonlocal discretization and construction of denominator function for NSFD models. We have discretized a couple of highly nonlinear continuous-time population models using these consistent rules. We give analytical proof in each case to show that the proposed NSFD model has identical dynamic properties to the continuous-time model. It is also shown that each NSFD system is positively invariant, and its dynamics do not depend on the step size. Numerical experiments have also been performed in favour of such claims.

AMS Subject Classifications: 37N25, 39A30, 92B05, 92D25, 92D40.

Keywords: Nonlocal discretization, denominator function, dynamic consistency, step-size independency, population models.

Contents

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1. Introduction

Nonlinear systems of ordinary differential equations are frequently used to unveil the underlying dynamics of physical, chemical and biological phenomena. In most cases, it becomes impossible to find the analytical solution of the system in a compact form. For this, the need for a numerical solution arises for which discretization of the continuous-time model is essential. Standard finite difference schemes, such as the Euler method, Runge-Kutta method etc., are commonly used discretization techniques for numerical solutions of both ordinary and partial differential equations [1–3]. However, there are significant drawbacks to these widely used discretization methods. First, the behaviours of standard finite difference schemes strictly depend on the step size and therefore, such schemes exhibit step-size dependent instability [4]. For example, the simple logistic equation in the continuous system and its corresponding Euler discrete equation are represented, respectively, by

$$\dot{x} = x(1-x), \;\; x(0) = x_0 > 0, \tag{1.1}$$

$$x_{t+1} = x_t + hx_t(1-x_t), \;\; x_0 > 0, \tag{1.2}$$

where $h > 0$ is the step-size. It is easy to show that the nontrivial fixed point $x = 1$ of the continuous system (1.1) is always stable. Still, for the discrete system (1.2), stability holds for $h < 2$ only and unstable if $h > 2$. The bifurcation diagram (Figure 1) of the system (1.2) with step-size h as the bifurcation parameter shows period-doubling bifurcation, leading to chaos [5]. Thus, the dynamics of the Euler discrete model (1.2) depend on the step size and exhibits spurious behaviours which are not observed in the corresponding continuous system (1.1).

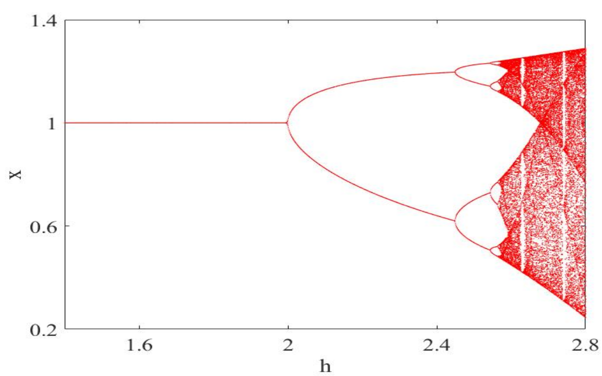

Figure 1: Bifurcation diagram of the discrete model (1.2) with respect to the step-size (h). The fixed point $x = 1$ is stable for $h < 2$ and unstable for $h > 2$. Chaos exists through period-doubling bifurcation for higher values of h, indicating a strong dependency on the step size.

Secondly, the positivity of the solutions of the discrete system may not be preserved for all step-size. For example, consider the continuous system

$$\dot{x} = -x, \; x(0) = x_0 > 0. \tag{1.3}$$

The solution of this equation $x(t) = x_0e^{-t}$ is always positive and monotonically converges to zero. However, the solution $x_t = (1-h)^t x_0$ of the corresponding Euler discrete system

$$x_{t+1} = (1-h)x_t, h > 0, h \neq 1, \tag{1.4}$$

is not always positive but may be negative also depending on the step size. In fact, the solution remains positive for $0 < h < 1$, $\forall t \geq 0$ and becomes alternatively positive and negative for $h > 1$ and $t \geq 0$ (Figure 2). In the latter case, all solutions having positive initial value converge to the fixed point $x = 0$ for any positive step-size $h < 2$. More precisely, solutions show oscillatory (taking positive and negative values in consecutive iterations) convergence for $1 < h < 2$ and oscillatory divergence for $h > 2$. Thus, huge differences exist in the dynamic behaviour between a continuous system and its corresponding discrete system. Any discrete system that permits negative solutions is supposed to show spurious dynamics, like bifurcation and chaos [2, 6, 7].

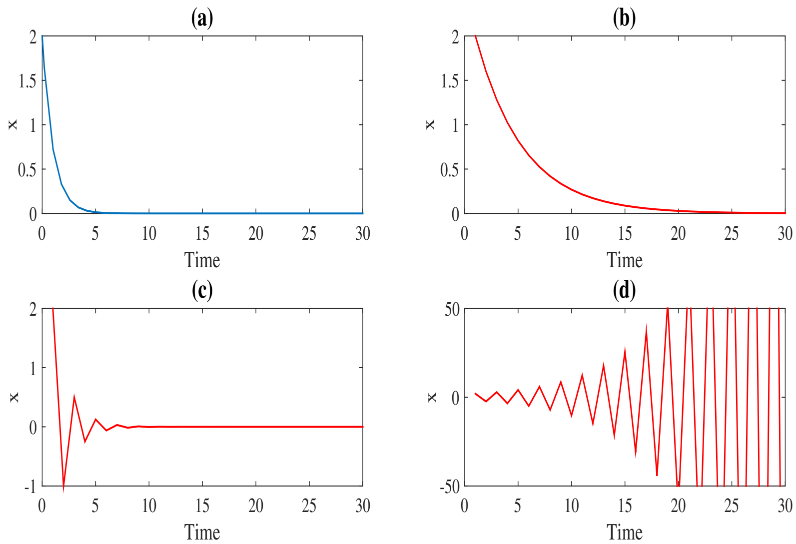

Figure 2: (a) Solution of the continuous model (1.3) converges exponentially to zero. Similar solutions of the Euler discrete system (1.4) are presented in Figure 2b-Figure 2d for different values of step-size. It shows different behaviours: (b) monotonic convergence for $h = 0.2$, (c) oscillatory convergence for $h = 1.5$ and (d) oscillatory divergence for $h = 2.2$.

One technique for avoiding such dynamic inconsistency is the nonstandard finite difference (NSFD) scheme introduced by Mickens [4, 5, 8] during $1989 - 1991$ and has been shown to have identical dynamics with its corresponding continuous model with zero truncation error [9]. It has also been shown that the dynamics of an NSFD discrete model are entirely independent of step size and do not produce spurious dynamics [5]. In the last few years, nonstandard methods have been successfully applied to various mathematical models in science and engineering [10–23] mainly because its solution does not depend on the step-size, maintains positivity and converges rapidly.

One of the most critical tasks in the NSFD scheme is to discretize the continuous system with nonlocal discrete terms [24–26]. For example, in a nonstandard finite difference scheme, the first derivative has to be discretized as $\frac{dx}{dt} \approx \frac{x_{k+1}-x_k}{\phi(h)}$, $h = \triangle t$, where $\phi(h)$ is a real, positive and monotonic function of the step-size (h), satisfying the condition $\phi(h) = h + O(h^2)$; and/or both the linear and nonlinear terms have to be represented nonlocally on the discrete computational lattice [5, 24, 26], e.g., $x = 2x - x \approx 2x_k - x_{k+1}$, $x^2 \approx x_k x_{k+1}$, $x^3 \approx 2x_k^3 - x_k^2 x_{k+1}$. Unfortunately, there is no general rule for constructing the denominator function as well as discretizing the nonlinear terms [5, 26]. In fact, one can construct different schemes for a given continuous-time model, but several of them can fail to converge and give desired results [27]. Some techniques for nonlocal discretization are given in [5, 26], and a methodology for calculating the form of the denominator function for the positive system is prescribed in [28]. Particular forms of the denominator function have been defined for continuous-time population models, where the total population is either constant (i.e., the system of differential equations can be expressed as $\frac{dL}{dt} = 0$, where L is the total population) or where total population asymptotically reaches to a constant value (i.e., the system can be expressed in the form $\frac{dL}{dt} = b - dL$, where b, d are constants). In the first case, we have to consider any equation of the given continuous system, where the first-order derivative has to be discretized by the Euler-forward method, and appropriate nonlocal approximations have to be given in the right-hand side of the equation so that positivity of the discrete system holds. Then rearrange this discrete equation as $(k+1)$-th time step dependent variable in terms of all k-th time step dependent variables. Thus if any term of the form $(1+\alpha h)$ occurs in the newly formed discrete equation, where α is composed of one or more system parameters and h is the step size, then the denominator function will be $\phi(h)=\frac{e^{\alpha h}-1}{\alpha}$. If, however, $\alpha=0$ then the denominator function can be taken as $\phi(h)=h$ (see pp. 677 in [28]). The denominator function for other equations of the system will be the same. In the second case, the denominator function has to be written as $\phi(h)=\frac{e^{dh}-1}{d}$. The denominator function will also be the same for all equations of this considered system [28, 29]. In other types of system equations, the denominator functions will be different for each equation of the continuous system, and these denominator functions can be obtained by doing the same steps as mentioned in the case of the conservative system [28]. We show that such a predetermined form of denominator function may not work for higher dimensional systems. Instead of considering a predetermined denominator function, it is better to choose a denominator function from the stability condition of the system. Here we also define some uniform rules for the nonlocal discretization of a continuous system to preserve the positivity and dynamic consistency of the discrete system with its continuous mother system. Several highly nonlinear systems from population biology have been considered to demonstrate the application of prescribed rules. In each example, we prove that the proposed NSFD models are positive for all step-size and dynamically consistent.

2. Nonlocal discretization techniques

One of the essential tasks in the NSFD method is the nonlocal representation of linear and nonlinear terms that appear in the differential equation. The primary goal of such discretization is to maintain the positivity of the constructed discrete system and to preserve the dynamics of the continuous system. We will demonstrate the nonlocal discretization technique with a two-dimension system for simplicity. The method, however, can be extended to any higher dimensional system of first-order difference equations.

Consider a two-dimensional continuous system of first-order differential equations:

$$\begin{aligned}\frac{dx}{dt}&=f(x,y),\\ \frac{dy}{dt}&=g(x,y),\end{aligned}\tag{2.1}$$

where f and g are C^1 functions. The following techniques may be adopted for dynamic preserving nonlocal discretization.

(R1) If in the first equation of (2.1), there is any constant term (say, α) with a negative (or positive) sign, then it would be discretized as $-\frac{\alpha x_{n+1}}{x_n}$ (or α).

(R2) If there is any linear term with a negative sign in the first equation, e.g., $-ax$, a being a positive constant, then it would be discretized as $-ax_{n+1}$ to keep the positivity for x_{n+1}. However, if the sign is positive, it would be discretized as ax_n.

(R3) For any higher degree term with a negative sign involving the first variable x only, e.g., $-ax^m(m>1)$, the nonlocal approximation would be $-ax_{n+1}x_n^{m-1}$. On the contrary, if the higher degree term appears with a positive sign, it would be expressed as ax_n^m.

(R4) If there is any product term containing first variable x and second variable y of the form $-axy$ (or axy) in the first equation, then it would be discretized by $-ax_{n+1}y_n$ (or ax_ny_n).

(R5) If any function $\phi(y)$ of the second variable appears alone (i.e., without involving the first variable x) in the first equation, then it will be discretized as $\frac{x_{n+1}\phi(y_n)}{x_n}$ (or $\phi(y_n)$) if there is a negative (or positive) sign before $\phi(y)$.

(R6) In the first equation, the second variable y will always be discretized by y_n and can't be y_{n+1} as we have to maintain a sequential form of calculation for using the initial condition. This rule is also valid for all other variables except the first one.

(R7) Similar terms appearing in different equations must be discretized similarly. For example, if the first equation contains the term axy and the second equation also contains axy then it will be replaced by ax_ny_n in both the equations. However, if the first equation contains $-axy$ and the second equation contains axy, then the nonlocal discretization will be $-ax_{n+1}y_n$ and $ax_{n+1}y_n$, respectively. If the term in the second equation is also negative, i.e., $-axy$, it would be discretized as $-ax_{n+1}y_{n+1}$. Note that y_n has to be changed by y_{n+1} as the term is placed in the second equation, and there is a negative sign before it, following (R2). Also, x_n in this term has to be expressed as x_{n+1} because it was written in the first equation. These rules are also applicable in discretizing other nonlinear terms.

(R8) For any rational function of the form $\frac{F(x,y)}{G(x,y)}$ $(G \neq 0)$, then the denominator function $G(x,y)$ will be replaced by $G(x_n, y_n)$ and the numerator function $F(x,y)$ will be discretized by the techniques prescribed in (R1) to (R7).

These rules are not unique, and one can find different nonlocal discretizations to construct an NSFD model for a given continuous system. What we have tried here is to define some uniform rules that one can follow while using the NSFD scheme of discretization. We here apply these rules to construct various NSFD models from their respective highly nonlinear continuous population models and show that they are dynamically consistent and the dynamics of these discrete systems are independent of the step size.

2.1. Example 1: Continuous-time epidemic model

Fayeldi et al. [30] have studied the following SIR (susceptible-infective-recovered) epidemic model with constant birth and nonmonotonic incidence rate:

$$\begin{aligned}\frac{dS}{dt} &= b - dS - \frac{kSI}{1+\alpha I^2},\\ \frac{dI}{dt} &= \frac{kSI}{1+\alpha I^2} - (d+\mu)I,\\ \frac{dR}{dt} &= \mu I - dR,\end{aligned} \tag{2.2}$$

where S, I and R denote the numbers of susceptible, infective and recovered individuals at time t. The parameters b and d represent, respectively, the recruitment and natural death rates of the host population; μ is the natural recovery rate of the infected individuals. The term $\frac{kSI}{1+\alpha I^2}$ is the nonmonotone incidence rate, where k is the disease transmission coefficient and α measures the inhibitory effect. Further description of the model can be seen in [30, 31].

Stability results of the continuous-time epidemic model

The model (2.2) has been analyzed in [30]. It has two equilibrium points, viz., the disease-free equilibrium point $E_1 = \left(\frac{b}{d}, 0, 0\right)$ and the interior fixed point $E^* = (S^*, I^*, R^*)$, where $S^* = \frac{1}{d}\{b - (d+\mu)I^*\}$, $I^* = \frac{-k+\sqrt{k^2-4d^2\alpha(1-R_0)}}{2\alpha d}$ and $R^* = \frac{\mu I^*}{d}$, where $R_0 = \frac{bk}{d(d+\mu)}$. Stability results of the equilibrium points are stated in the following theorems.

Theorem 2.1. *The continuous system* (2.2) *is locally asymptotically stable around the fixed point E_1 if $R_0 < 1$, and it is stable around the fixed point E^* if $R_0 > 1$.*

We now use the nonlocal discretization techniques (R1) to (R8) for the construction of the NSFD model corresponding to the continuous-time model (2.2).

Construction of NSFD model and its analysis

The first-order derivative $\frac{dS}{dt}$ will be replaced by $\frac{S_{n+1}-S_n}{\phi_1(h)}$, where $\phi_1(h) > 0$ and can be expressed as $\phi_1(h) = h + O(h^2)$. The constant term on the right-hand side will be left unaltered following (R1) because its sign is positive. Observe that S appears in the first equation of system (2.2) with a negative sign, indicating that it has to be replaced by S_{n+1}, following (R2). The nonlinear term $\frac{SI}{1+\alpha I^2}$ is present in both the first and second equations of system (2.2) with opposite signs. The negative sign of this term in the first equation indicates that we have to replace it by $\frac{S_{n+1}I_n}{1+\alpha I_n^2}$, following (R7) & (R8). Note that we can not replace I_n by I_{n+1} in the first equation because the sequential order will be lost. Similarly, the linear term I, which appears in the second and third equations of system (2.2) with opposite signs, has to be replaced by I_{n+1}, following (R2) and (R7). Also, to hold the positivity condition, the negative term $-dR$ in the third equation of system (2.2) has to be replaced by $-dR_{n+1}$, following (R2). Based on these nonlocal discretizations, we obtain the following discrete system corresponding to continuous system (2.2):

$$\begin{aligned}
\frac{S_{n+1}-S_n}{\phi_1(h)} &= b - dS_{n+1} - \frac{kS_{n+1}I_n}{1+\alpha I_n^2},\\
\frac{I_{n+1}-I_n}{\phi_2(h)} &= \frac{kS_{n+1}I_n}{1+\alpha I_n^2} - (d+\mu)I_{n+1},\\
\frac{R_{n+1}-R_n}{\phi_3(h)} &= \mu I_{n+1} - dR_{n+1},
\end{aligned} \tag{2.3}$$

where $\phi_i(h)$, $i = 1, 2, 3$, are denominator functions such that $\phi_i(h) > 0$ and $\phi_i(h) = h + O(h^2)$. After rearranging, one have

$$\begin{aligned}
S_{n+1} &= \frac{S_n + b\phi_1(h)}{1+\phi_1(h)\left(d + \frac{kI_n}{1+\alpha I_n^2}\right)},\\
I_{n+1} &= \frac{I_n\left(1 + \frac{\phi_2(h)kS_{n+1}}{1+\alpha I_n^2}\right)}{1+\phi_2(h)(d+\mu)},\\
R_{n+1} &= \frac{R_n + \phi_3(h)\mu I_{n+1}}{1+\phi_3(h)d}.
\end{aligned} \tag{2.4}$$

It is to be noted that all terms in the right-hand side of (2.4) are positive and therefore $S_n > 0$, $I_n > 0$, $R_n > 0$, for all n and any value of the step-size h when initial values are positive.

Next, we show that the fixed points of the discrete system (2.4) are the same as in the continuous system (2.2) and their linear stability properties are also the same. Equilibrium points or fixed points of (2.4) are determined by substituting $S_{n+1} = S_n$, $I_{n+1} = I_n$, $R_{n+1} = R_n$ in (2.4) and then solving the following simultaneous equations for S_n, I_n, R_n:

$$\begin{aligned}
S_n &= \frac{S_n + b\phi_1(h)}{1+\phi_1(h)\left(d + \frac{kI_n}{1+\alpha I_n^2}\right)},\\
I_n &= \frac{I_n\left(1 + \frac{\phi_2(h)kS_n}{1+\alpha I_n^2}\right)}{1+\phi_2(h)(d+\mu)},\\
R_n &= \frac{R_n + \phi_3(h)\mu I_n}{1+\phi_3(h)d}.
\end{aligned}$$

On simplifications, one can obtain the same equilibrium points E_1 and E^* as in the continuous case. The variational matrix at any arbitrary fixed point (S, I, R) of (2.4) is given by

$$J(S,I,R) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{2.5}$$

where

$$\begin{cases} a_{11} = \frac{1}{1+\phi_1(h)\left(d+\frac{kI}{1+\alpha I^2}\right)}, \;\; a_{12} = -\frac{\phi_1(h)k(S+b\phi_1(h))(1-\alpha I^2)}{\left\{1+\phi_1(h)\left(d+\frac{kI}{1+\alpha I^2}\right)\right\}^2(1+\alpha I^2)^2}, \\ a_{21} = \frac{\phi_2(h)kI}{\{1+\phi_2(h)(d+\mu)\}(1+\alpha I^2)} a_{11}, \\ a_{22} = \frac{1}{1+\phi_2(h)(d+\mu)}\left[1 + \frac{\phi_2(h)kI}{(1+\alpha I^2)} a_{12} + \frac{\phi_2(h)kS(1-\alpha I^2)}{(1+\alpha I^2)^2}\right], \\ a_{31} = \frac{\phi_3(h)\mu}{1+\phi_3(h)d} a_{21}, \;\; a_{32} = \frac{\phi_3(h)\mu}{1+\phi_3(h)d} a_{22}, \;\; a_{33} = \frac{1}{1+\phi_3(h)d}. \end{cases}$$

Definition 2.2. *[32] A fixed point of the system* (2.4) *is said to be locally asymptotically stable if* $|\lambda_i| < 1$ *and a source if* $|\lambda_i| > 1$*, where* λ_i*,* $i = 1, 2, 3$*, are the eigenvalues of the variational matrix* J *of system* (2.4) *evaluated at the fixed point.*

Lemma 2.3. *[32] Let* λ_1 *and* λ_2 *be the eigenvalues of a matrix* $\hat{J} = [\hat{a}_{ij}]$*,* $i, j = 1, 2$*. Then* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *iff (i)* $1 - \det(\hat{J}) > 0$*, (ii)* $1 - \mathrm{trace}(\hat{J}) + \det(\hat{J}) > 0$ *and (iii)* $1 + \mathrm{trace}(\hat{J}) + \det(\hat{J}) > 0$*.*

We have the following theorem about the stability of fixed points of (2.4).

Theorem 2.4. *The disease-free fixed point* $E_1 = \left(\frac{b}{d}, 0, 0\right)$ *is locally asymptotically stable if* $R_0 < 1$ *and the endemic fixed point* E^* *is stable if* $R_0 > 1$*, where* $R_0 = \frac{bk}{d(d+\mu)}$*.*

Proof. It is easy to check that the eigenvalues at E_1 are $\lambda_1 = \frac{1}{1+\phi_1(h)d}$, $\lambda_2 = \frac{1+\frac{bk\phi_2(h)}{d}}{1+\phi_2(h)(b+\mu)}$ and $\lambda_3 = \frac{1}{1+d\phi_3(h)}$. Here, $0 < |\lambda_{1,3}| < 1$ and $\lambda_2 > 0$ for any step-size $h > 0$. Thus, for any $h > 0$, $\lambda_2 < 1$ if $\frac{bk}{d} < d + \mu$, i.e., if $R_0 < 1$. Therefore, E_1 is stable if $R_0 < 1$.

At the endemic fixed point $E^* = (S^*, I^*, R^*)$, the variational matrix is given by

$$J(E^*) = \begin{pmatrix} a^*_{11} & a^*_{12} & 0 \\ a^*_{21} & a^*_{22} & 0 \\ a^*_{31} & a^*_{32} & a^*_{33} \end{pmatrix},$$

where

$$\begin{cases} a^*_{11} = \frac{1}{G}, \;\; a^*_{12} = -\frac{\phi_1(h)kS^*(1-\alpha I^{*2})}{(1+\alpha I^{*2})^2 G}, \;\; a^*_{21} = \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} a^*_{11}, \\ a^*_{22} = 1 + \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} a^*_{12} - \frac{2\phi_2(h)kS^*\alpha I^{*2}}{(1+\alpha I^{*2})^2 H}, \;\; a^*_{31} = \frac{\phi_3(h)\mu}{F} a^*_{21}, \;\; a^*_{32} = \frac{\phi_3(h)\mu}{F} a^*_{22}, \\ a^*_{33} = \frac{1}{F}, \;\; G = 1 + \frac{b\phi_1(h)}{S^*}, \;\; H = 1 + \frac{\phi_2(h)kS^*}{1+\alpha I^{*2}}, \;\; F = 1 + \frac{\phi_3(h)\mu I^*}{R^*}. \end{cases}$$

Note that $0 < a^*_{11} < 1$ and $0 < a^*_{22} < 1$ for any $h > 0$. Here one eigenvalue of the variational matrix $J(E^*)$ is $\lambda_3 = a^*_{33}$, which is always positive and less than unity for any $h > 0$. Other two eigenvalues $\lambda_i,\ i = 1, 2$, of $J(E^*)$ can be obtained by finding the eigenvalues of the matrix

$$J_1(E^*) = \begin{pmatrix} a^*_{11} & a^*_{12} \\ a^*_{21} & a^*_{22} \end{pmatrix}.$$

Here $\mathrm{trace}(J_1(E^*)) = a^*_{11} + a^*_{22}$ and

$$\begin{aligned} \det(J_1(E^*)) &= a^*_{11}a^*_{22} - a^*_{12}a^*_{21} \\ &= a^*_{11}\left\{1 + \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} a^*_{12} - \frac{2\phi_2(h)kS^*\alpha I^{*2}}{(1+\alpha I^{*2})^2 H}\right\} - \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} a^*_{11}a^*_{12} \end{aligned}$$

$= a_{11}^* \left(1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right).$

Since $0 < a_{11}^* < 1$ for any $h > 0$, so $\det(J_1(E^*)) < 1$ and the condition $1 - \det(J_1(E^*)) > 0$ always holds. Simple algebraic manipulations show that

$$\begin{aligned}
1 - \mathrm{trace}(J_1(E^*)) + \det(J_1(E^*)) &= 1 - (a_{11}^* + a_{22}^*) + a_{11}^* \left\{1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right\} \\
&= -\frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H}\left\{-\frac{\phi_1(h)kS^*}{(1+\alpha I^{*2})G} + \frac{2\phi_1(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 G}\right\} + \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\frac{b\phi_1(h)}{S^* G} \\
&= \frac{\phi_1(h)\phi_2(h)kI^*}{(1+\alpha I^{*2})^2 GH}\left[kS^* + 2\alpha I^*\left\{b - \frac{kS^* I^*}{(1+\alpha I^{*2})}\right\}\right] \\
&= \frac{\phi_1(h)\phi_2(h)kS^* I^*}{(1+\alpha I^{*2})^2 GH}(k + 2\alpha d I^*) > 0
\end{aligned}$$

and

$$\begin{aligned}
1 + \mathrm{trace}(J_1(E^*)) + \det(J_1(E^*)) &= 1 + (a_{11}^* + a_{22}^*) + a_{11}^* \left\{1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right\} \\
&= 1 + a_{11}^* + \left\{1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right\}(1 + a_{11}^*) - \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H}\frac{\phi_1(h)kS^*(1-\alpha I^{*2})}{(1+\alpha I^{*2})^2 G} \\
&= 1 + a_{11}^* + \left\{1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right\}(1 + a_{11}^*) \\
&\qquad - \frac{\phi_2(h)kS^*}{(1+\alpha I^{*2})H}\frac{\phi_1(h)kI^*}{(1+\alpha I^{*2})G}\left\{1 - \frac{2\alpha I^{*2}}{(1+\alpha I^{*2})}\right\} \\
&= a_{11}^* + \left[1 - \frac{2\phi_2(h)kS^*}{(1+\alpha I^{*2})H}\frac{\alpha I^{*2}}{(1+\alpha I^{*2})}\right](1 + a_{11}^*) + \frac{2\phi_1(h)\phi_2(h)k^2\alpha S^* I^{*3}}{(1+\alpha I^{*2})^3 GH} \\
&\qquad + \left\{1 - \frac{\phi_1(h)\left(\frac{b}{S^*} - d\right)}{G}\frac{\phi_2(h)\frac{kS^*}{(1+\alpha I^{*2})}}{H}\right\}.
\end{aligned} \tag{2.6}$$

Here we show the positivity of each term on the right hand side of (2.6). Note that $a_{11}^* = \frac{1}{G}$, so $0 < a_{11}^* < 1$. Using the value of G and H, one can check that $0 < \frac{\phi_1(h)\left(\frac{b}{S^*} - d\right)}{G} < 1$ and $0 < \frac{\phi_2(h)\frac{kS^*}{(1+\alpha I^{*2})}}{H} < 1$. It is then easy to see that the expression in curly bracket is positive. The third term is always positive as $\phi_1(h)$, $\phi_2(h)$, G and H are all positive. To prove that the expression in the third bracket is also positive, we note that $\frac{\alpha I^{*2}}{1+\alpha I^{*2}} < 1$. Thus, if $\frac{2\phi_2(h)kS^*}{(1+\alpha I^{*2})H} < 1$, then $\left\{1 - \frac{2\phi_2(h)kS^*}{(1+\alpha I^{*2})H}\frac{\alpha I^{*2}}{(1+\alpha I^{*2})}\right\} > 0$. The first term gives $\frac{2\phi_2(h)kS^*}{(1+\alpha I^{*2})} < H = 1 + \frac{\phi_2(h)kS^*}{(1+\alpha I^{*2})} \Rightarrow \frac{\phi_2(h)kS^*}{(1+\alpha I^{*2})} < 1 \Rightarrow \phi_2(h) < \frac{(1+\alpha I^{*2})}{kS^*} = \frac{1}{(d+\mu)}$. Therefore $1 + \mathrm{trace}(J_1(E^*)) + \det(J_1(E^*)) > 0$ if $\phi_2(h) < \frac{1}{(d+\mu)}$. One can then choose the denominator function as $\phi_2(h) = \frac{1-e^{-(d+\mu)h}}{(d+\mu)}$, so that $\phi_2(h) < \frac{1}{(d+\mu)}$ holds. Also, the denominator function is in the form $\phi_2(h) = h + O(h^2)$. It is to be noted that no restriction is required on $\phi_1(h)$ and $\phi_3(h)$ to hold the stability conditions of E^*, and therefore simplest form can be considered for $\phi_1(h)$ and $\phi_3(h)$ such that $\phi_1(h) = h = \phi_3(h)$. Therefore, following Lemma 2.3, $|\lambda_i| < 1,\ i = 1, 2$. By Definition 2.2, the endemic fixed point E^* is stable whenever it exists, i.e., if $R_0 > 1$. This completes the theorem. ■

Remark 2.5. *The system* (2.2) *can be written as*

$$\frac{dN}{dt} = b - dN, \tag{2.7}$$

where $N(t) = S(t) + I(t) + R(t)$ is the total population at time t. Following Mickens rule as described in [28], all the denominator functions $\phi_i(h)$, $i = 1, 2, 3$ will be same and it is $\phi_i(h) = \frac{e^{dh}-1}{d}$. It is to be noted *that the stability condition* $1+\mathrm{trace}(J(E^*))+\det(J(E^*))>0$ *does not hold for this choice of denominator function. However, one can easily determine the denominator function* $\phi_2(h)$ *as shown above such that the stability condition holds.*

Euler discrete-time epidemic model

Discretization of the continuous model (2.2) by Euler-forward technique gives the following system:

$$\begin{aligned} S_{n+1} &= S_n + bh - hS_n\left(d+\frac{kI_n}{1+\alpha I_n^2}\right), \\ I_{n+1} &= I_n + hI_n\left\{\frac{kS_n}{1+\alpha I_n^2}-(d+\mu)\right\}, \\ R_{n+1} &= \mu h I_n + R_n(1-dh), \end{aligned} \tag{2.8}$$

where $h(>0)$ is the step-size. Due to the presence of negative terms on the right-hand side, the solutions are not unconditionally positive as in the case of NSFD model (2.12). Such systems are prone to exhibit spurious dynamics. The following results are known for the Euler discrete system (2.8).

Theorem 2.6. *[30] The discrete system* (2.8) *is stable around the fixed point* E_1 *if* $R_0<1$, $h<\min\left\{\frac{2}{d},\frac{2}{(1-R_0)(d+\mu)},\frac{2}{\mu}\right\}$ *and it is locally asymptotically stable around the fixed point* E^* *if one of the following condition holds: (a)* $R_0>R_1>1$ *and* $h<\min\left\{h^*,\frac{2}{\mu}\right\}$, *or (b)* $1<R_0<R_1$ *and* $h<\min\left\{h_1,h^*,\frac{2}{\mu}\right\}$, *where* $R_1=\frac{kI^*}{\phi_e^*}\left\{1+\frac{k(d+\phi_e^*+p)^2}{4d(d+\mu)(2d\alpha I^{*2}+k)}\right\}$, $h^*=\frac{d+\phi_e^*+p}{dp+\phi_e^*(d+\mu)}$,
$h_1=h^*-\frac{\sqrt{4(d+\phi_e^*+p)^2-16\phi_e^*(d+\mu)\left(\frac{2d\alpha I^*}{k}+1\right)}}{2\phi_e^*(d+\mu)\left(\frac{2d\alpha I^*}{k}+1\right)}$, $\phi_e^*=\frac{kI^*}{1+\alpha I^{*2}}$, $p=\frac{2\alpha(d+\mu)I^*\phi_e^*}{k}$.

Numerical experiments

We perform numerical experiments to compare the dynamics and step-size dependency of the NSFD model (2.4) and Euler model (2.8). We have plotted bifurcation diagrams for both the systems (Figure 3) with respect to h. Population density remains at its steady-state value for all h, indicating consistent dynamics with its continuous counterpart. It shows that the dynamic behaviour of NSFD system (2.4) is independent of the step-size (Figure 3a). However, the dynamic behaviour of the Euler system (2.8) depends on the step size (Figure 3b). Here population density remains stable for $h<3.4647$ and becomes unstable for $h>3.4647$. In fact, it exhibits spurious dynamics as the step size is larger ($h>3.4647$).

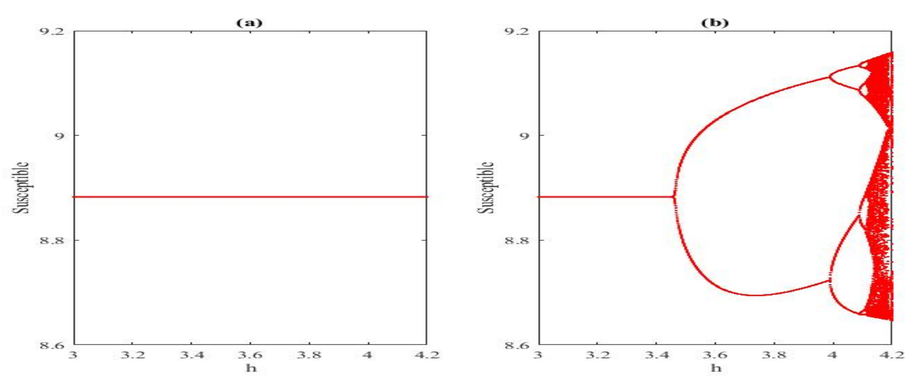

Figure 3: (a) Bifurcation diagram of the susceptible population of the NSFD system (2.4) with respect to the step size h. It shows no instability, and population density is always maintained at its stable value for all step-size. (b) A similar bifurcation diagram of Euler system (2.8) shows that population density remains stable for $h<3.4647$ and becomes unstable for $h>3.4647$. It shows chaotic dynamics as h is further increased. Parameters are [30]: $b=2,\ k=0.2,\ d=0.2,\ \mu=0.15,\ \alpha=10$.

2.2. Example 2: Continuous-time ecological model

Here we consider another population model in continuous time and construct the corresponding NSFD model using our nonlocal discretization technique. Chattopadhyay et al. [33] investigated the dynamics of following continuous-time plant-herbivore-parasite ecological model:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\alpha xy, \\ \frac{dy}{dt} &= -sy+\beta xy-\gamma yz, \\ \frac{dz}{dt} &= \delta yz-\mu z, \end{aligned} \tag{2.9}$$

where x, y and z represent, respectively, the densities of plant biomass, herbivore and parasite populations at time t. This model says that the plant population grows logistically to the environmental carrying capacity K with an intrinsic growth rate r when there is no herbivore. Herbivore eats plant population following mass action law with α as the rate constant. The parasite attacks herbivores, and the attack rate is proportional to the product of herbivore and parasite densities with γ as the proportionality constant. Natural death rates of herbivores and parasites are s and μ, respectively. The parameters β and δ represent the growth rates of herbivores and parasites. All parameters are positive. The following results [33] are known for the system (2.9).

Theorem 2.7. *The system* (2.9) *has four equilibrium points. (i) The equilibrium point* $E_0^P=(0,0,0)$ *is always unstable. (ii) The axial equilibrium point* $E_1^P=(K,0,0)$ *is stable if* $\beta K<s$. *(iii) The planar equilibrium point* $E_2^P=(\bar{x},\bar{y},0)$*, where* $\bar{x}=\frac{s}{\beta}$, $\bar{y}=\frac{r}{\alpha}\left(1-\frac{s}{\beta K}\right)$*, exists and is locally asymptotically stable if* $\beta K>s$ *and* $\delta<\frac{\beta K\alpha\mu}{r(\beta K-s)}$. *(iv) The interior equilibrium point* $E_P^*=(x_P^*,y_P^*,z_P^*)$*, where* $x_P^*=K\left(1-\frac{\alpha\mu}{r\delta}\right)$, $y_P^*=\frac{\mu}{\delta}$, $z_P^*=\frac{1}{\gamma}\left\{-s+\beta K\left(1-\frac{\alpha\mu}{r\delta}\right)\right\}$*, exists and is locally asymptotically stable if* $\beta K>s$ *and* $\delta>\frac{\beta K\alpha\mu}{r(\beta K-s)}$.

NSFD model and its analysis

For convenience, we rewrite the continuous model (2.9) as

$$\begin{aligned} \frac{dx}{dt} &= rx-\frac{r}{K}x^2-\alpha xy, \\ \frac{dy}{dt} &= -sy+\beta xy-\gamma yz, \\ \frac{dz}{dt} &= \delta yz-\mu z. \end{aligned} \tag{2.10}$$

The continuous system (2.10) is transformed to the following NSFD system using the previous nonlocal discretization techniques (R1) to (R7):

$$\begin{aligned} \frac{x_{n+1}-x_n}{\psi_1(h)} &= rx_n-\frac{rx_{n+1}x_n}{K}-\alpha x_{n+1}y_n, \\ \frac{y_{n+1}-y_n}{\psi_2(h)} &= -sy_{n+1}+\beta x_{n+1}y_n-\gamma y_{n+1}z_n, \\ \frac{z_{n+1}-z_n}{\psi_3(h)} &= \delta y_{n+1}z_n-\mu z_{n+1}, \end{aligned} \tag{2.11}$$

where $\psi_i(h)$, $i=1,2,3$, are such that $\psi_i(h)>0$ and $\psi_i(h)=h+O(h^2)$. Note that the similar term xy in the first & second equations and yz in the second & third equations have been discretized following the rule (R7).

Rearranging (2.11), we get

$$\begin{aligned} x_{n+1} &= \frac{x_n(1+r\psi_1(h))}{1+\psi_1(h)\left(\frac{rx_n}{K}+\alpha y_n\right)}, \\ y_{n+1} &= \frac{y_n(1+\beta\psi_2(h)x_{n+1})}{1+\psi_2(h)(s+\gamma z_n)}, \\ z_{n+1} &= \frac{z_n(1+\delta\psi_3(h)y_{n+1})}{1+\psi_3(h)\mu}. \end{aligned} \tag{2.12}$$

Thus, the solutions of the discrete system (2.12) remain positive for all step-size h whenever the initial values are positive.

As before, one can observe that the NSFD system (2.12) has the same four fixed points with the same existence conditions as it were in the continuous system (2.9). The variational matrix corresponding to the system (2.12) at any arbitrary fixed point (x, y, z) is given by

$$J(x,y,z) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{2.13}$$

where

$$\begin{cases} a_{11} = \frac{1+r\psi_1(h)}{1+\psi_1(h)\left(\frac{rx}{K}+\alpha y\right)} - \frac{x(1+r\psi_1(h))}{\left\{1+\psi_1(h)\left(\frac{rx}{K}+\alpha y\right)\right\}^2}\left(\frac{r\psi_1(h)}{K}\right), \quad a_{12} = -\frac{x(1+r\psi_1(h))}{\left\{1+\psi_1(h)\left(\frac{rx}{K}+\alpha y\right)\right\}^2}\alpha\psi_1(h), \\ a_{21} = \frac{\beta\psi_2(h)y}{1+\psi_2(h)(s+\gamma z)}a_{11}, \quad a_{22} = \frac{1+\beta\psi_2(h)x}{1+\psi_2(h)(s+\gamma z)} + \frac{\beta\psi_2(h)y}{1+\psi_2(h)(s+\gamma z)}a_{12}, \quad a_{23} = -\frac{y(1+\beta\psi_2(h)x)\gamma\psi_2(h)}{\{1+\psi_2(h)(s+\gamma z)\}^2}, \\ a_{31} = \frac{\delta\psi_3(h)z}{1+\psi_3(h)\mu}a_{21}, \quad a_{32} = \frac{\delta\psi_3(h)z}{1+\psi_3(h)\mu}a_{22}, \quad a_{33} = \frac{1+\delta\psi_3(h)y}{1+\psi_3(h)\mu} + \frac{\delta\psi_3(h)z}{1+\psi_3(h)\mu}a_{23}. \end{cases}$$

We have the following lemma in relation to the stability of system (2.12).

Lemma 2.8. *[34] Suppose the characteristic polynomial $p(\lambda)$ of the variational matrix* (2.13) *is given by*

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

Then the roots λ_i, $i = 1, 2, 3$, of $p(\lambda) = 0$ satisfy $|\lambda_i| < 1$, $i = 1, 2, 3$ iff

(i) $p(1) = 1 + a_1 + a_2 + a_3 > 0$,

(ii) $(-1)^3 p(-1) = 1 - a_1 + a_2 - a_3 > 0$,

(iii) $1 - (a_3)^2 > |a_2 - a_3 a_1|$.

Then the following results are true for the system (2.12).

Theorem 2.9. *(i) E_0^P is always an unstable fixed point. (ii) E_1^P is locally asymptotically stable if $\beta K < s$. (iii) E_2^P is stable if $\beta K > s$ and $\delta < \frac{\beta K\alpha\mu}{r(\beta K - s)}$. (iv) The interior fixed point E_P^* is always stable if $\beta K > s$ and $\delta > \frac{\beta K\alpha\mu}{r(\beta K - s)}$.*

Proof. At the trivial fixed point E_0^P, the eigenvalues are $\lambda_1 = 1 + r\psi_1(h)$, $\lambda_2 = \frac{1}{1+s\psi_2(h)}$ and $\lambda_3 = \frac{1}{1+\psi_3(h)\mu}$. As $\lambda_1 > 1$, E_0^P is always unstable $\forall\ h > 0$.

At E_1^P, the eigenvalues are given by $\lambda_1 = \frac{1}{1+r\psi_1(h)}$, $\lambda_2 = \frac{1+\beta K\psi_2(h)}{1+s\psi_2(h)}$ and $\lambda_3 = \frac{1}{1+\psi_3(h)\mu}$. Here λ_1 and λ_3 both are positive and less than unity. λ_2 will be positive and less than unity for all $h > 0$ if $\beta K < s$. Therefore, E_1^P is stable if $\beta K < s$.

At the boundary fixed point $E_2^P(\bar{x}, \bar{y}, 0)$, the variational matrix is given by

$$J(E_2^P) = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & 0 \\ \bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} \\ 0 & 0 & \bar{a}_{33} \end{pmatrix}, \tag{2.14}$$

where

$$\begin{cases} \bar{a}_{11} = 1 - \left(\frac{\bar{x}}{K}\right)\left(\frac{r\psi_1(h)}{1+r\psi_1(h)}\right), \;\; \bar{a}_{12} = -\frac{\bar{x}}{1+r\psi_1(h)}\alpha\psi_1(h), \;\; \bar{a}_{21} = \frac{\beta\psi_2(h)\bar{y}}{1+\beta\psi_2(h)\bar{x}}\bar{a}_{11}, \\ \bar{a}_{22} = 1 + \frac{\beta\psi_2(h)\bar{y}}{1+\beta\psi_2(h)\bar{x}}\bar{a}_{12}, \;\; \bar{a}_{23} = -\frac{\bar{y}}{1+\beta\psi_2(h)\bar{x}}\gamma\psi_2(h), \;\; \bar{a}_{33} = \frac{1+\delta\psi_3(h)\bar{y}}{1+\psi_3(h)\mu}. \end{cases}$$

One eigenvalue of the above variational matrix $J(E_2^P)$ is $\bar{a}_{33} = \frac{1+\delta\psi_3(h)\bar{y}}{1+\psi_3(h)\mu}$, which is always positive and less than unity if $\delta < \frac{\mu}{\bar{y}} = \frac{\beta K\alpha\mu}{r(\beta K - s)}$. Other two eigenvalues of the matrix $J(E_2^P)$ will be the characteristics roots of the matrix

$$J_1 = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix}.$$

From the existence condition of E_2^P, it is easy to see that $0 < \bar{a}_{11} < 1$. After some algebraic manipulations, one have

$$\bar{a}_{22} = 1 - \frac{\beta\psi_2(h)\bar{y}}{\{1+\beta\psi_2(h)\bar{x}\}}\frac{\alpha\psi_1(h)\bar{x}}{\{1+r\psi_1(h)\}} = 1 - \left(\frac{\beta\psi_2(h)\bar{x}}{1+\beta\psi_2(h)\bar{x}}\right)\left\{\frac{r\left(1-\frac{s}{\beta K}\right)\psi_1(h)}{1+r\psi_1(h)}\right\},$$

implying that $0 < \bar{a}_{22} < 1$. On substitution the values of $\bar{a}_{22}, \bar{a}_{21}$ and noting that $\bar{x} < K$, one can obtain

$$\begin{aligned} 1 - \det(J_1) &= 1 - \bar{a}_{11}\,\bar{a}_{22} + \bar{a}_{12}\,\bar{a}_{21} \\ &= 1 - \bar{a}_{11} - \frac{\beta\psi_2(h)\bar{y}}{1+\beta\psi_2(h)\bar{x}}\bar{a}_{11}\bar{a}_{12} + \frac{\beta\psi_2(h)\bar{y}}{1+\beta\psi_2(h)\bar{x}}\bar{a}_{11}\bar{a}_{12} = 1 - \bar{a}_{11} > 0, \\ 1 - \operatorname{trace}(J_1) + \det(J_1) &= 1 - (\bar{a}_{11} + \bar{a}_{22}) + \bar{a}_{11} \\ &= 1 - \bar{a}_{22} > 0 \text{ and } 1 + \operatorname{trace}(J_1) + \det(J_1) = 1 + 2\bar{a}_{11} + \bar{a}_{22} > 0. \end{aligned}$$

Thus, whenever it exists, E_2^P is locally asymptotically stable if $\delta < \frac{\beta K\alpha\mu}{r(\beta K - s)}$.

At the interior fixed point E_P^*, the variational matrix is given by

$$J(E_P^*) = \begin{pmatrix} a_{11}^* & a_{12}^* & 0 \\ a_{21}^* & a_{22}^* & a_{23}^* \\ a_{31}^* & a_{32}^* & a_{33}^* \end{pmatrix}, \tag{2.15}$$

where

$$\begin{cases} a_{11}^* = 1 - \left(\frac{x_P^*}{K}\right)\left(\frac{r\psi_1(h)}{G}\right) > 0, \;\; a_{12}^* = -\frac{x_P^*\alpha\psi_1(h)}{G} < 0, \;\; a_{21}^* = \frac{\beta\psi_2(h)y_P^*}{H}a_{11}^* > 0, \\ a_{22}^* = 1 + \frac{\beta\psi_2(h)y_P^*}{H}a_{12}^* = 1 - \frac{\beta\psi_2(h)x_P^*}{H} * \frac{\alpha\psi_1(h)y_P^*}{G} > 0, \;\; a_{23}^* = -\frac{y_P^*\gamma\psi_2(h)}{H} < 0, \\ a_{31}^* = \frac{\delta\psi_3(h)z_P^*}{E}a_{21}^* > 0, \;\; a_{32}^* = \frac{\delta\psi_3(h)z_P^*}{E}a_{22}^* > 0, \\ a_{33}^* = 1 + \frac{\delta\psi_3(h)z_P^*}{E}a_{23}^* = 1 - \left(\frac{\delta\psi_3(h)y_P^*}{E}\right)\left(\frac{z_P^*\gamma\psi_2(h)}{H}\right) > 0, \\ G = 1 + r\psi_1(h), \;\; H = 1 + \beta\psi_2(h)x_P^*, \;\; E = 1 + \delta\psi_3(h)y_P^*. \end{cases}$$

Following the existence conditions of the interior fixed point E_P^*, $0 < a_{ii}^* < 1$, $i = 1, 2, 3$. The characteristic equation corresponding to the matrix $J(E_P^*)$ has the form

$$p_1(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{2.16}$$

where the coefficients are

$A_1 = -\operatorname{trace}(J(E_P^*)) = -a_{11}^* - a_{22}^* - a_{33}^*,$

$A_2 =$ sum of principle minors of $J(E_P^*)$
$= (a_{11}^* a_{22}^* - a_{12}^* a_{21}^*) + (a_{22}^* a_{33}^* - a_{23}^* a_{32}^*) + a_{11}^* a_{33}^*,$
$A_3 = -\det(J(E_P^*)) = -a_{11}^*(a_{22}^* a_{33}^* - a_{23}^* a_{32}^*) + a_{12}^*(a_{21}^* a_{33}^* - a_{23}^* a_{31}^*) < 0.$
Simple manipulations give
$a_{11}^* a_{22}^* - a_{12}^* a_{21}^* = a_{11}^* + \frac{\beta\phi_2(h)y^*}{H} a_{11}^* a_{12}^* - \frac{\beta\phi_2(h)y^*}{H} a_{11}^* a_{12}^* = a_{11}^*,$
$a_{22}^* a_{33}^* - a_{23}^* a_{32}^* = a_{22}^*$ and $a_{21}^* a_{33}^* - a_{31}^* a_{23}^* = a_{21}^*.$
Thus, the coefficients simplify to
$A_1 = -a_{11}^* - a_{22}^* - a_{33}^* \ (< 0),\ A_2 = a_{11}^* + a_{22}^* + a_{11}^* a_{33}^* \ (> 0),\ A_3 = -a_{11}^* \ (< 0).$
Now our objective is to show that all the conditions of Lemma 2.8 are satisfied for the characteristic equation (2.16). One can compute
$p_1(1) = 1 + A_1 + A_2 + A_3 = 1 - a_{33}^* - a_{11}^* + a_{11}^* a_{33}^* = (1 - a_{11}^*)(1 - a_{33}^*),$
$(-1)^3 p_1(-1) = 1 - A_1 + A_2 - A_3.$
Noting the signs of a_{ij}^*, A_i and $a_{ii}^* < 1$, $i, j = 1, 2, 3$, one can easily observe that $p_1(1)$ and $(-1)^3 p_1(-1)$ both are positive. Thus, first two conditions of Lemma 2.8 are satisfied. For the third condition, we first note that $|A_2 - A_3 A_1| < 1 - A_3^2$ gives $A_2 - A_3 A_1 - A_3^2 + 1 > 0$ and $A_2 - A_3 A_1 + A_3^2 - 1 < 0$. Here

$$\begin{aligned} A_2 - A_3 A_1 - A_3^2 + 1 &= (a_{11}^* + a_{22}^* + a_{11}^* a_{33}^*) - a_{11}^*(a_{11}^* + a_{22}^* + a_{33}^*) - {a_{11}^*}^2 + 1 \\ &= (a_{11}^* + a_{22}^*)(1 - a_{11}^*) + (1 - {a_{11}^*}^2) = (1 - a_{11}^*)(1 + 2a_{11}^* + a_{22}^*), \\ A_2 - A_3 A_1 + A_3^2 - 1 &= (a_{11}^* + a_{22}^* + a_{11}^* a_{33}^*) - a_{11}^*(a_{11}^* + a_{22}^* + a_{33}^*) + {a_{11}^*}^2 - 1 \\ &= a_{11}^* + a_{22}^*(1 - a_{11}^*) - 1 = (1 - a_{11}^*)(a_{22}^* - 1). \end{aligned}$$

Observing the signs as before, one can then easily have

$$A_2 - A_3 A_1 - A_3^2 + 1 > 0 \ \text{ and } \ A_2 - A_3 A_1 + A_3^2 - 1 < 0.$$

Combining these two inequalities, we have $|A_2 - A_3 A_1| < 1 - A_3^2$. Thus, all three conditions of Lemma 2.8 hold and therefore, the interior fixed point E_P^* is locally asymptotically stable whenever it exists, i.e., $\beta K > s$ and $\delta > \frac{\beta K \alpha \mu}{r(\beta K - s)}$. Hence the theorem. ■

Remark 2.10. *It is to be noted that we do not need any restriction on $\psi_i(h)$, $i = 1, 2, 3$, to prove the positivity and dynamic consistency of the discrete system* (2.12)*. Therefore, $\psi_i(h)$ can take any form that satisfies $\psi_i(h) > 0$ and $\psi_i(h) = h + O(h^2)$, $i = 1, 2, 3$. In the simulations, we consider the simplest form of $\psi_i(h) = h$.*

Remark 2.11. *It is to be noted that the system* (2.9) *does not satisfy the conservation law. For this type of system, Mickens [28] defined a rule for choosing the denominator functions $\psi_i(h)$, $i = 1, 2, 3$. Following that rule, one has to use the Euler-forward scheme for the first derivative and nonlocal approximations for other terms in all three equations of system* (2.9)*. After doing this for the first equation of system* (2.9) *and then solving for x_{n+1}, one has*

$$x_{n+1} = \frac{x_n(1 + rh)}{1 + h\left(\frac{r x_n}{K} + \alpha y_n\right)}.$$

Since $(1 + rh)$ occurs [28], it implies that the denominator function will be $\psi_1(h) = \frac{e^{rh} - 1}{r}$. Similarly, from the other two equations of system (2.9)*, one can find the other two denominator functions as $\psi_2(h) = \frac{e^{sh} - 1}{s}$ and $\psi_3(h) = \frac{e^{\mu h} - 1}{\mu}$. Thus all three denominator functions have to be determined separately using the Euler forward scheme and nonlocal approximations if the continuous system is not conservative and the transformed nonlocal system contains terms like $(1 + rh)$. But such a choice of separate denominator function for each equation of a higher-order equation will multiply the complexity for analytical computation of stability conditions.*

Numerical experiments

For numerical comparison, we first write the Euler-forward discrete version of the continuous model (2.9):

$$\begin{aligned} x_{n+1} &= x_n + h\left\{rx_n\left(1-\frac{x_n}{K}\right) - \alpha x_n y_n\right\}, \\ y_{n+1} &= y_n + h(-sy_n + \beta x_n y_n - \gamma y_n z_n), \\ z_{n+1} &= z_n + h(\delta y_n z_n - \mu z_n). \end{aligned} \tag{2.17}$$

To compare the step-size independency and dynamic consistency of the NSFD model (2.12) with that of the Euler model (2.17), we have plotted two bifurcation diagrams (Figure 4) of plant biomass with respect to the step-size h. As there is no restriction on $\psi_i(h)$, we consider $\psi_i(h)$=h for all i in (2.12). Figure 4a shows that the dynamic behaviour of the NSFD system (2.12) is independent of the step-size, and Figure 4b depicts step-size dependent numerical instabilities in Euler system (2.17). In the last case, plant biomass population density remains stable for $h < 1.1113$ and shows instability for $h > 1.1113$.

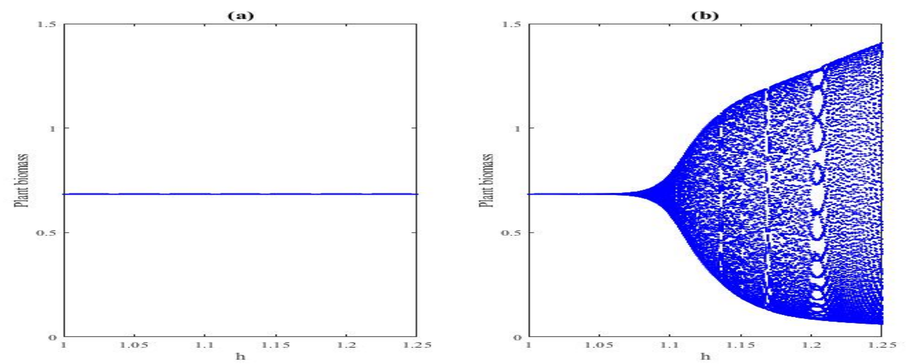

Figure 4: Bifurcation diagram of plant biomass (Fig. a) of the NSFD system (2.12) with varying step-size (h). This figure shows no instability in system (2.12) when step size is varied. Similar bifurcation diagram of Euler system (2.17) shows that the solution remains stable for $h < 1.1113$ and loses its stability for $h > 1.1113$. Parameters are $r = 0.95$, $K = 2.2$, $\alpha = 0.8$, $s = 0.25$, $\beta = 0.55$, $\gamma = 0.23$, $\mu = 0.09$, $\delta = 0.11$.

2.3. Example 3: Continuous-time epidemic model

O'Keefe [35] has investigated the dynamics of an epidemic model having frequency-dependent disease transmission. The model reads

$$\begin{aligned} \frac{dS}{dt} &= (S+\rho I)(1-S-I) - \frac{\beta SI}{S+I} - \mu S, \\ \frac{dI}{dt} &= \frac{\beta SI}{S+I} - (\alpha+\mu)I, \end{aligned} \tag{2.18}$$

where S and I represent, respectively, the densities of susceptible and infective hosts at time t. Here ρ $(0 \le \rho \le 1)$ is the fertility coefficient of infected hosts, and β is the disease transmission rate. μ represents the natural death rate of both hosts, and the additional death of infectives due to disease is represented by α. All parameters are non-negative from a biological point of view. The following stability results are known from [35].

Theorem 2.12. *The disease-free equilibrium point $E_1^e = (1-\mu, 0)$ always exists and it is locally asymptotically stable if $\mu < 1$, $\beta < (\alpha+\mu)$. The endemic (interior) equilibrium point $E^{e*} = (S^{e*}, I^{e*})$, where $S^{e*} = \frac{A(\alpha+\mu)}{B}$ and $I^{e*} = \frac{A(\beta-\alpha-\mu)}{B}$ with $A = -\alpha-\mu-\alpha(\alpha+\mu)+\beta(\alpha+\mu)+\rho(\alpha+\mu)-\beta\rho$, $B = \beta\{\rho(\alpha+\mu)-\alpha-\mu-\beta\rho\}$, exists and is locally asymptotically stable whenever $\beta > \alpha+\mu$, $A < 0$.*

We now construct the NSFD counterpart of the model (2.21) following the rules defined in Section 2.

NSFD model and its analysis

For convenience, we rewrite the continuous model (2.18) as

$$\begin{aligned} \frac{dS}{dt} &= S - S^2 - (1+\rho)SI + \rho I - \rho I^2 - \frac{\beta SI}{S+I} - \mu S, \\ \frac{dI}{dt} &= \frac{\beta SI}{S+I} - (\alpha+\mu)I. \end{aligned} \tag{2.19}$$

Using the previous nonlocal discretization techniques R1-R8, the continuous system (2.19) can easily be transformed to the following NSFD system:

$$\begin{aligned} \frac{S_{n+1}-S_n}{\xi_1(h)} &= S_n - S_nS_{n+1} - (1+\rho)S_{n+1}I_n + \rho I_n - \frac{\rho S_{n+1}I_n^2}{S_n} - \frac{\beta S_{n+1}I_n}{S_n+I_n} - \mu S_{n+1}, \\ \frac{I_{n+1}-I_n}{\xi_2(h)} &= \frac{\beta S_{n+1}I_n}{S_n+I_n} - (\alpha+\mu)I_{n+1}, \end{aligned} \tag{2.20}$$

where the denominator functions $\xi_i(h)$, $i = 1, 2$, are such that $\xi_i(h) > 0$, $\forall h > 0$ and $\xi_i(h) = h + O(h^2)$. One should notice that the terms ρI and ρI^2 of the first equation of (2.19) have been discretized following (R5).
Rearranging (2.20), we get

$$\begin{aligned} S_{n+1} &= \frac{S_n\left\{1+\xi_1(h)\left(1+\frac{\rho I_n}{S_n}\right)\right\}}{1+\xi_1(h)\left\{S_n+(1+\rho)I_n+\frac{\rho I_n^2}{S_n}+\frac{\beta I_n}{S_n+I_n}+\mu\right\}}, \\ I_{n+1} &= \frac{I_n\left(1+\xi_2(h)\frac{\beta S_{n+1}}{S_n+I_n}\right)}{1+\xi_2(h)(\alpha+\mu)}. \end{aligned} \tag{2.21}$$

As expected, the NSFD system (2.21) is positively invariant; therefore, all solutions remain positive if they start with a positive initial value. The discrete system (2.21) has the same equilibrium points as the continuous system (2.18). The variational matrix at any arbitrary fixed point (S, I) of (2.21) is given by

$$J(S,I) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.22}$$

where

$$\begin{cases} a_{11} = \frac{1+\xi_1(h)}{1+\xi_1(h)\left\{S+(1+\rho)I+\frac{\rho I^2}{S}+\frac{\beta I}{S+I}+\mu\right\}} - \frac{\left\{1+\xi_1(h)\left(1+\frac{\rho I}{S}\right)\right\}\xi_1(h)S\left(1-\frac{\rho I^2}{S^2}-\frac{\beta I}{(S+I)^2}\right)}{\left[1+\xi_1(h)\left\{S+(1+\rho)I+\frac{\rho I^2}{S}+\frac{\beta I}{S+I}+\mu\right\}\right]^2}, \\ a_{12} = \frac{\rho\xi_1(h)}{1+\xi_1(h)\left\{S+(1+\rho)I+\frac{\rho I^2}{S}+\frac{\beta I}{S+I}+\mu\right\}} - \frac{\left\{1+\xi_1(h)\left(1+\frac{\rho I}{S}\right)\right\}\xi_1(h)\left\{(1+\rho)S+2\rho I+\frac{\beta S^2}{(S+I)^2}\right\}}{\left[1+\xi_1(h)\left\{S+(1+\rho)I+\frac{\rho I^2}{S}+\frac{\beta I}{S+I}+\mu\right\}\right]^2}, \\ a_{21} = \frac{\xi_2(h)\frac{\beta I}{S+I}}{1+\xi_2(h)(\alpha+\mu)}a_{11} - \frac{\xi_2(h)\frac{\beta SI}{(S+I)^2}}{1+\xi_2(h)(\alpha+\mu)}, \\ a_{22} = \frac{1+\xi_2(h)\frac{\beta S}{S+I}}{1+\phi_2(\alpha+\mu)} + \frac{\xi_2(h)\frac{\beta I}{S+I}}{1+\xi_2(h)(\alpha+\mu)}a_{12} - \frac{\xi_2(h)\frac{\beta SI}{(S+I)^2}}{1+\xi_2(h)(\alpha+\mu)}. \end{cases}$$

The following stability results for the discrete system (2.21) can be proved.

Theorem 2.13. *The disease-free fixed point $E_1^e = (1-\mu, 0)$ is locally asymptotically stable if $\mu < 1$, $\beta < \alpha+\mu$ and the endemic equilibrium point $E^{e*} = (S^{e*}, I^{e*})$ is locally asymptotically stable if $\beta > \alpha+\mu$ and $A < 0$, where $A = -\alpha-\mu-\alpha(\alpha+\mu)+\beta(\alpha+\mu)+\rho(\alpha+\mu)-\beta\rho$, i.e., E^{e*} is stable whenever it exists.*

Proof. It is not a difficult tusk to check that the eigenvalues evaluated at E_1^e are $\lambda_1 = \frac{1+\xi_1(h)\mu}{1+\xi_1(h)}$ and $\lambda_2 = \frac{1+\xi_2(h)\beta}{1+\xi_2(h)(\alpha+\mu)}$. Note that $0 < \lambda_1 < 1$ as $0 < \mu < 1$, and $\lambda_2 > 0$ for any positive step-size. Thus, for any $h > 0$,

$\lambda_2 < 1$ if $\beta < \alpha + \mu$. Therefore, if E_1^e exists then it will be stable if $\beta < \alpha + \mu$. In this case, the interior equilibrium point E^{e*} does not exist.
At the interior equilibrium point $E^{e*} = (S^{e*}, I^{e*})$, the variational matrix is given by

$$J(E^{e*}) = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix},$$

where

$$\begin{cases} a_{11}^* = 1 - \frac{\xi_1(h)}{G}\left\{\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}}\right) - \frac{\rho I^{e*2}}{S^{e*}} - \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right\}, \\ a_{12}^* = \frac{\xi_1(h)}{G}\left\{\rho - S^{e*}(1+\rho) - 2\rho I^{e*} - \frac{\beta S^{e*2}}{(S^{e*}+I^{e*})^2}\right\}, \\ a_{21}^* = \frac{\xi_2(h)}{H}\left\{\frac{\beta I^{e*}}{S^{e*}+I^{e*}} a_{11}^* - \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right\}, \\ a_{22}^* = 1 - \frac{\xi_2(h)}{H}\left\{\frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2} - \frac{\beta I^{e*} a_{12}^*}{S^{e*}+I^{e*}}\right\}, \\ G = 1 + \xi_1(h)\left(1 + \frac{\rho I^{e*}}{S^{e*}}\right), \quad H = 1 + \xi_2(h)\frac{\beta S^{e*}}{S^{e*}+I^{e*}}. \end{cases}$$

We shall use Lemma 2.3 to prove the local stability of E^{e*}. One can evaluate

$$\begin{aligned} \mathrm{trace}(J(E^{e*})) &= a_{11}^* + a_{22}^* \\ &= \left\{1 - \frac{\xi_1(h)}{G}\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}}\right)\right\} + \frac{\xi_1(h)}{G}\left(\frac{\rho I^{e*2}}{S^{e*}} + \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right) + \left\{1 - \left(\frac{I^{e*}}{S^{e*}+I^{e*}}\right)\left(\frac{\beta S^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right)\right\} \\ &\quad + \frac{\beta I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})GH}\left\{\rho(1 - S^{e*} - I^{e*}) - S^{e*} - \left(\rho I^{e*} + \frac{\beta S^{e*2}}{(S^{e*}+I^{e*})^2}\right)\right\} \\ &= \left\{1 - \frac{\xi_1(h)}{G}\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}} + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right)\right\} + \left\{1 - \left(\frac{I^{e*}}{S^{e*}+I^{e*}}\right)\left(\frac{\beta S^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right)\right\} \\ &\quad + \frac{\xi_1(h)}{G}\left(\frac{\rho I^{e*2}}{S^{e*}} + \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right)\left(1 - \frac{\beta S^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right) + \frac{\xi_1(h)\xi_2(h)\beta I^{e*}}{(S^{e*}+I^{e*})GH}\rho(1 - S^{e*} - I^{e*}). \end{aligned}$$

Following the existence condition of E^{e*}, we have $S^{e*} + I^{e*} = \frac{\beta A}{B} < 1$ and then $S^{e*} + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H} = \frac{1}{H}(S^{e*} + \xi_2(h)\beta S^{e*}) < 1$ and also $\xi_1(h)\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}}\right) < G$ and $\frac{\xi_2(h)\beta S^{e*}}{S^{e*}+I^{e*}} < H$.
Thus, $\left\{1 - \frac{\xi_1(h)}{G}\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}} + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right)\right\} > 0$. Hence we get $\mathrm{trace}(J(E^{e*})) > 0$.
Also,

$$\begin{aligned} \det(J(E^{e*})) &= a_{11}^* a_{22}^* - a_{12}^* a_{21}^* \\ &= a_{11}^*\left[1 - \frac{\xi_2(h)}{H}\left\{\frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2} - \frac{\beta I^{e*}}{S^{e*}+I^{e*}} a_{12}^*\right\}\right] - a_{12}^*\frac{\xi_2(h)}{H}\left\{\frac{\beta I^{e*}}{S^{e*}+I^{e*}} a_{11}^* - \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right\} \\ &= a_{11}^* - \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}(a_{11}^* - a_{12}^*). \end{aligned}$$

Simple algebraic manipulations show that

$$\begin{aligned} 1 - \det(J(E^{e*})) &= 1 - a_{11}^* + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}(a_{11}^* - a_{12}^*) \\ &= \frac{\xi_1(h)}{G}\left\{S^{e*} + \frac{\rho I^{e*}}{S^{e*}}(1 - I^{e*}) - \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right\} + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}\left(1 - \frac{\rho\xi_1(h)}{G} + \frac{\beta S^{e*}\xi_1(h)}{(S^{e*}+I^{e*})G}\right) \\ &\quad + \frac{\beta S^{e*} I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})^2 GH}\left(\frac{\rho I^{e*2}}{S^{e*}} + \rho S^{e*} + \rho I^{e*}\right) \\ &= \frac{\xi_1(h)}{G}\left\{S^{e*} + \frac{\rho I^{e*}}{S^{e*}}(1 - I^{e*})\right\} + \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\frac{1}{GH}\left\{-H\xi_1(h) + G\xi_2(h)\right. \\ &\quad \left. + \xi_1(h)\xi_2(h)\left(-\rho + \frac{\beta S^{e*}}{S^{e*}+I^{e*}}\right)\right\} + \frac{\beta S^{e*} I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})^2 GH}\left(\frac{\rho I^{e*2}}{S^{e*}} + \rho S^{e*} + \rho I^{e*}\right) \\ &= \frac{\xi_1(h)}{G}\left\{S^{e*} + \frac{\rho I^{e*}}{S^{e*}}(1 - I^{e*})\right\} + \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\frac{1}{GH}\{\xi_2(h) - \xi_1(h)\} \\ &\quad + \frac{\beta S^{e*} I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})^2 GH}\left\{(1 - \rho) + \frac{\rho I^{e*}}{S^{e*}} + \frac{\rho I^{e*2}}{S^{e*}} + \rho S^{e*} + \rho I^{e*}\right\}. \end{aligned}$$

Again,

$$\begin{aligned} 1 - \mathrm{trace}(J(E^{e*})) + \det(J(E^{e*})) &= 1 - (a_{11}^* + a_{22}^*) + (a_{11}^* a_{22}^* - a_{12}^* a_{21}^*) \\ &= 1 - a_{22}^* - \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}(a_{11}^* - a_{12}^*) \\ &= \frac{\beta I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}\{S^{e*}(1 - a_{11}^*) - I^{e*} a_{12}^*\} \\ &= \frac{\beta I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})^2 GH}\left(S^{e*2} + S^{e*} I^{e*} + \rho S^{e*} I^{e*} + \rho I^{e*2}\right) > 0. \end{aligned}$$

One can easily check that $1 + \mathrm{trace}(J(E^{e*})) + \det(J(E^{e*})) > 0$, as $\mathrm{trace}(J(E^{e*})) > 0$ and also $1 - \mathrm{trace}(J(E^{e*})) + \det(J(E^{e*})) > 0$. If we choose the denominator functions $\xi_1(h)$ and $\xi_2(h)$ such that $\xi_2(h) \geq \xi_1(h)$, $\forall h > 0$, then $1 - \det(J(E^{e*}))$ is also positive. An obvious choice is $\xi_i(h) = h$, $i = 1, 2$, $\forall h > 0$. Thus, the interior equilibrium point E^{e*} is stable whenever it exists. Hence the theorem is proven. ■

Remark 2.14. *The system* (2.18) *does not satisfy the conservation law. In such a case, following Mickens [28] rules, the denominator functions for the first and second equations will be* $\xi_1(h) = \frac{e^{\mu h}-1}{\mu}$ *and* $\xi_2(h) = \frac{e^{(\alpha+\mu)h}-1}{\alpha+\mu}$*, respectively. To hold the condition* $1 - \det(J(E^{e*})) > 0$*, the denominator functions* $\xi_i(h)$*,* $i = 1, 2$*, have to satisfy* $\xi_2(h) \geq \xi_1(h)$*. However, as mentioned above, the denominator functions* $\xi_1(h)$ *and* $\xi_2(h)$ *do not satisfy this restriction for the nonzero value of* α*.*

Numerical experiments

Again we construct the following Euler discrete system for the continuous-time (2.18)

$$\begin{aligned} S_{n+1} &= S_n + h\left\{(S_n + \rho I_n)(1 - S_n - I_n) - \frac{\beta S_n I_n}{S_n + I_n} - \mu S_n\right\}, \\ I_{n+1} &= I_n + h\left\{\frac{\beta S_n I_n}{S_n + I_n} - (\alpha + \mu) I_n\right\}, \end{aligned} \tag{2.23}$$

and compare its dynamics with the NSFD discrete system (2.21). We have plotted bifurcation diagrams for both the systems taking h as the bifurcation parameter (Figure 5). It shows that the dynamics of NSFD system (2.21) is independent of the step-size (Figure 5a), but the Euler discrete system (2.23) shows step-size dependent dynamics (Figure 5b) and produces spurious behaviour for higher step-size. Therefore, the Euler-discrete model is dynamically inconsistent, but the NSFD model is dynamically consistent.

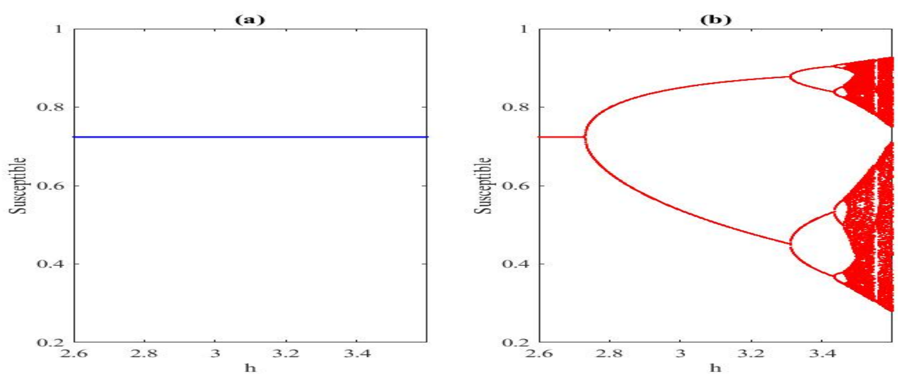

Figure 5: (a) Bifurcation diagram of the susceptible population with respect to step-size (h) for NSFD system (2.21). It shows that the population is stable for any positive value of the step size. (b) Bifurcation diagram of the susceptible population with respect to step-size (h) for Euler discrete system (2.23). It shows that the population becomes unstable as step-size h exceeds 2.73. Parameters are $\rho = 0.65$, $\beta = 0.45$, $\mu = 0.23$, $\alpha = 0.2$.

3. Summary

In the last two-three decades, nonstandard finite difference scheme has received significant interest in the discretization of the continuous system due to its superiority over other discretization techniques for various reasons. First, the transformed discrete system can be made positively invariant using proper nonlocal discretization techniques, though the standard discretization techniques often fail. Secondly, the NSFD model can be shown to be dynamically consistent with its continuous counterpart, which means the stability property of each equilibrium point of the continuous system remains the same for the NSFD model. However, in many cases, the discrete model formulated by the standard discretization technique shows (spurious) dynamics that are not at all the dynamics of the original continuous system. Another great advantage of the NSFD technique is that the dynamics, in this case, can be shown to be independent of the step size, which can reduce the computational cost. There are two main steps in the construction of an NSFD system from a given continuous system of first-order differential equations, viz. discretization of the first-order derivative of the continuous system, where one has to choose a denominator function and discretize the interaction terms, where one has to use nonlocal discretization for both the linear and nonlinear terms of the differential equation. Unfortunately, there is no general rule for both of these steps [5, 26]. However, some techniques have been defined [5, 26, 28] and successfully preserved both the positivity and dynamic properties of (relatively simple) continuous systems. However, previous techniques of choosing the denominator function may fail in many cases to preserve the dynamic properties of the continuous system. This study extends other studies mainly in two ways. First, we have defined some uniform rules for nonlocal discretization that one can follow while using the NSFD scheme. Secondly, the selection of the denominator function plays a crucial role in proving the dynamic consistency of the discrete model with its continuous systems. Mickens and others have defined some denominator functions for conservative and nonconservative systems. Such a predetermined form of the denominator function may not work well, and the dynamics of the discrete system constructed after nonlocal discretization may depend on the step size [36]. Instead of considering such a predetermined denominator function, we here show that the denominator function can be selected from the stability conditions of the transformed discrete system. Using our uniform rules for the nonlocal discretization of a continuous positive system, we have shown that highly complex population models not only preserve the positivity and dynamic consistency of the continuous system, but the dynamics also become independent of step-size, which has significant computational facility, especially for coupled systems.

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Fixed points of multiplicative closed graph operators on b-multiplicative metric spaces

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Abstract. This article discusses fixed point iterations of some multiplicative contraction mappings with variations in b-multiplicative metric space domains. A map that satisfies the multiplicative contraction condition has been proposed for an increasing sequence of subsets of a b-multiplicative metric space, where one element of the sequence is mapped into the following member of the sequence. Additionally, several fixed point conclusions are established for various multiplicative contraction mappings with multiplicative closed graphs.

AMS Subject Classifications: Primary 54E40; Secondary 54H25.

Keywords: Multiplicative contraction; b-Multiplicative metric space; Multiplicative closed graph, exponential transformation.

Contents

1. Introduction

W. A. Kirk et al. [10] proposed using a cycle of domains to derive various fixed point theorems for metric spaces. C. G. Moorthy and P. X. Raj considered an increasing sequence of subsets $\Xi_1 \subseteq \Xi_2 \subseteq ...$ of a metric space (Ξ, d), and a map $G : \Xi \to \Xi$ satisfying a contraction condition such that $G(\Xi_i) \subseteq \Xi_{i+1}, \forall i$, and $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$ in [11]. Also, the fixed point results of [11] are generalized in some articles [14–16].

A.E. Bashirov et al. introduced multiplicative metric space (also known as MMS) in [5]. M. Ozavsar and A. C. Cevikel [13] developed topological features of multiplicative metric spaces (or MMSs) and established fixed point findings in MMSs. There are numerous papers [1–3, 7–9, 12, 17] for fixed point theory in MMSs.

b-Metric space, a generalisation of a metric space, was first introduced by Czerwik [6]. b-MMS was introduced by M. U. Ali et al in [4]. There are some topological properties and fixed point results in b-MMSs.

By variations in b-MMS domains, we prove some more fixed point theorems for different types of multiplicative contraction mappings with multiplicative closed graphs. Also, we generalize a main result of [11] and we derive that result by using exponential transformation.

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2. b-Multiplicative metric spaces

Let us give some preliminary and known results in this section. See [4] for further information.

Definition 2.1. *[4] Assuming that $\Xi \neq \emptyset$ is a set and $s \in \mathbb{R}$ with $s \geq 1$. A multiplicative metric is a mapping $d : \Xi \times \Xi \to \mathbb{R}^+ = [0, \infty)$ satisfying the next four axioms.*

(i) $d(\kappa, \iota) \geq 1$, $\forall\, \kappa, \iota \in \Xi$,

(ii) $d(\kappa, \iota) = 1$ *if and only if(or, iff)* $\kappa = \iota$ *in* Ξ,

(iii) $d(\kappa, \iota) = d(\iota, \kappa)$, $\forall\, \kappa, \iota \in \Xi$,

(iv) $d(\kappa, \iota) \leq [d(\kappa, \rho) d(\rho, \iota)]^s$, $\forall\, \kappa, \iota, \rho \in \Xi$.

The triple (Ξ, d, s) is then referred to as a b-MMS.

Definition 2.2. *[4] Assuming that (Ξ, d, s) is a b-MMS, $\{\kappa_n\}$ is a sequence in Ξ, and $\kappa \in \Xi$. Then $\{\kappa_n\}$ is called multiplicative converging to κ, if for every multiplicative open ball $B_\epsilon(\kappa) = \{\iota : d(\kappa, \iota) < \epsilon\}$, $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $\kappa_n \in B_\epsilon(\kappa)$, $\forall\, n > N$. It is denoted by $\kappa_n \to \kappa (n \to \infty)$.*

Lemma 2.3. *[4] Assuming that (Ξ, d, s) is a b-MMS, $\{\kappa_n\}$ is a sequence in Ξ and $\kappa \in \Xi$. Then $\kappa_n \to \kappa (n \to \infty)$ iff $d(\kappa_n, \kappa) \to 1 (n \to \infty)$.*

Lemma 2.4. *[4] Assuming that (Ξ, d, s) is a b-MMS, and $\{\kappa_n\}$ is a sequence in Ξ. Then every multiplicative convergent sequence $\{\kappa_n\}$ has an unique multiplicative limit point.*

Definition 2.5. *[4] Assuming that (Ξ, d, s) is a b-MMS. The sequence $\{\kappa_n\} \in \Xi$ is called a multiplicative Cauchy sequence(or, MCS) if for every $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(\kappa_n, \kappa_m) < \epsilon, \forall m, n \geq N$.*

Lemma 2.6. *[4] Assuming that (Ξ, d, s) is a b-MMS and $\{\kappa_n\}$ is a sequence in Ξ. Then $\{\kappa_n\}$ is a MCS iff $d(\kappa_n, \kappa_m) \to 1 (m, n \to \infty)$.*

Definition 2.7. *[4] Assuming that (Ξ, d, s) is a b-MMS. Then (Ξ, d, s) is said to be multiplicative complete, if every MCS is multiplicative convergent in Ξ.*

Theorem 2.8. *[4] Assuming that (Ξ, d, s) is a b-MMS. Let $\{\kappa_n\}$ and $\{\iota_n\}$ be two sequences in Ξ such that $\kappa_n \to \kappa$, $\iota_n \to \iota$ $(n \to \infty)$, $\kappa, \iota \in \Xi$. Then $d(\kappa_n, \iota_n) \to d(\kappa, \iota)(n \to \infty)$.*

Definition 2.9. *Assuming that $G : (\Xi, d, s) \to (\Xi, d, s)$ is a self mapping on a b-MMS (Ξ, d, s). If whenever $\kappa_n \to \kappa_0$ and $G\kappa_n \to \iota_0$ for some sequence $\{\kappa_n\}$ in Ξ and some κ_0, ι_0 in Ξ, we have $\iota_0 = G\kappa_0$, then G is said to have a multiplicative closed graph(or, MCG).*

3. Main results

Let's prove some fixed point theorems for various multiplicative contractions on b-MMSs in this section.

Theorem 3.1. *Assuming that (Ξ, d, s) is a complete b-MMS, and $G : \Xi \to \Xi$ have a MCG. Let $\Xi_1 \subseteq \Xi_2 \subseteq ...$ be subsets of Ξ such that $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $d(Gt, Gz) \leq d(t, z)^{\xi_i}$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, \infty)$ are real positive constants such that $\sum_{n=1}^{\infty} s^n \xi_1 \xi_2 ... \xi_n < \infty$. Then, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ multiplicative converges to a fixed point.*
Moreover, if $\xi_i \in (0, 1)$, $\forall i$, then G has a unique fixed point(or, UFP) in Ξ.

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\, n = 1, 2, 3, ...$ Then we have,

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}} \\ &\leq d(Gt_1, t_1)^{\xi_{n+1}\xi_n\xi_{n-1}\ldots\xi_2}. \end{aligned}$$

Further, for $1 \leq n < m$, we have,

$$\begin{aligned} d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1}t_1)^{s^{m-1}} d(G^{m-1}t_1, G^{m-2}t_1)^{s^{m-2}} \ldots d(G^{n+1}t_1, G^n t_1)^{s^n} \\ &\leq d(Gt_1, t_1)^{\left(\sum\limits_{i=n}^{m-1} s^i \xi_2\xi_3\ldots\xi_{i+1}\right)}. \end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1$ $(m, n \to \infty)$. Since Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to t^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1}t_1\}_{m=1}^{\infty}$ is also an MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
Assuming additionally that $\xi_i \in (0, 1)$, $\forall i$.
If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) \leq d(t^*, z^*)^{\xi_n}.$$

Then, $d(t^*, z^*) \leq d(t^*, z^*)^{(\xi_n)^m}$, $\forall\, m \in \mathbb{N}$. Since $(\xi_n)^m \to 0$ as $m \to \infty$, $d(t^*, z^*) = 1$ and $t^* = z^*$. Hence, G has a UFP. ■

Corollary 3.2. *Assuming that (Ξ, D) is a complete metric space, and $G : \Xi \to \Xi$ have a closed graph. Let $\Xi_1 \subseteq \Xi_2 \subseteq \ldots$ be subsets of Ξ such that $\Xi = \bigcup\limits_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $D(Gt, Gz) \leq \xi_i D(t, z)$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, \infty)$ are real positive constants such that $\sum\limits_{n=1}^{\infty} \xi_1\xi_2\ldots\xi_n < \infty$.*
Then, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ converges to a fixed point. Also, if $\xi_i \in (0, 1)$, $\forall i$, then G has a UFP in Ξ.

Proof. Let $d = \exp D$. That is $d(t, z) = \exp D(t, z)$, $\forall t, z \in \Xi$. Then (Ξ, d) is a complete b-MMS with $s = 1$. Also, $d(Gt, Gz) \leq (d(t, z))^{\xi_i}$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, \infty)$ are real positive constants such that $\sum\limits_{n=1}^{\infty} \xi_1\xi_2\ldots\xi_n < \infty$. Theorem 3.1 now leads to Corollary 3.2. ■

The above Corollary is Theorem 2.1 of [11]

Example 3.3. *Let $\Xi = \left[\frac{1}{4}, \infty\right)$. Assuming that $d(t, z) = \max\{tz^{-1}, zt^{-1}\}$, $\forall t, z \in \Xi$.*
Then (Ξ, d, s) is a complete b-MMS with $s = 1$.
Assuming that $\Xi_n = [\frac{1}{4}, n]$, and $\xi_n = \frac{n^2}{(n+1)^2} \in [\frac{1}{4}, 1)$, for $n = 1, 2, 3, ...$ Then $\sum\limits_{n=1}^{\infty} s^n \xi_1\xi_2\ldots\xi_n < \infty$.
Define $G : \Xi \to \Xi$ by $Gt = t^{\frac{1}{4}}$, if $t \in \Xi_n$, for $n \in \mathbb{N}$.
For $t, z \in \Xi_n$, we get

$$\begin{aligned} d(Gt, Gz) &= \max\left\{\left(\frac{t}{z}\right)^{\frac{1}{4}}, \left(\frac{z}{t}\right)^{\frac{1}{4}}\right\} \\ &= (d(t, z))^{\frac{1}{4}} \\ &\leq d(t, z)^{\xi_n},\ \forall n \in \mathbb{N}. \end{aligned}$$

Theorem 3.1's hypotheses are then fulfilled. Moreover, the UFP is 1.

Theorem 3.4. *Assuming that* (Ξ, d, s) *is a complete b-MMS, and* $G : \Xi \to \Xi$ *have a MCG. Let* $\Xi_1 \subseteq \Xi_2 \subseteq ...$ *be subsets of* Ξ *such that* $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, *and* $d(Gt, Gz) \leq (d(Gt,t)d(Gz,z))^{\xi_i}$, $\forall t, z \in \Xi_i, \forall i$, *where* $\xi_i \in (0,1)$ *are real positive constants such that* $\sum_{n=1}^{\infty} s^n \vartheta_1 \vartheta_2 ... \vartheta_n < \infty$, *where* $\vartheta_i = \frac{\xi_i}{1-\xi_i}$, $\forall i$. *Then* G *has a UFP in* Ξ.
Moreover, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ *multiplicative converges to the UFP.*

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\ n = 1, 2, ...$ Then we have

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq (d(G^{n+1}t_1, G^n t_1) d(G^n t_1, G^{n-1} t_1))^{\xi_{n+1}} \\ &= d(G^{n+1}t_1, G^n t_1)^{\xi_{n+1}} d(G^n t_1, G^{n-1} t_1)^{\xi_{n+1}}. \end{aligned}$$

Now, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1} t_1)^{\frac{\xi_{n+1}}{1-\xi_{n+1}}} \\ &= d(G^n t_1, G^{n-1} t_1)^{\vartheta_{n+1}}, \\ &\leq d(Gt_1, t_1)^{\vartheta_{n+1}\vartheta_n \vartheta_{n-1} ... \vartheta_2}. \end{aligned}$$

Further, for $1 \leq n < m$, we have

$$\begin{aligned} d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1} t_1)^{s^{m-1}} d(G^{m-1} t_1, G^{m-2} t_1)^{s^{m-2}} ... d(G^{n+1} t_1, G^n t_1)^{s^n} \\ &\leq d(Gt_1, t_1)^{\left(\sum_{i=n}^{m-1} s^i \vartheta_2 \vartheta_3 ... \vartheta_{i+1} \right)}. \end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1 (m, n \to \infty)$. By Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to w^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1} t_1\}_{m=1}^{\infty}$ is also an MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) \leq (d(Gt^*, t^*) d(Gz^*, z^*))^{\xi_n} = 1.$$

Therefore $t^* = z^*$. Hence, G has a UFP. ■

Corollary 3.5. *Assuming that* (Ξ, D) *be a complete metric space, and* $G : \Xi \to \Xi$ *have a closed graph. Let* $\Xi_1 \subseteq \Xi_2 \subseteq ...$ *be subsets of* Ξ *such that* $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, *and* $D(Gt, Gz) \leq \xi_i (D(Gt, t) + D(Gz, z))$, $\forall t, z \in \Xi_i, \forall i$, *where* $\xi_i \in (0,1)$ *are real positive constants such that* $\sum_{n=1}^{\infty} \vartheta_1 \vartheta_2 ... \vartheta_n < \infty$, *where* $\vartheta_i = \frac{\xi_i}{1-\xi_i}$, $\forall i$. *Then* G *has a UFP in* Ξ. *Moreover, for any fixed* $t_1 \in \Xi$, $\{G^n t_1\}$ *converges to the UFP.*

Proof. Let $d = \exp D$. That is $d(t, z) = \exp D(t, z)$, $\forall t, z \in \Xi$. Then (Ξ, d) is a complete b-MMS with $s = 1$. Also, $d(Gt, Gz) \leq (d(Gt, t) d(Gz, z))^{\xi_i}$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, \infty)$ are real positive constants such that $\sum_{n=1}^{\infty} \vartheta_1 \vartheta_2 ... \vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i}{1-\xi_i}$, $\forall i$. Theorem 3.4 now leads to Corollary 3.5. ■

Theorem 3.6. *Assuming that* (Ξ, d, s) *is a complete b-MMS, and* $G : \Xi \to \Xi$ *have a MCG. Let* $\Xi_1 \subseteq \Xi_2 \subseteq ...$ *be subsets of* Ξ *such that* $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, *and* $d(Gt, Gz) \leq (d(Gt, z) d(Gz, t))^{\xi_i}$, $\forall t, z \in \Xi_i, \forall i$,

where $\xi_i \in (0, \frac{1}{2})$ are real positive constants such that $\sum_{n=1}^{\infty} s^n \vartheta_1 \vartheta_2 ... \vartheta_n < \infty$, where $\vartheta_i = \frac{s\xi_i}{1-s\xi_i}$, $\forall i$. Then G has a UFP in Ξ.
Moreover, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ multiplicative converges to the UFP.

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\ n = 1, 2, 3, ...$ Then we have

$$d(G^{n+1}t_1, G^n t_1) \leq (d(G^{n+1}t_1, G^{n-1}t_1) d(G^n t_1, G^n t_1))^{\xi_{n+1}}$$

Since $d(G^n t_1, G^n t_1) = 1$, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^{n+1}t_1, G^{n-1}t_1)^{\xi_{n+1}} \\ &\leq (d(G^{n+1}t_1, G^n t_1) d(G^n t_1, G^{n-1}t_1))^{s\xi_{n+1}} \\ &= d(G^{n+1}t_1, G^n t_1)^{s\xi_{n+1}} d(G^n t_1, G^{n-1}t_1)^{s\xi_{n+1}}. \end{aligned}$$

Now, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\left(\frac{s\xi_{n+1}}{1-s\xi_{n+1}}\right)} \\ &= d(G^n t_1, G^{n-1}t_1)^{\vartheta_{n+1}}, \\ &\leq d(Gt_1, t_1)^{\vartheta_{n+1}\vartheta_n \vartheta_{n-1} ... \vartheta_2}. \end{aligned}$$

Further, for $1 \leq n < m$, we have

$$\begin{aligned} d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1}t_1)^{s^{m-1}} d(G^{m-1}t_1, G^{m-2}t_1)^{s^{m-2}} ... d(G^{n+1}t_1, G^n t_1)^{s^n} \\ &\leq d(Gt_1, t_1)^{\left(\sum_{i=n}^{m-1} s^i \vartheta_2 \vartheta_3 ... \vartheta_{i+1}\right)}. \end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1\ (m, n \to \infty)$. By Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to w^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1}t_1\}_{m=1}^{\infty}$ is also a MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$\begin{aligned} 1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) &\leq (d(Gt^*, z^*) d(Gz^*, t^*))^{\xi_n} \\ &= d(t^*, z^*)^{2\xi_n}. \end{aligned}$$

Then, $d(t^*, z^*) \leq d(t^*, z^*)^{(2\xi_n)^m}$, $\forall\ m \in \mathbb{N}$. Since $(2\xi_n)^m \to 0$ as $m \to \infty$, $d(t^*, z^*) = 1$ and $t^* = z^*$. Hence, G has a UFP. ■

Corollary 3.7. *Assuming that (Ξ, D) be a complete metric space, and $G : \Xi \to \Xi$ have a closed graph. Let $\Xi_1 \subseteq \Xi_2 \subseteq ...$ be subsets of Ξ such that $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $D(Gt, Gz) \leq \xi_i(D(Gt, z) + D(Gz, t))$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, 1)$ are real positive constants such that $\sum_{n=1}^{\infty} \vartheta_1 \vartheta_2 ... \vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i}{1-\xi_i}$, $\forall i$. Then G has a UFP in Ξ. Moreover, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ converges to the UFP.*

Proof. Let $d = \exp D$. That is $d(t, z) = \exp D(t, z)$, $\forall t, z \in \Xi$. Then (Ξ, d) is a complete b-MMS with $s = 1$. Also, $d(Gt, Gz) \leq (d(Gt, z) d(Gz, t))^{\xi_i}$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, \infty)$ are real positive constants such that $\sum_{n=1}^{\infty} \vartheta_1 \vartheta_2 ... \vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i}{1-\xi_i}$, $\forall i$. Theorem 3.6 now leads to Corollary 3.7. ■

Theorem 3.8. *Assuming that* (Ξ, d, s) *is a complete b-MMS, and* $G : \Xi \to \Xi$ *have a MCG. Let* $\Xi_1 \subseteq \Xi_2 \subseteq \ldots$ *be subsets of* Ξ *such that* $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, *and* $d(Gt, Gz) \leq d(t,z)^{\xi_i} d(z, Gt)^{\mu_i}$, $\forall t, z \in \Xi_i, \forall i$, *where* $\xi_i, \mu_i \in (0,1)$ *are real positive constants such that* $\xi_i + \mu_i < 1$, $\forall i$, *and* $\sum_{n=1}^{\infty} s^n \vartheta_1 \vartheta_2 \ldots \vartheta_n < \infty$, *where* $\vartheta_i = \frac{\xi_i + s\mu_i}{1 - s\mu_i}$, $\forall i$. *Then* G *has a UFP in* Ξ.
Moreover, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ *multiplicative converges to the UFP.*

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\, n = 1, 2, 3, \ldots$. Then we have

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}} d(G^{n-1}t_1, G^{n+1}t_1)^{\mu_{n+1}} \\ &\leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}} d(G^{n-1}t_1, G^n t_1)^{s\mu_{n+1}} d(G^n t_1, G^{n+1}t_1)^{s\mu_{n+1}} \\ &= d(G^n t_1, G^{n-1}t_1)^{(\xi_{n+1} + s\mu_{n+1})} d(G^n t_1, G^{n+1}t_1)^{s\mu_{n+1}} \end{aligned}$$

Now, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\left(\frac{\xi_{n+1} + s\mu_{n+1}}{1 - s\mu_{n+1}}\right)} \\ &= d(G^n t_1, G^{n-1}t_1)^{\vartheta_{n+1}}, \\ &\leq d(Gt_1, t_1)^{\vartheta_{n+1}\vartheta_n \vartheta_{n-1} \ldots \vartheta_2}. \end{aligned}$$

Further, for $1 \leq n < m$, we have

$$\begin{aligned} d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1}t_1)^{s^{m-1}} d(G^{m-1}t_1, G^{m-2}t_1)^{s^{m-2}} \ldots d(G^{n+1}t_1, G^n t_1)^{s^n} \\ &\leq d(Gt_1, t_1)^{\left(\sum_{i=n}^{m-1} s^i \vartheta_2 \vartheta_3 \ldots \vartheta_{i+1}\right)}. \end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1 (m, n \to \infty)$. By Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to t^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1} t_1\}_{m=1}^{\infty}$ is also an MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
If $Gt^* = t^*, Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$\begin{aligned} 1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) &\leq d(t^*, z^*)^{\xi_n} d(z^*, Gt^*)^{\mu_n} \\ &\leq d(t^*, z^*)^{(\xi_n + \mu_n)}. \end{aligned}$$

Then, $d(t^*, z^*) \leq d(t^*, z^*)^{(\xi_n + \mu_n)^m}$, $\forall\, m \in \mathbb{N}$. Since $(\xi_n + \mu_n)^m \to 0$ as $m \to \infty$, $d(t^*, z^*) = 1$ and $t^* = z^*$. Hence, G has a UFP. ■

Remark 3.9. *In Theorem 3.8, replacement of the condition* $d(Gt, Gz) \leq d(t,z)^{\xi_i} d(z, Gt)^{\mu_i}$, $\forall t, z \in \Xi_i, \forall i$, *where* $\xi_i, \mu_i \in (0,1)$ *are real positive constants such that* $\xi_i + \mu_i < 1$, $\forall i$, *and* $\sum_{n=1}^{\infty} s^n \vartheta_1 \vartheta_2 \ldots \vartheta_n < \infty$, *where* $\vartheta_i = \frac{\xi_i + s\mu_i}{1 - s\mu_i}$, $\forall i$ *by the condition* $d(Gt, Gz) \leq d(Gt, Gz)^{\xi_i} d(z, Gt)^{\mu_i}$, $\forall t, z \in \Xi_i, \forall i$, *where* $\xi_i, \mu_i \in (0,1)$ *are real positive constants such that* $\xi_i + \mu_i < 1$, $\forall i$, *and* $\sum_{n=1}^{\infty} s^n \vartheta_1 \vartheta_2 \ldots \vartheta_n < \infty$, *where* $\vartheta_i = \frac{s\mu_{i+1}}{1 - s\mu_{i+1} - \xi_{i+1}}$, $\forall i$ *gives a UFP.*

Theorem 3.10. *Assuming that* (Ξ, d, s) *is a complete b-MMS, and* $G : \Xi \to \Xi$ *have a MCG. Let* $\Xi_1 \subseteq \Xi_2 \subseteq \ldots$ *be subsets of* Ξ *such that* $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, *and*

$d(Gt,Gz) \leq d(t,z)^{\xi_i}(d(Gt,t)d(Gz,z))^{\mu_i}(d(Gt,z)d(Gz,t))^{\nu_i}$, $\forall t,z \in \Xi_i, \forall i$, where $\xi_i, \mu_i, \nu_i \in (0,\frac{1}{2})$ are real positive constants such that $\xi_i + 2\nu_i < 1$, $\forall i$, and $\sum\limits_{n=1}^{\infty} s^n \vartheta_1\vartheta_2...\vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i+\mu_i+s\nu_i}{1-\mu_i-s\nu_i}$, $\forall i$. Then G has a UFP in Ξ.
Moreover, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ multiplicative converges to the UFP.

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\, n = 1,2,3,....$ Then we have

$$d(G^{n+1}t_1, G^n t_1) \leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}}(d(G^{n+1}t_1, G^n t_1)d(G^n t_1, G^{n-1}t_1))^{\mu_{n+1}} (d(G^{n+1}t_1, G^{n-1}t_1)d(G^n t_1, G^n t_1))^{\nu_{n+1}}.$$

Since $d(G^n t_1, G^n t_1) = 1$, we get

$$d(G^{n+1}t_1, G^n t_1) \leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}}(d(G^{n+1}t_1, G^n t_1)d(G^n t_1, G^{n-1}t_1))^{\mu_{n+1}} (d(G^{n+1}t_1, G^n t_1)d(G^n t_1, G^{n-1}t_1))^{s\nu_{n+1}}.$$

Now, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\left(\frac{\xi_{n+1}+\mu_{n+1}+s\nu_{n+1}}{1-\mu_{n+1}-s\nu_{n+1}}\right)} \\ &= d(G^n t_1, G^{n-1}t_1)^{\vartheta_{n+1}}, \\ &\leq d(Gt_1, t_1)^{\vartheta_{n+1}\vartheta_n\vartheta_{n-1}...\vartheta_2}. \end{aligned}$$

Further, for $1 \leq n < m$, we have

$$\begin{aligned} d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1}t_1)^{s^{m-1}} d(G^{m-1}t_1, G^{m-2}t_1)^{s^{m-2}}...d(G^{n+1}t_1, G^n t_1)^{s^n} \\ &\leq d(Gt_1, t_1)^{\left(\sum\limits_{i=n}^{m-1} s^i \vartheta_2\vartheta_3...\vartheta_{i+1}\right)}. \end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1 (m,n \to \infty)$. By Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to t^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1}t_1\}_{m=1}^{\infty}$ is also an MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$\begin{aligned} 1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) &\leq d(t^*, z^*)^{\xi_n}(d(Gt^*, t^*)d(Gz^*, z^*))^{\mu_n}(d(Gt^*, z^*)d(Gz^*, t^*))^{\nu_n} \\ &\leq d(t^*, z^*)^{(\xi_n+2\nu_n)}. \end{aligned}$$

Then, $d(t^*, z^*) \leq d(t^*, z^*)^{(\xi_n+2\nu_n)^m}$, $\forall\, m \in \mathbb{N}$. Since $(\xi_n + 2\nu_n)^m \to 0$ as $m \to \infty$, $d(t^*, z^*) = 1$ and $t^* = z^*$. Hence, G has a UFP. ∎

Theorem 3.11. *Assuming that (Ξ, d, s) is a complete b-MMS. Let $G : \Xi \to \Xi$ have a MCG. Let $\xi_i \in (0,1)$, $\forall i$, such that for $1 \leq n \leq m$, $\frac{s^n\xi_1\xi_2...\xi_m}{1-s^n\xi_n} \to 0$ as $n \to \infty$, and let $\Xi_1 \subseteq \Xi_2 \subseteq ...$ be subsets of Ξ such that $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $d(Gt,Gz) \leq d(t,z)^{\xi_i}$, $\forall t \in \Xi_i$, $\forall z \in \Xi, \forall i$. Suppose $t_1 \in \bigcup\limits_{j=1}^{\infty} \Xi_j$. Then $\{G^n t_1\}$ multiplicative converges to the fixed point of G in Ξ. If $\Xi = \bigcup\limits_{j=1}^{\infty} \Xi_j$, then G has a UFP in Ξ.*

Proof. Fix $t_1 \in \Xi$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\, n \in \mathbb{N}$. Then for each n, we have

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\xi_n} \\ &\leq d(Gt_1, t_1)^{\xi_n\xi_{n-1}...\xi_1}. \end{aligned}$$

Therefore, $d(t_{n+1}, t_n) \leq d(Gt_1, t_1)^{\xi_{n-1}\xi_{n-2}\ldots\xi_1}$. Further, for $1 \leq n < m$, we have

$$\begin{aligned} d(t_n, t_m) &\leq d(t_n, t_{n+1})^{s^n} d(t_{n+1}, t_{m+1})^{s^{n+1}} d(t_m, t_{m+1})^{s^m} \\ &\leq d(Gt_1, t_1)^{s^n \xi_1 \xi_2 \ldots \xi_{n-1}} d(t_n, t_m)^{s^{n+1}\xi_{n+1}} d(Gt_1, t_1)^{s^m \xi_1 \xi_2 \ldots \xi_{m-1}} \end{aligned}$$

so that,

$$d(t_n, t_m) \leq d(Gt_1, t_1)^{\frac{(s^n \xi_1 \xi_2 \ldots \xi_{n-1}) + (s^m \xi_1 \xi_2 \ldots \xi_{m-1})}{(1 - s^{n+1}\xi_{n+1})}}.$$

Therefore, $d(t_n, t_m) \to 1$ $(n, m \to \infty)$. Since Lemma 2.6, $\{t_n\}$ is a MCS. By Ξ is multiplicative complete, $\{t_n\} \to w^*$, for some t^* in Ξ. Then $\{Gt_n\} \to t^*$ and $Gt^* = t^*$, because G has a MCG. Hence, we obtained a fixed point t^* of G.

These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.

Assuming now $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$. If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$\begin{aligned} 1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) &\leq d(t^*, z^*)^{\xi_n} \\ &\leq d(t^*, z^*)^{(\xi_n)^m}, \ \forall\, m > 1. \end{aligned}$$

Therefore, $d(t^*, z^*) = 1$, because $(\xi_n)^m \to 0$ as $m \to \infty$. Hence, G has a UFP, when $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$. ■

Theorem 3.12. *Assuming that (Ξ, d, s) is a complete b-MMS, and $G : \Xi \to \Xi$ have a MCG. Suppose $d(t, z) \leq \alpha, \forall t, z \in \Xi$ and for some $\alpha \in [1, \infty)$. Let $\xi_i \in (0, 1)$, $\forall i$, be such that $\xi_1 \xi_2 \ldots \xi_n \to 0$ as $n \to \infty$. Suppose $\Xi_1 \subseteq \Xi_2 \subseteq \ldots$ be subsets of Ξ such that $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $d(Gt, Gz) \leq d(t, z)^{\xi_i}$, $\forall t \in \Xi_i$, $\forall z \in \bigcup_{j=1}^{\infty} \Xi_j$, $\forall i$. Let $t_1 \in \bigcup_{j=1}^{\infty} \Xi_j$. Then the sequence $\{G^n t_1\}$ multiplicative converges to a unique fixed point G in Ξ. If $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, then G has a UFP in Ξ.*

Proof. Fix $t_1, z_1 \in \Xi_1$. Set $t_{n+1} = Gt_n = G^n t_1$, and $z_{n+1} = Gz_n = G^n z_1$, $\forall\, n \in \mathbb{N}$. For $m < n$, we have

$$\begin{aligned} d(t_n, z_m) = d(Gt_{n-1}, Gz_{m-1}) &\leq d(t_{n-1}, z_{m-1})^{\xi_{m-1}} \\ &\leq d(t_{n-m+1}, z_1)^{\xi_{m-1}\xi_{m-2}\ldots\xi_2\xi_1} \\ &\leq \alpha^{\xi_{m-1}\xi_{m-2}\ldots\xi_2\xi_1}, \end{aligned}$$

because $d(t, z) \leq \alpha, \forall t, z \in \Xi$. Hence, $d(t_n, z_m) \to 1$ as $m, n \to \infty$. Also $d(t_n, t_m) \to 1$, and $d(z_n, z_m) \to 1$ as $m, n \to \infty$. So, $\{t_n\}$ and $\{z_n\}$ are multiplicative Cauchy sequences in Ξ, because of Lemma 2.6. By (Ξ, d) is multiplicative complete, $\{t_n\}$ and $\{z_n\}$ multiplicative converges to a unique point t^* in Ξ, because of Lemma 2.8. Since $\{t_n\} \to t^*$, we have $\{Gt_n\} \to t^*$. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.

These processes can be extended to the general case: $t_1, z_1 \in \Xi_n$, for some n.

Assuming now $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$. If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) \leq d(t^*, z^*)^{\xi_n} \leq d(t^*, z^*)^{(\xi_n)^m}, \ \forall\, m > 1.$$

So, $d(t^*, z^*) = 1$, because $(\xi_n)^m \to 0$ as $m \to \infty$. Therefore, G has a UFP, when $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$. ■

4. Conclusion

If $s = 1$ in a b-multiplicative metric space (Ξ, d, s), then it becomes a multiplicative metric space. All fixed point results can be converted from b-multiplicative metric spaces to metric spaces through exponential transformation. It has been illustrated in Corollary 3.2, Corollary 3.5, and Corollary 3.7. As a result, studies of fixed points of multiplicative contractions in b-multiplicative metric spaces are important.

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Characterizations for pseudoparalel submanifolds of Lorentz-Sasakian space forms

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Abstract. In this study, invariant total geodesic submanifolds, which are important submanifolds of Lorentz-Sasakian space forms, have been investigated. An important class of the considered invariant submanifolds, called pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and 2-Ricci generalized pseudoparallel invariant submanifolds, has been defined and the characterizations of Lorentz-Sasakian space forms for these types of invariant submanifolds have been revealed. Then, conditions are given for these obtained invariant submanifolds to be total geodesic by means of concircular and projective curvature tensors.

AMS Subject Classifications: 53C15; 53C44, 53D10.

Keywords: Lorentz-Sasakian Space Forms, Lorentzian Manifold, Total Geodesic Submanifold.

Contents

1. Introduction

$\phi-$sectional curvature plays the an important role for Sasakian manifold. If the $\phi-$sectional curvature of a Sasakian manifold is constant, then the manifold is a Sasakian-space-form [1]. P. Alegre and D. Blair described generalized Sasakian space forms [2]. P. Alegre and D. Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. P. Alegre and A. Carriazo later discussed generalized indefinite Sasakian space forms [3]. Generalized indefinite Sasakian space forms are also called Lorentz-Sasakian space forms, and Lorentz manifolds are of great importance for Einstein's theory of Relativity. Sasakian space forms, generalized Sasakian space forms and Lorentz-Sasakian space forms have been discussed by many scientists and important properties of these manifolds have been obtained ([4]-[8]).

Many mathematicians have considered the submanifolds of manifolds such as $K-$paracontak, Lorentzian para-Kenmotsu, almost Kenmotsu and studied their various characterizations ([9],[10],[11]).

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In this study, invariant total geodesic submanifolds, which are important submanifolds of Lorentz-Sasakian space forms, have been investigated. An important class of the considered invariant submanifolds, called pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and 2-Ricci generalized pseudoparallel invariant submanifolds, has been defined and the characterizations of Lorentz-Sasakian space forms for these types of invariant submanifolds have been revealed. Then, conditions are given for these obtained invariant submanifolds to be total geodesic by means of concircular and projective curvature tensors.

Starting from this part of the article, for the sake of brevity, Lorentz Sasakian space form with $\mathcal{LSS}$-form, pseudoparallel submanifold with $\mathcal{P}$-submanifolds, 2-pseudoparallel submanifold with 2-$\mathcal{P}$ submanifold, Ricci generalized pseudoparallel submanifold with $\mathcal{RGP}$-submanifold and 2-Ricci generalized pseudoparallel submanifold with 2- $\mathcal{RGP}$ submanifold will be shown.

2. Preliminary

Let $\tilde{\Psi}$ be a $(2m+1)-$dimensional Lorentz manifold. If the $\tilde{\Psi}$ Lorentz manifold with (ϕ, ξ, η, g) structure tensors satisfies the following conditions, this manifold is called a Lorentz-Sasakian manifold

$$\phi^2\Lambda_1 = -\Lambda_1 + \eta(\Lambda_1)\xi, \eta(\xi) = 1, \eta(\phi\Lambda_1) = 0,$$

$$g(\phi\Lambda_1, \phi\Lambda_2) = g(\Lambda_1, \Lambda_2) + \eta(\Lambda_1)\eta(\Lambda_2), \eta(\Lambda_1) = -g(\Lambda_1, \xi),$$

$$\left(\tilde{\nabla}_{\Lambda_1}\phi\right)\Lambda_2 = -g(\Lambda_1, \Lambda_2)\xi - \eta(\Lambda_2)\Lambda_1, \tilde{\nabla}_{\Lambda_1}\xi = -\phi\Lambda_1,$$

where, $\tilde{\nabla}$ is the Levi-Civita connection according to the Riemann metric g.

The plane section Π in $T_x\tilde{\Psi}$. If the Π plane is spanned by Λ_1 and $\phi\Lambda_1$, this plane is called the ϕ-section. The curvature of the ϕ-section is called the ϕ-sectional curvature. If the Lorentz-Sasakian manifold has a constant ϕ-sectional curvature, this manifold is called the $\mathcal{LSS}-$form and is denoted by $\tilde{\Psi}(c)$. The curvature tensor of the $\mathcal{LSS}-$form $\tilde{\Psi}(c)$ is defined as

$$\begin{aligned} \tilde{R}(\Lambda_1, \Lambda_2)\Lambda_3 &= \left(\tfrac{c-3}{4}\right)\{g(\Lambda_2, \Lambda_3)\Lambda_1 - g(\Lambda_1, \Lambda_3)\Lambda_2\} \\ &+ \left(\tfrac{c+1}{4}\right)\{g(\Lambda_1, \phi\Lambda_3)\phi\Lambda_2 - g(\Lambda_2, \phi\Lambda_3)\phi\Lambda_1 \\ &+2g(\Lambda_1, \phi\Lambda_2)\phi\Lambda_3 + \eta(\Lambda_2)\eta(\Lambda_3)\Lambda_1 - \eta(\Lambda_1)\eta(\Lambda_3)\Lambda_2 \\ &+g(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\xi - g(\Lambda_2, \Lambda_3)\eta(\Lambda_1)\xi\}, \end{aligned} \tag{1}$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \chi\left(\tilde{\Psi}\right)$.

Lemma 2.1. *Let $\tilde{\Psi}(c)$ be the $(2m+1)-$dimensional $\mathcal{LSS}-$form. The following relations are provided for the $\mathcal{LSS}-$forms.*

$$\tilde{\nabla}_{\Lambda_1}\xi = -\phi\Lambda_1, \tag{2}$$

$$\left(\tilde{\nabla}_{\Lambda_1}\phi\right)\Lambda_2 = -g(\Lambda_1, \Lambda_2)\xi - \eta(\Lambda_2)\Lambda_1, \tag{3}$$

$$\left(\tilde{\nabla}_{\Lambda_1}\eta\right)\Lambda_2 = g(\phi\Lambda_1, \Lambda_2), \tag{4}$$

$$\tilde{R}(\xi, \Lambda_2)\Lambda_3 = -g(\Lambda_2, \Lambda_3)\xi - \eta(\Lambda_3)\Lambda_2, \tag{5}$$

$$\tilde{R}(\xi, \Lambda_2)\xi = \eta(\Lambda_2)\xi - \Lambda_2, \tag{6}$$

$$\tilde{R}(\Lambda_1,\Lambda_2)\xi=\eta(\Lambda_2)\Lambda_1-\eta(\Lambda_1)\Lambda_2, \tag{7}$$

$$S(\Lambda_1,\xi)=-\left[\frac{(c+1)-4m}{2}\right]\eta(\Lambda_1), \tag{8}$$

where $\tilde{R}, S$ and Q are the Riemann curvature tensor, Ricci curvature tensor and Ricci operator of $\tilde{\Psi}(c)$, respectively.

Let Ψ be the immersed submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. Let the tangent and normal subspaces of Ψ in $\tilde{\Psi}(c)$ be $\Gamma(T\Psi)$ and $\Gamma\left(T^{\perp}\Psi\right)$, respectively. Gauss and Weingarten formulas for $\Gamma(T\Psi)$ and $\Gamma\left(T^{\perp}\Psi\right)$ are

$$\tilde{\bigtriangledown}_{\Lambda_1}\Lambda_2=\bigtriangledown_{\Lambda_1}\Lambda_2+h(\Lambda_1,\Lambda_2), \tag{9}$$

$$\tilde{\bigtriangledown}_{\Lambda_1}\Lambda_5=-A_{\Lambda_5}\Lambda_1+\bigtriangledown^{\perp}_{\Lambda_1}\Lambda_5, \tag{10}$$

respectively, for all $\Lambda_1,\Lambda_2\in\Gamma(T\Psi)$ and $\Lambda_5\in\Gamma\left(T^{\perp}\Psi\right)$, where $\bigtriangledown$ and $\bigtriangledown^{\perp}$ are the connections on Ψ and $\Gamma\left(T^{\perp}\Psi\right)$, respectively, h and A are the second fundamental form and the shape operator of Ψ. There is a relation

$$g(A_{\Lambda_5}\Lambda_1,\Lambda_2)=g(h(\Lambda_1,\Lambda_2),\Lambda_5) \tag{11}$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form h is defined as

$$\left(\tilde{\bigtriangledown}_{\Lambda_1}h\right)(\Lambda_2,\Lambda_3)=\bigtriangledown^{\perp}_{\Lambda_1}h(\Lambda_2,\Lambda_3)-h(\bigtriangledown_{\Lambda_1}\Lambda_2,\Lambda_3)-h(\Lambda_2,\bigtriangledown_{\Lambda_1}\Lambda_3). \tag{12}$$

Specifically, if $\tilde{\bigtriangledown}h=0$, Ψ is said to be in the parallel second fundamental form or $1-$parallel.

Let R be the Riemann curvature tensor of Ψ. In this case, the Gauss equation can be expressed as

$$\begin{aligned}\tilde{R}(\Lambda_1,\Lambda_2)\Lambda_3&=R(\Lambda_1,\Lambda_2)\Lambda_3+A_{h(\Lambda_1,\Lambda_3)}\Lambda_2-A_{h(\Lambda_2,\Lambda_3)}\Lambda_1\\&+\left(\tilde{\bigtriangledown}_{\Lambda_1}h\right)(\Lambda_2,\Lambda_3)-\left(\tilde{\bigtriangledown}_{\Lambda_1}h\right)(\Lambda_1,\Lambda_3).\end{aligned} \tag{13}$$

Let Ψ be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, the tensor field $Q(A,T)$ is defined as

$$\begin{aligned}Q(A,T)(\Lambda_1,...,X_k;X,Y)&=-T((X\wedge_A Y)\Lambda_1,...,X_k)\\&-...-T(\Lambda_1,...,X_{k-1},(X\wedge_A Y)X_k),\end{aligned} \tag{14}$$

where

$$(X\wedge_A Y)Z=A(Y,Z)X-A(X,Z)Y,$$

$k\geq 1,\Lambda_1,\Lambda_2,...,X_k,X,Y\in\Gamma(T\Psi)$

3. Invariant Pseudoparalel submanifolds of Lorentz-Sasakian space forms

Let Ψ be the immersed submanifold of a $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If $\phi(T_{x_1}\Psi)\subset T_{x_1}\Psi$ in every x_1 point, the Ψ manifold is called invariant submanifold. From this section of the article, we will assume that the manifold Ψ is the invariant submanifold of the $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. So it is clear from (3) and (9) that

$$h(\Lambda_1,\xi)=0, h(\phi\Lambda_1,\Lambda_2)=h(\Lambda_1,\phi\Lambda_2)=\phi h(\Lambda_1,\Lambda_2) \tag{15}$$

for all $\Lambda_1,\Lambda_2\in\Gamma(T\Psi)$.

Lemma 3.1. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. The second fundamental form h of the submanifold Ψ is parallel if and only if Ψ is the total geodesic submanifold.*

Proof. The proof of the theorem is easily obtained if we choose $\Lambda_3 = \xi$ in (12) and make the necessary adjustments. ■

Definition 3.2. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If $\tilde{R}.h$ and $Q(g,h)$ are linearly dependent, M is called $\mathcal{P}$-submanifold.*

Equivalent to this definition, it can be said that there is a function L_1 on the set $M_1 = \{\Lambda_1 \in \Psi | \, h(\Lambda_1) \neq g(\Lambda_1)\}$ such that

$$\tilde{R}.h = L_1 Q(g,h).$$

If $L_1 = 0$ specifically, Ψ is called a semiparallel submanifold.

Theorem 3.3. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ is $\mathcal{P}$-submanifold, then Ψ is either a total geodesic or $L_1 = -1$.*

Proof. Let's assume that Ψ is a $\mathcal{P}$-submanifold. So, we can write

$$\left(\tilde{R}(\Lambda_1,\Lambda_2)h\right)(\Lambda_4,\Lambda_5) = L_1 Q(g,h)(\Lambda_4,\Lambda_5;\Lambda_1,\Lambda_2),$$

that is

$$\begin{aligned} &\tilde{R}^{\perp}(\Lambda_1,\Lambda_2)h(\Lambda_4,\Lambda_5) - h(R(\Lambda_1,\Lambda_2)\Lambda_4,\Lambda_5) \\ &-h(\Lambda_4,R(\Lambda_1,\Lambda_2)\Lambda_5) = -\lambda_1\{h((\Lambda_1 \wedge_g \Lambda_2)\Lambda_4,\Lambda_5) \\ &+h(\Lambda_4,(\Lambda_1 \wedge_g \Lambda_2)\Lambda_5)\}, \end{aligned} \tag{16}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5 \in \Gamma(T\Psi)$. If we choose $\Lambda_5 = \xi$ in (16) and make use of (7), (15), we get

$$(1+L_1)\{\eta(\Lambda_2)h(\Lambda_4,\Lambda_1) - \eta(\Lambda_1)h(\Lambda_4,\Lambda_2)\} = 0. \tag{17}$$

IIf we choose $\Lambda_2 = \xi$ in (17), we obtain

$$(1+L_1)h(\Lambda_4,\Lambda_1) = 0.$$

This completes the proof. ■

Corollary 3.4. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. Ψ is semiparallel if and only if Ψ is total geodesic submanifold.*

Definition 3.5. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If $\tilde{R}.\tilde{\bigtriangledown}h$ and $Q\left(g,\tilde{\bigtriangledown}h\right)$ are linearly dependent, M is called 2-$\mathcal{P}$ submanifold.*

Equivalent to this definition, it can be said that there is a function L_2 on the set $M_2 = \left\{\Lambda_1 \in \Psi | \, \tilde{\bigtriangledown}h(\Lambda_1) \neq g(\Lambda_1)\right\}$ such that

$$\tilde{R}.\tilde{\bigtriangledown}h = L_2 Q\left(g,\tilde{\bigtriangledown}h\right).$$

If $L_2 = 0$ specifically, Ψ is called a 2-semiparallel submanifold.

Theorem 3.6. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ is 2-$\mathcal{P}$ submanifold, then Ψ is a total geodesic submanifold.*

Proof. Let's assume that Ψ is a 2-$\mathcal{P}$ submanifold. So, we can write

$$\left(\tilde{R}\left(\Lambda_1,\Lambda_2\right)\tilde{\bigtriangledown}h\right)\left(\Lambda_4,\Lambda_5,\Lambda_3\right)=L_2Q\left(g,\tilde{\bigtriangledown}h\right)\left(\Lambda_4,\Lambda_5,\Lambda_3;\Lambda_1,\Lambda_2\right), \tag{18}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5,\Lambda_3\in\Gamma\left(T\Psi\right)$. If we choose $\Lambda_1=\Lambda_3=\xi$ in (18), we can write

$$\begin{aligned}
&R^{\perp}\left(\xi,\Lambda_2\right)\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\xi\right)-\left(\tilde{\bigtriangledown}_{R(\xi,\Lambda_2)\Lambda_4}h\right)\left(\Lambda_5,\xi\right)\\
&-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(R\left(\xi,\Lambda_2\right)\Lambda_5,\xi\right)-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,R\left(\xi,\Lambda_2\right)\xi\right)\\
&=-L_2\left\{\left(\tilde{\bigtriangledown}_{(\xi\wedge_g\Lambda_2)\Lambda_4}h\right)\left(\Lambda_5,\xi\right)+\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\xi\right)\right.\\
&\left.+\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\left(\xi\wedge_g\Lambda_2\right)\xi\right)\right\}.
\end{aligned} \tag{19}$$

Let's calculate all the expressions in (19). So, we can write

$$\begin{aligned}
&R^{\perp}\left(\xi,\Lambda_2\right)\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\xi\right)=R^{\perp}\left(\xi,\Lambda_2\right)\left\{\bigtriangledown^{\perp}_{\Lambda_4}h\left(\Lambda_5,\xi\right)\right.\\
&\left.-h\left(\bigtriangledown_{\Lambda_4}\Lambda_5,\xi\right)-h\left(\Lambda_5,\bigtriangledown_{\Lambda_4}\xi\right)\right\}\\
&=R^{\perp}\left(\xi,\Lambda_2\right)\phi h\left(\Lambda_5,\Lambda_4\right),
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\left(\tilde{\bigtriangledown}_{R(\xi,\Lambda_2)\Lambda_4}h\right)\left(\Lambda_5,\xi\right)=\bigtriangledown^{\perp}_{R(\xi,\Lambda_2)\Lambda_4}h\left(\Lambda_5,\xi\right)-h\left(\bigtriangledown_{R(\xi,\Lambda_2)\Lambda_4}\Lambda_5,\xi\right)\\
&-h\left(\Lambda_5,\bigtriangledown_{R(\xi,\Lambda_2)\Lambda_4}\xi\right)=-\phi\eta\left(\Lambda_4\right)h\left(\Lambda_5,\Lambda_2\right),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(R\left(\xi,\Lambda_2\right)\Lambda_5,\xi\right)=\bigtriangledown^{\perp}_{\Lambda_4}h\left(R\left(\xi,\Lambda_2\right)\Lambda_5,\xi\right)-h\left(\bigtriangledown_{\Lambda_4}R\left(\xi,\Lambda_2\right)\Lambda_5,\xi\right)\\
&-h\left(R\left(\xi,\Lambda_2\right)\Lambda_5,\bigtriangledown_{\Lambda_4}\xi\right)=-\phi\eta\left(\Lambda_5\right)h\left(\Lambda_2,\Lambda_4\right),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,R\left(\xi,\Lambda_2\right)\xi\right)=\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\eta\left(\Lambda_2\right)\xi-\Lambda_2\right)\\
&=\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\eta\left(\Lambda_2\right)\xi\right)-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right)\\
&=-h\left(\Lambda_5,\Lambda_4\eta\left(\Lambda_2\right)\xi+\eta\left(\Lambda_2\right)\bigtriangledown_{\Lambda_4}\xi\right)-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right)\\
&=\eta\left(\Lambda_2\right)\phi h\left(\Lambda_5,\Lambda_4\right)-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right),
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\left(\tilde{\bigtriangledown}_{(\xi\wedge_g\Lambda_2)\Lambda_4}h\right)\left(\Lambda_5,\xi\right)=\bigtriangledown^{\perp}_{(\xi\wedge_g\Lambda_2)\Lambda_4}h\left(\Lambda_5,\xi\right)-h\left(\bigtriangledown_{(\xi\wedge_g\Lambda_2)\Lambda_4}\Lambda_5,\xi\right)\\
&-h\left(\Lambda_5,\bigtriangledown_{(\xi\wedge_g\Lambda_2)\Lambda_4}\xi\right)=\phi\eta\left(\Lambda_4\right)h\left(\Lambda_5,\Lambda_2\right),
\end{aligned} \tag{24}$$

$$\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\xi\right)=\nabla^{\perp}_{\Lambda_4}h\left(\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\xi\right)-h\left(\nabla_{\Lambda_4}\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\xi\right)$$
$$-h\left(\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\nabla_{\Lambda_4}\xi\right)=\phi\eta\left(\Lambda_5\right)h\left(\Lambda_2,\Lambda_4\right), \tag{25}$$

$$\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\left(\xi\wedge_g\Lambda_2\right)\xi\right)=\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,-\eta\left(\Lambda_2\right)\xi+\Lambda_2\right)$$
$$=\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,-\eta\left(\Lambda_2\right)\xi\right)-\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right) \tag{26}$$
$$=-\phi\eta\left(\Lambda_2\right)h\left(\Lambda_5,\Lambda_4\right)-\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right).$$

If we substitute $(20), (21), (22), (23), (24), (25), (26)$ for (19), we obtain

$$R^{\perp}\left(\xi,\Lambda_2\right)\phi h\left(\Lambda_5,\Lambda_4\right)+\phi\eta\left(\Lambda_4\right)h\left(\Lambda_5,\Lambda_2\right)+\phi\eta\left(\Lambda_5\right)h\left(\Lambda_2,\Lambda_4\right)$$
$$-\eta\left(\Lambda_2\right)\phi h\left(\Lambda_5,\Lambda_4\right)+\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right)=-L_2\left\{\phi\eta\left(\Lambda_4\right)h\left(\Lambda_5,\Lambda_2\right)\right. \tag{27}$$
$$\left.+\phi\eta\left(\Lambda_5\right)h\left(\Lambda_2,\Lambda_4\right)-\phi\eta\left(\Lambda_2\right)h\left(\Lambda_5,\Lambda_4\right)-\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right)\right\}$$

If we choose $\Lambda_5=\xi$ and use (15), we get

$$\phi h\left(\Lambda_2,\Lambda_4\right)+\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\xi,\Lambda_2\right)=-L_2\left\{\phi h\left(\Lambda_2,\Lambda_4\right)\right.$$
$$\left.-\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\xi,\Lambda_2\right)\right\}. \tag{28}$$

On the other hand, it is clear that

$$\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\xi,\Lambda_2\right)=\phi h\left(\Lambda_2,\Lambda_4\right). \tag{29}$$

If (29) is written instead of (28), we obtain

$$h\left(\Lambda_2,\Lambda_4\right)=0.$$

This completes the proof. ■

Corollary 3.7. *The total geodesic of the invariant 2-pseudoparalell submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form is independent of the choice of L_2.*

Definition 3.8. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If $\tilde{R}.h$ and $Q(S,h)$ are linearly dependent, M is called $\mathcal{RGP}$-submanifold.*

Equivalent to this definition, it can be said that there is a function L_3 on the set $M_3=\{\Lambda_1\in\Psi|\, h(\Lambda_1)\neq S(\Lambda_1)\}$ such that

$$\tilde{R}.h=L_3Q\left(S,h\right).$$

Theorem 3.9. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ is $\mathcal{RGP}$-submanifold, then Ψ is either a total geodesic or $L_3=\dfrac{-2}{(c+1)-4m}$ provided $4m\neq(c+1)$.*

Proof. Let's assume that Ψ is a $\mathcal{RGP}$-submanifold. So, we can write

$$\left(\tilde{R}\left(\Lambda_1,\Lambda_2\right)h\right)\left(\Lambda_4,\Lambda_5\right)=L_3Q\left(S,h\right)\left(\Lambda_4,\Lambda_5;\Lambda_1,\Lambda_2\right),$$

that is

$$\begin{aligned}&\tilde{R}^{\perp}\left(\Lambda_1,\Lambda_2\right)h\left(\Lambda_4,\Lambda_5\right)-h\left(R\left(\Lambda_1,\Lambda_2\right)\Lambda_4,\Lambda_5\right)\\&-h\left(\Lambda_4,R\left(\Lambda_1,\Lambda_2\right)\Lambda_5\right)=-\lambda_3\left\{h\left(\left(\Lambda_1\wedge_g\Lambda_2\right)\Lambda_4,\Lambda_5\right)\right.\\&\left.+h\left(\Lambda_4,\left(\Lambda_1\wedge_g\Lambda_2\right)\Lambda_5\right)\right\},\end{aligned}\tag{30}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5\in\Gamma\left(T\Psi\right)$. If we choose $\Lambda_1=\Lambda_5=\xi$ in (30) and make use of $(8),(15)$, we get

$$\left[1+\frac{(c+1)-4m}{2}L_3\right]h\left(\Lambda_4,\Lambda_2\right)=0.$$

It is clear from the last equation that either

$$h\left(\Lambda_4,\Lambda_2\right)=0,$$

or

$$L_3=\frac{-2}{(c+1)-4m}.$$

This completes the proof. ■

Definition 3.10. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}\left(c\right)$. If $\tilde{R}.\tilde{\triangledown}h$ and $Q\left(S,\tilde{\triangledown}h\right)$ are linearly dependent, M is called 2-$\mathcal{RGP}$-submanifold.*

Equivalent to this definition, it can be said that there is a function L_4 on the set $M_4=\left\{\Lambda_1\in\Psi|\,\tilde{\triangledown}h\left(\Lambda_1\right)\neq S\left(\Lambda_1\right)\right\}$ such that

$$\tilde{R}.\tilde{\triangledown}h=L_4Q\left(S,\tilde{\triangledown}h\right).$$

Theorem 3.11. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}\left(c\right)$. If Ψ is 2-$\mathcal{RGP}$-submanifold, then Ψ is either a total geodesic or $L_4=\dfrac{2}{4m-(c+1)}$ provided $4m\neq(c+1)$.*

Proof. Let's assume that Ψ is a 2-$\mathcal{RGP}$-submanifold. So, we can write

$$\left(\tilde{R}\left(\Lambda_1,\Lambda_2\right)\tilde{\triangledown}h\right)\left(\Lambda_4,\Lambda_5,\Lambda_3\right)=L_4Q\left(S,\tilde{\triangledown}h\right)\left(\Lambda_4,\Lambda_5,\Lambda_3;\Lambda_1,\Lambda_2\right),\tag{31}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5,\Lambda_3\in\Gamma\left(T\Psi\right)$. If we choose $\Lambda_1=\Lambda_5=\xi$ in (31), we can write

$$\begin{aligned}&R^{\perp}\left(\xi,\Lambda_2\right)\left(\tilde{\triangledown}_{\Lambda_4}h\right)\left(\xi,\Lambda_3\right)-\left(\tilde{\triangledown}_{R(\xi,\Lambda_2)\Lambda_4}h\right)\left(\xi,\Lambda_3\right)\\&-\left(\tilde{\triangledown}_{\Lambda_4}h\right)\left(R\left(\xi,\Lambda_2\right)\xi,\Lambda_3\right)-\left(\tilde{\triangledown}_{\Lambda_4}h\right)\left(\xi,R\left(\xi,\Lambda_2\right)\Lambda_3\right)\\&=-L_4\left\{\left(\tilde{\triangledown}_{(\xi\wedge_S\Lambda_2)\Lambda_4}h\right)\left(\xi,\Lambda_3\right)+\left(\tilde{\triangledown}_{\Lambda_4}h\right)\left(\left(\xi\wedge_S\Lambda_2\right)\xi,\Lambda_3\right)\right.\\&\left.+\left(\tilde{\triangledown}_{\Lambda_4}h\right)\left(\xi,\left(\xi\wedge_S\Lambda_2\right)\Lambda_3\right)\right\}.\end{aligned}\tag{32}$$

Let's calculate all the expressions in (32). So, we can write

$$\begin{aligned} &R^{\perp}(\xi,\Lambda_2)\left(\tilde{\nabla}_{\Lambda_4}h\right)(\xi,\Lambda_3) = R^{\perp}(\xi,\Lambda_2)\left\{\nabla^{\perp}_{\Lambda_4}h(\xi,\Lambda_3)\right. \\ &\left.-h(\nabla_{\Lambda_4}\Lambda_3,\xi) - h(\Lambda_3,\nabla_{\Lambda_4}\xi)\right\} \\ &= R^{\perp}(\xi,\Lambda_2)\,\phi h(\Lambda_3,\Lambda_4), \end{aligned} \tag{33}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{R(\xi,\Lambda_2)\Lambda_4}h\right)(\xi,\Lambda_3) = \nabla^{\perp}_{R(\xi,\Lambda_2)\Lambda_4}h(\xi,\Lambda_3) - h\left(\nabla_{R(\xi,\Lambda_2)\Lambda_4}\xi,\Lambda_3\right) \\ &-h\left(\xi,\nabla_{R(\xi,\Lambda_2)\Lambda_4}\Lambda_3\right) = -\phi\eta(\Lambda_4)\,h(\Lambda_2,\Lambda_3), \end{aligned} \tag{34}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{\Lambda_4}h\right)(R(\xi,\Lambda_2)\xi,\Lambda_3) = \left(\tilde{\nabla}_{\Lambda_4}h\right)(\eta(\Lambda_2)\xi-\Lambda_2,\Lambda_3) \\ &-\left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3) = \nabla^{\perp}_{\Lambda_4}h(\eta(\Lambda_2)\xi,\Lambda_3) - h(\nabla_{\Lambda_4}\eta(\Lambda_2)\xi,\Lambda_3) \\ &-h(\eta(\Lambda_2)\xi,\nabla_{\Lambda_4}\Lambda_3) - \left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3) \\ &= \phi\eta(\Lambda_2)\,h(\Lambda_4,\Lambda_3) - \left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3), \end{aligned} \tag{35}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{\Lambda_4}h\right)(\xi,R(\xi,\Lambda_2)\Lambda_3) = \nabla^{\perp}_{\Lambda_4}h(\xi,R(\xi,\Lambda_2)\Lambda_3) - h(\nabla_{\Lambda_4}\xi,R(\xi,\Lambda_2)\Lambda_3) \\ &-h(\xi,\nabla_{\Lambda_4}R(\xi,\Lambda_2)\Lambda_3) = -\phi\eta(\Lambda_3)\,h(\Lambda_4,\Lambda_2) \end{aligned} \tag{36}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{(\xi\wedge_S\Lambda_2)\Lambda_4}h\right)(\xi,\Lambda_3) = \nabla^{\perp}_{(\xi\wedge_S\Lambda_2)\Lambda_4}h(\xi,\Lambda_3) - h\left(\nabla_{(\xi\wedge_S\Lambda_2)\Lambda_4}\xi,\Lambda_3\right) \\ &-h\left(\xi,\nabla_{(\xi\wedge_S\Lambda_2)\Lambda_4}\Lambda_3\right) = \tfrac{(c+1)-4m}{2}\phi\eta(\Lambda_4)\,h(\Lambda_2,\Lambda_3), \end{aligned} \tag{37}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{\Lambda_4}h\right)((\xi\wedge_S\Lambda_2)\xi,\Lambda_3) = \left(\tilde{\nabla}_{\Lambda_4}h\right)(S(\Lambda_2,\xi)\xi - S(\xi,\xi)\Lambda_2,\Lambda_3) \\ &= \tfrac{(c+1)-4m}{2}\left\{\left(\tilde{\nabla}_{\Lambda_4}h\right)(-\eta(\Lambda_2)\xi+\Lambda_2,\Lambda_3)\right\} \\ &= \tfrac{(c+1)-4m}{2}\left\{-\nabla^{\perp}_{\Lambda_4}h(\eta(\Lambda_2)\xi,\Lambda_3) + h(\nabla_{\Lambda_4}\eta(\Lambda_2)\xi,\Lambda_3)\right. \\ &\left. h(\eta(\Lambda_2)\xi,\nabla_{\Lambda_4}\Lambda_3) + \left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3)\right\} \\ &= \tfrac{(c+1)-4m}{2}\left\{\left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3) - \phi\eta(\Lambda_2)\,h(\Lambda_4,\Lambda_3)\right\}, \end{aligned} \tag{38}$$

$$\begin{aligned}&\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi,\left(\xi \wedge_S \Lambda_2\right) \Lambda_3\right)=\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi, S\left(\Lambda_2, \Lambda_3\right) \xi-S\left(\xi, \Lambda_3\right) \Lambda_2\right)\\&=\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi, S\left(\Lambda_2, \Lambda_3\right) \xi\right)+\tfrac{(c+1)-4 m}{2}\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi, \eta\left(\Lambda_3\right), \Lambda_2\right)\\&=\tfrac{(c+1)-4 m}{2} \phi \eta\left(\Lambda_3\right) h\left(\Lambda_4, \Lambda_2\right) .\end{aligned} \tag{39}$$

If we substitute $(33),(34),(35),(36),(37),(38),(39)$ for (32), we obtain

$$\begin{aligned}&R^{\perp}\left(\xi, \Lambda_2\right) \phi h\left(\Lambda_3, \Lambda_4\right)+\phi \eta\left(\Lambda_4\right) h\left(\Lambda_2, \Lambda_3\right)-\phi \eta\left(\Lambda_2\right) h\left(\Lambda_4, \Lambda_3\right)\\&+\eta\left(\Lambda_3\right) \phi h\left(\Lambda_4, \Lambda_2\right)+\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\Lambda_2, \Lambda_3\right)=-L_4\left\{\tfrac{(c+1)-4 m}{2} \phi \eta\left(\Lambda_4\right) h\left(\Lambda_2, \Lambda_3\right)\right.\\&\left.-\tfrac{(c+1)-4 m}{2} \phi \eta\left(\Lambda_2\right) h\left(\Lambda_4, \Lambda_3\right)+\tfrac{(c+1)-4 m}{2} \phi \eta\left(\Lambda_3\right) h\left(\Lambda_4, \Lambda_2\right)+\tfrac{(c+1)-4 m}{2}\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\Lambda_2, \Lambda_3\right)\right\}\end{aligned} \tag{40}$$

If we choose $\Lambda_3=\xi$ in (40) and use (15), we get

$$\begin{aligned}&\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\Lambda_2, \xi\right)+\phi h\left(\Lambda_4, \Lambda_2\right)=-\tfrac{(c+1)-4 m}{2} L_4\left\{\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\Lambda_2, \xi\right)\right.\\&\left.+\phi h\left(\Lambda_4, \Lambda_2\right)\right\} .\end{aligned} \tag{41}$$

On the other hand, it is clear that

$$\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi, \Lambda_2\right)=\phi h\left(\Lambda_2, \Lambda_4\right) . \tag{42}$$

If (42) is written instead of (41), we obtain

$$2 \phi h\left(\Lambda_2, \Lambda_4\right)=[4 m-(c+1)] L_4 \phi h\left(\Lambda_2, \Lambda_4\right) .$$

It is clear from the last equality

$$h\left(\Lambda_2, \Lambda_4\right)=0 \text { or } L_4=\frac{2}{4 m-(c+1)} .$$

This completes the proof. ■

4. Total geodesic submanifolds on concircular and projective curvature tensor

In this section, the invariant submanifold Ψ of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$ will be considered with the concircular and projective curvature tensor. The concircular curvature tensor is defined as

$$\tilde{Z}\left(\Lambda_1, \Lambda_2\right) \Lambda_3=R\left(\Lambda_1, \Lambda_2\right) \Lambda_3-\frac{r}{2 m(2 m+1)}\left[g\left(\Lambda_2, \Lambda_3\right) \Lambda_1-g\left(\Lambda_1, \Lambda_3\right) \Lambda_2\right], \tag{43}$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \chi\left(\tilde{\Psi}\right)$. If we choose $\Lambda_1=\Lambda_3=\xi$ in (43) and use (6), we get

$$\tilde{Z}\left(\xi, \Lambda_2\right) \xi=-\left[1+\frac{r}{2 m(2 m+1)}\right]\left[-\eta\left(\Lambda_2\right) \xi+\Lambda_2\right] . \tag{44}$$

Theorem 4.1. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ satisfies the condition $\tilde{Z}\left(\Lambda_1, \Lambda_2\right) h=L_5 Q(g, h)$,then Ψ is either total geodesic or $L_5=-\left(1+\frac{r}{2 m(2 m+1)}\right)$.*

Proof. Let's assume that Ψ satisfies the condition

$$\left(\tilde{Z}\left(\Lambda_1, \Lambda_2\right) h\right)\left(\Lambda_4, \Lambda_5\right) = L_5 Q\left(g, h\right)\left(\Lambda_4, \Lambda_5; \Lambda_1, \Lambda_2\right), \tag{45}$$

for all $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5 \in \Gamma\left(T\Psi\right)$. If we choose $\Lambda_1 = \Lambda_5 = \xi$ in (45) and use (15), we get

$$-h\left(\Lambda_4, \tilde{Z}\left(\xi, \Lambda_2\right)\xi\right) = -L_5 h\left(\Lambda_4, \Lambda_2\right). \tag{46}$$

If we use (44) out of (46), we obtain

$$\left[\left(1 + \frac{r}{2m\left(2m+1\right)}\right) + L_5\right] h\left(\Lambda_4, \Lambda_2\right) = 0.$$

This completes the proof. ■

Theorem 4.2. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}\left(c\right)$. If Ψ satisfies the condition $\tilde{Z}\left(\Lambda_1, \Lambda_2\right) h = L_6 Q\left(S, h\right)$, then Ψ is total geodesic or $L_6 = \frac{2[r+2m(2m+1)]}{2m(2m+1)[(c+1)-4m]}$ and $(c+1) \neq 4m$.*

Proof. Let's assume that Ψ satisfies the condition

$$\left(\tilde{Z}\left(\Lambda_1, \Lambda_2\right) h\right)\left(\Lambda_4, \Lambda_5\right) = L_6 Q\left(S, h\right)\left(\Lambda_4, \Lambda_5; \Lambda_1, \Lambda_2\right), \tag{47}$$

for all $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5 \in \Gamma\left(T\Psi\right)$. If we choose $\Lambda_1 = \Lambda_5 = \xi$ in (47) and use (15), we get

$$-h\left(\Lambda_4, \tilde{Z}\left(\xi, \Lambda_2\right)\xi\right) = L_6 S\left(\xi, \xi\right) h\left(\Lambda_4, \Lambda_2\right). \tag{48}$$

If we use (44) and (8) out of (48), we obtain

$$\left[\left(1 + \frac{r}{2m\left(2m+1\right)}\right) + \left(\frac{(c+1)-4m}{2}\right) L_6\right] h\left(\Lambda_4, \Lambda_2\right) = 0.$$

This completes the proof. ■

The projective curvature tensor is defined as

$$P\left(\Lambda_1, \Lambda_2\right)\Lambda_3 = R\left(\Lambda_1, \Lambda_2\right)\Lambda_3 - \frac{1}{2m}\left[S\left(\Lambda_2, \Lambda_3\right)\Lambda_1 - S\left(\Lambda_1, \Lambda_3\right)\Lambda_2\right], \tag{49}$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \chi\left(\tilde{\Psi}\right)$. If we choose $\Lambda_1 = \Lambda_3 = \xi$ in (49) and use (6), (8), we get

$$P\left(\xi, \Lambda_2\right)\xi = \frac{c+1}{4m}\left[\eta\left(\Lambda_2\right)\xi - \Lambda_2\right]. \tag{50}$$

Theorem 4.3. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}\left(c\right)$. If Ψ satisfies the condition $P\left(\Lambda_1, \Lambda_2\right) h = L_7 Q\left(g, h\right)$, then Ψ is either total geodesic or $L_7 = -\frac{c+1}{4m}$.*

Proof. Let's assume that Ψ satisfies the condition

$$\left(P\left(\Lambda_1, \Lambda_2\right) h\right)\left(\Lambda_4, \Lambda_5\right) = L_7 Q\left(g, h\right)\left(\Lambda_4, \Lambda_5; \Lambda_1, \Lambda_2\right), \tag{51}$$

for all $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5 \in \Gamma\left(T\Psi\right)$. If we choose $\Lambda_1 = \Lambda_5 = \xi$ in (51) and use (15), we get

$$-h\left(\Lambda_4, P\left(\xi, \Lambda_2\right)\xi\right) = -L_7 h\left(\Lambda_4, \Lambda_2\right). \tag{52}$$

If we use (50) out of (52), we obtain

$$\left[\frac{c+1}{4m} + L_7\right] h\left(\Lambda_4, \Lambda_2\right) = 0.$$

This completes the proof. ■

Theorem 4.4. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ satisfies the condition $P(\Lambda_1, \Lambda_2) h = L_8 Q(S, h)$, then Ψ is either total geodesic or $L_8 = \frac{2(c+1)}{4m[4m-(c+1)]}$ and $(c+1) \neq 4m$.*

Proof. Let's assume that Ψ satisfies the condition

$$(P(\Lambda_1, \Lambda_2) h)(\Lambda_4, \Lambda_5) = L_8 Q(S, h)(\Lambda_4, \Lambda_5; \Lambda_1, \Lambda_2), \tag{53}$$

for all $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5 \in \Gamma(T\Psi)$. If we choose $\Lambda_1 = \Lambda_5 = \xi$ in (53) and use (15), we get

$$-h(\Lambda_4, P(\xi, \Lambda_2)\xi) = L_8 S(\xi, \xi) h(\Lambda_4, \Lambda_2). \tag{54}$$

If we use (50) and (8) out of (54), we obtain

$$\left[\frac{c+1}{4m} + \frac{[(c+1)-4m]}{2} L_8\right] h(\Lambda_4, \Lambda_2) = 0.$$

This completes the proof. ■

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Certain operator algebras of star-like reducible $P\omega_n^*$ transformations

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Abstract. Let $Z_n = 1, 2, 3, \ldots$ denote a distinct non-negative n-order collection of numbers, and $\alpha\omega_n^*$ denote a star-like transformation semigroup. The characterization of $P\omega_n^*$ star-like partial on the $\alpha\omega_n^*$ leads to the semigroup of linear operators. The research produced a completely new classical metamorphosis that was divided into inner product and norm parts. The study demonstrated that any specific star-like transformation $\lambda_i^*, \beta_j^* \in V^*$ is stable and uniformly continuous if there exists $T^{\vartheta^*} : (V^*, \langle v - \alpha^* u, u - \alpha^* v\rangle) \longrightarrow (V^*, \langle u - \alpha^* v, v - \alpha^* u\rangle)$ with a star-like polygon ϑ^* of $\vartheta^* V^*$ such that $T^{\vartheta^*}(v^*) = \vartheta^* V^*$. Every star-like composite vector space $V^* \in P\omega_n^*$ can be uniquely decomposed as the sum of subspaces $w_i^* \leq W_{i+1}^*$ and $s_j^* \leq S_{j+1}^*$ such that $W_{i+1}^* + S_{j+1}^* \subseteq V^* \in P\omega_n^*$. The study suggests that the research's findings be used to address issues in the mathematical disciplines of genetics, engineering, code theory, and telecommunications.

AMS Subject Classifications: 20M20, 20M20.

Keywords: Star-like transformation, semigroup, innerproduct, vector space.

Contents

1. Introduction and Background

The study of vector spaces and vector space functions is known as linear algebra. They form the fundamental objects of study in this paper. Once a star-like vector space is defined, its properties will be investigated. A non-empty star-like transformation $\alpha\omega_n^*$ on which a polygon

$$\vartheta^* : \alpha\omega_n^* \times \alpha\omega_n^* \longrightarrow \alpha\omega_n^*$$

is defined as a star-like groupoid $(\alpha\omega_n^*,\ \vartheta^*)$. Then, $(\alpha\omega_n^*,\ \vartheta^*)$ is a star-like semigroup if the operation ϑ^* disk associative.

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Similar to how sets play a vital role in mathematics, mapping also aids in understanding the relationships between various algebraic structures. Instead of using the term mapping, which refers to the former, transformation is utilized. The publications of [1] and [2] provide additional details on semigroup mappings. According to the terminology employed by [3] the domain and image set of any given transformation $\lambda_i^* \in \alpha\omega_n^*$ were indicated by $D(\lambda_i^*)$ and $I(\lambda_i^*)$ respectively.

Let (-) signify the empty set and $\begin{pmatrix} u & v & l \\ r & s & t \end{pmatrix}$ be depicted as $\begin{pmatrix} r & s & t \end{pmatrix}$ not $\begin{pmatrix} r, & s, & t \end{pmatrix}$ not making the cycle notation more complex to the point that any transformation that contains an empty map is referred to as a star-like reducible transformation $P\omega_n^*$.

If a mapping $P\omega_n^*$ is a star-like linear vector in a semigroup such that any star-like vector can be metricized using the Hamming distance function method for every $i, j \in Z_n; i \leq j \Rightarrow ri \leq jr$, then it is said to be star-like order-preserving. One of the most potential transformation families for the current and upcoming generations of academics is created by the associative function composition [1]. Hence, the new classical finite $\alpha\omega_n*$ transformation semigroups.

Assuming that $\lambda_i^* \in P\omega_n^*$ is a star-like transformation, under the composition of mapping it generates another star-like transformation of its form with trace of any composed star-like matrix $\beta_j^* \in M^* \subseteq B\omega_n^*$ where $tr(\beta_j*)$ stands for the sum of its star-like diagonal points consisting of a finitely star-like polygon $\vartheta^* \in \vartheta^* V^*$. Then the star-like polygon (transformation) $\vartheta^* : R_0^* \longrightarrow Q$ is a rule $f : A \longrightarrow Q$ for some $A \neq \emptyset, A \subseteq R_0^*$, where R_0^* is a star-like disk operator and $P\omega_n^*(R_0^*)$ denotes the set of all star-like reducible transformations whose domain and rank are subsets of R_0^*, then $\beta^* \times \lambda^*$ of $\beta^*, \lambda^* \in P\omega_n^*(R_0^*)$ is the transformation with domain

$$Q = (I(\beta^*) \cap D(\lambda^*))\beta^{-1*}$$

so that for each $r_0^* \in R_0^*$,

$$r_0^*(\beta^* \times \lambda^*) = (r_0^*\beta^*)\lambda^*.$$

Given two associated star-like subspaces of V^*, W_{i+1}^* and S_{j+1}^* with the rule $\vartheta^* : W_{i+1}^* \longrightarrow S_{j+1}^*$; $\beta_j^* \in P\omega_n^*(V^*)$. The domain and rank of β_j^* are subspaces of V^* vector space V^*, and a subspace W_{i+1}^* of V^* whenever $P\omega_n^*(V^*, W_{i+1}^* \longrightarrow S_{j+1}^*) = \{\beta^* \in P\omega_n^*(V^*) : \alpha^* V^* \subseteq \alpha\omega_n^*$ if the following conditions are satisfy

(i) the range space $W_{i+1}^*(\beta^*)$ of β^*, which consists of all $\beta^* u$ with u in V^*

(ii) the null space $S_{j+1}^*(\beta^*)$ of β^*, which consists of all u in V^* such that $u\beta^* = 0$.

If $\alpha\omega_n^*$ is considered to be star-like, then

$$|v - \alpha^* u| \leq |u - \alpha^* v| \tag{1.1}$$

for all $u, v \in D(\lambda_i^*, \beta_j^*)$ and $\alpha^* u, \alpha^* v \in I(\beta_j^*,\ \lambda_i^*)$ where

$$V^* = \begin{pmatrix} u & v & \dots & uv_{i,j} \\ \alpha^* u & \alpha^* v & \dots & \alpha^* uv_{i,j} \end{pmatrix} = (u, v, uv_{i,j},\ \alpha^* u, \alpha^* v, \alpha^* uv_{i,j}). \tag{1.2}$$

The inner product was characterized by star-like transformation semigroups $\alpha\omega_n^*$ such that for all $\beta^*, \lambda^* \in P\omega_n^*$

$$\lambda_i^* = \left\langle \begin{pmatrix} u & v & u \dots & uv_{i,j} \\ q_1 & q_2 & q_3 \dots & q_{i,j} \end{pmatrix} \right\rangle,\ and\ \beta_j^* = \left\langle \begin{pmatrix} v & u & v \dots & vu_{j,i} \\ k_1 & k_2 & k_3 \dots & k_{j,i} \end{pmatrix} \right\rangle$$

equals to

$$\langle \beta^* \rangle = \langle (\beta^* k_1, \beta^* k_2, \dots \beta^* k_{i,j}) \rangle\,,$$

$$\langle \lambda^* \rangle = \langle (\lambda^* q_1, \lambda^* q_2, \dots \lambda^* q_{j,i}) \rangle\,.$$

Then $U_j = \beta_j^*$ and $U_i = \lambda_i$ such that

$$U_i^* = (q+1, q+1-i),\ U_j^* = (k+1, k+1-j)$$

which implies

$$\langle U_i, U_j \rangle = \left\{ \begin{pmatrix} u_1\ u_2\ u_3 \dots\ u_{i+1} \\ k_1\ k_2\ k_3 \dots\ k_{i+1} \end{pmatrix}, \begin{pmatrix} v_1\ v_2\ v_3 \dots\ v_{j+1} \\ q_1\ q_2\ q_3 \dots\ q_{j+1} \end{pmatrix} \right\} \tag{1.3}$$

with the star-like disk operator $R_0^* \geq 0$ on the inner product of a star-like vector space $V^* : [0, \infty) \to [0, \infty)$ such that $0\alpha^* = 0$, and $\beta^*(R_0^*) \leq R_0^*$, in which

$$\|\langle R_0^*(k), (u) \rangle\| \leq \|\langle R_0^*(v), (q) \rangle\|, \tag{1.4}$$

such that $k_{i,j}$, and $q_{i,j}$ are lower diagonal elements and upper diagonal elements of $\lambda_i^*, \beta_j^* \in P\omega_n^*$ star-like reducible semigroup respectively.

Let $M\omega_n^*$ denote a star-like monoid semigroup with unique identity $1 \in M\omega_n^* : 1\lambda^* = \lambda^* = \lambda^* 1$ for all $\lambda^* \in M\omega_n^*$. Putting $\lambda^{0*} = 1$ (index law) holds for all b, d in $\mathbb{N} \cup \emptyset$ then $\alpha\omega_n^*$ contains a unique element 0 (zero):

$$\beta^*\lambda^* = \lambda^*\beta^*$$
$$0\beta^* = 0$$
$$\lambda^* 0 = 0$$

For all β^*, λ^* in $P\omega_n^*$ is disk associative, by equation (1.1) $P\omega_n^* \cup \emptyset$ is a semigroup obtained from $P\omega_n^*$ by adjoining zero where necessary. If a semigroup $P\omega_n^*$ has the property that for all $\beta^*, \lambda^* \in P\omega_n^*$:
$\lambda^{0*} = 1$, for all $b, d \in \mathbb{R}$. Then

$$\lambda^{b*}\lambda^{d*} = \lambda^{b+d*} \Longrightarrow (\lambda^b)^{d*} = \lambda^{bd*}.$$

Thus, equations (1.3) and equation give useful characterizations of inner product normed space on the star-like vector of a star-like mapping such that the star-like vector of order n in equations (1.1) and (1.2) for any given $\beta_i^* \in P\omega_n^*$ is given by

$$\beta_{i,\ j}^* = \begin{cases} \beta_{i,n}^* - D(\lambda_{i,n}^*) \\ V_{i,j}^* - n \end{cases} \tag{1.5}$$

such that

$$V_{i,j}^* = \begin{pmatrix} \beta_i^*\ \beta_j^* \ \dots\ \beta_n^* \\ \lambda_i^*\ \lambda_i^* \ \dots\ \lambda_n^* \end{pmatrix}$$

The star-like order reversing of $V_{i,\ j}^*$ in (1.5) generates elements of $P\omega_n^*$. Hence, a star-like vector space is a triple $(V^*,\ +, \times)$ over $P\omega_n^*(n,\ F)$ comprised of a set V^* and F^n along with the operation $'+'$ and $'\times'$ by real integers such that the operations most produce vectors in the space and the following statements must be true;

(i) if β^*, λ^* are vectors in V^* then $\beta^* + \lambda^*$ is a vector in V^*

(ii) if β^* is a vector in V^* and b is a star-like scalar in $P\omega_n^*(n,\ F) \in \mathbb{R}$ then $b\beta^*$ is a vector in V^*.

As a result, a star-like vector space is a triple $(V^*,\ +, \times)$ over $P\omega_n^*(n,\ F)$ consisting of a set V^* and F^n as well as the operations $'+'$ and $'\times'$ by real integer and star-like disk operator such that the operations most produce star-like vectors in the space and the following must be true:

(i) if β^*, λ^* are vectors in V^* then $\beta^* + \lambda^*$ is a vector in V^*

(ii) if β^* is a vector in V^* and b is a star-like scalar in $P\omega_n^*(n,\ F) \in \mathbb{R}$ then β^* is a vector in V^*.

That is, given two vectors β^*, λ^* in $V_{i,\ j}^*$ of equation (1.5), it associates a new vector in $V_{i,j}^*$ denoted by $\beta^* + \lambda^*$:

$$(+) : V_{i,\ j}^* \times V_{i,\ j}^* \longrightarrow V_{i,\ j}^*$$
$$(\beta^*, \lambda^*) \longrightarrow \beta^* + \lambda^*$$

and given a vector β^* in $V_{i,\ j}^*$ and a star-like disk operator $r_0^* \in R_0^*$, it associate a new vector in $r_0^*\beta^* \in V_{i,\ j}^*$ such that

$$(\times) : \mathbb{R} \times V_{i,\ j}^* \longrightarrow V_{i,\ j}^*$$
$$(r_0^*, v^*) \longrightarrow r_0^* v^*.$$

Let $V^* \in P\omega_n^*(n, F)$ represent a star-like vector space. A mapping $T^{\vartheta^*} : V^* \longrightarrow V^*$ is a star-like mapping if there exists a star-like disk operator (constant) $r_0^* \subseteq V^*$ with $0 \leq b \leq 1$ such that

$$V^*(T^{\vartheta^*}(\beta^*), \lambda^*) \leq r_0^* V^*(T^{\vartheta^*}(\lambda^*), \beta^*) \tag{1.6}$$

As a result, a star-like map points closer diagonally together. For every $\beta^*, \lambda^* \in V^*$ and $r \leq 0$, all points λ^* in the ball $B_r(\beta^*)$ are mapped diagonally into a ball $B_s(T^*(\beta^*))$ with $s \leq r$. This is depicted in 1. It also follows from equation (1.3) and (1.5) that a star-like mapping is uniformly continuous. If $T^{\vartheta^*} : V^* \longrightarrow V^*$ then a point $v^* \in V^*$ such that

$$\left|\left|T^{\vartheta^*}(v^*)\right|\right| = ||r_0^* V^*|| \tag{1.7}$$

is called a star-like fixed point of T^{ϑ^*}

The following is a partial list of papers and books:[4], [5], [6], [7], and [10] for basic and standard notions in transformation semigroup theory. [8] factorized assertions about the relationships between metric spaces, normed linear spaces, and inner product spaces. Refer to [9] for an introduction to functional analysis with algebraic applications. The characterization relations of algebraic structure to linear operators have not been investigated on $P\omega_n^*$, hence the need for this research.

That exists between metric spaces, normed linear spaces, and inner product spaces.

2. Preliminary

There is a need to demonstrate the application of algebraic theory to other relevant pure mathematical topics. The research study created mathematical relationships to connect some operator algebras with transformation semigroups. Some fundamental concepts and preliminary information that would be required in the following part were defined:

Definition 2.1. *Star-like Mapping* ($\longrightarrow^*$)*: Consider the star-like sets of disk operators* R_n^* *and* Q_n^* *to be non-empty. A star-like rule* $\vartheta^* : R_n^* \longrightarrow^* Q_n^*$ *is a function* ϑ^* *that transforms* Q_n^* *into* R_n^**.*

(i) $D(\vartheta^*) = R_n^*$

(ii) for every $r, n, k^{'}, l^{'} \in \rho^* \Longrightarrow r = k^{'},\ n = l^{'}$*.*

Definition 2.2. *Star-like fixed point: A fixed point element* $m^* \in I(\beta^*)$ *of* $P\omega_n^*$ *is a function* $\vartheta^* : \beta^* \longrightarrow \beta^*$ *such that* $f(\beta^*) = |m^*(\beta^*)|$*. It is read that* β^* *fixes* m^**.*

Definition 2.3. *Star-like vector space: A triple* $(V^*, +, \times)$ *is a star-like space* $V^* \subseteq P\omega_n^*$ *containing a set of mapping (vectors) and star-like operator* $+$ *and* $\times$ *by real integer as follows:*

(i) *Given two vectors $\beta^*, \lambda^* \in V^*$, a new star-like vector in V^* denoted by $\beta^* + \lambda^*$ is obtained*

$$\begin{aligned}(+) : V^* \times V^* &\longrightarrow V^* \\ (\beta^* \lambda^*) &\longrightarrow \beta^* + \lambda^*\end{aligned}$$

(ii) *Given a star-like vector $\beta^* \in V^*$ and a real star-like disk operator $r_0^* \in R_0^*$, associates a new star-like vector in V^* denoted by $r_0^* V^*$ and a real number $b \in \mathbb{R}$ so that*

$$\begin{aligned}(\times) : \mathbb{R} \times V^* \longrightarrow& V^* \\ (r_0^*,\ V^*) \longrightarrow& r_0^* V^*,\end{aligned}$$

Then, $(V^*,\ +, \times)$ is a (real) star-like vector space V^* if

(i) $(\beta^* + \lambda^*) + \gamma^* = \beta^* + (\lambda^* + \gamma^*)$

(ii) $0 \in V^*$ *such that* $\beta^* + 0 = 0 + \beta^* = \beta^*$

(iii) *If* $\beta^* \in V^*$ *there exists* $-\beta^* \subseteq V^*$ *which satisfies* $\beta^* + (-\beta^*) = 0$

(iv) $\beta^* + \lambda^* = \lambda^* + \beta^*$

(v) $(r_0^*\ b) \times \lambda^* = r_0^* \times (b \times \lambda^*)$, *for all* $r_0^*,\ b \in \mathbb{R}$

(vi) $r_0^* \times (\beta^* + \lambda^*) = r_0^* \beta^* + r_0^* \lambda^*$, *for every* $\beta^*, \lambda^* \subseteq V^*$

(vii) $(r_0^*\ b) \times \beta^* = r_0^* \times (b\beta^*)$

(viii) $1_{\beta^*} = \beta^*$ *for every* $\beta^*, \lambda^*, \gamma^* \subseteq V^*$

Definition 2.4. *Supplementary subspaces: Let $W_{i+1}^*, S_{j+1}^* \in V^*$ be two subspaces of a star-like vector space V^*, Then W_{i+1}^* and S_{j+1}^* are said to be supplementary subspaces if*

$$W_{i+1}^* + S_{j+1}^* = V^* and\ W_{i+1}^* \bigcap S_{j+1}^* = \langle 0 \rangle\,.$$

Definition 2.5. *Star-like inner product space: Let $(V*, +, \times)$ represent a star-like vector space over the field $P\omega_n(n,\ F)$. A star-like inner product is a space function $\langle \times, \times \rangle : V^* \times V^* \longrightarrow \mathbb{R}$ that assigns to each ordered pair (γ^*, λ^*) in V^* and a scalar (real number) given that the following propositions are true.*

(i) $\langle \gamma^*,\ \gamma^* \rangle \geq 0$ *and* $\langle \gamma^*,\ \gamma^* \rangle = 0$ *if and only if* $\gamma^* = 0$ *for all* $\gamma^* \in V^*$

(ii) $\langle r_0^* \gamma^*,\ \lambda^* \rangle = r_0^* \langle \gamma^*,\ \lambda^* \rangle$ *for* $r_0^* \in \mathbb{R}$

(iii) $\langle \gamma^*,\ \lambda^* \rangle = \langle \lambda^*,\ \gamma^* \rangle$ *for all* $\gamma^*,\ \lambda^* \in V^*$

(iv) $\langle \gamma^*,\ r_0^* \lambda^* + b\beta^* \rangle = r_0^* \langle \gamma^*,\ \lambda^* \rangle + b \langle \gamma^*,\ \beta^* \rangle$ *for every* $\beta^*,\ \lambda^*, \gamma^* \in V^*$.

Definition 2.6. *In the case of any particular star-like transformation $\lambda_i^* \in P\omega_n^*$, there exists a unique identity star-like rule $e_{\lambda_i^*}^* : \lambda_i^* \longrightarrow \lambda_i^*$ defined by $e_{\lambda_i^*}{}^*(u) = u$ for all $u \in \lambda_i^*$.*

3. Main Results

The following results explain how particular operator algebras affect star-like $P\omega_n^*$ reducible transformation semigroups.

Lemma 3.1. *A $R_0^* \subseteq V^*$ in $P\omega_n^*$ element is a star-like disk operator if and only if $R_0^* f(\lambda_i^*) \leq I(R_0^*)$.*

Proof. Suppose $R_0^* f(\lambda_i^*) \leq I(R_0^*)$, there exist $\lambda_i^* s_i^* \in I(R_0^*)$ such that $\lambda_i^* s_i^*(R_0^*) = \lambda_i^* s_i^*$.
If

$$I(R_0^*) = \{\lambda_i^* s_i^* R_0^* : \lambda_i^* s_i^* \in P\omega_n^*\},$$

Then

$$R_0^* \in P\omega_n^* \Longleftrightarrow |R_0^*(v) - R_0^*(\lambda_i^* \alpha^* u)| \leq |R_0^*(\lambda_i^* u) - R_0^*(\alpha^* v)| \leq R_0^*$$

Implies

$$\lambda_i^* v \{|R_0^*(v) - R_0^*(\lambda_i^* \alpha^* u)| \leq |R_0^*(\lambda_i^* u) - R_0^*(\alpha^* v)|\} \leq \lambda_i^* s_i^* R_0^*.$$

By (1.1)

$$|R_0^*(u) - R_0^*(\lambda_i^* \alpha^* u)| \leq |R_0^*(\lambda_i^* v) - R_0^*(\alpha^* v)| \leq K_{R_0^*}$$

shows that

$$(\lambda_i^* \alpha^* v K_{R_0^*}) R_0^* \leq \alpha^* u R_0^*$$

and

$$\lambda_i^* s_i^* R_0^* \leq \lambda_i^* s_i^*.$$

Thus, $R_0^* f(\lambda_i^*) \leq I(R_0^*)$, for every $\lambda_i^* s_i^* \in I(R_0^*)$. ■

Theorem 3.2. *Assume $\vartheta^* V^*$ is a star-like disknorm of $V^* \in \alpha\omega_n^*(n,\ F)$ such that $P\omega_n^* \subseteq \alpha\omega_n^*$ then the following are true:*

i Every element $\lambda_n^* \in P\omega_n^*$ is star-like reducible

ii $P\omega_n^*$ contains $w^+(\vartheta^* V^*) \leq w^-(\vartheta^* V^*)$

iii There exists a unique $r_0^* \in P\omega_n^*(V^*) : \{r_0^* = \langle Max(n, w^+ V^*) \times Min(n, w^- V^*)\rangle\}$ and $\langle (n, w^+(V^*), w^-(V^*))\rangle = \sum_{\vartheta_i^*=1}^{n} \begin{pmatrix} 2^{\vartheta^*-1} \\ \vartheta^* + n - 1 \end{pmatrix}$ such that $r_0^* \subseteq R_0^*$ is the star-like disk operator degree of $P\omega_n^*$.

Proof. $(i) \Longrightarrow (ii)$
If $r_0^* \in \vartheta^* V^*$ is a star-like reducible degree order, then $b \in Z_n$ is in the range set $d \in Z_n : \lambda_n^*(b)d$. Because $\vartheta^* V^*$ is a star-like vector

$$(\lambda_n^*(b)\vartheta^* V^*) = \lambda_n^*(b)\vartheta^* V^* \tag{3.1}$$

Implies

$$\lambda_n^*(d\vartheta^* V^*) = d\vartheta^* V^*. \tag{3.2}$$

Then

$$\langle Max(n, w^+ \vartheta^* V^*)\rangle \leq \langle Min(n, w^- \vartheta^* V^*)\rangle$$

for some $b,\ d \in D(\lambda_n^*)$ with a star-like disknorm

$$\lambda_n^*(bV^*) = dv^* : r_0^*(\lambda_n^*) \leqslant dV^*. \tag{3.3}$$

$(ii) \longrightarrow (iii)$
Let $\vartheta^* \in V_n^* : \vartheta^* = w^+(\lambda_n^*) \times w^-(\lambda_n^*)$.
By star-like folding principle and composition of star-like reducible transformation.

$$D \left\langle Max(n, w^+(V^*)) \right\rangle$$

gives the star-like order-reversing of

$$I \left\langle Min(n, w^-((V^*)) \right\rangle.$$

The theorem follows from 2

Since $\vartheta^* V^*$ possesses a reducible order of $F(r_0^*)$ in $V^* \subseteq P\omega_n^*$, then $F(n; w^+(\lambda_n^*), w^-(\lambda_n^*)$ generates a finitely reducible recurrence star-like disknorm:

$$\left\langle (n, w^+(V^*), w^-(V^*)) \right\rangle = \sum_{\vartheta_i^*=1}^{n} \binom{2^{\vartheta^*-1}}{\vartheta^* + n - 1} \tag{3.4}$$

$(iii) \longrightarrow (i)$
Assuming $\lambda_n^* \in P\omega_n^*$ is star-like reducible such that $\vartheta^* V^*$ is a star-like vector space with a star-like disknorm, then for any given star-like transformation

$$\lambda^* \in D(\vartheta^* V_n^*) : r_0^*(\lambda_n^*) \leq Z_n$$

such that $b_{i+1} - b_i$ is the domain and $d_{j+1} - d_j$ is the image order of $\lambda_n^* \subseteq \vartheta^* V^*$

$$\vartheta^* V_n^* \mid b_{i+1} - b_i \mid \leq \vartheta^* V_n^* \mid d_{j+1} - d_j \mid \tag{3.5}$$

then $r_0^* \in R_n^* : \vartheta^* V_n^* \times \vartheta^* V_n^* = \vartheta^* V_n^*$ which completes the proof. ■

Proposition 3.3. *Given a star-like vector space* $(V^*, +, \times)$*, the following statements are true:*

(i) $0 \times \lambda^* = 0$ *for any* $\lambda^* \in V^*$

(ii) $(-r_0^*) \times \lambda^* = r_0^* \times (\lambda^*)$ *for any* $r_0^* \in \mathbb{R}$

(iii) $r_0^* \times 0 = 0$ *for any* $r_0^* \in \mathbb{R}$

(iv) If $r_0^* \times \lambda^* = 0$ *then either* $r_0^* = 0$ *or* $\lambda^* = 0$.

Proof. (i) Suppose $\lambda^* \in V^*$ such that $V^* \in \lambda\omega_n^*(n,\ F)$, by definition 2.5 using (viii), (v) and (ii) gives

$$\begin{aligned} \lambda^* + 0 \times \lambda^* &= 1 \times \lambda^* + 0 \times \lambda^* \\ &= (1+0) \times \lambda^* = \lambda^* + 0. \end{aligned}$$

By adding $(-\lambda^*)$ to both sides of the equality: $-\lambda^* + \lambda^* + 0 \times \lambda^* = -\lambda^* + \lambda^* + 0$.
Thus, $0 \times \lambda^* = 0$.
(ii) Let $-(r_0^* \times \lambda^*)$ be an element in V^* that satisfies property (iii) in definition 2.5, replace λ^* by $r_0^* \times \lambda^*$.
Now $r_0^* \times 0 = 0$ for any $r_0^* \in R_0^*$ holds if $-(r_0^* \times \lambda^*) = (-r_0^*) \times \lambda^*$.
Using $(-r_0^*) \times \lambda^* = r_0^* \times (-\lambda^*)$ and $r_0^* \times \lambda^* = 0$
where $(-r_0^*) \times \lambda^* + r_0^* \times \lambda^* = (-r_0^* + r_0^*) \times \lambda^* = 0 \times \lambda^* = 0$.
Then, $(-r_0^*) \times \lambda^* = r_0^* \times (-\lambda^*) = 0$.
(iii) In general, if $\lambda_j^* \in V^*$ we see that:

$$\begin{aligned} r_0^* \times 0 &= r_0^* \times (\lambda_j^* - \lambda_j^*) = r_0^* \times \lambda_j^* + r_0^* \times (-\lambda_j^*) \\ &= r_0^* \times \lambda_j^* + r_0^* \times \left\{(-1) \times \lambda_j^*\right\} = r_0^* \times \lambda_j^* + (-r_0^*) \times \lambda_j^* \\ &= r_0^* \times \lambda_j^* - r_0^* \lambda_j^* = 0 \end{aligned}$$

Therefore, $r_0^* \times 0 = 0$ for any $\lambda_j^* \in \alpha\omega_n^*(n,\ F)$.
(iv) Suppose $r_0^* \neq 0$, and $r_0^*\lambda^* = 0$.
Consider $\frac{1}{r_0^*}$, then,

$$\lambda^* = 1\lambda^* = (\frac{1}{r_0^*}\, r_0^*)\lambda^*$$
$$= \frac{1}{r_0^*}\,(r_0^*\lambda^*) = \frac{1}{r_0^*}\,0\ = 0$$

Thus, if $r_0^* = 0$, then $\lambda^* = 0$ or r_0^*. A reducible star-like transformation of $P\omega_n^*$ is the subset $K^* \subset \alpha\omega_n^*$, which is closed by the same operation on $P\omega_n^*$. If $K^* \subset \alpha\omega_n^*$, then it is a legitimate star-like sub-vector of V_n^* that is not equivalent to $|P\omega_n^*|$. Then

$$\cap_{j\in J} K_j^* \neq \emptyset \Longrightarrow \cap_{j\in J} K_j^* \subseteq P\omega_n^*.$$

Similarly, $S_{j+1}^* \subseteq V^* \in \alpha\omega_n^*(n,\ F)$ is a star-like vector addition and scalar multiplication closed subspace of V^*.
Therefore a star-like subspace $S_{j+1}^* \subseteq V^*$ in any given star-like triple $(V^*,\ +,\ \times)$ is a vector subspace of V^* if it is a vector space with the induced operation and still satisfies properties (i) - (vii) of definition 2.5. ■

Proposition 3.4. *Any star-like subset S_{i+j}^* of V^* is a star-like subspace if and only if the following requirements are met:*

(i) $w_i^* + s_j^* \in V_{i,j}^*$, *for any* $w_i^*, s_j^* \in V^*$

(ii) $b\ s_{i+j}^* \in P\omega^*$ *for any* $b \in \mathbb{R}$.

Proof. (i) Given any $S_{i+j} \in V^*$ such that $w_i^* + s_j^* \subseteq S_{i+j}^*$ and any star-like real number $b \in \mathbb{R}$ such that $w_i^* + s_j^* \leq S_{i+j}^*$. As a result of the limited vectors being well defined on S_{i+j}^*, the outcome vector is still in S_{i+j}^*.
Furthermore, $w_i^* + s_j^* \in V^* : 0 = 0(w_i^* + s_j^*) \in S_{i+j}^*$ and $-S_{i+j}^* = (-1)w_i^* + s_j^* V^*$.

(ii) Suppose $V^* \in \alpha\omega_n^*(n,\ F)$ and $S_{i+j}^* \in V^*$ with $i, j \in \mathbb{Z}_i \cup \emptyset$; $Z_i(i = \{0, 1, 2, \times\}) : \emptyset \in \mathbb{R}$, by the properties of $V^* \in P\omega_n^*$ it is obvious that S_{i+j}^* is star-like subspace.
Therefore, by properties (i) - (iv) of definition 2.6, the proof is complete. ■

Remark 3.5. *A star-like subset containing only zero vector, $z^* \in V^* = \emptyset$, and the whole space V^* are trivial subspaces, in which z^* is the smallest possible star-like subspace and $V^* \subseteq P\omega^*$ is the largest one.*

Proposition 3.6. *Let $(V^*,\ +,\ \times)$ represent a star-like vector space and $W_{i+1}^*,\ S_{j+1}^*$ represent two star-like subspaces. The following are interchangeable:*

(i) $W_{i+1}^* \bigcap S_{j+1}^* = \langle 1 \rangle$

(ii) There exists a unique couple $(w_i^*,\ s_j^*) \in W_{i+1}^* \times S_{j+1}^*$ *for each* $r_0^* \in W_{i+1}^* + S_{j+1}^*$ *such that* $r_0^* = w_i^* + s_j^*$.

Proof. (i) $\Longrightarrow$ (ii) Assume a star-like vector operator $r_0^* \in W_{i+1}^* + S_{j+1}^*$ n be expressed in two paths: $r_0^* = u_{i,j}^* + v_{j,i}^*$ and $r_0^* = v_{i,j}^* + u_{j,i}^*$ with $uv_{i,j}^* \in W_{i+1}^*$, and $vu_{j,i}^* \in S_{j+1}^*$.
Take note that

$$u_{i,j}^* - v_{i,j}^* \leq v_{j,i}^* - u_{j,i}^* \in W_{i+1}^* \bigcap S_{j+1}^* = \langle 1 \rangle\,.$$

As a result, $r_0^* v \leq \alpha^* u$ and $r_0^* u \leq \alpha^* v$ are equal
(ii) $\Longrightarrow$ (i) suppose by contradiction, there exists $0 \neq r_0^* \in W_{i+1}^* \bigcap S_{j+1}^*$.
Therefore,

$$r_0^* = 0 + r_0^* \leq r_0^* + 0 \in W_{i+1}^* \bigcap S_{j+1}^*.$$

This contradict the initial proposed statement, because any star-like vector in the same transformation can be expressed uniquely as the combination of vectors in W_{i+1}^* and S_{j+1}^*, this means that $r_0^* V^*$ can be decomposed in two different ways as a vector of $W_{i+1}^* + S_{j+1}^* \in V^* \subseteq P\omega_n^*$. ■

Remark 3.7. *Every vector of equation* (1.5) *can be uniquely decomposed in proposition* **??** *as the combination of a star-like vector in* W_{i+1}^* *and* S_{j+1}^*.

Theorem 3.8. *Let* $\|V^*\|$ *denote a star-like norm vector space with the disk operator* r_0^* *and let* u^* *and* v^* *be any two star-like vectors in* V^* *then* $2d(u^*, v^*) = \|E^*(u^*) - F^*(v^*)\| + 2\phi r_0^*$.

Proof. By a star-like operator

$$r_0^*(u^*) = r_0^*(u^* - v^* + v^*) \leq r_0^*(u^* - u^* + v^*) + r_0^*(V^*)$$

which is equivalent to

$$\frac{E^* - F^*}{2} + \phi r_0^*(u^*) - r_0^*(v^*) \leq \frac{1}{2} V^* r_0^*(v^* - u^*)$$

Then,

$$r_0^*(v^*) - r_0^*(u^*) \leq r_0^*(u^* - v^*) = \frac{1}{2} V^* r_0^*(v^* u^*) \tag{3.6}$$

It was deduced in equation (1.6) that r_0^* is a continuous star-like disk when using it as a norm on the star-like vector space $r_0^* V^*$, using the absolute value as a norm on the real star-like space,

$$-2r_0^*(v^* - u^*) \leq E^* r_0^*(v^*) - F^* r_0^*(u^*)$$

Gives

$$|r_0^*(u^*) - v^*| \leq \frac{1}{2} V^* r_0^*(u^* - v^*).$$

Given a star-like mapping

$$T^{\vartheta^*} : (r_0^* V^*, \|\times\|) \longrightarrow (r_0^* V^*, \ \frac{1}{2} V^* \, \|\times\|)$$

such that a star-like operator $r_0^* < b \leq 1$. Then

$$\frac{1}{2} V^* \left\| E^*(T^{\vartheta^*}(v^*), F^*(u^*)) \right\| \leq \frac{1}{2} V^* b \left\| (T^{\vartheta^*}(u^*), T^{\vartheta^*}(v^*)) \right\| \tag{3.7}$$

As a result, the diagonal distance between two star-like vectors in a star-like disknorm space $\vartheta^* V^*$ is provided by

$$2d(u^*, v^*) = \|E^*(u^*) - F^*(v^*)\| + 2\phi r_0^*.$$

■

Example 3.9. *Consider a star-like 3- dimensional real space* $\mathbb{R}^3 \in V_{i,j}^*$ *such that*

$$V_{i,j}^* = \begin{pmatrix} \beta_i^* \ \beta_j^* \ \dots \ \beta_n^* \\ \lambda_i^* \ \lambda_i^* \ \dots \ \lambda_n^* \end{pmatrix}$$

Then

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{pmatrix} : \beta_1^*, \beta_2^*, \beta_3^* \in \mathbb{R} \right\}$$

with the usual operation $+$ *and* $\times$ *where*

$$\beta^* = \begin{pmatrix} \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{pmatrix},$$

$$\lambda^* = \begin{pmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_3^* \end{pmatrix}.$$

Then $\begin{pmatrix} \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{pmatrix} + \begin{pmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_3^* \end{pmatrix} = \begin{pmatrix} \beta_1^* + \lambda_1^* \\ \beta_2^* + \lambda_2^* \\ \beta_3^* + \lambda_3^* \end{pmatrix}$ *such that*

$$b \begin{pmatrix} \beta_1^* \\ \beta_2^* \\ \alpha_3^* \end{pmatrix} = \begin{pmatrix} b\beta_1^* \\ b\beta_2^* \\ b\beta_3^* \end{pmatrix}$$

for every $\beta^*, \lambda^* \in V_{i,j}^*$ *and* $b \in \mathbb{R}$.

Example 3.10. *: Let* $V_{i,j}^* = \mathbb{R}$ *be the star-like space of real number with usual star-like norm:* $T^{\vartheta^*} : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by*

$$T^{\vartheta^*}(\beta^*) = 2\beta^*$$

given that $\alpha^*, \beta^* \in V_{i,j}^*$

$$\left\| T^{\vartheta^*}(\beta^*) \leq V_{i,j}^* - \lambda^* \right\| = \|2\beta^* - 2\lambda^*\| = 2 \left\| V_{i,j}^* - 1 \right\|$$

So T^{ϑ^*} *is a star-like mapping shown in 1.*

Lemma 3.11. *Suppose* $\vartheta^* \subseteq R_0^*$ *is a star-like polygon with star-like inner angles of* $Area(R_0^*) = (\sum_{n=1}^{\infty} \lambda_n) - (n-2)\pi$. *Then the star-like norm* $\|\vartheta^*\| : \vartheta^* V^* \longrightarrow \mathbb{R}$ *is then continuous.*

Proof. Let ξ and δ be any star-like elements such that $\xi > 0$ and $\delta = \xi$. The star-like convex polygon in ϑ^* of R_0^* can be strictly accomplished by arranging ϑ^* so that the origin is in the interior of R_0^* and projecting the boundary of ϑ^* on T^{*2} using

$$\vartheta^*(i, j, k) = \frac{(i, j, k)}{\sqrt{i^2 + j^2 + k^2}}. \tag{3.8}$$

The vertices of ϑ_n^* correspond to a portion of T^{*2}, the edges correspond to a portion of great circles of (ϑ_n^*), and the faces correspond to the star-like polygon. The union of $\vartheta_0^*, + \cdots \vartheta_n^*$ forms a star-like polygon on T^{*2}.

$$U(R_0^*) + U(R_1^*) + \cdots + U(R_n^*) = Area(T^{*2}) \tag{3.9}$$

For each β^*, and λ^* in $\vartheta^* V^*$

$$\vartheta^*(\lambda^*,\ \beta^*) = \vartheta^* |\lambda^* - \beta^*|.$$

generates

$$\frac{1}{2} V^* = \frac{E^* - F^*}{2} + \vartheta^*$$

Such that every star-like edge is shared by two star-like polygons and $\||v - \alpha^* u| \leq |u - \alpha^* v|\| : V^* \longrightarrow \mathbb{R}$ gives

$$\bigcup_{i=1}^{n} \bigcup_{j=1}^{n} v - \alpha^* u_{ij} - \bigcup_{i=1}^{n} u - \alpha^* v + \bigcup_{i=1}^{n} 2T^{*2} = 4\pi \tag{3.10}$$

which shows that

$$\vartheta^* T^{*2}(\|\lambda^*\|,\ \|\beta^*\|) = \vartheta^* T^{*2}|\,\|\lambda^*\| - \vartheta^* T^{*2}\,\|\beta^*\|\,|. \\ \leq \vartheta^* T^{*2}\,\|\lambda^* - \beta^*\| < \xi = \delta$$

and since the sum of star-like polygons at each vertex is 2π we obtain

$$2\pi V^* - 2\pi E^* + 2\pi F^* = 4\pi$$

As a result, $\|\vartheta^*\|$ is continuous on $\vartheta^* V^*$. ∎

Theorem 3.12. *Let $V_{i,j}^* \in P\omega_n^*$ be a star-like symmetric reducible vector space, then for any $\vartheta^* \in V_{i,j}^*$ then $\langle ab\rangle \parallel \langle cd\rangle$ such that $\langle abc\rangle \leq \langle bcd\rangle$ for any a, b, c, d of $\vartheta^* \in V_{i,j}^*$.*

Proof. Given that a and d are on the star-like opposite side of the line bc in a star-like symmetric reducible vector space shown in 3 below,

Then, by using the folding principle of a star-like reducible transformation

$$\langle abc\rangle \leq |< abc\,| \longrightarrow \langle bcd\rangle \leq |< bcd\,|\ . \tag{3.11}$$

Then

$$\langle ab\rangle \leq \langle cd\rangle = \langle v - \alpha^* u\rangle \leq \langle u - \alpha^* v\rangle \tag{3.12}$$

Therefore, for any given reducible star-like vector space, the transverse of each $V_{i,j}^* \in P\omega_n^*$ makes an equal alternative angle on two sides because the lines of any reducible star-like vector space $V_{i,j}^*$ are always reducible. ∎

Theorem 3.13. *Assume V^* is a vector norm space with a star-like disknorm. Then, on V^*, every star-like mapping T^{ϑ^*} is uniformly continuous.*

Proof. Given the fact that $(V^*, \|\times\|)$ denotes a star-like vector normed space and $T^{\vartheta^*} : (V^*, \|\times\|) \longrightarrow (V^*, \|\times\|)$ represents a star-like map, so, by equation (1.3) a star-like disknorm $\vartheta^* \in \mathbb{R}$ is defined, with $0 < \vartheta^* < 2$. Where $\xi > 0$ denotes an arbitrary element and $\delta = \frac{\xi}{\beta^*} > 0, then$ Then $\left\|T^{\vartheta^*}(\beta^*), (\lambda^*)\right\| < \delta$ such that

$$\left\|T^{\vartheta^*}(\beta^*), (\lambda^*)\right\| < \vartheta^* \times \frac{\xi}{\alpha^*} = \xi$$

Then, according to equations (1.5) and (1.7), every star-like mapping is continuous, implying that T^{ϑ^*} is uniformly continuous on $\vartheta^* V^*$. As a result, a star-like inner product space $(V^*, \langle\vartheta^*, \vartheta^*\rangle)$ is a normed vector space with the disknorm $||\vartheta^*|| = \sqrt{\langle v - \alpha^* u, u - \alpha^* v\rangle}$. ∎

Theorem 3.14. *Let $\beta^*, \lambda^* \in V^*$ then $\langle\beta^*, \lambda^*\rangle = \left\langle \begin{pmatrix} q-k \\ k-1 \end{pmatrix} = \begin{pmatrix} q-(k-1) \\ q-k \end{pmatrix} \right\rangle$.*

Proof. ;
Suppose $D(\beta^*, \lambda^*) \subseteq Z_n$.If

$$F(q,k) = \langle \beta^*, \lambda^* \in V^* \subseteq P\omega_n^* : r(\beta^*, \lambda^*)\rangle = \langle I(\beta^*, \lambda^*)\rangle = k$$

Consider $u_{ij} v_{ji} \in D(\beta^*, \lambda^*)$ such that

$$u_{ij}\,\langle\beta^*, \lambda^*\rangle \leq \langle\lambda^*, \beta^*\rangle\, v_{ji} \tag{3.13}$$

Implies

$$\langle u_{ij}v_{ji}\rangle = 0$$

So, $u_{ij}v_{ji} \subseteq V^*$ has a $q-0+1$ disknorm degree of freedom with star-like order

$$\left\langle \binom{q-k}{q-1} = \binom{q-(k-1)}{q-k} \right\rangle = 1.$$

Therefore, since for star-like reducible transformation, $\vartheta^* V^*$ is a star-like subspace of all star-like vector space and that if $\langle u_{ij}v_{ji}\rangle \in V^* : r(\beta^*,\lambda^*) = k$, irrespective of the value of $q \geq 2$ whenever $q = (q-1)$, there are exactly two star-like disknorm of rank such that

$$\langle \beta^*, \lambda^* \rangle = \left\langle \binom{q-(k-1)}{q-k} \right\rangle$$

. ■

Theorem 3.15. *If $T^{\vartheta^*} : (V^*, \langle v-\alpha^* u, u-\alpha^* v\rangle) \longrightarrow (V^*, \langle u-\alpha^* v, v-\alpha^* u\rangle)$ is a star-like map, then for each positive integer $n \in Z_n$, $T^{\vartheta^*} n* : (V^*, \langle \vartheta^*, \vartheta^*\rangle) \longrightarrow (V^*, \langle \vartheta^*, \vartheta^*\rangle)$ is also a star-like map.*

Proof. Assume $T^{\vartheta^*} : (V^*, \langle v-\alpha^* u, u-\alpha^* v\rangle) \longrightarrow (V^*, \langle u-\alpha^* v, v-\alpha^* u\rangle)$. Because T^{ϑ^*} is a star-like map, there exists a positive real integer $b \in \mathbb{R}$ that satisfies $\left\langle T^{\vartheta^*}\vartheta^*(u_{ij}v_{ji}), \vartheta^*(v_{ji}u_{ij})\right\rangle \leq b \left\langle T^{\vartheta^*}\vartheta^*(u_{ij}), (v_{ji})\right\rangle$. Then

$$\begin{aligned}
\left\langle T^2\vartheta^*(u_{ij}), (v_{ji})\right\rangle &= \left\langle T^{\vartheta^*}(T^{\vartheta^*}(\vartheta^*(u_{ij}))), T^{\vartheta^*}(\vartheta^*(v_{ji}))\right\rangle \\
&\leq b \left\langle T^{\vartheta^*}(T^{\vartheta^*}(\vartheta^*(v_{ji}))), T^{\vartheta^*}(\vartheta^*(u_{ij}))\right\rangle \\
&\leq b^2 \left\langle T^{\vartheta^*}(\vartheta^*(u_{ij})), (\vartheta^*(v_{ji}))\right\rangle \\
&= d \left\langle T^{\vartheta^*}(\lambda^*), (\beta^*)\right\rangle .
\end{aligned}$$

Where $d = b^2 \leq 2$. So, for $n = 2$, see that

$$T^2\vartheta^* : (V^*, \langle v-\alpha^* u, u-\alpha^* v\rangle) \longrightarrow (V^*, \langle u-\alpha^* v, v-\alpha^* u\rangle)$$

is a star-like map. Now, for $n = \vartheta^*$

$$T^n : (V^*, \langle v-\alpha^* u, u-\alpha^* v\rangle) \longrightarrow (V^*, \langle u-\alpha^* v, v-\alpha^* u\rangle)$$

is a star-like map:

$$\langle T^n(v_{ji}), (u_{ij})\rangle \leq b^n \langle T^n((u_{ij}), (v_{ji}))\rangle$$

for every $\beta^*, \lambda^* \in V^*$. Then,

$$\begin{aligned}
\left\langle T^{\vartheta^*+1}(u_{ij}), (v_{ji})\right\rangle &= \left\langle T^{\vartheta^*+1}(v_{ji}), T^{\vartheta^*}(T^{\vartheta^*}(u_{ij}))\right\rangle \\
&\leq b \left\langle T^{\vartheta^*+1}(u_{ij}), T^{\vartheta^*}(v_{ji})\right\rangle \\
&\leq b^{\vartheta^*+1} \left\langle T^{\vartheta^*}(v_{ji}), (u_{ij})\right\rangle .
\end{aligned}$$

Hence, by mathematical induction, we deduced that

$$T^{\vartheta^*} : (V^*, \langle v-\alpha^* u, u-\alpha^* v\rangle) \longrightarrow (V^*, \langle u-\alpha^* v, v-\alpha^* u\rangle)$$

is a star-like map for all positive integers $Z_n = 1, 2, 3, \cdots$. ■

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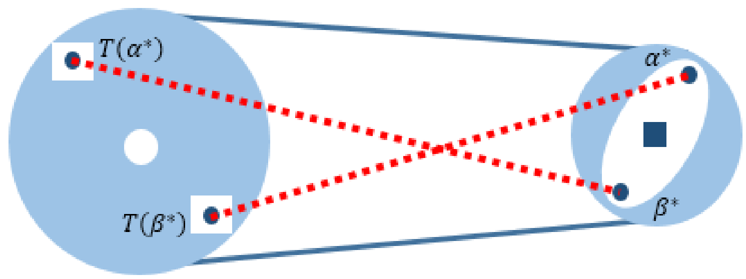

Figure 1: A star-like map $T^{\vartheta}*: T^{\vartheta}*(\upsilon^*) = \vartheta^* V^*$

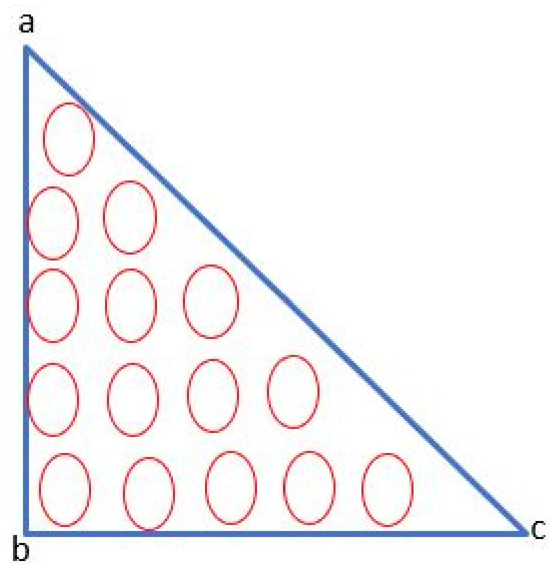

Figure 2:

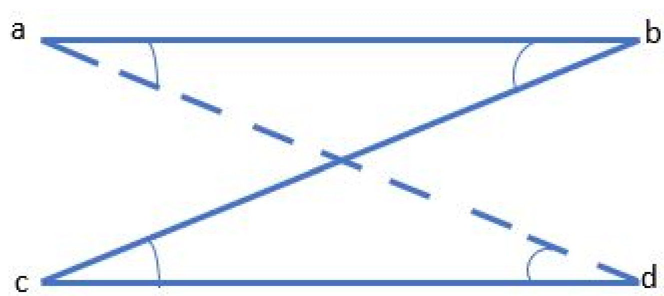

Figure 3:

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A quasistatic elastic-viscoplastic contact problem with wear and frictionless

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Abstract. We consider here a frictionless contact problem for elastic-viscoplastic materials, in a quasi-static process. The contact with a rigid base is modeled without friction with condition of wear and damage. The damage the elastic deformations of the material is modeled by an internal variable of the body called the damage field. The problem formula is given as a system that includes a variational equation with respect to the displacement field, and a variational inequality of the parabolic type with respect to the damage field. We prove a weak solution existence and uniqueness theorem relating to the problem. The methods utilised are grounded in the concept of monotonic operators, followed by fixed-point arguments.

AMS Subject Classifications: 74C10, 49J40, 74M15, 74R20

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Contents

1. Introduction

Contact-related problems, whether involving friction or not, between deformable bodies or between a rigid body and a deformable one, are frequently encountered in both industrial settings and everyday experiences. Considering the importance and the multitude of these phenomena, vast studies have been undertaken, also the literature concerning contact mechanics is vast and addresses as many different subjects as are modeling, mathematical analysis or approximation numerical contact problems, see the works [1, 2, 10, 11].

This paper explores an investigation concerning boundary conditions that mirror real-world phenomena like contact, material wear and damage. In our study, we adopt an elastic-viscoplastic constitutive law to describe the behavior of the material.

To illustrate the procedure of deformation of an elastic-viscoplastic body with wear when it contacts with a rigid body foundation, been touched on many quasi-static elastic-viscoplastic frictional Contact problems involving wear have been introduced and investigated under various conditions. For further details, we direct the reader to [5, 6] and the cited references therein.

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Chen et al.[4] were among the first to provide error estimates for fully discrete schemes designed to solve quasi-static viscoplastic frictional contact problems with wear. Gasinski et al. [7] introduced a mathematical model to describe quasi-static frictional contact with wear between a thermo-viscoelastic body and a moving foundation. In a recent development, Jureczka and Ochal [9] conducted numerical analysis and simulations for the quasi-static elastic frictional contact problem that accounts for wear.

There are other real phenomena which are very important. Such as material damage and body adhesion. The consideration of damage holds fundamental significance in the field of design engineering since it has a direct impact on the useful lifespan of the designed structure or component. There exists a substantial body of engineering literature devoted to this subject. Mathematical models that incorporate the influence of internal material damage on the contact process have been thoroughly examined. In [8], novel comprehensive damage models have been derived based on the principle of virtual power. Further mathematical analyses of one-dimensional problems related to this topic can be found in [3]. the material damage is described by capacity damage. The damage function α varies between 0 and 1. When $\alpha = 1$ there is no damage in the material, when $\alpha = 0$ the material is completely damaged, when $0 \prec \alpha \prec 1$ the damage is partial. This work is a continuation in this line of research to the mathematical study of a frictionlessly contact problem for Viscoplastic materials, in a quasi-static process. The contact with a rigid base is modeled without friction with condition of wear and damage. Our focus is to establish the existence of a unique weak solution for the abstract problem with regularized boundary conditions. The structure of the remainder of this paper is as follows: In Section 2, we provide an inventory of notations and outline the assumptions concerning the problem data. Additionally, we state our primary result regarding the existence and uniqueness of solutions. In Section 3, we delve into the proof of the theorem, where we consider the existence and uniqueness of the solution, utilizing arguments derived from the theory of monotonic operators and the Banach fixed-point theorem. In Section 4, we present an illustrative example that demonstrates the practical application of the abstract result.

Problem $\mathcal{P}$

Find the displacement field $\mathbf{u} : [0,T] \to V$, the stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$, the damage field $\alpha : [0,T] \to \mathbb{R}$.

$$\begin{aligned} &(A\dot{\mathbf{u}}(t), \mathbf{v})_V + (B\mathbf{u}(t), \mathbf{v})_V + \left(\int_0^t F(\boldsymbol{\sigma}(s) - A\dot{\mathbf{u}}(t), \mathbf{u}(s), \alpha(s))\, ds, \mathbf{v} \right)_{\mathcal{H}} \\ &\quad = (\boldsymbol{f}(t), \mathbf{v})_V \quad a.e.\ t \in (0,T), \end{aligned} \tag{1.1}$$

$$\begin{aligned} &(\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ &\quad \geq (S(\boldsymbol{\sigma}(s) - A\dot{\mathbf{u}}(t), \mathbf{u}(t), \alpha(t)), \xi - \alpha(t))_{L^2(\Omega)}, \xi \in K, \text{a.e } t \in (0,T), \end{aligned} \tag{1.2}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \alpha(0) = \alpha_0. \tag{1.3}$$

We have three spaces denoted as V, $\mathcal{H}$, and K. These spaces correspond to admissible displacements, stress, and damage, and they are all Hilbert spaces. Notably, K is a nonempty, closed, and convex set within the space V. It is defined as follows:

$$K = \{\zeta \in V \mid \quad 0 \leq \zeta(x) \leq 1 \text{ a.e. } x \in \Omega\}.$$

The operators A, B, and F are associated with the constitutive law governing an elastic-viscoplastic material with damage. The functional S is determined by the source function of the damage and the friction occurring on part Γ_3. The data $\boldsymbol{f}$ relates to both traction forces and body forces. The functions $\mathbf{u}_0$ and α_0 represent the initial data for displacement and damage, respectively. We denote the displacement field as $\mathbf{u}$ and the stress tensor field as $\boldsymbol{\sigma}$. The constitutive law applied here pertains to an elastic-viscoplastic material with damage. The interval $[0,T]$ signifies the time span of observation. A dot above $\mathbf{u}$ and α indicates the derivative of displacement $\mathbf{u}$ and the derivative of damage α with respect to the variable t.

2. Preliminaries and notion

In this section, we introduce important tools for our main results. Specifically, we denote:

$\mathbb{S}^d$ as the space comprising second-order symmetric tensors defined on $\Omega \subset \mathbb{R}^d$ (where $d = 2, 3$), and with a smooth boundary $\partial\Omega = \Gamma$. We designate Γ_3 as the boundary contact .

We define $\boldsymbol{\nu} = (\nu_i)$ as the unit outward normal vector, and $x \in \overline{\Omega} = \Omega \cup \partial\Omega$ represents the position vector. It's worth noting that unless specified otherwise, the indices i, j range from 1 to d, and we apply the summation convention to repeated indices. For the sake of simplicity, we do not explicitly indicate the variables' dependence on x.

The inner products and norms for $\mathbb{R}^d$ and $\mathbb{S}^d$ are denoted as follows:

$$\mathbf{u} \cdot \mathbf{w} = u_i w_i \quad \|\mathbf{w}\|_{\mathbb{R}^d} = (\mathbf{w}, \mathbf{w})^{1/2} \text{ for all } \mathbf{u} = (u_i), \mathbf{w} = (w_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}.\boldsymbol{\vartheta} = \sigma_{ij}\vartheta_{ij} \quad \|\boldsymbol{\vartheta}\|_{\mathbb{S}^d} = (\boldsymbol{\vartheta}, \boldsymbol{\vartheta})^{1/2} \text{ for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\vartheta} = (\vartheta_{ij}) \in \mathbb{S}^d,$$

We denote the following quantities:

$\mathbf{u} = (u_i)$ represents the displacement vector.

$\boldsymbol{\sigma} = (\sigma_{ij})$ denotes the stress tensor.

$\varepsilon(\mathbf{u}) = (\varepsilon(\mathbf{u}_{ij}))$ represents the linear strain tensor.

Furthermore, we use the following notation for components of displacement $\mathbf{u}$ on Γ:

Normal component: $u_\nu = u.\nu$

Tangential component: $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$

Similar notation is applied to $\dot{u_\nu}$ and $\dot{\mathbf{u}}_\tau$, which represent the normal and tangential velocities on the boundary, respectively.

Regarding the stress field $\boldsymbol{\sigma}$ on the boundary, we define its components as:

Normal component: $\sigma_\nu = (\boldsymbol{\sigma\nu}).\boldsymbol{\nu}$

Tangential component: $\boldsymbol{\sigma_\tau} = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$

We use the following notations

$$H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\right\}, \quad H_1 = \{\mathbf{u} = (u_i) \mid \varepsilon(\mathbf{u}) \in \mathcal{H}\},$$
$$\mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\sigma} \in H\}.$$

The deformation operator ε and the divergence operator Div are defined as follows:

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces H, H_1, $\mathcal{H}$, and $\mathcal{H}_1$ are real Hilbert spaces equipped with the canonical inner products defined as follows:

$$(\mathbf{u}, \mathbf{w})_H = \int u_i w_i dx, \quad \forall \mathbf{u}, \mathbf{w} \in H,$$
$$(\mathbf{u}, \mathbf{w})_{H_1} = (\mathbf{u}, \mathbf{w})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{w}))_{\mathcal{H}}, \forall \mathbf{u}, \mathbf{w} \in H_1,$$
$$(\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{\mathcal{H}} = \int \sigma_{ij}\vartheta_{ij} dx, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\vartheta} \in \mathcal{H},$$
$$(\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\vartheta})_H, \ \forall \boldsymbol{\sigma}, \boldsymbol{\vartheta} \in \mathcal{H}_1.$$

The associated norm in the space H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$, is denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}_1}$, respectively.

When $\boldsymbol{\sigma}$ is a regular function. The following Green-type formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{w}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \mathbf{w})_H = \int_\Gamma \boldsymbol{\sigma\nu}.\mathbf{w} da \quad \forall \mathbf{w} \in H_1. \tag{2.1}$$

For the displacement field, we necessitate the closed subspace of H_1 defined as

$$V = \{\mathbf{w} \in H_1 \mid \mathbf{w} = \mathbf{0},\ on\ \Gamma_1\}.$$

Given that $meas(\Gamma_1) > 0$, Korn's inequality is satisfied, and there exists a positive constant C_k, which solely depends on Ω and Γ_1, such that

$$\|\varepsilon(\mathbf{w})\|_{\mathcal{H}} \geq C_k \|\mathbf{w}\|_{H^1(\Omega)^d}, \quad \forall \mathbf{w} \in V.$$

We define inner product on V by

$$(\mathbf{u}, \mathbf{w})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{w}))_{\mathcal{H}}, \quad \|\mathbf{w}\|_V = \|\varepsilon(\mathbf{w})\|_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{w} \in V, \tag{2.2}$$

and let $\|.\|_V$ be the associated norm. Consequently, the norms $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_V$ are equivalent on V, and as a result, $(V, (,)_V)$ forms a real Hilbert space. Furthermore, in accordance with the Sobolev trace theorem, there exists a constant $\tilde{C}_0$, which relies solely on Ω, Γ_1, and Γ_3, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq \tilde{C}_0 \|\mathbf{v}\|_V, \quad \forall \mathbf{v} \in V. \tag{2.3}$$

We recall some spaces $W^{k,p}(0,T;V), H^k(0,T;V)$, $C(0;T;V)$ and $C^1(0;T;V)$ for a Banach space V equipped with the norm $\|.\|_V$ for $1 < p < +\infty$ and $k \geq 1$. Let $W^{k,p}(0,T;V)$ be the space of all functions from $[0,T]$ to V with the norm

$$\|\omega\|_{W^{k,p}(0,T;V)} = \begin{cases} \left(\displaystyle\int_0^T \sum_{1 \leq l \leq k} \left\|\partial_t^l \omega\right\|_V^p dt \right)^{1/p}, & \text{if } 1 \leq p < +\infty \\ \max_{0 \leq l \leq k_0 \leq t \leq T} \sup_t \left\|\partial_t^l \omega\right\|_V, & \text{if } p = +\infty. \end{cases}$$

When $p = 2$ or $k = 0$, $W^{k,2}([0,T];V)$ is written as $H^k([0,T];V)$ or $L^p([0,T];V)$, respectively. We denote by $C([0,T];V)$ the space of continuous functions from $[0,T]$ to V, and by $C^1(0,T;V)$ the space of continuously differentiable functions from $(0,T)$ to V. These spaces are equipped with the following norms:

$$\|\omega\|_{C([0,T];V)} = \max_{t \in [0,T]} \|\omega(t)\|_V.$$

$$\|\omega\|_{C^1([0,T];V)} = \max_{t \in [0,T]} \|\omega(t)\|_V + \max_{t \in [0,T]} \|\dot{\omega}(t)\|_V.$$

Clearly, $C([0,T];V)$, $W^{k,p}([0,T];V)$ and $H^k([0,T];V)$ are all Banach spaces when V is a Banach space.

In order to solve Problem $\mathcal{P}$, we impose the following assumptions.

We consider operators $A, B : V \to V$, $F : \mathcal{H} \times \mathcal{H} \times H^1(\Omega) \to V$, the damage source function $S : \mathcal{H} \times \mathcal{H} \times H^1(\Omega) \to \mathbb{R}$, and two initial values $u_0 \in V$ and $\alpha_0 \in K$. These operators and values satisfy the following properties

There exists a constant $M_A \succ 0$ such that

$$(A\mathbf{v}_1 - A\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \geq M_A \|\mathbf{v}_1 - \mathbf{v}_2\|^2, \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{2.4}$$

There exists a constant $L_A \succ 0$ such that

$$\|A\mathbf{v}_1 - A\mathbf{v}_2\|_{V'} \leq L_A \|\mathbf{v}_1 - \mathbf{v}_2\|_V, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{2.5}$$

There exists a constant $L_B \succ 0$ such that

$$\|B\mathbf{v}_1 - B\mathbf{v}_2\|_V \leq L_B \|\mathbf{v}_1 - \mathbf{v}_2\|, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{2.6}$$

The $\boldsymbol{f}$ function satisfies:

$$\boldsymbol{f} \in L^2(0,T;V). \tag{2.7}$$

There exists a constant $L_F \succ 0$ such that

$$\|F(\boldsymbol{\sigma}_1, \mathbf{u}_1, \zeta_1) - F(\boldsymbol{\sigma}_2, \mathbf{u}_2, \zeta_2)\| \leq L_F(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\| + \|\zeta_1 - \zeta_2\|), \tag{2.8}$$

for all $\boldsymbol{\sigma}_i \in \mathcal{H},\ \mathbf{u}_i \in V,\ \zeta_i \in H^1(\Omega),\ i = 1, 2.$

There exists $M_S \succ 0$ such that

$$\|S(\boldsymbol{\sigma}_1, \mathbf{u}_1, \zeta_1) - S(\boldsymbol{\sigma}_2, \mathbf{u}_2, \zeta_2)\| \leq M_S(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\| + \|\zeta_1 - \zeta_2\|), \tag{2.9}$$

for all $\boldsymbol{\sigma}_i \in \mathcal{H},\ \mathbf{u}_i \in V,\ \forall\zeta_i \in H^1(\Omega),\ i = 1, 2.$

Now let problem $\mathcal{P}_1$ as it follows

Problem $\mathcal{P}_1$

Find $\mathbf{u} \in C^1(0, T; V)$ such that

$$\begin{cases} A\mathbf{u}(t) = \mathbf{f}, \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{2.10}$$

Theorem 2.1. *If conditions (2.4),(2.5) and (2.7) are satisfied Then there exists $\mathbf{u} \in C^1(0, T; V)$ solution to the problem $\mathcal{P}_1$ satisfying*

$$\mathbf{u} \in H^1(0, T; V) \cap C^1(0, T; H). \tag{2.11}$$

The previous result is a special case of the Minty-Browder Theorem.

Problem $\mathcal{P}_2$

Find $\alpha(t) \in K$ such that

$$(\dot{\alpha}(t), \rho - \alpha(t))_{V' \times V} + a(\dot{\alpha}(t), \rho - \alpha(t)) \geq (S(t), \rho - \alpha(t))_{L^2(\Omega)},\ \forall\rho \in K, \tag{2.12}$$

$$\alpha(0) = \alpha_0. \tag{2.13}$$

We consider two real Hilbert spaces, denoted as V and H. It is important to note that V is densely embedded in H, and this injection map is continuous. Furthermore, we identify the space H with both its own dual and as a subspace of the dual space V' of V. In other words, we express this relationship as $V \subset H \subset V'$, and this set of inclusions is what defines a Gelfand triple.

The following is a well-established result for parabolic variational inequalities, and you can find it in standard references such as [12].

Theorem 2.2. *Consider a Gelfand triple $V \subset H \subset V'$, where K is a nonempty, closed, and convex set in V. Assume the existence of a continuous and symmetric bilinear form $a(.,.) : V \times V \to \mathbb{R}$ satisfying the following inequality for constants λ and γ:*

$$a(\alpha, \alpha) + \gamma\|\alpha\|_H^2 \geq \lambda\|\alpha\|_V^2, \quad \forall\alpha \in V.$$

Under these conditions, for any initial value $\alpha_0 \in K$ and source function $S \in L^2(0, T; H)$, there exists a unique function $\alpha \in H^1(0, T; H) \cap L^2(0, T; V)$ such that $\alpha(0) = \alpha_0$ and $\alpha(t) \in K$ for all $t \in [0, T]$. This α is the unique solution to Problem $\mathcal{P}_2$.

The next section is dedicated to investigating the existence of a unique solution to Problem $\mathcal{P}$.

3. Proof of the main result

Theorem 3.1. *Under the assumptions (2.4)-(2.9), there exists a unique solution of the problem $\mathcal{P}$, Moreover the solution satisfies:*

$$\mathbf{u} \in H^1(0,T;V) \cap C^1(0,T;H), \tag{3.1}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad \operatorname{Div} \boldsymbol{\sigma} \in L^2(0,T;H), \tag{3.2}$$

$$\alpha \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \tag{3.3}$$

The proof of Theorem 3.1 is conducted through several sequential steps and relies on the subsequent abstract result concerning evolutionary variational inequalities.

Suppose we have $\eta \in L^2(0,T;V)$, and let's consider the following problem

Problem $\mathcal{P}_\eta$

Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$, such that

$$\begin{cases} (A\dot{\mathbf{u}}_\eta(t), \mathbf{v})_V + (\boldsymbol{\eta}(t), \mathbf{v})_V = (\boldsymbol{f}, \mathbf{v})_V, \\ \quad \text{a.e. } t \in (0,T), \quad \forall \mathbf{v} \in V, \\ \quad \mathbf{u}_\eta(0) = \mathbf{u}_0. \end{cases} \tag{3.4}$$

Here is the given result concerning $\mathcal{P}_\eta$.

Lemma 3.2. *A unique solution $\mathbf{u}_\eta \in C^1(0,T;V)$ to the problem $\mathcal{P}_\eta$ exists, and it satisfies the condition (3.1).*

Proof. We apply Theorem 2.1, The Riesz representation theorem allows us to define $\boldsymbol{f}_\eta : [0,T] \to V$, by $\left(\boldsymbol{f}_\eta(t), \mathbf{v}\right)_V = (f(t) - \boldsymbol{\eta}(t), \mathbf{v})_V$. Using hypotheses (2.4)-(2.7), and $\mathbf{u}_\eta(t) = \mathbf{u}_0 + \int_0^t \dot{\mathbf{u}}_\eta(s)ds, \quad \forall t \in (0,T)$, we directly find the result. ■

Subsequently, introduce $\theta \in L^2(0,T;L^2(\Omega))$, and let's examine the following problem

Problem $\mathcal{P}_\theta$

Find the damage field $\alpha_\theta : [0,T] \to \mathbb{R}$,

$$\begin{aligned} &\alpha_\theta(t) \in K, (\dot{\alpha}_\theta(t), \rho - \alpha_\theta(t))_{L^2(\Omega)} + a(\alpha_\theta(t), \rho - \alpha_\theta(t)) \\ &\qquad \geq (\theta(t), \rho - \alpha_\theta(t))_{L^2(\Omega)}, \forall \rho \in K, \text{ a.e.t } \in (0,T), \end{aligned} \tag{3.5}$$

$$\alpha_\theta(0) = \alpha_0. \tag{3.6}$$

Lemma 3.3. *problem $\mathcal{P}_\theta$ has a unique solution α_θ such that*

$$\alpha_\theta \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \tag{3.7}$$

For the proof, we apply Theorem 2.2.

Finally, in the concluding step, formulate the subsequent Cauchy problem for the stress field

Problem $\mathcal{P}_{\eta,\theta}$

Find the stress field $\boldsymbol{\sigma}_{\eta,\theta} : (0,T) \to \mathcal{H}$, solution of the problem

$$\boldsymbol{\sigma}_{\eta,\theta}(t) = B\mathbf{u}_\eta(t) + \int_0^t F\left(\boldsymbol{\sigma}_{\eta,\theta}(s) - A\dot{\mathbf{u}}_\eta(s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds, \text{ a.e.t } \in (0,T). \tag{3.8}$$

Lemma 3.4. *The problem $\mathcal{P}_{\eta,\theta}$ has a unique solution. Additionally, if $\mathbf{u}_{\eta_i}$, α_{θ_i}, and $\boldsymbol{\sigma}_{\eta_i,\theta_i}$ represent the solutions to problems $\mathcal{P}_\eta$, $\mathcal{P}_\theta$, and $\mathcal{P}_{\eta,\theta}$ for $i = 1,2$, then there exists a positive constant C such that*

$$\begin{aligned}\|\boldsymbol{\sigma}_{\eta_1,\theta_1}(t) - \boldsymbol{\sigma}_{\eta_2,\theta_2}(t)\|^2_{\mathcal{H}} \leq & C\left(\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|^2_V + \int_0^t \|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|^2_V \, ds \right.\\ & \left. + \int_0^t \|\alpha_{\theta_1}(s) - \alpha_{\theta_2}(s)\|^2_{L^2(\Omega)} \, ds\right).\end{aligned} \tag{3.9}$$

Proof. Consider the mapping $\sum_{\eta,\theta} : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ defined as

$$\sum_{\eta,\theta} \boldsymbol{\sigma}_{\eta,\theta}(t) = B\mathbf{u}_\eta(t) + \int_0^t F\left(\boldsymbol{\sigma}_{\eta,\theta}(s) - A\dot{\mathbf{u}}_\eta(s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds. \tag{3.10}$$

let $\boldsymbol{\sigma}_i \in L^2(0,T;\mathcal{H})$, $\quad i = 1,2$ and $t_1 \in (0,T)$, we use the assumption (2.8) and the HÖlder inequality we find

$$\left\|\sum_{\eta,\theta} \boldsymbol{\sigma}_1(t_1) - \sum_{\eta,\theta} \boldsymbol{\sigma}_2(t_1)\right\|^2_{\mathcal{H}} \leq L_F^2 T \int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}} \, ds. \tag{3.11}$$

We have more

$$\begin{aligned}&\left\|\sum_{\eta,\theta}\left(\sum_{\eta,\theta} \boldsymbol{\sigma}_1(t_1)\right) - \sum_{\eta,\theta}\left(\sum_{\eta,\theta} \boldsymbol{\sigma}_2(t_1)\right)\right\|^2_{\mathcal{H}} \\ &\qquad \leq L_F^2 T \int_0^{t_1} \left\|\sum_{\eta,\theta} \boldsymbol{\sigma}_1(t_1) - \sum_{\eta,\theta} \boldsymbol{\sigma}_2(t_1)\right\|^2_{\mathcal{H}} dt_2 \\ &\qquad \leq L_F^4 T^2 \int_0^{t_1}\int_0^{t_2} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}} \, ds dt_2.\end{aligned}$$

By extending the inequality through recurrence, we deduce that for all $t_1, t_2, ..., t_n \in (0,T)$,

$$\left\|\sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_1(t_n) - \sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_2(t_n)\right\|^2_{\mathcal{H}} \leq L_F^{2n} T^n \int_0^{t_1}\int_0^{t_2} \dots \int_0^{t_n} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}} \, ds dt_n \dots dt_2.$$

Thus, we can deduce by integrating with respect to $(0,T)$ the following inequality

$$\left\|\sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_1 - \sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_2\right\|^2_{\mathcal{H}} \leq \frac{L_F^{2n} T^{2n}}{n!} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|^2_{\mathcal{H}}. \tag{3.12}$$

Then from (3.12), for n sufficiently large, the operator $\sum_{\eta,\theta}^{(n)}$, is a contraction on space $L^2(0,T;\mathcal{H})$ and according to the Banach fixed point theorem, there is a single element $\boldsymbol{\sigma}_{\eta,\theta} \in L^2(0,T;\mathcal{H})$ such that

$\sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_{\eta,\theta} = \boldsymbol{\sigma}_{\eta,\theta}$, which represents the unique solution of problem $\mathcal{P}_{\eta,\theta}$. Moreover, if $\mathbf{u}_{\eta_i}$, α_{θ_i} and $\boldsymbol{\sigma}_{\eta_i,\theta_i}$, represents the solutions of problem $\mathcal{P}_{\eta_i}$,$\mathcal{P}_{\theta_i}$ and $\mathcal{P}_{\eta_i,\theta_i}$ respectively. For $i = 1,2$. designate $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \boldsymbol{\sigma}_{\eta_i,\theta_i} = \boldsymbol{\sigma}_i, \alpha_{\theta_i} = \alpha_i$.

We have

$$\boldsymbol{\sigma}_i(t) = B\mathbf{u}_i(t) + \int_0^t F\left(\boldsymbol{\sigma}_i(s) - A\dot{\mathbf{u}}_i(s),\ \mathbf{u}_i(s), \alpha_i(s)\right) ds,\ \text{a.e. } t \in (0,T),$$

we use the assumption (2.6),(2.8)), we find

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}}^2 \le C \Bigg(&\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\, ds \\ &+ \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\, ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2\, ds \Bigg).\end{aligned}$$

We employ the Gronwall argument within the resulting inequality to derive (3.9). ■

Now, let's contemplate the mapping

$$\Lambda : L^2(0,T;\mathcal{H} \times L^2(\Omega)) \to L^2(0,T;\mathcal{H} \times L^2(\Omega)),$$

$$\Lambda\left(\boldsymbol{\eta},\theta\right)(t) = \left(\Lambda^1\left(\boldsymbol{\eta},\theta\right)(t), \Lambda^2\left(\boldsymbol{\eta},\theta\right)(t)\right), \tag{3.13}$$

defined by equalities

$$\Lambda^1(\boldsymbol{\eta},\theta)(t), = B\mathbf{u}_\eta(t) + \int_0^t F\left(\boldsymbol{\sigma}_{\eta,\theta}(s) - A\dot{\mathbf{u}}(s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds, \tag{3.14}$$

$$\Lambda^2(\boldsymbol{\eta},\theta)(t) = S\left(\left(\boldsymbol{\sigma}_{\eta,\theta}(t), \mathbf{u}_\eta(t)\right), \alpha_\theta(t)\right). \tag{3.15}$$

We have the following result.

Lemma 3.5. *For $(\boldsymbol{\eta},\theta) \in L^2\left(0,T;\mathcal{H} \times L^2(\Omega)\right)$, the operator $\Lambda(\boldsymbol{\eta},\theta) : [0,T] \to \mathcal{H} \times L^2(\Omega)$ have a unique fixed point denoted as $(\boldsymbol{\eta}^*,\theta^*) \in L^2\left(0,T;\mathcal{H} \times L^2(\Omega)\right)$, satisfying*

$$\Lambda\left(\boldsymbol{\eta}^*,\theta^*\right) = (\boldsymbol{\eta}^*,\theta^*).$$

Proof. Let $t \in (0,T)$ and $(\boldsymbol{\eta}_1,\theta_1), (\boldsymbol{\eta}_2,\theta_2) \in L^2\left(0,T;\mathcal{H} \times L^2(\Omega)\right)$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i,\ \dot{\mathbf{u}}_{\eta_i} = \dot{\mathbf{u}}_i,\ \alpha_{\eta_i} = \alpha_i,\ \boldsymbol{\sigma}_{\boldsymbol{\eta}_i,\theta_i} = \boldsymbol{\sigma}_i$,For $i = 1,2$ and using the assumptions (2.5),(2.6) and (2.8)

$$\begin{aligned}&\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}}^2 \\ &\qquad = \|B\mathbf{u}_1(t) + \int_0^t F\left(\boldsymbol{\sigma}_1(s) - A\dot{\mathbf{u}}_1(s), \mathbf{u}_1(s), \alpha_1(s)\right) ds \\ &\qquad - B\mathbf{u}_2(t) - \int_0^t F\left(\boldsymbol{\sigma}_2(s) - A\dot{\mathbf{u}}_2(s), \mathbf{u}_2(s), \alpha_2(s)\right) ds\|_{\mathcal{H}}^2 \\ &\qquad \le L_B\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + L_F\int_0^t (\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 + \\ &\qquad L_A\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds.\end{aligned}$$

We utilise the estimate (3.9) to derive

$$\begin{aligned}&\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}}^2 \\ &\qquad \le C(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ &\qquad + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2))ds.\end{aligned}$$

On the other hand, we know that $\mathbf{u}_i(t)=\mathbf{u}_0+\int_0^t \dot{\mathbf{u}}_i(s)ds$, for all $t\in(0,T)$

$$\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2\leq\int_0^t\|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s)\|_V^2ds. \tag{3.16}$$

By Apply the inequality (3.16) becomes

$$\begin{aligned}\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}}^2\leq C\int_0^t(\|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s)\|_V^2\\ +\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2+\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2)ds.\end{aligned} \tag{3.17}$$

By a similar argument, from (3.9),(3.15) and (2.9) it follows that

$$\begin{aligned}\|\Lambda^2(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^2(\boldsymbol{\eta}_2,\theta_2)(t)\|_{L^2(\Omega)}^2\leq C(\int_0^t(\|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s)\|_V^2\\ +\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2+\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2)ds\\ +\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V^2+\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Omega)}^2).\end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned}\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}^2\leq C(\int_0^t(\|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s)\|_V^2\\ +\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2+\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2)ds\\ +\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V^2+\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Omega)}^2).\end{aligned} \tag{3.19}$$

Combine the inequality (3.16) with (3.19) to obtain

$$\begin{aligned}\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}^2\leq C\int_0^t(\|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s)\|_V^2\\ +\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2+\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2)ds.\end{aligned} \tag{3.20}$$

Using the inequality (3.4), by adding the results obtained we have

$$(A\dot{\mathbf{u}}_1(t)-A\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t))_V=(\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t),\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t))_V,\ t\in(0,T), \tag{3.21}$$

using inequality (2.4), we find

$$M_A\|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\|_V^2\leq\|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\|_V\|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\|_V.$$

Therefore

$$\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\|_V\leq C\|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\|_V,\quad\forall t\in[0,T].$$

Let's use (3.16)

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\leq C\int_0^t\|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\|_Vds,\quad\forall t\in[0,T]. \tag{3.22}$$

Using (3.5) we find

$$(\dot{\alpha}_1-\dot{\alpha}_2,\alpha_1-\alpha_2)_{L^2(\Omega)}+a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\leq(\theta_1-\theta_2,\alpha_1-\alpha_2)_{L^2(\Omega)},$$

$$a\cdot e\cdot t\in(0,T),$$

By integrating the inequality with respect to time and incorporating the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$, along with the inequality $a(\alpha_1-\alpha_2, \alpha_1-\alpha_2) \geq 0$, we combine this inequality with Gronwall's lemma, resulting in the following result

$$\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)} \leq C\int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\, ds, \forall t \in [0,T]. \tag{3.23}$$

From the previous inequality and estimates (3.20), (3.22) and (3.23) it follows that now

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)}\\ &\qquad\leq C\left(\int_0^t \|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|^2_{\mathcal{H}\times L^2(\Omega)} ds\right).\end{aligned}$$

Let is introduce the following notations

$$\begin{cases} I_1 = \displaystyle\int_0^t \|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|_{\mathcal{H}\times L^2(\Omega)}\, ds,\\ \vdots\\ I_k = \displaystyle\int_0^t\int_0^{s_{k-1}}\cdots\int_0^{s_1} \|(\boldsymbol{\eta}_1,\theta_1)(r)-(\boldsymbol{\eta}_2,\theta_2)(r)\|_{\mathcal{H}\times L^2(\Omega)}, \end{cases}$$

Through an inductive process, denoting the m^{th} power of the operator Λ as Λ^m, we arrive at the following conclusion

$$\begin{aligned}&\|\Lambda^m(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^m(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}\\ &\qquad\leq C^m\left(\sum_{k=1}^m C_m^k I^{m-k}\|(\boldsymbol{\eta}_1,\theta_1)(t)-(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}\right),\end{aligned} \tag{3.24}$$

for all $t \in [0,T]$ and $m \in \mathbb{N}$,

$$\begin{aligned} I^{m-k}((\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)) &= \int_{(m-k)\text{ fois}} . \int \|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\|\\ &\leq \int_0^s\int\cdots\int_{(m-k)\text{ fois}} \|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\|_{L^2(0,T;\mathcal{H}\times L^2(\Omega))}\\ &\leq \frac{t^{m-k}}{k!}\|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\|_{L^2(0,T;\mathcal{H}\times L^2(\Omega))}\\ &\leq \frac{T^{m-k}}{k!}\|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\|_{L^2(0,T;\mathcal{H}\times L^2(\Omega))},\end{aligned}$$

Consequently,

$$\begin{aligned}&\|\Lambda^m(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^m(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{L^2(0,T;\mathcal{H}\times L^2(\Omega))}\\ &\qquad\leq C^m\left(\sum_{k=1}^m C_m^k\frac{T^{m-k}}{k!}\|(\boldsymbol{\eta}_1,\theta_1)(t)-(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{L^2(0,T;\mathcal{H}\times L^2(\Omega))}\right)\\ &\qquad\leq \frac{(CT)^m}{m!}\|(\boldsymbol{\eta}_1,\theta_1)(t)-(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{L^2(0,T;\mathcal{H}\times L^2(\Omega))},\end{aligned}$$

this implies that for m large enough, the operator Λ^m of Λ is a contraction on Banach space $L^2(0,T;\mathcal{H}\times L^2(\Omega))$. So Λ^m has a unique fixed point $(\boldsymbol{\eta}^*,\theta^*) \in L^2(0,T;\mathcal{H}\times L^2(\Omega))$, and therefore $(\boldsymbol{\eta}^*,\theta^*)$ is the only fixed point of Λ. ■

Existence

Let $(\boldsymbol{\eta}^*, \theta^*) \in L^2\left(0,T;\mathcal{H}\times L^2(\Omega)\right)$, be the fixed point of Λ defined by (3.14)-(3.15) and let $\mathbf{u}_\eta,\ \alpha_\theta$, be the solutions of problems $\mathcal{P}_\eta$, $\mathcal{P}_\theta$, for $\eta = \eta^*,\ \theta = \theta^*,\ \mathbf{u} = \mathbf{u}_{\eta^*}$, $\alpha = \alpha_{\theta^*}$, we find $(\mathbf{u}, \boldsymbol{\sigma}, \alpha)$ is a solution of problem $\mathcal{P}$. properties (3.1)-(3.3) follow from lemma 3.2, 3.3, 3.4.

Uniqueness

The uniqueness of the solution is a result of the uniqueness of the fixed point of operator Λ.

4. Application

In this section, we will utilise the main result from Section 3 to analyse a problem of contact without friction with condition of wear and damage. between an elastic-viscoplastic body and a rigid base in a quasistatic process. We provide the physical context for the contact problem and introduce certain notations that will be employed in the subsequent discussion. We consider a elastic-viscoplastic body which occupies a domain $\Omega \subset \mathbb{R}^d$, where $d = 2,3$, such that the boundary $\Gamma = \partial\Omega$ is Lipschitz continuous. The boundary $\partial\Omega$ is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 with $meas(\Gamma_1) > 0$. We are interested in an evolution of the body in a finite time interval $(0,T)$.

We consider the following classical formulation of the problem

Problem P

Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, the damage field $\alpha : \Omega \times [0,T] \to \mathbb{R}$.

$$0 = \mathrm{Div}\boldsymbol{\sigma} + f_0, \qquad \text{in } \Omega \times (0,T), \tag{4.1}$$

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{F}\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon\left(\mathbf{u}(s)\right), \alpha(s)\right) ds \qquad \text{in } \Omega \times (0,T), \tag{4.2}$$

$$\dot{\alpha} - k_0\Delta\alpha + \partial\varphi_K(\alpha) \ni \phi(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}), \alpha), \qquad \text{in } \Omega \times (0,T), \tag{4.3}$$

$$\mathbf{u} = 0, \qquad \text{on } \Gamma_1 \times (0,T), \tag{4.4}$$

$$\sigma\nu = f_2, \qquad \text{on } \Gamma_2 \times (0,T), \tag{4.5}$$

$$\begin{cases} -\sigma_\nu = k\|\dot{u}_\nu\| \\ \boldsymbol{\sigma}_\tau = 0 \end{cases} \qquad \text{on } \Gamma_3 \times (0,T), \tag{4.6}$$

$$\frac{\partial\alpha}{\partial\nu} = 0, \qquad \text{on } \Gamma \times (0,T), \tag{4.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \alpha(0) = \alpha_0, \qquad \text{in } \Omega. \tag{4.8}$$

Equation (4.1) describes the equation of motion, where f_0 stands for the density of the voluminal forces exerted upon the deformable body Ω. Equation (4.2) describes the constitutive law applicable to an elastic-viscoplastic material with damage, (4.3) represents a differential inclusion describing the evolution of the damage field where S is a damage source function. φ_K is the sub-differential of the indicator function of the set of admissible damage functions K. The conditions (4.4) and (4.5) are displacement-traction conditions, (4.6) represent the boundary contact conditions with wear and without friction. (4.7) represents the boundary condition of Neumann, Finally,(4.8) represents the initial conditions.

Next, we outline the assumptions concerning the data of the problem, starting with the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfied

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{A}} > 0 \quad \text{such that} \\ \|\mathcal{A}(\mathbf{x}, \boldsymbol{v}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{v}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|, \ \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \quad \text{such that} \\ (\mathcal{A}(\mathbf{x}, \boldsymbol{v}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{v}_2)).(\boldsymbol{v}_1 - \boldsymbol{v}_2) \geq m_{\mathcal{A}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2, \ \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (c) \text{ The mapping } \ \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{v}) \text{ is lebesgue measurable on } \Omega, \forall \boldsymbol{v} \in \mathbb{S}^d. \\ (d) \text{ The mapping } \ \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{4.9}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfied

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{B}} > 0 \quad \text{such that} \\ \|\mathcal{B}(\mathbf{x}, \boldsymbol{v}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{v}_2)\| \leq L_{\mathcal{B}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|, \quad \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{B}} > 0 \quad \text{such that} \\ (\mathcal{B}(\mathbf{x}, \boldsymbol{v}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{v}_2)).(\boldsymbol{v}_1 - \boldsymbol{v}_2) \geq m_{\mathcal{B}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2, \ \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (c) \text{ The mapping } \ \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{v}) \text{ is lebesgue measurable on } \Omega, \\ \qquad \forall \boldsymbol{v} \in \mathbb{S}^d. \\ (d) \text{ The mapping } \ \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{4.10}$$

The relaxation function $\mathcal{F} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{S}^d$, satisfied

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{F}} > 0 \quad \text{such that} \\ \|\mathcal{F}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{v}_1, \alpha_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{v}_2, \alpha_2)\| \leq \\ \qquad\qquad L_{\mathcal{F}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + |\boldsymbol{v}_1 - \boldsymbol{v}_2\| + \|\alpha_1 - \alpha_2\|) \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, \forall t \in [0, T], \ \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \mathbf{x} \mapsto \mathcal{F}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{v}, \alpha) \text{ is lebesgue measurable on } \Omega, \\ \forall \boldsymbol{\sigma}, \boldsymbol{v} \in \mathbb{S}^d, \ \forall t \in [0, T], \ \forall \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } x \mapsto \mathcal{F}(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0) \in \mathcal{H}, \forall t \in [0, T]. \end{cases} \tag{4.11}$$

The function describing the source of damages, denoted as $\phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{R}$, is satisfied

$$\begin{cases} (a) \text{ There exists } M_\phi > 0 \quad \text{such that} \\ \|\phi(\mathbf{x}, \boldsymbol{v}_1, \alpha_1) - \phi(\mathbf{x}, \boldsymbol{v}_2, \alpha_2)\| \leq M_\phi(\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| + \|\alpha_1 - \alpha_2\|), \\ \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \ \mathbf{x} \mapsto \phi(\mathbf{x}, \boldsymbol{v}, \alpha) \text{ is lebesgue measurable on } \Omega, \\ \forall \boldsymbol{v} \in \mathbb{S}^d, \forall \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } \ \mathbf{x} \mapsto \phi(\mathbf{x}, \mathbf{0}, 0) \in L^2(\Omega). \end{cases} \tag{4.12}$$

The body force $\mathbf{f}_0$, surface traction $\mathbf{f}_2$, coefficient of friction k, initial conditions u_0, have the following properties

$$\begin{cases} \mathbf{f}_0 \in L^2\left(0, T; H\right), \\ \mathbf{f}_2 \in L^2\left(0, T; L^2\left(\Gamma_2\right)^d\right), \\ k \in L^\infty\left(\Gamma_3\right), \quad k(x) \geq 0 \text{ for a.e. } x \in \Gamma_3, \\ \mathbf{u}_0 \in V. \end{cases} \tag{4.13}$$

We establish the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ as follows

$$a(\xi, \zeta) = k_0 \int_\Omega \nabla\xi \nabla\zeta dx \tag{4.14}$$

and the micro crack diffusion coefficient verifies $k_0 > 0$.

The initial damage α_0 field satisfies

$$\alpha_0 \in K. \tag{4.15}$$

To consider the field of displacements, we require the closed subspace V within the space H_1, defined by:

$$V = \{\mathbf{u} \in H_1 \mid \mathbf{u} = \mathbf{0} \ \ on\, \Gamma_1 \ \}\,. \tag{4.16}$$

Using Riesz's representation theorem, we find

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Gamma \mathbf{f}_0 \cdot \mathbf{v} dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in V, t \in [0, T]. \tag{4.17}$$

It's important to observe that condition (4.13) results in the implication that

$$\mathbf{f} \in L^2\left(0, T; V\right). \tag{4.18}$$

Now, consider the application $j : V \times V \to \mathbb{R}$, defined as follows

$$j\left(\mathbf{u}, \mathbf{v}\right) = \int\limits_{\Gamma_3} k \left\| u_\nu \right\| v_\nu da. \tag{4.19}$$

The variational formulation for problem P is presented as follows

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) \\ &\quad + \int_0^t \mathcal{F}\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon\left(\mathbf{u}(s)\right), \alpha(s)\right) ds \quad \text{a.e } .t \in (0, T), \end{aligned} \tag{4.20}$$

$$(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) = (\mathbf{f}, \mathbf{v})_V, \quad \forall \mathbf{v} \in V, \tag{4.21}$$

$$\begin{aligned} \alpha(t) \in K, \quad & (\dot{\alpha}(t), \zeta - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \zeta - \alpha(t)) \\ & \geq (\phi(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{u}(t)), \alpha(t)), \zeta - \alpha(t))_{L^2(\Omega)}, \quad \forall \zeta \in K, t \in [0, T], \end{aligned} \tag{4.22}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \alpha(0) = \alpha_0. \tag{4.23}$$

Utilising Riesz's representation theorem, we define the operator $A : V \to V$ as follows:

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}(\varepsilon(\mathbf{u})), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.24}$$

We will verify the hypotheses (2.4),(2.5). Let $\mathbf{u}_1, \mathbf{u}_2 \in V$. Using (4.9),(4.24) and the definition of j given by (4.19), we let's find

$$\begin{aligned} \|A\mathbf{u}_1 - A\mathbf{u}_2\|_V &= \|\mathcal{A}\varepsilon(\mathbf{u}_1) - \mathcal{A}\varepsilon(\mathbf{u}_2)\|_{\mathcal{H}} + C_0^2 \|k\|_{L^\infty(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_V \\ &\leq L_{\mathcal{A}} \|\varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2)\|_{\mathcal{H}} + C_0^2 \|k\|_{L^\infty(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_V \\ &= \left(L_{\mathcal{A}} + C_0^2 \|k\|_{L^\infty(\Gamma_3)}\right) \|\mathbf{u}_1 - \mathbf{u}_2\|_V, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in V. \end{aligned} \tag{4.25}$$

Similarly for all $\mathbf{u}_1, \mathbf{u}_2 \in V$ we have

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \geq \left(m_{\mathcal{A}} - C_0^2 \|k\|_{L^\infty(\Gamma_3)}\right) \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in V. \tag{4.26}$$

Let $\gamma_0 = \dfrac{m_{\mathcal{A}}}{C_0^2}$, it is clear that γ_0 is positive which depends on Ω_1,Γ_3, and $\mathcal{A}$. Then A is strongly monotonic on V if

$$\|k\|_{L^\infty(\Gamma_3)} < \gamma_0.$$

After confirming that all the assumptions of Theorem 3.1 are met, we can conclude that a unique weak solution to problem P exists, satisfying (4.20)-(4.23), along with the regularity conditions (3.1)-(3.3).

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Optimal strategy on inventory model under permissible delay in payments and return policy for deteriorating items with shortages

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Abstract. In this model, the manufacturer offers a trade credit policy to the retailer. Demand depends upon selling price and time for non-instantaneous deterioration items. The retailer offers the customer a returns policy. Customers can return the product to the retailer if the product is unsatisfactory for the customer. The retailer does not ultimately return the amount to its customer for the returned product. The manufacturer offers the retailer a trade credit policy. The retailer resale the returned products at the same selling price. A partially backlogged shortage is permitted and its rate is thought to depend on how long it takes for the following lot to arrive after a lot has been replenished. The main objective is to increase the retailers' overall profit by determining the optimal order quantity, optimal selling price, and optimal replenishment cycle. An EOQ is framed for analyzing the sample, which can obtain the optimal solution.

AMS Subject Classifications: 90-05, 90-06, 90-08, 90-10, 90-11.

Keywords: Non-instantaneous deteriorating items, Permissible delay in payments, resalable returns, Partial backlogging.

Contents

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1. Introduction

In the enterprise international, each production firm/provider usually continues the inventory so one can sell the product to their potential clients. There is a large question, how to keep the inventory degree and the way to sell the goods to their capacity clients? Basically, most profit or minimal loss relies upon the idea of these two questions. Here we are using some other factors such as controlling the deterioration rate and introducing some attractive offers that promote greater products. Deteriorating products are the greatest challenge to companies. Deterioration means damage, decay and spoilage of products from their condition initially. There is a fresh-product phase for some disintegrating products, during which they hold onto their original quality and worth before eventually degrading. These are non-instantaneous deteriorations. Permissible delay in payments is often used in most of business organizations. Trade credit is the arrangement to buy the goods on the account without making on the spot cash or cheque payments. Trade credit is a helpful device for developing companies. The retailer gets a trade credit policy from the manufacturer. The retailer has to pay the amount to the manufacturer by the next replenishment time. This helps the retailer to purchase products without paying immediately. Retailers must pay the price plus some interest if they don't pay within the allotted time. The retailer offered a return policy to the customers. This offer makes customers buy the products and return the product within a specific time period. For the returned product the retailer did not fully reimburse. Customer returns rise in proportion to both sales volume and product price. Duary et al. (2022) developed model for delay in payments and deteriorating items with partially backlogged shortages [1]. Geetha and Uthayakumar (2010) proposed the EOQ model for non-instantaneous deteriorating items with permissible delay in payments and partial backlogging [2]. Ghoreishi and Mirzazadeh (2013) studied the effect of inflation and customer returns on joint pricing and inventory control for deteriorating items [3]. Ghoreishi et al. (2015) developed an economic ordering policy model for non-instantaneous deteriorating items with selling price- and demand permissible delay in payments and customer returns [4]. Ghoreishi et al. (2013b) studied the optimal pricing and inventory control policy for non-instantaneously deteriorating items with the finite replenishment rate considering time- and price-dependent demand, customer returns and time value of money [5]. Jani et al. (2021) developed an EOQ model for customer returns and trade credit for deteriorating items with price sensitive demand [6]. Kumari and De investigated an EOQ model for deteriorating items analyzing retailer's optimal strategy under trade credit and return policy with nonlinear demand and resalable returns [7]. Maihami and Kamalabadi (2012) developed inventory control for non-instantaneous deteriorating items adopts a price and time dependent function with partially backlogged [8]. Mashud (2020) developed a deteriorating EOQ inventory model according to consideration of the price with shortage [9]. Musa and Sani (2010) developed a mathematical model on the inventory of deteriorating items that do not start deteriorating immediately they are stocked with permissible delay in payments [10]. Ouyang et al. (2006) investigated the inventory model for non-instantaneous deteriorating items considering permissible delay in payments [11]. Singh and Mishra (2022) developed an inventory model for deteriorating items [12]. Sundararajan et al. (2019) developed a deterministic inventory model for non-instantaneous deteriorating items with price and time-dependent demand with shortages [13] Yang et al. (2009) considered the optimal pricing and ordering strategies for non-instantaneous deteriorating items with partial backlogging and price dependent demand [14].

2. Assumptions

- The model includes a single non-instantaneous deteriorating item.
- Assume that the inventory system planning horizon is infinite.
- Demand rate is depends on time and selling price is given by:
  $D(y,t)=(\alpha-\beta y)e^{\eta t}$ where α is the demand scale, β represents price sensitivity, Demand is a linearly decreasing function of the price and decreases (increases) exponentially with time when $\eta<0(\eta>0)$.

- Shortages are permitted. The unsatisfied demand is backlogged, and the fraction of shortage backordered is $\zeta(x) = K_0 e^{-\mu x}$ $(\mu > 0, 0 < K_0 \leq 1)$, where x is a waiting time up to the upcoming replenishment and μ is a positive constant.

- It is plausible to say that buyer returns grow as more goods are sold. So,

$$\Lambda(y,t) = \nu D(y,t) \text{ where } 0 \leq \nu < 1.$$

The customers can return the products at any time in the replenishment cycle. But the retailer will not give the total amount of initial value, and the retailer will provide half the amount of initial value. The returned products can be resalable at the same selling price.

3. Notations

The terms used in the mathematical formulation are listed in the table 1.

Notation	**Unit**	**Description**
A	\$/ order	ordering cost
C_1	\$/ unit/unit time	holding cost
C_s	\$/ unit	shortage cost
C_p	\$/ unit	purchase price
Q		order quantity
θ		constant deterioration rate $0 < \theta < 1$
S	unit time	trade credit period offered by the manufacturer to the retailer
t_d	unit time	time at which deterioration starts
I_r	%/unit time	interest earned by the retailer
I_m	%/unit time	interest paid by the retailer to the manufacturer
R		maximum shortage level
Decision Variables		
y	\$/ unit	selling price
t_1	unit time	time at which inventory level reaches to zero
t_2	unit time	time at partially backlogged shortage
$TPC(y,t)$	\$/ unit time	Total profit

Table 1: Notations that are considered in the formulation of the inventory model

4. Model formulation

In this section, At the beginning of the cycle I_0 units of item arrive at the inventory system. In the time of interval, $(0, t_d)$, the inventory level depend upon demand and returns, at that time there is no deterioration. At $t = t_d$ the deterioration starts takes place. During the interval (t_d, t_1) the inventory level depends upon demand, returns and deterioration. At next stage, during the interval (t_1, t_2) shortage caused by partial backlogging and demand. In this research paper, it is assumed that the manufacturer offers permissible delay in payments to the retailer. The customers offered a product can return during the replenishment cycle to the retailer. The returned products can be sold again for the same price. And the retailer did not fully reimburse the amount of returned product to the customer. During the time interval $[0, t_d]$, the differential equation represents the inventory is given by

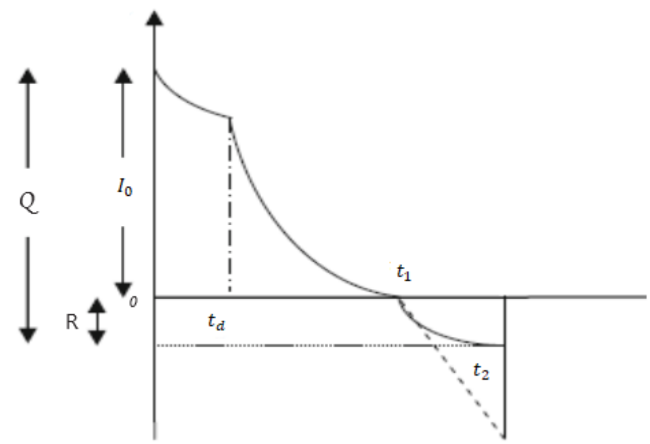

Figure 1:

$$\frac{dI_1(t)}{dt} = -(\alpha - \beta y)e^{\eta t} + \nu(\alpha - \beta y)e^{\eta t},\ 0 \leq t \leq t_d \tag{4.1}$$

When $t = 0$, put $I_1(0) = I_0$ in above equation we get

$$I_1(t) = I_0 + \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t} - 1)}{\eta}, 0 \leq t \leq t_d \tag{4.2}$$

The differential equation for the time interval $[t_d, t_1]$ is,

$$\frac{dI_2(t)}{dt} = -(\alpha - \beta y)e^{\eta t} + \nu(\alpha - \beta y)e^{\eta t} - \theta I(t),\ t_d \leq t \leq t_1 \tag{4.3}$$

When $t = t_1$, put $I_2(t_1) = 0$ in above equation we get

$$I_2(t) = \frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t}}{(\eta + \theta)}\left[e^{(\eta+\theta)t} - e^{(\eta+\theta)t_1}\right], t_d \leq t \leq t_1 \tag{4.4}$$

At $t = t_d$, the equations (2) and (4) becomes

$$\begin{aligned} I_0 =& \frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t_d}}{(\eta + \theta)}\left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] \\ &- \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t_d} - 1)}{\eta} \end{aligned} \tag{4.5}$$

Substitute equation (5) in equation (2) we get,

$$\begin{aligned} I_1(t) =& \frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t_d}}{(\eta + \theta)}\left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] \\ &- \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t_d} - 1)}{\eta} + \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t} - 1)}{\eta}, 0 \leq t \leq t_d \end{aligned} \tag{4.6}$$

In the time interval $[t_1, t_2]$, partially backlogged shortage occurs according to a fraction $\zeta(t_2 - t_1)$. Then the differential equation for the inventory level is given by

$$\frac{dI_3(t)}{dt} = -(\alpha - \beta y)e^{\eta t}\zeta(t_2 - t),\ t_1 \leq t \leq t_2 \tag{4.7}$$

When $t = t_1$ put $I_3(t_1) = 0$ in above equation we get

$$I_3(t) = \frac{(\alpha - \beta y)}{\eta + \mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t}\right],\ t_1 \leq t \leq t_2 \tag{4.8}$$

Put $t = t_2$ in eqn $I_3(t)$ where R is the maximum shortage level

$$R = -\frac{(\alpha - \beta y)}{\eta + \mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right] \tag{4.9}$$

The sum of R and I_0 is the Order Quantity per cycle (Q) is

$$\begin{aligned} Q =& R + I_0 \\ =& \frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t_d}}{(\eta + \theta)}\left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] - \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t_d} - 1)}{\eta} \\ & - \frac{(\alpha - \beta y)}{\eta + \mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right] \end{aligned} \tag{4.10}$$

This model's various costs are specified as follows.

(1) Ordering cost = A

(2) Purchasing cost

$$\begin{aligned} PC =& C_pQ \\ =& C_p\Bigg[\frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t_d}}{(\eta + \theta)}\left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] - \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t_d} - 1)}{\eta} \\ & - \frac{(\alpha - \beta y)}{\eta + \mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right]\Bigg] \end{aligned} \tag{4.11}$$

(3) Sales revenue

$$\begin{aligned} SR &= y\left[\int_0^{t_1} D(y,t)dt - \int_0^{t_1}\frac{\Lambda(y,t)}{2}dt + R\right] \\ &= y\left[\int_0^{t_1}(\alpha - \beta y)e^{\eta t}dt - \int_0^{t_1}\frac{\nu(\alpha - \beta y)e^{\eta t}}{2}dt - \frac{(\alpha - \beta y)}{\eta + \mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right]\right] \\ &= y\left[\left[\frac{(\alpha - \beta y)(e^{\eta t_1} - 1)}{\eta}\right]\left[1 - \frac{\nu}{2}\right] - \frac{(\alpha - \beta y)}{\eta + \mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right]\right] \end{aligned} \tag{4.12}$$

(4) Deterioration cost

$$\begin{aligned} DC &= C_p\int_{t_d}^{t_1}\theta I(t)dt \\ &= C_p\int_{t_d}^{t_1}\theta\frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t}}{(\eta + \theta)}\left[e^{(\eta+\theta)t} - e^{(\eta+\theta)t_1}\right]dt \\ &= \left[\frac{C_p(\alpha - \beta y)(\nu - 1)\left[(\theta + \eta)e^{\eta t_1} - \theta e^{\eta t_d} - \eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]}{\eta(\eta + \theta)}\right] \end{aligned} \tag{4.13}$$

(5) Holding cost

$$\begin{aligned}HC =&C_1\left[\int_0^{t_d} I(t)dt+\int_{t_d}^{t_1} I(t)dt\right]\\=&C_1\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}t_d\right.\\&+\left[(\nu-1)(\alpha-\beta y)\frac{(e^{\eta t_d}-\eta t_d-1)}{\eta^2}\right]-\left[\frac{(\alpha-\beta y)(\nu-1)}{\theta\eta(\theta+\eta)}\right]\\&\left.\left[\theta e^{\eta t_d}-(\theta+\eta)e^{\eta t_1}+\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]\right]\end{aligned}\tag{4.14}$$

(6) Shortage cost

$$\begin{aligned}SC &= c_2\left[\int_{t_1}^{t_2} -I(t)dt\right]\\&= c_2\left[\int_{t_1}^{t_2}\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t}\right]dt\right]\\&= c_2\left[\frac{(\alpha-\beta y)}{(\eta+\mu)^2}e^{-\mu t_2}\left[e^{(\mu+\eta)t_2}-e^{(\mu+\eta)t_1}\left[(t_2-t_1)(\mu+\eta)+1\right]\right]\right]\end{aligned}\tag{4.15}$$

(7) Permissible delay in payments:
The retailer gets a trade credit policy from the manufacturer. The retailer has to pay the amount to the manufacturer by the delay period S. We suggest three subcases for the delay period based on the values of S, t_d, and t_1.

(i) $0 < S \le t_d$

(ii) $t_d < S \le t_1$

(iii) $S > t_1$

Case (i) : Payment delays occur previous to time deterioration: $0 \le S \le t_d$
In this subcase, the retailer has to pay the amount before the deterioration starts. Otherwise, he have to pay the interest to the manufacturer. Interest earns is estimated as follows:

$$\begin{aligned}IR_1 =&yI_r\left[\int_0^S\int_0^t(\alpha-\beta y)e^{\eta u}dudt-\int_0^S\int_0^t\frac{\nu(\alpha-\beta y)e^{\eta u}}{2}dudt\right]\\=&yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta S-1\right]\left[1-\frac{\nu}{2}\right]\right\}\end{aligned}\tag{4.16}$$

Interest paid by the retailer to the manufacturer is estimated as follows:

$$\begin{aligned}IM_1 =&C_pI_m\left[\int_S^{t_d} I(t)dt+\int_{t_d}^{t_1} I(t)dt\right]\\=&C_pI_m\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}(t_d-S)\right.\\&+\left[\frac{(\nu-1)(\alpha-\beta y)}{\eta}\right]\left[S-t_d+\frac{(e^{\eta t_d}-e^{\eta S})}{\eta}\right]-\frac{e^{\eta t_d}}{\eta}-\frac{e^{(\eta+\theta)t_1}e^{-\theta t_d}}{\theta}\\&\left.+\left[\frac{(\nu-1)(\alpha-\beta y)}{(\eta+\theta)}\right]\left[\frac{e^{\eta t_1}}{\theta\eta}(\theta+\eta)\right]\right]\end{aligned}\tag{4.17}$$

The total profit per unit time is estimated as follows:

$$\begin{aligned}
&TPC_1(y,t_2)\\
=&\frac{SR-A-PC-DC-HC-SC-IM_1+IR_1}{t_2}\\
=&y\left[\left[\frac{(\alpha-\beta y)(e^{\eta t_1}-1)}{\eta}\right]\left[1-\frac{\nu}{2}\right]-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t_2}\right]\right]\\
&-A-C_p\left[\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right.\\
&\left.-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t}\right]\right]\\
&-\left[\frac{C_p(\alpha-\beta y)(\nu-1)\left[(\theta+\eta)e^{\eta t_1}-\theta e^{\eta t_d}-\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]}{\eta(\eta+\theta)}\right]\\
&-C_1\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}t_d\right.\\
&+\left[(\nu-1)(\alpha-\beta y)\frac{(e^{\eta t_d}-\eta t_d-1)}{\eta^2}\right]-\left[\frac{(\alpha-\beta y)(\nu-1)}{\theta\eta(\theta+\eta)}\right]\\
&\left.\left[\theta e^{\eta t_d}-(\theta+\eta)e^{\eta t_1}+\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]\right]\\
&-c_2\left[\frac{(\alpha-\beta y)}{(\eta+\mu)^2}e^{-\mu t_2}\left[e^{(\mu+\eta)t_2}-e^{(\mu+\eta)t_1}\left[(t_2-t_1)(\mu+\eta)+1\right]\right]\right]\\
&-C_pI_m\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}(t_d-S)\right.\\
&+\left[\frac{(\nu-1)(\alpha-\beta y)}{\eta}\right]\left[S-t_d+\frac{(e^{\eta t_d}-e^{\eta S})}{\eta}\right]-\frac{e^{\eta t_d}}{\eta}-\frac{e^{(\eta+\theta)t_1}e^{-\theta t_d}}{\theta}\\
&\left.+\left[\frac{(\nu-1)(\alpha-\beta y)}{(\eta+\theta)}\right]\left[\frac{e^{\eta t_1}}{\theta\eta}(\theta+\eta)\right]\right]+yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta S-1\right]\left[1-\frac{\nu}{2}\right]\right\}.
\end{aligned} \tag{4.18}$$

Case (ii) :Payment delays occur between deterioration time and before the inventory cycle. $t_d < S \leq t_1$

Interest paid by the retailer to the manufacturer

$$\begin{aligned}
IM_2&=C_pI_m\left[\int_S^{t_1}I(t)dt\right]\\
&=C_pI_m\left[\int_S^{t_1}\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t}}{(\eta+\theta)}\left[e^{(\eta+\theta)t}-e^{(\eta+\theta)t_1}\right]dt\right]\\
&=C_pI_m\left[\frac{(\nu-1)(\alpha-\beta y)}{(\eta+\theta)}\right]\left[\frac{e^{\eta t_1}}{\theta\eta}(\theta+\eta)-\frac{e^{\eta M}}{\eta}-\frac{e^{(\eta+\theta)t_1}e^{-\theta S}}{\theta}\right]
\end{aligned} \tag{4.19}$$

Interest earns is estimated as follows:

$$\begin{aligned}
IR_2=&yI_r\left[\int_0^S\int_0^t(\alpha-\beta y)e^{\eta u}dudt-\int_0^S\int_0^t\frac{\nu(\alpha-\beta y)e^{\eta u}}{2}dudt\right]\\
=&yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta M-1\right]\left[1-\frac{\nu}{2}\right]\right\}
\end{aligned} \tag{4.20}$$

The total profit per unit time is estimated as follows:

$$\begin{aligned}
&TPC_2(y,t_2)\\
=&\frac{SR-A-PC-DC-HC-SC-IM_1+IR_1}{t_2}\\
=&y\left[\left[\frac{(\alpha-\beta y)(e^{\eta t_1}-1)}{\eta}\right]\left[1-\frac{\nu}{2}\right]-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t_2}\right]\right]\\
&-A-C_p\left[\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right.\\
&\left.-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t}\right]\right]\\
&-\left[\frac{C_p(\alpha-\beta y)(\nu-1)\left[(\theta+\eta)e^{\eta t_1}-\theta e^{\eta t_d}-\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]}{\eta(\eta+\theta)}\right]\\
&-C_1\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}t_d\right.\\
&+\left[(\nu-1)(\alpha-\beta y)\frac{(e^{\eta t_d}-\eta t_d-1)}{\eta^2}\right]-\left[\frac{(\alpha-\beta y)(\nu-1)}{\theta\eta(\theta+\eta)}\right]\\
&\left.\left[\theta e^{\eta t_d}-(\theta+\eta)e^{\eta t_1}+\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]\right]\\
&-c_2\left[\frac{(\alpha-\beta y)}{(\eta+\mu)^2}e^{-\mu t_2}\left[e^{(\mu+\eta)t_2}-e^{(\mu+\eta)t_1}\left[(t_2-t_1)(\mu+\eta)+1\right]\right]\right]\\
&-C_pI_m\left[\frac{(\nu-1)(\alpha-\beta y)}{(\eta+\theta)}\right]\left[\frac{e^{\eta t_1}}{\theta\eta}(\theta+\eta)-\frac{e^{\eta M}}{\eta}-\frac{e^{(\eta+\theta)t_1}e^{-\theta S}}{\theta}\right]\\
&+yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta M-1\right]\left[1-\frac{\nu}{2}\right]\right\}.
\end{aligned}\tag{4.21}$$

Case (iii) : $S > t_1$

In this subcase, the delay period is greater than the time at which the amount of inventory reaches zero. During this time retailer pays completely all of his bills. Then

$$IM_3 = 0. \tag{4.22}$$

Interest earns is estimated as follows:

$$\begin{aligned}
IR_3 =&yI_r\left[\int_0^{t_1}\int_0^{t}(\alpha-\beta y)e^{\eta u}dudt-\int_0^{t_1}\int_0^{t}\frac{\nu(\alpha-\beta y)e^{\eta u}}{2}dudt\right.\\
&\left.+(S-t_1)\int_0^{t_1}(\alpha-\beta y)e^{\eta t}dt-(S-t_1)\int_0^{t_1}\frac{\nu(\alpha-\beta y)e^{\eta t}}{2}dt\right]\\
=&yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta S-1\right]\left[1-\frac{\nu}{2}\right]\left[\frac{(\alpha-\beta y)(S-t_2)(e^{\eta t_2}-1)}{\eta}\right]\right\}
\end{aligned}\tag{4.23}$$

The total profit per unit time is estimated as follows:

$$
\begin{aligned}
&TPC_3(y,t_2)\\
=&\frac{SR-A-PC-DC-HC-SC-IM_1+IR_1}{t_2}\\
=&y\left[\left[\frac{(\alpha-\beta y)(e^{\eta t_1}-1)}{\eta}\right]\left[1-\frac{\nu}{2}\right]-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t_2}\right]\right]\\
&-A-C_p\left[\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right.\\
&\left.-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t}\right]\right]\\
&-\left[\frac{C_p(\alpha-\beta y)(\nu-1)\left[(\theta+\eta)e^{\eta t_1}-\theta e^{\eta t_d}-\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]}{\eta(\eta+\theta)}\right]\\
&-C_1\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}t_d\right.\\
&+\left[(\nu-1)(\alpha-\beta y)\frac{(e^{\eta t_d}-\eta t_d-1)}{\eta^2}\right]-\left[\frac{(\alpha-\beta y)(\nu-1)}{\theta\eta(\theta+\eta)}\right]\\
&\left.\left[\theta e^{\eta t_d}-(\theta+\eta)e^{\eta t_1}+\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]\right]\\
&-c_2\left[\frac{(\alpha-\beta y)}{(\eta+\mu)^2}e^{-\mu t_2}\left[e^{(\mu+\eta)t_2}-e^{(\mu+\eta)t_1}\left[(t_2-t_1)(\mu+\eta)+1\right]\right]\right]\\
&+yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta S-1\right]\left[1-\frac{\nu}{2}\right]\left[\frac{(\alpha-\beta y)(S-t_2)(e^{\eta t_2}-1)}{\eta}\right]\right\}
\end{aligned}
\tag{4.24}
$$

5. Solution Procedure

The following method is used to resolve the aforementioned issue.

Step 1: Fill the equation with all of the values for the necessary parameters for the proposed model.

Step 2: Put $\frac{\partial TPC_i}{\partial y}=\frac{\partial TPC_i}{\partial t_1}=0$, where $i=1,2,3$

Step 3: Fix the optimization issue TPC_i for $i=1,2,3$ and hold the optimal values of y, t_1 and TPC

Step 4: Compare the values of TPC_1, TPC_2 and TPC_3

Step 5: Choose the highest value among TPC_1, TPC_2 and TPC_3.

Step 6: Stop.

6. Numerical Example

Consider a numerical example to demonstrate the model. The parameter values are as follows: $\alpha=290$; $\beta=4$; $t_d=1/12$; $t_1=0.765$; $C_s=2.5$; $I_r=10\%$ per year; $I_m=15\%$ per year;
$\nu=0.1$; $\mu=0.1$; $\theta=0.08$; $C_p=20$; $C_s=2.5$; $A=200$; $\eta=-0.98$; $C_1=1$;
If $S=0.08$;, then it is in the category of case (i), since $S<t_d$.
If $S=0.4$;, then it is in the category of case (ii), since $t_d<S\le t_1$.
If $S=0.91$;, then it is in the category of case (iii), since $S>t_1$.
Then we obtain the following results. $y=35.5357; t_2=0.9$.

Expressions	Case 1	Case 2	Case 3
S	0.08	0.4	0.91
Q	32.049	32.049	32.049
SR	1173.32	1173.32	1173.32
DC	11.77	11.77	11.77
HC	9.85	9.85	9.85
SC	0.5908	0.5908	0.5908
IR	0.609	13.769	60.435
IM	22.323	5.6123	0
TPC	320.435	353.626	411.713

Table 2: Results of numerical example

Parameter	% Changes in parameters	Case 1	Case 2	Case 3
I_r	-30	320.23	445.42	814.62
	-20	320.30	460.72	881.77
	-10	320.37	476.02	948.92
	10	320.50	355.16	418.43
	20	320.57	356.69	425.14
	30	320.64	358.22	431.86
I_m	-30	327.88	355.50	411.71
	-20	325.40	354.87	411.71
	-10	322.92	354.25	411.71
	10	317.96	353.00	411.71
	20	315.48	352.38	411.71
	30	312.99	351.76	411.71
C_p	-30	545.47	573.09	629.30
	-20	470.46	499.93	556.77
	-10	395.45	426.78	484.24
	10	245.43	280.47	339.18
	20	170.42	207.32	266.65
	30	95.40	134.17	194.12
A	-30	387.10	420.29	478.38
	-20	364.88	398.07	456.16
	-10	342.66	375.85	433.94
	10	298.21	331.40	389.49
	20	275.99	309.18	367.27
	30	253.77	286.96	345.05
C_1	-30	323.72	356.91	415.00
	-20	322.63	355.82	413.90
	-10	321.53	354.72	412.81
	10	319.34	352.53	410.62
	20	318.25	351.44	409.52
	30	317.15	350.34	408.43
α	-30	-242.32	-243.55	-245.70
	-20	-54.73	-44.49	-26.56
	-10	132.85	154.57	192.58
	10	508.02	552.68	630.85
	20	695.61	751.74	849.99
	30	883.19	950.80	1,069.13

Table 3: Results of sensitivity analysis

7. Sensitivity Analysis

Using the numerical example, we do sensitivity analyses for various parameters. In any circumstance requiring decision-making, uncertainty may cause parameter values to vary. Sensitivity analysis is given here for the three cases. The changes are made from -30 percent to $+30$ percent. The result of this analysis is in the following table 3 and table 4. The main conclusion from the sensitivity analysis are as follows:

- When α is increased (decreased), the total profit for the three cases increases(decreases).
- There is an increase (decrease) in the total profit for the three cases value when A, C_p, η and β are decreases(increases).
- I_r is less sensitive and ν, C_1 and θ are moderately sensitive.
- Other parameter modifications have minimal impact on the total profit.

Parameter	% Changes in parameters	Case 1	Case 2	Case 3
β	-30	720.39	778.05	878.95
	-20	587.08	636.57	723.20
	-10	453.76	495.10	567.46
	10	187.12	212.15	255.97
	20	53.80	70.68	100.22
	30	-79.52	-70.79	-55.52
θ	-30	328.57	361.65	419.72
	-20	325.87	358.98	417.06
	-10	323.16	356.31	414.39
	10	317.71	350.93	409.03
	20	314.97	348.23	406.33
	30	312.22	345.52	403.63
η	-30	384.32	420.86	484.94
	-20	361.85	397.22	459.21
	-10	340.58	374.83	434.83
	10	301.35	333.52	389.79
	20	283.25	314.45	368.98
	30	266.08	296.36	349.22
ν	-30	316.08	350.12	409.23
	-20	317.53	351.29	410.06
	-10	318.98	352.46	410.89
	10	321.89	354.80	412.54
	20	323.34	355.97	413.37
	30	324.80	357.14	414.20
μ	-30	320.56	353.75	411.83
	-20	320.52	353.71	411.79
	-10	320.48	353.67	411.75
	10	320.40	353.59	411.67
	20	320.36	353.55	411.63
	30	320.32	353.51	411.59

Table 4: Results of sensitivity analysis

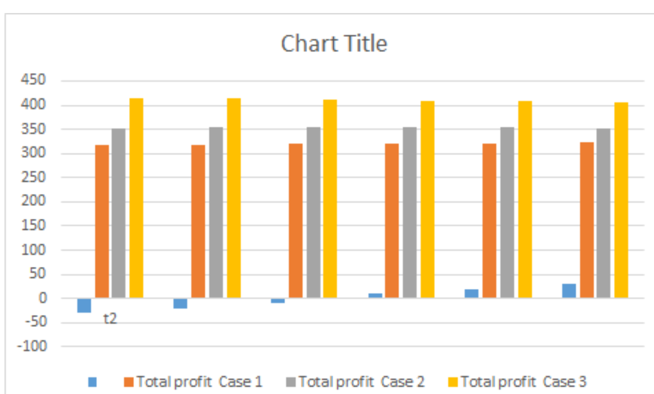

Figure 2:

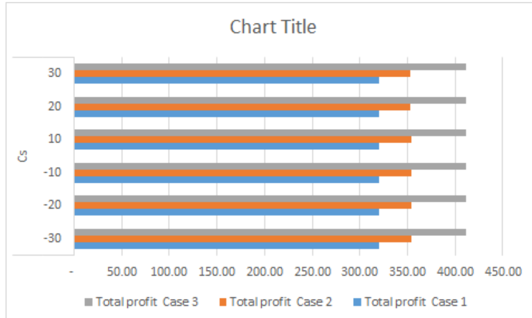

Figure 3:

Figure 4: Total profit by changing the parameters C_s and t_2

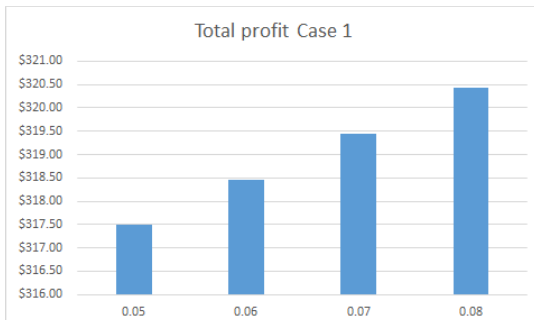

Figure 5:

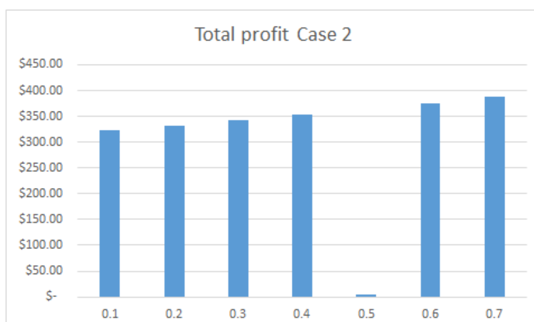

Figure 6:

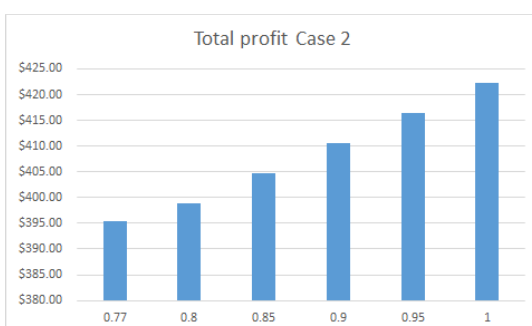

Figure 7:

8. Conclusion

In this work, an inventory model with a single item is developed for a non-instantaneous deterioration item with a return policy, allowable payment delays, and partial backlogging. Customers may return products at any time during the replenishment cycle. Products that have been returned may be resalable at the same selling price. During shortages partially backlogged is considered. This model maximizes the total profit by selling price and time. We investigated three cases. Solution procedure and numerical example are given. for future research, this model can be extended to advance payment with fully backlogged and instantly deteriorating items.

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On a conformable fractional differential equations with maxima

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Abstract. This study deals with the existence and uniqueness of solutions for a class of first order conformable fractional differential equations with maxima. We also provide some examples to illustrate the application of the results.

Keywords: Conformable, maxima, monotone, uniqueness, upper solutions, lower solutions.

Contents

1. Introduction

The purpose of this study the following problem

$$\begin{cases} \left(\mathfrak{D}_{\alpha}\mathfrak{u}\right)(\tau) = \mathfrak{f}(\tau, \mathfrak{u}(\tau), \max\limits_{s\in[\tau-r,\tau]} \mathfrak{u}(s)), \ \tau \in J = [0, \mathfrak{T}], \\ \mathfrak{u}(\tau) = \varphi(\tau), \ \tau \in [-r, 0], \end{cases} \tag{1.1}$$

where $\mathfrak{D}_{\alpha}$ represents the conformable fractional derivative of order α, $0 < \alpha \leq 1$, $\mathfrak{T} > 0$ and $r > 0$, $\mathfrak{f} : J \times \mathbb{R} \times C \to \mathbb{R}$ is continuous with $C = C([-r, \mathfrak{T}], \mathbb{R})$ and $\varphi : [-r, 0] \to \mathbb{R}$ continuous.

Conformable fractional derivative was first introduced in [23], later developed in [1] and it appears in many fields (see [2], [3], [11], [17], [25], [35] along with the cited references therein).

However differential equations with maxima and differential inequalities with maxima were initially used in automatic control and in the study stability of equations with retarded argument (see [30] and [19, Chapter 4 Section 5]). Nevertheless, a variety of fields, including there are a wide range of areas such as psychology (e.g., dynamic model for happiness), optimal control, theory of lateral inhibition, chemostat models and economy (see [5], [6], [8], [15], [18], [20], [21], [28] and [33]) use differential equations with maxima.

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Some authors have studied conformable fractional differential equations with deviating arguments using fixed point theorems, numerical methods, monotone iterative technique, and upper and lower solutions method see [14], [16], [22], [24] and [31]). Let us recall some of them.

In [14], the authors studied the problem

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{u})(\tau) = f(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\theta(\tau))), \ \tau \in J = [0, \mathfrak{T}], \\ \mathfrak{u}(0) = \mathfrak{u}(T), \end{cases} \tag{1.2}$$

where $0 < \alpha \leq 1$, $\mathfrak{T} > 0$, $f : J \times \mathbb{R} \times C(J, \mathbb{R}) \to \mathbb{R}$ and $\theta : J \to J$ are continuous with $\theta(J) \subseteq J$.

The authors used the monotone iterative technique to establish some sufficient conditions for the existence of extremal solutions for periodic boundary value problem (1.2).

In [16], the author studied the following problem

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{v})(\tau) = f(\tau, \mathfrak{v}(\tau), \mathfrak{v}(\theta(\tau))), \ \tau \in J = [0, \mathfrak{T}], \\ \mathfrak{v}(0) = g(\mathfrak{v}), \end{cases} \tag{1.3}$$

where $0 < \alpha \leq 1$, $\mathfrak{T} > 0$, $f : J \times \mathbb{R} \times C(J, \mathbb{R}) \to \mathbb{R}$ and $\theta : J \to J$ continuous with $\theta(J) \subseteq J$, and $g : C(J, \mathbb{R}) \to \mathbb{R}$ continuous increasing.

The author established the existence of minimal and maximal solutions for the problem (1.3) by combining the upper and lower solutions method with the monotone iterative technique.

In [22], the authors studied the following problem

$$\begin{cases} (\mathfrak{D}_\alpha y)(\tau) + y(\tau) = \mu y(\mu\tau), \ \tau > 0, \\ y(0) = \lambda, \end{cases} \tag{1.4}$$

where $0 < \alpha \leq 1$, λ and μ are real numbers with $\mu < 1$.

The approximate solution for problem (1.4) was provided by the authors using the homotopy perturbation method.

One well know that the existence of solutions for first order differential equations with maxima is proved using the monotone iterative technique (see [4], [7], [8, Chapter 6] and the references cited therein). The aim of this work is to demonstrate its successful application to problems of type (1.1).

This work is structured to the following plan. We provide some definitions and preliminaries results in Section 2. Section 3 presents and demonstrates the main results and finally Section 4 offers how our results are applied.

2. Definitions and Preliminary Results

Definition 2.1. *[23]Let $h : J \to \mathbb{R}$ continuous $0 < \alpha \leq 1$. The conformable fractional integral of order α of h is defined by*

$$(I_\alpha h)(\tau) = \int_0^\tau s^{\alpha-1} h(s)\, ds, \text{ for } \tau > 0.$$

Definition 2.2. *[23]Let $h : J \to \mathbb{R}$ and $0 < \alpha \leq 1$. The Conformable fractional derivative of order α of h is defined by*

$$\begin{cases} (\mathfrak{D}_\alpha h)(\tau) = \lim\limits_{\rho \to 0} \frac{h\left(\tau + \rho \tau^{1-\alpha}\right) - h(\tau)}{\rho}, \text{ for } \tau > 0, \\ (\mathfrak{D}_\alpha h)(0) = \lim\limits_{\tau \to 0^+} (D_\alpha h)(\tau). \end{cases} \tag{2.1}$$

Example 2.3. *We have*

(i) $(\mathfrak{D}_\alpha c)(\tau) = 0$*, where* $c \in \mathbb{R}$.

(ii) $(\mathfrak{D}_\alpha \tau^\lambda)(\tau) = \begin{cases} \lambda \tau^{\lambda-\alpha} \text{ if } \tau > 0, \\ \lambda \text{ if } \lambda = \alpha \text{ and } \tau = 0, \\ 0 \text{ if } \lambda > \alpha \text{ and } \tau = 0. \end{cases}$

(iii) $(\mathfrak{D}_\alpha e^{\tau^\alpha})(\tau) = \alpha e^{\tau^\alpha}$.

(iv) $(\mathfrak{D}_\alpha \sin\left(\frac{t^\alpha}{\alpha}\right))(\tau) = \cos\left(\frac{\tau^\alpha}{\alpha}\right)$.

(v) $(\mathfrak{D}_\alpha \cos\left(\frac{t^\alpha}{\alpha}\right))(\tau) = -\sin\left(\frac{\tau^\alpha}{\alpha}\right)$.

Theorem 2.4. *[23, Theorem 2.1]If* $\mathtt{h} : J \to \mathbb{R}$ *is* α*-differentiable at* $\tau_0 > 0$*, then* $\mathtt{h}$ *is continuous at* τ_0*.*

Lemma 2.5. *[23, Theorem 3.1]Let* $\mathtt{h} : J \to \mathbb{R}$ *be a continuous function and* $0 < \alpha \leq 1$*, then we have* $(\mathfrak{D}_\alpha \circ I_\alpha)\mathtt{h} = \mathtt{h}$*.*

Lemma 2.6. *[23, Theorem 2.4]Let* $\mathtt{h} : [a,b] \to \mathbb{R}$ *continuous with* $0 \leq a < b$ *and* $0 < \alpha \leq 1$*. If* $\mathtt{h}$ *is* α*-differentiable in* (a,b)*, then*

$$\mathtt{h}(b) - \mathtt{h}(a) = \left(\frac{b^\alpha - a^\alpha}{\alpha}\right)(\mathfrak{D}_\alpha \mathtt{h})(c),$$

with c in (a,b)*.*

Notation 2.7. *For* $0 < \alpha \leq 1$*, we define* $C^{\alpha,0}(J,\mathbb{R})$ *as follows*

$$C^{\alpha,0}(J,\mathbb{R}) = \{\mathtt{h} \in C(J,\mathbb{R}) : \mathfrak{D}_\alpha \mathtt{h} \in C(J,\mathbb{R})\}.$$

Lemma 2.8. *Let* $\mathtt{h} \in C^{\alpha,0}([a,b],\mathbb{R})$ *with* $0 \leq a < b$*. Then* $\mathfrak{D}_\alpha \mathtt{h} \equiv 0$ *in* $[a,b]$ *if and only if* $\mathtt{h} \equiv c$ *in* $[a,b]$*, where c is a real constant.*

Proof. Assume that $\mathtt{h} \in C^{\alpha,0}([a,b],\mathbb{R})$ with $0 \leq a < b$.

Suppose that $\mathfrak{D}_\alpha \mathtt{h} \equiv 0$ in $[a,b]$ and we put by definition

$$\mathtt{h}(\tau_0) = \min_{\tau \in [a,b]} \mathtt{h}(\tau) \text{ and } \mathtt{h}(\tau_1) = \max_{t \in [a,b]} \mathtt{h}(t).$$

From Lemma 2.6, one has

$$\mathtt{h}(\tau_0) = \mathtt{h}(\tau_1),$$

which means that

$$\mathtt{h} \equiv c \text{ in } [a,b], \text{ with } c \in \mathbb{R}.$$

Conversely if $\mathtt{h} \equiv c$ in $[a,b]$ with $c \in \mathbb{R}$, then by using the definition of Conformable fractional derivative, we obtain $\mathtt{h} \in C^{\alpha,0}([a,b],\mathbb{R})$. ■

Lemma 2.9. *Assume that* $\mathtt{h} \in C^{\alpha,0}(J,\mathbb{R})$*, then we have*

$$(I_\alpha \circ \mathfrak{D}_\alpha)(\mathtt{h}(\tau)) = \mathtt{h}(\tau) - h(0), \text{ for } \tau \in J.$$

Proof. We put by definition

$$\mathtt{g}(\tau) = (I_\alpha \circ \mathfrak{D}_\alpha)(\mathtt{h}(\tau)), \text{ for } t \in J.$$

From Lemma 2.5, we obtain

$$(\mathfrak{D}_\alpha \mathtt{g})(\tau) = (\mathfrak{D}_\alpha \mathtt{h})(\tau), \text{ for } \tau \in J,$$

which means that

$$(\mathfrak{D}_\alpha)(g - h)(t) = 0, \text{ for } t \in J,$$

and consequently since $g(0) = 0$ and from the preceding Lemma, we deduce that

$$\mathtt{g}(\tau) = \mathtt{h}(\tau) - \mathtt{h}(0), \text{ for } \tau \in J.$$

That is

$$(I_\alpha \circ \mathfrak{D}_\alpha)(\mathtt{h}(\tau)) = \mathtt{h}(\tau) - h(0), \text{ for } \tau \in J.$$

■

Lemma 2.10. *[32, Theorem 1 page 44] If the functions* $\mathtt{u} : [\mathfrak{c}, \mathfrak{d}] \to \mathbb{R}$ *and* $\mathtt{v} : [\mathfrak{c}, \mathfrak{d}] \to \mathbb{R}$ *are continuous on the segment* $[\mathfrak{c}, \mathfrak{d}]$*, then*

$$\max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} |\mathtt{u}(\tau) - \mathtt{v}(\tau)| \geq \left| \max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{u}(\tau) - \max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{v}(\tau) \right|.$$

Lemma 2.11. *If the functions* $\mathtt{u} : [\mathfrak{c}, \mathfrak{d}] \to \mathbb{R}$ *and* $\mathtt{v} : [\mathfrak{c}, \mathfrak{d}] \to \mathbb{R}$ *are continuous on the segment* $[\mathfrak{c}, \mathfrak{d}]$*, then*

$$\max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{u}(\tau) - \max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{v}(\tau) \geq \min_{\tau \in [\mathfrak{c}, \mathfrak{d}]} (\mathtt{u}(\tau) - \mathtt{v}(\tau)).$$

Proof. We have

$$\max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{u}(\tau) - \max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{v}(\tau) = \max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{u}(\tau) - \mathtt{v}(\varsigma),$$

where $\varsigma \in [\mathfrak{c}, \mathfrak{d}]$.

Which implies that

$$\begin{aligned} \max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{u}(\tau) - \max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{v}(\tau) &\geq \mathtt{u}(\varsigma) - \mathtt{v}(\varsigma) \\ &\geq \min_{\tau \in [\mathfrak{c}, \mathfrak{d}]} (\mathtt{u}(\tau) - \mathtt{v}(\tau)). \end{aligned}$$

That is

$$\max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{u}(\tau) - \max_{\tau \in [\mathfrak{c}, \mathfrak{d}]} \mathtt{v}(\tau) \geq \min_{\tau \in [\mathfrak{c}, \mathfrak{d}]} (\mathtt{u}(\tau) - \mathtt{v}(\tau)).$$

■

Now consider the problem

$$\begin{cases} (\mathfrak{D}_\alpha u)(\tau) = \widetilde{g}(\tau, \mathfrak{u}(\tau), \max\limits_{s \in [\tau - r, \tau]} \mathfrak{u}(\tau)), \ \tau \in J, \\ \mathfrak{u}(\tau) = \psi(t), \ \tau \in [-r, 0], \end{cases} \tag{2.2}$$

where $0 < \alpha \leq 1$, $\widetilde{g} : J \times \mathbb{R} \times C([-r, \mathfrak{T}], \mathbb{R}) \to \mathbb{R}$ continuous and $\psi \in C([-r, 0], \mathbb{R})$.

Notation 2.12. *For* $0 < \alpha \leq 1$ *the space* $C^\alpha([-r, \mathfrak{T}], \mathbb{R})$ *is defined as follows*

$$C^\alpha([-r, \mathfrak{T}], \mathbb{R}) = \{\mathfrak{u} \in C([-r, T], \mathbb{R}) : \ D_\alpha \mathfrak{u} \in C(J, \mathbb{R})\}.$$

The following result is an immediate consequence of Lemma 2.5 and Lemma 2.9.

Lemma 2.13. *Let* $0 < \alpha \leq 1$*. If* $\mathfrak{u} \in C^\alpha([-r, T], \mathbb{R})$*, then* $\mathfrak{u}$ *is a solution of the following integral equation*

$$\begin{cases} \mathfrak{u}(\tau) = \psi(0) + \int\limits_0^\tau s^{\alpha - 1} \widetilde{g}(s, \mathfrak{u}(s), \max\limits_{t \in [s - r, s]} \mathfrak{u}(t)) ds, \text{ for all } \tau \in J, \\ \mathfrak{u}(\tau) = \psi(\tau), \text{ for all } \tau \in [-r, 0], \end{cases}$$

if, and only if, $\mathfrak{u}$ *is a solution of the Cauchy problem (2.2).*

Now, we have the following result.

Theorem 2.14. *Assume that the following hypothesis are satisfied*

(H) There exists a positive constants L_1 and L_2 such that

$$\left|\widetilde{g}\left(t,\mathfrak{u}_1,\mathfrak{v}_1\right)-\widetilde{g}\left(t,\mathfrak{u}_2,\mathfrak{v}_2\right)\right|\leq L_1\left|\mathfrak{u}_1-\mathfrak{u}_2\right|+L_2\left|\mathfrak{v}_1-\mathfrak{v}_2\right|,$$

for all $t\in J$, $\mathfrak{u}_i\in\mathbb{R}$ and $\mathfrak{v}_i\in\mathbb{R}$ for $i=1,2$.

Then the problem (2.2) admits a unique solution $\mathfrak{u}\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$.

Proof. Let $\mathfrak{u}\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ and consider the following equation

$$\begin{cases}\mathfrak{u}\left(\tau\right)=\psi\left(0\right)+\int_0^{\tau}s^{\alpha-1}\widetilde{g}(s,\mathfrak{u}\left(s\right),\max\limits_{t\in[s-r,s]}\mathfrak{u}\left(t\right))ds, \text{ for all } \tau\in J,\\ \mathfrak{u}(\tau)=\psi\left(\tau\right), \text{ for all } \tau\in\left[-r,0\right].\end{cases}$$

Now we define the operator

$$\begin{aligned}A: C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)&\to C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)\\ u\quad&\mapsto\left(A\mathfrak{u}\right)\left(\tau\right)=\begin{cases}\psi\left(0\right)+\int_0^{\tau}s^{\alpha-1}\widetilde{g}(s,\mathfrak{u}\left(s\right),\max\limits_{t\in[s-r,s]}\mathfrak{u}\left(t\right))ds, \text{ for all } \tau\in J,\\ \mathfrak{u}(\tau)=\psi\left(\tau\right), \text{ for all } \tau\in\left[-r,0\right],\end{cases}\end{aligned}$$

and we define the following norm

$$\|v\|=\max_{\tau\in[-r,\mathfrak{T}]}e^{-\frac{\lambda}{\alpha}|\tau|^{\alpha}}\left|v\left(\tau\right)\right|,$$

where $v\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ and $\lambda>0$.

Since the norms $\|.\|_*$ and $\|.\|_0$ are equivalent, then $\left(C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right),\|.\|_*\right)$ is a Banach space.

Now let $\mathfrak{u}_1,\mathfrak{u}_2\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$, then for all $\tau\in J$, one has

$$\begin{aligned}&e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\left|\left(A\mathfrak{u}_1\right)\left(\tau\right)-\left(A\mathfrak{u}_2\right)\left(\tau\right)\right|\\ &=e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\left|\int_0^{\tau}s^{\alpha-1}\left(\widetilde{g}(s,\mathfrak{u}_1\left(s\right),\max_{t\in[s-r,s]}\mathfrak{u}_1\left(t\right))-\widetilde{g}(s,\mathfrak{u}_2\left(s\right),\max_{t\in[s-r,s]}\mathfrak{u}_2\left(t\right))\right)ds\right|\\ &\leq e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\int_0^{\tau}s^{\alpha-1}\left|\widetilde{g}(s,\mathfrak{u}_1\left(s\right),\max_{t\in[s-r,s]}\mathfrak{u}_1\left(t\right))-\widetilde{g}(s,\mathfrak{u}_2\left(s\right),\max_{t\in[s-r,s]}\mathfrak{u}_2\left(t\right))\right|ds\\ &\leq e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\int_0^{\tau}s^{\alpha-1}\left(L_1\left|\mathfrak{u}_1\left(s\right)-\mathfrak{u}_2\left(s\right)\right|+L_2\left|\max_{t\in[s-r,s]}\mathfrak{u}_1\left(t\right)-\max_{t\in[s-r,s]}\mathfrak{u}_2\left(t\right)\right|\right)ds.\end{aligned}$$

From Lemma 2.10, we obtain

$$
\begin{aligned}
& e^{-\frac{\lambda}{\alpha}\tau^{\alpha}} \left|(A\mathfrak{u}_1)(\tau) - (A\mathfrak{u}_2)(\tau)\right| \\
& e^{-\frac{\lambda}{\alpha}\tau^{\alpha}} \int_0^{\tau} s^{\alpha-1} \left(L_1 \left|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)\right| + L_2 \max_{t\in[s-r,s]} \left|\mathfrak{u}_1(t) - \mathfrak{u}_2(t)\right| \right) ds \\
& \leq e^{-\frac{\lambda}{\alpha}\tau^{\alpha}} (L_1 + L_2) \left\|\mathfrak{u}_1 - \mathfrak{u}_2\right\| \int_0^{\tau} e^{\frac{\lambda}{\alpha}s^{\alpha}} s^{\alpha-1} ds \\
& = e^{-\frac{\lambda}{\alpha}\tau^{\alpha}} (L_1 + L_2) \left\|\mathfrak{u}_1 - \mathfrak{u}_2\right\| \left(\frac{e^{\frac{\lambda}{\alpha}\tau^{\alpha}} - 1}{\lambda} \right) \\
& = (L_1 + L_2) \left\|\mathfrak{u}_1 - \mathfrak{u}_2\right\| \left(\frac{1 - e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}}{\lambda} \right) \\
& < \frac{(L_1 + L_2)}{\lambda} \left\|u_1 - u_2\right\|.
\end{aligned}
$$

If we choose $\lambda \geq (L_1 + L_2)$, we obtain A is a contraction on $\left(C^{\alpha}\left([-r, \mathfrak{T}], \mathbb{R}\right), \|.\|\right)$ and therefore by Banach's fixed point theorem, the operator A admits a unique fixed point and consequently from Lemma 2.13, it follows that the problem (2.2) admits a unique solution $u \in C^{\alpha}\left([-r, \mathfrak{T}], \mathbb{R}\right)$. ■

Lemma 2.15. *Let $u \in C^{\alpha}\left([-r, \mathfrak{T}], \mathbb{R}\right)$ satisfying*

$$
\begin{cases}
(\mathfrak{D}_{\alpha}\mathfrak{u})(\tau) \leq -M_1 \mathfrak{u}(\tau) - N_1 \min\limits_{s\in[\tau-r,\tau]} \mathfrak{u}(s), \ \tau \in J, \\
\mathfrak{u}(0) \leq \mathfrak{u}(\tau) \leq 0, \text{ for all } \tau \in [-r, 0],
\end{cases}
\tag{2.3}
$$

where $0 < \alpha \leq 1$ and M_1 and N_1 are positive real numbers.

If

$$
(M_1 + N_1) \frac{\mathfrak{T}^{\alpha}}{\alpha} \leq 1,
$$

then $\mathfrak{u} \leq 0$ in $[-r, \mathfrak{T}]$.

Proof. Assume that there exists $t_0 \in (0, \mathfrak{T}]$ such that

$$
u(t_0) > 0. \tag{2.4}
$$

We put by definition

$$
\mathfrak{u}(\eta) = \min_{t\in[-r,t_0]} u(t) \leq 0,
$$

where $\eta \in [0, t_0)$.

From Lemma 2.6, there exists $\sigma \in (\eta, t_0)$ such that

$$
\mathfrak{u}(t_0) - \mathfrak{u}(\eta) = \left(\frac{t_0^{\alpha} - \eta^{\alpha}}{\alpha} \right) (\mathfrak{D}_{\alpha} u)(\sigma).
$$

Then by using (2.3) and (2.4), we obtain

$$
-\mathfrak{u}(\eta) < -\left(M_1 \mathfrak{u}(\sigma) + N_1 \min_{s\in[\sigma-r,\sigma]} \mathfrak{u}(\sigma) \right) \left(\frac{t_0^{\alpha} - \eta^{\alpha}}{\alpha} \right).
$$

Which implies that

$$-\mathfrak{u}(\eta) < -(M_1+N_1)\,\mathfrak{u}(\eta)\left(\frac{t_0^\alpha-\eta^\alpha}{\alpha}\right)$$
$$< -(M_1+N_1)\,\mathfrak{u}(\eta)\,\frac{\mathfrak{T}^\alpha}{\alpha}.$$

That is

$$(M_1+N_1)\,\frac{\mathfrak{T}^\alpha}{\alpha} > 1 \text{ if } u(\eta) < 0.$$

Which is a contradiction with the assumption

$$(M_1+N_1)\,\frac{\mathfrak{T}^\alpha}{\alpha} \leq 1.$$

If $\mathfrak{u}(\eta)=0$, we obtain also a contradiction.

Then, we have

$$\mathfrak{u}(t) \leq 0, \text{ for all } t \in [-r,\mathfrak{T}].$$

■

Remark 2.16. *The idea of the proof of the preceding Lemma 2.15 is similar to that of [26, Lemma 2.1 part i)].*

Lemma 2.17. *Assume that* $\mathfrak{u} \in C^\alpha([-r,\mathfrak{T}],\mathbb{R})$ *satisfying*

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{u})(t) \leq -\widetilde{M_1}\mathfrak{u}(t) - \widetilde{N_1}\max\limits_{s\in[t-r,t]}\mathfrak{u}(s),\ t\in J, \\ \mathfrak{u}(t) \leq 0, \textit{ for all } t\in[-r,0], \end{cases}$$

where $0<\alpha\leq 1$, $\widetilde{M_1}\leq 0$ *and* $\widetilde{N_1}\leq 0$.

If

$$-\left(\widetilde{M_1}+\widetilde{N_1}\right)\frac{\mathfrak{T}^\alpha}{\alpha} < 1,$$

then $\mathfrak{u}(t)\leq 0$, *for all* $t\in[-r,\mathfrak{T}]$.

Proof. Assume that there exists $t_1 \in (0,T]$ such that

$$\mathfrak{u}(t_1) > 0.$$

We put by definition

$$\mathfrak{u}\left(\widetilde{t}\right) = \max_{t\in[-r,t_1]}\mathfrak{u}(t) > 0,$$

where $\widetilde{t}\in(0,t_1]$.

We have

$$(\mathfrak{D}_\alpha \mathfrak{u})(t) \leq -\widetilde{M_1}\mathfrak{u}(t) - \widetilde{N_1}\max_{s\in[t-r,t]}\mathfrak{u}(s),\ t\in J.$$

Which implies

$$(\mathfrak{D}_\alpha \mathfrak{u})(t) \leq -\left(\widetilde{M_1}+\widetilde{N_1}\right)\mathfrak{u}\left(\widetilde{t}\right)$$

Applying the operator I_α to the both sides of the previous inequality, we obtain

$$\mathfrak{u}\left(\widetilde{t}\right) - \mathfrak{u}(0) \leq -\left(\widetilde{M_1}+\widetilde{N_1}\right)u\left(\widetilde{t}\right)\int_0^{\widetilde{t}} s^{\alpha-1}ds$$

That is

$$\mathfrak{u}\left(\widetilde{t}\right)-\mathfrak{u}\left(0\right)\leq-\frac{\left(\widetilde{M_1}+\widetilde{N_1}\right)\mathfrak{u}\left(\widetilde{t}\right)}{\alpha}\widetilde{t}^{\alpha}.$$

Which implies

$$\mathfrak{u}\left(\widetilde{t}\right)\leq-\frac{\left(\widetilde{M_1}+\widetilde{N_1}\right)\mathfrak{u}\left(\widetilde{t}\right)}{\alpha}\mathfrak{T}^{\alpha}.$$

Since $\mathfrak{u}\left(\widetilde{t}\right)>0$, we obtain

$$1\leq-\frac{\left(\widetilde{M_1}+\widetilde{N_1}\right)}{\alpha}\mathfrak{T}^{\alpha}.$$

Which is a contradiction with the assumption

$$-\frac{\left(\widetilde{M_1}+\widetilde{N_1}\right)}{\alpha}\mathfrak{T}^{\alpha}<1,$$

and then, we get

$$\mathfrak{u}\left(t\right)\leq 0, \text{ for all } t\in\left[-r,\mathfrak{T}\right].$$

■

3. Main Results

Definition 3.1. *We say that* $\underline{\mathfrak{u}}\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ *is a lower solution of (1.1) if*

$$\begin{cases}(\mathfrak{D}_{\alpha}\underline{\mathfrak{u}})(\tau)\leq\mathfrak{f}(\tau,\underline{\mathfrak{u}}\left(\tau\right),\max\limits_{s\in[\tau-r,\tau]}\underline{\mathfrak{u}}\left(s\right)),\ \tau\in J,\\ \underline{\mathfrak{u}}\left(\tau\right)\leq\varphi\left(\tau\right),\ \tau\in\left[-r,0\right].\end{cases}$$

Definition 3.2. *We say that* $\overline{\mathfrak{u}}\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ *is an upper solution of (1.1) if*

$$\begin{cases}\left(\mathfrak{D}_{\alpha}\overline{\mathfrak{u}}\right)\left(\tau\right)\geq\mathfrak{f}\left(\tau,\overline{\mathfrak{u}}\left(\tau\right),\max\limits_{s\in[\tau-r,\tau]}\overline{\mathfrak{u}}\left(s\right)\right),\ \tau\in J,\\ \overline{\mathfrak{u}}\left(\tau\right)\geq\varphi\left(\tau\right),\ \tau\in\left[-r,0\right].\end{cases}$$

Definition 3.3. *If* $u\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ *and fulfills (1.1), then we say that* $\mathfrak{u}$ *is a solution of (1.1).*

We have the following result.

Theorem 3.4. *Assume that there two constants* $M\geq 0$, $N\geq 0$ *satisfying*

(H1) $\mathfrak{f}(\tau,x_1,y_1)-\mathfrak{f}\left(\tau,x_2,y_2\right)\geq-M\left(x_1-x_2\right)-N\left(y_1-y_2\right)$, *for all* $\tau\in J$, $\underline{\mathfrak{u}}\left(t\right)\leq x_2\leq x_1\leq\overline{\mathfrak{u}}\left(t\right)$ *and* $\max\limits_{s\in[t-r,t]}\underline{\mathfrak{u}}\left(s\right)\leq y_2\leq y_1\leq\max\limits_{s\in[t-r,t]}\overline{\mathfrak{u}}\left(s\right)$, *where* $\underline{\mathfrak{u}}$ *and* $\overline{\mathfrak{u}}$ *are lower and upper solutions respectively for problem (1.1) such that* $\underline{\mathfrak{u}}\leq\overline{\mathfrak{u}}$ *in* $[-r,\mathfrak{T}]$.

(H2) $\overline{\mathfrak{u}}\left(\tau\right)-\overline{\mathfrak{u}}\left(0\right)\leq\varphi\left(t\right)-\varphi\left(0\right)\leq\underline{\mathfrak{u}}\left(\tau\right)-\underline{\mathfrak{u}}\left(0\right)$, *for all* $\tau\in\left[-r,0\right]$.

(H3) $\left(M+N\right)\dfrac{\mathfrak{T}^{\alpha}}{\alpha}\leq 1$.

Then the problem (1.1) has a minimal solution $\mathfrak{u}_{-}$ *and a maximal solution* $\mathfrak{u}^{+}$ *such that for every solution* $\mathfrak{u}$ *of (1.1) with* $\underline{\mathfrak{u}}\leq\mathfrak{u}\leq\overline{\mathfrak{u}}$ *in* $[-r,\mathfrak{T}]$, *we have*

$$\underline{\mathfrak{u}}\leq\mathfrak{u}_{-}\leq\mathfrak{u}\leq\mathfrak{u}^{+}\leq\overline{\mathfrak{u}}\ \textit{in}\ \left[-r,\mathfrak{T}\right].$$

Proof. We take $\underline{\mathfrak{u}}_0 = \underline{\mathfrak{u}}$, and we define the sequences $(\underline{\mathfrak{u}}_n)_{n\geq 1}$ by

$$\begin{cases} (\mathfrak{D}_\alpha \underline{\mathfrak{u}}_{n+1})(\tau) + M\underline{\mathfrak{u}}_{n+1}(\tau) + N \min\limits_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_{n+1}(s) = \mathfrak{f}_n(\tau), \ \tau \in J, \\ \underline{\mathfrak{u}}_{n+1}(\tau) = \varphi(\tau), \ \tau \in [-r, 0], \end{cases} \tag{3.1}$$

where

$$\mathfrak{f}_n(\tau) = \mathfrak{f}(\tau, \underline{\mathfrak{u}}_n(\tau), \max_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_n(s)) + M\underline{\mathfrak{u}}_n(\tau) + N \min_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_n(s).$$

Analogously, we take $\overline{\mathfrak{u}}_0 = \overline{\mathfrak{u}}$ and we define the sequences $(\overline{\mathfrak{u}}_n)_{n\geq 1}$ by

$$\begin{cases} (\mathfrak{D}_\alpha \overline{\mathfrak{u}}_{n+1})(\tau) + M\overline{\mathfrak{u}}_{n+1}(\tau) + N \min\limits_{s\in[\tau-r,\tau]} \overline{\mathfrak{u}}_{n+1}(s) = \widetilde{f}_n(\tau), \ \tau \in J, \\ \overline{\mathfrak{u}}_{n+1}(\tau) = \varphi(\tau), \ \tau \in [-r, 0], \end{cases} \tag{3.2}$$

where

$$\widetilde{f}_n(\tau) = \mathfrak{f}(\tau, \overline{\mathfrak{u}}_n(\tau), \max_{s\in[\tau-r,\tau]} \overline{\mathfrak{u}}_n(s)) + M\overline{\mathfrak{u}}_n(\tau) + N \min_{s\in[\tau-r,\tau]} \overline{\mathfrak{u}}_n(s).$$

Step 1: For all $n \in \mathbb{N}$, we have

$$\underline{\mathfrak{u}}_n \leq \underline{\mathfrak{u}}_{n+1} \leq \overline{\mathfrak{u}}_{n+1} \leq \overline{\mathfrak{u}}_n \text{ in } [-r, \mathfrak{T}].$$

Let

$$\mathfrak{v}_0(\tau) := \underline{\mathfrak{u}}_0(\tau) - \underline{\mathfrak{u}}_1(\tau), \tau \in [-r, \mathfrak{T}].$$

By (3.1) and using the definition of lower solution and the hypothesis (H2), we have

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{v}_0)(\tau) + M\mathfrak{v}_0(\tau) + N\left(\max\limits_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_0(s) - \max\limits_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_1(s)\right) \leq 0, \tau \in J, \\ \mathfrak{v}_0(0) \leq \mathfrak{v}_0(\tau) \leq 0, \text{ for all } \tau \in [-r, 0]. \end{cases}$$

Then from Lemma 2.11, we obtain

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{v}_0)(\tau) + M\mathfrak{v}_0(\tau) + N \min\limits_{s\in[\tau-r,\tau]} \mathfrak{v}_0(s) \leq 0, \tau \in J, \\ v_0(0) \leq v_0(\tau) \leq 0, \text{ for all } \tau \in [-r, 0]. \end{cases}$$

From Lemma 2.15, one has

$$\mathfrak{v}_0 \leq 0 \text{ in } [-r, \mathfrak{T}].$$

Which means that

$$\underline{\mathfrak{u}}_0 \leq \underline{\mathfrak{u}}_1 \text{ in } [-r, \mathfrak{T}]. \tag{3.3}$$

Similarly, we can prove that

$$\overline{\mathfrak{u}}_1 \leq \overline{\mathfrak{u}}_0 \text{ in } [-r, \mathfrak{T}]. \tag{3.4}$$

Now, we put by definition

$$w_1(t) = \underline{u}_1(t) - \overline{u}_1(t), t \in [-r, \mathfrak{T}].$$

Using (3.1) and (3.2), we have

$$\begin{aligned} &(\mathfrak{D}_\alpha w_1)(\tau) + Mw_1(t) + N \min_{s\in[\tau-r,\tau]} w_1(s) \\ &= \mathfrak{f}_0(\tau) - \widetilde{f}_0(\tau) - N \max_{s\in[\tau-r,\tau]} \underline{u}_1(\tau) + N \max_{s\in[\tau-r,\tau]} \overline{u}_1(\tau) + N \min_{s\in[\tau-r,\tau]} w_1(s). \end{aligned}$$

From Lemma 2.11, we obtain

$$(\mathfrak{D}_\alpha w_1)(\tau) + Mw_1(\tau) + N \min_{s\in[\tau-r,\tau]} w_1(s) \leq \mathfrak{f}_0(\tau) - \widetilde{f}_0(\tau), \tau \in J.$$

Since $\underline{\mathfrak{u}}_0 = \underline{\mathfrak{u}} \leq \overline{\mathfrak{u}} = \overline{\mathfrak{u}}_0$ in $[-r, 0]$ and using the hypothesis (H1) , we obtain

$$(\mathfrak{D}_\alpha w_1)(\tau) + Mw_1(\tau) + N \min_{s \in [\tau - r, \tau]} w_1(s) \leq 0, \ \tau \in J. \tag{3.5}$$

On the other hand, we have

$$w_1(\tau) = 0, \text{ for all } \tau \in [-r, 0].$$

That is

$$w_1(0) = w_1(\tau) = 0, \text{ for all } \tau \in [-r, 0]. \tag{3.6}$$

By the previous equality and (3.5), we have

$$\begin{cases} (\mathfrak{D}_\alpha w_1)(\tau) + Mw_1(\tau) + N \min_{s \in [\tau - r, \tau]} w_1(s) \leq 0, \ \tau \in J, \\ w_1(0) = w_1(\tau) = 0, \text{ for all } \tau \in [-r, 0]. \end{cases}$$

Then by hypothesis (H3) Lemma 2.15 implies

$$w_1 \leq 0 \text{ in } [-r, \mathfrak{T}].$$

Which means that

$$\underline{\mathfrak{u}}_1 \leq \overline{\mathfrak{u}}_1 \text{ in } [-r, \mathfrak{T}]. \tag{3.7}$$

Then by (3.3), (3.4) and (3.7), we have

$$\underline{u}_0 \leq \underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_0 \text{ in } [-r, \mathfrak{T}].$$

Now we assume for fixed $n \geq 1$, we have

$$\underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \text{ in } [-r, \mathfrak{T}],$$

and we show that

$$\underline{u}_{n+1} \leq \underline{u}_{n+2} \leq \overline{u}_{n+2} \leq \overline{u}_{n+1} \text{ in } [-r, \mathfrak{T}].$$

We put by definition

$$v_{n+1}(\tau) := \underline{\mathfrak{u}}_{n+1}(\tau) - \underline{\mathfrak{u}}_{n+2}(\tau), \tau \in [-r, \mathfrak{T}].$$

By (3.1), we have

$$\begin{cases} (\mathfrak{D}_\alpha v_{n+1})(\tau) + Mv_{n+1}(\tau) + N \min_{s \in [\tau - r, \tau]} v_{n+1}(s) = \mathfrak{g}_n(\tau), \ \tau \in J, \\ v_{n+1}(0) = v_{n+1}(\tau) = 0, \tau \in [-r, 0], \end{cases}$$

where

$$\mathfrak{g}_n(\tau) = \mathfrak{f}_n(\tau) - \mathfrak{f}_{n+1}(\tau), \text{ for all } \tau \in J.$$

Since by the hypothesis of recurrence, we have $\underline{\mathfrak{u}}_n \leq \underline{\mathfrak{u}}_{n+1}$ in J and from Lemma 2.11 and using the hypothesis (H1), we obtain

$$\begin{cases} (\mathfrak{D}_\alpha v_{n+1})(\tau) + Mv_{n+1}(\tau) + N \min_{s \in [\tau - r, \tau]} v_{n+1}(s) \leq 0, \ \tau \in J, \\ v_{n+1}(0) = v_{n+1}(\tau) = 0, \tau \in [-r, 0], \end{cases}$$

and then from Lemma 2.15, we get

$$v_{n+1} \leq 0 \text{ in } [-r, \mathfrak{T}].$$

That is

$$\underline{\mathfrak{u}}_{n+1} \leq \underline{\mathfrak{u}}_{n+2} \text{ in } [-r, \mathfrak{T}]. \tag{3.8}$$

Similarly, we can prove that

$$\overline{\mathfrak{u}}_{n+2} \leq \overline{\mathfrak{u}}_{n+1} \text{ in } [-r, \mathfrak{T}], \tag{3.9}$$

and

$$\underline{\mathfrak{u}}_{n+2} \leq \overline{\mathfrak{u}}_{n+2} \text{ in } [-r, \mathfrak{T}]. \tag{3.10}$$

Then by (3.8), (3.9) and (3.10), we obtain

$$\underline{\mathfrak{u}}_{n+1} \leq \underline{\mathfrak{u}}_{n+2} \leq \overline{\mathfrak{u}}_{n+2} \leq \overline{\mathfrak{u}}_{n+1} \text{ in } [-r, \mathfrak{T}].$$

Hence for all $n \in \mathbb{N}$, we have

$$\underline{\mathfrak{u}}_{n} \leq \underline{\mathfrak{u}}_{n+1} \leq \overline{\mathfrak{u}}_{n+1} \leq \overline{\mathfrak{u}}_{n} \text{ in } [-r, \mathfrak{T}].$$

Step 2: The consequence $(\underline{\mathfrak{u}}_n)_{n\in\mathbb{N}}$ converges to a minimal solution of (1.1).

By **Step 1** and using Dini theorem, it follows that the sequence of functions $(\mathfrak{u}_n)_{n\in\mathbb{N}}$ converges uniformly to $\mathfrak{u}_-$.

Let $n \in \mathbb{N}^*$ and $t \in J$, then from Lemma 2.13 we get

$$\underline{\mathfrak{u}}_{n+1}(\tau) - \underline{\mathfrak{u}}_{n+1}(0) = \int_0^\tau s^{\alpha-1} \mathfrak{F}_n(s)\, ds,$$

where

$$\mathfrak{F}_n(s) = \mathfrak{f}_n(s) - M\underline{\mathfrak{u}}_{n+1}(s) - N \max_{t\in[s-r,s]} \underline{\mathfrak{u}}_{n+1}(t).$$

Now, as n tends to $+\infty$, we obtain

$$\mathfrak{F}_n(s) \to \mathfrak{f}(s, \mathfrak{u}_-(s), \max_{t\in[s-r,s]} \mathfrak{u}_-(s)).$$

Which implies

$$-u_-(\tau) - \mathfrak{u}_-(0) = \int_0^\tau s^{\alpha-1} \mathfrak{f}(s, \mathfrak{u}_-(s), \max_{t\in[s-r,s]} \mathfrak{u}_-(s)) ds,$$

and from Lemma 2.13, we deduce

$$(\mathfrak{D}_\alpha \mathfrak{u}_-)(t) = \mathfrak{f}(\tau, \mathfrak{u}_-(\tau), \max_{t\in[\tau-r,\tau]} \mathfrak{u}_-(t)),\ \tau \in J.$$

On the other hand, we have

$$\mathfrak{u}_- = \varphi \text{ in } [-r, 0],$$

and consequently it follows that $\mathfrak{u}_-$ is a solution of (1.1).

Now, we prove that if $\mathfrak{u}$ is another solution of (1.1) such that $\underline{\mathfrak{u}} \leq \mathfrak{u} \leq \overline{\mathfrak{u}}$, then $\mathfrak{u}_- \leq \mathfrak{u}$.

Since $\mathfrak{u}$ is an upper solution of (1.1), then by **Step 1**, we have

$$\forall n \in \mathbb{N},\ \underline{\mathfrak{u}}_n \leq \mathfrak{u}.$$

Which implies that

$$\mathfrak{u}_- = \lim_{n\to+\infty} \underline{\mathfrak{u}}_n \leq \mathfrak{u}.$$

This means that $\mathfrak{u}_-$ is a minimal solution of (1.1).

The second step's proof is finished.

In a similar way, we can prove that the sequence $(\overline{\mathfrak{u}}_n)_{n\in\mathbb{N}}$ converges to a maximal solution $\mathfrak{u}^+$ of (1.1).

The proof of Theorem 3.4 is complete. ■

For the uniqueness of solutions for the problem (1.1), it is necessary to impose additional conditions on $\mathfrak{f}$.

(H4) There exists a negative real number M_1 such that the function $x \longmapsto \mathfrak{f}(\tau, x, y) + M_1 y$ is decreasing for all $\tau \in J$ and $y \in \mathbb{R}$.

(H5) There exists a negative real number N_1 such that the function $y \longmapsto \mathfrak{f}(\tau, x, y) + N_1 y$ is decreasing for all $\tau \in J$ and $x \in \mathbb{R}$.

(H6) $-(M_1 + N_1)\dfrac{\mathfrak{T}^\alpha}{\alpha} < 1.$

We have the following result.

Theorem 3.5. *Assume that hypothesis (Hi) for i = 1,...,6 are satisfied, then the problem (1.1) admits a unique solution* $\mathfrak{u}$ *such that* $\underline{\mathfrak{u}} \le \mathfrak{u} \le \overline{\mathfrak{u}}$ *in* $[-r, \mathfrak{T}]$.

Proof. By Theorem 3.4, the problem (*1.1*) admits a minimal and a maximal solutions $\mathfrak{u}_-$ and $\mathfrak{u}^+$ such that

$$\underline{\mathfrak{u}} \le \mathfrak{u}_- \le \mathfrak{u}^+ \le \overline{\mathfrak{u}} \text{ in } [-r, \mathfrak{T}].$$

We put by definition

$$\mathfrak{z}(\tau) = \mathfrak{u}^+(\tau) - \mathfrak{u}_-(\tau), \tau \in [-r, \mathfrak{T}].$$

We have

$$\mathfrak{z} \ge 0 \text{ in } [-r, \mathfrak{T}]. \tag{3.11}$$

Now, we are going to prove that

$$\mathfrak{z} \le 0 \text{ in } [-r, \mathfrak{T}].$$

As we have

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{z})(\tau) = \mathfrak{f}(\tau, \mathfrak{u}^+(\tau), \max\limits_{t \in [\tau - r, \tau]} \mathfrak{u}^+(t)) - \mathfrak{f}(\tau, \mathfrak{u}_-(\tau), \max\limits_{t \in [\tau - r, \tau]} u_-(t)), \ \tau \in J, \\ \mathfrak{z}(0) = \mathfrak{z}(\tau) = 0, \ \tau \in [-r, 0]. \end{cases}$$

By using the hypothesis (H4), we obtain

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{z})(\tau) + M_1 \mathfrak{z}(\tau) \le \\ \mathfrak{f}(\tau, \mathfrak{u}_-(\tau), \max\limits_{t \in [\tau - r, \tau]} \mathfrak{u}^+(t)) - \mathfrak{f}(\tau, \mathfrak{u}_-(\tau), \max\limits_{t \in [\tau - r, \tau]} u_-(t)), \ \tau \in J, \\ \mathfrak{z}(0) = \mathfrak{z}(\tau) = 0, \ \tau \in [-r, 0]. \end{cases}$$

Now from Lemma 2.10, we have

$$\max_{t \in [\tau - r, \tau]} z(t) = \max_{t \in [\tau - r, \tau]} \left| \mathfrak{u}^+(t) - \mathfrak{u}_-(t) \right| \ge \max_{t \in [\tau - r, \tau]} \mathfrak{u}^+(t) - \max_{t \in [\tau - r, \tau]} u_-(t),$$

and then according to hypothesis (H4), we obtain

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{z})(\tau) + M_1 z(\tau) + N_1 \max\limits_{t \in [\tau - r, \tau]} \mathfrak{z}(t) \le 0, \ \tau \in J, \\ \mathfrak{z}(0) = \mathfrak{z}(\tau) = 0, \ \tau \in [-r, 0]. \end{cases}$$

From Lemma 2.17, we get

$$\mathfrak{z}(t) \le 0 \text{ in } [-r, \mathfrak{T}],$$

and therefore, there is a unique solution to problem (1.1).

■

4. Applications

4.1. Example 1

Consider the problem

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}}\mathfrak{u}(\tau) = \sqrt{\tau}\mathfrak{u}(\tau) - \max_{s\in[\tau-1,\tau]}\mathfrak{u}(s) + \cos\tau, \ \tau \in \left[0, \frac{1}{4}\right], \\ \mathfrak{u}(\tau) = \tau, \ \tau \in [-1, 0]. \end{cases} \tag{4.1}$$

Let $\underline{\mathfrak{u}}(\tau) = \tau$ and $\overline{\mathfrak{u}}(\tau) = \tau + 1$ in $\left[-1, \frac{1}{4}\right]$.

For the problem (4.1), $\underline{\mathfrak{u}}$ is a lower solution if

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}}\underline{\mathfrak{u}}(\tau) \leq \sqrt{\tau}\underline{\mathfrak{u}}(\tau) - \max_{s\in[\tau-1,\tau]}\underline{\mathfrak{u}}(s) + \cos\tau, \ \tau \in \left[0, \frac{1}{4}\right], \\ \underline{\mathfrak{u}}(\tau) \leq \tau, , \ \tau \in [-1, 0]. \end{cases}$$

That is

$$\begin{cases} \sqrt{\tau} \leq \tau^{\frac{3}{2}} - \tau + \cos\tau, \ \tau \in \left[0, \frac{1}{4}\right], \\ \tau \leq \tau, \ \tau \in [-1, 0]. \end{cases}$$

Since $\varphi_1(\tau) = \sqrt{\tau} - \tau^{\frac{3}{2}} + \tau - \cos\tau \leq 0$ in $\left[0, \frac{1}{4}\right]$,

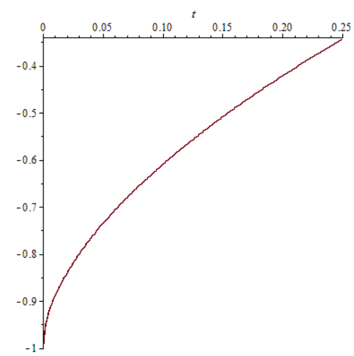

Figure 1: Graph of the function φ_1

we conclude that $\underline{\mathfrak{u}}$ is a lower solution for the problem (4.1).

Similarly if we have

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}}\overline{\mathfrak{u}}(\tau) \geq \sqrt{\tau}\overline{\mathfrak{u}}(\tau) - \max_{s\in[\tau-1,\tau]}\overline{\mathfrak{u}}(s) + \cos\tau, \ \tau \in \left[0, \frac{1}{4}\right], \\ \overline{\mathfrak{u}}(\tau) \geq \tau, \tau \in [-1, 0]. \end{cases}$$

we obtain $\overline{\mathfrak{u}}$ is an upper solution for the problem (4.1).
That is

$$\begin{cases} \tau^{\frac{3}{2}} - \tau - 1 + \cos\tau \leq 0, \ \tau \in \left[0, \frac{1}{4}\right], \\ \tau + 1 \geq \tau, \tau \in [-1, 0]. \end{cases}$$

Since $\varphi_2(\tau) = \tau^{\frac{3}{2}} - \tau - 1 + \cos\tau \leq 0$, for all $\tau \in \left[0, \frac{1}{4}\right]$,

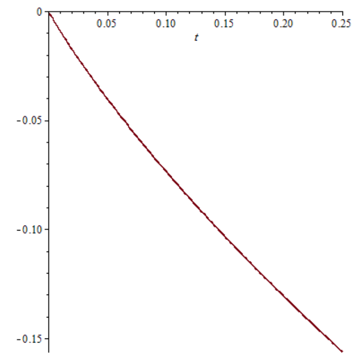

Figure 2: Graph of the function φ_2

we obtain the desired upper solution for the problem (4.1).

Now, if we select $N=1$ and $M=0$, then

$$(M+N)\,\frac{T^{\alpha}}{\alpha}=\frac{\left(\frac{1}{4}\right)^{\frac{1}{2}}}{\frac{1}{2}}=1\leq 1,$$

and if we choose $M_1=-\frac{1}{2}$ and $N_1=0$, we have

$$-\left(M_1+N_1\right)\frac{T^{\alpha}}{\alpha}=\frac{\left(\frac{1}{4}\right)^{\frac{1}{2}}}{2\times\frac{1}{2}}=\frac{1}{2}<1.$$

Nevertheless, it is evident that the function $\tau\mapsto\sqrt{\tau}u\left(\tau\right)-\max\limits_{s\in[\tau-1,\tau]}u\left(s\right)+\cos\tau$ satisfies the remaining assumptions of Theorem 3.5. As a result, the problem (4.1) admits a unique solution $\mathfrak{u}$ such that $\underline{\mathfrak{u}}\leq u\leq\overline{\mathfrak{u}}$.

4.2. Example 2

Consider the problem

$$\begin{cases}\mathfrak{D}_{\frac{2}{3}}\mathfrak{u}\left(\tau\right)=\dfrac{\tau^{\frac{2}{3}}}{4}\mathfrak{u}\left(\tau\right)-\dfrac{\max\limits_{s\in[\tau-1,\tau]}\mathfrak{u}\left(s\right)}{8}+\dfrac{2}{3}(\tau^{\frac{2}{3}}+1)+\dfrac{1}{8},\ \tau\in\left[0,\frac{1}{2}\right],\\ \mathfrak{u}(\tau)=\tau^{\frac{2}{3}},,\ \tau\in\left[-1,0\right].\end{cases}\tag{4.2}$$

Let $\underline{\mathfrak{u}}(\tau)=\tau^{\frac{2}{3}}$ and $\overline{\mathfrak{u}}(\tau)=2\tau^{\frac{2}{3}}+1$, in $\left[-1,\frac{1}{2}\right]$.

For the problem (4.2), $\underline{\mathfrak{u}}$ is a lower solution if we have

$$\begin{cases}\mathfrak{D}_{\frac{2}{3}}\underline{\mathfrak{u}}\left(\tau\right)\leq\dfrac{\tau^{\frac{2}{3}}}{4}\underline{\mathfrak{u}}\left(\tau\right)-\dfrac{\max\limits_{s\in[\tau-1,\tau]}\mathfrak{u}\left(s\right)}{8}+\dfrac{2}{3}(\tau^{\frac{2}{3}}+1)+\dfrac{1}{8},\ \tau\in\left[0,\dfrac{1}{2}\right],\\ \underline{\mathfrak{u}}(\tau)\leq\tau^{\frac{2}{3}},\tau\in\left[-1,0\right].\end{cases}$$

That is

$$\begin{cases}\dfrac{2}{3}\leq\dfrac{\tau^{\frac{4}{3}}}{4}-\dfrac{(\tau-1)^{\frac{2}{3}}}{8}+\dfrac{2}{3}(\tau^{\frac{2}{3}}+1)+\dfrac{1}{8},\ \tau\in\left[0,\dfrac{1}{2}\right],\\ \tau^{\frac{2}{3}}\leq\tau^{\frac{2}{3}},\tau\in\left[-1,0\right].\end{cases}$$

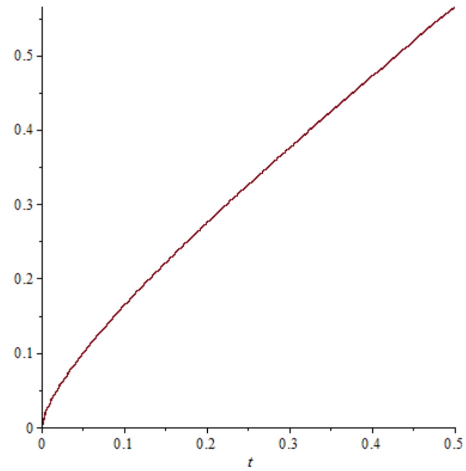

Figure 3: Graph of the function φ_3

Since $\varphi_3(\tau) = \frac{\tau^{\frac{4}{3}}}{4} - \frac{(\tau-1)^{\frac{2}{3}}}{8} + \frac{2}{3}\tau^{\frac{2}{3}} + \frac{1}{8} \geq 0$, for all $\tau \in \left[0, \frac{1}{2}\right]$,

we get $\underline{\mathrm{u}}$ is a lower solution for the problem (4.2).

Similarly if

$$\begin{cases} \mathfrak{D}_{\frac{2}{3}}\overline{\mathrm{u}}(\tau) \geq \frac{\tau^{\frac{2}{3}}}{4}\overline{\mathrm{u}}(\tau) - \frac{\max\limits_{s\in[\tau-1,\tau]} \overline{u}(s)}{8} + \frac{2}{3}(\tau^{\frac{2}{3}}+1) + \frac{1}{8},\ \tau \in \left[0, \frac{1}{2}\right], \\ \overline{\mathrm{u}}(\tau) \geq \tau^{\frac{2}{3}}, \tau \in [-1, 0]. \end{cases}$$

we obtain $\overline{\mathrm{u}}$ is an upper solution for the problem (4.2).
That is

$$\begin{cases} \frac{4}{3} \geq \frac{\tau^{\frac{2}{3}}}{4}\left(2\tau^{\frac{2}{3}}+1\right) - \left(\frac{2(\tau-1)^{\frac{2}{3}}+1}{8}\right) + \frac{2}{3}(t^{\frac{2}{3}}+1) + \frac{1}{8},\ \tau \in \left[0, \frac{1}{2}\right], \\ 2\tau^{\frac{2}{3}}+1 \geq \tau^{\frac{2}{3}},\ \tau \in [-1, 0]. \end{cases}$$

That is

$$\begin{cases} \frac{\tau^{\frac{4}{3}}}{2} + \frac{11}{8}\tau^{\frac{2}{3}} - \frac{(\tau-1)^{\frac{2}{3}}}{4} - \frac{2}{3} \leq 0,\ \tau \in \left[0, \frac{1}{2}\right], \\ \tau^{\frac{2}{3}}+1 \geq 0, ,\ \tau \in [-1, 0]. \end{cases}$$

Since $\varphi_4(\tau) = \frac{\tau^{\frac{4}{3}}}{2} + \frac{11}{8}\tau^{\frac{2}{3}} - \frac{(\tau-1)^{\frac{2}{3}}}{4} - \frac{2}{3} \leq 0$, for all $\tau \in \left[0, \frac{1}{2}\right]$,

we obtain the desired upper solution for the problem (4.2).

Now, if we select $M = 0$ and $N = \frac{1}{8}$, then

$$(M+N)\frac{T^{\alpha}}{\alpha} = \frac{1}{8}\frac{\left(\frac{1}{2}\right)^{\frac{2}{3}}}{\frac{2}{3}} = 0.11812 \leq 1,$$

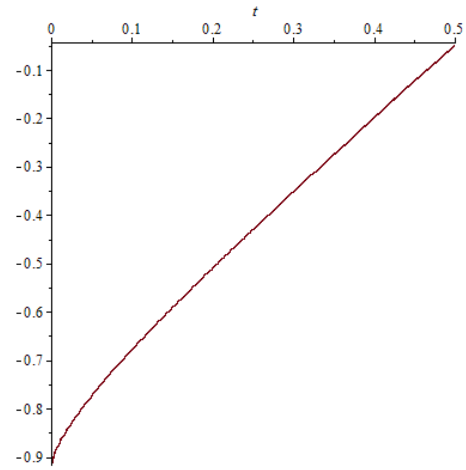

Figure 4: Graph of the function φ_4

and if we choose $M_1 = -\frac{1}{2}$ and $N_1 = 0$, we have

$$-\left(M_1 + N_1\right)\frac{T^\alpha}{\alpha} = \frac{\left(\frac{1}{2}\right)^{\frac{2}{3}}}{2 \times \frac{2}{3}} = 0.47247 < 1.$$

Nevertheless, it is evident that the function $\tau \mapsto \frac{\tau^{\frac{2}{3}}}{4} u\left(\tau\right) - \frac{\max\limits_{s \in [\tau-1,\tau]} u\left(s\right)}{8} + \frac{2}{3}(\tau^{\frac{2}{3}} + 1) + \frac{1}{8}$ satisfies the remaining assumptions of Theorem 3.5. As a result, the problem (4.2) admits a unique solution u such that $\underline{\mathfrak{u}} \leq \mathfrak{u} \leq \overline{\mathfrak{u}}$.

4.3. Example 3

Consider the problem

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}} \mathfrak{u}\left(\tau\right) = -\dfrac{\mathfrak{u}\left(\tau\right)}{2} - \dfrac{\max\limits_{s \in \left[\tau - \frac{\pi}{2}, \tau\right]} \mathfrak{u}\left(s\right)}{2} + \sqrt{\tau} \cos\left(\tau\right) + \sin\left(\tau\right), \ \tau \in \left[0, \dfrac{\pi}{16}\right], \\ \mathfrak{u}(\tau) = 1 + \tau, \tau \in \left[-\dfrac{\pi}{2}, 0\right]. \end{cases} \tag{4.3}$$

Let $\underline{\mathfrak{u}}(\tau) = \sin\left(\tau\right)$ and $\overline{\mathfrak{u}}(\tau) = 1$, for all $\tau \in \left[0, \frac{\pi}{16}\right]$.

First $\underline{u}$ is a lower solution if

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}} \underline{\mathfrak{u}}\left(\tau\right) \leq -\dfrac{\underline{\mathfrak{u}}\left(\tau\right)}{2} - \dfrac{\max\limits_{t \in \left[\tau - \frac{\pi}{2}, \tau\right]} \underline{\mathfrak{u}}\left(t\right)}{2} + \sqrt{\tau} \cos\left(\tau\right) + \sin\left(\tau\right), \ \tau \in \left[0, \dfrac{\pi}{16}\right], \\ \underline{\mathfrak{u}}\left(\tau\right) \leq 1 + \tau, \tau \in \left[-\dfrac{\pi}{2}, 0\right]. \end{cases}$$

That is

$$\begin{cases} \sqrt{\tau} \cos\left(\tau\right) \leq \left(1 - \dfrac{2}{2}\right) \dfrac{\sin\left(\tau\right)}{2} + \sqrt{\tau} \cos\left(\tau\right), \ \tau \in \left[0, \dfrac{\pi}{16}\right], \\ \sin\left(\tau\right) \leq 1 + \tau, \tau \in \left[-\dfrac{\pi}{2}, 0\right]. \end{cases}$$

Since $\sin(\tau)-1-\tau\leq 0$, for all $\tau\in\left[-\frac{\pi}{2},0\right]$, we conclude that $\underline{\mathfrak{u}}$ is a lower solution for the problem (4.3).

Similarly if we have,

$$\begin{cases}\mathfrak{D}_{\frac{1}{2}}\overline{\mathfrak{u}}\left(\tau\right)\geq-\dfrac{\overline{u}\left(\tau\right)}{2}-\dfrac{\max\limits_{s\in\left[\tau-\frac{\pi}{2},\tau\right]}\mathfrak{u}\left(s\right)}{2}+\sqrt{\tau}\cos\left(\tau\right)+\sin\left(\tau\right),\ \tau\in\left[0,\dfrac{\pi}{16}\right],\\ \overline{\mathfrak{u}}(\tau)\geq 1+\tau,\tau\in\left[-\dfrac{\pi}{2},0\right].\end{cases}$$

we obtain $\overline{\mathfrak{u}}$ is an upper solution for the problem (4.3).
That is

$$\begin{cases}0\geq-1+\sqrt{\tau}\cos\left(\tau\right)+\sin\left(\tau\right),\ \tau\in\left[0,\dfrac{\pi}{16}\right],\\ 1\geq 1+\tau,\tau\in\left[-\dfrac{\pi}{2},0\right].\end{cases}$$

Since $\varphi_5(\tau)=-1+\sqrt{\tau}\cos\left(\tau\right)+\sin\left(\tau\right)\leq 0$, for all $\tau\in\left[0,\frac{\pi}{16}\right]$,

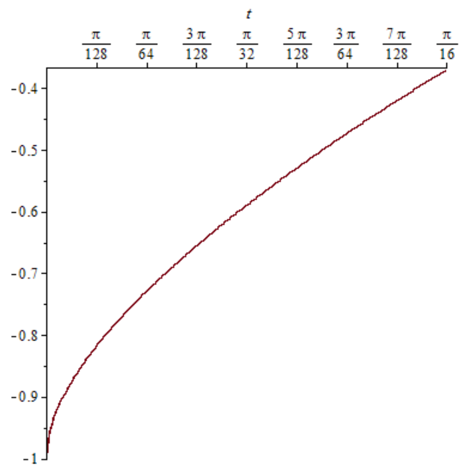

Figure 5: Graph of the function φ_5

we obtain the desired upper solution for the problem (4.3).

Now, if we select $M=N+\frac{1}{2}$, then $\dfrac{\left(\frac{\pi}{16}\right)^{\frac{1}{2}}}{\frac{1}{2}}=0.88623\leq 1$ and the function $\tau\mapsto-\dfrac{\mathfrak{u}\left(\frac{\tau}{2}\right)}{2\pi}+\dfrac{\cos\left(\sqrt{\tau}\right)}{2}+\sin\left(\sqrt{\frac{\tau}{2}}\right)$ satisfies the remaining assumptions of Theorem 3.5. As a result, the problem (4.3) admits a unique solution u such that $\underline{\mathfrak{u}}\leq\mathfrak{u}\leq\overline{\mathfrak{u}}$.

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Existence and regularity of solutions in α-norm for some second order partial neutral functional differential equations with finite delay in Banach spaces

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Abstract. In this paper, we investigate the regularity and existence of solutions in the α-norm for some second order partial neutral functional differential equations with finite delay in Banach spaces. To do this, we use the cosine families theory and Schauder's fixed point theorem to establish the existence of solutions and then we give some sufficient conditions that ensure the regularity of solutions. Finally, we give an example to illustrate the theoretical results.

AMS Subject Classifications: 47D09, 34K40, 45N05, 35A01.

Keywords: Cosine family, Mild and strict solutions, Neutral equations, α-norm, second order functional differential equations.

Contents

1. Introduction

The aim of this work, we to study the existence and regularity of solutions in α-norm for the following second order neutral partial functional differential equation

$$\begin{cases} \dfrac{d}{dt}[u'(t) - g(t, u_t)] = Au(t) + f(t, u_t, u'_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathcal{C}_\alpha, \\ u'_0 = \varphi' \in \mathcal{C}_\alpha, \end{cases} \tag{1.1}$$

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where A is the (possibly unbounded) infinitesimal generator of strongly continuous cosine family of linear operators in X. $\mathcal{C}_\alpha = C^1([-r,0],D((-A)^\alpha))$, $0<\alpha<1$, denotes the space of continuous differentiable functions from $[-r,0]$ into $D((-A)^\alpha)$, $(-A)^\alpha$ is the fractional α-power of A. This operator $((-A)^\alpha, D((-A)^\alpha))$ will be describe later. C_α is endowed with the following norm $\|h\|_{\mathcal{C}_\alpha} = \|h\|_\alpha + \|h'\|_\alpha$ for all $h \in \mathcal{C}_\alpha = C^1([-r,0],X_\alpha)$, where $\|h\|_\alpha = \sup\limits_{-r\leq 0\leq 0} |h(\theta)|_\alpha$. The norm $|.|_\alpha$ will be specified later. For $u \in C^1([-r,b],D((-A)^\alpha))$, $t\geq 0, b>0$, and $t\in[0,b]$ u_t denotes the history function of $\mathcal{C}_\alpha$ defined by

$$u_t(\theta) = u(t+\theta) \text{ for } \theta \in [-r,0],$$

$f : \mathbb{R}^+ \times \mathcal{C}_\alpha \times \mathcal{C}_\alpha \to X$ and $g : \mathbb{R}^+ \times \mathcal{C}_\alpha \to X_\alpha$ are given functions.
In [3] the authors study firstly the abstract semi-linear second order initial value problem and secondly they unify and simplify some ideas from strongly continuous cosine families of linear operators in Banach spaces.
In [7], the authors reveal three properties of cosine families, distinguishing them from semi-groups of operators.
In [1] by use of the theory of cosine families of linear operators in Banach space, the author studied the existence of solutions of following second order partial neutral functional differential equation

$$\begin{cases} \dfrac{d}{dt}[u'(t) - g(t,u_t)] = Au(t) + f(t,u_t,u'(t)),\ t\in J=[0,T] \\ \\ u_0 = \varphi \in \mathcal{B},\ u'(0) = z \in X. \end{cases} \tag{1.2}$$

To the best of the authors knowledge, the equation (1.2) and most similar other problems using cosine families theory are studied without delay arguments. However time-delay is known to have a significant impact on the asymptotic behavior and stability of these dynamic systems, it is inevitable that it be included in the mathematical description of phenomena. For this purpose, in [5], Zabsonre et al. studied the existence and regularity of solution for some nonlinear second order differential with finite delay in Banach spaces.

This present work is a generalization of [4] and a continuation of [1]. The neutral functional differential equations, on the other hand, received a lot of attention in recent years due to the fact that they are present in many areas of applied mathematics.
By use of the theory of strongly continuous cosine families of linear operator in Banach space, we will prove in this paper the existence of mild and strict solution.
The organization of this work as follows, in Section 2, we recall some preliminary results about cosine families theory and fractional α-power, in Section 3, we prove the existence and uniqueness of mild solution in the α-norm for (1.1). In Section 4, we study the regularity of solutions. Finally, we illustrate our results, in Section 5 by examining an example.

2. Preliminary Results

Let $(X,\|.\|)$ be a Banach space and α be a constant such that $0<\alpha<1$ and A be the infinitesimal generator of strongly continuous $(C(t))_{t\geq 0}$ on X. We assume without loss of generality that $0\in\rho(-A)$. Note that if the assumption $0\in\rho(-A)$ is not satisfied, one can substitute the operator $-A$ by the operator $(-A-\sigma I)$ with σ large enough such that $0\in\rho(-A-\sigma I)$. This allows us to define the fractional power $(-A)^\alpha$ for $0<\alpha<1$, as a closed linear invertible operator with domain $D((-A)^\alpha)$ dense in X. The closeness of $(-A)^\alpha$ implies that $D((-A)^\alpha)$, endowed with the graph norm of $(-A)^\alpha$, $|x| = \|x\| + \|(-A)^\alpha x\|$, is a Banach space. Since $(-A)^\alpha$ is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|(-A)^\alpha x\|$. Thus, $D((-A)^\alpha)$ equipped with the norm $|.|_\alpha$, is a Banach space, which we denote by X_α.

Definition 2.1. *[3] A one parameter family $\{C(t), t\in\mathbb{R}\}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if*

i) $C(s+t)+C(s-t)=2C(s)C(t)$ *for all* $s,t\in\mathbb{R}$
ii) $C(0)=I$
iii) $C(t)x$ *is continuous on* $\mathbb{R}$ *for each fixed* $x\in X$.

The strongly continuous sine family $\{S(t),t\in\mathbb{R}\}$ associated to the given strongly continuous cosine family $\{C(t),t\in\mathbb{R}\}$ by

$$S(t)x=\int_0^t C(s)xds,\ \text{for } x\in X, t\in\mathbb{R}. \tag{2.1}$$

Definition 2.2. *The infinitesimal generator of strongly continuous cosine family* $\{C(t),t\in\mathbb{R}\}$ *is the operator* $A:X\longrightarrow X$ *define by*

$$Ax=\frac{d^2C(t)x}{dt^2}\Big|_{t=0}.$$

$D(A)=\{x\in X:$ *C(t)x is a twice continuously differentiable function of t*$\}$.

We shall also make use of the set

$$E=\{x:C(t)x \text{ is a once continuously differentiable function of t}\}.$$

Lemma 2.3. *Let* $C(t),\in\mathbb{R}$ *be a strongly continuous cosine family in* X *with infinitesimal generator A. The following are true.*
i) $D(A)$ *is dense in* X *and* A *is closed operator in* X*;*

ii) if $x\in X$ *and* $s,r\in\mathbb{R}$ *then* $z=\int_s^r = S(u)xdu\in D(A)$ *and* $Az=C(s)x-C(r)x$*;*
iii) if $x\in X$, $s,r\in\mathbb{R}$ *then* $z=\int_0^s\int_0^r C(u)C(v)xdudv\in D(A)$ *and*
$Az=\frac{1}{2}(C(s+r)x-C(s-r)x)$*;*
iv) if $x\in X$, $S(t)x\in E$*;*
v) if $\in X$*, the* $S(t)x\in D(A)$ *and* $\frac{dC(t)}{dt}=AS(t)x$*:*

vi) if $x\in D(A)$*, then* $C(t)x\in D(A)$ *and* $\frac{d^2C(t)}{dt^2}=AC(t)x=C(t)Ax$*;*
vii) if $x\in E$*, then* $\lim\limits_{t\to 0}AS(t)=0$*;*
viii) if $x\in E$*, then* $S(t)x\in D(A)$ *and* $\frac{d^2S(t)}{dt^2}=AS(t)x$*;*
ix) if $x\in D(A)$*, then* $S(t)x\in D(A)$ *and* $AS(t)x=S(t)Ax$*;*
x) $C(t+s)+C(t-s)=2AS(t)S(s)$ *for all* $s,t\in\mathbb{R}$.

In [3], for $0<\alpha<1$ the fractional powers $(-A)^\alpha$ exist as closed linear operators in X,

$D((-A)^\alpha)\subset D((-A)^\beta)$ for $0\leq\beta\leq\alpha\leq 1$ and $(-A)^\alpha(-A)^\beta=(-A)^{\alpha+\beta}$ for $0\leq\alpha+\beta\leq 1$.

For our objective we assume that
($\mathbf{H}_0$) A is the infinitesimal generator of a strongly continuous cosine family of linear operators on a Banach space X.

By Lemma2.3, ($\mathbf{H}_{0)}$) implies that the operator A is densely defined in X, i.e $\overline{D(A)}=X$. We have the following result.

Lemma 2.4. *[3] Assume that* (H_0) *hols. Then there are constants* $M \geq 1$ *and* $\omega \geq 0$ *such that* $\|C(t)\| \leq Me^{\omega|t|}$ *and* $\|S(t_1) - S(t_2)\| \leq M\Big|\int_{t_1}^{t_2} e^{\omega|s|}ds\Big|$, *for all* $t_1, t_2 \in \mathbb{R}$.

From previous inequality, since $S(0) = 0$ we can deduce that

$$\|S(t)\| \leq \frac{M}{\omega}e^{\omega t} \text{ for } t \in \mathbb{R}^+.$$

In the sequel, let us pose $M_1 = \max\Big(M, \dfrac{M}{\omega}\Big)$.

Theorem 2.5. *[3] If* $k : \mathbb{R}^+ \to X$ *is continuous,* $h : \mathbb{R}^+ \to X$ *is continuous and* u *is a solution of equation* (1.1)*, then* u *is a solution of integral equation*

$$u(t) = C(t)x + S(t)y + \int_0^t C(t-s)k(s)ds + \int_0^t S(t-s)h(s)ds.$$

($\mathbf{A}_1$): For $0 < \alpha < 1$, $(-A)^\alpha$ maps onto X and $1-1$, so that $D((-A)^\alpha)$ endowed with the norm $|x|_\alpha = \|(-A)^\alpha x\|$ is a Banach space. We denote by X_α this space. In addition we assume that A^{-1} is compact. To establish our results, we need the following Lemmas.

Lemma 2.6. *[4] Assume that* $(\boldsymbol{H}_0)$ *holds. The following are true*
(i) *For* $0 < \alpha < 1$, $(-A)^{-\alpha}$ *is compact if and only if* A^{-1} *is compact.*
(ii) *For* $0 < \alpha < 1$, *and* $t \in \mathbb{R}$ $(-A)^{-\alpha}C(t) = C(t)(-A)^{-\alpha}$ *and* $(-A)^{-\alpha}S(t) = S(t)(-A)^{-\alpha}$.

Recall from [10], $(-A)^{-\alpha}$ is given by the following formula

$$(-A)^{-\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^{+\infty} t^{-\alpha}(tI - A)^{-1}dt.$$

Lemma 2.7. *[4] Assume that* $(\boldsymbol{H}_0)$ *holds. Let* $v : \mathbb{R} \longrightarrow x$ *such that* v *is continuously differentiable and let* $q(t) = \int_0^t S(t-s)v(s)ds$. *Then*
(i) q *is twice continuously differentiable and for* $t \in \mathbb{R}$, $q(t) \in D(A)$,

$$q'(t) = \int_0^t C(t-s)v(s)ds$$

and

$$q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t).$$

(ii) *For* $0 < \alpha < 1$ *and* $t \in \mathbb{R}$, $(-A)^{\alpha-1}q'(t) \in E$.

Theorem 2.8. *(Heine's theorem)*
Let f *be a continuous function on a compact set* K, *then* f *is uniformly continuous on* K.

Theorem 2.9. *(Arzela-Ascoli theorem)*
Let (X, d_X) *and* (Y, d_Y) *be compact metric spaces,* $C(X,Y)$ *be the set of continuous functions from* X *to* Y *and Let* $\mathcal{F}$ *be q subset of* $C(X,Y)$. *If* $\mathcal{F}$ *is closed and equicontinuous then, it is compact.*

Theorem 2.10. *(Schauder's fixed point theorem)*
Let X *be a locally convex topological vector space, and let* $K \subset X$ *be a non-empty, compact, and convex set. Then given any continuous mapping* $f : K \longrightarrow K$ *there exists* $x \in K$ *such that* $f(x) = x$.

3. Existence of mild solutions

Definition 3.1. *A continuous function* $u\ :]-r,+\infty[\to X_\alpha$ *is said a strict solution of equation* (1.1) *if the following conditions hold*

(i) $u \in C^1([0,+\infty[;X_\alpha)\cap C^2([0,\infty[;X_\alpha)$
(ii) u *satisfies equation* (1.1) *on* $[0,+\infty[$.
(iii) $u(\theta)=\varphi(\theta)$ *for* $-r\le\theta\le 0$.

Proposition 3.2. *Assume that* $(\boldsymbol{H}_{0)})$ *holds. If* u *is a strict solution of equation* (1.1)*, then*

$$u(t)=C(t)\phi(0)+S(t)(\phi'(0)-g(0,\varphi))+\int_0^t C(t-s)g(s,u_s)ds+\int_0^t S(t-s)f(s,u_s,u'_s)ds. \qquad (3.1)$$

Proof. It is just the consequence of Theorem 2.5. In fact, let us pose $k(t)=g(t,u_t)$ and $h(t)=f(t,u_t,u'_t)$ for $t\ge 0$. The we get the desired results.■

Remark 3.3. *The converse is not true. In fact if* u *satisfies equation* (3.1)*,* u *may be not twice continuously differentiable, that is why we distinguish between mild and strict solutions.*

Definition 3.4. *A continuous function* $u\ :]-r,+\infty[\to X_\alpha$*, for* $b>0$ *is said to a mild solution of equation* (1.1) *if*

$$\begin{cases} u(t)=C(t)\varphi(0)+S(t)(\varphi'(0)-g(0,\varphi))+\int_0^t C(t-s)g(s,u_s)ds+\int_0^t S(t-s)f(s,u_s,u'_s)ds \text{ for } t\in[0,b],\\ u_0=\varphi(0),\\ u'_0=\varphi'(0). \end{cases}$$

In the following, we give a local existence of mild solutions of equation(1.1). We will use the Schauder's fixed point theorem. For this purpose, we make this following assumptions.

$(\mathbf{H}_1)$The function $f:[0,b]\times\mathcal{C}_\alpha\to X$ satisfies the following conditions

i) $f:[0,b]\times\mathcal{C}_\alpha\times\mathcal{C}_\alpha\to X$ is continuously differentiable.

ii) There exists a continuous nondecreasing function $\beta:[0,b]\to\mathbb{R}^+$ such that

$\|f(t,\varphi,\varphi')\|\le\beta(t)\|\varphi\|_\alpha$ for $(t,\varphi)\in[0,b]\times\mathcal{C}_\alpha$.

$(\mathbf{H}_2)$ $g:[0,b]\times\mathcal{C}_\alpha\to X_\alpha$ is continuously differentiable and for each $b>0$ there exist $0<L_g<1$ such that

$$|g(t,\varphi)-g(t,\psi)|_\alpha\le L_g\|\varphi-\psi\|_\alpha \text{ for every } t\in[0,b] \text{ and } \varphi,\psi\in\mathcal{C}_\alpha.$$

$(\mathbf{H}_3)$ A^{-1} is compact on X.

Theorem 3.5. *Assume that* $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$ *and* $(\boldsymbol{H}_3)$ *hold. Let* $\varphi \in \mathcal{C}_\alpha$ *such that* $\varphi(0) \in D(A)$, $\varphi'(0) - g(0,\varphi) \in E$ *and assume that*

$$L_g M_1 e^{\omega b} + \|(-A)^{\alpha-1}\| \sup_{t\in[0,b]} \Big[\big(\beta(t)(1+2Me^{\omega b}) + Me^{\omega b}\Big] < 1.$$

Then equation (1.1) *has at least one mild solution on* $[0,b]$.

Proof. Let $k > \|\varphi\|_{\mathcal{C}_\alpha}$, we define the following set

$$B_k = \{u \in C([0,b], X_\alpha) : \ u(0) = \varphi(0) \text{ and } |u|_\infty \leq k\},$$

where $|u|_\infty = \sup\limits_{t\in[0,b]} |u(t)|_\alpha$. For $u \in B_k$, define the $\widetilde{u}(t) : [-r,b] \to X_\alpha$ by

$$\widetilde{u}(t) = \begin{cases} u(t) \text{ for } t \in [0,b] \\ \\ \varphi(t) \text{ for } t \in [-r,0]. \end{cases}$$

The function $t \to \widetilde{u}_t$ is continuous from $[0,b]$ to $\mathcal{C}_\alpha$. Now, define the operator $\mathcal{K}$ on B_k by

$$\mathcal{K}(u)(t) = C(t)\varphi(0) + S(t)(\varphi'(0)g(0,\varphi)) + \int_0^t C(t-s)g(s,\widetilde{u}_s)ds + \int_0^t S(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds \text{ for } t \in [0,b].$$

It is sufficient to show that $\mathcal{K}$ has a fixed point in B_k. We give the proof in several steps.

Step 1: There is a positive $k > \|\varphi\|_\alpha$ such that $\mathcal{K}(B_k) \subset B_k$.

If not, then for each $k > \|\varphi\|_{\mathcal{C}_\alpha}$, there exist $u_k \in B_k$ and $t_k \in [0,b]$ such that $|(\mathcal{K}u_k)(t_k)|_\alpha > k$.

$$\begin{aligned}
& k < |(\mathcal{K}u_k)(t_k)|_\alpha \\
&= \Big| C(t_k)\varphi(0) + S(t_k)(\varphi'(0) - g(0,\varphi)) + \int_0^{t_k} C(t_k-s)g(s,\widetilde{u}_s)ds + \int_0^{t_k} S(t_k-s)f(s,\widetilde{u}_s)ds \Big|_\alpha \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)(\varphi'(0) - g(0,\varphi))|_\alpha + \Big\| -(-A)^{\alpha-1} \int_0^{t_k} AS(t_k-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds \Big\| \\
&\quad + \Big| \int_0^{t_k} \frac{d}{ds}\big(S(s)g(t_k-s,\widetilde{u}_{t_k-s})\big)ds - \int_0^{t_k} S(s)\frac{d}{ds}\big(g(t_k-s,\widetilde{u}_{t_k-s})\big)ds \Big|_\alpha \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)(\varphi'(0) - g(0,\varphi))|_\alpha \\
&\quad + \Big| \int_0^{t_k} \frac{d}{ds}\big(S(s)g(t_k-s,\widetilde{u}_{t_k-s})\big)ds - \int_0^{t_k} S(s)\frac{d}{ds}\big(g(t_k-s,\widetilde{u}_{t_k-s})\big)ds \Big|_\alpha \\
&\quad + \Big\| (-A)^{\alpha-1} \Big[\int_0^{t_k} \frac{d}{ds}\big(C(t_k-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds - \int_0^{t_k} C(t_k-s)\frac{d}{ds}\big(f(s,\widetilde{u}_s,\widetilde{u'}_s)\big) \Big] \Big\|
\end{aligned}$$

$$< |C(t_k)\varphi(0)|_\alpha + |S(t_k)(\varphi'(0) - g(0,\varphi))|_\alpha + |S(t_k)g(0,\widetilde{u}_0)|_\alpha + M_1 e^{\omega b}|g(t_k, \widetilde{u}_{t_k}) - g(0,\widetilde{u}_0)|_\alpha$$

$$+\|(-A)^{\alpha-1}\|\Big(\|f(t_k,\widetilde{u}_{t_k},\widetilde{u'}_{t_k})\| + \|C(t_k)f(0,\widetilde{u}_0,\widetilde{u'}_0)\| + Me^{\omega b}\|f(t_k,\widetilde{u}_{t_k},\widetilde{u'}_{t_k}) - f(0,\widetilde{u}_0,\widetilde{u'}_0)\|\Big)$$

$$< M_1 e^{\omega b}\Big(|\varphi(0)|_\alpha + |(\varphi'(0) - g(0,\varphi))|_\alpha\Big) + M_1 e^{\omega b}\sup_{s\in[0,b]}|g(s,0)|_\alpha + M_1 e^{\omega b} L_g\|\widetilde{u}_{t_k}\|_\alpha$$

$$+2M_1 e^{\omega b}|g(0,\varphi)|_\alpha + \|(-A)^{\alpha-1}\|\Big[\big(\beta(t_k) + Me^{\omega b}\big)\|\widetilde{u}_{t_k}\|_\alpha + 2Me^{\omega b}\beta(0)\|\widetilde{u}_0\|_\alpha\Big].$$

Since $\|\widetilde{u}_t\|_\alpha \leq k$ for all $t \in [0,b]$ and $u \in B_k$. Then we have

$$k < M_1 e^{\omega b}\Big(|\varphi(0)|_\alpha + |(\varphi'(0) - g(0\varphi))|_\alpha\Big) + M_1 e^{\omega b} L_g k + M_1 e^{\omega b}\sup_{s\in[0,b]}|g(s,0)|_\alpha + 2M_1 e^{\omega b}|g(0,\widetilde{u}_0)|_\alpha$$

$$+\|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\big(\beta(t)(1+2Me^{\omega b}) + Me^{\omega b}\Big]k.$$

Dividing above sides of above inequality by k, it follows that

$$1 < \frac{M_1 e^{\omega b}\Big(|\varphi(0)|_\alpha + |(\varphi'(0) - g(0,\varphi))|_\alpha\Big)}{k} + L_g M_1 e^{\omega b} + \frac{M_1 e^{\omega b}\sup_{s\in[0,b]}|g(s,0)|_\alpha}{k} + \frac{2M_1 e^{\omega b}|g(0,\varphi)|_\alpha}{k} +$$

$$+\|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\big(\beta(t)(1+2Me^{\omega b}) + Me^{\omega b}\Big].$$

When $k \to 0$, we have

$$1 < L_g M_1 e^{\omega b} + \|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\big(\beta(t)(1+2Me^{\omega b}) + Me^{\omega b}\Big],$$

which gives contradiction.

Step 2: $\mathcal{K}$ is continuous.

Let $(u^n)_n \subset B_k$ with $u^n \to u$ and $u'^n \to u'$ in B_k. Then, the set

$$\Delta = \{(s,\widetilde{u}^n_s,\widetilde{u'}^n_s)),\ (s,\widetilde{u}_s,\widetilde{u}_s)) :\ s\in[0,b],\ n\geq 1\}$$

and

$$\wedge = \{(s,\widetilde{u}^n_s),\ (s,\widetilde{u}_s) :\ s\in[0,b],\ n\geq 1\}$$

are compact respectively in $[0,b]\times\mathcal{C}_\alpha\times\mathcal{C}_\alpha$ and $[0,b]\times\mathcal{C}_\alpha$. Heine's theorem implies that f and g are uniformly continuous respectively in Δ and $\wedge$. Then, we have

$$
\begin{aligned}
&|\mathcal{K}(u^n)(t) - \mathcal{K}(u)(t)|_\infty \\
\leq & \sup_{t\in[0,b]} \Big| \int_0^t C(t-s)\Big(g(s,\widetilde{u}_s^n) - g(s,\widetilde{u}_s)\Big)ds\Big|_\alpha \\
&+ \sup_{t\in[0,b]} \Big\| -(-A)^{\alpha-1}\int_0^t AS(t-s)\Big(f(s,\widetilde{u}_s^n,\widetilde{u'}_s^n) - f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big)\Big\| \\
\leq & \sup_{t\in[0,b]} \Big| \int_0^t \frac{d}{ds}\Big(S(s)g(t_k-s,\widetilde{u}_{t_k-s}^n) - g(t_k-s,\widetilde{u}_{t_k-s})\Big)ds \\
&- \int_0^t S(s)\frac{d}{ds}\Big(g(t_k-s,\widetilde{u}_{t_k-s}^n) - g(t_k-s,\widetilde{u}_{t_k-s})\Big)ds\Big|_\alpha \\
&+ \sup_{t\in[0,b]} \Big\|(-A)^{\alpha-1}\Big[\int_0^t \frac{d}{ds}\big(C(t-s)f(s,\widetilde{u}_s^n,\widetilde{u'}_s^n) - f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds \\
&- \int_0^t C(t-s)\frac{d}{ds}\big(f(s,\widetilde{u}_s^n,\widetilde{u'}_s^n) - f(s,\widetilde{x}_s,\widetilde{u'}_s)ds\big)\Big]\Big\| \\
\leq & \sup_{t\in[0,b]} \Big[|g(0,\widetilde{u}_0^n) - g(0,\widetilde{u}_0)|_\alpha + M_1e^{\omega b}\Big(|g(0,\widetilde{u}_0^n) - g(0,\widetilde{u}_0)|_\alpha + |g(t,\widetilde{u}_t^n) - g(t,\widetilde{u}_t)|_\alpha\Big] \\
&+ \sup_{t\in[0,b]} \|(-A)^{\alpha-1}\|\Big[\big(f(t,\widetilde{u}_t^n,\widetilde{u'}_t^n) - f(t,\widetilde{u}_t,\widetilde{u'}_t)\big) - C(t)\big(f(0,\widetilde{u}_0^n,\widetilde{u'}_0^n) - f(0,\widetilde{u}_0,\widetilde{u'}_0)\big)\big)\| \\
&+ Me^{\omega b}\|f(t,\widetilde{u}_t^n,\widetilde{u'}_t^n) - f(t,\widetilde{u}_t,\widetilde{u'}_t)\big) - \big(f(0,\widetilde{u}_0^n,\widetilde{u'}_0^n) - f(0,\widetilde{u}_0,\widetilde{u'}_0)\big)\big)\Big] \\
\leq & \sup_{t\in[0,b]} \Big[(1+Me^{\omega b})|g(0,\widetilde{u}_0^n) - g(0,\widetilde{u}_0)|_\alpha + M_1e^{\omega b}|g(t,\widetilde{u}_t^n) - g(t,\widetilde{u}_t)|_\alpha\Big] \\
&+ \sup_{t\in[0,b]} \|(-A)^{\alpha-1}\|\Big[(1+Me^{\omega b})\|f(t,\widetilde{u}_t^n,\widetilde{u'}_t^n) - f(t,\widetilde{u}_t,\widetilde{u'}_t)\| \\
&+ 2Me^{\omega b}\|f(0,\widetilde{u}_0^n,\widetilde{u'}_0^n) - f(0,\widetilde{u}_0,\widetilde{u'}_0)\|\Big] \to 0 \text{ as } n\to\infty,
\end{aligned}
$$

and this yield the continuity of $\mathcal{K}$ on B_k.

Step 3: The set $\{\mathcal{K}(u)(t) : \ u \in B_k\}$ is relatively compact for each $t \in]0,b]$.

Let $t \in]0,b]$ be fixed and $\gamma > 0$ be such that $\alpha < \gamma < 1$. Using the same reasoning like previously, it follows that

$$
\begin{aligned}
\|(-A)^\gamma\mathcal{K}(u)\| \leq \|(-A)^{\gamma-1}\|\Big[M_1e^{\omega b}\Big(\|A\varphi(0)\| + \|A(\varphi'(0) - g(0,\varphi))\| + \sup_{t\in[0,b]}\Big[\big(\beta(t)(1+2Me^{\omega b}) + Me^{\omega b}\Big]k \\
+ M_1e^{\omega b}\Big[L_gk + \sup_{s\in[0,b]}|g(s,0)|_\gamma + |g(0,\varphi)|_\gamma\Big] < \infty.
\end{aligned}
$$

Consequently for $t \in]0,b]$ fixed, the set $\{(-A)^\gamma\mathcal{K}(u)(t) : \ u \in B_k\}$ is bounded in X. By $(\mathbf{H}_3)$, we deduce that $(-A)^{-\gamma} : X \to X_\alpha$ is compact. It follows that the set $\{\mathcal{K}(u)(t) : \ u \in B_k\}$ is relatively compact for each $t \in]0,b]$ in X_α.

Step 4: The set $\{\mathcal{K}(u):\ u\in B_k\}$ is an equicontinuous family of functions.

Let $u\in B_k$ and $0\le\tau_1<\tau_2\le b$ then, we have

$$\begin{aligned}|\mathcal{K}(u)(\tau_2)-\mathcal{K}(u)(\tau_1)|_\alpha &\le |[C(\tau_2)-C(\tau_1)]\varphi(0)|_\alpha+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-g(0,\varphi))|_\alpha\\ &+\Big|\int_0^{\tau_2}C(\tau_2-s)g(s,\widetilde{u}_s)ds-\int_0^{\tau_1}C(\tau_1-s)g(s,\widetilde{u}_s)ds\Big|_\alpha\\ &+\Big|\int_0^{\tau_2}S(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds-\int_0^{\tau_2}S(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big|\\ &\le |[C(\tau_2)-C(\tau_1)](\varphi(0)-g(0,\varphi))|_\alpha+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-\eta)|_\alpha\\ &+\Big|\int_0^{\tau_1}[C(\tau_2-s)-C(\tau_1-s)]g(s,\widetilde{u}_s)ds-\int_{\tau_1}^{\tau_2}[C(\tau_2-s)g(s,\widetilde{u}_s)ds\Big|_\alpha\\ &+\Big|\int_0^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]f(s,\widetilde{x}_s,\widetilde{u'}_s)ds\Big|\\ &+\Big|\int_{\tau_2}^{\tau_2}S(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big|,\end{aligned}$$

it follows that

$$\begin{aligned}&|\mathcal{K}(u)(\tau_2)-\mathcal{K}(u)(\tau_1)|_\alpha\\ \le\ &|[C(\tau_2)-C(\tau_1)]\varphi(0)|_\alpha+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-g(0,\varphi))|_\alpha\\ &+\Big|\int_0^{\tau_1}\frac{d}{ds}\Big([S(\tau_2-s)-S(\tau_1-s)]g(s,\widetilde{u}_s)\Big)ds-\int_0^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]\frac{d}{ds}g(s,\widetilde{u}_s)ds\Big|_\alpha\\ &+\Big|\int_{\tau_1}^{\tau_2}\frac{d}{ds}\big(S(\tau_2-s)g(s,u_s)\big)ds-\int_{\tau_1}^{\tau_2}S(\tau_2-s)\frac{d}{ds}\big(g(s,u_s)\big)ds\Big|_\alpha\\ &+\Big\|(-A)^{\alpha-1}\Big[\int_0^{\tau_1}\frac{d}{ds}\Big([C(\tau_2-s)-C(\tau_1-s)]f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\\ &-\int_0^{\tau_1}[C(\tau_2-s)-C(\tau_1-s)]\frac{d}{ds}\big(]f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds\Big\|\\ &+\Big\|(-A)^{\alpha-1}\int_{\tau_1}^{\tau_2}\frac{d}{ds}\big(C(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds-\int_{\tau_1}^{\tau_2}C(\tau_2-s)\frac{d}{ds}\big(f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds\Big\|.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&|\mathcal{K}(u)(\tau_2)-\mathcal{K}(u)(\tau_1)|_\alpha\\ \le\ &|[C(\tau_2)-C(\tau_1)]\varphi(0)|_\alpha+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-g(0,\varphi))|_\alpha+|(S(\tau_2-\tau_1)g(\tau_1,\widetilde{u}_{\tau_1})|_\alpha\\ &+\|S(\tau_2)-S(\tau_1)\|\|g(0,\widetilde{u}_0)|_\alpha+|S(\tau_2)-S(\tau_1)\|\|(g(\tau_1,\widetilde{u}_{\tau_1}))-(g(0,\widetilde{u}_0))|_\alpha\end{aligned}$$

$$+M_1 e^{\omega b}|g(\tau_2,\widetilde{u}_{\tau_2})-g(\tau_1,\widetilde{u}_{\tau_1})|_\alpha+\|(-A)^{\alpha-1}\|\Big[\|(C(\tau_2-\tau_1)-I)f(\tau_1,\widetilde{x}_{\tau_1},\widetilde{u'}_{\tau_1})\|$$

$$+\|[C(\tau_2)-C(\tau_1)]f(0,\widetilde{u}_0,\widetilde{u'}_0)\|+\|f(\tau_2,\widetilde{u}_{\tau_2},\widetilde{u'}_{\tau_2})-C(\tau_2-\tau_1)f(\tau_1,\widetilde{u}_{\tau_1},\widetilde{u'}_{\tau_1})\|$$

$$+Me^{\omega b}\|f(\tau_2,\widetilde{u}_{\tau_2},\widetilde{u'}_{\tau_2)}-f(\tau_1,\widetilde{u}_{\tau_1},\widetilde{u'}_{\tau_1})\Big]\to 0 \text{ as } \tau_1\to\tau_2.$$

Since $(-A)^{\alpha-1}$ is compact from X to X and $(C(t)_{t\in\mathbb{R}})$ is uniformly continuous on compact subset of X. Thus $\mathcal{K}$ maps B_k into an equicontinuous family of functions.
So from **Step 1** to **Step 4** and by Ascoli-Arzela theorem, we can conclude that $\mathcal{K}:B_k\to B_k$ is completely continuous. Hence by Schauder's fixed point theorem, we conclude that $\mathcal{K}$ has least one fixed point in B_k which is a mild solution of equation (1.1) on $[0,b]$. ■

Our next objective is to prove the uniqueness of mild solution. For this purpose formulate the followings assumptions

($\mathbf{H}_4$): $f:[0,b]\times\mathcal{C}_\alpha\times\mathcal{C}_\alpha\to X$ is continuously differentiable and locally Lipschitzian with the respect on second variable. Then there exists $c_0(r)>0$ such that for $\varphi,\ \psi\in\mathcal{C}_\alpha$ with $\|\varphi\|_{\mathcal{C}_\alpha},\ \|\psi\|_{\mathcal{C}_\alpha}\le r$, we have

$$\|f(t,\varphi_1,\varphi_1',)-f(t,\varphi_2,\varphi_2',)\le c_0(r)\|\varphi_1-\varphi_2\|_{\mathcal{C}_\alpha} \text{ for } \in[0,b],\ \varphi_1,\varphi_2\in\mathcal{C}_\alpha.$$

($\mathbf{H}_5$) The maps $t\mapsto AC(t)$ is locally bounded.

Theorem 3.6. *Assume that ($\boldsymbol{H}_0$), ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$), ($\boldsymbol{H}_4$) and ($\boldsymbol{H}_5$) hold. Let $\varphi\in\mathcal{C}_\alpha$ such that $\varphi(0)\in D(A)$ and $\varphi'(0)-g(0,\varphi)\in E$. Assume that*

$$\Big[L_g\big(1+(Me^{\omega b}+\mu b)b\big)+\|(-A)^{\alpha-1}\|\mu c_0(r)b(1+b)\Big]<1.$$

Then Equation (1.1) *has unique mild solution.*

Proof. Let us consider the following set

$$\mathbb{F}(\varphi)=\{u\in C^1([0,b]),X_\alpha):u(0)=\varphi(0)\}.$$

For $u\in\mathbb{F}(\varphi)$ we define $\widetilde{u}:[-r,b]\to X_\alpha$ by

$$\widetilde{u}(t)=\begin{cases}u(t) \text{ for } t\in[0,b]\\ \varphi(t) \text{ for } t\in[-r,0].\end{cases}$$

Now, we define the operator $\Phi:\mathcal{F}(\varphi)\to\mathcal{F}(\varphi)$ by

$$\Phi(u)(t)=C(t)\varphi(0)+S(t)(\varphi'(0)-g(0,\varphi))+\int_0^t C(t-s)g(s,\widetilde{u}_s)ds+\int_0^t S(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds \text{ for } t\in[0,b].$$

We will show that Φ is a strict contraction. Let $u,v\in\mathbb{F}(\varphi)$ and μ be a positive real number such that $\|AC(t)\|\le\mu$ for $t\in[0,b]$. Then we have

$$\Phi(u)(t)-\Phi(v)(t)=\int_0^t C(t-s)[g(s,\widetilde{u}_s)-g(s,\widetilde{u}_s)]ds+\int_0^t S(t-s)[f(s,\widetilde{u}_s,\widetilde{u'}_s)-f(s,\widetilde{v}_s,\widetilde{v'}_s)]ds.$$

Then

$$\begin{aligned}
&|\Phi(u)(t)-\Phi(v)(t)|_\alpha\\
&\leq \Big|\int_0^t C(t-s)[g(s,\widetilde{u}_s)-g(s,\widetilde{u}_s)]ds\Big|_\alpha+\Big|\int_0^t S(t-s)[f(s,\widetilde{u}_s,\widetilde{u'}_s)-f(s,\widetilde{v}_s,\widetilde{v'}_s)]ds\Big|_\alpha\\
&\leq \Big|\int_0^t \Big(C(t-s)[g(s,\widetilde{u}_s)-g(s,\widetilde{u}_s)]\Big)ds\Big|_\alpha+\Big|\int_0^t\Big(\int_0^{t-s} C(\sigma)[f(s,\widetilde{u}_s,\widetilde{u'}_s)-f(s,\widetilde{v}_s,\widetilde{v'}_s)]d\sigma\Big)ds\Big|_\alpha\\
&\leq Me^{\omega b}\int_0^t\Big|g(s,\widetilde{u}_s)-g(s,\widetilde{u}_s)]\Big|_\alpha ds+\|(-A)^{\alpha-1}\|\mu b\int_0^t\|f(s,\widetilde{u}_s,\widetilde{u'}_s)-f(s,\widetilde{v}_s,\widetilde{v'}_s)\|ds\\
&\leq \Big(Me^{\omega b}Lgb+\|(-A)^{\alpha-1}\|\mu b^2c_0(r)\Big)\|u-\|_{\mathcal{C}_\alpha},
\end{aligned}$$

it follows that

$$|\Phi(u)(t)-\Phi(v)(t)|_\alpha\leq\Big(Me^{\omega b}Lgb+\|(-A)^{\alpha-1}\|\mu b^2c_0(r)\Big)\|u-\|_{\mathcal{C}_\alpha} \tag{3.2}$$

On the other hand, by use of Equation (2.1) and Proposition 2.3, we have

$$\begin{aligned}
(\phi(u))'(t)=AS(t)\varphi(0)+C(t)(\varphi'(0)-g(0,\varphi))+g(t,u_t)+\int_0^t AS(t-s)g(s,\widetilde{u}_s)ds\\
+\int_0^t C(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds.
\end{aligned}$$

Using the same reasoning like previously, then we have

$$|(\Phi(u))'(t)-(\Phi(v))'(t)|_\alpha\leq\Big[L_g+\mu L_g b^2+\|(-A)^{\alpha-1}\|\mu c_0(r)b\Big]\|u-v\|_{\mathcal{C}_\alpha}. \tag{3.3}$$

Adding equation (3.2) and equation (3.3), then we have

$$\|\Phi(u)(t)-\Phi(u)(t)\|_{\mathcal{C}_\alpha}\leq\Big[L_g\big(1+(Me^{\omega b}+\mu b)b\big)+\|(-A)^{\alpha-1}\|\mu c_0(r)b(1+b)\Big]\|u-v\|_{\mathcal{C}_\alpha}.$$

This means Φ is a strict contraction.Thus by Banach's fixed point theorem, we deduce that Φ has a unique fixed point in $\mathbb{F}(\varphi)$. Then Equation(1.1) has a unique mild solution on $[0,b]$ ■

4. Existence of strict solutions

Theorem 4.1. *Assume that* $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$ *and* $(\boldsymbol{H}_5)$ *hold and* f *is continuously differentiable. Moreover assume that the partial derivatives* D_1f *and* D_2f *are locally lipschitz in classical sens. Let* $\varphi\in C^3([-r,0],D((-A)^\alpha))$ *such that* $\varphi(0),\ \varphi''(0)\in D(A)$ *and* $\varphi'(0)-g(0,\varphi),\ \varphi^{(3)}(0)\in E$ *and*

$$\varphi''(0)-D_tg(0,\varphi)-D_\varphi g(0,\varphi)\varphi'=A\varphi(0)+f(\varphi,\varphi').$$

Then the corresponding of mild solution u *becomes a strict solution of equation* (1.1) *on* $[0,b]$.

Proof Let $\varphi\in C^3([-r,0],D((-A)^\alpha))$ such that $\varphi(0),\ \varphi''(0)\in D(A)$, $\varphi'(0)-g(0,\varphi),\ \varphi^{(3)}(0)\in E$ and

$\varphi''(0) - D_t g(0,\varphi) - D_\varphi g(0,\varphi)\varphi' = A\varphi(0) + f(\varphi,\varphi')$.

Let u be the corresponding mild solution of equation (1.1) which is defined on $[0,b]$. Consider

$$\begin{cases} v(t) = C(t)\Big[A\varphi(0) + f(\varphi,\varphi')\Big] + S(t)A(\varphi'(0) - g(0,\varphi)) \\ +[D_1 g(t,u_t) + D_2 g(t,u_t)u_t'] + \displaystyle\int_0^t AC(t-s)g(s,u_s)ds \\ \qquad\qquad + \displaystyle\int_0^t C(t-s)[D_1 f(u_s,u_s')u_s' + D_2 f(u_s,u_s')v_s]ds \\ v_0 = \varphi''. \end{cases}$$

Now, we define w by

$$\begin{cases} w(t) = \varphi'(0) + \displaystyle\int_0^t v(s)ds \text{ if } t \in [0,b] \\ w(t) = \varphi'(t) \ \text{ if } -r \le t \le 0 \\ w'(t) = \varphi''(t) \ \text{ if } -r \le t \le 0. \end{cases} \tag{4.1}$$

Then we can see that $w_t = \varphi' + \displaystyle\int_0^t v_s ds$ for $t \in [0,b]$.

Consequently the map $t \mapsto w_t$ and $t \mapsto \displaystyle\int_0^t C(t-s)f(u_s,w_s)ds$ are continuously differentiable. Then we have

$$\begin{aligned} \frac{d}{dt}\int_0^t C(t-s)f(u_s,w_s)ds &= \frac{d}{dt}\int_0^t C(s)f(u_{t-s},w_{t-s})ds \\ &= C(t)f(u_0,w_0) + \int_0^t C(t-s)\Big[D_1 fs(u_s,w_s)u_s' + D_2 f(u_s,w_s)v_s\Big]ds \\ &= C(t)f(\varphi,\varphi') + \int_0^t C(t-s)\Big[D_1 fs(u_s,w_s)u_s' + D_2 f(u_s,w_s)v_s\Big]ds, \end{aligned}$$

it follows that

$$\int_0^t C(s)f(\varphi,\varphi')ds = \int_0^t C(t-s)f(u_s,u_s')ds - \int_0^t\int_0^s C(s-\tau)\Big[D_1 f(u_\tau,w_\tau)u_\tau' + D_f(u_\tau,w_\tau)v_\tau\Big]d\tau ds.$$

On other hand by Lemma 2.7 one has

$$\int_0^t\int_0^s AC(s-\tau)g(\tau,u_\tau)d\tau ds = \int_0^t Aq'(s)ds = Aq(t) = \int_0^t AS(t-s)g(s,u_s)ds.$$

Consequently we have

$$w(t) = \varphi'(0) + \int_0^t S(s)A(\varphi'(0) - g(0,\varphi))ds + \int_0^t C(s)A\varphi(0)ds + \int_0^t C(t-s)f(u_s, w_s)ds + g(t,u_t) - g(0,\varphi)$$
$$- \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, w_\tau)u'_s + D_2 f(u_\tau, w_\tau)v_\tau\Big]d\tau ds$$
$$+ \int_0^t AS(t-s)g(s,u_s)ds + \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, u_\tau)u'_\tau + D_2 f(u_\tau, u_\tau)v_\tau\Big]d\tau ds.$$

Moreover by Lemma 2.3, we have

$$\int_0^t C(s)A\varphi(0)ds = S(t)A\varphi(0)$$

$$\int_0^t S(s)A(\varphi'(0) - g(0,\varphi))ds = C(t)(\varphi'(0) - g(0,\varphi) - (\varphi'(0) - g(0,\varphi)).$$

It follows that

$$w(t) = \varphi'(0) + C(t)(\varphi'(0) - g(0,\varphi)) + S(t)A\varphi(0) - (\varphi'(0) - g(0,\varphi)) + g(t,u_t) - g(0,\varphi)$$
$$+ \int_0^t AS(t-s)g(s,u_s)ds + \int_0^t C(t-s)f(u_s, w_s)ds$$
$$+ \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, u'_\tau)u'_s + D_2 f(u_\tau, u'_\tau)v_\tau\Big]d\tau ds$$
$$- \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, w_\tau)u'_\tau + D_2 f(u_\tau, w_\tau)v_\tau\Big]d\tau ds.$$

Furthermore for $t \geq 0$, we know that

$$u'(t) = AS(t)\varphi(0) + C(t)(\varphi'(0) - g(0,\varphi)) + g(t,u_t) + \int_0^t AS(t-s)g(s,u_s)ds + \int_0^t C(t-s)f(u_s, u'_s)ds,$$

then for $t \in [0,b]$, we have

$$u'(t) - w(t) = \int_0^t C(t-s)[f(u_s, u'_s) - f(u_s, w_s)]ds + \int_0^t \int_0^s C(s-\tau)\Big[(D_1 f(u_\tau, u'_\tau) - D_1 f(u_\tau, u'_\tau))u'_\tau$$
$$+(D_2 f(u_\tau, u'_\tau) - D_2 f(u_\tau, w_\tau))v_\tau d\tau\Big]ds.$$

$$|u'(t) - w(t)|_\alpha$$
$$\leq \int_0^t |C(t-s)[f(u_s, u'_s) - f(s, u_s, w_s)]|_\alpha ds + \int_0^t \int_0^s |C(s-\tau)(D_1 f(u_\tau, u'_\tau) - D_1 f(u_\tau, w_\tau))u'_\tau|_\alpha d\tau ds$$
$$+ \int_0^t \int_0^s |C(s-\tau)(D_2 f(u_\tau, u'_\tau) - D_2 f(u_\tau, w_\tau))v_\tau|_\alpha d\tau ds. \tag{4.2}$$

Let us choose $F = \Big\{u'_s, w_s : s \in [0,b]\Big\}$. Then F is compact set. It follows that $D_1 f$ and $D_2 f$ are globally Lipschitz on F. Let $L_1 > 0$ be such that for $t \in [0,b]$ and $x, y, x', y' \in H$, then we have

$$\|f(x,x')-f(x,y')\| \leq L_1\|x'-y'\|_\alpha$$
$$\|D_1f(x,x')-D_1f(x,y')\| \leq L_1\|x'-y'\|_\alpha$$
$$\|D_2f(x,x')-D_2f(x,y')\| \leq L_1\|x'-y'\|_\alpha.$$

Consequently, using equation (4.2), we one can find a positive Constance $k(b)$ such that by Gronwall's lemma,

$$\|u(t)-w(t)\|_\alpha \leq k(b)\int_0^t \|u'_s-w_s\|_\alpha ds,$$

then we deduce that $u'=w$. Consequently, we deduce that the mild solution is twice continuous differentiable from $[0,b]$ to X_α. Then functions $t\to g(t,u_t)$ and $t\to f(t,u_t,u'_t)$ are continuously differentiable on $[0,b]$. According to the Theorem 2.5, we conclude that u is a strict solution of equation (1.1) on $[0,b]$. ■

5. Application

For our illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} \frac{\partial}{\partial}[z'(t,x)-\int_{-r}^0 k(t,z(t+\theta,x))d\theta] = \frac{\partial^2}{\partial x^2}z(t,x)] \\ +\int_{-r}^0 h(t,\frac{\partial}{\partial x}z(t+\theta,x),\frac{\partial}{\partial x}z'(t+\theta,x))d\theta \text{ for } t\geq 0 \text{ and } x\in[0,\pi] \\ z(t,0)-\int_{-r}^0 k(t,z(t+\theta,x))d\theta = 0 \text{ for } t\geq 0 \\ z(t,\pi)-\int_{-r}^0 k(t,z(t+\theta,x))d\theta = 0 \\ z(\theta,x)=\varphi_0(\theta)(x) \text{ for } \theta\in[-r,0] \text{ and } x\in[0,\pi], \end{cases}$$

where $h:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and there exists a positive constant L such that for $x,y,x_1,y_1\in\mathbb{R}$,

$$|h(t,x,y)-h(t,x_1,y_2)| \leq L\Big(|x-x_1|+|y-y_1|\Big).$$

we can choose for example

$$h(t,x,y)=e^{-t^2}[\sin(\frac{x}{2})+\sin(\frac{y}{2})] \text{ for } (\theta,x,y)\in\mathbb{R}^-\times\mathbb{R}\times\mathbb{R}.$$

we can observe that

$$|h(t,x_1,y_1)-h(t,x_2,y_2)| \leq \frac{1}{2}\Big(|x_1-x_2|+|y_1-y_2|\Big)$$

and $k:\mathbb{R}^-\times\mathbb{R}\longrightarrow\mathbb{R}$ is Lipschizian with respect to the second argument.
In the order to rewrite equation (5.1) in the abstract form, we introduce the space $X=L^2([0,\pi];\mathbb{R})$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x\in X$,

$$\|x\|_{L^2}=\Big(\int_0^\pi |x(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Let $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $x \in [0,\pi]$, $n \geq 1$, then $(e_n)_{n\geq 1}$ is an orthogonal base for X.
Let $A : X \to X$ be defined by

$$\begin{cases} Ay = y'' \\ \\ D(A) = \big\{y \in X : y, y' \text{ are absolutely continuous},\ y'' \in X,\ y(0) = y(\pi) = 0\big\} \end{cases}$$

Then the operator is computed by

$$Ay = \sum_{n=1}^{+\infty} -n^2(y, e_n)e_n,\ y \in D(A),$$

where

$$(u, v) = \int_0^{\pi} u(s)v(s)ds \ \text{ for } u, v \in X.$$

It is well known that A is the infinitesimal generator of strongly continuous cosine family $C(t)$, $\in \mathbb{R}$ in X which is given by

$$C(t)y = \sum_{n=1}^{+\infty} \cos nt(y, e_n)e_n,\ y \in X$$

and that the associated sine family is given by

$$S(t)y = \sum_{n=1}^{+\infty} \frac{1}{n}\sin nt(y, e_n)e_n,\ y \in X.$$

If we choose $\alpha = \frac{1}{2}$. then $(\mathbf{H}_0)$ is satisfied since

$$(-A)^{\frac{1}{2}}y = \sum_{n=1}^{+\infty}(y, e_n)e_n,\ y \in D((-A)^{\frac{1}{2}}).$$

and

$$(-A)^{-\frac{1}{2}}y = \sum_{n=1}^{+\infty}\frac{1}{n}(y, e_n)e_n,\ y \in X.$$

From [4], the compactness of A^{-1} follows from Lemma 2.6 and the fact that the eigenvalues of $(-A)^{-\frac{1}{2}}$ are $\lambda_n = \frac{1}{n}$, $\ n = 1, 2 \ldots$, the $(\mathbf{H}_3)$ is satisfied.

We define the space

$$\mathcal{C}_{\frac{1}{2}} = C^1([-r, 0], X_{\frac{1}{2}}),$$

where $C^1([-r, 0], X_{\frac{1}{2}})$ is the space of bounded uniformly continuous differentiable from $[-r, 0]$ into $X_{\frac{1}{2}}$, where $X_{\frac{1}{2}}$ is endowed with the norm

$$|\varphi|_{\frac{1}{2}} = \sup_{-r\leq\theta\leq 0}|\varphi(\theta)|.$$

Let $f : \mathbb{R} \times \mathcal{C}_{\frac{1}{2}} \times \mathcal{C}_{\frac{1}{2}} \longrightarrow X$ and $g : \mathbb{R} \times \mathcal{C}_{\frac{1}{2}}$ define by

$$f(t, \varphi, \varphi')(x) = \int_{-r}^{0} h(t, \frac{\partial}{\partial x}\varphi(\theta)(x), \frac{\partial}{\partial x}\varphi'(\theta)(x))d\theta \text{ for } x \in [0, \pi], t \geq 0,\ \varphi, \in \mathcal{C}_{\frac{1}{2}}$$

and

$$g(t,\varphi,)(x)=\int_{-r}^{0}k(t,\varphi(\theta)(x))d\theta \text{ for } x\in[0,\pi], t\geq 0,\ \varphi,\in\mathcal{C}_{\frac{1}{2}}$$

where $\varphi,\in\mathcal{C}_{\frac{1}{2}}$ define by

$$\varphi(\theta)(x)=\varphi_0(\theta,x)$$

and the norm in $\mathcal{C}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\mathcal{C}_{\frac{1}{2}}}=\sup_{\theta\in[-r,0]}\Big(\int_0^\pi\big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]\big|^2dx\Big)^{\frac{1}{2}}+\sup_{\theta\in[-r,0]}\Big(\int_0^\pi\big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]\big|^2dx\Big)^{\frac{1}{2}}.$$

Let us pose $v(t)=z(t,x)$. Then equation (5.1) takes the following abstract form

$$\begin{cases}\dfrac{d}{dt}[v'(t)-g(t,v_t)]=Av(t)+f(t,v_t,v'_t) \text{ for } t\geq 0\\ v_0=\varphi\in\mathcal{C}_{\frac{1}{2}}\\ v'_0=\varphi'\in\mathcal{C}_{\frac{1}{2}}.\end{cases}\tag{5.1}$$

From [4], for all $y\in X_{\frac{1}{2}}$, y is absolutely continuous and $|y|_{\frac{1}{2}}=|y|_{L^2}$ Let $\varphi,\psi\in C^1([-r,0],X_{\frac{1}{2}})$, since $|h(t,x_1,y_1)-h(t,x_2,y_2)|\leq\frac{1}{2}\Big(|x_1-x_2|+\|y_1-y_2|\Big)$, we have

$$\begin{aligned}|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2}&\leq\Big(\int_0^\pi\Big(\int_{-r}^0h(t,\frac{\partial}{\partial x}[\varphi(\theta)(x)],\frac{\partial}{\partial x}[\varphi'(\theta)(x)]d\theta\Big)\\&\quad+\Big(\int_0^\pi\Big(\int_{-r}^0h(t,\frac{\partial}{\partial x}[\psi(\theta)(x)],\frac{\partial}{\partial x}[\psi'(\theta)(x)])d\theta\Big)^2dx\Big)^{\frac{1}{2}}\\&\leq\frac{1}{2}r\Big[\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\\&\quad+\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\Big].\end{aligned}$$

By Minkowski's Lemma, we have

$$\begin{aligned}|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2}&\leq\frac{1}{2}r\Big[\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\\&\quad+\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\Big]\\&\leq\frac{1}{2}r\Big[\sup_{\theta\in[-r,0]}\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\\&\quad+\sup_{\theta\in[-r,0]}\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\Big],\end{aligned}$$

which implies that

$$|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2}\leq\frac{1}{2}r\|\varphi-\psi\|_{\mathcal{C}_{\frac{1}{2}}}.$$

Consequently the function f satisfies $(\mathbf{H}_4)$.

$(\mathbf{H}_7)$ $0 < rL_k < 1$.
We claim that g is a contraction function with respect to the second argument with value in $X_{\frac{1}{2}}$. Indeed let $\varphi_1, \varphi_2 \in \mathcal{C}_{\frac{1}{2}}$ and L_k the constant Lipschitz of k. Then we have

$$|g(t,\varphi) - g(t,\psi)|_{\frac{1}{2}} \leq rL_k \|\varphi - \psi\|_{\mathcal{C}_{\frac{1}{2}}}.$$

Then, assumption $(\mathbf{H}_7)$ implies that g is a strict contraction. Moreover the boundedness of $(-A)^{-\frac{1}{2}}$ implies that g stays in $X_{\frac{1}{2}}$. Consequently g satisfies $(\mathbf{H}_2)$.

We have the following result.

Proposition 5.1. *Under the above assumptions, equation* (5.1) *has a unique mild solution which is defined for all* $t \geq 0$*.*

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Conclusion

In this paper we study the existence and regularity of solutions for some nonlinear neutral functional differential equations with finite delay by use of the cosine family theory. Some results of this study when the delay is infinite will be presented in next works.

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